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THÈSE DE DOCTORAT DE L'UNIVERSITÉ PARIS VI

Spécialité :
Mathématiques

présentée par
Grégory MIERMONT

pour obtenir le grade de DOCTEUR de l'UNIVERSITÉ PARIS VI

Sujet de la thèse :

**Coalescence et fragmentation
stochastiques,
arbres aléatoires et processus de Lévy**

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M. Alain ROUAULT

Soutenue le 16 décembre 2003 devant le jury composé de

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M. Jean-François LE GALL (Examineur)
M. Yves LE JAN (Examineur)
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Table des matières

1	Introduction	7
1.1	Objets étudiés	8
1.2	Aperçu des résultats	20
2	Ordered additive coalescent and Lévy processes	39
2.1	Introduction	40
2.2	Ordered additive coalescent	41
2.3	The Lévy fragmentation	51
2.4	The fragmentation semigroup	55
2.5	The left-most fragment	59
2.6	The mixing of extremal additive coalescents	61
2.7	Proof of Vervaat's Theorem	63
2.8	Concluding remarks	70
3	Fragmentations and stable subordinators	73
3.1	Introduction	73
3.2	A Poisson process construction of self-similar fragmentations	78
3.3	Stable subordinators	79
3.4	Small-time behavior of self-similar fragmentations	83
3.5	Large-time behavior of self-similar fragmentations	90
3.6	One-dimensional distributions	91
3.7	Mass of a tagged fragment	94
4	Self-similar fragmentations of the stable tree I	97
4.1	Introduction	97
4.2	Preliminaries	101
4.3	Study of F^-	108
4.4	Small-time asymptotics	117
4.5	On continuous-state branching processes...	120
5	Self-similar fragmentations of the stable tree II	127
5.1	Introduction	127
5.2	Some facts about Lévy processes	131
5.3	The stable tree	136
5.4	Study of F^+	142
5.5	Study of F^\natural	149
5.6	Asymptotics	156

6	The genealogy of self-similar fragmentations	159
6.1	Introduction	159
6.2	The CRT \mathcal{T}_F	162
6.3	Hausdorff dimension of \mathcal{T}_F	169
7	The exploration process of the ICRT	183
7.1	Introduction	184
7.2	Constructing X^θ and Y^θ	189
7.3	p -trees and associated processes	193
7.4	Convergence of p -trees to the ICRT	199
7.5	Height profile	204
7.6	The exploration process	206
7.7	Miscellaneous comments	216
8	Asymptotics for Random p-mappings	219
8.1	Introduction	219
8.2	Background	221
8.3	Proof of Theorem 8.1	227
8.4	Final comments	233
	Bibliographie	235

Chapitre 1

Introduction

Table des matières

1.1 Objets étudiés	8
1.1.1 Processus de Lévy complètement asymétriques, subordinateurs	9
1.1.2 Coalescence additive	11
1.1.3 Fragmentations auto-similaires	13
1.1.4 Arbres aléatoires	15
1.2 Aperçu des résultats	20
1.2.1 Coalescent additif ordonné associé à certains processus de Lévy	20
1.2.2 Fragmentations auto-similaires et subordinateurs stables	22
1.2.3 Deux fragmentations de l'arbre stable	24
1.2.4 Généalogie des fragmentations auto-similaires d'indice négatif	28
1.2.5 Processus d'exploration des ICRT	31
1.2.6 Propriétés asymptotiques des applications aléatoires	35

Cette thèse est une contribution à un approfondissement des liens nombreux existant entre les processus de coalescence et de fragmentation stochastiques, les arbres aléatoires et les processus de Lévy. Nos travaux portent sur trois sujets principaux :

- La coalescence additive, qui est un modèle décrivant l'évolution d'un ensemble d'objets qui s'agglomèrent à un taux égal à la somme de leurs masses. Ces processus ont été étudiés par Evans et Pitman [52] puis par une série d'articles d'Aldous et Pitman [10, 12] et de Bertoin [21, 22], où ils sont reliés d'une part à des arbres aléatoires continus, dont nous parlons plus loin, et d'autre part à des trajectoires de processus de Lévy.
- Les fragmentations auto-similaires, où au contraire des objets se disloquent aléatoirement au cours du temps à un taux pouvant dépendre de leur taille. Ces processus ont été introduits et étudiés très en détail par Bertoin [23, 26, 27, 25], et peuvent être considérés comme des processus duaux (en un sens très faible) de processus de coalescence échangeables introduits et étudiés par Pitman [89], Schweinsberg [98, 99], Bertoin et Le Gall [29].
- Les arbres continus aléatoires (Continuum Random Trees, ou CRT). Ces arbres, dans un formalisme introduit par Aldous [5], sont un modèle d'espace métrique aléatoire possédant une structure arborescente ainsi qu'une *mesure de masse* qui évalue la densité des feuilles de ces arbres. Ils apparaissent notamment dans la généalogie

des processus de branchements continus, des superprocessus et des serpents associés (voir par exemple [76, 49]) ainsi que comme modèles limites pour des systèmes discrets (amas de percolation en grande dimension, arbres de Galton-Watson, applications aléatoires).

Cette thèse se compose de sept chapitres :

- Un article intitulé *Ordered additive coalescent and fragmentations associated to Lévy processes with no positive jumps*. Paru à *Electronic Journal of Probability*, **6**, 2001 (33 pp.)
- Un article intitulé *Self-similar fragmentations and stable subordinators*, écrit en collaboration avec Jason Schweinsberg. À paraître au *Séminaire de Probabilités XXXVII*, Springer, 2003.
- Deux articles complémentaires intitulés *Self-similar fragmentations derived from the stable tree I & II : Splitting at heights et Splitting at nodes*. Le premier est paru à *Probability Theory and Related Fields*, **127** n.3, pp. 423–454.
- Une version partielle d'un article intitulé *The genealogy of self-similar fragmentations as a continuum random tree*, écrit en collaboration avec Bénédicte Haas.
- Un article intitulé *The exploration process of inhomogeneous continuum random trees, and an extension of Jeulin's local time identity*, écrit en collaboration avec David Aldous et Jim Pitman, à paraître à *Probability Theory and Related Fields*.
- Un article intitulé *Brownian bridge asymptotics for random \mathbf{p} -mappings*, écrit en collaboration avec David Aldous et Jim Pitman.

(Les versions qui apparaissent ici sont parfois légèrement différentes des versions publiées.) Ces chapitres peuvent eux-mêmes se regrouper en trois parties, selon la nature des résultats qui y sont démontrés. La première partie (chapitres 2 et 3) traite de propriétés des lois de certaines versions du processus de coalescence additive et de certaines fragmentations auto-similaires. La seconde (chapitres 4, 5 et 6) est consacrée à l'étude de fragmentations auto-similaires liées à certains modèles de CRT, dont les récents "arbres stables" de Duquesne et Le Gall [49]. Enfin, la troisième (chapitres 7 et 8) porte sur un modèle de CRT obtenu comme limite d'arbres discrets (les \mathbf{p} -arbres) ; ces CRT, appelés ICRT (pour "Inhomogeneous"), interviennent à la fois dans les processus de coalescence additive considérés au chapitre 2 [12] et dans les propriétés asymptotiques de certaines applications aléatoires, qui sont étudiées au chapitre 8 et qui sont le seul sujet de cette thèse n'ayant pas pour motivation l'étude de processus de fragmentation ou de coalescence.

Dans la suite de l'introduction, nous définissons d'abord les objets mathématiques étudiés dans cette thèse, après quoi nous donnons un aperçu des résultats ainsi que de leur démonstration.

1.1 Objets étudiés

Nous introduisons ici les différentes définitions des processus de coalescence et de fragmentation, et les modèles d'arbres aléatoires cités ci-dessus, en commençant par certains résultats utiles sur les processus de Lévy.

1.1.1 Processus de Lévy complètement asymétriques, subordinateurs

Les processus de Lévy sont des outils récurrents dans l'étude des processus de fragmentation et de coalescence ainsi que des arbres aléatoires. La plupart des résultats de cette section proviennent de [19, 97]. Un processus de Lévy est un processus réel càdlàg $(X_s, s \geq 0)$ issu de 0, dont les accroissements sont indépendants et stationnaires. Si on suppose de plus que X n'a que des sauts positifs ou que des sauts négatifs, on dit qu'il est complètement asymétrique. Nous nous concentrons dans cette introduction sur les processus n'ayant que des sauts positifs, le cas symétrique s'en déduisant simplement en remplaçant X par $-X$. Il est connu qu'un processus de Lévy est caractérisé par ses lois marginales de dimension 1 (et même entièrement par celle de X_s pour un $s > 0$). Si en plus il n'a pas de saut négatif, l'exposant de Laplace de X_s existe et est donné par la formule de Lévy-Khintchine :

$$s\psi(\lambda) = \log E[\exp(-\lambda X_s)] = s \left(\alpha\lambda + \frac{Q\lambda^2}{2} + \int_0^\infty L(dx)(e^{-\lambda x} - 1 + \lambda x) \right) \quad \lambda, s \geq 0,$$

où $L(dx)$ est une mesure sur \mathbb{R}_+^* , dite *mesure de Lévy*, vérifiant $\int_0^\infty L(dx)(1 \wedge x^2) < \infty$, $\alpha \in \mathbb{R}$ et $Q \geq 0$ est la composante gaussienne de X . Les processus de Lévy à sauts positifs considérés dans cette thèse satisfont de plus une hypothèse de variation infinie, ce qui se traduit par $Q > 0$ ou $\int_0^\infty L(dx)(1 \wedge x) = \infty$, ainsi que l'hypothèse que X oscille ou dérive vers $-\infty$, ce qui implique à la fois $\int_0^\infty L(dx)(x \wedge x^2) < \infty$, et que $E[X_1] = -\alpha \leq 0$. Enfin, on évitera les cas pathologiques en supposant l'existence de densités bicontinues pour les lois de $X_s, s > 0$, et on notera $p_s(x) = P(X_s \in dx)/dx$.

Subordinateurs Un subordinateur est un processus de Lévy $(\sigma_s, s \geq 0)$ qui est croissant. Sa loi est à son tour caractérisée par l'exposant de Laplace

$$s\Phi(\lambda) = -\log E[\exp(-\lambda\sigma_s)] = s \left(d\lambda + \int_0^\infty l(dx)(1 - e^{-\lambda x}) \right) \quad \lambda, s \geq 0,$$

où $d \geq 0$ est un coefficient de dérive et $l(dx)$ est une mesure (dite de Lévy) sur \mathbb{R}_+ vérifiant $\int_0^\infty l(dx)(1 \wedge x) < \infty$. Lorsque $d = 0$, le processus σ est de saut pur (il est égal à la somme de ses sauts), et plus précisément on a la construction suivante (décomposition de Lévy-Itô). Si on se donne $(\Delta_s, s \geq 0)$ un processus de Poisson ponctuel d'intensité $ds \otimes l(dx)$, alors

$$\sigma_s \stackrel{d}{=} ds + \sum_{0 \leq u \leq s} \Delta_u \quad s \geq 0.$$

Par la suite, on notera $\Delta\sigma_{[0,s]}$ la suite classée par ordre décroissant des sauts de σ accomplis dans l'intervalle $[0, s]$.

Si X est un processus de Lévy sans saut négatif, satisfaisant l'hypothèse de variation infinie et $E[X_1] \leq 0$, on définit son processus de l'infimum passé par $\underline{X}_s = \inf_{0 \leq u \leq s} X_u$. Le processus inverse à droite

$$T_s = \inf\{u \geq 0 : \underline{X}_u < -s\} \quad s \geq 0$$

est alors un subordinateur sans dérive ($d = 0$), dont l'exposant de Laplace Φ est la fonction inverse de ψ , l'exposant de Laplace de X .

Processus stables Un cas particulier de processus de Lévy qui sera crucial plus loin est le cas *stable* d'indice $\alpha \in (1, 2)$, où $L(dx)$ est multiple de la mesure $x^{-1-\alpha}dx$. Dans le cas $\alpha = 2$, on prend $L(dx) = 0$ et $Q > 0$, de sorte que X est un mouvement brownien. La transformée de Laplace est alors de la forme $E[\exp(-\lambda X_s)] = \exp(cs\lambda^\alpha)$ pour un $c > 0$, et on en déduit la propriété dite d'invariance par changement d'échelle (*scaling*) :

$$\frac{1}{\lambda^{1/\alpha}} X_{\lambda s} \stackrel{d}{=} X_s \quad s \geq 0$$

(quand $\alpha = 2$ on retrouve le *scaling* habituel du mouvement brownien). De même, un subordonateur stable d'indice $\alpha \in (0, 1)$ est un subordonateur dont la mesure de Lévy est proportionnelle à $x^{-1-\alpha}dx$. Son exposant de Laplace est alors de la forme $\Phi(\lambda) = c\lambda^\alpha$ avec $c > 0$.

On déduit des rappels précédents que si X est un processus stable sans saut négatif d'exposant de Laplace $\psi(\lambda) = \lambda^\alpha$ (avec $\alpha \in (1, 2]$), son subordonateur inverse T est un subordonateur stable d'indice $1/\alpha \in [1/2, 1)$.

Ponts et excursions Nous faisons dans cette thèse un usage important des ponts et des excursions des processus de Lévy. On se donne un processus de Lévy X satisfaisant les hypothèses ci-dessus. Rappelons que le pont de X de 0 à $z \in \mathbb{R}$ de longueur t est un processus $X_{0 \rightarrow z}^t$ dont la loi est une version de la loi conditionnelle de $(X_s, 0 \leq s \leq t)$ sachant $X_t = z$. Comme cet événement a une probabilité nulle, ce conditionnement est singulier, mais on peut définir la loi de $(X_{0 \rightarrow z}^t(s), 0 \leq s \leq t - \varepsilon)$ par absolue continuité pour tout $\varepsilon > 0$ par (rappelons que $p_s(x) = P(X_s \in dx)/dx$)

$$E[F(X_{0 \rightarrow z}^t(s), 0 \leq s \leq t - \varepsilon)] = E \left[F(X_s, 0 \leq s \leq t - \varepsilon) \frac{p_\varepsilon(z - X_{t-\varepsilon})}{p_t(z)} \right].$$

On peut montrer que cette relation définit bien une loi unique pour chaque t, z , qui est une version de la loi conditionnelle recherchée. De plus, il est aisé de voir que c'est une "bonne version" au sens où elle est continue en t, z pour la convergence faible des mesures.

Par ailleurs, on sait par la théorie d'Itô que si on note $\varepsilon_x(u) = X_{T_x+u} - x, 0 \leq u \leq T_x - T_{x-}$ l'excursion de X au-dessus de son infimum au niveau $x < 0$, il existe une mesure σ -finie N sur l'espace des "excursions" (processus càdlàg strictement positifs sur un intervalle de la forme $(0, \zeta)$ et nuls ailleurs) telle que le processus $(\varepsilon_x, x \geq 0)$ est un processus de Poisson ponctuel d'intensité $dx \otimes N(dx)$. Pour $v > 0$ nous notons $N^{(v)}$ une version de la probabilité conditionnelle $N(\cdot | \zeta = v)$, la mesure des excursions de durée v . Encore une fois, ce conditionnement est singulier, mais on peut construire ainsi une famille adéquate. On prend le pont $X_{0 \rightarrow 0}^v$ de longueur v et on note s_{\min} l'instant où $X_{0 \rightarrow 0}^v$ atteint son minimum. On peut montrer que cet instant est unique et correspond à un point de continuité de $X_{0 \rightarrow 0}^v$ presque-sûrement. On pose alors

$$V X_{0 \rightarrow 0}^v(s) = X_{0 \rightarrow 0}^v(s + s_{\min}) - X_{0 \rightarrow 0}^v(s_{\min}) \quad 0 \leq s \leq v$$

sa *transformée de Vervaat*, où les additions ci-dessus sont prises modulo v . Nous montrons au chapitre 2 que la loi $N^{(v)}$ de $V X_{0 \rightarrow 0}^v$ donne bien la loi recherchée, en généralisant, par des méthodes identiques, un résultat dû à Vervaat [103] pour le mouvement brownien et Chaumont [43] pour les processus stables. Cette version est à nouveau agréable car continue en $v > 0$; c'est celle que nous utiliserons.

1.1.2 Coalescence additive

Informellement, un processus de coalescence est un processus aléatoire qui décrit l'évolution au cours du temps d'un ensemble d'objets susceptibles de fusionner. Le premier modèle que nous considérons fait entièrement abstraction de la configuration spatiale des objets, c'est-à-dire que l'on connaît uniquement la suite de leurs masses. On suppose également que le système est à *volume fini*, c'est-à-dire que la somme des masses des objets est finie, on supposera en fait qu'elle vaut 1. On note $\mathbb{N} = \{1, 2, \dots\}$ l'ensemble des entiers non nuls (suivant la notation anglo-saxonne), et on introduit l'espace

$$S = \left\{ \mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i \geq 1} s_i \leq 1 \right\}$$

des suites indicées par \mathbb{N} décroissantes positives de somme plus petite que 1, que l'on munit de la topologie de la convergence terme à terme. On notera également $S^1 \subset S$ le sous-ensemble des suites de S de somme égale à 1. Si $\mathbf{s} \in S$, et $i, j \geq 1$ sont deux entiers non nuls distincts, on note

$$\mathbf{s}^{\oplus(i,j)} = (s_i + s_j, s_k : k \in \mathbb{N} \setminus \{i, j\})^\downarrow$$

la suite de S obtenue en fusionnant le i -ième et le j -ième terme de \mathbf{s} et en réordonnant par ordre décroissant.

Définition :

Un processus $(C(t), t \geq 0) = ((C_1(t), C_2(t), \dots), t \geq 0)$ à valeurs dans S^1 est un *coalescent additif* si c'est un processus de Markov homogène tel que le taux de saut d'un élément \mathbf{s} à un élément \mathbf{s}' est donné par

$$q(\mathbf{s}, \mathbf{s}') = \begin{cases} s_i + s_j & \text{s'il existe } i \neq j \text{ tels que } \mathbf{s}' = \mathbf{s}^{\oplus(i,j)} \\ 0 & \text{sinon} \end{cases}.$$

Une construction Dans le cas où l'état initial $C(0)$ du processus est une partition de masse "finie" (au sens où $s_i = 0$ à partir d'un certain rang), $(C(t), t \geq 0)$ est une chaîne de Markov à espace d'états finis, que l'on peut construire ainsi. On note $C(0) = (s_1, s_2, \dots, s_n, 0, \dots)$ avec $s_n > 0$, et pour chaque couple (i, j) avec $1 \leq i \neq j \leq n$ on se donne une variable exponentielle $e_{i,j}$ de taux $s_i + s_j$, de sorte que les $n(n-1)$ variables soient indépendantes. Presque-sûrement, il existe un unique couple (i_0, j_0) tel que $e_{i_0, j_0} = \inf_{(i,j)} e_{i,j}$. On pose alors $C(t) = C(0)$ pour $0 \leq t < e_{i_0, j_0}$, et $C(e_{i_0, j_0}) = \mathbf{s}^{\oplus(i_0, j_0)}$. Après quoi, on itère le procédé en utilisant de nouvelles variables exponentielles, jusqu'à ce que le système parvienne à l'état absorbant $(1, 0, 0, \dots)$. Nous verrons plus loin comment cette construction peut être "améliorée" de diverses façons en tenant compte de certaines informations supplémentaires, permettant de donner aux amas du processus des structures ordonnées ou arborescentes au lieu de ne distinguer que leurs masses.

Lorsque l'état initial du processus est "infini" ($C_i(0) > 0$ pour tout i), la définition ci-dessus devient problématique. Evans et Pitman [52] ont montré qu'il existe bien un unique processus de Hunt satisfaisant à cette définition quel que soit l'état de départ dans S^1 , mais la preuve, utilisant une approximation par des coalescents issus d'états finis, est loin d'être triviale.

Frontière d'entrée du coalescent additif Un des aspects étonnants du coalescent additif (contrairement à d'autres coalescents comme le coalescent de Kingman [69], pour lequel les taux de coalescence sont tous égaux à 1) est qu'il possède une *frontière d'entrée* extrêmement riche. En l'occurrence, il existe un grand nombre de processus *éternels* $(C(t), -\infty < t < \infty)$ de lois différentes, dont le semigroupe est celui du coalescent additif. On interprète ces processus comme des coalescents issus au temps $-\infty$ d'une "poussière" dont la nature peut varier. Nous expliquerons plus loin comment on peut obtenir les versions *extrêmes* de ce processus à l'aide d'une fragmentation d'arbres continus inhomogènes.

Coalescent additif standard La version la plus naturelle de tels processus est la version "standard", obtenue comme limite lorsque $n \rightarrow \infty$ du coalescent $(C^n(t), t \geq -\log(n)/2)$ issu au temps $-\log(n)/2$ de l'état $(1/n, 1/n, \dots, 1/n, 0, \dots)$ (n fois). Evans et Pitman [52] ont été les premiers à prouver l'existence de cette version standard. Elle a été ensuite étudiée par Aldous et Pitman [10], qui prouvèrent qu'elle peut être obtenue comme processus dual d'un processus de fragmentation de l'*arbre continu brownien*, sur lequel nous reviendrons. La construction la plus élémentaire de ce processus est due à Bertoin [21] et utilise des excursions browniennes plutôt que des arbres continus. Cette construction a été également étudiée par Chassaing et Louchard [41], qui la relie à certaines fonctions de parking et au hachage. Considérons une excursion brownienne normalisée (c'est-à-dire conditionnée à avoir une durée de vie égale à 1) $(B_s^{\text{exc}}, 0 \leq s \leq 1)$. Pour chaque $t \geq 0$ on note $B_s^{(t)} = B_s^{\text{exc}} - ts$ pour $0 \leq s \leq 1$, ainsi que $\underline{B}_s^{(t)} = \inf_{0 \leq u \leq s} B_u^{(t)}$ pour $0 \leq s \leq 1$ son minimum avant l'instant s . Le processus $(\underline{B}_s^{(t)}, 0 \leq s \leq 1)$ est décroissant continu, et découpe l'intervalle $(0, 1)$ en intervalles disjoints maximaux $I_j^{(t)}, j \geq 1$ sur lesquels $\underline{B}^{(t)}$ est constant. De plus, on vérifie que la somme des mesures de Lebesgue $\sum_j |I_j^{(t)}|$ vaut 1, et si on note $F_{\text{AP}}(t)$ le réarrangement décroissant de la suite $(|I_1^{(t)}|, |I_2^{(t)}|, \dots)$, alors on a que $(F_{\text{AP}}(\exp(-t)), -\infty < t < \infty)$ est le coalescent additif standard.

Cette représentation permet d'effectuer des calculs de lois sur le coalescent additif standard. Ainsi, on obtient que la loi de F_{AP} au temps $t > 0$ est

$$P(F_{\text{AP}}(t) \in ds) = P(\Delta\sigma_{[0,t]} \in ds | \sigma_t = 1), \quad (1.1)$$

où σ est un subordonateur stable d'indice $1/2$. On peut également prouver que si l'on se donne une variable U^* indépendante uniforme sur $[0, 1]$, le fragment $F_{\text{AP}}^*(t)$ contenant U^* au temps t (un tirage aléatoire biaisé par la taille parmi les fragments de $F_{\text{AP}}(t)$) possède la représentation en loi suivante :

$$(F_{\text{AP}}^*(t), t \geq 0) \stackrel{d}{=} ((1 + \sigma_t)^{-1}, t \geq 0) \quad (1.2)$$

où σ est à nouveau un subordonateur stable d'indice $1/2$. Ces deux représentations (ou des variantes proches) ont été découvertes par Aldous et Pitman [10]. Bertoin [21] a également montré que ce dernier processus a même loi que le fragment le plus à gauche, c'est-à-dire que si on note $I_g^{(t)}$ l'intervalle $I_j^{(t)}$ le plus à gauche, alors $(|I_g^{(t)}|, t \geq 0)$ et $(F_{\text{AP}}^*(t), t \geq 0)$ ont même loi. Enfin, on déduit de ce calcul de loi le comportement asymptotique en 0 de F_{AP} :

$$t^{-1/2}(1 - F_{\text{AP}}^1(t), F_{\text{AP}}^2(t), F_{\text{AP}}^3(t), \dots) \xrightarrow{d} (\sigma_1, \Delta_1(1), \Delta_2(1), \dots), \quad (1.3)$$

où $F_{AP}^1(t), F_{AP}^2(t) \dots$ sont les composantes de $F_{AP}(t)$, σ est un subordonateur stable d'indice $1/2$, et $\Delta_1(1), \dots$ sont ses sauts sur l'intervalle $[0, 1]$, classés par ordre décroissant. Nous verrons dans quelle mesure de tels résultats peuvent être généralisés à d'autres versions de la coalescence additive et à d'autres fragmentations que F_{AP} .

1.1.3 Fragmentations auto-similaires

À l'inverse des processus de coalescence, un processus de fragmentation décrit un système d'objets qui se disloquent au cours du temps, une propriété naturelle, dite "de fragmentation", étant que deux objets différents se fragmentent en suivant deux mécanismes indépendants. La dynamique d'un processus de fragmentation est souvent bien plus aisée à étudier que celle d'un processus de coalescence du fait de cette propriété. Notons que l'une des motivations pour l'étude des coalescents stochastiques est qu'ils sont l'analogue aléatoire d'équations déterministes dites de Smoluchowski ; ces équations, non-linéaires, admettent des analogues linéaires si on remplace la coalescence par de la fragmentation. De fait, de nombreux résultats fins portant sur des processus de coalescence ont été obtenus parce qu'il était possible de renverser le temps et d'obtenir un processus dual de fragmentation, comme cela a été vu plus haut pour le coalescent additif standard.

Nous nous intéressons à un certain modèle de fragmentations, dites auto-similaires. Supposons à nouveau que l'état du système est déterminé par le vecteur des masses des objets, à nouveau pris dans S . On se donne $\beta \in \mathbb{R}$.

Définition :

Un processus $(F(t), t \geq 0)$ à valeurs dans S et issu de $F(0) = (1, 0, 0, \dots)$ est dit *processus de fragmentation auto-similaire d'indice β* si c'est un processus de Markov continu en probabilité, et satisfaisant la propriété de fragmentation suivante. Conditionnellement à $F(t) = \mathbf{s}$, la loi de $F(t + t')$ est celle du réarrangement décroissant des suites $s_i F^{(i)}(s_i^\beta t')$, $i \geq 1$, où les $F^{(i)}$ sont des copies de F indépendantes.

Ces processus ont été introduits par Bertoin [23, 26], qui a montré qu'on pouvait en simplifier l'étude en "discrétisant l'espace", et en commençant par considérer des fragmentations non pas à valeurs dans S , mais à valeurs dans les partitions de \mathbb{N} . La théorie des partitions échangeables de Kingman permet alors de dériver un grand nombre de propriétés fondamentales des fragmentations auto-similaires.

Fragmentations de partitions Soit \mathcal{P}_∞ l'espace des partitions de \mathbb{N} en sous-ensembles A_1, A_2, \dots deux à deux disjoints de réunion \mathbb{N} . Nous supposons ces partitions non ordonnées, et la numérotation choisie pour A_1, A_2, \dots est par convention suivant l'ordre du plus petit élément. Il est facile de doter \mathcal{P}_∞ d'une métrique faisant de lui un espace compact. Une variable aléatoire π à valeurs dans \mathcal{P}_∞ est dite *échangeable* si pour toute permutation σ de \mathbb{N} , $\sigma\pi$ a même loi que π , où $(\sigma, \pi) \mapsto \sigma\pi$ désigne l'action naturelle de σ sur \mathcal{P}_∞ :

$$i \stackrel{\sigma\pi}{\sim} j \iff \sigma(i) \stackrel{\pi}{\sim} \sigma(j) \quad i, j \in \mathbb{N}$$

et $i \stackrel{\pi}{\sim} j$ signifie que i et j sont dans le même bloc de π .

L'exemple fondamental de partition échangeable est la *boîte de peinture* de Kingman. On se donne une boîte de peinture $\mathbf{s} \in S$ (s_i mesure la quantité de couleur i , et $s_0 = 1 - \sum_i s_i$ représente une absence de couleur). Soit alors une suite de variables aléatoires

dans $\mathbb{N} \cup \{0\}$ notées X_1, X_2, \dots , indépendantes de même loi $P(X_1 = i) = s_i, i \geq 0$. On définit alors une partition π_s de \mathbb{N} par la relation d'équivalence $i \stackrel{\pi_s}{\sim} j$ si $i \in \mathbb{N}$ et

$$i \stackrel{\pi_s}{\sim} j \iff X_i = X_j > 0 \quad i \neq j \in \mathbb{N}.$$

Le théorème de Kingman [68] (voir [2] pour une preuve élégante) énonce que toute partition échangeable π de \mathbb{N} est un mélange de boîtes de peinture, au sens où il existe une loi $\mu(ds)$ sur S telle que $P(\pi \in d\gamma) = \int_S \mu(ds) P(\pi_s \in d\gamma)$. En particulier, une application aisée de la loi des grands nombres implique que tous les blocs d'une partition échangeable $\pi = (A_1, A_2, \dots)$ admettent presque-sûrement des *fréquences asymptotiques*

$$|A_i| = \lim_{n \rightarrow \infty} \frac{\text{Card}(A_i \cap \{1, 2, \dots, n\})}{n} \quad i \geq 1,$$

telles que le réordonnement décroissant $|\pi|^\downarrow = (|A_i|, i \geq 1)^\downarrow$ soit dans S . Enfin, on dit qu'un processus $(\Pi(t), t \geq 0)$ à valeurs dans \mathcal{P}_∞ est échangeable si $(\sigma\Pi(t), t \geq 0)$ a même loi que $(\Pi(t), t \geq 0)$ pour toute permutation σ .

Définition :

Un processus échangeable $(\Pi(t), t \geq 0)$ à valeurs dans \mathcal{P}_∞ et satisfaisant la condition initiale $\Pi(0) = (\mathbb{N}, \emptyset, \emptyset, \dots)$ est un processus de fragmentation auto-similaire d'indice $\beta \in \mathbb{R}$ si c'est un processus de Markov satisfaisant les conditions suivantes.

- (i) Presque-sûrement, $\Pi(t)$ admet des fréquences asymptotiques pour tout t .
- (ii) Le processus $(|\Pi(t)|^\downarrow, t \geq 0)$ est continu en probabilité.
- (iii) Sachant $\Pi(t) = (A_1, A_2, \dots)$, la partition $\Pi(t + t')$ a même loi que la partition aléatoire dont les blocs sont ceux des partitions $A_i \cap \Pi^{(i)}(|A_i|^{\beta t'})$ de A_i , où $(\Pi^{(i)}(|A_i|^{\beta t'}), i \geq 1)$ est une suite de copies indépendantes de $\Pi(|A_i|^{\beta t'})$.

Mesure de dislocation Bertoin [23, 26] a montré que les lois de telles fragmentations à valeurs dans les partitions sont entièrement déterminées par un triplet (β, c, ν) où β est l'indice d'auto-similarité, c est un *coefficient d'érosion* positif et ν est une mesure σ -finie sur S , vérifiant $\nu\{(1, 0, \dots)\} = 0$ et $\int_S (1 - s_1) \nu(ds) < \infty$. Berestycki [17] a montré qu'il en est de même pour les fragmentations auto-similaires à valeurs dans S , c'est-à-dire que les lois de ces deux types de fragmentations sont en correspondance bijective.

Intuitivement, β est un paramètre de *vitesse* du système : plus β est grand et plus les différents objets se disloquent lentement au fil du temps. Le coefficient c témoigne de la présence d'un phénomène d'*érosion* dans la fragmentation, c'est-à-dire que chaque fragment perd continuellement de la masse au fil du temps. Enfin, ν décrit la façon dont les objets se disloquent instantanément : informellement, un objet de masse r va se briser en une suite d'objets de masses rs , pour un $s \in S$, à un taux $r^\beta \nu(ds)$.

Cas de la fragmentation d'Aldous-Pitman Il est tout à fait frappant de constater que la fragmentation d'Aldous-Pitman F_{AP} introduite plus haut est auto-similaire d'indice $1/2$, ce qui est obtenu par la théorie des excursions browniennes et le théorème de Girsanov, qui implique qu'une excursion d'un brownien avec drift au-dessus de son processus d'infimum passé et conditionnée à avoir une durée $v > 0$ est une excursion brownienne de durée v .

Bertoin [26] a montré que la mesure de dislocation $\nu_{\text{AP}}(\text{ds})$ de F_{AP} satisfait

$$\nu_{\text{AP}}(s_1 \in dx) = \frac{C_{\text{AP}} dx}{\sqrt{x^3(1-x)^3}} \mathbb{1}_{\{x>1/2\}} \quad (1.4)$$

pour une constante $C_{\text{AP}} > 0$, et que $\nu_{\text{AP}}\{\mathbf{s} : s_1 + s_2 < 1\} = 0$, ce qui caractérise entièrement ν_{AP} . On dit qu'une fragmentation auto-similaire dont la mesure de dislocation vérifie cette dernière propriété est une fragmentation *binnaire* (un objet ne peut se fragmenter instantanément qu'en au plus deux objets).

Une fragmentation duale Une autre propriété frappante de la fragmentation d'Aldous-Pitman a été également remarquée par Bertoin dans [26]. Considérons à nouveau une excursion brownienne normalisée B^{exc} , et notons

$$I(t) = \{s \in [0, 1] : 2B_s^{\text{exc}} > t\}, \quad t \geq 0.$$

En tant qu'ouvert de $[0, 1]$, on peut écrire cet ensemble comme unique réunion dénombrable d'intervalles ouverts (dans $[0, 1]$) disjoints $(I_1(t), I_2(t), \dots)$, classés de sorte que la suite $F_{\text{B}}(t) = (|I_1(t)|, |I_2(t)|, \dots) \in S$. On montre alors, en utilisant à nouveau la théorie des excursions browniennes, que $(F_{\text{B}}(t), t \geq 0)$ est un processus de fragmentation auto-similaire d'indice $-1/2$. Le plus étonnant est que sa mesure de dislocation ν_{B} est égale à ν_{AP} . On peut comprendre ceci en notant que, pour F_{AP} aussi bien que pour F_{B} , les dislocations soudaines sont créées par le "franchissement" d'un minimum local de l'excursion brownienne, qui partage l'objet en deux sous-objets selon la même "loi". Comme nous allons le voir plus bas, la fragmentation F_{B} a une interprétation en termes d'un procédé de fragmentation de l'arbre continu brownien, différent de celui employé pour obtenir F_{AP} . Nous avons généralisé ce genre de propriété de "dualité" entre deux fragmentations d'un même arbre au contexte plus général des arbres stables de Duquesne et Le Gall.

Perte de masse Notons également que pour $t > 0$, on a $F_{\text{AP}}(t) \in S^1$ mais $F_{\text{B}}(t) \notin S^1$ presque-sûrement, c'est-à-dire que F_{B} subit une perte de masse au cours du temps (et en fait finit par disparaître, c'est-à-dire à être égale à $(0, 0, \dots)$). Ce phénomène est étranger au phénomène d'érosion mentionné plus haut (en fait on peut montrer que l'érosion c est nulle pour F_{AP} et F_{B}). Il est dû au fait que l'indice de F_{B} , égal à $-1/2$, est strictement négatif, et donc que les fragments petits ont tendance à se disloquer de plus en plus rapidement. Bertoin a montré que si l'indice est strictement négatif, il se produit presque-sûrement une accumulation des temps de fragmentation, et les objets disparaissent (se réduisent en "poussière") en un temps fini. Ce phénomène nous sera utile pour décrire la généalogie des fragmentations auto-similaires d'indice négatif.

1.1.4 Arbres aléatoires

Nous nous intéressons à deux types d'arbres aléatoires : arbres discrets et arbres continus.

Arbres discrets et processus de contour Pour $n \in \mathbb{N}$, on considère deux types d'arbres (graphes connexes sans cycles) enracinés (un des nœuds est distingué et appelé la racine) à n nœuds. La racine définit une orientation du graphe (par exemple, on demande que toutes les arêtes de l'arbre pointent dans la direction opposée à la racine), et chaque nœud v possède un certain nombre d'*enfants* (nombre d'arêtes pointant depuis v), on note c_v ce nombre ($c_v + 1$ est le degré de v dans le graphe, sauf si v est la racine). Un nœud v tel que $c_v = 0$ est appelé une *feuille*. Un arbre *étiqueté enraciné* est un arbre enraciné dont les nœuds sont numérotés, par exemple par $1, 2, \dots, n$. En revanche, on décide d'ignorer une quelconque structure d'ordre entre les enfants d'un même nœud. On note \mathbf{T}_n l'ensemble des arbres étiquetés à n nœuds. Un *arbre enraciné ordonné* (ou *planaire*) est un arbre enraciné tel que pour chaque sommet v avec $c_v > 0$, les c_v enfants de v sont distingués comme étant le premier, second, etc. On peut les considérer comme des ensembles particuliers de suites finies à valeurs dans \mathbb{N} : la suite vide \emptyset est la racine et $(u_1, u_2, \dots, u_i, k)$ est le k -ième enfant de $v = (u_1, \dots, u_i)$. La distance de v à la racine (la longueur de la suite) est appelée *hauteur* du nœud et notée $\text{ht}(v)$. On note \mathbf{T}_n^o l'ensemble des arbres planaires à n nœuds.

Pour chaque $\mathbf{t}^o \in \mathbf{T}_n^o$, il existe un ordre naturel sur \mathbf{t}^o , correspondant à l'ordre lexicographique si on adopte la représentation précédente, appelé ordre de *parcours en profondeur* (*depth-first order*). Si on note v_1, v_2, \dots, v_n les nœuds de \mathbf{t}^o classés dans cet ordre (v_1 est donc la racine), et si on suppose qu'à chaque v_i est associé un poids $p_i > 0$ tel, on peut définir un processus, appelé processus de contour de l'arbre, par

$$H_s(\mathbf{t}^o) = \text{ht}(v_i) \quad \text{pour} \quad \sum_{1 \leq j \leq i-1} p_j \leq s < \sum_{1 \leq j \leq i} p_j,$$

défini sur $[0, \sum_i p_i]$ (on le prolonge par continuité à gauche en $\sum_i p_i$). Nous mettons en valeur les liens pouvant exister entre arbres discrets et processus de Lévy en citant un cas particulier d'un résultat dû à Aldous [5], et que nous généralisons au chapitre 7. Prenons un arbre \mathbf{t} au hasard uniformément dans \mathbf{T}_n . Pour chaque v tel que $c_v > 0$, on ordonne aléatoirement les enfants de v , uniformément parmi les $c_v!$ ordres possibles, de façon indépendante selon les nœuds conditionnellement à \mathbf{t} , et on efface les numéros. On obtient un arbre aléatoire $\mathbf{t}^* \in \mathbf{T}_n^o$ (c'est en fait un arbre de Galton-Watson conditionné à être de taille n , où chaque individu a un nombre d'enfants dont la loi est celle de Poisson de moyenne 1). Supposons que tous les poids valent $p_i = 1/n$. Alors

$$(n^{-1/2} H_s(\mathbf{t}^*), 0 \leq s \leq 1) \xrightarrow[n \rightarrow \infty]{d} (2B_s^{\text{exc}}, 0 \leq s \leq 1). \quad (1.5)$$

En d'autres termes, le processus de contour de l'arbre \mathbf{t}^* , où l'on a renormalisé les longueurs des arêtes à valoir $1/\sqrt{n}$, converge vers deux fois l'excursion brownienne normalisée.

p-arbres Un modèle d'arbre aléatoire qui sera utile par la suite est celui des **p-arbres** (ou "arbres d'anniversaire"). On suppose donnée une probabilité $\mathbf{p} = (p_1, \dots, p_n)$ sur $\{1, \dots, n\}$ avec $p_i > 0$ pour tout $1 \leq i \leq n$. On montre que la formule

$$P(\mathcal{T}^{\mathbf{p}} = \mathbf{t}) = \prod_{1 \leq i \leq n} p_i^{c_i(\mathbf{t})} \quad \mathbf{t} \in \mathbf{T}_n$$

définit une probabilité sur \mathbf{T}_n (on suppose que les nœuds sont les éléments de $\{1, \dots, n\}$, et $c_i(\mathbf{t})$ est le nombre d'enfants de i dans \mathbf{t}). Pitman [88] a montré que $\mathcal{T}^{\mathbf{p}}$ est l'arbre obtenu

en observant un coalescent additif issu de l'état $(p_1, \dots, p_n, 0, \dots)^\downarrow$ et en créant de façon adéquate une arête à chaque collision entre deux amas pour conserver une structure d'arbre. À l'inverse, si on enlève une à une uniformément les $n - 1$ arêtes d'un \mathbf{p} -arbre, et qu'on regarde à chaque instant le vecteur constitué des \mathbf{p} -masses des composantes connexes obtenues, on obtient une fragmentation qui est la retournée en temps du coalescent additif issu de $(p_1, \dots, p_n, 0, \dots)^\downarrow$.

Remarquons que lorsque $p_i = 1/n$ pour tout $1 \leq i \leq n$, $\mathcal{T}^{\mathbf{P}}$ est l'arbre aléatoire pris uniformément dans \mathbf{T}_n (ce qui donne la formule de Cayley bien connue $\text{Card } \mathbf{T}_n = n^{n-1}$ puisqu'un arbre à n nœuds possède $n - 1$ arêtes).

Arbres continus Pour rendre compte de la structure limite de certains arbres discrets, et typiquement, pour essayer de comprendre la convergence (1.5) en termes d'arbres, Aldous a été amené ([5]) à introduire la notion d'*arbre continu aléatoire* (CRT). Un \mathbb{R} -arbre (dans la terminologie de Dress et Terhalle [46], Aldous les appelle plutôt ensembles d'arbres continus) est un espace métrique complet (T, d) tel que :

- pour chaque $v, w \in T$, il existe une unique isométrie $\varphi_{v,w} : [0, d(v, w)] \rightarrow T$ avec $\varphi_{v,w}(0) = v$ et $\varphi_{v,w}(d(v, w)) = w$, on nomme $[[v, w]]$ son image, et
- si $(v_s, 0 \leq s \leq 1)$ est un chemin injectif avec $v_0 = v$ et $v_1 = w$, alors $\{v_s : 0 \leq s \leq 1\} = [[v, w]]$.

On suppose de plus T enraciné (un de ses éléments est distingué, on l'appelle \emptyset). Une *feuille* de T est un nœud qui n'appartient à aucun chemin de la forme $[[\emptyset, v]] = \varphi_{\emptyset, v}([0, \text{ht}(v)])$ pour $v \in T$, où l'on note $\text{ht}(v) = d(\emptyset, v)$ la hauteur de v . On note $\mathcal{L}(T)$ (pour *Lipton*) l'ensemble des feuilles de T .

Définition :

On appelle arbre continu un couple (T, μ) où T est un \mathbb{R} -arbre enraciné, et μ est une mesure de probabilité sans atome, telle que $\mu(\mathcal{L}(T)) = 1$ et pour tout $v \notin \mathcal{L}(T)$, $\mu\{w \in \mathcal{L}(T) : [[\emptyset, v]] \cap [[\emptyset, w]] = [[\emptyset, v]]\} > 0$. Un arbre continu aléatoire (CRT) est une variable aléatoire $\omega \mapsto (\mathcal{T}(\omega), \mu(\omega))$ sur un espace probabilisé (Ω, \mathcal{A}, P) telle que $(\mathcal{T}(\omega), \mu(\omega))$ est un arbre continu pour chaque $\omega \in \Omega$.

La définition précédente est légèrement floue du fait, bien sur, que l'ensemble des arbres continus n'est pas muni d'une tribu, et même, n'est pas parfaitement défini. Ce problème est en fait secondaire, car les arbres continus que nous considérerons seront toujours "codés" par des variables aléatoires *ad hoc*. Par exemple, il suffit en fait de connaître les "marginales" de ces arbres, c'est-à-dire les sous-arbres engendrés par la racine et une suite finie de feuilles indépendantes de loi μ sachant μ , pour plonger, par une construction "spéciale", l'arbre continu dans ℓ^1 (l'espace des suites de somme finie).

Soit T un \mathbb{R} -arbre enraciné. Un *point de branchement* est un nœud b pour lequel il existe v, w tels que $[[\emptyset, b]] = [[\emptyset, v]] \cup [[\emptyset, w]]$. Si $\mathcal{V} \subseteq T$, on appelle enfin arbre engendré par la racine et \mathcal{V} l'ensemble des nœuds $v \in T$ tels qu'il existe $w \in \mathcal{V}$ avec $v \in [[\emptyset, w]]$.

Arbres stables Les arbres stables sont une famille de CRT appartenant à une classe importante d'arbres dérivés de processus de Lévy introduits par Duquesne et Le Gall [49]. On peut les interpréter comme les limites possibles d'arbres de Galton-Watson renormalisés (où la loi du nombre de fils d'un individu est toujours la même), c'est-à-dire que de tels arbres encodent la généalogie de certains processus de branchement continus. Nous donnons ici la construction, qui est motivée par une analogie avec des relations bien connues

entre processus de contour d'arbres de Galton-Watson avec certaines marches aléatoires (voir par exemple [5, 77, 16],...).

Soit X un processus de Lévy stable d'indice $\alpha \in (1, 2]$ n'ayant que des sauts positifs, on le suppose normalisé de sorte que $\psi(\lambda) = \lambda^\alpha$ (remarquons que si $\alpha = 2$ il s'agit de $\sqrt{2}B$ où B est un mouvement brownien standard). Pour chaque $t \geq 0, s \in [0, t]$, on note $\widehat{X}_s^t = X_t - X_{(t-s)-}$ le processus retourné à partir de t . Il est connu que \widehat{X}^t a même loi que le processus X stoppé au temps t . On considère alors le processus du supremum $\widehat{S}_s^t = \sup_{0 \leq u \leq s} \widehat{X}_u^t$, et on note \widehat{L}_s^t le temps local en 0 à l'instant $s \leq t$ du processus $\widehat{S}^t - \widehat{X}^t$ (normalisé de façon à être égal à la densité de la mesure d'occupation). On pose alors $H_t^X = \widehat{L}_t^t, t \geq 0$, le *processus de hauteur*. Par théorie des excursions et *scaling*, on peut également définir l'excursion normalisée du processus de hauteur $(H_s, 0 \leq s \leq 1)$, à partir de l'excursion normalisée du processus stable (si $\alpha = 2$ on a juste $H = \sqrt{2}B^{\text{exc}}$).

On définit alors un arbre continu à partir de ce processus. Notons D la pseudo-métrique sur $[0, 1]$ définie par $D(s, s') = H_s + H_{s'} - 2 \inf_{u \in [s, s']} H_u$. On note \mathcal{T}^α l'espace métrique quotient $[0, 1]/\equiv$ où $s \equiv s'$ si et seulement si $D(s, s') = 0$. On peut montrer que cet espace est un \mathbb{R} -arbre. Enfin, on note μ^α la mesure de Borel induite sur \mathcal{T}^α par la mesure de Lebesgue sur $[0, 1]$. Alors $(\mathcal{T}^\alpha, \mu^\alpha)$ est un CRT, appelé arbre stable. Une de ses caractéristiques est que chacun de ses points de branchement est adjacent à un nombre infini de branches, une propriété qui découle du fait que le processus des hauteurs possède des infima locaux qui sont atteints une infinité de fois. Plus précisément, si le processus de Lévy X accomplit un saut $\Delta X_s > 0$ au temps s , on peut constater que $H_u \geq H_s$ pour chaque $u \leq \inf\{r \geq s : X_r = X_{s-} = X_s - \Delta X_s\}$, et que $H_u = H_s$ pour de tels u vérifiant en plus $X_u = \inf\{X_r : s \leq r \leq u\}$.

Montrons comment on peut obtenir les "marginales" des arbres stables, c'est-à-dire les sous-arbres de \mathcal{T}^α engendrés par la racine et une suite L_1, \dots, L_n de feuilles prises indépendamment avec la loi μ^α conditionnellement à μ^α . L'équivalent de μ^α pour le processus des hauteurs est la mesure de Lebesgue sur $[0, 1]$, on prend donc une suite de variables uniformes indépendantes U_1, U_2, \dots sur $[0, 1]$, et on définit récursivement l'arbre réduit $\mathcal{R}(B)$ comme suit pour $B \subset \mathbb{N}$ fini. Si $B = \{i\}$ avec $i \in \mathbb{N}$, $\mathcal{R}(B)$ est restreint à une branche $[[\emptyset, L_i]]$ de taille $\text{ht}(L_i) = H_{U_i}$. Ensuite, si on sait construire $\mathcal{R}(B)$ et $\mathcal{R}(B')$ avec B et B' disjoints, on pose

$$m(B, B') = \inf \left\{ H_s : s \in \left[\min_{i \in B \cup B'} U_i, \max_{i \in B \cup B'} U_i \right] \right\},$$

et on construit un arbre en traçant une branche $[[\emptyset, b]]$ avec $\text{ht}(b) = m(B, B')$, et en plantant sur b deux arbres $\mathcal{R}(B) - m(B, B'), \mathcal{R}(B') - m(B, B')$, où par exemple $\mathcal{R}(B) - m(B, B')$ signifie qu'on a enlevé une longueur $m(B, B')$ à la branche de $\mathcal{R}(B)$ issue de la racine. Enfin, on pose $\mathcal{R}(k) = \mathcal{R}(\{1, \dots, k\})$, et la construction précédente est en fait une construction graphique à partir du processus des hauteurs.

Les ICRT Nous terminons ces préliminaires par la description d'un autre modèle d'arbres continus aléatoires introduits par Aldous, Camarri et Pitman [39, 12]. Il s'agit en fait d'une famille de lois dépendant d'un paramètre $\theta = (\theta_1, \theta_2, \dots)$, suite décroissante telle que $\sum_i \theta_i^2 \leq 1$ et vérifiant $\theta_0 = (1 - \sum_i \theta_i^2)^{1/2} > 0$ ou $\sum_i \theta_i = \infty$. La description la plus aisée de cet arbre, noté \mathcal{T}^θ , est de donner ses marginales, c'est-à-dire de le construire branche par branche. Pour ce faire, on considère un processus de Poisson ponctuel $\{(U_i, V_i), i \geq 1\}$ sur

le premier octant $\text{Oct} = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\}$, dont l'intensité est $\theta_0 dx dy \mathbb{1}_{\text{Oct}}(x, y)$. Notons que la projection $(U_i, i \geq 1)$ est de ce fait un processus de Poisson ponctuel sur \mathbb{R}_+ d'intensité $\theta_0 dx$. Par ailleurs, on se donne une famille de processus de Poisson ponctuels indépendants (et indépendants du premier) $(\xi_{i,j}, j \geq 1)$ pour $i \geq 1$, d'intensité $\theta_i dx$ sur \mathbb{R}_+ . Les points $(V_i, i \geq 1)$ ainsi que les points $(\xi_{i,1}, i \geq 1)$ sont distingués des autres et sont appelés *points de jonction*, tandis que les points $U_i, i \geq 1, \xi_{i,j}, i \geq 1, j \geq 2$ sont appelés *points de coupure*. Si η est un point de coupure, on notera η^* le point de jonction associé, c'est-à-dire $U_i^* = V_i$ et $\xi_{i,j}^* = \xi_{i,1}$.

On montre grâce à l'hypothèse faite sur θ que l'on peut presque-sûrement ordonner les points de coupure en $0 < \eta_1 < \eta_2 < \dots$, et on construit un arbre $\mathcal{R}(k)$ récursivement comme suit. On coupe l'intervalle $(0, \infty)$ en "branches" $(\eta_i, \eta_{i+1}]$. L'arbre $\mathcal{R}(1)$ est constitué de la première branche $(0, \eta_1]$ de longueur η_1 , enracinée en 0. Ensuite, connaissant $\mathcal{R}(k)$, on doit placer la nouvelle branche $(\eta_k, \eta_{k+1}]$ quelque part sur $\mathcal{R}(k)$, et on plante l'extrémité η_k sur le point de jonction η_k^* : notons que $\eta_k^* < \eta_k$ et donc le point de jonction a déjà été placé quelque part sur l'arbre). On définit ainsi (en passant quelques détails techniques) des \mathbb{R} -arbres emboîtés, et on pose

$$\mathcal{T}^\theta = \overline{\bigcup_{k \geq 1} \mathcal{R}(k)},$$

où \bar{A} est la complétion de l'espace métrique A . On montre que $(\mathcal{T}^\theta, \mu^\theta)$ est un CRT, où μ^θ est définie comme la limite de la mesure empirique sur les feuilles de $\mathcal{R}(k)$ (cet arbre, défini à une isométrie près, a alors la même loi que le sous-arbre de \mathcal{T}^θ engendré par la racine 0 et une suite de k feuilles indépendantes de loi μ^θ sachant μ^θ). Qualitativement, un arbre inhomogène possède deux types de branchements : des branchements binaires (correspondant aux points de jonction V_i , qui ne servent à planter qu'une seule branche) et des branchements infinis, comme pour les arbres stables, qui correspondent aux points de jonction $\xi_{i,1}$. Il y a donc autant de points de branchement infinis que de $i \geq 1$ tels que $\theta_i > 0$. Lorsque $\theta_0 = 1$ ($\theta_i = 0, i \geq 1$), l'arbre ainsi construit n'est autre que l'arbre brownien (l'arbre stable \mathcal{T}^2 défini plus haut), cette construction étant due à Aldous [3].

Nous présentons deux motivations pour l'utilisation de ces arbres. La première est que les arbres inhomogènes ICRT sont les limites possibles des \mathbf{p} -arbres introduits plus haut, au sens suivant. Soit $\mathbf{p} = \mathbf{p}_n$ une suite de probabilités sur \mathbb{N} avec $p_{n,i} > 0, 1 \leq i \leq n$ et $p_{n,i} = 0, i > n$. On suppose que \mathbf{p} remplit la condition

$$\max_{i \in \mathbb{N}} p_i \xrightarrow{n \rightarrow \infty} 0 \quad \text{et} \quad \frac{p_i}{\sigma(\mathbf{p})} \rightarrow \theta_i, \quad i \geq 1, \quad \text{où} \quad \sigma(\mathbf{p}) = \sqrt{\sum_{i \geq 1} p_i^2}. \quad (1.6)$$

Soient alors X_1, X_2, \dots, X_k des points tirés au hasard suivant \mathbf{p} , indépendants d'un \mathbf{p} -arbre $\mathcal{T}^{\mathbf{p}}$. On considère le sous-arbre de $\mathcal{T}^{\mathbf{p}}$ engendré par la racine et les nœuds X_1, \dots, X_k , et on le transforme en un \mathbb{R} -arbre en attribuant une longueur $\sigma(\mathbf{p})$ à chaque arête et en effaçant les nœuds qui ne sont ni des X_i , ni des points de branchement dans l'arbre réduit (de sorte qu'une ligne de r arêtes consécutives sans branchement a une longueur $r\sigma(\mathbf{p})$). Alors, lorsque $n \rightarrow \infty$, ce \mathbb{R} -arbre converge en loi vers $\mathcal{R}(k)$ (cette convergence porte sur des arbres finis et correspond au produit de la topologie discrète sur les arbres par la topologie de la convergence des longueurs de chacune des arêtes). Lorsque $\theta_1 = 0$, on peut retrouver le résultat de convergence des arbres de Galton-Watson avec distribution poissonnienne du

nombre d'enfants vers l'arbre continu brownien en prenant $\mathbf{p} = (1/n, \dots, 1/n)$. Réciproquement, on peut montrer [39] qu'essentiellement toute limite de \mathbf{p} -arbres renormalisés doit être un ICRT.

La seconde motivation est que les ICRT ont permis à Aldous et Pitman [12] de déterminer la frontière d'entrée du coalescent additif. Intuitivement, comme on a vu qu'une fragmentation uniforme sur les \mathbf{p} -arbres permettait de retrouver un coalescent additif, il se trouve qu'on peut également fragmenter les ICRT limites à l'aide d'un processus de Poisson. Précisément, on se donne un processus de Poisson ponctuel d'intensité λ par unité de longueur de l'arbre (soit un processus de Poisson homogène d'intensité λ sur chacune des branches de $\mathcal{R}(k)$ pour $k \geq 1$), de façon couplée lorsque λ varie. Pour chaque λ , on peut noter $(\mathcal{T}_1^\theta(\lambda), \mathcal{T}_2^\theta(\lambda), \dots)$ les composantes connexes de la forêt obtenue lorsqu'on a coupé l'arbre \mathcal{T}^θ en tous les points du processus de Poisson au niveau λ , où les $\mathcal{T}_i^\theta(\lambda)$ sont classés par ordre décroissant de leur μ^θ -masses. Alors, le processus

$$C^\theta(t) = (\mu^\theta(\mathcal{T}_1^\theta(e^{-t})), \mu^\theta(\mathcal{T}_2^\theta(e^{-t})), \dots) \quad -\infty < t < \infty$$

est un coalescent additif éternel. De plus, chaque coalescent additif éternel est obtenu par "mélange", c'est-à-dire est de la forme $C^\theta(t - t_0)$ avec θ et t_0 aléatoires. Lorsque $\theta_0 = 1$, on trouve le coalescent additif standard introduit plus haut, et qui correspond donc bien à une fragmentation de l'arbre brownien (les connexions avec la construction par l'excursion brownienne de Bertoin sont encore mal comprises malgré les nombreux liens existant entre l'arbre et l'excursion).

1.2 Aperçu des résultats

1.2.1 Coalescent additif ordonné associé à certains processus de Lévy

Ce premier travail [79] est constitué de deux parties complémentaires. Il est motivé par la construction du coalescent additif standard et de toutes les versions éternelles du coalescent additif à l'aide de processus à accroissements échangeables à variation infinie, par Bertoin [22]. Pour fixer les idées, nous avons vu plus haut deux constructions différentes du coalescent additif standard, l'une à l'aide des trajectoires de l'excursion brownienne, l'autre via une fragmentation de l'arbre continu brownien. On voit que la première construction possède une structure additionnelle, car elle ordonne les fragments les uns par rapport aux autres, cet ordre étant induit par $[0, 1]$.

Un *ordre partiel* sur un ensemble E est un sous-ensemble $O \subset E \times E$, tel que $(i, i) \in O$ pour tout i , si $(i, j) \in O$ et $(j, k) \in O$, alors $(i, k) \in O$, et enfin si $(i, j) \in O$ et $(j, i) \in O$ alors $i = j$. L'ordre est dit total si pour tout i, j , on a $(i, j) \in O$ ou $(j, i) \in O$. On s'intéresse plus spécifiquement aux ordres O sur $[n]$ ou \mathbb{N} tels qu'il existe une partition $\pi(O)$ de $[n]$ ou \mathbb{N} telle que les restrictions de O aux blocs B_i de π (c'est-à-dire les ensembles $(B_i \times B_i) \cap O$) ne sont pas comparables entre elles (si $i \in B_k, j \in B_l$ avec $k \neq l$ alors $(i, j) \notin O$), et tels que ces restrictions sont des ordres totaux sur les B_i . Nous appelons *amas* (*clusters*) ces restrictions. Si O est un tel ordre et \mathcal{I}, \mathcal{J} sont deux amas, nous définissons un ordre $O_{\mathcal{I}\mathcal{J}}$ par concaténation de ces amas avec \mathcal{I} "à gauche" : $(i, j) \in O_{\mathcal{I}\mathcal{J}}$ si (i, j) est dans \mathcal{I} ou \mathcal{J} , ou si $i \in p(\mathcal{I})$ et $j \in p(\mathcal{J})$, où p est la projection sur la première coordonnée.

Nous construisons une version ordonnée du coalescent additif en partant d'abord d'un nombre fini de fragments. On se donne donc un ordre $O = O(0)$ sur $[n]$, possédant m

amas, et on attribue à chaque entier $i \leq n$ une masse $s_i > 0$. Pour chaque cluster \mathcal{I} de O , on note $s_{\mathcal{I}} = \sum_{j \in p(O)} s_j$ sa masse. Rappelons que l'on peut décrire le premier temps de coalescence comme l'infimum de $n(n-1)$ variables exponentielles indépendantes de taux $s_i + s_j$, $i \neq j$. Nous modifions la construction en "découplant" les variables $e_{i,j}$, c'est-à-dire en les remplaçant par $\min(e_{i,j}^i, e_{i,j}^j)$, où ces deux variables sont exponentielles indépendantes de taux s_i et s_j . On voit alors que l'on peut adopter le point de vue alternatif suivant : on se donne des variables exponentielles $e_{\mathcal{I}}$, où \mathcal{I} décrit les amas de O , indépendantes de taux respectifs $s_{\mathcal{I}}$, il existe alors p.s. un unique \mathcal{I}_0 tel que $T_1 = e_{\mathcal{I}_0}/m = \inf_{\mathcal{I}} e_{\mathcal{I}}/m$. On note $O(t) = O(0)$ pour $0 \leq t \leq T_1$, et au temps T_1 on change $O(T_1-)$ en $O(T_1) = O_{\mathcal{I}_0 \mathcal{J}_0}$, où \mathcal{J}_0 est tiré au hasard uniformément parmi les $m-1$ clusters de O distincts de \mathcal{I} . On continue jusqu'à ce que $O(t)$ soit un ordre total sur $[n]$. On voit alors que le processus $s_{O(t)} = (s_{\mathcal{I}}, \mathcal{I} \in O(t)), t \geq 0$ est un coalescent additif. La même méthode que celle employée par Evans et Pitman permet de montrer que l'on peut étendre le processus $(O(t), t \geq 0)$ en un processus de Feller défini sur l'espace des ordres de \mathbb{N} , qui soit consistant avec ce processus, c'est-à-dire que si l'on effectue une opération de projection sur les ordres de $[n]$, on obtient le processus ci-dessus.

Nous relierons alors une construction de Bertoin [22] du coalescent additif à ce processus de coalescence additive ordonnée. On se donne (s_1, s_2, \dots) décroissante de somme 1 avec $s_i > 0$ pour tout i (le résultat est plus pénible à énoncer si s_i est presque nulle). On considère des variables $(U_i, i \geq 1)$ uniformes sur $[0, 1]$, indépendantes. On pose alors

$$b_s(u) = \sum_{i=1}^{\infty} s_i (\mathbb{1}_{\{U_i \leq u\}} - u), \quad 0 \leq u \leq 1,$$

de sorte que b_s est un *pont à accroissements échangeables* : il est continu en 0 et 1 et s'y annule, et si $[a_n, b_n], 1 \leq n \leq N$ sont deux-à-deux disjoints de même longueur, la suite $(b_s(b_n) - b_s(a_n), 1 \leq n \leq N)$ est échangeable. On note ε_s sa transformée de Vervaat : on montre qu'il existe p.s. un unique u_{\min} , qui est un temps de saut de b_s , tel que $b_s(u_{\min}-) = \inf_{u \in [0,1]} b_s(u)$, et on note $\varepsilon_s(u) = b_s(u + u_{\min}) - b_s(u_{\min}-)$, où l'addition $u + u_{\min}$ est modulo 1, on note également $V_i = U_i - s_{\min}$ modulo 1. Pour $t \geq 0$ soit $\varepsilon_s^{(t)}(u) = \varepsilon_s(u) - tu$. Le processus $\underline{\varepsilon}_s^{(t)}(u) = \inf_{r \in [0,u]} \varepsilon_s^{(t)}(r)$ possède des paliers disjoints, notons-les $[a_n^t, b_n^t], n \geq 1$. On note $F(t)$ l'ordre sur \mathbb{N} tel que $(i, j) \in O$ si et seulement si $V_i \leq V_j$ et V_i, V_j appartiennent au même intervalle $[a_n^t, b_n^t]$. Notons que chaque amas de $F(t)$ possède un plus petit élément.

Théorème 1.1 :

Le processus $O^s(t) = F^s(e^{-t}/(1 - e^{-t})), t \geq 0$ est un coalescent additif ordonné issu de l'ordre dont les amas sont les $\{(i, i)\}, i \in \mathbb{N}$, et avec des masses initiales s_i .

Cette construction est obtenue comme processus limite d'un système de *serveurs agrégatifs*, que l'on peut construire à l'aide de processus à accroissements échangeables à variation finie.

D'autre part, nous considérons une généralisation de la construction du coalescent additif standard par l'excursion brownienne, en utilisant des excursions de processus de Lévy sans sauts positifs en dessous de leur supremum. Soit donc X un processus de Lévy sans sauts positifs, on suppose en plus que X_s possède une densité p_s pour chaque $s > 0$, de sorte que $(p_s(x), s > 0, x \in \mathbb{R})$ soit bicontinu. On note \bar{X} son processus du supremum, et T le subordonateur inverse continu à droite. De même, nous notons $X_s^{(t)} = X_s + ts, s \geq 0$,

$\overline{X}^{(t)}$ son processus du supremum, et $T^{(t)}$ son inverse continu à droite. Par ailleurs, nous notons ε l'excursion de X de longueur 1 sous son supremum (ce processus prend des valeurs négatives). On pose $\varepsilon_s^{(t)} = \varepsilon_s + ts$, et $\overline{\varepsilon}^{(t)}$ son processus du supremum. Enfin, nous notons $F(t)$ la suite décroissante des longueurs des paliers de $\overline{\varepsilon}^{(t)}$. En généralisant la méthode de Bertoin [21], nous montrons que $(F(t), t \geq 0)$ est un processus de fragmentation (au sens où les évolutions futures de deux fragments distincts sont indépendantes), et

Théorème 1.2 :

La loi de $F(t)$ est celle de $\Delta T_{[0,t]}^{(t)}$ sachant $T_t^{(t)} = 1$. De plus, $(F(e^{-t}), -\infty < t < \infty)$ est un coalescent additif éternel.

Pour ce faire, nous mettons en correspondance les excursions de X avec celles du processus $(x - tT_x^{(t)}, x \geq 0)$. Ceci fournit donc une généralisation de (1.1). Notons que, du fait de l'absence d'un théorème de Girsanov pour les processus de Lévy autres que le mouvement brownien, on ne peut pas ramener cette loi à celle de $\Delta T_{[0,t]}$ sachant $T_t = 1$. De ce fait, sauf dans le cas brownien, la fragmentation n'est pas auto-similaire. Nous montrons également des résultats généralisant le cas brownien sur le fragment "le plus à gauche" (le premier palier de $\overline{\varepsilon}^{(t)}$), en utilisant le théorème du scrutin pour les processus à accroissements échangeables, de telles méthodes ont été également employées dans des buts proches par Schweinsberg [100]. Nous montrons que ce fragment (en tant que processus) est un tirage aléatoire biaisé par la taille parmi les fragments de $F(t)$, et nous donnons son semigroupe ; ceci généralise dans une certaine mesure le résultat (1.2).

Notons que les versions éternelles du coalescent additif que nous obtenons ainsi sont exactement reliées aux solutions éternelles de l'équation de coagulation de Smoluchowski avec noyau additif [24].

1.2.2 Fragmentations auto-similaires et subordinateurs stables

Dans ce travail avec Jason Schweinsberg [82], nous nous sommes intéressés à la question de Pitman qui suit :

Pour $\alpha \in (0, 1)$, peut-on trouver un processus de fragmentation $(F(t), t \geq 0)$ tel que la loi de $F(t)$ soit égale à celle de $(\Delta\sigma_{[0,t]} | \sigma_t = 1)$ pour tout t , où σ est un subordinateur stable d'indice α ?

Nous avons vu que si $\alpha = 1/2$, le problème admet une réponse positive, une telle fragmentation étant F_{AP} . Nous avons vu précédemment que les généralisations Lévy de la fragmentation d'Aldous et Pitman ont des lois faisant intervenir des subordinateurs qui ne sont pas des subordinateurs stables du fait de l'absence de théorème de Girsanov. Enfin, les semigroupes qui interviendront dans les fragmentations décrites plus bas font tous intervenir des subordinateurs stables, mais d'une façon toujours compliquée. Ceci souligne le fait que, si l'on peut assez aisément déterminer une fragmentation via des caractéristiques locales (par exemple l'indice, le coefficient d'érosion et surtout la mesure de dislocation pour une fragmentation auto-similaire), il est beaucoup plus difficile d'obtenir des théorèmes généraux sur son semigroupe. En particulier, si la ressemblance entre les processus de fragmentation auto-similaires et les processus de Lévy saute aux yeux, nous n'avons pas à notre disposition d'équivalent de la formule de Lévy-Khintchine.

Comme F_{AP} est auto-similaire d'indice $1/2$, il semble naturel d'essayer en premier lieu de chercher une telle fragmentation qui soit auto-similaire (avec un indice positif et une

érosion nulle pour éviter toute perte de masse). Hélas, les résultats que nous avons obtenus sont pour l'essentiel des résultats négatifs. Le premier résultat montre que (1.1) n'admet pas de généralisation stricte dans notre contexte.

Théorème 1.3 :

Soit $\alpha \in]0, 1[$ et σ un subordonateur stable d'indice α . Soit $(F(t), t \geq 0)$ une fragmentation auto-similaire de caractéristiques $(\beta, 0, \nu)$ avec $\beta \geq 0$. On suppose que pour tout $t \geq 0$, la loi de $F(t)$ est celle de $\Delta\sigma_{[0,t]}$ sachant $\sigma_t = 1$. Alors $\alpha = \beta = 1/2$, et il existe $C > 0$ telle que $\nu = C\nu_{AP}$ où ν_{AP} est définie en (1.4), c'est-à-dire que la fragmentation est celle d'Aldous-Pitman à un changement de temps près.

Ce théorème se montre en confrontant le comportement asymptotique en $t \rightarrow 0$ et en $t \rightarrow +\infty$ des fragmentations auto-similaires avec le comportement de la loi de $\Delta\sigma_{[0,t]}$ sachant $\sigma_t = 1$. En 0, nous montrons le résultat suivant :

Proposition 1.1 :

Soit $(F(t), t \geq 0)$ une fragmentation auto-similaire de caractéristiques $(\beta, 0, \nu)$ avec $\beta \geq 0$. Alors pour toute fonction G sur S positive continue, nulle sur un voisinage de $(1, 0, \dots)$, on a

$$\frac{1}{t}E[G(F(t))] \xrightarrow{t \rightarrow 0} \nu(G).$$

A l'aide d'équivalents explicites pour les densités des subordonateurs stables [101], on peut en déduire que si une fragmentation satisfait aux hypothèses du Théorème 1.3, alors nécessairement ν est binaire ($\nu\{\mathbf{s} : s_1 + s_2 < 1\} = 0$) et est caractérisée par

$$\nu(s_1 \in dx) = Cx^{-1-\alpha}(1-x)^{-1-\alpha}\mathbb{1}_{[1/2,1]}(x)dx$$

(où $C > 0$). D'autre part, un théorème dû à Bertoin [27] montre que pour une très grande catégorie de fragmentations auto-similaires d'indice $\beta \geq 0$ (dans laquelle entre la fragmentation considérée comme on le vérifie sur la formule donnant ν), le plus grand fragment $F_1(t)$ se comporte comme $t^{-1/\beta}$ si $t \rightarrow \infty$ ($\exp(-t)$ si $\beta = 0$). Pour notre problème, en comparant à nouveau avec des estimations pour les subordonateurs stables, on obtient que nécessairement $\beta = 1 - \alpha > 0$. Enfin, le théorème de Bertoin montre que la mesure de probabilités $\mu_t = \sum_{i \geq 1} F_i(t)\delta_{t^{1/\beta}F_i(t)}$ converge en loi vers une mesure limite dont les moments sont explicites, et en comparant encore une fois avec les lois stables, on obtient que nécessairement $\alpha = 1/2$, ce qui démontre le théorème.

Par ailleurs, rappelons que dans une partition échangeable sans singletons, le bloc contenant 1 est un bloc tiré de façon biaisée par la taille parmi les blocs de la partition. Le théorème montre donc que (1.2) n'a pas de généralisation aisée.

Proposition 1.2 :

Soit $(\Pi(t), t \geq 0)$ une fragmentation auto-similaire binaire à valeurs dans les partitions de \mathbb{N} . Soit $\alpha \in]0, 1[$ et σ un subordonateur stable d'indice α . Soit $\lambda(t)$ la fréquence asymptotique du block de $\Pi(t)$ contenant 1. S'il existe une fonction croissante g telle que

$$(g(\sigma_t), t \geq 0) \stackrel{d}{=} (\lambda(t), t \geq 0),$$

alors $\alpha = 1/2$, $g(x) = (1 + Kx)^{-1}$ pour un $K > 0$, et la fragmentation à valeurs dans S ($(\Pi(t))^\downarrow, t \geq 0$) est égale en loi à $(F_{AP}(Ct), t \geq 0)$ pour un $C > 0$.

Cette proposition se montre en utilisant le fait ([26]) que le processus du fragment marqué est un processus de Markov semi-stable, ce qui impose à g d'avoir une expression

particulière de la forme $g(s) = (1 + Ks)^{\alpha/\beta}$. On conclut que nécessairement, $\lambda(t)$ peut s'écrire $(1 + K\sigma(t))^{\alpha/\beta}$, et on en déduit la forme que doit avoir la mesure de dislocation (binaire) par la Proposition 1.1. La formule obtenue doit alors avoir une symétrie qui impose $\beta = \alpha = 1/2$, et la seule forme possible pour ν est alors $C\nu_{AP}$.

En revanche, nous montrons que le comportement en 0 dans (1.3) admet une généralisation à un certain nombre de fragmentations auto-similaires : par un procédé de couplage de processus de Poisson, nous établissons le résultat suivant. Si $\alpha \in]0, 1[$ et $(F(t), t \geq 0)$ est une fragmentation auto-similaire binaire de caractéristiques $(\beta, 0, \nu)$ avec $\beta \geq 0$ et

$$\nu(s_2 \in dx) = Cx^{-1-\alpha}s(x)\mathbb{1}_{[0,1/2]}(x)dx,$$

où $s(x) \geq 0$ vérifie $s(x) \rightarrow 1$ quand $x \downarrow 0$, alors

$$t^{-1/\alpha}(1 - F_1(t), F_2(t), F_3(t), \dots) \xrightarrow[t \downarrow 0]{d} (\sigma(1), \Delta_1(1), \Delta_2(1), \dots), \quad (1.7)$$

où σ est un subordonateur stable d'indice α et $\Delta_1(1) \geq \Delta_2(1) \geq \dots \geq 0$ sont les sauts $\Delta\sigma_{[0,1]}$. Pour mieux comprendre le rôle de $\nu(s_2 \in dx)$, notons qu'en temps petit, le plus gros fragment a une taille très proche de 1, et de plus, les tailles des fragments qui se détachent du plus gros fragment sont très peu modifiées. Comme en plus les dislocations sont binaires, les second, troisième, ... fragments proviennent chacun d'une dislocation de ce plus gros fragment, c'est-à-dire qu'ils sont proches des premiers, second, ... plus grands fragments qui se sont détachés du plus gros fragment. On se ramène donc à l'étude du processus d'apparition de ces fragments, qui est en fait Poissonnien à un changement de temps près. Ce résultat ainsi que sa démonstration font assez largement écho à l'article de Berestycki [17, Proposition 4.1 et Remark 4.5], avec la différence notable que ce même article ne considère que des fragmentations homogènes, et donc que nous tenons compte en plus du changement de temps nécessaire pour passer au cas auto-similaire d'indice positif. Bien sûr, le cas non binaire semble beaucoup plus difficile à traiter.

1.2.3 Deux fragmentations de l'arbre stable

Dans les chapitres 4 et 5 [80, 81], nous généralisons le résultat souligné plus haut de "dualité" entre les fragmentations F_{AP} et F_B . Comme nous l'avons dit, l'une s'obtient en coupant le squelette de l'arbre continu brownien à l'aide d'un processus de Poisson ponctuel homogène, tandis que l'autre s'obtient en jetant tous les sommets de l'arbre brownien qui se situent en dessous de la hauteur t , et en faisant varier t . Les deux fragmentations sont auto-similaires sans érosion et avec la même mesure de dislocation, mais leurs indices sont opposés ($1/2$ et $-1/2$). Peut-on en faire de même pour d'autres modèles d'arbres continus ? Les arbres stables sont évidemment de bons candidats puisqu'ils font intervenir des processus de Lévy stables, possédant une propriété d'auto-similarité. Il est donc très naturel de considérer en premier lieu la fragmentation F^- suivante.

Soit $(H_s, 0 \leq s \leq 1)$ le processus des hauteurs de l'arbre stable d'indice $\alpha \in]1, 2[$, et $I^-(t)$ l'ouvert de $[0, 1]$ défini par $\{s \in [0, 1] : H_s > t\}$. Il est aisé de voir que $(I^-(t), t \geq 0)$ est une fragmentation d'intervalles au sens où $I^-(t + t') \subset I^-(t)$ pour $t, t' \geq 0$. On note $F^-(t)$ la suite décroissante des longueurs des composantes connexes de $I^-(t)$. Nous

prouvons alors le résultat suivant au chapitre 4 :

Théorème 1.4 :

Le processus $(F^-(t), t \geq 0)$ est une fragmentation auto-similaire d'indice $1/\alpha - 1 \in]-1/2, 0[$, de coefficient d'érosion $c = 0$, et de mesure de dislocation ν_α caractérisée par : pour toute fonction G positive mesurable sur S ,

$$\nu_\alpha(G) = D_\alpha E[T_1 G(\Delta T_{[0,1]})],$$

où T est un subordonateur stable d'exposant de Laplace $\Phi(\lambda) = \lambda^{1/\alpha}$ et $D_\alpha > 0$ dépend de α .

Pour démontrer ce théorème, nous prouvons tout d'abord la propriété d'auto-similarité du processus de hauteur qui suit : sachant que $F^-(t) = (s_1, s_2, \dots)$, les excursions du processus de hauteur H au-dessus du niveau t sont des copies indépendantes renormalisées

$$(s_i^{1-1/\alpha} H^{(i)}(u/s_i), 0 \leq u \leq s_i), \quad i \geq 1$$

du processus de hauteur de durée 1. Ceci s'obtient grâce à la théorie des excursions d'Itô et en s'appuyant sur des résultats de Duquesne et Le Gall [49] sur le comportement du processus de hauteur au-dessus d'un niveau fixé. Avec un peu plus de travail, on peut également en déduire le semigroupe de F^- en considérant le temps local du processus de hauteur au niveau t . A partir de cette formule et à l'aide de la Proposition 1.1, on peut par le calcul émettre une conjecture sur l'allure de la mesure de dislocation ν , et on trouve ν_α . Cependant, certaines étapes du calcul sont délicates à justifier du fait d'une multitude de conditionnements singuliers, et de plus, la Proposition 1.1 n'a pas été énoncée pour les indices négatifs (quoiqu'il serait surprenant qu'elle devienne fautive dans ce cas-là). Le reste de l'étude consiste donc à *vérifier* que ν_α est bien la mesure de dislocation de F^- .

Pour obtenir la mesure de dislocation, nous utilisons une fragmentation à valeurs partitions associée de la façon suivante à F^- . On considère, connaissant la réalisation de l'arbre stable \mathcal{T}^α , une suite de feuilles L_1, L_2, \dots indépendantes et identiquement distribuées selon la mesure de masse μ^α . Pour $t \geq 0$ et $n \in \mathbb{N}$ on note $\Pi(t)$ la partition de \mathbb{N} telle que i et j sont dans le même bloc de $\Pi(t)$ si et seulement si le point de branchement de L_i et L_j a une hauteur $> t$ (L_i et L_j sont dans la même composante connexe de l'arbre tronqué en dessous du niveau t). On peut alors ramener l'étude de la mesure de dislocation ν à celle des processus restreints $\Pi_n(t) = [n] \cap \Pi(t), t \geq 0$. En fait, il est aisé de montrer à partir des résultats de [26] que si T_n est le premier temps tel que $\Pi_n(t)$ n'est pas constituée d'un seul bloc non-vide, la suite $\rho_n(d\pi)$ des lois de $\Pi_n(T_n)$ caractérise la mesure de dislocation ν à une constante multiplicative près.

Or, la partition $\Pi_n(T_n)$ a une interprétation simple en termes d'arbres. Rappelons que l'on peut définir à partir des feuilles L_1, \dots, L_n un arbre "réduit" (avec une terminologie différent de Duquesne et Le Gall, qui parlent plutôt de "marginales") $\mathcal{R}(n)$ engendré par la racine de l'arbre stable et ces feuilles. Cet arbre a donc n feuilles, et de la racine naît une unique branche de hauteur h_{root} en haut de laquelle viennent se brancher d'autres arbres (au moins deux si $n \geq 2$). Si on retire cette branche, on disconnecte ces arbres, et leurs feuilles induisent une partition de $[n]$, qui n'est autre que $\Pi_n(T_n)$. Or, la loi explicite de la forme τ de l'arbre réduit à n feuilles est donnée dans [49, Theorem 3.3.3] : si $\mathbf{t}^\circ \in \mathbf{T}^\circ$ est un arbre à n feuilles tel que $c_v \geq 2$ pour tout sommet $v \in \mathcal{N}$ dans l'ensemble des sommets

qui ne sont pas des feuilles, alors

$$P(\tau = \mathbf{t}^o) = \frac{n!}{\prod_{v \in \mathcal{N}} c_v!} \frac{\prod_{v \in \mathcal{N}} |(\alpha - 1)(\alpha - 2) \dots (\alpha - c_v + 1)|}{(\alpha - 1)(2\alpha - 1) \dots ((n - 1)\alpha - 1)}.$$

Un calcul permet finalement de déterminer la loi $\rho_n(d\pi)$ à partir de cette formule, et on vérifie que le résultat concorde avec la mesure ν_α , à l'aide de calculs sur les subordinateurs stables.

Il reste enfin à déterminer la constante multiplicative “flottante” en face de ν_α , et ceci se fait par un calcul explicite faisant intervenir la taille de la branche issue de la racine dans l'arbre réduit $\mathcal{R}(1)$, que l'on peut relier à fragment marqué de façon biaisée par la taille dans F^- . Une étude proche de celle de Bertoin [26, Section 4] permet de conclure.

Notons que la mesure ν_α ressemble de très près aux mesures de Poisson-Dirichlet à deux paramètres et aux partitions (“restaurants chinois”) associées, qui ont été introduites par Pitman [92]. De façon abusive, on peut considérer que ν_α est une loi de Poisson-Dirichlet de paramètres $(1/\alpha, -1)$, bien que ces paramètres soient incompatibles selon les notations de [92] (d'ailleurs, la mesure n'est pas une mesure de probabilité, elle est infinie).

Il est d'autre part intuitivement plus compliqué d'essayer de construire une fragmentation de l'arbre stable de même mesure de dislocation que F^- , mais d'indice positif. La tentative naïve, mimer la fragmentation d'Aldous-Pitman et couper l'arbre stable par un processus de Poisson homogène sur son squelette, échoue car la fragmentation obtenue est alors clairement binaire (un sommet pris au hasard sur le squelette est un point de branchement avec probabilité zéro), ce qui n'est pas le cas de F^- (sauf si $\alpha = 2$, auquel cas l'arbre est l'arbre brownien...). La seule possibilité est donc d'essayer de couper l'arbre au niveau des points de branchement, puisqu'en enlevant un de ces points on partage l'arbre en une infinité de fragments. Il reste à trouver à quelle “fréquence” on doit couper ces nœuds, c'est-à-dire, suffisamment lentement pour ne pas disloquer l'arbre instantanément, mais d'une façon appropriée pour que les fragments obtenus soient des copies renormalisées de l'arbre initial. Pour cela, il faut pouvoir mesurer la “taille” d'un nœud, et ceci est fait à travers la notion de “temps local d'un nœud” que nous décrivons maintenant.

Le processus des hauteurs de l'arbre stable possède, comme l'ont montré Duquesne et Le Gall, un processus de temps local $(L_s^t, 0 \leq s \leq 1, t \geq 0)$, obtenu comme la densité de la mesure d'occupation associée à H . Dans le cas de l'arbre stable, on peut montrer en utilisant ce processus que les nœuds b ont un temps local

$$L(b) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mu^\alpha \{v \in \mathcal{T}_b : d(b, v) < \varepsilon\} > 0 \quad p.s.,$$

où \mathcal{T}_b est le sous-arbre de \mathcal{T} enraciné en b . Plus précisément, rappelons que l'on construit le processus des hauteurs à partir de l'excursion normalisée $(X_s^{\text{exc}}, 0 \leq s \leq 1)$ du processus stable au-dessus de son infimum, et que les sommets de \mathcal{T}^α sont encodés par les points de $[0, 1]$. On montre que chaque temps de saut de l'excursion X^{exc} code exactement un point de branchement de l'arbre stable, et que son temps local correspond à la taille du saut correspondant. Plus précisément, si τ est un temps de saut et $\sigma = \inf\{s \geq \tau : X_s = X_{\tau-}\}$, alors l'ensemble des sommets de l'arbre situé au-dessus du nœud codé par τ est codé par l'intervalle $[\tau, \sigma]$.

Il est alors plus intuitif que l'on doive adopter la stratégie suivante pour couper l'arbre stable : chaque nœud b doit être coupé à un taux $L(b)$. On ne peut pas par exemple couper

chaque nœud avec le même taux constant, car alors au temps $0+$ on aurait coupé l'arbre stable "selon un temps local total infini", puisque les sauts de l'excursion X^{exc} ne sont pas sommables. En revanche, ils sont de carré sommable, et le temps local total moyen des nœuds enlevés avec le taux $L(b)$ est fini, d'ordre $t \sum_b L(b)^2 = t \sum_{0 \leq s \leq 1} \Delta X_s^2$. Ce qui tranche définitivement en faveur de cette stratégie est la propriété suivante. Pour $t \geq 0$ marquons chaque saut $\Delta X_s > 0$ de X indépendamment avec probabilité $1 - \exp(-t \Delta X_s)$. Notons M_t l'ensemble des temps des sauts u marqués, $Z_s^{(t)} = \sum_{u \in M_t} \Delta X_u$, et $X^{(t)} = X - Z^{(t)}$. Alors pour chaque s , $(X_u^{(t)}, 0 \leq u \leq s)$ a une loi absolument continue par rapport à celle de X , dont la densité ne dépend que de X_s . Ceci montre que si l'on marque les sauts de X à taux égal à leur taille, le processus où on a enlevé ces sauts est absolument continu par rapport à X , et en particulier, on montre que les excursions de $X^{(t)}$ au-dessus de son infimum sont les mêmes que celles de X . Il s'agit là de la clef de la propriété d'auto-similarité (rappelons que le "miracle" du fait que la fragmentation d'Aldous-Pitman est auto-similaire tient au théorème de Girsanov, ici la bonne méthode est de retirer des sauts plutôt qu'ajouter une dérive). D'après notre analyse, marquer des sauts revient à marquer des nœuds de l'arbre stable, et intuitivement, les excursions du processus tronqué doivent correspondre aux composantes connexes de l'arbre fragmenté.

Nous pouvons à présent faire la synthèse de notre analyse. Conditionnellement à la réalisation de \mathcal{T}^α , on se donne un processus de Poisson ponctuel $(b(s), s \geq 0)$ d'intensité $ds \otimes \sum_b L(b) \delta_b(dv)$, où la somme porte sur les nœuds de \mathcal{T}^α . On note ensuite $v \sim_t w$ pour $v, w \in \mathcal{T}^\alpha$ si pour tout $s \in [0, t]$, $b(s)$ n'est pas dans $[[v, w]]$. Ceci divise \mathcal{T} en classes d'équivalence dont on montre qu'elles sont mesurables. Soit enfin $F^+(t)$ la suite décroissante des μ^α -masses des classes d'équivalence de \sim_t (les composantes connexes de l'arbre \mathcal{T}^α d'où l'on a ôté les nœuds $b(s), 0 \leq s \leq t$).

Théorème 1.5 :

Le processus $(F^+(t), t \geq 0)$ est une fragmentation auto-similaire d'indice $1/\alpha$, de coefficient d'érosion $c = 0$, et dont la mesure de dislocation est égale à celle de F^- .

La preuve de ce résultat repose sur deux ingrédients. Le premier, un peu délicat, consiste à extraire directement de l'excursion X^{exc} à partir de laquelle on a construit le processus de hauteur les différents morceaux de l'arbre fragmenté au temps t , c'est-à-dire que pour chaque temps τ_i^t de saut marqué au temps t , en notant $\sigma_i^t = \inf\{s \geq \tau_i^t : X_s = X_{\tau_i^t-}\}$, on retire la portion de X située dans l'intervalle $[\tau_i^t, \sigma_i^t]$. Ceci implique des décompositions trajectorielles de processus qui sont au fond élémentaires mais un peu intriquées (du fait que l'on doit enlever une infinité de morceaux et que ces morceaux sont eux-mêmes emboîtés). Les processus extraits sont alors des excursions de $X^{(t)}$ conditionnées par leur durée, et donc ce sont aussi des excursions de X conditionnées, ceci permet de conclure à l'auto-similarité grâce à une propriété de *scaling* des excursions du processus stable. Le second ingrédient, intuitivement plus simple car il ne fait intervenir qu'une seule décomposition trajectorielles, consiste à trouver la mesure de dislocation en analysant l'effet créé par un unique point de coupe (un unique saut de l'excursion) tiré selon la "loi" $m(dv) = \sum_b L(b) \delta_b(dv)$. À l'aide du théorème de Vervaat, on relie ceci à l'effet d'une certaine décomposition du pont $X_{0 \rightarrow 0}^1$ en un saut choisi selon la loi $\sum_s \Delta X_{0 \rightarrow 0}^1(s) \delta_s(du)$, on trouve que la dislocation "générique" a bien la même intensité que pour F^- .

Une autre propriété notable de la fragmentation F^+ est qu'elle admet une autre représentation, plus simple (sans décompositions trajectorielles) et permettant un calcul de son semigroupe. Cette représentation rappelle le fait qu'il existe (au moins) deux re-

présentations de la fragmentation F_{AP} , l'une en fragmentant l'arbre brownien, l'autre en ajoutant une dérive à l'excursion brownienne. Ici, comme nous l'avons noté, il est plus adapté de marquer et retirer des sauts plutôt qu'ajouter une dérive. Rappelons que l'on peut marquer chaque saut ΔX_s^{exc} de l'excursion X^{exc} avec probabilité $1 - \exp(-t\Delta X_s^{\text{exc}})$, et définir le processus $Z_{\text{exc}}^{(t)}$ qui cumule les sauts marqués. On définit ensuite le processus $X_{\text{exc}}^{(t)} = X^{\text{exc}} - Z_{\text{exc}}^{(t)}$. Soit $\underline{X}_{\text{exc}}^{(t)}$ le processus de l'infimum passé, et notons $F^{\natural}(t)$ la suite décroissante des longueurs des intervalles maximaux sur lesquels $\underline{X}_{\text{exc}}^{(t)}$ est constante.

Théorème 1.6 :

| Le processus $(F^{\natural}(t), t \geq 0)$ a même loi que F^+ .

Nous montrons ce résultat par le calcul de son semigroupe (ce qui donne donc au passage le semigroupe de F^+), ce qui est rendu possible par la relation d'absolue continuité entre $X^{(t)}$ et X évoquée plus haut, ainsi que le théorème de Vervaat et une décomposition du pont de type "décomposition de Williams" en son minimum, due à Chaumont [42]. On retrouve ensuite la mesure de dislocation par la Proposition 1.1.

Nous donnons également un certain nombre de résultats asymptotiques sur les fragmentations F^- et F^+ . On remarque en particulier que leurs comportements en temps petit, s'il fait intervenir des sauts de subordinateurs, sont un peu plus compliqués que ceux qui interviennent pour les fragmentations binaires (1.7). On les démontre à l'aide de la forme explicite des semigroupes de F^- et F^+ . Il est à noter que pour F^- , on voit apparaître des processus de branchements continus, ce qui est une conséquence du théorème de Ray-Knight [49, Theorem 1.4.1] reliant le temps local des arbres stables avec les processus de branchement continu. Les autres résultats asymptotiques (en temps grand, comportement des petits fragments) sont pour l'essentiel des applications directes des résultats de Bertoin [27, 25].

1.2.4 Généalogie des fragmentations auto-similaires d'indice négatif

La motivation de ce travail avec Bénédicte Haas [59] est de déterminer dans quelle mesure une fragmentation auto-similaire admet une représentation du type de la fragmentation F^- étudiée ci-dessus. En d'autres termes, une fragmentation auto-similaire F est-elle une fragmentation d'un arbre continu (\mathcal{T}, μ) , au sens où F a la même loi que la suite décroissante des masses des composantes connexes de $\mathcal{T}^t = \{v \in \mathcal{T} : \text{ht}(v) > t\}$? Les arbres continus que nous avons considérés jusqu'ici sont compacts (ils sont encodés par des fonctions continues de $[0, 1]$), et on voit donc que $\mu(\mathcal{T}^t) = 0$ pour t assez grand. On s'intéresse donc *a priori* à des fragmentations d'indice strictement négatif (qui sont non-constantes), autorisant donc une perte de masse, bien que l'on puisse certainement élargir le formalisme des arbres continus à des arbres dont les feuilles sont "à l'infini", et pour lesquels $\mu(\mathcal{T}^t) = 1$ pour tout t .

Il est alors assez naturel de construire un arbre généalogique de la fragmentation F de la façon qui suit. On suppose que F n'a pas d'érosion (ceci correspond au fait que la mesure de masse d'un arbre continu ne doit pas charger le squelette) et que la mesure de dislocation ν vérifie $\nu\{\mathbf{s} \in S : \sum_i s_i < 1\} = 0$ (pour éviter que la mesure de masse ait des atomes). On se donne une représentation "partitions" $(\Pi(t), t \geq 0)$ de F , c'est-à-dire que Π est une fragmentation à valeurs dans les partitions de \mathbb{N} de mêmes caractéristiques que F . A chaque $i \in \mathbb{N}$, on associe un "temps de vie" $D_i = \inf\{t \geq 0 : \{i\} \in \Pi(t)\}$ qui marque la fin de la lignée de l'individu i , et la structure généalogique est la structure

naturelle : deux fragments (blocs) B_1 et B_2 au temps $t + t'$ qui faisaient partie du même bloc au temps t sont des fils de ce bloc. Plus précisément, pour $B \subset \mathbb{N}$ fini, on construit des arbres $\mathcal{R}(B)$ par récurrence sur la taille de B . Si $B = \{i\}$, l'arbre $\mathcal{R}(\{i\})$ est constitué d'une unique branche de taille D_i . On pose alors pour chaque $B \subset \mathbb{N}$ de cardinal ≥ 2 $D_B = \inf\{t \geq 0 : B \cap \Pi(t) \neq B\}$ le premier instant de séparation de ce bloc, et on note B_1, B_2, \dots, B_n les blocs non-vides de $B \cap \Pi(D_B)$. Enfin, on pose

$$\mathcal{R}(B) = \text{MERGE}((\mathcal{R}(B_1) - D_B), \dots, (\mathcal{R}(B_n) - D_B); D_B),$$

où la notation ci-dessus signifie que l'on branche, tout en haut d'un segment de taille D_B , les arbres $\mathcal{R}(B_i), 1 \leq i \leq n$ déjà construits, mais auxquels on a retiré une longueur D_B au segment issu de la racine.

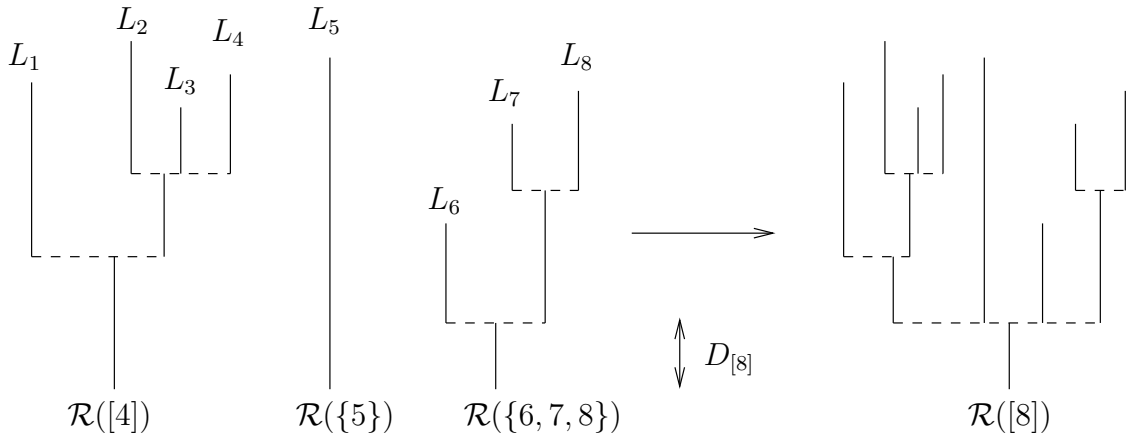


Fig. 1.1 – L'opération MERGE

En notant $\mathcal{R}(k) = \mathcal{R}([k])$, on construit ainsi une famille consistante de \mathbb{R} -arbres, et un théorème d'Aldous [5] permet de conclure que l'adhérence de la "réunion" de ces arbres (pour une construction *ad hoc*) est un arbre continu aléatoire, que nous notons (\mathcal{T}_F, μ_F) , où μ_F est la limite de la mesure empirique des feuilles de $\mathcal{R}(k)$. Par des résultats d'échangeabilité, on montre que \mathcal{T}_F répond bien à la question posée. Ce résultat admet une forme de réciproque, dans le sens où tout arbre continu aléatoire possédant une certaine forme simple d'auto-similarité est de la forme \mathcal{T}_F pour une fragmentation auto-similaire F satisfaisant aux hypothèses ci-dessus.

