L'équation de Hamlilton-Jacobi en contrôle optimal : dualité et géodésiques

Chadi Nour

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L'équation de Hamilton-Jacobi en contrôle optimal : dualité et géodésiques

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À mes parents

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Notation

$\mathbb{R}^n$	Euclidean n-dimensional space
$\|\cdot\|$	Euclidean norm
$\langle\cdot,\cdot\rangle$	Euclidean inner product
$\mathcal{D}$	The diagonal $\{(\alpha,\alpha) : \alpha \in \mathbb{R}^n\}$
B (resp. $\bar{B}$)	Open (resp. closed) unit ball in $\mathbb{R}^n$
$B(x_0;\rho)$	The open ball $\{x \in \mathbb{R}^n : \|x_0 - x\| < \rho\}$
$\bar{B}(x_0;\rho)$	The closed ball $\{x \in \mathbb{R}^n : \|x_0 - x\| \leq \rho\}$
$\operatorname{int} S$	Interior of the set S
$\bar{S}$	Closure of the set S
$\operatorname{co} S$	Convex hull of the set S
$\overline{\operatorname{co}}\, S$	Closed convex hull of the set S
$\operatorname{comp} S$	Complement of the set S
∂S	Boundary of the set S
$d_S(x)$	Euclidean distance of x from S

$\mathrm{proj}_S(x)$	The set of all points of S which realize the distance between x and S
$I_S(x)$	Indicator function of the set S
$N_S^P(s)$	Proximal normal cone to S at s
$N_S^L(s)$	Limiting normal cone to S at s
$\mathrm{dom}\, f$	Effective domain of f
$\mathrm{epi}\, f$	Epigraph of f
$\mathcal{F}(U)$	The set of all functions $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ which are lower semicontinuous and not indentically $+\infty$ on the open set U
$\partial_P f(x)$ (resp. $\partial^P f(x)$)	Proximal subdifferential (resp. superdifferential) of f at x
$\partial_L f(x)$	Limiting subdifferential of f at x
$\partial f(x)$	Generalized gradient of f at x
$f'(x; v)$	Directional derivative of f at x in direction v
$f^\circ(x; v)$	Generalized directional derivative of f at x in direction v
$f'_G(x)$	Gâteaux derivative to f at x
$f'(x)$	Fréchet derivative of f at x
$C^1(\Omega)$ (resp. $C^2(\Omega)$)	The space of continuously (resp. twice continuously) differentiable functions from the open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}$
h (resp. H)	Lower (resp. Upper) Hamiltonian
iff	If and only if

Abstract

The Hamliton-Jacobi equation in optimal control: duality and geodesics

Abstract. The main object of this thesis is the application of new methods from nonsmooth analysis and which use the Hamilton-Jacobi equation for the study of certain problems in control theory. There are three parts in our work:
• In the first part we develop a new duality result in control theory. This result generalizes, in a number of ways, the Vinter's duality (1993) and gives a new characterization of the minimal time function.
• The second part is devoted to the study of the Hamilton-Jacobi equation of minimal time, but in a domain which contains the origin. We prove the existence of (minimal) solutions of this equation and we show that these solutions are closely linked to global geodesics trajectories.
• In the third part, we study the existence of minimal loop trajectories for a control system. We give a necessary and sufficient conditions for the existence of this type of trajectories at a given point.

Key words. optimal control, nonsmooth analysis, Hamilton-Jacobi equation, duality, minimal time function, optimal trajectories.

L'équation de Hamlilton-Jacobi en contrôle optimal : dualité et géodésiques

Résumé. L'objet principal de cette thèse est l'application de méthodes nouvelles inspirées de l'analyse non-lisse et impliquant l'équation Hamilton-Jacobi pour l'étude de certains problèmes en théorie du contrôle. Notre travail se compose de trois parties :
• La première partie est consacrée à la généralisation d'un résultat célèbre de R. Vinter (1993) qui porte sur la dualité non-convexe en contrôle optimal. Entre autre, ceci mène à une nouvelle caractérisation de la fonction temps minimal.
• Dans la deuxième partie, nous étudions l'équation classique d'Hamilton-Jacobi de la fonction temps minimal mais dans un domaine contenant l'origine. Nous démontrons l'existence de solutions et même d'une solution minimale de cette équation, et établissons des liens avec les trajectoires géodésiques.
• La dernière partie de cette thèse est consacrée à l'étude des boucles minimales pour les systèmes de contrôle. Nous donnons des conditions nécessaires et suffisantes pour l'existence de ces boucles en un point donné.

Mots-clés. contrôle optimal, analyse non-lisse, équations de Hamilton-Jacobi, dualité, fonction temps minimal, trajectoires optimales.

Introduction

L'analyse non-lisse concerne la description locale des fonctions non différentiables et des ensembles avec frontières non différentiables, en termes de généralisations des concepts classiques de dérivés, de normales et de tangentes. Depuis son apparition au début des années 70, il y a une relation cruciale entre l'analyse non-lisse et la théorie de la commande optimale. L'objet principal de cette thèse est l'application de méthodes nouvelles inspirées de l'analyse non-lisse et impliquant l'équation Hamilton-Jacobi pour l'études de certains problèmes d'optimalités en théorie du contrôle. Nous présentons dans cette introduction nos principaux résultats mais nous commençons tout d'abord à introduire quelques outils d'analyse non-lisse.

Soit $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ une fonction semi-continue inférieurement et soit $x \in \operatorname{dom} f := \{x' : f(x') < +\infty\}$. On dit que ξ est un sous-gradient proximal de f en x si et seulement si (noté ssi) il existe $\sigma \geq 0$ tel que

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \xi, y - x \rangle,$$

pour tout y dans un voisinage de x.
L'ensemble de tous ces ξ est noté par $\partial_P f(x)$, sous-différentiel proximal de f en x. On peut définir encore le sous-différentiel limite de f en x par

$$\partial_L f(x) = \{\zeta \ : \text{ il existe } x_i \xrightarrow{f} x,\ \zeta_i \longrightarrow \zeta \text{ tel que } \zeta_i \in \partial_P f(x_i) \text{ pour tout } i\},$$

où $x_i \xrightarrow{f} x$ signifie que $x_i \longrightarrow x$ et $f(x_i) \longrightarrow f(x)$.
Nous pouvons de manière équivalente, lorsque $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{-\infty\}$ est semi-continue supérieurement, définir son sur-différentiel proximal. En effet $-f$ est une fonction semi-continue inférieurement, on peut alors définir le sur-différentiel proximal (noté par $\partial^P f(x)$) tout simplement par $\partial^P f(x) := -\partial_P(-f)(x)$.

Passons maintenant à la présentation de nos résultats.

Dualité lisse et non-lisse en contrôle optimal

En 1993, Vinter a considéré dans son papier [66] le problème de commande optimale suivant :

$$(Q)\ \begin{cases} \text{Minimiser } \ell(T, x(T)), \\ T \in [0,1], \\ \dot{x}(t) \in F(t, x(t)) \ \ p.p.\ t \in [0,T], \\ x(0) = x_0, \\ (t, x(t)) \in A \subset [0,1] \times \mathbb{R}^n \ \ \forall t \in [0,T], \\ (T, x(T)) \in C \subset [0,1] \times \mathbb{R}^n, \end{cases}$$

où la fonction $\ell : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, le point x_0, la multifonction F, les ensembles A et C de $\mathbb{R} \times \mathbb{R}^n$ sont les données du problème (Q). Vinter a démontré en utilisant l'analyse convexe (*dualité convexe*) et le concept de "generalized flow" que le minimum de (Q) peut être représenté comme l'enveloppe supérieure des sous-solutions lisses de l'équation d'Hamilton-Jacobi (c'est ce qu'on appelle *dualité lisse*). Ceci lui a permis de trouver une condition nécessaire et suffisante d'optimalité pour le problème (Q). Cette méthode appelée "convex duality" a été introduite pour la première fois par Vinter et Lewis dans [68] et [69].

Nous remarquons que le problème (Q) est un problème en temps libre mais avec horizon fini ($T \in [0,1]$). De plus, Vinter affirme dans [67] que ses méthodes ne s'appliquent pas pour les problèmes en temps libre avec horizon infini ($T \in [0,+\infty[$) et n'aboutissent pas à une caractérisation du minimum comme l'enveloppe supérieure des sous-solutions lisses d'une équation d'Hamilton-Jacobi *autonome*. Une question importante se pose : peut-on trouver des méthodes directes pour retrouver le résultat de Vinter et puis l'étendre pour les problémes en temps libre avec horizon infini ?

Dans le Chapitre 1, nous considérons le problème de commande opti-

male suivant :

$$(P)\ \begin{cases} \text{Minimiser } \ell(T, x(T)), \\ T \in [0, +\infty[, \\ \dot{x}(t) \in F(x(t)) \ \ p.p.\ t \in [0, +\infty[, \\ x(0) = x_0, \\ x(t) \in A \ \ \forall t \in [0, T], \\ x(T) \in C. \end{cases}$$

Nous remarquons tout d'abord que les problèmes en temps fixe et en temps libre (avec horizon fini ou bien infini) sont tous des cas spéciaux de notre problème (P). Par exemple, pour que (P) soit un problème en temps fixe, il suffit de prendre $\ell(T, x)$ égale à $+\infty$ tant que T diffère de l'horizon donné T_0. Pour les hypothèses de (P), nous supposons que la fonction $\ell : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ est semi-continue inférieurement et qu'elle satisfait la condition de croissance suivante :

$$\lim_{t \to +\infty} \inf_x \ell(t, x) = +\infty.$$

Pour la multifonction F, nous supposons qu'elle prend des valeurs convexes, compactes et non vides, qu'elle est semi-continue supérieurement et qu'elle vérifie la condition de croissance linéaire suivante : $\exists \gamma, c \geq 0$ tels que pour tout $x \in \mathbb{R}^n$ on a

$$v \in F(x) \Longrightarrow \|v\| \leq \gamma \|x\| + c.^1$$

Finalement, nous supposons que A et C sont respectivement fermé et compact et que le problème (P) est non-trivial, c'est à dire qu'il existe au moins une trajectoire admissible pour (P).

Notons maintenant par h l'Hamiltonien inférieur associé à F ($h(x, p) := \min_{v \in F(x)} \langle v, p \rangle$, pour tout $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$), alors le résultat principal démontré dans le Chapitre 1 est le théorème suivant :

Théorème (*Dualité non-lisse*)

$$\min(P) = \sup_{\psi \in \Psi} \psi(0, x_0),$$

où Ψ est l'ensemble des fonctions $\psi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ qui vérifient :

[1] Lorsqu'une multifonction F satisfait ces hypothèses alors on dit que F satisfait les *hypothèses standard*.

- ψ *est localement Lipschitzienne.*
- $\bar{h}(x, \partial_L \psi(t,x)) \geq 0 \ \ \forall (t,x) \in \mathbb{R} \times A.$
- $\psi(t,x) \leq \ell(t,x) \ \ \forall (t,x) \in \mathbb{R} \times C.$

La démonstration de ce théorème repose sur des techniques d'analyse non-lisse (analyse proximale, inf-convolution....) et sur la théorie de monotonicité des trajectoires. Les détails sont donnés dans la Section 1.2 du Chapitre 1.

Ce théorème unifie et étend d'un certain nombre de manières, plusieurs résultats dans la littérature, qui traitent dans la plupart des cas de problèmes en temps fixe, (voir par exemple [6], [9], [22, Chapter 4], [36], [37], [57], [58] et [71]). Il est encore nouveau par ses hypothèses qui sont très faibles, en effet il n'est pas nécessaire que F et ℓ soient continues. Le fait que des fonctions localement Lipschitziennes figurent dans notre dualité ainsi que l'utilisation du sous-gradient limite, nous permet de retrouver le résultat de Vinter et même de l'étendre aux problèmes en temps libre (horizon infini) en obtenant dans ce cas une caractérisation en terme de sous-solutions lisses d'une équation d'Hamilton-Jacobi *autonome*. Une conséquence importante de cette dualité lisse est la démonstration d'une nouvelle caractérisation de la fonction temps minimal. Pour plus des détails, voir la Section 1.3 du Chapitre 1.

L'équation Hamilton-Jacobi de la fonction temps minimal

Soit F une multifonction de $\mathbb{R}^n$ dans $\mathbb{R}^n$ qui satisfait les hypothèses standard et qui de plus est localement Lipschitzienne. Nous définissons la fonction temps minimal $T(\cdot, 0)$ associée à l'origine par :

$$T(\alpha, 0) := \begin{cases} \text{Inf } T \geq 0, \\ \dot{x}(t) \in F(x(t)) \ p.p. \ t \in [0,T], \\ x(0) = \alpha \ \text{ et } \ x(T) = 0, \end{cases}$$

pour tout α dans $\mathbb{R}^n$. Si l'origine n'est pas accessible par α alors $T(\alpha, 0)$ prend la valeur $+\infty$.

Le problème de temps minimal est l'un des problèmes les plus célèbres en

commande optimale et il a été largement étudié dans la littérature. L'une des propriétés les plus importantes de la fonction $T(\cdot,0)$ est sa caractérisation comme solution non-lisse de l'équation d'Hamilton-Jacobi. Nous trouvons dans la littérature plusieurs types de solutions non-lisses : solutions de viscosité [30], solutions minimax [61], solutions généralisées [28] et solutions proximales [21]. Nous nous intéressons ici à la solution proximale. Il est facile de démontrer en utilisant la monotonicité des trajectoires (voir [22] et [71]) que $T(\cdot,0)$ est la solution de l'équation d'Hamilton-Jacobi suivante :

$$1 + h(x, \partial_P \varphi(x)) = 0 \ \ \forall x \in \mathbb{R}^n \setminus \{0\}, \ \ \varphi(0) = 0,$$

où h est l'Hamiltonien inférieur associé à F. Nous remarquons que la fonction $T(\cdot,0)$ n'est pas une solution de l'équation précédente dans un domaine qui contient l'origine. En effet, on a $0 \in \partial_P T(\cdot,0)(0)$ et $h(0,0) = 0$. Le but du Chapitre 3 est d'étudier cette équation dans un domaine contenant l'origine. Une nouvelle fonction va jouer un rôle très important dans cette étude. Cette fonction appelée la *fonction bilatérale de temps minimal* et notée par $T(\cdot,\cdot)$ est définie par :

$$T(\alpha,\beta) := \begin{cases} \text{Inf } T \geq 0, \\ \dot{x}(t) \in F(x(t)) \ p.p. \ t \in [0,T], \\ x(0) = \alpha \ \text{ et } \ x(T) = \beta, \end{cases}$$

pour tout $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$. S'il n'y a pas de trajectoires entre α et β alors $T(\alpha,\beta)$ prend la valeur $+\infty$. Le Chapitre 2 de cette thèse est consacré à l'étude de la fonction $T(\cdot,\cdot)$. Nous commençons tout d'abord par étudier la régularité de cette fonction. Nous démontrons qu'elle est continue (resp. localement Lipschitzienne) dans son domaine de définition $\mathcal{R}$ ssi elle est continue (resp. localement Lipschitzienne) en chaque point de la diagonale $\mathcal{D} := \{(\alpha,\alpha) : \alpha \in \mathbb{R}^n\}$. D'autres conditions nécessaires et suffisantes sont encore données pour ces mêmes propriétés. Dans le cas où la dynamique est linéraire nous démontrons que $T(\cdot,\cdot)$ est semi-convexe. Pour plus de détails, voir la Section 2.4 du Chapitre 2. À la fin de cette section nous calculons le sous-différentiel proximal de $T(\cdot,\cdot)$, puis nous déduisons une caractérisation de cette fonction comme sous-solution proximale d'un système d'équations d'Hamilton-Jacobi partielles.

Passons maintenant à la présentation des résultats du Chapitre 3. Nous considérons dans ce chapitre l'équation d'Hamilton-Jacobi suivante :

$$1 + h(x, \partial_P \varphi(x)) = 0 \ \ \forall x \in \mathcal{R}^0_+, \ \ \varphi(0) = 0 \qquad (*)$$

où $\mathcal{R}_+^0 := \{\alpha : T(0,\alpha) < +\infty\}$ est l'ensemble des points accessibles par l'origine. Une solution de l'équation $(*)$ désigne une fonction semi-continue inférieurement $\varphi : \mathcal{R}_+^0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ telle que $\varphi(0) = 0$ et pour tout $x \in \mathcal{R}_+^0$, pour tout $\zeta \in \partial_P \varphi(x)$ (s'il existe) on a $h(x,\zeta)+1 = 0$. Comme nous l'avons déjà mentionné, la fonction $T(\cdot,0)$ n'est pas solution de l'équation $(*)$ car $0 \in \mathcal{R}_+^0$. Alors plusieurs questions se posent : cette équation possède t-elle des solutions ? Possède t-elle une solution minimale ou bien maximale? Quelle est la relation entre une solution de cette équation et la fonction temps minimal ?

Nous démontrons dans le Chapitre 3 l'existence des solutions et même d'une solution minimale pour l'équation $(*)$ qui sont liées aux trajectoires géodésiques. Notre principale hypothèse est que la dynamique $-F$ est 0-STLC (*small-time locally controllable*), c'est à dire que la fonction $T(0,\cdot)$ est continue en 0. Sous cette hypothèse nous aurons que :

- $\mathcal{R}_+^0$ est ouvert.
- $T(0,\cdot)$ est continue dans $\mathcal{R}_+^0$.
- Pour chaque $\alpha_0 \in \partial\mathcal{R}_+^0$ on a

$$\lim_{\alpha \longrightarrow \alpha_0} T(0,\alpha) = +\infty.$$

Nous entrons maintenant dans les détails. Notons par $\mathcal{G}$ l'ensemble suivant

$$\{\Gamma \subset \mathcal{R}_+^0 : \text{il existe une suite } \beta_i \in \Gamma \text{ telle que } T(0,\beta_i) \longrightarrow +\infty\}.$$

Il est clair que sous nos hypothèses, cet ensemble n'est pas vide. Pour chaque $\Gamma \in \mathcal{G}$ nous définissons la fonction $\varphi_\Gamma : \mathcal{R}_+^0 \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ par :

$$\varphi_\Gamma(\alpha) := \liminf_{\substack{\alpha' \longrightarrow \alpha,\ \beta \in \Gamma \\ T(0,\beta) \longrightarrow +\infty}} [T(\alpha',\beta) - T(0,\beta)].$$

Alors nous avons le théorème suivant :

Théorème *Pour chaque $\Gamma \in \mathcal{G}$, la fonction φ_Γ est une solution de $(*)$. De plus si φ_0 est la fonction φ_Γ qui correspond au choix $\mathcal{R}_+^0$ de Γ, alors φ_0 est la solution minimale de $(*)$.*

La démonstration de ce théorème repose sur la monotonicité des trajectoires. Les détails sont présentés dans la Section 3.3 du Chapitre 3. Plusieurs exemples sont encore donnés à la fin de cette section dans lesquels nous démontrons que l'equation $(*)$ ne possède pas nécessairement de solution maximale.

Nous disons qu'une trajectoire $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ (resp. $x :]-\infty, +\infty[\longrightarrow \mathbb{R}^n$) de F est une semi-géodésique (resp. géodésique) ssi $T(x(s), x(t)) = t - s$ pour tout $s \leq t \in [0, +\infty[$ (resp. pour tout $s \leq t \in]-\infty, +\infty[$). La proposition suivante présente la relation entre une solution de $(*)$ et les semi-géodésiques.

Proposition *Soit φ une solution de $(*)$ et soit $\alpha \in \mathrm{dom}\,\varphi$. Alors il existe une semi-géodésique x qui part de α (c'est à dire $x(0) = \alpha$) telle que*

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

Nous remarquons que dans la plupart des exemples, la fonction φ_0 coïncide avec la fonction $-T(0, \cdot)$. De plus nous pouvons même montrer (voir Section 3.4) que dans le cas linéaire, φ_0 est exactement la fonction $-T(0, \cdot)$. Alors une question importante se pose : est-ce que φ_0 coïncide toujours avec $-T(0, \cdot)$? Sinon, existe t-il un exemple dans lequel φ_0 est différente de $-T(0, \cdot)$?

Avant de répondre à ces questions nous démontrons dans la Section 3.5 le théorème suivant :

Théorème *Soit $\alpha \in \mathcal{R}^0_+$. Alors les assertions suivantes sont équivalentes:*

1. *Il existe une semi-géodésique qui part de l'origine et qui passe par α.*
2. *$\varphi_0(\alpha) = -T(0, \alpha)$.*

En utilisant ce théorème, nous réussissons à construire un exemple (voir Example 3.5.8) dans lequel φ_0 prend des valeurs strictements positives et est donc différente de $-T(0, \cdot)$.

D'une façon naturelle nous définissons dans la Section 3.6 une équation duale pour $(*)$ (nous remplaçons ∂_P par ∂^P et $\mathcal{R}^0_+$ par $\mathcal{R}^0_- := \{\alpha \in \mathbb{R}^n : T(\alpha, 0) < +\infty\}$). L'étude de cette équation avec

l'équation $(*)$ nous permet de trouver des conditions nécessaires et suffisantes pour l'existence de géodésiques passant par l'origine.

À la fin du Chapitre 3, nous étudions le cas où S n'est pas nécessairement un singleton. En imposant quelques hypothèses sur S, nous généralisons les résultats de la Section 3.4 (le cas linéaire) et quelques résultats de la Section 3.5. Nous démontrons encore que cette généralisation ne s'étend pas pour les résultats de la Section 3.3 (existence des solutions). La généralisation de ces résultats reste une question ouverte.

Les boucles minimales en théorie du contrôle

Soit F une multifonction de $\mathbb{R}^n$ dans $\mathbb{R}^n$ localement Lipschitzienne et qui satisfait les hypothèses standard. Après avoir étudié dans le Chapitre 3 l'existence des semi-géodésiques et des géodésiques d'un système dynamique, nous étudions dans le Chapitre 4 l'existence d'un autre type de trajectoire, les boucles minimales ("minimal loop trajectories"). Une boucle est une trajectoire x de F qui commence en un point α et se termine en temps strictement positif T en ce même point α. Nous disons qu'elle est minimale ssi elle est minimale entre $x(s)$ et $x(t)$ $(T(x(s), x(t)) = t - s)$ pour tout $s \leq t \in [0, T[$ et pour tout $s \leq t \in]0, T]$. Ce type de trajectoire est utilisé dans [33] pour démontrer l'existence de sous-solutions lisses de l'équation d'Hamilton-Jacobi

$$H(x, u'(x)) \geq c[0],$$

où $c[0]$ est la valeur critique de Mañé.[2]

Avant de commencer à donner des conditions nécessaires et suffisantes pour l'existence des boucles minimales en un point donné α, nous définissons

[2] Soit $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ un Hamiltonien donné et soit L le Lagrangien correspondant. Alors la valeur critique de Mañé est définie par

$$c[0] := -\inf\{\frac{1}{T}\int_0^T L(x(t), \dot{x}(t))\, dt : x \in X,\ T > 0,\ x(0) = x(T)\},$$

où X est l'ensemble des fonctions absolument continues $x : [0, +\infty[\longrightarrow \mathbb{R}^n$.

une nouvelle fonction $L(\cdot)$ par :

$$L(\alpha) := \begin{cases} \text{Inf } T > 0, \\ \dot{x}(t) \in F(x(t)) \ p.p. \ t \in [0,T], \\ x(0) = x(T) = \alpha, \end{cases}$$

pour tout $\alpha \in \mathbb{R}^n$. S'il n'existe pas de boucles en α alors $L(\alpha)$ prend la valeur $+\infty$. Cette fonction qu'on appelle "minimal loop function" va jouer un rôle important dans la suite, et c'est pourquoi nous consacrons la Section 4.3 à son étude.

Il est clair que l'existence d'une boucle minimale en un point α implique que les fonctions $T(\cdot,\alpha)$, $T(\alpha,\cdot)$ et $T(\cdot,\cdot)$ sont respectivement discontinue en α, α et (α,α). De plus nous avons la proposition suivante qui donne une condition nécessaire et suffisante à l'existence d'une boucle minimale en un point α mais qui n'est pas un point d'équilibre ($0 \notin F(\alpha)$).

Proposition *Soit $\alpha \in \mathbb{R}^n$ tel que $0 \notin F(\alpha)$. Alors les assertions suivantes sont équivalentes :*

1. *Il existe une boucle minimale en α.*
2. $\liminf_{\beta \neq \alpha \ \beta \longrightarrow \alpha} [T(\alpha,\beta) + T(\beta,\alpha)] < +\infty.$

Dans le cas où α est un point d'équilibre, un simple exemple montre que la condition $\liminf_{\beta \neq \alpha \ \beta \longrightarrow \alpha} [T(\alpha,\beta) + T(\beta,\alpha)] < +\infty$ n'est pas une condition suffisante pour l'existence d'une boucle minimale en α. Nous démontrons dans la Proposition 4.4.4 qu'en ajoutant une autre hypothèse, cette condition devient suffisante. On peut trouver encore dans la Section 4.4 du Chapitre 4 d' autres conditions nécessaires et suffisantes à l'existence des boucles minimales.

Nous finissons la Section 4.4 par un théorème qui montre que sous certaines hypothèses chaque point $\alpha \in \mathbb{R}^n$ vérifie au moins une des propriétés suivantes :

- $0 \in F(\alpha)$.
- Il existe une semi-géodésique qui part de α.

- Il existe une boucle minimale en α.

À la fin du Chapitre 4 nous donnons quelques exemples qui aident à la compréhension de nos résultats.

Notons à la fin de cette introduction que nous présentons dans le premier chapitre (Chapitre 0) des outils et des résultats d'analyse non-lisse et de théorie des inclusions différentielles, indispensables à la bonne compréhension de cette thèse.

Chapter 0

Preliminaries

0.1 Introduction

Nonsmooth analysis concerns the local description of nondifferentiable functions and sets lacking smooth boundaries, in terms of generalizations of classical concepts of derivatives, normals and tangents. An important branch of nonsmooth analysis is *proximal analysis*. Proximal analysis refers to the calculus associated to proximal normals and subgradients (see below for the definitions). It offers the most natural and also the most powerful way to develop nonsmooth analysis in a Hilbert space setting. The infinite-dimensional theory is an important branch of proximal analysis but for the purposes of our work we limit attention to $\mathbb{R}^n$. We present in this chapter some tools and results from proximal analysis which play an important role in this thesis, see Section 0.2. We also define other basic constructs of nonsmooth analysis (limiting subgradient, generalized gradient...) and then we explore the relationships between them. We end Section 0.2 by studying some useful classes of functions: convex, semiconvex and regular functions. At the end of this chapter and since all our control systems in our work are governed by a differential inclusion, we give a short background on the differential inclusion theory, see Section 0.3.

Our general reference in this chapter is the book of Clarke *et al.* "*Nonsmooth Analysis and Control Theory*" [22] which gives a fuller development of nonsmooth analysis based on the proximal concepts. See also [2], [14], [19], [31], [53] and [65].

0.2 Nonsmooth analysis

0.2.1 Proximal analysis

Proximal normal cone

Let S be a nonempty closed subset of $\mathbb{R}^n$. The distance function $d_S : \mathbb{R}^n \longrightarrow \mathbb{R}$ associated to S is defined by

$$d_S(x) := \inf_{s \in S} \|x - s\| \quad \forall x \in \mathbb{R}^n.$$

It is easy to verify that the function $d_S(\cdot)$ is 1-Lipschitz on $\mathbb{R}^n$. Now let $x \in \mathbb{R}^n$. Since S is nonempty and closed, it is easy to verify that there exists at least one point $s \in S$ such that

$$d_S(x) = \|x - s\|.$$

This point s is called *closest point* or *projection* of x onto S. The set of all closest points is denoted by $\mathrm{proj}_S(x)$. For $s \in \mathrm{proj}_S(x)$, the vector $x - s$ determines what we call a *proximal normal direction* to S at s; any nonnegative multiple $\zeta = t(x-s)$, $t \geq 0$, of such a vector will be called a proximal normal to S at s. The set of all ζ obtainable in this manner is termed the *proximal normal cone* to S at s, and is denoted by $N_S^P(s)$. Suppose that $s \in S$ such that $s \notin \mathrm{proj}_S(x)$ for all x not in S (which is certainly the case if s lies in $\mathrm{int}\, S$). Then $N_S^P(s) = \{0\}$. It is easy to prove that for all $s \in S$ we have

$$N_S^P(s) = \{\zeta : \exists t > 0 \text{ so that } d_S(s + t\zeta) = t\|\zeta\|\}.$$

The following proposition gives an important characterization of proximal normals. The proof is left to the reader, (see [22]).

Proposition 0.2.1 *Let S be a nonempty closed set of $\mathbb{R}^n$ and let $s \in S$. Then for all $\zeta \in \mathbb{R}^n$, the following assertions are equivalent:*

1. *$\zeta \in N_S^P(s)$.*
2. *There exists $t > 0$ such that $proj_S(s + t\zeta) = \{s\}$.*
3. *There exists $t > 0$ such that for all $s' \in S$ we have $t\|\zeta\| \leq \|s + t\zeta - s'\|$.*
4. *There exists $\sigma \geq 0$ such that for all $s' \in S$ we have*

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2. \tag{1}$$

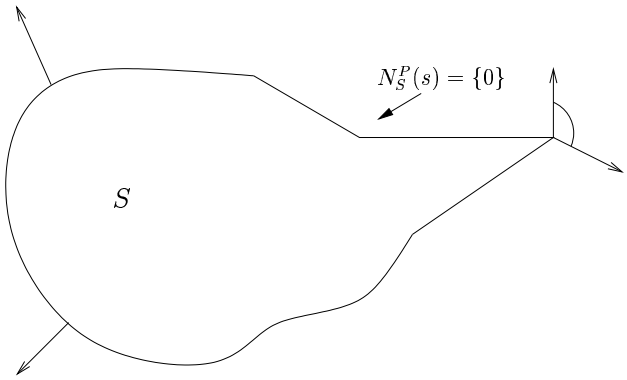

Figure 1: Proximal normal cone

The inequality (1) is called the *proximal normal inequality*. Using this inequality, it is clear that $N_S^P(s)$ is a convex cone; however, it need be neither open nor closed. We can easily remark that the structure of the proximal cone $N_S^P(s)$ depends only of the form of S in a neighborhood of s and then we have the following proposition.

Proposition 0.2.2 *Let S be a nonempty closed set of $\mathbb{R}^n$ and let $s \in S$. Then for any given $\delta > 0$, we have $\zeta \in N_S^P(s)$ iff there exists $\sigma = \sigma(\zeta, s) > 0$ such that*

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2 \ \ \forall s' \in S \cap B(s; \delta).$$

We note that $N_S^P(s)$ can be trivial (reduce to $\{0\}$) even when s lies in ∂S. For example if we consider in $\mathbb{R}^2$ the following set

$$S := \{(x, y) \in \mathbb{R}^n : y \geq -|x|\},$$

then we can easily prove that $N_S^P(0, 0) = \{0\}$. The special case in which S is convex in an important one.

Proposition 0.2.3 *Assume that S is convex and let $s \in S$. Then*

1. *$\zeta \in N_S^P(s)$ iff*
$$\langle \zeta, s' - s \rangle \leq 0 \ \ \forall s' \in S.$$
2. *If $s \in \partial S$ then $N_S^P(s) \neq \{0\}$.*

The statement 1) of the preceding proposition follows immediately from the proximal normal inequality. The statement 2) follows by 1) and using the fact that the set $\{s \in \text{proj}_S(x) : x \in \mathbb{R}^n \backslash S\}$ is dense in ∂S.

Proximal subgradients

Let U be an open set of $\mathbb{R}^n$. We denote by $\mathcal{F}(U)$ the class of all functions $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ which are lower semicontinuous on U and such that

$$\operatorname{dom} f \cap U \neq \emptyset,$$

where $\operatorname{dom} f$ is the *effective* domain of f defined by

$$\operatorname{dom} f := \{x : f(x) < +\infty\}.$$

If $U = \mathbb{R}^n$, then we simply write $\mathcal{F}$ for $\mathcal{F}(\mathbb{R}^n)$.
Now let $f \in \mathcal{F}$ and let $x \in \operatorname{dom} f$. A vector $\xi \in \mathbb{R}^n$ is a *proximal subgradient* of f at x provided $(\xi, -1) \in N^P_{epi(f)}(x, f(x))$, where $\operatorname{epi}(f)$ denotes the epigraph $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$ of f, which is a closed subset of $\mathbb{R}^{n+1}$ since f is lower semicontinuous. The set (which could be empty) of all proximal subgradients of f at x is denoted by $\partial_P f(x)$ and is referred to as the *proximal subdifferential*. The following analytic description of the proximal subgradient is often useful.

Proposition 0.2.4 *Let $f \in \mathcal{F}$ and let $x \in dom\, f$. Then $\xi \in \partial_P f(x)$ iff there exist positive numbers σ and ν such that*

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \xi, y - x \rangle \ \ \forall y \in B(x; \nu). \tag{2}$$

The inequality (2) is called the *proximal subgradient inequality* and its proof (we omit the details) relies on Proposition 0.2.1, (see [22]). Using this inequality we can develop several important characterizations for the proximal subgradient and particularly the links with the classical differentiability. Then we have the following proposition.

Proposition 0.2.5 *Let $f \in \mathcal{F}$ and let $x \in dom\, f$. Then*

1. *If f has a local minimum at x, then $0 \in \partial_P f(x)$.*
2. *If f is Gâteaux differentiable at x, then $\partial_P f(x) \subset \{f'_G(x)\}$.*
3. *If $f \in C^2(U)$ where U is an open set of $\mathbb{R}^n$. Then*

$$\partial_P f(x) = \{f'(x)\} \ \ \forall x \in U.$$

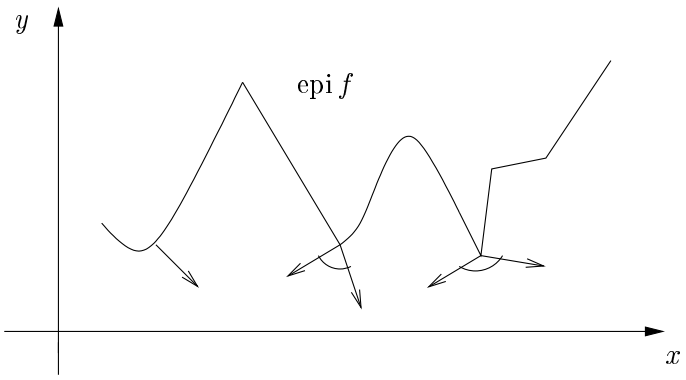

Figure 2: Proximal subdifferential

As mentioned above the proximal subdifferential could be an empty set, for example the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = -|x|$ is a simple example of a continuous function having $\partial_P f(0) = \emptyset$. The following theorem is called *The Density Theorem*. This important theorem proves that the set $\mathrm{dom}(\partial_P f)$ of points in $\mathrm{dom}\, f$ at which at least one proximal subgradient exists is dense in $\mathrm{dom}\, f$. Since the proof is very technical and long, we don't give here the details. We invite the reader to see [22] for the proof.

Theorem 0.2.6 *Let $f \in \mathcal{F}$ and let $x \in dom\, f$. Then for all $\varepsilon > 0$, there exists a point $y \in B(x_0; \varepsilon)$ satisfying $\partial_P f(y) \neq \emptyset$ and*

$$f(x_0) - \varepsilon \leq f(y) \leq f(x_0).$$

In particular, $dom(\partial_P f)$ is dense in $dom\, f$.

The statement 1) of Proposition 0.2.5 is the equivalent form of the Fermat's Rule in the nonsmooth case. In fact, we can find in [22] several important extensions of the (classical) differential calculus for the proximal subgradient. For example, in [22] Clarke *et al.* give a proximal version of the mean value theorem and then a sum rule and a chain rule for the proximal subgradient (we don't give the details here). An application of this (nonsmooth) differential calculus is the following proposition.

Proposition 0.2.7 *Let $U \subset \mathbb{R}^n$ be an open and convex set and let $f \in \mathcal{F}(U)$. Then f is K-Lipschitz on U iff*

$$\|\zeta\| \leq K \quad \forall \zeta \in \partial_P f(x), \quad \forall x \in U.$$

For an upper semicontinuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{-\infty\}$, we can define the *proximal superdifferential* $\partial^P f(x)$ simply as $-\partial_P(-f)(x)$. As for the proximal subdifferential we can prove an analogous results for the proximal superdifferential but we replace "sub" by "super", "$\leq$" by "$\geq$" and "minimum" by "maximum". Moreover, using Proposition 0.2.4 we can easily prove the following proposition.