Nous montrons par ailleurs un résultat sur la dimension de Hausdorff de l'arbre \mathcal{T}_F (voir par exemple [53] pour les résultats mentionnés ci-dessous sur la dimension de Hausdorff). Rappelons que si (E, d) est un espace métrique, on définit

$$m_\gamma(E) = \sup_{\varepsilon > 0} \inf_{R_\varepsilon(E)} \sum_{B \in R_\varepsilon(E)} \text{diam}(B)^\gamma$$

où l'infimum porte sur l'ensemble des recouvrements $R_\varepsilon(E)$ de E par des ensembles de diamètre $\text{diam}(B) < \varepsilon$. On appelle *dimension de Hausdorff* de E , et on note $\dim_{\mathcal{H}}(E)$, le nombre

$$\dim_{\mathcal{H}}(E) = \inf\{\gamma > 0 : m_\gamma(E) = 0\} \in [0, \infty].$$

Nous montrons le résultat suivant :

Théorème 1.7 :

Soit F une fragmentation auto-similaire de caractéristiques $(\beta, 0, \nu)$, avec $\beta < 0$ et $\nu\{\mathbf{s} : \sum_i s_i < 1\} = 0$. On suppose que pour un (et donc tout) $\varepsilon \in]0, 1[$, on a

$$\int_S s_1^{-2} \mathbb{1}_{\{s_1 < 1-\varepsilon\}} \nu(d\mathbf{s}) < \infty.$$

Alors la dimension de Hausdorff de l'arbre \mathcal{T}_F est, presque-sûrement, $\dim_{\mathcal{H}}(\mathcal{T}_F) = (1/|\beta|) \wedge 1$.

En particulier, notons qu'il est trivial que $\dim_{\mathcal{H}}(\mathcal{T}_F) \geq 1$ puisque l'arbre contient toujours au moins une branche. Par ailleurs, l'hypothèse sur ν est toujours vérifiée dans le cas où la fragmentation est binaire, ou plus généralement dans le cas où elle est "au plus n -aire", au sens où $\nu\{\mathbf{s} : s_1 + \dots + s_n < 1\} = 0$ pour un $n \geq 2$.

Comme souvent pour les résultats sur les dimensions de Hausdorff, la preuve se découpe en deux parties, une majoration et une minoration, la seconde étant un peu plus délicate. Pour la majoration, on doit chercher un recouvrement approprié de l'arbre. L'idée grossière est de couper en "tranches horizontales" de taille $\varepsilon > 0$ l'arbre \mathcal{T}_F . Si un bloc a "traversé" une tranche de taille ε et est "mort" dans la tranche immédiatement supérieure, la propriété d'auto-similarité ainsi qu'un contrôle exponentiel [57] sur la queue du temps de vie de la fragmentation permet de déduire que la taille du bloc est d'ordre plus grand que $\varepsilon^{1/\beta}$. En choisissant convenablement les boules, on peut alors recouvrir l'arbre avec $\varepsilon^{-1/\beta}$ boules de rayon d'ordre ε , et on voit donc que $\gamma = -1/\beta$ est la valeur critique dans la définition de la mesure de Hausdorff. Ce raisonnement heuristique ne permet pas de voir où intervient la distinction $\beta \leq -1, \beta > -1$, qui n'apparaît que dans le détail des calculs. On note que la majoration est vraie sans hypothèse supplémentaire sur F .

Pour la minoration, la méthode la plus naturelle est la méthode de l'énergie de Frostman. Rappelons que si (E, d) est un espace métrique et si μ est une mesure (non nulle) finie sur E , alors

$$\iint \frac{d\mu(x)d\mu(y)}{d(x, y)^\gamma} < \infty \implies \dim_{\mathcal{H}}(E) \geq \gamma.$$

La tentative la plus naïve est bien sûr de prendre $E = \mathcal{T}_F$ et $\mu = \mu_F$ la mesure de masse. On essaie alors de montrer que

$$E \left[\iint \frac{d\mu_F(v)d\mu_F(w)}{d(v, w)^\gamma} \right] = E [d(L_1, L_2)^{-\gamma}] < \infty,$$

où conditionnellement à (\mathcal{T}_F, μ_F) , L_1 et L_2 sont indépendantes de loi μ_F . Du fait de la construction de \mathcal{T}_F comme "limite" des arbres réduits $\mathcal{R}(k), k \geq 1$, la distance $d(L_1, L_2)$ est en fait égale en loi à la distance entre les deux feuilles de $\mathcal{R}(2)$, c'est-à-dire à $D_1 + D_2 - 2D_{\{1,2\}}$ avec les notations ci-dessus. Par une propriété de Markov au temps $D_{\{1,2\}}$ et par auto-similarité, on peut réécrire ceci $\tilde{D}_1 \lambda_1(D_{\{1,2\}})^\beta + \tilde{D}_2 \lambda_2(D_{\{1,2\}})^\beta$, où $\lambda_i(t)$ est la fréquence asymptotique du fragment qui contient $i \in \{1, 2\}$, et \tilde{D}_1, \tilde{D}_2 sont indépendants de même loi que D_1 , indépendants de $\lambda_1(D_{\{1,2\}}), \lambda_2(D_{\{1,2\}})$. On peut alors terminer le calcul par des relations déjà évoquées plus haut entre les fragments marqués et certains subordinateurs. Hélas, ceci échoue en général : la minoration que l'on obtient diffère de la borne supérieure dans un grand nombre de cas. Ce à l'exception d'un cas notable : celui d'une mesure de dislocation finie et au plus n -aire au sens ci-dessus.

Ceci donne donc une piste pour trouver une mesure de Frostman plus adaptée : on cherche, par un procédé de troncation, à “transformer” la fragmentation initiale en une fragmentation $F^{N,\varepsilon}$ au plus N -aire et de mesure de dislocation finie, pour ce faire, on ignore les dislocations pour lesquelles le plus gros fragment est $> 1 - \varepsilon$, ainsi que les sort des $N + 1, N + 2, \dots$ -ième plus gros fragments à chaque dislocation. On peut associer à cette nouvelle fragmentation une mesure de masse $\mu_F^{N,\varepsilon}$ portée par \mathcal{T}_F , mais singulière par rapport à μ_F . En appliquant la méthode de Frostman et pour des valeurs grandes de N et $1/\varepsilon$, on peut sous les hypothèses du Théorème 1.7 trouver une minoration arbitrairement proche de la majoration.

L'un des intérêts du Théorème 1.7 est qu'il peut s'appliquer au cas de l'arbre stable ($\alpha \in]1, 2[$), puisqu'on a clairement que l'arbre stable \mathcal{T}^α a même loi que \mathcal{T}_{F^-} avec les notations ci-dessus. On peut vérifier les hypothèses du Théorème 1.7 et en déduire

Corollaire 1.1 :

| La dimension de l'arbre stable α est égale à $\alpha/(\alpha - 1)$ presque-sûrement.

Ce résultat a été obtenu indépendamment par Duquesne et Le Gall [50].

1.2.5 Processus d'exploration des ICRT

Dans l'introduction, nous avons présenté les ICRT par une méthode permettant de construire leurs “marginales”, et nous n'avons pas défini ces arbres par leur “processus de hauteur”, contrairement aux arbres stables. C'est-à-dire, nous n'avons pas exhibé de fonction aléatoire $(H_s^\theta, 0 \leq s \leq 1)$ telle que, si D est la pseudo-métrique sur $[0, 1]$ définie par $D(s, s') = H_s^\theta + H_{s'}^\theta - 2 \inf_{u \in [s, s']} H_u^\theta$, alors l'arbre \mathcal{T}^θ est isomorphe au quotient $[0, 1]/\equiv$ (où $s \equiv s'$ si et seulement si $D(s, s') = 0$), et μ^θ est la mesure associée à la mesure de Lebesgue sur $[0, 1]$

Nous nous proposons dans un travail avec Aldous et Pitman [7] de combler cette lacune. Soit $\theta \in \Theta$ défini ci-dessus, et soit B^{br} un pont brownien standard. On se donne une suite $(U_i, i \geq 1)$ de variables aléatoires uniformes sur $[0, 1]$, indépendantes, et indépendantes de B^{br} . On définit un *pont à accroissements échangeables*

$$X_s^{\text{br},\theta} = \theta_0 B_s^{\text{br}} + \sum_{i \geq 1} \theta_i (\mathbb{1}_{\{U_i \leq s\}} - s), \quad 0 \leq s \leq 1. \quad (1.8)$$

Un théorème de Kallenberg [65] montre que $X^{\text{br},\theta}$ est bien défini et est càdlàg, c'est un *pont* au sens où $X_0^{\text{br},\theta} = X_1^{\text{br},\theta} = 0$ et $X^{\text{br},\theta}$ est continu en 0 et 1. Un théorème dû à Knight [71] montre qu'il existe p.s. un unique temps $s_{\min} \in [0, 1]$ où $X^{\text{br},\theta}$ atteint son minimum, de plus Bertoin a montré que $X^{\text{br},\theta}$ y est continu. Nous notons $X_s^\theta = X_{s+s_{\min}}^{\text{br},\theta} - X_{s_{\min}}^{\text{br},\theta}$, où l'addition est modulo 1, la transformée de Vervaat du pont $X^{\text{br},\theta}$. Enfin, nous définissons un autre processus $(H_s^\theta, 0 \leq s \leq 1)$ à partir de X^θ . En notant $t_i = U_i - s_{\min}$ (modulo 1) le temps du saut d'amplitude θ_i pour X^θ , et $T_i = \inf s \geq t_i : X_s^\theta = X_{t_i}^\theta$, on remplace sur l'intervalle $[t_i, T_i]$ le processus X par le processus réfléchi $X_s^\theta - \inf_{t_i \leq u \leq s} X_u^\theta$. On note Y^θ le processus obtenu. De façon plus synthétique, si on note $|A|$ la mesure de Lebesgue de l'ensemble $A \subset \mathbb{R}$, on a la formule suivante :

$$Y_s^\theta = \left| \left\{ \inf_{r \leq u \leq s} X_u^\theta : 0 \leq r \leq s \right\} \right|.$$

Théorème 1.8 :

Soit $\theta \in \Theta$ avec $\theta_0 > 0$. Alors le processus $H^\theta = 2\theta_0^{-2}Y^\theta$ est le processus des hauteurs de l'ICRT \mathcal{T}^θ .

On s'intéresse également au processus de "largeur" de l'ICRT. Ce processus se décrit ainsi : si (\mathcal{T}, μ) est un CRT, notons $\bar{W}_s = \mu\{v \in \mathcal{T} : \text{ht}(v) \leq s\}$, appelé "processus de largeur cumulatif". Cette fonction est croissante, et si $d\bar{W}$ est absolument continue par rapport à la mesure de Lebesgue, nous notons $d\bar{W}_s = W_s ds$. Le processus W est alors le processus de largeur, il décrit heuristiquement l'épaisseur des couches de l'arbre (pour l'arbre stable, ce processus est bien sûr le processus de temps local du processus de hauteur, mais ici la dimension est intrinsèque à l'arbre). Nous donnons également une description du processus de largeur de l'ICRT, cette fois sans restriction sur θ . On note \bar{W}^θ le processus de largeur cumulatif, et $(\bar{W}^\theta)^{-1}$ son inverse continu à droite.

Théorème 1.9 :

Pour tout $\theta \in \Theta$, le processus de largeur de l'ICRT \mathcal{T}^θ existe, on le note W^θ . De plus, on a

$$(W^\theta((\bar{W}^\theta)^{-1}(s)), 0 \leq s \leq 1) \stackrel{d}{=} (X_s^\theta, 0 \leq s \leq 1).$$

En d'autres termes, W^θ est à un changement près de type "changement de temps de Lamperti" égal en loi au processus X^θ . On note en particulier que W^θ a des sauts de taille θ_i , ce qui correspond à de brusques densifications des couches de l'arbre. Ces densifications correspondent bien sûr aux nœuds de degré infini, et on peut rapprocher la présente discussion à celle du temps local des nœuds de l'arbre stable. La combinaison des deux derniers théorèmes donne un résultat qui ne fait plus intervenir d'arbres, et qui évoque un théorème de Jeulin sur le temps local d'une excursion brownienne. Si $\theta_0 > 0$, on déduit que la mesure d'occupation du processus H^θ a une densité par rapport à la mesure de Lebesgue : pour toute fonction f positive mesurable,

$$\int_0^1 f(H_s^\theta) ds = \int_0^\infty f(u) W_u^\theta du,$$

où W^θ a même loi que $(X_{\tau(u)}^\theta, u \geq 0)$, où

$$\tau(u) = \inf \left\{ r \geq 0 : \int_0^r \frac{ds}{X_s^\theta} > u \right\}.$$

Le théorème de Jeulin s'obtient dans le cas où $\theta_0 = 1$, c'est-à-dire où $X^\theta = Y^\theta = B^{\text{exc}}$ est une excursion brownienne normalisée, et W^θ est le temps local de $2B^{\text{exc}}$. On peut un peu abusivement interpréter ce résultat comme un théorème de Ray-Knight conditionné¹.

Donnons un aperçu rapide de la démonstration de ces résultats, qui utilisent dans les deux cas une approximation par des processus associés aux \mathbf{p} -arbres. Ces processus sont proches (mais cependant différents) des fonctions de parking utilisées par Chassaing et Louchard [41], qui codent l'arbre uniforme à n sommets, et de la file d'attente LIFO utilisée dans [77] pour coder les versions "à variation finie" des arbres Lévy. On se donne \mathbf{p}

¹Dans le cas $\theta_0 = 1$ on a l'interprétation suivante en termes de processus de branchement continu : le temps local de l'excursion brownienne est un processus de branchement continu quadratique, issu de 0 mais conditionné à avoir une population totale de taille 1

une probabilité sur $[n]$ avec $p_i > 0, 0 \leq i \leq n$. Nous construisons un pont à accroissements échangeables

$$F^{\mathbf{P}}(s) = \sum_{i=1}^n p_i (\mathbb{1}_{U_i \leq s} - s), \quad 0 \leq s \leq 1,$$

où U_1, \dots, U_n sont indépendantes uniformes sur $[0, 1]$. Alors il existe un $s_{\min} \in [0, 1]$, unique presque-sûrement, tel que $F^{\mathbf{P}}(s_{\min}-) = \inf_{s \in [0, 1]} F^{\mathbf{P}}(s)$, de plus $F^{\mathbf{P}}$ est discontinue en s_{\min} . On note donc $F^{\text{exc}, \mathbf{P}}(s) = F^{\mathbf{P}}(s + s_{\min}) - F^{\mathbf{P}}(s_{\min}-), 0 \leq s \leq 1$ la transformée de Vervaat de $F^{\mathbf{P}}$. On note $V_i = U_i - s_{\min}$ modulo 1. À partir de $F^{\text{exc}, \mathbf{P}}$, nous construisons de deux façons différentes un arbre $\mathcal{T}^{\mathbf{P}}$ dont la loi est celle du \mathbf{p} -arbre, et la propriété clef est que sous les hypothèses de convergence du \mathbf{p} -arbre vers l'ICRT \mathcal{T}^θ (1.6), on montre la convergence en loi dans l'espace de Skorokhod

$$(\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}}(s), 0 \leq s \leq 1) \xrightarrow{d} (X_s^\theta, 0 \leq s \leq 1). \quad (1.9)$$

D'une part, on sait que $0 = V_{i_1}$ pour un certain $i_1 \in [n]$ puisque $F^{\mathbf{P}}$ est discontinue en s_{\min} . On définit i_1 comme étant la racine de $\mathcal{T}^{\mathbf{P}}$. On note alors $\sigma(i_1) = 1$ et $\sigma(i_k) = k$ si V_{i_k} est le temps du $(k-1)$ -ième saut de $F^{\text{exc}, \mathbf{P}}$ dans l'ordre induit par $[0, 1]$. On note $y(k) = \sum_{r=1}^k p_{i_r}$. On construit alors $\mathcal{T}^{\mathbf{P}}$ par la règle : les enfants de i_k sont les j tels que $V_j \in (y(k-1), y(k)]$. On montre facilement que $\mathcal{T}^{\mathbf{P}}$ est un \mathbf{p} -arbre, dont la construction est inspirée par un "parcours en largeur" (les nœuds i_1, i_2, \dots, i_n parcourent tour à tour les différentes couches de l'arbre en partant de la racine et en remontant vers la cime). On s'aperçoit que $\mathcal{T}^{\mathbf{P}}$ est un \mathbf{p} -arbre, intuitivement cela provient du fait que les intervalles $(y(k-1), y(k)]$ ont des longueurs $p_i, 1 \leq i \leq n$, et que pour que i ait k enfants dans $\mathcal{T}^{\mathbf{P}}$, il faut donc que k des variables $U_j, 1 \leq j \leq n$ tombent dans un intervalle de longueur p_i , ce qui arrive avec probabilité p_i^k . Soit $\bar{u}(h) = \sum_{\text{ht}(i) \leq h-1} p_i$ la masse des $h-1$ premières couches de $\mathcal{T}^{\mathbf{P}}$, et $u(h) = \sum_{\text{ht}(i)=h} p_i$ la masse de la h -ième couche. La clef de la preuve du Théorème 1.9 est la relation

$$F^{\text{exc}, \mathbf{P}}(\bar{u}(h)) = u(h), \quad h \geq 0,$$

qui, du fait de (1.9), de la convergence de $\mathcal{T}^{\mathbf{P}}$ vers \mathcal{T}^θ sous l'hypothèse (1.6) et au prix d'un certain nombre d'arguments techniques entraîne l'existence du processus de largeur W^θ ainsi que la relation (avec des notations abusives)

$$X^\theta(\bar{W}^\theta(h)) = W^\theta(h), \quad h \geq 0.$$

La preuve du Théorème 1.8 est plus subtile. Tout d'abord, on construit différemment le \mathbf{p} -arbre à partir de $F^{\text{exc}, \mathbf{P}}$, cette fois par un procédé d'exploration en profondeur plutôt qu'en largeur. Avec les notations ci-dessus, i_1 reste la racine, mais on construit $\mathcal{T}^{\mathbf{P}}$ de façon récursive en implémentant ainsi. La racine i_1 est la première à être examinée, et on pose $y^*(1) = 0$. Sachant i_1, \dots, i_k et $y^*(1), \dots, y^*(k)$, on pose $y^*(k+1) = y^*(k) + p_{i_k}$, et les fils de i_k sont les j tels que V_j tombe dans $(y^*(k), y^*(k+1)]$, et on les ordonne par ordre d'apparition dans cet intervalle. On pose alors i_{k+1} le premier fils de i_k s'il y en a, sinon, le premier fils du parent de i_k qui n'a pas été examiné, s'il y en a, sinon, le premier fils du grand-père de i_k qui n'a pas été examiné, s'il y en a, et ainsi de suite. Cet ordre d'exploration de l'arbre correspond à l'ordre d'exploration en profondeur défini plus haut pour les arbres de \mathbf{T}^o . Soit $\mathcal{T}_o^{\mathbf{P}}$ l'arbre obtenu par cette construction, considéré

comme à la fois étiquetés (par $[n]$) et ordonné (l'ordre sur les enfants d'un sommet est celui induit par l'ordre d'exploration i_1, i_2, \dots, i_n). Pour les mêmes raisons que ci-dessus, si l'on oublie cet ordre et qu'on considère $\mathcal{T}^{\mathbf{p}}$ l'arbre étiqueté non-ordonné associé, alors $\mathcal{T}^{\mathbf{p}}$ est un \mathbf{p} -arbre. La propriété clef de ce codage est la suivante. Soit $v \in [n]$ un sommet de $\mathcal{T}^{\mathbf{p}}$, on note $i_0 = v_0, v_1, \dots, v_j = v$ le chemin menant de la racine à v . Pour $0 < k \leq j$ on note $v_{k,1}, v_{k,2}, \dots$ les plus jeunes frères de v_k (ceux qui sont nés après), et on note $v_{j+1,1}, v_{j+1,2}, \dots$ les fils de v_j . Alors, si on pose

$$\mathcal{N}(v) = \bigcup_{1 \leq k \leq j+1} \{v_{k,1}, v_{k,2}, \dots\},$$

on a la propriété

$$F^{\text{exc}, \mathbf{p}}(e(v)) = p(\mathcal{N}(v))$$

pour tout v , où $e(v) = y^*(k)$ si $v = i_k$ (le moment où v va être examiné). On prouve d'abord le Théorème 1.8 dans le cas où $\theta = (\theta_0, \theta_1, \dots, \theta_I, 0, \dots)$ avec $\theta_I > 0$, c'est-à-dire que la suite approximante \mathbf{p} a I valeurs "grandes" p_1, \dots, p_I , de l'ordre de $\sigma(\mathbf{p})$, et le reste est $\max_{[n] \setminus [I]} p_i = o(\sigma(\mathbf{p}))$. Le théorème s'en déduit par un argument d'approximation. De plus, on peut choisir la suite \mathbf{p} comme on le veut à condition qu'elle vérifie (1.6), on demande donc que les "petites" valeurs $p_i, i > I$ soient proches. Dans ce cas, on a une sorte de loi des grands nombres sur l'arbre $\mathcal{T}^{\mathbf{p}}$ lorsque $n \rightarrow \infty$. La quantité $p(\mathcal{N}(v))$ peut se scinder en deux parties $m_1 + m_2$: m_1 , la masse des fils des "petits sommets" v_k tels que $v_k > I$ (notons les A), et m_2 , celle des "gros" $v_k \leq I$. Par la construction de l'arbre à l'aide du processus $F^{\text{exc}, \mathbf{p}}$, on voit heuristiquement que le nombre et la masse des fils de A , s'il n'est "pas trop gros" (de masse $O(\sigma(\mathbf{p}))$), sont proportionnellement proche de la masse de ce sous-ensemble (on regarde combien parmi les variables uniformes U_i tombent dans un ensemble de masse $p(A)$). Il faut cependant exclure de ces fils les sommets qui sont trop "gros", c'est-à-dire ceux de $[I]$, mais pour n grand ils tombent avec une faible probabilité dans un ensemble de masse $O(\sigma(\mathbf{p}))$. Avec un bon choix de \mathbf{p} , on a alors que la masse des fils de A est d'ordre $\sigma(\mathbf{p})\text{Card}(A)$, qui est proche de $C\sigma(\mathbf{p})\text{ht}(v)$ pour une constante C (que l'on montre être θ_0^2). Comme en plus on ne considère que les fils de A qui tombent à droite de la lignée de v , on voit que m_1 est proche de $\theta_0^2\sigma(\mathbf{p})\text{ht}(v)/2$. Par conséquent, en prenant v au hasard avec la loi \mathbf{p} (pour cela on prend U uniforme sur $[0, 1]$ et on pose v le nœud tel que $U \in (e(v), e(v) + p_v)$), la convergence de $\sigma(\mathbf{p})^{-1}F^{\text{exc}, \mathbf{p}}$ vers X^θ , la convergence des \mathbf{p} -arbres vers \mathcal{T}^θ (et donc de $\text{ht}(v)$ vers la hauteur d'une feuille L de \mathcal{T}^θ de loi μ^θ) et l'analyse ci-dessus indiquent que (avec des notations abusives)

$$X_U^\theta = \frac{\theta_0^2}{2}\text{ht}(L) + \lim \sigma(\mathbf{p})^{-1}m_2.$$

On montre alors que la limite de $\sigma(\mathbf{p})^{-1}m_2$ existe et égale $X_U^\theta - Y_U^\theta$, ce qui termine la preuve (la vraie démonstration est bien sûr plus précise et donne en fait un contrôle sur tout $[0, 1]$ plutôt qu'à travers une seule variable U).

Au cours de la démonstration, nous prouvons un résultat intéressant sur la convergence du processus de contour des \mathbf{p} -arbres. Rappelons que nous pouvons construire un \mathbf{p} -arbre $\mathcal{T}_o^{\mathbf{p}}$ qui est ordonné (ceci revient à prendre un \mathbf{p} arbre $\mathcal{T}^{\mathbf{p}}$ et à mettre chaque ensemble d'enfants de chaque sommet en ordre échangeable sachant $\mathcal{T}^{\mathbf{p}}$). On peut lui associer son processus de contour $(H_s(\mathcal{T}_o^{\mathbf{p}}), 0 \leq s \leq 1)$, où le poids associé au sommet i est, ou bien sa \mathbf{p} -masse p_i , ou bien, ce qui peut sembler plus naturel, la masse uniforme $1/n$.

Théorème 1.10 :

Supposons que \mathbf{p} (classée par ordre décroissant pour plus de facilité) vérifie (1.6) avec $\theta_I > 0, \theta_{I+1} = 0$ pour un $I \geq 0$. Alors, sous des hypothèses techniques (pas de \mathbf{p} -masses exponentiellement petites devant $\sigma(\mathbf{p})$, et hypothèse de concentration des petites \mathbf{p} -masses autour de $\sigma(\mathbf{p})^2$), on a la convergence en loi suivante pour la topologie de Skorokhod (ou de la norme uniforme) :

$$\sigma(\mathbf{p})H(\mathcal{T}_o^{\mathbf{p}}) \xrightarrow[n \rightarrow \infty]{} H^\theta.$$

Il est à noter que ce théorème n'est pas vrai en toute généralité, c'est-à-dire qu'on peut trouver \mathbf{p} satisfaisant (1.6) avec $\theta_I > 0, \theta_{I+1} = 0$ mais tels que l'on n'ait pas convergence de $\sigma(\mathbf{p})H(\mathcal{T}_o^{\mathbf{p}})$ vers H^θ dans l'espace de Skorokhod (voir [14]), du fait de la présence de trop nombreuses valeurs de \mathbf{p} "minuscules", qui s'empilent et forment des "pics" d'aire négligeable mais de hauteur non-négligeable devant $\sigma(\mathbf{p})^{-1}$. On montre en revanche que l'on a toujours convergence en loi pour une topologie un peu plus forte que la convergence L^1 des processus, dont nous reparlons dans la section suivante.

1.2.6 Propriétés asymptotiques des applications aléatoires

Ce second travail avec Aldous et Pitman [6] met en relation les résultats de convergence des \mathbf{p} -arbres vers les ICRT (en fait seulement le CRT brownien ici, le cas général sera traité dans [8]) avec les propriétés asymptotiques quand $n \rightarrow \infty$ d'une application de $[n]$ dans lui-même, prise au hasard avec une certaine probabilité. Nous prouvons ainsi de façon "conceptuelle" un résultat dû à Aldous et Pitman [9], datant de 1994, mais dont la preuve est technique, peu visuelle, et difficilement généralisable.

À toute application $m : [n] \rightarrow [n]$ est associé son graphe orienté, dont les flèches sont $i \rightarrow m(i)$. Un point i est dit *cyclique* s'il existe $k \geq 1$ tel que $m^k(i) = i$, où m^k désigne la k -ième itérée de m . Soit $\mathcal{C}(m)$ l'ensemble des points cycliques de m , notons que pour tout i , $m^k(i) \in \mathcal{C}(m)$ pour k assez grand. On définit de plus une relation d'équivalence par $i \sim j$ si et seulement s'il existe k, k' tels que $m^k(i) = m^{k'}(j)$. Les classes associées à cette relation sont appelées *bassins d'attraction* de m . Si on ordonne ces bassins $\mathcal{B}_1(m), \dots, \mathcal{B}_k(m)$ d'une certaine façon, on découpe alors $\mathcal{C}(m)$ en cycles disjoints $\mathcal{C}_i(m) = \mathcal{B}_i(m) \cap \mathcal{C}(m)$. On voit enfin que si l'on efface les arêtes reliant les points cycliques entre eux, chaque point cyclique $c \in \mathcal{C}(m)$ est la racine d'un arbre² (où chaque sommet pointe vers la racine). On note \mathcal{T}_c cet arbre.

Nous nous intéressons au modèle aléatoire suivant d'application de $[n]$ dans $[n]$: soit \mathbf{p} une probabilité sur $[n]$ avec $p_i > 0$ pour $1 \leq i \leq n$, on définit M en associant à chaque $i \in [n]$ un point $j \in [n]$ tiré selon \mathbf{p} , indépendamment quand i varie. Appelons M la \mathbf{p} -application. Lorsque $\mathbf{p} = (1/n, \dots, 1/n)$ est la probabilité uniforme, M est l'application uniforme de $[n]$ dans $[n]$ parmi les n^n possibles. Pour analyser les propriétés asymptotiques de M (taille des bassins, des cycles, diamètre $\sup_{j \in [n]} \inf\{k \geq 1 : m^k(j) \in \{j, m(j), \dots, m^{k-1}(j)\}\}$), on va associer à M une marche aléatoire. On pourrait le faire d'une façon déterministe, mais il existe une façon aléatoire qui nous sera utile. Soit X_2, X_3, \dots une suite de variables indépendante de M , de même loi \mathbf{p} . On ordonne les bassins et les cycles dans l'ordre de découverte par la suite X_2, \dots , c'est-à-dire, $\mathcal{B}_1(M)$

²Les combinatoristes disent que l'espèce des applications est la composée de celle des forêts étiquetées enracinées et de celle des permutations (on regroupe les arbres de la forêt en cycles d'arbres disjoints).

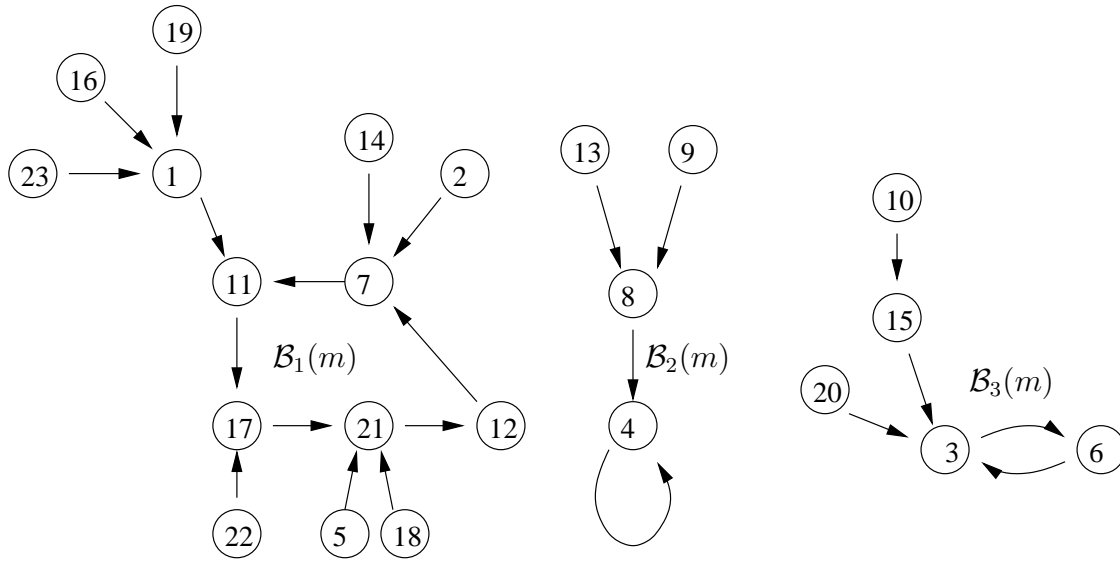


Fig. 1.2 – Le graphe orienté d'une application de [23] sur lui-même

est le bassin qui contient X_2 , on note $\tau_1 = 2$, et connaissant τ_i et $\mathcal{B}_1(M), \dots, \mathcal{B}_i(M)$ avec $\bigcup_{1 \leq j \leq i} \mathcal{B}_j(M) \neq [n]$, on pose

$$\tau_{i+1} = \inf \left\{ j \geq \tau_i : X_j \notin \bigcup_{1 \leq j \leq i} \mathcal{B}_j(M) \right\},$$

et on note $\mathcal{B}_{i+1}(M)$ le bassin contenant $X_{\tau_{i+1}}$. Pour chaque bassin non-vide $\mathcal{B}_i(M)$, $1 \leq i \leq k$, notons c^i le point cyclique tel que \mathcal{T}_{c^i} contient X_{τ_i} . Au sein du cycle $\mathcal{C}_i(M)$, on place les points cycliques dans l'ordre $M(c^i), M^2(c^i), \dots, M^{k_i-1}(c^i), c^i$, où $k_i = \text{Card}(\mathcal{C}_i(M))$. De concert avec l'ordre sur les cycles, ceci fournit un ordre sur tous les points cycliques, que nous notons à présent c_1, c_2, \dots, c_k (ainsi $c_1 = M(c^1), \dots, c_{k_1} = c^1, c_{k_1+1} = M(c^2)$, etc). Pour chaque i , on transforme l'arbre étiqueté \mathcal{T}_{c_i} en un arbre ordonné $\mathcal{T}_{c_i}^o$ en mettant chaque ensemble d'enfants de chaque sommet dans un ordre aléatoire uniforme conditionnellement à \mathcal{T}_{c_i} . On peut alors associer à $\mathcal{T}_{c_i}^o$ son processus de contour $H(\mathcal{T}_{c_i}^o)$, où le poids associé à i est sa \mathbf{p} -masse p_i . On pose alors, pour $0 \leq s < 1$,

$$H_s(M) = H_{s - \sum_{j < i} p(\mathcal{T}_{c_j})}(\mathcal{T}_{c_i}) \quad \text{où} \quad \sum_{j < i} p(\mathcal{T}_{c_j}) \leq s < \sum_{j \leq i} p(\mathcal{T}_{c_j}),$$

et $H_1(M) = H_{1-}(M)$. En d'autres termes, $H(M)$ est la concaténation des marches $H(\mathcal{T}_{c_i})$, $1 \leq i \leq k$. Par ailleurs, on ajoute des marques à cette marche, en posant $Z_0 = 0$, et $Z_i = \sum_{1 \leq j \leq i} p(\mathcal{B}_j(M))$. De la sorte, $Z_i - Z_{i-1} = p(\mathcal{B}_i(M))$, et la marche aléatoire "décrit" le i -ième bassin dans l'intervalle $[Z_{i-1}, Z_i)$. Enfin, pour garder la trace des points cycliques, nous notons $\ell_s(M)$ le nombre de points cycliques explorés par la marche $H(M)$ avant le temps s . Ceci est relié (mais non nécessairement égal) au nombre de paliers en 0 de $H(M)$ avant le temps s .

Par ailleurs, nous considérons un pont brownien réfléchi standard $B^{|br|}$, c'est-à-dire que $B^{|br|}$ est la valeur absolue d'un pont brownien standard. Nous notons $(L_s, 0 \leq s \leq 1)$ son

temps local en 0, normalisé de façon à être égal à la **moitié** de la version continue de la densité d'occupation de $B^{|br|}$ en 0. On marque également le pont brownien de la façon suivante : on prend U_1, U_2, \dots une suite de variables aléatoires uniformes indépendantes, indépendantes de $B^{|br|}$. On pose $D_0 = 0$ et, sachant (D_0, \dots, D_i) , $D_{i+1} = \inf\{s \geq D_i + (1 - D_i)U_i : B_s^{|br|} = 0\}$.

Enfin, nous définissons une topologie sur $\mathbb{D}[0, 1]$: nous disons que f_n converge vers $f \in C[0, 1]$ (il y a des subtilités si la limite n'est pas continue) pour la topologie $*$ si f_n converge vers f dans L^1 et si pour tous $a < b$, $\inf_{s \in [a, b]} f_n(s)$ converge vers $\inf_{s \in [a, b]} f(s)$. Nous pouvons maintenant énoncer le résultat principal.

Théorème 1.11 :

Supposons que \mathbf{p} vérifie le régime asymptotique (1.6) avec $\theta_0 = 1, \theta_2 = 0, \dots$. Alors on a convergence en loi pour la topologie $*$:

$$\sigma(\mathbf{p})H(M) \rightarrow 2B^{|br|}.$$

Cette convergence peut être transformée en une convergence pour la topologie de Skorokhod sous les hypothèses techniques supplémentaires évoquées au Théorème 1.10, par exemple, si \mathbf{p} est la loi uniforme sur $[n]$. De plus, conjointement à cette convergence, on a convergence en loi des marques (Z_1, Z_2, \dots) vers (D_1, D_2, \dots) . Enfin, on a également la convergence jointe dans l'espace de Skorokhod

$$\sigma(\mathbf{p})\ell(M) \rightarrow L.$$

En particulier, on a donc par exemple

$$(p(\mathcal{B}_i(M)), \sigma(\mathbf{p})\text{Card}(\mathcal{C}_i(M)), i \geq 1) \xrightarrow{d} (D_i - D_{i-1}, L_{D_i} - L_{D_{i-1}}, i \geq 1).$$

En revanche, il n'est pas toujours vrai sans hypothèses supplémentaires sur \mathbf{p} que, par exemple, $\sigma(\mathbf{p}) \sup_i \inf\{k : M^k(i) \in \mathcal{C}(M)\}$ converge en loi vers $\sup_{s \in [0, 1]} 2B_s^{|br|}$. Une motivation pour généraliser le résultat à des \mathbf{p} -applications provient de l'article de O'Connell et Pokrovski [85]. La méthode que nous utilisons peut se généraliser à des régimes asymptotiques pour \mathbf{p} plus généraux, la différence étant que l'on doit remplacer $B^{|br|}$ par un processus dérivé du processus d'exploration H^θ , de l'ICRT ce que nous expliquons dans un travail en préparation [8].

L'argument central repose sur une bijection due à Joyal [64] entre les applications de $[n]$ dans $[n]$ et les arbres étiquetés doublement enracinés, c'est-à-dire avec un deuxième sommet distingué (qui peut être aussi la racine). On voit d'ailleurs que ces deux ensembles ont le même cardinal par la formule de Cayley (c'est d'ailleurs un moyen de la démontrer). Nous modifions légèrement la bijection de Joyal pour en faire une "bijection aléatoire" qui associe la loi de la \mathbf{p} -application à celle du \mathbf{p} -arbre. Soit $\mathcal{T}^{\mathbf{p}}$ un \mathbf{p} -arbre, on note X_0 sa racine, et on prend X_1 de loi \mathbf{p} indépendamment de $\mathcal{T}^{\mathbf{p}}$ comme deuxième racine. On oriente les arêtes de façon à ce qu'elles pointent vers la racine. Nous appelons le chemin $X_0 = c_1, c_2, \dots, c_k = X_1$ entre les deux racines le *tronc (spine)* de $\mathcal{T}^{\mathbf{p}}$. Si l'on efface les arêtes entre les sommets du tronc, on obtient des arbres enracinés en chaque c_i , notons-les \mathcal{T}_{c_i} . Soit ensuite X_2, X_3, \dots une suite iid indépendante de $\mathcal{T}^{\mathbf{p}}$, de loi commune \mathbf{p} . On pose c^1 l'unique c_j tel que $X_2 \in \mathcal{T}_{c_j}$, et $\tau_1 = 2$. On construit un premier bassin \mathcal{B}_1 en coupant l'arête du tronc qui pointe vers c^1 , en inversant le sens des arêtes du tronc en dessous de c^1 , et en créant une nouvelle arête $c^1 \rightarrow c_1$. Ensuite, connaissant $\mathcal{B}_1, \dots, \mathcal{B}_j$, et τ_j , on pose

$\tau_{j+1} = \inf\{r \geq \tau_j : X_r \notin \bigcup_{1 \leq l \leq j} \mathcal{B}_l\}$, c^{j+1} l'unique c_r tel que $X_{\tau_{j+1}} \in \mathcal{T}_{c_r}$. On construit \mathcal{B}_{j+1} en coupant l'arête du tronc au-dessus de c^{j+1} , en inversant le sens des arêtes du tronc entre le sommet $c_{r(j)}$ du tronc qui pointait vers c^j et c^{j+1} et en créant une arête $c^{j+1} \rightarrow c_{r(j)}$. On s'arrête après l'étape où $c^{j+1} = X_1$. Le graphe orienté que l'on obtient définit une application de $[n]$ dans $[n]$, et on montre que c'est une \mathbf{p} -application, avec un ordonnement des bassins de même loi que lors de la construction ci-dessus.

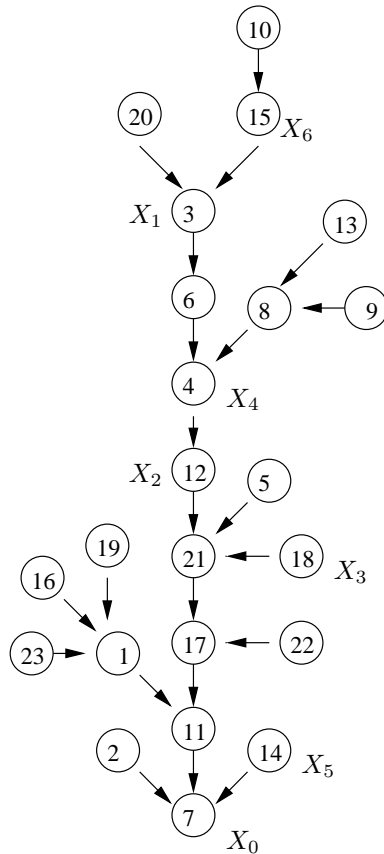


Fig. 1.3 – Un arbre doublement enraciné et une suite X_2, \dots donnant l'application de la Fig. 1.2 par la bijection de Joyal

Nous savons donc construire une \mathbf{p} -application à l'aide d'un \mathbf{p} -arbre et de sommets pris au hasard dans cet arbre. Nous en déduisons une fonction qui transforme la marche de contour du \mathbf{p} -arbre ordonné $\mathcal{T}_o^{\mathbf{p}}$ en la marche $H(M)$, et qui associe les marques à des variables indépendantes U_1, U_2, U_3, \dots uniformes sur $[0, 1]$ (on code X_i à l'aide de U_i). On vérifie que cette fonction a la propriété de continuité nécessaire pour passer à la limite, et enfin qu'elle transforme bien l'excursion brownienne en un pont brownien réfléchi (rappelons que $2B^{ex}$ est la limite de la marche associée à $\mathcal{T}_o^{\mathbf{p}}$ sous les hypothèses du Théorème 1.11).

Chapter 2

Ordered additive coalescent and fragmentations derived from Lévy processes with no positive jumps

Contents

2.1	Introduction	40
2.2	Ordered additive coalescent	41
2.2.1	Finite-state case	41
2.2.2	Bridge representation	43
2.2.3	The ballot Theorem	46
2.2.4	Infinite state case : fragmentation with erosion	48
2.3	The Lévy fragmentation	51
2.3.1	Lévy processes with no positive jumps	52
2.3.2	The fragmentation property	53
2.4	The fragmentation semigroup	55
2.4.1	Densities for the jumps of a subordinator	56
2.4.2	A useful process	57
2.4.3	Proof of Theorem 2.2	59
2.5	The left-most fragment	59
2.6	The mixing of extremal additive coalescents	61
2.7	Proof of Vervaat's Theorem	63
2.7.1	Unbounded variation case	64
2.7.2	Bounded variation case	66
2.8	Concluding remarks	70

2.1 Introduction

This paper is centered on the study of the additive coalescent which has been constructed, as a large class of coalescent Markovian processes, by Evans and Pitman in [52]. They describe the dynamics of a system of clusters with finite total mass, in which pairs of clusters merge into one bigger cluster at a rate given by the sum of the two masses (this is made rigorous by giving the associated Lévy system). They also present a construction of Markovian “eternal” coalescent processes on the whole real line which are starting from some infinitesimal masses.

One standard way to study such processes is to consider the dual fragmentation processes that split each cluster into smaller clusters. Coalescent processes are then obtained by appropriate time-reversal. Aldous and Pitman, successively in [10] and [12], have given a construction of the standard additive coalescent, which is the limit as n tends to ∞ of a coalescent process starting at time $-\frac{1}{2}\log(n)$ with n clusters of masses $1/n$, by time-reversing a fragmentation process obtained by logging Aldous’ CRT (*Continuum Random Tree*) by a certain family of Poisson processes on its skeleton. They also constructed more general fragmentation processes by using inhomogeneous generalizations of the CRT, and gave the exact entrance boundary of the additive coalescent. Bertoin [21, 22] gave a different approach to obtain the same fragmentation processes, by using partitions of intervals induced by some bridges with exchangeable increments (the Brownian bridge in the case of the standard additive coalescent, giving the “Brownian fragmentation”).

Our goal in this paper is to investigate the fragmentation processes that can be associated to Lévy processes with no positive jumps in a way similar to the Brownian fragmentation. One main motivation for studying this “path representation” approach is that it induces a particular ordering on the coalescing clusters that is not seen directly with the CRT construction. We study this order in section 2.2 in the most general setting, and construct the so-called *ordered additive coalescent*. Our study is naturally connected to the so-called fragmentation with erosion ([12, 22]), which in turn is related to non-eternal coalescents. Then we study the Lévy fragmentation processes in sections 2.3 and 2.4 in the case where the Lévy process has unbounded variation, and we give in particular the law of the process at a fixed time $t \geq 0$, which is interpreted as a conditioned sequence of jumps of some subordinators, generalizing a result in [10]. We also give in section 2.5 a description of the left-most fragment induced by the ordering on the real line, which is similar to [10] and [21]. Then, we show that the fragmentation processes associated to Lévy processes are up to a proper time-reversal eternal additive coalescents, and we study the mixing of the extremal eternal additive coalescent appearing in the time-reversed process in section 2.6.

For this, we rely in particular on the ballot theorem (Lemma 2.2), which Schweinsberg [100] has also used in a similar context. We also use a generalization of Vervaat’s Theorem (Proposition 2.1) for the Brownian bridge, which is proved in section 2.7.

As a conclusion, we make some comments on the case where the Lévy process has bounded variation in section 2.8.

2.2 Ordered additive coalescent

There are various ways to construct additive coalescent processes. Among them, we may recall three rather different approaches :

- The construction by Evans and Pitman [52] of the additive coalescent semigroup and Lévy system, when the coalescent takes values either in the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$ or in $S^\downarrow = \{x_1 \geq x_2 \geq \dots \geq 0, \sum x_i = 1\}$.
- The description of *eternal* additive coalescent on the whole real line \mathbb{R} by Aldous and Pitman [10, 12], by time-reversing a fragmentation process obtained by splitting the skeleton of some continuum random tree.
- The representation by Bertoin [22] of these fragmentation processes by path transformation on bridges with exchangeable increments and excursion-type functions.

We may also mention the method of Chassaing and Louchard [41], which is quite close to Bertoin's and is based on *parking schemes*.

In this paper, we shall mostly focus on the third method. To begin with, we stress that in contrast to the first two constructions, the third method naturally induces a puzzling natural order on the fragments (which are sub-intervals of $[0, 1]$ with total length 1). Indeed, one may wonder for instance why there should be a fragment (the "left-most fragment", see [21]) that always coalesces by the left? We shall answer this question by constructing a more precise process we call *ordered additive coalescent*, in which the initial coalescing fragments can merge in different ways (and which is *not* the same as the ranked coalescent of [52]). This study is naturally connected to the "fragmentation with erosion".

2.2.1 Finite-state case

First recall the dynamics of the additive coalescent starting from a finite number of clusters with masses $m_1, m_2, \dots, m_n > 0$ (we sometimes designate the clusters by their masses even if there may be some ambiguity). We stress that the sequence $\mathbf{m} = (m_1, \dots, m_n)$ need not be ranked in decreasing order. For each pair of indices (i, j) with $i < j$ let e_{ij} be an exponential r.v. with parameter $m_i + m_j$ (we say that $\kappa(x, y) = x + y$ is the *additive coalescent kernel*). Suppose that these variables are independent. At time $e = \inf_{i < j} e_{ij}$ the clusters with masses m_{i_0} and m_{j_0} merge into a unique cluster of mass $m_{i_0} + m_{j_0}$ where $i_0 < j_0$ are the a.s. unique integers such that $e = e_{i_0 j_0}$. Then the system evolves in the same way, and it stops when only one cluster with mass 1 remains. In the sequel, we will always suppose for convenience that $m_1 + m_2 + \dots + m_n = 1$. The general case follows under some change of the time scale.

These dynamics do not induce any ordering on the clusters. We now introduce a natural order which is closely related to the *additive* property of the coalescent kernel. Indeed, one can view each variable e_{ij} as the minimum of two independent exponential variables e_{ij}^i and e_{ij}^j with respective parameters m_i and m_j . The first time e of coalescence corresponds to some e_{ij}^k with $k \in \{i, j\}$, meaning that clusters i and j have merged at time e , and then we decide to say that cluster k has absorbed the other one. Alternatively, the system evolves as if we took n exponential variables e_1, \dots, e_n with respective rates $(n-1)m_1, \dots, (n-1)m_n$ and by declaring that, at time $e_{i^*} = \min_{1 \leq i \leq n} e_i$, the cluster labeled i^* absorbs one of the

$n - 1$ other clusters which is picked at random uniformly. We are going to make this more precise in the sequel.

First we introduce a more “accurate” space. Call *total order* on a set M a subset O of $M \times M$ which satisfies the following properties

1. If $(i, j) \in O$ and $(j, k) \in O$ then $(i, k) \in O$.
2. $\forall i \in M, (i, i) \in O$.
3. $\forall i, j \in M, (i, j) \in O$ or $(j, i) \in O$.
4. If $(i, j) \in O$ and $(j, i) \in O$ then $i = j$.

A *partial order* on M is a subset of $M \times M$ that satisfies only properties 1, 2 and 4. For any partial order O on M and any subset $M' \subset M$, we can define the restriction of O to M' which is the intersection of O with $M' \times M'$.

Let \mathcal{O}_∞ (resp. $\mathcal{O}_n, n \geq 1$) be the set of all partial orders O on \mathbb{N} (resp. $\mathbb{N}_n = \{1, \dots, n\}$) such that

$$iRj \iff (i, j) \in O \text{ or } (j, i) \in O$$

defines an equivalence relation. This is equivalent to saying that there exists a partition w of \mathbb{N} (resp. \mathbb{N}_n) such that the restrictions of O to the blocks of w are total orders, and that two integers in disjoint blocks of w are not comparable. We may also write O uniquely in the form $(w, (O_I)_{I \in w})$ where w is a partition of \mathbb{N} (resp. \mathbb{N}_n) and for each $I \in w, O_I$ is a total order on I (the restriction of O to I). We call the O_I 's *clusters*, and by abuse of notation, by $\mathcal{I} \in O$ we mean that \mathcal{I} is a cluster of O . For O in \mathcal{O}_n ($1 \leq n \leq \infty$) and $1 \leq k \leq n$ we call k_O the cluster in which k is appearing.

If O is in \mathcal{O}_n ($n \leq \infty$), we define for any pair $(\mathcal{I}, \mathcal{J})$ of distinct clusters in O the order $O_{\mathcal{I}\mathcal{J}} \in \mathcal{O}_n$ which has the same clusters as O except that the clusters \mathcal{I} and \mathcal{J} merge into $\mathcal{I}\mathcal{J}$ where

$$(i, j) \in \mathcal{I}\mathcal{J} \iff \begin{cases} (i, j) \in \mathcal{I} \text{ or} \\ (i, j) \in \mathcal{J} \text{ or} \\ i \in \pi(\mathcal{I}) \text{ and } j \in \pi(\mathcal{J}) \end{cases}$$

and π is the projection on the first coordinate axis. Let also

$$S_1^{+,n} = \{\mathbf{m} = (m_1, \dots, m_n), m_i > 0 \forall 1 \leq i \leq n, \sum_{i=1}^n m_i = 1\}$$

and

$$S_1^{+,\infty} = \{\mathbf{m} = (m_1, m_2, \dots), m_i > 0 \forall i \geq 1, \sum_{i=1}^{+\infty} m_i = 1\}.$$

Last, for $n \leq \infty, \mathbf{m} \in S_1^{+,n}$ and $O \in \mathcal{O}_n$, we call *mass of cluster* $\mathcal{I} \in O$ the number $m_{\mathcal{I}} = \sum_{i \in \pi(\mathcal{I})} m_i$.

For $n \in \mathbb{N}$, we now describe the dynamics of the so-called \mathcal{O}_n -additive coalescent with “proto-galaxy masses” $\mathbf{m} = (m_1, \dots, m_n) \in l_1^{+,n}$. Let $O \in \mathcal{O}_n$ be the current state of the process, and $\#O \geq 2$ the number of clusters. Consider n exponential r.v.'s $(e_k)_{1 \leq k \leq n}$ with respective parameters $(m_k)_{1 \leq k \leq n}$. There is a.s. a unique k^* such that $e_{k^*} = \min_{1 \leq k \leq n} e_k$. At time $e_{k^*}/(\#O - 1)$, which is exponential with parameter $\#O - 1$, cluster k^*_O merges with

one of the $\#O - 1$ other clusters \mathcal{I}^* picked at random uniformly. The state of the process then turns to $O_{k_O^* \mathcal{I}^*}$, and the system evolves similarly until only one cluster remains. The dynamics of the process of the ranked sequence of the clusters' masses are then the same as the additive coalescent described above : it is easily seen that two clusters $\mathcal{I}, \mathcal{J} \in O$ merge together at rate $m_{\mathcal{I}} + m_{\mathcal{J}}$. Indeed, $P[k_O^* = \mathcal{I}] = P[k^* \in \mathcal{I}] = m_{\mathcal{I}}$. We call $\mathbb{P}_O^{\mathbf{m}}$ the law of the process with initial state O .

We say that the cluster k_O^* *absorbs* cluster \mathcal{I}^* (more generally we designate the order in every cluster \mathcal{I} by the binary relation “ i absorbs j ”). When the system stops evolving, it is constituted of a single cluster O_∞ with mass $\sum m_i$, which is a total order on \mathbb{N}_n . There is always a left-most fragment (here we call fragment any integer) $\min O_\infty$, which is the fragment that has absorbed all the others. Notice that the process is increasing in the sense of the inclusion of sets.

Remarks. • This construction has to be compared with Construction 3 in [52], but where the system keeps the memory of the orders of coalescence by labeling the edges of the resulting tree in their order of appearance. It would be interesting to give a description of a limit labeled tree in an asymptotic regime such as in [12].

- It is immediate that, when ignoring the ordering, the evolution of the cluster masses starting at m_1, \dots, m_n is a finite-state additive coalescent evolution (in fact, the ordered coalescent is not so different of the classical coalescent, it only contains an extra information at each of the coalescence times). If we had replaced the time $e_{k^*}/(\#O - 1)$ of first coalescence by e_{k^*} above, the evolution of the clusters' masses would give the *aggregating server* evolution described in [22].

- One may notice that this way of ordering the clusters can be seen as a particular case of Norris [84] who studies coagulation equations by finite-state Markov processes approximation, and where clusters may coagulate in different manners depending on their shapes. In this direction, the “shape” of a cluster is simply its order.

2.2.2 Bridge representation

We now give a representation of the ordered coalescent process by using aggregative server systems coded by bridges with exchangeable increments as in [22]. Let $n < \infty$ and \mathbf{m} be in $S_1^{+,n}$.

Let $b_{\mathbf{m}}$ be the bridge with exchangeable increments on $[0, 1]$ defined by

$$b_{\mathbf{m}}(s) = \sum_{i=1}^n m_i (\mathbb{1}_{\{U_i \leq s\}} - s), \quad 0 \leq s \leq 1, \quad (2.1)$$

where the $(U_i)_{1 \leq i \leq n}$ are independent uniform r.v.'s on $[0, 1]$.

Definition :

Let f be a *bridge* in the Skorohod space $\mathbb{D}([0, \ell])$, $\ell > 0$, i.e. $f(0) = f(\ell) = f(\ell-) = 0$. Let x_{\min} be the location of the right-most minimum of f , that is, the largest x such that $f(x-) \wedge f(x) = \inf f$. We call *Vervaat transform* of f , or *Vervaat excursion* obtained from f , the function $Vf \in \mathbb{D}([0, \ell])$ defined by

$$Vf(x) = f(x + x_{\min}[\text{mod } \ell]) - \inf_{[0, \ell]} f, \quad x \in [0, \ell]$$

and $Vf(\ell) = \lim_{x \rightarrow \ell-} Vf(x)$ (see Figure 2.1).

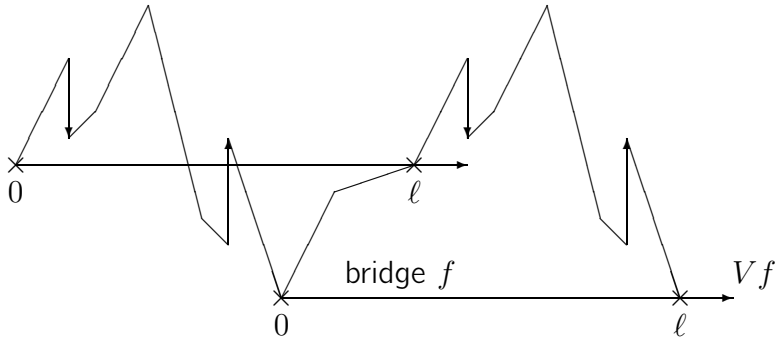


Figure 2.1: Vervaat's transform

Throughout this paper, the functions f we will consider will be sample path of some processes with exchangeable increments that attain a.s. their minimum at a unique location, so that we could have omitted to take the largest location of the minimum in the definition. See [71] for details.

Let $\varepsilon_{\mathbf{m}} = Vb_{\mathbf{m}}$ be the Vervaat excursion obtained from $b_{\mathbf{m}}$, and s_{\min} the location of the infimum of $b_{\mathbf{m}}$, which is a.s. unique by [71]. Let $V_i = U_i - s_{\min}[\text{mod } 1]$, $1 \leq i \leq n$ be the jump times of $\varepsilon_{\mathbf{m}}$, and remark that s_{\min} is itself one of the U_i 's. For $t \geq 0$ and $0 \leq s \leq 1$, let $\varepsilon_{\mathbf{m}}^{(t)}(s) = -\varepsilon_{\mathbf{m}}(s) + ts$,

$$\bar{\varepsilon}_{\mathbf{m}}^{(t)}(s) := \sup_{0 \leq s' \leq s} (-\varepsilon_{\mathbf{m}}(s') + ts'), \quad 0 \leq s \leq 1,$$

its supremum process, and

$$J(t) := ([a_1(t), b_1(t)], [a_2(t), b_2(t)], \dots, [a_k(t), b_k(t)])$$

be the sequence of its intervals of constancy ranked in decreasing order of their lengths (see Figure 2.2). Let $\#(t) = k$ their number.

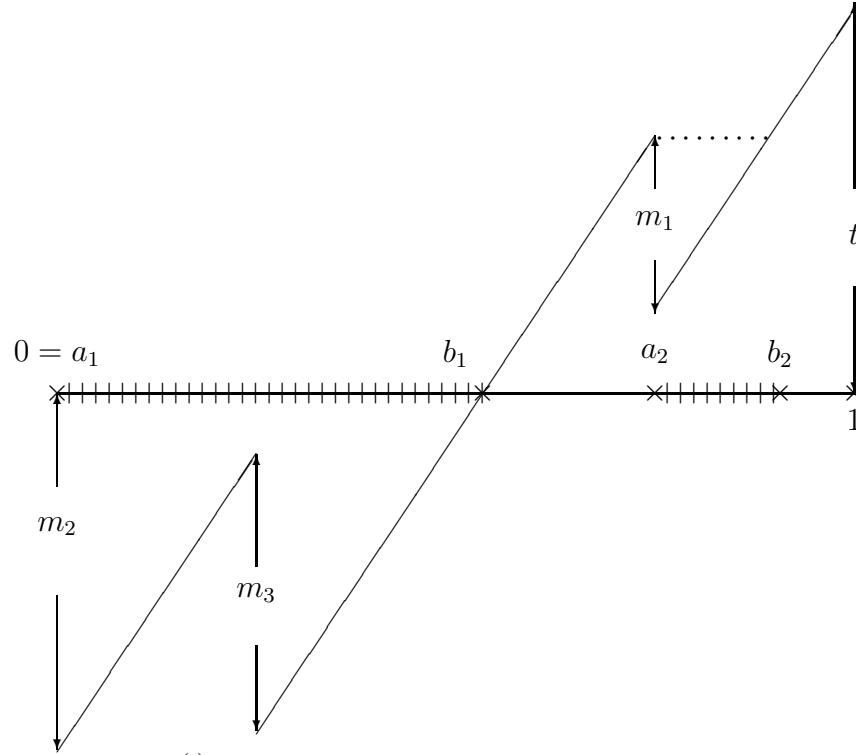
Let

$$F(t) = (1+t).(b_1 - a_1, \dots, b_{\#(t)} - a_{\#(t)}, 0, 0, \dots)$$

be the sequence of the corresponding lengths, renormalized by the proper constant so that their sum is 1 (that this constant equals $1+t$ is a consequence of the fact that $\varepsilon_{\mathbf{m}}^{(t)}$ has a slope $1+t$). Also, let $F(\infty) = (m_1, \dots, m_n)$, which is equal to $F(t)$ for t sufficiently large, a.s. Then by [22] the sequence of the distinct states of $(F(t))_{t \geq 0}$ is equal in law to the time-reversed sequence of the distinct states of the additive coalescent starting at (m_1, \dots, m_n) (it is in particular easy to observe that the terms in $F(t)$ are constituted of sums of subfamilies extracted from \mathbf{m}). On the other hand it is easy to see that every jump time V_j , $1 \leq j \leq n$ belongs to some $[a_i, b_i)$, and that the left bounds $(a_i(t), 1 \leq i \leq \#(t))$ are all equal to some V_j . Hence the $(V_j)_{1 \leq j \leq n}$ induce on the intervals of constancy a random order $O_{\varepsilon}^{\mathbf{m}}(t)$ at time t :

$$(i, j) \in O_{\varepsilon}^{\mathbf{m}}(t) \iff V_i \leq V_j \text{ and } V_i, V_j \in [a_k(t), b_k(t)] \text{ for some } k \leq \#(t).$$

We thus deduce a process $(O_{\varepsilon}^{\mathbf{m}}(t))_{t \geq 0}$ with values in \mathcal{O}_n . By convention, let $O_{\varepsilon}^{\mathbf{m}}(\infty)$ be the element of \mathcal{O}_n constituted of the singletons $\{1\}, \dots, \{n\}$. It is easy to see that it is indeed equal to $O_{\varepsilon}^{\mathbf{m}}(t)$ for t sufficiently large.


 Figure 2.2: $\varepsilon_{\mathbf{m}}^{(t)}$ and intervals of constancy of its supremum

Now we show how to recover the \mathcal{O}_n -coalescent with proto-galaxy masses m_1, \dots, m_n from the bridge $b_{\mathbf{m}}$. In [22] the bridge $b_{\mathbf{m}}$ is defined in a different way : if $(s_i)_{1 \leq i \leq n}$ are independent standard exponential r.v.'s, the jump times U'_1, \dots, U'_n of $b_{\mathbf{m}}$ are defined by $U'_{k+1} - U'_k [\text{mod } 1] = s_k / (s_1 + \dots + s_n)$ and U'_1 independent uniform on $[0, 1]$. At time U'_i , the bridge has a positive jump $m_{\sigma(i)}$ where σ is a uniform random permutation on \mathbb{N}_n . It is easy to see that the bridge defined in this way has the same law as $b_{\mathbf{m}}$. Let $A_n = s_1 + \dots + s_n$, which is independent of the bridge (since it is independent of the jump times). We then know from Propositions 1 and 2 of [22] that $(F(t^{-1}A_n - 1))_{0 \leq t < A_n}$ (with $F(+\infty) = (m_1, \dots, m_n)$) is the aggregative server system described in the second remark of section 2.2.1.

Therefore, let $T(t)$ be the time-change defined as follows. For $0 \leq k \leq n$ let $t_k = \sup\{t \geq 0, \#(t) = n - k\}$ with $t_0 = \infty$, $t_n = 0$ and

$$A_k = \frac{A_n}{1 + t_k}$$

be the first time when $(F(t^{-1}A_n - 1))_{0 < t < A_n}$ has $n - k$ components. For $0 \leq i \leq n - 2$ let also

$$I(t) = \frac{t - A_i}{n - i - 1} + \sum_{j=0}^{i-1} \frac{A_{j+1} - A_j}{n - j - 1} \text{ if } t \in [A_i, A_{i+1}],$$

so that $I(t)$ defines a continuous increasing function on $[0, A_{n-1}]$ whose inverse is denoted by I^{-1} . Last, for $0 \leq t \leq I(A_{n-1})$ we set

$$T(t) = \frac{A_n}{I^{-1}(t)} - 1, \quad T(0) = \infty.$$

We then have the

Lemma 2.1 :

The process

$$O_t^m = \begin{cases} O_\varepsilon^m(T(t)) & \text{if } 0 \leq t \leq I(A_{n-1}) \\ O_\varepsilon^m(T(I(A_{n-1}))) & \text{if } t > I(A_{n-1}) \end{cases}$$

is an ordered additive coalescent in \mathcal{O}_n with proto-galaxy masses m_1, \dots, m_n .

// The discrete state-evolution has the proper law, since given that two clusters with mass m_i and m_j coalesce, the probability that the mass m_i is at the left of the mass m_j is $m_i/(m_i + m_j)$, which is obtained from the calculation of Proposition 2 in [22]. This proposition also shows that (A_1, \dots, A_n) are the n first jumps of a Poisson process with intensity 1. Together with the definition of $I(t)$ and $T(t)$, this implies that the time-evolution also has the appropriate law : when the system is constituted of k clusters, the time until the following coalescence is exponential with parameter $k - 1$. //

Remark. In particular, we easily get that the final order $O_{I(A_{n-1})}^m = O_\infty^m$ is defined by $(i, j) \in O_{I(A_{n-1})}^m \iff V_i \leq V_j$, where $(V_j)_{1 \leq j \leq n}$ is a cyclic permutation of the uniformly distributed $(U_j)_{1 \leq j \leq n}$ which is determined by the location of the minimum of the bridge b_m .

2.2.3 The ballot Theorem

Before examining ordered coalescent any further, we present a tool that will be seen to be very useful in the sequel.

The most famous version of the ballot theorem is doubtless its discrete version (see Takács [102]). When we pick the balls one by one without replacement in a box containing a red balls and b green balls, $a > b$, the probability that the red balls are always leading is $(a - b)/(a + b)$. There exists a continuous version, which can be deduced from the discrete one by approximation, as it is done in [102], for non-decreasing processes with derivative a.s. 0, with exchangeable increments. We give an alternative version.

Let $\ell > 0$. From [65] we know that every process b with exchangeable increments on $[0, \ell]$ with bounded variation may be represented in the form

$$b(x) = \alpha x + \sum_{i=1}^{\infty} \beta_i (\mathbb{1}_{\{x \geq U_i\}} - \frac{x}{\ell}), \quad 0 \leq x \leq \ell \tag{2.2}$$

where $(U_i)_{i \geq 1}$ is i.i.d. uniform on $[0, \ell]$, and $\alpha, \beta_1, \beta_2, \dots$ are (not necessary independent) r.v.'s which are independent of the sequence $(U_i)_{i \geq 1}$, and satisfy $\sum_{i=1}^{\infty} |\beta_i| < \infty$ a.s. We call it the *Kallenberg bridge* with drift coefficient α and jumps β_1, \dots

Lemma 2.2 (ballot Theorem) :

Suppose that the jumps β_1, \dots are negative. Then we have

$$\mathbb{P} \left[b(x) \geq 0 \ \forall x \in [0, \ell] \mid b(\ell), \beta_1, \beta_2, \dots \right] = \max \left(\frac{b(\ell)}{b(\ell) - \sum_{i=1}^{\infty} \beta_i}, 0 \right)$$

In particular, conditionally on $\sum_{i=1}^{\infty} \beta_i$ and $b(\ell)$, the event $\{b(x) \geq 0 \forall 0 \leq x \leq \ell\}$ is independent of the sequence of the jumps $(\beta_1, \beta_2, \dots)$.

// Denote $-\frac{\sum_{i=1}^{\infty} \beta_i}{\ell}$ by Σ . For each i we define the following process on $(0, \ell)$:

$$M_x^i = \mathbb{1}_{\{U_i \leq x\}},$$

so that

$$b(x) = (\alpha + \Sigma)x + \sum_{i=1}^{\infty} \beta_i M_x^i$$

Let $(\mathcal{F}_x)_{x \geq 0}$ be the filtration generated by all the processes $M_{(\ell-x)-}^i$ and enlarged with the variables α and β_i 's. Then the process

$$\frac{M_{(\ell-x)-}^i}{(\ell-x)}, \quad 0 \leq x \leq \ell$$

is a martingale with respect to this filtration, and if $\mathbf{M}_x = -\sum_{i=1}^{\infty} \beta_i M_x^i$, so is

$$\frac{\mathbf{M}_{(\ell-x)-}}{\ell-x}, \quad 0 \leq x \leq \ell$$

which has Σ for starting point. Moreover we remark that it tends to 0 at ℓ with probability 1, as a consequence of Theorem 2.1 (ii) in [66], with $f(t) = t$ (in other words, processes with exchangeable increments with no drift have a.s. a zero derivative at 0). The hypothesis that the jumps β_i are negative enables us to apply the optional sampling theorem which thus gives that, conditionally on the β_i 's and α , \mathbf{M}_x stays below $(\alpha + \Sigma)x$ on $(0, \ell)$ with probability $\alpha/(\alpha + \Sigma)$. The second assertion follows. //

As a consequence of this lemma, we obtain that when $\alpha = 0$ a.s., b attains its minimum at a jump time. Indeed, for $i \geq 0$, let $v_i b(x) = b(x + U_i[\text{mod } 1]) - b(U_i)$ the process obtained by splitting the bridge at U_i , and modified at time ℓ so that it is continuous at this time. Since the variables $U_j - U_i$ for $j \neq i$ are also uniform independent, it is easy to see that $v_i b$ is the Kallenberg bridge with jumps β_j , $j \neq i$ and drift coefficient β_i/ℓ . Lemma 2.2 implies that conditionally on β_i and on $\sum_{i=1}^{\infty} \beta_j$, it is positive (i.e. U_i is the location of the minimum of b , or also that $v_i b$ is equal to Vb) with probability $\beta_i/\sum_{j=1}^{\infty} \beta_j$. Since the sum of these probabilities is 1, we can conclude.

On the other hand, we obtain by a simple time-reversal on $[0, \ell]$ the same result for processes with exchangeable increments and positive jumps. Moreover, if b has **positive** jumps, we have the

Corollary 2.1 :

Conditionally on the sequence $(\beta_i, i \geq 1)$, the first jump of Vb is β_i with probability $\beta_i/\sum_{j=1}^{\infty} \beta_j$, that is, it has the law of a *size biased pick* from the sequence β_1, β_2, \dots

// It is immediate from our discussion above when considering $b(\ell - x)$, which has negative jumps. //

An important consequence is that the left-most fragment $\min O_{\infty}^{\mathbf{m}}$ is a **m**-size biased pick from $\{1, \dots, n\}$, that is

$$P[\min O_{\infty}^{\mathbf{m}} = i] = m_i.$$

2.2.4 Infinite state case : fragmentation with erosion

We now give a generalization of the previous results to additive coalescents with an infinite number of clusters. We will use approximation methods that are very close to [22], with the difference that the processes we are considering have bounded variation, which makes the approximations technically more difficult (in particular the functionals of trajectories such as the Vervaat's transform are not continuous).

It is conceptually easy to generalize the construction by Evans and Pitman [52] of partition valued additive coalescents. We are following the same approach by replacing the set of partitions \mathcal{P}_∞ by the set \mathcal{O}_∞ .

We endow \mathcal{O}_∞ with a topology as in [52] : first endow the finite set \mathcal{O}_n with the discrete topology. For $n \leq \alpha \leq \infty$, call π^n the function from \mathcal{O}_α to \mathcal{O}_n corresponding to the restriction to \mathbb{N}_n . Then the topology on \mathcal{O}_∞ is that generated by $(\pi^n)_{n \geq 1}$. It is a compact totally disconnected metrizable space, and the distance $d(O, O') = \sup_{n \geq 1} 2^{-n} \mathbb{1}_{\{\pi^n(O) \neq \pi^n(O')\}}$ induces the same topology.

We also denote by $(O_t)_{t \geq 0}$ the canonical process associated to càdlàg functions on \mathcal{O}_∞ . Last, for $O \in \mathcal{O}_\infty$ let $\mu^{\mathbf{m}}(O, \cdot)$ be the measure that places, for each pair of distinct clusters $(\mathcal{I}, \mathcal{J})$ mass $m_{\mathcal{I}}$ at $O_{\mathcal{I}\mathcal{J}}$. We wish to show that there exist for each $\mathbf{m} \in S_1^{+, \infty}$, a family of laws $(\mathbb{P}_O^{\mathbf{m}})_{O \in \mathcal{O}_\infty}$ such that

- If $O \in \mathcal{O}_\infty$ contains the cluster $(n, n+1, n+2, \dots)$ with any order on it (for example, the natural order on \mathbb{N}) for some $n \geq 1$, then $(\pi^n(O_t))_{t \geq 0}$ is under $\mathbb{P}_O^{\mathbf{m}}$ a \mathcal{O}_n -coalescent with proto-galaxy masses $\mathbf{m}^{[n]} = (m_1, \dots, m_{n-1}, \sum_{i=n}^\infty m_i)$ started at $\pi^n(O)$.
- Under $\mathbb{P}_O^{\mathbf{m}}$, the canonical process is a Feller process with Lévy system given by the jump kernel $\mu^{\mathbf{m}}$.
- The map $(\mathbf{m}, O) \mapsto \mathbb{P}_O^{\mathbf{m}}$ from $S_1^{+, \infty} \times \mathcal{O}_\infty$ to the space of measures on $\mathbb{D}(\mathbb{R}_+, \mathcal{O}_\infty)$, is weakly continuous (where we endow $S_1^{+, \infty}$ with the usual l_1 topology).

Again, we could state the result for more general \mathbf{m} 's (with finite sum $\neq 1$), but this would only require an easy time-change.

We claim that these properties follow from the same arguments as in [52]. Indeed, Theorem 10, Lemma 14 and Lemma 15 there still hold when \mathcal{P}_∞ (the space of partitions on \mathbb{N}) is replaced by the topologically very similar space \mathcal{O}_∞ . We have, however, to check that the construction of a "coupled family of coalescents" as in [52] (Definition 12 and Lemmas 14 and 15 there), still exists in our ordered setting. For this we adapt the arguments of Lemma 16 there: we use the same construction with the help of Poisson measures, but we do not neglect the orientation of the edges in the random birthday trees $\mathcal{T}(Y_j^{n, \mathbf{m}, O})$ we obtain in a similar way. In this way, we construct from these birthday trees ordered coalescents instead of the ordinary coalescents (see the first remark of section 2.2.1).

We will give a description of the \mathcal{O}_∞ -additive coalescent processes with the help of bridges extending that in section 2.2.2.

Let now \mathbf{m} be in $S_1^{+, \infty}$, and let $b_{\mathbf{m}}$ be the Kallenberg bridge with jumps m_1, m_2, \dots constructed from an i.i.d. sequence of uniform variables U_1, U_2, \dots in $[0, 1]$. Let U^* be the a.s. unique ([71]) location of the minimum of $b_{\mathbf{m}}$. We know from Lemma 2.2 that it is a jump time for $b_{\mathbf{m}}$, that is, $U^* = U_i$ for some i a.s. Let $\varepsilon_{\mathbf{m}}$ be the associated Vervaat excursion. Last, for $i \in \mathbb{N}$ let $V_i = U_i - U^*[\text{mod } 1]$ be the jump times of $\varepsilon_{\mathbf{m}}$.

Similarly as above let

$$\bar{\varepsilon}_{\mathbf{m}}^{(t)}(s) = \sup_{0 \leq s' \leq s} (ts - \varepsilon_{\mathbf{m}}(s')), \quad 0 \leq s \leq 1.$$

We denote by $([a_i(t), b_i(t)])_{i \geq 1}$ the intervals of constancy of $\bar{\varepsilon}_{\mathbf{m}}^{(t)}$. We may thus construct an \mathcal{O}_{∞} -valued process $(O_{\varepsilon}^{\mathbf{m}}(t))_{t \geq 0}$ which consists on the order induced by the V_j 's on each of the $[a_i(t), b_i(t)]$. Remark that every $a_i(t)$ corresponds to some U_j . Indeed, the bridge $b_{\mathbf{m}}$ leaves its local minima by a jump, otherwise by exchangeability of the increments the minimum would be attained continuously with positive probability. As a consequence, every cluster of $O_{\varepsilon}^{\mathbf{m}}(t)$ has a minimum, the "left-most fragment". Denote by O_{Sing} the element of \mathcal{O}_{∞} constituted of the singletons $\{1\}, \{2\}, \dots$, and let $O_{\varepsilon}^{\mathbf{m}}(\infty) = O_{\text{Sing}}$. Our claim is that

Theorem 2.1 :

The process

$$O_t^{\mathbf{m}} = O_{\varepsilon}^{\mathbf{m}} \left(\frac{e^{-t}}{1 - e^{-t}} \right) \quad t \geq 0$$

has law $\mathbb{P}_{O_{\text{Sing}}}^{\mathbf{m}}$.

Moreover, $O_t^{\mathbf{m}}$ has a limit $O_{\infty}^{\mathbf{m}} = O_{\varepsilon}^{\mathbf{m}}(0)$ at $+\infty$, and the left-most fragment $\min O_{\infty}^{\mathbf{m}}$ is a \mathbf{m} -size biased pick from \mathbb{N} .

// We prove this theorem by using a limit of the bridge representation of the ordered additive coalescent described in section 2.2.2 and the weak continuity properties of $P_{\mathcal{O}}^{\mathbf{m}}$. Recall that $(U_i)_{i \geq 1}$ are the jump times of $b_{\mathbf{m}}$. Let $b_{\mathbf{m}}^n$ be the bridge defined as in (2.1) with jumps $m_1/S_n, \dots, m_n/S_n$ where $S_n = \sum_{i=1}^n m_i$ and with jump times U_1, \dots, U_n . Let $\varepsilon_{\mathbf{m}}^n$ be the associated Vervaat excursion. Last, let $\bar{\varepsilon}_{\mathbf{m}}^{(n,t)}$ be the supremum process of $(ts - \varepsilon_{\mathbf{m}}^n(s))_{0 \leq s \leq 1}$. We will need the following technical lemma:

Lemma :

Almost surely, we may extract a subsequence of $(\varepsilon_{\mathbf{m}}^n)_{n \geq 1}$ which converges uniformly to $\varepsilon_{\mathbf{m}}$ as n goes to infinity.

// It is trivial that $b_{\mathbf{m}}^n$ converges uniformly to $b_{\mathbf{m}}$ since the jump times coincide (the bridges are built on the same U_i 's). To get the uniform convergence of the Vervaat excursions, it suffices to show that a.s. for n sufficiently large and up to the extraction of subsequences, the location of the minimum of the bridge $b_{\mathbf{m}}^n$, which is some jump time U_n^* , remains unchanged.

For $i \neq j \in \mathbb{N}$, consider the probability $p_{i,j}^n = \mathbb{P}[U_n^* = U_i, U^* = U_j]$. Fix i . Then for $j \neq i$, on $\{U^* = U_j\}$, there is a.s. some $\eta > 0$ such that $b_{\mathbf{m}}(U_i-) \geq b_{\mathbf{m}}(U_j-) + \eta$. Since we have uniform convergence of $b_{\mathbf{m}}^n$ to $b_{\mathbf{m}}$, this implies that $p_{i,j}^n$ tends to 0, and also

$$\forall \epsilon > 0, \forall k \in \mathbb{N}, \exists n_{\epsilon,k} \in \mathbb{N}, \forall n \geq n_{\epsilon,k}, \forall j \in \mathbb{N}_k, j \neq i, p_{i,j}^n \leq \frac{\epsilon}{2k}.$$

Next, independently of n let k be sufficiently large so that

$$\sum_{j=k+1}^{\infty} \mathbb{P}[U^* = U_j] = \sum_{j=k+1}^{\infty} m_j < \frac{\epsilon}{2},$$

where the equality is obtained from Corollary 2.1. In this case we obtain that for $\epsilon > 0$ and n large,

$$\sum_{j \neq i} p_{i,j}^n = \mathbb{P}[U_n^* = U_i \neq U^*] < \epsilon.$$

By dominated convergence, this implies that

$$\lim_{n \rightarrow \infty} \mathbb{P}[U_n^* \neq U^*] = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \mathbb{P}[U_n^* = U_i \neq U^*] = 0,$$

since

$$\mathbb{P}[U_n^* = U_i \neq U^*] \leq \mathbb{P}[U_n^* = U_i] = \frac{m_i}{m_1 + \dots + m_n} \leq \frac{m_i}{m_1}.$$

Where the last equality above is also obtained from Corollary 2.1. From this we deduce that up to extraction of a subsequence, $U_n^* = U^*$ for n sufficiently large. $\parallel\parallel\parallel$

Next we associate to each integer n an ordered additive coalescent process $(O_t^n)_{t \geq 0}$ taking values in \mathcal{O}_∞ as follows. Let $([a_i^n(t), b_i^n(t)])_{1 \leq i \leq \#(n,t)}$ be the intervals of constancy of $\bar{\varepsilon}_{\mathbf{m}}^{(n,t)}$ (the process defined as $\bar{\varepsilon}_{\mathbf{m}}^{(t)}$, but for the bridge $b_{\mathbf{m}}^n$). We know from section 2.2.2 how to obtain an ordered coalescent in \mathcal{O}_n from $b_{\mathbf{m}}^n$, with proto-galaxy masses m_1, \dots, m_n , and starting from the singletons $\{1\}, \dots, \{n\}$ by a proper time-change $T_n(t)$ from the process $(O_\varepsilon^n(t))_{t \geq 0}$, with obvious notations (each T_n requires the choice of a variable $A_n^{(n)}$ with law $\text{Gamma}(1, n)$, so we take independent variables $(A_n^{(n)})_{n \in \mathbb{N}}$ with the proper distributions). We now just turn $\{n\}$ into the cluster $(n, n+1, \dots)$ with the natural order induced by \mathbb{N} in the initial state, and assign mass $m_n/2^{i+1}$ to the integer $n+i$. We thus obtain a \mathcal{O}_∞ coalescent with proto-galaxy masses $m_1, \dots, m_{n-1}, m_n/2, m_n/4, \dots$. This last sequence converges in l_1 norm to \mathbf{m} . Hence the coalescent converges in law to the ordered coalescent starting from all singletons, with proto-galaxy masses \mathbf{m} , in virtue of the weak continuity property for $(\mathbb{P}_{\mathcal{O}}^{\mathbf{m}})_{\mathbf{m} \in S_1^{+, \infty}, \mathcal{O} \in \mathcal{O}_\infty}$.

Now, we have a.s. that, if $[a, b]$ is an interval of constancy for the process $(\bar{\varepsilon}_{\mathbf{m}}^{(t)}(s))_{0 \leq s \leq 1}$, then for every $s \in]a, b[$,

$$\max(\bar{\varepsilon}_{\mathbf{m}}^{(t)}(s-), \bar{\varepsilon}_{\mathbf{m}}^{(t)}(s)) < \bar{\varepsilon}_{\mathbf{m}}^{(t)}(a).$$

This is proved in [22], Lemma 7. This assertion combined with the uniform convergence in the previous lemma implies, up to extraction of subsequences, the pointwise convergence of $(a_i^n(t))_{i \geq 1}$ (resp. $(b_i^n(t))_{i \geq 1}$) to $(a_i(t))_{i \geq 1}$ (resp. $(b_i(t))_{i \geq 1}$) as n goes to infinity, for every $i \geq 0$. Moreover, this gives that the b_i 's are not equal to some of the V_j 's (else the process would jump at the end of an interval of constancy of its supremum process, which would be absurd).

More precisely, $a_i^n(t) \leq a_i(t)$ holds for n sufficiently large, for every t . Indeed, the process $(ts - \varepsilon_{\mathbf{m}}(s))_{0 \leq s \leq 1}$ jumps at $a_i(t) = V_j$ for some j and

$a_i^n(t)$ tends to $a_i(t)$ so that if $a_i^n(t)$ was greater than $a_i(t)$ for arbitrarily large n , $a_i^n(t)$ would not be the beginning of an interval of constancy for $\bar{\varepsilon}_m^n$, for some $n \geq j$.

From this we deduce that (still up to extraction) $(O_\varepsilon^n(t))_{t \geq 0}$ converges in \mathcal{O}_∞ to $(O_\varepsilon^m(t))_{t \geq 0}$ in the sense of finite-dimensional distributions. Indeed, it suffices to show that for every $m \geq 0$ the restriction of the order $O_\varepsilon^n(t)$ to \mathbb{N}_m is equal to the restriction of the order $O_\varepsilon^m(t)$ for n sufficiently large. For this it suffices to choose n such that $[a_i^n(t), a_i(t)]$ does not contain a V_j with label $j \leq m$, and that such V_j 's does not fall between any $b_i^n(t)$ and $b_i(t)$. This is possible according to the above remarks. For such n the orders induced by the V_j 's in each $[a_i(t), b_i(t)]$, and restricted to \mathbb{N}_m are the same.

Similarly, we have that the process $(O_\varepsilon^m(t))_{t \geq 0}$ is continuous in probability at every time t . For this we use the fact that for every i , $a_i(t') = a_i(t)$ a.s. for t' close to t , which is a consequence of [66] Theorem 2.1 (ii) for $f(t) = t$. Indeed, this theorem shows that the bridge b^m has a derivative equal to -1 at 0, and hence by exchangeability b^m has a left derivative equal to -1 at any jump time since we would not lose the exchangeability by suppressing the corresponding jump.

Last, it is easily seen that the time-change $T_n(t)$ converges in probability to $e^{-t}/(1 - e^{-t})$. Together with the above, this ends the proof of the first assertion of the theorem. Together with Corollary 2.1, we get the second one. //

Remark also that the mass of the i -th heavier cluster is given by

$$\frac{b_i(s) - a_i(s)}{1 - e^{-t}} \quad \text{for } s = \frac{e^{-t}}{1 - e^{-t}} \quad (2.3)$$

and can be read directly on the intervals of constancy of $\bar{\varepsilon}_m$. In the following studies, we will turn our study to these lengths of constancy intervals.

We conclude this section with a remark concerning the so-called *fragmentation with erosion* ([22, 12]) that typically appears in such ‘‘bounded variation’’ settings as the one in this part (b_m has bounded variation a.s.). The ‘‘erosion’’ comes from the fact that the total sum of the intervals of constancy of $\bar{\varepsilon}_m$ is less than 1. Yet, in our study, we have shown that the erosion is deterministic, and that when we compensate it by the proper multiplicative constant, we obtain after appropriate time-changing an additive coalescent starting at time 0 (at the opposite of eternal coalescent obtained in the infinite variation case that starts from time $-\infty$). We have not pursued it, but we believe that in the ICRT context in [12], the ICRT obtained in the equivalent ‘‘bounded variation’’ setting ($\sum \theta_i^2 = 1$ and $\sum \theta_i < \infty$ with the notations therein) is somehow equivalent to the birthday tree with probabilities m_1, m_2, \dots so that Poisson logging on its skeleton just gives a process which is somehow isomorphic to a non-eternal additive coalescent.

2.3 The Lévy fragmentation

We now turn to the study of fragmentation processes associated to Lévy processes. We are motivated by the fact that it is known from [21] and [22] how to obtain eternal coalescent

processes from certain types of bridges with exchangeable increments. Following a remark of Doney we may use the methods described in [21] in much more general context. For example, it is natural to wonder what kind of processes can be obtained in the same way from Lévy bridges, which are important examples of bridges with exchangeable increments. Moreover, the following is to be read at the light of the preceding section, which gives an interpretation of the ordering naturally induced on the fragments by their respective places on the real line. Our goal is to make “explicit” the law of the fragmentation process at a fixed time in terms of the hitting times process of the Lévy process.

We begin by giving the setting of our study, and by recalling some properties on Lévy processes with no positive jumps and the excursions of their reflected processes. Most of them can be found in [19]. From now on in this paper, X designates a Lévy process with no positive jumps.

2.3.1 Lévy processes with no positive jumps, bridges and reflected process

Let ν be the Lévy measure of X . We will make the following assumptions :

- X has no positive jumps.
- X does not drift to $-\infty$, i.e. (together with the above hypothesis) X has first moments and $E[X_1] \geq 0$.
- X has a.s. unbounded variation.

We set

$$X_s^{(t)} = X_s + ts,$$

and

$$\overline{X}_s^{(t)} = \sup_{0 \leq s' \leq s} X_{s'}^{(t)}$$

its supremum process (sometimes called “supremum” by a slight abuse). Let $T_x^{(t)}$ the first hitting time of x by $X^{(t)}$. Recall that the fact that $X^{(t)}$ has no positive jumps implies that the process of first hitting times defined by $T_x^{(t)} = \inf\{u \geq 0, X_u^{(t)} > x\}$ is a subordinator, and the fact that X , and then $X^{(t)}$, does not drift to $-\infty$ implies that $T^{(t)}$ is not killed. Moreover, $T^{(t)}$ is pure jump since $X^{(t)}$ has infinite variation. When $t = 0$ we will drop the exponent (0) in the notation.

The reason why we impose infinite variation is that we are going to study eternal coalescent processes. Nevertheless, we make some comments on the bounded variation case in section 2.8.

Last, we suppose for technical reasons that for all $s > 0$, the law of X_s has a continuous density w.r.t. Lebesgue measure. We call its density $q_s(x) = \mathbb{P}[X_s \in dx]/dx$. For comfort, we suppose that $q_s(x)$ is bi-continuous in $s > 0$ and $x \in \mathbb{R}$. This is not the weakest hypothesis that we can assume, but it makes some definitions clearer, and some proofs simpler (Vervaat’s Theorem,...). In particular, for every $\ell > 0$ we may define the law of the *bridge of X from 0 to 0 with length ℓ* as the limit

$$P_{0,0}^\ell(\cdot) = \lim_{\varepsilon \rightarrow 0} P^\ell[\cdot \mid |X_\ell| < \varepsilon], \quad (2.4)$$

where P^ℓ is the law of the process X stopped at time ℓ , see [55], [71].

Let us now present some facts about reflected processes and excursion theory.

We call reflected process (below its supremum) of X the process $\bar{X} - X$. The general theory of Lévy processes gives that this process is Markov, and since X has no positive jumps, the process \bar{X} is a local time at 0 for the reflected process, and T is the inverse local time process. This local time enable us to apply the Itô excursion theory to the excursions of the reflected process away from 0. The excursion at level $x \geq 0$ is the process

$$\varepsilon^x = (\bar{X}_{T_{x-}+u} - X_{T_{x-}+u}, 0 \leq u \leq T_x - T_{x-}).$$

Itô theory implies that the excursion process is a Poisson point process: there exists a σ -finite measure n , called the excursion measure, which satisfies the Master Formula

$$\mathbb{E} \left(\sum_{x>0} H(T_{x-}(\omega), \omega, \varepsilon_x(\omega)) \right) = \mathbb{E} \left(\int_0^\infty d\bar{X}_s(\omega) \int n(d\varepsilon) H(s, \omega, \varepsilon) \right)$$

where H is a positive functional which is jointly predictable with respect to its first two components and measurable with respect to the third.

Let $D = \inf\{u > 0 : \varepsilon(u) = 0\}$ be the death time of an excursion ε . We are going to give a “good” representation of the excursion measure conditioned by the duration with the help of the following generalization of Vervaat’s Theorem, that we will still call “Vervaat’s Theorem” :

Proposition 2.1 (Vervaat’s Theorem) :

Let b_X^ℓ be the bridge of X on $[0, \ell]$ from 0 to 0. Let Γ_ℓ be the law of the process $(Vb_X^\ell(\ell - x))_{0 \leq x \leq \ell}$. Then the family $(\Gamma_\ell)_{\ell>0}$ is a regular version for the “conditional law” $n(\cdot|D)$ in the sense that

$$n(d\varepsilon) = \int_0^\infty n(D \in d\ell) \Gamma_\ell(d\varepsilon) = \int_0^\infty \frac{d\ell}{\ell} q_\ell(0) \Gamma_\ell(d\varepsilon)$$

Hence, we will always refer to the time-reversed Vervaat Transform of the bridge of X from 0 to 0 with length $\ell > 0$ as the *excursion of X below its supremum, with duration ℓ* . The explanation for the second equality above is in the first assertion of Lemma 2.6, and the proof of this proposition is postponed to section 2.7.

2.3.2 The fragmentation property

We now state a definition of what we call an (inhomogeneous) fragmentation process, following [21]. Let S^\downarrow be the space of all decreasing positive real sequences with finite sum, and for all $\ell > 0$, let $S_\ell^\downarrow \subset S^\downarrow$ be the space of the elements of S^\downarrow with sum ℓ . Then, for $0 \leq t \leq t'$, consider for each ℓ an “elementary” probability measure $\kappa_{t,t'}(\ell)$ on S_ℓ^\downarrow . Next, for all $L = (\ell_1, \ell_2, \dots) \in S^\downarrow$, let L_1, L_2, \dots be independent sequences with respective laws $\kappa_{t,t'}(\ell_1), \kappa_{t,t'}(\ell_2), \dots$ and define $\kappa_{t,t'}(L, \cdot)$ as the law of the decreasing arrangement of the elements of L_1, L_2, \dots .

Definition :

We call fragmentation process a (*a priori* not homogeneous in time) Markov process with transition kernel $(\kappa_{t,t'}(L, dL'))_{t < t', L \in S^\downarrow}$.

Informally, conditionally on the current state of the fragmentation, a fragment then splits in a way depending only on its length, independently of the others, a property which is usually referred to as the *fragmentation property*.

As in the Brownian case developed in [21], the process that we are now defining is a fragmentation process of the **random** interval $[0, T_1]$.

Definition :

We call *Lévy fragmentation* associated to X , and we denote it by $(F^X(t))_{t \geq 0}$, the process such that for each t , $F^X(t)$ is the decreasing sequence of the lengths of the interval components of the complementary of the support of the measure $\mathbb{1}_{[0, T_1]} d\bar{X}^{(t)}$ (in other terms, of constancy intervals of the supremum of $X^{(t)}$ before T_1).

Since the support of the measure $d\bar{X}^{(t)}$ is precisely the range of the subordinator $T^{(t)}$, we see that the sum of the “fragments” at a time $t > 0$ is T_1 . Indeed, since $X^{(t)}$ has infinite variation, 0 is regular for itself (see Corollary VII,5 in [19]), and it easily follows that the closure of the range of $T^{(t)}$ has zero Lebesgue measure.

The purpose of this section is to prove that the Lévy fragmentation is indeed a fragmentation in the sense of Definition 2.3.2. For any $\ell > 0$ we consider the transition kernel $\varphi_{t,t'}(\ell)$ defined as follows:

Definition :

For any $t' > t \geq 0$, let $(\varepsilon_\ell^{(t)}(s), 0 \leq s \leq \ell)$ be the generic excursion with duration ℓ of the reflected process $\bar{X}^{(t)} - X^{(t)}$. We denote by $\varphi_{t,t'}(\ell)$ the law of the sequence of the lengths of the constancy intervals for the supremum process of $(s(t' - t) - \varepsilon_\ell^{(t)}(s), 0 \leq s \leq \ell)$, arranged in decreasing order.

By convention let $\varphi_{t,t'}(0)$ be the Dirac mass on $(0, 0, \dots)$.

Remark. In fact one could define a fragmentation-type process from any excursion-type function f defined on $[0, \ell]$ (that is, which is positive, null at 0 and ℓ and with only positive jumps), deterministic or not, by declaring that $F^f(t)$ is the decreasing sequence of the lengths of the intervals of constancy of the supremum process of $(st - f(s), 0 \leq s \leq \ell)$. We will sometimes refer to it as the fragmentation process associated to f .

Proposition 2.2 :

The process $(F^X(t))_{t \geq 0}$ is a fragmentation process with kernels $\varphi_{t,t'}(L, dL')$ ($0 \leq t < t'$, $L \in S^\downarrow$). In other words, for $t' > t \geq 0$, conditionally on $F^X(t) = (\ell_1, \ell_2, \dots)$, if we consider a sequence of independent random sequences F_1, F_2, \dots with respective laws $\varphi_{t,t'}(\ell_1), \varphi_{t,t'}(\ell_2), \dots$, then the law of $F^X(t')$ is the one of the sequence obtained by rearranging the elements of F_1, F_2, \dots in decreasing order.

Remarks. From the definition of F^X we can see that it is a fragmentation beginning at a *random* state which corresponds to the sequence of the jumps of (T_x) for $x \leq 1$. But we can also define the Lévy fragmentation beginning at $(\ell, 0, \dots)$ by applying the transition mechanism explained in Proposition 2.2 to this sequence. We will denote the derived Markov process by F^{ε_ℓ} , the fragmentation beginning from fragment ℓ , which is equal in law to the fragmentation process associated to ε_ℓ by virtue of Proposition 2.2. Even if the definition of F^X is simpler than that of F^{ε_ℓ} , we will rather study the latter in the sequel since the behaviour of F^X can be deduced from that of the F^{ε_ℓ} 's as we will see at the beginning of section 2.4.

In order to prove Proposition 2.2 we will use, as mentioned above, the same methods as in [21]. Since the proofs are almost the same, we will be a bit sketchy.

First we remark that a “Skorohod-like formula” holds for the supremum processes $\overline{X}^{(t)}$ and $\overline{X}^{(t')}$. This formula is at the heart of the fragmentation property.

Lemma 2.3 :

For any $t' \geq t \geq 0$ we have

$$\overline{X}_u^{(t)} = \sup_{0 \leq v \leq u} (\overline{X}_v^{(t')} - (t' - t)v) \quad (2.5)$$

This property holds for any process X which has a.s. no positive jumps; the proof of [21] applies without change.

Remark. In fact we have that

$$\overline{X}_u^{(t)} = \sup_{g \leq v \leq u} (\overline{X}_v^{(t')} - (t' - t)v) \quad (2.6)$$

where u belongs to the interval of constancy $[g, d]$ of $\overline{X}^{(t)}$ (in other terms, g is the time in $[0, u]$ where $X^{(t)}$ is maximal).

// *Proof of Proposition 2.2.* Following [21] we deduce from Lemma 2.3 that if \mathcal{G}_t is the σ -field generated by the process $\overline{X}^{(t)}$, then $(\mathcal{G}_t)_{t \geq 0}$ is a filtration. Indeed, the Skorohod-like formula shows that $\overline{X}^{(t)}$ is measurable w.r.t. $\overline{X}^{(t')}$ for any $t' > t$.

Now suppose that K is a \mathcal{G}_t -measurable positive r.v., and let us denote by $\varepsilon_{1,K}^{(t)}, \varepsilon_{2,K}^{(t)}, \dots$ the sequence of the excursions accomplished by $X^{(t)}$ below its supremum, ranked by decreasing order of duration (we call $\ell_{1,K}^{(t)}, \ell_{2,K}^{(t)}, \dots$ the sequence of their respective durations), before time $T_K^{(t)}$. If $n^{(t)}$ is the corresponding excursion measure, and if $n^{(t)}(\ell)$ is the law of the excursion of X below its supremum with duration ℓ , we have the analogous for Lemma 4 in [21]: conditionally on \mathcal{G}_t , the excursions $\varepsilon_{1,K}^{(t)}, \varepsilon_{2,K}^{(t)}, \dots$ are independent random processes with respective distributions $n^{(t)}(\ell_{1,K}^{(t)}), n^{(t)}(\ell_{2,K}^{(t)}), \dots$.

Again, the proof is identical to [21], with the only difference that $n^{(t)}(\ell)$ cannot be replaced by $n(\ell)$, the law of the excursion of X below its supremum with duration ℓ (which stems from Girsanov's theorem in the case of Brownian motion).

Applying this result to $K = \overline{X}_{T_1}^{(t)}$ and using the forthcoming Lemma 2.4 which will show that $T_K^{(t)} = T_1$, it is now easy to see that $(F(t), t \geq 0)$ has the desired fragmentation property and transition kernels. //

2.4 The fragmentation semigroup

Our next task is to characterize the semigroup of the fragmentation process at a fixed time. In this direction, it suffices to characterize the semigroup of $F^{\varepsilon_\ell}(t)$ for fixed $t > 0$ and $\ell > 0$, since conditionally on the jumps $\ell_1 > \ell_2 > \dots$ of T before level 1, the fragmentation $F^X(t)$ at time t comes from the independent fragmentations $F^{\varepsilon_{\ell_1}}, F^{\varepsilon_{\ell_2}}, \dots$ at time t .

Our main result is Theorem 2.2, a generalization of the result of Aldous and Pitman [10] for the Brownian fragmentation. The conditioning mentioned in the statement is

explained immediately below (equations (2.9) and (2.10)). Recall that $q_t(\cdot)$ is the density of X_t .

Theorem 2.2 :

The following assertions hold :

- (i) The process $(\Delta T_x^{(t)})_{0 \leq x \leq t\ell}$ of the jumps of $T^{(t)}$ before the level $t\ell$ is a Poisson point process on $(0, \infty)$ with intensity measure $t\ell z^{-1}q_z(-tz)dz$.
- (ii) For any $t > 0$, the law of $F^{\varepsilon\ell}(t)$ is that of the decreasing sequence of the jumps of $T^{(t)}$ before time $t\ell$, conditioned on $T_{t\ell}^{(t)} = \ell$.

The first assertion is well-known, and will be recalled in Lemma 2.6. We essentially focus on the second assertion.

2.4.1 Densities for the jumps of a subordinator

We recall some results on the law of the jumps of a subordinator that can be found in Perman [86]. From now on in this paper, we will often have to use them.

We consider a subordinator T with no drift, and infinite Lévy measure $\pi(dz)$. We assume that the Lévy measure is absolutely continuous with density $h(z) = \pi(dz)/dz$ that is continuous on $(0, \infty)$. It is then known in particular that for each level x , T_x has a density f , which is characterized by its Laplace transform

$$\int_0^{+\infty} e^{-\lambda u} f(u) du = \exp \left(-x \int_0^{\infty} (1 - e^{-\lambda z}) h(z) dz \right). \quad (2.7)$$

Next, for all $v > 0$, let $f_v(x)$ denote the density at level x (which is known to exist) of the subordinator T^v which Lévy measure is $h(z)\mathbb{1}_{\{z < v\}}dz$. It is characterized by its Laplace transform

$$\int_0^{+\infty} e^{-\lambda u} f_v(u) du = \exp \left(-x \int_0^v (1 - e^{-\lambda z}) h(z) dz \right) \quad (2.8)$$

We also denote by $(\Delta_i)_{i \geq 1}$ the decreasing sequence of the jumps of $(T_y)_{0 \leq y \leq x}$.

By [86], the k -tuple $(\Delta_1, \dots, \Delta_k)$ admits for every k a density:

$$\frac{\mathbb{P}[\Delta_1 \in du_1, \dots, \Delta_k \in du_k]}{du_1 \dots du_k} = x^k h(u_1) \dots h(u_k) \exp \left(-x \int_{u_k}^{\infty} h(z) dz \right) \quad (2.9)$$

which we denote by $p(u_1, \dots, u_k)$.

Besides, the $k + 1$ -tuple $(\Delta_1, \dots, \Delta_k, T_x)$ has density

$$\frac{\mathbb{P}[\Delta_1 \in du_1, \dots, \Delta_k \in du_k, T_x \in ds]}{du_1 \dots du_k ds} = p(u_1, \dots, u_k) f_{u_k}(s - u_1 - \dots - u_k). \quad (2.10)$$

The proof relies on the general fact for Poisson measures that, conditionally on the k largest jumps $(\Delta_1, \dots, \Delta_k)$, the sequence $(\Delta_i)_{i \geq k+1}$ is equal in law to the decreasing sequence of the jumps of the subordinator T^{Δ_k} before time x , that is, the atoms of a Poisson point process with intensity $h(z)\mathbb{1}_{\{z < v\}}dz$ at time x . Since T_x has a density, these formulas give the conditional law of the jumps of T before time x given $T_x = s$ for every $s > 0$.

Remark. In the case of a Lévy measure π with finite mass a (compound Poisson case) admitting a density, first condition by the fact that there are k jumps in $[0, x]$ (the probability is $e^{-a} a^k / k!$), then by the size of the jumps, which are independent with law π/a .

2.4.2 A useful process

Now to prove Theorem 2.2, we introduce a process with bounded variation that is containing in a practical way all the information on the fragmentation before a fixed time t . In [21] as well as in [20], Bertoin uses the process $(x - tT_x^{(t)})_{x \geq 0}$, that we will denote here by $Y^{(t)}$, for any $t > 0$.

It is clear that $Y^{(t)}$ is a Lévy process with bounded variation and with no positive jumps. As the subordinator $T^{(t)}$ can be recovered from $Y^{(t)}$, the sigma-field generated by the latter coincides with \mathcal{G}_t , and in particular it should be possible to deduce $(F^X(s))_{0 \leq s \leq t}$ from it.

We begin with a lemma which is related to Lemma 7 in [21]. We denote by $\sigma^{(t)}$ the inverse of $Y^{(t)}$, in particular $\sigma^{(t)}$ is a subordinator.

Lemma 2.4 :

For any $t > 0$, $F^X(t)$ has the law of the decreasing sequence of the jumps of $(T_x^{(t)})$ for $x \leq \bar{X}_{T_1}^{(t)} = 1 + tT_1$, that is, the jumps of $(-Y_x^{(t)}/t)$ for $x \leq \sigma_1^{(t)}$.

Moreover we have for any y ,

$$\sigma_y^{(t)} = y + tT_y \quad (2.11)$$

// We prove the second assertion, the first one being a straightforward consequence. We have

$$\begin{aligned} \sigma_x^{(t)} &= \inf\{z \geq 0 : z - tT_z^{(t)} > y\} \\ &= \inf\{\bar{X}_u^{(t)} : \bar{X}_u^{(t)} - tu > y\} \\ &= \bar{X}^{(t)}(\inf\{u \geq 0 : \bar{X}_u^{(t)} - tu > y\}) \\ &= \bar{X}^{(t)}(\inf\{u \geq 0 : \sup_{0 \leq v \leq u} (\bar{X}_v^{(t)} - tv) > y\}) \\ &= \bar{X}^{(t)}(\inf\{u \geq 0 : \bar{X}_u > y\}) \\ &= \bar{X}_{T_y}^{(t)} \end{aligned}$$

where we used the fact that $\bar{X}_u^{(t)} - tu$ is non-increasing on a constancy interval of $\bar{X}_u^{(t)}$, the continuity of $\bar{X}^{(t)}$ which follows from the fact that X has no positive jumps, and formula (2.5).

We then note that

$$\sigma_{\bar{X}_u}^{(t)} = \bar{X}_{T_{\bar{X}_u}}^{(t)} = \bar{X}_u + tT_{\bar{X}_u}$$

which follows from the fact that $X_s \leq \bar{X}_u$ for $s \leq T_{\bar{X}_u}$, and $X_s^{(t)} \leq \bar{X}_u + tu \leq \bar{X}_u + tT_{\bar{X}_u} = X_{T_{\bar{X}_u}}^{(t)}$. Applying this for $u = T_y$ we finally obtain

$$\sigma_y^{(t)} = \bar{X}_{T_y}^{(t)} = y + tT_y. \quad //$$

Remark that this last result implies that the process of first hitting times of $Y^{(t)}$ is not killed, so that $Y^{(t)}$ is oscillating or drifting to ∞ . Moreover, from the fact that the Laplace

exponents of $X^{(t)}$ and $T^{(t)}$ are inverse functions, we obtain that $\mathbb{E}[T_1^{(t)}] = 1/\mathbb{E}[X_1^{(t)}] = 1/(\mathbb{E}[X_1] + t)$ so that

$$\mathbb{E}[Y_1^{(t)}] = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1] + t},$$

and $Y^{(t)}$ oscillates if and only if X does so.

Lemma 2.4 also shows that the information of $F^X(t)$ for fixed t is (very simply) connected to the process $Y^{(t)}$, but also gives us a tool for studying the law of $F^{\varepsilon_\ell}(t)$. Indeed, we know that $(\overline{X}_u)_{u \geq 0} = (T_u^{-1})_{u \geq 0}$ is a local time for the reflected process of X , so that the previous lemma implies that, up to a multiplicative constant and a drift, X and $Y^{(t)}$ share for all $t > 0$ the same inverse local time processes.

Recall that n is the excursion measure of $\overline{X} - X$ and that V is the lifetime of the canonical process. If ε is an excursion-type function we denote by $\overline{\varepsilon}^{(t)}$ the supremum process of $ts - \varepsilon_s$. We are now able to state the

Lemma 2.5 :

The “law” under $n(d\varepsilon)$ of the decreasing lengths of the constancy intervals of $\overline{\varepsilon}^{(t)}$ is the same as the “law” under the excursion measure of $\overline{Y}^{(t)} - Y^{(t)}$ of the jumps of the canonical process, multiplied by $1/t$ and ranked in the decreasing order. The same holds for the conditioned law $n(d\varepsilon|V = \ell)$ and the corresponding law of the excursion of $\overline{Y}^{(t)} - Y^{(t)}$ with duration $t\ell$.

// From the above remark, to the excursion ε^x of the reflected process of X at level x (that is, the excursion of X below its supremum and starting at T_x , $\varepsilon^x(u) = x - X_{T_{x-}+u}$ for $0 \leq u \leq T_x - T_{x-}$) we can associate the excursion $\gamma_x^{(t)}$ of $Y^{(t)}$ below its supremum, at level x , given by

$$\gamma_x^{(t)}(u) = x - Y_{u+\sigma_{x-}^{(t)}}^{(t)} = x - u - (x + tT_{x-}) + tT_{u+x+tT_{x-}}^{(t)} \tag{2.12}$$

for $0 \leq u \leq \sigma_x^{(t)} - \sigma_{x-}^{(t)}$. We underline that the random times $-tT_{x-} + T_{u+x+tT_{x-}}^{(t)}$ appearing in the formula only depend on the process $x - X$ between the times T_{x-} and T_x , that is, of ε^x . Indeed, Lemma 2.4 implies that

$$\overline{X}_{T_{x-}}^{(t)} = x + tT_{x-}, \quad \overline{X}_{T_x}^{(t)} = x + tT_x$$

so that the values taken by the jumps of $T^{(t)}$ between T_{x-} and T_x are exactly the length of the constancy intervals of the supremum of $(-\varepsilon_x(u) + tu)_{0 \leq u \leq T_x - T_{x-}}$. The proof then follows. //

Remarks. • More precisely, if we call V_k the location of the k -th largest jump of $\gamma_x^{(t)}$, then we have that the i -th largest constancy interval of the supremum process of $(ts - \varepsilon^x(s))_{0 \leq s \leq T_x - T_{x-}}$ is at the left of the j -th one if and only if $V_i < V_j$. This is a straightforward consequence of elementary sample path properties of X . In particular, there a.s. exists a left-most interval of constancy of the supremum process of $(ts - \varepsilon_\ell(s))_{0 \leq s \leq \ell}$ if and only if the excursion of $Y^{(t)}$ with length $t\ell$ a.s. begins by a jump. This is to be related, of course, with section 2.2, but also with the forthcoming section 2.5

• The last proof also implies the fact (that could easily be guessed on a drawing) that if ε is the excursion of X below its supremum with duration 1 and if for $0 \leq x \leq t$, $T'_x = \inf\{u \geq 0, -\varepsilon_u + tu > x\}$, then $x - tT'_x$, $0 \leq x \leq t$ has the law of the excursion of $Y^{(t)}$ below its supremum with duration t .

2.4.3 Proof of Theorem 2.2

The last step before the proof is a lemma that gives explicit densities for the characteristics of $T^{(t)}$.

Lemma 2.6 :

The Lévy measure $\pi^{(t)}(dz)$ of $T^{(t)}$ is absolutely continuous w.r.t. Lebesgue measure, with density

$$h^{(t)}(z) = \frac{1}{z} q_z(-tz) \mathbb{1}_{\{z>0\}}. \quad (2.13)$$

Moreover, for every $x > 0$, $\mathbb{P}[T_x^{(t)} \in ds]$ has density

$$\mathbb{P}[T_x^{(t)} \in ds]/ds = \frac{x}{s} q_s(x - st) \quad (2.14)$$

// Following from the fact that $X^{(t)}$ is a Lévy process with no positive jumps, we have the well-known result ($x, s \in \mathbb{R}_+$)

$$x\mathbb{P}[X_s^{(t)} \in dx]ds = s\mathbb{P}[T_x^{(t)} \in ds]dx, \quad (2.15)$$

see Corollary VII,3 in [19] for example. From this we deduce (2.14), as $q_s(x - st)$ is the density of $X^{(t)}$.

Next, we know from Corollary 8.8 page 45 in [97] that the Lévy measure of $T^{(t)}$ is on (a, ∞) the weak limit of $(1/\varepsilon)\mathbb{P}[T_\varepsilon^{(t)} \in ds]$ for any $a > 0$. We thus obtain (2.13). //

// *Proof of Theorem 2.2.* From Lemma 2.5 we know that the law of $F^{\varepsilon_\ell}(t)$ is equal to the law of the decreasing sequence of sizes of the jumps of the excursion of $(\bar{Y}^{(t)} - Y^{(t)})/t$ with length ℓ . But Vervaat's Theorem (Proposition 2.1) implies that this excursion has the same law as $Vy_{t\ell}^{(t)}(\ell - \cdot)$ where $(y_{t\ell}^{(t)}(x))_{0 \leq x \leq \ell}$ is the bridge with length $t\ell$ of $Y^{(t)}$ from 0 to 0. Since $T_{t\ell}^{(t)}$, and hence $Y_{t\ell}^{(t)}$, has a continuous density by Lemma 2.6, the law of $y_{t\ell}^{(t)}$ is defined as the limit as $\varepsilon \rightarrow 0$ of the law of $Y^{(t)}$ before time $t\ell$ conditioned on $|t\ell - tT_{t\ell}^{(t)}| < \varepsilon$. Now since the Lévy measure of $T^{(t)}$ also has a continuous density by Lemma 2.6, we get that under this limit probability, the jumps of the canonical process are the same as the jumps of $T^{(t)}/t$ conditioned on $T_{t\ell}^{(t)} = \ell$ in the sense of Perman [86]. This concludes the proof. //

Notice that formulas (2.10), (2.13) and (2.14) make the densities of the first k terms of $F^{\varepsilon_\ell}(t)$ ($k \in \mathbb{N}$) explicit in terms of $(q_t(x), t > 0, x \in \mathbb{R})$.

2.5 The left-most fragment

In the two preceding sections, we did not consider specifically the sample path properties of the excursion-type functions ε_ℓ that we used to describe the Lévy fragmentation. As a link with section 2.2 we are now studying some properties of the order induced by $[0, \ell]$ on the constancy intervals of the supremum process of $(st - \varepsilon_\ell(s))_{0 \leq s \leq \ell}$.

We can be a bit more precise than in the proof of Theorem 2.2 in our description of the bridge of $Y^{(t)}$. The Lévy-Itô decomposition of subordinators imply that $(T_x^{(t)})_{0 \leq x \leq t\ell}$ may be written in the form

$$T_x^{(t)} = \sum_{i=0}^{\infty} \Delta_i \mathbb{1}_{\{x \geq U_i\}}$$

where $(\Delta_i)_{i \geq 1}$ is the sequence of the jumps of $T^{(t)}$ before time $t\ell$ ranked in decreasing order, and $(U_i)_{i \geq 1}$ is an i.i.d. sequence of r.v.'s uniform on $[0, t\ell]$, independent of $(\Delta_i)_{i \geq 1}$. Thus the bridge of $Y^{(t)}$ with length $t\ell$ from 0 to 0 may be written in the form $x - t \sum_{i=0}^{\infty} \delta_i \mathbb{1}_{\{x \geq U_i\}}$ where $(\delta_i)_{i \geq 1}$ has the conditional law of $(\Delta_i)_{i \geq 1}$ given $T_{t\ell}^{(t)} = \ell$. We recognize a Kallenberg bridge with exchangeable increments, zero drift coefficient, finite variation and only negative random jumps $(t\delta_i)_{i \geq 0}$.

Recall from Lemma 2.2 that bridges with exchangeable increments, finite variation and only positive jumps a.s. attain their minimum at a jump, so that their Vervaat's Transforms begin with a jump. Hence Vervaat's Theorem applied to $Y^{(t)}$ combined with the discussion after Lemma 2.2 gives that the excursions of $Y^{(t)}$ below its supremum a.s. start by a jump, which is a size-biased pick from the variables $(\delta_i)_{i \geq 1}$. In fact this is proved by other means to give an alternative proof of Vervaat's Theorem itself in the bounded variation case, see 2.7.2 below. Nonetheless, it is clearer in our setting to present it rather as a consequence of Vervaat's Theorem. We denote this first jump by $tH^\ell(t)$, and according to the remark in section 2.4.2, $H^\ell(t)$ is equal in law to the left-most constancy interval of the supremum process of $(ts - \varepsilon_\ell(s))_{0 \leq s \leq \ell}$.

As a consequence of Corollary 2.1, we also have that $tH^\ell(t)$ has the law of a size-biased pick from the jumps of the bridge of $Y^{(t)}$ with length $t\ell$ from 0 to 0. Equivalently, $H^\ell(t)$ has the law of a size-biased pick from the jumps of $T^{(t)}$ before time $t\ell$ conditioned on $T_{t\ell}^{(t)} = \ell$. It has been already proved by Schweinsberg in [100] in the Brownian case, by similar methods. As noticed in this article, remark that $H^\ell(t)$ has the law of a size-biased without being a size-biased itself. Generalizing a result of Bertoin in the Brownian case, we can do even better, that is, to show that the process $(H^\ell(t))_{t \geq 0}$ has the law of a size-biased marked fragment in a sense we precise here.

Let U be uniform on $[0, \ell)$, independent of ε_ℓ , and let $H_*^\ell(t)$ the length of the constancy interval of the supremum process of $(ts - \varepsilon_\ell(s))_{0 \leq s \leq \ell}$ that contains U . At every time $t > 0$, $H_*^\ell(t)$ has the law of a size-biased pick from the elements of $F^{\varepsilon_\ell}(t)$.

Theorem 2.3 :

The processes H^ℓ and H_*^ℓ have the same law; they are both (in general time-inhomogeneous) Markov processes with transition kernel Q given by (for $t \leq t'$, $h' \leq h$):

$$Q_{t,t'}(h, dh') = \frac{(t' - t)h q_{h'}(-t'h') q_{h-h'}(t'h' - th)}{(h - h') q_h(-th)} dh'. \quad (2.16)$$

// Recall from section 2.3.2 that conditionally on the length $H^\ell(t) = h$ of the left-most fragment, at time t , the law of the fragmentation starting with this fragment is $\varphi_{t,t'}(h)$. Remark also that the left-most fragment at time t' comes from the fragmentation of the left-most fragment at time t . By the fact that $H^\ell(t)$ is a size-biased pick from the jumps of $T^{(t)}$ before $t\ell$ conditioned by $T_{t\ell}^{(t)} = \ell$, and by replacing time 0 by time t , time t by time $t' - t$, and X by $X^{(t)}$, we obtain that the left-most fragment at time t' given

its value h at time t has the law of a size-biased of the jumps of $T^{(t')}$ before time $(t' - t)h$ conditionally on $T_{(t'-t)h}^{(t')} = h$

Now recall from [94] the explicit law $F(dz)$ of a size-biased pick from jumps of a subordinator T before a fixed time x , conditionally on the value s of the subordinator at this time :

$$F(dz) = \frac{zxh(z)f(s-z)}{sf(s)}dz \quad (2.17)$$

where h and f are respectively the (continuous) density of the Lévy measure of T , and the (continuous) density of the law of T_x . Thanks to this formula and the expressions of the density of $T^{(t')}$ and its Lévy measure in terms of q in Lemma 2.6 we obtain (2.16).

Next, it is trivial that the initial states for H^ℓ and H_*^ℓ are the same (namely ℓ). It just remains to prove that H_*^ℓ is a Markov process with the same transition function as H^ℓ .

For this, let us condition on $H_*^\ell(t) = h$ at time t . It means that U belongs to an interval of excursion of $X^{(t)}$ below its supremum with length h . But then, U is uniform on this interval and independent of this excursion conditionally on its length $H_*^\ell(t)$. The value at time t' of the process H_*^ℓ is thus by definition a size-biased from the fragments of the Lévy fragmentation process with initial state $(h, 0, \dots)$. It thus has the same law that $H^\ell(t')$ given $H^\ell(t) = h$, that is, precisely $Q_{t,t'}(h, dh')$. //

Remark. The fact that $H^\ell(t)$ is non zero a.s. gives in particular an interesting property for the excursions : The excursions of the reflected process of X out of 0 (under the excursion measure, or with fixed duration) have at 0 an “infinite slope” in the sense that

$$\frac{\varepsilon(s)}{s} \xrightarrow{s \rightarrow 0} +\infty$$

Indeed, the only other possibility is that they begin with a jump ($\varepsilon(0) > 0$), but it is never the case when X has infinite variation. This result could be also be deduced from Millar [83] who shows that X “moves away” from a local maximum faster than does the supremum process from 0 at time 0.

2.6 The mixing of extremal additive coalescents

In this section, we associate by time-reversal a coalescent process to each Lévy fragmentation. We will show that it is a ranked eternal additive coalescent as described by Evans and Pitman [52] and Aldous and Pitman [12], where the initial random data (at time $-\infty$) depend on X . It is thus a mixing of the extremal coalescents of [12], and we will give the exact law of this mixing.

The most natural way to identify the mixing is to use the representation of the extremal coalescents by Bertoin [22], with the help of Vervaat’s Transforms of some bridges with exchangeable increments and deterministic jumps, by noticing that the bridge of X with length 1 is such a bridge, but with random jumps. We will focus on the case where the total mass is 1, so that we do not have to introduce too many time-changes.

Definition :

Recall the definition of F^{ε_1} from section 2.3. Call *Lévy coalescent* derived from X the process defined on the whole real line by

$$C^{\varepsilon_1}(t) = F^{\varepsilon_1}(e^{-t}), \quad t \in \mathbb{R} \quad (2.18)$$

More generally, if $(F(t), t \geq 0)$ is some fragmentation-type process, we interpret the process $(F(e^{-t}), t \in \mathbb{R})$ as the “associated coalescent”.

We first recall the results of [22]. Let l_2^\downarrow be the set of (non-negative) decreasing l_2 sequences. For $a \geq 0$ and $\theta \in l_2^\downarrow$ define the bridge

$$b_{a,\theta} = ab_s + \sum_{i=1}^{\infty} \theta_i (\mathbb{1}_{\{U_i \leq s\}} - s)$$

where b is a standard Brownian bridge on $[0, 1]$ and the U_i are as usual independent uniform r.v. on $[0, 1]$. We call $b_{a,\theta}$ the *Kallenberg bridge* with jumps θ and Brownian bridge component ab (it is a particular case of the general representation for bridges with exchangeable increments, see [65]).

Let (ϑ_i) be a decreasing positive sequence such that $\sum \vartheta_i^2 \leq 1$ (we call the corresponding space $l_2^{1,\downarrow}$), and

$$\varsigma = \sqrt{1 - \sum_{i=1}^{\infty} \vartheta_i^2}. \quad (2.19)$$

Consider the fragmentation $F^{\vartheta}(t)$ associated to the excursion $Vb_{\varsigma,\vartheta}$ (it consists on the lengths of the intervals of constancy of the supremum process of $ts - Vb_{\varsigma,\vartheta}(s)$). Let $C^{\vartheta}(t) = F^{\vartheta}(e^{-t})$ be the associated coalescent process. Then ([22], Theorem 1 and [12], Theorems 10 and 15) it is an extreme eternal additive coalescent process, the mapping

$$\vartheta \in l_2^{1,\downarrow} \longmapsto C^{\vartheta}$$

is one-to-one, and every extreme eternal additive coalescent (where the total mass of the clusters is 1) can be represented in this way up to a deterministic time-translation. We call \mathbb{P}^{ϑ} its law, and for $t_0 \in \mathbb{R}$ we denote by $\mathbb{P}^{\vartheta,t_0}$ the law of the time translated coalescent $(C^{\vartheta}(t - t_0), t \in \mathbb{R})$. In this way, the law of any extreme eternal additive coalescent is of this form.

We now denote by σ the Gaussian component of X . For $\theta \in l_2^\downarrow$, let $k(\theta) \geq 0$ be such that

$$k(\theta)^2 \sigma^2 = 1 - k(\theta)^2 \sum \theta_i^2,$$

that is

$$k(\theta) = \frac{1}{\sqrt{\sigma^2 + \sum_{i=1}^{\infty} \theta_i^2}}.$$

Remark that $k(\theta) \cdot \theta$ is in $l_2^{1,\downarrow}$ and that $k(\theta) \sigma = \sqrt{1 - \sum_{i=1}^{\infty} (k(\theta) \theta_i)^2}$ is the corresponding “ ς ”.

Now the bridge b_X of X from 0 to 0 and length 1 has exchangeable increments, and as such the ranked sequence of its jumps is a random element of l_2^\downarrow (see Kallenberg [65]).

Let $\Theta_X(d\theta)$ be its law. It is not difficult to see that if θ^* has law $\Theta_X(d\theta)$, then $b_X(1 - \cdot)$ has the same law as b_{σ, θ^*} . Let $\tilde{\Theta}_X(d\vartheta, dt_0)$ be the image of $\Theta_X(d\theta)$ by the mapping

$$\theta \mapsto (\vartheta, t_0) = (k(\theta)\theta, \log k(\theta)).$$

Proposition 2.3 :

The Lévy coalescent C^{ε_1} associated to X is an additive coalescent, and its law is given by the mixing

$$\int_{(\vartheta, t_0) \in l_2^{1,1} \times \mathbb{R}} \mathbb{P}^{\vartheta, t_0}(\cdot) \tilde{\Theta}_X(d\vartheta, dt_0). \quad (2.20)$$

// Consider the fragmentation F^{ε_1} . It is associated to $Vb_X(1 - \cdot)$, which is equal in law to Vb_{σ, θ^*} where θ^* has law Θ_X . Hence for $t \geq 0$, $F^{\varepsilon_1}(t)$ has the law of the intervals of constancy of the supremum process of $(ts - Vb_{\sigma, \theta^*}(s), 0 \leq s \leq 1)$. This last process is equal to

$$\frac{1}{k(\theta^*)} (k(\theta^*)ts - Vb_{k(\theta^*)\sigma, k(\theta^*)\theta^*}(s)), \quad 0 \leq s \leq 1,$$

so that the supremum processes of the processes $(ts - Vb_{\sigma, \theta^*}(s), 0 \leq s \leq 1)$ and $(k(\theta^*)ts - Vb_{k(\theta^*)\sigma, k(\theta^*)\theta^*}(s), 0 \leq s \leq 1)$ share the same constancy intervals. Hence, by definition, $F^{\varepsilon_1}(t) = F^{k(\theta^*)\theta^*}(k(\theta^*)t)$, and this means that the associated coalescent is $C^{k(\theta^*)\theta^*, \log k(\theta^*)}$. The law of C^{ε_1} is thus

$$\int_{\theta \in l_2^1} \mathbb{P}^{k(\theta)\theta, \log k(\theta)}(\cdot) \Theta_X(d\theta) \quad (2.21)$$

and we conclude by a change of variables. //

Remark. We stress that, whether the Lévy measure of X integrates $|x| \wedge 1$ or not, the typical mixings that appear are not the same : the configurations where $\sum \theta_i = \infty$ have no “weight” in the first case, whereas $\sum \theta_i < \infty$ does not happen in the second. Moreover, under some more hypotheses on X (e.g. that its Lévy measure $\nu(du)$ is absolutely continuous with respect to Lebesgue measure, and that the Lévy process with truncated Lévy measure $\mathbb{1}_{\{u \leq x\}} \nu(du)$ has densities), one can make more “explicit” the law Θ_X by the same arguments of conditioned Poisson measures as in section 2.4.1 above.

2.7 Proof of Vervaat's Theorem

We are going to give two proofs of Proposition 2.1, the first one being quite technical, and essentially devoted to the unbounded variation case since we have not found how to prove it with simple arguments. Of course this proof applies also in the bounded variation case. The second proof only works for Lévy processes with bounded variation, but uses only tools that are directly connected to this work, such as the ballot theorem.

2.7.1 Unbounded variation case

Let $(\mathcal{F}_t^0)_{t \geq 0}$ the natural filtration on the space of càdlàg functions on \mathbb{R}_+ . Let \widehat{P} be the law of the spectrally **positive** Lévy process $\widehat{X} = -X$. Without risk of ambiguity, \widehat{X} will also denote the canonical process on $\mathbb{D}([0, \infty))$. Recall the definition of the law $P_{0,0}^t$ of the bridge of X with length $t > 0$ starting and ending at 0 from (2.4). Let P^t be the law of the process X killed at time t , and $(\mathcal{F}_t)_{t \geq 0}$ be the P -completed filtration.

Let also $P^J = P^{\tau_{J^c}}$ where $\tau_{J^c} = \inf\{s \geq 0, X_s \notin J\}$ for any interval J . Recall that n is the excursion measure of the reflected process $\overline{X} - X = \widehat{X} - \underline{\widehat{X}}$ where $\underline{\widehat{X}}_t = \inf_{0 \leq s \leq t} \widehat{X}_s$. Since \widehat{X} oscillates or drifts to $-\infty$, every excursion of the process has a finite lifetime D . Let n^u be the measure associated to the excursion killed at time $u \wedge D$. Remark that the measure $n(\cdot, t \leq D)$ is a finite measure, with total mass $\pi((t, \infty))$ where π is the Lévy measure of the subordinator $(\widehat{T}_{-y})_{y \geq 0}$ (where $\widehat{T}_x = \inf\{s \geq 0, \widehat{X}_s = x\}$). We already saw that the inverse local time process of $\widehat{X} - \underline{\widehat{X}}$ is $(\widehat{T}_{-y})_{y \geq 0}$, and that it has Lévy measure $q_v(0)dv/v$.

The demonstration that we are giving is close to the method used by Biane [32] for Brownian motion and Chaumont [43] for stable processes. It involves a path decomposition of the trajectories of \widehat{X} under P^t at its minimum. We will first need the following result (see [42]) which is an application of Maisonneuve's formula. Chaumont stated the result only for oscillating Lévy processes, but the proof applies without change to processes drifting to $-\infty$.

// Let k_t be the standard killing operator at time t , ζ the life of the canonical process, θ_t be the shift operator and θ'_t be defined by $\widehat{X}_s \circ \theta'_t = \widehat{X}_{s+t} - \widehat{X}_t$. Last, let g_t be the right-most instant at which \widehat{X} attains its minimum on $[0, t]$. Then, under the measure $\int_0^\infty dt P^t$, the pair $(\widehat{X} \circ k_{g_\zeta}, \widehat{X} \circ \theta'_{g_\zeta})$ has the "law"

$$\int_0^\infty dx P^{(-x, \infty)} \otimes \int_0^\infty du n^u(\cdot, u < D)$$

In other terms, if H and H' are positive measurable functionals, that can be taken of the form

$$H = \mathbb{1}_{\{t_n < \zeta\}} \prod_{i=1}^n f_i(\widehat{X}_{t_i}), \quad 0 \leq t_1 \leq \dots \leq t_n,$$

then

$$\int_0^\infty dt P^t (H \circ k_{g_\zeta} H' \circ k_{\zeta - g_\zeta} \circ \theta'_{g_\zeta}) = \int_0^\infty dx P^{(-x, \infty)}(H) \int_0^\infty du n^u(H, u < D)$$

Let H and H' be continuous measurable positive bounded functionals (that can be of the form above with the f_i continuous, positive and bounded) such that $H \circ k_u$ is integrable w.r.t. the measure $n(d\omega)du \mathbb{1}_{\{u < D\}}$, and f a continuous positive real function with compact support which does not contain 0. We then have by dominated convergence and by definition of

$$\begin{aligned}
P_{0,0}^t &: \\
&\int_0^\infty dt f(t) q_t(0) P_{0,0}^t (H \circ k_{g_\zeta} H' \circ k_{\zeta-g_\zeta} \circ \theta'_{g_\zeta}) \\
&= \lim_{\epsilon \downarrow 0} \int_0^\infty dt f(t) q_t(0) P^t (H \circ k_{g_\zeta} H' \circ k_{\zeta-g_\zeta} \circ \theta'_{g_\zeta} | |\widehat{X}_\zeta| < \epsilon),
\end{aligned}$$

This is thus equal to

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \int_0^\infty dy \int_0^\infty du \int \int dP^{(-y,\infty)}(\omega') dn(\omega) \mathbb{1}_{\{u \leq D\}} H(\omega') \\
&\quad \times \mathbb{1}_{\{|\omega_u - y| < \epsilon\}} \frac{H' \circ k_u(\omega) f(\widehat{T}_{-y}(\omega') + u) q_{\widehat{T}_{-y}(\omega') + u}(0)}{\int_{-\epsilon}^\epsilon q_{\widehat{T}_{-y}(\omega') + u}(z) dz}
\end{aligned}$$

Now remark that the measure $\mathbb{1}_{\{|\omega_u - y| < \epsilon\}} dy / 2\epsilon$ converges weakly to $\delta_{\omega_u}(dy)$ when $\epsilon \downarrow 0$.

Moreover we have that the family of probability measures $(P^{(y,\infty)})_{y < 0}$ is weakly continuous in the sense that for any continuous bounded functional F , $P^{(y,\infty)}(F)$ is continuous in y . This follows from the a.s. continuity of the subordinator $(\widehat{T}_{-y})_{y \geq 0}$ at a fixed y . The continuity of H and H' , $q_x(s)$ and of the killing operator thus implies that the limit we are studying is equal to

$$\int n(d\omega) \int_0^D du H' \circ k_u(\omega) \int dP^{(-\omega_u,\infty)}(\omega') H(\omega') f(\widehat{T}_{-\omega_u}(\omega') + u)$$

or

$$n \left(\int_0^D du H' \circ k_u E^{(-\omega_u,\infty)} [H f(\zeta + u)] \right) \quad (2.22)$$

In other means, using the Markov property of the excursion measure, we have for H, H' two measurable positive functionals,

$$\begin{aligned}
&\int n(d\omega) f(D) \int_0^D du H \circ k_u H' \circ \theta_u \\
&= \int n(d\omega) \int_0^D du H \circ k_u(\omega) \int dP_{\omega_u}^{(0,\infty)}(\omega') H'(\omega') f(u + \widehat{T}_0(\omega'))
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^\infty \frac{dv}{v} q_v(0) f(v) n \left(\int_0^D du H \circ k_u H' \circ \theta_u \Big| D = v \right) \\
&= n \left(\int_0^D du H \circ k_u E_{\omega_u}^{(0,\infty)} [H' f(u + \zeta)] \right)
\end{aligned}$$

By comparing this with (2.22) we obtain the desired result : by interverting and sticking together the pre- and post- minimum processes of the bridge with length t , we obtain a regular version of the conditional “law” $n(\cdot | D)$, such that the sticking point u is uniform on the interval $(0, D)$ (this

gives also a way to recover the law of the bridge by splitting the excursion at an independent uniform point). This ends the proof. //

Remark. Recently, Chaumont [44] has given a generalization of the Vervaat’s Transformation by constructing processes with cyclically exchangeable increments conditioned to spend a fixed time $s > 0$ below 0. This conditional law converges when s goes to 0, and would give a (certainly more general!) proof of the Vervaat Theorem if one could precisely and properly identify the limit law (“process conditioned to spend time 0 below 0”) as that of the excursion.

2.7.2 Bounded variation case

There is a more natural way to approach Vervaat’s Theorem in the setting of this paper. Let $Y_t = ct - \tau_t$ be a spectrally negative Lévy process with bounded variation, where c is a positive constant and τ is a strict subordinator without drift (which means that it is pure jump and that it is not killed). Without loss of generality we suppose that $c = 1$. Let \bar{Y} be the supremum process. We suppose that Y oscillates or drifts to $+\infty$ (that is, $\bar{Y}_\infty = +\infty$ a.s.), which happens if and only if $1 \geq \mathbb{E}[\tau_1]$. We suppose also that the Lévy measure $\kappa(dz)$ of τ and the law of τ_t for any $t > 0$ have continuous densities, $\kappa(dz) = h(z)dz$, $\mathbb{P}[\tau_t \in ds]/ds = p_t(s)$ so that we may apply our results on the densities of the jumps of this subordinator.

Let us study the law of the excursions of the reflected process of Y . Let $P_u^\dagger(\cdot)$ be the law of the process $(u - Y_x)_{0 \leq x \leq \zeta}$ killed at the time ζ when it reaches 0. Since we know that the process Y oscillates or drifts to $+\infty$, we have $\zeta < \infty$ a.s. The following description then holds:

Proposition 2.4 :

An excursion of Y below its supremum a.s. begins by a jump, and its Itô excursion measure is given by

$$\tilde{n}(\cdot) = \int_{\mathbb{R}_+} P_u^\dagger(\cdot)h(u)du \tag{2.23}$$

In other words, such an excursion begins with a jump with “law” $h(u)du$ and evolves as the dual process of Y killed when it reaches 0.

// We first show that an excursion of Y below its supremum a.s. begins by a jump. From [104] we prove this by time-reversal arguments : if excursions could begin by a jump with positive probability, then an independent exponential time T would belong with positive probability to the interval of life of such an excursion. But the fact that this excursions starts “continuously” would imply that the time-reversed process $(Y_T - Y_{(T-t)-})_{0 \leq t \leq T}$ would not jump to its overall infimum with positive probability, which is impossible since for this process, 0 is irregular for $(-\infty, 0)$.

Now we remark that

$$\bar{Y}_s = \int_0^s \mathbb{1}_{\{Y_u = \bar{Y}_u\}} du = \int_0^s \mathbb{1}_{\{Y_{u-} = \bar{Y}_u\}} du$$

which follows from the fact that Y has a slope 1, and where the second

equality follows from the fact that the set $\{u : Y_u \neq Y_{u-}\}$ has zero Lebesgue measure a.s. Thus we a.s. have $d\bar{Y}_u = \mathbb{1}_{\{Y_{u-} = \bar{Y}_u\}} du$

For any positive measurable functional F we then have by compensation formula for the Poisson point process of the excursions (recall that \bar{Y} is a local time for the reflected process of Y):

$$\begin{aligned} \mathbb{E} \left[\int_0^1 d\bar{Y}_u \int \tilde{n}(d\gamma) F(\gamma) \right] &= \mathbb{E} \left[\sum_{0 < x \leq 1} F(\gamma_x) \right] \\ &= \mathbb{E} \left[\sum_{0 < s} \mathbb{1}_{\{\bar{Y}_s \leq 1\}} \mathbb{1}_{\{Y_{s-} = \bar{Y}_s\}} F \left((Y_{s-} - Y_{s+u})_{0 \leq u \leq \tilde{T}_{\{0\}} \right) \right] \end{aligned}$$

where $\tilde{T}_{\{0\}}$ is the first time when the process $(Y_{s-} - Y_{s+u})_{u \geq 0}$ hits 0. We also remark that the process \tilde{Y} defined by

$$\tilde{Y}_u = Y_s - Y_{s+u}$$

has the same law as Y and is independent of $(Y_u)_{0 \leq u \leq s}$. As such, it is independent of $\Delta Y_s = Y_s - Y_{s-}$. Noting that

$$Y_{s-} - Y_{s+u} = \Delta Y_s - \tilde{Y}_s$$

The strong Markov property then gives, since every excursion begins with a jump

$$\mathbb{E} \left[\sum_{0 < x \leq 1} F(\gamma_x) \right] = \mathbb{E} \left[\sum_{0 < s} \mathbb{1}_{\{\bar{Y}_s \leq 1\}} \frac{d\bar{Y}_s}{ds} \tilde{E}_{-\Delta Y_s} \left[F \left((-\tilde{Y}_u)_{0 \leq u \leq \tilde{T}_{\{0\}} \right) \right] \right]$$

Where \tilde{E}_x is the expectation with respect to the process \tilde{Y} , starting at $-x$.

To conclude, we apply the compensation formula for the Poisson point process $(-\Delta Y_s)_{s \geq 0}$ which has Lévy measure $h(z)dz$:

$$\begin{aligned} E[\bar{Y}_1] \int \tilde{n}(d\gamma) F(\gamma) &= \\ \mathbb{E} \left[\int_0^{+\infty} ds \mathbb{1}_{\{\bar{Y}_s \leq 1\}} \frac{d\bar{Y}_s}{ds} \int h(z) dz E_z \left[F \left((-\tilde{Y}_u)_{0 \leq u \leq \tilde{T}_{\{0\}} \right) \right] \right] \end{aligned}$$

which finally gives

$$\int \tilde{n}(d\gamma) F(\gamma) = \int h(z) dz \hat{E}_z^\dagger [F((\cdot)_{0 \leq u \leq \zeta})]. \quad //$$

Let us then try to give more detail about the “law” under \tilde{n} of the excursion with duration ℓ . Recall that D denotes the lifetime of the excursion, and that the lifetimes are the jumps of a Poisson point process with intensity $(p_t(t)/t)dt$ according to Lemma 2.6 (with notations adapted to Y). Hence we have

$$\tilde{n}(D \in d\ell) = \frac{p\ell(\ell)}{\ell} d\ell$$

According to the preceding lemma, it seems natural to try to give sense to the conditional probability law $P_u^\dagger(\cdot|\zeta = \ell)$ for every $\ell > 0$. For this we introduce another probability law $P_{u,\ell}$ which is the law of the process $\tilde{Y} = u - Y$ given that $u - Y_\ell = 0$. This “bridge-type” conditional law is well-defined by the methods described in this article (Kallenberg bridges and joint law of the jumps of a subordinator). It can be rewritten as being the process $u - Y$ before time ℓ with jumps conditioned to have sum $\ell - u$, and it has exchangeable increments. This is also the case for the process obtained by reversing the time at ℓ , which has exchangeable increments, drift coefficient u/ℓ and slope 1, and as such is positive with positive probability u/ℓ from Lemma 2.2. This allows us to give a “good” regular version for $P_u^\dagger(\cdot|\zeta)$.

Lemma 2.7 :

| The probability law $P_{u,\ell}[\cdot|\tilde{Y}_s \geq 0 \forall s \in [0, \ell]]$ is a regular version for $P_u^\dagger(\cdot|\zeta = \ell)$.

// By Lebesgue's derivation theorem, for every measurable positive functional H , we have that for a.e. $\ell > 0$, $E_u^\dagger[H|\zeta = \ell]$ is the limit of $E_u^\dagger[H|\zeta \in [\ell, \ell + \epsilon]]$ as $\epsilon \rightarrow 0$. On the other hand, Markov's property gives that

$$\begin{aligned} & E_u^\dagger[H, \zeta \in [\ell, \ell + \epsilon]] \\ &= E_u[H, \tilde{Y}_s \geq 0 \forall s \in [0, \ell], \exists s \in [\ell, \ell + \epsilon], \tilde{Y}_s = 0] \\ &= \int_{w \geq 0} E_u[H, \tilde{Y}_s \geq 0 \forall s \in [0, \ell] | \tilde{Y}_\ell = w] P_u[\tilde{Y}_\ell \in dw] P_w^\dagger[\zeta \leq \epsilon] \\ &= \int_{w \geq 0} E_u[H | \tilde{Y}_s \geq 0 \forall s \in [0, \ell], \tilde{Y}_\ell = w] P_u[\tilde{Y}_s \geq 0 \forall s \in [0, \ell] | \tilde{Y}_\ell = w] \\ &\times P_u^\dagger[\tilde{Y}_\ell \in dw] P_w[\zeta \leq \epsilon] \end{aligned}$$

Then divide by

$$P_u^\dagger[\zeta \in [\ell, \ell + \epsilon]] = \int_{w \geq 0} P_u[\tilde{Y}_s \geq 0 \forall s \in [0, \ell] | \tilde{Y}_\ell = w] P_u^\dagger[\tilde{Y}_\ell \in dw] P_w[\zeta \leq \epsilon],$$

and notice, from the fact that if the process \tilde{Y} has its first zero at a time in $[z, z + \epsilon]$, it can not be greater than ϵ at time z , that the measure

$$P_u[\tilde{Y}_s \geq 0 \forall s \in [0, \ell] | \tilde{Y}_\ell = w] P_u^\dagger[\tilde{Y}_\ell \in dw] P_w[\zeta \leq \epsilon]$$

only charges $[0, \epsilon]$. We then obtain that the limit is $P_u[\cdot | \tilde{Y}_s \geq 0 \forall s \in [0, \ell], \tilde{Y}_\ell = 0]$. //

Remark. Notice that formula (2.15), in the case where X has bounded variation, is an immediate consequence of this lemma and the ballot Theorem.

It is then easy to define the excursion with duration ℓ . Indeed, we obtain from (2.23) and our last discussion

$$\begin{aligned} \tilde{n}(\cdot) &= \int_{\mathbb{R}_+} h(u) du \int_u^{+\infty} P_u^\dagger(\cdot|\zeta = \ell) \frac{u}{\ell} p_\ell(\ell - u) d\ell \\ &= \int_{\mathbb{R}_+} \frac{p_\ell(\ell)}{\ell} d\ell \int_0^\ell \frac{u h(u) p_\ell(\ell - u)}{p_\ell(\ell)} P_u^\dagger(\cdot|\zeta = \ell) du \\ &= \int_{\mathbb{R}_+} \tilde{n}(D \in d\ell) \int_0^\ell \mu_\ell(du) P_u^\dagger(\cdot|\zeta = \ell), \end{aligned}$$

where

$$\mu_\ell(du) = \frac{uh(u)p_\ell(\ell - u)}{p_\ell(\ell)} du.$$

As a consequence,

$$\tilde{n}(\cdot | D = \ell) = \int_0^\ell \mu_\ell(du) P_u^\dagger(\cdot | \zeta = \ell) \quad (2.24)$$

which means that the excursion with life ℓ begins with a jump distributed by μ_ℓ and evolves as the dual process of $Y^{(t)}$ to which we added u , conditioned to first hit 0 at time ℓ . Notice from (2.17) that μ_ℓ is a probability measure, which is the law of a size-biased pick from the jumps of τ before time ℓ conditionally on $\tau_\ell = \ell$.

Let us sum up our study of the excursions of the reflected process of Y with duration ℓ . It begins with a jump u with law μ_ℓ , and then conditionally on this jump it evolves as the Kallenberg bridge on $[0, \ell]$ starting at u , which jumps are that of the subordinator τ before time ℓ given $\tau_\ell = \ell - u$, and conditioned to stay positive on $[0, \ell]$. To complete the proof of Vervaat's Theorem, it suffices to identify this description as that of the Vervaat's Transform of the bridge of Y with length ℓ from 0 to 0.

We recall a lemma that can be found in Pitman-Yor [94]: it states that if (δ_i) is the decreasing sequence of the atoms of a Poisson measure such that $\sum \delta_i < \infty$ a.s. then conditionally on $\sum \delta_i = \ell$, if δ^* is a size-biased from the (δ_i) , then given δ^* the law of the other atoms is the law of (δ_i) given $\sum \delta_i = \ell - \delta^*$.

Let $(\delta_i)_{i \geq 1}$ have the law of the decreasing sequence of the jumps of τ before time ℓ given $\tau_\ell = \ell$, so that the bridge y of Y with length ℓ from 0 to 0 is equal in law to the Kallenberg bridge on $[0, \ell]$ with zero drift and jumps $-(\delta_i)_{i \geq 1}$. Recall from Corollary 2.1 that the final value of Vy has the law of a size-biased pick from the sequence (δ_i) . Together with the lemma recalled in the preceding paragraph, we have the following description for the Vervaat's transform of y : conditionally on the value z of a r.v. with law μ_ℓ (a size-biased pick from the δ_i 's), Vy has the law of the bridge of Y starting at 0 and ending at z , conditioned to stay positive. It is thus the time-reversed excursion of Y below its supremum with life ℓ , which ends the proof of Vervaat's Theorem.

We end this section with a slight digression about some interesting properties of the processes $Y^{(t)}$ defined in 2.4.2, and which could imply Vervaat's Theorem in the general case.

From the bounded variation proof above we know that Vervaat's Theorem holds also for the bridge of the process $(Y_{tx}^{(t)})_{x \geq 0}$.

We have the

Lemma 2.8 :

We have

$$(Y_{tx}^{(t)})_{x \geq 0} \xrightarrow[t \rightarrow +\infty]{(\mathcal{L})} X,$$

where this notation means convergence in law in the Skorohod space \mathbb{D} .

// First, notice that from the results of section 2.4, the two reflected processes of $Y_t^{(t)}$ and X share, up to a drift coefficient, the same inverse local time processes, and in particular, the lengths of their excursions are the same.

Then we have $Y_{tx}^{(t)} = t(x - T_{tx}^{(t)})$, where for a fixed x ,

$$\begin{aligned} T_{tx}^{(t)} &= \inf\{u \geq 0 : X_u + tu > tx\} \\ &= \inf\{u \geq 0 : X_u > t(x - u)\} \end{aligned}$$

which means that $T_{tx}^{(t)}$ is the infimum abscissa of the intersection points of the trajectory of X and of the line $y = t(u - x)$ (which intuitively tends to $u = x$ when t tends to infinity, which permits to say that $T_{tx}^{(t)}$ should converge to x). The fact that X has no positive jumps implies that these intersection points always exist. Moreover, the a.s. continuity of X at x gives indeed that $T_{tx}^{(t)}$ a.s. converges to x . Then from $X_{T_{tx}^{(t)}} = tx$, we get $X_{T_{tx}^{(t)}} = Y_{tx}^{(t)}$ and a.s. convergence of this last r.v. to X_x , using again the a.s. continuity of X at x .

Since the processes we are considering are Lévy processes, this last convergence implies convergence in law for the processes. //

It seems that this lemma may imply the Vervaat's Theorem for processes with infinite variation, if we could prove that the laws of the bridge and the excursion below its supremum of $(Y_{tx}^{(t)})_{x \geq 0}$ converge weakly to the associated laws for X . But even it is intuitively clear, it seems that it can not be obtained without imposing more regularity properties for the law of X_s , $s > 0$.

2.8 Concluding remarks

To conclude, we make some comments about the bounded variation case to enlighten the links between section 2.2 and the rest of the paper. We will give no proof as arguments similar to the ones already used apply.

Recall that a Lévy process with no positive jumps and bounded variation can be written in the form $Y_s = cs - \tau_s$, where τ is a strict subordinator and $c \in \mathbb{R}$. Moreover, if Y does not drift to $-\infty$ we must have $c \geq E[\tau_1]$, and for convenience (i.e. up to change the time scale in the sequel) we suppose that $c = 1$. Suppose that the Lévy measure of τ is infinite. Let y be the bridge of Y with length 1 from 0 to 0, so that Vy is the time-reversed excursion ε of Y below its supremum with duration 1 (again, we could consider the case where the duration is ℓ by changing the time). As in section 2.2.4 let $\bar{\varepsilon}^{(t)}$ be the supremum process of $(ts - \varepsilon(s))_{0 \leq s \leq \ell}$ and $F(t)$ be the decreasing sequence of the lengths of its intervals of constancy, multiplied by $1 + t$ so that their sum is 1. Also, the locations of the jump times of ε on $[0, 1]$ induce an order the intervals of constancy, which we call $O^Y(t) \in \mathcal{O}_\infty$, so that $(i, j) \in O^Y(t)$ if the i -th and j -th largest jumps of ε occur in the same constancy interval of $\bar{\varepsilon}^{(t)}$, and the i -th is located on the left of the j -th.

The following assertion should be clear from section 2.2 :

Proposition 2.5 :

$$F\left(\frac{e^{-t}}{1 - e^{-t}}\right), \quad t \geq 0$$

is an additive coalescent starting from the random element $(\delta_i)_{i \geq 1}$ which law is that of the jumps of τ before time 1 given $\tau_1 = 1$ (and ranked in decreasing order).

Also, the process $(O^Y(e^{-t}/(1 - e^{-t})), t \geq 0)$ is somehow a mixture of ordered additive coalescents starting from O_{Sing} , and must be interpreted as an “ordered additive coalescent with random proto-galaxy masses”, determining the order of the coalescences in the process of Proposition 2.5. Last, if τ is a compound Poisson process, there is a similar result for the processes that start from finite number of fragments, but where the time-change that is appearing is random.

Now let as before ε_1 be the normalized excursion of a spectrally negative Lévy process X with infinite variation under its supremum. For any $t \geq 0$ and $0 \leq s \leq t$, since the constancy intervals of $\bar{\varepsilon}_1^{(t)}$ partition that of $\bar{\varepsilon}_1^{(s)}$, there is a natural order $O^{\varepsilon_1, t}(s)$ defined as follows: say that $(i, j) \in O^{\varepsilon_1, t}(s)$ if the i -th and j -th largest constancy intervals of $\bar{\varepsilon}_1^{(t)}$ are included in the same constancy interval of $\bar{\varepsilon}_1^{(s)}$, and the i -th is located on the left of the j -th. Write $(\mathbf{t}_x^{(t)}, 0 \leq x \leq t)$ for the right inverse of the process $\bar{\varepsilon}_1^{(t)}$, write $\kappa_x^{(t)} = x - t\mathbf{t}_x^{(t)}$ and notice that $(\kappa_x^{(t)}, 0 \leq x \leq t)$ has the law of the excursion of $(Y_x^{(t)}, x \geq 0) = (x - tT_x^{(t)}, x \geq 0)$ below its supremum with duration t , because of the one-to-one correspondence of the excursions of this process with that of X . Moreover, a computation similar to that of Lemma 2.4 and using Lemma 2.3 shows that for any $0 \leq s \leq t$,

$$\bar{\varepsilon}_1^{(s)}(r) = (t - s)\bar{\kappa}_{\bar{\varepsilon}_1^{(t)}(r)/t}^{(t, s/(t-s))}, \quad 0 \leq r \leq 1,$$

where $\bar{\kappa}^{(t, s/(t-s))}$ is the supremum process of $(t^{-1}\kappa_{tx}^{(t)} + xs/(t - s), 0 \leq x \leq 1)$. Since $(\kappa_{tx}^{(t)}/t, 0 \leq x \leq 1)$, is the excursion with duration 1 of a Lévy process with finite variation, we can associate an order $O^{\kappa, t}(s)$ to it as above. But then, it is not difficult to deduce from the previous displayed equation that in fact $O^{\kappa, t}(s/(t - s)) = O^{\varepsilon_1, t}(s)$. Therefore, $O^{\varepsilon_1, t}(\exp(-v - \log t)) = O^{\kappa, t}(\exp(-v - \log t)/(1 - \exp(-v - \log t)))$ for every $t, v \geq -\log t$, so by Proposition 2.5 the process $(O^{\varepsilon_1, t}(\exp(-v - \log t)), v \geq -\log t)$ is an ordered additive coalescent started at time $-\log t$.

As a conclusion, we see that, by enriching the structure of the additive coalescent $(O^{\varepsilon_1}(s), s \geq -\log t)$ defined above by taking into account the positions of constancy intervals associated to the underlying Lévy process, we turn it into an ordered additive coalescent $(O^{\varepsilon_1, t}(\exp(-s - \log t)), s \geq -\log t)$, this for any t (we cannot define an ordered additive coalescent on the whole time-interval \mathbb{R} , though, which can be seen from the fact that the orders defined for two different values of t are not consistent, i.e. $O^{\varepsilon_1, t}(0)$ need not contain $O^{\varepsilon_1, t'}(0)$ for some $t < t'$).

Chapter 3

Self-similar fragmentations and stable subordinators

Contents

3.1	Introduction	73
3.2	A Poisson process construction of self-similar fragmentations	78
3.3	Stable subordinators	79
3.4	Small-time behavior of self-similar fragmentations	83
3.5	Large-time behavior of self-similar fragmentations	90
3.6	One-dimensional distributions	91
3.7	Mass of a tagged fragment	94

3.1 Introduction

Fragmentation processes describe an object that breaks into smaller pieces in a random way as time moves forward. *Ranked fragmentations* are Markov processes taking their values in the set $\Delta = \{(x_i)_{i=1}^{\infty} : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1\}$. If $(X(t), t \geq 0)$ is a ranked fragmentation, we can regard the terms in the sequence $X(t)$ as being the masses of the components into which the object has fragmented after time t , with the masses being ranked in decreasing order. Alternatively, one can study *partition-valued fragmentations*, which take their values in the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$. If $(\Pi(t), t \geq 0)$ is a partition-valued fragmentation and $s < t$, then the partition $\Pi(t)$ is a refinement of the partition $\Pi(s)$.

In recent years, a fragmentation introduced in [10] by Aldous and Pitman, which we call the *Aldous–Pitman fragmentation*, has been studied extensively. Aldous and Pitman first constructed this fragmentation process from the Brownian continuum random tree (CRT) of Aldous (see [3, 4, 5]). The CRT is equipped with a finite “mass measure” concentrated on the leaves of the tree and a σ -finite “length measure” on the skeleton of the tree. When the CRT is cut at various points along the skeleton, the tree is split into components whose masses sum to one. Aldous and Pitman defined a ranked fragmentation process $(Y(t), t \geq 0)$ such that $Y(t)$ consists of the ranked sequence of masses of tree

components after the CRT has been subjected to a Poisson process of cuts at rate t per unit length.

One can also obtain a partition-valued fragmentation $(\Pi(t), t \geq 0)$ by first picking leaves U_1, U_2, \dots independently from the mass measure of the CRT, and then declaring i and j to be in the same block of $\Pi(t)$ if and only if the leaves U_i and U_j are in the same tree component at time t . To see how this process is related to $(Y(t), t \geq 0)$, we first give a definition. If $B \subset \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{j \in B\}}$$

exists, then this limit is called the *asymptotic frequency* of B . If π is a partition of \mathbb{N} , let $\Lambda(\pi)$ be the sequence consisting of the asymptotic frequencies of the blocks of π ranked in decreasing order (whenever these frequencies exist). Then $(\Lambda(\Pi(t)), t \geq 0) =_d (Y(t), t \geq 0)$.

The Aldous–Pitman fragmentation has arisen in a variety of contexts. Aldous and Pitman showed in [10] that if $X(t) = Y(e^{-t})$, then the process $(X(t), -\infty < t < \infty)$ is a version of the *standard additive coalescent*. Loosely speaking, the standard additive coalescent is a coalescent process with the property that fragments of masses x and y are merging together at the rate $x + y$. See [52], [12], and [22] for more results related to the additive coalescent. Chassaing and Louchard [41] related the process $(Y(t), t \geq 0)$ to parking functions in combinatorics. Also, Bertoin [21, 22] showed that $(Y(t), t \geq 0)$ can be constructed from a Brownian motion with drift and that the so-called *eternal* versions of the additive coalescent could be constructed in a similar way from excursions of processes with exchangeable increments. Miermont [79] used this method to generalize [21] by studying a larger class of fragmentation processes, related to the additive coalescent, which can be obtained by adding drift to a general Lévy process with no positive jumps, implying several explicit laws for certain versions of the additive coalescent. The use of the ballot theorem therein was motivated by a similar approach of Schweinsberg [100] to analyze some functionals of the Brownian excursion.

The starting point for the present paper is the following theorem due to Aldous and Pitman, which shows three ways in which the Aldous–Pitman fragmentation is related to the stable subordinator of index $1/2$.

Theorem 3.1 :

Let $(Y(t), t \geq 0)$ be the Aldous–Pitman fragmentation, and write the components of the fragmentation as $Y(t) = (Y_1(t), Y_2(t), \dots)$. Also, let $(\Pi(t), t \geq 0)$ be a partition-valued fragmentation with the property that $(\Lambda(\Pi(t)), t \geq 0) =_d (Y(t), t \geq 0)$. Let $Y^*(t)$ be the asymptotic frequency of the block of $\Pi(t)$ containing the integer 1. Let $(\sigma_{1/2}(t), t \geq 0)$ be a stable subordinator of index $1/2$. Then, the following hold:

1. For every $t \geq 0$, we have $Y(t) =_d (J_1, J_2, \dots | \sigma_{1/2}(t) = 1)$, where J_1, J_2, \dots are the jump sizes of $\sigma_{1/2}$ up to time t , ranked in decreasing order.

2. We have

$$(Y^*(t), t \geq 0) =_d \left(\frac{1}{1 + \sigma_{1/2}(t)}, t \geq 0 \right).$$

3. As $t \rightarrow 0$, we have $t^{-2}(1 - Y_1(t), Y_2(t), Y_3(t), \dots) \rightarrow_d (\sigma_{1/2}(1), J_1, J_2, \dots)$.

Part 1 of the theorem can easily be obtained from Theorem 4 of [10] and scaling properties of stable subordinators. Part 2 is Theorem 6 of [10]. Part 3 is Corollary 13 of [10].

It is natural to ask whether there are other fragmentation processes related to the stable subordinator of index $\alpha \in (0, 1)$ in the same ways that the Aldous–Pitman fragmentation is related to the stable subordinator of index $1/2$. In [26], Bertoin constructed a family of fragmentation processes, called self-similar fragmentations, which satisfy a scaling property. Because the Aldous–Pitman fragmentation is self-similar, one might expect the family of self-similar fragmentations to include fragmentations with properties that generalize properties of the Aldous–Pitman fragmentation. The purpose of this paper is to consider separately the three parts of Theorem 3.1 and to determine whether there are other self-similar fragmentations for which similar results hold, with the stable subordinator of index $1/2$ replaced by the stable subordinator of index α . Our conclusion, made precise by Theorem 3.2 and Propositions 3.1 and 3.2 below, is that only part 3 relating to asymptotics as $t \rightarrow 0$ can be easily generalized. Parts 1 and 2 of the theorem describe special properties of the $\alpha = 1/2$ case which do not extend, at least not in the most natural way, to other $\alpha \in (0, 1)$.

Before stating these results, we will define self-similar fragmentations and review some of their properties. For $0 \leq l \leq 1$, define $\Delta_l = \{(x_i)_{i=1}^\infty : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^\infty x_i \leq l\}$. Note that $\Delta = \Delta_1$. We will denote points in Δ by $x = (x_1, x_2, \dots)$. Suppose $\kappa_t(l)$ is a probability measure on Δ_l for all $0 \leq l \leq 1$ and $t \geq 0$. For each $L = (l_1, l_2, \dots) \in \Delta$, let $\kappa_t(L)$ denote the distribution of the decreasing rearrangement of the terms of independent sequences L_1, L_2, \dots , where L_i has the distribution $\kappa_t(l_i)$ for all i . For each $t \geq 0$, denote by κ_t the family of distributions $(\kappa_t(L), L \in \Delta)$, which we call the *fragmentation kernel* generated by $(\kappa_t(l), 0 \leq l \leq 1)$. A time-homogeneous, Δ -valued Markov process whose transition semigroup is given by fragmentation kernels is called a *fragmentation process* or *ranked fragmentation*. This definition is essentially taken from [21], although we allow the sum of the masses of the fragments to decrease over time as in [17].

For $0 \leq l \leq 1$, let $g_l : \Delta \rightarrow \Delta_l$ be the map defined by $g_l(x_1, x_2, \dots) = (lx_1, lx_2, \dots)$. A ranked fragmentation is said to be a *homogeneous fragmentation* if, for all $0 \leq l \leq 1$ and $t > 0$, the probability measure $\kappa_t(l)$ is the image under g_l of the probability measure $\kappa_t(1)$. Notice that the term “homogeneous” does not refer to the assumed homogeneous Markov property of the semigroup. We call the fragmentation process a *self-similar fragmentation of index $\beta \in \mathbb{R}$* if, for all $0 \leq l \leq 1$ and $t > 0$, $\kappa_t(l)$ is the image under g_l of $\kappa_r(1)$, where $r = tl^\beta$. Note that a self-similar fragmentation of index 0 is a homogeneous fragmentation.

Bertoin formulated definitions of homogeneity and self-similarity for partition-valued fragmentations that are analogous to the definitions given above for ranked fragmentations. In [23], Bertoin showed that all homogeneous partition-valued fragmentations can be described in terms of an erosion rate $c \geq 0$ and a measure ν on $\Delta \setminus (1, 0, 0, \dots)$, called the Lévy measure (or dislocation measure), which satisfies

$$\int_{\Delta} (1 - x_1) \nu(dx) < \infty. \quad (3.1)$$

In [26], Bertoin showed that all self-similar fragmentations can be obtained from homogeneous fragmentations by a random time change which is determined by β . Consequently, all self-similar partition-valued fragmentations are fully described by their *characteristics* (β, c, ν) . For each triple (β, c, ν) , Bertoin constructs a self-similar fragmentation with

these characteristics from a Poisson process. We will present this construction in the next section. The erosion rate c describes the rate at which singletons break away from larger blocks of the partition, and the Lévy measure governs the rates of other fragmentation events. If $\nu(\{x : x_1 + x_2 < 1\}) = 0$, then no block will break into more than two blocks at any given time. We then call the process a *binary* fragmentation.

If $(\Pi(t), t \geq 0)$ is a self-similar partition-valued fragmentation, then $\Pi(t)$ is an exchangeable random partition for all t . It follows from results of Kingman [69] that almost surely each block of $\Pi(t)$ has an asymptotic frequency. By Theorem 3 of [23], we have the stronger result that if $(\Pi(t), t \geq 0)$ is homogeneous, then almost surely all blocks of $\Pi(t)$ have asymptotic frequencies for all t . We can then see from the construction described in section 3 of [26] (and recalled in Sect. 3.2 below) that there exists a version $(\Pi(t), t \geq 0)$ of any self-similar fragmentation process such that almost surely all blocks of $\Pi(t)$ have asymptotic frequencies for all t . Furthermore, if $(\Pi(t), t \geq 0)$ denotes this version of a self-similar partition-valued fragmentation (which we will always suppose in the sequel when considering self-similar fragmentations), then $(\Lambda(\Pi(t)), t \geq 0)$ is a self-similar ranked fragmentation with the same index of self-similarity. Berestycki [17] showed conversely that if $(X(t), t \geq 0)$ is a self-similar ranked fragmentation, then there exists a self-similar partition-valued fragmentation $(\Pi(t), t \geq 0)$ such that $(\Lambda(\Pi(t)), t \geq 0) = (X(t), t \geq 0)$. Consequently, self-similar ranked fragmentations are also in one-to-one correspondence with triples (β, c, ν) , where $\beta \in \mathbb{R}$, $c \geq 0$, and ν is a measure on $\Delta \setminus (1, 0, 0, \dots)$ satisfying (3.1). Thus, we may work either with partition-valued fragmentations or ranked fragmentations, and both will be useful later in the paper.

Several examples of self-similar fragmentations have been studied. In [37] and [38], Brennan and Durrett studied a family of self-similar fragmentations. In the same context, see also Filippov [54]. Bertoin [26] considered an example that is related to Brownian excursions. Bertoin also observed in [26] that the Aldous–Pitman fragmentation is the binary self-similar fragmentation with characteristics $(1/2, 0, \nu)$, where the restriction of ν to the first coordinate has density $h(x) = (2\pi)^{-1/2} x^{-3/2} (1-x)^{-3/2} \mathbb{1}_{[1/2, 1]}(x)$.

The following theorem, which is our main result, is related to part 1 of Theorem 3.1 about one-dimensional distributions. Here, and throughout the rest of the paper, $\sigma_\alpha = (\sigma_\alpha(t), t \geq 0)$ denotes a stable subordinator of index α .

Theorem 3.2 :

Let $(X(t), t \geq 0)$ be a self-similar fragmentation, and let $\alpha \in (0, 1)$. Let $J_1(t) \geq J_2(t) \geq \dots$ be the ranked jump sizes of σ_α between times 0 and t . If

$$X(t) =_d (J_1(t), J_2(t), \dots | \sigma_\alpha(t) = 1) \quad (3.2)$$

for all t , then $\alpha = 1/2$ and $(X(t), t \geq 0)$ is the Aldous–Pitman fragmentation.

The distributions on the right-hand side of (3.2) are part of a larger family of distributions studied in [92, 86]. Suppose $J_1 \geq J_2 \geq \dots$ is the ranked sequence of points from a Poisson process with intensity measure Θ on $(0, \infty)$, where Θ has density $\theta(x)$ and integrates $1 \wedge x$. Let $T = \sum_{i=1}^{\infty} J_i$. Then $(J_i/T)_{i=1}^{\infty}$ is a random point in Δ . Its distribution is called the *Poisson–Kingman distribution with Lévy density θ* and is denoted by $\text{PK}(\theta)$. The conditional distribution of $(J_i/T)_{i=1}^{\infty}$ given $T = t$ is denoted by $\text{PK}(\theta|t)$. Since $\sigma_\alpha(t) =_d t^{1/\alpha} \sigma_\alpha(1)$ by scaling properties of stable subordinators, we have

$$(J_1(t), J_2(t), \dots | \sigma_\alpha(t) = 1) =_d (t^{1/\alpha} J_1(1), t^{1/\alpha} J_2(1), \dots | \sigma_\alpha(1) = t^{-1/\alpha}). \quad (3.3)$$

For $\alpha \in (0, 1)$, let θ_α be the Lévy density given by $\theta_\alpha(x) = C_\alpha x^{-\alpha-1}$, where C_α is the constant defined later in (3.8). If $J_1(t) \geq J_2(t) \geq \dots$ are the ranked jump sizes of σ_α between times 0 and t , then the distribution of (J_1, J_2, \dots) has the same distribution as the ranked sequence of points of a Poisson point process with Lévy density $t\theta_\alpha$. Therefore (3.3) implies that the $\text{PK}(t\theta_\alpha|1)$ distribution is the same as the $\text{PK}(\theta_\alpha|t^{-1/\alpha})$ distribution. Theorem 3.1 therefore shows that if $(Y(t), t \geq 0)$ is the Aldous–Pitman fragmentation, then $Y(t)$ has the $\text{PK}(\theta_{1/2}|t^{-2})$ distribution. Theorem 3.2 shows that there is no other self-similar fragmentation $(X(t), t \geq 0)$ such that the distribution of $X(t)$ is $\text{PK}(\theta_\alpha|t^{-1/\alpha})$ for all t . We have not, however, ruled out the possibility that a fragmentation which is not self-similar may have this property. In general, it remains an open problem to characterize the Lévy densities θ for which there exists a fragmentation process $(Z(t), t \geq 0)$ and a function $f : (0, \infty) \rightarrow (0, \infty)$ such that $Z(t)$ has the $\text{PK}(\theta|f(t))$ distribution for all $t > 0$. However, we note that Miermont, in [79], has studied fragmentation processes that are not self-similar whose one-dimensional distributions are those of jump sizes for conditioned subordinators with varying Lévy measure, and one can show that a subclass of these fragmentations satisfy the asymptotics (3.4) below.

We now turn to a result for partition-valued fragmentations that pertains to the distribution of the mass of the block containing 1, which we sometimes call a “tagged fragment”. The distribution of the mass of this block at time t is the same as the distribution of a size-biased pick from the sizes of the fragments of the corresponding ranked fragmentation at time t , provided that the sum of the sizes of the fragments at time t is 1 almost surely.

Proposition 3.1 :

Let $(\Pi(t), t \geq 0)$ be a partition-valued binary self-similar fragmentation. Let $\alpha \in (0, 1)$. Let $\lambda(t)$ be the asymptotic frequency of the block of $\Pi(t)$ containing the integer 1. If for some decreasing function g ,

$$(\lambda(t), t \geq 0) =_d (g(\sigma_\alpha(t)), t \geq 0),$$

then $\alpha = 1/2$, $g(x) = (1 + Kx)^{-1}$ for some $K > 0$ and $(\Lambda(\Pi(t)), t \geq 0)$ is the Aldous–Pitman fragmentation, up to a multiplicative time constant.

Our next result gives, for each $\alpha \in (0, 1)$, a family of binary self-similar fragmentations whose asymptotics as $t \rightarrow 0$ are related to the stable subordinator of index α .

Proposition 3.2 :

Fix $\alpha \in (0, 1)$, and let $C_\alpha = \alpha/(\Gamma(1 - \alpha) \cos(\frac{\pi\alpha}{2}))$. Let ν be a Lévy measure on Δ such that $\nu(\{x : x_1 + x_2 < 1\}) = 0$ and the restriction ν_2 of ν to the second coordinate has density h , where

$$h(x) = C_\alpha x^{-1-\alpha} s(x) \mathbb{1}_{[0, 1/2]}(x)$$

for some positive function s satisfying $\lim_{x \rightarrow 0} s(x) = 1$. Let $\beta \geq 0$. Let $(X(t), t \geq 0)$ be the self-similar fragmentation with characteristics $(\beta, 0, \nu)$. Write $X(t) = (X_1(t), X_2(t), \dots)$. Then, as $t \rightarrow 0$, we have

$$t^{-1/\alpha}(1 - X_1(t), X_2(t), X_3(t), \dots) \rightarrow_d (\sigma_\alpha(1), J_1(1), J_2(1), \dots), \quad (3.4)$$

where $J_1(1) \geq J_2(1) \geq \dots$ are the jump sizes of σ_α up to time 1.

Another connection between the self-similar fragmentations in Proposition 3.2 and

stable subordinators can be deduced from Bertoin's work [25] regarding the small masses in self-similar fragmentations. Consider a binary self-similar fragmentation $(X(t), t \geq 0)$ with characteristics $(\beta, 0, \nu)$, where $\beta \geq 0$. Let ν_2 be the restriction of ν to the second coordinate. Let

$$N(\varepsilon, t) = \max\{i : X_i(t) > \varepsilon\}$$

be the number of components in the fragmentation at time t whose size is greater than ε . Let

$$M(\varepsilon, t) = \sum_{i=1}^{\infty} X_i(t) \mathbb{1}_{\{X_i(t) < \varepsilon\}}$$

be the total mass of the fragments at time t of size less than ε . Define $\phi(\varepsilon) = \nu_2([\varepsilon, 1/2])$ and $f(\varepsilon) = \int_0^\varepsilon x \nu_2(dx)$. It follows from Theorem 1 of [25] that ϕ is regularly varying as $\varepsilon \downarrow 0$ with index $-\alpha$ if and only if f is regularly varying as $\varepsilon \downarrow 0$ with index $1 - \alpha$. It also follows from Theorem 1 of [25] that if these regular variation conditions hold and $\beta = 1 - \alpha$, then for all $t > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{N(\varepsilon, t)}{\phi(\varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{M(\varepsilon, t)}{f(\varepsilon)} = t$$

with probability one. Therefore, a straightforward calculation shows that if $(X(t), t \geq 0)$ satisfies the conditions of Proposition 3.2 with $\beta = 1 - \alpha$, then $N(\varepsilon, t) \sim C_\alpha \alpha^{-1} t \varepsilon^{-\alpha}$ and $M(\varepsilon, t) \sim C_\alpha (1 - \alpha)^{-1} t \varepsilon^{1-\alpha}$ with probability one for all $t > 0$, where \sim means that the ratio of the two sides tends to 1 as $\varepsilon \downarrow 0$. For a stable subordinator of index α with Lévy measure $\eta(dx) = C_\alpha x^{-1-\alpha} dx$, the expected number of jumps of size larger than ε before time t is $C_\alpha \alpha^{-1} t \varepsilon^{-\alpha}$, and the expected value of the sum of the sizes of the jumps of size less than ε before time t is $C_\alpha (1 - \alpha)^{-1} t \varepsilon^{1-\alpha}$. Thus, $N(\varepsilon, t)$ behaves like the number of jumps of a stable subordinator of index α that have size larger than ε , while $M(\varepsilon, t)$ behaves like the sum of the sizes of the small jumps of a stable subordinator of index α .

The rest of this paper is organized as follows. In Sect. 2, we present the Poisson process construction of self-similar fragmentations given by Bertoin in [26]. In Sect. 3, we establish some relevant facts about stable subordinators. In Sect. 4, we relate the small-time behavior of self-similar fragmentations to the dislocation measure (Proposition 3.3) and prove Proposition 3.2. We review some of Bertoin's results on the large-time behavior of self-similar fragmentations in Sect. 5. Section 6 is devoted to the proof of Theorem 3.2, and Sect. 7 is devoted to the proof of Proposition 3.1.

3.2 A Poisson process construction of self-similar fragmentations

In [26], Bertoin shows how to construct an arbitrary partition-valued self-similar fragmentation with characteristics (β, c, ν) from a Poisson process. The conventions we are using here (for labelling partitions, and for taking reduced partitions in property 3 below) are actually those used in [17], but by exchangeability arguments explained therein they do indeed give the same distributional object as the construction in [23, 26].

Let ε_n be the partition of \mathbb{N} into the two blocks $\{n\}$ and $\mathbb{N} \setminus \{n\}$. Given $x = (x_1, x_2, \dots) \in \Delta$, let P^x be the distribution of the random partition Π obtained by first

defining an i.i.d. sequence of random variables $(Z_i)_{i=1}^\infty$ such that $P(Z_i = j) = x_j$ and $P(Z_i = 0) = 1 - \sum_{j=1}^\infty x_j$, and then defining Π to be the partition with the property that i and j are in the same block if and only if $Z_i = Z_j \geq 1$. Let κ be the measure on the set \mathcal{P} of partitions of \mathbb{N} defined such that for all Borel subsets B of \mathcal{P} , we have

$$\kappa(B) = \int_{\Delta} P^x(B) \nu(dx) + c \sum_{i=1}^{\infty} \mathbb{1}_{\{\varepsilon_n \in B\}}. \quad (3.5)$$

Now, let $\#$ denote counting measure on \mathbb{N} , and let $((\Gamma_t, k_t), t \geq 0)$ be a Poisson point process on $\mathcal{P} \times \mathbb{N}$ with intensity measure $\kappa \otimes \#$. We can use this Poisson point process to construct a partition-valued self-similar fragmentation with characteristics (β, c, ν) . The first step is to construct a homogeneous fragmentation with characteristics $(0, c, \nu)$. Let A_N consist of all partitions in \mathcal{P} such that not all the integers $\{1, \dots, N\}$ are in the same block. Then $\kappa(A_N) < \infty$ for all N , so $(\Gamma_t, k_t) \in A_N \times \{1, \dots, N\}$ for only a discrete set of times, which we can enumerate as $t_1 < t_2 < \dots$. Define $(\Pi_N(t), t \geq 0)$ to be the unique process taking its values in the set of partitions of $\{1, \dots, N\}$ that satisfies the following three properties:

1. $\Pi_N(0)$ is the trivial partition of $\{1, \dots, N\}$.
2. Π_N is constant on $[t_{i-1}, t_i)$ for all $i \in \mathbb{N}$, where we set $t_0 = 0$.
3. Integers i and j are in distinct blocks of $\Pi_N(t_i)$ if and only if either i and j are in distinct blocks of $\Pi_N(t_{i-1})$, or i and j are in distinct blocks of Γ_{t_i} and both i and j are in a block of $\Pi_N(t_{i-1})$ whose smallest integer is k_{t_i} .

If π is a random partition of \mathbb{N} , let $R_N\pi$ be the random partition of $\{1, \dots, N\}$ such that if $1 \leq i, j \leq N$, then i and j are in the same block of $R_N\pi$ if and only if they are in the same block of π . The processes Π_N are consistent as N varies, so there exists a unique process $(\Pi(t), t \geq 0)$ such that $(R_N\Pi(t), t \geq 0) = (\Pi_N(t), t \geq 0)$ for all N . Then $(\Pi(t), t \geq 0)$ is a homogeneous fragmentation with characteristics $(0, c, \nu)$, as discussed in [23].

In [26], Bertoin shows that any self-similar fragmentation can be constructed from a homogeneous fragmentation by a random time change. Let $I_n(t)$ be the asymptotic frequency of the block of $\Pi(t)$ containing n . Define

$$T_n^{(\beta)}(t) = \inf \left\{ u \geq 0 : \int_0^u |I_n(r)|^{-\beta} dr > t \right\}. \quad (3.6)$$

Define the process $(\Pi^{(\beta)}(t), t \geq 0)$ such that i and j are in the same block of $\Pi^{(\beta)}(t)$ if and only if i and j are in the same block of $\Pi(T_i^{(\beta)}(t))$. It is shown in [26] that $(\Pi^{(\beta)}(t), t \geq 0)$ is a self-similar partition-valued fragmentation with characteristics (β, c, ν) . Therefore, $(\Lambda(\Pi^{(\beta)}(t)), t \geq 0)$ is a self-similar ranked fragmentation with characteristics (β, c, ν) .

3.3 Stable subordinators

An \mathbb{R} -valued stochastic process $X = (X_t, t \geq 0)$ is called a *subordinator* if it is nondecreasing and has stationary independent increments. If X is a subordinator, then for all $\lambda \geq 0$, we have

$$-\frac{1}{t} \log E[e^{-\lambda X_t}] = d\lambda + \int_0^\infty (1 - e^{-\lambda x}) \eta(dx),$$

where $d \geq 0$ is the *drift coefficient* and η is the *Lévy measure* on $(0, \infty)$, which must satisfy $\int_0^\infty (1 \wedge x) \eta(dx) < \infty$. The process X is said to be a stable subordinator of index $\alpha \in (0, 1)$ if $d = 0$ and

$$\eta(dx) = C_\alpha x^{-1-\alpha} dx \quad (3.7)$$

for some constant C_α . Since changing the constant C_α just changes time by a constant factor, we lose no generality by considering just one value for C_α . We will therefore take

$$C_\alpha = \frac{\alpha}{\Gamma(1-\alpha) \cos(\pi\alpha/2)}. \quad (3.8)$$

We will denote by $(\sigma_\alpha(t), t \geq 0)$ a subordinator whose Lévy measure is given by (3.7) and (3.8). The stable subordinator of index α satisfies the scaling property

$$(\lambda^{1/\alpha} \sigma_\alpha(t), t \geq 0) =_d (\sigma_\alpha(\lambda t), t \geq 0) \quad \text{for every } \lambda > 0. \quad (3.9)$$

It is shown, for example, in chapter 17 of [56] that the characteristic function of $\sigma_\alpha(1)$ is given by

$$\phi(t) = \exp\left(-|t|^\alpha \left(1 - i \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right).$$

Remark. Proposition 11 in chapter 17 of [56] actually gives this result when $C_\alpha = 2\alpha\Gamma(\alpha) \sin(\pi\alpha/2)/\pi$, but this is equivalent to (3.8) because of the duplication formula $\Gamma(\alpha)\Gamma(1-\alpha) \sin(\frac{\pi\alpha}{2}) \cos(\frac{\pi\alpha}{2}) = \pi/2$ for all $\alpha \in (0, 1)$.

Let f_t be the density function of $\sigma_\alpha(t)$, and let $f = f_1$. It follows from the formulas given in [101] that if $A = \alpha^{1/2(1-\alpha)} (\cos \frac{\pi\alpha}{2})^{-1/(2(1-\alpha))} [2\pi(1-\alpha)]^{-1/2}$ and $B = (1-\alpha)\alpha^{\alpha/(1-\alpha)} (\cos(\frac{\pi\alpha}{2}))^{-1/(1-\alpha)}$, then

$$f(x) \sim Ax^{-1-\alpha/(2(1-\alpha))} \exp(-Bx^{-\alpha/(1-\alpha)}), \quad (3.10)$$

where \sim means that the ratio of the two sides goes to 1 as $x \rightarrow 0$.

To get asymptotics for large x , note that [101] gives

$$f(x) = \sum_{n=1}^{\infty} a_n x^{-1-\alpha n},$$

where

$$a_n = \frac{(-1)^{n-1} \Gamma(n\alpha + 1)}{n! \pi} \left(1 + \tan^2\left(\frac{\pi\alpha}{2}\right)\right)^{n/2} \sin(n\pi\alpha). \quad (3.11)$$

Stirling's formula gives $\lim_{n \rightarrow \infty} a_n = 0$, so there exists a constant D such that if we write

$$f(x) = a_1 x^{-1-\alpha} (1 + r(x)), \quad (3.12)$$

then $|r(x)| \leq Dx^{-\alpha}$ for all x .

It is well-known that σ_α is a pure-jump process. The sequence consisting of the jump sizes of σ_α between times 0 and t , ranked in decreasing order, has the same distribution as the ranked sequence of points from a Poisson random measure on $(0, \infty)$ with intensity measure $\rho_t(x) dx$, where $\rho_t(x) = C_\alpha t x^{-1-\alpha}$. It will be useful to consider size-biased picks

from the jump sizes of σ_α . We will use the following lemma, which can be deduced from Lemma 2.1 of [87].

Lemma 3.1 :

Fix $t > 0$. Let $J_1(t) \geq J_2(t) \geq \dots$ be the jump sizes of σ_α between times 0 and t . Let $J_1^*(t)$ be a size-biased pick from these jump sizes, and then let $J_2^*(t)$ be a size-biased pick from the remaining jump sizes. Then,

$$P(J_1^* \in dx | \sigma_\alpha(t) = z) = \frac{x \rho_t(x) f_t(z-x)}{z f_t(z)} dx,$$

and the joint density of $(\sigma_\alpha(t), J_1^*(t), J_2^*(t))$ is given by

$$h(z, x, y) = \frac{(x \rho_t(x))(y \rho_t(y)) f_t(z-x-y)}{z(z-x)}. \quad (3.13)$$

This Lemma implies the following result about the distribution as $t \rightarrow \infty$ of a size-biased pick from the jump sizes of $\sigma_\alpha(t)$, conditional on $\sigma_\alpha(t) = 1$.

Lemma 3.2 :

Let $J_1^*(t)$ be a size-biased pick from the jump sizes of σ_α between times 0 and t . Let μ_t denote the conditional distribution of $t^{1/(1-\alpha)} J_1^*(t)$ given $\sigma_\alpha(t) = 1$. As $t \rightarrow \infty$, μ_t converges weakly to the Gamma($1 - \alpha, B\alpha/(1 - \alpha)$) distribution.

// It follows from Lemma 3.1 that $P(J_1^*(t) \in dx | \sigma_\alpha(t) = 1) = g_t(x) dx$, where the density g_t is given by

$$g_t(x) = \frac{x \rho_t(x) f_t(1-x)}{f_t(1)} = \frac{C_\alpha t x^{-\alpha} f_t(1-x)}{f_t(1)} \quad (3.14)$$

for $x \in (0, 1)$. It follows from (3.9) that $f_t(x) = f(t^{-1/\alpha} x) t^{-1/\alpha}$ for all $x > 0$. Using this fact and (3.14), we see that μ_t has density

$$\begin{aligned} h_t(x) &= g_t(t^{-1/(1-\alpha)} x) t^{-1/(1-\alpha)} \\ &= \frac{C_\alpha t (t^{-1/(1-\alpha)} x)^{-\alpha} f(t^{-1/\alpha} (1 - t^{-1/(1-\alpha)} x))}{f(t^{-1/\alpha})} t^{-1/(1-\alpha)} \\ &= \frac{C_\alpha x^{-\alpha} f(t^{-1/\alpha} (1 - t^{-1/(1-\alpha)} x))}{f(t^{-1/\alpha})} \end{aligned}$$

for $0 < x < t^{1/(1-\alpha)}$. Using (3.10), it follows that for each $x > 0$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} h_t(x) &= \lim_{t \rightarrow \infty} \frac{C_\alpha x^{-\alpha} (1 - t^{-1/(1-\alpha)} x)^{-1-\alpha/(2(1-\alpha))}}{\exp(-Bt^{1/(1-\alpha)})} \\ &\quad \times \exp(-B(t^{-1/\alpha} (1 - t^{-1/(1-\alpha)} x))^{-\alpha/(1-\alpha)}) \\ &= \lim_{t \rightarrow \infty} C_\alpha x^{-\alpha} \exp(-Bt^{1/(1-\alpha)} ((1 - t^{-1/(1-\alpha)} x)^{-\alpha/(1-\alpha)} - 1)) \\ &= C_\alpha x^{-\alpha} e^{-B\alpha x/(1-\alpha)}. \end{aligned}$$

Note that if $\lambda = B\alpha/(1 - \alpha)$, then $\lambda^{1-\alpha} = \alpha/\cos(\frac{\pi\alpha}{2})$, and thus $C_\alpha = \lambda^{1-\alpha}/\Gamma(1-\alpha)$. Thus, h_t converges pointwise to the Gamma($1 - \alpha, \lambda$) density as $t \rightarrow \infty$. The result of the lemma then follows from Scheffé's Theorem. //

If Z has a Gamma($1 - \alpha, B\alpha/(1 - \alpha)$) distribution, then for all $r \geq 0$,

$$E[Z^r] = \frac{\Gamma(r + 1 - \alpha)}{\Gamma(1 - \alpha)} \left(\frac{B\alpha}{1 - \alpha} \right)^{-r} = \frac{\Gamma(r + 1 - \alpha)}{\Gamma(1 - \alpha)} \left(\frac{\cos(\frac{\pi\alpha}{2})}{\alpha} \right)^{\frac{r}{1-\alpha}}. \quad (3.15)$$

We will need these moments in Sect. 3.6.

We now consider small-time asymptotics.

Lemma 3.3 :

Let $J_1(t) \geq J_2(t) \geq \dots$ be the jump sizes of σ_α between times 0 and t . Let $J_1^*(t)$ be a size-biased pick from these jump sizes. If A is a Borel subset of $[0, 1 - a]$ for some $a > 0$, then

$$\lim_{t \rightarrow 0} t^{-1} P(J_1^*(t) \in A | \sigma_\alpha(t) = 1) = \int_A C_\alpha x^{-\alpha} (1 - x)^{-1-\alpha} dx. \quad (3.16)$$

If B is a Borel subset of $[1/2, 1 - a]$, then

$$\lim_{t \rightarrow 0} t^{-1} P(J_1(t) \in B | \sigma_\alpha(t) = 1) = \int_B C_\alpha x^{-1-\alpha} (1 - x)^{-1-\alpha} dx. \quad (3.17)$$

// For all $t > 0$ and all Borel subsets A of $[0, 1 - a]$, Lemma 3.1 implies that

$$\begin{aligned} t^{-1} P(J_1^*(t) \in A | \sigma_\alpha(t) = 1) &= \int_A \frac{x \rho_t(x) f_t(1 - x)}{t f_t(1)} dx \\ &= \int_A \frac{C_\alpha x^{-\alpha} f_t(1 - x)}{f_t(1)} dx. \end{aligned}$$

By (3.9), $f_t(1)t^{1/\alpha} = f(t^{-1/\alpha})$ and $f_t(1-x)t^{1/\alpha} = f(t^{-1/\alpha}(1-x))$. Therefore, (3.12) implies that $a_1 t^{1+1/\alpha} (1 - Dt) \leq f_t(1)t^{1/\alpha} \leq a_1 t^{1+1/\alpha} (1 + Dt)$ and $a_1 t^{1+1/\alpha} (1-x)^{-1-\alpha} (1 - Dt(1-x)^{-\alpha}) \leq f_t(1-x)t^{1/\alpha} \leq a_1 t^{1+1/\alpha} (1-x)^{-1-\alpha} (1 - Dt(1-x)^{-\alpha})$. It follows that for all $x \in [0, 1 - a]$, we have

$$\begin{aligned} (1 - x)^{-1-\alpha} \left(\frac{1 - Dt(1 - a)^{-\alpha}}{1 + Dt} \right) &\leq \frac{f_t(1 - x)}{f_t(1)} \\ &\leq (1 - x)^{-1-\alpha} \left(\frac{1 + Dt(1 - a)^{-\alpha}}{1 - Dt} \right). \end{aligned}$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{t \rightarrow 0} t^{-1} P(J_1^*(t) \in A | \sigma_\alpha(t) = 1) = \int_A C_\alpha x^{-\alpha} (1 - x)^{-1-\alpha} dx,$$

which is (3.16).

If $J_1^*(t) > 1/2$, then $J_1^*(t) = J_1(t)$. Therefore, it follows from the definition of a size-biased pick from a sequence that for $x \in [1/2, 1 - a]$,

$$P(J_1(t) \in dx | \sigma_\alpha(t) = 1) = x^{-1} P(J_1^*(t) \in dx | \sigma_\alpha(t) = 1).$$

Therefore, if B is a Borel subset of $[1/2, 1]$, then

$$\begin{aligned} t^{-1}P(J_1(t) \in B | \sigma_\alpha(t) = 1) &= \int_B \frac{\rho_t(x)f_t(1-x)}{tf_t(1)} dx \\ &= \int_B \frac{C_\alpha x^{-1-\alpha} f_t(1-x)}{f_t(1)} dx . \end{aligned}$$

Equation (3.17) follows from the Dominated Convergence Theorem as in the proof of (3.16). //

Lemma 3.4 :

Let $J_1(t) \geq J_2(t) \geq \dots$ be the jump sizes of σ_α between times 0 and t . Let $J_1^*(t)$ be a size-biased pick from these jump sizes, and then let $J_2^*(t)$ be a size-biased pick from the remaining jump sizes. Let A be a Borel subset of $[0, 1]^2$ such that $A \subset \{(x, y) \in [0, 1]^2 : 0 < x + y < 1 - a\}$ for some $a > 0$. Then

$$\begin{aligned} \lim_{t \rightarrow 0} t^{-2}P((J_1^*(t), J_2^*(t)) \in A | \sigma_\alpha(t) = 1) \\ = \int_A \frac{C_\alpha^2 x^{-\alpha} y^{-\alpha} (1-x-y)^{-1-\alpha}}{1-x} dx dy . \end{aligned} \quad (3.18)$$

// Using (3.13), we see that

$$\begin{aligned} t^{-2}P((J_1^*(t), J_2^*(t)) \in A | \sigma_\alpha(t) = 1) &= \int_A \frac{t^{-2}x\rho_t(x)y\rho_t(y)f_t(1-x-y)}{(1-x)f_t(1)} dx dy \\ &= \int_A \frac{C_\alpha^2 x^{-\alpha} y^{-\alpha} f_t(1-x-y)}{(1-x)f_t(1)} dx dy . \end{aligned}$$

Equation (3.12) gives

$$\begin{aligned} (1-x-y)^{-1-\alpha} \left(\frac{1-Dt(1-a)^{-\alpha}}{1+Dt} \right) \\ \leq \frac{f_t(1-x-y)}{f_t(1)} \leq (1-x-y)^{-1-\alpha} \left(\frac{1+Dt(1-a)^{-\alpha}}{1-Dt} \right) . \end{aligned}$$

The lemma now follows from the Dominated Convergence Theorem. //

3.4 Small-time behavior of self-similar fragmentations

The proofs of Theorem 3.2 and Proposition 3.1 will use results on the small-time behavior of self-similar fragmentations. In this section, we record some results that we will need, and then we prove Proposition 3.2. First we give a way to recover the dislocation measure ν of a self-similar fragmentation with positive index and no erosion from its semigroup.

Proposition 3.3 :

Let $(X(t), t \geq 0)$ be a Δ -valued self-similar fragmentation with characteristics $(\beta, 0, \nu)$, where $\beta \geq 0$. For all $t > 0$, let μ_t be the measure on Δ defined by $\mu_t(A) = t^{-1}P(X(t) \in A)$ for all Borel measurable subsets A of Δ . Then μ_t converges weakly to ν as $t \rightarrow 0$ on any subset of Δ that is the complement of an open neighborhood of $(1, 0, 0, \dots)$.