Proposition 0.2.8 *Suppose $U \subset \mathbb{R}^n$ is open, $x \in U$, $f : U \longrightarrow \mathbb{R}$ is continuous on U and both $\partial_P f(x)$ and $\partial^P f(x)$ are nonempty. Then f is differentiable at x, and we have $\partial_P f(x) = \partial^P f(x) = \{f'(x)\}$.*

Limiting subdifferential

Certain results in proximal analysis are conveniently expressed in limiting terms, and the following constructs are convenient. Let $S \subset \mathbb{R}^n$ be a nonempty closed set and let $s \in S$. We define the *limiting normal cone* $N_S^L(s)$ by

$$N_S^L(s) := \{\zeta \ : \text{ there exists } s_i \xrightarrow{S} s,\ \zeta_i \longrightarrow \zeta \text{ such that } \zeta_i \in N_S^P(s_i) \text{ for all } i\},$$

where $s_i \xrightarrow{S} s$ signifies that $s_i \longrightarrow s$ and that $s_i \in S$ for all i. A similar procedure defines the *limiting subdifferential* ($f \in \mathcal{F}$ and $x \in \operatorname{dom} f$) by:

$$\partial_L f(x) := \{\zeta \ : \ (\zeta, -1) \in N_{epi(f)}^L(x, f(x))\}.$$

A vector ζ in $\partial_L f(x)$ is called a *limiting subgradient* of f at x. Clearly we have:

$$\partial_L f(x) = \{\zeta \ : \text{ there exists } x_i \xrightarrow{f} x,\ \zeta_i \longrightarrow \zeta \text{ such that } \zeta_i \in \partial_P f(x_i) \text{ for all } i\},$$

where $x_i \xrightarrow{f} x$ signifies that $x_i \longrightarrow x$ and $f(x_i) \longrightarrow f(x)$.

Using the results presented in the preceding subsection for the proximal subgradient, we can show the following proposition which gives some properties for $\partial_L f$.

Proposition 0.2.9 *Let $f \in \mathcal{F}$ and let $x \in \operatorname{dom} f$. Then*

1. *If f is Lipschitz near x then $\partial_L f(x) \neq \emptyset$.*

2. *If f is differentiable at x then $f'(x) \in \partial_L f(x)$.*

3. *If f is locally Lipschitz on an open set $U \subset \mathbb{R}^n$, then f is C^1 on U iff $\partial_L f(y)$ reduces to a singleton for every $y \in U$.*

4. *Assume that f is Lipschitz near x and let x_i be a sequence converging to x. If ζ_i is a sequence which converges to ζ, where $\zeta_i \in \partial_L f(x_i)$ for each i, then $\zeta \in \partial_L f(x)$.*

0.2.2 Inf-convolution

The inf-convolution of f and g is another function h defined as follows:

$$h(x) := \inf_{y \in \mathbb{R}^n} \{f(y) + g(x - y)\}.$$

Our interest here involves only such inf-convolutions formed between a function f and the quadratic function $k\|x\|^2$, where $k \in \mathbb{N}^*$. Given an bounded below function $f \in \mathcal{F}$ and $k \in \mathbb{N}^*$, we define $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$f_k(x) := \inf_{y \in \mathbb{R}^n} \{f(y) + k\|x - y\|^2\}. \tag{3}$$

The sequence $(f_k)_k$ is called the quadratic inf-convolution sequence associated to f. The following proposition shows that this sequence has surprisingly far-reaching properties.

Proposition 0.2.10 *Suppose $f \in \mathcal{F}$ such that f is bounded below by some constant ω. Then for all $k \in \mathbb{N}^*$:*

1. *$f_k(\cdot) \leq f(\cdot)$ and the set of minimizing points y in (3) is nonempty.*

2. *f_k is locally Lipschitz and bounded below by ω.*

3. *For all $x \in \mathbb{R}^n$,*

$$\lim_{k \longrightarrow +\infty} f_k(x) = f(x).$$

0.2.3 Some regularity properties

In this subsection we present some regularity properties for a function f. We begin by some definitions and notations.
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $x \in \operatorname{dom} f$. The *directional derivative* of f at x in the direction $v \in \mathbb{R}^n$ is defined by

$$f'(x;v) := \lim_{t \longrightarrow 0} \frac{f(x+tv) - f(x)}{t},$$

when the limits exists. For f Lipschitz near x and given $v \in \mathbb{R}^n$, one defines the *generalized directional derivative* of f at x in the direction v as

$$f^\circ(x;v) := \limsup_{\substack{y \longrightarrow x \\ t \downarrow 0}} \frac{f(y+tv) - f(y)}{t}.$$

The *generalized gradient* of f at x (f still assumed Lipschitz near x), is the following (nonempty) set

$$\partial f(x) := \{\xi \in \mathbb{R}^n : f^\circ(x;v) \geq \langle \xi, v \rangle \ \ \forall v \in \mathbb{R}^n\}.$$

For more information about the preceding definitions see [19], [22] and [65].

Convex functions

Let $U \subset \mathbb{R}^n$ be an open convex set. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be convex on U provided

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \ \ \forall x, y \in U, \ 0 < t < 1.$$

A function f which is convex on $\mathbb{R}^n$ is simply said to be convex. Note that $\operatorname{dom} f$ is necessarily a convex set if f is convex. Clearly a function f is convex on $\mathbb{R}^n$ iff $\operatorname{epi} f$ is a convex set of $\mathbb{R}^n \times \mathbb{R}$. The following proposition follows from Propositions 0.2.3 and 0.2.4.

Proposition 0.2.11 *Let $f \in \mathcal{F}$ and let $x \in \operatorname{dom} f$. Then*

1. *If f is convex, then $\zeta \in \partial_P f(x)$ iff*

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle \ \ \forall y \in \mathbb{R}^n.$$

2. *If f is convex, then $\partial_P f(x) = \partial_L f(x) = \partial f(x)$.*

3. *If f is convex and $0 \in \partial_P f(x)$, then x is a global minimum of f.*

Semiconvex functions

Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \longrightarrow \mathbb{R}$ be a continuous function. We say that f is semiconvex if for all $x_0 \in U$ there exist δ, $C > 0$ such that

$$2f(\frac{x+y}{2}) - f(x) - f(y) \leq C\|x-y\|^2 \ \ \forall x, y \in B(x_0;\delta).^3$$

Clearly when f is a continuous convex function then it is semiconvex. Moreover, it is easy to see that when f is semiconvex on U then for all $x_0 \in U$, the function

$$x \mapsto f(x) + \frac{C}{2}\|x\|^2$$

is convex on $B(x_0;\delta)$ (C and δ is as in the definition of the semiconvexity). This remark with Proposition 0.2.11 imply the following.

Proposition 0.2.12 *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \longrightarrow \mathbb{R}$ be a semiconvex function on U. Then f is locally Lipschitz and*

$$\partial_P f(x) = \partial_L f(x) = \partial f(x) \ \ \forall x \in U.$$

When $f : U \longrightarrow \mathbb{R}$ is in $C^2(U)$ then we can easily prove that f is semiconvex. Moreover, we have the following (see [14] for the proof).

Proposition 0.2.13 *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \longrightarrow \mathbb{R}$ be a continuous function. If f is the upper envelope of C^2 functions which have second derivatives uniformly bounded, then f is semiconvex.*

We can also define the semiconcavity property. We say that a continuous function $f : U \longrightarrow \mathbb{R}$ is semiconcave if the function $-f$ is semiconvex. Then f is semiconcave if for all $x_0 \in U$ there exist δ, $C > 0$ such that

$$f(x) + f(y) - 2f(\frac{x+y}{2}) \leq C\|x-y\|^2 \ \ \forall x, y \in B(x_0;\delta).$$

Clearly we can adopt an analogue version of the preceding propositions for the semiconcave case.

[3] There are several names for this type of functions. The first name "fonctions sous-linéarisables" is given by Janin [43]. We also say "weak convexity", "convexity up to a square", "lower C^2 functions", and so on.

Regular functions

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $x \in \operatorname{dom} f$. We say that f is regular at x if it is Lipschitz near x and admits directional derivatives $f'(x;v)$ at x for all v, with $f'(x;v) = f^\circ(x;v)$. If $U \subset \mathbb{R}^n$ is an open, then we say that f is regular in U if it is regular at x for all $x \in U$. The following is proven in [22].

Proposition 0.2.14 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $x \in \operatorname{dom} f$. Then*

1. *If f is regular at x then we have $\partial_L f(x) = \partial f(x)$.*
2. *Assume that one of the following conditions holds:*
   (a) *f is convex and finite near x.*
   (b) *f is semiconvex near x.*
   (c) *f is continuously differentiable near x.*

   Then f is regular at x.

0.3 Differential inclusions

We are given a multifunction (we also say multivalued function) F mapping $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$, and a time interval $[a,b]$. Associated with F is the differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad a.e.\ t \in [a,b]. \tag{4}$$

A *solution* $x(\cdot)$ of (4) is taken to mean an absolutely continuous function $x : [a,b] \longrightarrow \mathbb{R}^n$ which, together with $\dot{x}$, its derivative with respect to t, satisfies (4). For brevity, we will refer to any absolutely continuous function x from $[a,b]$ to $\mathbb{R}^n$ as an *arc* on $[a,b]$. We also refer to an arc x satisfying (4) as a *trajectory* of F.
The concept of differential inclusion subsumes that of *a standard control system*

$$\dot{x}(t) = f(x(t), u(t)), \tag{5}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, and where the control function u takes values in some prescribed subset U of $\mathbb{R}^m$; simply consider $F(x) := f(x,U)$. Filippov's

Lemma (see [22, Chapter 3]) implies that under mild hypotheses on f, an arc x satisfies (4) iff there is measurable function $u(\cdot)$ with values in U such that (5) holds.
The *lower Hamiltonian* $h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (resp. *upper Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$) corresponding to F is defined by follows

$$h(x,p) := \min_{v \in F(x)} \langle p, v \rangle \quad (\text{resp. } H(x,p) := \max_{v \in F(x)} \langle p, v \rangle).$$

We assume in this section that the multifunction F satisfies the following standing hypotheses:

(a) For every $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.

(b) The linear growth condition: For some positive constants γ and c, and for all $x \in \mathbb{R}^n$,

$$v \in F(x) \Longrightarrow \|v\| \leq \gamma \|x\| + c.$$

(c) F is upper semicontinuous; that is, for every $x \in \mathbb{R}^n$, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x' - x\| < \delta \Longrightarrow F(x') \subset F(x) + \varepsilon B.$$

We note that in the presence of hypotheses (a) and (b), property (c) is equivalent to the graph of F ($\operatorname{gr} F := \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n : v \in F(x)\}$) being closed. The role of the *linear growth* condition in the classical theory of differential equations is predicated on the a priori bounds on solutions to which it gives rise. We will benefit from it in precisely the same way. The following is known as *Gronwall's Lemma*. For the proof see [22] and [3].

Lemma 0.3.1 *Let $x(\cdot)$ be a trajectory of F on $[a,b]$ then:*

$$\|x(t) - x(a)\| \leq (e^{\gamma(t-a)} - 1)(\|x(a)\| + c/\gamma).$$

Under our hypotheses on F, any trajectory can be extend indefinitely both forward and backward, so all trajectories can be considered as being defined on $]-\infty, +\infty[$. We recall that a multifunction F is said to be locally Lipschitz if every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that

$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

The following important proposition asserts that upper semicontinuous multifunctions can be approximated "from above" by Lipschitz one. The proof follows using a special covering of $\mathbb{R}^n$ and then the existence of a locally Lipschitz *partition of unity* for this covering. For details, see [22].

Proposition 0.3.2 *There exists a sequence of locally Lipschitz multifunctions $\{F_k\}$ also satisfying the hypotheses of F such that:*

- *For each $k \in \mathbb{N}$, for every $x \in \mathbb{R}^n$,*

$$F(x) \subseteq F_{k+1}(x) \subseteq F_k(x) \subseteq \overline{co}\, F(x + 3^{-k+1}B).$$

- $\bigcap_{k\geq 1} F_k(x) = F(x)\ \forall x \in \mathbb{R}^n$.

0.3.1 Compactness of trajectories

The following result is fundamental to differential inclusion theory and is referred to as "the compactness of trajectories" theorem. See ([22] and [19]) for the proof.

Theorem 0.3.3 *Let x_i be a sequence of arcs on $[a,b]$ such that the set $\{x_i(a)\}$ is bounded, and satisfying:*

$$\dot{x}_i(t) \in F(x_i(t) + y_i(t)) + r_i(t)B \ \ a.e.,$$

where y_i and r_i are sequences of measurable functions on $[a,b]$, such that y_i converges to 0 in L^2 and $r_i \geq 0$ converge to 0 in L^2. Then there is a subsequence of x_i which converges uniformly to an arc x which is a trajectory of F, and whose derivatives converge weakly to $\dot{x}$.

Corollary 0.3.4 *We have the following:*

1. *Let x_i be a sequence of trajectories of F on $[a_i, b_i]$, where $a_i \longrightarrow a$, $b_i \longrightarrow b$, $a < b$, and where the sequence $x_i(a_i)$ is bounded. Let the x_i be continued to $]-\infty, +\infty[$. Then there exists a subsequence x_{i_j} of x_i such that for some trajectory $\bar{x}$ of F on $[a,b]$, x_{i_j} converges uniformly to $\bar{x}$ on $[a,b]$.*

2. *Let x_i be a sequence of trajectories of F on $[a, +\infty[$ such that the set $\{x_i(a)\}$ is bounded. Then there exists a trajectory $\bar{x}$ of F on $[a, +\infty[$, and there is a subsequence x_{i_j} of x_i such that for all $T > 0$, x_{i_j} converges uniformly to $\bar{x}$ on $[a,T]$.*

0.3.2 Dependence on initial conditions

Let $t_0 \in \mathbb{R}$ and let $x_0 \in \mathbb{R}^n$. The attainable set $A(t_0, x_0; T)$, for $T \geq t_0$, is the set of all points of the form $x(T)$, where x is a trajectory for F on $[t_0, +\infty[$ satisfying $x(t_0) = x_0$. The theorem by which the solution of a differential equation depends regularly on initial conditions is a well-known and useful result. The following theorem gives its counterpart for differential inclusions. The proof relies on the *proximal aiming* technique introduced in [22].

Theorem 0.3.5 *Let F be locally Lipschitz and let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then for any fixed $T \in \mathbb{R}$, the multifunction $(t_0, x_0) \mapsto A(t_0, x_0; T)$ is locally Lipschitz on $]-\infty, T] \times \mathbb{R}^n$.*

0.3.3 Monotonicity of trajectories

Let Ω be an open subset of $\mathbb{R}^n$ and let $\varphi \in \mathcal{F}(\Omega)$. We say that (φ, F) is *strongly increasing* on Ω if for any trajectory x on an interval $[a, b]$ for which $x([a, b]) \subset \Omega$, we have

$$\varphi(x(s)) \leq \varphi(x(t)) \ \ \forall s, t \in [a, b], s \leq t.$$

The system (φ, F) is said to be *weakly decreasing* on Ω if for every $\alpha \in \Omega$ there is a trajectory x on a nontrivial interval $[a, b]$ satisfying

$$x(a) = \alpha, \varphi(x(t)) \leq \varphi(\alpha) \quad \forall t \in [a, b];$$

by reducing b if necessary we may also arrange to have $x([a, b]) \subset \Omega$. In each case, one obtains an equivalent definition by requiring the inequality to hold on $[a, \tau[$, where $\tau \in]a, +\infty]$ is the *exit time* of the trajectory x from Ω: the supremum of all $T > 0$ having the property that $x([a, T]) \subset \Omega$. The following proposition is proven in [22].

Proposition 0.3.6 *Let F be locally Lipschitz and let $\varphi \in \mathcal{F}(\Omega)$. Then the system (φ, F) is strongly increasing on Ω iff*

$$h(x, \partial_P \varphi(x)) \geq 0 \quad \forall x \in \Omega,$$

and weakly decreasing on Ω iff

$$h(x, \partial_P \varphi(x)) \leq 0 \quad \forall x \in \Omega.$$

The proof of the preceding proposition also relies on the proximal aiming technique for trajectory construction developed in [22].

Chapter 1

Nonconvex duality in optimal control*

1.1 Introduction

We consider in this chapter the following optimal control problem in Mayer form:

$$(P)\ \begin{cases} \text{Minimize } \ell(T, x(T)), \\ T \in [0, +\infty[, \\ \dot{x}(t) \in F(x(t)) \ \ a.e.\ t \in [0, +\infty[, \\ x(0) = x_0, \\ x(t) \in A \ \ \forall t \in [0, T], \\ x(T) \in C, \end{cases}$$

where the given data is a point x_0, the extended-valued function $\ell : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the multivalued function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and the sets A and C. We assume along this chapter that the set A is closed, that C is compact, and that ℓ is lower semicontinuous and satisfies the following growth condition (GC):

$$\lim_{t \to +\infty} \inf_x \ell(t, x) = +\infty.$$

As for the multivalued function F, we assume that it takes nonempty compact convex values, has closed graph, and satisfies a linear growth condition: for some positive constants γ and c, and for all $x \in \mathbb{R}^n$,

$$v \in F(x) \Longrightarrow \|v\| \leq \gamma \|x\| + c.$$

*This chapter is based on joint work [25] with F. H. Clarke.

Finally, we assume that (P) is nontrivial in the sense that there is at least one admissible trajectory for which the cost is finite.

We recall that the lower Hamiltonian associated to F is denoted by h. There is a well-known relationship between the value of the problem (P) on the one hand and certain solutions of a Hamilton-Jacobi inequality in terms of h on the other; let us recall it now. Let $\psi(t,x)$ be a smooth function satisfying

$$\psi_t(t,x) + h(x,\psi_x(t,x)) \geq 0 \ \ \forall (t,x), \quad \psi(t,x) \leq \ell(t,x) \text{ if } x \in C.$$

Then it follows from the Hamiltonian inequality that for any trajectory $x(\cdot)$ of the differential inclusion, the function $t \mapsto \psi(t,x(t))$ is nondecreasing. Accordingly, for any $T \geq 0$ we have $\psi(T,x(T)) \geq \psi(0,x(0))$. If we now restrict the trajectories to those that are admissible for (P), we deduce

$$\ell(T,x(T)) \geq \psi(0,x_0).$$

There results a lower bound for the value of (P):

$$\inf(P) \geq \sup \psi(0,x_0),$$

where the supremum is taken over all functions ψ as described above.

The term 'nonconvex duality' has been applied in optimal control to situations in which equality holds in this last relation. The basic idea is at the heart of Carathéodory's method in the calculus of variations, which is also known as that of 'verification functions'. It is also related to the 'generalized flows' of L.C. Young, as observed by Vinter *et al.* ([66], [68] and [69]) who have extended this duality to the setting of optimal control.

In addition to equality above, it is natural to ask under what conditions the supremum on the right is attained. Since this fails in general for smooth functions ψ, this question led Clarke *et al.* to introduce a generalized solution concept for nonsmooth solutions of the Hamilton-Jacobi equation ([17], [28], [40] and [50]); these have turned out to be what is now known as viscosity semisolutions. The method of verification functions extended in this way has been used to solve explicitly a number of problems in optimal control; we refer to Clarke [18] for a thorough discussion of the method, and for examples.

The issue of generalized solutions inevitably involves nonsmooth analysis, and here we employ the tools of proximal analysis. We remark that one can use the proximal subdifferential to define generalized solutions of the Hamilton-Jacobi equation, and the resulting solutions coincide (in the present context) with the viscosity solutions of Crandall and Lions and the minimax solutions of Subbotin (see for example [21] and [22]). Let the extended Hamiltonian $\bar{h}$ be the function given by

$$\bar{h}(x, \theta, \zeta) := \theta + h(x, \zeta).$$

We define Ψ to be the set of all locally Lipschitz functions ψ on $\mathbb{R} \times \mathbb{R}^n$ that satisfy the limiting Hamilton-Jacobi inequality

$$\bar{h}(x, \partial_L \psi(t, x)) \geq 0 \ \ \forall (t, x) \in \mathbb{R} \times A$$

as well as the boundary condition

$$\psi(t, x) \leq \ell(t, x) \ \ \forall (t, x) \in \mathbb{R} \times C.$$

The following is the main result.

Theorem 1.1.1

$$\min(P) = \sup_{\psi \in \Psi} \psi(0, x_0).$$

This result unifies, and extends in a number of ways, the ones in the literature, which treat for the most part the fixed-horizon case (see for example [6], [9], [22, Chapter 4], [36], [37], [57], [58] and [71]). We remark that the fixed-horizon case is obtained by taking $\ell(T, x)$ equal to $+\infty$ whenever T differs from the given horizon T_0; see Section 1.3 below for a discussion of this and other special cases. Our theorem, whose proof is self-contained modulo some basic facts from proximal analysis, is also new with respect to its very mild regularity hypotheses on F (which need not even be continuous), as well as the presence of a unilateral state constraint. The fact that locally Lipschitz functions figure in our duality also gives easy access to smooth duality of the type found by Vinter [66]. Furthermore, we extend his result by obtaining in the case of the minimal time problem a duality in which feature only smooth solutions of an *autonomous* Hamilton-Jacobi inequality. Section 1.3 is devoted to specializations such as these, while the next section gives the proof of the theorem.

1.2 Proof of Theorem 1.1.1

First we note that under our hypotheses on F, any trajectory can be extend indefinitely both forward and backward, so all trajectories can be considered as being defined on $]-\infty,+\infty[$. Using (GC) and since C is compact, we show that ℓ is bounded below over $[0,+\infty[\times C$ then we can assume that ℓ is bounded below over $\mathbb{R}\times\mathbb{R}^n$ by a constant ω.

We begin by proving that the problem (P) admits a solution. Let $(T_i, x_i(\cdot))$ be a minimizing sequence for the problem (P) then there exists a positive sequence $\varepsilon_i \longrightarrow 0$ such that

$$\ell(T_i, x_i(T_i)) = \inf(P) + \varepsilon_i.$$

By (GC) we can assume, by passing to subsequence, that there exists $b \geq 0$ such that $T_i \in [0,b]$ and $T_i \longrightarrow \bar{T}$. Using the compactness property of trajectories, there exists a trajectory $\bar{x}$ of F on $[0,+\infty[$ which satisfies $\bar{x}(0) = x_0$, and there is a subsequence x_{i_j} of x_i such that x_{i_j} converge uniformly to $\bar{x}$ on $[0,b]$. Then

$$\ell(\bar{T},\bar{x}(\bar{T})) \leq \liminf_{(\tau,y)\longrightarrow(\bar{T},\bar{x}(\bar{T}))} \ell(\tau,y) \leq \liminf_{j\longrightarrow+\infty} \ell(T_{i_j}, x_{i_j}(T_{i_j})) = \inf(P).$$

But $(\bar{T},\bar{x}(\cdot))$ is an admissible trajectory for the problem (P), then $(\bar{T},\bar{x}(\cdot))$ is a solution for this problem.
Now we use the Proposition 0.3.2 which asserts that upper semicontinuous multifunctions can be approximated "from above" by Lipschitz one. By this proposition there exists a sequence of locally Lipschitz multifunctions $\{F_k\}$ also satisfying the hypotheses of F such that:

- For each $k \in \mathbb{N}$, for every $x \in \mathbb{R}^n$,

$$F(x) \subseteq F_{k+1}(x) \subseteq F_k(x) \subseteq \overline{\text{co}}\, F(x + 3^{-k+1}B).$$

- $\bigcap_{k\geq 1} F_k(x) = F(x)\ \forall x \in \mathbb{R}^n$.

A well known method of approximating the terminally constrained problem (P) by a problem free of such constraints involves the imposition of a penalty term. To use this technique, the inf-convolution technique and the preceding approximation we consider for all $k \geq 1$ the following optimal control problems:

$$(P_k)\ \begin{cases} \text{Minimize } \ell_k(T,x(T)) + kd_C(x(T)) + k\int_0^T d_A(x(t))\,dt, \\ T \geq 0, \\ \dot{x}(t) \in F_k(x(t)) \ \ a.e.\ t \in [0,+\infty[, \\ x(0) = x_0, \end{cases}$$

where $(\ell_k)_k$ is the quadratic inf-convolution sequence of the function ℓ, see Proposition 0.2.10.

Lemma 1.2.1 *There exists a sequence λ_n strictly increasing in $\mathbb{N}^*$ such that:*

$$\lim_{n\longrightarrow+\infty} \min(P_{\lambda_n}) = \min(P)$$

Proof. We denote by $(\bar{T},\bar{x}(\cdot))$ a solution of the problem (P), and by $(T_k,\bar{y}_k(\cdot))$ a solution of the problem (P_k), (the existence of the solutions is easy to check). For all $k \in \mathbb{N}^*$ we have

$$\min(P_k) = \ell_k(T_k,\bar{y}_k(T_k)) + kd_C(\bar{y}_k(T_k)) + k\int_0^{T_k} d_A(\bar{y}_k(t))\,dt$$

and

$$\min(P) = \ell(\bar{T},\bar{x}(\bar{T})).$$

Since $F \subset F_k$, $(\bar{T},\bar{x}(\cdot))$ is admissible for the problem (P_k) then we have:

$$\begin{aligned} \min(P_k) &= \ell_k(T_k,\bar{y}_k(T_k)) + kd_C(\bar{y}_k(T_k)) + k\int_0^{T_k} d_A(\bar{y}_k(t))\,dt \\ &\leq \ell_k(\bar{T},\bar{x}(\bar{T})) \\ &\leq \ell(\bar{T},\bar{x}(\bar{T})) \\ &= \min(P) \end{aligned}$$

hence

$$\omega \leq \ell_k(T_k,\bar{y}_k(T_k)) \leq \ell(\bar{T},\bar{x}(\bar{T})) \tag{1.1}$$

and

$$kd_C(\bar{y}_k(T_k)) + k\int_0^{T_k} d_A(y_k(t)) \leq \ell(\bar{T},\bar{x}(\bar{T})) - \omega. \tag{1.2}$$

By Proposition 0.2.10 there exists $(s_k,z_k) \in \mathbb{R}\times\mathbb{R}^n$ such that

$$\ell_k(T_k,\bar{y}_k(T_k)) = \ell(s_k,z_k) + k\|(s_k,z_k) - (T_k,\bar{y}_k(T_k))\|^2 \tag{1.3}$$

hence

$$k\|(s_k, z_k) - (T_k, \bar{y}_k(T_k))\|^2 \leq \ell_k(\bar{T}, \bar{x}(\bar{T})) - \omega \leq \ell(\bar{T}, \bar{x}(\bar{T})) - \omega. \qquad (1.4)$$

We claim that

$$\lim_{k \longrightarrow +\infty} T_k \neq +\infty.$$

Indeed, by (1.4) we have that there exists a sequence a_n strictly increasing in $\mathbb{N}^*$ such that

$$\lim_{n \longrightarrow +\infty} (s_{a_n} - T_{a_n}) = 0$$

then if

$$\lim_{n \longrightarrow +\infty} T_{a_n} = +\infty$$

we get

$$\lim_{n \longrightarrow +\infty} s_{a_n} = +\infty.$$

Since ℓ satisfies (GC) we have

$$\lim_{n \longrightarrow +\infty} \ell(s_{a_n}, z_{a_n}) = +\infty$$

hence by (1.3)

$$\lim_{n \longrightarrow +\infty} \ell_{a_n}(T_{a_n}, \bar{y}_{a_n}(T_{a_n})) = +\infty$$

but this contradicts (1.1). Then we can assume that there exists $b \geq 0$ such that $T_k \in [0, b]$.
Now we use the following lemma (compactness of trajectories). For the proof see [3] and [31].

Lemma 1.2.2 *Let δ_i be a sequence such that $\delta_i \longrightarrow 0$, and let x_i be a sequence of uniformly bounded arcs on $[a, b]$ such that*

$$\dot{x}_i(t) \in \overline{\text{co}}\, F(x_i(t) + \delta_i \bar{B}) + \delta_i \bar{B} \ \ a.e.,$$

then there is a subsequence of x_i which converges uniformly to an arc x which is a trajectory of F on $[a, b]$.

Using the preceding lemma and since $F_k(\cdot) \subseteq \overline{\text{co}}\, F(\cdot + 3^{-k+1}B)$, there exists a trajectory $\bar{y}(\cdot)$ of F on $[0, +\infty[$ which satisfies $\bar{y}(0) = x_0$, and there is a subsequence $\bar{y}_{k_i}$ of $\bar{y}_k$ such that $\bar{y}_{k_i}$ converge uniformly to $\bar{y}(\cdot)$ on $[0, b]$. Moreover, since $T_k \in [0, b]$, there exists a subsequence of T_k, which converge to a point in $[0, b]$. These considerations with (1.1), (1.2) and (1.4) give that there exists a sequence λ_n strictly increasing in $\mathbb{N}^*$ such that:

- The sequence T_{λ_n} converges to a $\bar{T}_0 \in [0,b]$.
- The sequence $\bar{y}_{\lambda_n}$ converges uniformly on $[0,b]$ to the trajectory $\bar{y}(\cdot)$.
- The sequence $\lambda_n \|(s_{\lambda_n}, z_{\lambda_n}) - (T_{\lambda_n}, \bar{y}_{\lambda_n}(T_{\lambda_n}))\|^2$ is convergent.
- The sequence $\lambda_n d_C(\bar{y}_{\lambda_n}(T_{\lambda_n})) + \lambda_n \int_0^{T_{\lambda_n}} d_A(\bar{y}_{\lambda_n}(t))$ is convergent.
- The sequence $\ell_{\lambda_n}(T_{\lambda_n}, \bar{y}_{\lambda_n}(T_{\lambda_n}))$ is convergent.

Since the sequence λ_n is strictly increasing in $\mathbb{N}^*$ we get

$$\lim_{n \longrightarrow +\infty} z_{\lambda_n} = \bar{y}(\bar{T}_0)$$

and

$$\lim_{n \longrightarrow +\infty} s_{\lambda_n} = \bar{T}_0$$

hence by the convergence of the sequence $\lambda_n d_C(\bar{y}_{\lambda_n}(T_{\lambda_n})) + \lambda_n \int_0^{T_{\lambda_n}} d_A(\bar{y}_{\lambda_n}(t))$ and using Lebesgue's theorem we have

$$\bar{y}(\bar{T}_0) \in C$$

and

$$\bar{y}(t) \in A, \ \ \forall t \in [0, \bar{T}_0].$$

Then $(\bar{T}_0, \bar{y}(\cdot))$ is admissible for the problem (P). Hence

$$\begin{aligned}
\min(P) &= \ell(\bar{T}, \bar{x}(\bar{T})) \\
&\geq \lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) \\
&\geq \lim_{n \longrightarrow +\infty} \ell_{\lambda_n}(T_{\lambda_n}, \bar{y}_{\lambda_n}(T_{\lambda_n})) \\
&= \lim_{n \longrightarrow +\infty} \ell(s_{\lambda_n}, z_{\lambda_n}) + \lambda_n \|(s_{\lambda_n}, z_{\lambda_n}) - (T_{\lambda_n}, \bar{y}_{\lambda_n}(T_{\lambda_n}))\|^2 \\
&\geq \liminf_{(s',z') \longrightarrow (\bar{T}_0, \bar{y}(\bar{T}_0))} \ell(s', z') \\
&\geq \ell(\bar{T}_0, \bar{y}(\bar{T}_0)) \\
&\geq \min(P).
\end{aligned}$$

Then

$$\min(P) = \lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) = \ell(\bar{T}_0, \bar{y}(\bar{T}_0))$$

which completes the proof of the lemma. □
We continue the proof and we remark that the problem (P_{λ_n}) is exactly the following problem:

$$\begin{cases} \text{Minimize } \hat{\ell}_{\lambda_n}(T, z(T)), \\ T \geq 0, \\ \dot{z}(t) \in \hat{F}_{\lambda_n}(z(t)) \ \ a.e.\ t \in [0, +\infty[, \\ z(0) = (0, x_0), \end{cases}$$

where $\hat{F}_{\lambda_n}$ is an augmented locally Lipschitz multifunction defined as follows $\forall (y,x) \in \mathbb{R} \times \mathbb{R}^n$, $\hat{F}_{\lambda_n}(y,x) = \{\lambda_n d_A(x)\} \times F_{\lambda_n}(x)$, and $\hat{\ell}_{\lambda_n}$ is the locally Lipschitz function defined by $\hat{\ell}_{\lambda_n}(t,y,x) = \ell_{\lambda_n}(t,x) + \lambda_n d_C(x) + |y|$, $\forall (t,y,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$.
Let $\hat{V}_{\lambda_n} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be the value function of the problem (P_{λ_n}); that is, for every $(\tau, \beta, \alpha) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, $\hat{V}_{\lambda_n}(\tau, \beta, \alpha)$ is the minimum of the following problem:

$$\begin{cases} \text{Minimize } \hat{\ell}_{\lambda_n}(T, z(T)), \\ T \geq \tau, \\ \dot{z}(t) \in \hat{F}_{\lambda_n}(z(t)) \ \ a.e.\ t \in [\tau, +\infty[, \\ z(\tau) = (\beta, \alpha). \end{cases}$$

Now we show the following lemma which is a slightly strengthened version of Theorem 0.3.5. We use this lemma to show that $\hat{V}_{\lambda_n}$ is locally Lipschitz.

Lemma 1.2.3 *Assume that F is locally Lipschitz. Then $\forall b \in \mathbb{R}$, $\forall a < b$, $\forall \alpha \in \mathbb{R}^n$, $\exists\ \rho = \rho(a,b,\alpha)$ such that:*

$$A(\tau_1, \alpha_1; c) \subseteq A(\tau_2, \alpha_2; c) + \rho \|(\tau_1 - \tau_2, \alpha_1 - \alpha_2)\| \bar{B}$$

$\forall c \in]a,b]$, $\forall (\tau_1, \alpha_1), (\tau_2, \alpha_2) \in [a,c] \times B(\alpha; 1)$.