// We will need the following lemma in the course of the proof:

Lemma :

Let $(\xi_t, t \geq 0)$ be a subordinator with Lévy measure $L(dx)$. Then the measure $t^{-1}P(\xi_t \in dx)$ converges to $L(dx)$ as $t \rightarrow 0$ weakly on any set of the form $(a, +\infty)$ with $a > 0$. Moreover, denoting the jump $\xi_u - \xi_{u-}$ at time u by $\Delta\xi_u$, one has, as $t \rightarrow 0$,

$$P(\xi_t \geq a \text{ and } \Delta\xi_u < a \text{ for all } u \in [0, t]) = o(t).$$

// The first part is classical, see e.g. [19]. For the second part, standard properties of Poisson measures give

$$P(\xi_t \geq a \text{ and } \Delta\xi_u \geq a \text{ for some } u \in [0, t]) = tL([a, \infty)) + o(t). \quad (3.19)$$

On the other hand, the Portmanteau theorem (see [36]) and the first part imply

$$\limsup_{t \rightarrow 0} \frac{1}{t} P(\xi_t \geq a) \leq L([a, \infty)). \quad (3.20)$$

Hence, dividing (3.19) by t and subtracting from (3.20) gives

$$\limsup_{t \rightarrow 0} \frac{1}{t} P(\xi_t \geq a \text{ and } \Delta\xi_u < a \text{ for all } u \in [0, t]) \leq 0. \quad //$$

Now let $A_\delta = \{x \in \Delta : x_1 \leq 1 - \delta\}$. Any subset of Δ that is the complement of an open neighborhood of $(1, 0, 0, \dots)$ is a subset of A_δ for some $\delta > 0$. Therefore, it suffices to show that μ_t converges weakly to ν on A_δ for all $\delta > 0$. Fix $\delta > 0$, and let G be a positive, bounded, continuous function on Δ such that $G(x) = 0$ for $x \notin A_\delta$. By the definition of μ_t and the definition of weak convergence, we need to show that

$$\lim_{t \rightarrow 0} t^{-1} E[G(X(t))] = \int_{\Delta} G(s) \nu(ds). \quad (3.21)$$

Without loss of generality, suppose that $X(t) = (X_1(t), X_2(t), \dots) = \Lambda(\Pi^{(\beta)}(t))$ for a partition-valued fragmentation process $\Pi^{(\beta)}$ with the same characteristics as X . We may also assume that $\Pi^{(\beta)}$ is constructed by time-changing a partition-valued fragmentation Π with characteristics $(0, 0, \nu)$ as in Sect. 3.2. That is, if $I_n(t)$ is the asymptotic frequency of the block of $\Pi(t)$ containing n and $T_n^{(\beta)}(t)$ is defined as in (3.6), then i and j are in the same block of $\Pi^{(\beta)}(t)$ if and only if i and j are in the same block of $\Pi(T_i^{(\beta)}(t))$. Also, we suppose that Π is constructed out of a Poisson point process $((\Gamma_t, k_t), t \geq 0)$ with intensity $\kappa \otimes \#$ as in Sect. 3.2. Notice that for every i and $t \geq 0$, we have $T_i^{(\beta)}(t) \leq t$ because $\beta > 0$. It follows that $(\Pi^{(\beta)}(u), 0 \leq u \leq t)$ is completely determined by the process $((\Gamma_u, k_u), 0 \leq u \leq t)$.

Let $(\Theta_t, t \geq 0)$ be the process such that $\Theta_t = \Gamma_t$ whenever (Γ_t, k_t) is a point of the Poisson process such that k_t is the least element of the block with maximal asymptotic frequency of $\Pi(t)$ at time $t-$. If two or more

blocks are tied for having the largest asymptotic frequency, we rank the blocks according to their smallest elements. As a consequence of Lemma 10 in [17], Θ is a Poisson point process with intensity κ .

Let N_t be the cardinality of $\{s \in [0, t] : \Lambda(\Theta_s) \in A_\delta\}$. Note that N_t has a Poisson distribution with mean $t\nu(A_\delta)$. Therefore,

$$\lim_{t \rightarrow 0} t^{-1} E[G(X(t)) \mathbb{1}_{\{N_t \geq 2\}}] \leq \lim_{t \rightarrow 0} t^{-1} \|G\|_\infty P(N_t \geq 2) = 0.$$

Next, note that $E[G(X(t)) \mathbb{1}_{\{N_t=0\}}] \leq \|G\|_\infty P(\{X_1(t) \leq 1 - \delta\} \cap \{N_t = 0\})$. If π is a partition of \mathbb{N} , let $\Lambda_j(\pi)$ denote the asymptotic frequency of the block of π having the j th-largest asymptotic frequency. Since $\beta \geq 0$, we have

$$X_1(t) \geq \prod_{0 \leq u \leq t} \Lambda_1(\Theta_u) \geq 1 - \sum_{0 \leq u \leq t} (1 - \Lambda_1(\Theta_u)).$$

Since $t \mapsto \sum_{0 \leq u \leq t} (1 - \Lambda_1(\Theta_u))$ is a subordinator, it follows from the intermediate lemma that $P(\{X_1(t) \leq 1 - \delta\} \cap \{N_t = 0\}) = o(t)$. Therefore,

$$\lim_{t \rightarrow 0} t^{-1} E[G(X(t)) \mathbb{1}_{\{N_t=0\}}] = 0.$$

Thus, to prove (3.21), it remains only to show that

$$\lim_{t \rightarrow 0} t^{-1} E[G(X(t)) \mathbb{1}_{\{N_t=1\}}] = \int_{\Delta} G(s) \nu(ds). \quad (3.22)$$

Let $0 < \varepsilon < 1/2$, and let $\eta > 0$. Then there exists a positive number t_0 such that $P(I_i(t_0) < 1 - \varepsilon) < \eta$ for every $i \geq 1$. Fix $t < t_0$. On the event $\{N_t = 1\}$, define U such that $\Lambda(\Theta_{tU}) \in A_\delta$. Note that U has a uniform distribution on $[0, 1]$. Define B to be the event that $U \leq (1 - \varepsilon)^\beta$. Let B_0 be the event that $I_1(tU-) \geq 1 - \varepsilon$. Fix $J \in \mathbb{N}$. For $1 \leq j \leq J$, let i_j be the smallest integer in the block of $\Pi(tU)$ having the j th-largest asymptotic frequency, provided that integer is in the same block as 1 at time $tU-$; otherwise, define $i_j = 0$. Let B_j be the event that either $i_j = 0$ or $|I_{i_j}(T_{i_j}^{(\beta)}(t)) - I_{i_j}(tU)| \leq \varepsilon$.

We have $P(B|N_t = 1) = (1 - \varepsilon)^\beta$. Also,

$$P(B_0|N_t = 1) \geq P(I_1(t) \geq 1 - \varepsilon) \geq 1 - \eta.$$

If B and B_0 occur, then

$$\int_0^{tU} I_1(s)^{-\beta} ds \leq tU(1 - \varepsilon)^{-\beta} \leq t,$$

which implies that $T_1^{(\beta)}(t) \geq tU$. If, in addition, $i_j > 0$, then $tU \leq T_{i_j}^{(\beta)}(t) \leq t$. In this case $|I_{i_j}(T_{i_j}^{(\beta)}(t)) - I_{i_j}(tU)| \leq |I_{i_j}(t) - I_{i_j}(tU)|$ which, conditional on B , B_0 , and $N_t = 1$, is less than or equal to ε with probability at least $1 - \eta$. Thus,

$$P(B \cap B_0 \cap B_1 \cap \cdots \cap B_J | N_t = 1) \geq (1 - \varepsilon)^\beta - (J + 1)\eta.$$

Suppose B, B_0, B_1, \dots, B_J all occur. If $i_j = 0$, then $X_j(t) < \varepsilon$ and $\Lambda_j(\Theta_{tU}) \leq \varepsilon/(1 - \varepsilon)$, so $|X_j(t) - \Lambda_j(\Theta_{tU})| \leq 2\varepsilon$. If $i_j > 0$, then

$$\begin{aligned} |I_{i_j}(T_{i_j}^{(\beta)}(t)) - \Lambda_j(\Theta_{tU})| &\leq |I_{i_j}(T_{i_j}^{(\beta)}(t)) - I_{i_j}(tU)| + |I_{i_j}(tU) - \Lambda_j(\Theta_{tU})| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Since the block of $\Pi^{(\beta)}(t)$ containing the integer i_j has asymptotic frequency $I_{i_j}(T_{i_j}^{(\beta)}(t))$, it follows that $|X_j(t) - \Lambda_j(\Theta_{tU})| \leq 2\varepsilon$. Thus, for $t < t_0$,

$$P(|X_j(t) - \Lambda_j(\Theta_{tU})| \leq 2\varepsilon \text{ for } j = 1, \dots, J | N_t = 1) \geq (1 - \varepsilon)^\beta - (J + 1)\eta.$$

By letting $\varepsilon, \eta \rightarrow 0$ and applying Theorem 3.1 of [36], we can see that the conditional distribution of $(X_1(t), \dots, X_J(t))$ given $N_t = 1$ converges to the distribution of $(\Lambda_1(\Theta_{tU}), \dots, \Lambda_J(\Theta_{tU}))$. By properties of weak convergence in Δ (see chapter 4 of [36]), it follows that the conditional distribution of $X(t)$ given $N_t = 1$ converges as $t \rightarrow 0$ to the distribution of $\Lambda(\Theta_{tU})$, which does not depend on t . Thus,

$$\begin{aligned} \lim_{t \rightarrow 0} t^{-1} E[G(X(t)) \mathbb{1}_{\{N_t=1\}}] &= \lim_{t \rightarrow 0} t^{-1} P(N_t = 1) E[G(X(t)) | N_t = 1] \\ &= \lim_{t \rightarrow 0} \nu(A_\delta) e^{-t\nu(A_\delta)} E[G(\Lambda(\Theta_{tU}))] \\ &= \lim_{t \rightarrow 0} \nu(A_\delta) e^{-t\nu(A_\delta)} \int_{\Delta} \frac{G(s)}{\nu(A_\delta)} \nu(ds), \\ &= \int_{\Delta} G(s) \nu(ds), \end{aligned}$$

which is (3.22). //

Remark. In this proposition and the following corollary, the assumption that $c = 0$, $\beta \geq 0$ could be avoided. When $c > 0$, we may follow essentially the same reasoning as above because the drift at rate c has little effect on the block sizes for small t . When $\beta < 0$, however, the proof requires a more careful analysis of the time-changes $T_i^{(\beta)}$. We thus omit the proof here, as we are only concerned with positive self-similarity indices.

From Proposition 3.3, we get the following result concerning the small-time behavior of the asymptotic frequency of the block containing 1 in a partition-valued fragmentation.

Corollary 3.1 :

Let $(\Pi(t), t \geq 0)$ be a partition-valued self-similar fragmentation with characteristics (β, c, ν) . Let $\lambda(t)$ be the asymptotic frequency of the block containing 1 at time t . For all $t > 0$, let γ_t be the measure on $[0, 1]$ defined by $\gamma_t(A) = t^{-1} P(\lambda(t) \in A)$. Let ν_i be the restriction of ν to the i th coordinate. Let γ be the measure on $[0, 1]$ defined by

$$\gamma(A) = \sum_{i=1}^{\infty} \int_A x \nu_i(dx) \quad (3.23)$$

for all A . Then, γ_t converges weakly to γ as $t \rightarrow 0$ on $[a, 1 - a]$ for all $a > 0$.

// Let μ_t be the measure on Δ defined by $\mu_t(A) = t^{-1} P(\Lambda(\Pi(t)) \in A)$ for all Borel measurable subsets A of Δ . Let $\mu_{t,i}$ be the restriction of μ_t to the

i th coordinate. Then,

$$\gamma_t(A) = \sum_{i=1}^{\infty} \int_A x \mu_{t,i}(dx)$$

Let f be a bounded continuous function defined on $[a, 1-a]$. By Proposition 3.3, $\mu_{t,i}$ converges weakly on $[a, 1-a]$ to ν_i for all i . Therefore,

$$\begin{aligned} \lim_{t \rightarrow 0} \int_a^{1-a} f(x) \gamma_t(dx) &= \lim_{t \rightarrow 0} \sum_{i=1}^{\infty} \int_a^{1-a} x f(x) \mu_{t,i}(dx) \\ &= \sum_{i=1}^{\infty} \lim_{t \rightarrow 0} \int_a^{1-a} x f(x) \mu_{t,i}(dx) = \sum_{i=1}^{\infty} \int_a^{1-a} x f(x) \nu_i(dx) \\ &= \int_a^{1-a} f(x) \gamma(dx), \end{aligned}$$

which implies the conclusion of the corollary. Note that interchanging the limit and the sum is justified because $\mu_{t,i}([a, 1-a]) = 0$ for all t whenever $i > 1/a$, so only finitely many terms in the sum are nonzero. //

We now prove Proposition 3.2, which shows that the small-time behavior of some self-similar fragmentations is related to the stable subordinator of index α . In the case of homogeneous fragmentations, our results are similar to the results in section 4 of [17]. Our arguments are also similar to those in section 4 of [17], but we work here with partition-valued fragmentations rather than ranked fragmentations and prove the result for self-similar fragmentations with a positive index of self-similarity in addition to homogeneous fragmentations.

// Since the fragmentation $(X(t), t \geq 0)$ is a binary fragmentation with no erosion and positive index β , we have that $1 - X_1(t) = \sum_{i=2}^{\infty} X_i(t)$ for all t . Also, since σ_α is a pure-jump process, $\sigma_\alpha(1) = \sum_{i=1}^{\infty} J_i(1)$. Therefore, to show (3.4), it suffices to show that

$$t^{-1/\alpha}(X_2(t), X_3(t), \dots) \rightarrow_d (J_1(1), J_2(1), \dots).$$

Therefore, it suffices to show that

$$t^{-1/\alpha}(X_2(t), \dots, X_{n+1}(t)) \rightarrow_d (J_1(1), \dots, J_n(1)) \quad (3.24)$$

for all $n \in \mathbb{N}$.

As in the proof of Proposition 3.3 above, we may suppose that $X(\cdot) = \Lambda(\Pi^{(\beta)}(\cdot))$, where $\Pi^{(\beta)}$ is the partition-valued fragmentation with characteristics $(\beta, 0, \nu)$ that is obtained from a homogeneous fragmentation Π , being constructed out of a Poisson process $((\Gamma_t, k_t), t \geq 0)$ with intensity $\kappa \otimes \#$ as in Sect. 3.2.

For all $k \in \mathbb{N}$, let $(r_t^{(k)}, t \geq 0)$ be the Poisson point process on $[0, 1/2]$ with the property that $r_t^{(k)} = r$ if and only if $(\Gamma_t, k_t) = (\pi, k)$ for some $\pi \in \mathcal{P}$ such that the block of π with the second-largest asymptotic frequency has

asymptotic frequency r . Note that for all k , the Poisson point process $(r_t^{(k)}, t \geq 0)$ has characteristic measure $\nu_2(dx)$, where ν_2 is the restriction of ν to the second coordinate. For all j , let $K_j(t)$ be the j th-largest point of $(r_s^{(1)}, 0 \leq s \leq t)$. Let $\tau_j(t)$ be the time such that $r_{\tau_j(t)}^{(1)} = K_j(t)$. Let $N_j(t)$ be the smallest integer which is in the same block as 1 in the partition $\Pi(\tau_j(t)-)$ but is not in the same block as 1 in $\Pi(\tau_j(t))$.

Define another Poisson point process $(\Theta_t, t \geq 0)$ whose characteristic measure has density

$$q(x) = C_\alpha (s(x) \vee 1) x^{-1-\alpha}.$$

We now construct two new Poisson point processes by marking, as described in chapter 5 of [70]. Let $(\Theta_t^{(1)}, t \geq 0)$ consist of the marked points of $(\Theta_t, t \geq 0)$ when a point at x is marked with probability $1/(s(x) \vee 1)$. Let $(\Theta_t^{(2)}, t \geq 0)$ consist of the marked points of $(\Theta_t, t \geq 0)$ when a point at x is marked with probability $s(x) \mathbb{1}_{[0,1/2]}(x)/(s(x) \vee 1)$. Then, $(\Theta_t^{(1)}, t \geq 0)$ is a Poisson point process whose characteristic measure has density $q_1(x) = C_\alpha x^{-1-\alpha}$, and $(\Theta_t^{(2)}, t \geq 0)$ is a Poisson point process whose characteristic measure has density $q_2(x) = C_\alpha x^{-1-\alpha} s(x) \mathbb{1}_{[0,1/2]}(x)$.

Let $L_j(t)$ denote the j th largest point of $(\Theta_s^{(1)}, 0 \leq s \leq t)$, and let $\tilde{K}_j(t)$ denote the j th largest point of $(\Theta_s^{(2)}, 0 \leq s \leq t)$. If the n largest points of $(\Theta_s, 0 \leq s \leq t)$ are also points in both $(\Theta_s^{(1)}, 0 \leq s \leq t)$ and $(\Theta_s^{(2)}, 0 \leq s \leq t)$, then $L_j(t) = \tilde{K}_j(t)$ for $j = 1, \dots, n$. For all $x > 0$, the probability that the largest point of $(\Theta_s, 0 \leq s \leq t)$ is less than x approaches 1 as $t \rightarrow 0$. Since $\lim_{x \rightarrow 0} s(x) = 1$, we have $\lim_{x \rightarrow 0} s(x) \mathbb{1}_{[0,1/2]}(x)/(s(x) \vee 1) = 1$ and $\lim_{x \rightarrow 0} 1/(s(x) \vee 1) = 1$. It follows from these observations that

$$\lim_{t \rightarrow 0} P(L_j(t) = \tilde{K}_j(t) \text{ for } j = 1, \dots, n) = 1. \quad (3.25)$$

Note that $(L_1(t), \dots, L_n(t))$ has the same distribution as the sizes of the n largest jumps of $(\sigma_\alpha(s), 0 \leq s \leq t)$. By scaling properties of the stable subordinator of index α , it follows that

$$t^{-1/\alpha}(L_1(t), \dots, L_n(t)) =_d (J_1(1), \dots, J_n(1)). \quad (3.26)$$

Since $(\tilde{K}_1(t), \dots, \tilde{K}_n(t)) =_d (K_1(t), \dots, K_n(t))$, It follows from equations (3.25) and (3.26), and Theorem 3.1 in [36] that

$$t^{-1/\alpha}(K_1(t), \dots, K_n(t)) \rightarrow_d (J_1(1), \dots, J_n(1)). \quad (3.27)$$

as $t \rightarrow 0$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. We will show next that for all $n \in \mathbb{N}$, we have

$$\lim_{t \rightarrow 0} P(|t^{-1/\alpha} K_j(t) - t^{-1/\alpha} X_{j+1}(t)| < \varepsilon \text{ for } j = 1, \dots, n) = 1. \quad (3.28)$$

Equations (3.27) and (3.28), combined with Theorem 3.1 of [36], establish (3.24), which suffices to prove Proposition 3.2.

Given $0 < \delta < 1/2$ and $i \in \mathbb{N}$, let λ_t^i be asymptotic frequency of the set of all integers m such that m is in the same block as i in every partition π for which $\Gamma_s = \pi$ and $k_s = i$ for some $s \in [0, t]$. Let $A_{\delta,t}^i$ be the event that $\lambda_t^i > 1 - \delta$. If $A_{\delta,t}^1$ occurs, then the block of $\Pi(t)$ containing 1 has asymptotic frequency at least $1 - \delta$. Also, since $\beta \geq 0$, it follows that $T_i^\beta(t) \leq t$ for every $i \geq 1$. Therefore, taking $i = 1$, if $A_{\delta,t}^1$ occurs, the block of $\Pi^{(\beta)}(t)$ containing 1 has asymptotic frequency at least $1 - \delta$.

Let $B_{j,t}$ be the event that $\tau_j(t) \leq T_1^\beta(t)$. Note that this is the same as the event that $\tau_j(t) \leq T_{N_j(t)}^\beta(t)$ because 1 and $N_j(t)$ are in the same block before time $\tau_j(t)$. Suppose $A_{\delta,t}^1$ occurs, and suppose $A_{\delta,t}^{N_j(t)}$ and $B_{j,t}$ occur for $j = 1, \dots, n$. Then, for $j = 1, \dots, n$, the block of $\Pi(\tau_j(t)-)$ containing $N_j(t)$ has asymptotic frequency between $1 - \delta$ and 1, the block of $\Pi(\tau_j(t))$ containing $N_j(t)$ has asymptotic frequency between $K_j(t)(1 - \delta)$ and $K_j(t)$, and the asymptotic frequency of the block of $\Pi^{(\beta)}(t)$ containing $N_j(t)$ has asymptotic frequency between $K_j(t)(1 - \delta)^2$ and $K_j(t)$. Furthermore, the largest of all blocks of $\Pi^{(\beta)}(t)$ not containing any of the integers $\{1, N_1(t), \dots, N_n(t)\}$ has asymptotic frequency at most $\max\{\delta K_1(t), K_{n+1}(t)\}$. Indeed, this block could be obtained from the $n + 1$ -th largest fragmentation of the fragment containing 1, which since $\delta < 1/2$ is also the largest one, in which case its asymptotic frequency is at most $K_{n+1}(t)$. Alternatively, it could be obtained from one of the fragments containing some $N_j(t)$ for $1 \leq j \leq n$. Since we assume that $A_{\delta,t}^{N_j(t)}$ occurs for every $1 \leq j \leq n$, the size of these fragments can not be reduced by more than a factor of $1 - \delta$. Therefore, at time t , the fragments that do not contain any of the $N_j(t)$, for $1 \leq j \leq n$, but are obtained by splitting the blocks containing one of the $N_j(t)$, $1 \leq j \leq n$, have asymptotic frequency smaller than $\delta K_j(t) \leq \delta K_1(t)$.

Therefore, if in addition $\delta K_1(t) < K_n(t)$, then

$$K_j(t)(1 - \delta)^2 \leq X_{j+1}(t) \leq K_j(t) \quad (3.29)$$

for $j = 1, \dots, n$.

Note that $\lim_{t \rightarrow 0} P(A_{\delta,t}^1) = 1$ for all $\delta \in (0, 1/2)$. Likewise, for all $j \in \mathbb{N}$ and $\delta \in (0, 1/2)$, we have $\lim_{t \rightarrow 0} P(A_{\delta,t}^{N_j(t)}) = 1$. We now prove that

$$\lim_{t \rightarrow 0} P(B_{j,t}) = 1. \quad (3.30)$$

Let $\varepsilon > 0$. Choose δ small enough that $1 - (1 - \delta)^\beta < \varepsilon/2$. Then choose t small enough that $P(A_{\delta,t}^1) > 1 - \varepsilon/2$. Suppose $A_{\delta,t}^1$ occurs. Then the fragment of $\Pi(s)$ containing 1 has asymptotic frequency larger than $1 - \delta$ for $0 \leq s \leq t$. It follows from (3.6) that

$$(1 - \delta)^\beta t \leq T_1^\beta(t) \leq t.$$

Since $\tau_j(t)$ is uniform on $(0, t)$, we have $P(B_{j,t} | A_{\delta,t}^1) > 1 - \varepsilon/2$. Since $P(A_{\delta,t}^1) > 1 - \varepsilon/2$, it follows that $P(B_{j,t}) > 1 - \varepsilon$, which implies (3.30). Last, by (3.27),

$$\lim_{\delta \rightarrow 0} \lim_{t \rightarrow 0} P(\delta K_1(t) < K_n(t)) = 1.$$

These results, combined with (3.29), prove (3.28). //

3.5 Large-time behavior of self-similar fragmentations

In [27], Bertoin studied the asymptotic behavior of self-similar fragmentations as $t \rightarrow \infty$. Using facts from [28] about semi-stable processes, he proved the following result.

Lemma 3.5 :

Let $(X(t), t \geq 0) = ((X_1(t), X_2(t), \dots), t \geq 0)$ be a self-similar fragmentation with characteristics (β, c, ν) . Suppose $\nu(\{x : \sum_{i=1}^{\infty} x_i < 1\}) = 0$. Also assume that there exists no $r > 0$ such that the size of every fragment at time $t > 0$ lies in the set $\{e^{-kr} : k = 0, 1, \dots\}$. For $q \geq 0$, define

$$\Phi(q) = c(q+1) + \int_{\Delta} \left(1 - \sum_{i=1}^{\infty} x_i^{q+1}\right) \nu(dx). \quad (3.31)$$

Assume that

$$\Phi'(0+) = c + \sum_{i=1}^{\infty} \int_{\Delta} x_i \log(1/x_i) \nu(dx) < \infty. \quad (3.32)$$

If $\beta = 0$, then $\lim_{t \rightarrow \infty} t^{-1} \log(X_1(t))$ exists and is finite almost surely. If $\beta > 0$ and $c = 0$, define

$$\mu_t = \sum_{i=1}^{\infty} X_i(t) \delta_{t^{1/\beta} X_i(t)}.$$

Then, the random probability measures μ_t converge in probability as $t \rightarrow \infty$ to a deterministic limit μ_{∞} , for the weak topology on measures. Furthermore, for $k \in \mathbb{N}$, we have

$$\int_0^{\infty} y^{\beta k} \mu_{\infty}(dy) = \frac{1}{\beta \Phi'(0+)} \prod_{i=1}^{k-1} \frac{i}{\Phi(i\beta)}. \quad (3.33)$$

Suppose the hypotheses of Lemma 3.5 are satisfied, and that $\beta > 0$ and $c = 0$. Let $\lambda(t)$ be a size-biased pick from the sequence $X(t) = (X_1(t), X_2(t), \dots)$. Note that μ_t is the conditional distribution of $t^{1/\beta} \lambda(t)$ given $X(t)$. The proof of the convergence in probability of μ_t to μ_{∞} in [27] actually shows that for every f continuous and bounded, we have

$$\lim_{t \rightarrow \infty} E \left[\int_0^{\infty} f(y) \mu_t(dy) \right] = \int_0^{\infty} f(y) \mu_{\infty}(dy).$$

Therefore, the unconditional distributions γ_t of $t^{1/\beta} \lambda(t)$, given by

$$\gamma_t(B) = E[\mu_t(B)],$$

converge weakly to μ_{∞} as $t \rightarrow \infty$.

3.6 One-dimensional distributions

Our goal in this section is to prove Theorem 3.2. The first step is Lemma 3.6. Once this lemma is proved, there is, for each $\alpha \in (0, 1)$, only one remaining candidate for a self-similar fragmentation that could satisfy (3.2). To prove Theorem 3.2, we then only need to show that this fragmentation does indeed satisfy (3.2) only when $\alpha = 1/2$.

Lemma 3.6 :

Fix $\alpha \in (0, 1)$. Suppose $(X(t), t \geq 0)$ is a self-similar fragmentation with characteristics (β, c, ν) such that (3.2) holds. Then $\beta = 1 - \alpha$ and $c = 0$. Also, $(X(t), t \geq 0)$ is binary, and the restriction ν_1 of ν to the first coordinate has density $h_\alpha(x) = C_\alpha x^{-1-\alpha}(1-x)^{-1-\alpha} \mathbb{1}_{[1/2, 1]}(x)$.

// Write the components of $X(t)$ as $(X_1(t), X_2(t), \dots)$. Let $J_1(t), J_2(t), \dots$ be the jump sizes of σ_α up to time t . Note that $\sum_{i=1}^{\infty} J_i(t) = \sigma_\alpha(t)$, so the conditional distribution of $\sum_{i=1}^{\infty} J_i(t)$ given $\sigma_\alpha(t) = 1$ is a unit mass at 1. Therefore, if (3.2) holds, we must have $\sum_{i=1}^{\infty} X_i(t) = 1$ almost surely. It follows from the construction in Sect. 3.2 that $c = 0$. Also, by section 3.3 of [27], we have $\beta \geq 0$.

Let $\lambda(t)$ be a size-biased pick from the sequence $(X_1(t), X_2(t), \dots)$. It follows from Lemma 3.2 that if (3.2) holds, then the distribution of $t^{1/(1-\alpha)}\lambda(t)$ converges to a nondegenerate limit. Combining this result with Lemma 3.5, we get $\beta = 1 - \alpha$.

Suppose, for some $a > 0$, we have $\nu(\{x \in \Delta : x_1 + x_2 < 1 - a\}) = b > 0$. Then, by Proposition 3.3 and the Portmanteau Theorem,

$$\liminf_{t \rightarrow 0} t^{-1} P(X_1(t) + X_2(t) < 1 - a) \geq b.$$

Therefore, if (3.2) holds, then

$$\liminf_{t \rightarrow 0} t^{-1} P(J_1(t) + J_2(t) < 1 - a) \geq b.$$

Let $J_1^*(t)$ be a size-biased pick from the jump sizes $J_1(t), J_2(t), \dots$, and let $J_2^*(t)$ be a size-biased pick from the remaining jump sizes. Note that $J_1^*(t) + J_2^*(t) \leq J_1(t) + J_2(t)$, so

$$\liminf_{t \rightarrow 0} t^{-1} P(J_1^*(t) + J_2^*(t) < 1 - a) \geq b. \quad (3.34)$$

However, Lemma 3.4 implies that if $A = \{(x, y) \in [0, 1]^2 : 0 < x + y < 1 - a\}$, then

$$\lim_{t \rightarrow 0} t^{-2} P(J_1^*(t) + J_2^*(t) < 1 - a) = \int_A \frac{C_\alpha^2 x^{-\alpha} y^{-\alpha} (1 - x - y)^{-1-\alpha}}{1 - x} dx dy < \infty,$$

which contradicts (3.34). We conclude that $\nu(\{x \in \Delta : x_1 + x_2 < 1 - a\}) = 0$ for all $a > 0$, which means X is a binary self-similar fragmentation.

Let μ_t be the measure on Δ defined by $\mu_t(A) = t^{-1} P(X(t) \in A)$. By Proposition 3.3, as $t \rightarrow 0$, μ_t converges weakly on complements of open neighborhoods of $(1, 0, \dots)$ to ν . Let $\tilde{\mu}_t$ be the measure defined by $\tilde{\mu}_t(B) =$

$t^{-1}P(X_1(t) \in B)$. Let ν_1 be the restriction of ν to the first coordinate. Then $\tilde{\mu}_t$ converges weakly on $[0, a]$ to ν_1 as $t \rightarrow 0$ for any $a < 1$. It follows that

$$\lim_{t \rightarrow 0} t^{-1}P(X_1(t) \in [0, a]) = \lim_{t \rightarrow 0} \mu_t([0, a]) = \nu_1([0, a]) \quad (3.35)$$

for all $a \in [0, 1)$ (the only interesting case is $a > 1/2$ since ν_1 does not support $[0, 1/2]$) such that the function $x \mapsto \nu_1([0, x])$ is continuous at a . If (3.2) holds, then we can combine (3.35) with (3.17) to obtain

$$\int_0^a h_\alpha(x) dx = \nu_1([0, a])$$

for all $a \in (1/2, 1)$ such that $x \mapsto \nu_1([0, x])$ is continuous at a . Thus, h_α is the density of ν_1 . //

The binary self-similar fragmentation whose characteristics are $(1/2, 0, \nu)$, where the restriction of ν to the first coordinate has density $h_{1/2}$, is the Aldous–Pitman fragmentation. Therefore, Theorem 3.2 follows immediately from Lemma 3.6 and the following lemma.

Lemma 3.7 :

Let $(X(t), t \geq 0)$ be a binary self-similar fragmentation with characteristics $(1 - \alpha, 0, \nu)$, where the restriction of ν to the first coordinate has density h_α . If (3.2) holds, then $\alpha = 1/2$.

// Let $\lambda(t)$ be a size-biased pick from the sequence of $X(t)$. Let $\beta = 1 - \alpha$. Let γ_t be the law of $t^{1/(1-\alpha)}\lambda(t) = t^{1/\beta}\lambda(t)$. Then, by results in Sect. 3.5, γ_t converges weakly to some measure μ_∞ as $t \rightarrow \infty$. Also, for all $k \in \mathbb{N}$, (3.33) gives

$$\int_0^\infty y^{\beta k} \mu_\infty(dy) = \frac{1}{\beta\Phi'(0+)} \prod_{i=1}^{k-1} \frac{i}{\Phi(i\beta)},$$

where Φ is the function defined in (3.31).

Suppose (3.2) holds. By Lemma 3.2, μ_∞ is the Gamma($1 - \alpha, B\alpha/(1 - \alpha)$) distribution. By (3.15),

$$\int_0^\infty y^{\beta k} \mu_\infty(dy) = \frac{\Gamma(\beta k + 1 - \alpha)}{\Gamma(1 - \alpha)} \left(\frac{\cos(\frac{\pi\alpha}{2})}{\alpha} \right)^{\frac{\beta k}{1-\alpha}} = \frac{\Gamma(\beta k + \beta)}{\Gamma(\beta)} \left(\frac{\cos(\frac{\pi\alpha}{2})}{\alpha} \right)^k$$

for all $k \in \mathbb{N}$. It follows that

$$\frac{1}{\beta\Phi'(0+)} \prod_{i=1}^{k-1} \frac{i}{\Phi(i\beta)} = \frac{\Gamma(\beta k + \beta)}{\Gamma(\beta)} \left(\frac{\cos(\frac{\pi\alpha}{2})}{\alpha} \right)^k \quad (3.36)$$

for all $k \in \mathbb{N}$. By considering (3.36) for $k + 1$ and k and taking the ratio of the two equations, we get

$$\frac{\Gamma(\beta k + 2\beta)}{\Gamma(\beta k + \beta)} \left(\frac{\cos(\frac{\pi\alpha}{2})}{\alpha} \right) = \frac{k}{\Phi(k\beta)}. \quad (3.37)$$

Since $\alpha / \cos(\frac{\pi\alpha}{2}) = C_\alpha \Gamma(1 - \alpha) = C_\alpha \Gamma(\beta)$ by (3.8), equation (3.37) implies

$$\Phi(k\beta) = \frac{C_\alpha k \Gamma(\beta) \Gamma(\beta k + \beta)}{\Gamma(\beta k + 2\beta)}. \quad (3.38)$$

By Stirling's Formula,

$$\lim_{k \rightarrow \infty} \frac{(k\beta)^\beta \Gamma(\beta k + \beta)}{\Gamma(\beta k + 2\beta)} = 1.$$

Combining this result with (3.38), we get

$$\lim_{k \rightarrow \infty} k^{\beta-1} \Phi(k\beta) = C_\alpha \Gamma(\beta) \beta^{-\beta}. \quad (3.39)$$

We will now compute $\lim_{k \rightarrow \infty} k^{\beta-1} \Phi(k\beta)$ directly from (3.31). We will show that the result agrees with the right-hand side of (3.39) only when $\beta = 1/2$, which will prove the lemma. Using the definitions of ν and h_α , equation (3.31), and the fact that $c = 0$, we have

$$\begin{aligned} \Phi(k\beta) &= \int_{\Delta} \left(1 - \sum_{i=1}^{\infty} x_i^{k\beta+1} \right) \nu(dx) \\ &= C_\alpha \int_0^{1/2} (1 - x^{k\beta+1} - (1-x)^{k\beta+1}) x^{\beta-2} (1-x)^{\beta-2} dx. \end{aligned}$$

By making the substitution $y = kx$, we get

$$\begin{aligned} k^{\beta-1} \Phi(k\beta) &= C_\alpha \int_0^{k/2} (1 - (k^{-1}y)^{k\beta+1} - (1 - k^{-1}y)^{k\beta+1}) \\ &\quad \times y^{\beta-2} (1 - k^{-1}y)^{\beta-2} dy. \end{aligned}$$

Note that for each fixed $y > 0$,

$$\begin{aligned} \lim_{k \rightarrow \infty} (1 - (k^{-1}y)^{k\beta+1} - (1 - k^{-1}y)^{k\beta+1}) y^{\beta-2} (1 - k^{-1}y)^{\beta-2} \mathbb{1}_{[0, k/2]}(y) \\ = (1 - e^{-\beta y}) y^{\beta-2}. \end{aligned}$$

If $0 \leq y \leq k/2$, then $(1 - k^{-1}y)^{\beta-2} \leq 2^{2-\beta} \leq 4$. Also, if $0 < y < 1/2$, then $k \mapsto (1 - k^{-1}y)^{k\beta+1}$ is an increasing function, and therefore $1 - (k^{-1}y)^{k\beta+1} - (1 - k^{-1}y)^{k\beta+1} \leq 1 - (1-y)^{\beta+1} \leq 1 - (1-y)^2 \leq 2y$. Therefore, for all $k \in \mathbb{N}$,

$$\begin{aligned} (1 - (k^{-1}y)^{k\beta+1} - (1 - k^{-1}y)^{k\beta+1}) \\ \times y^{\beta-2} (1 - k^{-1}y)^{\beta-2} \mathbb{1}_{[0, k/2]}(y) \leq 4(2y \wedge 1) y^{\beta-2}, \end{aligned}$$

and $\int_0^\infty 4(2y \wedge 1) y^{\beta-2} dy < \infty$. Hence, by the Dominated Convergence Theorem,

$$\lim_{k \rightarrow \infty} k^{\beta-1} \Phi(k\beta) = C_\alpha \int_0^\infty (1 - e^{-\beta y}) y^{\beta-2} dy.$$

Integrating by parts, we get

$$\lim_{k \rightarrow \infty} k^{\beta-1} \Phi(k\beta) = \frac{C_\alpha \beta}{1 - \beta} \int_0^\infty y^{\beta-1} e^{-\beta y} dy = \frac{C_\alpha \beta}{1 - \beta} \Gamma(\beta) \beta^{-\beta}. \quad (3.40)$$

Combining (3.39) and (3.40), we get $\beta/(1 - \beta) = 1$, which means $\beta = 1/2$ and therefore $\alpha = 1/2$, as claimed. //

3.7 Mass of a tagged fragment

Our goal in this section is to prove Proposition 3.1, which pertains to the distribution of the asymptotic frequency of the block containing 1 in a partition-valued self-similar fragmentation or, equivalently, the distribution of a size-biased pick from a self-similar ranked fragmentation.

According to [26], the tagged fragment in a self-similar fragmentation with index β has to be the inverse of some increasing semi-stable Markov process of index $1/\beta$ started at 1. A semi-stable Markov process with index $1/\beta > 0$ is a real-valued strong Markov process X satisfying the following self-similarity property. If, for $x > 0$, P_x denotes the law of X starting from $X_0 = x$, then for every $k > 0$, the law of the process $(kX(k^{-\beta}s), s \geq 0)$ under P_x is the same as the law of $(X(s), s \geq 0)$ under P_{kx} .

Lemma 3.8 :

Let $G(x, s)$ be a function defined on $[0, \infty)^2$ which is increasing in x and s . Suppose that there exists a semi-stable Markov process X with index $1/\beta$ such that $(G(x, \sigma_\alpha(t)), t \geq 0)$ has the law of X started at x . Then G is of the form

$$G(x, s) = (x^{\frac{\beta}{\alpha}} + Ks)^{\frac{\alpha}{\beta}}$$

for some $K > 0$.

// By the scaling property, we have

$$(kG(x, \sigma_\alpha(k^{-\beta}t)), t \geq 0) =_d (kG(x, k^{-\beta/\alpha}\sigma_\alpha(t)), t \geq 0) \quad (3.41)$$

for all $k > 0$. Since X is a semi-stable Markov process with index $1/\beta$, we have for all $k > 0$,

$$(kG(x, \sigma_\alpha(k^{-\beta}t)), t \geq 0) =_d (G(kx, \sigma_\alpha(t)), t \geq 0). \quad (3.42)$$

Given k and x , define $f_1(s) = kG(x, k^{-\beta/\alpha}s)$ and $f_2(s) = G(kx, s)$. Then, f_1 and f_2 are increasing functions, and equations (3.41) and (3.42) imply that $f_1(\sigma_\alpha(t)) =_d f_2(\sigma_\alpha(t))$ for all $t > 0$. For an increasing function f , define $f^{-1}(z) = \sup\{x : f(x) \leq z\}$. We have $P(f_1(\sigma_\alpha(t)) \leq z) = P(f_2(\sigma_\alpha(t)) \leq z)$, which means $P(\sigma_\alpha(t) \leq f_1^{-1}(z)) = P(\sigma_\alpha(t) \leq f_2^{-1}(z))$. Since for all $t > 0$ the density of $\sigma_\alpha(t)$ is positive on $(0, \infty)$, it follows that $f_1^{-1}(z) = f_2^{-1}(z)$ for all z . Therefore, if $f_1(s) < f_2(s)$, then $f_2(u) \leq f_1(s)$ for all $u < s$, and $f_1(u) \geq f_2(s)$ for all $u > s$. It follows that both f_1 and f_2 have a jump at s . Thus, $f_1(s) = f_2(s)$ for all but countably many s . Let $g(s) = G(1, s)$. Then, for all x , we have $G(x, s) = xg(x^{-\beta/\alpha}s)$ for all but countably many s . For any fixed $s > 0$, we have $P(\sigma_\alpha(t) \neq s \text{ for all } t) = 1$. Therefore, with probability one, $G(x, \sigma_\alpha(t)) = xg(x^{-\beta/\alpha}\sigma_\alpha(t))$ for all t . Thus, X under P_x has the same law as $(xg(x^{-\beta/\alpha}\sigma_\alpha(t)), t \geq 0)$ for some increasing function g .

By the Markov property,

$$P_y(X(t) \leq z) =_d P_x(X(s+t) \leq z | X(s) = y).$$

We have $P_y(X(t) \leq z) = P(\sigma_\alpha(t) \leq y^{\beta/\alpha} g^{-1}(z/y))$. Also,

$$\begin{aligned} P_x(X(s+t) \leq z | X(s) = y) &= P(\sigma_\alpha(s+t) \leq x^{\beta/\alpha} g^{-1}(z/x) | \sigma_\alpha(s) = x^{\beta/\alpha} g^{-1}(z/y)) \\ &= P(\sigma_\alpha(t) \leq x^{\beta/\alpha} (g^{-1}(z/x) - g^{-1}(y/x))) . \end{aligned}$$

It follows that for every $x \leq y \leq z$, we have

$$x^{\beta/\alpha} \left(g^{-1} \left(\frac{z}{x} \right) - g^{-1} \left(\frac{y}{x} \right) \right) = y^{\beta/\alpha} g^{-1} \left(\frac{z}{y} \right) .$$

Writing $u = y/x$ and $v = z/y$ gives that for every $u, v \geq 1$, we have

$$g^{-1}(uv) = u^{\beta/\alpha} g^{-1}(v) + g^{-1}(u) .$$

Taking $u = x$ and $v = 2$, we get $g^{-1}(2x) = x^{\beta/\alpha} g^{-1}(2) + g^{-1}(x)$. Taking $u = 2$ and $v = x$, we get $g^{-1}(2x) = 2^{\beta/\alpha} g^{-1}(x) + g^{-1}(2)$. It follows that $g^{-1}(x)(2^{\beta/\alpha} - 1) = g^{-1}(2)(x^{\beta/\alpha} - 1)$, which means $g^{-1}(x) = L(x^{\beta/\alpha} - 1)$ for some $L > 0$. Thus, $g(s) = G(1, s) = (1 + Ks)^{\alpha/\beta}$ for all s , where $K = L^{-1}$. It follows that for all x , we have $G(x, s) = xG(1, x^{-\beta/\alpha}(s)) = (x^{\beta/\alpha} + Ks)^{\alpha/\beta}$ for all but countably many s . Since G is increasing in x and s , we conclude that $G(x, s) = (x^{\beta/\alpha} + Ks)^{\alpha/\beta}$ for all x and s . //

We are now able to prove Proposition 3.1.

// Suppose that $\lambda(t)$ is of the form $g(\sigma_\alpha(t))$ for some decreasing function g . By Lemma 3.8 and the preceding discussion, g must be of the form $g(x) = (1 + Kx)^{-\alpha/\beta}$ for some $\beta > 0$.

Set $h(x) = g^{-1}(x) = K^{-1}(x^{-\beta/\alpha} - 1)$. Then $h'(x) = -\beta K^{-1} x^{-(\beta/\alpha)-1} / \alpha$. Let f_t be the density of $\sigma_\alpha(t)$, and let $f = f_1$. Then the density of $\lambda(t)$ is given by

$$k(x) = f_t(h(x)) |h'(x)| = f(t^{-1/\alpha} K^{-1}(x^{-\beta/\alpha} - 1)) t^{-1/\alpha} \frac{\beta}{\alpha} K^{-1} x^{-(\beta/\alpha)-1}$$

for all $x \in (0, 1)$. Note that

$$(t^{-1/\alpha} K^{-1}(x^{-\beta/\alpha} - 1))^{-1-\alpha} t^{-1/\alpha} K^{-1} x^{-(\beta/\alpha)-1} = t K^\alpha x^{\beta-1} (1 - x^{\beta/\alpha})^{-1-\alpha} .$$

Therefore, it follows from (3.12) and the Dominated Convergence Theorem that if A is a Borel subset of $[a, 1 - a]$ where $a > 0$, then

$$\lim_{t \rightarrow 0} t^{-1} P(\lambda(t) \in A) = \int_A a_1 \frac{K^\alpha \beta}{\alpha} x^{\beta-1} (1 - x^{\beta/\alpha})^{-1-\alpha} dx , \quad (3.43)$$

where a_1 is given in (3.11).

Let ν_i be the restriction of ν to the i th coordinate, and let γ be the measure defined by (3.23). By (3.43) and Corollary 3.1, γ is the measure on $[0, 1]$ with density $a_1 K^\alpha \beta \alpha^{-1} x^{\beta-1} (1 - x^{\beta/\alpha})^{-1-\alpha}$ for $x \in (0, 1)$. Since Π is a binary fragmentation process,

$$\gamma(A) = \int_{A \cap [1/2, 1]} x \nu_1(dx) + \int_{A \cap [0, 1/2]} x \nu_2(dx) .$$

Therefore, ν_1 has density $k_1(x) = a_1 K^\alpha \beta \alpha^{-1} x^{\beta-2} (1 - x^{\beta/\alpha})^{-1-\alpha} \mathbb{1}_{[1/2,1]}(x)$, while ν_2 has density $k_2(x) = a_1 K^\alpha \beta \alpha^{-1} x^{\beta-2} (1 - x^{\beta/\alpha})^{-1-\alpha} \mathbb{1}_{[0,1/2]}(x)$. However, since ν is concentrated on the set $\{x : x_1 + x_2 = 1\}$, we must have $k_1(x) = k_2(1 - x)$ for all x . This gives that

$$\left(\frac{1 - x^{\beta/\alpha}}{1 - (1 - x)^{\beta/\alpha}} \right)^{-1-\alpha} = \frac{(1 - x)^{\beta-2}}{x^{\beta-2}}.$$

Comparing asymptotic behavior as $x \rightarrow 0$, we get $\beta = \alpha$ and then $\alpha = 1/2$. Note that $a_1 = (2\pi)^{-1/2} = C_{1/2}$ when $\alpha = 1/2$. Thus, $\nu_1(dx) = (2\pi)^{-1/2} K^{1/2} x^{-3/2} (1 - x)^{-3/2} \mathbb{1}_{[1/2,1]}(x) dx$, which means that $(\Lambda(\Pi(t)), t \geq 0)$ is the Aldous–Pitman fragmentation up to a multiplicative time constant, as claimed. //

Chapter 4

Self-similar fragmentations derived from the stable tree I: splitting at heights

Contents

4.1	Introduction	97
4.2	Preliminaries	101
4.2.1	Stable processes, excursions, conditioned inverse subordinator	101
4.2.2	The stable tree	103
4.2.3	Some results on self-similar fragmentations	105
4.3	Study of F^-	108
4.3.1	Self-similarity and semigroup	108
4.3.2	Erosion and first properties of the dislocation measure	112
4.3.3	Dislocation measure	114
4.3.4	The missing constant	116
4.4	Small-time asymptotics	117
4.5	On continuous-state branching processes...	120

4.1 Introduction

The recent advances in the study of coalescence and fragmentation processes pointed at the key role played by tree structures in this topic, both at the discrete and continuous level [52, 10, 12]. Our goal here is to push further the investigation, begun in [10, 26], of a category of fragmentations obtained by cutting a certain continuum random tree. The tree that was fragmented in the latter articles is the Brownian Continuum Random Tree of Aldous, and the fragmentation is related to the so-called *standard additive coalescent*. The family of trees we consider is a natural but technically involved “Lévy generalization” of the Brownian tree. It has been introduced in Duquesne and Le Gall [49], and implicitly considered in the previous work of Kersting [67]. Some of these trees, which we call the *stable trees*, enjoy certain self-similar properties as their Brownian companion. In the

present work the crucial property is that when removing the vertices of the stable tree located under a fixed *height* (or distance to the root), the remaining object is a forest of smaller trees that have the same law as the original one up to rescaling. This is formalized in Lemma 4.3 below. This way of logging the stable tree induces a *fragmentation process* which by the property explained above turns out to be a *self-similar fragmentation*, the theory of such processes being extensively studied by Bertoin [23, 26, 27]. The goal of this paper is to describe the characteristics and give some properties of this fragmentation process. We will have to use stochastic processes and combinatorial approaches in the same time; in particular, we will encounter σ -finite generalizations of the (α, θ) -partitions of [91], which are distributions on the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$, as well as we will need the construction of the stable tree out of stable Lévy processes and its connection to continuous-state branching processes (CSBP) explained in [49].

In a companion paper [81] we will consider another way of obtaining a self-similar fragmentation by another cutting device on the stable tree, using the heuristic fact that when cutting at random one node (or “hub”) in the stable tree, the trunk and branches that have been separated are scaled versions of the initial tree. Surprisingly, although this other device looks quite different from the first (no mass is lost when cutting hubs, whereas there is a loss of mass when we throw everything that is located under the height h), it turns out that the only difference between these two fragmentations is the speed at which fragments decay, hence generalizing a “duality” relation stressed by Bertoin in [26] between two different fragmentations of the Brownian tree (one of these fragmentations being a direct analog of the fragmentation F^- considered here).

To state our main results, let us introduce quickly the already mentioned tree structures and fragmentation processes, postponing the details to a further section.

Let $S = \{\mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i \geq 1} s_i \leq 1\}$, endowed with the topology of pointwise convergence. A ranked self-similar fragmentation process $(F(t), t \geq 0)$ with index $\beta \in \mathbb{R}$ is a S -valued Markov process that is continuous in probability, such that $F(0) = (1, 0, 0, \dots)$ and such that conditionally on $F(t) = (x_1, x_2, \dots)$, $F(t + t')$ has the law of the decreasing arrangement of the sequences $x_i F^{(i)}(x_i^\beta t')$, where the $F^{(i)}$ are independent with the same law as F . That is, after time t , the different fragments evolve independently with a speed that depends on their size. It has been shown in [26] that such fragmentations are characterized by a 3-tuple (β, c, ν) , where β is the index, $c \geq 0$ is an “erosion” real constant saying that the fragments may melt continuously at some rate depending on c , and ν is a σ -finite measure on S that attributes mass 0 to $(1, 0, \dots)$ and that integrates $\mathbf{s} \mapsto (1 - s_1)$. This measure governs the sudden dislocations in the fragmentation process, and the integrability assumption ensures that these dislocations do not occur too quickly, although the fragmentation epochs may form a dense subset of \mathbb{R}_+ as soon as $\nu(S) = +\infty$. When $\beta < 0$, a positive fraction of the mass can disappear within a finite time, even though there is no loss of mass due to erosion nor to sudden dislocations. This phenomenon will be crucial in the fragmentation F^- below.

The trees we are considering are continuum random trees. Intuitively, they are metric spaces with an “infinitely ramified” tree structure, which can be considered as genealogical structures combined with two measures: a σ -finite *length measure* supported by the “skeleton” of the tree and a finite *mass measure* supported by its leaves, which are everywhere dense in the tree. These trees can be defined in several equivalent ways:

- as a weak limit of Galton-Watson trees

- through its *height process* H , which is a positive continuous process on $[0, 1]$. To a point $u \in [0, 1]$ corresponds a vertex of the tree with height (distance to the root) equal to H_u , and the mass measure on the tree is represented by Lebesgue's measure on $[0, 1]$
- through its explicit "marginals", that is, the laws of subtrees spanned by a random sample of leaves.

We will have to use the second (stochastic process) and third (combinatorial) points of view. We know from the works of Duquesne and Le Gall [49] and Duquesne [48] that one may define a particular instance of tree, called the *stable tree with index* α (for some $\alpha \in (1, 2]$). When $\alpha = 2$, the stable tree is equal to the Brownian CRT of Aldous [5], in which case the height process is a Brownian excursion conditioned to have duration 1. We will recall the rigorous construction of the height process of the stable tree in Sect. 4.2.2, but let us state our results now. Fix $\alpha \in (1, 2)$ and let $(H_s, 0 \leq s \leq 1)$ be the height process of the stable tree with index α .

The fragmentation process, that we call F^- , is defined as follows. For each $t \geq 0$, let $I_-(t)$ be the open subset of $(0, 1)$ defined by

$$I_-(t) = \{s \in (0, 1) : H_s > t\}.$$

With our intuitive interpretation of the height process, $I_-(t)$ is the set of vertices of the tree with height $> t$. We denote by $F^-(t)$ the decreasing sequence of the lengths of the connected components of $I_-(t)$. Hence, $F^-(t)$ is the sequence of the masses of the tree components obtained by cutting the stable tree below height t . The boundedness of H implies that $F^-(t) = (0, 0, \dots)$ as soon as $t \geq \max_{0 \leq s \leq 1} H_s$. As mentioned above, F^- is a direct generalization of the fragmentation F in [26, Section 4]. However, the nature of F^- strongly differs from that of F , because the latter is *binary* (a fragment breaks into exactly two fragments when a sudden dislocation occurs, which one expresses by $\nu\{\mathbf{s} : s_1 + s_2 < 1\} = 0$ where ν is the dislocation measure of F), while F^- is *infinitary* (the dislocation measure ν_- satisfies $\nu_-\{\mathbf{s} : s_N = 0\} = 0$ for every $N \geq 1$). This difference is due to the fact that the Brownian tree is itself binary, a property one can deduce from the well-known fact that local infima of the Brownian motion are pairwise distinct. By contrast, as explained below, the local infima of the height process of (non-Brownian) stable trees are attained at an infinite number of locations, so the stable trees are infinitary (see the construction of the tree out of its height process in Sect. 4.2.2).

Proposition 4.1 :

The process F^- is a ranked self-similar fragmentation with index $1/\alpha - 1 \in (-1/2, 0)$ and erosion coefficient 0.

Notice that, as mentioned before, F^- loses some mass, and eventually disappears completely in finite time even though the erosion is 0. This is due, of course, to the fact that the self-similarity index is negative.

Our main result is a description of the dislocation measure $\nu_-(ds)$ of F^- . Let us introduce some notation. For $\alpha \in (1, 2)$, let $(T_x, x \geq 0)$ be a stable subordinator with Laplace exponent $\lambda^{1/\alpha}$, that is, $(T_x, x \geq 0)$ has the same law as $(\sum_{y_i \leq x} r_i, x \geq 0)$, where $(y_i, r_i, i \geq 1)$ are the atoms of a Poisson point measure on $\mathbb{R}_+ \times (0, \infty)$ with intensity $c_\alpha dy dr / r^{1+1/\alpha}$, where $c_\alpha = (\alpha \Gamma(1 - 1/\alpha))^{-1}$. We denote by $\Delta T_x = T_x - T_{x-}$ the jump

at level x and by $\Delta T_{[0,x]}$ the sequence of the jumps of T before time x , and ranked in decreasing order. Define the measure ν_α on S by

$$\nu_\alpha(ds) = E \left[T_1 ; \frac{\Delta T_{[0,1]}}{T_1} \in ds \right] \tag{4.1}$$

where the last expression means that for any positive measurable function G , the quantity $\nu_\alpha(G)$ is equal to $E[T_1 G(T_1^{-1} \Delta T_{[0,1]})]$.

Theorem 4.1 :

The dislocation measure of F^- is $\nu_- = D_\alpha \nu_\alpha$, where

$$D_\alpha = \frac{\alpha(\alpha - 1)\Gamma(1 - \frac{1}{\alpha})}{\Gamma(2 - \alpha)} = \frac{\alpha^2\Gamma(2 - \frac{1}{\alpha})}{\Gamma(2 - \alpha)}.$$

Some comments about this. First, the dislocation measure charges only the sequences s for which $\sum_{i \geq 1} s_i = 1$, that is, no mass can be lost within a sudden dislocation. Second, we recognize an expression close to [92], of a Poisson-Dirichlet type distribution. Recall from [95, 92] that the (β, θ) Poisson-Dirichlet distribution is the law on S given by

$$PD(\beta, \theta)(ds) = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\beta + 1)} E \left[(T_1^\beta)^{-\theta}; \frac{\Delta T_{[0,1]}^\beta}{T_1^\beta} \in ds \right], \tag{4.2}$$

where T^β is a β -stable subordinator with Laplace exponent λ^β , and the definition makes sense if $\beta \in (0, 1)$ and $\theta > -\beta$. With this notation, $\nu_\alpha(ds)$ looks like a “renormalized Poisson-Dirichlet $(1/\alpha, -1)$ distribution”. However, it has to be noticed that this corresponds to a forbidden parametrization $\theta = -1$, and indeed, the measure that we obtain is infinite since $E[T_1] = \infty$. This measure integrates $s \mapsto 1 - s_1$ though, just as it has to. Indeed, $E[T_1 - \Delta_1]$ is finite if Δ_1 denotes the largest jump of T before time 1. To see this, notice that $\Delta_1 \geq \Delta_1^*$ where Δ_1^* is a size-biased pick from the jumps of T before time 1, and it follows from Lemma 4.1 in Sect. 4.2.1 below and scaling arguments that $T - \Delta_1^*$ has finite expectation.

The rest of the paper is organized as follows. In Sect. 4.2 we first recall some facts about Lévy processes, excursions, and conditioned subordinators. Then we give the rigorous description of the stable tree, and state some properties of the height process that we will need. Last we recall some facts about self-similar fragmentations. We then obtain the characteristics of F^- in Sect. 4.3 and derive its semigroup. We insist on the fact that knowing explicitly the semigroup of a fragmentation process is in general a very complicated problem, see [82] for somehow surprising negative results in this vein. However, most of the fragmentation processes that have been extensively studied in recent years [10, 21, 79, 26] do have known, and sometimes strange-looking semigroups involving conditioned Poisson clouds. And as a matter of fact, the fragmentation F^+ considered in the companion paper [81] has also an explicit semigroup. We end the study of F^- by giving asymptotic results for small times in Sect. 4.4. These results need some properties of conditioned continuous-state branching processes, which are in the vein of Jeulin’s results for the rescaled Brownian excursion and its local times. We prove these properties in Sect. 4.5, where we give the rigorous definition of some processes that are used heuristically in Sect. 4.3 to conjecture the form of the dislocation measure.

4.2 Preliminaries

4.2.1 Stable processes, excursions, conditioned inverse subordinator

Throughout the paper, we let $(X_s, s \geq 0)$ be the canonical process in the Skorokhod space $\mathbb{D}([0, \infty))$ of càdlàg paths on $[0, \infty)$. Recall that a Lévy process is a real-valued càdlàg process with independent and stationary increments. We fix $\alpha \in (1, 2)$. Let P be the law that makes X a stable Lévy process with no negative jumps and Laplace exponent $E[\exp(-\lambda X_s)] = \exp(-\lambda^\alpha s)$ for $s, \lambda \geq 0$, where E is the expectation associated with P . Such a process has infinite variation and satisfies $E[X_1] = 0$. When there is no ambiguity, we may sometimes speak of X as being itself the Lévy process with law P . Writing this in the form of the Lévy-Khintchine formula, we have :

$$E[\exp(-\lambda X_s)] = \exp\left(s \int_0^\infty \frac{C_\alpha dx}{x^{1+\alpha}} (e^{-\lambda x} - 1 + \lambda x)\right), \quad s, \lambda \geq 0, \quad (4.3)$$

where $C_\alpha = \alpha(\alpha - 1)/\Gamma(2 - \alpha)$, and we say that the Lévy measure of X under P is $C_\alpha x^{-1-\alpha} dx \mathbb{1}_{\{x>0\}}$. An important property of X is then the scaling property: under P ,

$$\left(\frac{1}{\lambda^{1/\alpha}} X_{\lambda s}, s \geq 0\right) \stackrel{d}{=} (X_s, s \geq 0) \quad \text{for all } \lambda > 0.$$

It is known [101] that under P , X_s has a density $(p_s(x), x \in \mathbb{R})$ for every $s > 0$, such that $p_s(x)$ is jointly continuous in x and s .

Excursions Let \underline{X} be the infimum process of X , defined for $s \geq 0$ by

$$\underline{X}_s = \inf\{X_u, 0 \leq u \leq s\}.$$

By Itô's excursion theory for Markov processes, the excursions away from 0 of the process $X - \underline{X}$ under P are distributed according to a Poisson point process that can be described by the Itô excursion measure, which we call N . We now either consider the process X under the law P that makes it a Lévy process starting at 0, or under the σ -finite measure N under which the sample paths are excursions with finite lifetime ζ (since $E[X_1] = 0$). Let $N^{(v)}$ be a regular version of the probability law $N(\cdot | \zeta = v)$, which is weakly continuous in v . That is, for any positive continuous functional G ,

$$N(G) = \int_{(0, \infty)} N(\zeta \in dv) N^{(v)}(G)$$

and $\lim_{w \rightarrow v} N^{(w)}(G) = N^{(v)}(G)$ as $w \rightarrow v$. Such a version can be obtained by scaling: for any fixed $\eta > 0$, the process

$$\left((v/\zeta)^{1/\alpha} X_{\zeta s/v}, 0 \leq s \leq v\right) \quad \text{under } N(\cdot | \zeta > \eta) = \frac{N(\cdot, \zeta > \eta)}{N(\zeta > \eta)}$$

is $N^{(v)}$. See [43] for this and other interesting ways to obtain processes with law $N^{(v)}$ by path transformations. In particular, one has the scaling property at the level of conditioned excursions: under $N^{(v)}$, $(v^{-1/\alpha} X_{vs}, 0 \leq s \leq 1)$ has law $N^{(1)}$.

First-passage subordinator Let T be the right-continuous inverse of the increasing process $-\underline{X}$, that is,

$$T_x = \inf\{s \geq 0 : \underline{X}_s < -x\}.$$

Then it is known that under P , T is a subordinator, that is, an increasing Lévy process. According to [19, Theorem VII.1.1], its Laplace exponent ϕ is the inverse function of the restriction of the Laplace exponent of X to \mathbb{R}_+ . Thus $\phi(\lambda) = \lambda^{1/\alpha}$, and T is a stable subordinator with index $1/\alpha$, as defined above. The Lévy-Khintchine formula gives, for $\lambda, x \geq 0$,

$$E[\exp(-\lambda T_x)] = \exp(-x\lambda^{1/\alpha}) = \exp\left(-x \int_0^\infty \frac{c_\alpha dy}{y^{1+1/\alpha}}(1 - e^{-\lambda y})\right).$$

where c_α has been defined in the introduction. Recall that X has a marginal density $p_s(\cdot)$ at time s under P . Then under P , the inverse subordinator T has also jointly continuous densities, given by (see e.g. [19, Corollary VII.1.3])

$$q_x(s) = \frac{P(T_x \in ds)}{ds} = \frac{x}{s} p_s(x). \tag{4.4}$$

This equation can be derived from the ballot theorem of Takács [102].

Let us now discuss the conditioned forms of distributions of the sequence $\Delta T_{[0,x]}$ given T_x . An easy way to obtain nice regular versions for these conditional laws is developed in [87, 92], and uses the notion of size-biased fragment. Precisely, the range of any subordinator, with drift 0 say (which we will assume in the sequel), between times 0 and x , induces a partition of $[0, T_x]$ into subintervals with sum T_x . Consider a sequence $(U_i, i \geq 1)$ of independent uniform $(0, 1)$ variables, independent of T , and let $\Delta_1^*(x), \Delta_2^*(x), \dots$ be the sequence of the lengths of these intervals in the order in which they are discovered by the U_i 's. That is, $\Delta_1^*(x)$ is the length of the interval in which $T_x U_1$ falls, $\Delta_2^*(x)$ is the length of the first interval different from the one containing $T_x U_1$ in which $T_x U_i$ falls, and so on. Then Palm measure results for Poisson clouds give the following result (specialized to the case of stable subordinators).

Lemma 4.1 :

The joint law under P of $(\Delta_1^*(x), T_x)$ is

$$P(\Delta_1^*(x) \in dy, T_x \in ds) = \frac{c_\alpha x q_x(s-y)}{s y^{1/\alpha}} dy ds, \tag{4.5}$$

and more generally for $j \geq 1$,

$$P(\Delta_j^*(x) \in dy | T_x = s_0, \Delta_k^*(x) = s_k, 1 \leq k \leq j-1) = \frac{c_\alpha x q_x(s-y)}{s y^{1/\alpha} q_x(s)} dy,$$

where $s = s_0 - s_1 - \dots - s_{j-1}$.

This gives a nice regular conditional version for $(\Delta_i^*(x), i \geq 1)$ given T_x , and thus induces a conditional version for $\Delta T_{[0,x]}$ given T_x , by ranking, where $\Delta T_{[0,x]}$ is the sequence of jumps of T before x , ranked in decreasing order of magnitude.

4.2.2 The stable tree

We now introduce the models of trees we will consider. This section is mainly inspired by [49, 48]. With the notations of Sect. 4.2.1, for $u \geq 0$, let $R^{(u)}$ be the time-reversed process of X at time u :

$$R_s^{(u)} = X_u - X_{(u-s)-}, \quad 0 \leq s \leq u.$$

It is standard that this process has the same law as X killed at time u under P . Let also

$$\overline{R}_s^{(u)} = \sup_{0 \leq v \leq s} R_v^{(u)}, \quad 0 \leq s \leq u$$

be its supremum process. We let H_u be the local time at 0 of the process $R^{(u)}$ reflected under its supremum $\overline{R}^{(u)}$ up to time u . The normalization can be chosen so that we have the limit in probability

$$H_u = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^u \mathbb{1}_{\{\overline{R}_s^{(u)} - R_s^{(u)} \leq \varepsilon\}} ds.$$

It is known by [49, Theorem 1.4.3] that H admits a continuous version, with which we shall work in the sequel. It has to be noticed that H is not a Markov process (the only exception in the theory of Lévy trees is the Brownian tree obtained when P is the law of Brownian motion with drift, which has been excluded in our discussion). As a matter of fact, it can be checked that under P , H admits local minima that are attained an infinite number of times, a property that strongly contrasts with Brownian motion or Lévy processes with infinite variation. To see this, consider a jump time t of X , and let $t_1, t_2 > t$ so that $\inf_{t \leq u \leq t_i} X_u = X_{t_i}$ and $X_{t-} < X_{t_i} < X_t$, $i \in \{1, 2\}$. Then it is easy to see that $H_t = H_{t_1} = H_{t_2}$ and that one may in fact find an infinite number of distinct t_i 's satisfying the properties of t_1, t_2 . On the other hand, it is not difficult to see that H_t is a local minimum of H . One can in fact deduce from the fact that F^- is infinitary that every local minimum is attained an infinite number of times, as mentioned in the introduction.

It is shown in [49] that the definition of H still makes sense under the σ -finite measure N rather than the probability law P . The process H is then defined only on $[0, \zeta]$, and we call it the *excursion of the height process*. One can define without difficulty, using the scaling property, the height process under the laws $N^{(v)}$: this is simply the law of

$$((v/\zeta)^{1-1/\alpha} H_{\zeta t/v}, 0 \leq t \leq v) \quad \text{under } N(\cdot | \zeta > \eta).$$

Call it the law of the excursion of the height process with duration v . The following scaling property is the key for the self-similarity of F^- : for every $x > 0$,

$$(v^{1/\alpha-1} H_{sv}, 0 \leq s \leq 1) \text{ under } N^{(v)} \stackrel{d}{=} (H_s, 0 \leq s \leq 1) \text{ under } N^{(1)}. \quad (4.6)$$

This property is inherited from the scaling property of X , and it is easily obtained e.g. by the above definition of H as an approximation.

An important tool for studying the height process is its *local time process*, or *width process*, which we will denote by $(L_s^t, t \geq 0, s \geq 0)$. It can be obtained for every fixed s, t by the limit in probability

$$L_s^t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbb{1}_{\{t < H_u \leq t + \varepsilon\}} du.$$

L_s^t is then the density of the occupation measure of H at level t and time s . For $t = 0$, one has that $(L_s^0, s \geq 0)$ is the inverse of the subordinator T , which is a reminiscent of the fact that the excursions of the height process are in one-to-one correspondence with excursions of X with the same lengths. According to the Ray-Knight theorem [49, Theorem 1.4.1], for every $x > 0$, the process $(L_{T_x}^t, t \geq 0)$ is a continuous-state branching process with branching mechanism λ^α , in short α -CSBP. We will recall basic and less basic features about this processes in Sect. 4.5, where in particular an interpretation for the law of the process $(L_1^t, t \geq 0)$ under $N^{(1)}$ will be given. For now we just note that for every x the process $(L_{T_x}^t, t \geq 0)$ is a process with no negative jumps, and a jump of this process at time t corresponds precisely to one of the infinitely often attained local infima of the height process. With the forthcoming interpretation of the tree encoded within excursions of the height process, this means that there is a branchpoint with infinite degree at level t . It is again possible to define the local time process under the excursion measure N , and by scaling it is also possible to define the local time process under $N^{(v)}$.

Let us now motivate the term of “height process” for H . Under the σ -finite “law” N , we define a tree structure following [5, 75].

First we introduce some extra vocabulary. Let \mathbf{T} be the set of finite rooted plane trees, that is, for any $\mathcal{T} \in \mathbf{T}$, each set of children of a vertex $v \in \mathcal{T}$ is ordered as first, second, ..., last child. Let $\mathbf{T}^* \subset \mathbf{T}$ be those rooted plane trees for which the out-degree (number of children) of vertices is never 1. Let \mathbf{T}_n and \mathbf{T}_n^* be the corresponding sets of trees that have exactly n leaves (vertices with out-degree 0). A marked tree ϑ is a pair $(\mathcal{T}, \{h_v, v \in \mathcal{T}\})$ where $\mathcal{T} \in \mathbf{T}$ and $h_v \geq 0$ for every vertex v of \mathcal{T} (which we denote by $v \in \mathcal{T}$). The tree \mathcal{T} is called the skeleton of ϑ , and the h_v 's are the marks. These marks induce a distance on the tree, given by $d_\vartheta(v, v') = \sum_{w \in [[v, v']]} h_w$ if $v, v' \in \mathcal{T}$ are two vertices of the marked tree, where $[[v, v']]$ is the set of vertices of the path from v to v' in the skeleton. The distance of a vertex to the root will be called its *height*. Let \mathbb{T}_n^* be the set of marked trees with n leaves and no out-degree equal to 1.

Let $(U_i, i \geq 1)$ be independent random variables with uniform law on $(0, 1)$ and independent of the excursion H of the height process. One may define a random marked tree $\vartheta(U_1, \dots, U_k) = \vartheta_k \in \mathbb{T}_k^*$, as follows. For $u, v \in [0, \zeta]$ let $m(u, v) = \inf_{s \in [u, v]} H_s$. Roughly, the key fact about ϑ_k is that the height of the i -th leaf to the root is $H_{U_{(i)}}$, where $(U_{(i)}, 1 \leq i \leq k)$ are the order statistics of $(U_i, 1 \leq i \leq k)$, and the ancestor of the i -th and j -th leaves has height $m(\zeta U_{(i)}, \zeta U_{(j)})$ for every i, j . This allows to build recursively a tree by first putting the mark $h_{\text{root}} = \inf_{1 \leq i \leq j \leq k} m(U_i, U_j)$ on a root vertex. Let c_{root} be the number of excursions of H above level h_{root} in which at least one ζU_i falls. Attach c_{root} vertices to the root, and let the i -th of these vertices be the root of the tree embedded in the i -th of these excursions above level h_{root} . Go on until the excursions separate the variables U_i . By construction $\vartheta_k \in \mathbb{T}_k^*$. Adding a $(k + 1)$ -th variable U_{k+1} to the first k just adds a new branch to the tree in a consistent way as k varies.

As noted above, we may as well define the trees $(\vartheta_k, k \geq 0)$ under the law $N^{(1)}$ by means of scaling.

Definition :

 The family of marked trees $(\vartheta_k, k \geq 1)$ associated with the height process under the law $N^{(1)}$ is called the stable tree.

Remark. The previous definition is not the only way to characterize the same object. Alternatively, one easily sees that the marked tree ϑ_k can be interpreted as a subset of l^1 ,

each new branch going in a direction orthogonal to the preceding branches, in a consistent way as k varies. Then it makes sense to take the metric completion of $\cup_{k \geq 1} \vartheta_k$, which we could also call the stable tree, and one can check that the branchpoints of this tree all have infinite degree because the local minima of H are attained an infinite number of times. This object is also isometric to the space obtained by taking the quotient of $[0, 1]$ endowed with the pseudo-metric

$$d(u, v) = H_u + H_v - 2m(u, v), \quad u, v \in [0, 1],$$

with respect to the equivalence relation $u \equiv v \iff d(u, v) = 0$. With this way of looking at things, the leaves of the tree are uncountable and everywhere dense in the tree, and the empirical distribution on the leaves of ϑ_k converges weakly to a probability measure on the stable tree, called the mass measure. Then it turns out that ϑ_k is equal in law to the subtree of the stable tree that is spanned by the root and k independent leaves distributed according to the mass measure. Hence, the mass measure is represented by Lebesgue measure on $[0, 1]$ in the coding of the stable tree through its height process. This is coherent with the definition of $F^-(t)$ as the “masses of the tree components located above height t ”. The equivalence between these possible definitions is discussed in [5].

The key property for obtaining the dislocation measure of F^- is the following description of the law of the skeleton of ϑ_n , and the mark of the root of ϑ_1 . For $\mathcal{T} \in \mathbf{T}$ let $\mathcal{N}_{\mathcal{T}}$ be the set of non-leaf vertices of \mathcal{T} and for $v \in \mathcal{N}_{\mathcal{T}}$ let $c_v(\mathcal{T})$ be the number of children of v . From the more complete description of the marked trees in [49, Theorem 3.3.3], we recall that

Proposition 4.2 :

The probability that the skeleton of ϑ_n is $\mathcal{T} \in \mathbf{T}_n^*$ is

$$\frac{n!}{(\alpha - 1)(2\alpha - 1) \dots ((n - 1)\alpha - 1)} \prod_{v \in \mathcal{N}_{\mathcal{T}}} \frac{|(\alpha - 1)(\alpha - 2) \dots (\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!}.$$

Moreover, the law of the mark of the root in ϑ_1 is

$$N^{(1)}(H_{U_1} \in dh) = \alpha \Gamma \left(1 - \frac{1}{\alpha} \right) \chi_{\alpha h}(1) dh,$$

where $(\chi_x(s), s \geq 0)$ is the density of the stable $1 - 1/\alpha$ subordinator (with Laplace exponent equal to $\lambda^{1-1/\alpha}$) at time x .

4.2.3 Some results on self-similar fragmentations

In this section we are going to recall some basic facts about the theory of self-similar fragmentations, and also introduce some useful ways to recover the characteristics of these fragmentations. We will suppose that the fragmentations we consider are not trivial, that is, they are not equal to their initial state for every time. It will be useful to consider not only S -valued (or *ranked*) fragmentations, but also fragmentations with values in the set of open subsets of $(0, 1)$ and in the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$, respectively called *interval* and *partition-valued* fragmentations. As established in [26, 17], there is a one-to-one mapping between the laws of the three kinds of fragmentation when they satisfy a

self-similarity property that is similar to that of the ranked fragmentations. That is, each of them is characterized by the same 3-tuple (β, c, ν) introduced above. To be completely accurate, we should stress that there actually exist several versions of interval partitions that give the same ranked or partition-valued fragmentation, but all these versions have the same characteristics (β, c, ν) . Let us make the terms precise.

Let \mathcal{P} be the set of unordered partitions of \mathbb{N} . An exchangeable partition Π is a \mathcal{P} -valued random variable whose restriction Π_n to $[n] = \{1, \dots, n\}$ has an invariant law under the action of the permutations of $[n]$, for every n . By Kingman's representation theorem [68, 2], the blocks of exchangeable partitions of \mathbb{N} admit almost-sure asymptotic frequencies, that is, if $\Pi = \{B_1, B_2, \dots\}$ where the B_i 's are listed by order of their least element, then

$$\Lambda(B_i) = \lim_{n \rightarrow \infty} \frac{\text{Card}(B_i \cap [n])}{n}$$

exists a.s. for every $i \geq 0$. Denoting by $\Lambda(\Pi)$ the ranked sequence of these asymptotic frequencies, $\Lambda(\Pi)$ is then a \mathcal{S} -valued random variable, whose law characterizes that of Π .

A self-similar partition-valued fragmentation $(\Pi(t), t \geq 0)$ with index β is a \mathcal{P} -valued càdlàg process that is continuous in probability, exchangeable, meaning that for every permutation σ of \mathbb{N} , $(\sigma\Pi(t), t \geq 0)$ and $(\Pi(t), t \geq 0)$ have the same law, and such that given $\Pi(t) = \{B_1, B_2, \dots\}$, the variable $\Pi(t+t')$ has the law of the partition with blocks $\Pi^{(i)}(\Lambda(B_i)^{\beta}t') \circ B_i$ where the $\Pi^{(i)}$ are independent copies of Π . Here, the operation \circ is the natural "fragmentation" operation of a set by a partition: if $\Pi = \{B_1, B_2, \dots\}$ and $C \subset \mathbb{N}$, then $\Pi \circ C$ is the partition of C with blocks $B_i \cap C$.

A self-similar interval partition $(I(t), t \geq 0)$ with index β is a process with values in the open subsets \mathcal{O} of $(0, 1)$ which is right-continuous and continuous in probability for the usual Hausdorff distance between the complementary sets $[0, 1] \setminus \mathcal{O}$, with the property that given $I(t) = \cup_{i \geq 1} I_i$ say, where the I_i are the disjoint connected components of $I(t)$, the set $I(t+t')$ has the law of $\cup_{i \geq 1} g_i(I^{(i)}(t'|I_i|^\beta))$, where $|I_i|$ is the length of I_i , g_i is the affine transformation that maps $(0, 1)$ to I_i and conserves orientation and the $I^{(i)}$ are independent copies of I .

Consider an interval self-similar fragmentation $(I(t), t \geq 0)$, with characteristics $(\beta, 0, \nu)$ (the case when $c > 0$ would be similar, but we do not need it in the sequel). Let $U_i, i \geq 1$ be independent uniform random variables on $(0, 1)$. These induce a partition-valued fragmentation $(\Pi(t), t \geq 0)$ by letting $i \stackrel{\Pi(t)}{\sim} j$ iff U_i and U_j are in the same connected component of $I(t)$. It is known [26] that Π is a self-similar fragmentation with values in the set of partitions of \mathbb{N} and characteristics $(\beta, 0, \nu)$. For $n \geq 2$ let \mathcal{P}_n^* be the set of partitions of \mathbb{N} whose restriction to $[n]$ is non-trivial, i.e. different from $\{[n]\}$. Then there is some random time $t_n > 0$ such that the restriction of $\Pi(t)$ to $[n]$ jumps from the trivial state $\{[n]\}$ to some non-trivial state at time t_n . Let $\rho(n)$ be the law of the restriction of $\Pi(t_n)$ to $[n]$. The next lemma states that the knowledge of the family $(\rho(n), n \geq 2)$ almost determines the dislocation measure ν of the fragmentation. Precisely, we introduce from [23] the notion of characteristic measure of the fragmentation. This measure, denoted by κ , is a σ -finite measure supported by the non-trivial partitions of \mathbb{N} , which is determined by the dislocation measure of the fragmentation. This measure may be written as

$$\kappa(d\pi) = \int_{\mathcal{S}} \nu(ds) \kappa_s(d\pi),$$

where κ_s is the law of the exchangeable partition of \mathbb{N} with ranked asymptotic frequencies given by s . Conversely, this measure characterizes the dislocation measure ν (simply by taking the asymptotic frequencies of the generic partition under κ).

Lemma 4.2 :

The restriction of κ to the non-trivial partitions of $[n]$, for $n \geq 2$, equals $q(n)\rho(n)$, for some sequence $(q(n), n \geq 2)$ of strictly positive numbers. As a consequence, the dislocation measure of the fragmentation I is characterized by the sequence of laws $(\rho(n), n \geq 2)$, up to a multiplicative constant.

Otherwise said, and using the correspondence between self-similar fragmentations with same dislocation measure and different indices established by Bertoin [26] by introducing the appropriate time-changes, if we have two interval-valued self-similar fragmentations I and I' with the same index and no erosion, and with the same associated probabilities $\rho(n)$ and $\rho'(n)$, $n \geq 1$, then there exists $K > 0$ such that $(I(Kt), t \geq 0)$ has the same dislocation measure as I' .

// Suppose $\beta = 0$, then the result is almost immediate by the results of [23] on homogeneous fragmentation processes. In this case $q(n)$ is the inverse of the expected jump time of Π in \mathcal{P}_n^* , and the restriction of the measure $q(n+1)\rho(n+1)$ to the set of non-trivial partitions of $[n]$ is $q(n)\rho(n)$, for every $n \geq 1$. Hence, it is easy to see that the knowledge on $\rho(n)$ determines uniquely the sequence $(q(n), n \geq 1)$, up to a multiplicative positive constant: one simply has $q(n)/q(n+1) = \rho(n+1)(\pi|_{[n]} : \pi \in \mathcal{P}_n^*)$, where $\pi|_{[n]}$ denotes the restriction of π to $[n]$. It remains to notice that the sequence of restrictions $(q(n)\rho(n), n \geq 2)$ characterizes κ .

When $\beta \neq 0$, we obtain the same results by noticing that the law $\rho(n)$ still equals the law of the restriction to $[n]$ of the exchangeable partition with limiting frequencies having the “law” ν and restricted to \mathcal{P}_n^* , up to a multiplicative constant. Indeed, let $I^*(t)$ be the subinterval of $I(t)$ containing U_1 at time t , and recall [26] that if

$$a(t) = \inf \left\{ u \geq 0 : \int_0^u |I^*(v)|^\beta dv > t \right\},$$

then $(|I^*(a(t))|, t \geq 0)$ evolves as the fragment containing U_1 in an interval fragmentation with characteristics $(0, 0, \nu)$. Now, before time t_n , the fragment containing U_1 is the same as that containing all the $(U_i, 1 \leq i \leq n)$. Hence, $a(t_n)$ is the first time when Π' jumps in \mathcal{P}_n^* for some homogeneous partition-valued fragmentation process Π' with characteristics $(0, 0, \nu)$, and the law of $\Pi'(a(t_n))$ restricted to $[n]$ is $\rho(n)$. Hence the result. //

We also cite the following result [82, Proposition 3] which allows to recover the dislocation measure of a self-similar fragmentation with positive index out of its semigroup. We will not use this proposition in a proof, but it is useful to keep it in mind to conjecture the form of the dislocation measure of F^- , as it will be done below.

Proposition 4.3 :

Let $(F(t), t \geq 0)$ be a ranked self-similar fragmentation with characteristics $(\beta, 0, \nu)$, $\beta \geq 0$. Then for every continuous bounded function G on S which is null on an open

neighborhood of $(1, 0, \dots)$, one has

$$\frac{1}{t} E[G(F(t))] \xrightarrow[t \downarrow 0]{} \nu(G).$$

4.3 Study of F^-

We now specifically turn to the study of F^- defined in the introduction. Although some of the results below may be easily generalized to a broader ‘‘Lévy context’’, we will suppose in this section that X is a stable process with index $\alpha \in (1, 2)$, with first-passage subordinator T . The references to height processes, excursion measures and so on, will always be with respect to this process, unless otherwise specified. Also, for the needs of the proofs below, we define the process $(F^-(t), t \geq 0)$ not only under the law $N^{(1)}$ used to define the stable tree, but also for all the excursion measures $N^{(v)}$ and N . Under $N^{(v)}$, let $F^-(t)$ be the decreasing sequence of lengths of the constancy intervals of $I_-(t) = \{s \in (0, v) : H_s > t\}$ (v is replaced by ζ under N). To avoid confusions, we will always mention in Sect. 4.3.1 the measure we are working with, but this formalism will be abandoned in the following sections where no more use of $N^{(v)}$ is made with $v \neq 1$.

The study contains four steps. First we prove the self-similarity property for F^- and make its semigroup explicit. Heuristic arguments based on generators of conditioned CSBP’s allow us to conjecture the rough shape of the dislocation measure. Then we prove that the erosion coefficient is 0 by studying the evolution of a tagged fragment. We then apply Lemma 4.2, giving us the dislocation measure up to a constant, and we finally recover the constant by re-obtaining the results needed in the second step by another computation.

4.3.1 Self-similarity and semigroup

The self-similarity and the description of the semigroup rely strongly on the following result, which is a variant of [49, Proposition 1.3.1]. For $t, s \geq 0$ let

$$\gamma_s^t = \inf\{u \geq 0 : \int_0^u \mathbb{1}_{\{H_v > t\}} dv > s\}$$

and

$$\tilde{\gamma}_s^t = \inf\{u \geq 0 : \int_0^u \mathbb{1}_{\{H_v \leq t\}} dv > s\}.$$

Denote by \mathcal{H}_t the sigma-field generated by the process $(H_{\gamma_s^t}, s \geq 0)$ and the P -negligible sets. Let also $(H_s^t, s \geq 0)$ be the process $(H_{\gamma_s^t} - t, s \geq 0)$. Then under P , H^t is independent of \mathcal{H}_t , and its law is the same as that of H under P .

As a first consequence, we obtain that the excursions of H above level t , that is, the excursions of H^t above level 0, are, conditionally on their durations, independent excursions of H . This simple result allows us to state the Markov property and self-similarity of F^- . In the following statement, it has to be understood that we work under the probability

$N^{(1)}$ and that the process H that is considered is the same that is used to construct F^- .

Lemma 4.3 :

Conditionally on $F^-(t) = (x_1, x_2, \dots)$, the excursions of H above level t , that is, of H^t away from 0, are independent excursions with respective laws $N^{(x_1)}, N^{(x_2)}, \dots$

As a consequence, the process F^- is a self-similar fragmentation process with index $1/\alpha - 1$.

// By the previous considerations on H^t , we have that under P , given that the lengths of interval components of the set $\{s \in [0, T_1] : H_s > t\}$ ranked in decreasing order are equal to (x_1, x_2, \dots) , the excursions of the killed process $(H(t), 0 \leq t \leq T_1)$ above level t are independent excursions of H with durations x_1, x_2, \dots . The first part of the statement follows by considering the first excursion of H (or of X) that has duration greater than some $v > 0$, which gives the result under the measure $N(\cdot, \zeta > v)$, hence for N , hence for $N^{(v)}$ for almost all v , and then for $v = 1$ by continuity of the measures $N^{(v)}$.

Thus, conditionally on $F^-(t) = (x_1, x_2, \dots)$, the process $(F^-(t + t'), t \geq 0)$ has the same law as the random sequence obtained by taking independent excursions $H^{(x_1)}, H^{(x_2)}, \dots$ with durations x_1, x_2, \dots of the height process, and then arranging in decreasing order the lengths of constancy intervals of the sets

$$\{s \in [0, x_i] : H_s^{(x_i)} > t'\}.$$

It thus follows from the scaling property (4.6) of the excursions of H that given $F^-(t) = (x_1, x_2, \dots)$, the process $(F^-(t + t'), t' \geq 0)$ has the same law as the decreasing rearrangement of the processes $(x_i F_{(i)}^-(x_i^{1/\alpha-1} t'), t' \geq 0)$, where the $F_{(i)}^-$'s are independent copies of F^- . The fact that F^- is a Markov process that is continuous in probability easily follows, as does the self-similar fragmentation property with the index $1/\alpha - 1$. //

We now turn our attention to the semigroup of F^- .

Proposition 4.4 :

For every $t \geq 0$ one has

$$\begin{aligned} N^{(1)}(F^-(t) \in ds) & \hspace{15em} (4.7) \\ &= \int_{\mathbb{R}_+ \times [0,1]} N^{(1)} \left(L_1^t \in d\ell, \int_t^\infty db L_1^b \in dz \right) P(\Delta T_{[0,\ell]} \in ds | T_\ell = z), \end{aligned}$$

with the convention that the law $P(\Delta T_{[0,0]} \in ds | T_0 = z)$ is the Dirac mass at the sequence $(z, 0, 0, \dots)$ for every $z \geq 0$.

// It suffices to prove the result for some fixed $t > 0$. Let $\omega(t) = \inf\{s \geq 0 : H_s > t\}$, $d_{\omega(t)} = \inf\{s \geq \omega(t) : X_s = \underline{X}_s\}$ and $g_{\omega(t)} = \sup\{s \leq \omega(t) : X_s = \underline{X}_s\}$. Call $\mathcal{F}^-(t)$ the ranked sequence of the lengths of the interval components of the set $\{s \in [\omega(t), d_{\omega(t)}] : H_s > t\}$. Notice that under the law $N^{(1)}$, \mathcal{F}^- would be F^- , but we will first define \mathcal{F}^- under P . By the definition of H , $\omega(t)$ and $d_{\omega(t)}$ are stopping times with respect to the natural

filtration generated by X . In fact, it also holds that $\omega(t)$ is a *terminal time*, that is,

$$\omega(t) = s + \inf\{u \geq 0 : H_{s+u} > t\} \quad \text{on } \{\omega(t) > s\}.$$

Moreover, $0 < \omega(t) < \infty$ P -a.s., because of the continuity of H and the fact that excursions of H have a positive probability to hit level t (which follows e.g. by scaling). Recall the notations at the beginning of the section, and denote by A^t and \tilde{A}^t the right-continuous inverses of γ^t and $\tilde{\gamma}^t$. Then the local time $L_{d_{\omega(t)}}^t$ is the local time at level 0 and time $A_{d_{\omega(t)}}^t$ of the process H^t . This is also equal to the local time of $(H_{\tilde{\gamma}_s^t}, s \geq 0)$ at level t and time $\tilde{A}_{d_{\omega(t)}}^t$. This last time is \mathcal{H}_t -measurable, as it is the first time the process $(H_{\tilde{\gamma}_s^t}, s \geq 0)$ hits back 0 after first hitting t . Hence $L_{d_{\omega(t)}}^t$ is \mathcal{H}_t -measurable, hence independent of H^t . Let T^t be the inverse local time of H^t at level 0, which is $\sigma(H^t)$ -measurable, hence independent of \mathcal{H}_t , and has same law as T since H^t has same law as H under P . Notice that $\mathcal{F}^-(t)$ equals the sequence $\Delta T_{[0, L_{d_{\omega(t)}}^t]}^t$, and that the $\sigma(H^t)$ -measurable random variable $\int_t^\infty db L_{d_{\omega(t)}}^b = T^t(L_{d_{\omega(t)}}^t)$. Thus, conditionally on $L_{d_{\omega(t)}}^t = \ell$ and $\int_t^\infty db L_{d_{\omega(t)}}^b = z$, $\mathcal{F}^-(t)$ has law $P(\Delta T_{[0, \ell]} \in ds | T_\ell = z)$. Hence

$$P(\mathcal{F}^-(t) \in ds) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} P\left(L_{d_{\omega(t)}}^t \in d\ell, \int_t^\infty db L_{d_{\omega(t)}}^b \in dz\right) P(\Delta T_{[0, \ell]} \in ds | T_\ell = z),$$

and also, since $d_{\omega(t)} - g_{\omega(t)} = \int_0^\infty db (L_{d_{\omega(t)}}^b - L_{g_{\omega(t)}}^b)$ and since $\int_0^t db (L_{d_{\omega(t)}}^b - L_{g_{\omega(t)}}^b)$ is independent of $\sigma(H^t)$, the result also holds conditionally on $d_{\omega(t)} - g_{\omega(t)}$, namely

$$\begin{aligned} P(\mathcal{F}^-(t) \in ds | d_{\omega(t)} - g_{\omega(t)}) &= \int_{\mathbb{R}_+ \times \mathbb{R}_+} P\left(L_{d_{\omega(t)}}^t \in d\ell, \int_t^\infty db L_{d_{\omega(t)}}^b \in dz \middle| d_{\omega(t)} - g_{\omega(t)}\right) \\ &\times P(\Delta T_{[0, \ell]} \in ds | T_\ell = z). \end{aligned}$$

Now notice that the excursion of H straddling time $\omega(t)$ is the first excursion of H that attains level t , and apply [96, Proposition XII.3.5] to obtain that

$$\begin{aligned} P(\mathcal{F}^-(t) \in ds | d_{\omega(t)} - g_{\omega(t)} = v) \\ = N^{(v)}(\zeta > \omega(t))^{-1} N^{(v)}(F_1^-(t) \in ds, v > \omega(t)), \end{aligned}$$

and similarly

$$\begin{aligned} P\left(L_{d_{\omega(t)}}^t \in d\ell, \int_t^\infty db L_{d_{\omega(t)}}^b \in dz \middle| d_{\omega(t)} - g_{\omega(t)} = v\right) \\ = N^{(v)}(\zeta > \omega(t))^{-1} N^{(v)}\left(L_v^t \in d\ell, \int_t^\infty db L_v^b \in dz, v > \omega(t)\right), \end{aligned}$$

for almost every v . Finally, notice that $\mathcal{F}^-(t) = F^-(t)$ under N and the $N^{(v)}$'s and that we may remove the indicator of $v > \omega(t)$ since a.s. under $N^{(v)}$, $L_v^t = 0$ if and only if $\max H \leq t$, to obtain

$$N^{(v)}(F^-(t) \in ds) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} N^{(v)}\left(L_v^t \in d\ell, \int_t^\infty db L_v^b \in dz\right) P(\Delta T_{[0, \ell]} \in ds | T_\ell = z).$$

Using scaling allows to take $v = 1$, entailing the claim. //

As a consequence of this result we may conjecture the shape of the dislocation measure of F^- . The next subsections will give essentially the rigorous proof of this conjecture, but finding ν_- directly from the forthcoming computations would certainly have been tricky without any former intuition. Roughly, suppose that the statement of Proposition 4.3 remains true for negative self-similarity indices (which is probably true, but we will not need it anyway). Then take G a bounded continuous function that is null on a neighborhood of $(1, 0, \dots)$ and write

$$N^{(1)}(G(F^-(t))) = \int_{\mathbb{R}_+ \times [0,1]} N^{(1)} \left(L_1^t \in dx, \int_t^\infty db L_1^b \in dz \right) E[G(\Delta T_{[0,x]}) | T_x = z].$$

Call $J(x, z)$ the expectation in the integral on the right hand side. Dividing by t and letting $t \downarrow 0$ should yield the generator of the \mathbb{R}_+^2 -valued process $((L_1^t, \int_t^\infty db L_1^b), t \geq 0)$, evaluated at the function J and at the starting point $(0, 1)$. Now, we interpret (see Sect. 4.5 for definitions) the process $(L_1^t, t \geq 0)$ under $N^{(1)}$ as the α -CSBP conditioned both to start at 0 and stay positive, and to have a total progeny equal to 1. It is thus heuristically a Doob h -transform of the initial CSBP with harmonic function $h(x) = x$, and conditioned to come back near 0 when its integral comes near 1. Now as a consequence of Lamperti's time-change between CSBP's and Lévy processes, the generator of the CSBP started at x is $x\mathcal{L}(x, dy)$ where \mathcal{L} is the generator of the stable Lévy process with index α :

$$\mathcal{L}f(x) = \int_0^\infty \frac{C_\alpha dy}{y^{\alpha+1}} (f(x+y) - f(x) - yf'(x)),$$

where f stands for a generic function in the Schwartz space. This, together with well-known properties for generators of h -transforms allows to conjecture that the generator \mathcal{L}' of the CSBP conditioned to stay positive and started at 0 is given by

$$\mathcal{L}'f(0) = \int_0^\infty \frac{C_\alpha dy}{y^\alpha} (f(y) - f(0)),$$

for a certain class of nice functions f , so roughly, the conditioned CSBP jumps at time $0+$ to level y at rate $C_\alpha y^{-\alpha} dy$. On the other hand, conditioning to come back to 0 when the progeny reaches 1 should introduce the extra term $q_y(1)$ (recall its definition (4.4)) in the integral with a certain coefficient, since the total progeny of a CSBP started at y is equal in law to T_y , as a consequence of Ray-Knight's theorem. To be a bit more accurate, the CSBP starting at y and conditioned to stay positive should be in $[0, \varepsilon]$ when its integral equals 1 with probability close to $g(\varepsilon)y^{-1}q_y(1)$ for some positive g with $g(\varepsilon) \rightarrow 0$ as $\varepsilon \downarrow 0$. Indeed, by the conditioned form of Lamperti's theorem of [73] to be recalled in Sect. 4.5, this is the same as the probability that the Lévy process started at y and conditioned to stay positive is in $[0, \varepsilon]$ at time 1. With the notations of Sect. 4.5, this is

$$P_y^\dagger(X_1 \leq \varepsilon) = \int_0^\varepsilon xy^{-1}P_y(X_1 \in dx, T_0 > 1).$$

We may expect that the quantity $P_y(X_1 \in dx, T_0 > 1)$ can be expressed as $r(y, x)dx$ with $r(y, x) \sim g'(x)q_y(1)$ as $x \downarrow 0$ for some g' vanishing at 0. Consequently, we expect that

under $N^{(1)}$, the process $(L_1^t, t \geq 0)$ jumps at time $0+$ to level $y > 0$ at rate $Cy^{-\alpha-1}q_y(1)dy$ for some $C > 0$. This, thanks to Lemma 4.3, allows to conjecture the form of the dislocation measure as

$$\nu_-(G) = C \int_0^\infty \frac{dy q_y(1)}{y^{\alpha+1}} E[G(\Delta T_{[0,y]}) | T_y = 1]$$

for some $C > 0$, that can be shown to be equal to αD_α with some extra care, but we do not need it at this point. It is then easy to reduce this to the form of Theorem 4.1: by using the scaling identities and changing variables $u = y^{-\alpha}$, we have

$$\begin{aligned} \int_0^\infty \frac{dy q_y(1)}{y^{\alpha+1}} E[G(\Delta T_{[0,y]}) | T_y = 1] &= \int_0^\infty \frac{dy q_1(y^{-\alpha})}{y^{2\alpha-1}} E[G(y^\alpha \Delta T_{[0,1]}) | y^\alpha T_1 = 1] \\ &= \int_0^\infty \alpha^{-1} du u q_1(u) E[G(u^{-1} \Delta T_{[0,1]}) | T_1 = u] \\ &= \alpha^{-1} E[T_1 G(T_1^{-1} \Delta T_{[0,1]})], \end{aligned}$$

as wanted.

This very rough program of proof could probably be “upgraded” to a real rigorous proof, but the technical difficulties on generators of processes would undoubtedly make it quite involved. We are going to use a path that uses more the structure of the stable tree.

4.3.2 Erosion and first properties of the dislocation measure

From this section on, F^- is exclusively defined under $N^{(1)}$, so that we may use the nicer notations $P(F^-(t) \in ds)$ or $E[G(F^-(t))]$ instead of $N^{(1)}(F^-(t) \in ds)$ or $N^{(1)}(G(F^-(t)))$ if there is no ambiguity.

Lemma 4.4 :

The erosion coefficient c of F^- is 0, and the dislocation measure $\nu_-(ds)$ charges only $\{s \in S : \sum_{i=1}^{+\infty} s_i = 1\}$.

// We will follow the analysis of Bertoin [26], using the law of the time at which a tagged fragment vanishes. Let U be uniform on $(0, 1)$ and independent of the height process of the stable tree. Recall the definition of $F^-(t)$ out of the open set $I_-(t)$ and let $\lambda(t) = |I^*(t)|$ be the size of the interval $I_-^*(t)$ of $I_-(t)$ that contains U . As in Sect. 4.2.3, if we define

$$a(t) = \inf \left\{ u \geq 0 : \int_0^u \lambda(v)^{1/\alpha-1} dv > t \right\}, \quad t \geq 0,$$

then $(-\log(\lambda(a(t))), t \geq 0)$ is a subordinator with Laplace exponent

$$\Phi(r) = -\log E[\lambda(a(t))^r] = c(r+1) + \int_S \left(1 - \sum_{n=1}^{+\infty} s_n^{r+1} \right) \nu_-(ds). \quad (4.8)$$

Moreover, if $\xi = H_U$ is the lifetime of the tagged fragment, then

$$E[\xi^k] = \frac{k!}{\prod_{i=1}^k \Phi\left(i\left(1 - \frac{1}{\alpha}\right)\right)}. \quad (4.9)$$

For the computation, recall that the density $(\chi_x(s), s \geq 0)$ introduced in Proposition 4.2 is characterized by its Laplace transform

$$\int_0^{+\infty} e^{-\mu s} \chi_x(s) ds = \exp(-x\mu^{1-1/\alpha}). \tag{4.10}$$

We may now compute the moments of ξ . By Proposition 4.2,

$$E[\xi^k] = \int_0^{+\infty} h^k \alpha \Gamma\left(1 - \frac{1}{\alpha}\right) \chi_{\alpha h}(1) dh = \frac{\Gamma\left(1 - \frac{1}{\alpha}\right)}{\alpha^k} \int_0^{+\infty} x^k \chi_x(1) dx.$$

To compute this we use (4.10) and Fubini's theorem to get

$$\int_0^{+\infty} ds e^{-\mu s} \int_0^{+\infty} dx \chi_x(s) x^k = \int_0^{+\infty} x^k \exp(-x\mu^{1-1/\alpha}) dx = \frac{k!}{\mu^{(k+1)(1-1/\alpha)}},$$

and then the last term above is equal to

$$\frac{k!}{\Gamma\left(\left(k+1\right)\left(1 - \frac{1}{\alpha}\right)\right)} \int_0^{+\infty} du e^{-\mu u} u^{(k+1)(1-1/\alpha)-1}.$$

Inverting Laplace transforms and taking $u = 1$ thus give

$$\int_0^{+\infty} x^k \chi_x(1) dx = \frac{k!}{\Gamma\left(\left(k+1\right)\left(1 - \frac{1}{\alpha}\right)\right)},$$

hence we finally get

$$E[\xi^k] = \frac{k! \Gamma\left(1 - \frac{1}{\alpha}\right)}{\alpha^k \Gamma\left(\left(k+1\right)\left(1 - \frac{1}{\alpha}\right)\right)}.$$

Using (4.9) we now obtain that

$$\Phi\left(k\left(1 - \frac{1}{\alpha}\right)\right) = \alpha \frac{\Gamma\left(\left(k+1\right)\left(1 - \frac{1}{\alpha}\right)\right)}{\Gamma\left(k\left(1 - \frac{1}{\alpha}\right)\right)}, \quad k = 1, 2, \dots$$

Thus, for r of the form $k(1 - 1/\alpha)$,

$$\Phi(r) = \alpha \frac{\Gamma\left(r + 1 - \frac{1}{\alpha}\right)}{\Gamma(r)} = \frac{r}{\Gamma\left(1 + \frac{1}{\alpha}\right)} B\left(r + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right). \tag{4.11}$$

It is not difficult, using the integral representation of the function B , then changing variables and integrating by parts, to write this in Lévy-Khintchine form, that is, for every $r \geq 0$,

$$\frac{r}{\Gamma\left(1 + \frac{1}{\alpha}\right)} B\left(r + 1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right) = \int_0^\infty dx \frac{\left(1 - \frac{1}{\alpha}\right) e^x}{\Gamma\left(1 + \frac{1}{\alpha}\right) (e^x - 1)^{2-1/\alpha}} (1 - e^{-xr}), \tag{4.12}$$

and it follows that (4.11) remains true for every $r \geq 0$, because $\lambda(a(t))^{1-1/\alpha}$ is characterized by its moments, hence generalizing Equation (12) in [26] in the Brownian case. It also gives the formula

$$L(dx) = \frac{\left(1 - \frac{1}{\alpha}\right) e^x dx}{\Gamma\left(1 + \frac{1}{\alpha}\right) (e^x - 1)^{2-1/\alpha}}$$

for the Lévy measure $L(dx)$ of Φ , hence generalizing Equation (11) in [26].

To conclude, we just notice that $\Phi(0) = 0$, which by (4.8) gives both $c = 0$ and $\int_S \nu_-(ds) (1 - \sum_{i=1}^\infty s_i) = 0$, implying the result. //

4.3.3 Dislocation measure

The dislocation measure of F^- will now be obtained by explicitly computing the law of the first fragmentation of the fragments marked by n independent uniform variables U_1, \dots, U_n on $(0, 1)$, as explained in Sect. 4.2.3. This is going to be a purely combinatorial computation based on the first formula of Proposition 4.2. What we want to compute is the law of the partition of $[n]$ induced by the partition $I_-(t_n)$ and the variables U_1, \dots, U_n at the time t_n when they are first separated. We want to evaluate the probability $\rho_-(n)(\{\pi_n\})$ that the partition induced by $I_-(t_n)$ and the variables (U_1, \dots, U_n) equals some non-trivial partition π_n of $[n]$ with blocks A_1, \dots, A_k having sizes n_1, \dots, n_k with sum n ($n, k \geq 2$). In terms of the stable tree described in Sect. 4.2.2, this is simply the probability that the skeleton of the marked tree ϑ_n is such that the root has out-degree k , and the k trees that are rooted at the children of the root have n_1, n_2, \dots, n_k leaves, times $n_1! \dots n_k! / n!$, which is the probability that labeling by i the leaf associated to the variable U_i , for $1 \leq i \leq n$, induces the partition π_n (where i and j are in the same block if the leaves labeled i, j share the same child of the root as a common ancestor). Let $\mathbf{T}_{n_1, \dots, n_k}^*$ be the set of trees of \mathbf{T}_n^* that have this last property. For $x \geq 0$ and $n \geq 0$ we denote by $[x]_n$ the quantity $\prod_{i=0}^{n-1} (x+i) = \Gamma(x+n)/\Gamma(x)$.

Lemma 4.5 :

Let π_n be a partition of $[n]$ with $k \geq 2$ blocks having sizes n_1, n_2, \dots, n_k . Then

$$\rho_-(n)(\{\pi_n\}) = \frac{D_\alpha \Gamma(k - \alpha)}{\alpha^k \Gamma(n - \frac{1}{\alpha})} \prod_{i=1}^k \left[1 - \frac{1}{\alpha} \right]_{n_i - 1}.$$

// Recall that we want to compute the probability that the skeleton of the marked tree ϑ_n has a root with k children, and the fringe subtrees spanned by these children are trees of $\mathbf{T}_{n_i}^*$ for $1 \leq i \leq k$. The fact that the first displayed quantity in Proposition 4.2 defines a probability on \mathbf{T}_n^* implies

$$\begin{aligned} \sum_{\mathcal{T} \in \mathbf{T}_n^*} \prod_{v \in \mathcal{N}_{\mathcal{T}}} \frac{|(\alpha - 1)(\alpha - 2) \dots (\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!} &= \frac{(\alpha - 1)(2\alpha - 1) \dots ((n - 1)\alpha - 1)}{n!} \\ &= \frac{\alpha^{n-1}}{n!} \left[1 - \frac{1}{\alpha} \right]_{n-1}. \end{aligned}$$

Now we compute, using Proposition 4.2,

$$\begin{aligned} \rho_-(n)(\{\pi_n\}) &= \sum_{\mathcal{T} \in \mathbf{T}_{n_1, \dots, n_k}^*} \frac{n! n_1! \dots n_k!}{\alpha^{n-1} [1 - \frac{1}{\alpha}]_{n-1} n!} \prod_{v \in \mathcal{N}_{\mathcal{T}}} \frac{|(\alpha - 1)(\alpha - 2) \dots (\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!} \\ &= \frac{n_1! \dots n_k! |(\alpha - 1)(\alpha - 2) \dots (\alpha - k + 1)|}{\alpha^{n-1} k! [1 - \frac{1}{\alpha}]_{n-1}} \\ &\times \sum_{\mathcal{T} \in \mathbf{T}_{n_1, \dots, n_k}^*} \prod_{v \in \mathcal{N}_{\mathcal{T}} \setminus \{\text{root}\}} \frac{|(\alpha - 1)(\alpha - 2) \dots (\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!} \end{aligned}$$

By definition of $\mathbf{T}_{n_1, \dots, n_k}^*$, this is

$$\begin{aligned} \rho_-^n(\{\pi_n\}) &= \frac{(\alpha - 1)\Gamma(k - \alpha)\Gamma\left(1 - \frac{1}{\alpha}\right)}{k!\alpha^{n-1}\Gamma(2 - \alpha)\Gamma\left(n - \frac{1}{\alpha}\right)} \\ &\times k!n_1! \dots n_k! \prod_{i=1}^k \sum_{T \in \mathbf{T}_{n_i}^*} \prod_{v \in \mathcal{N}_T} \frac{|(\alpha - 1)(\alpha - 2) \dots (\alpha - c_v(T) + 1)|}{c_v(T)!}, \end{aligned}$$

where the factor $k!$ appears because the k fringe subtrees spanned by the sons of the root may appear in any order. By the first formula of the proof this now reduces to

$$\rho_-(n)(\{\pi_n\}) = \frac{D_\alpha \Gamma(k - \alpha) \prod_{i=1}^k n_i!}{\alpha^n \Gamma\left(n - \frac{1}{\alpha}\right)} \prod_{i=1}^k \frac{\alpha^{n_i-1}}{n_i!} \left[1 - \frac{1}{\alpha}\right]_{n_i-1},$$

giving the result. //

Comparing with Lemma 4.2 implies, since $c = 0$, that the dislocation measure ν_- of F^- is thus determined up to a multiplicative constant. Since we have a conjectured form $D_\alpha \nu_\alpha$ for the dislocation measure ν_- of F^- , we just have to compute the quantity $\kappa_-(\pi)$ for κ_- the exchangeable measure on \mathcal{P} with frequencies given by the conjectured ν_- . Precisely, we have

Lemma 4.6 :

Let π_n be a partition of $[n]$ with $k \geq 2$ blocks and block sizes n_1, \dots, n_k . Then

$$\kappa_-^n(\{\pi_n\}) := \kappa_-(\{\pi \in \mathcal{P} : \pi|_{[n]} = \pi_n\}) = \frac{D_\alpha \Gamma(k - \alpha)}{\alpha^{k-1} \Gamma(n - 1)} \prod_{i=1}^k \left[1 - \frac{1}{\alpha}\right]_{n_i-1}$$

// To prove this we first state from (74) in section 6 of [92] (notice that the α there is our $1/\alpha$):

Proposition 4.5 :

Let $\theta > -1/\alpha$ and recall (4.2) the definition of the Poisson-Dirichlet $\text{PD}(1/\alpha, \theta)$ distribution. Let π_n be a partition of $[n]$ with non-void block sizes n_1, \dots, n_k . Then the probability that the restriction to $[n]$ of the exchangeable partition of \mathcal{P} with frequencies having law $\text{PD}(1/\alpha, \theta)(ds)$ is π_n is given by

$$p_\theta(n_1, \dots, n_k) = \frac{[\alpha\theta + 1]_{k-1}}{\alpha^{k-1}[\theta + 1]_{n-1}} \prod_{i=1}^k \left[1 - \frac{1}{\alpha}\right]_{n_i-1}$$

The computation of the κ_-^n associated with the conjectured dislocation measure ν_- can go through the same lines as the proof of this proposition given in [92], using the explicit densities for size-biased picks among the jumps of the subordinator T . However, we use the following more direct proof. For $\theta \geq -1$ write

$$\nu_\theta = D_\alpha E \left[T_1^{-\theta}; \frac{\Delta T_{[0,1]}}{T_1} \in ds \right],$$

so $\nu_\theta = D_\alpha(\Gamma(\alpha\theta + 1)/\Gamma(\theta + 1))\text{PD}(1/\alpha, \theta)$ for $\theta > -1/\alpha$. Recall from the above the notation $\kappa_{\mathbf{s}}(d\pi)$ for the law of the exchangeable partition of \mathbb{N} with ranked asymptotic frequencies given by \mathbf{s} . Define

$$\kappa_\theta(d\pi) = \int_S \nu_\theta(d\mathbf{s})\kappa_{\mathbf{s}}(d\pi) = D_\alpha E \left[T_1^{-\theta} \kappa_{\Delta T_{[0,1]}/T_1}(d\pi) \right], \quad (4.13)$$

and for π_n a partition of $[n]$ with block sizes n_1, \dots, n_k write $\kappa_\theta^n(\{\pi_n\}) = \kappa_\theta(\{\pi \in \mathcal{P} : \pi|_{[n]} = \pi_n\})$. Notice that when $n, k \geq 2$ and $\mathbf{s} \in S$, we have $\kappa_{\mathbf{s}}(\{\pi \in \mathcal{P} : \pi|_{[n]} = \pi_n\}) \leq n(1 - s_1)$ (this is easy by Kingman’s exchangeable partitions representation theorem, see e.g. [23, p. 310]). Moreover, the fact that ν_- integrates $\mathbf{s} \mapsto 1 - s_1$ is easily generalized to ν_θ for $\theta > -1$. We deduce that the map $\theta \mapsto \kappa_\theta^n(\{\pi_n\})$ is analytic on $\{\theta \in \mathbb{C} : \text{Re}(\theta) > -1\}$. The same holds for

$$D_\alpha \frac{\Gamma(\alpha\theta + 1)}{\Gamma(\theta + 1)} p_\theta(n_1, \dots, n_k) = \frac{D_\alpha \Gamma(\alpha\theta + k)}{\alpha^{k-1} \Gamma(\theta + n)} \prod_{i=1}^k \left[1 - \frac{1}{\alpha} \right]_{n_i-1} \quad (4.14)$$

provided $k \geq 2$, and by Proposition 4.5 they are equal on $\theta \in (-1/\alpha, \infty)$. Thus they are equal on $\{\theta \in \mathbb{C} : \text{Re}(\theta) > -1\}$, so the limits as $\theta \in \mathbb{R} \downarrow -1$ of $\kappa_\theta^n(\{\pi_n\})$ and of (4.14) coincide. Using (4.13) and a dominated convergence argument we have $\kappa_\theta^n(\{\pi_n\}) \rightarrow \kappa_-^n(\{\pi_n\})$ as $\theta \downarrow -1$, so

$$\kappa_-^n(\{\pi_n\}) = \frac{D_\alpha \Gamma(k - \alpha)}{\alpha^{k-1} \Gamma(n - 1)} \prod_{i=1}^k \left[1 - \frac{1}{\alpha} \right]_{n_i-1},$$

as wanted. //

Remark. By analogy with the EPPF (exchangeable partition probability function) that allows to characterize the law of exchangeable partitions, expressions such as in Lemma 4.6 could be called “exchangeable partition distribution functions”, as they characterize σ -finite exchangeable measures on the set of partitions of \mathbb{N} . The expression in Lemma 4.6 should be interpreted as an EPDF for a generalized $(1/\alpha, \theta)$ partition (see [91]), for $\theta = -1$. One certainly could imagine more general exchangeable partitions as θ goes further in the negative axis: this would impose more and more stringent constraints on the number of blocks of the partitions.

Therefore, we obtain that

$$\kappa_-^n = \alpha(\Gamma(n - 1/\alpha)/\Gamma(n - 1))\rho_-(n)$$

on the set of non-trivial partitions of $[n]$. Lemma 4.2 implies that the dislocation measure of F^- is equal to the conjectured ν_- up to a multiplicative constant. We are going to recover the missing information with the help of the computation of Φ above.

4.3.4 The missing constant

In this section, we compute the Laplace exponent Φ of the subordinator $-\log(\lambda(a(\cdot)))$ of Sect. 4.3.2, whose value is indicated in (4.11), directly from formulas (4.8) and (4.1). Let

$$\Phi_0(r) = \int_S \left(1 - \sum_{n=1}^{\infty} s_n^{r+1} \right) \nu_-(d\mathbf{s}),$$

where ν_- is the measure given in Theorem 4.1. If we can prove that $\Phi_0(r) = \Phi(r)$ for every $r \geq 0$, we will therefore have established that the normalization of ν_- is the appropriate one. By (4.1),

$$\begin{aligned}\Phi_0(r) &= D_\alpha E \left[T_1 \left(1 - \sum_{0 \leq x \leq 1} \left(\frac{\Delta T_x}{T_1} \right)^{r+1} \right) \right] \\ &= D_\alpha \int_0^\infty du u q_1(u) E \left[1 - \sum_{0 \leq x \leq 1} \left(\frac{\Delta T_x}{u} \right)^{r+1} \middle| T_1 = u \right] \\ &= D_\alpha \int_0^\infty du u q_1(u) E \left[1 - \left(\frac{\Delta_1^*}{u} \right)^r \right]\end{aligned}$$

where Δ_1^* is a size-biased pick from the jumps of T_x , for $0 \leq x \leq 1$, conditionally on $T_1 = u$. Using formula (4.5) and recalling that T has Lévy measure $c_\alpha x^{-1-1/\alpha} dx$, we can write

$$\begin{aligned}\Phi_0(r) &= D_\alpha \int_0^\infty du u q_1(u) \int_0^u dx (1 - (x/u)^r) \frac{c_\alpha q_1(u-x)}{u x^{1/\alpha} q_1(u)} \\ &= D_\alpha \int_0^\infty du \int_0^1 dy c_\alpha u^{1-1/\alpha} q_1(u(1-y)) \frac{1-y^r}{y^{1/\alpha}} \\ &= D_\alpha \int_0^1 dy \frac{c_\alpha (1-y^r)}{y^{1/\alpha} (1-y)^{2-1/\alpha}} \int_0^\infty du u^{1-1/\alpha} q_1(u)\end{aligned}$$

as obtained by Fubini's theorem, and linear changes of variables. The integral in du equals $\mathbb{E}[T_1^{1-1/\alpha}]$, which is $\Gamma(2-\alpha)/\Gamma(1/\alpha)$ (see e.g. (43) in [91]). Using the expressions for D_α , c_α and the identity $\alpha^{-1}\Gamma(1/\alpha) = \Gamma(1+1/\alpha)$, it remains to compute the quantity

$$\frac{1 - \frac{1}{\alpha}}{\Gamma(1 + \frac{1}{\alpha})} \int_0^1 dy \frac{y^{-1/\alpha} (1-y^r)}{(1-y)^{2-1/\alpha}}.$$

But this is exactly the expression (4.12) after changing variables $y = e^{-x}$, and it is thus equal to $rB(r+1-1/\alpha, 1/\alpha)/\Gamma(1+1/\alpha)$, which is (4.11) as wanted, thus completing the proof of Theorem 4.1.

4.4 Small-time asymptotics

In this section we study the asymptotic behavior of F^- for small times. Precisely, let $M(t) = \sum_{i \geq 1} F_i^-(t)$ denote the total mass of F^- at time t . Let $(Y_x, x \geq 0)$ denote an α -CSBP, started at 0 and conditioned to stay positive. See the following section for definitions. We have the following result, that generalizes and mimics somehow results from [10, 17, 82]. However, these results dealt with self-similar fragmentations with positive indices, and also, the occurrence of the randomization introduced by Y_1 below is

somehow unusual.

Proposition 4.6 :

The following convergence in law holds:

$$t^{\alpha/(1-\alpha)}(M(t) - F_1^-(t), F_2^-(t), F_3^-(t), \dots) \xrightarrow[t \downarrow 0]{d} (T_{Y_1}, \Delta_1, \Delta_2, \dots)$$

where T is the stable $1/\alpha$ subordinator as above, independent of Y , and $\Delta_1, \Delta_2, \dots$ are the jumps of $(T_x, 0 \leq x \leq Y_1)$ ranked in decreasing order of magnitude.

// For this we are going to use the following lemma, which resembles the result of Jeulin in [62] relating a scaled normalized Brownian excursion and a 3-dimensional Bessel process. The proof is postponed to the following section. Recall that $(L_1^t, t \geq 0)$ stands for the local time of the height process up to time 1.

Lemma 4.7 :

The following convergence in law holds:

$$\text{Under } N^{(1)}, \quad (t^{1/(1-\alpha)} L_1^{tx}, x \geq 0) \xrightarrow[t \downarrow 0]{d} (Y_x, x \geq 0),$$

and this last limit is independent of the initial process $(L_1^t, t \geq 0)$. In particular, $t^{1/(1-\alpha)} L_1^t$ converges in distribution to Y_1 as $t \downarrow 0$.

In the sequel let (y_t, \bar{y}_t) have the law of $(L_1^t, \int_t^\infty db L_1^b)$ under $N^{(1)}$. Following the method of Aldous and Pitman [10], we are actually going to prove that for every k ,

$$t^{\alpha/(1-\alpha)}(M(t) - F_1^*(t), F_2^*(t), F_3^*(t), \dots, F_k^*(t)) \xrightarrow[t \downarrow 0]{d} (T_{Y_1}, \Delta_1^*, \Delta_2^*, \dots, \Delta_{k-1}^*), \quad (4.15)$$

for every $k \geq 1$, where the quantities with the stars are the size-biased quantities associated with the ones of the statement, and this is sufficient. We are going to proceed by induction on k . To start the induction, let g be a continuous function with compact support and write, using Lemma 4.1, Proposition 4.4, then changing variables and using scaling identities,

$$\begin{aligned} & E[g(t^{\alpha/(1-\alpha)}(M(t) - F_1^*(t)))] \\ &= E \left(\int_0^{\bar{y}_t} du \frac{c_\alpha y_t q_{y_t}(\bar{y}_t - u)}{\bar{y}_t u^{1/\alpha} q_{y_t}(\bar{y}_t)} g(t^{\alpha/(1-\alpha)}(\bar{y}_t - u)) \right) \\ &= E \left(\int_0^{t^{\alpha/(1-\alpha)} \bar{y}_t} dv \frac{t^{\alpha/(\alpha-1)} c_\alpha y_t q_1 \left(\frac{v}{t^{\alpha/(1-\alpha)} y_t^\alpha} \right)}{(\bar{y}_t - t^{\alpha/(\alpha-1)} v)^{1/\alpha} \bar{y}_t q_1 \left(\frac{\bar{y}_t}{y_t^\alpha} \right)} g(v) \right). \end{aligned} \quad (4.16)$$

By making use of Skorokhod's representation theorem, we may suppose that the convergence of $(t^{1/(1-\alpha)} y_t, t^{\alpha/(1-\alpha)} \bar{y}_t)$ to (Y_1, ∞) is almost-sure. Now the integral inside the expectation is the integral according to a probability law, hence it is dominated by the supremum of $|g|$, so it suffices to show that the integral converges a.s. to apply dominated convergence. For almost every ω ,

there exists ε such that if $t < \varepsilon$, $t^{\alpha/(1-\alpha)}\bar{y}_t(\omega) > K$ where K is the right-end of the support of g . For such an ω and t , the integral is thus

$$\begin{aligned} & \int_0^K dv g(v) \frac{c_\alpha t^{\alpha/(\alpha-1)} y_t q_1(v(t^{1/(1-\alpha)} y_t)^{-\alpha})}{\bar{y}_t^{1+1/\alpha} (1 - t^{\alpha/(\alpha-1)} v / \bar{y}_t)^{1/\alpha} q_1(\bar{y}_t y_t^{-\alpha})} \\ & \leq M \frac{t^{\alpha/(\alpha-1)} y_t}{\bar{y}_t^{1+1/\alpha} q_1(\bar{y}_t y_t^{-\alpha})} \int_0^K dv q_1\left(\frac{v}{t^{\alpha/(1-\alpha)} y_t^\alpha}\right) \end{aligned}$$

for some constant M not depending on t . Now we use the fact from [101] that q_1 is bounded and

$$q_1(x) \underset{x \rightarrow \infty}{=} c_\alpha x^{-1-1/\alpha} + O(x^{-1-2/\alpha}).$$

This allows to conclude by dominated convergence that the integral in (4.16) a.s. goes to

$$\int_0^K dv g(v) \frac{q_1(v/Y_1^\alpha)}{Y_1^\alpha} = \int_0^K dv g(v) q_{Y_1}(v),$$

and by dominated convergence its expectation converges to the expectation of the above limit, that is $E[g(T_{Y_1})]$.

To implement the recursive argument, suppose that (4.15) holds for some $k \geq 1$. Let g and h be continuous bounded functions on \mathbb{R}_+ and \mathbb{R}_+^k respectively. Write $(y_t, \bar{y}_t, \Delta_1(t), \Delta_2(t), \dots)$ for a sequence with the same law as $(L_1^t, \int_t^\infty ds L_1^s, \Delta T'_{[0, L_1^t]})$ given $T'_{L_1^t} = \int_t^\infty ds L_1^s$, where L_1 is taken under $N^{(1)}$ and T' is a stable $1/\alpha$ subordinator, taken independent of L . Last, let $\Delta_1^*(t), \Delta_2^*(t), \dots$ be the size-biased permutation associated with $\Delta_1(t), \Delta_2(t), \dots$. By Proposition 4.4, conditioning, and using Lemma 4.1, we have

$$\begin{aligned} & E[g(t^{\alpha/(1-\alpha)} F_{k+1}^*(t)) h(t^{\alpha/(1-\alpha)} (M(t) - F_1^*(t), F_2^*(t), \dots, F_k^*(t)))] \\ & = E \left[h(t^{\alpha/(1-\alpha)} (\bar{y}_t - \Delta_1^*(t), \Delta_2^*(t), \dots, \Delta_k^*(t))) \int_0^{\bar{y}_t - \sum_{i=1}^k \Delta_i^*(t)} du g(t^{\alpha/(1-\alpha)} u) \right. \\ & \quad \left. \times \frac{c_\alpha y_t q_{y_t} (\bar{y}_t - \sum_{i=1}^k \Delta_i^*(t) - u)}{u^{1/\alpha} (\bar{y}_t - \sum_{i=1}^k \Delta_i^*(t)) q_{y_t} (\bar{y}_t - \sum_{i=1}^k \Delta_i^*(t))} \right] \end{aligned}$$

Similarly as above, we show by changing variables and then using the scaling identities and the asymptotic behavior of q_1 that this converges to

$$E \left[h(T_{Y_1}, \Delta_1^*, \dots, \Delta_{k-1}^*) \int_0^{T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^*} dv g(v) \frac{c_\alpha Y_1 q_{Y_1} (T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^* - v)}{v^{1/\alpha} (T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^*) q_{Y_1} (T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^*)} \right]$$

and by Lemma 4.1 this is $E[h(T_{Y_1}, \Delta_1^*, \dots, \Delta_{k-1}^*) g(\Delta_k^*)]$. This finishes the proof. //

The same method also allows to show that the rescaled remaining mass $t^{\alpha/(1-\alpha)}(1 - M(t))$ converges in distribution to $\int_0^1 Y_v dv$ jointly with the vector of the proposition.

4.5 On continuous-state branching processes...

In this section we develop the material needed to prove Lemma 4.7. In the course, we will give an analog of Jeulin's theorem [63] linking the local time process of a Brownian excursion to another time-changed Brownian excursion. To stay in the line of the present paper, we will suppose that the laws we consider are associated to stable processes, but all of the results (except the proof of Lemma 4.7 which strongly uses scaling) can be extended to more general Lévy processes and their associated CSBP's. To avoid confusions, we will denote by $(Z_t, t \geq 0)$ the different CSBP's we will consider, or to be more precise, we let $(Z_t, t \geq 0)$ instead of $(X_s, s \geq 0)$ be the canonical process on $\mathbb{D}([0, \infty))$ when dealing with the laws $\mathbb{P}_x, \mathbb{P}_x^\dagger, \dots$ associated to CSBP's.

Definition :

For any $x > 0$, let \mathbb{P}_x be the unique law on $\mathbb{D}([0, \infty))$ that makes the canonical process $(Z_t, t \geq 0)$ a right-continuous Markov process starting at x with transition probabilities characterized by

$$\mathbb{E}[\exp(-\lambda Z_{t+r}) | Z_t = y] = \exp(-y u_r(\lambda)),$$

where $u_r(\lambda) = (\lambda^{1-\alpha} + (\alpha - 1)r)^{1/(1-\alpha)}$ is determined by the equation

$$\int_{u_r(\lambda)}^\lambda \frac{dv}{v^\alpha} = r.$$

Then \mathbb{P}_x is called the law of of the α -CSBP started at x .

Remark. For more general branching mechanisms, the definition of $u_r(\lambda)$ is modified by replacing v^α by $\psi(v)$, where ψ is the Laplace exponent of a spectrally positive Lévy process with infinite variation that oscillates or drifts to $-\infty$.

Recall the setting of Sect. 4.2.1, and let P_x be law under which X is the spectrally positive stable process with Laplace exponent λ^α and started at $x > 0$, that is, the law of $x + X$ under P . Let E_x be the corresponding expectation. Define the time-change $(\tau_t, t \geq 0)$ by

$$\tau_t = \inf \left\{ u \geq 0 : \int_0^u \frac{dv}{X_{v \wedge h_0}} > t \right\},$$

where $h_0 = \inf\{s > 0 : X_s = 0\}$ is the first hitting time of 0. This definition makes sense either under the law P_x , for $x > 0$, or the σ -finite excursion measure N (we will see below that under N , τ is not the trivial process identical to 0).

Theorem 4.2 :

We have the following identities in law: for every $x > 0$,

$$(L_{T_x}^t, t \geq 0) \text{ under } P \stackrel{d}{=} (X_{\tau_t}, t \geq 0) \text{ under } P_x,$$

and both have law \mathbb{P}_x . Moreover,

$$(L_\zeta^t, t \geq 0) \text{ under } N \stackrel{d}{=} (X_{\tau_t}, t \geq 0) \text{ under } N.$$

The first part is already known and is a conjunction of Lamperti's theorem (stating that $(X_{\tau_t}, t \geq 0)$ under P_x has law \mathbb{P}_x) and the Ray-Knight theorem mentioned in Sect.

4.2.2. We will use it to prove the second part. First we introduce some notations, which were already used in a heuristic way above.

For $x > 0$ one can define the law P_x^\uparrow of the stable process started at x and conditioned to stay positive by means of Doob's theory of harmonic h -transforms. It is characterized by the property

$$E_x^\uparrow[F(X_s, 0 \leq s \leq K)] = E_x \left[\frac{X_K}{x} F(X_s, 0 \leq s \leq K), K < T_0 \right]$$

for any positive measurable functional F . Here T_0 denotes as above the first hitting time of 0 by X . It can be shown (see e.g. [43]) that P_x^\uparrow has a weak limit as $x \rightarrow 0$, which we call P^\uparrow , the law of the stable process conditioned to stay positive.

Similarly, we define the CSBP conditioned to stay positive according to [73], by letting \mathbb{P}_x be the law of the CSBP started at $x > 0$, then setting

$$\mathbb{E}_x^\uparrow[F(Z_t, 0 \leq t \leq K)] = \mathbb{E}_x \left[\frac{Z_K}{x} F(Z_s, 0 \leq s \leq K) \right].$$

We want to show that a $x \downarrow 0$ limit also exists in this case. This is made possible by the interpretation of [73] of the law \mathbb{P}_x^\uparrow in terms of a CSBP with *immigration*. To be concise, we have

Lemma 4.8 :

For $x > 0$, the law \mathbb{P}_x^\uparrow is the law of the α -CSBP with immigration function $\alpha\lambda^{\alpha-1}$ and started at x . That is, under \mathbb{P}_x^\uparrow , $(Z_t, t \geq 0)$ is a Markov process starting at x and with transition probabilities

$$\mathbb{E}_x^\uparrow[\exp(-\lambda Z_{t+r}) | Z_t = y] = \exp \left(-y u_r(\lambda) - \int_0^r \alpha u_v(\lambda)^{\alpha-1} dv \right).$$

As a consequence, the laws \mathbb{P}_x^\uparrow converge weakly as $x \downarrow 0$ to a law $\mathbb{P}_0^\uparrow = \mathbb{P}^\uparrow$, which is the law of a Markov process with same transition probabilities and whose entrance law is given by the above formula, taking $t = y = x = 0$. It is also easy that the law \mathbb{P}^\uparrow is that of a Feller process according to the definition for $u_r(\lambda)$.

It is shown in [73] that Lamperti's correspondence is still valid between conditioned processes started at $x > 0$: the process $(X_{\tau_t}, t \geq 0)$ under the law P_x^\uparrow has law \mathbb{P}_x^\uparrow . To be more accurate, the exact statement is that if the process $(Z_t, t \geq 0)$ has law \mathbb{P}_x^\uparrow , then the process $(Z_{C_s}, s \geq 0)$ has law P_x^\uparrow where

$$C_s = \inf \left\{ u \geq 0 : \int_0^u dv Z_v > s \right\},$$

but this is the second part of Lamperti's transformation, which is easily inverted (see also the comment at the end of the section). We generalize this to

Lemma 4.9 :

The process $(X_{\tau_t}, t \geq 0)$ under the law P^\uparrow has law \mathbb{P}^\uparrow . Part of this lemma is that $\tau_t > 0$ for every t .

// For fixed $\eta > 0$, let

$$\tau_t^\eta = \inf \left\{ u : \int_\eta^{u \vee \eta} \frac{dv}{X_v} > t \right\}.$$

This is well defined under P^\uparrow since $X_t > 0$ for all $t > 0$ a.s. under this law. Then since $\int_\eta^{u \vee \eta} dv / X_v = \int_0^{u-\eta} dv / X_{\eta+v}$ whenever $u \geq \eta$ and is null else, we have that

$$\tau_t^\eta = \eta + \inf \left\{ u \geq 0 : \int_0^u \frac{dv}{X_{\eta+v}} > t \right\}.$$

That is, $\tau^\eta - \eta$ equals the time-change τ defined above, but associated to the process $(X_{\eta+t}, t \geq 0)$ (notice that h_0 plays no role here since we are dealing with processes that are strictly positive on $(0, \infty)$). Under P^\uparrow , this process is independent of $(X_s, 0 \leq s \leq \eta)$ conditionally on X_η and has law $P_{X_\eta}^\uparrow$. Hence, by Lamperti's identity, conditionally on $(X_s, 0 \leq s \leq \eta)$ under P^\uparrow , the process $(X_{\tau_t^\eta}, t \geq 0)$ has law $\mathbb{P}_{X_\eta}^\uparrow$. Hence, for any continuous bounded functional G on the paths defined on $[0, K]$ for some $K > 0$,

$$E^\uparrow[G(X_{\tau_t^\eta}, 0 \leq t \leq K)] = E^\uparrow[\mathbb{E}_{X_\eta}^\uparrow[G(Z_t, 0 \leq t \leq K)]].$$

Now, it is not difficult to see that τ^η decreases to the limit τ uniformly on compact sets. Thus, using the right-continuity of X on the one hand, and the Feller property on the other (in fact, less than the Feller property is needed here), we obtain by letting $\eta \downarrow 0$ in the above identity

$$E^\uparrow[G(X_{\tau_t}, 0 \leq t \leq K)] = \mathbb{E}^\uparrow[G(Z_t, 0 \leq t \leq K)],$$

which is the desired identity. In particular, τ cannot be identically 0. //

Remark. Notice that the fact that the time-change τ_t is still well-defined under the law P^\uparrow can be double-checked by a law of the iterated logarithm for the law P^\uparrow . See also the end of the section.

Motivated by the definition in Pitman-Yor [93] for the excursion measure away from 0 of continuous diffusions for which 0 is an exit point (and initially by Itô's description of the Brownian excursion measure linking the three-dimensional Bessel process semigroup to the entrance law of Brownian excursions), we now state the following

Proposition 4.7 :

The process $(L_\zeta^t, t \geq 0)$ under the measure N is governed by the excursion measure of the CSBP with characteristic λ^α . That is, its entrance law $N(L_\zeta^t \in dy)$ for $t > 0$ is equal to $y^{-1} \mathbb{P}^\uparrow(Z_t \in dy)$ for $y > 0$ (and it puts mass ∞ on $\{0\}$), and given $(L_\zeta^u, 0 \leq u \leq t)$, the process $(L_\zeta^{t+t'}, t' \geq 0)$ has law $\mathbb{P}_{L_\zeta^t}$.

The use of the height process and its local time under N , and hence of an "excursion measure" associated to the genealogy of CSBP's, snakes and superprocesses, is a very natural tool, however it does not seem that the above proposition, which states that this notion of "excursion measure" is the most natural one, has been checked somewhere. However, as noticed in [93], since the point 0 is not an entrance point for the initial CSBP, one cannot define a reentering diffusion by sticking the atoms of a Poisson measure with intensity given by this excursion measure, because the durations are almost never summable.

// The law $\mathbb{P}^\uparrow(Z_t \in dy)$ is the weak limit of $\mathbb{P}_x^\uparrow(Z_t \in dy) = x^{-1} y \mathbb{P}_x(Z_t \in dy)$ as $x \rightarrow 0$. Since by the properties of the CSBP mentioned in Sect. 4.2.2,

we have $\mathbb{E}_x[\exp(-\lambda Z_t)] = \exp(-xu_t(\lambda))$, we obtain

$$\int_0^\infty \frac{\mathbb{P}_x^\dagger(Z_t \in dy)}{y} (1 - e^{-\lambda y}) = \int_0^\infty \frac{\mathbb{P}_x(Z_t \in dy)}{x} (1 - e^{-\lambda y}) = \frac{1 - e^{-xu_t(\lambda)}}{x}.$$

This converges to $u_t(\lambda)$ as $x \rightarrow 0$, and thanks to the proof of [49, Theorem 1.4.1], this equals $N(1 - \exp(-\lambda L_\zeta^t))$. This gives the identity of the entrance laws. For the Markov property we use excursion theory and Ray-Knight's theorem. Let $0 < t_1 < \dots < t_n < t$, then Markov's property for $(L_{T_1}^t, t \geq 0)$ entails that for every $\lambda_1, \dots, \lambda_n, \lambda \geq 0$,

$$E \left[\exp \left(- \sum_{i=1}^n \lambda_i L_{T_1}^{t_i} - \lambda L_{T_1}^t \right) \right] = E \left[\exp \left(- \sum_{i=1}^{n-1} \lambda_i L_{T_1}^{t_i} - (\lambda_n + u_{t-t_n}(\lambda)) L_{T_1}^{t_n} \right) \right].$$

On the other hand, we may write $L_{T_1}^t = \sum_{0 < s \leq 1} (L_{T_s}^t - L_{T_{s-}}^t)$ so that the Laplace exponent identity for Poisson point processes applied to both sides of the above displayed expression gives after taking logarithms:

$$N \left(1 - \exp \left(- \sum_{i=1}^n \lambda_i L_\zeta^{t_i} - \lambda L_\zeta^t \right) \right) N \left(1 - \exp \left(- \sum_{i=1}^{n-1} \lambda_i L_\zeta^{t_i} - (\lambda_n + u_{t-t_n}(\lambda)) L_\zeta^{t_n} \right) \right),$$

so that a substraction gives

$$\begin{aligned} & N \left(\exp \left(- \sum_{i=1}^n \lambda_i L_\zeta^{t_i} \right) (1 - \exp(-\lambda L_\zeta^t)) \right) \\ &= N \left(\exp \left(- \sum_{i=1}^n \lambda_i L_\zeta^{t_i} \right) (1 - \exp(-u_{t-t_n}(\lambda) L_\zeta^{t_n})) \right) \\ &= N \left(\exp \left(- \sum_{i=1}^n \lambda_i L_\zeta^{t_i} \right) \mathbb{E}_{L_\zeta^{t_n}} [1 - \exp(-\lambda Z_{t-t_n})] \right). \end{aligned}$$

Hence the Markov property. //

// *Proof of Theorem 4.2.* It just remains to prove the second statement. For this we let $\eta > 0$ and we define as above the time change τ_t^η . Using the Markov property under the measure N , we again have that under N , $(X_{\eta+s}, s \geq 0)$ is independent of $(X_s, 0 \leq s \leq \eta)$ conditionally on X_η and has the law $P_{X_\eta}^{h_0}$ of the stable process started at X_η and killed at time h_0 . Hence, by Lamperti's identity, under N and conditionally on $(X_s, 0 \leq s \leq \eta)$, the process $(X_{\tau_t^\eta}, t \geq 0)$ has law \mathbb{P}_{X_η} . Thus if $\eta < t_1 < \dots < t_n < t$ and if g_1, \dots, g_n, g are positive continuous functions with compact support that does not contain 0, then

$$\begin{aligned} N \left(\prod_{i=1}^n g_i(X_{\tau_{t_i}^\eta}) g(X_{\tau_t^\eta}) \right) &= \int_0^\infty N(X_\eta \in dx) \mathbb{E}_x \left[\prod_{i=1}^n g_i(Z_{t_i-\eta}) g(Z_{t-\eta}) \right] \\ &= \int_0^\infty N(X_\eta \in dx) \mathbb{E}_x \left[\prod_{i=1}^n g_i(Z_{t_i-\eta}) \mathbb{E}_{Z_{t_n-\eta}} [g(Z_{t-t_n})] \right]. \end{aligned}$$

As for the CSBP, the entrance law $N(X_\eta \in dx)$ equals $x^{-1}P^\uparrow(X_\eta \in dx)$ for $x > 0$. So we recast the last expression as

$$\begin{aligned} & \int_0^\infty P^\uparrow(X_\eta \in dx) \mathbb{E}_x \left[\frac{\prod_{i=1}^n g_i(Z_{t_i-\eta})}{x} \mathbb{E}_{Z_{t_n-\eta}}[g(Z_{t-t_n})] \right] \\ &= \int_0^\infty P^\uparrow(X_\eta \in dx) \mathbb{E}_x^\uparrow \left[\frac{\prod_{i=1}^n g_i(Z_{t_i-\eta})}{Z_{t_n-\eta}} \mathbb{E}_{Z_{t_n-\eta}}[g(Z_{t-t_n})] \right]. \end{aligned}$$

Now we let $\eta \downarrow 0$, using the right continuity and the Feller property of the CSBP, to obtain

$$N \left(\prod_{i=1}^n g_i(X_{\tau_{t_i}}) g(X_{\tau_t}) \right) = \mathbb{E}^\uparrow \left[\frac{\prod_{i=1}^n g_i(Z_{t_i})}{Z_{t_n}} \mathbb{E}_{Z_{t_n}}[g(Z_{t-t_n})] \right].$$

Hence, thanks to Proposition 4.7 we obtain that under N the process $(X_{\tau_t}, t \geq 0)$ has the same entrance law and Markov property as $(L_\zeta^t, t \geq 0)$, hence the same law. //

// *Proof of Lemma 4.7.* Let G be a continuous bounded functional on the paths with lifetime K . We want to show that $N^{(1)}[G(t^{1/(1-\alpha)}L_1^{tx}, 0 \leq x \leq K)]$ goes to $E^\uparrow[G(X_{\tau_x}, 0 \leq x \leq K)]$. By Theorem 4.2, the process $(L_v^x, x \geq 0)$ under $N^{(v)}$ is equal to the process $(X_{\tau_x}, x \geq 0)$ under the law $N^{(v)}$ for almost every v , and we can take $v = 1$ by the usual scaling argument. By [43], the law $N^{(1)}$ can be obtained as the bridge with length 1 of the stable process conditioned to stay positive, and there exists a positive measurable space-time harmonic function $(h_r(x), 0 < r < 1, x \geq 0)$ such that for every functional J and every $r < 1$,

$$N^{(1)}[J(X_s, 0 \leq s \leq r)] = E^\uparrow[h_r(X_r)J(X_s, 0 \leq s \leq r)].$$

We now use essentially the same proof as in [34, Lemma 6]. Let $\varepsilon > 0$. Since $\tau_{tK} \wedge \varepsilon$ is a stopping time for the natural filtration of X ,

$$\begin{aligned} & N^{(1)}[G(t^{1/(1-\alpha)}X_{\tau_{tK} \wedge \varepsilon}, 0 \leq x \leq K)] \\ &= E^\uparrow[h_\varepsilon(X_\varepsilon)G(t^{1/(1-\alpha)}X_{\tau_{tK} \wedge \varepsilon}, 0 \leq x \leq K)] \\ &= E^\uparrow[E^\uparrow[h_\varepsilon(X_\varepsilon)|X_{\tau_{tK} \wedge \varepsilon}]G(t^{1/(1-\alpha)}X_{\tau_{tK} \wedge \varepsilon}, 0 \leq x \leq K)]. \end{aligned}$$

Since $\tau_{tK} \rightarrow 0$ a.s. as $t \downarrow 0$, we obtain the same limit if we remove the ε in the left-hand side, hence giving $\lim N^{(1)}[G(t^{1/(1-\alpha)}L_1^{tx}, 0 \leq x \leq K)]$ by Theorem 4.2. Using the backwards martingale convergence theorem we obtain that the conditional expectation on the right-hand side converges to $E^\uparrow[h_\varepsilon(X_\varepsilon)] = 1$. So

$$\lim_{t \downarrow 0} N^{(1)}[G(t^{1/(1-\alpha)}L_1^{tx}, 0 \leq x \leq K)] = \lim_{t \downarrow 0} E^\uparrow[G(t^{1/(1-\alpha)}X_{\tau_{tK}}, 0 \leq x \leq K)]$$

and the last expression is constant, equal to $E^\uparrow[G(X_{\tau_x}, 0 \leq x \leq K)]$ by scaling, hence the result by Lamperti's transform. The independence with

the initial process is a refinement of the argument above, using the Markov property at the time $\tau_{tK} \wedge \varepsilon$. //

One final comment. It may look quite strange in the proofs above that the *a priori* ill-defined time τ_t under the laws P^\uparrow or N somehow has to be non-degenerate by the proofs we used, even though no argument on the path behavior near 0 has been given for these laws. As a matter of fact, things are maybe clearer when considering also the inverse Lamperti transform. As above, for some process Z that is strictly positive on a set of the form $(0, K)$, $K > 0$, we let

$$C_s = \inf \left\{ u \geq 0 : \int_0^u dv Z_v > s \right\}.$$

Define the process X by $X_s = Z_{C_s}$. Then we claim that the map $s \mapsto 1/X_s$ is integrable on a neighborhood of 0 and that $X_{\tau_t} = Z_t$. Indeed, by a change of variables $w = C_v$, one has:

$$\int_0^u \frac{dv}{X_v} = \int_0^u \frac{dv}{Z_{C_v}} = \int_0^{C_u} \frac{Z_w dw}{Z_w} = C_u < \infty,$$

as long as $u < C^{-1}(\infty) = \inf\{s : X_s = 0\}$, which is strictly positive by the hypothesis made on Z . This kind of arguments also shows that as soon as we have one side of Lamperti's theorem, i.e. $X_s = Z_{C_s}$ or $Z_t = X_{\tau_t}$, with non-degenerate C or τ , then the other side is true. In particular, Theorem 4.2 and Lemma 4.9 could be restated with the inverse statement giving the Lévy process by time-changing the CSBP with C .

Chapter 5

Self-similar fragmentations derived from the stable tree II: splitting at nodes

Contents

5.1	Introduction	127
5.2	Some facts about Lévy processes	131
5.2.1	Stable processes, inverse subordinators	131
5.2.2	Marked processes	132
5.2.3	Bridges, excursions	134
5.3	The stable tree	136
5.3.1	Height Process, width process	136
5.3.2	The tree structure	137
5.3.3	A second way to define F^+	139
5.4	Study of F^+	142
5.4.1	Self-similarity	142
5.4.2	Splitting rates and dislocation measure	146
5.5	Study of F^\natural	149
5.5.1	The self-similar fragmentation property	149
5.5.2	The semigroup	150
5.5.3	Proof of Theorem 5.2	155
5.6	Asymptotics	156
5.6.1	Small-time asymptotics	156
5.6.2	Large-time asymptotics	157

5.1 Introduction

The goal of this paper is to investigate a Markovian fragmentation of the so-called *stable tree*, which is a model of *continuum random tree* (CRT) depending on a parameter $\alpha \in$

(1, 2], that has been introduced by Duquesne and Le Gall [49]. When $\alpha = 2$ this is the Brownian CRT of Aldous [5]. In a companion paper [80], we already studied such a fragmentation process called $(F^-(t), t \geq 0)$ with values in

$$S := \left\{ \mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\},$$

which roughly consisted in putting aside the vertices of the stable tree (with $\alpha < 2$) at height less than t and recording the sizes of the tree components of the resulting forest in decreasing order of magnitude. Such a fragmentation was shown to have a self-similarity property with self-similarity index $1/\alpha - 1$. Precisely, given that the state of the fragmentation at time t is $F^-(t) = (x_1, x_2, \dots)$, the law of $F^-(t + t')$ is that of the decreasing rearrangement of the sequences $F_{(i)}^-(x_i^{1/\alpha-1}t')$ for $i \geq 1$, where the $F_{(i)}^-$'s are independent copies of F^- . Call any S -valued Markov process $(F(t), t \geq 0)$ with such a property, where the exponent $1/\alpha - 1$ is replaced by some $\beta \in \mathbb{R}$, and which is continuous in probability (S is endowed with the pointwise convergence topology), a (ranked) *self-similar fragmentation*. Such fragmentations have been introduced and extensively studied by Bertoin in [23, 26]. By [17], the laws of the self-similar fragmentations are characterized by a 3-tuple (β, c, ν) , where β is the self-similarity index, $c \geq 0$ is an erosion coefficient and, more importantly, ν is a σ -finite *dislocation measure* on S that integrates the map $\mathbf{s} \mapsto 1 - s_1$. This measure ν describes the ‘‘jumps’’ of the fragmentation process, i.e. the way sudden dislocations occur. Roughly speaking, $x^\alpha \nu(d\mathbf{s})$ is the instantaneous rate at which an object with size x fragments to form objects with sizes $x\mathbf{s}$. In the present paper, the only things we need to know about ν are simple variants of the following proposition from [82], which formalizes the preceding rough statement for a single object with size 1.

Proposition 5.1 :

Let $(F(t), t \geq 0)$ be a self-similar fragmentation with index $\beta \geq 0$ and erosion coefficient $c = 0$. Then for every function G that is continuous and null on a neighborhood of $(1, 0, \dots)$ in S ,

$$t^{-1} E[G(F(t))] \xrightarrow[t \downarrow 0]{} \nu(G).$$

See [80] (or Theorem 5.1 below, but let us not anticipate) for an explicit formula for the dislocation measure of F^- when $\alpha \in (1, 2)$. When $\alpha = 2$, the fragmentation F^- corresponds to a fragmentation of the Brownian CRT, or equivalently of the normalized Brownian excursion, that has been studied in [26]. It was shown that the self-similarity index of F^- is $-1/2$, which agrees with the above statement, and the dislocation measure was given explicitly. It turned out that this measure also arose in another self-similar fragmentation of the Brownian CRT introduced by Aldous and Pitman [10], which is related to the so-called standard additive coalescent. This fragmentation has index $1/2$ and the same dislocation measure, up to a multiplicative constant (which can be set equal to 1 up to a linear time-change of the fragmentation).

The motivation of the present paper is to look for possible generalizations of such a result to the other stable trees: does there exist another way as F^- to log the stable tree, that would induce a self-similar fragmentation with the same dislocation measure but positive index? The naïve approach of this problem would be to mimic the description of the Aldous-Pitman fragmentation. This approach fails because the Aldous-Pitman

fragmentation, which uses a Poisson cutting along the skeleton of the Brownian CRT, is binary (when a fragment dislocates, it gives birth to exactly two fragments), and it is not difficult to see that it is also the case when trying to generalize it to other stable trees, because the cutpoints of the Poisson processes have zero chance to fall on branchpoints of the tree. When $\alpha = 2$ this is not a problem since the associated F^- is also binary, a property it inherits from the fact that the Brownian CRT is a binary tree. But when $\alpha < 2$, the situation is completely different and the branchpoints of the stable tree all have infinite degree, which implies that for F^- , every dislocation involves infinitely many fragments.

With these heuristics, it is natural to look for a fragmentation of the stable tree that would cut only at the branchpoints of the tree, which we call “hubs” in the sequel, because each branchpoint of the stable tree has an infinite degree. It is not difficult to see that one should cut these hubs at different rates according to their “magnitude” to obtain a self-similarity property, because some of the hubs are somehow more “surrounded” by leaves than others. The correct notion is the following.

We denote by \mathcal{T} the stable CRT, which is a random metric space with respect to a certain distance d , whose elements v are called *vertices*. One of these vertices is distinguished and called the *root*. This space is a tree in the sense that for v, w two vertices there is a unique non-self-crossing path $[[v, w]]$ from v to w in \mathcal{T} , whose length equals $d(v, w)$. For every $v \in \mathcal{T}$, call *height* of v in \mathcal{T} and denote by $\text{ht}(v)$ the distance of v to the root. The *leaves* of \mathcal{T} are those vertices that do not belong to the interior of any path leading from one vertex to another, and the *skeleton* of the tree is the set of non-leaf vertices. The *branchpoints* (hubs) are the vertices b so that there exist $v \neq b, w \neq b$ such that $[[\text{root}, v]] \cap [[\text{root}, w]] = [[\text{root}, b]]$. Call $\mathcal{H}(\mathcal{T})$ the set of hubs of \mathcal{T} . With each realization of \mathcal{T} is associated a probability measure μ , called the *mass measure*, that is supported by \mathcal{T} and that attributes zero mass to the skeleton. This measure allows to evaluate the magnitude of hubs as follows. For every $b \in \mathcal{H}(\mathcal{T})$, consider the *fringe subtree* \mathcal{T}_b rooted at b , i.e. the subset $\{v \in \mathcal{T} : b \in [[\text{root}, v]]\}$. Then one can define the *local time*, or *width* of the hub b as the limit

$$L(b) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mu\{v \in \mathcal{T}_b : d(v, b) < \varepsilon\} \quad (5.1)$$

which exists a.s. and is positive (see Proposition 5.3 below).

Now given a realization of \mathcal{T} and for every $b \in \mathcal{H}(\mathcal{T})$ take a standard exponential random variable e_b , so that the variables e_b are independent as b varies (notice that $\mathcal{H}(\mathcal{T})$ is countable). For all $t \geq 0$ define an equivalence relation \sim_t on \mathcal{T} by saying that $v \sim_t w$ if and only if the path $[[v, w]]$ does not contain any hub b for which $e_b < tL(b)$. Alternatively, following more closely the spirit of Aldous-Pitman’s fragmentation, we can also say that we consider Poisson point process $(b(t), t \geq 0)$ on the set of hubs with intensity $dt \otimes \sum_{b \in \mathcal{H}(\mathcal{T})} L(b) \delta_b(dv)$, and for each t we let $v \sim_t w$ if and only if no atom of the Poisson process that has appeared before time t belongs to the path $[[v, w]]$. We let $\mathcal{T}_1^t, \mathcal{T}_2^t, \dots$ be the distinct equivalence classes for \sim_t , ranked according to the decreasing order of their μ -masses (provided these are well-defined quantities). It is easy to see that these sets are trees (in the same sense as \mathcal{T}), and that the families $(\mathcal{T}_i^t, i \geq 1)$ are nested as t varies, that is, for every $t' > t$ and $i \geq 1$, there exists $j \geq 1$ such that $\mathcal{T}_i^{t'} \subset \mathcal{T}_j^t$. If we let $F^+(t) = (\mu(\mathcal{T}_1^t), \mu(\mathcal{T}_2^t), \dots)$, F^+ is thus a fragmentation process in the sense that $F^+(t')$ is obtained by splitting at random the elements of $F^+(t)$. We mention that

the fragmentation F^+ is also considered and studied in the work in preparation [1], with independent methods.

Let

$$D_\alpha = \frac{\alpha(\alpha-1)\Gamma(1-\frac{1}{\alpha})}{\Gamma(2-\alpha)} = \frac{\alpha^2\Gamma(2-\frac{1}{\alpha})}{\Gamma(2-\alpha)}.$$

We now state our main result, see the following section for definitions and properties of stable subordinators.

Theorem 5.1 :

The process F^+ is a self-similar fragmentation with index $1/\alpha \in (1/2, 1)$ and erosion coefficient $c = 0$. Its dislocation measure ν_α is characterized by

$$\nu_\alpha(G) = D_\alpha E [T_1 G(T_1^{-1} \Delta T_{[0,1]})]$$

for any positive measurable function G , where $(T_x, 0 \leq x \leq 1)$ is a stable subordinator with index $1/\alpha$, characterized by the Laplace transform

$$E[\exp(-\lambda T_1)] = \exp(-\lambda^{1/\alpha}) \quad \lambda \geq 0,$$

and $\Delta T_{[0,1]}$ is the sequence of the jumps of T , ranked by decreasing order of magnitude.

Comparing this result with [80, Theorem 1], we see that the dislocation measure is the same as that of F^- , and our question admits a positive answer.

Let us now present a second motivation for studying the fragmentation F^+ . As the rest of the paper will show, our proofs involve a lot the theory of Lévy processes, and compared with the study of F^- , which made a consequent place to combinatoric tree structures, the study of F^+ will be mainly “analytic”. The fact that Lévy processes may be involved in fragmentation processes is not new. According to [21] and [79], adding a drift to a certain class of Lévy processes allows to construct interesting fragmentations related to the entrance boundary of the stochastic additive coalescent. Here, rather than adding a drift, which by analogy between [12] and [21] amounts to cut the skeleton of a continuum random tree with a homogeneous Poisson process, we will perform a “removing the jumps” operation analog to our inhomogeneous cutting on the hubs of the tree.

Precisely, let $(X_s, s \geq 0)$ be the canonical process in the Skorokhod space $\mathbb{D}([0, \infty))$ and let P be the law of the stable Lévy process with index $\alpha \in (1, 2)$, upward jumps only, characterized by the Laplace exponent

$$E[\exp(-\lambda X_1)] = \exp(\lambda^\alpha).$$

As we will see in the following section, we may define the law $N^{(1)}$ of the excursion with unit duration of this process above its infimum process. Under this law, $X_s = 0$ for $s > 1$, so we let $\Delta X_{[0,1]}$ be the sequence of the jumps $\Delta X_s = X_s - X_{s-}$ for $s \in (0, 1]$, ranked in decreasing order of magnitude. Consider the following marking process on the jumps: conditionally on X , let $(e_s, s : \Delta X_s > 0)$ be a family of independent random variables with standard exponential distribution, indexed by the countable set of jump-times of X . For every $t \geq 0$ let

$$Z_s^{(t)} = \sum_{0 \leq u \leq s} \Delta X_u \mathbb{1}_{\{e_u < t \Delta X_u\}}.$$

That is, each jump with magnitude Δ is marked with probability $1 - \exp(-t\Delta)$ independently of the other jumps and consistently as t varies, and $Z^{(t)}$ is the process that sums the marked jumps. We will see that $Z^{(t)}$ is finite a.s., so we may define $X^{(t)} = X - Z^{(t)}$ under $N^{(1)}$. Let

$$\underline{X}_s^{(t)} = \inf_{0 \leq u \leq s} X_u^{(t)}, \quad 0 \leq s \leq 1,$$

and let $F^{\natural}(t)$ be the sequence of lengths of the constancy intervals of the process $\underline{X}^{(t)}$, ranked in decreasing order.

Theorem 5.2 :

| The process $(F^{\natural}(t), t \geq 0)$ has the same law as $(F^+(t), t \geq 0)$.

We organize the paper as follows. In Sect. 5.2 we recall some facts about Lévy processes, excursions, and conditioned subordinators that will be crucial for our study. In Sect. 5.3 we give the rigorous description of Duquesne and Le Gall's Lévy trees, and rephrase the definition of F^+ given above in terms of a partition of the unit interval associated to a certain marked excursion of a stable Lévy process. Sections 5.4 and 5.5 are then respectively dedicated to the study of F^+ and F^{\natural} . Asymptotic results are finally given concerning the behavior at small and large times of F^+ in Sect. 5.6.

5.2 Some facts about Lévy processes

5.2.1 Stable processes, inverse subordinators

Let $(X_s, s \geq 0)$ be the canonical process in the Skorokhod space $\mathbb{D}([0, \infty))$ of càdlàg paths on $[0, \infty)$. We fix $\alpha \in (1, 2)$. Let P be the law on $\mathbb{D}([0, \infty))$ that makes X the spectrally positive stable process with index α , that is, X has independent and stationary increments under P , it has only positive jumps, and its marginal law at some (and then all) $s > 0$ has Laplace transform given by the Lévy-Khintchine formula:

$$E[e^{-\lambda X_s}] = \exp(s\lambda^\alpha) = \exp\left(s \int_0^\infty \frac{C_\alpha dx}{x^{1+\alpha}} (e^{-\lambda x} - 1 + \lambda x)\right), \quad \lambda \geq 0,$$

where $C_\alpha = \alpha(\alpha - 1)/\Gamma(2 - \alpha)$. A fundamental property of X under P is the *scaling property*

$$\left(\frac{1}{\lambda^{1/\alpha}} X_{\lambda s}, s \geq 0\right) \stackrel{d}{=} (X_s, s \geq 0) \quad \text{for all } \lambda > 0.$$

We let $(p_s(x), s > 0, x \in \mathbb{R})$ be the density with respect to Lebesgue measure of the law $P(X_s \in dx)$, which is known to exist and to be jointly continuous in s and x .

Denote by \underline{X} the infimum process of X defined by

$$\underline{X}_s = \inf_{0 \leq u \leq s} X_u, \quad s \geq 0.$$

Let T be the right-continuous inverse of the increasing process $-\underline{X}$ defined by

$$T_x = \inf\{s \geq 0 : \underline{X}_s < -x\}.$$

Then it is known that under P , T is a stable subordinator with index $1/\alpha$, that is, an increasing Lévy process with Laplace exponent

$$E[e^{-\lambda T_x}] = \exp(-x\lambda^{1/\alpha}) = \exp\left(-x \int_0^\infty \frac{c_\alpha dy}{y^{1+1/\alpha}}(1 - e^{-\lambda y})\right) \quad \text{for } \lambda, x \geq 0,$$

where $c_\alpha = (\alpha\Gamma(1 - 1/\alpha))^{-1}$. We denote by $(q_x(s), x, s > 0)$ the family of densities with respect to Lebesgue measure of the law $P(T_x \in ds)$, by [19, Corollary VII.1.3] they are given by

$$q_x(s) = \frac{x}{s} p_s(-x). \tag{5.2}$$

We also introduce the notations P^s for the law of the processes X under P , killed at time s , and $P^{(-x, \infty)} := P^{T_x}$ for the law of the process killed when it first hits $-x$.

Let us now discuss the conditioned forms of distributions of jumps of subordinators. An easy way to obtain regular versions for these conditional laws is developed in [87, 92]. First, we define the size-biased permutation of the sequence $\Delta T_{[0,x]}$ of the ranked jumps of T in the interval $[0, x]$ as follows. Write $\Delta T_{[0,x]} = (\Delta_1(x), \Delta_2(x), \dots)$ with $\Delta_1(x) \geq \Delta_2(x) \geq \dots$, and recall that $T_x = \sum_i \Delta_i(x)$. Then let 1^* be a r.v. such that

$$P(1^* = i | \Delta T_{[0,x]}) = \frac{\Delta_i(x)}{T_x}$$

for all $i \geq 1$, and set $\Delta_1^*(x) = \Delta_{1^*}(x)$. Recursively, let k^* be such that

$$P(k^* = i | \Delta T_{[0,x]}, (j^*, 1 \leq j \leq k-1)) = \frac{\Delta_i(x)}{T_x - \Delta_1^*(x) - \dots - \Delta_{k-1}^*(x)}$$

for $i \geq 1$ distinct of the j^* , $1 \leq j \leq k-1$, and let $\Delta_k^*(x) = \Delta_{k^*}(x)$. Then

Lemma 5.1 :

(i) For $k \geq 1$,

$$P(\Delta_k^*(x) \in dy | T_x, (\Delta_j^*(x), 1 \leq j \leq k-1)) = \frac{c_\alpha x q_x(s-y)}{s y^{1/\alpha} q_x(s)} dy$$

where $s = T_x - \Delta_1^*(x) - \dots - \Delta_{k-1}^*(x)$.

(ii) Consequently, given $T_x = t$, $\Delta_1^*(x) = y$, the sequence $(\Delta_2^*(x), \Delta_3^*(x), \dots)$ has the same law as $(\Delta_1^*(x), \Delta_2^*(x), \dots)$ given $T_x = t - y$. Conversely, if we are given a random variable Y with same law as $\Delta_1^*(x)$ given $T_x = t$ and, given $Y = y$, a sequence (Y_1, Y_2, \dots) with same law as $(\Delta_1^*(x), \Delta_2^*(x), \dots)$ given $T_x = t - y$, then (Y, Y_1, Y_2, \dots) has same law as $(\Delta_1^*(x), \Delta_2^*(x), \dots)$ given $T_x = t$.

This gives a regular conditional version for $(\Delta_i^*(x), i \geq 1)$ given T_x , and thus induces a conditional version for $\Delta T_{[0,x]}$ given T_x by ranking.

5.2.2 Marked processes

We are now going to enlarge the original probability space to mark the jumps of the stable process. We let M_X be the law of a sequence $\mathbf{e} = (e_s, s : \Delta X_s > 0)$ of independent standard exponential random variables, indexed by the (countable) set of times where the

canonical process X jumps. We let $\mathbf{P}(dX, de) = P(dX) \otimes M_X(de)$. This probability allows to mark the jumps of X , precisely we say that a jump occurring at time s is marked at level $t \geq 0$ if $e_s < t\Delta X_s$. Write

$$Z_s^{(t)} = \sum_{0 \leq u \leq s} \Delta X_u \mathbb{1}_{\{e_u < t\Delta X_u\}}$$

for the cumulative process of marked jumps at level t . We also let $X^{(t)} = X - Z^{(t)}$. We know that the process $(\Delta X_s, s \geq 0)$ of the jumps of X is under P a Poisson point process with intensity $C_\alpha x^{-1-\alpha} dx$ on $(0, \infty)$, it is then standard that the process $(\Delta Z_s^{(t)}, s \geq 0)$ is a Poisson point process with intensity $C_\alpha x^{-\alpha-1}(1 - e^{-tx}) dx$, meaning that under \mathbf{P} , $Z^{(t)}$ is a subordinator with no drift and Lévy measure $C_\alpha x^{-\alpha-1}(1 - e^{-tx}) dx$, more precisely its Laplace transforms are given by

$$\mathbf{E}[e^{-\lambda Z_s^{(t)}}] = \exp\left(-s \int_0^\infty C_\alpha (1 - e^{-tx}) \frac{1 - e^{-\lambda x}}{x^{\alpha+1}} dx\right) = \exp(-s(\lambda + t)^\alpha + s\lambda^\alpha + st^\alpha).$$

We denote by $(\rho_s^{(t)}(x), s, x \geq 0)$ the densities of the laws $P(Z_s^{(t)} \in dx)$. It can be checked by [97, Proposition 28.3] from the expression of the Lévy measure of $Z^{(t)}$ that these densities exist and are jointly continuous. Likewise, the process $X^{(t)}$ is under \mathbf{P} a Lévy process with Lévy measure $C_\alpha e^{-tx} x^{-\alpha-1} dx$, and the Laplace transform of $X_s^{(t)}$ is given by

$$\mathbf{E}[e^{-\lambda X_s^{(t)}}] = \exp\left(s\lambda^\alpha t^{\alpha-1} + s \int_0^\infty C_\alpha e^{-tx} \frac{dx}{x^{\alpha+1}} (e^{-\lambda x} - 1 + \lambda x)\right) = \exp(s(\lambda + t)^\alpha - st^\alpha),$$

which is obtained by dividing the Laplace exponent of X_s by that of $Z_s^{(t)}$.

We now state an absolute continuity result that is analogous to Cameron-Martin's formula for Brownian motion with drift.

Proposition 5.2 :

For every $t, s \geq 0$, we have the following absolute continuity relation: for every positive measurable functional F ,

$$\mathbf{E}[F(X_u^{(t)}, 0 \leq u \leq s)] = E[\exp(-st^\alpha - tX_s) F(X_u, 0 \leq u \leq s)].$$

// By the expression for the Laplace exponent of $X^{(t)}$, we get

$$\mathbf{E}[e^{-\lambda X_s^{(t)}}] = e^{-st^\alpha} E[e^{-(\lambda+t)X_s}],$$

hence giving $\mathbf{P}(X_s^{(t)} \in dx) = e^{-st^\alpha - tx} P(X_s \in dx)$. The result easily follows by the Markov property. //

As a first consequence, it immediately follows that $X^{(t)}$ also have jointly continuous densities under P , which are given by

$$p_s^{(t)}(x) = \frac{\mathbf{P}(X_s^{(t)} \in dx)}{dx} = \exp(-st^\alpha - tx) p_s(x).$$

Let $\underline{X}^{(t)}$ be the infimum process of $X^{(t)}$ and $T^{(t)}$ the right-inverse process of $-\underline{X}^{(t)}$, defined as we did above define \underline{X} and T .

It is easily obtained that for every $t \geq 0$, the process $(X, Z^{(t)})$ is again a Lévy process under the law \mathbf{P} . We will also denote by $\mathbf{P}^s, \mathbf{P}^{(-x, \infty)}$ the laws derived from P^s and $P^{(-x, \infty)}$ by marking the jumps with M_X ; $Z^{(t)}$ and $X^{(t)}$ are then defined as before.

5.2.3 Bridges, excursions

For $r \in \mathbb{R}$ and $s > 0$ we will denote by $P_{0 \rightarrow r}^s$ the law of the *stable bridge from 0 to r with length s*, so the family $(P_{0 \rightarrow r}^s, r \in \mathbb{R})$ forms a regular conditional version for $P^s(\cdot | X_s = r)$. By [55], a regular version (which is the one we will always consider) is obtained as the unique law on the Skorokhod space $\mathbb{D}([0, s])$ that satisfies the following absolute continuity relation: for every $a \in (0, s)$ and any continuous functional F ,

$$P_{0 \rightarrow r}^s(F(X_u, 0 \leq u \leq s - a)) = E \left[F(X_u, 0 \leq u \leq s - a) \frac{p_a(r - X_{s-a})}{p_s(r)} \right]. \quad (5.3)$$

We let $\mathbf{P}_{0 \rightarrow r}^s$ be the marked analog of $P_{0 \rightarrow r}^s$ on an enriched probability space. Notice that Proposition 5.2 immediately implies that the laws bridges for the process $X^{(t)}$ under \mathbf{P} are the same as those of X . Stable bridges from 0 to 0 satisfy the following scaling property: under $P_{0 \rightarrow 0}^v$, the process $(v^{-1/\alpha} X_{vs}, 0 \leq s \leq 1)$ has law $P_{0 \rightarrow 0}^1$.

Lemma 5.2 :

The following formula holds for any positive measurable f, g, H :

$$\begin{aligned} & E_{0 \rightarrow 0}^1 \left[H(X) \sum_{0 \leq s \leq 1} \Delta X_s f(s) g(\Delta X_s) \right] \\ &= \int_0^1 ds f(s) \int_0^\infty dx \frac{C_\alpha p_1(-x)}{x^\alpha p_1(0)} g(x) E_{0 \rightarrow -x}^1 [H(X \oplus (s, x))], \end{aligned}$$

where $X \oplus (s, x)$ is the process X to which has been added a jump at time s with magnitude x . Otherwise said, a stable bridge from 0 to 0 together with a jump $(s, \Delta X_s)$ picked according to the σ -finite measure $m(ds, dx) = \sum_{u: \Delta X_u > 0} \Delta X_u \delta_{(u, \Delta X_u)}(ds, dx)$ is obtained by taking a stable bridge from 0 to $-x$ and adding a jump with magnitude x at time s , where s is uniform in $(0, 1)$ and x is independent with σ -finite “law” $C_\alpha p_1(-x) p_1(0)^{-1} x^{-\alpha} dx$.

// By the Lévy-Itô decomposition of Lévy processes, one can write, under P , that X_s is the compensated sum

$$X_s = \lim_{\varepsilon \rightarrow 0} \left(\sum_{0 \leq u \leq s} \Delta X_u \mathbb{1}_{\{\Delta X_u > \varepsilon\}} - (\alpha - 1)^{-1} C_\alpha \varepsilon^{1-\alpha} s \right), \quad s \geq 0,$$

where $(\Delta X_u, u \geq 0)$ is a Poisson point process with intensity $C_\alpha x^{-\alpha-1} dx$, and where the convergence is almost sure. By the Palm formula for Poisson processes, we obtain that for positive measurable f, g, h, H :

$$\begin{aligned} & E^1 \left[h(X_1) H(X) \sum_{0 \leq s \leq 1} \Delta X_s f(s) g(\Delta X_s) \right] \\ &= \int_0^1 ds f(s) \int_0^\infty dx \frac{C_\alpha}{x^\alpha} g(x) E^1 [h(x + X_1) H(X \oplus (s, x))]. \end{aligned}$$

The result is then obtained by disintegrating with respect to the law of X_1 . //

We now state a useful decomposition of the standard stable bridge. Recall $(\rho_s^{(t)}(x), x \geq 0)$ is the density of $Z_s^{(t)}$ under \mathbf{P} and that $X_1^{(t)} + Z_1^{(t)} = X_1$, which is a sum of two independent variables. From this we conclude that $(p_1(0)^{-1}p_1^{(t)}(-x)\rho_1^{(t)}(x), x \geq 0)$ is a probability density on \mathbb{R}_+ .

Lemma 5.3 :

Take a random variable \mathcal{Z} with law $P(\mathcal{Z} \in dz) = p_1^{(t)}(-z)\rho_1^{(t)}(z)p_1(0)^{-1}dz$. Conditionally on $\mathcal{Z} = z$, take X' with law $P_{0 \rightarrow -z}^1$ and Z with law $\mathbf{P}^1(Z^{(t)} \in \cdot | Z_1^{(t)} = z)$, independently. That is, Z is the bridge of $Z^{(t)}$ with length 1 from 0 to z . Then $X' + Z$ has law $P_{0 \rightarrow 0}^1$.

Remark. The definition for the bridges of $Z^{(t)}$ under \mathbf{P}^1 has not been given before. One can either follow an analogous definition as (5.3), or use Lemma 5.1 about conditioned jumps of subordinators. Precisely, take $(\Delta_i, i \geq 1)$ a sequence whose law is that of the jumps $\Delta T_{[0,1]}$ of T under P before time 1, ranked in decreasing order, and conditioned by $T_1 = z$, in the sense of Lemma 5.1. Take also a sequence $(U_i, i \geq 1)$ of independent uniformly distributed random variables on $[0, 1]$, independent of $\Delta T_{[0,1]}$. Then one checks from the Lévy-Itô decomposition for Lévy processes that the law Q_z of the process $Z_s = \sum \Delta_i \mathbb{1}_{\{s \geq U_i\}}$, with $0 \leq s \leq 1$, defines as z varies a regular version of the conditional law $\mathbf{P}^1(Z^{(t)} \in \cdot | Z_1^{(t)} = z)$.

// Recall that under \mathbf{P}^1 , X can be written as $X^{(t)} + Z^{(t)}$ with $X^{(t)}$ and $Z^{(t)}$ independent. Consequently, for f and G positive continuous, we have

$$E^1[f(X_1)G(X)] = \mathbf{E}^1[f(X_1^{(t)} + Z_1^{(t)})G(X^{(t)} + Z^{(t)})]$$

so

$$\int_{\mathbb{R}} dx p_1(x) f(x) E_{0 \rightarrow x}^1[G(X)] = \int_{\mathbb{R}} dx p_1(x) \int_0^\infty dz \frac{p_1^{(t)}(x-z)\rho_1^{(t)}(z)}{p_1(x)} f(z) \times \mathbf{E}^1[G(X^{(t)} + Z^{(t)}) | X_1^{(t)} = x-z, Z_1^{(t)} = z].$$

Thus, for (Lebesgue) almost every x , the bridge with law $P_{0 \rightarrow x}^1$ is obtained by taking a bridge of $X^{(t)}$ (or X by previous remarks) from 0 to $-\mathcal{Z}_x$ and an independent bridge of $Z^{(t)}$ from 0 to \mathcal{Z}_x , where \mathcal{Z}_x is a r.v. with law $dz p_1(x)^{-1}p_1^{(t)}(x-z)\rho_1^{(t)}(z)$ on \mathbb{R}_+ . We extend this result to every $x \in \mathbb{R}$ by an easily checked continuity result for the laws of bridges which stems from (5.3) and the continuity of the densities. Taking $x = 0$ gives the result. //

We now turn our attention to excursions. The fact that X has no negative jumps implies that $-\underline{X}$ is a local time at 0 for the reflected process $X - \underline{X}$. Let N be the Itô excursion measure of $X - \underline{X}$ away from 0, so that the path of $X - \underline{X}$ is obtained by concatenation of the atoms of a Poisson measure with intensity $N(dX) \otimes dt$ on $\mathbb{D}^\zeta[0, \infty) \times \mathbb{R}_+$, where $\mathbb{D}^\zeta[0, \infty)$ denotes the Skorokhod space of paths that are killed at some time ζ . Under N , almost every path X starts at 0, is positive on an interval $(0, \zeta)$ and dies at the first time $\zeta(X) \in (0, \infty)$ it hits 0 again. We let \mathbf{N} be the enriched law with marked jumps. It follows from excursion theory that the Lévy process $(X, Z^{(t)})$ under \mathbf{P} is obtained by taking a Poisson point measure $\sum_{i \in I} \delta_{X^i, e^i, s^i}$ with intensity $\mathbf{N}(dX, de) \otimes ds$, writing $Z^{(t), i}$

for the cumulative process of marked jumps for X^i and letting

$$X_s = -s^i + X^i \left(s - \sum_{j:s^j < s^i} \zeta_j(X^j) \right)$$

and

$$Z_s^{(t)} = \sum_{j:s^j < s^i} Z_{\zeta_j(X^j)}^{(t),j} + Z^{(t),i} \left(s - \sum_{j:s^j < s^i} \zeta_j(X^j) \right),$$

whenever $\sum_{j:s^j < s^i} \zeta_j(X^j) \leq s \leq \sum_{j:s^j \leq s^i} \zeta_j(X^j)$.

If X is stopped at some time s , for any $u \in [0, s]$ we define the rotated process

$$V_u X(r) = (X_{r+u} - X_u) \mathbb{1}_{\{0 \leq u < s-u\}} + (X(r-s+u) + X_s - X_u) \mathbb{1}_{\{s-u \leq r \leq s\}}.$$

Let $m_s = -\underline{X}_s$ and suppose that this minimum is attained only once on $[0, s]$. We define the Vervaat transform of X as $VX = V_{T(m_s-)} X$, the rotation of X at the time where it attains its infimum. Provided that $X_0 = 0$ and $X_s = X_{s-} = 0$ (say that X is a bridge), VX is then an excursion-like function, starting and ending at 0, and staying positive in the meanwhile.

We will denote by $N^{(v)}$ the law of VX under $P_{0 \rightarrow 0}^v$, and $\mathbf{N}^{(v)}$ the corresponding ‘‘marked’’ version. Call it the law of the *excursion of X with duration v* . The ‘‘Vervaat theorem’’ in [43] shows that $N^{(v)}$ is indeed a regular conditional version for the ‘‘law’’ $N(\cdot | \zeta = v)$: for any positive measurable functional F and function f ,

$$N(F(X_s, 0 \leq s \leq \zeta) f(\zeta)) = \int_{\mathbb{R}_+} f(\zeta) N(\zeta \in dv) N^{(v)}(F(X_s, 0 \leq s \leq v)).$$

As for bridges, we also have the scaling property at the level of conditioned excursions: under $N^{(v)}$, $(v^{-1/\alpha} X_{vs}, 0 \leq s \leq 1)$ has law $N^{(1)}$. Notice also (either by Vervaat’s theorem or directly, using Proposition 5.2) that the excursions of $X^{(t)}$ under \mathbf{P} , conditioned to have a fixed duration v are the same as that of X under $N^{(v)}$.

5.3 The stable tree

5.3.1 Height Process, width process

We now introduce the rigorous definition and useful properties of the stable tree. This section is mainly inspired by [49, 48]. With the notations of section 5.2, and for $t \geq 0$, let $R^{(t)}$ be the time-reversed process of X at time t :

$$R_s^{(t)} = X_t - X_{(t-s)-} \quad 0 \leq s \leq t.$$

It is standard that this process has the same law as X killed at time t under P . Let $\overline{R}^{(t)}$ be its supremum process, and $\widehat{L}^{(t)}$ be the local time process at level 0 of the reflected process $\overline{R}^{(t)} - R^{(t)}$. We let $H_t = \widehat{L}_t^{(t)}$. The normalization for $\widehat{L}^{(t)}$ is chosen so that

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{\overline{R}_s^{(t)} - R_s^{(t)} \leq \varepsilon\}} ds,$$

in probability for every t . It is proved in [49] that H admits a continuous modification, which is the one we are going to work with from now on. It has to be noticed that H_t is not a Markov process, except in the case where X is Brownian motion. As a matter of fact, it can be noticed that H admits infinitely many local minima attaining the same value as soon as X has jumps. To see this, consider a jump time t of X , and let $t_1, t_2 > t$ so that $\inf_{t \leq u \leq t_i} X_u = X_{t_i}$ and $X_{t-} < X_{t_i} < X_t$, $i \in \{1, 2\}$. Then it is easy to see that $H_t = H_{t_1} = H_{t_2}$ and that one may in fact find an infinite number of distinct t_i 's satisfying the properties of t_1, t_2 . On the other hand, it is not difficult to see that H_t is a local minimum of H , see Proposition 5.3 below.

It is shown in [49] that the definition of H still makes sense under the σ -finite measure N rather than the probability law P . The process H is then defined only on $[0, \zeta]$, and we call it the *excursion of the height process*. Using the scaling property, one can then define the height process under the laws $N^{(v)}$. Call it the law of the excursion of the height process with duration v .

The key tool for defining the local time of hubs is the local time process of the height process. We will denote by $(L_s^t, t, s \geq 0)$. It can be obtained a.s. for every fixed s, t by

$$L_s^t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbb{1}_{\{t < H_u \leq t + \varepsilon\}} du. \tag{5.4}$$

That is, L_s^t is the density of the occupation measure of H at level t and time s . For $t = 0$, one gets $(L_s^0, s \geq 0) = (\underline{X}_s, s \geq 0)$, which is a reminiscence of the fact that the excursions of the height process are in one-to-one correspondence with excursions of X with the same lengths.

It is again possible to define the local time process under the excursion measures N and $N^{(v)}$. Duquesne and Le Gall [49] have shown that under P , the process $(L_{T_x}^t, t \geq 0)$ has the law of the continuous-state branching process starting at $x > 0$, with stable (α) branching mechanism. One can get interpretations for the process $(L_\zeta^t, t \geq 0)$ under the measure N or of $(L_v^t, t \geq 0)$ under $N^{(v)}$ in terms of conditioned continuous-state branching processes, see [80].

5.3.2 The tree structure

Let us motivate the term of “height process” for H by embedding a tree inside H , following [74, 5]. Consider the height process H under the law $N^{(1)}$. We can define a pseudo metric D on $[0, 1]$ by letting $D(s, s') = H_s + H_{s'} - 2 \inf_{u \in [s, s']} H_u$ (with the convention that $[s, s'] = [s', s]$ if $s' < s$). Let $s \equiv s'$ if and only if $D(s, s') = 0$.

Definition :

The *stable tree* (\mathcal{T}, d) is the quotient of the pseudo-metric space $([0, 1], D)$ by \equiv . The *root* of \mathcal{T} is the equivalence class of 0. The *mass measure* μ is the Borel measure induced on \mathcal{T} by Lebesgue’s measure on $[0, 1]$ (so its support is \mathcal{T}).

In the sequel, we will often identify \mathcal{T} with $[0, 1]$, even if the correspondence is not one-to-one. Some comments on this definition. First, the way the tree is embedded in the function H can seem quite intricate. It is not difficult, however, to see what its “marginals” look like. For any finite set of vertices $s_1, s_2, \dots, s_k \in [0, 1]$, one recovers the structure of the subtree spanned by the root and s_1, s_2, \dots, s_k , according to the following simple rules:

- The height of s is $\text{ht}(s) = H_s$.

- The common ancestor of s_1, \dots, s_k is $b = b(s_1, \dots, s_k) \in [\min_{1 \leq i \leq k} s_i, \max_{1 \leq i \leq k} s_i]$ such that $H_b = \inf\{H_s : s \in [\min_{1 \leq i \leq k} s_i, \max_{1 \leq i \leq k} s_i]\}$.

Notice that all such b are equivalent with respect to \equiv . The fact that (\mathcal{T}, d) is indeed a tree (a complete metric space such that the only path leading from a vertex to another is the geodesic) is intuitive and proven in [50]. It follows from the construction of “marginals” of \mathcal{T} in [49] that given μ , μ -a.e. vertex is a leaf of \mathcal{T} .

We now relate properties of the stable tree to path properties of the underlying Lévy process we started with to construct the height process. We understand here that X and H are defined under $N^{(1)}$. Recall that \mathcal{T}_b stands for the fringe subtree rooted at b .

Proposition 5.3 :

- (i) Each hub $b \in \mathcal{H}(\mathcal{T})$ with height h , is encoded by exactly one time $\tau(b) \in [0, 1]$ such that $L(b) = \Delta X_{\tau(b)} > 0$, and $L(b)$ is given by (5.1) a.s.
- (ii) If $\sigma(b) = \inf\{s \geq \tau(b) : X_s = X_{\tau(b)-}\}$, then $\mathcal{T}_b = [\tau(b), \sigma(b)] / \equiv$.
- (iii) More precisely, let $\mathcal{T}_b^1, \mathcal{T}_b^2, \dots$ be the connected components of $\mathcal{T}_b \setminus \{b\}$, arranged in decreasing order of mass. Let $([\tau_i(b), \sigma_i(b)], i \geq 1)$ be the constancy intervals of the infimum process of $(X_s - X_{\tau(b)}, \tau(b) \leq s \leq \sigma(b))$, and ranked in decreasing order of length. Then $\mathcal{T}_b^i = (\tau_i(b), \sigma_i(b)) / \equiv$.

// (i) Working first under P , fix $\ell > 0$ and let $\tau_\ell = \inf\{s \geq 0 : \Delta X_s > \ell\}$. Then τ_ℓ is a stopping time for the natural filtration associated to X , as well as $\sigma_\ell = \inf\{s > \tau_\ell : X_s = X_{\tau_\ell-}\}$. By the Markov property, the process $X_{[\tau_\ell, \sigma_\ell]} = (X_{\tau_\ell+s} - X_{\tau_\ell}, 0 \leq s \leq \sigma_\ell - \tau_\ell)$ is independent of $(X_{s+\sigma_\ell} - X_{\sigma_\ell}, s \geq 0)$, which has the same law as X , and of $(X_s, 0 \leq s \leq \tau_\ell)$ conditionally on its final jump ΔX_{τ_ℓ} . Now if we remove this jump, that is, if we let $(\tilde{X}_s, 0 \leq s \leq \tau_\ell)$ be the modification of $(X_s, 0 \leq s \leq \tau_\ell)$ that is left-continuous at τ_ℓ , then \tilde{X} has the law of a stable Lévy process killed at some independent exponential time, and conditioned to have jumps with magnitude less than ℓ . Also, conditionally on $\Delta X_{\tau_\ell} = x$, $X_{[\tau_\ell, \sigma_\ell]}$ has the law $P^{(-x, \infty)}$ of the stable process killed when it first hits $-x$. Hence, by the additivity of the local time and the definition of H , one has that for every $s \in [\tau_\ell, \sigma_\ell]$, $H_s = H_{\tau_\ell} + \tilde{H}_{s-\tau_\ell}$, where \tilde{H} is an independent copy of H , killed when its local time at 0 attains x . Consequently, one has $H_s \geq H_{\tau_\ell}$ for every $s \in [\tau_\ell, \sigma_\ell]$ and $H_{\sigma_\ell} = H_{\tau_\ell}$, moreover, one has that for every $\varepsilon > 0$,

$$\inf_{(\tau_\ell - \varepsilon) \vee 0 \leq s \leq \tau_\ell} H_s \vee \inf_{\sigma_\ell \leq s \leq \sigma_\ell + \varepsilon} H_s < H_{\tau_\ell}, \tag{5.5}$$

as a consequence of the following fact. By the left-continuity of X at τ_ℓ , for any $\varepsilon > 0$ we may find $s \in [\tau_\ell - \varepsilon, \tau_\ell]$ such that $\inf_{u \in [s, \tau_\ell]} X_u = X_s$. This implies $H_s = H_{\tau_\ell} - \widehat{L}_{\tau_\ell - s}^{(\tau_\ell)}$, and this last term is a.s. strictly less than H_{τ_ℓ} because 0 is a.s. not a holding point for $(\widehat{L}_s^{(\tau_\ell)}, 0 \leq s \leq \tau_\ell)$. This last fact is obtained by a time-reversal argument, using the fact that the points of increase of the local time $\widehat{L}^{(t)}$ correspond to that of the supremum process of $R^{(t)}$. Moreover, the fact that X has only positive jumps under P implies that for some suitable $\varepsilon' > 0$, one can find some $s' \in [\sigma_\ell, \sigma_\ell + \varepsilon']$ and $s'' \in [\tau_\ell - \varepsilon, \tau_\ell]$ such that $H_u \geq H_{s'} = H_{s''}$ for every $u \in [s', s'']$, and such

that again $\inf_{u \in [s'', \tau_\ell]} X_u = X_{s''}$. Thus the claimed inequality. In terms of the structure of the stable tree, (5.5) implies that a node b of the tree is present at height h , which is encoded by all the $s \in [\tau_\ell, \sigma_\ell]$ such that $H_s = H_{\tau_\ell}$, i.e. such that $X_s = \inf_{u \in [\tau_\ell, s]} X_u$ (there is always an infinite number of them). By definition, the mass measure of the vertices in \mathcal{T}_b at distance less than ε of b is exactly the Lebesgue measure of $\{s \in [\tau_\ell, \sigma_\ell] : \tilde{H}_{s-\tau_\ell} < \varepsilon\}$. Thus by (5.4) we can conclude that $L(b)$ defined at (5.1) exists and equals $\tilde{L}_{\sigma_\ell-\tau_\ell}^0 = x$ where \tilde{L} is the local time associated to \tilde{H} . The same argument allows to handle the second, third, ... jumps that are $> \ell$. Letting $\ell \downarrow 0$ implies that to any jump of X with magnitude x corresponds a hub of the stable tree with local time x . By excursion theory and scaling, the same property holds under N and $N^{(1)}$.

Conversely, suppose that b is a node in the stable tree. This means that there exist times $s_1 < s_2 < s_3$ such that $H_{s_1} = H_{s_2} = H_{s_3}$ and $H_s \geq H_{s_1}$ for every $s \in [s_1, s_3]$. Let

$$\tau(b) = \inf\{s \leq s_2 : H_s = H_{s_2} \text{ and } H_u \geq H_{s_2} \forall u \in [s, s_2]\}$$

and

$$\sigma(b) = \sup\{s \geq s_2 : H_s = H_{s_2} \text{ and } H_u \geq H_{s_2} \forall u \in [s_2, s]\}$$

(which are not stopping times). If $\Delta X_{\tau(b)} > 0$, we are in the preceding case. Suppose that $\Delta X_{\tau(b)} = 0$, then by the same arguments as above, $X_s \geq X_{\tau(b)}$ for $s \in [\tau(b), \sigma(b)]$, else we could find some $s' \in [\tau(b), \sigma(b)]$ such that $H_{s'} < H_{\tau(b)}$. Also, the points $s \in [\tau(b), \sigma(b)]$ such that $H_s = H_{\tau(b)}$ must then satisfy $X_s = X_{\tau(b)}$ (else there would be a strict increase of the local time of the reversed process). This implies that $X_{\tau(b)}$ is a local infimum of X , attained at s . By standard considerations, such local infima cannot be attained more than three times on the interval $[\tau(b), \sigma(b)]$, a.s. But if it was attained exactly three times, then the node would have degree 3, which is impossible according to the analysis of F^- in [80], which implies that all hubs of the stable tree have infinite degree.

Assertion (ii) follows easily from this, and (iii) comes from the fact that the points $u \in [\tau(b), \sigma(b)]$ with $H_u = H_{\tau(b)}$ are exactly those points where $\inf_{r \in [\tau(b), u]} X_r = X_u$, and the definition of the mass measure on \mathcal{T} . //

5.3.3 A second way to define F^+

We will now give some elementary properties of F^+ and rephrase its definition directly from the excursion of the underlying stable excursion X rather than the tree itself. First recall that given \mathcal{T} , we can define a marking procedure on $\mathcal{H}(\mathcal{T})$ by taking a Poisson process $(b(t), t \geq 0)$ with intensity $dt \otimes \sum_{b \in \mathcal{H}(b)} L(b) \delta_b(dv)$, and by saying that b is marked at level

t if $b \in \{b(s), 0 \leq s \leq t\}$. Let us state a useful lemma.

Lemma 5.4 :

Let $s \in [0, 1]$, and write $v(s)$ for the vertex of \mathcal{T} encoded by s . Then almost-surely,

$$\sum_{b \in \mathcal{H}(\mathcal{T}) \cap [[\text{root}, v]]} L(b) < \infty.$$

In particular, almost surely, for every hub $b \in \mathcal{H}(\mathcal{T})$ and $t \geq 0$, there is a finite number of marked hubs at level t on the path $[[\text{root}, b]]$.