Proof. Let $b \in \mathbb{R}$, $a < b$, $c \in]a,b]$ and $\alpha \in \mathbb{R}^n$. By the proof of Theorem 0.3.5, there exists $\sigma = 2ke^{K(b-c)}$ such that:

$$A(\tau_1, \alpha_1; c) \subseteq A(\tau_2, \alpha_2; c) + \sigma \|(\tau_1 - \tau_2, \alpha_1 - \alpha_2)\| \bar{B} \qquad (1.5)$$

$\forall (\tau_1, \alpha_1), (\tau_2, \alpha_2) \in [a,c] \times B(\alpha; 1)$, where $k \geq 1$ is a common Lipschitz constant for all trajectories of F on $[a,c]$ with initial-value in $B(\alpha; 1)$, and K is a Lipschitz constant for F on an appropriately large ball (the radius

of the ball is independent of c). Let $\rho := 2k_1 e^{K(b-a)}$, where $k_1 \geq 1$ is a common Lipschitz constant for all trajectories of F on $[a,b]$ with initial-value in $B(\alpha;1)$, then by (1.5)

$$A(\tau_1,\alpha_1;c) \subseteq A(\tau_2,\alpha_2;c) + \rho\|(\tau_1-\tau_2,\alpha_1-\alpha_2)\|\bar{B}$$

$\forall(\tau_1,\alpha_1),(\tau_2,\alpha_2) \in [a,c]\times B(\alpha;1)$. The result follows. □

Lemma 1.2.4 *The value function $\hat{V}_{\lambda_n}$ is locally Lipschitz on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$.*

Proof. First, it is easy to show that the function $\hat{\ell}_{\lambda_n}$ satisfies like ℓ the growth condition (GC). We define the augmented multifunction $\bar{F}_{\lambda_n}$ which satisfies the same hypotheses as $\hat{F}_{\lambda_n}$ by: $\bar{F}_{\lambda_n}(x^0,x) = \{1\}\times\hat{F}_{\lambda_n}(x)$. We remark that for all $(\tau,\beta,\alpha)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$, $\hat{V}_{\lambda_n}(\tau,\beta,\alpha)$ is the minimum of the following problem:

$$(\bar{P}_{\lambda_n}(\tau,\beta,\alpha))\ \begin{cases} \text{Minimize } \hat{\ell}_{\lambda_n}(T,w(T)),\\ T\geq 0,\\ \dot{w}(t)\in\bar{F}_{\lambda_n}(w(t)) \ \ a.e.\ t\in[0,+\infty[,\\ w(0)=(\tau,\beta,\alpha),\end{cases}$$

Let $(\tau,\beta,\alpha)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$, by (GC) there exists $n_{(\tau,\beta,\alpha)}>0$ and $0<r_{(\tau,\beta,\alpha)}<1$ such that for every $\gamma\in B((\tau,\beta,\alpha);r_{(\tau,\beta,\alpha)})$ and for every trajectory $(T_\gamma,w(\cdot))$ solution for the problem $(\bar{P}_{\lambda_n}(\gamma))$, we have $T_\gamma\in[0,n_{(\tau,\beta,\alpha)}]$. By Lemma 0.3.1 there exists $\nu_{(\tau,\beta,\alpha)}>0$ such that $w(t)\in\bar{B}(0;\nu_{(\tau,\beta,\alpha)})$, $\forall t\in[0,n_{(\tau,\beta,\alpha)}]$ $\forall\gamma\in B((\tau,\beta,\alpha);r_{(\tau,\beta,\alpha)})$, $\forall\ w(\cdot)$ a trajectory of $\bar{F}_{\lambda_n}$ on $[0,+\infty[$ which satisfies $w(0)=\gamma$.
By Lemma 1.2.3 ($b=n_{(\tau,\beta,\alpha)}, a=0$ and applied for $\bar{F}_{\lambda_n}$), there exists ρ such that:

$$A(0,(\tau_1,\beta_1,\alpha_1);T)\subseteq$$
$$A(0,(\tau_2,\beta_2,\alpha_2);T)+\rho\|(\tau_1,\beta_1,\alpha_1)-(\tau_2,\beta_2,\alpha_2)\|\bar{B} \qquad (1.6)$$

$\forall(\tau_1,\beta_1,\alpha_1)$, $(\tau_2,\beta_2,\alpha_2)\in B((\tau,\beta,\alpha);r_{(\tau,\beta,\alpha)})$, $\forall\, T\in[0,n_{(\tau,\beta,\alpha)}]$.
Let K_0 be a Lipschitz constant of $\hat{\ell}_{\lambda_n}$ on the closed ball $\bar{B}(0;\nu_{(\tau,\beta,\alpha)})$. We claim that $\hat{V}_{\lambda_n}$ is $K_0\rho$-Lipschitz on the ball $B((\tau,\beta,\alpha);r_{(\tau,\beta,\alpha)})$. Indeed, let $(\tau_1,\beta_1,\alpha_1)$ and $(\tau_2,\beta_1,\alpha_2)$ in $B((\tau,\beta,\alpha);r_{(\tau,\beta,\alpha)})$ and let $(\bar{T}_1,\bar{w}_1(\cdot))$ be a solution of the problem $(\bar{P}_{\lambda_n}(\tau_1,\beta_1,\alpha_1))$, we have $\bar{T}_1\leq n_{(\tau,\beta,\alpha)}$ then by (1.6) there exists a trajectory $w_2(\cdot)$ of $\bar{F}_{\lambda_n}$ on $[0,+\infty[$ which satisfies $w_2(0)=(\tau_2,\beta_2,\alpha_2)$ and such that:

$$\|\bar{w}_1(\bar{T}_1)-w_2(\bar{T}_1)\|\leq\rho\|(\tau_1,\beta_1,\alpha_1)-(\tau_2,\beta_2,\alpha_2)\|$$

then

$$|\hat{\ell}_{\lambda_n}(\bar{w_1}(\bar{T_1})) - \hat{\ell}_{\lambda_n}(w_2(\bar{T_1}))| \leq K_0\rho\|(\tau_1,\beta_1,\alpha_1) - (\tau_2,\beta_2,\alpha_2)\|$$

hence

$$\hat{V}_{\lambda_n}(\tau_1,\beta_1,\alpha_1) \geq \hat{V}_{\lambda_n}(\tau_2,\beta_2,\alpha_2) - K_0\rho\|(\tau_1,\beta_1,\alpha_1) - (\tau_2,\beta_2,\alpha_2)\|$$

using the same techniques as above, we show that

$$\hat{V}_{\lambda_n}(\tau_2,\beta_2,\alpha_2) \geq \hat{V}_{\lambda_n}(\tau_1,\beta_1,\alpha_1) - K_0\rho\|(\tau_1,\beta_1,\alpha_1) - (\tau_2,\beta_2,\alpha_2)\|$$

therefore

$$\|\hat{V}_{\lambda_n}(\tau_2,\beta_2,\alpha_2) - \hat{V}_{\lambda_n}(\tau_1,\beta_1,\alpha_1)\| \leq K_0\rho\|(\tau_1,\beta_1,\alpha_1) - (\tau_2,\beta_2,\alpha_2)\|$$

and this completes the proof of the lemma. □

Moreover, we have

$$\hat{V}_{\lambda_n}(\cdot) \leq \hat{\ell}_{\lambda_n}(\cdot) \tag{1.7}$$

and

$$\hat{V}_{\lambda_n}(\tau,\beta,\alpha) = \hat{V}_{\lambda_n}(\tau,0,\alpha) + \beta \qquad \forall(\tau,\beta,\alpha) \in \mathbb{R} \times [0,+\infty[\times\mathbb{R}^n. \tag{1.8}$$

Using the logic known as the *principle of optimality* the system $(\hat{V}_{\lambda_n}, \hat{F}_{\lambda_n})$ is strongly increasing. Then by Proposition 0.3.6 we have

$$\theta + \lambda_n d_A(x)\xi + h_{\lambda_n}(x,\zeta) \geq 0,^2 \tag{1.9}$$

$\forall(\theta,\xi,\zeta) \in \partial_P\hat{V}_{\lambda_n}(t,y,x)$, $\forall(t,y,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$.
If we consider $\psi_{\lambda_n} : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\psi_{\lambda_n}(t,x) = \hat{V}_{\lambda_n}(t,0,x)$, $\forall(t,x) \in \mathbb{R}\times\mathbb{R}^n$, then since $\hat{V}_{\lambda_n}$ is locally Lipschitz and by (1.7), (1.8) and (1.9) we have:

1. ψ_{λ_n} is locally Lipschitz on $\mathbb{R}\times\mathbb{R}^n$,
2. $\psi_{\lambda_n}(0,x_0) = \hat{V}_{\lambda_n}(0,0,x_0) = \min(P_{\lambda_n})$,
3. $\theta + \lambda_n d_A(x) + h_{\lambda_n}(x,\zeta) \geq 0$, $\forall(\theta,\zeta) \in \partial_P\psi_{\lambda_n}(t,x)$, $\forall(t,x) \in \mathbb{R}\times\mathbb{R}^n$
4. $\psi_{\lambda_n}(t,x) \leq \ell_{\lambda_n}(t,x)$, $\forall(t,x) \in \mathbb{R}\times C$.

[2] h_{λ_n} is the lower Hamiltonian corresponding to F_{λ_n}.

Lemma 1.2.5 $\psi_{\lambda_n} \in \Psi$.

Proof. By the preceding 1) we have ψ_{λ_n} is locally Lipschitz on $\mathbb{R} \times \mathbb{R}^n$ and by 4) and since $\ell_{\lambda_n}(\cdot) \leq \ell(\cdot)$ we get that $\psi_{\lambda_n}(t,x) \leq \ell_{\lambda_n}(t,x)$, $\forall (t,x) \in \mathbb{R} \times C$. Now we show that ψ_{λ_n} satisfies the limiting Hamilton-Jacobi equation. Let $(t,x) \in \mathbb{R} \times A$ and let $(\theta, \zeta) \in \partial_L \psi_{\lambda_n}(t,x)$, then there exists two sequences $(t_i, x_i) \in \mathbb{R} \times \mathbb{R}^n$ such that $(t_i, x_i) \longrightarrow (t,x)$ and $(\theta_i, \zeta_i) \longrightarrow (\theta, \zeta)$. By 3) we have

$$\theta_i + \lambda_n d_A(x_i) + h_{\lambda_n}(x_i, \zeta_i) \geq 0$$

then

$$\theta_i + \lambda_n \|x - x_i\| + h_{\lambda_n}(x_i, \zeta_i) \geq 0.$$

Taking $i \longrightarrow +\infty$ we find that

$$\theta + h_{\lambda_n}(x, \zeta) \geq 0,$$

the result follows. □

Since $\psi_{\lambda_n}(0, x_0) = \min(P_{\lambda_n})$ and using the preceding lemma we get

$$\sup_{\psi \in \Psi} \psi(0, x_0) \geq \psi_{\lambda_n}(0, x_0) = \min(P_{\lambda_n})$$

then

$$\min(P) = \lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) \leq \sup_{\psi \in \Psi} \psi(0, x_0).$$

Now we show the reverse inequality. We make the *temporary hypothesis* that F is locally Lipschitz. Let $\psi \in \Psi$ then we have the following lemma.

Lemma 1.2.6 *For all open and bounded subset $S \subset \mathbb{R}^{n+1}$, for all $\varepsilon > 0$, there exists a neighborhood U of A such that*

$$\theta + h(x, \zeta) \geq -\varepsilon,$$

$\forall (\theta, \zeta) \in \partial_P \psi(t,x)$, $\forall (t,x) \in S \cap \{\mathbb{R} \times U\}$.

Proof. We reason by the absurd. We assume that there exist a open and bounded subset $S \subset \mathbb{R}^{n+1}$ and a $\varepsilon > 0$ such that for all neighborhood U of A, there exist $(t,x) \in S \cap \{\mathbb{R} \times U\}$ and $(\theta, \zeta) \in \partial_P \psi(t,x)$ such that

$$\theta + h(x, \zeta) < -\varepsilon.$$

Then there exist two sequences $(t_n, x_n) \in S$ and (θ_n, ζ_n) such that

$$(t_n, x_n) \longrightarrow (t_0, x_0) \in \mathbb{R} \times A,$$

$$(\theta_n, \zeta_n) \in \partial_P \psi(t_n, x_n),$$

and

$$\theta_n + h(x_n, \zeta_n) < -\varepsilon. \tag{1.10}$$

Since ψ is locally Lipschitz and S is bounded, the sequence (θ_n, ζ_n) is bounded and then we can assume that it converges to a point (θ_0, ζ_0). By the definition of ∂_L we get that $(\theta_0, \zeta_0) \in \partial_L \psi(t_0, x_0)$. By (1.10) and since F is locally Lipschitz we find that

$$\theta_0 + h(x_0, \zeta_0) \leq -\varepsilon$$

and this gives a contradiction since $\psi \in \Psi$. □

Now let $(\bar{T}, \bar{x}(\cdot))$ be a solution of the problem (P) then by Lemma 0.3.1 there exists $\rho > 0$ such that $\bar{x}(t) \in B(0;\rho)$, $\forall t \in [0, \bar{T}]$. We apply the preceding lemma for $S =]-1, \bar{T}+1[\times B(0;\rho)$ and for $\varepsilon > 0$, we get that there exists a neighborhood U_ε of A such that

$$\theta + h(x, \zeta) \geq -\varepsilon$$

$\forall (\theta, \zeta) \in \partial_P \psi(t, x), \forall (t, x) \in S \cap \{\mathbb{R} \times U_\varepsilon\}$.
But $S \cap \{\mathbb{R} \times U_\varepsilon\} =]-1, \bar{T}+1[\times \{B(0;\rho) \cap U_\varepsilon\}$ then since $[0, \bar{T}] \subset]-1, \bar{T}+1[$, $\bar{x}(t) \in B(0;\rho) \cap U_\varepsilon$ $\forall t \in [0, \bar{T}]$ and by Proposition 0.3.6 we get that

$$\psi(0, x_0) \leq \varepsilon \bar{T} + \psi(\bar{T}, \bar{x}(\bar{T}))$$

moreover $\psi(\bar{T}, \bar{x}(\bar{T})) \leq \ell(\bar{T}, \bar{x}(\bar{T})) = \min(P)$ then

$$\psi(0, x_0) \leq \min(P) + \varepsilon \bar{T}$$

hence by taking $\varepsilon \longrightarrow 0$ we get

$$\psi(0, x_0) \leq \min(P)$$

therefore

$$\min(P) \geq \sup_{\psi \in \Psi} \psi(0, x_0).$$

To remove the need for the locally Lipschitz hypothesis on F we use the sequence F_k. First we have the following lemma.

Lemma 1.2.7 *For all $n \in \mathbb{N}$ there exists $k_n \geq n$ such that*

$$\theta + h_{k_n}(x,\zeta) \geq -\frac{1}{n},$$

$\forall (\theta,\zeta) \in \partial_L \psi(t,x), \forall (t,x) \in]-1,\bar{T}+1[\times\{A \cap \bar{B}(0;\rho)\}$.

Proof. We also reason by the absurd. Assume that there exists $n_0 \in \mathbb{N}$ such that for all $k > n_0$ there exists $(t_k,x_k) \in]-1,\bar{T}+1[\times\{A\cap\bar{B}(0;\rho)\}$ and $(\theta_k,\zeta_k) \in \partial_L\psi(t_k,x_k)$ such that

$$\theta_k + h_k(x_k,\zeta_k) < -\frac{1}{n_0}. \tag{1.11}$$

Since the sequence (t_k,x_k) is bounded, we assume that it converges to a point $(t_0,x_0) \in \mathbb{R}\times A$. In the other hand, the sequence (θ_k,ζ_k) is also bounded since ψ is locally Lipschitz, then we can assume that it converges to a point (θ_0,ζ_0). Using Proposition 0.2.9 we have $(\theta_0,\zeta_0) \in \partial_L\psi(t_0,x_0)$. Now let $\varepsilon > 0$, then since F is upper semicontinuous we have that for k sufficiently large $F(x_k + 3^{-k+1}B) \subset F(x_0) + \varepsilon\bar{B}$. Hence $F_k(x_k) \subset F(x_0)+\varepsilon\bar{B}$. Using the definition of the lower Hamiltonian, we get that $h_k(x_k,\zeta_k) \geq h(x_0,\zeta_k) - \varepsilon\|\zeta_k\|$. Then by (1.11), we have

$$\theta_k + h(x_0,\zeta_k) - \varepsilon\|\zeta_k\| < -\frac{1}{n_0}.$$

But h is continuous in the second variable, then if we take $k \longrightarrow +\infty$ in the preceding inequality we get

$$\theta_0 + h(x,\zeta_0) - \varepsilon\|\zeta_0\| \leq -\frac{1}{n_0}.$$

Since ε is arbitrary we find that

$$\theta_0 + h(x,\zeta_0) \leq -\frac{1}{n_0}$$

and this gives a contradiction with $\psi \in \Psi$. □

By the preceding lemma there exist a subsequence F_{k_i} of F_k and a sequence $\varepsilon_i > 0$ such that $\varepsilon_i \longrightarrow 0$ and

$$\theta + h_{k_i}(x,\zeta) \geq -\varepsilon_i,$$

$\forall (\theta,\zeta) \in \partial_L\psi(t,x), \forall (t,x) \in]-1,\bar{T}+1[\times\{A\cap\bar{B}(0;\rho)\}$.
We continue as in the Lipschitz case and we find the result. □

Remark 1.2.8 *If we suppose in the definition of Ψ that the functions ψ are continuous and satisfies the proximal Hamilton-Jacobi equation (∂_P instead of ∂_L) then the inequality*

$$\min(P) \leq \sup_{\psi \in \Psi} \psi(0, x_0)$$

remains true and the equality remains true if we assume in addition that $A = \mathbb{R}^n$ and F is locally Lipschitz.

1.3 Consequences and special cases

In this section we present some applications of our main result. In the next subsection we give necessary and sufficient optimality conditions. In Subsection 1.3.2, we treat several types of optimal control problems (fixed time problem and free time problem with finite and infinite horizon). Vinter's smooth duality is studied in Subsection 1.3.3.

1.3.1 Characterization of optimality

Theorem 1.1.1 leads directly to the following optimality conditions.

Corollary 1.3.1 *Let $(\bar{T}, \bar{x}(\cdot))$ be an admissible trajectory for the problem (P). Then $(\bar{T}, \bar{x}(\cdot))$ is a minimizer for the problem (P) iff there exists a sequence of functions $\{\psi_i\}$ in Ψ such that*

$$\lim_{i \longrightarrow +\infty} \psi_i(0, x_0) = \ell(\bar{T}, \bar{x}(\bar{T})).$$

Proof. Let $(\bar{T}, \bar{x}(\cdot))$ be an admissible trajectory for (P) and assume that $(\bar{T}, \bar{x}(\cdot))$ is a minimizer for (P). Then by Theorem 1.1.1 we have

$$\ell(\bar{T}, \bar{x}(\bar{T})) = \sup_{\psi \in \Psi} \psi(0, x_0).$$

Then the necessary condition follows by taking a maximizing sequence $\{\psi_i\}$ for the supremum $\sup_{\psi \in \Psi} \psi(0, x_0)$.

For the sufficient condition, let $(\bar{T}, \bar{x}(\cdot))$ be an admissible trajectory for (P) and suppose that there exists a sequence of functions $\{\psi_i\}$ in Ψ such that

$$\lim_{i \longrightarrow +\infty} \psi_i(0, x_0) = \ell(\bar{T}, \bar{x}(\bar{T})).$$

Since $\psi_i \in \Psi$ and by Theorem 1.1.1 we have

$$\lim_{i \longrightarrow +\infty} \psi_i(0, x_0) \leq \min(P).$$

But $(\bar{T}, \bar{x}(\cdot))$ is an admissible trajectory for (P), then

$$\min(P) \geq \lim_{i \longrightarrow +\infty} \psi_i(0, x_0) = \ell(\bar{T}, \bar{x}(\bar{T})) \geq \min(P),$$

then

$$\ell(\bar{T}, \bar{x}(\bar{T})) = \min(P),$$

which completes the proof. □

1.3.2 Special cases

1. *Fixed time problem:* If we take $\ell(t, x) = I_{\{T_0\} \times C}(t, x) + \ell_0(x)$ where $T_0 \geq 0$ is fixed and $\ell_0 : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, then we can treat the fixed time case and obtain the following duality:

   $$\min(P) = \sup_{\psi \in \Psi} \psi(0, x_0)$$

   where Ψ is the set of all functions $\psi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

   - ψ is locally Lipschitz on $\mathbb{R} \times \mathbb{R}^n$,
   - $\bar{h}((x, \partial_L \psi(t, x)) \geq 0$, $\forall (t, x) \in \mathbb{R} \times A$, $\forall v \in F(x)$,
   - $\psi(T_0, x) \leq \ell_0(x)$ $\forall x \in C$.

   If we assume that $A = \mathbb{R}^n$ and F is locally Lipschitz, then we can consider in the preceding duality continuous functions ψ and we can replace ∂_L by ∂_P, and this gives a slightly strengthened version of [22, Theorem 4.7.7][3].

2. *Free time problem with finite horizon:* Problems in which T varies in a compact interval (as is the case in Vinter's work) can be treated by taking (for example) $\ell(t, x) = I_{[0,1] \times C}(t, x) + \ell_0(x)$ where $\ell_0 : \mathbb{R}^n \longrightarrow$

[3] In [22, Theorem 4.7.7], ℓ is continuous and the functions ψ are defined on $]-\infty, T_0] \times \mathbb{R}^n$.

$\mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and $I_{[0,1]\times C}$ denotes the indicator function of $[0,1] \times C$. Then we find using our main result the following duality:

$$\min(P) = \sup_{\psi \in \Psi} \psi(0, x_0)$$

where Ψ is the set of all functions $\psi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- ψ is locally Lipschitz on $\mathbb{R} \times \mathbb{R}^n$,
- $\bar{h}(x, \partial_L \psi(t,x)) \geq 0$, $\forall (t,x) \in \mathbb{R} \times A$, $\forall v \in F(x)$,
- $\psi(t,x) \leq \ell_0(x)$ $\forall (t,x) \in [0,1] \times C$.

3. *Free time problem with infinite horizon:* We now consider the free time problem in its most familiar form, where T is completely unrestricted. We suppose that in (P) we have $\ell(t,x) = t + \ell_0(x)$ $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$ where $\ell_0 : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function bounded below. In this case and by Theorem 1.1.1 we have that

$$\min(P) = \sup_{\psi \in \Psi} \psi(x_0)$$

where Ψ is the set of all functions $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- ψ is locally Lipschitz on $\mathbb{R}^n$,
- $1 + h(x, \partial_L \psi(x)) \geq 0$, $\forall x \in A$, $\forall v \in F(x)$,
- $\psi(x) \leq \ell_0(x)$ $\forall x \in C$.

We remark that in this result, only autonomous functions ψ contribute to the upper envelope. This follows since in this case we can take in the proof of Theorem 1.1.1, $\ell_k := t + \ell_0^k$ where ℓ_0^k is the quadratic inf-convolution of ℓ_0 and then we get that the function $\hat{V}_{\lambda_n}(\cdot,\cdot,\cdot)$ satisfies

$$\hat{V}_{\lambda_n}(\tau, \beta, \alpha) = \tau + \hat{V}_{\lambda_n}(0,0,\alpha) + \beta \qquad \forall (\tau, \beta, \alpha) \in \mathbb{R} \times [0, +\infty[\times \mathbb{R}^n.$$

1.3.3 Smooth duality

Another important application of our main result is the smooth duality studied by Vinter in [66]. In this paper, Vinter considers the following optimal

control problem:

$$(Q)\ \begin{cases} \text{Minimize } \ell(T, x(T)), \\ T \in [0,1], \\ \dot{x}(t) \in F(t, x(t)) \ \ a.e.\ t \in [0,T], \\ x(0) = x_0, \\ (t, x(t)) \in A \subset [0,1] \times \mathbb{R}^n \ \ \forall t \in [0,T], \\ (T, x(T)) \in C \subset [0,1] \times \mathbb{R}^n. \end{cases}$$

He formulated a problem (W), associated with (Q), of convex optimization, namely, the minimization of a linear functional under linear constraints of equality type on the set $\mathcal{W}$ of generalized flows, a weak*-convex compact set of a space of Radon measures also associated with problem (Q). Based on the apparatus of convex analysis and, in particular, on convex duality, he established a very close interconnection between problems (Q) and (W). He proved that the set $\mathcal{W}$ is the convex closure of the set of admissible arcs of the original problem (Q), and also that both problems are solvable and that, moreover, their values coincide. This makes it possible to prove a necessary and sufficient condition for optimality for problem (Q) related to well-known sufficient conditions, referred to as verification theorems, in dynamic optimization, see [22] and [28]. Simultaneously the value of problem (Q) is represented in terms of the upper envelope of smooth subsolutions of the Hamilton-Jacobi equation. This so-called "convex duality" method was first introduced by Vinter and Lewis [68], [69]. For more information about the possibility of approaching control problems via duality theory in abstract spaces, see ([34], [35], [39], [44], [66], [68] and [69]).

We remark that the problem (Q) treated by Vinter is an optimal control problem with finite horizon ($T \in [0,1]$). Moreover, Vinter affirms in [67] that his methods (generalized flows approach) do not extend to free time problems with infinite horizon ($T \in [0,+\infty[$) and do not lead an upper envelope characterization of the minimum cost, in term of smooth solutions of autonomous Hamilton-Jacobi inequality. In this subsection we present how we can using our main result prove and extend the Vinter's smooth duality for fixed and free time problems with infinite horizon. For this, we need the following technical extension of Theorem 1.1.1:

Corollary 1.3.2 *There exists $\delta_{x_0} > 0$ such that*

$$\min(P) = \sup_{\varphi \in \Phi} \varphi(0, x_0)$$

where Φ *is the set of all functions* $\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *which satisfy:*

- $\varphi \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$,
- $\varphi_t(t,x) + \langle \varphi_x(t,x), v \rangle \geq 0$, $\forall (t,x) \in \mathbb{R} \times A$, $\forall v \in F(x)$,
- $\varphi(t,x) \leq \ell(t,x)$ $\forall (t,x) \in [0, \delta_{x_0}] \times C$.

Proof. We fix a constant δ_{x_0} such that for any trajectory $(\bar{T}, \bar{x}(\cdot))$ solving the problem (P) we have $\bar{T} \leq \delta_{x_0}$ (the existence of δ_{x_0} follows from (GC)). Now let $\varphi \in \Phi$. Since φ is a C^1 function on $\mathbb{R}^{n+1}$, we have

$$\partial_L \varphi(t,x) = \{\varphi'(t,x)\}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Then we have $\varphi \in \Psi$[4]. By Theorem 1.1.1 we get

$$\min(P) = \sup_{\psi \in \Psi} \psi(0, x_0) \geq \sup_{\varphi \in \Phi} \varphi(0, x_0).$$

For the reverse inequality, let $\psi \in \Psi$. Then we have the following lemma.

Lemma 1.3.3 *Let* $(\tau, \alpha) \in \mathbb{R} \times A$ *such that* ψ *is differentiable at* (τ, α). *Then*

$$\psi_t(\tau,\alpha) + \langle \psi_x(\tau,\alpha), v \rangle \geq 0, \ \forall v \in F(\alpha).$$

The preceding lemma follows from the fact that if ψ is differentiable at (τ, α) then $\psi'(\tau,\alpha) \in \partial_L \psi(\tau,\alpha)$.
Since ψ is locally Lipschitz and by Rademacher's theorem we have that ψ is differentiable a.e $(\tau,\alpha) \in \mathbb{R} \times \mathbb{R}^n$. Now we make the *temporary hypothesis* that F is Lipschitz and $A = \mathbb{R}^n$. Then by Lemma 1.3.3 and using a standard mollification technique (convolution with mollifier sequence) there exist a sequence $\delta_i \in \mathbb{R}$ and a sequence ψ^i such that

1. $\delta_i \longrightarrow 0$,
2. $\psi_i \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$,
3. $\psi^i_t(t,x) + \langle \psi^i_x(t,x), v \rangle \geq \delta_i$, $\forall (t,x) \in \mathbb{R} \times A$, $\forall v \in F(x)$,
4. $\psi^i(t,x) \longrightarrow \psi(t,x)$ uniformly on compact sets.

[4]We can replace in Theorem 1.1.1 the condition $\psi(t,x) \leq \ell(t,x)$ $\forall (t,x) \in \mathbb{R} \times C$ by $\psi(t,x) \leq \ell(t,x)$ $\forall (t,x) \in [0, \delta_{x_0}] \times C$

Let $\varepsilon > 0$, then there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ we have

1. $\psi^i(0, x_0) \geq \psi(0, x_0) - \dfrac{\varepsilon}{2}$,
2. $\psi^i(t, x) \leq \psi(t, x) + \dfrac{\varepsilon}{2} \leq \ell(t, x) + \dfrac{\varepsilon}{2}$ for all $(t, x) \in [0, \delta_{x_0}] \times C$.

If we consider the sequence ψ^i_ε defined by $\psi^i_\varepsilon(t, x) = \psi^i(t, x) - t\delta_i - \dfrac{\varepsilon}{2} - \delta_{x_0}|\delta_i|$ then for $i \geq i_0$ we have

1. $\psi^i_\varepsilon(0, x_0) = \psi^i(0, x_0) - \dfrac{\varepsilon}{2} - \delta_{x_0}|\delta_i| \geq \psi(0, x_0) - \varepsilon - \delta_{x_0}|\delta_i|$,
2. $\psi^i_\varepsilon \in C^1(\mathbb{R}^{n+1}, \mathbb{R})$,
3. $(\psi^i_\varepsilon)_t(t, x) + \langle (\psi^i_\varepsilon)_x(t, x), v\rangle \geq 0$, $\forall (t, x) \in \mathbb{R}\times A$, $\forall v \in F(x)$,
4. $\psi^i_\varepsilon(t, x) \leq \ell(t, x) - t\delta_i - \delta_{x_0}|\delta_i| \leq \ell(t, x)$ for all $(t, x) \in [0, \delta_{x_0}] \times C$.

Hence for $i \geq i_0$

1. $\psi^i_\varepsilon(0, x_0) \geq \psi(0, x_0) - \varepsilon - \delta_{x_0}|\delta_i|$,
2. $\psi^i_\varepsilon \in \Phi$.

Therefore for all $i \geq i_0$ we have

$$\sup_{\varphi\in\Phi} \varphi(0, x_0) \geq \psi^i_\varepsilon(0, x_0) \geq \psi(0, x_0) - \varepsilon - \delta_{x_0}|\delta_i|$$

taking $i \longrightarrow +\infty$ we get that

$$\sup_{\varphi\in\Phi} \varphi(0, x_0) \geq \psi(0, x_0) - \varepsilon$$

and this is true for all $\psi \in \Psi$ and for all $\varepsilon > 0$.
Hence

$$\min(P) = \sup_{\psi\in\Psi} \psi(0, x_0) \leq \sup_{\varphi\in\Phi} \varphi(0, x_0).$$

To remove the need for the Lipschitz hypothesis on F and the assumption $A = \mathbb{R}^n$, it is sufficient to use the sequence F_k and the penalization term $k\int_0^T d_A(x(t))\,dt$ as in the proof of Theorem 1.1.1. For details, see Appendix A.1. □

It is clear that Corollary 1.3.2 leads to a version of the necessary and sufficient conditions of Corollary 1.3.1 in which only smooth semisolutions are used. Let us now examine more closely this type of 'smooth duality' in the three special cases of Subsection 1.3.2.

1. *Fixed time problem*: We take $\ell(t,x) = I_{\{T_0\}\times C}(t,x)+\ell_0(x)$ where $T_0 \geq 0$ is fixed and $\ell_0 : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous function, then by Corollary 1.3.2 ($\delta_{x_0} = T_0$) we obtain the following smooth duality:

$$\min(P) = \sup_{\varphi\in\Phi} \varphi(0,x_0)$$

where Φ is the set of all functions $\varphi : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^{n+1},\mathbb{R})$,
- $\varphi_t(t,x) + \langle\varphi_x(t,x), v\rangle \geq 0$, $\forall (t,x)\in\mathbb{R}\times A$, $\forall v\in F(x)$,
- $\varphi(T_0,x) \leq \ell_0(x)$ $\forall x\in C$.

2. *Free time problem with finite horizon*: We take $\ell(t,x) = I_{[0,1]\times C}(t,x) + \ell_0(x)$ where $\ell_0 : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous function. By Corollary 1.3.2 ($\delta_{x_0} = 1$) we obtain

$$\min(P) = \sup_{\varphi\in\Phi} \varphi(0,x_0)$$

where Φ is the set of all functions $\varphi : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^{n+1},\mathbb{R})$,
- $\varphi_t(t,x) + \langle\varphi_x(t,x), v\rangle \geq 0$, $\forall (t,x)\in\mathbb{R}\times A$, $\forall v\in F(x)$,
- $\varphi(t,x) \leq \ell_0(x)$ $\forall (t,x)\in[0,1]\times C$.

We remark that we obtain the autonomous version of Vinter's duality (F and ℓ_0 are independent of t). There is a well-known route for getting the nonautonomous results from the autonomous case. This technique is called *state augmentation*, [22, Chapter 4]. For details, see Appendix A.2.

3. *Free time problem with infinite horizon*: We take $\ell(t,x) = t + \ell_0(x)$ $\forall (t,x)\in\mathbb{R}\times\mathbb{R}^n$ where $\ell_0 : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous function bounded below. This is the most familiar case of the minimal time problem, and Vinter has remarked [67] that his generalized flows approach does not appear to yield a duality involving

solutions of the autonomous Hamilton-Jacobi inequality (as one would hope). However, we obtain the following[5]:

$$\min(P) = \sup_{\varphi \in \Phi} \varphi(x_0)$$

where Φ is the set of all functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$,
- $1 + \langle \varphi'(x), v \rangle \geq 0$, $\forall x \in A$, $\forall v \in F(x)$,
- $\varphi(x) \leq \ell_0(x)$ $\forall x \in C$.

A well-known and more special case of the present framework involves the minimal time function associated to the target C and under the state constraint A:

$$T_A(\alpha, C) := \begin{cases} \inf T \geq 0, \\ \dot{x}(t) \in F(x(t)) \ \ a.e.\ t \in [0, T] \\ x(0) = \alpha, \\ x(t) \in A \ \ \forall t \in [0, T], \\ x(T) \in C. \end{cases}$$

We then obtain the following characterization of $T_A(\cdot, C)$, which appear to be new at a technical level:

Corollary 1.3.4

$$T_A(\alpha, C) = \sup_{\varphi \in \Phi} \varphi(\alpha)$$

where Φ is the set of all functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$,
- $1 + \langle \varphi'(x), v \rangle \geq 0$, $\forall x \in A$ $\forall v \in F(x)$,
- $\varphi(x) \leq 0$ $\forall x \in C$.

[5]This smooth duality follows using our nonsmooth duality presented in Subsection 3.2 for the free time with infinite horizon case, and using the same techniques as in Corollary 1.3.2. For details, see Appendix A.3.

1.4 Appendix A

Appendix A.1 Here we present how we can prove in Corollary 1.3.2 the inequality

$$\min(P) \leq \sup_{\varphi \in \Phi} \varphi(0, x_0),$$

when F is upper semicontinuous and in the presence of the state constraint A. First, we consider for all $k \in \mathbb{N}$ the following optimal control problem:

$$(P_k) \begin{cases} \text{Minimize } \ell(T, x(T)) + k \int_0^T d_A(x(t))\, dt, \\ T \geq 0, \\ \dot{x}(t) \in F_k(x(t)) \ \ a.e.\ t \in [0, +\infty[, \\ x(0) = x_0, \\ x(T) \in C. \end{cases}$$

Then as Lemma 1.2.1 we have the following.

Lemma 1.4.1 *There exists a sequence λ_n strictly increasing in $\mathbb{N}^*$ such that:*

$$\lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) = \min(P)$$

Moreover, we can show (see the proof of Lemma 1.2.1) that there exists a sequence $(T_{\lambda_n}, x_{\lambda_n}(\cdot))$ of solution of (P_{λ_n}) such that

- $T_{\lambda_n} \longrightarrow \bar{T}$.
- $x_{\lambda_n} \longrightarrow \bar{x}$ uniformly on compact interval.
- $(\bar{T}, \bar{x}(\cdot))$ is a solution of (P).

Then we can assume that $T_{\lambda_n} \leq \delta_{x_0}$ for all $n \in \mathbb{N}$. We take $M_{x_0} > 0$ such that $x(t) \in \bar{B}(0; M_{x_0})$ for all $t \in [0, \delta_{x_0}]$ and for all $x(\cdot)$ a trajectory of F with $x(0) = x_0$. It is easy to verify that (P_{λ_n}) is exactly the following problem:

$$\begin{cases} \text{Minimize } \hat{\ell}(T, z(T)), \\ T \geq 0, \\ \dot{z}(t) \in \hat{F}_{\lambda_n}(z(t)) \ \ a.e.\ t \in [0, +\infty[, \\ z(0) = (0, x_0), \\ z(T) \in [0, M_{\lambda_n}] \times C, \end{cases}$$

where $\hat{F}_{\lambda_n}$ is an augmented locally Lipschitz multifunction defined as follows $\forall (y,x) \in \mathbb{R} \times \mathbb{R}^n$, $\hat{F}_{\lambda_n}(y,x) = \{\lambda_n d_A(x)\} \times F_{\lambda_n}(x)$, $\hat{\ell}$ is the lower semicontinuous function defined by $\hat{\ell}(t,y,x) = \ell(t,x) + y$, $\forall (t,y,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ and $M_{\lambda_n} := \lambda_n \delta_{x_0} \sup_{\alpha \in \bar{B}(0;M_{x_0})} d_A(\alpha)$. Hence we have

$$\min(P_{\lambda_n}) = \sup_{\varphi \in \hat{\Phi}_{\lambda_n}} \varphi(0,0,x_0)^6$$

where $\hat{\Phi}$ is the set of all functions $\varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$
- $\varphi_t(t,y,x) + \lambda_n d_A(x) \varphi_y(t,y,x) + \langle \varphi_x(t,y,x), v \rangle \geq 0$, $\forall (t,y,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, $\forall v \in F(x)$,
- $\varphi(t,y,x) \leq \ell(t,x) + y$, $\forall (t,y,x) \in [0,\delta_{x_0}] \times [0, M_{\lambda_n}] \times C$.

Now let $\varphi \in \Phi_{\lambda_n}$. We consider the function $\hat{\varphi} : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\hat{\varphi}(t,x) = \varphi(t,0,x)$. Then clearly we have

- $\hat{\varphi}(0,x_0) = \varphi(0,0,x_0)$.
- $\hat{\varphi} \in \Phi$.

Then for all $n \in \mathbb{N}$ we have

$$\sup_{\varphi \in \Phi_{\lambda_n}} \varphi(0,0,x_0) \leq \sup_{\varphi \in \Phi} \varphi(0,x_0),$$

hence

$$\min(P) = \lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) = \lim_{n \longrightarrow +\infty} \sup_{\varphi \in \Phi_{\lambda_n}} \varphi(0,0,x_0) \leq \sup_{\varphi \in \Phi} \varphi(0,x_0),$$

which completes the proof.

Appendix A.2 We present in this appendix how we can get the (nonautonomous) duality presented in [66] using our autonomous smooth duality.

[6] This follows since $\hat{F}_{\lambda_n}$ is Lipschitz and we don't have a state constraint in (P_{λ_n}), see the proof of Corollary 1.3.2 when we make the temporary hypotheses F is Lipschitz and $A = \mathbb{R}^n$.

We recall that if we take $\ell(t,x) = I_{[0,1]\times C} + \ell_0(x)$, in our problem (P) then our smooth duality give that

$$\min(P) = \sup_{\varphi\in\Phi} \varphi(0,x_0)$$

where Φ is the set of all functions $\varphi : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^{n+1},\mathbb{R})$,
- $\varphi_t(t,x) + \langle \varphi_x(t,x), v\rangle \geq 0$, $\forall (t,x)\in \mathbb{R}\times A$, $\forall v\in F(x)$,
- $\varphi(t,x) \leq \ell_0(x)$ $\forall (t,x)\in [0,1]\times C$.

Now we consider the nonautonomous problem (Q) in which F and ℓ depend on t as well as on x. We proceed to define an augmented multifunction $\bar{F}$ which satisfies the same hypotheses as F :

$$\bar{F}(\bar{x}) = \bar{F}(x^0,x) = \{1\}\times F(x^0,x).$$

Then if $\bar{x}(\cdot) = (x^0(\cdot), x(\cdot))$ is a trajectory for $\bar{F}$, with $\bar{x}(a) = \bar{x}_0 = (x_0^0, x_0)$, it follows that x is a trajectory for F with $x(a) = x_0$, and that $x^0(t) = x_0^0 + t - a$. Conversely, if x is a trajectory for F, we augment it to trajectory $\bar{x}$ for $\bar{F}$ by setting $x^0(t) = x_0^0 + t - a$ (for any choice of x_0^0). It is easy to see that (Q) is exactly the following autonomous problem:

$$\begin{cases} \text{Minimize } \ell(\bar{x}(T)), \\ \dot{\bar{x}}(t) \in \bar{F}(\bar{x}(t)) \ a.e\ t\in[0,+\infty[, \\ \bar{x}(0) = x_0, \\ \bar{x}(t)\in A \ \ \forall t\in[0,\,T], \\ \bar{x}(T)\in C. \end{cases}$$

Then by our smooth duality we have

$$\min(Q) = \min(\bar{Q}) = \sup_{\varphi\in\bar{\Phi}} \varphi(0,0,x_0) \tag{1.12}$$

where $\bar{\Phi}$ is the set of all functions $\varphi : \mathbb{R}\times\mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}\times\mathbb{R}^{n+1},\mathbb{R})$
- $\varphi_t(t,\bar{x}) + \langle \varphi_{\bar{x}}(t,\bar{x}), v\rangle \geq 0$, $\forall (t,\bar{x})\in \mathbb{R}\times A$, $\forall v\in \bar{F}(\bar{x})$,

- $\varphi(t,\bar{x}) \leq \ell(\bar{x})$, $\forall (t,\bar{x}) \in [0,1] \times C$.

For any $\varphi \in \bar{\Phi}$, if we consider $\hat{\varphi} : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\hat{\varphi}(t,x) = \varphi(t,t,x)$, $\forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$, then we get:

- $\hat{\varphi} \in \Phi$,
- $\hat{\varphi}(0,x_0) = \varphi(0,0,x_0)$,

where Φ is the set of all functions $\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$,
- $\varphi_t(t,x) + \langle \varphi_x(t,x), v\rangle \geq 0$, $\forall (t,x) \in A$, $\forall v \in F(t,x)$,
- $\varphi(t,x) \leq \ell(t,x)$, $\forall (t,x) \in C$.

Therefore

$$\sup_{\varphi\in\bar{\Phi}} \varphi(0,0,x_0) \leq \sup_{\varphi\in\Phi} \varphi(0,x_0) \tag{1.13}$$

by (1.12) and (1.13) we get that

$$\min(Q) \leq \sup_{\varphi\in\Phi} \varphi(0,x_0).$$

For the reverse inequality, let $\varphi \in \Phi$. As above, we consider $\hat{\varphi} : \mathbb{R}\times\mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ such that $\hat{\varphi}(t,\bar{x}) = \varphi(\bar{x})$, $\forall (t,\bar{x}) \in \mathbb{R} \times \mathbb{R}^{n+1}$. Then we have:

- $\hat{\varphi} \in \bar{\Phi}$,
- $\hat{\varphi}(0,0,x_0) = \varphi(0,x_0)$.

Hence

$$\sup_{\varphi\in\bar{\Phi}} \varphi(0,0,x_0) \geq \sup_{\varphi\in\Phi} \varphi(0,x_0) \tag{1.14}$$

by (1.12) and (1.14) we get that

$$\min(Q) \leq \sup_{\varphi\in\Phi} \varphi(0,x_0).$$

The nonautonomous case follows.

Appendix A.3 Here we give the complete proof of our smooth duality for the free time problem with infinite horizon case. We recall that in

this case we have $\ell(t,x) = t + \ell_0(x)\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n$ and our nonsmooth duality gives that

$$\min(P) = \sup_{\psi \in \Psi} \psi(x_0)$$

where Ψ is the set of all functions $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- ψ is locally Lipschitz on $\mathbb{R}^n$,
- $1 + h(x, \partial_L \psi(x)) \geq 0$, $\forall x \in A$, $\forall v \in F(x)$,
- $\psi(x) \leq \ell_0(x)\ \forall x \in C$.

Our goal is to show that

$$\min(P) = \sup_{\varphi \in \Phi} \varphi(x_0)$$

where Φ is the set of all functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$,
- $1 + \langle \varphi'(x), v \rangle \geq 0$, $\forall x \in A$, $\forall v \in F(x)$,
- $\varphi(x) \leq \ell_0(x)\ \forall x \in C$.

First we note that the inequality

$$\min(P) \geq \sup_{\varphi \in \Phi} \varphi(x_0)$$

follows exactly as in the proof of Corollary 1.3.2.
For the reverse inequality, we take $\psi \in \Psi$. We proceed as in the proof of Corollary 1.3.2 and then we get that there exist a sequence $\delta_i \in \mathbb{R}$ and a sequence ψ^i such that

1. $\delta_i \longrightarrow 0$,
2. $\psi_i \in C^1(\mathbb{R}^n, \mathbb{R})$,
3. $1 + \langle (\psi^i)'(x), v \rangle \geq \delta_i \geq -|\delta_i|$, $\forall x \in A$, $\forall v \in F(x)$,
4. $\psi^i(x) \longrightarrow \psi(x)$ uniformly on compact sets.

Now let $\varepsilon > 0$, then there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ we have

1. $\psi^i(x_0) \geq \psi(x_0) - \dfrac{\varepsilon}{2}$,
2. $\psi^i(x) \leq \psi(x) + \dfrac{\varepsilon}{2}$ for all $x \in C$.

We consider the sequence ψ^i_ε defined by

$$\psi^i_\varepsilon(x) = \frac{\psi^i(x)}{1+|\delta_i|} - \frac{M|\delta_i|}{1+|\delta_i|} - \frac{\varepsilon}{2(1+|\delta_i|)},$$

where $M := \max_{x \in C} -\psi(x)$. Then for $i \geq i_0$ we have

1. $\psi^i_\varepsilon(x_0) = \dfrac{\psi^i(x_0)}{1+|\delta_i|} - \dfrac{M|\delta_i|}{1+|\delta_i|} - \dfrac{\varepsilon}{2(1+|\delta_i|)} \geq \dfrac{\psi(x_0)}{1+|\delta_i|} - \dfrac{M|\delta_i|}{1+|\delta_i|} - \dfrac{\varepsilon}{1+|\delta_i|}$,
2. $\psi^i_\varepsilon \in C^1(\mathbb{R}^n, \mathbb{R})$,
3. $1 + \langle (\psi^i_\varepsilon)'(x), v\rangle \geq 0$, $\forall x \in A$, $\forall v \in F(x)$,
4. $\psi^i_\varepsilon(x) \leq \dfrac{\psi(x)}{1+|\delta_i|} - \dfrac{M|\delta_i|}{1+|\delta_i|} = \psi(x) - \dfrac{\psi(x)|\delta_i|}{1+|\delta_i|} - \dfrac{M|\delta_i|}{1+|\delta_i|} \leq \ell_0(x)$ for all $x \in C$.

Hence for $i \geq i_0$

1. $\psi^i_\varepsilon(x_0) \geq \dfrac{\psi(x_0)}{1+|\delta_i|} - \dfrac{M|\delta_i|}{1+|\delta_i|} - \dfrac{\varepsilon}{1+|\delta_i|}$,
2. $\psi^i_\varepsilon \in \Phi$.

Therefore for all $i \geq i_0$ we have

$$\sup_{\varphi \in \Phi} \varphi(x_0) \geq \psi^i_\varepsilon(x_0) \geq \frac{\psi(x_0)}{1+|\delta_i|} - \frac{M|\delta_i|}{1+|\delta_i|} - \frac{\varepsilon}{1+|\delta_i|},$$

taking $i \longrightarrow +\infty$ we get that

$$\sup_{\varphi \in \Phi} \varphi(x_0) \geq \psi(x_0) - \varepsilon$$

and this is true for all $\psi \in \Psi$ and for all $\varepsilon > 0$.
Hence

$$\min(P) = \sup_{\psi \in \Psi} \psi(x_0) \leq \sup_{\varphi \in \Phi} \varphi(x_0).$$

The proof is achieved.

Chapter 2

The bilateral minimal time function

2.1 Introduction

Let F be a multifunction mapping points x in $\mathbb{R}^n$ to subsets $F(x)$ of $\mathbb{R}^n$ and let S be a nonempty subset of $\mathbb{R}^n$. The minimal time control problem associated to the target set S is a problem in which the goal is to steer an initial point α to S along a trajectory of the system F in minimal time. The minimal time value is denoted by $T(\alpha, S)$, which could be $+\infty$ if no trajectory from α can reach S.

The minimal time control problem is one of the most classical problems in control theory. It appears already in Carathéodory's book [15] and it has a large literature. The function $T(\cdot, S)$ is well studied and the property of *small time controllability* plays an important role in this study. Indeed, this property is equivalent to the continuity of $T(\cdot, S)$. There is a considerable literature devoted to local controllability, see [63]. The Lipschitz continuity of $T(\cdot, S)$ is first studied in [51] for $S = \{0\}$. In this paper Petrov defined the *Petrov condition* and showed the equivalence between this condition and the Lipschitz continuity of $T(\cdot, S)$. This result was extended to arbitrarily closed subsets by Soravia in [56]. In [64], Veliov gives a general result for the Lipschitz continuity of $T(\cdot, S)$, allowing the multifunction F to be nonautonomous and depend measurably on t. On the other hand, simple examples show that $T(\cdot, S)$ fails to be everywhere differentiable, in general. Differentia-

bility results for $T(\cdot, S)$ have been proved for linear systems if ∂S is smooth, see [11] and [38]. In [13], Cannarsa and Sinestrari study the semiconcavity of this function in analogy with the distance function $d_S(\cdot)$. We also find in this paper a semiconvexity result for the case where S is convex and the control system is linear. For a thorough history of such results, we invite the reader to see [6, Chapter 4].

Another important result for the minimal time function is the characterization as a solution of a Hamilton-Jacobi equation. Solving the Hamilton-Jacobi equation in some nonclassical sense has developed into an active research area with several different schools of thought participating. In this chapter, we are interested in *proximal solutions*. This concept of solution appeared in Clarke and Ledyaev [21], where the various concepts were also unified. We can find in the literature many results concerning the characterization of $T(\cdot, S)$ as a solution of a Hamilton-Jacobi equation. The first result in this direction was found by Bardi in [5] using the viscosity methods. In [57], Soravia extended these results to allow for noncontrollability and more general boundary conditions. Other related results are proved in [71]. In this paper Wolenski and Zhuang show using an invariance-based approach and without controllability assumptions that $T(\cdot, \beta)$ is the unique proximal solution of the Hamilton-Jacobi equation that satisfies certain boundary conditions, see [71, Theorem 3.2]. For more information about the possibility of characterizing $T(\cdot, S)$ as a solution of a Hamilton-Jacobi equation, see [6], [7], [8], [16] and [32].

In this chapter, we study the minimal time function as a function of two variables. This *bilateral minimal time function*, denoted by $T(\cdot, \cdot)$, is defined as follows. For $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$, $T(\alpha, \beta)$ is the minimum time taken by a trajectory to go from α to β (when no such trajectory exists, $T(\alpha, \beta)$ is taken to be $+\infty$). The (unilateral) minimal time function associated to $S := \{\beta\}$ is $T(\cdot, \beta)$. The purpose of this chapter is to study the properties of $T(\cdot, \cdot)$. We give a necessary and sufficient conditions for the continuity and the Lipschitz continuity, and we show a semiconvexity result in the linear case. We calculate the proximal subgradient of this function and then we give a characterization as a proximal solution of a system of partial Hamilton-Jacobi equations. In the next chapters, we use this new function for the construction of solutions of Hamilton-Jacobi equations and to study the existence of semigeodesic and minimal loop trajectories at a given point α.

In the next section we give some definitions and establish some notation. We present some known results for the minimal time function in Section 2.3. Section 2.4 is devoted to the bilateral minimal time function $T(\cdot,\cdot)$.

2.2 Definitions and notations

Let F be a multifunction mapping $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$. We assume that F satisfies the following hypotheses:

- For every $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.
- The linear growth condition: For some positive constants γ and c, and for all $x \in \mathbb{R}^n$,
$$v \in F(x) \Longrightarrow \|v\| \leq \gamma\|x\| + c.$$
- F is locally Lipschitz; that is, every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that
$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

The bilateral minimal time function $T(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, +\infty]$ is defined as follows:

$$T(\alpha,\beta) := \begin{cases} \text{Inf } T \geq 0, \\ \dot{x}(t) \in F(x(t)) \text{ a.e. } t \in [0,T], \\ x(0) = \alpha \text{ and } x(T) = \beta. \end{cases}$$

If no trajectory between α and β exists, then $T(\alpha,\beta) = +\infty$. Clearly we have $T(\alpha,\alpha) = 0$ for all $\alpha \in \mathbb{R}^n$. We define

$$\mathcal{R}_+^\beta(t) := \{\alpha \in \mathbb{R}^n \ : \ T(\beta,\alpha) < t\}, \quad t > 0,$$

the set of points reachable from β in time less than t.
Similarly, we introduce

- $\mathcal{R}_+^\beta := \bigcup_{t>0} \mathcal{R}_+^\beta(t) = \{\alpha \in \mathbb{R}^n \ : \ T(\beta,\alpha) < +\infty\}$,
- $\mathcal{R}_-^\beta(t) := \{\alpha \in \mathbb{R}^n \ : \ T(\alpha,\beta) < t\}, \quad t > 0,$
- $\mathcal{R}_-^\beta := \bigcup_{t>0} \mathcal{R}_-^\beta(t) = \{\alpha \in \mathbb{R}^n \ : \ T(\alpha,\beta) < +\infty\}$,

- $\mathcal{R}(t) := \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ T(\alpha, \beta) < t\}, \quad t > 0,$
- $\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t) = \{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ T(\alpha, \beta) < +\infty\}.$

It is easy to see that we have

1. $T(\cdot, \cdot)$ and $T(\cdot, \beta)$ are lower semicontinuous.
2. If $T(\alpha, \beta) < +\infty$ then the minimum defining $T(\alpha, \beta)$ is attained.
3. For all (α, β, γ) we have the following triangle inequality:

$$T(\alpha, \beta) \leq T(\alpha, \gamma) + T(\gamma, \beta).$$

Now we give some definitions which will play an important role in what follows. We characterize these properties in the next section.

Definition 2.2.1 *We say that F:*

- *is β-LC (β-locally controllable), if $\beta \in int\mathcal{R}_-^\beta$.*
- *is β-STLC (β-small-time locally controllable), if $\beta \in int\mathcal{R}_-^\beta(t)\ \forall\ t > 0$; that is, $\forall\ t > 0\ \exists\ \delta > 0$ such that $T(\cdot, \beta) < t$ on $B(\beta; \delta)$.*
- *satisfies the "positive basis condition" at β, if $h(\beta, \gamma) < 0$ for any unit vector γ.*

2.3 The (unilateral) minimal time function

In this section, we give some known results about the minimal time function $T(\cdot, \beta)$. Our principal reference is [6, Chapter 4], see also [22] and [71]. We begin by the following proposition.

Proposition 2.3.1 *Let $\beta \in \mathbb{R}^n$. Then*

1. *F is β-LC iff $\mathcal{R}_-^\beta$ is open.*
2. *F is β-STLC $\Longrightarrow h(\beta, \gamma) \leq 0$ for any unit vector $\gamma \Longleftrightarrow 0 \in F(\beta)$.*
3. *The following statements are equivalent:*

(a) *F satisfies the positive basis condition on β.*

(b) *There exist $r > 0$ and $\delta > 0$ such that for any $\beta' \in B(\beta; r)$ and for any unit vector γ we have $h(\beta', \gamma) < -\delta$.*

(c) *$0 \in int\, F(\beta)$.*

Proof. 1) Clearly we have: $\mathcal{R}_-^\beta$ is open $\Longrightarrow$ F is β-LC. For the converse, let $\alpha \in \mathcal{R}_-^\beta$. Then there exists a trajectory $\bar{x} : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F such that $\bar{x}(0) = \alpha$ and $\bar{x}(T(\alpha, \beta)) = \beta$ ($\bar{x}$ is the minimal trajectory between α and β). By Lemma 1.2.3 there exists $\rho > 0$ such that

$$A(0, \alpha; T(\alpha, \beta)) \subseteq A(0, \alpha'; T(\alpha, \beta)) + \rho \|\alpha - \alpha'\| \bar{B} \tag{2.1}$$

for all $\alpha' \in B(\alpha; 1)$. Since F is β-LC, there exists $\varepsilon > 0$ such that $B(\beta; \varepsilon) \subset \mathcal{R}_-^\beta$. Then for $0 < \delta < \min\{1, \dfrac{\varepsilon}{2\rho}\}$ and by (2.1) we get that for all $\alpha' \in B(\alpha; \delta)$ there exists a trajectory $y : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F such that $y(0) = \alpha'$ and $y(T(\alpha, \beta)) \in B(\beta; \varepsilon)$. The result follows.

2) It easy to show that: $[h(\beta, \gamma) \leq 0$ for any unit vector $\gamma] \Longleftrightarrow 0 \in F(\beta)$. Now we show the first implication. We reason by the absurd. Let $\beta \in \mathbb{R}^n$ and assume that there exist a unit vector γ and $\varepsilon > 0$ such that $h(\beta, \gamma) > \varepsilon$. We consider the sequence $\alpha_n := \beta + \dfrac{\gamma}{n}$. There exist a sequence $x_n : [0, +\infty[$ of trajectories of F and $T_n \geq 0$ such that $x_n(0) = \alpha_n$, $x_n(T_n) = \beta$ and $T(\alpha_n, \beta) \leq T_n < T(\alpha_n, \beta) + \dfrac{1}{n}$. We have

$$\frac{\gamma}{n} = \alpha_n - \beta = -\int_0^{T_n} \dot{x}_n(t)\, dt. \tag{2.2}$$

In the other hand, using Lemma 0.3.1 and since F is β-STLC ($T(\alpha_n, \beta) \longrightarrow 0$) there exists a constant $M > 0$ such that

$$\|x_n(t) - \beta\| \leq \frac{1}{n} + MT_n \tag{2.3}$$

for all $t \in [0, T_n]$. Moreover, since F is locally Lipschitz, there exists $K > 0$ such that

$$\dot{x}_n(t) \in F(\beta) + K\|x(t) - \beta\| \bar{B}, \tag{2.4}$$

for all $t \in [0, T_n]$. Then by (2.2), (2.3), (2.4) and using the fact that $h(\beta, \gamma) > \varepsilon$ we get

$$\frac{1}{n} \leq T_n(K(\frac{1}{n} + MT_n) - \varepsilon)$$

but $T_n \longrightarrow 0$, hence the right-hand side can be made negative for n sufficiently large, a contradiction.

3) $(a) \Longleftrightarrow (b)$: Follows by a simple continuity argument.

$(a) \Longrightarrow (c)$: We reason by the absurd. Assume that $0 \notin \operatorname{int} F(\beta)$. Then there exists a sequence $v_n \in \mathbb{R}^n$ such that $v_n \longrightarrow 0$ and $v_n \notin F(\beta)$. Since $F(\beta)$ is a compact and convex set, there exists a sequence $p_n \in \mathbb{R}^n$ such that $\|p_n\| = 1$ and $h(\beta, p_n) > \langle p_n, v_n \rangle$, see [22, Exercise 4.1.15]. We can assume that $p_n \longrightarrow p$ such that $\|p\| = 1$. Hence $0 > h(\beta, p) \geq 0$, contradiction.

$(c) \Longrightarrow (a)$: Let γ be a unit vector in $\mathbb{R}^n$. Since $0 \in \operatorname{int} F(\beta)$, there exist $r > 0$ such that $\bar{B}(0; r) \subset F(\beta)$ and then $-\gamma r \in F(\beta)$. Hence, $h(\beta, \gamma) \leq \langle -\gamma r, \gamma \rangle = -r < 0$. The result follows. □

Proposition 2.3.2 *Let $\beta \in \mathbb{R}^n$ and suppose that $0 \in int F(\beta)$. Then there exist $r > 0$ and $\delta > 0$ such that $T(\alpha, \beta) \leq \dfrac{\|\alpha - \beta\|}{\delta}$ for all $\alpha \in B(\beta; r)$.*

Proof. Let $\beta \in \mathbb{R}^n$ and suppose that $0 \in \operatorname{int} F(\beta)$. Then by Proposition 2.3.1 there exist $r > 0$ and $\delta > 0$ such that for any $\beta' \in B(\beta; r)$ and for any unit vector γ we have $h(\beta', \gamma) < -\delta$. We set $V(\cdot) = \dfrac{1}{\delta}\| \cdot - \beta\|$. Then we have $h(\alpha, \partial_P V(\alpha)) < -1$ for all $\alpha \in B(\beta; r) \backslash \{\beta\}$. Hence the system $(t+V, \{1\} \times F)$ is weakly decreasing on $\mathbb{R} \times B(\beta; r) \setminus \{\beta\}$. Now let $\alpha \in B(\beta; r) \setminus \{\beta\}$, by the weak decrease property, there exists a trajectory $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F having the property that for any interval $[0, T]$ for which $x([0, T]) \subset B(\beta; r) \setminus \{\beta\}$, we have

$$V(\alpha) \geq t + V(x(t)) \ \forall t \in [0, T].$$

Let $\bar{T} := \inf\{t \geq 0 : x(t) \in \operatorname{comp}\{B(\beta; r) \setminus \{\beta\}\}\}$. We claim that $\bar{T} < +\infty$. Indeed, if not then for all $t \geq 0$ we have $x(t) \in B(\beta; r) \setminus \{\beta\}$. Hence

$$V(\alpha) \geq t + V(x(t))$$

for all $t \geq 0$, which gives a contradiction. Therefore $\bar{T} < +\infty$. Moreover we have :

- $\bar{T} \neq 0$.

- $x(\bar{T}) \in \text{comp}\{B(\beta; r) \setminus \{\beta\}\}$.
- For all $T < \bar{T}$ we have $x(T) \in B(\beta; r) \setminus \{\beta\}$.

Hence

$$V(\alpha) \geq \bar{T} + V(x(\bar{T})). \tag{2.5}$$

We claim that $x(\bar{T}) = \beta$. Indeed, if not then $\|\beta - x(\bar{T})\| \geq r$. By (2.5) we have

$$\frac{r}{\delta} > V(\alpha)) \geq \bar{T} + V(x(\bar{T})) \geq \bar{T} + \frac{r}{\delta}$$

which gives a contradiction. Therefore $x(\bar{T}) = \beta$. Using again (2.5) we get that

$$V(\alpha) \geq \bar{T} + V(x(\bar{T})) = \bar{T} \geq T(\alpha, \beta)$$

which completes the proof. □

Proposition 2.3.3 *Let $\beta \in \mathbb{R}^n$. Then*

$$0 \in \mathit{int}\, F(\beta) \Longrightarrow F \textit{ is } \beta\textit{-STLC} \Longrightarrow F \textit{ is } \beta\textit{-LC}.$$

Proof. It is clear that: F is β-STLC $\Longrightarrow$ F is β-LC. The first implication follows from Proposition 2.3.2. □

Remark 2.3.4 *We note that $0 \in F(\beta)$ is not a necessary condition for β-LC. See Example 4.5.2.*

The following proposition gives a necessary and sufficient condition for the continuity of $T(\cdot, \beta)$.

Proposition 2.3.5 *Let $\beta \in \mathbb{R}^n$. Then the following statements are equivalent:*

(i) F is β-STLC.

(ii) $T(\cdot, \beta)$ is continuous at β.

(iii) $\mathcal{R}_-^\beta$ is open, $T(\cdot, \beta)$ is continuous in $\mathcal{R}_-^\beta$ and for any $\alpha_0 \in \partial\mathcal{R}_-^\beta$ we have

$$\lim_{\alpha \longrightarrow \alpha_0} T(\alpha, \beta) = +\infty.$$

Proof. Clearly we have $(i) \Longleftrightarrow (ii)$ and $(iii) \Longrightarrow (ii)$. Let us show that $(ii) \Longrightarrow (iii)$. We have that $\mathcal{R}_-^\beta$ is open since $(i) \Longleftrightarrow (ii)$. Let $\alpha \in \mathcal{R}_-^\beta \setminus \{\beta\}$. We will show that $T(\cdot, \beta)$ is continuous at α. By Lemma 1.2.3 ($a = 0$, $b = T(\alpha, \beta)$) there exists $\rho > 0$ such that for all α_1, $\alpha_2 \in B(\alpha; 1)$ we have

$$A(0, \alpha_1; T(\alpha, \beta)) \subseteq A(0, \alpha_2; T(\alpha, \beta)) + \rho \|\alpha_1 - \alpha_2\| \bar{B}. \tag{2.6}$$

Let α_n be a sequence such that $\alpha_n \longrightarrow \alpha$. By (2.6) ($c = T(\alpha, \beta)$) and for n sufficiently large, there exists a trajectory x_n of F on $[0, +\infty[$ such that $x_n(0) = \alpha_n$ and

$$\|\beta - x_n(T(\alpha, \beta))\| \leq \rho \|\alpha_n - \alpha\|.$$

We set $\beta_n = x_n(T(\alpha, \beta))$ then for n sufficiently large, we have

$$T(\alpha_n, \beta_n) \leq T(\alpha, \beta). \tag{2.7}$$

By the triangle inequality and using (2.7) we have

$$T(\alpha_n, \beta) \leq T(\alpha_n, \beta_n) + T(\beta_n, \beta) \leq T(\alpha, \beta) + T(\beta_n, \beta). \tag{2.8}$$

For $\varepsilon > 0$ and since $T(\cdot, \beta)$ is lower semicontinuous in $\mathbb{R}^n$ and continuous at β and using (2.8) we get that

$$-\varepsilon + T(\alpha, \beta) \leq T(\alpha_n, \beta) \leq T(\alpha, \beta) + \varepsilon.$$

The continuity follows.
Now let $\alpha_0 \in \partial \mathcal{R}_-^\beta$ and suppose that there exist a constant M and a sequence $\alpha_n \in \mathcal{R}_-^\beta$ such that $\|\alpha_n - \alpha_0\| \leq \dfrac{1}{n}$ and $T(\alpha_n, \beta) \leq M$. We consider the minimal trajectory x_n between α_n and β. We have $x_n(0) = \alpha_n$ and $x_n(T(\alpha_n, \beta)) = \beta$. By the compactness of trajectories, there exist a trajectory $\bar{x}$ of F and a subsequence (we do not relabel) of x_n having the property that x_n converges uniformly to $\bar{x}$ on any interval $[0, b]$. Since $0 \leq T(\alpha_n, \beta) \leq M$ we can assume that $T(\alpha_n, \beta) \longrightarrow T \in [0, M]$. Then we have $\bar{x}(0) = \alpha_0$ and $\bar{x}(T) = \beta$ and hence $\alpha_0 \in \mathcal{R}_-^\beta$ which gives a contradiction since $\mathcal{R}_-^\beta$ is an open subset. □
For the Lipschitz continuity, we have the following proposition.

Proposition 2.3.6 *Let $\beta \in \mathbb{R}^n$, then the following statements are equivalent:*

(i) $\mathcal{R}_-^\beta$ *is open and* $T(\cdot,\beta)$ *is locally Lipschitz in* $\mathcal{R}_-^\beta$.

(ii) $T(\cdot,\beta)$ *is Lipschitz near* β.

(iii) $0 \in int F(\beta)$.

Proof. Clearly we have (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iii): We proceed as in the proof of 2) of Proposition 2.3.1. But here we have that $h(\beta,\gamma) \geq 0$ and since $T(\cdot,\beta)$ is Lipschitz near β, there exists $C > 0$ such that $T_n \leq \dfrac{1}{n}(C+1)$. Then we find that

$$\frac{1}{n} \leq \frac{1}{n^2}(C+1)K(1+M(C+1))$$

and then

$$1 \leq \frac{1}{n}(C+1)K(1+M(C+1))$$

which gives the required contradiction.

(iii)$\Longrightarrow$(i): Since $0 \in \operatorname{int} F(\beta)$ and by Proposition 2.3.3, $\mathcal{R}_-^\beta$ is open. By Proposition 2.3.2 there exist $r > 0$ and $\delta > 0$ such that $T(\alpha,\beta) \leq \dfrac{\|\alpha-\beta\|}{\delta}$ for all $\alpha \in B(\beta;r)$. Now let $\alpha \in \mathcal{R}_-^\beta$. By Lemma 1.2.3 ($a=-1$, $b = T(\alpha,\beta)+1$), there exist ρ such that for all $c \in]-1, T(\alpha,\beta)+1]$ and for all α_1, $\alpha_2 \in B(\alpha;1)$ we have

$$A(0,\alpha_1;c) \subseteq A(0,\alpha_2;c) + \rho\|\alpha_1-\alpha_2\|\bar{B}. \tag{2.9}$$

Since $T(\cdot,\beta)$ is continuous at α, there exists $\mu > 0$ such that on $B(\alpha;\mu) \subset \mathcal{R}_-^\beta$ we have $T(\cdot,\beta) \leq T(\alpha,\beta)+1$.
Let $\nu := \min\{\dfrac{r}{4\rho}, \mu, 1\}$, and let α_1, $\alpha_2 \in B(\alpha;\nu)$ then by 2.9 ($c = T(\alpha_1,\beta)$), there exists a trajectory x of F such that $x(0) = \alpha_2$ and

$$\|\beta - x(T(\alpha_1,\beta))\| \leq \rho\|\alpha_1-\alpha_2\| \leq \frac{r}{2}. \tag{2.10}$$

Then we have

$$\begin{aligned} T(\alpha_2,\beta) &\leq T(\alpha_1,\beta) + T(x(T(\alpha_1,\beta)),\beta) \\ &\leq T(\alpha_1,\beta) + \frac{\|x(T(\alpha_1,\beta)) - \beta\|}{\delta} \\ &\leq T(\alpha_1,\beta) + \frac{\rho}{\delta}\|\alpha_1-\alpha_2\|. \end{aligned}$$

By interchanging the role of α_1 and α_2, we get that $T(\cdot,\beta)$ is $\frac{\rho}{\delta}$-Lipschitz on $B(\alpha;\nu)$. The result follows. □

Remark 2.3.7 *Here we give a new proof of (iii)$\Longrightarrow$(ii) of the preceding proposition using our Corollary 1.3.4. Let $\beta \in \mathbb{R}^n$ and assume that $0 \in int\, F(\beta)$. Then by Proposition 2.3.3, $\mathcal{R}_-^\beta$ is open. By Corollary 1.3.4, we have that for all $\alpha \in \mathcal{R}_-^\beta$:*

$$T(\alpha,\beta) = \sup_{\varphi\in\Phi} \varphi(\alpha)$$

where Φ is the set of all functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$,
- $1 + \langle \varphi'(x), v\rangle \geq 0$, $\forall x \in \mathbb{R}^n$ $\forall v \in F(x)$,
- $\varphi(\beta) \leq 0$.

Then it is sufficient to show that there exist $\rho > 0$ and $C > 0$ such that for all $\varphi \in \Phi$ we have:

$$\|\varphi'(\alpha)\| \leq C \quad \forall x \in B(\beta;\rho).$$

Since $0 \in int\, F(\beta)$, F is locally Lipschitz and since $\mathcal{R}_-^\beta$ is open, there exist r, $K > 0$ such that

$$\bar{B}(0;r) \subset \mathcal{R}_-^\beta$$

and

$$F(\beta) \subset F(\alpha) + K\|\alpha - \beta\|\bar{B} \ \ \forall \alpha \in B(0;r).$$

Let $\alpha \in B(0;r)$ and let $\varphi \in \Phi$. We assume that $\varphi'(\alpha) \neq 0$, then

$$\frac{-\varphi'(\alpha)}{\|\varphi'(\alpha)\|} r \in F(\beta) + K\|\alpha - \beta\|\bar{B}.$$

Hence there exist $v \in F(\beta)$ and $\omega \in \bar{B}$ such that

$$\frac{-\varphi'(\alpha)}{\|\varphi'(\alpha)\|} r = v + K\|\alpha - \beta\|\omega$$

but $1 + \langle \varphi'(x), v\rangle \geq 0$ then we get that

$$1 - \|\varphi'(\alpha)\| + K\|\alpha\|\|\varphi'(\alpha)\| \geq 0. \tag{2.11}$$

We take $\rho > 0$ *such that* $\rho < \min\{r, \frac{1}{K}\}$ *then by (2.11) we find that for all* $\alpha \in B(0;\rho)$ *we have*

$$\|\varphi'(\alpha)\| \leq \frac{1}{1 - K\rho}$$

which completes the proof since ρ *depends only by* F *and* β.

2.4 The bilateral minimal time function

In this section, we give some properties of the function $T(\cdot,\cdot)$. First we study the regularity of this function. We show that if $T(\cdot,\cdot)$ is continuous (resp. locally Lipschitz) at every point of the diagonal $\mathcal{D} := \{(\alpha,\alpha) : \alpha \in \mathbb{R}^n\}$, then it is continuous (resp. locally Lipschitz) everywhere in $\mathcal{R}$. We also give a semiconvexity result in the linear case. We calculate the proximal subgradient and then we characterize this function as a proximal solution of a system of partial Hamilton-Jacobi equations at the end of this section.

2.4.1 Regularity

We begin by the following proposition which gives a necessary and sufficient condition for $\mathcal{R}$ to be an open set.

Proposition 2.4.1 *We have the following statements:*

(i) For $(\alpha,\beta) \in \mathcal{R}$*:*

$$F \text{ and } -F \text{ are respectively } \alpha\text{-LC and } \beta\text{-LC} \Longrightarrow (\alpha,\beta) \in int\,\mathcal{R}.$$

(ii) The following statements are equivalent:

1. $\mathcal{R}$ *is open.*
2. $\mathcal{D} \subset int\,\mathcal{R}$.
3. F *and* $-F$ *are* α*-LC, for all* $\alpha \in \mathbb{R}^n$.

Proof. (i) Let $(\alpha,\beta) \in \mathcal{R}$ and assume that F and $-F$ are α-LC and β-LC respectively. Then by Proposition 2.3.5 we have $\mathcal{R}^\alpha_-$ and $\mathcal{R}^\beta_+$ are open. Then using the fact that $(\alpha,\beta) \in \mathcal{R}$ we get that $(\alpha,\beta) \in \mathcal{R}^\alpha_- \times \mathcal{R}^\beta_+ \subset \mathcal{R}$. The result follows.

(ii) Clearly we have 1)$\Longrightarrow$2).

2)$\Longrightarrow$3): Assume that $\mathcal{D} \subset \operatorname{int} \mathcal{R}$ and let $\alpha \in \mathbb{R}^n$. Then $(\alpha, \alpha) \in \operatorname{int} \mathcal{R}$ and this gives the existence of $r > 0$ such that $(\alpha, \alpha) \in B(\alpha; r) \times B(\alpha; r) \subset \mathcal{R}$. Hence $\alpha \in B(\alpha; r) \subset \mathcal{R}_+^\alpha$ and $\alpha \in B(\alpha; r) \subset \mathcal{R}_-^\alpha$. Therefore F and $-F$ are α-LC.

3)$\Longrightarrow$1): Follows from (i). □

Proposition 2.4.2 *Let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have:*

(i) $T(\cdot, \cdot)$ is continuous at $(\alpha, \alpha) \Longleftrightarrow$ F and $-F$ are α-STLC.

(ii) Assume that one of the following conditions holds:

1. *$(\alpha, \beta) \in \mathcal{R}$, F is α-STLC and $-F$ is β-STLC.*
2. *F and $-F$ are α-STLC and $\beta \in \mathcal{R}_+^\alpha$.*
3. *F and $-F$ are β-STLC and $\alpha \in \mathcal{R}_-^\beta$.*

Then $T(\cdot, \cdot)$ is continuous at (α, β).

Proof. (i) Let $\alpha \in \mathbb{R}^n$, by Proposition 2.3.5 we have

$T(\alpha, \cdot)$ and $T(\cdot, \alpha)$ are continuous in $\alpha \Longleftrightarrow$ F and $-F$ are α-STLC.