// Let s be the leftmost time in $[0, 1]$ that encodes v . It follows from Proposition 5.3 (ii) (and the fact that a.s. under P , every excursion of $R^{(s)}$ below $\bar{R}^{(s)}$ ends by a jump) that the hubs b in the path $[[\text{root}, v]]$ are all encoded by the times $s' < s$ such that $\bar{R}^{(s)}$ jumps at time $s - s'$. This jump corresponds to a jump of the reversed process $R^{(s)}$, whose magnitude $\Delta R_{s-s'}^{(s)} \geq \Delta \bar{R}_{s-s'}^{(s)}$ equals $L(b)$ by Proposition 5.3 (i). Therefore, we have to show that the sum of these jumps is finite a.s. By excursion theory and time-reversal, it suffices to show that under P , letting \bar{X} be the supremum process of X ,

$$\sum_{0 \leq s' \leq s: \Delta \bar{X}_{s'} > 0} \Delta X_{s'} < \infty, \quad s \geq 0.$$

Now by excursion theory and basic fluctuation theory (see e.g. the proof of [49, Lemma 1.1.2]), after appropriate time-change, the jumps $\Delta X_{s'}$ above a previous supremum form a Poisson point process with intensity $C_\alpha x^{-\alpha} dx$, so the sum defines a time-changed subordinator, which is a.s. finite at all times. The statement on hubs follows since for any hub b encoded by a jump-time $\tau(b)$, there is a rational number $r' \in [\tau(b), \sigma(b)]$ which encodes some vertex v in the fringe subtree rooted at b . //

By definition, two vertices $v, w \in \mathcal{T}$ satisfy $v \sim_t w$ if and only if $\{b(s) : 0 \leq s \leq t\} \cap [[v, w]] = \emptyset$. Let $\mathcal{H}_t = \{b(s) : 0 \leq s \leq t\}$. For $b \in \mathcal{H}_t$, let $\mathcal{T}_b^1, \mathcal{T}_b^2, \dots$ be the connected components of $\mathcal{T}_b \setminus \{b\}$ ranked in decreasing order of total mass. We know that these trees are encoded by intervals of the form $(\tau_i(b), \sigma_i(b))$ whose union is $[\tau(b), \sigma(b)] \setminus \{u : u \equiv b\}$. Define

$$C(t, b, i) = \mathcal{T}_b^i \setminus \bigcup_{b' \in \mathcal{H}_t \cap \mathcal{T}_b^i} \mathcal{T}_{b'}$$

Plainly, $C(t, b, i)$ is an equivalence class for \sim_t for every $b \in \mathcal{H}_t$ and $i \geq 1$. By (iii) in Proposition 5.3, with obvious notations,

$$C(t, b, i) \equiv (\tau_i(b), \sigma_i(b)) \setminus \bigcup_{b' \in \mathcal{T}_b^i \cap \mathcal{H}_t} [\tau(b'), \sigma(b')].$$

We also let $C(t, \emptyset)$ be the set of vertices whose path to the root does not cross any marked hub at level t , which is equivalent to $[0, 1] \setminus \bigcup_{b \in \mathcal{H}_t} [\tau(b), \sigma(b)]$. Then $C(t, \emptyset)$ is also an equivalence class for \sim_t . A moment's thought shows that the classes $C(t, \emptyset)$

and $C(t, b, i)$ for b a hub are the only equivalence classes for \sim_t that possibly have a positive weight, so we may write $F^+(t)$ as the decreasing rearrangement of the sequence $(\mu(C(t, \emptyset)), \mu(C(t, b, i)), b \in \mathcal{H}(\mathcal{T}), i \geq 1)$. We will see later that the rest is a set of leaves of mass zero, so $\sum_i F_i^+(t) = 1$ a.s.

Let us now translate the relation \sim_t in terms of the stable excursion X under $\mathbf{N}^{(1)}$. Let $s, s' \in [0, 1]$ encode respectively the vertices $v \neq w \in \mathcal{T}$. Again by Proposition 5.3 (ii), the branchpoint $b(v, w)$ of v and w is encoded by the largest u such that the processes $(\overline{R}_{s-u+r}^{(s)}, 0 \leq r \leq u)$ and $(\overline{R}_{s'-u+r}^{(s')}, 0 \leq r \leq u)$ coincide. Let $u(s, s')$ be the jump-time of X that encodes this branchpoint. Then $v \sim_t w$ if and only if the (left-continuous) processes $(\overline{R}_{s-r}^{(s)}, u(v, w) \leq r \leq s)$ and $(\overline{R}_{s'-r}^{(s')}, u(v, w) \leq r \leq s')$ never jump at times when marked jumps at level t for X occur.

In particular, we may rewrite the equivalence classes $C(t, b, i)$ and $C(t, \emptyset)$ as follows. Let $z_1^t \geq z_2^t \geq \dots \geq 0$ be the marked jumps of X at level t under $\mathbf{N}^{(1)}$, ranked in decreasing order, and let $\tau_1^t, \tau_2^t, \dots$ the corresponding jump times (i.e. such that $\Delta Z_{\tau_i^t}^{(t)} = z_i^t$). For every i , let

$$\sigma_i^t = \inf\{s > \tau_i^t : X_s = X_{\tau_i^t-} = X_{\tau_i^t} - z_i^t\}$$

be the first return time to level $X_{\tau_i^t-}$ after time τ_i^t . Define the intervals $I_i^t = [\tau_i^t, \sigma_i^t]$ (so I_i^t / \equiv is the fringe subtree of the marked hub that has width z_i^t). Moreover, for each i , the jump with magnitude z_i^t gives rise to a family of excursions of X above its minimum. Precisely, let $(X_{i,1}^t, X_{i,2}^t, \dots)$ the sequence of excursions above its infimum of the process

$$X_i^t(s) = X_{\tau_i^t+s} - X_{\tau_i^t} \quad \tau_i^t \leq s \leq \sigma_i^t,$$

where the $(X_{i,j}^t, j \geq 1)$ are arranged by decreasing order of duration. Let also $I_{i,j}^t = [\tau_{i,j}^t, \sigma_{i,j}^t]$ be the interval in which $X_{i,j}^t$ appears in X , so that $\bigcup_j I_{i,j}^t = I_i^t$. Consider the set

$$C_{i,j}^t = I_{i,j}^t \setminus \bigcup_{k: I_k^t \subsetneq I_i^t} I_k^t.$$

By Lemma 5.4, there exists some set of indices k' such that $I_{k'}^t \subsetneq I_{i,j}^t$ and so that the $I_{k'}^t$'s are maximal with this property (else we could find an infinite number of marked hubs on a path from the root to one of the hubs encoded by the left-end of some $I_k^t \subsetneq I_{i,j}^t$). The Lebesgue measure of $C_{i,j}^t$ is thus equal to

$$|C_{i,j}^t| = \sigma_{i,j}^t - \tau_{i,j}^t - \sum (\sigma_k^t - \tau_k^t),$$

where the sum is over the k 's such that $I_k^t \subsetneq I_i^t$ and the I_k^t 's are maximal with this property. Writing $C_0^t = [0, 1] \setminus \bigcup_{i=1}^{\infty} I_i^t$, we finally get (identifying Borel subsets of $[0, 1]$ with Borel subsets of \mathcal{T}):

Lemma 5.5 :

The sets C_0^t and $C_{i,j}^t$, for $i, j \geq 1$, are a relabeling of the sets $C(t, \emptyset)$ and $C(t, b, i)$, so $F^+(t)$ is the decreasing rearrangement of the sequence $(|C_0^t|, |C_{i,j}^t|, i, j \geq 1)$.

Notice also that another consequence of Lemma 5.4 is that F^+ is continuous in probability at time 0. Indeed, as $t \downarrow 0$, the component $C(t, \emptyset)$ of the fragmented tree containing the root increases to $C(0+, \emptyset)$. Suppose $\mu(C(0+, \emptyset)) < 1$ with positive probability. Given \mathcal{T} take L_1, L_2, \dots independent with law μ . By the law of large numbers, with positive

probability a positive proportion of the L_i 's are separated from the root at time $0+$. However, as a consequence of Lemma 5.4, a.s. for every $n \geq 1$ and t small enough, there is no marked hub on the paths $[[\text{root}, L_i]], 1 \leq i \leq n$, hence a contradiction.

5.4 Study of F^+

The goal of this section is to study the fragmentation F^+ through the representation given in the last section. The first step is to study the behavior of the excursion on the equivalence classes $C_{i,j}^t$ and C_0^t defined previously.

5.4.1 Self-similarity

This section is devoted to the proof that F^+ is a self-similar fragmentation with index $1/\alpha$ and no erosion.

Let us first introduce some notation. Let $(f(x), 0 \leq x \leq \zeta)$ be a càdlàg function on $[0, \zeta]$ with $\zeta \in [0, \infty)$. By convention we let $f(x) = f(\zeta)$ for $x > \zeta$. We define the unplugging operation UNPLUG as follows. Let $([a_n, b_n], n \geq 1)$ be a sequence of disjoint closed intervals with non-empty interior, such that $0 < a_n < b_n < \zeta$ for every n . Define the increasing continuous function

$$x^{-1}(s) = s - \sum_{n \geq 1} (s \wedge b_n - a_n)^+, \quad s \geq 0,$$

where $a^+ = a \vee 0$ and where the sum converges uniformly on $[0, \zeta]$ since $\sum_n (b_n - a_n) < \zeta$. Define x as the right-continuous inverse of x^{-1} , then $f \circ x$ is càdlàg (notice that $(f \circ x)(s-) = f(x(s-)-)$), call it UNPLUG($f, [a_n, b_n], n \geq 1$) (the action of UNPLUG is to remove the bits of the path of f that are included in $[a_n, b_n]$). We say that the intervals $[a_n, b_n]$ are *separated* if $x(a_n) < x(a_m)$ for every $n \neq m$ such that $a_n < a_m$. This is equivalent to the fact that for every $n \neq m$ with $a_n < a_m$, the set $[a_n, a_m] \setminus \bigcup_i [a_i, b_i]$ has positive Lebesgue measure. Last, if we are given intervals $[a_n, b_n]$ that are not overlapping (i.e. such that $a_n < a_m < b_n < b_m$ does not happen for $n \neq m$, though we might have $[a_n, b_n] \subset [a_m, b_m]$), but such that there is a subsequence $[a_{\phi(n)}, b_{\phi(n)}], n \geq 1$ of maximal intervals that covers $\bigcup_n [a_n, b_n]$, we similarly define the unplugging operation by simply ignoring the non-maximal intervals.

Lemma 5.6 :

Let $([a_n, b_n], n \geq 1)$ be a sequence of separated intervals. Then as $N \rightarrow \infty$, UNPLUG($f, [a_n, b_n], 1 \leq n \leq N$) converges to UNPLUG($f, [a_n, b_n], n \geq 1$) in the Skorhod space.

// Define x_N^{-1} as above by truncating the sum at N . The separation of intervals ensures that every jump of x corresponds to a jump of x_N for some large N . Since $f \circ x$ is càdlàg with duration $\zeta' = \zeta - \sum_n (b_n - a_n)$, for every N we may find a sequence of times $0 = s_0 < s_1 < s_2 < \dots < s_{k(N)} = \zeta'$ such that the oscillation

$$\omega(f \circ x, [s_i, s_{i+1})) = \sup_{s, s' \in [s_i, s_{i+1}))} |f \circ x(s) - f \circ x(s')| < 2^{-N}.$$

Let also s_i^N be the corresponding times for $f \circ x_N$, that is, $s_i^N = x_N^{-1}(x(s_i))$. We build a time change λ_N (a strictly increasing continuous function) by setting $\lambda_N(s_i) = s_i^N$ for $1 \leq i \leq k(N)$, and interpolating linearly between these times. Then $|\lambda_N(s_i) - s_i| < \sum_{n>N} (b_n - a_n) \rightarrow 0$, and it follows that λ_N converges pointwise and uniformly to the identity function of $[0, \zeta']$. On the other hand, we have that $f \circ x(s_i) = f \circ x_N \circ \lambda_N(s_i)$, and for $s \in (s_i, s_{i+1})$,

$$|f \circ x_N \circ \lambda_N(s) - f \circ x(s)| \leq \omega(f \circ x, [s_i, s_{i+1})) + \sup_{n>N} (\omega(f, [a_n, b_n]) + f(a_n) - f(a_n-)).$$

We can conclude that $f \circ x_N \circ \lambda_N$ converges uniformly to $f \circ x$ since the oscillation $\omega(f, [a_n, b_n])$ converges to 0 uniformly in $n \geq N$ as $N \rightarrow \infty$, as does the jump $f(a_n) - f(a_n-)$. //

Under the law $\mathbf{P}^{(-z, \infty)}$ under which X is killed when it first attains $-z$, for every $t > 0$ we let $z_1^t \geq z_2^t \geq \dots \geq 0$ be the marked jumps of X at level t , ranked in decreasing order of magnitude, and τ_i^t be the time of occurrence of the jump with magnitude z_i^t , while σ_i^t is the first time after τ_i^t when X hits level $X_{\tau_i^t-}$ (notice that τ_i^t, σ_i^t are not stopping times). Similarly as before, we let $I_i^t = [\tau_i^t, \sigma_i^t]$.

Lemma 5.7 :

For every $z, t \geq 0$, the process $\text{UNPLUG}(X, (I_i^t : i \geq 1))$ has same law as $X^{(t)}$ under \mathbf{P} , killed when it first hits $-z$.

Part of this lemma is that it makes sense to apply the unplugging operation with the intervals I_i^t , that is, that these intervals admit a separated covering maximal subfamily.

// The fact that the intervals I_i^t admit a covering maximal subfamily is obtained by re-using the proof of Lemma 5.4 and an argument in the preceding section. Next, write $X = X^{(t)} + Z^{(t)}$. For $a > 0$, let $\tau_1^{t,a}$ be the time of the first jump of $Z^{(t)}$ that is $> a$, and let $\sigma_1^{t,a} = \inf\{u \geq \tau_1^{t,a} : X_u = X_{\tau_1^{t,a}-}\}$. Recursively, let $\tau_{i+1}^{t,a} = \inf\{u \geq \tau_i^{t,a} : \Delta Z_u^{(t)} > a\}$ and $\sigma_{i+1}^{t,a} = \inf\{u \geq \tau_{i+1}^{t,a} : X_u = X_{\tau_{i+1}^{t,a}-}\}$. Let $Z_s^{(t,a)} = \sum_{u \leq s} \Delta Z_u^{(t)} \mathbb{1}_{\{\Delta Z_u^{(t)} \leq a\}}$. The $\tau_i^{t,a}$'s are stopping times for the filtration generated by $(X^{(t)}, Z^{(t)})$, as well as the $\sigma_i^{t,a}$'s. By a repeated use of the Markov property at these times we get

$$\text{UNPLUG}(X; (I_i^t : z_i^t > a)) \stackrel{d}{=} X^{(t)} + Z^{(t,a)},$$

where this last process is killed at the time $T_z^{(t,a)}$ when it first hits $-z$. In particular, $T_z - \sum_i (\sigma_i^{t,a} - \tau_i^{t,a})$ has the same law as $T_z^{(t,a)}$, which converges in law to $T_z^{(t)}$ as $a \downarrow 0$ because $Z^{(t,a)}$ converges to 0 uniformly on compacts, and $X^{(t)}$ enters $(-\infty, -z)$ immediately after $T_z^{(t)}$ by the Markov property and the fact that 0 is a regular point for Lévy processes with infinite total variation. Therefore, writing $|I_i^t|$ for the Lebesgue measure of I_i^t , $T_z - \sum_k |I_k^t|$ (where the sum is over the I_k^t that are maximal) has same law as $T_z^{(t)}$, and in particular it is nonzero a.s. Now to check that the intervals I_i^t are separated (we are only interested by those which are maximal), consider two left-ends of such intervals such as $\tau_i^{t,a} < \tau_j^{t,a}$ (where a is small enough). The regularity of 0 for the Lévy process X implies that $\inf_{s \in [\sigma_i^{t,a}, \tau_j^{t,a}]} X_s < X_{\sigma_i^{t,a}}$, so by the

same arguments as above and the Markov property at $\sigma_i^{t,a}$, there exists a (random) $\varepsilon_{i,j}^a > 0$ such that given $\varepsilon_{i,j}^a$,

$$\tau_j^{t,a} - \sigma_i^{t,a} - \sum_{I_k^t \subset [\sigma_i^{t,a}, \tau_j^{t,a}]} |I_k^t| \mathbb{1}_{\{I_k^t \text{ maximal}\}}$$

is stochastically larger than $T_{\varepsilon_{i,j}^a}^{(t)}$. This ensures the a.s. separation of the I_k^t 's, so the a.s. convergence of $\text{UNPLUG}(X, (I_i^t : z_i^t > a))$ to $\text{UNPLUG}(X, (I_i^t, i \geq 1))$ as $a \downarrow 0$ comes from Lemma 5.6. Identifying the limiting law follows from the above discussion. //

Let now $X_i^t(s) = X_{\tau_i^t+s} - X_{\tau_i^t}$ for $0 \leq s \leq \sigma_i^t - \tau_i^t$ and $i \geq 0$, where by convention $\tau_0^t = 0$, and $\sigma_0^t = T_1$. We write $-\tau_k^t + I_k^t = [0, \sigma_k^t - \tau_k^t]$. The next lemma does most of the job to extract the different tree components of the logged stable tree at time t .

Lemma 5.8 :

- (i) Under the law $P^{(-1, \infty)}$, as $a \downarrow 0$, the processes $\text{UNPLUG}(X_i^t, (-\tau_k^t + I_k^t, k : I_k^t \subsetneq I_i^t \text{ and } z_k^t > a)), i \geq 1$ converge in \mathbb{D} to the processes $Y_i^t = \text{UNPLUG}(X_i^t, (-\tau_k^t + I_k^t, k : I_k^t \subsetneq I_i^t)), i \geq 1$.
- (ii) The process Y_i^t has the same law as $z_i^t + X^{(t)}$ under \mathbf{P} , killed when it first hits 0, and these processes are independent conditionally on $(z_i^t, i \geq 1)$.
- (iii) The sum of the durations of $Y_i^t, i \geq 1$ equals T_1 a.s.

// (i) Fix $a > 0$, we modify slightly the notations of the preceding proof by letting $\tau_1^{t,a} < \dots < \tau_{k(a)}^{t,a}$ be the times when $Z^{(t)}$ accomplishes a jumps that is $> a$, and letting $\sigma_i^{t,a} = \inf\{u \geq \tau_i^{t,a} : X_u = X_{\tau_i^{t,a}-}\}$. Let also $\tau_0^{t,a} = 0, \sigma_0^{t,a} = T_1$. Write $I_i^{t,a} = [\tau_i^{t,a}, \sigma_i^{t,a}]$, and let $X_i^{t,a}(s) = X_{\tau_i^{t,a}+s} - X_{\tau_i^{t,a}}$ for $0 \leq s \leq \sigma_i^{t,a} - \tau_i^{t,a}$. By the Markov property at times $\tau_i^{t,a}, \sigma_i^{t,a}$, we obtain that for every i , $X_i^{t,a}$ is independent of $\text{UNPLUG}(X, I_i^{t,a})$ given the jump $\Delta X_{\tau_i^{t,a}}$. By a repeated use of the Markov property, we obtain the independence of the processes $\text{UNPLUG}(X_i^{t,a}, (I_k^{t,a} : I_k^{t,a} \subsetneq I_i^{t,a}))$ given $(\Delta X_{\tau_i^{t,a}}, 1 \leq i \leq k(a))$, and moreover, the law of $X_i^{t,a}$ given $\Delta X_{\tau_i^{t,a}}$ is that of X under P , killed when it first hits $-\Delta X_{\tau_i^{t,a}}$. Letting $a \downarrow 0$ and applying Lemma 5.7 finally gives the convergence to the processes $\text{UNPLUG}(X_i^t, (I_k^t : I_k^t \subsetneq I_i^t))$, as well as the conditional independence and the distribution of the processes, giving also (ii).

(iii) Let us introduce some extra notation. Say that the marked jump with magnitude z_i^t is of the j -th kind if and only if the future infimum process $(\inf_{s \leq u \leq \tau_i^t} X_u, 0 \leq s \leq \tau_i^t)$ accomplishes exactly j jumps at times that correspond to marked jumps of X . Write $|I_i^t|$ for the duration of X_i^t and let A_j be the set of indices i such that τ_i^t is a jump time of the j -th kind. By a variation of Lemma 5.4 already used above, every marked jump is of the j -th kind for some j a.s. By Lemma 5.7 the duration of Y_0^t is $T_1 - \sum_{i \in A_1} |I_i^t|$, similarly, one has that if $i \in A_j$, the duration of Y_i^t equals $|I_i^t| - \sum_{k \in A_{j+1}} |I_k^t| \mathbb{1}_{\{I_k^t \subset I_i^t\}}$. Therefore, proving that the sum of durations of Y_i^t equals T_1 amounts to showing that $\sum_{i \in A_j} |I_i^t| \rightarrow 0$ in probability as $j \rightarrow \infty$. But the sum of the

marked jumps is finite a.s., since conditionally on a marked jump z_i^t , the duration of the corresponding X_i^t has same law as $T_{z_i^t}$, and since we have independence as i varies. Hence this sum is (conditionally on $(z_i^t, i \geq 1)$) equal in law to $T_{\sum_{i \in A_j} z_i^t}$ under P , and it converges to 0. \parallel

Lemma 5.9 :

The process $(F^+(t), t \geq 0)$ is a Markovian self-similar fragmentation with index $1/\alpha$. Its erosion coefficient is 0

\parallel For every $v > 0$, define the processes X_i^t under $\mathbf{N}^{(v)}$ as in the preceding section, replacing the duration 1 by v . By virtue of the Lemma 5.8 and by excursion theory, we obtain that for almost every $v > 0$, and for all t in a dense countable subset of \mathbb{R}_+ , under $\mathbf{N}^{(v)}$, the processes $\text{UNPLUG}(X_i^t, (k : I_k^t \subsetneq I_i^t \text{ and } z_k^t > a))$ converge as $a \downarrow 0$ to processes Y_i^t that are independent conditionally on the z_i^t 's and on their durations, and whose durations sum to v (by convention we let $X_0^t = X$). By scaling, this statement remains valid for $v = 1$. We then extend it to all $t \geq 0$ by a continuity argument. The case $t = 0$ is obvious, so take $t_0 > 0$ and $t \uparrow t_0$ in the dense subset of \mathbb{R}_+ . Almost surely, t_0 is not a time at which a new hub is marked, so $X_i^{t_0} = X_i^t$ for t close enough of t_0 , and by Lemma 5.6 and the fact that $\{I_i^t, i \geq 0\} \subset \{I_i^{t_0}, i \geq 0\}$ for $t \leq t_0$,

$$Y_i^{t_0} = \text{UNPLUG}(X_i^t, I_k^{t_0} \subsetneq I_i^{t_0}) = \lim_{t \uparrow t_0} \text{UNPLUG}(X_i^t, I_k^t \subsetneq I_i^t).$$

Now write $Y_{i,j}^t$ for the excursions of Y_i^t above its infimum, ranked in decreasing order of durations. Then by the same arguments as in the proof of Lemma 5.7, the joint law of the durations of $Y_0^t, Y_{i,j}^t, i \geq 1, j \geq 1$ equals the law of $(|C_0^t|, |C_{i,j}^t|, i \geq 1, j \geq 1)$ with notations above. Hence, by Lemma 5.5 and the fact that excursions of $X^{(t)}$ with prescribed duration are stable excursions, it holds that conditionally on $F^+(t) = (x_1, x_2, \dots)$, the excursions $Y_{i,j}^t$ are independent stable excursions with respective durations x_1, x_2, \dots .

Now let $\sim_{t'}^{t,i,j}$ be the equivalence relation defined for the excursion $Y_{i,j}^t$ in a similar way as \sim_t for the normalized excursion of X . Write also $j_t(u) = u - \sum_{k: I_k^t \subsetneq I_i^t, \sigma_k^t < u} |I_k^t|$ for $u \in [0, 1]$. Then it is clear that if $x, y \in C_{i,j}^t$, one has also $x \sim_{t+t'} y$ if and only if $j_t(x) \sim_{t'}^{t,i,j} j_t(y)$. By the scaling property, a stable excursion ε^x with duration x where every jump with magnitude ℓ is marked with probability $1 - \exp(-t'\ell)$ is obtained by taking a normalized excursion $(\varepsilon_s^1, 0 \leq s \leq 1)$, marking every jump with magnitude ℓ independently with probability $1 - \exp(t'x^{1/\alpha}\ell)$, and then letting $\varepsilon_s^x = x^{1/\alpha}\varepsilon_{s/x}^1$ for $0 \leq s \leq x$; the marked jumps of ε^x occurring at the times sx whenever s is a marked jump time for ε^1 . This means that given $F^+(t) = (x_1, \dots)$, the process $(F^+(t + t'), t' \geq 0)$ has the same law as $((x_1 F^{+,1}(x_1^{1/\alpha} t'), x_2 F^{+,2}(x_2^{1/\alpha} t'), \dots)^\downarrow, t' \geq 0)$ where the $F^{+,i}$'s are independent copies of F^+ . This entails both the Markov property and the self-similar property, the self-similarity index being $1/\alpha$. Moreover, Lemma 5.8 (iii) shows that the sum of durations of $Y_{i,j}^t$ is 1 a.s. under $\mathbf{N}^{(1)}$, so $\sum_i F_i^+(t) = 1$ a.s. and the erosion coefficient must be 0 according to [26].

To conclude, we notice that the previous result of continuity in probability of F^+ at time 0 extends to any time $t \geq 0$ by the self-similar fragmentation property. //

5.4.2 Splitting rates and dislocation measure

To complete the study of the characteristics of F^+ , we must identify the dislocation measure. This is done by computing the *splitting rate* of the stable tree, that is, the rate at which the tree with mass 1 instantaneously splits into a sequence of subtrees with given masses $s_1 \geq s_2, \dots$ with $\sum_i s_i = 1$, by analogy with the splitting rate of the Brownian CRT in [10]. Recall that our marking process on the hubs of the tree amounts to taking a Poisson process with intensity $m(dv) = \sum_b L(b)\delta_b(dv)$ on \mathcal{T} , where the sum is over hubs $b \in \mathcal{T}$. For $v \in \mathcal{T}$, let $\mathcal{T}_1(v), \mathcal{T}_2(v), \dots$ be the tree components of the forest obtained when removing v , arranged by decreasing order of masses, and let

$$r(ds) = \mathbf{E}m\{v \in \mathcal{T} : (\mu(\mathcal{T}_1(v)), \mu(\mathcal{T}_2(v)), \dots) \in ds\}$$

be the rate at which a m -picked vertex splits \mathcal{T} into trees with masses in a volume element ds . It is quite intuitive that the splitting rate equals the dislocation measure of F^+ , and Theorem 5.1 reduces to the two following lemmas:

Lemma 5.10 :

| The splitting rate $r(ds)$ equals the dislocation measure of F^+ .

// For $t \geq 0$ we let $\mathcal{T}(t)$ be the forest obtained by our logging procedure of the stable tree at time t . Let $n \geq 2$, and consider n leaves $L_1, \dots, L_n \in \mathcal{T}$ that are independent and distributed according to the mass measure μ , conditionally on μ (we are implicitly working on an enlarged probability space). Write $\Pi_n(t)$ for the partition of $[n] = \{1, \dots, n\}$ obtained by letting i and j be in the same block of $\Pi_n(t)$ if and only if L_i and L_j belong to the same tree component of $\mathcal{T}(t)$. For $K > 2$ let $\Lambda_K^n(t)$ be the event that at time t , the leaves L_1, \dots, L_n are all contained in tree components of $\mathcal{T}(t)$ with masses $> 1/K$. Write \mathcal{P}_n^* for the set of partitions π of $[n] = \{1, \dots, n\}$ with at least two non void blocks A_1, \dots, A_k (for some arbitrary ordering convention). Given $F^+(t) = \mathbf{s} = (s_1, s_2, \dots)$, the probability that $\Pi_n(t)$ equals some partition $\pi \in \mathcal{P}_n^*$ and that $\Lambda_K^n(t)$ happens is

$$G_K(\mathbf{s}) = P(\Pi_n(t) = \pi, \Lambda_K^n(t) | F^+(t) = \mathbf{s}) = \sum_{i_1, \dots, i_k}^{*K} \prod_{j=1}^k s_{i_j}^{\#A_j},$$

the sum being over pairwise distinct i_j 's such that $s_{i_j} > 1/K$. This last function is continuous and null on a neighborhood of $(1, 0, \dots)$, so Proposition 5.1 (which use is enabled by Lemma 5.9) gives

$$\lim_{t \downarrow 0} t^{-1} P(\Pi_n(t) = \pi, \Lambda_K^n(t)) = \int_S \nu(ds) \sum_{i_1, \dots, i_k}^{*K} \prod_{j=1}^k s_{i_j}^{\#A_j}. \tag{5.6}$$

We claim that knowing this quantity for every n, π, K characterizes ν . One can obtain this by first letting $K \rightarrow \infty$ by monotone convergence, and then using an argument based on exchangeable partitions as in [68, p. 378] (a Stone-Weierstrass argument can also work).

On the other hand, for any b in the set $\mathcal{B}(\mathcal{T})$ of branchpoints of \mathcal{T} , let π_n^b be the partition of $[n]$ obtained by letting i and j be in the same block if and only if b is not on the path from L_i to L_j . Let also $\mathcal{T}_{L_i}(b)$ be the tree component of the forest obtained by removing b from \mathcal{T} that contains L_i . For $K \in (2, \infty]$ and $\pi \in \mathcal{P}_n^*$, let $\Psi_K^n(\pi)$ be the set of branchpoints $b \in \mathcal{T}$ such that $\pi_n^b = \pi$ and such that $\mu(\mathcal{T}_{L_i}(b)) > 1/K$ for $1 \leq i \leq n$, and let $\Psi_K^n = \bigcup_{\pi \in \mathcal{P}_n^*} \Psi_K^n(\pi)$. Recall that we may construct the fragmentation F^+ by cutting the stable tree at the points of a Poisson point process $(b(s), s \geq 0)$ with intensity $ds \otimes m(db)$. Now for $\Pi_n(t) = \pi$ to happen, it is plainly necessary that at least one $b(s)$ falls in Ψ_∞^n for some $s \in [0, t]$, if in addition $\Lambda_K^n(t)$ happens then no $b(s), 0 \leq s \leq t$ must fall in $\Psi_\infty^n \setminus \Psi_K^n$. Therefore,

$$P(\Pi_n(t) = \pi, \Lambda_K^n(t)) = P(\exists! s \in [0, t] : b(s) \in \Psi_\infty^n, \text{ and } b(s) \in \Psi_K^n(\pi), \Lambda_K^n(t)) + R(t), \quad (5.7)$$

where the residual $R(t)$ is bounded by the probability that $b(s)$ falls in Ψ_∞^n for at least two $s \in [0, t]$. Hence $R(t) = o(t)$ by standard properties of Poisson processes provided we can show that $E[m(\Psi_\infty^n)] < \infty$. This could be shown using the forthcoming lemma, but we may also just notice that if $E[m(\Psi_\infty^n)]$ was infinite, then there would be arbitrarily many $b(s), 0 \leq s \leq t$ falling in $\Psi_\infty^n \setminus \Psi_K^n$ for some appropriately large K , and the above probability would be 0, which is impossible from the beginning of this proof and since F^+ is a self-similar fragmentation with nonzero dislocation measure (because it has erosion coefficient 0 and it is not constant). On the other hand, conditionally on the event on the right-hand side of (5.7), the $b(s), 0 \leq s \leq t$ that do not fall in Ψ_∞^n (call them $b'(s)$) form an independent Poisson point process with intensity $m(\cdot \cap \mathcal{B}(\mathcal{T}) \setminus \Psi_\infty^n)$. Therefore, the size of the tree component of the forest obtained when removing the points $b'(s), 0 \leq s \leq t$ that contains L_1 converges a.s. to 1 as $t \downarrow 0$ (so it also contains the other L_i 's for small t a.s.), as it is stochastically bigger than the component of $\mathcal{T}(t)$ containing L_1 , and since $F^+(t) \rightarrow (1, 0, \dots)$ in probability as $t \downarrow 0$. It follows that one can remove $\Lambda_K^n(t)$ from the right-hand side of (5.7), and basic properties of Poisson measures finally give $t^{-1}P(\Pi_n(t) = \pi, \Lambda_K^n(t)) \rightarrow E[m(\Psi_K^n(\pi))]$, which is equal to $\int_S r(ds) \sum_{i_1, \dots, i_k}^{*K} \prod_{j=1}^k s_{i_j}^{\#A_j}$ since L_i belongs to $B \subset \mathcal{T}$ with probability $\mu(B)$ that is equal to the Lebesgue measure of the subset of $[0, 1]$ encoding B . Identifying with (5.6) gives the claim. //

Lemma 5.11 :

| One has $r(ds) = \nu_\alpha(ds)$ with the notations of Theorem 5.1.

// We must see what is the effect of splitting \mathcal{T} at a hub b picked according to $m(dv)$. Recall that m picks a hub proportionally to its local time, and that hubs are in one-to-one correspondence with jumps of the stable excursion with duration 1. More precisely, if b is the hub that has been picked and with

the notations $\tau(b), \sigma(b)$ above, the masses of the tree components obtained when removing b are equal to the lengths of the constancy intervals of the infimum process of $(X_{\tau(b)+s} - X_{\tau(b)}, 0 \leq s \leq \sigma(b) - \tau(b))$, and the extra term $1 - (\sigma(b) - \tau(b))$. Now by Vervaat's theorem, we may suppose that the excursion is the Vervaat transform of a stable bridge and that the marked jump in the excursion corresponds to a jump $(s, \Delta X_s)$ of the bridge picked according to the σ -finite measure $\sum_{u: \Delta X_u > 0} \Delta X_u \delta_{u, \Delta X_u}(ds, dx)$. By Lemma 5.2, the bridge has the same law as $X \oplus (s, x)$, where (s, x) is independent of X , with a certain σ -finite "law" and X has law $P_{0 \rightarrow -x}^1$. Now a bridge is invariant under cyclic shift, so we may suppose without change that the extra jump with size x occurs at time 0. This shows that the sizes of the components of the split CRT have the same law as the sequence constituted of $1 - T_x$ and the lengths of the constancy intervals of the infimum process of $(X_u, 0 \leq u \leq T_x)$, under the law $P_{0 \rightarrow -x}^1$.

It is now easy that conditionally on $x, T_x = t$ these constancy intervals have the same law as $\Delta T_{[0,x]}$ given $T_x = t$ under P (one checks that $(X_u, 0 \leq u \leq T_x)$ is the first-passage bridge with law $P_{0 \downarrow -x}^t$ below). The law of $1 - T_x$ given x is simply obtained by using the definition of bridges and the Markov property: for $a < 1$ and positive measurable f ,

$$\begin{aligned} E_{0 \rightarrow -x}^1[f(1 - T_x) \mathbb{1}_{\{T_x < a\}}] &= E^1[f(1 - T_x) \mathbb{1}_{\{T_x < a\}} p_1(-x)^{-1} p_{1-a}(-x - X_a)] \\ &= \int_0^a ds q_x(s) f(1 - s) p_1(-x)^{-1} \int dy p_{a-s}(y) p_{1-a}(-y) \\ &\xrightarrow{a \rightarrow 1} \int_0^1 ds q_x(s) f(1 - s) p_{1-s}(0) p_1(-x)^{-1}. \end{aligned}$$

In the last integral, change variables $1 - s \rightarrow s$, use $p_1(-x) = x^{-1} q_x(1)$, check by scaling that $p_s(0) = s^{-1/\alpha} p_1(0)$, and conclude by identifying with Lemma 5.1 that $1 - T_x$ under $P_{0 \rightarrow -x}^1$ has same law as a size-biased pick from $\Delta T_{[0,x]}$ given $T_x = 1$ under P (notice that in particular we must have $p_1(0) = c_\alpha$). By Lemma 5.1 (ii), it follows that given the local time x of the marked hub b , the law of the sizes of the stable tree split at this hub is the same as that of $\Delta T_{[0,x]}$ given $T_x = 1$ under P .

Putting pieces together and recalling the distribution of the marked jump x from Lemma 5.2 we obtain the formula

$$r(ds) = \int_0^\infty dx \frac{C_\alpha p_1(-x)}{x^\alpha p_1(0)} P(\Delta T_{[0,x]} \in ds | T_x = 1).$$

By using the scaling property for T and its density ($q_x(1) = x^{-\alpha} q_1(x^{-\alpha})$), formula (5.2) and a change of variables, we obtain

$$\begin{aligned} r(ds) &= \int_0^\infty dx \frac{C_\alpha q_1(x^{-\alpha})}{c_\alpha x^{2\alpha+1}} P(T_1^{-1} \Delta T_{[0,1]} \in ds | T_1 = x^{-\alpha}) \\ &= \alpha^{-1} c_\alpha^{-1} C_\alpha \int_0^\infty du u q_1(u) P(T_1^{-1} \Delta T_{[0,1]} \in ds | T_1 = u), \end{aligned}$$

which gives the desired formula, after checking that $\alpha^{-1} c_\alpha^{-1} C_\alpha = D_\alpha$. //

5.5 Study of F^{\natural}

Recall the construction of F^{\natural} from Sect. 5.1. As noticed above, this fragmentation process somehow generalizes the one considered in [21, 79] (we could actually build it in an analogous way for a large class of Lévy processes with no negative jumps, though the resulting fragmentations would not be self-similar due to the absence of scaling). Notice that none of the fragmentations of [79] are self-similar, but for the Brownian case. The reason for this was a lack of a Girsanov-type theorem saying that a Lévy process plus drift has a law that is absolutely continuous with the initial process, but for the Brownian case. Here, this is fixed by Lemma 5.2, but where the operation is removing jumps rather than adding a drift.

5.5.1 The self-similar fragmentation property

For any $t' > t \geq 0$ let $\mu_t(x, ds)$ be a kernel from \mathbb{R}_+^* to S defined as follows: $\mu_t(x, ds)$ is the law of the ranked lengths of the constancy intervals of the process $\underline{X}^{(t)}$ under $\mathbf{N}^{(x)}$. Moreover, define $F^{\natural,1}$ exactly as F^{\natural} , but where X is under the law $\mathbf{P}^{(-1,\infty)}$. In particular, $F^{\natural,1}(t)$ is not S -valued (the sum of its components is random).

Proposition 5.4 :

- (i) The processes $F^{\natural,1}$ and F^{\natural} enjoy the fragmentation property, with fragmentation kernel $\mu_t(x, ds)$. That is, conditionally on $F^{\natural,1}(t) = (x_1, x_2, \dots)$ (resp. $F^{\natural}(t)$), $F^{\natural,1}(t+t')$ (resp. $F^{\natural}(t+t')$) has the same law as the decreasing rearrangement of independent sequences s_i with respective laws $\mu_{t'}(x_i, ds)$.
- (ii) The process F^{\natural} is a self-similar fragmentation with index $1/\alpha$, and no erosion.

The fact that F^{\natural} is a fragmentation process directly comes from the fact that the processes $X^{(t')} - X^{(t)} = Z^{(t)} - Z^{(t')}$ are non-increasing. We now prove the fragmentation property. The key lies in a Skorokhod-like relation that is analogous to that in [21] and generalized in [79].

Lemma 5.12 :

For every $t, t' \geq 0$ with $t < t'$ and $s \geq 0$, one has

$$\underline{X}_s^{(t)} = \inf_{0 \leq u \leq s} (\underline{X}_u^{(t')} + (Z_u^{(t')} - Z_u^{(t)})).$$

The proof can be done following exactly the same lines as in [21, Lemma 2]. As a consequence, we obtain that the sigma-field $\mathcal{G}_t = \sigma\{X^{(t)}, (Z^{(s)}, 0 \leq s \leq t)\}$ induces a filtration, with respect to which $F^{\natural,1}$ is adapted.

The end of the proof of the fragmentation property in Proposition 5.4 also goes as in [21]. For any variable K that is \mathcal{G}_t -measurable, the excursions of $X^{(t)}$ above its infimum and before time $T_K^{(t)}$ are independent excursions conditionally on \mathcal{G}_t , respectively conditioned to have durations $\ell_{1,X}^{(t)}, \ell_{2,X}^{(t)}, \dots$ where the last family is the decreasing sequence of constancy intervals of $\underline{X}^{(t)}$ before time $T_K^{(t)}$. Take $K = \underline{X}_{T_1}^t$, which is measurable with respect to \mathcal{G}_t by virtue of the Skorokhod property. Then $T_K^{(t)} = T_1$, which gives readily that conditionally on \mathcal{G}_t , the excursions of $X^{(t)}$ above $\underline{X}^{(t)}$ are independent with durations $(F_i^{\natural,1}(t), i \geq 0)$.

To conclude, it remains to notice that the lack of memory of the exponential law implies that the jumps that are unmarked at time t but that are marked at time $t+t'$ can be obtained also by marking with probability $1 - e^{-t'\ell}$ any unmarked jump at time t that

has magnitude ℓ . Thus, conditionally on $F^{\natural,1}(t)$, we obtain a sequence with the same law as $F^{\natural,1}(t + t')$ by taking independent sequences $(s_i, i \geq 1)$ with laws $\mu_{t'}(F_i^{\natural,1}(t), ds)$ and rearranging, as claimed. This remains true for F^{\natural} by excursion theory and scaling.

To show the self-similarity for F^{\natural} , it then suffices to check, using the scaling property of the excursions of stable processes, that $\mu_t(x, ds)$ is the image of $\mu_{tx^{1/\alpha}}(1, ds)$ by $s \mapsto xs$. The fact that F^{\natural} has no erosion again comes from the fact that $\sum_i F_i^{\natural}(t) = 1$ a.s.

5.5.2 The semigroup

According to the preceding section, and since plainly there is no loss of mass in the fragmentation F^{\natural} (so the erosion coefficient is 0), proving Theorem 5.2 requires only to check that the dislocation measure of F^{\natural} equals that of F^+ . It is intuitively straightforward that this is the case, by looking at the procedure we use for deleting jumps, and indeed we could easily follow the same lines as above and compute a “splitting rate” for the bridge, when the “first” marked jump is deleted. However, a nice feature of this fragmentation is that we can compute explicitly its semigroup (hence that of F^+), as will follow. The semigroup then gives enough information to re-obtain the dislocation measure, and this will prove Theorem 5.2. Recall from Sect. 5.2 that $\rho_1^{(t)}$ is the density of $Z_1^{(t)}$ under \mathbf{P} .

Proposition 5.5 :

The semigroup of F^{\natural} is given by

$$P(F^{\natural}(t) \in ds) = \int_0^\infty dz \frac{p_1^{(t)}(-z)\rho_1^{(t)}(z)}{p_1(0)} P(\Delta T_{[0,z]} \in ds | T_z = 1).$$

We will need a couple of intermediate lemmas. Since $Z^{(t)}$ is non-decreasing, under the law $\mathbf{N}^{(1)}$, the process $X^{(t)}$ starts at 0 and hits $-Z_1^{(t)}$ at time 1 for the first time. Since we are interested in the constancy intervals of $\underline{X}^{(t)}$, and thanks to Vervaat’s theorem, we would like to relate these constancy intervals to the bridge of X . We now work under the law of the bridge with unit duration $\mathbf{P}_{0 \rightarrow 0}^1$, so we may suppose that the excursion of X with duration 1 is equal to the Vervaat transform VX . Let $m = -\underline{X}_1$ be the absolute value of the minimum of X , and $\tau_2 = T_{m-}$ be the (a.s. unique) time when X attains this minimum, so $VX = V^{\tau_2}X$. Decompose X as $X^{(t)} + Z^{(t)}$ where $Z^{(t)}$ is the cumulative process of marked jumps. Then $VX = V^{\tau_2}X^{(t)} + V^{\tau_2}Z^{(t)}$, and by independence of the marking procedure of jumps we can consider that $V^{\tau_2}Z^{(t)}$ is the cumulative process of marked jumps for the excursion VX . The problem is now to describe the law of lengths of the constancy intervals of the process $\underline{V^{\tau_2}X^{(t)}}$. Let $m^{(t)} = -\underline{X}_1^{(t)}$ be the absolute value of the minimum of $X^{(t)}$ and $\tau_3 = T_{m^{(t)}-}^{(t)}$ be the (a.s. unique) time when $X^{(t)}$ attains this minimum. Let also $\tau_1 = T_{m^{(t)}-Z_1^{(t)}}^{(t)}$ be the first time when $X^{(t)}$ attains the value $Z_1^{(t)} - m^{(t)}$. The following lemma is somehow “deterministic”. For $a < b$, write $X_{[a,b]}$ for the process $(X_{a+s} - X_a, 0 \leq s \leq b - a)$.

Lemma 5.13 :

One has $\tau_1 \leq \tau_2 \leq \tau_3$ a.s., and the sequence of lengths of the constancy intervals of $\underline{V^{\tau_2}X^{(t)}}$, ranked in decreasing order, are equal to that of the process $\underline{X}_{[\tau_1, \tau_3]}^{(t)}$, to which has been added (at the appropriate rank) the extra term $1 - \tau_3 + \tau_1$.

// Since $Z^{(t)}$ is an increasing process, one has $X_{\tau_2}^{(t)} = X_{\tau_2} - Z_{\tau_2}^{(t)} \leq X_s - Z_s^{(t)}$

for any $s \leq \tau_2$. Hence, $X_{\tau_2}^{(t)} = \underline{X}_{\tau_2}^{(t)}$ which implies $\tau_2 \leq \tau_3$. On the other hand, one has $-m(t) = X_{\tau_3} - Z_{\tau_3}^{(t)} \geq -m - Z_1^{(t)}$ and thus $m(t) - Z_1^{(t)} \leq m$, implying $\tau_1 \leq \tau_2$.

For convenience, if $(f(x), 0 \leq x \leq \zeta)$ and $(f'(x), 0 \leq x \leq \zeta')$ are two càdlàg functions, we let $f \bowtie f'$ be the concatenation of the paths of f and f' , defined by

$$f \bowtie f'(s) = \begin{cases} f(s) & \text{if } 0 \leq s < \zeta \\ f'(s - \zeta) + f(\zeta) & \text{if } \zeta \leq s \leq \zeta + \zeta' \end{cases} .$$

We let $Y^1 = X_{[0, \tau_2]}^{(t)}$, $Y^2 = X_{[\tau_2, \tau_3]}^{(t)}$ and $Y^3 = X_{[\tau_3, 1]}^{(t)}$, so $X^{(t)} = Y^1 \bowtie Y^2 \bowtie Y^3$, and $V^{\tau_2} X^{(t)} = Y^2 \bowtie Y^3 \bowtie Y^1$.

Observing that Y_3 is non-negative, we obtain that $\underline{Y^2 \bowtie Y^3} = \underline{Y^2} \bowtie \mathbf{0}_{[0, 1 - \tau_3]}$ where $\mathbf{0}_{[0, a]}$ is the null process on $[0, a]$. Since the final value of Y_3 is $m(t) - Z_1^{(t)}$, we obtain that

$$\underline{V^{\tau_2} X^{(t)}} = \underline{Y^2} \bowtie \mathbf{0}_{[0, 1 - \tau_3]} \bowtie \mathbf{0}_{[0, \tau_1]} \bowtie \underline{X_{[\tau_1, \tau_2]}^{(t)}} = \underline{Y^2} \bowtie \mathbf{0}_{[0, 1 - \tau_3 + \tau_1]} \bowtie \underline{X_{[\tau_1, \tau_3]}^{(t)}} .$$

It follows that the constancy intervals of $\underline{V^{\tau_2} X^{(t)}}$ are the same as those of $\underline{X^{(t)}}$, except for the first and last constancy intervals of $\underline{X^{(t)}}$ which are merged to form the constancy interval with length $1 - \tau_3 + \tau_1$. //

The rest of the section is devoted to the study of these constancy intervals. Recall from Lemma 5.3 that under $\mathbf{P}_{0 \rightarrow 0}^1$, the process $X^{(t)}$ has law $P_{0 \rightarrow -z}^1$, where \mathcal{Z} is an independent random variable with law $P(\mathcal{Z} \in dz) = p_1(0)^{-1} p_1^{(t)}(-z) \rho_1^{(t)}(z) dz$. It thus suffices to analyze the constancy intervals of $\underline{X_{[\tau_1, \tau_3]}^{(t)}}$ under the law $P_{0 \rightarrow -z}^1$ for fixed $z > 0$, where we now call $m = -\underline{X}_1$, τ_1 the time when X first hits level $z - m$ and τ_3 the first time when X attains level $-m$.

For $z > 0$, let $(P_{0 \downarrow -z}^v, v > 0)$ be a regular version of the conditional law $P^{-z, \infty}[\cdot | T_z = v]$. Call this the law of the first-passage bridge from 0 to $-z$ with length v . A consequence of the Markov property is

Lemma 5.14 :

Let $a, b > 0$. For (Lebesgue) almost every $v > 0$, under the law $P_{0 \downarrow -(a+b)}^v$, the law of T_a is given by

$$P_{0 \downarrow -(a+b)}^v(T_a \in ds) = ds \frac{q_a(s) q_b(v - s)}{q_{a+b}(v)} .$$

Moreover, conditionally on T_a , the paths $(X_s, 0 \leq s \leq T_a)$ and $(X_{s+T_a} - a, 0 \leq s \leq T_{a+b} - T_a)$ are independent with respective laws $P_{0 \downarrow -a}^{T_a}$ and $P_{0 \downarrow -b}^{v - T_a}$.

We also state a generalization of Williams' decomposition of the excursion of Brownian motion at the maximum, given in Chaumont [42]. We need to make a step out of the world of probability and consider σ -finite measures instead of probability laws. Recall that $m_v = -\underline{X}_s$ is the absolute value of the minimum before time s , and with our notations $T_{m(v)-}$ is the first time (and a.s. last before v) when X attains this value. Write

$$\begin{aligned} \underline{X}_s &= X_s & 0 \leq s \leq T_{m(v)-}, \\ \underline{X}_s &= m_v + X_{s+T_{m(v)-}} & 0 \leq s \leq v - T_{m(v)-} \end{aligned}$$

for the pre- and post- minimum processes of X before time v . Then by [42],

Lemma 5.15 :

One has the identity for σ -finite measures

$$\int_0^\infty dv P^v(\underline{X} \in d\omega, \underline{X} \in d\omega') = \int_0^\infty dx P^{(-x, \infty)}(d\omega) \otimes \int_0^\infty du N^{>u}(d\omega'),$$

where $N^{>u}(d\omega') := N^u(d\omega', \zeta > u)$. This in turn determines entirely the laws P^v for $v > 0$.

Loosely speaking, if v is “random” with “law” the Lebesgue measure on $(0, \infty)$, the pre- and post- minimum processes are independent with respective “laws” $\int_0^\infty dx P^{(-x, \infty)}(d\omega)$ and $\int_0^\infty du N^{>u}(dX')$, where $N^{>u}$ is the finite measure characterized by $N^{>u}(F(X)) = N(F(X_s, 0 \leq s \leq u), \zeta(X) > u)$. As a consequence of this identity, we have that under P^v for some fixed $v > 0$, conditionally on m_v and $T_{m(v)-} = \tau$, the processes \underline{X} and \underline{X} are independent with respective laws $P_{0 \downarrow -m(v)}^\tau(d\omega)$ and $(N^{>v-\tau}(1))^{-1} N^{>v-\tau}(d\omega')$.

Lemma 5.16 :

Let $z > 0$. Under the probability $P_{0 \rightarrow -z}^1$, conditionally on $\tau_3 - \tau_1 = t$, the ranked sequence of lengths of the constancy intervals of the infimum process of $(X_{s+\tau_1}, 0 \leq s \leq \tau_3 - \tau_1)$ have the same law as $\Delta T_{[0, z]}$ given $T_z = t$ under P .

// We first condition by the value of (m, τ_3) . Then by Lemma 5.15 the path \underline{X} has the law $P_{0 \downarrow -m}^{\tau_3}$ of the first-passage bridge from 0 to $-z$ with lifetime τ_3 . Applying Lemma 5.14 and the Markov property we obtain that conditionally on τ_1 the path $(X_{s+\tau_1} + m - z, 0 \leq s \leq \tau_3 - \tau_1)$ is a first passage bridge ending at $-z$ at time $\tau_3 - \tau_1$. Since it depends only on $\tau_3 - \tau_1$, we have obtained the conditional distribution given $\tau_3 - \tau_1$. Hence, the sequence defined in the lemma’s statement has the same conditional law as the ranked lengths of the constancy intervals of the infimum process of such a first-passage bridge, that is, it has the same law as $\Delta T'_{[0, z]}$ given $T'_z = \tau_3 - \tau_1$, with T' as in the statement. //

The last lemma gives an explicit form for the law of the remaining length $1 - \tau_3 + \tau_1$ under $P_{0 \rightarrow -z}^1$.

Lemma 5.17 :

One has

$$P_{0 \rightarrow -z}^1(1 - \tau_3 + \tau_1 \in ds) = ds \frac{c_\alpha z q_z (1 - s)}{s^{1/\alpha} q_z(1)},$$

which is the law of a size-biased pick of the sequence $\Delta T_{[0, z]}$ given $T_z = 1$ under P .

// By Lemma 5.15, if s is “distributed” according to Lebesgue measure on \mathbb{R}_+ , then under P^s , the processes \underline{X} and \underline{X} are independent with respective “laws” $\int_0^\infty dx P^{(-x, \infty)}(d\omega)$ and $\int_0^\infty du N^{>u}(d\omega')$. Our first task is to disintegrate these laws to obtain a relation under $P_{0 \rightarrow -z}^1$. Let H and H' be two continuous bounded functionals and f be continuous with a compact support

on $(0, \infty)$. Then, letting $T_x^\omega = \inf\{s \geq 0 : \omega(s) < \cdot\}$, we have

$$\begin{aligned} & \int_0^\infty ds f(s) E^s [H(\underline{X}) H'(\underline{X}) \mid |X_s + z| < \varepsilon] \\ &= \int_0^\infty dx \int_0^\infty du \iint P^{(-x, \infty)}(d\omega) N^{>u}(d\omega') f(T_x^\omega + u) H(\omega) H'(\omega') \frac{\mathbb{1}_{\{|z-x+\omega'(u)| < \varepsilon\}}}{P(|X_{T_x^\omega+u} + z| < \varepsilon)} \end{aligned}$$

which equals

$$\int_0^\infty du \int N^{>u}(d\omega') H'(\omega') \int_0^\infty dx \frac{\mathbb{1}_{\{|z-x+\omega'(u)| < \varepsilon\}}}{2\varepsilon} \int P^{(-x, \infty)}(d\omega) \frac{f(T_x^\omega + u) H(\omega)}{(2\varepsilon)^{-1} P(|X_{T_x^\omega+u} + z| < \varepsilon)}.$$

The measure $(2\varepsilon)^{-1} \mathbb{1}_{\{|z-x+\omega'(u)| < \varepsilon\}} dx$ converges weakly as $\varepsilon \rightarrow 0$ to the Dirac mass at $z + \omega'_u$. Recall that the family of probability measures $P^{(-x, \infty)}$ is continuous as x varies. Since f has compact support, we can restrain $T_x^\omega + u$ to stay in a compact set. Then, the denominator in the last integral, which converges to $p_{T_x^\omega+u}(-z)$, remains bounded and converges uniformly in x and u . Then the boundedness of H implies that the two last integrals converge to

$$\int P^{(-z-\omega'_u, \infty)}(d\omega) \frac{f(T_{z+\omega'(u)}^\omega + u)}{p_{T_x^\omega+u}(-z)}.$$

Now, the measure $N^{>u}$ is a finite measure, so the fact that u actually stays in a compact set and the fact that the two last integrals above remain bounded allow to apply the dominated convergence theorem to obtain

$$\begin{aligned} & \int_0^\infty ds f(s) P_{0 \rightarrow -z}^s (H(\underline{X}) H'(\underline{X})) \\ &= \int_0^\infty du \int N^{>u}(d\omega') H'(\omega') \int P^{(-z-\omega'(u), \infty)}(d\omega) \frac{f(T_{z+\omega'(u)}^\omega(\omega) + u)}{p_{T_x^\omega+u}(-z)} \end{aligned}$$

Now we disintegrate this relation by taking $f(s) = (2\varepsilon)^{-1} \mathbb{1}_{[1-\varepsilon, 1+\varepsilon]}(s)$, so a similar argument as above gives that the left hand side converges to $P_{0 \rightarrow -z}^1(H(\underline{X}) H'(\underline{X}))$ as $\varepsilon \downarrow 0$, whereas the right hand side is

$$\int_0^\infty du \int N^{>u}(d\omega') H'(\omega') \int P^{(-z-\omega'(u), \infty)} H(\omega) \frac{\mathbb{1}_{[1-\varepsilon, 1+\varepsilon]}(T_{z+\omega'(u)}^\omega + u)}{2\varepsilon p_{T_{z+\omega'(u)}^\omega+u}(-z)}.$$

The third integral may be rewritten as

$$\frac{P(|T_{z+\omega'(u)}^\omega + u - 1| < \varepsilon)}{2\varepsilon} E^{(-z-\omega'(u), \infty)} \left[\frac{H(\omega)}{p_{T_{z+\omega'(u)}^\omega+u}(-z)} \mid |T_{z+\omega'(u)}^\omega + u - 1| < \varepsilon \right],$$

with a slightly improper writing (the ω 's should not appear in the expectation, but we keep them to keep the distinction with the expectation with respect to ω'). Similar arguments as above imply that the limit we are looking for is

$$P_{0 \rightarrow -z}^1(H(\underline{X}) H'(\underline{X})) = p_1(-z)^{-1} \int_0^1 du N^{>u} \left[H'(\omega') q_{z+\omega'(u)}(1-u) E_{0 \downarrow -(z+\omega'(u))}^{1-u} [H(\omega)] \right].$$

This in turn completely determines the law of the bridge by a monotone class argument. A careful application of the above identity thus gives

$$E_{0 \rightarrow -z}^1[f(1 - (\tau_3 - \tau_1))] = p_1(-z)^{-1} \int_0^1 du N^{>u} \left[q_{z+\omega'(u)}(1-u) E_{0 \downarrow -(z+\omega'(u))}^{1-u}[f(u + T_{\omega'(u)}^\omega)] \right].$$

Applying Lemma 5.14 to the rightmost expectation term, this is equal to

$$\begin{aligned} & p_1(-z)^{-1} \int_0^1 du N^{>u} \left[q_{z+\omega'(u)}(1-u) \int_0^{1-u} dv \frac{q_{\omega'(u)}(v) q_z(1-u-v)}{q_{\omega'(u)+z}(1-u)} f(u+v) \right] \\ &= p_1(-z)^{-1} \int_0^1 du \int_u^1 ds f(s) q_z(1-s) N^{>u} [q_{\omega'(u)}(s-u)] \\ &= z q_z(1)^{-1} \int_0^1 ds f(s) q_z(1-s) \int_0^s du N^{>u} [q_{\omega'(u)}(s-u)] \end{aligned}$$

It remains to compute the second integral. Using scaling identities for $N^{>u}$ and $q_x(s)$ we have

$$\begin{aligned} \int_0^s du N^{>u} [q_{\omega'(u)}(s-u)] &= \int_0^1 dr N^{>sr} [q_{\omega'(sr)}(s(1-r))] \\ &= s^{-1/\alpha} \int_0^1 dr s^{1/\alpha} N^{>sr} [q_{s^{-1/\alpha} \omega'(sr)}(1-r)] \\ &= s^{-1/\alpha} \int_0^1 dr N^{>r} [q_{\omega'(r)}(1-r)], \end{aligned}$$

where the last equality is an easy consequence of the scaling property. Finally, the integral in the right hand side does not depend on s , we call it c and obtain

$$E_{0 \rightarrow -z}^1[f(1 - (\tau_3 - \tau_1))] = \int_0^1 ds f(s) \frac{cz q_z(1-s)}{s^{1/\alpha} q_z(1)}.$$

So we necessarily have $c = c_\alpha$, and the claim follows. //

The proof of Proposition 5.5. is now easily obtained by combining the last lemmas:

// Under $\mathbf{P}_{0 \rightarrow 0}^1$, conditionally on $Z_1^{(t)} = z$, the law of the lengths of constancy intervals of $\underline{V}^{\tau_2} \underline{X}^{(t)}$ is obtained by adjoining the term $1 - (\tau_3 - \tau_1)$ to a sequence which, conditionally on $1 - (\tau_3 - \tau_1) = t$, has same law as $\Delta T_{[0,z]}$ given $T_z = 1 - t$ under P (Lemma 5.16). By Lemma 5.17, $1 - (\tau_3 - \tau_1)$ has itself the law of a size-biased pick from $\Delta T_{[0,z]}$ given $T_z = 1$ under P , so Lemma 5.1 shows the whole sequence has the law of $\Delta T_{[0,z]}$ given $T_z = 1$. Last, by Lemma 5.3, $Z_1^{(t)}$ has density $p_1^{(t)}(-z) \rho_1^{(t)}(z) p_1(0)^{-1} dz$, entailing the claim. //

5.5.3 Proof of Theorem 5.2

To recover the dislocation measure of F^\natural , we use the following variation of Proposition 5.1 and [82, Corollary 1]. For details on size-biased versions of measures on S , see e.g. [45], which deals with probability measures, but the results we mention are easily extended to σ -finite measures.

Proposition 5.6 :

Let $(F(t), t \geq 0)$ be a ranked self-similar fragmentation with characteristics $(\beta, 0, \nu)$, $\beta \geq 0$. For every t , let $F_*(t)$ be a random size-biased permutation of the sequence $F(t)$ (defined on a possibly enlarged probability space). Let G be a continuous bounded function on the set of non-negative sequences with sum ≤ 1 , depending only on the first I terms of the sequence, with support included in a set of the form $\{s_i \in [\eta, 1 - \eta], 1 \leq i \leq I\}$. Then

$$\frac{1}{t} E[G(F_*(t))] \xrightarrow[t \downarrow 0]{} \nu_*(G),$$

where ν_* is the size-biased version of ν characterized by

$$\nu_*(G) = \int_S \nu(ds) \sum_{j_1, \dots, j_I} G(s_{j_1}, \dots, s_{j_I}) s_{j_1} \frac{s_{j_2}}{1 - s_{j_1}} \cdots \frac{s_{j_I}}{1 - s_{j_1} - \dots - s_{j_{I-1}}},$$

where the sum is on all possible distinct j_1, \dots, j_I . Moreover, ν can be recovered from ν_* .

We are now able to prove Theorem 5.2:

// Let G be a function of the form $G(x) = f_1(x_1) \dots f_I(x_k)$ for $x = (x_1, x_2, \dots)$ and $\sum_i x_i \leq 1$, with f_1, \dots, f_I continuous bounded functions on $[0, 1]$ that are null on a set of the form $[0, 1] \setminus [\eta, 1 - \eta]$. Let $\Delta^* T_{[0, z]}$ be the sequence of the jumps of T on the interval $[0, z]$, listed in size-biased order (which involves some enlargement of the probability space). Using Lemma 5.1, it is easy that $z \mapsto E[G(\Delta^* T_{[0, z]}) | T_z = 1]$ is a continuously differentiable function with derivative bounded by some $M > 0$. Let also $F_*^+(t)$ be the sequence $F^+(t)$ listed in size-biased order. Now by Proposition 5.5,

$$E \left[\frac{G(F_*^+(t))}{t} \right] = \frac{1}{t} E \left[e^{-t^\alpha + tZ_1^{(t)}} p_1(-Z_1^{(t)}) p_1(0)^{-1} E \left[G(\Delta T_{[0, Z_1^{(t)}]}^*) \middle| T_{Z_1^{(t)}} = 1 \right] \right].$$

Consider a function $f(t, z)$ that is continuous in t and x and null at $(t, 0)$ for every $t \geq 0$. Then the compensation formula applied the subordinator $Z^{(t)}$ between times 0 and 1 gives

$$\begin{aligned} \frac{1}{t} E[f(t, Z_1^{(t)})] &= \frac{1}{t} \int_0^1 dx \int C_\alpha (1 - e^{-ts}) s^{-\alpha-1} ds E[f(t, Z_x^{(t)} + s) - f(t, Z_x^{(t)})] \\ &\xrightarrow[t \rightarrow 0]{} C_\alpha \int_0^1 dx \int s^{-\alpha} ds f(0, s) = C_\alpha \int s^{-\alpha} ds f(0, s), \end{aligned}$$

as soon as we may justify the convergence above. Take

$$f(t, z) = \exp(-t^\alpha + tz) p_1(-z) p_1(0)^{-1} E[G(\Delta T_{[0, z]}^*) | T_z = 1],$$

then we have to check that $s^{-\alpha}E[|f(t, Z_x^{(t)} + s) - f(t, Z_x^{(t)})|]$ is bounded independently on $x \in [0, 1]$. By the hypotheses on G , it is again true that $z \mapsto f(t, z)$ is a continuously differentiable function with uniformly bounded derivative, when t stays in a neighborhood of 0. Hence the expectation above is bounded by $(M's \wedge M'')s^{-\alpha}$ for some $M', M'' > 0$, which allows to apply the dominated convergence theorem. By Proposition 5.6, we obtain that

$$\begin{aligned} t^{-1}E[G(F_*^+(t))] &\xrightarrow{t \rightarrow 0} \int_S \nu_+(ds)G(s_{j_1}, \dots, s_{j_I}) \sum_{j_1, \dots, j_I} s_{j_1} \frac{s_{j_2}}{1 - s_{i_1}} \cdots \frac{s_{j_I}}{1 - s_{j_1} - \dots - s_{j_{I-1}}} \\ &= C_\alpha \int_0^\infty ds \frac{s^{-\alpha} p_1(-s)}{p_1(0)} E[G(\Delta T_{[0,s]}^*) | T_s = 1], \end{aligned}$$

allowing to conclude with the same computations as in the proof of Lemma 5.11. //

5.6 Asymptotics

5.6.1 Small-time asymptotics

Proposition 5.7 :

Let $(Z_x, x \geq 0)$ be a stable subordinator with Laplace exponent $\alpha\lambda^{\alpha-1}$. Denote by $\Delta_1, \Delta_2, \dots$ the ranked jumps of $(T_x, 0 \leq x \leq Z_1)$, where T is as before the stable $1/\alpha$ subordinator, which is taken independent of Z . Then

$$t^{\alpha/(1-\alpha)}(1 - F_1^+(t), F_2^+(t), F_3^+(t), \dots) \xrightarrow[t \rightarrow 0^+]{d} (T_{Z_1}, \Delta_1, \Delta_2, \dots).$$

// Notice that the limiting sequence differs from the jumps of $(T_{Z_x}, 0 \leq x \leq 1)$, which by Bochner's subordination is a stable $1 - 1/\alpha$ subordinator. However, the first component is equal in law to σ_1 where σ is a stable $1 - 1/\alpha$ subordinator. We first need the

Lemma :

Let $Z_1^{(t)}$ have the law $\rho_1^{(t)}(s)ds$ above, then

$$t^{1/(1-\alpha)}T_1^{(t)} \xrightarrow[t \rightarrow 0^+]{d} Z_1.$$

// Recall that $Z^{(t)}$ is a subordinator with characteristic exponent given by

$$\mathbf{E}[e^{-\lambda Z_1^{(t)}}] = \exp\left(-\int_0^\infty \frac{C_\alpha(1 - e^{-tx})dx}{x^{\alpha+1}}(1 - e^{-\lambda x})\right).$$

Therefore, evaluating the Laplace exponent at the point $t^{1/(1-\alpha)}\lambda$, changing variables and using dominated convergence entails

$$\mathbf{E}[\exp(-\lambda t^{1/(1-\alpha)} Z_1^{(t)})] \xrightarrow[t \rightarrow 0^+]{d} \exp\left(-\int_0^\infty \frac{C_\alpha dy}{y^\alpha}(1 - e^{-\lambda y})\right).$$

Thus the convergence to some limiting Z_1 . Using now the explicit value for C_α , we see that the Laplace exponent of Z_1 has to be $\alpha\lambda^{\alpha-1}$, as claimed. \parallel

The proof of Proposition 5.7 follows the same lines as for Proposition 6 in [80], so we will only sketch it. One first begins with proving that if \mathcal{Z} is as in Lemma 5.3 a random variable distributed according to the law that has density $\rho_1^{(t)}(z)p_1^{(t)}(-z)dz/p_1(0)$, then $t^{1/(1-\alpha)}\mathcal{Z}$ converges in law to Z_1 . This is a consequence of the preceding lemma, since as $t \rightarrow 0$, $X^{(t)}$ converges to X , so one can write

$$\mathbf{E}[g(t^{1/(1-\alpha)}\mathcal{Z})] = \mathbf{E}[g(t^{1/(1-\alpha)}Z_1^{(t)})p_1^{(t)}(-Z_1^{(t)})/p_1(0)],$$

where $Z_1^{(t)}$ is distributed as above. By Skorokhod's embedding theorem, we may suppose that $t^{1/(1-\alpha)}Z_1^{(t)}$ converges a.s. to its limit in law Z_1 . So it remains to show that a.s. $p_1^{(t)}(-Z_1^{(t)}) \rightarrow p_1(0)$ as $t \rightarrow 0$ to apply dominated convergence, and this is done by recalling that $p_1^{(t)}(z) = e^{-t^\alpha - tz}p_1(z)$. Then one reasons by induction just as in [80, Proposition 6], using the explicit form of the semigroup of F^+ . \parallel

5.6.2 Large-time asymptotics

By a direct application of Theorem 3 in [27], one gets the large t asymptotic behavior for F^+ . Recall that the Gamma law with parameters (a, b) is the law with density proportional to $x^{a-1}e^{-bx}$ on \mathbb{R}_+ . The moments of this law are given, for $r > -a$, by

$$\frac{b^a}{\Gamma(a)} \int_0^\infty x^{r+a-1}e^{-bx} dx = \frac{\Gamma(a+r)}{\Gamma(a)b^r}.$$

Proposition 5.8 :

Define

$$\rho_t(dy) = \sum_{i=1}^\infty F_i(t)\delta_{t^\alpha F_i(t)}(dy),$$

then ρ_t is a probability measure that converges in law as $t \rightarrow \infty$ to the deterministic Gamma law with parameter $1 - 1/\alpha$.

\parallel We know by [27, Theorem 3] that ρ_t converges to some probability ρ_∞ that is characterized by its moments,

$$\int_0^\infty y^{k/\alpha} \rho_\infty(dy) = \frac{\alpha(k-1)!}{\Phi'(0+)\Phi(\frac{1}{\alpha}) \dots \Phi(\frac{k-1}{\alpha})}$$

for every $k \geq 1$, where Φ is the Laplace exponent of a subordinator related to a tagged fragment of the process F^+ . This exponent depends only on the dislocation measure (and not the index), so it is the same as for F_- in [80]. By taking the explicit value of Φ (Section 3.2 therein), we easily get

$$\int_0^\infty y^{k/\alpha} \rho_\infty(dy) = \left(\frac{\alpha\Gamma(1 + \frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha})} \right)^k \frac{\Gamma(1 + \frac{k-1}{\alpha})}{\Gamma(1 - \frac{1}{\alpha})}.$$

The first constant is equal to 1, and by replacing k by αk , one can recognize the moments of the law Gamma with the claimed parameter. //

Chapter 6

The genealogy of self-similar fragmentations as a continuum random tree

Contents

6.1	Introduction	159
6.2	The CRT \mathcal{T}_F	162
6.2.1	Partitions-valued self-similar fragmentations	163
6.2.2	Trees with edge-lengths	165
6.2.3	Building the CRT	166
6.3	Hausdorff dimension of \mathcal{T}_F	169
6.3.1	Upper bound	169
6.3.2	A first lower bound	172
6.3.3	A subtree of \mathcal{T}_F and a reduced fragmentation	173
6.3.4	Lower bound	176
6.3.5	Dimension of the stable tree	181

6.1 Introduction

Self-similar fragmentation processes describe the evolution of an object that falls apart, so that different fragments keep on collapsing independently with a rate that depends on their sizes to a certain power, called the *index* of the self-similar fragmentation. A genealogy is naturally associated to such fragmentation processes, by saying that the common ancestor of two fragments is the block that included these fragments for the last time, before a dislocation had definitely separated them. With an appropriate coding of the fragments, one guesses that there should be a natural way to define a genealogy tree, rooted at the initial fragment, associated to any such fragmentation. It would be natural to put a metric on this tree, e.g. by letting the distance from a fragment to the root of the tree be the time at which the fragment disappears.

Conversely, it turns out that trees have played a key role in models involving self-similar fragmentations, notably, Aldous and Pitman [10] have introduced a way to log the so-called Brownian *Continuum Random Tree* (CRT) [5] that is related to the standard additive coalescent. Bertoin [26] has shown that a fragmentation that is somehow dual to the Aldous-Pitman fragmentation can be obtained as follows. Let \mathcal{T}_B be the Brownian CRT, which is considered as an “infinite tree with edge-lengths” (formal definitions are given below). Let $\mathcal{T}_t^1, \mathcal{T}_t^2, \dots$ be the distinct tree components of the forest obtained by removing all the vertices of \mathcal{T} that are at distance less than t from the root, and arranged by decreasing order of “size”. Then the sequence $F_B(t)$ of these sizes defines as t varies a self-similar fragmentation. A moment of thought points out that the notion of genealogy defined above precisely coincides with the tree we have fragmented in this way, since a split occurs precisely at branchpoints of the tree. Fragmentations of CRT’s that are different from the Brownian one and that follow the same kind of construction have been studied in [80].

The goal of this paper is to show that any self-similar fragmentation process with negative index can be obtained by a similar construction as above, for a certain instance of CRT. We are interested in negative indices, because in most interesting cases when the self-similarity index is non-negative, all fragments have an “infinite lifetime”, meaning that the pieces of the fragmentation remain macroscopic at all times. In this case, the family tree defined above will be unbounded and without endpoints, hence looking completely different from the Brownian CRT. By contrast, as soon as the self-similarity index is negative, a loss of mass occurs, that makes the fragments disappear in finite time (see [27]). In this case, the metric family tree will be a bounded object, and in fact, a CRT. To state our results, we first give a rigorous definition of the involved objects. Call

$$S = \left\{ \mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0; \sum_{i \geq 1} s_i \leq 1 \right\},$$

and endow it with the topology of pointwise convergence.

Definition :

A Markovian S -valued process $(F(t), t \geq 0)$ is a ranked self-similar fragmentation with index $\alpha \in \mathbb{R}$ if it is continuous in probability and satisfies the following fragmentation property. For every $t, t' \geq 0$, given $F(t) = (x_1, x_2, \dots)$, $F(t + t')$ has the same law as the decreasing rearrangement of the sequences $x_1 F^{(1)}(x_1^\alpha t'), x_2 F^{(2)}(x_2^\alpha t'), \dots$, where the $F^{(i)}$ ’s are independent copies of F .

By a result of Bertoin [26] and Berestycki [17], the laws of such fragmentation processes are characterized by a 3-tuple (α, c, ν) , where α is the index, $c \geq 0$ is an “erosion” constant, and ν is a σ -finite measure on S that integrates $\mathbf{s} \mapsto 1 - s_1$ such that $\nu(\{(1, 0, 0, \dots)\}) = 0$. Informally, c measures the rate at which fragments melt continuously (a phenomenon we will not be much interested in here), while ν measures instantaneous breaks of fragments: a piece with size x breaks into fragments with masses $x\mathbf{s}$ at rate $x^\alpha \nu(d\mathbf{s})$. Notice that some mass can be lost within a sudden break: this happens as soon as $\nu(\sum_i s_i < 1) \neq 0$, but we will not be interested in this phenomenon here either. The loss of mass phenomenon stated above is completely different from erosion or sudden loss of mass: it is due to the fact that small fragments tend to decay faster when $\alpha < 0$.

On the other hand, let us define the notion of CRT. An \mathbb{R} -tree (with the terminology of Dress and Terhalle [46]; it is called *continuum tree set* in Aldous [5]) is a complete

metric space (T, d) , whose elements are called *vertices*, which satisfies the following two properties:

- For $v, w \in T$, there exists a unique geodesic $[[v, w]]$ going from v to w , i.e. there exists a unique isomorphism $\varphi_{v,w} : [0, d(v, w)] \rightarrow T$ with $\varphi_{v,w}(0) = v$ and $\varphi_{v,w}(d(v, w)) = w$, and its image is called $[[v, w]]$.
- For any $v, w \in T$, the only non-self-intersecting path going from v to w is $[[v, w]]$, i.e. for any continuous injective function $s \mapsto v_s$ from $[0, 1]$ to T with $v_0 = v$ and $v_1 = w$, $\{v_s : s \in [0, 1]\} = [[v, w]]$.

We will furthermore consider \mathbb{R} -trees that are *rooted*, that is, one vertex is distinguished as being the root, and we call it \emptyset . A *leaf* is a vertex which does not belong to $[[\emptyset, w]] := \varphi_{\emptyset,w}([0, d(\emptyset, w)])$ for any vertex w . Call $\mathcal{L}(T)$ the set of leaves of T , and $\mathcal{S}(T) = T \setminus \mathcal{L}(T)$ its skeleton. An \mathbb{R} -tree is *leaf-dense* if T is the closure of $\mathcal{L}(T)$. We also call *height* of a vertex v the quantity $\text{ht}(v) = d(\emptyset, v)$. Last, for T an \mathbb{R} -tree and $a > 0$, we let $a \otimes T$ be the \mathbb{R} -tree in which all distances are multiplied by a .

Definition :

A continuum tree is a pair (T, μ) where T is an \mathbb{R} -tree and μ is a probability measure on T , called the mass measure, which is non-atomic and satisfies $\mu(\mathcal{L}(T)) = 1$ and such that for every non-leaf vertex w , $\mu\{v \in T : [[\emptyset, v]] \cap [[\emptyset, w]] = [[\emptyset, w]]\} > 0$. The set of vertices just defined is called the fringe subtree rooted at w . A CRT is a random variable $\omega \mapsto (T(\omega), \mu(\omega))$ on a probability space (Ω, \mathcal{F}, P) which values are continuum trees.

Notice that the definition of a continuum tree implies that the \mathbb{R} -tree T satisfies certain extra properties, for example, its set of leaves must be uncountable and have no isolated point.

For (T, μ) a continuum tree, and for every $t \geq 0$, let $T_1(t), T_2(t), \dots$ be the tree components of $\{v \in T : \text{ht}(v) > t\}$, ranked by decreasing order of μ -mass. A continuum random tree (T, μ) is said to be *self-similar* with index $\alpha < 0$ if for every $t \geq 0$, conditionally on $(\mu(T_i(t)), i \geq 1)$, $(T_i(t), i \geq 1)$ has the same law as $(\mu(T_i(t))^{-\alpha} \otimes T^{(i)}, i \geq 1)$ where the $T^{(i)}$'s are independent copies of T .

Our first result is

Theorem 6.1 :

Let F be a ranked self-similar fragmentation process with characteristic 3-tuple (α, c, ν) , with $\alpha < 0$. Suppose also that F is not constant, that $c = 0$ and $\nu(\sum_i s_i < 1) = 0$. Then there exists an α -self-similar CRT (\mathcal{T}_F, μ_F) such that, writing $F'(t)$ for the decreasing sequence of masses of connected components of the open set $\{v \in \mathcal{T}_F : \text{ht}(v) > t\}$, the process $(F'(t), t \geq 0)$ has the same law as F . The tree \mathcal{T}_F is leaf-dense if and only if ν has infinite total mass.

The next statement is a kind of converse to this theorem.

Proposition 6.1 :

Let (\mathcal{T}, μ) be a self-similar CRT with index $\alpha < 0$. Then the process $F(t) = ((\mu(\mathcal{T}_i(t)), i \geq 1), t \geq 0)$ is a ranked self-similar fragmentation with index α , it has no erosion and its dislocation measure ν satisfies $\nu(\sum_i s_i < 1) = 0$. Moreover, \mathcal{T}_F and \mathcal{T} have the same law.

These two results are proved in the next section. There probably exists some notion of continuum random tree extending the former which would include fragmentations with erosion or with sudden loss of mass, but such fragmentations usually are less interesting.

The next result, which is proved in Sect. 6.3, deals with the Hausdorff dimension of the CRT \mathcal{T}_F .

Theorem 6.2 :

Let F be a ranked self-similar fragmentation with characteristics (α, c, ν) satisfying the hypotheses of Theorem 6.1. Writing $\dim_{\mathcal{H}}$ for Hausdorff dimension, one has

$$\dim_{\mathcal{H}}(\mathcal{T}_F) = \frac{1}{|\alpha|} \vee 1 \text{ a.s.} \quad (6.1)$$

as soon as $\int_S \left(\frac{1}{s_1^2} - 1\right) \nu(ds) < \infty$.

Some comments about this formula. First, notice that the fact that $\dim_{\mathcal{H}}(\mathcal{T}_F)$ is greater or equal to 1 is obvious from the fact that \mathcal{T}_F contains at least one “line” (the path from the root to a vertex) as soon as it is not empty or reduced to its root. Next, we see that the value -1 is critical for α , since the above formula shows that the dimension of \mathcal{T}_F is to be 1 as soon as $\alpha \leq -1$. It was shown in a previous work by Bertoin [27] that when $\alpha < -1$, for every fixed t the number of fragments at time t is a.s. finite, so that -1 is indeed the threshold under which fragments decay incredibly fast. One should then picture the CRT \mathcal{T}_F as a very slim tree looking like a handful of thin sticks connected to each other. Last, the integrability assumption in the theorem seems to be reasonably mild; its heuristic meaning is that when a fragmentation occurs, the largest resulting fragment is not too small. In particular, it is always satisfied in the case of fragmentations for which $\nu(s_{N+1} > 0) = 0$, since then $s_1 > 1/N$ for ν -a.e. s .

It is worth noting that these results allow as a special case to compute the Hausdorff dimension of the so-called *stable trees* of Duquesne and Le Gall, which were used to construct fragmentations in the manner of Theorem 6.1 in [80]. The dimension of the stable tree (as well as finer results of Hausdorff measures on more general Lévy trees) has been obtained independently in [50]. The stable tree is a CRT whose law depends on parameter $\beta \in (1, 2]$, and it satisfies the required self-similarity property of Proposition 6.1 with index $1/\beta - 1$. We check that the associated dislocation measure satisfies the integrability condition of Theorem 6.2 in Sect. 6.3.5, so that

Corollary 6.1 :

Fix $\beta \in (1, 2]$. The β -stable tree has Hausdorff dimension $\beta/(\beta - 1)$.

6.2 The CRT \mathcal{T}_F

Building the CRT \mathcal{T}_F associated to a ranked fragmentation F will be done by determining its “marginals”, i.e. the subtrees spanned by a finite but arbitrary number of randomly chosen leaves. To this purpose, it will be useful to use partitions-valued fragmentations, which we first define, as well as a certain family of trees with edge-lengths.

6.2.1 Exchangeable partitions and partitions-valued self-similar fragmentations

Let \mathcal{P}_∞ be the set of (unordered) partitions of $\mathbb{N} = \{1, 2, \dots\}$ and $[n] = \{1, 2, \dots, n\}$. We adopt the following ordering convention: for $\pi \in \mathcal{P}_\infty$, we let (π_1, π_2, \dots) be the blocks of π , so that π_i is the block containing i provided that i is the smallest integer of the block and $\pi_i = \emptyset$ otherwise. We let $\mathbb{O} = \{\{1\}, \{2\}, \dots\}$ be the partition of \mathbb{N} into singletons. If $B \subset \mathbb{N}$ and $\pi \in \mathcal{P}_\infty$ we let $\pi \cap B$ (or $\pi|_B$) be the restriction of π to B , i.e. the partition of B whose collection of blocks is $\{\pi_i \cap B, i \geq 1\}$. If $\pi \in \mathcal{P}_\infty$ and $B \in \pi$ is a block of π , we let

$$|B| = \lim_{n \rightarrow \infty} \frac{\#(B \cap [n])}{n}$$

be the asymptotic frequency of the block B , whenever it exists. A random variable π with values in \mathcal{P}_∞ is called *exchangeable* if its law is invariant by the natural action of permutations of \mathbb{N} on \mathcal{P}_∞ . By a theorem of Kingman [68, 2], all the blocks of such random partitions admit asymptotic frequencies a.s. For π whose blocks have asymptotic frequencies, we let $|\pi| \in S$ be the decreasing sequence of these frequencies. Kingman's theorem more precisely says that the law of any exchangeable random partition π is a (random) "paintbox process", a term we now explain. Take $\mathbf{s} \in S$ (the paintbox) and consider a sequence U_1, U_2, \dots of i.i.d. variables in $\mathbb{N} \cup \{0\}$ (the colors) with $P(U_1 = j) = s_j$ for $j \geq 1$ and $P(U_1 = 0) = 1 - \sum_k s_k$. Define a partition π on \mathbb{N} by saying that $i \neq j$ are in the same block if and only if $U_i = U_j \neq 0$ (i.e. i and j have the same color, where 0 is considered as colorless). Call $\rho_{\mathbf{s}}(d\pi)$ its law, the *s-paintbox law*. Kingman's theorem says that the law of any random partition is a mixing of paintboxes, i.e. it has the form $\int_{\mathbf{s} \in S} m(d\mathbf{s}) \rho_{\mathbf{s}}(d\pi)$ for some probability measure m on S . A useful consequence is that the block of an exchangeable partition π containing 1, or some prescribed integer i , is a *size-biased pick* from the blocks of π , i.e. the probability it equals a non-singleton block π_j conditionally on $(|\pi_j|, j \geq 1)$ equals $|\pi_j|$. Similarly,

Lemma 6.1 :

Let π be an exchangeable random partition which is a.s. different from the trivial partition \mathbb{O} , and B an infinite subset of \mathbb{N} . For any $i \in \mathbb{N}$, let

$$\tilde{i} = \inf\{j \geq i : j \in B \text{ and } \{j\} \notin \pi\},$$

then $\tilde{i} < \infty$ a.s. and the block $\tilde{\pi}$ of π containing \tilde{i} is a size-biased pick among the non-singleton blocks of π , i.e. if we denote these by π'_1, π'_2, \dots ,

$$P(\tilde{\pi} = \pi'_k | (|\pi'_j|, j \geq 1)) = |\pi'_k| / \sum_j |\pi'_j|.$$

For any sequence of partitions $(\pi^{(i)}, i \geq 1)$, define $\pi = \bigcap_{i \geq 1} \pi^{(i)}$ by

$$k \stackrel{\pi}{\sim} j \iff k \stackrel{\pi^{(i)}}{\sim} j \quad \forall i \geq 1.$$

Lemma 6.2 :

Let $(\pi^{(i)}, i \geq 1)$ be a sequence of independent exchangeable partitions and set

$\pi := \bigcap_{i \geq 1} \pi^{(i)}$. Then, a.s. for every $j \in \mathbb{N}$,

$$|\pi_j| = \prod_{i \geq 1} |\pi_{k(i,j)}^{(i)}|,$$

where $(k(i,j), j \geq 1)$ is defined so that $\pi_j = \bigcap_{i \geq 1} \pi_{k(i,j)}^{(i)}$.

// First notice that $k(i,j) \leq j$ for all $i \geq 1$ a.s. This is clear when $\pi_j \neq \emptyset$, since $j \in \pi_j$ and then $j \in \pi_{k(i,j)}^{(i)}$. When $\pi_j = \emptyset$, $j \in \pi_m$ for some $m < j$ and then m and j belong to the same block of $\pi^{(i)}$ for all $i \geq 1$. Thus $k(i,j) \leq m < j$. Using then the paintbox construction of exchangeable partitions explained above and the independence of the $\pi^{(i)}$'s, we see that the r.v. $\prod_{i \geq 1} \mathbb{1}_{m \in \pi_{k(i,j)}^{(i)}}$, $m \geq j+1$, are iid conditionally on $(|\pi_{k(i,j)}^{(i)}|, i \geq 1)$ with a mean equal to $\prod_{i \geq 1} |\pi_{k(i,j)}^{(i)}|$. The law of large numbers therefore gives

$$\prod_{i \geq 1} |\pi_{k(i,j)}^{(i)}| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j+1 \leq m \leq n} \prod_{i \geq 1} \mathbb{1}_{\{m \in \pi_{k(i,j)}^{(i)}\}} \text{ a.s.}$$

On the other hand, the random variables $\prod_{i \geq 1} \mathbb{1}_{\{m \in \pi_{k(i,j)}^{(i)}\}} = \mathbb{1}_{\{m \in \pi_j\}}$, $m \geq j+1$, are i.i.d. conditionally on $|\pi_j|$ with mean $|\pi_j|$ and then the limit above converges a.s. to $|\pi_j|$, again by the law of large numbers. //

We now turn our attention to partitions-valued fragmentations.

Definition :

Let $(\Pi(t), t \geq 0)$ be a Markovian \mathcal{P}_∞ -valued process with $\Pi(0) = \{\mathbb{N}, \emptyset, \emptyset, \dots\}$ that is continuous in probability and exchangeable as a process (meaning that the law of Π is invariant by the action of permutations). Call it a partition-valued self-similar fragmentation with index $\alpha \in \mathbb{R}$ if moreover $\Pi(t)$ admits asymptotic frequencies for all t , a.s., if the process $(|\Pi(t)|, t \geq 0)$ is continuous in probability, and if the following fragmentation property is satisfied. For $t, t' \geq 0$, given $\Pi(t) = (\pi_1, \pi_2, \dots)$, the sequence $\Pi(t+t')$ has the same law as the partition with blocks $\pi_1 \cap \Pi^{(1)}(|\pi_1|^{\alpha t'})$, $\pi_2 \cap \Pi^{(2)}(|\pi_2|^{\alpha t'})$, \dots , where $(\Pi^{(i)}, i \geq 1)$ are independent copies of Π .

Bertoin [26] has shown that any such fragmentation is also characterized by the same 3-tuple (α, c, ν) as above, meaning that the laws of partition-valued and ranked self-similar fragmentations are in a one-to-one correspondence. In fact, for every (α, c, ν) , one can construct a version of the partition-valued fragmentation Π with parameters (α, c, ν) , and then $(|\Pi(t)|, t \geq 0)$ is the ranked fragmentation with parameters (α, c, ν) . Let us build this version now. It is done following [23, 26] by a Poissonian construction. Recall the notation $\rho_s(d\pi)$, and define $\kappa_\nu(d\pi) = \int_S \nu(ds) \rho_s(d\pi)$. Let $\#$ be the counting measure on \mathbb{N} and let (Δ_t, k_t) be a $\mathcal{P}_\infty \times \mathbb{N}$ -valued Poisson point process with intensity $\kappa_\nu \otimes \#$. We may construct a process $(\Pi^0(t), t \geq 0)$ by letting $\Pi^0(0)$ be the trivial partition $(\mathbb{N}, \emptyset, \emptyset, \dots)$, and saying that Π^0 jumps only at times t when an atom (Δ_t, k_t) occurs. When this is the case, Π^0 jumps from the state $\Pi^0(t-)$ to the following partition $\Pi^0(t)$: replace the block $\Pi_{k_t}^0(t-)$ by $\Pi_{k_t}^0(t-) \cap \Delta_t$, and leave the other blocks unchanged. Such a construction can be made rigorous by considering restrictions of partitions to the first n integers and

by a consistency argument. Then Π^0 has the law of the fragmentation with parameters $(0, 0, \nu)$.

Out of this “homogeneous” fragmentation, we construct the $(\alpha, 0, \nu)$ -fragmentation by introducing a time-change. Call $\lambda_i(t)$ the asymptotic frequency of the block of $\Pi^0(t)$ that contains i , and write

$$T_i(t) = \inf \left\{ u \geq 0 : \int_0^u \lambda_i(r)^{-\alpha} dr > t \right\}. \quad (6.2)$$

Last, for every $t \geq 0$ we let $\Pi(t)$ be the random partition such that i, j are in the same block of $\Pi(t)$ if and only if they are in the same block of $\Pi^0(T_i(t))$, or equivalently of $\Pi^0(T_j(t))$. Then $(\Pi(t), t \geq 0)$ is the wanted version.

When $\alpha < 0$, the loss of mass in the ranked fragmentations shows up at the level of partitions by the fact that a positive fraction of the blocks of $\Pi(t)$ are singletons for some $t > 0$. This last property of self-similar fragmentations with negative index allows to build a collection of trees with edge-lengths.

6.2.2 Trees with edge-lengths

A tree is a finite connected graph with no cycles. It is *rooted* when a particular vertex (the root) is distinguished from the others, in this case the edges are by convention oriented, pointing from the root, and we define the out-degree of a vertex v as being the number of edges that point outward v . A *leaf* in a rooted tree is a vertex with out-degree 0. For $k \geq 1$, let \mathbf{T}_k be the set of rooted trees with exactly k labeled leaves (the names of the labels may change according to what we see fit), the other vertices (except the root) begin unlabeled, and such that the root is the only vertex that has out-degree 1. If $\mathbf{t} \in \mathbf{T}_k$, we let $E(\mathbf{t})$ be the set of its edges.

A *tree with edge-lengths* is a pair $\vartheta = (\mathbf{t}, \mathbf{e})$ for $\mathbf{t} \in \bigcup_{k \geq 1} \mathbf{T}_k$ and $\mathbf{e} = (e_i, i \in E(\mathbf{t})) \in (\mathbb{R}_+ \setminus \{0\})^{E(\mathbf{t})}$. Call \mathbf{t} the *skeleton* of ϑ . Such a tree is naturally equipped with a distance $d(v, w)$ on the set of its vertices, by adding the lengths of edges that appear in the unique path connecting v and w in the skeleton (which we still denote by $[[v, w]]$). The height of a vertex is its distance to the root. We let \mathbb{T}_k be the set of trees with edge-lengths whose skeleton is in \mathbf{T}_k . For $\vartheta \in \mathbb{T}_k$, let e_{root} be the length of the unique edge connected to the root, and for $e < e_{\text{root}}$ write $\vartheta - e$ for the tree with edge-lengths that has same skeleton and same edge-lengths as ϑ , but for the edge pointing outward the root which is assigned length $e_{\text{root}} - e$.

We also define an operation **MERGE** as follows. Let $n \geq 2$ and take $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ respectively in $\mathbb{T}_{k_1}, \mathbb{T}_{k_2}, \dots, \mathbb{T}_{k_n}$, with leaves $(L_i^1, 1 \leq i \leq k_1), (L_i^2, 1 \leq i \leq k_2), \dots, (L_i^n, 1 \leq i \leq k_n)$ respectively. Let also $e > 0$. The tree with edge-lengths $\text{MERGE}((\vartheta_1, \dots, \vartheta_n); e) \in \mathbb{T}_{\sum_i k_i}$ is defined by merging together the roots of $\vartheta_1, \dots, \vartheta_n$ into a single vertex \bullet , and by drawing a new edge $\text{root} \rightarrow \bullet$ with length e .

Last, for $\vartheta \in \mathbb{T}_k$ and i vertices v_1, \dots, v_i , define the subtree spanned by the root and v_1, \dots, v_i as follows. For every $p \neq q$, let $b(v_p, v_q)$ be the branchpoint of v_p and v_q , that is, the highest point in the tree that belongs to $[[\text{root}, v_p]] \cap [[\text{root}, v_q]]$. The spanned tree is the tree with edge-lengths whose vertices are the root, the vertices v_1, \dots, v_i and the branchpoints $b(v_p, v_q)$, $1 \leq p \neq q \leq i$, and whose edge-lengths are given by the respective distances between this subset of vertices of the original tree.

6.2.3 Building the CRT

Now for $B \subset \mathbb{N}$ finite, define $\mathcal{R}(B)$, a random variable with values in $\mathbb{T}_{\#B}$, whose leaf-labels are of the form L_i for $i \in \mathbb{N}$, as follows. Let $D_i = \inf\{t \geq 0 : \{i\} \in \Pi(t)\}$ be the first time when $\{i\}$ “disappears”, i.e. is isolated in a singleton of $\Pi(t)$. For B a finite subset of \mathbb{N} with at least two elements, let $D_B = \inf\{t \geq 0 : \#(B \cap \Pi(t)) \neq 1\}$ be the first time when the restriction of $\Pi(t)$ to B is non-trivial, i.e. has more than one block. By convention, $D_{\{i\}} = D_i$. For every $i \geq 1$, define $\mathcal{R}(\{i\})$ as a single edge root $\rightarrow L_i$, and assign this edge the length D_i . For B with $\#B \geq 2$, let B_1, \dots, B_i be the non-empty blocks of $B \cap \Pi(D_B)$, arranged in increasing order of least element, and define a tree $\mathcal{R}(B)$ recursively by

$$\mathcal{R}(B) = \text{MERGE}((\mathcal{R}(B_1) - D_B, \dots, \mathcal{R}(B_i) - D_B); D_B).$$

Last, define $\mathcal{R}(k) = \mathcal{R}([k])$. Notice that by definition of the distance, the distance between L_i and L_j in $\mathcal{R}(k)$ for any $k \geq i \vee j$ equals $D_i + D_j - 2D_{\{i,j\}}$.

We now state the key lemma that allows to describe the CRT out of the family $(\mathcal{R}(k), k \geq 1)$ which is the candidate for the marginals of \mathcal{T}_F . By Aldous [5], it suffices to check two properties, called *consistency* and *leaf-tightness*. Notice that in [5], only binary trees (in which branchpoint have out-degree 2) are considered, but as noticed therein, this translates to our setting with minor changes.

Lemma 6.3 :

(i) The family $(\mathcal{R}(k), k \geq 1)$ is consistent in the sense that for every k and $j \leq k$, $\mathcal{R}(j)$ has the same law as the subtree of $\mathcal{R}(k)$ spanned by the root and j distinct leaves L_1^k, \dots, L_j^k taken uniformly at random from the leaves L_1, \dots, L_k of $\mathcal{R}(k)$, independently of $\mathcal{R}(k)$.

(ii) The family $(\mathcal{R}(k), k \geq 1)$ is leaf-tight, that is, with the above notations,

$$\min_{2 \leq j \leq k} d(L_1^k, L_j^k) \xrightarrow{P} 0.$$

// The consistency property is an immediate consequence of the fact that the process Π is exchangeable. Taking j leaves uniformly out of the k ones of $\mathcal{R}(k)$ is just the same as if we had chosen exactly the leaves L_1, L_2, \dots, L_j , which give rise to the tree $\mathcal{R}(j)$, and this is (i).