But using the triangle inequality we also have

$T(\alpha, \cdot)$ and $T(\cdot, \alpha)$ are continuous at $\alpha \Longleftrightarrow T(\cdot, \cdot)$ is continuous at (α, α).

The result follows.

(ii) 1) Let $(\alpha, \beta) \in \mathcal{R}$ and suppose that F and $-F$ are respectively α-STLC and β-STLC. By (i) of Proposition 2.4.1, we have that $(\alpha, \beta) \in \operatorname{int} \mathcal{R}$. Now let (α_n, β_n) be a sequence such that $(\alpha_n, \beta_n) \longrightarrow (\alpha, \beta)$. By the triangle inequality we have

$$T(\alpha_n, \beta_n) \leq T(\alpha_n, \alpha) + T(\alpha, \beta) + T(\beta, \beta_n). \tag{2.12}$$

Then by the continuity of $T(\cdot, \alpha)$ and $T(\beta, \cdot)$ we get that $T(\cdot, \cdot)$ is upper semicontinuous and hence continuous.

2) Clearly we have $(\alpha,\beta) \in \operatorname{int}\mathcal{R}$ since $(\alpha,\beta) \in \mathcal{R}_-^\alpha \times \mathcal{R}_+^\alpha \subset \mathcal{R}$. Now let (α_n,β_n) be a sequence such that $(\alpha_n,\beta_n) \longrightarrow (\alpha,\beta)$. By the triangle inequality we have

$$T(\alpha_n,\beta_n) \leq T(\alpha_n,\alpha) + T(\alpha,\beta_n).$$

Since $T(\cdot,\alpha)$ and $T(\alpha,\cdot)$ are continuous in $\mathcal{R}_-^\alpha$ and $\mathcal{R}_+^\alpha$ respectively, the result follows as above.

3) We proceed as in 2) and we find the result. □
Now we give a necessary and sufficient condition for $T(\cdot,\cdot)$ to be continuous in $\mathcal{R}$.

Proposition 2.4.3 *The following statements are equivalent:*

(i) $\mathcal{R}$ is open, $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$ and for any $(\alpha_0,\beta_0) \in \partial\mathcal{R}$ we have

$$\lim_{(\alpha,\beta)\longrightarrow(\alpha_0,\beta_0)} T(\alpha,\beta) = +\infty.$$

(ii) $T(\cdot,\cdot)$ is continuous at (α,α) for all $\alpha \in \mathbb{R}^n$.

(iii) F and $-F$ are β-STLC for all $\beta \in \mathbb{R}^n$.

Proof. Clearly we have (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iii): Follows from (i) of Proposition 2.4.2.

(iii)$\Longrightarrow$(i): The first part ($\mathcal{R}$ is open and $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$) follows from (ii) of Proposition 2.4.2.
Now we show the second part. Let $(\alpha_0,\beta_0) \in \partial\mathcal{R}$. Suppose that there exist a sequence $(\alpha_n,\beta_n) \in \mathcal{R}$ and a constant K such that $(\alpha_n,\beta_n) \longrightarrow (\alpha_0,\beta_0)$ and $T_n := T(\alpha_n,\beta_n) \leq K$. Since $0 \leq T_n \leq K$ we can assume that T_n converges to $\bar{T} \in [0,K]$. Now let x_n be a trajectory of F on $[0,+\infty[$ which satisfies $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$. By the compactness property of trajectories, there exists a subsequence of x_n (we do not relabel) which converges uniformly on $[0,\bar{T}]$ to a trajectory $\bar{x}$ of F. Hence $\bar{x}(0) = \alpha_0$ and $\bar{x}(\bar{T}) = \beta_0$. Therefore $(\alpha_0,\beta_0) \in \mathcal{R}$ and this contradicts the fact that $\mathcal{R}$ is open. □
We proceed to study the Lipschitz continuity of $T(\cdot,\cdot)$. We begin by the following (local) proposition.

Proposition 2.4.4 *Let $(\alpha,\beta)\in\mathbb{R}^n\times\mathbb{R}^n$. Then we have:*

1. *If $(\alpha,\beta)\in\mathcal{R}$ then*

$$[\ 0\in \mathit{int}\,F(\alpha)\ \mathit{and}\ 0\in \mathit{int}\,F(\beta)\ ]\Longrightarrow T(\cdot,\cdot)\ \mathit{is\ Lipschitz\ near}\ (\alpha,\beta),$$

2. $0\in \mathit{int}\,F(\alpha)\Longleftrightarrow T(\cdot,\cdot)\ \mathit{is\ Lipschitz\ near}\ (\alpha,\alpha)$,

Proof. 1) Let $(\alpha,\beta)\in\mathcal{R}$ and assume that $0\in\operatorname{int}F(\alpha)$ and $0\in\operatorname{int}F(\beta)$. By Proposition 2.4.1 we have $(\alpha,\beta)\in\operatorname{int}\mathcal{R}$ and by Proposition 2.3.1 there exist $r,\delta>0$ such that

$$h(x,\gamma)<-\delta$$

for $x\in B(\alpha;r)\cup B(\beta;r)$, $\|\gamma\|=1$.
Let $W(\cdot):=\dfrac{1}{\delta}\|\cdot-\alpha\|$ and $V(\cdot):=\dfrac{1}{\delta}\|\cdot-\beta\|$ then by Proposition 2.3.2 we have that $T(\cdot,\alpha)\le W(\cdot)$ on $B(\alpha;r)$ and $T(\cdot,\beta)\le V(\cdot)$ on $B(\beta;r)$.
Moreover we have by the same argument that $T(\alpha,\cdot)\le W(\cdot)$ on $B(\alpha;r)$ and $T(\beta,\cdot)\le V(\cdot)$ on $B(\beta;r)$. Now we assume that $r<1$. Let $M:=\dfrac{2r}{\delta}+T(\alpha,\beta)$, $k\ge 1$ a common Lipschitz constant for all trajectories of F on $[0,M]$ with initial-values in $B(\alpha;1)$, K a Lipschitz constant for F on an appropriately large ball and $r':=\dfrac{r}{8k}e^{-KM}$. We claim that $T(\cdot,\cdot)$ is Lipschitz on $B((\alpha,\beta);r')$. Indeed, let (α',β'), $(\alpha'',\beta'')\in B((\alpha,\beta);r')$ and let $x(\cdot)$ be a minimal trajectory between α' and β', that is, $x(\cdot)$ is a trajectory of F on $[0,+\infty[$ which satisfies $x(0)=\alpha'$ and $x(T(\alpha',\beta'))=\beta'$. By Theorem 0.3.5 (dependence on initial conditions) and since

$$\begin{aligned}
T(\alpha',\beta') &\le T(\alpha',\alpha)+T(\alpha,\beta)+T(\beta,\beta')\\
&\le W(\alpha')+T(\alpha,\beta)+V(\beta')\\
&\le \frac{2r}{\delta}+T(\alpha,\beta)\\
&= M,
\end{aligned}$$

there exists a trajectory $y(\cdot)$ of F on $[0,+\infty[$ which satisfies $y(0)=\alpha''$ and

$$\|y(T(\alpha',\beta'))-\beta'\|\le 2ke^{KM}\|\alpha'-\alpha''\|$$

but $\|\alpha'-\alpha''\|<2r'$ then

$$y(T(\alpha',\beta'))\in B(\beta';\frac{r}{2}). \tag{2.13}$$

Since $\|\beta' - \beta\| < r' < \frac{r}{2}$ we have that $h(x,\gamma) < -\delta$ for $x \in B(\beta', r)$, $\|\gamma\| = 1$. Then as above we get that $T(\cdot, \beta') \leq \frac{\|\cdot - \beta'\|}{\delta}$ and $T(\beta', \cdot) \leq \frac{\|\cdot - \beta'\|}{\delta}$ on $B(\beta'; \frac{r}{2})$.
Hence by (2.13) and since $\|\beta' - \beta''\| < \frac{r}{2}$ we have

$$T(y(T(\alpha', \beta')), \beta') \leq \frac{1}{\delta}\|y(T(\alpha', \beta')) - \beta'\|$$

and

$$T(\beta', \beta'') \leq \frac{1}{\delta}\|\beta' - \beta''\|.$$

Moreover $T(\alpha'', \beta'') \leq T(y(T(\alpha', \beta')), \beta'') + T(\alpha', \beta')$ then

$$\begin{aligned} T(\alpha'', \beta'') - T(\alpha', \beta') &\leq T(y(T(\alpha', \beta')), \beta'') \\ &\leq T(y(T(\alpha', \beta'), \beta') + T(\beta', \beta'') \\ &\leq \frac{\|y(T(\alpha', \beta')) - \beta'\|}{\delta} + \frac{\|\beta' - \beta''\|}{\delta} \\ &\leq \frac{2ke^{KM}}{\delta}\|\alpha' - \alpha''\| + \frac{1}{\delta}\|\beta' - \beta''\| \\ &\leq K(\alpha, \beta)\|(\alpha' - \alpha'', \beta' - \beta'')\|. \end{aligned}$$

By interchanging the roles of (α', β'), (α'', β''), the proof is completed.

2) The necessary condition follows from 1) and the sufficient condition follows from the fact that if $T(\cdot, \cdot)$ is Lipschitz near (α, α) then $T(\cdot, \alpha)$ is Lipschitz near α and using Proposition 2.3.6. □
The following proposition gives a necessary and sufficient conditions for $T(\cdot, \cdot)$ to be locally Lipschitz in $\mathcal{R}$.

Proposition 2.4.5 *The following statements are equivalent:*

(i) $\mathcal{R}$ *is open and* $T(\cdot, \cdot)$ *is locally Lipschitz in* $\mathcal{R}$.

(ii) $T(\cdot, \cdot)$ *is Lipschitz near* (α, α) *for all* $\alpha \in \mathbb{R}^n$.

(iii) $0 \in int\, F(\beta)$ *for all* $\beta \in \mathbb{R}^n$.

Proof. Clearly we have (i)$\Longrightarrow$(ii).

(ii)$\Longrightarrow$(iii): Follows from Proposition 2.4.4.

(iii)$\Longrightarrow$(i): By Proposition 2.4.3 we have $\mathcal{R}$ is open, and by Proposition 2.4.4, $T(\cdot,\cdot)$ is locally Lipschitz in $\mathcal{R}$. □

It is well-known that in the linear case and under some hypotheses, the (unilateral) minimal time function $T(\cdot,\beta)$ is semiconvex, see [13]. In the following theorem, we show an analogous result for the bilateral minimal time function.

Theorem 2.4.6 *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set. Assume that $T(\cdot,\cdot)$ is locally Lipschitz in an open set $\Omega \subset \mathcal{R}$. Then $T(\cdot,\cdot)$ is semiconvex in Ω.

Proof. Let $\Omega \subset \mathcal{R}$ in which $T(\cdot,\cdot)$ is locally Lipschitz and let $(\alpha,\beta) \in \Omega$. Then there exists $r, k_1 > 0$ such that $T(\cdot,\cdot)$ is k_1-Lipschitz on $B((\alpha,\beta);r) \subset \Omega \subset \mathcal{R}$. Let $0 < r' < r$ and let $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in B((\alpha,\beta);r')$. We suppose for instance that $T(\alpha_1,\beta_1) \leq T(\alpha_2,\beta_2)$ and $T(\alpha_2,\beta_2) \neq 0$[1]. We consider $y(\cdot)$ a trajectory which realizes the minimum time between α_2 and β_2. We set $z(\cdot) = (y(\cdot),\beta_2)$ and $w(\cdot)$ the solution of the following differential equation:

$$\dot{w}(t) = (A,0)w(t) + (u(2t),0), \quad w(0) = (\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}),$$

where $u(\cdot)$ is the optimal control which realizes the minimum time between α_2 and β_2. We define

- $(\alpha_3,\beta_2) = z(T(\alpha_2,\beta_2) - T(\alpha_1,\beta_1))$,
- $(\alpha_4, \frac{\beta_1+\beta_2}{2}) = w(\frac{T(\alpha_2,\beta_2) - T(\alpha_1,\beta_1)}{2})$.

[1] If $T(\alpha_2,\beta_2) = 0$ then $T(\alpha_1,\beta_1) = T(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}) = 0$, and the desired inequality follows immediately.

Then by the principle of optimality we have that

$$T(\alpha_3, \beta_2) = T(\alpha_1, \beta_1), \tag{2.14}$$

and

$$T(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}) \leq T(\alpha_4, \frac{\beta_1+\beta_2}{2}) + \frac{T(\alpha_2,\beta_2) - T(\alpha_1,\beta_1)}{2}. \tag{2.15}$$

Moreover,

$$\begin{aligned} \|2(\alpha_4, \frac{\beta_1+\beta_2}{2}) &- (\alpha_1,\beta_1) - (\alpha_3,\beta_2)\| \\ &= \|2\int_0^{\frac{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)}{2}} \dot{w}(t)\,dt - \int_0^{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)} \dot{z}(t)\,dt\| \\ &= \|\int_0^{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)} (A,0)(w(\frac{t}{2}) - (y(t),\beta_2))\,dt \\ &\leq (T(\alpha_2,\beta_2) - T(\alpha_1,\beta_1))\|A\|M, \end{aligned}$$

where $M := \max_{t\in[0,T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)]} \|x(\frac{t}{2}) - y(t)\|$ with $x(\cdot)$ the solution of the following differential equation:

$$\dot{x}(t) = Ax(t) + u(2t), \quad x(0) = \frac{\alpha_1+\alpha_2}{2}.$$

Since $T(\cdot,\cdot)$ is k_1-Lipschitz on $B((\alpha,\beta);r)$ and $\dot{x}(\cdot)$, $\dot{y}(\cdot)$ are bounded on $[0, T(\alpha_2,\beta_2) - T(\alpha_1,\beta_1)]$, there exists a $k_2 > 0$ (depends only by r, k_1 and (α,β)) such that

$$M \leq k_2\|(\alpha_1-\alpha_2, \beta_1-\beta_2)\|.$$

Hence

$$\|2(\alpha_4, \frac{\beta_1+\beta_2}{2}) - (\alpha_1,\beta_1) - (\alpha_3,\beta_2)\| \leq k_1k_2\|A\|\|(\alpha_1-\alpha_2, \beta_1-\beta_2)\|^2.$$

Choosing r' very small, we can assume that $(\alpha_4, \frac{\beta_1+\beta_2}{2})$, $(\frac{\alpha_1+\alpha_3}{2}, \frac{\beta_1+\beta_2}{2})$ are in the ball $B((\alpha,\beta);r)$ and then since $T(\cdot,\cdot)$ is k_1-Lipschitz on $B((\alpha,\beta);r)$ we get that:

$$T(\alpha_4, \frac{\beta_1+\beta_2}{2}) \leq T(\frac{\alpha_1+\alpha_3}{2}, \frac{\beta_1+\beta_2}{2}) + K\|(\alpha_1-\alpha_2, \beta_1-\beta_2)\|^2, \tag{2.16}$$

where $K := \dfrac{k_1^2 k_2 \|A\|}{2}$. By (2.14) and using the convexity of U we can easily show that

$$T(\frac{\alpha_3+\alpha_1}{2}, \frac{\beta_1+\beta_2}{2}) \leq T(\alpha_1,\beta_1)^2. \tag{2.17}$$

Then by (2.15), (2.16) and (2.17) we find that

$$T(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}) \leq \frac{T(\alpha_1,\beta_1)+T(\alpha_2,\beta_2)}{2} + K\|(\alpha_1-\alpha_2, \beta_1-\beta_2)\|^2.$$

Then $T(\cdot,\cdot)$ is semiconvex near (α,β) which completes the proof. □
We have proved in Proposition 2.4.4, that if $0 \in \operatorname{int} F(\alpha)$ (resp. $0 \in \operatorname{int} F(\alpha)$ and $0 \in \operatorname{int} F(\beta)$) then $T(\cdot,\cdot)$ is Lipschitz near (α,α) (resp. near (α,β)). The following corollary shows that in the linear case we obtain that $T(\cdot,\cdot)$ is semiconvex near (α,α) (resp. near (α,β)).

Corollary 2.4.7 *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set. Let $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have the following statements:

1. *If $(\alpha,\beta) \in \mathcal{R}$ then*

$$[\, -A\alpha,\, -A\beta \in int U \,] \Longrightarrow T(\cdot,\cdot) \text{ is semiconvex near } (\alpha,\beta),$$

2. $-A\alpha \in int U \iff T(\cdot,\cdot)$ *is semiconvex near* (α,α),

Proof. Follows from Proposition 2.4.4 and Theorem 2.4.6. □

Example 2.4.8 *For $n=1$, let $F(x) = -x + [-1,1]$. It is easy to prove that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where*

- $\mathcal{R}_1 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, -1 < y \leq x,\ \ y \neq 1,\ \ x \neq 1\} \cup \{(1,1)\}$,
- $\mathcal{R}_2 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, x \leq y < 1,\ \ y \neq -1,\ \ x \neq -1\} \cup \{(-1,-1)\}$.

[2] We take $x_1(\cdot)$ (resp. $x_1(\cdot)$) a minimal trajectory between α_3 and β_2 (resp. α_1 and β_1). We define $x_3(\cdot) = \dfrac{x_1(\cdot)+x_2(\cdot)}{2}$. By the convexity of U, $x_3(\cdot)$ is a trajectory of F. Moreover, $x_3(0) = \dfrac{\alpha_1+\alpha_3}{2}$ and $x_3(T(\alpha_1,\beta_1)) = \dfrac{\beta_1+\beta_2}{2}$. The result follows.

We calculate $T(\cdot,\cdot)$ *in* $]-1,1[\times]-1,1[$ *and we find that:*

$$T(x,y)=\begin{cases}\ln(\dfrac{1+x}{1+y}) & \text{if } -1<y\leq x<1,\\ \ln(\dfrac{1-x}{1-y}) & \text{if } -1<x\leq y<1.\end{cases}$$

We remark that

$$T(x,y)=\max\{\ln(\frac{1+x}{1+y}),\ln(\frac{1-x}{1-y})\},$$

for all $(x,y)\in]-1,1[\times]-1,1[$. *Hence* $T(\cdot,\cdot)$ *is the maximum of two* C^2 *functions and then it is semiconvex on* $]-1,1[\times]-1,1[$, *see Proposition 0.2.13. We can easily deduce this from Corollary 2.4.7 since for all* $x\in]-1,1[$ *we have* $x\in int([-1,1])$. *We note that this example shows that* $T(\cdot,\cdot)$ *is not necessarily convex under our hypotheses.*

2.4.2 Proximal subgradients

In [71], Wolenski and Zhuang calculate the proximal subgradients of the (unilateral) minimal time function. In the following proposition we give an analogous result for the bilateral minimal function $T(\cdot,\cdot)$. This result will play an important role in the characterization of $T(\cdot,\cdot)$ as the solution of a system of partial Hamilton-Jacobi inequations.

Theorem 2.4.9 *We have:*

1. *For all* $\alpha\in\mathbb{R}^n$, *we have*

$$\partial_P T(\alpha,\alpha)=\{(\xi,-\xi)\in\mathbb{R}^n\times\mathbb{R}^n:\ \ h(\alpha,\xi)\geq -1\}.$$

2. *For all* $(\alpha,\beta)\in\mathcal{R}$ *with* $\alpha\neq\beta$, *we have*

$$\partial_P T(\alpha,\beta)=N^P_{\mathcal{A}(r)}(\alpha,\beta)\cap\{(\xi,\theta)\in\mathbb{R}^n\times\mathbb{R}^n:\ \ h(\alpha,\xi)=h(\beta,-\theta)=-1\},$$

where $r:=T(\alpha,\beta)$ *and* $\mathcal{A}(r):=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n: T(x,y)\leq r\}$.

Proof. 1) Suppose $\alpha \in \mathbb{R}^n$ and $(\xi, \theta) \in \partial_P T(\alpha, \alpha)$. Then there exist $\sigma > 0$ and $\nu > 0$ such that

$$T(\alpha', \beta') \geq -\sigma \|(\alpha' - \alpha, \beta' - \alpha)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \beta' - \alpha) \rangle,$$

for all $(\alpha', \beta') \in B((\alpha, \alpha); \nu)$. We take $\alpha' = \beta'$ and we get that

$$0 \geq -\sigma \|(\alpha' - \alpha, \alpha' - \alpha)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \alpha' - \alpha) \rangle,$$

for all $\alpha' \in B(\alpha; \nu)$. Let $v \in \mathbb{R}^n$ and $\alpha_n = \alpha + \dfrac{v}{n}$ for all $n \in \mathbb{N}^*$. There exists n_0 such that for $n \geq n_0$ we have

$$0 \geq \frac{-\sigma}{n} \|(v, v)\|^2 + \langle (\xi, \theta), (v, v) \rangle$$

hence

$$\langle (\xi, \theta), (v, v) \rangle \leq 0$$

and this is true for all $v \in \mathbb{R}^n$, then $\theta = -\xi$.
It is well-known that for $\zeta \in \partial_P T(\cdot, \alpha)(\alpha)$ we have $h(\alpha, \zeta) \geq -1$ ($(t + T(\cdot, \alpha), \{1\} \times F)$ is strongly increasing on $]0, +\infty[\times \mathbb{R}^n$). But

$$\partial_P T(\alpha, \alpha) \subset \partial_P T(\cdot, \alpha)(\alpha) \times \partial_P T(\alpha, \cdot)(\alpha),$$ [3]

hence $h(\alpha, \xi) \geq -1$. Therefore

$$\partial_P T(\alpha, \alpha) \subset \{(\xi, -\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \ \ h(\alpha, \xi) \geq -1\}.$$

For the opposite inclusion, suppose now that $(\alpha, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $h(\alpha, \xi) \geq -1$. We will show that $(\xi, -\xi) \in \partial_P T(\alpha, \alpha)$. Suppose the contrary, then there exists a sequence $(\alpha_n, \beta_n) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$(\alpha_n, \beta_n) \neq (\alpha, \alpha),$$

$$(\alpha_n, \beta_n) \longrightarrow (\alpha, \alpha)$$

and

$$T_n = T(\alpha_n, \beta_n) < -n \|(\alpha_n - \alpha, \beta_n - \alpha)\|^2 + \langle (\xi, -\xi), (\alpha_n - \alpha, \beta_n - \alpha) \rangle \quad (2.18)$$

[3] See [22, Exercise 1.2.9].

for all $n \in \mathbb{N}^*$.
Then we have

$$0 < T_n < 2\|\xi\|.\|(\alpha_n - \alpha, \beta_n - \alpha)\|. \tag{2.19}$$

Since $T_n < +\infty$ there exists a trajectory x_n of F on $[0, +\infty[$ such that $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$. Therefore

$$\beta_n - \alpha_n = \int_0^{T_n} \dot{x}_n(t)\, dt. \tag{2.20}$$

Let $p_n(t) := \mathrm{proj}_{F(\alpha)}(\dot{x}_n(t))$, then since $h(\alpha, \xi) \geq -1$ we have

$$\int_0^{T_n} \langle \xi, p_n(t) \rangle\, dt > -T_n. \tag{2.21}$$

By Lemma 0.3.1 and since T_n is bounded there exists $M > 0$ such that $\forall n$, $\forall t \in [0, T_n]$ we have

$$\|x_n(t) - \alpha\| \leq \|(\alpha_n - \alpha, \beta_n - \alpha)\| + MT_n \tag{2.22}$$

moreover

$$\langle \xi, \int_0^{T_n} \dot{x}_n(t) - p_n(t),\ dt \rangle \geq -K\|\xi\| \int_0^{T_n} \|x_n(t) - \alpha\|\, dt \tag{2.23}$$

where K is a Lipschitz constant for F on an appropriately large ball.
Using (2.19), (2.22) and (2.23) there exists $K' > 0$ such that

$$\langle \xi, \int_0^{T_n} \dot{x}_n(t) - p_n(t),\ dt \rangle \geq -K'\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2. \tag{2.24}$$

By (2.20) and (2.21) we get that

$$T_n - \langle \xi, \alpha_n - \beta_n \rangle \geq -K'\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2$$

and this contradicts (2.18) since $\langle (\xi, -\xi), (\alpha_n - \alpha, \beta_n - \alpha) \rangle = \langle \xi, \alpha_n - \beta_n \rangle$ and this finishes the proof.

2) Let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\alpha \neq \beta$ and $r = T(\alpha, \beta)$. Let $(\xi, \theta) \in \partial_P T(\alpha, \beta)$. Then there exists $\sigma > 0$ and $\nu > 0$ such that

$$T(\alpha', \beta') \geq r - \sigma\|(\alpha' - \alpha, \beta' - \beta)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \rangle,$$

for all $(\alpha', \beta') \in B((\alpha, \beta); \nu)$. If we take $(\alpha', \beta') \in \mathcal{A}(r) \cup B((\alpha, \beta); \nu)$ we get

$$0 \geq -\sigma \|(\alpha' - \alpha, \beta' - \beta)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \rangle,$$

hence $(\xi, \theta) \in N^P_{\mathcal{A}(r)}(\alpha, \beta)$. It is well-known that for $\zeta \in \partial_P T(\cdot, \beta)(\alpha)$ with $\alpha \neq \beta$ we have $h(\alpha, \zeta) = -1$ ($(t + T(\cdot, \alpha), \{1\} \times F)$ is strongly increasing and weakly decreasing on $]0, +\infty[\times \mathbb{R}^n \setminus \{\alpha\}$). But

$$\partial_P T(\alpha, \beta) \in \partial_P T(\cdot, \beta)(\alpha) \times \partial_P T(\alpha, \cdot)(\beta),$$

then $h(\alpha, \xi) = h(\beta, -\theta) = -1$.
The proof of the opposite inclusion is similar to that of 1). □

Remark 2.4.10 *In this remark we give a simple proof of 1)of Proposition 2.4.4, using the preceding result.*
Let $(\alpha, \beta) \in \mathcal{R}$ and suppose that $0 \in int\, F(\alpha)$ and $0 \in int\, F(\beta)$. Then by Proposition 2.3.1 there exists $r > 0$ and $\delta > 0$ such that for any $(\alpha', \beta') \in B(\alpha; r) \times B(\beta; r)$ and for any unit vector γ we have $h(\alpha, \gamma) < -\delta$ and $h(\beta, \gamma) < -\delta$. Hence, for $(\alpha', \beta') \in B(\alpha; r) \times B(\beta; r)$ and by Theorem 2.4.9 we have

$$\frac{-1}{\|\xi\|} \leq h(\alpha', \frac{\xi}{\|\xi\|}) < -\delta$$

and

$$\frac{-1}{\|\theta\|} \leq h(\beta', \frac{\theta}{\|\theta\|}) < -\delta$$

for all $(\xi, \theta) \in \partial_P T(\alpha', \beta')$ with $\xi \neq 0$ and $\theta \neq 0$. Then $\partial_P T(\cdot, \cdot)$ is bounded on $B(\alpha; r) \times B(\beta; r)$. By Proposition 0.2.7, $T(\cdot, \cdot)$ is Lipschitz on $B(\alpha; r) \times B(\beta; r)$.

2.4.3 The Hamilton-Jacobi equation

The following theorem gives a characterization of $T(\cdot, \cdot)$ as the solution of a system of partial Hamilton-Jacobi equations.

Theorem 2.4.11 *$T(\cdot, \cdot)$ is the unique lower semicontinuous function bounded below on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfying the following:*

1. $\forall \alpha \in \mathbb{R}^n$, $T(\alpha, \alpha) = 0$.

2. $\forall \alpha \neq \beta \in \mathbb{R}^n$, $\forall (\xi, \theta) \in \partial_P T(\alpha, \beta)$

$$h(\alpha, \xi) = h(\beta, -\theta) = -1.$$

3. $\forall \alpha \in \mathbb{R}^n$, $\forall (\xi, \theta) \in \partial_P T(\alpha, \alpha)$ *we have*

$$\theta = -\xi \ \ \textit{and} \ \ h(\alpha, \xi) \geq -1.$$

Proof. For all $\alpha \in \mathbb{R}^n$, $T(\alpha, \alpha) = 0$ and by Proposition 2.4.9, $T(\cdot, \cdot)$ satisfies 2) and 3).
To prove uniqueness, let $\psi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function bounded below and satisfies 1), 2) and 3) and let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then for all $(\xi, \theta) \in \partial_P \psi(\alpha, \beta)$ we have

$$1 + h(\alpha, \xi) \geq 0.$$

This gives that the system $(t + \psi, \{1\} \times F \times \{0\})$ is strongly increasing on $\mathbb{R} \times \mathbb{R}^{2n}$. Hence for $(\alpha, \beta) \in \mathcal{R}$ and for $\bar{x}(\cdot)$ a trajectory which realizes the minimal time from α to β we have

$$0 + \psi(z(0)) \leq T(\alpha, \beta) + \psi(z((T(\alpha, \beta))) \tag{2.25}$$

where $z(\cdot)$ is the trajectory of $F \times \{0\}$ on $[0, T(\alpha, \beta)]$ defined by $z(t) = (\bar{x}(t), \beta)$. By (2.25) we get that $\psi(\alpha, \beta) \leq T(\alpha, \beta)$. Therefore $\psi(\cdot, \cdot) \leq T(\cdot, \cdot)$ on $\mathcal{R}$ and then on $\mathbb{R}^n \times \mathbb{R}^n$.
Now we show the reverse inequality. Let $(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \mathcal{D}$. Then for all $(\xi, \theta) \in \partial_P \psi(\alpha, \beta)$ we have

$$1 + h(\alpha, \xi) \leq 0.$$

Hence the system $(t + \psi, \{1\} \times F \times \{0\})$ is weakly decreasing on $\mathbb{R} \times \mathbb{R}^{2n} \setminus \mathcal{D}$. Then for $=(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\alpha \neq \beta$ there exists a trajectory $z(\cdot)$ of $F \times \{0\}$ on $[0, +\infty[$ such that $z(0) = (\alpha, \beta)$ and

$$t + \psi(z(t)) \leq 0 + \psi(\alpha, \beta) \ \ \forall t \in [0, b]$$

where $[0, b]$ is any subinterval of $[0, +\infty[$ upon which $z(t) \notin \mathcal{D}, \forall t \in [0, b]$.
There are two cases to consider.
<u>*Case 1:*</u> $z(t) \notin \mathcal{D}, \forall t \in]0, +\infty[$.
Then $\psi(\alpha, \beta) \geq t + \psi(z(t)), \forall t \geq 0$. Since ψ is bounded below we get that

$\psi(\alpha,\beta) = +\infty$ hence $\psi(\alpha,\beta) \geq T(\alpha,\beta)$.
Case 2: There exists $a \in]0,+\infty[$ such that $z(a) \in \mathcal{D}$.
Let $\bar{a} := \inf\{a \in]0,+\infty[: z(a) \in \mathcal{D}\}$. Since $z(0) = (\alpha,\beta) \notin \mathcal{D}$ we have $\bar{a} \neq 0$.
Then $\forall t \in [0,\bar{a}[, \quad t + \psi(z(t)) \leq \psi(\alpha,\beta)$. Therefore

$$\psi(\alpha,\beta) \geq \liminf_{t \longrightarrow \bar{a}} t + \psi(z(t)) \geq \bar{a} \geq T(\alpha,\beta)$$

hence

$$T(\alpha,\beta) \leq \psi(\alpha,\beta)$$

which completes the proof since $\psi(\cdot,\cdot) = T(\cdot,\cdot) = 0$ on $\mathcal{D}$. □

Remark 2.4.12 *We remark (using the preceding proof) that we can replace the conditions 2) and 3) in Theorem 2.4.11 by*

- $\forall \alpha \neq \beta \in \mathbb{R}^n$, $\forall (\xi,\theta) \in \partial_P T(\alpha,\beta)$

$$h(\alpha,\xi) = -1.$$

- $\forall \alpha \in \mathbb{R}^n$, $\forall (\xi,\theta) \in \partial_P T(\alpha,\alpha)$ *we have*

$$h(\alpha,\xi) \geq -1,$$

or by

- $\forall \alpha \neq \beta \in \mathbb{R}^n$, $\forall (\xi,\theta) \in \partial_P T(\alpha,\beta)$

$$h(\beta,-\theta) = -1.$$

- $\forall \alpha \in \mathbb{R}^n$, $\forall (\xi,\theta) \in \partial_P T(\alpha,\alpha)$ *we have*

$$h(\beta,-\xi) \geq -1.$$

Chapter 3

The Hamilton-Jacobi equation of minimal time control*

3.1 Introduction

Let F be a multifunction mapping $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ and let h be the lower Hamiltonian corresponding to F. This chapter focuses on the following problem for the Hamilton-Jacobi equation:

$$1 + h(x, \partial_P \varphi(x)) = 0 \ \ \forall x \in \mathcal{R}^0_+, \ \ \varphi(0) = 0 \qquad (*)$$

where $\mathcal{R}^0_+$ is defined as in Chapter 2. A solution of $(*)$ means a lower semi-continuous function $\varphi : \mathcal{R}^0_+ \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that $\varphi(0) = 0$ and for every $x \in \mathcal{R}^0_+$, for every $\zeta \in \partial_P \varphi(x)$ (if any), we have $h(x, \zeta) + 1 = 0$. This is equivalent to the statement that φ is a viscosity solution (see [30]) of the following Hamilton-Jacobi equation:

$$H(x, -\varphi'(x)) - 1 = 0 \ \ \forall x \in \mathcal{R}^0_+, \ \ \varphi(0) = 0,$$

see [21] and [22] for the proof of the equivalence. When F admits a standard control representation $F(x) = f(x, U)$, then this assumes the form

$$\max\{\langle f(x,u), -\varphi'(x)\rangle \ : \ u \in U\} - 1 = 0,$$

a familiar object of study in connection with the dynamic programming approach to optimal control, see for example [6].

*This chapter is based on joint work [26] with F. H. Clarke.

When $(*)$ is modified by deleting the origin from the domain of the problem, there results a well-studied problem. Essentially one finds that the solution is the familiar (unilateral) minimum time function $T(\cdot, 0)$. However, $T(\cdot, 0)$ is *never* a solution on a set containing the origin, since necessarily we have $0 \in \partial_P T(\cdot, 0)(0)$, and since $h(0, 0) = 0$. We refer the reader to [6], [14] and [71] for discussions of this case.

In this chapter we explore (apparently for the first time) the consequence of including the origin in the domain. Our goal is to identify a framework in which such *global* solutions can be usefully studied. In contrast to the classical case, it turns out to be the *bilateral* minimal time function that plays a central role in determining the solutions of $(*)$ which, as we shall see, are closely linked to global geodesic trajectories.

The chapter is organized as follows. In the next section we present our hypotheses. In Section 3.3, we show the existence of (minimal) solutions of $(*)$. We study the regularity of solutions and the linear case in Section 3.4. The relation between semigeodesic trajectories and solutions is examined in Section 3.5. In Section 3.6, we define a dual equation for $(*)$ and we give a necessary and sufficient conditions for the existence of a geodesic passing through the origin. The Section 3.7 is devoted to the generalization for a target set.

3.2 Hypotheses

In this chapter we use the same notations as in Chapter 2 and we assume that the multifunction F satisfies the following hypotheses:

- For every $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.
- The linear growth condition: For some positive constants γ and c, and for all $x \in \mathbb{R}^n$,
$$v \in F(x) \Longrightarrow \|v\| \leq \gamma \|x\| + c.$$
- F is locally Lipschitz; that is, every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that
$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

- $-F$ is 0-STLC.

We note that under these hypotheses and by Proposition 2.3.5 we have:

- $\mathcal{R}_+^0$ is open.
- $T(0,\cdot)$ is continuous in $\mathcal{R}_+^0$.
- For any $\alpha_0 \in \partial\mathcal{R}_+^0$ we have

$$\lim_{\alpha \longrightarrow \alpha_0} T(0,\alpha) = +\infty.$$

3.3 Existence of solutions

Let φ be a solution of $(*)$. Applying the monotonicity theory to the function $\varphi + t$ and the multifunction $F(x) \times \{1\}$ (and $\Omega := \mathcal{R}_+^0 \times \mathbb{R}$), we deduce that the system $(\varphi + t, F \times \{1\})$ is both weakly and strongly increasing. Because of the nature of the t-dependence here, the latter property (for example) amounts to saying that for any trajectory x of F on an interval $[0,T]$ such that $x([0,T]) \subset \mathcal{R}_+^0$ (this being equivalent to $x(0) \in \mathcal{R}_+^0$), we have

$$\varphi(x(s)) + s \leq \varphi(x(t)) + t \ \ \forall s,t \in [0,T], s \leq t.$$

The following proposition gives some important properties of a solution of $(*)$.