For (ii), first notice that we may suppose by exchangeability that $L_1^k = L_1$. The only point is then to show that the minimal distance of this leaf to the leaves L_2, \dots, L_k tends to 0 in probability as $k \rightarrow \infty$. Fix $\eta > 0$ and for $\varepsilon > 0$ write $t_\varepsilon^1 = \inf\{t \geq 0 : |\Pi_1(t)| < \varepsilon\}$, where $\Pi_1(t)$ is the block of $\Pi(t)$ containing 1. Then t_ε^1 is a stopping time with respect to the natural filtration $(\mathcal{F}_t, t \geq 0)$ associated to Π and $t_\varepsilon^1 \uparrow D_1$ as $\varepsilon \downarrow 0$. By the strong Markov property and exchangeability, one has that if $K(\varepsilon) = \inf\{k > 1 : k \in \Pi_1(t_\varepsilon^1)\}$, then $P(D_1 + D_{K(\varepsilon)} - 2t_\varepsilon^1 < \eta) = E[P_{\Pi(t_\varepsilon^1)}(D_1 + D_{K(\varepsilon)} < \eta)]$ where P_π is the law of the fragmentation Π started at π (the law of Π under P_π is the same as that of the family of partitions $(\{\text{blocks of } \pi_1 \cap \Pi^{(1)}(|\pi_1|^\alpha t), \pi_2 \cap \Pi^{(2)}(|\pi_2|^\alpha t), \dots\}, t \geq 0)$ where the $\Pi^{(i)}$'s, $i \geq 1$, are independent copies of Π under $P_{\{\mathbb{N}, \emptyset, \emptyset, \dots\}}$). By the self-similar fragmentation property and exchangeability this is greater than

$P(D_1 + D_2 < \varepsilon^\alpha \eta)$, which in turn is greater than $P(2\tau < \varepsilon^\alpha \eta)$ where τ is the first time where $\Pi(t)$ becomes the partition into singletons, which by [27] is finite a.s. This last probability thus goes to 1 as $\varepsilon \downarrow 0$. Taking $\varepsilon = \varepsilon(n) \downarrow 0$ quickly enough as $n \rightarrow \infty$ and applying the Borel-Cantelli lemma, we a.s. obtain a sequence $K(\varepsilon(n))$ such that $d(L_1, L_{K(\varepsilon(n))}) \leq D_1 + D_{K(\varepsilon(n))} - 2t_{\varepsilon(n)} < \eta$. Hence the result. //

For a rooted \mathbb{R} -tree T and k vertices v_1, \dots, v_k , we define exactly as for marked trees the subtree spanned by the root and v_1, \dots, v_k , as an element of \mathbb{T}_k . A consequence of [5, Theorem 3] is then:

Lemma 6.4 :

There exists a CRT $(\mathcal{T}_\Pi, \mu_\Pi)$ such that if Z_1, \dots, Z_k is a sample of k leaves picked independently according to μ_Π conditionally on μ_Π , the subtree of \mathcal{T}_Π spanned by the root and Z_1, \dots, Z_k has the same law as $\mathcal{R}(k)$.

In the sequel, sequences like (Z_1, Z_2, \dots) will be called *exchangeable sequences with directing measure μ_Π* .

// *Proof of Theorem 6.1.* We have to check that the tree \mathcal{T}_Π of the preceding lemma gives rise to a fragmentation process with the same law as $F = |\Pi|$. By construction, we have that for every $t \geq 0$ the partition $\Pi(t)$ is such that i and j are in the same block of $\Pi(t)$ if and only if L_i and L_j are in the same connected component of $\{v \in \mathcal{T}_\Pi : \text{ht}(v) > t\}$. Hence, the law of large numbers implies that if $F'(t)$ is the decreasing sequence of the μ -masses of these connected components, then $F'(t) = F(t)$ a.s. for every t . Hence, F' is a version of F , so we can set $\mathcal{T}_F = \mathcal{T}_\Pi$. That \mathcal{T}_F is α -self-similar is an immediate consequence of the fragmentation and self-similar properties of F .

We now turn to the last statement of Theorem 6.1. With the notation of Lemma 6.4 we will show that the path $[[\emptyset, Z_1]]$ is almost-surely in the closure of the set of leaves of \mathcal{T}_F if and only if $\nu(S) = \infty$. Then it must hold by exchangeability that so do the paths $[[\emptyset, Z_i]]$ for every $i \geq 1$, and this is sufficient because the definition of the CRTs imply that $\mathcal{S}(\mathcal{T}_F) = \bigcup_{i \geq 1} [[\emptyset, Z_i]]$, see [5, Lemma 6] (the fact that \mathcal{T}_F is a.s. compact will be proved below). To this end, it suffices to show that for any $a \in (0, 1)$, the point aZ_1 of $[[\emptyset, Z_1]]$ that is at a proportion a from \emptyset (the point $\varphi_{\emptyset, Z_1}(ad(\emptyset, Z_1))$ with the above notations) can be approached closely by leaves, that is, for $\eta > 0$ there exists $j > 1$ such that $d(aZ_1, Z_j) < \eta$. It thus suffices to check that for any $\delta > 0$

$$P(\exists 2 \leq j \leq k : |D_{\{1,j\}} - aD_1| < \delta \text{ and } D_j - D_{\{1,j\}} < \delta) \xrightarrow[k \rightarrow \infty]{} 1,$$

with the above notations derived from Π (this is a slight variation of [5, (iii) a). Theorem 15]).

Suppose that $\nu(S) = \infty$. Then for every rational $r > 0$ such that $|\Pi_1(r)| \neq 0$ and for every $\delta > 0$, the block containing 1 undergoes a fragmentation in the time-interval $r, r + \delta/2$. This is obvious from the Poisson construction of the self-similar fragmentation Π given above, because ν is

an infinite measure so there is an infinite number of atoms of (Δ_t, k_t) with $k_t = 1$ in any time-interval with positive length. It is then easy that there exists an infinite number of elements of $\Pi_1(r)$ that are isolated in singletons of $\Pi(r + \delta)$, e.g. because of Proposition 6.2 below which asserts that only a finite number of the blocks of $\Pi(r + \delta/2)$ “survive” at time $r + \delta$, i.e. is not completely reduced to singletons. Thus, an infinite number of elements of $\Pi_1(r)$ correspond to leaves of some $\mathcal{R}(k)$ for k large enough. By taking r close to aD_1 we thus have the result.

On the other hand, if $\nu(S) < \infty$, it follows from the Poisson construction that the state $(1, 0, \dots)$ is a holding state, so the first fragmentation occurs at a positive time, so the root cannot be approached by leaves. //

Remark. We have seen that we may actually build simultaneously the trees $(\mathcal{R}(k), k \geq 1)$ on the same probability space as a measurable functional of the process $(\Pi(t), t \geq 0)$. This yields, by redoing the “special construction” of Aldous [5], a *stick-breaking construction* of the tree \mathcal{T}_F , by now considering the trees $\mathcal{R}(k)$ as \mathbb{R} -trees obtained as finite unions of segments rather than trees with edge-lengths. The special CRT thus constructed is a subset of ℓ^1 in [5], but we consider it as a universal, i.e. up to isomorphism. The tree $\mathcal{R}(k + 1)$ is then obtained from $\mathcal{R}(k)$ by branching a new segment with length $D_{k+1} - \max_{B \subset [k], B \neq \emptyset} D_{B \cup \{k\}}$, and \mathcal{T}_F can be reinterpreted as the completion of the metric space $\bigcup_{k \geq 1} \mathcal{R}(k)$, moreover the tips of the branches of $\bigcup_{k \geq 1} \mathcal{R}(k)$, which we still call L_1, L_2, \dots are distributed as an i.i.d. leaf sample with the mass measure as common distribution. We will allow such identifications of $\mathcal{R}(k)$ as a subtree of \mathcal{T}_F in the sequel.

// *Proof of Proposition 6.1.* The fact that the process F defined out of a CRT (\mathcal{T}, μ) with the stated properties is a S -valued self-similar fragmentation with index α is straightforward and left to the reader. The treatment of the erosion and sudden loss of mass is a little more subtle. Let Z_1, Z_2, \dots be an exchangeable sample directed by the measure μ , and for every $t \geq 0$ define a random partition $\Pi(t)$ by saying that i and j are in the same block of $\Pi(t)$ if Z_i and Z_j fall in the same tree component of $\{v \in \mathcal{T} : \text{ht}(v) > t\}$. By the arguments above, Π defines a self-similar partition-valued fragmentation such that $|\Pi(t)| = F(t)$ for every t . Notice that if we show that the erosion coefficient $c = 0$ and that no sudden loss of mass occur, it will immediately follow that \mathcal{T} has the same law as \mathcal{T}_F .

Now suppose that $\nu(\sum_i s_i < 1) \neq 0$. Then (e.g. by the Poisson construction of fragmentations described above) there exists a.s. two distinct integers i and j and a time D such that i and j are in the same block of $\Pi(D-)$ but $\{i\} \in \Pi(D)$ and $\{j\} \in \Pi(D)$. This implies that $Z_i = Z_j$, so μ has a.s. an atom and (\mathcal{T}, μ) cannot be a CRT. On the other hand, suppose that the erosion coefficient $c > 0$. Again from the Poisson construction, we see that there a.s. exists a time D such that $\{1\} \notin \Pi(D-)$ but $\{1\} \in \Pi(D)$, and nevertheless $\Pi(D) \cap \Pi_1(D-)$ is not the trivial partition \mathbb{O} . Taking j in a non-trivial block of this last partition and denoting its death time by D' , we obtain that the distance from Z_1 to Z_j is $D' - D$, while the height of Z_1 is D and that of Z_j is D' . This implies that Z_1 is a.s. not in the set of leaves of \mathcal{T} , again contradicting the definition of a CRT. //

6.3 Hausdorff dimension of \mathcal{T}_F

Let (M, d) be a compact metric space. For $\mathcal{E} \subseteq M$, the Hausdorff dimension of \mathcal{E} is the real number

$$\dim_{\mathcal{H}}(\mathcal{E}) := \inf \{ \gamma > 0 : m_{\gamma}(\mathcal{E}) = 0 \}, \quad (6.3)$$

where

$$m_{\gamma}(\mathcal{E}) := \sup_{\varepsilon > 0} \inf \sum_i \Delta(E_i)^{\gamma}, \quad (6.4)$$

the infimum being taken over all collections $(E_i, i \geq 1)$ of subsets of \mathcal{E} with diameter $\Delta(E_i) \leq \varepsilon$, whose union covers \mathcal{E} . This dimension is meant to measure the “fractal size” of the considered set. For background on this subject, we mention [53] (in the case $M = \mathbb{R}^n$, but the generalization to general metric spaces of the results we will need is straightforward). We will need in particular that the m_{γ} 's are measures on M and, consequently, that $\dim_{\mathcal{H}}(\cup_i \mathcal{E}_i) = \sup_i \dim_{\mathcal{H}}(\mathcal{E}_i)$ for each countable union of subsets $\mathcal{E}_i \subset M$.

The goal of this Section is to prove Theorem 2, and the proof is divided in the two usual upper and lower bound parts. First, we prove that \mathcal{T}_F is indeed compact and that $\dim_{\mathcal{H}}(\mathcal{T}_F) \leq (1/|\alpha|) \vee 1$ a.s., which is true without the extra integrability assumption on ν . The lower bound $\dim_{\mathcal{H}}(\mathcal{T}_F) \geq (1/|\alpha|) \vee 1$ a.s. will be obtained by using appropriate subtrees of \mathcal{T}_F (we will see that the most naive way to apply Frostman's energy method with the mass measure μ_F fails in general). That Theorem 2 applies to stable trees is proved in Sect. 6.3.5.

6.3.1 Upper bound

To prove the majoration, we will build a “good” covering of the support of the mass measure μ (denoted here by $\text{supp}(\mu)$). This will be constructed in the following proposition.

Proposition 6.2 :

For all $\varepsilon > 0$, there exists a covering of the support $\text{supp}(\mu)$ by N_{ε} balls of radius 5ε such that $\liminf_{\varepsilon \rightarrow 0} (N_{\varepsilon} \varepsilon^{(1/|\alpha| \vee 1) + \eta}) = 0$ a.s. for all $\eta > 0$. In particular, $\text{supp}(\mu)$ and \mathcal{T}_F are a.s. compact.

// For $t \geq 0$ and $\varepsilon > 0$, denote by N_t^{ε} the number of blocks of $\Pi(t)$ not reduced to singletons that are not entirely reduced to dust at time $t + \varepsilon$. We first prove that N_t^{ε} is a.s. finite. Let $(\Pi_i(t), i \geq 1)$ be the blocks of $\Pi(t)$, and $(|\Pi_i(t)|, i \geq 1)$, their respective asymptotic frequencies. For integers i such that $|\Pi_i(t)| > 0$, that is $\Pi_i(t) \neq \emptyset$ and $\Pi_i(t)$ is not reduced to a singleton, let $\tau_i := \inf \{ s > t : \Pi_i(t) \cap \Pi(s) = \emptyset \}$ be the first time at which the block $\Pi_i(t)$ is entirely reduced to dust. Applying the fragmentation and scaling properties at time t , we may write τ_i as $\tau_i = t + |\Pi_i(t)|^{|\alpha|} \tilde{\tau}_i$ where $\tilde{\tau}_i$ is a r.v. independent of $\mathcal{G}(t)$ that has same distribution as $\tau = \inf \{ t \geq 0 : \Pi(t) = \emptyset \}$, the first time at which the fragmentation is entirely reduced to dust. Now, fix $\varepsilon > 0$. The number of blocks of $\Pi(t)$ that are not entirely reduced to dust

at time $t + \varepsilon$, which could be *a priori* infinite, is then given by

$$N_t^\varepsilon = \sum_{i: |\Pi_i(t)| > 0} \mathbb{1}_{\{|\Pi_i(t)|^\alpha \tilde{\tau}_i > \varepsilon\}}.$$

From Proposition 15 in [57], we know that there exist two constants C_1, C_2 such that $P(\tau > t) \leq C_1 e^{-C_2 t}$ for all $t \geq 0$. Consequently, for all $\delta > 0$,

$$\begin{aligned} E[N_t^\varepsilon \mid \mathcal{G}(t)] &\leq C_1 \sum_{i: |\Pi_i(t)| > 0} e^{-C_2 \varepsilon |\Pi_i(t)|^\alpha} \\ &\leq C(\delta) \varepsilon^{-\delta} \sum_i |\Pi_i(t)|^{\alpha|\delta}, \end{aligned} \quad (6.5)$$

where $C(\delta) = \sup_{x \in \mathbb{R}^+} \{C_1 x^\delta e^{-C_2 x}\} < \infty$. Since $\sum_i |\Pi_i(t)| \leq 1$ a.s., this shows by taking $\delta = 1/|\alpha|$ that $N_t^\varepsilon < \infty$ a.s.

Let us now construct a covering of $\text{supp}(\mu)$ with balls of radius 5ε . Recall that we may suppose that the tree \mathcal{T}_F is constructed together with an exchangeable leaf sample (L_1, L_2, \dots) directed by μ_F . For each $l \in \mathbb{N} \cup \{0\}$, we introduce the set

$$B_l^\varepsilon = \{k \in \mathbb{N} : \{k\} \notin \Pi(l\varepsilon), \{k\} \in \Pi((l+1)\varepsilon)\},$$

some of which may be empty when $\nu(S) < \infty$, since the tree is not leaf-dense. For $l \geq 1$, the number of blocks of the partition $B_l^\varepsilon \cap \Pi((l-1)\varepsilon)$ of B_l^ε is less than or equal to $N_{(l-1)\varepsilon}^\varepsilon$ and so is a.s. finite. Since the fragmentation is entirely reduced to dust at time $\tau < \infty$ a.s., $N_{l\varepsilon}^\varepsilon$ is equal to zero for $l \geq \tau/\varepsilon$ and then, defining

$$N_\varepsilon := \sum_{l=0}^{\lceil \tau/\varepsilon \rceil} N_{l\varepsilon}^\varepsilon$$

we have $N_\varepsilon < \infty$ a.s. ($\lceil \tau/\varepsilon \rceil$ denotes here the largest integer smaller than τ/ε). Now, consider a finite random sequence of pairwise distinct integers $\sigma(1), \dots, \sigma(N_\varepsilon)$ such that for each $1 \leq l \leq \lceil \tau/\varepsilon \rceil$ and each non-empty block of $B_l^\varepsilon \cap \Pi((l-1)\varepsilon)$, there is a $\sigma(i), 1 \leq i \leq N_\varepsilon$, in this block. Then each leaf L_j belongs then to a ball of center $L_{\sigma(i)}$, for an integer $1 \leq i \leq N_\varepsilon$, and of radius 4ε . Indeed, fix $j \geq 1$. It is clear that the sequence $(B_l^\varepsilon)_{l \in \mathbb{N} \cup \{0\}}$ forms a partition of \mathbb{N} . Thus, there exists a unique block B_l^ε containing j and in this block we consider the integer $\sigma(i)$ that belongs to the same block as j in the partition $B_l^\varepsilon \cap \Pi(((l-1) \vee 0)\varepsilon)$. By definition (see Section 2.3), the distance between the leaves L_j and $L_{\sigma(i)}$ is $d(L_j, L_{\sigma(i)}) = D_j + D_{\sigma(i)} - 2D_{\{j, \sigma(i)\}}$. By construction, j and $\sigma(i)$ belong to the same block of $\Pi(((l-1) \vee 0)\varepsilon)$ and both die before $(l+1)\varepsilon$. In other words, $\max(D_j, D_{\sigma(i)}) \leq (l+1)\varepsilon$ and $D_{\{j, \sigma(i)\}} \geq ((l-1) \vee 0)\varepsilon$, which implies that $d(L_j, L_{\sigma(i)}) \leq 4\varepsilon$. Therefore, we have covered the set of leaves $\{L_j, j \geq 1\}$ by at most N_ε balls of radius 4ε . Since the sequence $(L_j)_{j \geq 1}$ is dense in $\text{supp}(\mu)$, this induces by taking balls with radius 5ε instead of 4ε a covering of $\text{supp}(\mu)$ by N_ε balls of radius 5ε . This holds for all $\varepsilon > 0$ so $\text{supp}(\mu)$ is a.s. compact. The compactness of \mathcal{T}_F follows immediately.

It remains to prove that $\liminf_{\varepsilon \rightarrow 0} (N_\varepsilon \varepsilon^{((1/|\alpha|) \vee 1) + \eta}) = 0$ a.s. for all $\eta > 0$. By definition of N_ε and (6.5) we have that for all $K > 0$ and all $\delta > 0$

$$E [N_\varepsilon \mathbb{1}_{\{\tau < K\}}] \leq C(\delta) \varepsilon^{-\delta} \sum_{l=0}^{\lfloor K/\varepsilon \rfloor} E \left[\sum_i |\Pi_i(l\varepsilon)|^{|\alpha|\delta} \right],$$

where $C(\delta)$ is a finite constant depending only on δ . Replacing ε by $1/k$ ($k \in \mathbb{N}$) in this inequality gives

$$E \left[\frac{N_{1/k} \mathbb{1}_{\{\tau < K\}}}{k^{1+\delta}} \right] \leq \frac{C(\delta)}{k} \sum_{l=0}^{Kk} E \left[\sum_i |\Pi_i(l/k)|^{|\alpha|\delta} \right].$$

The quantity $\frac{1}{k} \sum_{l=0}^{Kk} E[\sum_i |\Pi_i(l/k)|^{|\alpha|\delta}]$ is a Riemann sum converging to $\int_0^K E[\sum_i |\Pi_i(t)|^{|\alpha|\delta}] dt$ as soon as this integral is finite. By the proof of [27, Proposition 2] (and more precisely by Equation (10) therein), the quantity $\int_0^\infty E[\sum_i |\Pi_i(t)|^{|\alpha|\delta}] dt$ is finite when $|\alpha|\delta > 1 + \alpha$. Therefore, for $\delta > (1/|\alpha| - 1) \vee 0$, $E[k^{-1-\delta} N_{1/k} \mathbb{1}_{\{\tau < K\}}]$ is bounded in k and then, by Fatou's Lemma,

$$\liminf_{k \rightarrow \infty} k^{-1-\delta} N_{1/k} \mathbb{1}_{\{\tau < K\}} < \infty \text{ a.s.}$$

This holds for all $K > 0$, implying $\liminf_{k \rightarrow \infty} k^{-1-\delta} N_{1/k} < \infty$ a.s. when $\delta > (1/|\alpha| - 1) \vee 0$, hence the result. //

// *Proof of Theorem 6.2: upper bound.* To prove the upper bound in Theorem 6.2 we first point out that the all work consists in bounding the Hausdorff dimension of $\text{supp } \mu_F$. Indeed, the CRT \mathcal{T}_F can be written as the disjoint union of its skeleton and its leaves $\mathcal{T}_F = \mathcal{S}(\mathcal{T}_F) \cup \mathcal{L}(\mathcal{T}_F)$. By Lemma 6 of [5] and the fact that $\text{supp } (\mu_F)$ is a.s. compact, we have $\text{supp } (\mu_F) = \mathcal{L}(\mathcal{T}_F)$. Thus, $\dim_{\mathcal{H}}(\mathcal{T}_F) = \max(\dim_{\mathcal{H}}(\mathcal{S}(\mathcal{T}_F)), \dim_{\mathcal{H}}(\text{supp } (\mu_F)))$. Lemma 5 of [5] asserts that the skeleton $\mathcal{S}(\mathcal{T}_F) = \cup_{i \geq 1} [[\emptyset, L_i]]$. Since the Hausdorff dimension of a path $[[\emptyset, L_i]]$ is equal to one, we have then $\dim_{\mathcal{H}}(\mathcal{S}(\mathcal{T}_F)) = 1$ and so, $\dim_{\mathcal{H}}(\mathcal{T}_F) = \max(1, \dim_{\mathcal{H}}(\text{supp } (\mu_F)))$.

To estimate the upper bound of $\dim_{\mathcal{H}}(\text{supp } (\mu_F))$, fix $\gamma > 0$. For all $\varepsilon > 0$, by considering the covering of Proposition 6.2 we have

$$\inf_{\substack{\text{coverings of } \text{supp } \mu \text{ by} \\ \text{sets } E_i \text{ of diameter } \Delta(E_i) \leq 10\varepsilon}} \sum_i \Delta(E_i)^\gamma \leq \liminf_{\varepsilon \rightarrow 0} 10^\gamma \varepsilon^\gamma N_\varepsilon,$$

which is a.s. equal to 0 for all $\varepsilon > 0$ as soon as $\gamma > (1/|\alpha|) \vee 1$. In other words, by (6.4), $m_\gamma(\text{supp } (\mu_F)) = 0$ a.s. when $\gamma > (1/|\alpha|) \vee 1$ and consequently, by (6.3), $\dim_{\mathcal{H}}(\text{supp } (\mu_F)) \leq (1/|\alpha|) \vee 1$ a.s. //

6.3.2 A first lower bound

Recall that Frostman's energy method to prove that $\dim_{\mathcal{H}}(\mathcal{E}) \geq \gamma$ where \mathcal{E} is a subset of a metric space (M, d) is to find a nonzero positive measure $\eta(x)$ on \mathcal{E} such that $\int_{\mathcal{E}} \int_{\mathcal{E}} \frac{\eta(dx)\eta(dy)}{d(x,y)^\gamma} < \infty$. A naive approach for finding a lower bound of the Hausdorff dimension of \mathcal{T}_F is thus to apply this method by taking $\eta = \mu_F$ and $\mathcal{E} = \mathcal{T}_F$. The result states as follows. Notice that this first lower bound does not take into account the assumption $\int_S (s_1^{-2} - 1) \nu(ds) < \infty$.

Lemma 6.5 :

For any fragmentation process F satisfying the hypotheses of Theorem 6.1, one has

$$\dim_{\mathcal{H}}(\mathcal{T}_F) \geq \frac{A}{|\alpha|} \wedge \left(1 + \frac{p}{|\alpha|}\right),$$

where

$$p := -\inf \left\{ q : \int_S \left(1 - \sum_{i \geq 1} s_i^{q+1}\right) \nu(ds) > -\infty \right\} \in [0, 1],$$

and

$$A := \sup \left\{ a : \int_S \sum_{1 \leq i < j} s_i^{1-a} s_j \nu(ds) < \infty \right\} \in [0, 1].$$

// By Lemma 6.4 (recall that $(\mathcal{T}_{\Pi}, \mu_{\Pi}) = (\mathcal{T}_F, \mu_F)$ by Theorem 1) we have

$$\int_{\mathcal{T}_F} \int_{\mathcal{T}_F} \frac{\mu_F(dx)\mu_F(dy)}{d(x,y)^\gamma} \stackrel{a.s.}{=} E \left[\frac{1}{d(L_1, L_2)^\gamma} \mid \mathcal{T}_F, \mu_F \right]$$

so that

$$E \left[\int_{\mathcal{T}_F} \int_{\mathcal{T}_F} \frac{\mu_F(dx)\mu_F(dy)}{d(x,y)^\gamma} \right] = E \left[\frac{1}{d(L_1, L_2)^\gamma} \right]$$

and by definition, $d(L_1, L_2) = D_1 + D_2 - 2D_{\{1,2\}}$. Applying the scaling and strong fragmentation properties at the stopping time $D_{\{1,2\}}$, we can rewrite D_1 and D_2 as

$$D_1 = D_{\{1,2\}} + \lambda_1^{|\alpha|}(D_{\{1,2\}})\tilde{D}_1 \quad D_2 = D_{\{1,2\}} + \lambda_2^{|\alpha|}(D_{\{1,2\}})\tilde{D}_2$$

where $\lambda_1(D_{\{1,2\}})$ (resp. $\lambda_2(D_{\{1,2\}})$) is the asymptotic frequency of the block containing 1 (resp. 2) at time $D_{\{1,2\}}$ and \tilde{D}_1 and \tilde{D}_2 are independent with the same law as D_1 and independent of $\mathcal{G}(D_{\{1,2\}})$. Therefore,

$$d(L_1, L_2) = \lambda_1^{|\alpha|}(D_{\{1,2\}})\tilde{D}_1 + \lambda_2^{|\alpha|}(D_{\{1,2\}})\tilde{D}_2,$$

and

$$E \left[\frac{1}{d(L_1, L_2)^\gamma} \right] \leq 2E \left[\lambda_1^{\alpha\gamma}(D_{\{1,2\}}); \lambda_1(D_{\{1,2\}}) \geq \lambda_2(D_{\{1,2\}}) \right] E \left[D_1^{-\gamma} \right]. \quad (6.6)$$

By [58, Lemma 3] the first expectation in the right-hand side of inequality (6.6) is finite as soon as $|\alpha|\gamma < A$, while by [57, Sect. 4.2.1] the second expectation is finite as soon as $\gamma < 1 + \underline{p}/|\alpha|$. That $\dim_{\mathcal{H}}(\mathcal{T}_F) \stackrel{a.s.}{\geq} ((A/|\alpha|) \wedge (1 + \underline{p}/|\alpha|))$ follows. \parallel

Let us make some comments about this bound. First, for dislocation measures such that $\nu(s_{N+1} > 0) = 0$ for some $N \geq 1$, the constant A equals 1 since for all $a > -1$,

$$\int_S \sum_{i < j} s_i^{1+a} s_j \nu(ds) \leq \int_S (N-1) \sum_{2 \leq j \leq N} s_j \nu(ds) \leq (N-1) \int_S (1-s_1) \nu(ds) < \infty.$$

Moreover, when $\underline{p} = 1$, the term $1 + \underline{p}/|\alpha|$ is larger than $A/|\alpha|$. A typical setting in which this holds is when $\nu(S) < \infty$ and $\nu(s_{N+1} > 0) = 0$ and therefore, for such dislocation measures the “naive” lower bound is also the best possible.

6.3.3 A subtree of \mathcal{T}_F and a reduced fragmentation

In the general case, in order to improve this lower bound, we will thus try to transform the problem on F into a problem on an auxiliary fragmentation that satisfies the hypotheses above. The idea is as follows: fix an integer N and $0 < \varepsilon < 1$. Consider the subtree $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ constructed from \mathcal{T}_F by keeping, at each branchpoint, the N largest fringe subtrees rooted at this branchpoint (that is the subtrees with the largest masses) and discarding the others in order to yield a tree in which branchpoints have out-degree at most N . Also, we remove the accumulation of fragmentation times by discarding all the fringe subtrees rooted at the branchpoints but the largest one, as soon as the proportion of its mass compared to the others is larger than $1 - \varepsilon$. Then there exists a probability $\mu_F^{N,\varepsilon}$ such that $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is a CRT, to which we will apply the energy method.

Let us make the definition precise. Define $\mathcal{L}^{N,\varepsilon} \subset \mathcal{L}(\mathcal{T}_F)$ to be the set of leaves L such that for every branchpoint $b \in [[\emptyset, L]]$, $L \in \mathcal{F}_b^{N,\varepsilon}$ with $\mathcal{F}_b^{N,\varepsilon}$ defined by

$$\begin{cases} \mathcal{F}_b^{N,\varepsilon} = \mathcal{T}_b^1 \cup \dots \cup \mathcal{T}_b^N & \text{if } \mu_F(\mathcal{T}_b^1)/\mu_F(\bigcup_{i \geq 1} \mathcal{T}_b^i) \leq 1 - \varepsilon \\ \mathcal{F}_b^{N,\varepsilon} = \mathcal{T}_b^1 & \text{if } \mu_F(\mathcal{T}_b^1)/\mu_F(\bigcup_{i \geq 1} \mathcal{T}_b^i) > 1 - \varepsilon \end{cases}, \quad (6.7)$$

where $\mathcal{T}_b^1, \mathcal{T}_b^2, \dots$ are the connected components of the fringe subtree of \mathcal{T}_F rooted at b , from whom b has been removed (the connected components of $\{v \in \mathcal{T}_F : \text{ht}(v) > b\}$) and ranked in decreasing order of μ_F -mass. Then let $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ be the subtree of \mathcal{T}_F spanned by the root and the leaves of $\mathcal{L}^{N,\varepsilon}$, i.e.

$$\mathcal{T}_F^{N,\varepsilon} = \{v \in \mathcal{T}_F : \exists L \in \mathcal{L}^{N,\varepsilon}, v \in [[\emptyset, L]]\}.$$

The set $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ is plainly connected and closed in \mathcal{T}_F , thus an \mathbb{R} -tree.

Now let us try to give a sense to “taking at random a leaf in $\mathcal{T}_F^{N,\varepsilon}$ ”. In the case of \mathcal{T}_F , it was easy because, from the partition-valued fragmentation Π , it sufficed to look at the fragment containing 1 (or some prescribed integer). Here, it is not difficult (as we will see later) that the corresponding leaf L_1 a.s. never belongs to $\mathcal{T}_F^{N,\varepsilon}$ when the dislocation measure ν charges the set $\{s_1 > 1 - \varepsilon\} \cup \{s_{N+1} > 0\}$. Therefore, we will have to use several random leaves of \mathcal{T}_F . For any leaf $L \in \mathcal{L}(\mathcal{T}_F) \setminus \mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$ let $b(L)$ be the highest vertex v of $[[\emptyset, L]]$ such that $v \in \mathcal{T}_F^{N,\varepsilon}$. Call it the branchpoint of L and $\mathcal{T}_F^{N,\varepsilon}$.

Now take at random a leaf Z_1 of \mathcal{T}_F with law μ_F conditionally on μ_F , and define recursively a sequence $(Z_n, n \geq 1)$ with values in \mathcal{T}_F as follows. Let Z_{n+1} be independent of Z_1, \dots, Z_n conditionally on $(\mathcal{T}_F, \mu_F, b(Z_n))$, and take it with conditional law

$$P(Z_{n+1} \in \cdot | \mathcal{T}_F, \mu_F, b(Z_n)) = \mu_F(\cdot \cap \mathcal{F}_{b(Z_n)}^{N,\varepsilon}) / \mu_F(\mathcal{F}_{b(Z_n)}^{N,\varepsilon}).$$

Lemma 6.6 :

Almost surely, the sequence $(Z_n, n \geq 1)$ converges to a random leaf $Z^{N,\varepsilon} \in \mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$. If $\mu_F^{N,\varepsilon}$ denotes the conditional law of $Z^{N,\varepsilon}$ given (\mathcal{T}_F, μ_F) , then $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is a CRT, provided ε is small enough.

To prove this and for later use we first reconnect this discussion to partition-valued fragmentations. Recall from Sect. 6.2.1 the construction of the homogeneous fragmentation Π^0 with characteristics $(0, 0, \nu)$ out of a $\mathcal{P}_\infty \times \mathbb{N}$ -valued Poisson point process $((\Delta_t, k_t), t \geq 0)$ with intensity $\kappa_\nu \otimes \#$. For any partition $\pi \in \mathcal{P}_\infty$ that admits asymptotic frequencies whose ranked sequence is \mathbf{s} , write π_i^\downarrow for the block of π with asymptotic frequency s_i (with some convention for ties, e.g. taking the order of least element). We define a function $\text{GRIND}^{N,\varepsilon} : \mathcal{P}_\infty \rightarrow \mathcal{P}_\infty$ that reduces the smallest blocks of the partition to singletons as follows. If π does not admit asymptotic frequencies, let $\text{GRIND}^{N,\varepsilon}(\pi) = \pi$, else let

$$\text{GRIND}^{N,\varepsilon}(\pi) = \begin{cases} (\pi_1^\downarrow, \dots, \pi_N^\downarrow, \text{singletons}) & \text{if } s_1 \leq 1 - \varepsilon \\ (\pi_1^\downarrow, \text{singletons}) & \text{if } s_1 > 1 - \varepsilon. \end{cases}$$

Now for each $t \geq 0$ write $\Delta_t^{N,\varepsilon} = \text{GRIND}^{N,\varepsilon}(\Delta_t)$, so $((\Delta_t^{N,\varepsilon}, k_t), t \geq 0)$ is a $\mathcal{P}_\infty \times \mathbb{N}$ -valued Poisson point process with intensity measure $\kappa_{\nu^{N,\varepsilon}} \otimes \#$, where $\nu^{N,\varepsilon}$ is the image of ν by the function

$$\mathbf{s} \in S \mapsto \begin{cases} (s_1, \dots, s_N, 0, \dots) & \text{if } s_1 \leq 1 - \varepsilon \\ (s_1, 0, \dots) & \text{if } s_1 > 1 - \varepsilon. \end{cases}$$

From this Poisson point process we construct first a version $\Pi^{0,N,\varepsilon}$ of the $(0, 0, \nu^{N,\varepsilon})$ fragmentation, as explained in Section 2.1. For every time $t \geq 0$, the partition $\Pi^{0,N,\varepsilon}(t)$ is finer than $\Pi^0(t)$ and the blocks of $\Pi^{0,N,\varepsilon}(t)$ non-reduced to singleton are blocks of $\Pi^0(t)$. Next, using the times-change (6.2), we construct from $\Pi^{0,N,\varepsilon}$ a version of the $(\alpha, 0, \nu^{N,\varepsilon})$ fragmentation, that we denote by $\Pi^{N,\varepsilon}$.

Note that for dislocation measures ν such that $\nu^{N,\varepsilon}(\sum s_i < 1) = 0$, the Theorem 6.2 is already proved, by the previous subsection. For the rest of this subsection and next subsection, we shall thus focus on dislocation measures ν such that $\nu^{N,\varepsilon}(\sum s_i < 1) > 0$. In that case, in $\Pi^{0,N,\varepsilon}$ (unlike for Π^0) each integer i is eventually isolated in a singleton a.s. within a sudden break and this is why a μ_F -sampled leaf on \mathcal{T}_F cannot be in $\mathcal{T}_F^{N,\varepsilon}$, in other words, μ_F and $\mu_F^{N,\varepsilon}$ are a.s. singular. Recall that we may build \mathcal{T}_F together with an exchangeable μ_F -sample of leaves L_1, L_2, \dots on the same probability space as Π (or Π^0). We are going to use a subfamily of (L_1, L_2, \dots) to build a sequence with the same law as $(Z_n, n \geq 1)$ built above. Let $i_1 = 1$ and

$$i_{n+1} = \inf\{i > i_n : L_{i_{n+1}} \in \mathcal{F}_{b(L_{i_n})}^{N,\varepsilon}\}.$$

It is easy that $(L_{i_n}, n \geq 1)$ has the same law as $(Z_n, n \geq 1)$. From this, we build a decreasing family of blocks $B^{0,N,\varepsilon}(t) \in \Pi^0(t)$, $t \geq 0$, by letting $B^{0,N,\varepsilon}(t)$ be the unique block of $\Pi^0(t)$ that contains all but a finite number of elements of $\{i_1, i_2, \dots\}$.

Here is a useful alternative description of $B^{0,N,\varepsilon}(t)$. Let $D_i^{0,N,\varepsilon}$ be the death time of i for the fragmentation $\Pi^{0,N,\varepsilon}$ that is

$$D_i^{0,N,\varepsilon} = \inf\{t \geq 0 : \{i\} \in \Pi^{0,N,\varepsilon}(t)\}.$$

By exchangeability the $D_i^{0,N,\varepsilon}$'s are identically distributed and $D_1^{0,N,\varepsilon} = \inf\{t \geq 0 : k_t = 1 \text{ and } \{1\} \in \Delta_t^{N,\varepsilon}\}$ so it has an exponential law with parameter $\int_S (1 - \sum_i s_i) \nu^{N,\varepsilon}(ds)$. Then notice that $B^{0,N,\varepsilon}(t)$ is the block admitting i_n as least element when $D_{i_n}^{0,N,\varepsilon} \leq t < D_{i_{n+1}}^{0,N,\varepsilon}$. Indeed, by construction we have

$$i_{n+1} = \inf\{i \in B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon} -) : \{i\} \notin \Pi^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})\}.$$

Moreover, the asymptotic frequency $\lambda_1^{0,N,\varepsilon}(t)$ of $B^{0,N,\varepsilon}(t)$ exists for every t and equals the μ_F -mass of the tree component of $\{v \in \mathcal{T}_F : \text{ht}(v) > t\}$ containing L_{i_n} for $D_{i_n}^{0,N,\varepsilon} \leq t < D_{i_{n+1}}^{0,N,\varepsilon}$.

Notice that at time $D_{i_n}^{0,N,\varepsilon}$, either one non-singleton block coming from $B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon} -)$, or up to N non-singleton blocks may appear; by Lemma 6.1, $B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})$ is then obtained by taking at random one of these blocks with probability proportional to its size.

// *Proof of Lemma 6.6.* For $t \geq 0$ let $\lambda^{0,N,\varepsilon}(t) = |B^{0,N,\varepsilon}(t)|$ and

$$T^{0,N,\varepsilon}(t) := \inf\left\{u \geq 0 : \int_0^u (\lambda^{0,N,\varepsilon}(r))^{-\alpha} dr > t\right\} \quad (6.8)$$

and write $B^{N,\varepsilon}(t) := B^{0,N,\varepsilon}(T^{0,N,\varepsilon}(t))$, for $T^{0,N,\varepsilon}(t) < \infty$ and $B^{N,\varepsilon}(t) = \emptyset$ otherwise, so for all $t \geq 0$, $B^{N,\varepsilon}(t) \in \Pi^{N,\varepsilon}(t)$. Let also $D_{i_n}^{N,\varepsilon} := T^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})$ be the death time of i_n in the fragmentation $\Pi^{N,\varepsilon}$. It is easy that $b_n = b(L_{i_n})$ is the branchpoint of the paths $[[\emptyset, L_{i_n}]]$ and $[[\emptyset, L_{i_{n+1}}]]$, so the path $[[\emptyset, b_n]]$ has length $D_{i_n}^{N,\varepsilon}$. The “edges” $[[b_n, b_{n+1}]]$, $n \in \mathbb{N}$, have respective lengths $D_{i_{n+1}}^{N,\varepsilon} - D_{i_n}^{N,\varepsilon}$, $n \in \mathbb{N}$. Since the sequence of death times $(D_{i_n}^{N,\varepsilon}, n \geq 1)$ is increasing and bounded by τ (the first time at which Π is entirely reduced to singletons), the sequence $(b_n, n \geq 1)$ is Cauchy, so it converges by completeness of \mathcal{T}_F . Now it is easy that $D_{i_n}^{0,N,\varepsilon} \rightarrow \infty$ as $n \rightarrow \infty$ a.s., so $\lambda^{0,N,\varepsilon}(t) \rightarrow 0$ as $t \rightarrow \infty$ a.s. (see also the next Lemma). Therefore, it is easy by the fragmentation property that $d(L_{i_n}, b_n) \rightarrow 0$ a.s. so L_{i_n} is also Cauchy, with the same limit, and that the limit has to be a leaf which we denote $L^{N,\varepsilon}$ (of course it has same distribution as the $Z^{N,\varepsilon}$ of the lemma’s statement). The fact that $L^{N,\varepsilon} \in \mathcal{T}_F^{N,\varepsilon}$ a.s. is obtained by checking (6.7), which is true since it is verified for each branchpoint $b \in [[\emptyset, b_n]]$ for every $n \geq 1$ by construction.

We now sketch the proof that $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is indeed a CRT, leaving details to the reader. We need to show non-atomicity of $\mu_F^{N,\varepsilon}$, but it is clear that when performing the recursive construction of $Z^{N,\varepsilon}$ twice with independent variables, $(Z_n, n \geq 1)$ and $(Z'_n, n \geq 1)$ say, there exists a.s. some n such that Z_n and Z'_n end up in two different fringe subtrees rooted at some of the branchpoints b_n , provided that ε is small enough so that $\nu(1 - s_1 \geq \varepsilon) \neq 0$ (see also below the explicit construction of two independently $\mu_F^{N,\varepsilon}$ -sampled leaves). On the other hand, all of the subtrees of \mathcal{T}_F rooted at

the branchpoints of $\mathcal{T}_F^{N,\varepsilon}$ have positive μ_F -mass, so they will end up being visited by the intermediate leaves used to construct a $\mu_F^{N,\varepsilon}$ -i.i.d. sample, so the condition $\mu_F^{N,\varepsilon}(\{v \in \mathcal{T}_F^{N,\varepsilon} : [[\emptyset, v]] \cap [[\emptyset, w]] = [[\emptyset, w]]\}) > 0$ for every $w \in \mathcal{S}(\mathcal{T}_F^{N,\varepsilon})$ is satisfied. \parallel

It will also be useful to sample two leaves $(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$ that are independent with same distribution $\mu_F^{N,\varepsilon}$ conditionally on $\mu_F^{N,\varepsilon}$ out of the exchangeable family L_1, L_2, \dots . A natural way to do this is to use the family (L_1, L_3, L_5, \dots) to sample the first leaf in the same way as above, and to use the family (L_2, L_4, \dots) to sample the other one. That is, let $j_1^1 = 1, j_1^2 = 2$ and define recursively $(j_n^1, j_n^2, n \geq 1)$ by letting

$$\begin{cases} j_{n+1}^1 = \inf\{j \in 2\mathbb{N} + 1, j > j_n^1 : L_j \in \mathcal{F}_{b(L_{j_n^1})}^{N,\varepsilon}\} \\ j_{n+1}^2 = \inf\{j \in 2\mathbb{N}, j > j_{n+1}^1 : L_j \in \mathcal{F}_{b(L_{j_n^2})}^{N,\varepsilon}\} \end{cases}.$$

It is easy to check that $(L_{j_n^1}, n \geq 1)$ and $(L_{j_n^2}, n \geq 1)$ are two independent sequences distributed as (Z_1, Z_2, \dots) of Lemma 6.6. Therefore, these sequences a.s. converge to limits $L_1^{N,\varepsilon}, L_2^{N,\varepsilon}$, and these are independent with law $\mu_F^{N,\varepsilon}$ conditionally on $\mu_F^{N,\varepsilon}$. We let $\mathcal{D}_k = \text{ht}(L_k^{N,\varepsilon}), k = 1, 2$.

Similarly as above, for every $t \geq 0$ we let $B_k^{0,N,\varepsilon}(t), k = 1, 2$ (resp. $B_k^{N,\varepsilon}(t)$) be the block of $\Pi^0(t)$ (resp. $\Pi(t)$) that contains all but the first few elements of $\{j_1^k, j_2^k, \dots\}$, and we call $\lambda_k^{0,N,\varepsilon}(t)$ (resp. $\lambda_k^{N,\varepsilon}(t)$) its asymptotic frequency. Last, let $\mathcal{D}_{\{1,2\}}^0 = \inf\{t \geq 0 : B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t) = \emptyset\}$ (and define similarly $\mathcal{D}_{\{1,2\}}$). Notice that for $t < \mathcal{D}_{\{1,2\}}^0$, we have $B_1^{0,N,\varepsilon}(t) = B_2^{0,N,\varepsilon}(t)$, and by construction the two least elements of the blocks $(2\mathbb{N} + 1) \cap B_1^{0,N,\varepsilon}(t)$ and $(2\mathbb{N}) \cap B_1^{0,N,\varepsilon}(t)$ are of the form j_n^1, j_m^2 for some n, m . On the other hand, for $t \geq \mathcal{D}_{\{1,2\}}^0$, we have $B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t) = \emptyset$, and again the least elements of $(2\mathbb{N} + 1) \cap B_1^{0,N,\varepsilon}(t)$ and $(2\mathbb{N}) \cap B_2^{0,N,\varepsilon}(t)$ are of the the form j_n^1, j_m^2 for some n, m . In any case, we let $j^1(t) = j_n^1, j^2(t) = j_m^2$ for these n, m .

6.3.4 Lower bound

Now let F be a fragmentation process that satisfies the extra integrability condition of the statement of Theorem 6.2. We want to show that for every $a < 1$, the integral $\int_{\mathcal{T}_F^{N,\varepsilon}} \int_{\mathcal{T}_F^{N,\varepsilon}} \frac{\mu_F^{N,\varepsilon}(dx)\mu_F^{N,\varepsilon}(dy)}{d(x,y)^{a/|\alpha|}}$ is a.s. finite for suitable N and ε . So consider $a < 1$, and note that, with the above notations,

$$E \left[\int_{\mathcal{T}_F^{N,\varepsilon}} \int_{\mathcal{T}_F^{N,\varepsilon}} \frac{\mu_F^{N,\varepsilon}(dx)\mu_F^{N,\varepsilon}(dy)}{d(x,y)^{a/|\alpha|}} \right] = E \left[\frac{1}{d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})^{a/|\alpha|}} \right].$$

By definition, $d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon}) = \mathcal{D}_1 + \mathcal{D}_2 - 2\mathcal{D}_{\{1,2\}}$, with notations above. The fragmentation and scaling properties at the stopping time $\mathcal{D}_{\{1,2\}}$ lead to

$$\mathcal{D}_k = \mathcal{D}_{\{1,2\}} + \lambda_k^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})^{|\alpha|} \tilde{\mathcal{D}}_k, \quad k = 1, 2,$$

where $\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2$ are independent with the same distribution as \mathcal{D} , the height of the leaf $L^{N,\varepsilon}$ constructed above, and independent of $\mathcal{G}(\mathcal{D}_{\{1,2\}})$. Therefore, the distance $d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$ can be rewritten as

$$d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon}) = \left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right)^{|\alpha|} \tilde{\mathcal{D}}_1 + \left(\lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right)^{|\alpha|} \tilde{\mathcal{D}}_2$$

and

$$E \left[d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})^{-a/|\alpha|} \right] \leq 2E \left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right)^{-a}; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right] E \left[\mathcal{D}^{-a/|\alpha|} \right].$$

Therefore, Theorem 6.2 is directly implied by the following Lemmas.

Lemma 6.7 :

| The quantity $E[\mathcal{D}^{-\gamma}]$ is finite for every $\gamma < 1/|\alpha|$.

// The proof uses the following technical lemma. Recall that $\lambda^{N,\varepsilon}(t) = |B^{N,\varepsilon}(t)|$.

Lemma :

One can write $\lambda^{N,\varepsilon} = \exp(-\xi_{\rho(\cdot)})$, where ξ (tacitly depending on N, ε) is a subordinator with Laplace exponent

$$\begin{aligned} \Phi_\xi(q) &= \int_S \left((1 - s_1^q) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \right. \\ &\quad \left. + \sum_{i=1}^N (1 - s_i^q) \frac{s_i \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \dots + s_N} \right) \nu(ds), \quad q \geq 0, \end{aligned} \quad (6.9)$$

and ρ is the time-change

$$\rho(t) = \inf \left\{ u \geq 0 : \int_0^u \exp(\alpha \xi_r) dr > t \right\}, \quad t \geq 0.$$

// Recall the construction of the process $B^{0,N,\varepsilon}$ from Π^0 , which itself was constructed from a Poisson process $(\Delta_t, k_t, t \geq 0)$. From the definition of $B^{0,N,\varepsilon}(t)$, we have

$$B^{0,N,\varepsilon}(t) = \bigcap_{0 \leq s \leq t} \bar{\Delta}_s^{N,\varepsilon},$$

where the sets $\bar{\Delta}_s^{N,\varepsilon}$ are defined as follows. For each $s \geq 0$, let $i(s)$ be the least element of the block $B^{0,N,\varepsilon}(s-)$ (so that $B^{0,N,\varepsilon}(s-) = \Pi_{i(s)}^0(s-)$), so $(i(s), s \geq 0)$ is an $(\mathcal{F}(s-), s \geq 0)$ -adapted jump-hold process, and the process $\{\Delta_s : k_s = i(s), s \geq 0\}$ is a Poisson point process with intensity κ_ν . Then for each s such that $k_s = i(s)$, $\bar{\Delta}_s^{N,\varepsilon}$ consists in a certain block of Δ_s , and precisely, $\bar{\Delta}_s^{N,\varepsilon}$ is the block of Δ_s containing

$$\inf \{ i \in B^{0,N,\varepsilon}(s-) : \{i\} \notin \Delta_s^{N,\varepsilon} \},$$

the least element of $B^{0,N,\varepsilon}(s-)$ which is not isolated in a singleton of $\Delta_s^{N,\varepsilon}$ (such an integer must be of the form i_n for some n by definition). Now $B^{0,N,\varepsilon}(s-)$ is $\mathcal{F}(s-)$ -measurable, hence independent of Δ_s . By Lemma 6.1, $\bar{\Delta}_s^{N,\varepsilon}$ is thus a size-biased pick among the non-void blocks of $\Delta_s^{N,\varepsilon}$, and by definition of the function $\text{GRIND}^{N,\varepsilon}$, the process $(|\bar{\Delta}_s^{N,\varepsilon}|, s \geq 0)$ is a $[0, 1]$ -valued Poisson point process with intensity $\omega(s)$ characterized by

$$\int_{[0,1]} f(s) \omega(ds) = \int_S \left(\mathbb{1}_{\{s_1 > 1-\varepsilon\}} f(s_1) + \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}} \sum_{i=1}^N f(s_i) \frac{s_i}{s_1 + \dots + s_N} \right) \nu(ds),$$

for every positive measurable function f . Applying Lemma 6.2 twice then implies that $|B^{0,N,\varepsilon}(t)| = \prod_{0 \leq s \leq t} |\bar{\Delta}_s^{N,\varepsilon}|$ a.s. for every $t \geq 0$. To see this, denote for every $k \geq 1$ by $\Delta_{s_1}^{N,\varepsilon,k}, \Delta_{s_2}^{N,\varepsilon,k}, \dots$ the atoms $\Delta_s^{N,\varepsilon}$, $s \leq t$, such that $|\Delta_s^{N,\varepsilon}|_1 \in [1 - k^{-1}, 1 - (k+1)^{-1}]$. Complete this a.s. finite sequence of partitions by partitions $\mathbf{1}$ and call $\Gamma^{(k)}$ their intersection, i.e. $\Gamma^{(k)} := \bigcap_{i \geq 1} (\Delta_{s_i}^{N,\varepsilon,k})$. By Lemma 6.2, $|\Gamma_{n_k}^{(k)}| \stackrel{a.s.}{=} \prod_{i \geq 1} |\bar{\Delta}_{s_i}^{N,\varepsilon,k}|$, where n_k is the index of the block $\bigcap_{i \geq 1} \bar{\Delta}_{s_i}^{N,\varepsilon,k}$ in the partition $\Gamma^{(k)}$. These partitions $\Gamma^{(k)}$, $k \geq 1$, are exchangeable and clearly independent. Applying again Lemma 6.2 gives $|\bigcap_{k \geq 1} \Gamma_{n_k}^{(k)}| \stackrel{a.s.}{=} \prod_{k \geq 1} \prod_{i \geq 1} |\bar{\Delta}_{s_i}^{N,\varepsilon,k}|$, which is exactly the equality mentioned above. The exponential formula for Poisson processes then shows that $(\xi_t, t \geq 0) = (-\log(\lambda^{0,N,\varepsilon}(t)), t \geq 0)$ is a subordinator with Laplace exponent Φ_ξ . The result is now obtained by noticing that (6.2) rewrites $\lambda^{N,\varepsilon}(t) = \lambda^{0,N,\varepsilon}(\rho(t))$ in our setting. \parallel

By this lemma, $\mathcal{D} = \inf\{t \geq 0 : \lambda^{N,\varepsilon}(t) = 0\}$, which equals $\int_0^\infty \exp(\alpha \xi_t) dt$ by definition of ρ . According to Theorem 25.17 in [97], if for some positive γ the quantity

$$\Phi_\xi(-\gamma) := \int_S \left((1 - s_1^{-\gamma}) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} + \sum_{i=1}^N (1 - s_i^{-\gamma}) \frac{s_i \mathbb{1}_{\{s_i > 0\}} \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \dots + s_N} \right) \nu(ds)$$

is finite, then $E[\exp(\gamma \xi_t)] < \infty$ for all $t \geq 0$ and it equals $\exp(-t \Phi_\xi(-\gamma))$. Notice that $\Phi_\xi(-1) > -\infty$, indeed first

$$\int_S (s_1^{-1} - 1) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(ds) \leq \frac{1}{1-\varepsilon} \int_S (1 - s_1) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(ds) < \infty$$

and second

$$\int_S \left(\sum_{i=1}^N (1 - s_i) \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \dots + s_N} \right) \nu(ds) \leq N \int_S \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1} \nu(ds),$$

which is finite by the assumption on ν . This implies in particular that ξ_t has finite expectation for every t , and it follows by [40] that $E[\mathcal{D}^{-1}] < \infty$. Then, following the proof of Proposition 2 in [31] and using again that $\Phi_\xi(-1) > -\infty$,

$$E \left[\left(\int_0^\infty \exp(\alpha \xi_t) dt \right)^{-k-1} \right] = \frac{-\Phi_\xi(-|\alpha|k)}{k} E \left[\left(\int_0^\infty \exp(\alpha \xi_t) dt \right)^{-k} \right]$$

for every integer $k < 1/|\alpha|$. Hence, using induction, $E[(\int_0^\infty \exp(\alpha \xi_t) dt)^{-k-1}]$ is finite for $k = [1/|\alpha|]$ if $1/|\alpha| \notin \mathbb{N}$ and for $k = 1/|\alpha| - 1$ else. In both cases, we see that $E[\mathcal{D}^{-\gamma}] < \infty$ for every $\gamma < 1/|\alpha|$. \parallel

Lemma 6.8 :

For any $a < 1$, there exists N, ε such that

$$E \left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right)^{-a} ; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right] < \infty.$$

// The ingredient is the following lemma, which uses the notations around the construction of the leaves $(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$.

Lemma :

With the convention $\log(0) = -\infty$, the process

$$\sigma(t) = -\log \left| B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t) \right|, \quad t \geq 0$$

is a killed subordinator (its death time is $\mathcal{D}_{\{1,2\}}^0$) with Laplace exponent

$$\begin{aligned} \Phi_\sigma(q) &= \mathbf{k}^{N,\varepsilon} + \int_S \left((1 - s_1^q) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \right. \\ &\quad \left. + \sum_{i=1}^N (1 - s_i^q) \frac{s_i^2 \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \dots + s_N)^2} \right) \nu(ds), \quad q \geq 0, \end{aligned} \quad (6.10)$$

where the killing rate $\mathbf{k}^{N,\varepsilon} := \int_S \sum_{i \neq j} s_i s_j \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \dots + s_N)^2} \nu(ds) \in]0, \infty[$. Moreover, the pair

$$(l_1^{N,\varepsilon}, l_2^{N,\varepsilon}) = \exp(\sigma(\mathcal{D}_{\{1,2\}}^0 -)) (\lambda_1^{0,N,\varepsilon}(\mathcal{D}_{\{1,2\}}^0), \lambda_2^{0,N,\varepsilon}(\mathcal{D}_{\{1,2\}}^0))$$

is independent of $\sigma(\mathcal{D}_{\{1,2\}}^0 -)$ with law characterized by

$$\begin{aligned} E \left[f \left(l_1^{N,\varepsilon}, l_2^{N,\varepsilon} \right) \right] \\ = \frac{1}{\mathbf{k}^{N,\varepsilon}} \int_S \sum_{1 \leq i \neq j \leq N} f(s_i, s_j) \frac{s_i s_j \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}} \mathbb{1}_{\{s_i > 0\}} \mathbb{1}_{\{s_j > 0\}}}{(s_1 + \dots + s_N)^2} \nu(ds) \end{aligned}$$

for any positive measurable function f .

// We again use the Poisson construction of Π^0 out of $(\Delta_t, k_t, t \geq 0)$ and follow closely the proof of the intermediate lemma used in the proof of Lemma 6.7. For every $t \geq 0$ we have

$$B_k^{0,N,\varepsilon}(t) = \bigcap_{0 \leq s \leq t} \bar{\Delta}_s^k, \quad k = 1, 2,$$

where $\bar{\Delta}_s^k$ is defined as follows. Let $J^k(s), k = 1, 2$ be the integers such that $B_k^{0,N,\varepsilon}(s-) = \Pi_{J^k(s)}^0(s-)$, so $\{\Delta_s : k_s = J^k(s), s \geq 0\}$, $k = 1, 2$ are two Poisson processes with same intensity κ_ν , which are equal for s in the interval $[0, \mathcal{D}_{\{1,2\}}^0)$. Then for s with $k_s = J^k(s)$, let $\bar{\Delta}_s^k$ be the block of Δ_s containing $j^k(s)$. If $B_1^{0,N,\varepsilon}(s-) = B_2^{0,N,\varepsilon}(s-)$ notice that $j^1(s), j^2(s)$ are the two least integers of $(2\mathbb{N} + 1) \cap B_1^{0,N,\varepsilon}(s-)$ and $(2\mathbb{N}) \cap B_2^{0,N,\varepsilon}(s-)$ respectively that are not isolated as singletons of $\Delta_s^{N,\varepsilon}$, so $\bar{\Delta}_s^1 = \bar{\Delta}_s^2$ if these two integers fall in the same block of $\Delta_s^{N,\varepsilon}$. Hence by a variation of Lemma 6.1, $(|\bar{\Delta}_s^1 \cap \bar{\Delta}_s^2|, s \geq 0)$ is a Poisson process whose intensity is the image measure of $\kappa_{\nu^{N,\varepsilon}}(\pi \mathbb{1}_{\{1 \sim 2\}})$ by the map $\pi \mapsto |\pi|$, and killed at an independent exponential time (namely $\mathcal{D}_{\{1,2\}}^0$) with parameter $\kappa_{\nu^{N,\varepsilon}}(1 \sim 2)$ (here $1 \sim 2$ means that 1 and 2 are in the same block of π). This implies (6.10).

The time $\mathcal{D}_{\{1,2\}}^0$ is the first time when the two considered integers fall into two distinct blocks of $\Delta_s^{N,\varepsilon}$. It is then easy by the Poissonian construction and the paintbox representation to check that these blocks have asymptotic frequencies $(l_1^{N,\varepsilon}, l_2^{N,\varepsilon})$ which are independent of $\sigma(\mathcal{D}_{\{1,2\}}^0 -)$, and have the claimed law. $\parallel\parallel\parallel$

First notice, from the fact that self-similar fragmentations are time-changed homogeneous fragmentations, that

$$(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}), \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})) \stackrel{d}{=} (\lambda_1^{0,N,\varepsilon}(\mathcal{D}_{\{1,2\}}^0), \lambda_2^{0,N,\varepsilon}(\mathcal{D}_{\{1,2\}}^0)).$$

Thus, with the notations of the intermediate lemma,

$$\begin{aligned} & E \left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right)^{-a} ; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right] \\ &= E \left[\exp(a\sigma(\mathcal{D}_{\{1,2\}}^0 -)) \right] E \left[\left(l_1^{N,\varepsilon} \right)^{-a} ; l_1^{N,\varepsilon} \geq l_2^{N,\varepsilon} \right]. \end{aligned}$$

First, define $\Phi_\sigma(-a)$ by replacing q by $-a$ in (6.10) and then remark that $\Phi_\sigma(-a) > -\infty$, since

$$\int_S (s_1^{-a} - 1) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(ds) \leq \frac{1}{(1-\varepsilon)^a} \int_S (1-s_1) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(ds), \quad (6.11)$$

and, because $\sum_{1 \leq i \leq N} s_i^{2-a} \leq (\sum_{1 \leq i \leq N} s_i^{2-a}) (2-a \geq 1)$,

$$\sum_{1 \leq i \leq N} (s_i^{2-a} - s_i^2) \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \dots + s_N)^2} \leq \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1^2} \quad (6.12)$$

which, by assumption, is integrable with respect to ν . Then, consider the subordinator $\tilde{\sigma}$ with Laplace transform $\Phi_\sigma - \mathbf{k}^{N,\varepsilon}$ and independent of $\mathcal{D}_{\{1,2\}}^0$, such that $\sigma = \tilde{\sigma}$ on $(0, \mathcal{D}_{\{1,2\}}^0)$. As in the proof of Lemma 6.7, we use Theorem 25.17 of [97], which gives $E[\exp(a\tilde{\sigma}(t))] = \exp(-t(\Phi_\sigma(-a) - \mathbf{k}^{N,\varepsilon}))$ for all $t \geq 0$. Hence, by independence of $\tilde{\sigma}$ and $\mathcal{D}_{\{1,2\}}^0$,

$$E[\exp(a\sigma(\mathcal{D}_{\{1,2\}}^0 -))] = \mathbf{k}^{N,\varepsilon} \int_0^\infty \exp(-t\mathbf{k}^{N,\varepsilon}) \exp(-t(\Phi_\sigma(-a) - \mathbf{k}^{N,\varepsilon})) dt,$$

which is finite if and only if $\Phi_\sigma(-a) > 0$. Recall that $\Phi_\sigma(-a)$ is equal to

$$\int_S (1-s_1^{-a}) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(ds) + \int_S \left(\sum_{1 \leq i \neq j \leq N} s_i s_j + \sum_{1 \leq i \leq N} (s_i^2 - s_i^{2-a}) \right) \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \dots + s_N)^2} \nu(ds).$$

The first term converges to 0 as $\varepsilon \rightarrow 0$, by (6.11). In the second term, notice that

$$\sum_{1 \leq i \neq j \leq N} s_i s_j + \sum_{1 \leq i \leq N} (s_i^2 - s_i^{2-a}) = \left(\sum_{1 \leq i \leq N} s_i \right)^2 - \sum_{1 \leq i \leq N} s_i^{2-a},$$

which converges to $1 - \sum_i s_i^{2-a} > 0$ as $N \rightarrow \infty$. So, by (6.12) the dominated convergence theorem shows that the second term converges to $\int_S (1 - \sum_i s_i^{2-a}) \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}} \nu(ds)$ as $N \rightarrow \infty$ and this limit is strictly positive. Hence $E[\exp(a\sigma(\mathcal{D}_{\{1,2\}}^0 -)))] < \infty$ for N and $1/\varepsilon$ large enough.

On the other hand, the intermediate lemma implies that the finiteness of $E[(l_1^{N,\varepsilon})^{-a} \mathbb{1}_{\{l_1^{N,\varepsilon} \geq l_2^{N,\varepsilon}\}}]$ is equivalent to $\int_S \sum_{1 \leq i \neq j \leq N} s_i^{1-a} s_j \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \dots + s_N)^2} \nu(ds) < \infty$. The latter holds for every integers N and every $0 < \varepsilon < 1$, since $\sum_{1 \leq i \neq j \leq N} s_i^{1-a} s_j \leq N^2$ and ν integrates $s_1^{-2} \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}$. Hence the result. //

6.3.5 Dimension of the stable tree

This section is devoted to the proof of Corollary 6.1. Recall from [80] that the fragmentation F_- associated to the β -stable tree has index $1/\beta - 1$ (where $\beta \in (1, 2]$). In the case $\beta = 2$, the tree is the Brownian CRT and the fragmentation is binary (it is the fragmentation F_B of the Introduction), so that the integrability assumption of Theorem 2 is satisfied and then the dimension is 2. So suppose $\beta < 2$. The main result of [80] is that the dislocation measure $\nu(\mathbf{s})$ of F_- has the form

$$\nu(d\mathbf{s}) = C(\beta) E \left[T_1; \frac{\Delta T_{[0,1]}}{T_1} \in d\mathbf{s} \right]$$

for some constant $C(\beta)$, where $(T_x, x \geq 0)$ is a stable subordinator with index $1/\beta$ and $\Delta T_{[0,1]} = (\Delta_1, \Delta_2, \dots)$ is the decreasing rearrangement of the sequence of jumps of T accomplished within the time-interval $[0, 1]$ (so that $\sum_i \Delta_i = T_1$). By Theorem 6.2, to prove Corollary 6.1 it thus suffices to check that $E[\mathbb{1}_{\{\Delta_1/T_1 \leq 1-\varepsilon\}} T_1^3 / \Delta_1^2]$ is finite for any $\varepsilon \in (0, 1)$. The problem is that computations involving jumps of subordinators are often quite involved; they are sometimes eased by using size-biased picked jumps, whose laws are more tractable. However, one can check that if Δ_* is a size-biased picked jump among $(\Delta_1, \Delta_2, \dots)$, the quantity $E[\mathbb{1}_{\{\Delta_*/T_1 \leq 1-\varepsilon\}} T_1^3 / \Delta_*^2]$ is infinite, therefore we really have to study the joint law of (T_1, Δ_1) . This has been done in Perman [86], but we will re-explain all the details we need here.

Recall that the process $(T_x, x \geq 0)$ can be put in the Lévy-Itô form $T_x = \sum_{0 \leq y \leq x} \Delta(y)$, where $(\Delta(y), y \geq 0)$ is a Poisson point process with intensity $c u^{-1-1/\beta} du$ (the Lévy measure of T) for some constant $c > 0$. Therefore, the law of the largest jump of T before time 1 is characterized by

$$P(\Delta_1 < v) = P \left(\sup_{0 \leq y \leq 1} \Delta(y) < v \right) = \exp(-c\beta v^{-1/\beta}) \quad v > 0,$$

and by the restriction property of Poisson processes, conditionally on $\Delta_1 = v$, one can write $T_1 = v + T_1^{(v)}$, where $(T_x^{(v)}, x \geq 0)$ is a subordinator with Lévy measure $c u^{-1-1/\beta} \mathbb{1}_{\{0 \leq u \leq v\}} du$. The Laplace transform of $T_x^{(v)}$ is given by the Lévy-Khintchine formula

$$E[\exp(-\lambda T_x^{(v)})] = \exp \left(-x \int_0^v \frac{c(1 - e^{-\lambda u})}{u^{1+1/\beta}} du \right) \quad \lambda, x \geq 0,$$

in particular, $T_1^{(v)}$ admits moments of all order (by differentiating in λ) and $v^{-1}T_1^{(v)}$ has the same law as $T_{v^{-1/\beta}}^{(1)}$ (by changing variables). We then obtain

$$\begin{aligned}
& E \left[\mathbb{1}_{\{\Delta_1/T_1 \leq 1-\varepsilon\}} T_1^3 / \Delta_1^2 \right] \\
&= E \left[\Delta_1 \mathbb{1}_{\{\Delta_1/(\Delta_1+T_1^{(\Delta_1)}) \leq 1-\varepsilon\}} \left(\frac{\Delta_1 + T_1^{(\Delta_1)}}{\Delta_1} \right)^3 \right] \\
&= K_1 \int_{\mathbb{R}_+} dv v^{-1/\beta} e^{-\beta cv^{-1/\beta}} E \left[\left(1 + \frac{T_1^{(v)}}{v} \right)^3 \mathbb{1}_{\{v^{-1}T_1^{(v)} \geq \varepsilon(1-\varepsilon)^{-1}\}} \right] \\
&= K_1 \int_{\mathbb{R}_+} dv v^{-1/\beta} e^{-\beta cv^{-1/\beta}} E \left[\left(1 + T_{v^{-1/\beta}}^{(1)} \right)^3 \mathbb{1}_{\{T_{v^{-1/\beta}}^{(1)} \geq \varepsilon(1-\varepsilon)^{-1}\}} \right] \tag{6.13}
\end{aligned}$$

where $K_1 = K(\beta) > 0$. To prove this integral is finite, we study the behavior for v near 0 and ∞ . When v is small we can omit the indicator in the expectation and notice that it is dominated by $K_2 E[(T_{v^{-1/\beta}}^{(1)})^3]$ for some $K_2 > 0$; since $T_1^{(1)}$ has a moment of order 3, this is dominated by some $K_3 v^{-3/\beta}$. So the integrand in (6.13) is dominated near 0 by $K_3 v^{-4/\beta} \exp(-\beta cv^{-1/\beta})$, which is integrable. On the other hand, we see by the Hölder inequality that the expectation in (6.13) is bounded by

$$E \left[\left(1 + T_{v^{-1/\beta}}^{(1)} \right)^{3p} \right]^{1/p} P \left(T_{v^{-1/\beta}}^{(1)} > \varepsilon(1-\varepsilon)^{-1} \right)^{1/q}$$

for every $(p, q) \in (1, \infty)^2$ with $p^{-1} + q^{-1} = 1$. The expectation on the left converges to 1 by dominated convergence as $v \rightarrow \infty$, and the probability on the right is bounded by

$$\varepsilon^{-1/q} (1-\varepsilon)^{1/q} v^{-1/(\beta q)} E[T_1^{(1)}]^{1/q},$$

so that the integrand in (6.13) is bounded by $K_4 v^{-1/\beta-1/(\beta q)}$ near ∞ . By taking q such that $(1/\beta)(1+1/q) > 1$ (this is possible since $\beta < 2$) we see (6.13) is indeed finite.

Chapter 7

The exploration process of inhomogeneous continuum random trees, and an extension of Jeulin's local time identity

Contents

7.1	Introduction	184
7.1.1	Statement of results	184
7.1.2	Discussion	188
7.2	Constructing X^θ and Y^θ	189
7.3	p-trees and associated processes	193
7.3.1	The breadth-first construction	193
7.3.2	The depth-first construction	196
7.4	Convergence of p-trees to the ICRT	199
7.4.1	Proof of Theorem 7.1	202
7.4.2	Skorokhod convergence of the discrete exploration process	203
7.5	Height profile	204
7.5.1	Continuity of the cumulative height profile	204
7.5.2	Proof of Theorem 7.2	205
7.6	The exploration process	206
7.6.1	Convergence of $\sigma(\mathbf{p})^{-1}G_T^{\mathbf{p}}$ to Y^θ	207
7.6.2	Proof of Theorem 7.3	210
7.7	Miscellaneous comments	216

7.1 Introduction

This paper completes one circle of ideas (describing the inhomogeneous continuum random tree) while motivated by another (limits of non-uniform random \mathbf{p} -mappings which are essentially different from the uniform case limit). Along the way, a curious extension of Jeulin's result on total local time for standard Brownian excursion will be established.

Consider a continuous function $f : [0, 1] \rightarrow [0, \infty)$ which is an "excursion" in the sense

$$f(0) = f(1) = 0; \quad f(u) > 0, \quad 0 < u < 1.$$

Use f to make $[0, 1]$ into the pseudo-metric space with distance

$$d(u_1, u_2) := (f(u_1) - \inf_{u_1 \leq u \leq u_2} f(u)) + (f(u_2) - \inf_{u_1 \leq u \leq u_2} f(u)), \quad u_1 \leq u_2. \quad (7.1)$$

After taking the quotient by identifying points of $[0, 1]$ that are at d -pseudo distance 0, this space is a *tree* in that between any two points there is a unique path; it carries a *length measure* induced by the distance d , and a *mass measure*, with unit total mass, induced from Lebesgue measure on $[0, 1]$. An object with these properties can be abstracted as a *continuum tree*. Using a random excursion function yields a *continuum random tree* (CRT): Aldous [4, 5]. The construction of a continuum random tree \mathcal{T} via a random function f , in this context called the *exploration process* of \mathcal{T} (in Le Gall et al. [77, 49], it is instead called *height process* while the term exploration process is used for a related measure-valued process), is not the only way of looking at a CRT; there are also

- (a) constructions via line-breaking schemes
- (b) descriptions via the spanning subtrees on k random points chosen according to mass measure
- (c) descriptions as weak or strong $n \rightarrow \infty$ limits of rescaled n -vertex discrete random trees.

As discussed in [4, 5] the fundamental example is the *Brownian CRT*, whose exploration process is twice standard Brownian excursion (this was implicit in Le Gall [74]), with line-breaking construction given in Aldous [3], spanning subtree description in Aldous [5] and Le Gall [75], and weak limit (for conditional Galton-Watson trees) behavior in [4, 5] (see Marckert and Mokkadem [78] for recent review). A more general model, the *inhomogeneous continuum random tree* (ICRT) \mathcal{T}^θ , arose in Camarri and Pitman [39] as a weak limit in a certain model (\mathbf{p} -trees) of discrete random trees. The definition and simplest description of \mathcal{T}^θ is via a line-breaking construction based on a Poisson point process in the plane (Aldous and Pitman [12]), which we recall below. The spanning subtree description is set out in Aldous and Pitman [11], and the main purpose of this paper is to complete the description of \mathcal{T}^θ by determining its exploration process (Theorem 7.1).

7.1.1 Statement of results

The parameter space Θ of the ICRT \mathcal{T}^θ is defined [12] to consist of sequences $\theta = (\theta_0, \theta_1, \theta_2, \dots)$ such that

- (i) $\theta_0 \geq 0; \quad \theta_1 \geq \theta_2 \geq \dots \geq 0;$
- (ii) $\sum_i \theta_i^2 = 1;$

(iii) if $\sum_{i=1}^{\infty} \theta_i < \infty$ then $\theta_0 > 0$.

We will often consider the *finite-length* subspace Θ_{finite} of Θ for which $\theta_i = 0 \forall i > I$, for some $I \geq 0$, calling I the *length* of θ . Note that $\theta \in \Theta_{\text{finite}}$ can be specified by specifying a decreasing sequence $(\theta_1, \dots, \theta_I)$ for which $\sum_{i=1}^I \theta_i^2 < 1$; then set $\theta_0 = \sqrt{1 - \sum_{i \geq 1} \theta_i^2} > 0$.

Let $\{(U_i, V_i), i \geq 1\}$ be a Poisson measure on the first octant $\{(x, y) : 0 \leq y \leq x\}$, with intensity θ_0^2 per unit area. For every $i \geq 1$ let also $(\xi_{i,j}, j \geq 1)$ be a Poisson process on the positive real line with intensity θ_i per unit length. The hypotheses on θ entail that the set of points $\{U_i, i \geq 1, \xi_{i,j}, i \geq 1, j \geq 2\}$ is discrete and can be ordered as $0 < \eta_1 < \eta_2 < \dots$, we call them *cutpoints*. It is easy to see that $\eta_{k+1} - \eta_k \rightarrow 0$ as $k \rightarrow \infty$. By convention let $\eta_0 = 0$. Given a cutpoint $\eta_k, k \geq 1$, we associate a corresponding *joinpoint* η_k^* as follows. If the cutpoint is of the form U_i , then $\eta_k^* = V_i$. If it is of the form $\xi_{i,j}$ with $j \geq 2$, we let $\eta_k^* = \xi_{i,1}$. The hypothesis $\theta_0 > 0$ or $\sum_{i \geq 1} \theta_i = \infty$ implies that joinpoints are a.s. everywhere dense in $(0, \infty)$.

The tree is then constructed as follows. Start with a branch $[0, \eta_1]$, and recursively, given the tree is constructed at stage J , add the line segment $(\eta_J, \eta_{J+1}]$ by branching its left-end to the joinpoint η_J^* (notice that $\eta_J^* < \eta_J$ a.s. so that the construction is indeed recursive as J increases). When all the branches are attached to their respective joinpoints, relabel the joinpoint corresponding to some $\xi_{j,1}$ as joinpoint j , and forget other labels (of the form η_i or η_i^*). We obtain a metric tree (possibly with marked vertices $1, 2, \dots$), whose completion we call \mathcal{T}^θ .

A heuristic description of the structure of the ICRT goes as follows. When $\theta_0 = 1$ and hence $\theta_i = 0$ for $i \geq 1$, the tree is the Brownian CRT, it has no marked vertex and it is a.s. binary, meaning that branchpoints have degree 3. It is the only ICRT for which the width process defined below is continuous, and for which no branchpoint has degree more than 3. When $\theta \in \Theta_{\text{finite}}$ has length $I \geq 1$, the structure looks like that of the CRT, with infinitely many branchpoints with degree 3, but there exist also exactly I branchpoints with infinite degree which we call *hubs*, and these are precisely the marked vertices $1, 2, \dots, I$ corresponding to the joinpoints $\xi_{1,1}, \dots, \xi_{I,1}$ associated to the Poisson processes with intensities $\theta_1, \dots, \theta_I$ defined above. The width process defined below has I jumps with respective sizes $\theta_1, \dots, \theta_I$, which occur at distinct times a.s. These jump-sizes can be interpreted as the *local time* of the different hubs – see remark following Theorem 7.2. When $\theta \notin \Theta_{\text{finite}}$, then the hubs become everywhere dense on the tree. Whether there exists branchpoints with degree 3 or not depends on whether $\theta_0 \neq 0$ or $\theta_0 = 0$. Also, the tree can become unbounded.

It turns out that the relevant exploration process is closely related to processes recently studied for slightly different purposes. The Brownian CRT in [10], and then the ICRT in [12], were used by Aldous and Pitman to construct versions of the *standard*, and then the *general, additive coalescent*, and its dual *fragmentation process*, which are Markov processes on the state space Δ of sequences $\{(x_1, x_2, \dots) : x_i \geq 0, \sum_i x_i = 1\}$. In [10, 12] the time- t state $\mathbf{X}(t)$ is specified as the vector of masses of tree-components in the forest obtained by randomly cutting the Brownian CRT or ICRT at some rate depending on t . Bertoin [21] gave the following more direct construction. Let $(B_s^{\text{exc}}, 0 \leq s \leq 1)$ be standard Brownian excursion. For fixed $t \geq 0$ consider the process of height-above-past-minimum of

$$B_s^{\text{exc}} - ts, \quad 0 \leq s \leq 1.$$

Then its vector of excursion lengths is Δ -valued, and this process (as t varies) can be

identified with the standard case of the additive coalescent. More generally, for $\theta \in \Theta$ consider the "bridge" process

$$\theta_0 B_s^{\text{br}} + \sum_{i=1}^{\infty} \theta_i (1_{(U_i \leq s)} - s), \quad 0 \leq s \leq 1$$

where (U_i) are independent random variables with uniform law on $(0, 1)$. Use the *Vervaat transform* – relocate the space-time origin to the location of the infimum – to define an "excursion" process $X^\theta = (X_s^\theta, 0 \leq s \leq 1)$ which has positive but not negative jumps. Bertoin [22] used X^θ to construct the general additive coalescent, and Miermont [79] continued the study of fragmentation processes by this method. In this paper we use X^θ to construct a *continuous* excursion process Y^θ ; here is the essential idea. A jump of X^θ at time t_J defines an interval $[t_J, T_J]$ where $T_J := \inf\{t > t_J : X_t^\theta = X_{t_J-}^\theta\}$. Over that interval, replace X_s^θ by $X_{t_J-}^\theta + X_s^\theta - \inf_{t_J \leq u \leq s} X_u^\theta$. Do this for each jump, and let Y^θ be the resulting process. To write it in a more compact way, the formula

$$Y_s^\theta = m \left\{ \inf_{u \leq r \leq s} X_r^\theta : 0 \leq u \leq s \right\} \tag{7.2}$$

holds, where m is Lebesgue's measure on \mathbb{R} . Details are given in section 7.2. We can now state our main result.

Theorem 7.1 :

Suppose $\theta \in \Theta$ satisfies $\sum_i \theta_i < \infty$. Then the exploration process of the ICRT \mathcal{T}^θ is distributed as $\frac{2}{\theta_0^2} Y^\theta$.

As will be recalled in Sect. 7.3, the precise meaning of this theorem is: let U_1, U_2, \dots be independent uniform variables on $[0, 1]$, independent of Y^θ , and as around (7.1), replacing f by $\frac{2}{\theta_0^2} Y^\theta$, endow $[0, 1]$ with a pseudo-distance d , so that the natural quotient gives a tree $\mathcal{T}^{2\theta_0^{-2}Y}$ where $Y = Y^\theta$. Then for every $J \in \mathbb{N}$, the subtree spanned by the root (the class of 0) and the (classes of) U_1, \dots, U_J has the same law as the tree \mathcal{T}_J^θ obtained by performing the stick-breaking construction until the J -th step. Since $(U_i, i \geq 1)$ is a.s. dense in $[0, 1]$ and by uniqueness of the metric completion, \mathcal{T}^θ and $\mathcal{T}^{2\theta_0^{-2}Y}$ indeed encode the same random topological space. We also note that our proofs easily extend to showing that the hub with extra label i is associated to the class of t_i or T_i , and this class is exactly $\{s \in [t_i, T_i] : Y_s^\theta = Y_{t_i}^\theta\}$. To avoid heavier notations, we will not take these extra labels into account from now on.

When $\sum_i \theta_i = \infty$, the exploration process of the ICRT, if it exists, can be obtained as a certain weak limit of processes of the form $\frac{2}{(\theta_0^n)^2} Y^{\theta_n}$ for approximating sequences $\theta^n \in \Theta_{\text{finite}}$, and in particular, when $\theta_0 > 0$ one guesses that the exploration process of \mathcal{T}^θ will still be $\frac{2}{\theta_0^2} Y^\theta$, but we will not concentrate on this in the present paper.

Remark. Formula (7.2) is inspired by the work of Duquesne and Le Gall [49], in which continuum random trees ("Lévy trees") are built out of sample paths of Lévy processes. Our work suggest that there are many similarities between ICRTs and Lévy trees. In fact, Lévy trees turn out to be "mixings" of ICRTs in an analogous way that Lévy bridges are mixing of extremal bridges with exchangeable increments. This will be pursued elsewhere.

In principle Theorem 7.1 should be provable within the continuous-space context, but we do not see such a direct proof. Instead we use weak convergence arguments. As

background, there are many ways of coding discrete trees as walks. In particular, one can construct a Galton-Watson tree with offspring distribution ξ in terms of an excursion of the discrete-time integer-valued random walk with step distribution $\xi - 1$. In fact there are different ways to implement the same construction, which differ according to how one chooses to order vertices in the tree, and the two common choices are the *depth-first* and the *breadth-first* orders. In section 7.3 we give a construction of a random n -vertex \mathbf{p} -tree, based on using n i.i.d. uniform(0, 1) random variables to define an excursion-type function with drift rate -1 and with n upward jumps, and again there are two ways to implement the construction depending on choice of vertex order. These constructions seem similar in spirit to, but not exactly the same as, those used in the *server system* construction in [22] or the *parking process* construction in Chassaing and Louchard [41]. When $\theta \in \Theta_{\text{finite}}$, by analyzing asymptotics of the (appropriately rescaled) discrete excursion using depth-first order, in the asymptotic regime where convergence to the ICRT holds, we get weak convergence to the process Y^θ , and we show that this discrete excursion asymptotically agrees with $\theta_0^2/2$ times the discrete exploration process; we extend this to the case $\sum_i \theta_i < \infty$ by approximating the tree \mathcal{T}^θ by the tree \mathcal{T}^{θ^n} associated to the truncated sequence $(\theta_1, \dots, \theta_n, 0, \dots)$, and that is the proof of Theorem 7.1. It is a curious feature of the convergence of approximating \mathbf{p} -trees to \mathcal{T}^θ that the rescaled discrete approximation process converges to $\frac{2}{\theta_0^2} Y^\theta$ for a topology which is weaker than the usual Skorokhod topology. In the course of proving Theorem 7.1, we will give sufficient conditions for this stronger convergence to happen.

For any continuum tree with mass measure μ , we can define

$$\bar{W}(h) = \mu\{x : \text{ht}(x) \leq h\}, \quad h \geq 0$$

where the *height* $\text{ht}(x)$ of x is its distance to the root. If $\bar{W}(h) = \int_0^h W(y) dy, h \geq 0$ then $W(y)$ is the "width" or "height profile" of the tree (analogous to the size of a particular generation in a branching process model). The time-changed function $(W(\bar{W}^{-1}(u)), 0 \leq u \leq 1)$ can be roughly interpreted as the width of the layer of the tree containing vertex u , where vertices are labelled by $[0, 1]$ in breadth-first order. Parallel to (but simpler than) the proof of Theorem 7.1 sketched above, we show that excursions coding \mathbf{p} -trees using *breadth-first* order converge to X^θ , and agree asymptotically with the height profile (sizes of successive generations) of the \mathbf{p} -tree. In other words

Theorem 7.2 :

Let $\theta \in \Theta$. For the ICRT \mathcal{T}^θ the width process $W(y) = W^\theta(y)$ exists, and

$$(W^\theta((\bar{W}^\theta)^{-1}(u)), 0 \leq u \leq 1) \stackrel{d}{=} (X^\theta(u), 0 \leq u \leq 1).$$

Qualitatively, in breadth-first traversal of the ICRT, when we encounter a hub at some $0 < u < 1$ we expect the time-changed width function $W(\bar{W}^{-1}(u))$ to jump by an amount representing a "local time" measuring relative numbers of edges at that hub. Theorem 7.2 shows these jump amounts are precisely the θ -values of the hubs.

When $\sum_i \theta_i < \infty$, combining Theorems 7.1 and 7.2 gives a result whose statement does not involve trees:

Corollary 7.1 :

Let $\theta \in \Theta$ satisfy $\sum_i \theta_i < \infty$. The process $\frac{2}{\theta_0^2} Y^\theta$ has an occupation density

$$\left| \begin{array}{l} (W^\theta(y), 0 \leq y < \infty) \text{ satisfying} \\ (W^\theta((\bar{W}^\theta)^{-1}(u)), 0 \leq u \leq 1) \stackrel{d}{=} (X^\theta(u), 0 \leq u \leq 1). \end{array} \right.$$

Note that the ‘‘Lamperti-type’’ relation between W^θ and X^θ is easily inverted as

$$(X_{L^{-1}(y)}^\theta, y \geq 0) \stackrel{d}{=} (W^\theta(y), y \geq 0), \tag{7.3}$$

where

$$L(t) := \int_0^t \frac{ds}{X_s^\theta} \in [0, \infty], \quad 0 \leq t \leq 1.$$

This provides a generalization of the following result of Jeulin [63] (see also Biane-Yor [35]), which from our viewpoint is the Brownian CRT case where $\theta_0 = 1$. Let $(l_u, 0 \leq u < \infty)$ be occupation density for $(B_s^{\text{exc}}, 0 \leq s \leq 1)$. Then

$$(\frac{1}{2}l_{u/2}, 0 \leq u < \infty) \stackrel{d}{=} (B_{L^{-1}(u)}^{\text{exc}}, 0 \leq u < \infty)$$

where $L(t) := \int_0^t \frac{1}{B_s^{\text{exc}}} ds$. One might not have suspected a possible generalization of this identity to jump processes without the interpretation provided by the ICRT.

Theorem 7.2 has the following other corollary:

Corollary 7.2 :

$$\left| \begin{array}{l} \text{For any } \theta \in \Theta, \text{ the height } \sup_{v \in \mathcal{T}^\theta} \text{ht}(v) \text{ of the ICRT } \mathcal{T}^\theta \text{ has the same law as} \\ \int_0^1 \frac{ds}{X_s^\theta}. \end{array} \right.$$

7.1.2 Discussion

As formulated above, the purpose of this paper is to prove Theorems 7.1 and 7.2 concerning the ICRT. But we have further motivation. As ingredients of the proof, we take a known result (Proposition 7.1) on weak convergence of random \mathbf{p} -trees to the ICRT, and improve it to stronger and more informative versions (Propositions 7.2 and 7.3). The Theorems and these ingredients will be used in a sequel [8] studying asymptotics of random \mathbf{p} -mappings. By using Joyal’s bijection between mappings and trees, one can in a sense reduce questions of convergence of \mathbf{p} -mappings to convergence of random \mathbf{p} -trees. In particular, under a *uniform asymptotic negligibility* hypothesis which implies that the exploration process of \mathbf{p} -trees converges to Brownian excursion, one can use a ‘‘continuum Joyal functional’’ (which takes Brownian excursion to reflecting Brownian motion) to show [6] that the exploration process of the random \mathbf{p} -mappings converges to reflecting Brownian bridge. The results of the present paper give the limit exploration process Y^θ for more general sequences of \mathbf{p} -trees, and to deduce convergence of the associated random \mathbf{p} -mappings we need to understand how the continuum Joyal functional acts on Y^θ . This is the subject of the sequel [8].

7.2 Constructing X^θ and Y^θ

Let $\theta \in \Theta$, and consider a standard Brownian bridge B^{br} , and independent uniformly distributed random variables $(U_i, i \geq 1)$ in $[0, 1]$, independent of B^{br} . Define

$$X_t^{\text{br},\theta} = \theta_0 B_t^{\text{br}} + \sum_{i=1}^{\infty} \theta_i (1_{\{U_i \leq t\}} - t), \quad 0 \leq t \leq 1. \quad (7.4)$$

From Kallenberg [65], the sum on the right converges a.s. uniformly on $[0, 1]$. Then $X^{\text{br},\theta}$ has exchangeable increments and infinite variation, and by Knight [71] and Bertoin [22] it attains its overall minimum at a unique location t_{\min} , which is a continuity point of $X^{\text{br},\theta}$. Consider the Vervaat transform X^θ of $X^{\text{br},\theta}$, defined by

$$X_t^\theta = X_{t+t_{\min}}^{\text{br},\theta} - X_{t_{\min}}^{\text{br},\theta}, \quad 0 \leq t \leq 1, \quad (7.5)$$

where the addition is modulo 1. Then X^θ is an excursion-type process with infinite variation, and a countable number of upward jumps with magnitudes equal to $(\theta_i, i \geq 1)$. See Figure 7.1. Write $t_j = U_j - t_{\min} \pmod{1}$ for the location of the jump with size θ_j in X^θ .

For each $j \geq 1$ such that $\theta_j > 0$, write $T_j = \inf\{s > t_j : X_s^\theta = X_{t_j}^\theta\}$, which exists because the process X has no negative jumps. Notice that if for some $i \neq j$ one has $t_j \in (t_i, T_i)$, then one also has $T_j \in (t_i, T_i)$, so the intervals (t_i, T_i) are nested. Given a sample path of X^θ , for $0 \leq u \leq 1$ and $i \geq 1$ such that $\theta_i > 0$, let

$$R_i^\theta(u) = \begin{cases} \inf_{t_i \leq s \leq u} X_s^\theta - X_{t_i}^\theta & \text{if } u \in [t_i, T_i] \\ 0 & \text{else.} \end{cases} \quad (7.6)$$

If $\theta_i = 0$ then let R_i^θ be the null process on $[0, 1]$. We then set

$$Y^\theta = X^\theta - \sum_{i \geq 1} R_i^\theta, \quad (7.7)$$

which is defined as the pointwise decreasing limit of $X^\theta - \sum_{1 \leq i \leq n} R_i^\theta$ as $n \rightarrow \infty$. See Figure 7.2. It is immediate that Y^θ is a non-negative process on $[0, 1]$. More precisely, for any $0 \leq u \leq s \leq 1$ and i such that $u \geq t_i$, $R_i^\theta(u)$ is equal to the magnitude of the jump (if any) accomplished at time t_i by the increasing process

$$\underline{X}_s^\theta(u) = \inf_{u \leq r \leq s} X_r^\theta, \quad 0 \leq u \leq s.$$

Since the Lebesgue measure of the range of an increasing function $(f(s), 0 \leq s \leq t)$ is $f(t) - f(0)$ minus the sum of sizes of jumps accomplished by f , we obtain that

$$Y_s^\theta = m\{\underline{X}_s^\theta(u) : 0 \leq u \leq s\} \quad 0 \leq s \leq 1, \quad (7.8)$$

where m is Lebesgue measure. This easily implies that Y^θ is a continuous (possibly null) process, and since the largest jump of $X^\theta - \sum_{1 \leq i \leq n} R_i^\theta$ is θ_{n+1} , which tends to 0 as $n \rightarrow \infty$, a variation of Dini's theorem implies that (7.7) holds in the sense of uniform convergence.

The process Y^θ is an excursion-type process on $(0, 1)$. Moreover, since by classical properties of Brownian bridges the local infima of $X^{\text{br},\theta}$ are all distinct, the only local

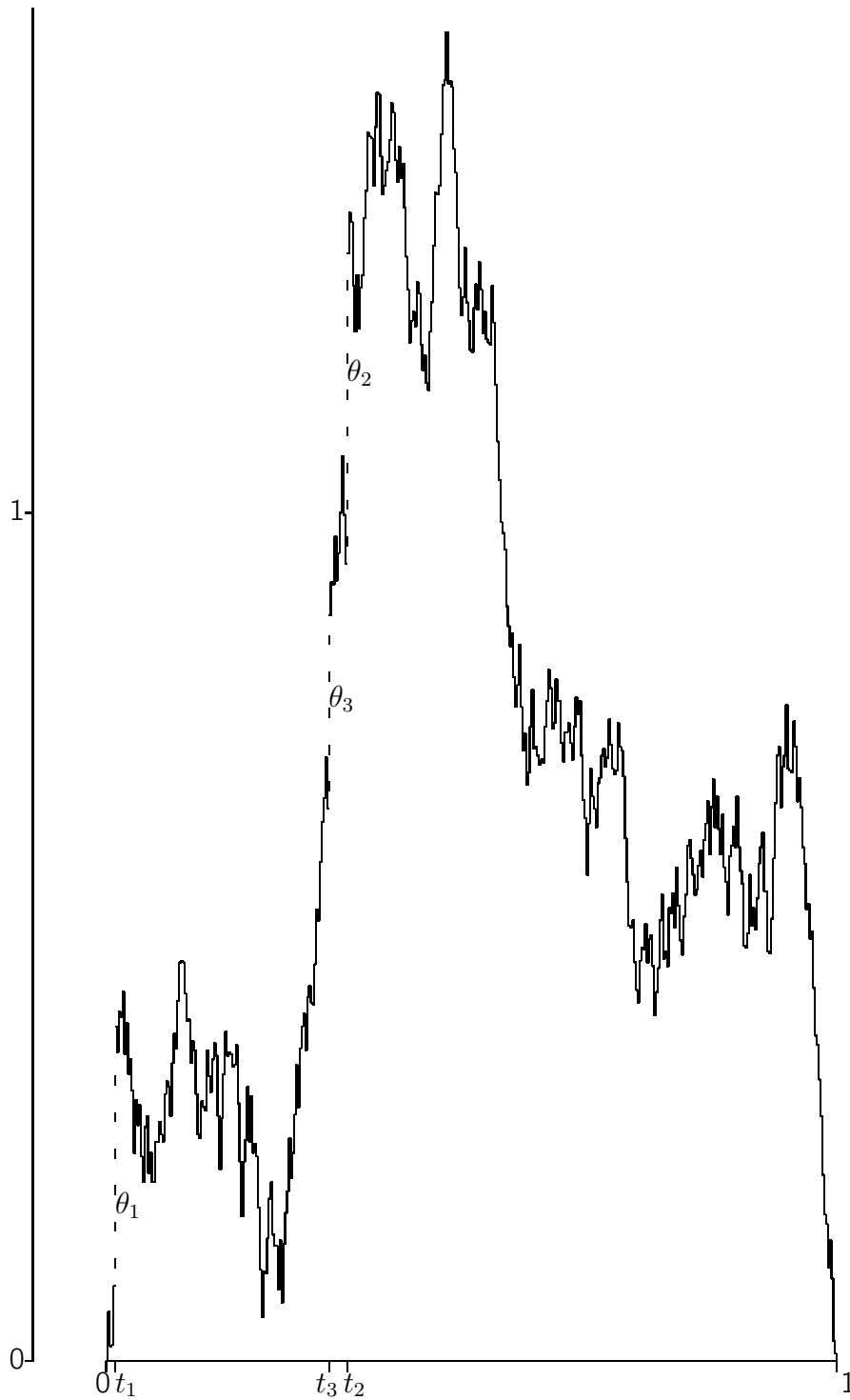


Figure 7.1: A realization of $(X_s^\theta, 0 \leq s \leq 1)$ with $I = 3$ and $(\theta_0, \theta_1, \theta_2, \theta_3) = (0.862, 0.345, 0.302, 0.216)$ ($I = 3$). The jumps are marked with dashed lines; the jump of height θ_i occurs at time t_i .