Proposition 3.3.1 *Let φ a solution of $(*)$. Then we have:*

1. *$T(\alpha,\beta) + \varphi(\beta) \geq \varphi(\alpha)$, for all α, $\beta \in \mathcal{R}_+^0$.*
2. *$T(\alpha,0) \geq \varphi(\alpha) \geq -T(0,\alpha)$, for all $\alpha \in \mathcal{R}_+^0$.*
3. *$\mathcal{R}_-^0 \cap \mathcal{R}_+^0 \subset dom\,\varphi$.*
4. *For every $\alpha \in dom\,\varphi$ there exists a trajectory x of F such that $x(0) = \alpha$ and*

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

Proof. The statements 2) and 3) follow immediately from 1). For the proof of 1), let φ a solution of $(*)$ and let $\alpha,\ \beta \in \mathcal{R}_+^0$. We can assume that $(\alpha,\beta) \in \mathcal{R}$, then let $\bar{x} : [0,+\infty[\longrightarrow \mathbb{R}^n$ be a trajectory which realizes the (finite) minimal time from α to β, that is, $\bar{x}(0) = \alpha$ and $\bar{x}(T(\alpha,\beta)) = \beta$. Because the system $(\varphi + t, F \times \{1\})$ is strongly increasing in the sense explained above, we get that

$$\varphi(\alpha) = \varphi(\bar{x}(0)) \leq T(\alpha,\beta) + \varphi(\bar{x}(T(\alpha,\beta))) = T(\alpha,\beta) + \varphi(\beta).$$

The statement 1) follows.
Now we give the proof of 4). We consider $\alpha \in \operatorname{dom}\varphi$. Since the system $(t+\varphi, \{1\} \times F)$ is strongly increasing and weakly decreasing and using the fact that every trajectory which begins at α remains in $\mathcal{R}_+^0$, there exists a trajectory $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F in $\mathcal{R}_+^0$ such that $x(0) = \alpha$ and

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

This completes the proof. □
Now we define the following set:
$\mathcal{G} := \{\Gamma \subset \mathcal{R}_+^0 : \text{there exists a sequence } \beta_i \in \Gamma \text{ such that } T(0,\beta_i) \longrightarrow +\infty\}$, which is nonempty since for any $\alpha_0 \in \partial\mathcal{R}_+^0$ we have

$$\lim_{\alpha \longrightarrow \alpha_0} T(0,\alpha) = +\infty.^2$$

The following theorem implies that the set of solutions of $(*)$ is nonempty. We use the set $\mathcal{G}$ and the function $T(\cdot,\cdot)$ for the construction of solutions.

Theorem 3.3.2 *Let $\Gamma \in \mathcal{G}$ and let $\varphi_\Gamma : \mathcal{R}_+^0 \longrightarrow \mathbb{R} \cup \{-\infty,+\infty\}$ be the function defined as follows:*

$$\varphi_\Gamma(\alpha) := \liminf_{\substack{\alpha' \longrightarrow \alpha,\ \beta \in \Gamma \\ T(0,\beta) \longrightarrow +\infty}} [T(\alpha',\beta) - T(0,\beta)].$$

Then φ_Γ is a solution of $(*)$.

Proof. By the triangle inequality we have that for all $\alpha',\ \beta \in \mathcal{R}_+^0$

$$T(\alpha',\beta) - T(0,\beta) \geq -T(0,\alpha'), \tag{3.1}$$

then using the continuity of $T(0,\cdot)$ we get that

[2] If $\mathcal{R}_+^0$ is unbounded then we have $\lim_{\|\alpha\| \longrightarrow +\infty} T(0,\alpha) = +\infty$.

- $\varphi_\Gamma(\alpha) > -\infty$ for all $\alpha \in \mathcal{R}^0_+$,
- $\varphi_\Gamma(0) \geq 0$.

But clearly we have $\varphi_\Gamma(0) \leq 0$, then $\varphi_\Gamma(0) = 0$. The function φ_Γ is lower semicontinuous by the definition. Let us show that this function satisfies the Hamilton-Jacobi equation. It is sufficient to show that the function $t + \varphi_\Gamma$ is strongly increasing and weakly decreasing. We begin by weakly decreasing. Let $\alpha \in \operatorname{dom} \varphi_\Gamma$. Then there exist two sequences α_i and β_i in $\mathcal{R}^0_+$ such that $\alpha_i \longrightarrow \alpha$, $T(0, \beta_i) \longrightarrow +\infty$ and

$$\varphi_\Gamma(\alpha) = \lim_{i \longrightarrow +\infty} [T(\alpha_i, \beta_i) - T(0, \beta_i)].$$

Then $T(\alpha_i, \beta_i) \longrightarrow +\infty$ and for i sufficiently large there exists a trajectory x_i of F on $[0, +\infty[$ such that $x_i(0) = \alpha_i$ and $x_i(T(\alpha_i, \beta_i)) = \beta_i$ (x_i realizes the minimum time between α_i and β_i). Then for i sufficiently we have

$$T(x_i(t), \beta_i) = T(\alpha_i, \beta_i) - t, \ \ \forall t \in [0, 1]$$

and then

$$T(x_i(t), \beta_i) - T(0, \beta_i) = T(\alpha_i, \beta_i) - T(0, \beta_i) - t, \ \ \forall t \in [0, 1]. \tag{3.2}$$

By the compactness of trajectories we can assume that there exists a trajectory $\bar{x}$ of F on $[0, +\infty[$ such that x_i converges uniformly to $\bar{x}$ on $[0, 1]$. Then

$$\bar{x}(0) = \lim_{i \longrightarrow +\infty} x_i(0) = \lim_{i \longrightarrow +\infty} \alpha_i = \alpha.$$

Moreover, taking $i \longrightarrow +\infty$ in (3.2) we get that

$$\varphi_\Gamma(\bar{x}(t)) + t \leq \varphi_\Gamma(\alpha), \ \ \forall t \in [0, 1].$$

The weak decrease follows.
To prove strong increase, we consider an interval $[a, b] \subset]-\infty, +\infty[$ and a trajectory x of F such that $x([a, b]) \subset \mathcal{R}^0_+$. It is sufficient to show that

$$\varphi_\Gamma(x(b)) + b \geq \varphi_\Gamma(x(t)) + t, \ \ \forall t \in [a, b].$$

We set $\alpha = x(b)$ and we consider the sequence α_i and β_i as above. Let $t \in [a, b]$, then by Lemma 1.2.3 (applied for $-F$) there exists a sequence x_i

of trajectories of F on $[t,b]$ such that $x_i(b)=\alpha_i$ and $\lim\limits_{i\longrightarrow+\infty} x_i(t)=x(t)$. By the triangle inequality we have

$$T(x_i(t),\beta_i)\leq b-t+T(\alpha_i,\beta_i)$$

then

$$T(x_i(t),\beta_i)-T(0,\beta_i)+t\leq T(\alpha_i,\beta_i)-T(0,\beta_i)+b.$$

Taking $i\longrightarrow+\infty$ we get

$$\varphi_\Gamma(x(t))+t\leq\varphi_\Gamma(x(b))+b.$$

The strong increase follows. □
We denote by φ_0 the function φ_Γ corresponding to the choice $\mathcal{R}_+^0$ of Γ.

Theorem 3.3.3 *The function φ_0 defined above is the minimal solution of $(*)$.*

Proof. By Theorem 3.3.2, φ_0 is a solution of $(*)$. For the minimality, let φ be a solution of $(*)$ and let $\alpha\in\operatorname{dom}\varphi$ (we can take $\alpha\in\operatorname{dom}\varphi$ since we need to show that $\varphi_0(\alpha)\leq\varphi(\alpha)$). By Proposition 3.3.1, there exists a trajectory $x:[0,+\infty[\longrightarrow\mathbb{R}^n$ of F in $\mathcal{R}_+^0$ such that $x(0)=\alpha$ and

$$\varphi(x(t))+t=\varphi(\alpha),\quad\forall t\geq 0.$$

Hence

$$T(0,x(t))\geq-\varphi(x(t))=t-\varphi(\alpha),\quad\forall t\geq 0,$$

and then

$$\lim_{t\longrightarrow+\infty}T(0,x(t))=+\infty.$$

Therefore

$$\varphi_0(\alpha)\leq\lim_{t\longrightarrow+\infty}[T(\alpha,x(t))-T(0,x(t))]\leq\lim_{t\longrightarrow+\infty}[t-t+\varphi(\alpha)]=\varphi(\alpha),$$

which completes the proof. □

Now we give some examples. We show in the first that the Hamilton-Jacobi equation $(*)$ does not necessarily admit a maximal solution.

Example 3.3.4 *For $n \in \mathbb{N}^*$, let $F(x) := C$, for all $x \in \mathbb{R}^n$, where $C \subset \mathbb{R}^n$ is a nonempty, convex and compact set with $0 \in intC$. In this case $h(x,\zeta) = h_C(\zeta)$, $\forall (x,\zeta) \in \mathbb{R}^n \times \mathbb{R}^n$, where h_C is the lower support function of C:*

$$h_C(\zeta) := \min\{\langle \zeta, c\rangle \ : \ c \in C\}.$$

The bilateral minimal time function is defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$T(\alpha,\beta) = g_C(\beta - \alpha),$$

for all $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$, where g_C is the Minkowski gauge of C:

$$g_C(x) := \min\{\lambda \geq 0 \ : \ \frac{x}{\lambda} \in C\}.$$

Then $\mathcal{R}^0_+ = \mathbb{R}^n$ and the Hamilton-Jacobi equation $(*)$ becomes the "eikonal equation" :

$$1 + h_C(\partial_P \varphi(x)) = 0, \ \forall x \in \mathbb{R}^n, \ \ \varphi(0) = 0.$$

For every $\theta \in \mathbb{R}^n \setminus \{0\}$, the function

$$\varphi_\theta(x) = \langle \frac{-\theta}{h_C(\theta)}, x\rangle$$

is evidently a solution of $(*)$. The upper envelope of all such solutions is seen to be the function $x \mapsto g_C(-x) = T(x,0)$. In view of Proposition 3.3.1, this coincides with the upper envelope of all solutions of $(*)$. It follows that no maximal solution to $(*)$ exists.
The lower envelope of all solutions exhibited above is given by

$$\varphi_0(\cdot) = -g_C(\cdot) = -T(0,\cdot).$$

This is a concave function nondifferentiable at 0, so that $\partial_P \varphi_0(x) \neq 0$ implies $x \neq 0$ and $\partial_P \varphi_0(x) = \{\varphi_0'(x)\}$. But it is known from convex duality that $\varphi_0'(x)$ satisfies

$$1 + h_C(-\varphi_0'(x)) = 0,$$

so that φ_0 is a solution of $(*)$. Since φ_0 is a lower bound on all solutions of $(*)$, it is revealed as the minimal solution.

Example 3.3.5 *For $n = 1$ we consider $F(x) := [-|x-1|, +|x-1|]$. In this case $\mathcal{R}_+^0 = \{\alpha \in \mathbb{R} : \alpha < 1\}$ and $h(x,\zeta) = -|\zeta||x-1|$, $\forall (x,\zeta) \in \mathbb{R} \times \mathbb{R}$. For $\alpha \in \mathcal{R}_+^0$ we calculate*

$$T(\alpha,\beta) = \begin{cases} \ln(1-\alpha) - \ln(1-\beta) & \text{if } \beta > \alpha \\ \ln(1-\beta) - \ln(1-\alpha) & \text{if } \beta < \alpha \end{cases}$$

Then for $\Gamma_1 = [0,1[$ (resp. $\Gamma_2 =]-\infty, 0]$) the corresponding solution of $(*)$, obtained as in Theorem 3.3.2, is:

$$\varphi_1(x) = \ln(1-x) \quad (\text{resp. } \varphi_2(x) = -\ln(1-x)),$$

and the minimal solution φ_0 is calculated to be $-|\ln(1-x)|$.

3.4 Regularity of solutions and the linear case

In this section we study the regularity of a solution of $(*)$ using the regularity results obtained in the preceding chapter for the bilateral minimal time function. We begin by the following proposition which gives a sufficient condition for the continuity and the Lipschitz continuity of a solution φ of $(*)$ at a point $\alpha \in \operatorname{dom}\varphi$.

Proposition 3.4.1 *Let φ be a solution of $(*)$ and let $\alpha \in \operatorname{dom}\varphi$. Then we have:*

1. *F is α-STLC $\Longrightarrow$ φ is continuous at α.*
2. *$0 \in \operatorname{int} F(\alpha) \Longrightarrow$ φ is Lipschitz near α.*

Proof. 1) Since $\mathcal{R}_+^0$ is open and F is α-STLC there exists $\rho > 0$ such that

$$B(\alpha;\rho) \subset \mathcal{R}_+^0 \quad \text{and} \quad B(\alpha;\rho) \subset \mathcal{R}_-^\alpha.$$

Then for $\beta \in B(\alpha;\rho)$ and by Proposition 3.3.1 we have

$$\varphi(\beta) \leq \varphi(\alpha) + T(\beta,\alpha) < +\infty, \tag{3.3}$$

hence φ is finite on $B(\alpha;\rho)$. Now let α_i be a sequence such that $\alpha_i \longrightarrow \alpha$ and let $\varepsilon > 0$. By the lower semicontinuity of φ it is sufficient to prove that for i sufficiently large we have

$$\varphi(\alpha_i) \leq \varphi(\alpha) + \varepsilon,$$

but this follows immediately from (3.3) since $T(\cdot,\alpha)$ is continuous in $\mathcal{R}^{\alpha}_{-}$.

2) Since $0 \in \operatorname{int} F(\alpha)$ and by Proposition 2.4.4 we have that $T(\cdot,\cdot)$ is Lipschitz near (α,α). Hence there exist $\rho > 0$ and $K > 0$ such that $T(\cdot,\cdot)$ is K-Lipschitz on $B(\alpha;\rho) \times B(\alpha;\rho)$. By 1) and since $[0 \in \operatorname{int} F(\alpha) \Longrightarrow F$ is α-STLC], we can assume that $B(\alpha;\rho) \subset \mathcal{R}^0_+$ and that φ is finite on $B(\alpha;\rho)$. We claim that φ is K-Lipschitz on $B(\alpha;\rho)$. Indeed, let α_1, $\alpha_2 \in B(\alpha;\rho)$. By Proposition 3.3.1 we have

$$-T(\alpha_1,\alpha_2) \leq \varphi(\alpha_1) - \varphi(\alpha_2) \leq T(\alpha_2,\alpha_1),$$

but since $T(\cdot,\cdot)$ is K-Lipschitz on $B(\alpha;\rho) \times B(\alpha;\rho)$ we have

$$|T(\alpha_2,\alpha_1)| = |T(\alpha_2,\alpha_1) - T(\alpha_1,\alpha_1)| \leq K\|\alpha_1 - \alpha_2\|,$$

and

$$|T(\alpha_1,\alpha_2)| = |T(\alpha_1,\alpha_2) - T(\alpha_1,\alpha_1)| \leq K\|\alpha_1 - \alpha_2\|.$$

Then

$$-K\|\alpha_1 - \alpha_2\| \leq \varphi(\alpha_1) - \varphi(\alpha_2) \leq K\|\alpha_1 - \alpha_2\|,$$

the Lipschitz continuity follows. □

Proposition 3.4.2 *We have the following statements:*

1. *Assume that F is β-STLC for all $\beta \in \mathcal{R}^0_+ \cap \mathcal{R}^0_-$. Then all solutions of $(*)$ are continuous in the open set $\mathcal{R}^0_+ \cap \mathcal{R}^0_-$.*

2. *Assume that $0 \in int\, F(\beta)$ for all $\beta \in \mathcal{R}^0_+ \cap \mathcal{R}^0_-$. Then all solutions of $(*)$ are locally Lipschitz in the open set $\mathcal{R}^0_+ \cap \mathcal{R}^0_-$.*

Proof. Since $0 \in \mathcal{R}^0_+ \cap \mathcal{R}^0_-$, we have that F is 0-STLC in 1) and 2). Then $\mathcal{R}^0_-$ is open and hence $\mathcal{R}^0_+ \cap \mathcal{R}^0_-$ is open. By Proposition 3.3.1 we have $\mathcal{R}^0_+ \cap \mathcal{R}^0_- \subset \operatorname{dom}\varphi$ for all φ a solution of $(*)$. Then by Proposition 3.4.1 we find the two statements. □

We proceed to introduce a property that will play an important role. A continuous function φ is said to be *mildly regular* at a point x if it satisfies

$$\partial^P \varphi(x) \subset \partial_L \varphi(x).$$

The following proposition gives sufficient conditions for a continuous function to be mildly regular at a point x.

Proposition 3.4.3 *Let φ be continuous in a neighborhood of a point x. Then if one of the following conditions holds:*

1. *φ is regular at x,*
2. *φ is differentiable at x,*
3. *$\partial_P\varphi(x)$ is nonempty,*

then φ is mildly regular at x.

Proof. 1) If φ is regular at x then we have

$$\partial^P\varphi(x) = -\partial_P(-\varphi)(x) \subset -\partial_C(-\varphi)(x) = \partial_C\varphi(x) = \partial_L\varphi(x),$$

so φ is mildly regular at x.

2) If φ is differentiable at x, then by Propositions 0.2.5 and 0.2.9 we have that $\partial^P\varphi(x) \subset \{\varphi'(x)\}$ and $\varphi'(x) \in \partial_L\varphi(x)$. Then $\partial^P\varphi(x) \subset \partial_L\varphi(x)$ and this gives that φ is mildly regular at x.

3) Suppose that $\partial_P\varphi(x)$ is nonempty. Then there are two cases. First, if $\partial^P\varphi(x)$ is empty, then φ is mildly regular at x. Second, if $\partial^P\varphi(x)$ is nonempty, then $\partial^P\varphi(x)$ and $\partial_P\varphi(x)$ are simultaneously nonempty, which implies that φ is differentiable at x. By 2), φ is mildly regular at x. □

We remark that the property of mild regularity, like the stronger ones of regularity or semiconvexity, can be thought of intuitively as one which rules out "concave corners". Its first use here is in the following theorem which gives an important characterization for the function φ_0.

Theorem 3.4.4 *Suppose that the function $T(0,\cdot)$ is mildly regular on $\mathcal{R}^0_+ \setminus \{0\}$. Then φ_0 coincides with $-T(0,\cdot)$.*

Proof. By Proposition 3.3.1 we have that $\varphi_0 \geq -T(0,\cdot)$ then by the minimality of φ_0 it is sufficient to prove that $-T(0,\cdot)$ is a solution of $(*)$. First we have that $-T(0,0) = 0$. Let us show that $-T(0,\cdot)$ satisfies the Hamilton-Jacobi equation of $(*)$. Let $\alpha \in \mathcal{R}^0_+$, then there are two cases:
Case 1: $\alpha \neq 0$.
Let $\zeta \in \partial_P(-T(0,\cdot))(\alpha)$. Then $-\zeta \in \partial^P T(0,\cdot)(\alpha) \subset \partial_L T(0,\cdot)(\alpha)$, since $T(0,\cdot)$ is mildly regular at α. But it is well-known that we have

$$1 + h_{-F}(\alpha, \partial_P T(0,\cdot)(\alpha)) = 0,^3$$

[3]This is a well-known characterization of the minimal time function but here applied for the dynamic $-F$. We note that h_{-F} is the lower Hamiltonian corresponding to $-F$.

Hence since h_{-F} is continuous and ∂_L is constructed from ∂_P by a limiting process we get that

$$1 + h_{-F}(\alpha, -\zeta) = 0,$$

and then

$$1 + h(\alpha, \zeta) = 0.$$

Case 2: $\alpha = 0$.
We claim that $\partial_P(-T(0,\cdot))(0) = \emptyset$. Indeed, if not then $\partial^P T(0,\cdot)(0) \neq \emptyset$. Hence since $0 \in \partial_P T(0,\cdot)(0)$ we get that $T(0,\cdot)$ is differentiable at 0 and we have

$$\partial_P T(0,\cdot)(0) = \{0\},$$

and this gives a contradiction since $0 \in \operatorname{int} \partial_P T(0,\cdot)(0)$[4]. □
The following corollary shows that in the linear case, φ_0 coincides with $-T(0,\cdot)$.

Corollary 3.4.5 *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set such that $0 \in \operatorname{int} U$. Then φ_0 is semiconcave on $\mathcal{R}^0_+$ and coincides with $-T(0,\cdot)$. Moreover, if we assume that ∂U is of class C^1 then $\varphi_0 \in C^1(\mathcal{R}^0_+ \setminus \{0\})$.

Proof. Clearly F satisfies our hypotheses ($-F$ is 0-STLC since $0 \in \operatorname{int} F(0)$) and when F has the stated form, it is known that the function $T(0,\cdot)$ is semiconvex on $\mathcal{R}^0_+$, see [13, Theorem 4.1]. Then $T(0,\cdot)$ is mildly regular on $\mathcal{R}^0_+$ and by Theorem 3.4.4 we find the result. When ∂U is of class C^1 it is known that $T(0,\cdot) \in C^1(\mathcal{R}^0_+ \setminus \{0\})$ (see [13, Corollary 5.10])[5] then $\varphi_0(\cdot) = -T(0,\cdot) \in C^1(\mathcal{R}^0_+ \setminus \{0\})$. □
It is known that if ∂U is not of class C^1 in the preceding theorem then $T(0,\cdot)$ (and thus φ_0) fails in general to be in $C^1(\mathcal{R}^0_+ \setminus \{0\})$. The following simple example shows this fact.

Example 3.4.6 *For $n = 2$, we consider the multifunction F defined as follows*

$$F(x, y) = [-1, 1] \times [-1, 1],$$

for all $(x, y) \in \mathbb{R}^2$. This is a special case of Example 3.3.4 ($C = [-1, 1] \times [-1, 1]$). Then we have

[4] $\partial_P T(0,\cdot)(0) = \{\zeta \in \mathbb{R}^n : h(0,\zeta) \geq -1\}$, see [71, Theorem 5.1].
[5] This result was first conjectured in [38] and then proved in [11].

- $\mathcal{R}^0_+ = \mathbb{R}^2$.
- $\varphi_0(\alpha, \beta) = -T((0,0),(\alpha,\beta)) = \min\{-|\alpha|, -|\beta|\}$.

Clearly the function $\varphi_0(\cdot)$ is not differentiable at the points (α, β) with $|\alpha| = |\beta|$.

Remark 3.4.7 *Let us return to Example 3.3.4 and Example 3.3.5. We remark that in this two examples we have $\varphi_0(\cdot) = -T(0,\cdot)$. This follows from the fact that in these examples the function $T(0,\cdot)$ is mildly regular. In the next section we give an example in which $\varphi_0(\cdot)$ does not coincide with $-T(0,\cdot)$, see Example 3.5.8.*

3.5 Semigeodesics

Let $\alpha \in \mathbb{R}^n$. A trajectory $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F is a semigeodesic from α iff $x(0) = \alpha$ and $T(x(s), x(t)) = t - s$ for all $s \leq t \in [0, +\infty[$. In this section we present the relationship between the solutions of $(*)$ and semigeodesic trajectories. The following proposition affirms that for a given solution φ of $(*)$ (there exist such solutions by Theorem 3.3.2), there exists a semigeodesic from every point $\alpha \in \operatorname{dom}\varphi$.

Proposition 3.5.1 *Let φ be a solution of $(*)$. Then for every $\alpha \in dom\varphi$ there exists a semigeodesic x from α such that*

$$\varphi(x(t)) + t = \varphi(\alpha), \quad \forall t \geq 0.$$

Proof. Let φ be a solution of $(*)$ and let $\alpha \in \operatorname{dom}\varphi$. By Proposition 3.3.1, there exists a trajectory $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F in $\mathcal{R}^0_+$ such that $x(0) = \alpha$ and

$$\varphi(x(t)) + t = 0, \quad \forall t \geq 0. \tag{3.4}$$

We claim that x is a semigeodesic from α. Indeed, let $s \leq t \in [0, +\infty[$, then by (3.4) and Proposition 3.3.1 we have

$$T(x(s), x(t)) \geq \varphi(x(s)) - \varphi(x(t)) = t - s,$$

but

$$T(x(s), x(t)) \leq t - s,$$

therefore $T(x(s), x(t)) = t - s$. □

Remark 3.5.2 *The preceding proposition implies that under our hypotheses, there exists at least one semigeodesic from the origin. This can be deduced directly as follows. We consider a sequence α_n in $\mathcal{R}_+^0$ such that $T(0,\alpha_n) \longrightarrow +\infty$ (this sequence exists since $-F$ is 0-STLC). Let x_n be the trajectory of F which realizes the minimum time between 0 and α_n. By the compactness property of trajectories, there exists a trajectory x of F on $[0,+\infty[$ such that $x(0) = 0$ and x_n converges uniformly to x on compact interval. We claim that x is a semigeodesic from 0. Indeed, let $t \in [0,+\infty[$, then since $T(0,\alpha_n) \longrightarrow +\infty$ there exists n_t such that for $n \geq n_t$ we have*

$$T(0, x_n(t)) = t.$$

Using the continuity of $T(0,\cdot)$ we get that

$$T(0,x(t)) = \lim_{n\longrightarrow+\infty} T(0,x_n(t)) = \lim_{n\longrightarrow+\infty} t = t,$$

and this shows that x is a semigeodesic from 0.

Remark 3.5.3 *Using Proposition 3.5.1, we can show that under our hypotheses the Hamilton-Jacobi equation $(*)$ does not necessarily admit a solution on $\mathbb{R}^n$ (if we consider lower semicontinuous functions and exclude the value $-\infty$). Let us give an example. We consider the same data of Example 2.4.8; that is, for $n = 1$ we consider $F(x) = -x + [-1,1]$ for all $x \in \mathbb{R}$. In this example, we have $\mathcal{R}_+^0 =]-1,1[$ and*

$$T(0,x) = \begin{cases} -\ln(1-x) & \text{if } 0 \leq x < 1, \\ -\ln(1+x) & \text{if } -1 < x \leq 0. \end{cases}$$

There exist only two semigeodesics from the origin, namely

- $x(t) = 1 - e^{-t}$, $t \geq 0$.
- $y(t) = e^{-t} - 1$, $t \geq 0$.

Now assume that there exists a lower semicontinuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ which is a solution of $(*)$ on $\mathbb{R}$, we shall derive a contradiction. By Proposition 3.5.1, there exists a semigeodesic z from the origin such that

$$\varphi(z(t)) + t = 0 \quad \forall t \geq 0.$$

We assume that $z = x$ (the case $z = y$ follows using the same argument). Then we have

$$\varphi(1 - e^{-t}) = -t \quad \forall t \geq 0.$$

Hence if $t \longrightarrow +\infty$ then

$$\varphi(1 - e^{-t}) \longrightarrow -\infty,$$

and this gives a contradiction since $(1 - e^{-t}) \longrightarrow 1$ and φ is lower semicontinuous on $\mathbb{R}$.

The following theorem proves that semigeodesics from 0 are closely related to the minimal solution φ_0 of $(*)$.

Theorem 3.5.4 *Let $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ be a trajectory of F from 0. Then the following statements are equivalent:*

1. *The trajectory x is a semigeodesic from 0.*
2. *For all $t \geq 0$, we have $\varphi_0(x(t)) + t = 0$.*

Proof. 2)$\Longrightarrow$1): Follows immediately by Proposition 3.3.1 (as in the proof of the preceding proposition).

1)$\Longrightarrow$2): Since $T(0, x(t)) = t$ and by the definition of φ_0 we have that for all $s \geq 0$

$$\varphi_0(x(s)) \leq \liminf_{t \longrightarrow +\infty} [T(x(s), x(t)) - T(0, x(t))] = \liminf_{t \longrightarrow +\infty} [t - s - t] = -s,$$

then

$$\varphi_0(x(s)) + s \leq 0.$$

The reverse inequality follows by the strong increase property. □

In the following proposition, we present the relationship between an arbitrary solution of $(*)$ and a solution of the type φ_Γ.

Proposition 3.5.5 *Let φ be any solution of $(*)$. Then there exists a solution φ_Γ of $(*)$ of the type provided by Theorem 3.3.2 such that $\varphi_\Gamma \geq \varphi \geq \varphi_0$, and a semigeodesic from 0 along which φ, φ_Γ and φ_0 all coincide.*

Proof. Let φ be any solution of $(*)$, then by Proposition 3.5.1 there exists a semigeodesic x from 0 such that

$$\varphi(x(t)) + t = 0, \quad \forall t \geq 0.$$

Let $\Gamma := \{x(t) : t \geq 0\}$, then since $T(0, x(t)) = t$ we have $\Gamma \in \mathcal{G}$. We consider the solution φ_Γ of $(*)$ corresponding to Γ. Then we have:

$$\varphi_\Gamma(\alpha) = \liminf_{\substack{\alpha' \longrightarrow \alpha \\ t \longrightarrow +\infty}} [T(\alpha', x(t)) - T(0, x(t))].$$

But using Proposition 3.3.1 and since $T(0, x(t)) = t$ and $\varphi(x(t)) + t = 0$ we get that

$$T(\alpha', x(t)) - T(0, x(t)) \geq \varphi(\alpha') - \varphi(x(t)) - t = \varphi(\alpha').$$

Then $\varphi_\Gamma(\alpha) \geq \varphi(\alpha)$ since φ is lower semicontinuous.
By Theorem 3.5.4, φ and φ_0 agree along $x(\cdot)$. But for any $\tau > 0$, we have

$$\begin{aligned} \varphi_\Gamma(x(\tau)) &\leq \liminf_{t \longrightarrow +\infty} [T(x(\tau), x(t)) - T(0, x(t))] \\ &= \liminf_{t \longrightarrow +\infty} [t - \tau - t] \\ &= -\tau \\ &= \varphi(x(\tau)) \leq \varphi_\Gamma(x(\tau)), \end{aligned}$$

which establishes that φ and φ_Γ agree along $x(\cdot)$. □
In Corollary 3.4.5, we have proved that in the linear case the function φ_0 coincides with $-T(0, \cdot)$. For $\alpha \in \mathcal{R}^0_+$, the following theorem gives a necessary and sufficient conditions for $\varphi_0(\alpha)$ to be equal to $-T(0, \alpha)$ at a given point α.

Theorem 3.5.6 *Let $\alpha \in \mathcal{R}^0_+$. Then the following statements are equivalent:*

1. *The point α lies on a semigeodesic from 0.*
2. $\varphi_0(\alpha) = -T(0, \alpha)$.

Proof. 1)$\Longrightarrow$2): Let $\alpha \in \mathcal{R}^0_+$ and assume that there exists a semigeodesic x from 0 and $t \geq 0$ such that $x(t) = \alpha$. By Theorem 3.5.4, we have $\varphi_0(x(t)) + t = 0$. But $T(0, \alpha) = t$, then $\varphi_0(\alpha) = -T(0, \alpha)$.

2)$\Longrightarrow$1): Let $\alpha \in \mathcal{R}_+^0$ and assume that $\varphi_0(\alpha) = -T(0,\alpha)$. We can assume that $\alpha \neq 0$. Since $\varphi_0(\alpha)$ is finite and by Proposition 3.5.1 there exists a semigeodesic y from α such that

$$\varphi_0(y(t)) + t = \varphi_0(\alpha) = -T(0,\alpha), \quad \forall t \geq 0. \tag{3.5}$$

Let $x : [0, T(0,\alpha)] \longrightarrow \mathbb{R}^n$ be the minimal trajectory between 0 and α and let z be the trajectory of F on $[0, +\infty[$ obtained by concatenating x and y. We claim that z is the required semigeodesic from 0. Indeed, by Theorem 3.5.4 it is sufficient to prove that

$$\varphi_0(x(t)) + t = 0.$$

We note that z is defined as follows, $z(t) = x(t)$ for $t \in [0, T(0,\alpha)]$ and $z(t) = y(t - T(0,\alpha))$ for $t \geq T(0,\alpha)$. Then by (3.5) we have

$$\varphi_0(z(t)) + t = 0, \quad \forall t \geq T(0,\alpha).$$

For $t \in [0, T(0,\alpha)[$ and by Proposition 3.3.1, we have

$$\varphi_0(z(t)) \geq -T(0, z(t)) = -T(0, x(t)) = -t,$$

hence

$$\varphi_0(z(t)) + t \geq 0 = \varphi_0 + T(0,\alpha).$$

The reverse inequality follows by the strong increase property.
This completes the proof. □

Corollary 3.5.7 *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set such that $0 \in \mathrm{int}U$. Then every point in $\mathcal{R}_+^0$ lies on a semigeodesic from 0.

Proof. By Corollary 3.4.5 we have that $\varphi_0(\alpha) = -T(0,\alpha)$, for all $\alpha \in \mathcal{R}_+^0$. Then by Theorem 3.5.6, every point in $\mathcal{R}_+^0$ lies on a semigeodesic from 0. □
In the following example, we show that $\varphi_0(\cdot)$ does not always coincide with $-T(0,\cdot)$. We also prove that $\varphi_0(\cdot)$ can take positive values.

Example 3.5.8 *We take $n = 2$ and we define the following two multifunctions:*

1. $$F_1(x,y) = \begin{cases} \{(\frac{x^2-y^2}{x^2+y^2}, \frac{2xy}{x^2+y^2})\} & \text{if } y \neq 0 \\ \{(1,0)\} & \text{if } y = 0 \end{cases}$$

2. $F_2(x,y) = \bar{B}$ *for all* $(x,y) \in \mathbb{R}^2$.

Now we consider the multifunction F defined by the following:

- $F(x,y) = F_2(x,y)$ *for all* (x,y) *such that* $\|(x,y)\| \leq 1$.
- $F(x,y) = F_1(x,y)$ *for all* (x,y) *such that* $\|(x,y)\| \geq 2$.
- $F(x,y) = \{(2-r)v_2 + (r-1)v_1 : v_1 \in F_1(x,y) \text{ and } v_2 \in F_2(x,y)\}$ *for all* (x,y) *such that* $1 < \|(x,y)\| := r < 2$.

Clearly F satisfies our hypotheses. We note that for all $(x,y) \in \mathbb{R}^2 \setminus \{(\alpha, 0) : \alpha \in \mathbb{R}\}$, $F_1(x,y)$ is the unit tangent vector at (x,y) (pointing clockwise if $y < 0$ and counterclockwise if $y > 0$) to the unique circle centered on the y-axis and passing through both (x,y) and the origin. This implies that in the region $\|(x,y)\| > 2$, the trajectories of F move along such circles.

Claim 1: $\mathcal{R}_+^0 = \mathbb{R}^2 \setminus]-\infty, -2] \times \{0\}$.

Proof. *It can be seen without much difficulity that we have*

$$\mathbb{R}^2 \setminus]-\infty, -1[\times \{0\} \subset \mathcal{R}_+^0 \subset \mathbb{R}^2 \setminus]-\infty, -2] \times \{0\}.$$

Let us prove that the points of the form $(-a, 0)$ where $1 < a < 2$ are in $\mathcal{R}_+^0$. We fix $\varepsilon \in]0,1[$ and we consider the multifunction F_ε defined exactly like F but we replace $\bar{B}$ by $(1-\varepsilon)\bar{B}$. We can easily verify that we have:

- $F_\varepsilon(\cdot) \subset F(\cdot)$.
- *There exist $\omega > 0$ and $\nu > 0$ such that*

$$F_\varepsilon(x,y) + \nu\bar{B} \subset F(x,y) \ \ \forall (x,y) \in B((-a,0);\omega),$$

and $B((-a,0);\omega) \subset \{(\alpha,\beta) \ : \ 1 < \alpha^2 + \beta^2 < 4\}$.

Now let $K > 1$ be the Lipschitz constant of F_ε on $\bar{B}((-a,0);\omega)$ and let $\mu > 0$ such that $\mu \leq \{\frac{\nu}{2K}, \frac{\omega}{2}\}$.
We consider a point (x,y) and an arc $z(\cdot)$ which verify:

- $\|(x,y)-(-a,0)\| = \frac{\mu}{2}$, $x < -a$ *and* $y > 0$.
- $z : [0,+\infty[\longrightarrow \mathbb{R}^2$ *and satisfies the following differential inclusion:*

$$\dot{z}(t) \in F_1(z(t)) \ \text{ a.e. } t \in [0,+\infty[, \ \ z(0) = (x,y).$$

We take $T > 0$ and $(x',y') \in \mathbb{R}^2$ such that $(x',y') := z(T) \in \{(-a,\beta) : \beta > 0\}$. Clearly we have $T \geq \|(x,y)-(x',y')\|$ and $z(t) \in \bar{B}((-a,0);\frac{\mu}{2})$ for all $t \in [0,T]$. Then by the definition of F_ε, the arc $z(\cdot)$ is a trajectory of F_ε and taking (x,y) sufficiently near $(-a-\frac{\mu}{2},0)$ (see Figure 3.1 (2)) we can assume that

$$\frac{\|(x',y')-(-a,0)\|}{T} \leq \frac{\|(x',y')-(-a,0)\|}{\|(x,y)-(x',y')\|} \leq \frac{\nu}{2}. \tag{3.6}$$

Now we define the arc $w(\cdot)$ by

$$w(t) := z(t) + \frac{(-a,0)-(x',y')}{T}t \ \ \forall t \in [0,T].$$

Then for all $t \in [0,T]$ we have

$$\|w(t)-(-a,0)\| \leq \|z(t)-(-a,0)\| + \|(-a,0)-(x',y')\| < \mu + \mu \leq \omega,$$

hence

$$w(t) \in B((-a,0);\omega) \ \ \forall t \in [t_0,T].$$

Using the fact that F_ε is K-Lipschitz on $\bar{B}((-a,0);\omega)$ we get that for all $t \in [0,T]$

$$\begin{aligned} F_\varepsilon(z(t)) &\subset F_\varepsilon(w(t)) + K\|z(t)-w(t)\|\bar{B} \\ &\subset F_\varepsilon(w(t)) + K\mu\bar{B} \\ &\subset F_\varepsilon(w(t)) + \frac{\nu}{2}\bar{B}. \end{aligned}$$

This gives that

$$F_\varepsilon(z(t)) + \frac{\nu}{2}\bar{B} \subset F_\varepsilon(w(t)) + \nu\bar{B} \subset F(w(t)) \ \ \forall t \in [0,T]. \tag{3.7}$$

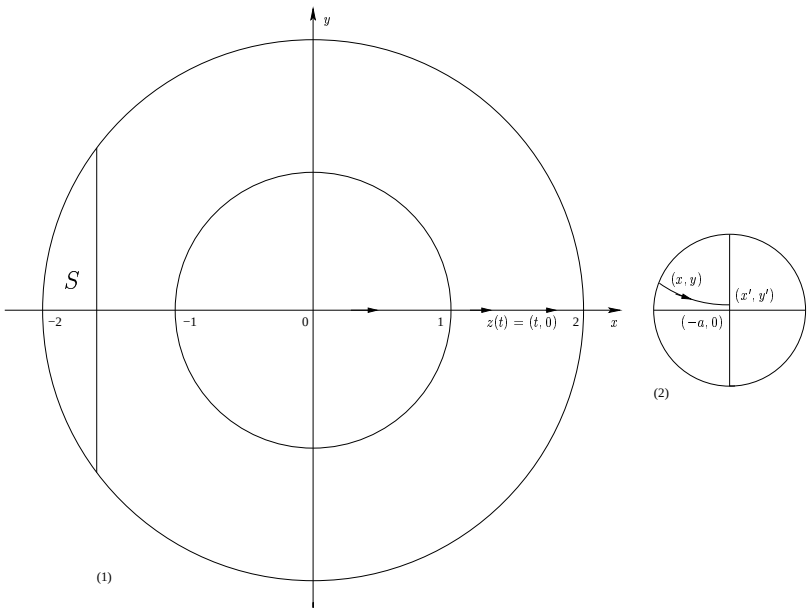

Figure 3.1: Example 3.5.8

In the other hand, we have

$$\dot{w}(t) = \dot{z}(t) + \frac{(-a,0) - (x',y')}{T} \quad a.e.\ t \in [0,T].$$

Then by (3.6) and (3.7), we get that a.e. $t \in [0,T]$ *we have*

$$\dot{w}(t) = \dot{z}(t) + \frac{(-a,0) - (x',y')}{T} \in F_\varepsilon(z(t)) + \frac{\nu}{2}\bar{B} \subset F(w(t)),$$

and this shows that $w(\cdot)$ *is a trajectory of* F *on* $[0,T]$*. By the definition of* $w(\cdot)$ *we have* $w(0) = (x,y)$ *and* $w(T) = (-a,0)$*, whence* $(-a,0) \in \mathcal{R}_+^{(x,y)}$*. Since* $(x,y) \in \mathcal{R}_+^0$ *we get that* $(-a,0) \in \mathcal{R}_+^0$*, and this completes the proof of the claim.* □

Claim 2: *The trajectory* $z(t) = (t,0)$, $t \in [0 + \infty[$ *is the unique semigeodesic from the origin.*

Proof. *Clearly the trajectory* $z(t) = (t,0)$, $t \in [0+\infty[$ *is a semigeodesic from the origin since all velocities are bounded by* 1 *in norm, and no arc*

between two points is shorter than a straight line. To prove the uniqueness, first we remark that there exists $-2 < b < -1$ *such that the set* $S := \{(\alpha, \beta) : \alpha^2 + \beta^2 < 4$ *and* $-2 < \alpha < b\}$ *satisfies the following: for all* $(x, y) \in S$ *and for all* $v = (v_1, v_2) \in F(x, y)$ *we have* $v_1 > 0$. *Using this fact, the continuity of* $T(0, \cdot)$ *and the fact that if* z *is a semigeodesic from* 0 *then* $T(0, z(t)) \longrightarrow +\infty$, *we can easily establish our claim (details are omitted).* □

Claim 3: φ_0 *does not coincide with* $-T((0,0), \cdot)$

Proof. *Since* $z(t) = (t, 0)$, $t \in [0 + \infty[$ *is the unique semigeodesic from the origin and by Theorem 3.5.6 we get that* φ_0 *agrees with* $-T((0,0), \cdot)$ *at points of the form* $(x, 0)$ $(x \geq 0)$, *and is strictly greater otherwise.* □

Claim 4: $\varphi_0(\cdot)$ *takes a positive value.*

Proof. *We consider the point* $(-\frac{1}{2}, 0)$, *and we remark that as for the origin, there exists only one semigeodesic from this point, namely the trajectory* $w(t) = (t - \frac{1}{2}, 0)$, $t \in [0 + \infty[$. *Then since* $(-\frac{1}{2}, 0) \in \mathcal{R}^0_+ \cap \mathcal{R}^0_- \subset dom\,\varphi_0$ *and by Proposition 3.5.1 we have*

$$\varphi_0(-\frac{1}{2}, 0) = t + \varphi_0(w(t)),$$

hence for $t = \frac{1}{2}$ *we get that* $\varphi_0(-\frac{1}{2}, 0) = \frac{1}{2} + \varphi_0(0, 0) = \frac{1}{2} > 0.$ □

3.6 Geodesics and the dual problem

A trajectory $x :]-\infty, +\infty[\longrightarrow \mathbb{R}^n$ of F is a geodesic iff

$$T(x(s), x(t)) = t - s,$$

for all $s \leq t \in]-\infty, +\infty[$. We have proved in the preceding section that the solutions of $(*)$ are closely linked to semigeodesic trajectories. A naturel question concerns the relationship between solutions of $(*)$ and geodesic trajectories.
We remark that in each of the three examples above (3.3.4, 3.3.5 et 3.5.8),

there is in fact a geodesic through the origin[6]. We now give an example to show that such a geodesic need not exist in general.

Example 3.6.1 *We take $n = 2$ and we define the following two multifunctions:*

1. *$F_1(x,y)$ is the closed segment between the points $\frac{(x,y)}{\sqrt{x^2+y^2}}$ and $\frac{(y,-x)}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and the set $\{(0,0)\}$ if $(x,y) = (0,0)$.*
2. *$F_2(x,y) = \bar{B}$ for all $(x,y) \in \mathbb{R}^2$.*

Now we consider the multifunction F defined by the following:

- *$F(x,y) = F_2(x,y)$ for all (x,y) such that $\|(x,y)\| \leq 1$.*
- *$F(x,y) = F_1(x,y)$ for all (x,y) such that $\|(x,y)\| \geq 2$.*
- *$F(x,y) = \{(2-r)v_2 + (r-1)v_1 : v_1 \in F_1(x,y) \text{ and } v_2 \in F_2(x,y)\}$ for all (x,y) such that $1 < \|(x,y)\| := r < 2$.*

Clearly F satisfies our hypotheses and it is easily seen that we have

$$\mathcal{R}^0_+ = \mathbb{R}^2, \quad \mathcal{R}^0_- = 2B.$$

Now assume that there exists $(x(t), y(t))$ a geodesic passing through the origin at $t = 0$; we shall derive a contradiction. Since points in the complement of $2B$ cannot be steered to $(0,0)$ we have that

$$\|(x(t), y(t))\| < 2, \ \forall t < 0.$$

Since $(0,0) \in \operatorname{int} F(0,0)$ and by Proposition 2.3.6 we have $T((0,0),\cdot)$ is continuous on $\mathbb{R}^2$ and then bounded above on $2B$. Then there exists a first $b > 0$ such that $\|(x(b), y(b))\| = 2$. But for all $(u,v) \in 2B$ we have

$$\begin{aligned} T((u,v),(x(b),y(b)) &\leq T((u,v),(u',v')) + T((u',v'),(x(b),y(b)) \\ &\leq 2 + 4\pi, \end{aligned}$$

where (u',v') is as in Figure 3.2.
It follows that $T((x(t),y(t)),(x(b),y(b))) = b - t$ is bounded for $t < 0$ and this gives the desired contradiction.

[6]See Remark 3.6.7 for the proof of the existence of a geodesic through the origin for these examples.

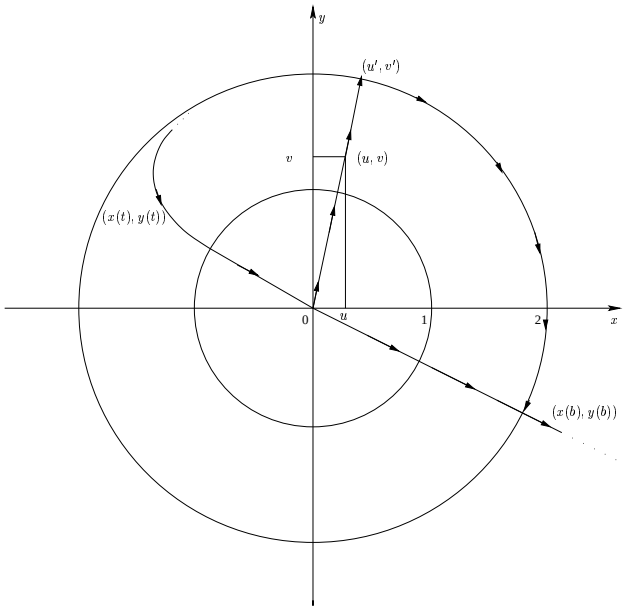

Figure 3.2: Example 3.6.1

We refer to the following as the *dual equation* to $(*)$:

$$1 + h(x, \partial^P \psi(x)) = 0, \ \forall x \in \mathcal{R}^0_-, \ \ \varphi(0) = 0, \qquad (*-).$$

A solution of $(*-)$ refers to an upper semicontinuous function. It is easy to see that ψ satisfies $(*-)$ iff $\varphi = -\psi$ is a solution of the version of $(*)$ obtained by replacing F by $-F$. Since the trajectories of $-F$ correspond to trajectories of F in reversed time, our previous results, applied to $(*-)$, give rise to consequences for F with respect to past (rather than future) time. For this purpose, the following hypothesis is relevant:

"F is 0-STLC"

To illustrate the use of the dual problem, suppose that ψ is a solution of $(*-)$, and that F is 0-STLC. Then we deduce the existence of a trajectory x of F on $]-\infty, 0]$ such that $x(0) = 0$ and

$$\psi(x(t)) + t = 0, \ \forall t \leq 0.$$

In the following theorem we use the dual problem to show the existence of a geodesic passing through the origin.

Theorem 3.6.2 *Assume that the following hypotheses hold:*

1. *F is is 0-STLC.*
2. *$\mathcal{R}^0_- \subset \mathcal{R}^0_+$.*
3. *There exist solutions φ and ψ of $(*)$ and $(*-)$ respectively such that $\varphi \geq \psi$ on $\mathcal{R}^0_-$.*

Then there exists a geodesic passing through 0.

Proof. There exists a trajectory x with $x(0) = 0$ such that

$$\varphi(x(t)) + t = 0, \ \forall t \geq 0,$$

and a trajectory y with $y(0) = 0$ such that

$$\psi(y(t)) + t = 0, \ \forall t \leq 0.$$

We have then

$$\varphi(y(t)) + t \geq 0, \ \forall t \leq 0,$$

but the opposite inequality holds by strong increase. Then the trajectory z defined on $]-\infty, +\infty]$ by concatenating y and x satisfies

$$\varphi(z(t)) + t = 0, \ \forall t \in \mathbb{R}.$$

This gives using Proposition 3.3.1 that z is a geodesic. □

Remark 3.6.3 *We can replace the second and third hypotheses of the preceding theorem by the following hypotheses and we find the same result:*

1. *$\mathcal{R}^0_+ \subset \mathcal{R}^0_-$.*
2. *There exist a solutions φ and ψ of $(*)$ and $(*-)$ respectively such that $\varphi \leq \psi$ on $\mathcal{R}^0_+$.*

Remark 3.6.4 *In Example 3.6.1 we have that F and $-F$ are 0-STLC ($0 \in int F(0)$) and $\mathcal{R}^0_- \subset \mathcal{R}^0_+$, but there is no geodesic passing through* 0. *This shows the necessity of the third hypothesis in Theorem 3.6.2.*

Corollary 3.6.5 *Assume that the following hypotheses hold:*

1. *F is is 0-STLC.*

2. $\mathcal{R}^0_- \subset \mathcal{R}^0_+$.

3. *There exists a continuous solution φ of $(*)$ which is mildly regular on $\mathcal{R}^0_+$.*

Then there exists a geodesic passing through 0.

Proof. The function φ satisfies

$$1 + h(x, \partial_L \psi(x)) = 0, \ \forall x \in \mathcal{R}^0_+, \ \ \varphi(0) = 0,$$

since h is continuous, and since ∂_L is constructed from ∂_P by a limiting process. Then since φ is mildly regular on $\mathcal{R}^0_+$, we get that φ is a solution of $(*-)$. The result follows from Theorem 3.6.2. □

The pointwise upper envelope $E(\cdot)$ of all solutions of $(*)$ defines a lower semi-continuous function on $\mathcal{R}^0_+$ that is bounded above by $T(\cdot, 0)$. This function will play an important role to give a necessary and sufficient condition for the existence of a geodesic passing through the origin.

Theorem 3.6.6 *Assume that the following hypotheses hold:*

1. $\mathcal{R}^0_- \subset \mathcal{R}^0_+$.

2. *F and $-F$ are β-STLC for all $\beta \in \mathcal{R}^0_-$.*

Then the following statements are equivalent:

(i) There exists a geodesic passing through 0.

(ii) $\limsup_{\substack{\alpha \in \mathcal{R}^0_- \\ T(\alpha,0) \longrightarrow +\infty}} [E(\alpha) - T(\alpha, 0)] = 0.$

Proof. First we show that under our hypotheses we have that $T(\cdot, \cdot)$ is continuous on $\mathcal{R}^0_- \times \mathcal{R}^0_+$. Let $(\alpha, \beta) \in \mathcal{R}^0_- \times \mathcal{R}^0_+$, then we have:

- F and $-F$ are α-STLC.
- $\beta \in \mathcal{R}^\alpha_+$.

Hence by Proposition 2.4.2, $T(\cdot,\cdot)$ is continuous at (α,β).[7]
(i)$\Longrightarrow$(ii): Let x be a geodesic passing through the origin. Set $\Gamma := \{x(t) : t \geq 0\}$, and consider the solution φ_Γ of $(*)$. Set $\alpha_i = x(-i)$. Then for any $\alpha \in \mathcal{R}^0_+$ and for $t > 0$ we have

$$\begin{aligned} T(\alpha, x(t)) - T(0, x(t)) &= T(\alpha_i, x(t)) - T(0, x(t)) + T(\alpha, x(t)) - T(\alpha_i, x(t)) \\ &\geq T(\alpha_i, 0) - T(\alpha_i, \alpha). \end{aligned}$$

Using the definition of φ_Γ and since $T(\cdot,\cdot)$ is continuous at $(\alpha_i,\alpha_i) \in \mathcal{R}^0_- \times \mathcal{R}^0_+$, we get that

$$\varphi_\Gamma(\alpha_i) \geq T(\alpha_i, 0).$$

Then $E(\alpha_i) = T(\alpha_i, 0)$. Since $T(\alpha_i, 0) \longrightarrow +\infty$ and $\alpha_i \in \mathcal{R}^0_-$, the result follows.

(ii)$\Longrightarrow$(i): Let α_i be a sequence in $\mathcal{R}^0_-$ such that $T(\alpha_i, 0) \longrightarrow +\infty$ and $E(\alpha_i) - T(\alpha_i, 0) \longrightarrow 0$. Then for each i there exists a solution φ_i of $(*)$ such that

$$\varphi_i(\alpha_i) \geq T(\alpha_i, 0) - \varepsilon_i,$$

where ε_i is a positive sequence converging to 0. Set $\tau_i = T(\alpha_i, 0)$, and let x_i be an optimal trajectory on the interval $[-\tau_i, 0]$ joining α_i to 0. By Proposition 3.5.1 we can extend x_i to $[0, +\infty[$ by a trajectory satisfying

$$\varphi_i(x_i(t)) + t = 0,\ \forall t \geq 0.$$

Since $x_i(\cdot) \in \mathcal{R}^0_- \subset \mathcal{R}^0_+$ and by the strong increasing property, we have that for any $t \in [\tau_i, 0]$,

$$\begin{aligned} 0 = \varphi(x_i(0)) + 0 &\geq \varphi_i(x_i(t)) + t \\ &\geq \varphi_i(x_i(-\tau_i)) - \tau_i \\ &= \varphi_i(\alpha_i) - T(\alpha_i, 0) \\ &\geq -\varepsilon_i. \end{aligned}$$

We deduce that

$$-\varepsilon_i \leq \varphi_i(x_i(t)) + t \leq 0,\ \forall t \in [-\tau_i, +\infty[.$$

[7] Under hypothesis 1), the hypothesis 2) is equivalent to the continuity of $T(\cdot,\cdot)$ on $\mathcal{R}^0_- \times \mathcal{R}^0_+$.

By Proposition 3.3.1, we get that for any two points $s \leq t \in [-\tau_i, +\infty[$ we have

$$t - s \geq T(x_i(s), x_i(t)) \geq t - s - \varepsilon_i. \tag{3.8}$$

By the compactness property of trajectories, we can assume that the sequence x_i converges uniformly on bounded intervals to a trajectory x. We claim that x is a geodesic. Indeed, let $s \in]-\infty, 0]$ and let $t \in [0, +\infty[$. We have $x(s) \in \mathcal{R}^0_-$ and $x(t) \in \mathcal{R}^0_+$. Then by the continuity of $T(\cdot,\cdot)$ in $\mathcal{R}^0_- \times \mathcal{R}^0_+$ and by (3.8) we get that

$$T(x(s), x(t)) = t - s,$$

which completes the proof. □

Remark 3.6.7 *In this remark, we show how to prove the existence of a geodesic passing through the origin for the examples 3.3.4, 3.3.5 and 3.5.8.*

1. **Example 3.3.4** *In this example we have that:*

   - $\mathcal{R}^0_+ = \mathcal{R}^0_- = \mathbb{R}^n$.
   - $T(\cdot,\cdot)$ *is continuous in* $\mathcal{R} = \mathbb{R}^2 \times \mathbb{R}^2$.
   - $E(\cdot) = T(\cdot, 0)$.

   Then by Theorem 3.6.6 there exists a geodesic passing through the origin.

2. **Example 3.3.5** *In this example we have that:*

   - F *is* 0-$STLC$ $(0 \in int\, F(0))$.
   - $\mathcal{R}^0_+ = \mathcal{R}^0_- = \{\alpha \in \mathbb{R} : \alpha < 1\}$.
   - *The function* $\varphi(\cdot) = \ln(1 - \cdot)$ *is a solution of* $(*)$ *which is mildly regular on* $\mathcal{R}^0_+$.

   Then by Corollary 3.6.5 there exists a geodesic passing through the origin.

3. **Example 3.5.8** *In this example, clearly the trajectory* $z(t) = (t, 0)$ *is a geodesic passing through the origin.*

3.7 Generalization for a target set

In this section the multifunction F satisfies all our hypotheses except the hypothesis $-F$ is 0-STLC. We consider a target set $S \subset \mathbb{R}^n$ and the minimal time function $T(\cdot, S)$ associated to S. This well-known function is defined as follows:

$$T(\alpha, S) := \begin{cases} \inf T \geq 0, \\ \dot{x}(t) \in F(x(t)) \ \ a.e.\ t \in [0, T] \\ x(0) = \alpha, \\ x(T) \in S. \end{cases}$$

Like in the case $S = \{0\}$, this function is a solution on $\mathbb{R}^n \backslash S$ of the following Hamilton-Jacobi equation:

$$1 + h(x, \partial_P \varphi(x)) = 0, \quad \varphi(S) = 0.$$

An important question follows, can we extend the preceding results for the following Hamilton-Jacobi equation:

$$1 + h(x, \partial_P \varphi(x)) = 0, \ \forall x \in \mathcal{R}_+^S, \ \ \varphi(S) = 0, \qquad (**)$$

where $\mathcal{R}_+^S$ is defined like $\mathcal{R}_+^0$ but replacing 0 by S.

In this section we generalize the results of Section 3.4 (the linear case) and some results of Section 3.5. We also give an example (see Example 3.7.9) in which we show that this generalization does not extend to the results of Section 3.3 (existence of solutions). The generalization of these results remains an open question.

In the next subsection we present our hypotheses on S and the generalization is presented in Subsection 3.6.2.

3.7.1 Hypotheses

The minimal time function associated to S for the dynamic $-F$ is denoted by $T(S, \cdot)$. As in Section 2.2 we define the sets: $\mathcal{R}_+^S(t)$, $\mathcal{R}_+^S$, $\mathcal{R}_-^S(t)$ and $\mathcal{R}_-^S$. As mentioned in the beginning of this section the multifunction F satisfies all the hypotheses presented in Section 3.2 but we replace the hypothesis $-F$ is 0-STLC by the following one: "$-F$ is S-STLC", that is, for all $t > 0$, we have $S \subset \operatorname{int} \mathcal{R}_+^S(t)$. Under this hypothesis, $\mathcal{R}_+^S$ is open and $T(S, \cdot)$ is

continuous on $\mathcal{R}_+^S$. Moreover, we assume that the target S satisfies the following hypotheses:

1. S is compact.
2. $\operatorname{int} S = \emptyset$.

Remark 3.7.1 *We note that we must assume that $\operatorname{int} S = \emptyset$ since if not then a solution φ of $(**)$ vanishes on $\operatorname{int} S$ and then $0 \in \partial_P \varphi(x)$ for all $x \in \operatorname{int} S$ which gives a contradiction since $h(\cdot, 0) = 0$.*

3.7.2 Generalization

We begin this subection by the following proposition which gives some properties of a solution of $(**)$. The proof follows excatly like that of Proposition 3.3.1.

Proposition 3.7.2 *Let φ a solution of $(**)$. Then we have:*

1. *$T(\alpha, \beta) + \varphi(\beta) \geq \varphi(\alpha)$, for all α, $\beta \in \mathcal{R}_+^S$.*
2. *$T(\alpha, S) \geq \varphi(\alpha) \geq -T(S, \alpha)$, for all $\alpha \in \mathcal{R}_+^S$.*
3. *For every $\alpha \in \operatorname{dom} \varphi$ there exists a trajectory x of F such that $x(0) = \alpha$ and $\varphi(x(t)) + t = \varphi(\alpha) \quad \forall t \geq 0$.*

Using the preceding proposition, clearly we have that if $-T(S, \cdot)$ is a solution of $(**)$ then it is minimal. The following theorem gives a necessary condition for $-T(S, \cdot)$ to be the minimal solution of $(**)$. This is a generalization of Theorem 3.4.4.

Theorem 3.7.3 *Assume that the function $T(S, \cdot)$ is mildly regular on $\mathcal{R}_+^S \setminus S$. Assume further that $\operatorname{dom} N_S^P = \partial S$ ($\operatorname{dom} N_S^P := \{s \in S : N_S^P(s) \neq \{0\}\}$). Then $-T(S, \cdot)$ is the minimal solution of $(**)$.*

Proof. By the definition, we have $-T(S, x) = 0$ for all $x \in S$. Let us show that $-T(S, \cdot)$ satisfies the Hamilton-Jacobi equation of $(**)$. Let $\alpha \in \mathcal{R}_+^S$, then there exist two cases.
Case 1: $\alpha \notin S$.
Then using the same technique as in the proof of Theorem 3.4.4 we find that $-T(S, \cdot)$ satisfies the Hamilton-Jacobi equation of $(**)$ at α.

Case 2: $\alpha \in S$.
We claim that $\partial_P(-T(S,\cdot))(\alpha) = \emptyset$. Indeed, if not then since $0 \in \partial_P T(S,\cdot)(\alpha)$ we get that $T(S,\cdot)$ is differentiable at α and $\partial_P T(S,\cdot)(\alpha) = \{0\}$. But by [71, Theorem 5.1] we have $\partial_P T(S,\cdot)(\alpha) = N_S^P(\alpha) \cap \{\zeta \in \mathbb{R}^n \ : \ h(x,\zeta) \geq -1\}$, then since $N_S^P(\alpha) \neq \{0\}$ we find a contradiction. □

Remark 3.7.4 *A sufficient condition for the condition* $\operatorname{dom} N_S^P = \partial S$ *is the convexity of* S, *see Proposition 0.2.3. We note that the condition* $\operatorname{int} S = \emptyset$ *does not necessarily implies that* $\operatorname{dom} N_S^P = \partial S$. *Indeed, for*

$$S = \{0\} \cup \{-1/n \ : \ n \in \mathbb{N}^*\} \cup \{1/n \ : \ n \in \mathbb{N}^*\},$$

we have that $\operatorname{int} S = \emptyset$ *but* $N_S^P(0) = \{0\}$.

Corollary 3.7.5 *Let* F *admit a representation of the form*

$$F(x) = \{Ax + u \ : \ u \in U\},$$

where A *is an* $n \times n$ *matrix and* U *is a compact and convex set. Assume further that we have the following hypotheses:*

(i) S *is a compact and convex set.*

(ii) $\operatorname{int} S = \emptyset$.

(iii) $\forall r > 0 \ \exists \mu_r, \sigma_r > 0$ *such that*

$$\forall x \in (S_{\sigma_r} \setminus S) \cap B(0;r) \ \exists v \in F(x):$$

$$\langle v, \frac{x - p(x)}{\|x - p(x)\|} \rangle \leq -\mu_r$$

for some $p(x) \in S$ *such that* $\|x - p(x)\| = d_S(x)$.[8]

Then $-T(S,\cdot)$ *is semiconcave and it is the minimal solution of* $(**)$.

[8] $S_{\sigma_r} := \{x \in \mathbb{R}^n : d_S(x) < \sigma_r\}$. This condition reduces to $0 \in \operatorname{int} F(\alpha)$ if $S = \{\alpha\}$. For more informations about this hypothesis see [13].

Proof. First we verify that all the hypotheses of Theorem 3.7.3 are satisfied. By hypothesis (iii) we have that F is S-STLC, see [13, Proposition 2.2]. Since S is convex, we have that $\operatorname{dom} N_S^P = \partial S$. Moreover, by [13, Theorem 4.1], the function $T(S,\cdot)$ is semiconvex on $\mathcal{R}_+^S$ and then mildly regular on $\mathcal{R}_+^S$. Hence we can apply Theorem 3.7.3 and find that $-T(S,\cdot)$ (which is semiconcave) is the minimal solution of $(**)$. □

A semigeodesic $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F is a semigeodesic from S if and only if $x(0) \in S$. Like Proposition 3.5.1, the following proposition presents the relationship between a solution of $(**)$ and semigeodesic trajectories.

Proposition 3.7.6 *Let φ be a solution of $(**)$. Then for every $\alpha \in \operatorname{dom}\varphi$ there exists a semigeodesic x from α such that*

$$\varphi(x(t)) + t = \varphi(\alpha), \ \forall t \geq 0.$$

Moreover, if $\alpha \in S$ then we have $T(S, x(t)) = t$, $\forall t \geq 0$.

The following theorem gives a necessary an sufficient condition for $-T(S,\cdot)$ to be the minimal solution of $(**)$. This theorem generalizes the Theorem 3.5.6.

Theorem 3.7.7 *The following statements are equivalent:*

1. *$-T(S,\cdot)$ is the minimal solution of $(**)$.*
2. *Every point of $\mathcal{R}_+^S$ lies on a semigeodesic $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ from S which satisfies: $T(S, x(t)) = t, \ \forall t \geq 0$.*

Proof. (1) $\Longrightarrow$ (2): Let $\alpha \in \mathcal{R}_+^S$, then by Proposition 3.7.6 there exists a trajectory $x : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F such that $x(0) = \alpha$ and

$$T(S, x(t)) - t = T(S, \alpha) \ \ \forall t \in [0,+\infty[.$$

Since $T(S,\alpha) < +\infty$, there exists a trajectory $y : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F such that $y(0) = a \in S$ and $y(T(S,\alpha)) = \alpha$ (this is the minimal trajectory between S and α). Clearly we have

$$T(S, y(t)) = t \quad \forall t \in [0, T(S,\alpha)].$$

We consider the trajectory $z : [0,+\infty[\longrightarrow \mathbb{R}^n$ of F defined by $z(t) = y(t)$ for all $t \in [0, T(S,\alpha)]$, and $z(t) = x(t - T(S,\alpha))$ for all $t \in [T(S,\alpha), +\infty[$. Then we have

$$T(S, z(t)) = t \quad \forall t \in [0,+\infty[.$$

Let $s \leq t \in [0, +\infty[$. Then

$$T(S, z(t)) = t, \quad T(S, z(s)) = s \quad \text{and} \quad T(z(s), z(t)) \leq t - s.$$

But for $b \in S$ we have by the triangle inequality that

$$T(S, z(t)) \leq T(b, z(t)) \leq T(b, z(s)) + T(z(s), z(t)),$$

then

$$T(S, z(t)) \leq \inf_{b \in S} T(b, z(s)) + T(z(s), z(t)) = T(S, z(s)) + T(z(s), z(t))$$

hence

$$T(z(s), z(t)) \geq T(S, z(t)) - T(S, z(s)) = t - s.$$

Therefore

$$T(z(s), z(t)) = t - s,$$

and this shows that $z(\cdot)$ is the desired semigeodesic.
(2) $\Longrightarrow$ (1): It is sufficient to show that $-T(S, \cdot)$ is a solution of $(**)$; that is, the system $(t - T(S, \cdot), \{1\} \times F)$ is strongly increasing and weakly decreasing in $\mathbb{R} \times \mathcal{R}_+^S$. The strong increase property follows immediately from the definition of $T(S, \cdot)$. For the weak decrease, let $\alpha \in \mathcal{R}_+^S$. Then there exists a semigeodesic $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ from S which satisfies: $T(S, x(t) = t, \ \forall t \geq 0$. We consider the trajectory $y : [0, +\infty[\longrightarrow \mathbb{R}^n$ of F defined by $z(t) = x(t + T(S, \alpha))$ for all $t \in [0, +\infty[$. Clearly we have: $y(0) = \alpha$ and y remains in $\mathcal{R}_+^S$. Now let $t \in [0, +\infty[$, then

$$T(S, y(t)) = T(S, x(t + T(S, \alpha)) = t + T(S, \alpha),$$

and hence

$$0 - T(S, y(0)) = t - T(S, y(t)).$$

The weak decrease follows. □

Remark 3.7.8 *Unsing the preceding theorem we can add in Theorem 3.7.3 and Corollary 3.7.5 that every point of $\mathcal{R}_+^S$ lies on a semigeodesic $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ from S which satisfies: $T(S, x(t) = t, \ \forall t \geq 0$.*

We proceed to treat the following question: can we generalize under our hypotheses the results of Section 3.3 ? In the following example we show that we can not do this. Indeed, we prove that a solution of the Hamilton-Jacobi equation $(**)$, need not exist in general.

Example 3.7.9 *We consider the same data of Example 3.5.8. Let* φ_0 *be the minimal solution of* $(*)$ *and let* $S := [-\frac{1}{2}, 0] \times \{0\}$. *Clearly* F *and* S *satisfies all our hypotheses. Now we assume that the Hamilton-Jacobi equation* $(**)$ *admits a solution* φ, *we shall derive a contradiction. Since* $\mathcal{R}^0_+ = \mathcal{R}^S_+$ *and* $(0,0) \in S$ *we have that* φ *is a solution of* $(*)$. *Then* $\varphi_0(\cdot) \leq \varphi(\cdot)$ *and hence* $\varphi_0(-\frac{1}{2}, 0) \leq \varphi(-\frac{1}{2}, 0)$. *But we have* $\varphi_0(-\frac{1}{2}, 0) = \frac{1}{2}$, *then* $\varphi(-\frac{1}{2}, 0) \geq \frac{1}{2} > 0$. *This gives a contradiction since* $\varphi(-\frac{1}{2}, 0) = 0$.

Chapter 4

Minimal loop trajectories in control theory

4.1 Introduction

Let F be a multifunction mapping $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ and let $\alpha \in \mathbb{R}^n$. We have studied in Chapter 3 the existence of a semigeodesic at α and proved that the existence of this special trajectory is closely linked to the existence of a solution of certain Hamilton-Jacobi equation. Our principal hypothesis in Chapter 3 is that $-F$ is α-STLC and then $0 \in F(\alpha)$ and $T(\alpha, \cdot)$ is continuous in her domain of definition. The question now is, what type of trajectories can we find at a point α if we suppose for example that the functions $T(\alpha, \cdot)$ or/and $T(\cdot, \alpha)$ are not continuous at α?

In this chapter we study the existence of a new type of trajectories, the *minimal loop trajectory*. A loop trajectory[1] at α is a trajectory x of F which begin from α and terminates at α in a finite and strictly positive time T, and it is *minimal* if it is minimal between $x(s)$ and $x(t)$ $(T(x(s), x(t)) = t - s)$ for all $s \leq t \in [0, T[$ and for all $s \leq t \in]0, T]$. This type of trajectories is used in [33] to show the existence of a C^1 subsolutions of the Hamilton-Jacobi equation

$$H(x, u'(x)) \geq c[0],$$

[1] We also say "closed trajectory".

where $c[0]$ is the Mañé critical value.[2]

We show that the existence of such trajectories at α is closely linked to the discontinuous of the functions $T(\alpha, \cdot)$, $T(\cdot, \alpha)$ and $T(\cdot, \cdot)$ and then we give a necessary and sufficient conditions for the existence. To do this we define a new function $L(\cdot)$, the *minimal loop function*. This function is defined as follows, $L(\alpha)$ is the minimal time in which we go along a trajectory of F from α to α, see Section 4.3.

This chapter is an introduction for the study of this type of trajectories and we think that much research in this area remains to be done.

The layout of this chapter is as follows. We give some definitions and notations in the next section. In Section 4.3, we define and study some properties of the minimal loop function. Necessary end sufficient conditions for the existence of semigeodesics and loop trajectories is given in Section 4.4. We give some examples in Section 4.5.

4.2 Definitions and notations

In this chapter we use the same notations as in Chapter 2 and we assume that the multifunction F satisfies the following hypotheses:

- For every $x \in \mathbb{R}^n$, $F(x)$ is a nonempty compact convex set.
- The linear growth condition: For some positive constants γ and c, and for all $x \in \mathbb{R}^n$,
$$v \in F(x) \Longrightarrow \|v\| \leq \gamma\|x\| + c.$$
- F is locally Lipschitz; that is, every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that
$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

[2]Let $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a given Hamiltonian and let L be the corresponding Lagrangian. Then the Mañé critical value $c[0]$ is defined by

$$c[0] := -\inf\{\frac{1}{T}\int_0^T L(x(t), \dot{x}(t))\, dt : x \in X,\, T > 0,\, x(0) = x(T)\},$$

where X denotes the class of absolutely continuous functions $x : [0, +\infty[\longrightarrow \mathbb{R}^n$.

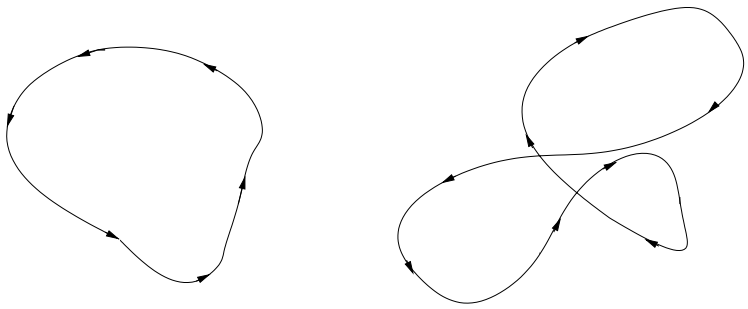

Figure 4.1: Simple and general closed curves

Definition 4.2.1 *Let $x(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ be a trajectory of F. We say that x is a loop trajectory iff $x(0) = x(T)$. It is a loop trajectory at α iff $x(0) = x(T) = \alpha$.*

A trivial loop trajectory at α is a trajectory $x(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ such that $x(t) = \alpha$ for all $t \in [0,T]$. By the definition it is easy to see that, there exists a trivial loop trajectory at α iff $0 \in F(\alpha)$.

Definition 4.2.2 *Let $x(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ be a loop trajectory at $\alpha \in \mathbb{R}^n$. We say that x is a minimal loop trajectory at α iff it satisfies:*

1. *$T(x(s), x(t)) = t - s$ for all $s \leq t \in [0,T[$,*
2. *$T(x(s), x(t)) = t - s$ for all $s \leq t \in]0,T]$.*

A loop trajectory may occur as a simple closed curve or as a complicated configuration (closed curves with self-intersections), see Figure 4.1. By the definition of a minimal loop trajectory, clearly this type of trajectories is a simple closed curve.

Remark 4.2.3 *It is easy to see that we have:*

1. *Let $x(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ be a loop trajectory (resp. minimal loop trajectory) at $\alpha \in \mathbb{R}^n$ for the dynamic F, then the trajectory $y(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ defined by $y(t) = x(T-t)$ for all $t \in [0,T]$ is a loop trajectory (resp. minimal loop trajectory) at $\alpha \in \mathbb{R}^n$ for the dynamic $-F$.*
2. *There exists a trivial loop trajectory at α iff $0 \in F(\alpha)$.*

3. $\mathcal{R}^{\alpha}_{-} \cap \mathcal{R}^{\alpha}_{+} = \{\alpha\} \cup \{\beta \in \mathbb{R}^n$ *: there exists a loop trajectory at* α *passing through* $\beta\}$.

4. *There exists a nontrivial loop trajectory at* α *iff* $\mathcal{R}^{\alpha}_{-} \cap \mathcal{R}^{\alpha}_{+} \neq \{\alpha\}$.

5. *Assume that* F *and* $-F$ *are* $\alpha-LC$. *Then* $\mathcal{R}^{\alpha}_{-} \cap \mathcal{R}^{\alpha}_{+}$ *is open and hence, there exists a nontrivial loop trajectory at* α.

6. *Let* $x(\cdot) : [0,T] \longrightarrow \mathbb{R}^n$ *be a loop trajectory at* $\alpha \in \mathbb{R}^n$. *Then* $x(\cdot)$ *is minimal iff* $T(\alpha, x(t)) + T(x(t), \alpha) = T$ *for all* $t \in]0,T[$.

4.3 Minimal loop function

We consider the following optimal control problem:

$$P(\alpha) \begin{cases} \text{Minimize } T > 0, \\ \dot{x}(t) \in F(x(t)) \text{ a.e. } t \in [0,T], \\ x(0) = x(T) = \alpha. \end{cases}$$

We denote by $L(\cdot) : \mathbb{R}^n \longrightarrow [0, +\infty]$ the value function of $P(\alpha)$, that is, $L(\alpha) = \inf P(\alpha)$ if there exists a loop trajectories at α and $L(\alpha) = +\infty$ otherwise. L is the *minimal loop function* for the dynamic F. We denote by $\mathcal{L}$ the $\operatorname{dom} L$. By Remark 4.2.3 we have that

$$\mathcal{L} = \{\alpha \in \mathbb{R}^n \ : \ 0 \in F(\alpha) \text{ or } \mathcal{R}^{\alpha}_{-} \cap \mathcal{R}^{\alpha}_{+} \neq \{\alpha\}\}.$$

By Remark 4.2.3 we have also that the dynamics F and $-F$ have the same minimal loop function. Now we give some important properties for the minimal loop function. We begin by the following proposition.

Proposition 4.3.1 *Let* $\alpha \in \mathbb{R}^n$, *then we have:*

1. $0 \in F(\alpha) \Longleftrightarrow L(\alpha) = 0$.

2. *If* $L(\alpha) \in]0, +\infty[$ *then there exists a solution* $\bar{x}(\cdot)$ *for the problem* $P(\alpha)$ *and* $\bar{x}(\cdot)$ *is a minimal loop trajectory at* α.

Proof. 1) Clearly we have $0 \in F(\alpha) \Longrightarrow L(\alpha) = 0$. Let us show the necessary condition. Assume that $L(\alpha) = 0$ then there exists a sequence

$x_n(\cdot) : [0, T_n] \longrightarrow \mathbb{R}^n$ of loop trajectories at α such that $\lim\limits_{n \longrightarrow +\infty} T_n = 0$. We have

$$0 = \frac{x_n(T_n) - x_n(0)}{T_n} = \frac{1}{T_n} \int_0^{T_n} \dot{x}_n(t)\, dt, \tag{4.1}$$

where $\dot{x}_n(t) \in F(x_n(t))$, $0 \leq t \leq T_n$. Now for any given $\rho > 0$, for all n sufficiently large, the set $\{x_n(t) : 0 \leq t \leq T_n\}$ lies in $B(\alpha; \rho)$. Further, given $\varepsilon > 0$, taking ρ small enough will ensure that $x \in B(\alpha; \rho)$ implies $F(x) \in F(\alpha) + \varepsilon B$. The upshot of all this, in view of (4.1), is that for all i large enough we have

$$\frac{x_n(T_n) - x_n(0)}{T_n} \in F(\alpha) + \varepsilon B.$$

Thus $0 \in F(\alpha) + \varepsilon B$. Since ε is arbitrary, $0 \in F(\alpha)$ as required.

2) Let $\alpha \in \mathcal{L}$ such that $L(\alpha) \neq 0$, then there exists a sequence of loop trajectories $x_n : [0, T_n] \longrightarrow \mathbb{R}^n$ at α and a sequence $\varepsilon_n \longrightarrow 0$ such that

$$L(\alpha) \leq T_n = L(\alpha) + \varepsilon_n.$$

We extend the trajectory x_n on $[0, +\infty[$ and then by the compactness property of trajectories there exists a trajectory $\bar{x} : [0, +\infty[\longrightarrow \mathbb{R}^n$ and a subsequence (we eschew relabeling) x_n having the property that for any $b > 0$, x_n converges uniformly to $\bar{x}$ on $[0, b]$. The trajectory $\bar{x} : [0, L(\alpha)] \longrightarrow \mathbb{R}^n$ is a loop trajectory at α since $\bar{x}(0) = \lim\limits_{n \longrightarrow +\infty} x_n(0) = \alpha$ and $\bar{x}(L(\alpha)) = \lim\limits_{n \longrightarrow +\infty} x_n(T_n) = \alpha$. Then $\bar{x}$ is a solution of $P(\alpha)$. Using the minimality of $\bar{x}$, it is clear that this trajectory is a minimal loop trajectory at α. □

Remark 4.3.2 *When $L(\alpha) \in]0, +\infty[$, we have proved in the preceding proposition that there exists a solution for the problem $P(\alpha)$ which is a minimal loop trajectory at α. We show in Example 4.5.2 that this solution is not necessarily unique and then we can find more than one minimal loop trajectory at a point α.*

In the following proposition we show that $L(\cdot)$ is lower semicontinuous and we give another characterizations for this function.

Proposition 4.3.3 *We have the following statements:*

1. *L is lower semicontinuous in $\mathbb{R}^n$.*

2. *If $T(\cdot,\cdot)$ is continuous at (α,α) then $L(\cdot)$ is continuous at α.*

3. *If $L(\alpha) \in]0,+\infty[$ then $L(\alpha) = \liminf_{\beta\neq\alpha,\ \beta\longrightarrow\alpha}[T(\alpha,\beta)+T(\beta,\alpha)]$.*

4. *For all $\alpha \in \mathbb{R}^n$ we have $h(\alpha,\partial_P L(\alpha)) \leq 0$ and $H(\alpha,\partial_P L(\alpha)) \geq 0$.*

Proof. 1) It is sufficient to show that $\operatorname{epi} L := \{(\alpha,r) \in \mathcal{L}\times\mathbb{R} : r \geq L(\alpha)\}$ is closed. Let (α_n,r_n) be a sequence of $\operatorname{epi} L$ such that $(\alpha_n,r_n) \longrightarrow (\alpha,r)$. There are two cases:
Case 1: There exists a subsequence of α_n (we eschew relabeling) such that $L(\alpha_n) = 0$.
Then $0 \in F(\alpha_n)$ for all n and this easily gives that $0 \in F(\alpha)$. Therefore

$$r = \lim_{n\longrightarrow+\infty} r_n \geq \lim_{n\longrightarrow+\infty} L(\alpha_n) = 0 = L(\alpha).$$

Case 2: There exists a subsequence of α_n (we do not relabel) such that $L(\alpha_n) \neq 0$.
Then there exists a sequence of minimal loop trajectories $x_n : [0, L(\alpha_n)] \longrightarrow \mathbb{R}^n$ such that $x_n(0) = x_n(L(\alpha_n)) = \alpha_n$. Since $r_n \geq L(\alpha_n)$ and $r_n \longrightarrow r$, we can assume that the sequence $L(\alpha_n)$ converges. There are two cases:

Case 2.1: $L(\alpha_n) \longrightarrow 0$.
Then using the same technique as in 1) of Proposition 4.3.1, we show that $0 \in F(0)$ and then $L(\alpha) = 0 = \lim_{n\longrightarrow+\infty} L(\alpha_n) \leq \lim_{n\longrightarrow+\infty} r_n = r$.

Case 2.2: $L(\alpha_n) \longrightarrow T > 0$.
As above and by the compactness property of trajectories there exists a loop trajectory $\bar{x} : [0,T] \longrightarrow \mathbb{R}^n$ such that $\bar{x}(0) = \bar{x}(T) = \alpha$. Hence $L(\alpha) \leq T = \lim_{n\longrightarrow+\infty} L(\alpha_n) \leq \lim_{n\longrightarrow+\infty} r_n = r$.
The result follows.

2) Let $\alpha \in \mathbb{R}^n$ and assume that $T(\cdot,\cdot)$ is continuous at (α,α). By Proposition 2.4.2, F and $-F$ are α-STLC, then $\mathcal{R}^\alpha_+ \cap \mathcal{R}^\alpha_-$ is open and $L(\alpha) = 0$ ($0 \in F(\alpha)$ since F is α-STLC). But $\mathcal{R}^\alpha_+ \cap \mathcal{R}^\alpha_- \subset \mathcal{L}$ then $\alpha \in \operatorname{int}\mathcal{L}$. Now, let $\alpha_n \in \mathcal{L}$ be a sequence such that $\alpha_n \longrightarrow \alpha$. There are two cases:
Case 1: There exits a subsequence α_{n_k} such that $L(\alpha_{n_k}) = 0$ for all k.

Then $\lim_{k\longrightarrow+\infty} L(\alpha_{n_k}) = L(\alpha) = 0$.
Case 2: There exits a subsequence α_{n_k} such that $L(\alpha_{n_k}) \neq 0$ for all k.
By Proposition 4.3.1 there exists $\bar{x}_{n_k}$ a solution for the problem $P(\alpha_{n_k})$ which is a minimal loop trajectory at α_{n_k}. We take $t_{n_k} \in]0, L(\alpha_{n_k})[$ such that $\|\bar{x}_{n_k}(t_{n_k}) - \alpha_{n_k}\| \leq \dfrac{1}{n_k}$. Since $\bar{x}_{n_k}$ is a minimal loop trajectory we have

$$L(\alpha_{n_k}) = T(\alpha_{n_k}, \bar{x}_{n_k}(t_{n_k})) + T(\bar{x}_{n_k}(t_{n_k}), \alpha_{n_k}),$$

then by the continuity of $T(\cdot,\cdot)$ at (α,α) we get that

$$\lim_{k\longrightarrow+\infty} L(\alpha_{n_k}) = 0 = L(\alpha).$$

In the two cases, we show that there exists a subsequence α_{n_k} of α_n such that $\lim_{k\longrightarrow+\infty} L(\alpha_{n_k}) = L(\alpha)$. The result follows.

3) Let $\alpha \in \mathcal{L}$ such that $L(\alpha) \neq 0$. It is easy to verify that for all $\beta \in \mathbb{R}^n$ we have $L(\alpha) \leq T(\alpha,\beta) + T\beta,\alpha)$. Then $L(\alpha) \leq \liminf_{\beta\neq\alpha\ \beta\longrightarrow\alpha}[T(\alpha,\beta) + T(\beta,\alpha)]$. Since $L(\alpha) \neq 0$, there exists a minimal loop trajectory $\bar{x} : [0, L(\alpha)] \longrightarrow \mathbb{R}^n$ at α. We take $\beta_\varepsilon = \bar{x}(T-\varepsilon)$, then $T(\alpha,\beta_\varepsilon)+T(\beta_\varepsilon,\alpha) = L(\alpha)-\varepsilon+\varepsilon = L(\alpha)$. Therefore $L(\alpha) \geq \liminf_{\beta\neq\alpha\ \beta\longrightarrow\alpha}[T(\alpha,\beta) + T(\beta,\alpha)]$. The result follows.

4) By Proposition 0.3.6 it is sufficient to show that the systems (L,F) and $(L,-F)$ are weakly decreasing on $\mathbb{R}^n$. We begin showing that (L,F) is weakly decreasing. Let $\alpha \in \mathcal{L}$,[3] there exists two cases:
Case 1: $L(\alpha) = 0$.
Then we consider the constant trajectory $x(t) = \alpha$ for all $t \in [0,+\infty[$ [4] and we get that $L(x(t)) \leq L(\alpha)$, for all $t \in [0,+\infty[$.
Case 2: $L(\alpha) > 0$.
Then we consider the solution $\bar{x} : [0, L(\alpha)] \longrightarrow \mathbb{R}^n$ of the problem $P(\alpha)$ and clearly we have $L(x(t)) \leq L(\alpha)$.
Therefore the system (L,F) is weakly decreasing and then by 1) the system $(L,-F)$ is also weakly decreasing. □

[3] The case $\alpha \notin \mathcal{L}$ follows immediately.
[4] There exists a such trajectory since $0 \in F(\alpha)$.

4.4 Necessary and sufficient conditions

In this section we give a necessary and sufficient conditions for the existence of minimal loop trajectories at a given point α. We begin by the following proposition which gives a necessary condition.

Proposition 4.4.1 *Assume that there exists a minimal loop trajectory at $\alpha \in \mathbb{R}^n$ then there exist a constant $M > 0$ and two sequences α_n and β_n such that*

$$\alpha_n \longrightarrow \alpha, \quad \beta_n \longrightarrow \alpha, \tag{4.2}$$

and

$$M < T(\alpha, \alpha_n) < +\infty, \quad M < T(\beta_n, \alpha) < +\infty \quad \forall n. \tag{4.3}$$

(This gives that $T(\alpha, \cdot)$ and $T(\cdot, \alpha)$ are not continuous at α).

Proof. Let $\alpha \in \mathbb{R}^n$ and assume that there exists a minimal loop trajectory $x : [0, T] \longrightarrow \mathbb{R}^n$ at α. We take for $n > \frac{1}{T}$, $\alpha_n = x(T - \frac{1}{n})$ and $\beta_n = x(\frac{1}{n})$. Hence

$$T(\alpha, \alpha_n) = T(\beta_n, \alpha) = T - \frac{1}{n}.$$

Then for $n > \frac{2}{T}$ and $M = \frac{T}{2}$ we find the result. □

The following proposition gives a necessary and sufficient condition for the existence of minimal loop trajectory at a point α which satisfies the condition $0 \notin F(\alpha)$.

Proposition 4.4.2 *Let $\alpha \in \mathbb{R}^n$ and assume that $0 \notin F(\alpha)$. Then the following statements are equivalent.*

1. *There exists a minimal loop trajectory at α.*
2. $\liminf_{\beta \neq \alpha \ \beta \longrightarrow \alpha} [T(\alpha, \beta) + T(\beta, \alpha)] < +\infty$.

Proof. Follows from Propositions 4.3.1 and 4.3.3. □

Clearly the discontinuity of functions $T(\alpha, \cdot)$ and $T(\cdot, \alpha)$ at α is not a sufficient condition for the existence of a minimal loop trajectories. A sufficient condition for the existence of a constant $M > 0$ and sequences α_n and β_n which satisfy (4.2) and (4.3) is the follows: $0 \notin F(\alpha)$ and F and $-F$ are α-LC. The following proposition shows that this condition is sufficient for the existence of a minimal loop trajectory.

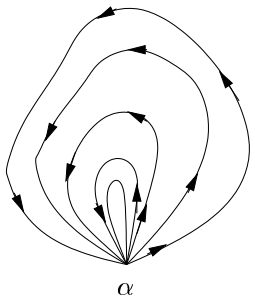

Figure 4.2: Nontrivial loop trajectories vanishing at α

Proposition 4.4.3 *Let $\alpha \in \mathbb{R}^n$ and assume that $0 \notin F(\alpha)$. Assume further that F and $-F$ are α-LC. Then there exists a minimal loop trajectory at α.*

Proof. Since F and $-F$ are α-LC and by Remark 4.2.3 there exists a nontrivial loop trajectory at α. Then $\liminf_{\beta \neq \alpha \ \beta \longrightarrow \alpha} [T(\alpha, \beta) + T(\beta, \alpha)] < +\infty$. The result follows form Proposition 4.4.2. □
We note that the condition

$$\liminf_{\beta \neq \alpha \ \beta \longrightarrow \alpha} [T(\alpha, \beta) + T(\beta, \alpha)] < +\infty, \tag{4.4}$$

remains a necessary condition for the existence of a minimal loop trajectory if we have $0 \in F(\alpha)$, but fails to be a sufficient condition, see the case 3) of Example 4.5.1. In Proposition 4.4.4 we show that if we add another assumption for the condition (4.4) then we can affirm the existence of a minimal loop trajectory at a point α which satisfies $0 \in F(\alpha)$.
Let $x_i : [0, T_i] \longrightarrow \mathbb{R}^n$ be a sequence of nontrivial loop trajectories at a point α and assume that the sequence T_i converges to a $T \geq 0$. We extend x_i to a periodic trajectory on $[0, +\infty[$ (period T_i). If there exists a trajectory x of F on $[0, +\infty[$ such that x_i converges uniformly to x on bounded interval and $x(t) = \alpha$ for all $t \in [0, +\infty[$, then we say that the sequence x_i vanishes at α, see Figure 4.2.

Proposition 4.4.4 *Let $\alpha \in \mathbb{R}^n$ and assume that $0 \in F(\alpha)$. Assume further that the following two hypotheses hold:*

1. *There is not a sequence of nontrivial loop trajectories at α which vanishes at α.*

2. $\liminf_{\beta \neq \alpha \ \beta \longrightarrow \alpha} [T(\alpha, \beta) + T(\beta, \alpha)] < +\infty$.

Then there exists a minimal loop trajectories at α.

Proof. For all $\beta \in \mathbb{R}^n$ we consider the following optimal control problem:

$$Q(\beta) \begin{cases} \text{Minimize } T > 0, \\ \dot{x}(t) \in F(x(t)) \ a.e. \ t \in [0, T], \\ x(0) = x(T) = \beta \ \text{ and } x(\cdot) \text{ is nontrivial.} \end{cases}$$

It is easy to prove that we have:

$$\inf Q(\beta) = \liminf_{\beta' \neq \beta \ \beta' \longrightarrow \beta} [T(\beta, \beta') + T(\beta', \beta)],$$

for all $\beta \in \mathbb{R}^n$. Then we have that $\inf Q(\alpha) < +\infty$. Hence there exists a sequence $x_i : [0, T_i]$ of nontrivial loop trajectory at α and a sequence $\varepsilon_i > 0$ such that $\varepsilon_i \longrightarrow 0$ and

$$\inf Q(\alpha) \leq T_i < \inf Q(\alpha) + \varepsilon_i.$$

We extend x_i to a periodic trajectory on $[0, +\infty[$ (period T_i). Since T_i is bounded and by the compactness property of trajectories we can assume that $T_i \longrightarrow T > 0$ and there exists a trajectory x of F on $[0, +\infty[$ such that x_i converges uniformly to x on bounded interval. Using our hypothesis 1), we have $T > 0$ and the trajectory $x : [0, T] \longrightarrow \mathbb{R}^n$ is a nontrivial loop trajectory at α. Since $T = \inf Q(\alpha)$ we get that $x(\cdot)$ is a solution of the problem $Q(\alpha)$. By the minimality of $x(\cdot)$, this trajectory is a minimal loop trajectory at α. □

We recall that a trajectory $x : [0, +\infty[\longrightarrow \mathbb{R}^n$ is a semigeodesic from α iff $x(0) = \alpha$ and $T(x(s), x(t)) = t - s$ for all $s \leq t \in [0, +\infty[$. In the following theorem, we show that under some hypotheses, there exists a semigeodesic or a minimal loop trajectory at a point α.

Theorem 4.4.5 *Let* $\alpha \in \mathbb{R}^n$ *and assume that we have the following hypotheses:*

1. *There exist a constant* $M > 0$ *and a sequence* α_n *such that* $\alpha_n \longrightarrow \alpha$ *and* $M < T(\alpha, \alpha_n) < +\infty \ \forall n$.

2. $\forall (\beta, \gamma) \in \mathcal{R} \cap (\mathcal{R}^\alpha_+ \times \mathcal{R}^\alpha_+) \setminus \mathcal{D}$, $\forall (\beta_i, \gamma_i) \in \mathcal{R} \cap (\mathcal{R}^\alpha_+ \times \mathcal{R}^\alpha_+) \setminus \mathcal{D}$ *we have*

$$[(\beta_i, \gamma_i) \longrightarrow (\beta, \gamma)] \Longrightarrow [T(\beta_i, \gamma_i) \longrightarrow T(\beta, \gamma)].$$

Then one of the following statements is true:

1. $0 \in F(\alpha)$.
2. *There exists a semigeodesic from* α.
3. *There exists a minimal loop trajectory at* α.

Proof. We consider the sequence $x_n : [0, T_n] \longrightarrow \mathbb{R}^n$ such that x_n is the minimal trajectory between α and α_n where $T_n = T(\alpha, \alpha_n)$. We have $M < T_n$. There exist two cases:
Case 1: $\lim_{n \longrightarrow +\infty} T_n = +\infty$.
As above there exist a trajectory $\bar{x} : [0, +\infty[\longrightarrow \mathbb{R}^n$ and a subsequence of x_n (we do not relabel) such that for every $b > 0$, x_n converges uniformly to $\bar{x}$ on $[0, b]$. There exist tow cases:

Case 1.1: There exists $s \in]0, +\infty[$ such that $\bar{x}(s) = \alpha$.
We claim that $\bar{x}(t) = \alpha$ for all $t \in [0, +\infty[$. Indeed, assume that $\bar{x}(t) \neq \alpha$ for $t \in]0, +\infty[$. There are two cases to consider:

Case 1.1.1: $t < s$.
Since $T_n \longrightarrow +\infty$, there exists a n_0 such that $t < s < T_n$ for $n > n_0$. Then $T(x_n(t), \alpha_n) = T(x_n(t), x_n(T_n)) = T_n - t$ and since the sequence $(x_n(t), \alpha_n)$ and $(\bar{x}(t), \alpha)$ are in $\mathcal{R} \cap (\mathcal{R}^\alpha_+ \times \mathcal{R}^\alpha_+) \setminus \mathcal{D}$ ($\bar{x}(0) = \alpha$, $\bar{x}(s) = \alpha$ and $0 < t < s$) we get that

$$T(\bar{x}(t), \alpha) = \lim_{n \longrightarrow +\infty} T(x_n(t), \alpha_n) = \lim_{n \longrightarrow +\infty} T_n - t = +\infty.$$

This gives a contradiction, since $(\bar{x}(t), \alpha) \in \mathcal{R}$.

Case 1.1.2: $t > s$.
Since $T_n \longrightarrow +\infty$, there exists a n_0 such that $t < s < T_n$ for $n > n_0$. Then $T(\alpha, x_n(t)) = T(x_n(0), x_n(t)) = t$. In the other hand $T(x_n(s), x_n(t)) = t - s$. Since the sequences $(\alpha, x_n(t))$ and $(x_n(s), x_n(t))$, and $(\alpha, \bar{x}(t))$ are in $\mathcal{R} \cap (\mathcal{R}^\alpha_+ \times \mathcal{R}^\alpha_+) \setminus \mathcal{D}$ we get that

$$T(\alpha, \bar{x}(t)) = \lim_{n \longrightarrow +\infty} T(\alpha, x_n(t)) = t$$

and

$$T(\alpha, \bar{x}(t)) = \lim_{n \longrightarrow +\infty} T(x_n(s), x_n(t)) = t - s.$$

And this gives a contradiction.
Then $\bar{x}(t) = \alpha$ for all $t \in [0, +\infty[$. Therefore $0 \in F(\alpha)$.

Case 1.2: For all $s \in]0, +\infty[$, $x(s) \neq \alpha$.
We show that $\bar{x}$ is a semigeodesic form α. First, we claim that $\bar{x}(s) \neq \bar{x}(t)$ for all $s < t \in]0, +\infty[$. Indeed, assume that there exists $s \leq t \in]0, +\infty[$ such that $\bar{x}(s) = \bar{x}(t)$. Since $T_n \longrightarrow +\infty$, there exists a n_0 such that $s \leq t < T_n$ for $n > n_0$. Then $T(\alpha, x_n(s)) = T(x_n(0), x_n(s)) = s$ and $T(\alpha, x_n(t)) = T(x_n(0), x_n(t)) = t$. By our hypothesis 2) we find that $s = t$. Now let $s \leq t \in [0, +\infty[$. Since $T_n \longrightarrow +\infty$, there exists a n_0 such that $s \leq t < T_n$ for $n > n_0$. Then $T(x_n(s), x_n(t)) = t - s$ and since the sequence $(x_n(s), x_n(t))$ and $(\bar{x}(s), \bar{x}(t))$ are in $\mathcal{R} \cap (\mathcal{R}^{\alpha}_{+} \times \mathcal{R}^{\alpha}_{+}) \setminus \mathcal{D}$ we get that

$$T(\bar{x}(s), \bar{x}(t)) = \lim_{n \longrightarrow +\infty} T(x_n(s), x_n(t)) = t - s.$$

Then $\bar{x}$ is a semigeodesic form α.
Case 2: $\lim_{n \longrightarrow +\infty} T_n \neq +\infty$.
Then there exits a subsequence (we do not relabel) of T_n such that $T_n \longrightarrow T > M > 0$. There exist a trajectory $\bar{x} : [0, +\infty[\longrightarrow \mathbb{R}^n$ and a subsequence of x_n (we do not relabel) such that for every $b > 0$, x_n converges uniformly to $\bar{x}$ on $[0, b]$. First we have that $\bar{x} : [0, T] \longrightarrow \mathbb{R}^n$ is a loop trajectory at α since $\bar{x}(0) = \alpha$ and $\bar{x}(T) = \lim_{n \longrightarrow +\infty} x_n(T_n) = \lim_{n \longrightarrow +\infty} \alpha_n = \alpha$. There exist tow cases:

Case 2.1: There exists $s \in]0, T[$ such that $\bar{x}(s) = \alpha$.
Then we show as in Case 1.1 that $\bar{x}(t) = \alpha$ for all $t \in [0, T]$ and then $0 \in F(\alpha)$.

Case 2.2: For all $s \in]0, T[$ we have $\bar{x}(s) \neq \alpha$.
Then we show as in Case 1.2 that $T(\alpha, \bar{x}(t)) + T(\bar{x}(t), \alpha) = T$. This gives using Remark 4.2.3 that $\bar{x}$ is a minimal loop trajectory at α. □

Remark 4.4.6
• *In the preceding theorem, we can replace the function $T(\alpha, \cdot)$ by $T(\cdot, \alpha)$ but in this case we obtain a semigeodesic from α for the dynamic $-F$.*
• *If we assume in Theorem 4.4.5 that the sequence $T(\alpha, \alpha_n)$ is bounded above by some constant $N > 0$, then the conclusion of the theorem becomes: $0 \in F(\alpha)$ or there exists a minimal loop trajectory at α.*

A sufficient condition for the existence of a constant $M > 0$ an a sequence α_n such that $\alpha_n \longrightarrow \alpha$ and $M < T(\alpha, \alpha_n) < +\infty$ is the following: $0 \notin F(\alpha)$ and $-F$ is α-LC. Then we have the following corollary.

Corollary 4.4.7 *Let $\alpha \in \mathbb{R}^n$ and assume that we have the following hypotheses:*

1. *$0 \notin F(\alpha)$ and $-F$ is α-LC.*
2. *$\forall(\beta, \gamma) \in \mathcal{R} \cap (\mathcal{R}^\alpha_+ \times \mathcal{R}^\alpha_+) \setminus \mathcal{D}$, $\forall(\beta_i, \gamma_i) \in \mathcal{R} \cap (\mathcal{R}^\alpha_+ \times \mathcal{R}^\alpha_+) \setminus \mathcal{D}$ we have*

$$[(\beta_i, \gamma_i) \longrightarrow (\beta, \gamma)] \Longrightarrow [T(\beta_i, \gamma_i) \longrightarrow T(\beta, \gamma)].$$

Then one of the following statements is true:

1. *There exists a semigeodesic from α.*
2. *There exists a minimal loop trajectory at α.*

Remark 4.4.8 *If we assume in the preceding corollary that $T(\alpha, \cdot)$ is bounded above near α then the conclusion of this corollary becomes that there exists a minimal loop trajectory at α.*

4.5 Examples

In this section we give some examples. We begin by the following example.

Example 4.5.1 *For $n = 2$, we define $F(x, y) := \{(y, -x)u : u \in U \subset \mathbb{R}\}$ and we consider these three cases:*

1. *$U = [\frac{1}{2}, 1]$. Then for $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ it is easy to see that every trajectory from (a, b) is in the circle centered at 0 and passes through (a, b). We have:*
   (a) *$\mathcal{R}^{(a,b)}_+ = \mathcal{R}^{(a,b)}_- = \{(x, y) : x^2 + y^2 = a^2 + b^2\}$ and $\mathcal{R}^{(0,0)}_+ = \mathcal{R}^{(0,0)}_- = \{(0, 0)\}$.*
   (b) *$\mathcal{R} = \bigcup_{(a,b)\in\mathbb{R}^2} \mathcal{R}^{(a,b)}_+ \times \mathcal{R}^{(a,b)}_+$.*

For all $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$*, all the hypotheses of Theorem 4.4.5 are satisfied. Moreover* $(0,0) \notin F(a,b)$ *then by Remark 4.4.6 there exists a minimal loop trajectory at* (a,b)*. This trajectory is*

$$(x(t), y(t)) = (a\cos t + b\sin t, -a\sin t + b\cos t), \quad t \in [0, 2\pi]. \tag{4.5}$$

The functions $T((0,0),\cdot)$ *and* $T(\cdot,(0,0))$ *are discontinuous at* $(0,0)$ *but we can not find a minimal loop trajectory at* $(0,0)$*. We calculate* $L(\cdot)$ *and we find that* $\mathcal{L} = \mathbb{R}^2$ *and*

$$L(a,b) = \begin{cases} 2\pi & \text{if } (a,b) \neq (0,0), \\ 0 & \text{if } (a,b) = (0,0), \end{cases}$$

and then $L(\cdot)$ *is discontinuous at* $(0,0)$*. We remark that for* $(a,b) \neq (0,0)$ *there exists only one solution for* $P(\alpha,\beta)$ *which is the minimal loop trajectory (4.5).*

2. $U = [0,1]$*. In this case* $\mathcal{R}_+^{(a,b)}$*,* $\mathcal{R}_-^{(a,b)}$ *and* $\mathcal{R}$ *remain as above but we find that* $L(\cdot) = 0$ *since* $0 \in F(\cdot)$*. For all* $(a,b) \neq (0,0)$ *and by Proposition 4.4.4 there exists a minimal loop trajectory which is exactly (4.5).*

3. $U = [-1,1]$*. In this case, the sets* $\mathcal{R}_+^{(a,b)}$*,* $\mathcal{R}_-^{(a,b)}$ *and* $\mathcal{R}$ *remain as in case 1). The function* $L(\cdot) = 0$ *since* $0 \in F(\cdot)$ *but using Proposition 4.4.1 we can not find any minimal loop trajectory.*

Example 4.5.2 *We take* $n = 2$ *and we define the following two multifunctions:*

1. $F_1(x,y) = \bar{B}$ *for all* $(x,y) \in \mathbb{R}^2$*.*
2. $F_2(x,y) = \{(1,0)\}$ *for all* $(x,y) \in \mathbb{R}^2$*.*

Now we consider the multifunction F *defined by the following:*

- $F(x,y) = F_2(x,y)$ *for all* (x,y) *such that* $\|(x,y)\| \leq 1$*.*
- $F(x,y) = F_1(x,y)$ *for all* (x,y) *such that* $\|(x,y)\| \geq 2$*.*
- $F(x,y) = \{(2-r)v_2 + (r-1)v_1 : v_1 \in F_1(x,y) \text{ and } v_2 \in F_2(x,y)\}$ *for all* (x,y) *such that* $1 < \|(x,y)\| := r < 2$*.*

It is easy to verify that we have:

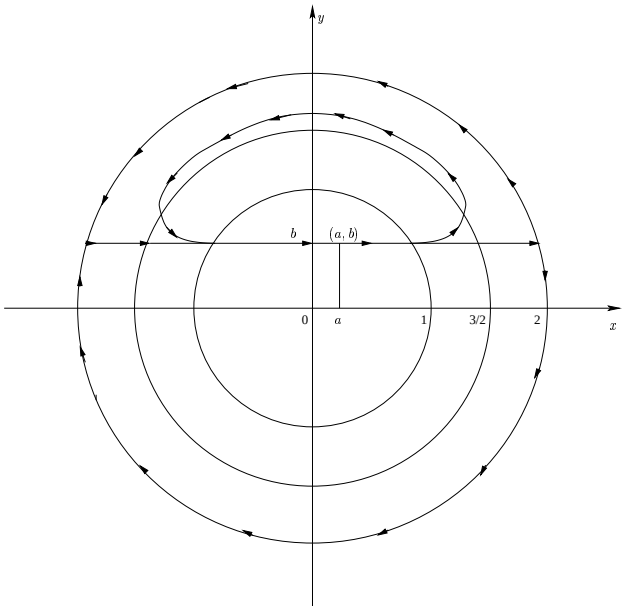

Figure 4.3: Example 4.5.2

- $0 \in F(x,y)$ *iff* $\|(x,y)\| \geq \dfrac{3}{2}$,
- $0 \in intF(x,y)$ *iff* $\|(x,y)\| > \dfrac{3}{2}$.

Hence by Propositions 2.3.5, 4.3.1 and 4.4.1 we get that:

- $L(x,y) = 0$ *iff* $\|(x,y)\| \geq \dfrac{3}{2}$.
- *We can not find a minimal loop trajectory at a point* $(a,b) \in \mathbb{R}^2$ *which satisfies* $\|(a,b)\| > \dfrac{3}{2}$

It is easily seen that for $\|(a,b)\| < \dfrac{3}{2}$ *there exists a loop trajectory at* (a,b)*, see Figure 4.3. Then* $\mathcal{L} = \mathbb{R}^2$ *and by Proposition 4.3.1 there exists a minimal loop trajectory at every point* (a,b) *which satisfies* $\|(a,b)\| < \dfrac{3}{2}$*. For a point* (a,b) *which satisfies* $\|(a,b)\| = \dfrac{3}{2}$*, there exist a several cases; for example we can show using Proposition 4.4.4 that there exists a minimal loop trajectory*

at $(\frac{3}{2},0)$ *but no minimal loop trajectories at* $(0,\frac{3}{2})$ *since* F *is* $(0,\frac{3}{2})$*-STLC. The function* $L(\cdot)$ *is continuous on* $\mathbb{R}^2 \setminus \{(x,y) : \|(x,y)\| = \frac{3}{2}\}$ *and have a discontinuous points in the set* $\{(x,y) : \|(x,y)\| = \frac{3}{2}\}$*. For a point* $(a,0)$ *where* $a \in [-1,1]$*, we can find two solutions for the problem* $P(a,0)$ *and then two minimal loop trajectories.*
We note that for all $(a,b) \in \mathbb{R}^2$ *we have:*

- $\mathcal{R}_+^{(a,b)} = \mathcal{R}_-^{(a,b)} = \mathbb{R}^2$,[5]
- $\mathcal{R} = \mathbb{R}^2 \times \mathbb{R}^2$,

and we can find a relation between $L(\cdot)$ *and* $T(\cdot,\cdot)$*. For example, for all* $\|(a,b)\| < 1$ *we have:*

$$L(a,b) = 2\sqrt{1-b^2} + T((\sqrt{1-b^2},b),(-\sqrt{1-b^2},b)),$$

and this shows that $L(\cdot)$ *does not depend on the variable* a *if* $\|(a,b)\| < 1$.

[5]This gives that $0 \in F(\alpha)$ is not a necessary condition for α-LC.

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