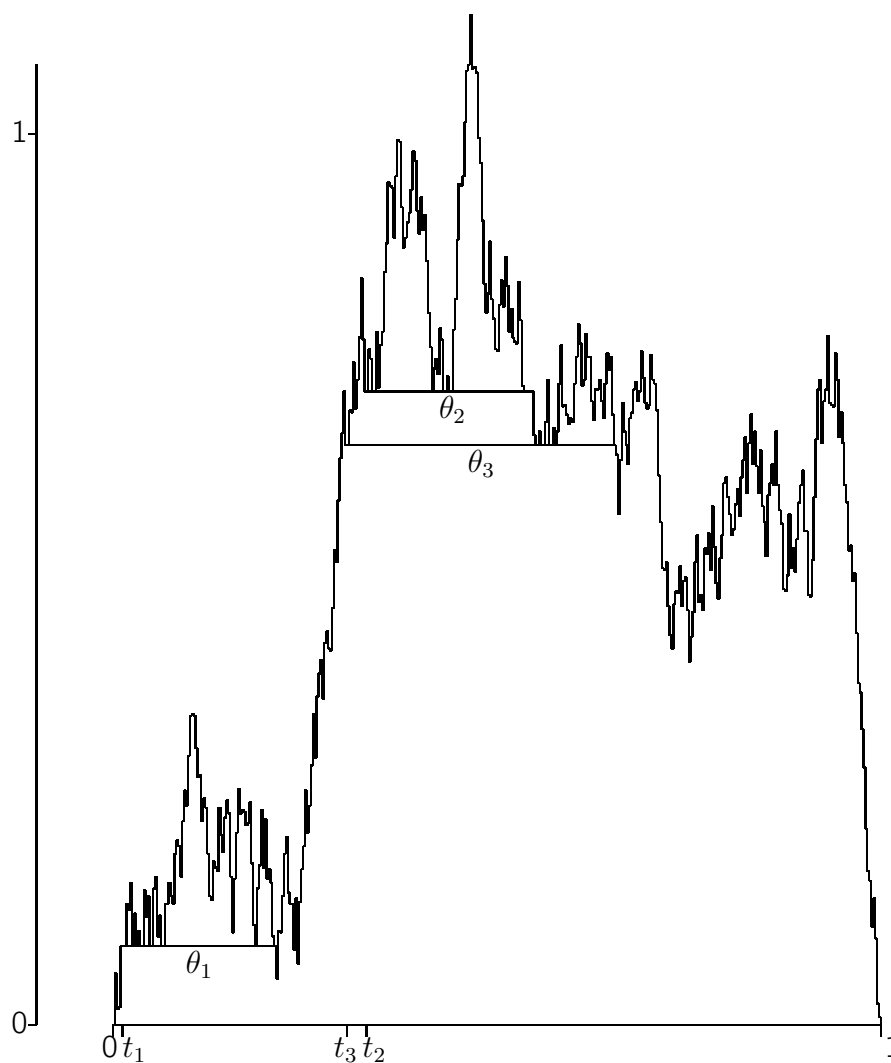


Figure 7.2: The process $(Y_s^\theta, 0 \leq s \leq 1)$ constructed from the process $(X_s^\theta, 0 \leq s \leq 1)$ in Figure 7.1. The “reflecting” portions of the path corresponding to jumps of X^θ are marked by the θ_i

infima that Y^θ attains an infinite number of times are in the intervals $[t_i, T_i]$. Let us record some other sample path properties of Y^θ .

Lemma 7.1 :

Suppose $\theta \in \Theta$ has length $I \in \mathbb{N} \cup \{\infty\}$ and $\theta_0 > 0$. Almost surely, the values $(X_{t_i}^\theta, i \geq 1)$ taken by X^θ at its jump times are not attained at local minima of X^θ . Also, the times t_i are a.s. not right-minima of X^θ in the sense that there does not exist $\varepsilon > 0$ such that $X_s^\theta \geq X_{t_i}^\theta$ for $s \in [t_i, t_i + \varepsilon]$.

Proof. Let $X_i^{\text{br},\theta}(s) = X_s^{\text{br},\theta} - \theta_i(1_{\{U_i \leq s\}} - s)$, which is independent of U_i . The shifted process $X_i^{\text{br},\theta}(\cdot + t) - X_i^{\text{br},\theta}(t)$ (with addition modulo 1) has same law as $X_i^{\text{br},\theta}$ for every t , so the fact that 1 is not the time of a local extremum for $X_i^{\text{br},\theta}$ and that $|X_i^{\text{br},\theta}(1-t)|/t \rightarrow \infty$ as $t \rightarrow 0$ (e.g. by [66, Theorem 2.2 (i)] and time-reversal) implies by adding back $\theta_i(1_{\{U_i \leq \cdot\}} - \cdot)$ to $X_i^{\text{br},\theta}$ that U_i is a.s. not a local minimum of $X^{\text{br},\theta}$. The statement about right-minima is obtained similarly, using the behavior of $X_i^{\text{br},\theta}$ at 0 rather than 1.

Next, since $X^{\text{br},\theta}$ is the sum of a Brownian bridge B^{br} and an independent process, the increments of $X^{\text{br},\theta}$ have continuous densities, as does the Brownian bridge (except of course the increment $X^{\text{br},\theta}(1) - X^{\text{br},\theta}(0) = 0$ a.s.). The probability that the minimum of $X^{\text{br},\theta}$ in any interval $[a, b]$ with distinct rational bounds not containing U_i equals $X_{U_i}^{\text{br},\theta}$ is therefore 0. This finishes the proof. \square

The following lemma will turn out to be useful at the end of the proof of Theorem 7.1.

Lemma 7.2 :

Let $\theta \in \Theta$ satisfy $\sum_i \theta_i < \infty$, and write $\theta^n = (\theta_0, \theta_1, \dots, \theta_n)$. Define X^{θ^n} as above, but where the sum defining X^{br,θ^n} is truncated at n . Last, define Y^{θ^n} as in (7.8) with X^{θ^n} instead of X^θ . Then Y^{θ^n} converges a.s. uniformly to Y^θ as $n \rightarrow \infty$.

Proof. We want to estimate the uniform norm $\|Y^{\theta^n} - Y^\theta\|$, which by definition is $\|X^{\theta^n} - X^\theta - \sum_{i \geq 1} (R_i^{\theta^n} - R_i^\theta)\|$ with obvious notations. The first problem is that X^{br,θ^n} may not attain its overall infimum at the same time as $X^{\text{br},\theta}$, so that jump times for X^{θ^n} and X^θ may not coincide anymore. So, rather than using X^{θ^n} we consider $X'_n(s) = X^{\text{br},\theta^n}(s + t_{\min}) - X^{\text{br},\theta^n}(t_{\min})$ (with addition modulo 1) where t_{\min} is the time at which $X^{\text{br},\theta}$ attains its infimum. Then $X'_n \rightarrow X^\theta$ uniformly. Define $R'_{n,i}$ as in (7.6) but for the process X'_n and write $Y'_n = X'_n - \sum_{1 \leq i \leq n} R'_{n,i}$. Notice that Y'_n is just a slight space-time shift of Y^{θ^n} , so by continuity of Y^{θ^n} and Y^θ it suffices to show that $Y'_n \rightarrow Y^\theta$ uniformly. It is thus enough to show that $\|\sum_{1 \leq i \leq n} (R'_{n,i} - R_i^\theta)\| \rightarrow 0$ as $n \rightarrow \infty$. It is easy that for each $i \geq 1$, one has uniform convergence of $R'_{n,i}$ to R_i^θ . Therefore, it suffices to show that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| \sum_{k \leq i \leq n} R'_{n,i} \right\| = 0,$$

which is trivial because $\|R'_{n,i}\| \leq \theta_i$, and $\sum_i \theta_i < \infty$ by hypothesis.

Remark. Again, one guesses that the same result holds in the general $\theta_0 > 0$ case, so that the proof of Theorem 7.1 should extend to this case. However, the fact that $\sum_i \theta_i$ might be infinite does not *a priori* prevent vanishing terms of the sum $\sum_{1 \leq i \leq n} R'_{n,i}$ to accumulate, so the proof might become quite technical.

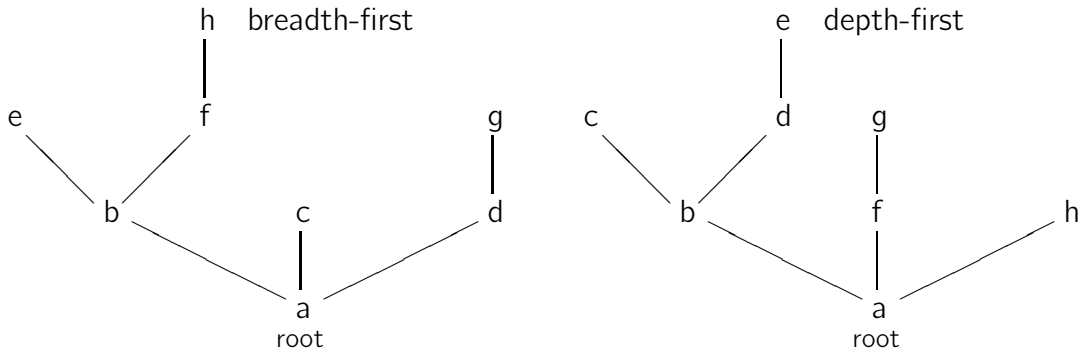


Figure 7.3: A planar tree, with the two orderings of vertices as a, b, c, d, e, f, g, h

7.3 Constructions of p-trees and associated excursion processes

Write \mathbf{T}_n for the set of rooted trees \mathbf{t} on vertex-set $[n]$, where \mathbf{t} is directed towards its root. Fix a probability distribution $\mathbf{p} = (p_1, \dots, p_n)$. Recall that associated with \mathbf{p} is a certain distribution on \mathbf{T}_n , the \mathbf{p} -tree

$$P(\mathcal{T} = \mathbf{t}) = \prod_v p_v^{d_v}, \quad d_v \text{ in-degree of } v \text{ in } \mathbf{t}. \tag{7.9}$$

See [90] for systematic discussion of the \mathbf{p} -tree model. We shall define two maps $\psi_{\mathbf{p}} : [0, 1]^n \rightarrow \mathbf{T}_n$ such that, if (X_1, \dots, X_n) are independent $U(0, 1)$ then each $\psi_{\mathbf{p}}(X_1, \dots, X_n)$ has the distribution (7.9). The two definitions are quite similar, but the essential difference is that $\psi_{\mathbf{p}}^{\text{breadth}}$ uses a *breadth-first* construction whereas $\psi_{\mathbf{p}}^{\text{depth}}$ uses a *depth-first* construction.

7.3.1 The breadth-first construction

The construction is illustrated in Figure 7.4. Fix distinct $(x_1, \dots, x_n) \in [0, 1]^n$. Picture this as a configuration of particles on the circle of unit circumference, where particle i is at position x_i and has a "weight" p_i associated with it. Define

$$F^{\mathbf{p}}(u) = -u + \sum_i p_i \mathbf{1}_{(x_i \leq u)}, \quad 0 \leq u \leq 1. \tag{7.10}$$

There exists some particle v such that $F^{\mathbf{p}}(x_v -) = \inf_u F^{\mathbf{p}}(u)$: assume the particle is unique. Let $v = v_1, v_2, \dots, v_n$ be the ordering of particles according to the natural ordering of positions $x_{v_1} < x_{v_2} < \dots$ around the circumference of the circle. (In Figure 7.4 we have $v_1 = 4$ and the ordering is 4, 8, 2, 3, 7, 1, 5, 6). Write $y(1) = x_{v_1}$ and for $2 \leq j \leq n$ let $y(j+1) = y(j) + p_{v_j} \text{ mod } 1$. So $y(n+1) = y(1)$ and the successive intervals $[y(j), y(j+1)], 1 \leq j \leq n$ are adjacent and cover the circle. We assert

$$x_{v_j} \in [y(1), y(j)), \quad 2 \leq j \leq n. \tag{7.11}$$

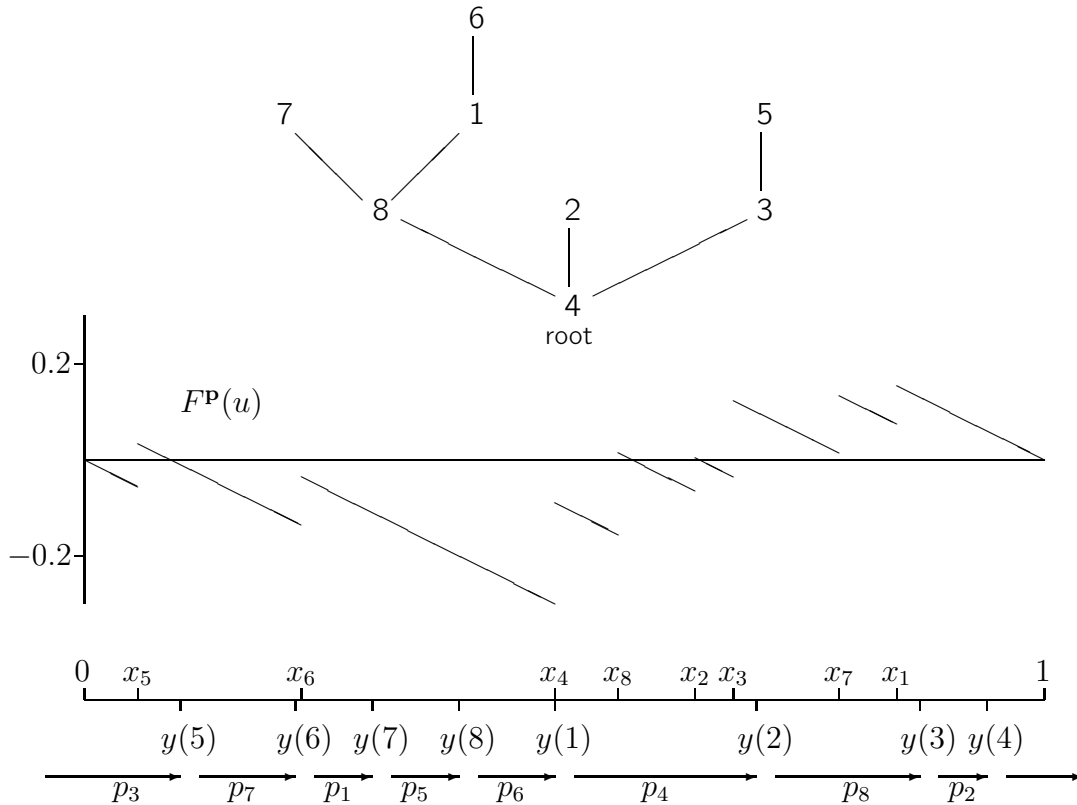


Figure 7.4: The construction of the tree $\psi_{\mathbf{p}}^{\text{breadth}}(x_1, \dots, x_8)$

To argue by contradiction, suppose this fails first for j . Then $[y(1), y(j)]$, interpreted mod 1, contains particles v_1, \dots, v_{j-1} only. Since $y(j) - y(1) = p_{v_1} + \dots + p_{v_{j-1}}$ this implies $F^{\mathbf{P}}(y(j)-) = F^{\mathbf{P}}(y(1)-)$, contradicting uniqueness of the minimum.

We specify the tree $\psi_{\mathbf{p}}^{\text{breadth}}(x_1, \dots, x_n)$ by:

- v_1 is the root
- the children of v_j are the particles v with $x_v \in (y(j), y(j+1))$.

By (7.11), any child v_k of v_j has $k > j$, so the graph cannot contain a cycle. If it were a forest and not a single tree, then the component containing the root v_1 would consist of vertices v_1, \dots, v_j for some $j < n$. Then the interval $[y(1), y(j+1)]$ would contain only the particles v_1, \dots, v_j , contradicting (7.11) for $j+1$.

Thus the construction does indeed give a tree. From the viewpoint of this construction it would be natural to regard the tree as *planar* (or *ordered*: the d_v children of v are distinguished as first, second, etc) but we disregard order and view trees in \mathbf{T}_n as unordered.

Now consider the case where $(x_1, \dots, x_n) = (X_1, \dots, X_n)$ are independent $U(0, 1)$. Fix an unordered tree \mathbf{t} and write v_1 for its root. Fix an arbitrary $x_{v_1} \in (0, 1)$ and condition on $X_{v_1} = x_{v_1}$. Consider the chance that the construction yields the particular tree \mathbf{t} . For this to happen, the particles corresponding to the d_{v_1} children of v_1 must fall into the interval $[x_{v_1}, x_{v_1} + p_{v_1}]$, which has chance $p_{v_1}^{d_{v_1}}$. Inductively, for each vertex v an interval of length p_v is specified and it is required that d_v specified particles fall into that interval, which has chance $p_v^{d_v}$. So the conditional probability of constructing \mathbf{t} is indeed the probability in (7.9), and hence so is the unconditional probability.

Remark. Note that in the argument above we do not start by conditioning on $F^{\mathbf{P}}$ having its minimum at x_{v_1} , which would affect the distribution of the (X_i) .

We now derive an interpretation (7.13,7.14) of the function $F^{\mathbf{P}}$ at (7.10), which will be used in the asymptotic setting later. From now on we also suppose that for $j \geq 2$, $y(j)$ is not a jump time for $F^{\mathbf{P}}$ to avoid needing the distinction between $F^{\mathbf{P}}(y(j))$ and $F^{\mathbf{P}}(y(j)-)$; this is obviously true a.s. when the jump times are independent uniform, which will be the relevant case.

For $2 \leq j \leq n$, vertex v_j has some parent $v_{z(j)}$, where $1 \leq z(j) < j$. By induction on j ,

$$F^{\mathbf{P}}(y(j)) - F^{\mathbf{P}}(y(1)-) = \sum_{i:j>j, z(i)\leq j} p_{v_i}.$$

In words, regarding \mathbf{t} as ordered, the sum is over vertices i which are in the same generation as j but *later* than j ; and over vertices i in the next generation whose parents are *before* j or are j itself. For $h \geq 1$, write $t(h)$ for the number of vertices at height $\leq h-1$. The identity above implies

$$F^{\mathbf{P}}(y(t(h)+1)) - F^{\mathbf{P}}(y(1)-) = \sum_{v:\text{ht}(v)=h} p_v.$$

Also by construction

$$y(t(h)+1) - y(1) \bmod 1 = \sum_{v:\text{ht}(v)\leq h-1} p_v.$$

We can rephrase the last two inequalities in terms of the "excursion" function

$$F^{\text{exc},\mathbf{P}}(u) := F^{\mathbf{P}}(y(1) + u \bmod 1) - F^{\mathbf{P}}(y(1)-), \quad 0 \leq u \leq 1 \quad (7.12)$$

and of $u(h) := y(t(h)+1) - y(1) \bmod 1$. Then

$$u(h) = \sum_{v:\text{ht}(v)\leq h-1} p_v \quad (7.13)$$

$$F^{\text{exc},\mathbf{P}}(u(h)) = \sum_{v:\text{ht}(v)=h} p_v. \quad (7.14)$$

So the weights of successive generations are coded within $F^{\text{exc},\mathbf{P}}(\cdot)$, as illustrated in Figure 7.5. Note that to draw Figure 7.5 we replace x_i by

$$x'_i := x_i - y(1) \bmod 1.$$

Remark. There is a queuing system interpretation to the breadth-first construction, which was pointed out to us by a referee. In this interpretation, the customer labelled i arrives at time x'_i and requires a total service time p_i . If customers are served according to the FIFO rule (first-in first-out) then $F^{\text{exc},\mathbf{P}}(u)$ is the remaining amount of time needed to serve the customers in line at time u .

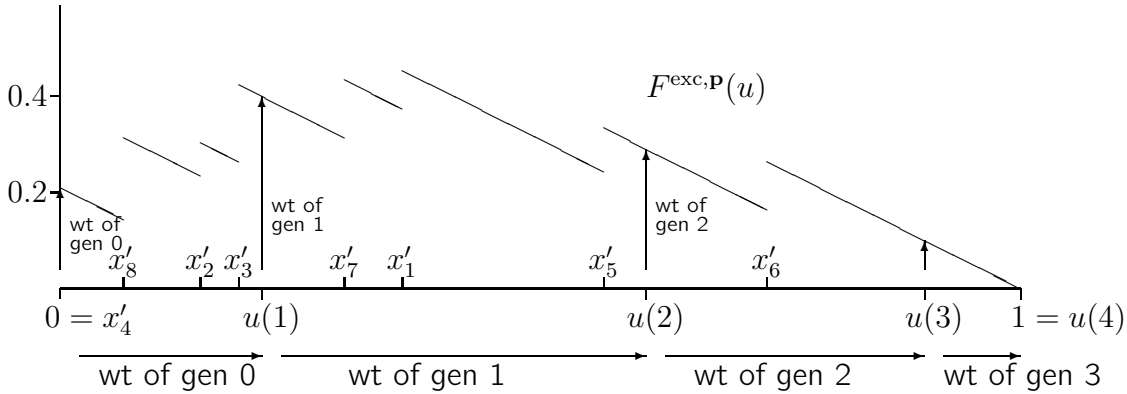


Figure 7.5: $F^{\text{exc}, \mathbf{P}}(\cdot)$ codes the weights of successive generations (*wt of gen*) of the \mathbf{p} -tree in Figure 7.4

7.3.2 The depth-first construction

The construction is illustrated in Figure 7.6, using the same (x_i) and (p_i) as before, and hence the same $F^{\mathbf{P}}(u)$. In the previous construction we “examined” particles in the order v_1, v_2, \dots, v_n ; we defined $y(1) = v_1$ and inductively

- $y(j + 1) = y(j) + p_{v_j} \bmod 1$
- the children of v_j are the particles v with $x_v \in (y(j), y(j + 1))$.

In the present construction we shall examine particles in a different order w_1, w_2, \dots, w_n and use different $y'(j)$ to specify the intervals which determine the offspring of a parent. Start as before with $w_1 = v_1$ and $y'(1) = x_{w_1}$. Inductively set

- $y'(j + 1) = y'(j) + p_{w_j} \bmod 1$
- the children of w_j are the particles v with $x_v \in (y'(j), y'(j + 1))$.
- w_{j+1} is
 - the first child of w_j , if any; else
 - the next unexamined child of parent(w_j), if any; else
 - the next unexamined child of parent(parent(w_j)), if any; else
 - and so on.

Here “unexamined” means “not one of w_1, \dots, w_j ” and “next” uses the natural order of children of the same parent.

Figure 7.6 and its legend talk through the construction in a particular example, using the same (x_i) and (p_i) as in Figure 7.4. Checking that $\psi_{\mathbf{P}}^{\text{depth}}(X_1, \dots, X_n)$ has distribution (7.9), i.e. is a random \mathbf{p} -tree, uses exactly the same argument as before.

As in Figure 7.4, we next examine the first child $w_2 = 8$ of the root, set $y'(3) = y'(2) + p_8$, and let the children of 8 be the vertices $\{7, 1\}$ for which $x_v \in (y'(2), y'(3))$. At this stage the constructions differ. We next examine vertex 7, being the first child of vertex 8, by setting $y'(4) = y'(3) + p_7$; the children of vertex 8 are the vertices v with $x_v \in (y'(3), y'(4))$, and it turns out there are no such vertices. We continue examining vertices in the depth-first order 4, 8, 7, 1, 5, 2, 6, 3.

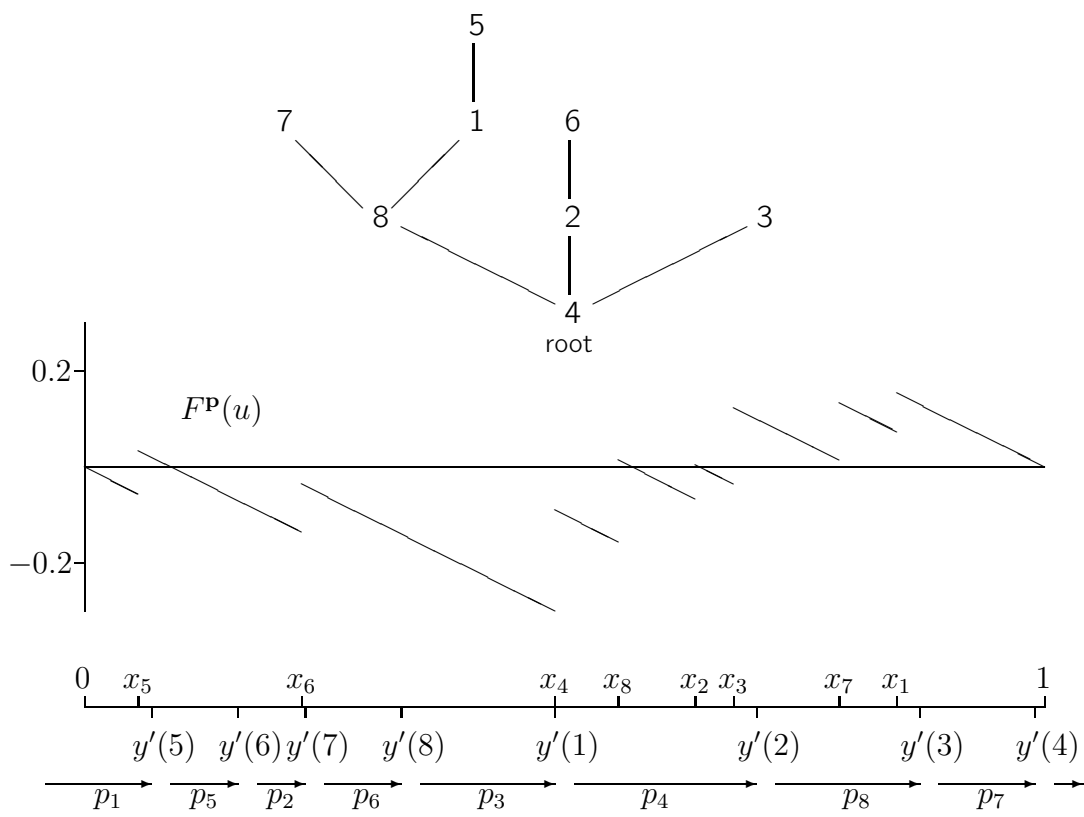


Figure 7.6: The construction of the tree $\psi_{\mathbf{p}}^{\text{depth}}(x_1, \dots, x_n)$

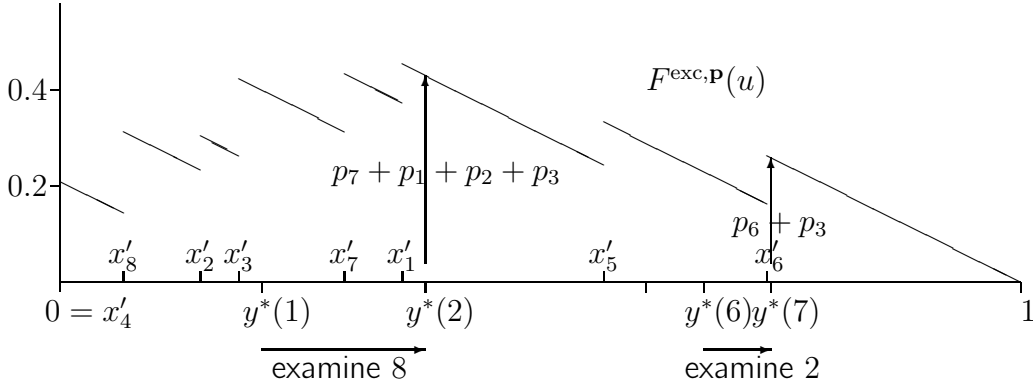


Figure 7.7: Relation (7.15) in the depth-first construction

As with the breadth-first construction, the point of the depth-first construction is that the excursion function $F^{\text{exc}, \mathbf{P}}(\cdot)$ tells us something about the distribution of the tree. For each vertex v of $\psi_{\mathbf{p}}^{\text{depth}}(x_1, \dots, x_n)$ there is a path $\text{root} = y_0, y_1, \dots, y_j = v$ from the root to v . For each $0 \leq i < j$ the vertex y_{i+1} is a child of vertex y_i ; let $y_{i,1}, y_{i,2}, \dots$ be the later children of y_i , and let $y_{j,1}, y_{j,2}, \dots$ be all children of v . Write $\mathcal{N}(v) = \cup_{0 \leq i \leq j} \{y_{i,1}, y_{i,2}, \dots\}$.

In the u -scale of $F^{\text{exc}, \mathbf{P}}(u)$, we finish "examining" vertex w_i at time $y^*(i) := y'(i) - y'(1)$. For vertex $v = w_i$ set $e(v) = y^*(i)$. Then the relevant property of $F^{\text{exc}, \mathbf{P}}$ is

$$F^{\text{exc}, \mathbf{P}}(e(v)) = \sum_{w \in \mathcal{N}(v)} p_w, \quad \forall v. \tag{7.15}$$

See Figure 7.7 for illustration. As before, in Figure 7.7 the position of the jump of height p_i is moved from x_i to $x'_i := x_i - y'(1) \bmod 1$. At first sight, relation (7.15) may not look useful. But we shall see in section 7.6.2 that in the asymptotic regime the right side of (7.15) can be related to $\sum_w \text{ancestor of } v p_w$ which in turn relates to the height of v .

Remark. We might alternatively have defined the tree $\psi_{\mathbf{p}}^{\text{depth}}(x_1, \dots, x_n)$ in a way that would have been less suited for the forthcoming analysis, but which is worth mentioning. It is based on the LIFO-queuing system construction of Galton-Watson trees in Le Gall-Le Jan [77] which we sketch here. Imagine vertex i is a customer in a line which requires a treatment time p_i . The customer i arrives at time x_i and customers are treated according to the Last In First Out rule. After relocating the the time-origin is at the time when the minimum of the bridge $F^{\mathbf{P}}$ is attained, the first customer in line will also be the last to get out. Then we say that vertex i is a parent of vertex j if customer j arrives in a time-interval when i was being treated. Notice that the tree thus defined is in general different from $\psi_{\mathbf{p}}^{\text{depth}}(x_1, \dots, x_n)$.

It is easy to see, using induction and the same kind of arguments as above, that taking x_1, \dots, x_n to be independent uniform random variables builds a \mathbf{p} -tree (in order that i has k children, k uniform random variables must interrupt the service of i which takes total time p_i , so this has probability p_i^k). It is also easy that the order of customer arrivals (after

relocating the time origin) corresponds to the depth-first order on the tree. In particular, the cyclic depth-first random order of vertices in a \mathbf{p} -tree is the uniform cyclic order on the n vertices.

7.4 Convergence of \mathbf{p} -trees to the ICRT

Here we review known results concerning convergence of \mathbf{p} -trees to the ICRT, and spotlight what new results are required to prove Theorems 7.1 and 7.2.

The general notion (7.1) of exploration process of a continuum random tree can be reinterpreted as follows. Fix $J \geq 1$. Let $(U_j, 1 \leq j \leq J)$ be independent $U(0, 1)$ r.v.s and let $U_{(1)} < U_{(2)} < \dots < U_{(J)}$ be their order statistics. To an excursion-type process $(H_s, 0 \leq s \leq 1)$ associate the random $2J - 1$ -vector

$$\left(H_{U_{(1)}}, \inf_{U_{(1)} \leq s \leq U_{(2)}} H_s, H_{U_{(2)}}, \inf_{U_{(2)} \leq s \leq U_{(3)}} H_s, \dots, H_{U_{(J)}} \right). \quad (7.16)$$

This specifies a random tree-with-edge-lengths, with J leaves, as follows.

- The path from the root to the i 'th leaf has length $H_{U_{(i)}}$.
- The paths from the root to the i 'th leaf and from the root to the $(i + 1)$ 'st leaf have their branchpoint at distance $\inf_{U_{(i)} \leq s \leq U_{(i+1)}} H_s$.

Now label the i 'th leaf as vertex i' , where $U_{(i)} = U_{i'}$. Write the resulting tree as \mathcal{T}_J^H . Call this the *sampling a function* construction.

On the other hand one can use a continuum random tree \mathcal{T} to define a random tree-with-edge-lengths \mathcal{T}_J as follows.

- Take a realization of \mathcal{T} .
- From the mass measure on that realization, pick independently J points and label them as $\{1, 2, \dots, J\}$.
- Construct the spanning tree on those J points and the root; this is the realization of \mathcal{T}_J . Call this the *sampling a CRT* construction.

As discussed in detail in [5], the relationship

the exploration process of \mathcal{T} is distributed as $(H_s, 0 \leq s \leq 1)$

is equivalent to

$$\mathcal{T}_J \stackrel{d}{=} \mathcal{T}_J^H, \quad \forall J \geq 1,$$

(the background hypotheses in [5] were rather different, assuming path-continuity for instance, but the ideas go through to our setting.) In our setting, there is an explicit description of the distribution of the spanning tree \mathcal{T}_J^θ derived from the ICRT \mathcal{T}^θ (see [11]), so to prove Theorem 7.1 it is enough to verify

$$\mathcal{T}_J^\theta \stackrel{d}{=} \mathcal{T}_J^{2Y/\theta_0^2}, \quad \forall J \geq 1 \quad (7.17)$$

for $Y = Y^\theta$ defined at (7.7). In principle one might verify (7.17) directly, but this seems difficult even in the case $J = 1$. Instead we shall rely on weak convergence arguments, starting with the known Proposition 7.1 below.

Consider a probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ which is *ranked*: $p_1 \geq p_2 \geq \dots \geq p_n > 0$. In the associated \mathbf{p} -tree (7.9), pick J vertices independently from distribution \mathbf{p} , label them as $[J]$ in order of pick, take the spanning tree on the root and these J vertices, regard each edge as having length 1, and then delete degree-2 vertices to form edges of positive integer length. Call the resulting random tree $\mathcal{S}_J^{\mathbf{p}}$. Define $\sigma(\mathbf{p}) := \sqrt{\sum_i p_i^2}$. Now consider a sequence $\mathbf{p}_n = (p_{ni})$ of ranked probability distributions which satisfy

$$\lim_n \sigma(\mathbf{p}_n) = 0; \quad \lim_n p_{ni}/\sigma(\mathbf{p}_n) = \theta_i, \quad 1 \leq i \leq I; \quad \lim_n p_{ni}/\sigma(\mathbf{p}_n) = 0, \quad i > I \quad (7.18)$$

for some limit $\boldsymbol{\theta} = (\theta_0, \dots, \theta_I) \in \Theta_{\text{finite}}$. For a tree \mathbf{t} and a real constant $\sigma > 0$ define $\sigma \otimes \mathbf{t}$ to be the tree obtained from \mathbf{t} by multiplying edge-lengths by σ . The following result summarizes Propositions 2, 3 and 5(b) of [12]. Recall \mathcal{T}_J^θ is obtained by sampling the ICRT \mathcal{T}^θ .

Proposition 7.1 :

For a sequence $\mathbf{p} = \mathbf{p}_n$ satisfying (7.18), as $n \rightarrow \infty$

$$\sigma(\mathbf{p}) \otimes \mathcal{S}_J^{\mathbf{p}} \xrightarrow{d} \mathcal{T}_J^\theta, \quad J \geq 1.$$

The tree $\mathcal{S}_J^{\mathbf{p}}$ may not be well-defined because two of the J sampled vertices may be the same; but part of Proposition 7.1 is that this probability tends to zero.

Now consider the "bridge" process $F^{\mathbf{p}}$ at (7.10), where from now on the jump times x_1, \dots, x_n are uniformly distributed independent random variables. Standard results going back to Kallenberg [65] show that, under the asymptotic regime (7.18),

$$(\sigma^{-1}(\mathbf{p})F^{\mathbf{p}}(s), \quad 0 \leq s \leq 1) \xrightarrow{d} (X_s^{\text{br},\theta}, \quad 0 \leq s \leq 1),$$

where $X^{\text{br},\theta}$ is defined at (7.4). It follows by an argument that can be found e.g. in [22] (using the continuity of the bridge process at its minimum) that the associated excursion process $F^{\text{exc},\mathbf{p}}$ at (7.12) satisfies

$$(\sigma^{-1}(\mathbf{p})F^{\text{exc},\mathbf{p}}(s), \quad 0 \leq s \leq 1) \xrightarrow{d} (X_s^\theta, \quad 0 \leq s \leq 1) \quad (7.19)$$

for X^θ defined at (7.5).

Recall from section 7.2 how (Y_s^θ) is constructed as a modification of (X_s^θ) . We next describe a parallel modification of $F^{\text{exc},\mathbf{p}}$ to construct a process $G_I^{\mathbf{p}}$. Given a realization of the \mathbf{p} -tree obtained via the depth-first construction illustrated in Figure 7.7, and given $I \geq 0$, let $\mathcal{B}_i \subseteq [n]$ be the set of vertices which are the child of some vertex i in from $\{1, \dots, I\}$. In the setting of the depth-first construction of the \mathbf{p} -tree from $F^{\text{exc},\mathbf{p}}$, illustrated in Figure 7.7, for every vertex $v \in \mathcal{B}_i$, define

$$\begin{aligned} \rho_v(u) &= 0 & 0 \leq u \leq x'_i \\ &= p_v & x'_i < u \leq e(v) - p_v \\ &= e(v) - u & e(v) - p_v \leq u \leq e(v) \\ &= 0 & e(v) \leq u \leq 1. \end{aligned} \quad (7.20)$$

and then let

$$r_i^{\mathbf{p}}(u) = \sum_{v \in \mathcal{B}_i} \rho_v(u) \quad (7.21)$$

and

$$G_I^{\mathbf{p}}(u) = F^{\text{exc}, \mathbf{p}}(u) - \sum_{i=1}^I r_i^{\mathbf{p}}(u). \quad (7.22)$$

We will show in section 7.6.1 that (7.19) extends to

Proposition 7.2 :

For a sequence $\mathbf{p} = \mathbf{p}_n$ satisfying (7.18) with limit $\boldsymbol{\theta} = (\theta_0, \dots, \theta_I)$, as $n \rightarrow \infty$

$$(\sigma^{-1}(\mathbf{p})G_I^{\mathbf{p}}(u), 0 \leq u \leq 1) \xrightarrow{d} (Y^{\boldsymbol{\theta}}(u), 0 \leq u \leq 1)$$

for $Y^{\boldsymbol{\theta}}$ defined at (7.7), for the Skorokhod topology.

We finally come to the key issue; we want to show that $G_I^{\mathbf{p}}(\cdot)$ approximates the (discrete) exploration process. In the depth-first construction of the \mathbf{p} -tree \mathcal{T} from $F^{\text{exc}, \mathbf{p}}$, we examine vertex w_i during $(y^*(i-1), y^*(i)]$. Define

$$H^{\mathbf{p}}(u) := \text{height of } w_i \text{ in } \mathcal{T}; \quad u \in (y^*(i-1), y^*(i)]. \quad (7.23)$$

Roughly, we show that *realizations* of $\frac{\theta_0^2}{2}\sigma(\mathbf{p})H^{\mathbf{p}}(\cdot)$ and of $\sigma^{-1}(\mathbf{p})G_I^{\mathbf{p}}(\cdot)$ are close. Precisely, we will prove the following in section 7.6.2

Proposition 7.3 :

Let $\boldsymbol{\theta} \in \Theta_{\text{finite}}$. There exists a sequence $\mathbf{p} = \mathbf{p}_n$ satisfying (7.18) with limit $\boldsymbol{\theta}$, such that as $n \rightarrow \infty$,

$$\sup_{u \in [0,1]} \left| \frac{\theta_0^2}{2}\sigma(\mathbf{p})H^{\mathbf{p}}(u) - \sigma(\mathbf{p})^{-1}G_I^{\mathbf{p}}(u) \right| \xrightarrow{p} 0.$$

The next result, Lemma 7.3, relates the exploration process $H^{\mathbf{p}}$ at (7.23) to the spanning trees $\mathcal{S}_j^{\mathbf{p}}$. This idea was used in ([14]; proof of Proposition 7) but we say it more carefully here. Given $u^1 \in (0, 1)$ define, as in (7.23),

$$w^1 = w_i \text{ for } i \text{ specified by } u^1 \in (y^*(i-1), y^*(i)].$$

Given $0 < u^1 < u^2 < 1$, define w^2 similarly, and let vertex b be the branchpoint of the paths from the root to vertices w^1 and w^2 . Distinguish two cases.

Case (i): $w^1 = w^2$ or w^1 is an ancestor of w^2 . In this case $b = w^1$ and so trivially $\text{ht}(b) = \min_{u^1 \leq u \leq u^2} H^{\mathbf{p}}(u)$.

Case (ii): otherwise, b is a strict ancestor of both w^1 and w^2 . In this case we assert

$$\text{ht}(b) = \min_{u^1 \leq u \leq u^2} H^{\mathbf{p}}(u) - 1,$$

because vertex b appears, in the depth-first order, strictly before vertex w^1 . Then consider the set of vertices between w^1 and w^2 (inclusive) in the depth first order. This set contains the child w^* of b which is an ancestor of w^2 or is w^2 itself, and $\text{ht}(w^*) = \text{ht}(b) + 1$. But the set cannot contain any vertex of lesser height.

Now the length of the interval $(y^*(i-1), y^*(i)]$ equals p_{w_i} by construction. So if U^1 has uniform distribution on $(0, 1)$ then the corresponding vertex W^1 at (7.23) has distribution \mathbf{p} . Combining with the discussion above regarding branchpoint heights gives

Lemma 7.3 :

Fix \mathbf{p} , make the depth-first construction of a \mathbf{p} -tree and define $H^{\mathbf{p}}$ by (7.23). Fix J . Take U_1, \dots, U_J independent uniform $(0, 1)$ and use them and $H^{\mathbf{p}}$ to define a tree-with-edge-lengths $\mathcal{T}_J^{\mathbf{p}}$ via the "sampling a function" construction below (7.16). Then this tree agrees, up to perhaps changing heights of branchpoints by 1, with a tree distributed as the tree $\mathcal{S}_J^{\mathbf{p}}$ defined above (7.18).

7.4.1 Proof of Theorem 7.1

We now show how the ingredients above (of which, Proposition 7.2 and Proposition 7.3 remain to be proved later) are enough to prove Theorem 7.1.

Let $\mathbf{p} = \mathbf{p}_n$ satisfy (7.18) with limit $\boldsymbol{\theta} \in \Theta_{\text{finite}}$. Fix J and take independent U_1, \dots, U_J with uniform $(0, 1)$ distribution. Proposition 7.2 implies that as $n \rightarrow \infty$

$$\begin{aligned} & \sigma^{-1}(\mathbf{p}) \left(G_I^{\mathbf{p}}(U_{(1)}), \inf_{U_{(1)} \leq s \leq U_{(2)}} G_I^{\mathbf{p}}(s), G_I^{\mathbf{p}}(U_{(2)}), \inf_{U_{(2)} \leq s \leq U_{(3)}} G_I^{\mathbf{p}}(s), \dots, G_I^{\mathbf{p}}(U_{(J)}) \right) \\ & \xrightarrow{d} \left(Y^{\boldsymbol{\theta}}(U_{(1)}), \inf_{U_{(1)} \leq s \leq U_{(2)}} Y^{\boldsymbol{\theta}}(s), Y^{\boldsymbol{\theta}}(U_{(2)}), \inf_{U_{(2)} \leq s \leq U_{(3)}} Y^{\boldsymbol{\theta}}(s), \dots, Y^{\boldsymbol{\theta}}(U_{(J)}) \right). \end{aligned}$$

By making the particular choice of (\mathbf{p}_n) used in Proposition 7.3,

$$\begin{aligned} & \frac{1}{2} \theta_0^2 \sigma(\mathbf{p}) \left(H^{\mathbf{p}}(U_{(1)}), \inf_{U_{(1)} \leq s \leq U_{(2)}} H^{\mathbf{p}}(s), H^{\mathbf{p}}(U_{(2)}), \inf_{U_{(2)} \leq s \leq U_{(3)}} H^{\mathbf{p}}(s), \dots, H^{\mathbf{p}}(U_{(J)}) \right) \\ & \xrightarrow{d} \left(Y^{\boldsymbol{\theta}}(U_{(1)}), \inf_{U_{(1)} \leq s \leq U_{(2)}} Y^{\boldsymbol{\theta}}(s), Y^{\boldsymbol{\theta}}(U_{(2)}), \inf_{U_{(2)} \leq s \leq U_{(3)}} Y^{\boldsymbol{\theta}}(s), \dots, Y^{\boldsymbol{\theta}}(U_{(J)}) \right). \quad (7.24) \end{aligned}$$

Appealing to Lemma 7.3, this implies

$$\frac{1}{2} \theta_0^2 \sigma(\mathbf{p}) \otimes \mathcal{S}_J^{\mathbf{p}} \xrightarrow{d} \mathcal{T}_J^Y$$

where the right side denotes the tree-with-edge-lengths obtained from sampling the function $Y^{\boldsymbol{\theta}}$, and where convergence is the natural notion of convergence of shapes and edge-lengths ([14] sec. 2.1). Rescaling by a constant factor,

$$\sigma(\mathbf{p}) \otimes \mathcal{S}_J^{\mathbf{p}} \xrightarrow{d} \mathcal{T}_J^{2\theta_0^{-2}Y}.$$

But Proposition 7.1 showed

$$\sigma(\mathbf{p}) \otimes \mathcal{S}_J^{\mathbf{p}} \xrightarrow{d} \mathcal{T}_J^{\boldsymbol{\theta}}$$

where the right side is the random tree-with-edge-lengths obtained by sampling the ICRT $\mathcal{T}^{\boldsymbol{\theta}}$. So we have established (7.17) and thereby proved Theorem 7.1 in the case $\boldsymbol{\theta} \in \Theta_{\text{finite}}$.

In the case $\sum_i \theta_i < \infty$, write $\boldsymbol{\theta}^n$ for the truncated sequence $(\theta_0, \dots, \theta_n, 0, \dots)$, and recall from Lemma 7.2 that $Y^n = Y^{\boldsymbol{\theta}^n}$ converges uniformly to $Y^{\boldsymbol{\theta}}$. By previous considerations this entails

$$\mathcal{T}_J^{2\theta_0^{-2}Y^n} \xrightarrow{d} \mathcal{T}_J^{\boldsymbol{\theta}}$$

for every $J \geq 1$. On the other hand, we have proved that the left-hand term has the same law as $c(\boldsymbol{\theta}^n)^{-1} \otimes \mathcal{T}_J^{c(\boldsymbol{\theta}^n)\boldsymbol{\theta}^n}$ where $c(\boldsymbol{\theta}^n) = (\sum_{0 \leq i \leq n} \theta_i^2)^{-1/2}$ is the renormalization constant so that $c(\boldsymbol{\theta}^n)\boldsymbol{\theta}^n \in \Theta$. It thus remain to show that this converges to \mathcal{T}^θ . Plainly the term $c(\boldsymbol{\theta}^n)$ converges to 1 and is unimportant. The result is then straightforward from the line-breaking construction of the ICRT: \mathcal{T}_J^θ can be build out of the first (at most) $2J$ points (cutpoints and their respective joinpoints) of the superimposition of infinitely many Poisson point processes on the line $(0, \infty)$. It is easily checked that taking only the superimposition of the n first Poisson processes allows us to construct jointly a reduced tree with same law as $c(\boldsymbol{\theta}^n)^{-1} \mathcal{T}_J^{c(\boldsymbol{\theta}^n)\boldsymbol{\theta}^n}$ on the same probability space. So for n large the first $2J$ points of both point processes coincide and we have actually $c(\boldsymbol{\theta}^n)^{-1} \mathcal{T}_J^{c(\boldsymbol{\theta}^n)\boldsymbol{\theta}^n} = \mathcal{T}_J^\theta$ on this probability space. \square

Remark. Theorem 7.1 essentially consists of an “identify the limit” problem, and that is why we are free to choose the approximating \mathbf{p}_n in Proposition 7.3. But having proved Theorem 7.1, we can reverse the proof above to show that (7.24) holds true for *any* \mathbf{p} satisfying (7.18) with limiting $\boldsymbol{\theta} \in \Theta_{\text{finite}}$. Indeed, the convergence in (7.24) is equivalent to that of $\sigma(\mathbf{p}) \otimes \mathcal{S}_J^\mathbf{p}$ to \mathcal{T}_J^θ for every J .

7.4.2 Skorokhod convergence of the discrete exploration process

Suppose again that the ranked probability \mathbf{p} satisfies (7.18) with limit $\boldsymbol{\theta} \in \Theta_{\text{finite}}$ with length I . As observed in [14] (Theorem 5 and Proposition 7), the convergence in (7.24) is equivalent to weak convergence of the rescaled exploration process to Y^θ , but using a certain topology on function space which is weaker than the usual Skorokhod topology. As noted in [14] Example 28, assumption (7.18) is paradoxically not sufficient to ensure convergence in the usual Skorokhod topology; the obstacle in that example was the presence of exponentially many (in terms of $1/\sigma(\mathbf{p})$) exponentially small p -values. In this section we present some crude sufficient conditions (7.25, 7.26); Proposition 7.3 will be a natural consequence of the proof in section 7.6.2. The hypotheses are as follows.

First, we prevent very small p -values by making the assumption

$$1/p_* = o(\exp(\alpha/\sigma(\mathbf{p}))) \text{ for all } \alpha > 0 \quad (7.25)$$

where

$$p_* := \min_i p_i.$$

Second, we will assume that most of the small $p(\cdot)$ -weights, as compared with the I first, are of order $\sigma(\mathbf{p})^2$. Write $\bar{\mathbf{p}} = (0, 0, \dots, p_{I+1}, \dots, p_n)$ for the sequence obtained from \mathbf{p} by truncating the first I terms. Let ξ have distribution \mathbf{p} on $[n]$, and write $\bar{p}(\xi)$ for the r.v. \bar{p}_ξ . We assume that there exists some r.v. $0 \leq Q < \infty$ such that the following “moment generating function” convergence holds:

$$\lim_{n \rightarrow \infty} E \left[\exp\left(\frac{\lambda \bar{p}(\xi)}{\sigma(\mathbf{p})^2}\right) \right] = E [\exp \lambda Q] < \infty, \quad (7.26)$$

for every λ in some neighborhood of 0. This implies that $\bar{p}(\xi)/\sigma(\mathbf{p})^2 \xrightarrow{d} Q$, and also that the moments of all order exist and converge to those of Q .

Then we have

Theorem 7.3 :

Suppose \mathbf{p} satisfies (7.18) with limit $\theta \in \Theta_{\text{finite}}$. Under extra hypotheses (7.25,7.26),

$$\sigma(\mathbf{p})H^{\mathbf{p}} \xrightarrow{d} \frac{2}{\theta_0^2} Y^\theta \quad (7.27)$$

in the usual Skorokhod topology.

Remark. The proof (section 7.6.2) rests upon applying the elementary large deviation inequality $P(S > s) \leq e^{-\lambda s} E \exp(\lambda S)$ to the independent sums involved in (7.39,7.41). Hypothesis (7.26) is designed to make the application very easy; it could surely be replaced by much weaker assumptions, such as plain moment convergence conditions.

We would also guess that the convergence in (7.27) also holds with $H^{\mathbf{p}}$ replaced by more general exploration processes, and in particular the “classical” one, where each vertex v is visited during an interval of length $1/n$ instead of p_v , or the Harris (or contour) walk on the tree (see e.g. [49, Chapter 2]). We can easily verify the first guess. Consider the \mathbf{p} -tree $\psi_{\mathbf{p}}^{\text{depth}}(X_1, \dots, X_n)$ defined as in section 7.3.2 out of uniformly distributed independent r.v. Write w_1, \dots, w_n for the vertices in depth-first order, and let $H^n(t)$ be the height of the w_i for which $i/n \leq t < (i+1)/n$ (and with the convention $H^n(1) = H^n(1-)$).

Corollary 7.3 :

Suppose \mathbf{p} satisfies (7.18) with limit $\theta \in \Theta_{\text{finite}}$. Under extra hypotheses (7.25,7.26),

$$\sigma(\mathbf{p})H^n \xrightarrow{d} \frac{2}{\theta_0^2} Y^\theta \quad (7.28)$$

in the usual Skorokhod topology.

Proof. By the functional weak law of large numbers for sampling without replacement, we know that if π is a uniform random permutation of the n first integers, the fact that $\max_i p_{n,i} \rightarrow 0$ as $n \rightarrow \infty$ implies that if $(S_n^0(t), 0 \leq t \leq 1)$ is the linear interpolation between points $((i/n, \sum_{1 \leq k \leq i} p_{\pi(i)}), 0 \leq i \leq n)$ then $\sup_{0 \leq t \leq 1} |S_n^0(t) - t| \rightarrow 0$ in probability. Now by the remark at the end of Sect. 7.3.2, the cyclic order on vertices associated to the depth-first order is uniform, so with the above notation for $i = w_1, \dots, w_n$ the linear interpolation S_n between points $((i/n, \sum_{1 \leq k \leq i} p_{w_k}), 0 \leq i \leq n)$ converges uniformly to the identity in probability, since it is a (random) cyclic permutation of a function distributed as S_n^0 . Noticing that $H^n = H^{\mathbf{p}} \circ S_n$, the result follows. \square

The convergence of the Harris walk follows from this proposition by the arguments in [49, Chapter 2.4].

7.5 Height profile

This section is devoted to the proof of Theorem 7.2. In this section, we do not assume that $\theta \in \Theta$ has finite length nor that $\theta_0 > 0$.

7.5.1 Continuity of the cumulative height profile

We first prove the following intermediate lemma. Recall that the *cumulative height process* of the \mathcal{T}^θ is defined as $\bar{W}^\theta(\cdot) = \mu^\theta \{v \in \mathcal{T}^\theta : \text{ht}(v) \leq \cdot\}$, where μ^θ is the mass measure of

\mathcal{T}^θ .

Lemma 7.4 :

The cumulative height process \bar{W}^θ is continuous for a.a. realizations of \mathcal{T}^θ . Moreover, it has no flat interval, except its (possibly empty) final constancy interval, equal to $[\sup_{v \in \mathcal{T}^\theta} \text{ht}(v), \infty)$.

Proof of Lemma 7.4. Recall the recursive line-breaking construction of \mathcal{T}^θ in the introduction, and the fact from [12] that the tree constructed at stage J is distributed as the reduced tree \mathcal{T}_J^θ of Sect. 7.3. From this, we see that the leaves labelled $1, 2, \dots$ are a.s. at pairwise different heights, meaning that the measure $d\bar{W}^\theta$ has no atom. Moreover, if \bar{W}^θ had a flat interval (other than the final constancy interval), this would mean that for some $h < \sup_{v \in \mathcal{T}^\theta}$, no leaf picked according to the mass measure can have a height in say $(h - \epsilon, h + \epsilon)$ for some $\epsilon > 0$. But let v be a vertex of \mathcal{T}^θ at height h . By the line-breaking construction, the fact that branches have size going to 0 and the “dense” property of joinpoints, we can find a joinpoint η^* at a distance $< \epsilon/2$ of v and so that the corresponding branch has length $\eta < \epsilon/2$. Since the leaves that are at the right-end of branches of the line-breaking construction are distributed as independent sampled leaves from the mass measure, this contradicts the above statement. \square

7.5.2 Proof of Theorem 7.2

The reader can consult [67] for a similar treatment of convergence of the height profile of Galton-Watson trees to a time-changed excursion of a stable Lévy process.

Suppose that $\mathbf{p} = \mathbf{p}^n$ satisfies the asymptotic regime (7.18). Let $\mathcal{T}^{\mathbf{p}}$ be the \mathbf{p} -tree, and \mathcal{T}^θ the limiting ICRT. Define \bar{W}^θ as above and recall the notation $u(h)$ in (7.13). For $h \geq 0$ let

$$W^{\mathbf{p}}(h) = \sum_{v \in \mathcal{T}^{\mathbf{p}}, \text{ht}(v) = \lfloor \frac{h}{\sigma(\mathbf{p})} \rfloor} p_v = u\left(\left\lfloor \frac{h}{\sigma(\mathbf{p})} \right\rfloor + 1\right) - u\left(\left\lfloor \frac{h}{\sigma(\mathbf{p})} \right\rfloor\right), \quad h \geq 0$$

and $\bar{W}^{\mathbf{p}}(h) = u(\lfloor h/\sigma(\mathbf{p}) \rfloor)$. Now let U_1, U_2, \dots be independent uniform(0,1) random variables. The sequence $((\bar{W}^{\mathbf{p}})^{-1}(U_j), j \geq 1)$ has the law of the heights of an i.i.d. random sample of vertices of $\mathcal{T}^{\mathbf{p}}$, chosen according to \mathbf{p} , and the same holds for $((\bar{W}^\theta)^{-1}(U_j), j \geq 1)$ and the tree \mathcal{T}^θ , with the mass measure μ^θ as common law. For $J \geq 1$ let $\bar{W}_J^{\mathbf{p}}(h)$ be the associated empirical distribution of the first J terms, defined by

$$\bar{W}_J^{\mathbf{p}}(h) = \frac{1}{J} \sum_{i=1}^J 1_{\{(\bar{W}^{\mathbf{p}})^{-1}(U_i) \leq h\}} = \frac{1}{J} \sum_{i=1}^J 1_{\{U_i \leq \bar{W}^{\mathbf{p}}(h)\}},$$

and define $\bar{W}_J^\theta(h)$ in a similar way.

By Proposition 7.1, we have that the random Stieltjes measure $d\bar{W}_J^{\mathbf{p}}$ converges in law to $d\bar{W}_J^\theta$ as $n \rightarrow \infty$ for every $J \geq 1$. Moreover, the empirical measure of an i.i.d. J -sample of leaves distributed according to μ^θ converges to μ^θ , implying $d\bar{W}_J^\theta \xrightarrow{d} d\bar{W}^\theta$ as $J \rightarrow \infty$. Thus, for $h \geq 0$ and $J_n \rightarrow \infty$ slowly enough,

$$d\bar{W}_{J_n}^{\mathbf{p}} \xrightarrow{d} d\bar{W}^\theta.$$

Now let $F_J(x) = J^{-1} \sum_{i=1}^J 1_{\{U_i \leq x\}}$ be the empirical distribution associated to the uniform variables U_1, \dots, U_J . Then $\sup_{h \geq 0} |\bar{W}_J^{\mathbf{P}}(h) - \bar{W}^{\mathbf{P}}(h)| \leq \sup_{x \in [0,1]} |F_J(x) - x|$, which by the Glivenko-Cantelli Theorem converges to 0 as $J \rightarrow \infty$, and this convergence is uniform in n . Hence the random measure $d\bar{W}^{\mathbf{P}}$ converges in distribution to $d\bar{W}^\theta$ for the weak topology on measures. Thanks to Lemma 7.4 we may improve this to

$$\bar{W}^{\mathbf{P}}(\cdot) \xrightarrow{d} \bar{W}^\theta(\cdot)$$

where the convergence is weak convergence of processes for the topology of uniform convergence. It is then an elementary consequence of Lemma 7.4 that $\bar{W}^{\mathbf{P}}((\bar{W}^{\mathbf{P}})^{-1}(\cdot))$ converges in law for the uniform convergence topology to the identity function on $[0, 1]$.

Equation (7.14) can be rewritten as

$$W^{\mathbf{P}}(h) = F^{\text{exc}, \mathbf{P}}(\bar{W}^{\mathbf{P}}(h)), \quad h \geq 0, \quad (7.29)$$

so the convergence in distribution of $\bar{W}^{\mathbf{P}}$, the fact that its limit is strictly increasing and continuous, and (7.19) imply that the sequence of random processes $(\sigma(\mathbf{p})^{-1}W^{\mathbf{P}})$ is tight. Moreover the limit in law of $\bar{W}^{\mathbf{P}}$ is continuous, thus by [61, p. 353, Corollary 3.33], the pair $(\sigma(\mathbf{p})^{-1}W^{\mathbf{P}}, \bar{W}^{\mathbf{P}})$ is tight, and up to extraction of a subsequence, we can suppose that $(\sigma(\mathbf{p})^{-1}W^{\mathbf{P}}, \bar{W}^{\mathbf{P}}) \xrightarrow{d} (W, \bar{W}')$ for some process W , and where \bar{W}' has the same law as \bar{W}^θ . Suppose further by Skorokhod's embedding theorem that the convergence is almost-sure. By definition

$$\int_0^h \frac{W^{\mathbf{P}}(u)}{\sigma(\mathbf{p})} du = \bar{W}^{\mathbf{P}}(h - \sigma(\mathbf{p})) + R(n, h)$$

where $R(n, h) \leq \bar{W}^{\mathbf{P}}(h) - \bar{W}^{\mathbf{P}}(h - \sigma(\mathbf{p}))$ goes to 0 uniformly as $n \rightarrow \infty$ by continuity of the limiting \bar{W}' . So necessarily,

$$\int_0^h W(u) du = \bar{W}'_h, \quad h \geq 0$$

for every $h \geq 0$, so that the only possible limit W is the density of $d\bar{W}'$. Therefore, the height profile W^θ of the ICRT exists and $(\sigma(\mathbf{p})^{-1}W^{\mathbf{P}}, \bar{W}^{\mathbf{P}}) \xrightarrow{d} (W^\theta, \bar{W}^\theta)$. Looking back at (7.29) we have

$$\sigma(\mathbf{p})^{-1}W^{\mathbf{P}}((\bar{W}^{\mathbf{P}})^{-1}(u)) = \sigma(\mathbf{p})^{-1}F^{\text{exc}, \mathbf{P}}(\bar{W}^{\mathbf{P}}((\bar{W}^{\mathbf{P}})^{-1}(u))), \quad 0 \leq u \leq 1,$$

so by the convergence of $\bar{W}^{\mathbf{P}}((\bar{W}^{\mathbf{P}})^{-1}(\cdot))$ and (7.19), we obtain convergence in distribution of the right-hand side to X^θ . By the convergence in law of $W^{\mathbf{P}}$ this finally implies that $W^\theta((\bar{W}^\theta)^{-1}(\cdot)) \stackrel{d}{=} X^\theta(\cdot)$ and Theorem 7.2 is proved. \square

Proof of Corollary 7.2. By the proof of Lemma 7.4, the only constant interval of the width process of the ICRT is $[\sup_{v \in \mathcal{T}^\theta} \text{ht}(v), \infty)$. Thus the height of the tree, $\sup_{v \in \mathcal{T}^\theta} \text{ht}(v)$, is the first point after which the width process remains constant. By (7.3), this point has same law as $\int_0^1 ds/X_s^\theta$. \square

7.6 The exploration process

To shorten notation, for $A \subseteq [n]$ we write $p(A)$ for the quantity $\sum_{j \in A} p_j$.

7.6.1 Convergence of $\sigma(\mathbf{p})^{-1}G_I^{\mathbf{p}}$ to Y^θ

This subsection is devoted to the proof of Proposition 7.2. Let \mathbf{p} satisfy (7.18) for some limiting $\theta \in \Theta_{\text{finite}}$, with length I . In this subsection we suppose that the \mathbf{p} -tree $\mathcal{T}^{\mathbf{p}}$ is constructed from the process $F^{\text{exc},\mathbf{p}}$ by the depth-first search construction of section 7.3. Moreover, since we have (7.19) the convergence in law $\sigma(\mathbf{p})^{-1}F^{\text{exc},\mathbf{p}} \xrightarrow{d} X^\theta$, we suppose by Skorokhod's representation theorem that our probability space is such that the convergence holds almost surely. Recall that in the depth-first search construction of the \mathbf{p} -tree out of the process $F^{\text{exc},\mathbf{p}}$, the i -th examined vertex $v = w_i$ is examined during an interval $[e(v) - p_v, e(v))$, during which the labels of jumps of $F^{\text{exc},\mathbf{p}}$ determine the set \mathcal{B}_v of children of v .

We begin with two useful observations. First, if v is a vertex of $\mathcal{T}^{\mathbf{p}}$ and if $\mathcal{T}_v^{\mathbf{p}}$ denotes the *fringe subtree* of $\mathcal{T}^{\mathbf{p}}$ rooted at v , that is, the subtree of descendants of v , then for every vertex w of $\mathcal{T}_v^{\mathbf{p}}$ one has

$$F^{\text{exc},\mathbf{p}}(e(w)) \geq F^{\text{exc},\mathbf{p}}(e(v)) - p(\mathcal{B}_v). \quad (7.30)$$

To argue this, simply recall formula (7.15) and notice that $\mathcal{N}(v) \subseteq \mathcal{N}(w) \cup \mathcal{B}_v$.

Second, notice that since $\max_j p_j \rightarrow 0$ and the limiting process X^θ is continuous except for a finite number I of upward jumps, we must necessarily have that a.s. as $n \rightarrow \infty$,

$$\eta_n := \max_{v \in [n]} \left| \inf_{u \in [e(v) - p_v, e(v))} (F^{\text{exc},\mathbf{p}}(u) - F^{\text{exc},\mathbf{p}}(e(v) - p_v)) \right| = o(\sigma(\mathbf{p})). \quad (7.31)$$

Lemma 7.5 :

Almost surely

$$\max_{j \in [n]} \sigma(\mathbf{p})^{-1} |p_j - p(\mathcal{B}_j \setminus [I])| \rightarrow 0.$$

Proof. As mentioned, for every vertex $v \in [n]$,

$$F^{\text{exc},\mathbf{p}}(e(v)) - F^{\text{exc},\mathbf{p}}(e(v) - p_v) = p(\mathcal{B}_v) - p_v.$$

Consider the process $F^{\mathbf{p}\downarrow}$ defined by

$$F^{\mathbf{p}\downarrow}(s) = F^{\text{exc},\mathbf{p}}(s) - \sum_{1 \leq i \leq I} p_i 1\{s \geq x'_i\}$$

where as above x'_i is the time when $F^{\text{exc},\mathbf{p}}$ has its jump with size p_i . Easily, $\sigma(\mathbf{p})^{-1}F^{\mathbf{p}\downarrow}$ converges in the Skorokhod space to the process $X^{\theta\downarrow}$ defined by

$$X_s^{\theta\downarrow} = X_s^\theta - \sum_{1 \leq i \leq I} \theta_i 1\{s \geq t_i\}$$

where t_i is the time when X^θ jumps by θ_i . This process is continuous, hence $\max_j p_j \rightarrow 0$ implies

$$\sigma(\mathbf{p})^{-1} \max_v |F^{\mathbf{p}\downarrow}(e(v)) - F^{\mathbf{p}\downarrow}(e(v) - p_v)| \rightarrow 0.$$

Now the quantity $F^{\mathbf{p}\downarrow}(e(v)) - F^{\mathbf{p}\downarrow}(e(v) - p_v)$ equals

$$\begin{aligned} & F^{\text{exc},\mathbf{p}}(e(v)) - F^{\text{exc},\mathbf{p}}(e(v) - p_v) - \sum_{1 \leq i \leq I} p_i 1\{x'_i \in (e(v) - p_v, e(v))\} \\ &= p(\mathcal{B}_v) - p_v - p(\mathcal{B}_v \cap [I]) \end{aligned}$$

implying the lemma. \square

Now, for v a non-root vertex of $\mathcal{T}^{\mathbf{P}}$ let $f(v)$ be its parent. For $i \in [I]$ and n large enough, i is not the root (since the limiting X^θ does not begin with a jump), so $f(i)$ exists.

Lemma 7.6 :

Let $i \in I$. Let $\mathcal{M}(i)$ be the set of descendents of $f(i)$ that come strictly before i in depth-first order. Suppose that $f(i) \notin [I]$ for n large enough. Then as $n \rightarrow \infty$, $p(\mathcal{M}(i)) \rightarrow 0$ almost surely.

Proof. A variation of (7.30) implies for any $v \in \mathcal{M}(i)$ and n large that

$$F^{\text{exc}, \mathbf{P}}(e(v)) \geq F^{\text{exc}, \mathbf{P}}(e(f(i))) - p(\mathcal{B}_{f(i)} \setminus [I]). \quad (7.32)$$

Indeed, it is clear that for n large the sets $\mathcal{B}_v \cap [I]$ contain at most one element, otherwise the Skorokhod convergence $\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}} \rightarrow X^\theta$ would fail as two or more upward jumps of non-negligible sizes could occur in an ultimately negligible interval. Moreover, for $v \in \mathcal{M}(i)$, it is clear that $\mathcal{N}(v)$ contains i , hence (7.32). Thus

$$F^{\text{exc}, \mathbf{P}}(u) \geq F^{\text{exc}, \mathbf{P}}(e(f(i))) - p(\mathcal{B}_{f(i)} \setminus [I]) - \eta_n,$$

$\inf_{e(f(i)) \leq u \leq e(f(i)) + p(\mathcal{M}(i))}$

with η_n defined at (7.31), since the vertices of $\mathcal{M}(i)$ are visited during the interval $[e(f(i)), e(f(i)) + p(\mathcal{M}(i))]$. Since $\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}}(e(f(i)))$ is easily seen to converge to $X_{t_i}^\theta$, by (7.31), Lemma 7.5 and the fact that $f(i) \notin [I]$ for n large, if $p(\mathcal{M}(i))$ did not converge to 0, by extracting along a subsequence we could find an interval $[t_i, t_i + \varepsilon]$ with $\varepsilon > 0$ where $X_u^\theta \geq X_{t_i}^\theta$, and this is a.s. impossible by Lemma 7.1. \square

The assumption that $f(i) \notin [I]$ may look strange since it is intuitive that the child of some $i \in [I]$ is very unlikely to be in $[I]$ for n large (e.g. by Theorem 7.2). We actually have:

Lemma 7.7 :

For every $i \in [I]$, almost surely, $\mathcal{B}_i \cap [I] = \emptyset$ for n large, and

$$\sigma(\mathbf{p})^{-1} p(\mathcal{B}_i) \rightarrow \theta_i.$$

Proof. By Lemma 7.5 it suffices to prove that a.s. for large n , $\mathcal{B}_i \cap [I] = \emptyset$. Suppose that there exist $i, j \in [I]$ such that j is the child of i in the \mathbf{p} -tree infinitely often. Since $I < \infty$, we may further suppose that $f(i) \notin [I]$ by taking (up to extraction) the least such i in depth-first order. By definition, $F^{\text{exc}, \mathbf{P}}$ has a jump with size i in the interval $[e(f(i)) - p_{f(i)}, e(f(i))]$. Moreover, it follows from the definition of $\mathcal{M}(i)$ that $e(i) - p_i = e(f(i)) + p(\mathcal{M}(i))$. Since the vertex i is examined in the interval $[e(i) - p_i, e(i)]$ and $p(\mathcal{M}(i)) \rightarrow 0$ by the preceding lemma, the fact that $f(j) = i$ implies that the jumps with size p_i and p_j occur within a vanishing interval $[e(f(i)) - p_{f(i)}, e(i)]$. Therefore, the Skorokhod convergence of $\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}}$ to X^θ would fail. \square

Now recall the definition (7.21) of the processes $r_i^{\mathbf{P}}$ used to build $G_I^{\mathbf{P}}$ in section 7.4, and that x'_i is the time when $F^{\text{exc}, \mathbf{P}}$ jumps by p_i .

Lemma 7.8 :

For every $i \in [I]$, as $n \rightarrow \infty$, we have

$$\sigma(\mathbf{p})^{-1} \left| \inf_{x'_i \leq u \leq s} F^{\text{exc}, \mathbf{P}}(u) - F^{\text{exc}, \mathbf{P}}(x'_i-) - r_i^{\mathbf{P}}(s) \right| \rightarrow 0$$

| a.s. uniformly in $s \in [x'_i, e(i) + p(\mathcal{T}_i^{\mathbf{P}})]$.

Proof. Let $i \in [I]$, and let $\mathcal{B}_i = \{v_1, v_2, \dots, v_k\}$ (with $k = |\mathcal{B}_i|$) where v_1, v_2, \dots are in depth-first order. For $1 \leq j \leq k$ let also v'_j be the last examined vertex of $\mathcal{T}_{v'_j}^{\mathbf{P}}$ in depth-first order, that is, the predecessor of v_{j+1} if $j < k$. Then one has, for every $1 \leq j \leq k$ and $w \in \mathcal{T}_{v'_j}^{\mathbf{P}}$

$$F^{\text{exc}, \mathbf{P}}(e(w)) \geq F^{\text{exc}, \mathbf{P}}(e(v_j)) - p(\mathcal{B}_{v_j}),$$

as follows from (7.30). Rewrite this as

$$F^{\text{exc}, \mathbf{P}}(e(w)) \geq F^{\text{exc}, \mathbf{P}}(e(i)) - \sum_{1 \leq r \leq j-1} p_{v_r}$$

and check that the right hand side equals $F^{\text{exc}, \mathbf{P}}(e(v'_{j-1}))$. In particular, we obtain

$$\left| \inf_{v: e(v) \in [e(i), e(w)]} F^{\text{exc}, \mathbf{P}}(e(v)) - F^{\text{exc}, \mathbf{P}}(e(i)) + \sum_{1 \leq r \leq j-1} p_{v_r} \right| \leq \max_{1 \leq j \leq k} p_{v_j}.$$

Now check that for w a vertex of $\mathcal{T}_{v'_j}^{\mathbf{P}}$, one has $r_i^{\mathbf{P}}(e(w)) = \sum_{j \leq r \leq k} p_{v_r}$. For s as in the statement of the lemma deduce, for n large (since $\mathcal{B}_i \cap [I] = \emptyset$ by Lemma 7.7),

$$\left| \inf_{u \in [x'_i, s]} F^{\text{exc}, \mathbf{P}}(u) - F^{\text{exc}, \mathbf{P}}(e(i)) + p(\mathcal{B}_i) - r_i^{\mathbf{P}}(u) \right| \leq 2 \max_{j \notin [I]} p_j + \eta_n + \eta'_n$$

where

$$\eta'_n = \max_{x'_i \leq u \leq e(i)} |F^{\text{exc}, \mathbf{P}}(u) - F^{\text{exc}, \mathbf{P}}(e(i))|$$

which is $o(\sigma(\mathbf{p}))$ by Lemma 7.6 and the convergence $\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}} \rightarrow X^\theta$. We conclude, using the fact that $\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}}(x'_i -) \rightarrow X_{t_i -}^\theta$, which is equal to the limit of $\sigma(\mathbf{p})^{-1} (F^{\text{exc}, \mathbf{P}}(e(i)) - p(\mathcal{B}_i))$, as follows from Lemmas 7.6 and 7.7. \square

Proof of Proposition 7.2. We prove that the process $\sigma(\mathbf{p})^{-1} r_i^{\mathbf{P}}$ converges to the R_i^θ of section 7.2 in the Skorokhod topology, for every i . In view of Lemma 7.8, and since by definition of r_i one has $r_i(u) = 0$ for $u \geq e(i) + p(\mathcal{T}_i^{\mathbf{P}})$, the only thing to do is to show that $e(v'_k) = e(i) + p(\mathcal{T}_i^{\mathbf{P}})$ converges to the T_i of section 7.2. Since $e(v'_k) \geq \inf\{s \geq x'_i : r_i^{\mathbf{P}}(s) = 0\}$, we obtain that $\liminf e(v'_k) \geq T_i$. Suppose $\ell = \limsup e(v'_k) > T_i$, and up to extraction suppose that ℓ is actually the limit of $e(v'_k)$. From the fact that $F^{\text{exc}, \mathbf{P}}(e(v'_k)) = F^{\text{exc}, \mathbf{P}}(e(i)) - p(\mathcal{B}_i)$, hence $\sigma(\mathbf{p})^{-1} F^{\text{exc}, \mathbf{P}}(e(v'_k))$ converges to $X_{t_i -}^\theta$ by Lemmas 7.6 and 7.7, we would find $\ell > T_i$ with $X_\ell^\theta = X_{t_i -}^\theta$ and $X_s^\theta \geq X_{t_i -}^\theta$ for $s \in [T_i, \ell]$, and this is almost surely impossible by Lemma 7.1 as $X_{t_i -}^\theta$ would be a local minimum of X^θ , attained at time T_i .

Without extra argument we cannot conclude that the sum $\sigma(\mathbf{p})^{-1} (F^{\text{exc}, \mathbf{P}} - \sum_{i=1}^I r_i^{\mathbf{P}})$ converges to $X^\theta - \sum_{i=1}^I R_i^\theta$, but this is nonetheless true for the following reason. The process R_i^θ is continuous except for one jump at t_i , and the process $r_i^{\mathbf{P}}$ has precisely one jump with size $p(\mathcal{B}_i)$ at time x'_i , that is, at the *same* time as the jump of $F^{\text{exc}, \mathbf{P}}$ with size p_i . Together with Lemma 7.7, we obtain the Skorokhod convergence $\sigma(\mathbf{p})^{-1} G_I^{\mathbf{P}} \rightarrow Y^\theta$. \square

7.6.2 Proof of Theorem 7.3

As above, we suppose that \mathbf{p} is a ranked probability distribution satisfying (7.18) for some limiting θ with length I , and we suppose that the \mathbf{p} -tree $\mathcal{T}^{\mathbf{p}}$ is obtained by the depth-first construction of section 7.3 out of the process $F^{\text{exc}, \mathbf{p}}$. We are going to show the following result:

Proposition 7.4 :

Under extra hypotheses (7.25,7.26) on $(\mathbf{p}^{(n)})$, as $n \rightarrow \infty$

$$\max_v \left| \frac{\theta_0^2 \sigma(\mathbf{p})}{2} \text{ht}(v) - \sigma(\mathbf{p})^{-1} G_I^{\mathbf{p}}(e(v)) \right| \xrightarrow{p} 0.$$

We first show how Theorem 7.3 and Proposition 7.3 are easy consequences of Proposition 7.4.

Proof of Theorem 7.3. Since $\sigma(\mathbf{p})^{-1} G_I^{\mathbf{p}}$ converges uniformly in distribution to a continuous process, and since $H^{\mathbf{p}}$ does not vary in the intervals $[e(v) - p_v, e(v))$, the last displayed convergence extends to

$$\max_{u \in [0,1]} \left| \frac{\theta_0^2 \sigma(\mathbf{p})}{2} H^{\mathbf{p}}(u) - \sigma(\mathbf{p})^{-1} G_I^{\mathbf{p}}(u) \right| \xrightarrow{p} 0, \quad (7.33)$$

and then Proposition 7.2 implies Theorem 7.3. \square

Proof of Proposition 7.3. For Proposition 7.3, we choose the following approximating sequence $\mathbf{p}^{(n+I)}$ for $\theta \in \Theta_{\text{finite}}$ with length I . Given n , let $z_n = \sqrt{n}/\theta_0$, $s_n = n + z_n \sum_{1 \leq i \leq I} \theta_i$ and

$$\begin{cases} p_i = \frac{z_n \theta_i}{s_n} & \text{if } 1 \leq i \leq I \\ p_i = \frac{1}{s_n} & \text{if } I+1 \leq i \leq n+I. \end{cases} \quad (7.34)$$

It is trivial to see that this sequence fulfills hypotheses (7.25,7.26). Hence (7.33) is satisfied, and Proposition 7.3 is an immediate consequence. \square

We now mention three consequences of hypotheses (7.25,7.26) that will be used later. First, notice that $p_* \leq 1/n$ since \mathbf{p} is a probability on $[n]$, so (7.25) implies

$$n = o(\exp(\alpha/\sigma(\mathbf{p}))) \text{ for all } \alpha > 0. \quad (7.35)$$

Second, (7.26) implies convergence of all moments of $\bar{p}(\xi)/\sigma(\mathbf{p})^2$, and in particular

$$\begin{aligned} E \left(\frac{\bar{p}(\xi)}{\sigma(\mathbf{p})^2} \right) &= \sum_{i \notin [I]} p_i^2 / \sigma(\mathbf{p})^2 \\ &= 1 - \sum_{i \in [I]} p_i^2 / \sigma(\mathbf{p})^2 \\ &\xrightarrow[n \rightarrow \infty]{} \theta_0^2 = E(Q). \end{aligned} \quad (7.36)$$

Third, for every λ in a neighborhood of 0,

$$\sigma(\mathbf{p})^2 \sum_{i \notin [I]} \left[\exp\left(\frac{\lambda p_i}{\sigma^2}\right) - 1 - \frac{\lambda p_i}{\sigma^2} \right] \xrightarrow[n \rightarrow \infty]{} E \frac{1}{Q} \left[\exp(\lambda Q) - 1 - \lambda Q \right] < \infty. \quad (7.37)$$

Indeed, the left side can be rewritten as $E \left(\frac{\sigma(\mathbf{p})^2}{\bar{p}(\xi)} \left[\exp\left(\frac{\lambda \bar{p}(\xi)}{\sigma^2}\right) - 1 - \frac{\lambda \bar{p}(\xi)}{\sigma^2} \right] \right)$, where the function $f(x) = (e^{\lambda x} - 1 - \lambda x)/x$ is understood to equal its limit 0 at 0. Since it is

bounded in a neighborhood of 0 and dominated by $e^{\lambda x}$ near ∞ , the convergence of this expectation is an easy consequence of (7.26).

The first step in the proof of Proposition 7.4 is to relate $H(\cdot)$ to another function $\mathcal{G}(\cdot)$ measuring "sum of small \mathbf{p} -values along path to root". Let $\mathcal{A}(v)$ be the set of ancestors of v in the \mathbf{p} -tree, and let

$$\mathcal{G}(v) := p(\mathcal{A}(v) \setminus [I]). \quad (7.38)$$

Lemma 7.9 :

Under extra hypotheses (7.25,7.26), as $n \rightarrow \infty$ for fixed $K > 0$

$$\max_{v: \text{ht}(v) \leq K/\sigma(\mathbf{p})} |\sigma(\mathbf{p})\theta_0^2 \text{ht}(v) - \sigma(\mathbf{p})^{-1}\mathcal{G}(v)| \xrightarrow{p} 0.$$

Proof. Let V be a \mathbf{p} -distributed random vertex. Fix $\varepsilon > 0$. It is enough to prove that as $n \rightarrow \infty$

$$P(|\sigma(\mathbf{p})\theta_0^2 \text{ht}(V) - \sigma(\mathbf{p})^{-1}\mathcal{G}(e(V))| > \varepsilon, \sigma(\mathbf{p})\text{ht}(V) \leq K) = o(p_*).$$

Let ξ have distribution \mathbf{p} on $[n]$ and let $(\xi_i, i \geq 1)$ be i.i.d. By the "birthday tree" construction of the \mathbf{p} -tree [39, Corollary 3] we have equality of joint distributions

$$(\text{ht}(V), \mathcal{G}(V)) \stackrel{d}{=} (T - 2, \sum_{i=1}^{T-1} \bar{p}(\xi_i))$$

where

$$T := \min\{j \geq 2 : \xi_j = \xi_i \text{ for some } 1 \leq i < j\}$$

is the first repeat time in the sequence ξ_i . So it is enough to prove

$$P\left(\left|\sigma(\mathbf{p})\theta_0^2(T - 2) - \sigma(\mathbf{p})^{-1} \sum_{i=1}^{T-1} \bar{p}(\xi_i)\right| > \varepsilon, \sigma(\mathbf{p})(T - 2) \leq K\right) = o(p_*).$$

We may replace $T - 2$ by $T - 1$ and θ_0^2 by $E(\frac{\bar{p}(\xi)}{\sigma(\mathbf{p})^2})$ by the above remark. Rewriting in terms of $\tilde{p}(i) := \frac{\bar{p}(i)}{\sigma(\mathbf{p})^2} - E(\frac{\bar{p}(\xi)}{\sigma(\mathbf{p})^2})$, we need to prove

$$P\left(\left|\sum_{i=1}^{T-1} \tilde{p}(\xi_i)\right| > \varepsilon/\sigma(\mathbf{p}), T - 1 \leq K/\sigma(\mathbf{p})\right) = o(p_*).$$

Now we are dealing with a mean-zero random walk, and classical fluctuation inequalities (e.g. [51] Exercise 1.8.9) reduce the problem to proving the fixed-time bound

$$P\left(\left|\sum_{1 \leq i \leq K/\sigma(\mathbf{p})} \tilde{p}(\xi_i)\right| \geq \varepsilon/\sigma(\mathbf{p})\right) = o(p_*). \quad (7.39)$$

We now appeal to assumption (7.26), which basically says that the sums in question behave as if the summands had distribution $Q - \theta_0^2$ not depending on n . More precisely,

the elementary large deviation inequality applied to the probability in (7.39) but without the absolute values implies that for any small $\lambda > 0$,

$$\log P \left(\sum_{i=1}^{K/\sigma(\mathbf{p})} \tilde{p}(\xi_i) \geq \varepsilon/\sigma(\mathbf{p}) \right) \leq -\frac{\lambda\varepsilon}{\sigma(\mathbf{p})} + \frac{K}{\sigma(\mathbf{p})} \log(E(\exp(\lambda\tilde{p}(\xi)))).$$

Assumption (7.26) and the convergence of the expectation of $\bar{p}(\xi)$ allows us to rewrite the log term on the right as

$$\frac{K}{\sigma(\mathbf{p})} \log E(\exp(\lambda(Q - \theta_0^2))) + \frac{K\eta_\lambda(n)}{\sigma(\mathbf{p})E(\exp(\lambda(Q - \theta_0^2)))},$$

where $\eta_\lambda(n) \rightarrow 0$ as $n \rightarrow \infty$ for any fixed λ . We now choose λ small enough so that $-\lambda\varepsilon + K \log E(\exp(\lambda(Q - \theta_0^2))) = -\delta < 0$ and we let $n \rightarrow \infty$, obtaining the bound $\exp(-\delta'/\sigma(\mathbf{p}))$, for some $\delta' > 0$, for the probability in (7.39) without absolute values, but the other side of the inequality is similar. Now assumption (7.25) gives the desired bound (7.39). \square

The next, rather strange-looking lemma does most of the work in relating the processes $G_I^{\mathbf{p}}(\cdot)$ and $\mathcal{G}(\cdot)$.

Given a probability distribution \mathbf{p} on $[n]$ and given a subset $A \subset [n]$, let \mathbf{q} be the probability distribution obtained by lumping the points A into a single point; that is, $q_1 = p(A)$ and the multiset $\{q_i, i \geq 2\}$ is the multiset $\{p_i, i \notin A\}$. We also let \mathcal{I} be the set of “large” \mathbf{q} -values, except q_1 . Precisely, \mathcal{I} is such that the multisets $\{p_v, v \in [I] \setminus A\} = \{q_v, v \in \mathcal{I}\}$ are equal. Then

Lemma 7.10 :

Suppose $\mathbf{p} = \mathbf{p}^{(n)}$ satisfies the regime (7.18) and extra hypotheses (7.25,7.26). Let $A = A^{(n)} \subset [n]$ and define \mathbf{q} as above. Define a random variable $X = X(\mathbf{q})$ as follows. Take a \mathbf{q} -tree, condition on vertex 1 being the root. Let \mathcal{B}_1 be the set of children of 1, and for each $v \in \mathcal{B}_1$ toss two coins c_1 and c_2 , c_1 a fair coin and $P(c_2 = \text{Heads}) = p(A \setminus [I])/p(A)$, and set

$$X := \sum \{q_v : v \in \mathcal{B}_1 \setminus \mathcal{I}, \text{ coins } c_1 \text{ and } c_2 \text{ land Heads}\}.$$

Suppose $q_1 \leq K\sigma(\mathbf{p})$ and set $\bar{q}_1 = p(A \setminus [I])$. Then for fixed $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, K) > 0$ with

$$P(|X - \frac{1}{2}\bar{q}_1| > \varepsilon\sigma(\mathbf{p})) \leq \exp(-\delta/\sigma(\mathbf{p})) = o(1/n),$$

where the $o(1/n)$ is thus uniform over $q_1 \leq K\sigma(\mathbf{p})$.

Proof. Consider the random variable

$$Y := \sum_{i \notin A \cup [I]} p_i 1_{(U_i \leq \bar{q}_1/2)}$$

where the (U_i) are independent uniform(0, 1). The key relation is

$$P(X \in \cdot) \leq \frac{1}{q_1} P(Y \in \cdot). \quad (7.40)$$

This follows from the *breadth-first* construction of the \mathbf{p} -trees. In that construction of a \mathbf{q} -tree, vertices i are associated with uniform $(0, 1)$ r.v.'s U'_i in such a way that, if vertex 1 happens to be the root, then the children v of 1 are the vertices v for which $U'_v := U'_v - U'_1 \bmod 1$ falls within $(0, q_1)$. Thus, writing

$$X' := \sum \{q_v : v \in \mathcal{B}_1 \setminus \mathcal{I}\}$$

$$Y' := \sum_{i \notin AU[I]} p_i 1_{(U_i \leq q_1)}$$

we have

$$X' = Y' \text{ on the event } \{ \text{vertex 1 is root} \}.$$

So

$$P(X' \in \cdot \mid 1 \text{ is root}) \leq \frac{P(Y' \in \cdot)}{P(1 \text{ is root})} = \frac{1}{q_1} P(Y' \in \cdot).$$

The stated inequality (7.40) follows by applying an independent Bernoulli $(\bar{q}_1/(2q_1))$ thinning procedure to both sides.

Now write $c = \bar{q}_1/2$ and let us study the centered version of Y :

$$\tilde{Y} := \sum_{i \notin AU[I]} p_i (1_{(U_i \leq c)} - c). \quad (7.41)$$

The elementary large deviation bound, applied to $\tilde{Y}/\sigma(\mathbf{p})^2$, is: for arbitrary $\lambda > 0$,

$$\log P(\tilde{Y} > \varepsilon \sigma(\mathbf{p})) \leq \frac{-\lambda \varepsilon}{\sigma(\mathbf{p})} + \log E \exp(\lambda \tilde{Y} / \sigma(\mathbf{p})^2).$$

We calculate

$$\begin{aligned} & \log E \exp(\lambda \tilde{Y} / \sigma(\mathbf{p})^2) \\ &= \sum_{i \notin AU[I]} \left\{ \frac{-\lambda p_i}{\sigma(\mathbf{p})^2} c + \log \left[1 + c(e^{\lambda p_i / \sigma(\mathbf{p})^2} - 1) \right] \right\} \\ &\leq c \sum_{i \in [n]} \left\{ e^{\lambda p_i / \sigma(\mathbf{p})^2} - 1 - \frac{\lambda p_i}{\sigma(\mathbf{p})^2} \right\}, \end{aligned}$$

since the quantities we are summing are positive, and by (7.37) the bound is asymptotic to $c\sigma(\mathbf{p})^{-2}\Phi(\lambda)$ for

$$\Phi(\lambda) := E \frac{1}{Q} [\exp(\lambda Q) - 1 - \lambda Q].$$

By hypothesis $c := \bar{q}_1/2 \leq K\sigma(\mathbf{p})$, so $c\sigma(\mathbf{p})^{-2} \leq K\sigma(\mathbf{p})^{-1}$. So there is a constant $C_1 = C_1(K)$ such that

$$\log P(\tilde{Y} > \varepsilon \sigma(\mathbf{p})) \leq \frac{1}{\sigma(\mathbf{p})} (-\lambda \varepsilon + C_1 \Phi(\lambda)).$$

But $\Phi'(0) = 0$ and so $\Phi(\lambda) = o(\lambda)$ as $\lambda \downarrow 0$, so the right side is strictly negative for small $\lambda > 0$. So there exists $\delta_1 = \delta_1(\varepsilon, K) > 0$ such that

$$P(\tilde{Y} > \varepsilon \sigma(\mathbf{p})) \leq \exp(-\delta_1 / \sigma(\mathbf{p})).$$

Since $Y - \tilde{Y} = c \sum_{i \notin A \cup [I]} p_i \leq \bar{q}_1/2$ we have established the one-sided inequality

$$P(Y - \frac{1}{2}\bar{q}_1 > \varepsilon\sigma(\mathbf{p})) \leq \exp(-\delta_1/\sigma(\mathbf{p})).$$

The other side of the inequality is similar except for this last step: we cannot bound so easily the quantity $\tilde{Y} - Y$. However, by (7.18),

$$\sum_{i \notin A \cup [I]} p_i = 1 - p(A \cup [I]) \geq 1 - q_1 - \sum_{i=1}^I p_i \geq 1 - C_2\sigma(\mathbf{p})$$

for some $C_2 = C_2(K) < \infty$. Thus $Y - \tilde{Y} \geq c(1 - C_2\sigma(\mathbf{p}))$ and we can conclude as above by the existence of $\delta_2 = \delta_2(\varepsilon, K)$ satisfying

$$P(\frac{1}{2}\bar{q}_1 - Y > \varepsilon\sigma(\mathbf{p})) \leq \exp(-\delta_2/\sigma(\mathbf{p})).$$

So, letting $\delta' = \delta_1 \wedge \delta_2$,

$$P(|Y - \frac{1}{2}\bar{q}_1| > \varepsilon\sigma(\mathbf{p})) \leq 2 \exp(-\delta'/\sigma(\mathbf{p})).$$

Now (7.40) and hypothesis (7.25) and its consequence (7.35) establish Lemma 7.10 (with any $\delta < \delta'$). \square

For the next lemma, recall the definition of $\mathcal{N}(v)$ around (7.15) and let $\mathcal{N}^*(v)$ be the subset of vertices of $\mathcal{N}(v)$ which are not in $[I]$ and whose parent is not in $[I]$ either.

Lemma 7.11 :

Fix $j \in [n]$ and a subset $A \subset [n]$ with $j \in A$. Take a random \mathbf{p} -tree and condition on $\mathcal{A}(j) = A$. Let also v_1, \dots, v_k be the children of j that are not in $[I]$ and let $c^*(j) = \sum_{1 \leq l \leq k} b_l p_{v_l}$, where the b_l 's are independent Bernoulli random variables with parameter $1/2$, independent of the \mathbf{p} -tree. Define

$$X^* := p(\mathcal{N}^*(j)) - c^*(j).$$

Then X^* is distributed as the random variable X in Lemma 7.10.

Proof. Order A as v_0, v_1, \dots, j , arbitrarily except for ending with j . Let \mathbf{T}^* be the set of rooted trees on $[n]$ with root v_0 whose path to j is the path v_0, v_1, \dots, j . Let \mathbf{T}^\oplus be the set of rooted trees on $[n] \setminus A \cup \{\oplus\}$ with root \oplus . There is a natural map $\mathbf{T}^* \rightarrow \mathbf{T}^\oplus$: "lump the vertices in A together into a single vertex \oplus ". It is straightforward to check, from the combinatorial definition (see e.g. [90]) of \mathbf{p} -tree, that this map takes the distribution of \mathbf{p} -tree (conditioned to \mathbf{T}^*) into the distribution of a \mathbf{q} -tree (conditioned on having root \oplus). Also, we have the extra constraint in X^* that the parents of the vertices we are summing on are not in $[I]$, but conditionally on the fact that v has some parent in A , it is easy that the parent is in $[I]$ with probability $p(A \cap [I])/p(A)$. This corresponds to the biased coin-tosses in Lemma 7.10. And the fair coin-tosses in Lemma 7.10 reflect the random ordering of branches used in defining the depth-first order, as can be seen from the definition in Section 7.3 (the set of children of any vertex is put in exchangeable random order). The only exception is on children of j itself, which are all in $\mathcal{N}^*(v)$, so the b_l 's are designated to artificially remove each of them with probability $1/2$. This establishes the lemma. \square

The importance of the lemma is explained by the following formula

$$\max_v |G_I^{\mathbf{p}}(e(v)) - p(\mathcal{N}^*(v)) - c^*(v)| = o(\sigma(\mathbf{p})) \text{ in probability.} \quad (7.42)$$

Since asymptotically we know that children of $i \in [I]$ are not in $[I]$, and since by Lemmas 7.5 and 7.7:

$$\sigma(\mathbf{p})^{-1} \max_{v \notin [I]} p(\mathcal{B}_v \setminus [I]) \xrightarrow{p} 0, \quad (7.43)$$

so in particular $\max_j \sigma(\mathbf{p})^{-1} c^*(j) \rightarrow 0$ in probability with the notations above, this is a straightforward consequence of Lemma 7.7 and

Lemma 7.12 :

Suppose that no vertex $i \in [I]$ has a child that is also in $[I]$, then we have for every

$$v \quad G_I^{\mathbf{p}}(e(v)) = p(\mathcal{N}^*(v)) - \sum_{i \in \mathcal{N}(v) \cap [I]} (p_i - p(\mathcal{B}_i)). \quad (7.44)$$

Proof. Recall by definition (7.20) of the processes ρ_k that if k is a child of some $i \in [I]$, $\rho_k(e(v)) = p_k$ whenever v is examined after the parent $f(i)$ of i and strictly before k in depth-first order, and $\rho_k(e(v)) = 0$ otherwise. As a consequence of (7.15), we thus have

$$G_I^{\mathbf{p}}(e(v)) = p(\mathcal{N}(v)) - \sum_{i \in [I], k \in \mathcal{B}_i} p_k 1\{e(f(i)) \leq e(v) < e(k)\}.$$

A careful examination of this formula shows that a term in the sum on the right is not zero if either v has some ancestor $i \in [I]$, or some ancestor of v has a child $i \in [I]$ that is after v in depth-first order, and these situations are exclusive by the assumption that vertices of $[I]$ do not have children in $[I]$. In the first case, the formula says that we remove all the \mathbf{p} -values of children of i that are after v in depth-first order, in the second case, it says that we remove the \mathbf{p} -values of all the children of i , implying (7.44). \square

Proof of Proposition 7.4. Fix $\varepsilon > 0$ and consider arbitrary $v \in [n]$. Recall the definition of $\mathcal{A}(v), \mathcal{G}(v), \mathcal{N}^*(v), c^*(v)$. We assert, from Lemmas 7.10 and 7.11, that for any $K > 0$ there exists $\delta = \delta(\varepsilon, K)$ with

$$P(|p(\mathcal{N}^*(v)) - c^*(v) - \frac{1}{2}\mathcal{G}(v)| > \varepsilon \sigma(\mathbf{p}) | \mathcal{A}(v)) \leq \exp(-\delta/\sigma(\mathbf{p})) \text{ on } \{\mathcal{G}(v) \leq (K+2\varepsilon)\sigma(\mathbf{p})\}. \quad (7.45)$$

To argue (7.45), note that conditioning on the set $A = \mathcal{A}(v)$ of vertices in the path from the root to v determines the value $\mathcal{G}(v) := p(\mathcal{A}(v) \setminus [I]) = \bar{q}_1$ say. Then Lemmas 7.10, 7.11 imply that the conditional distribution of $p(\mathcal{N}^*(v)) - c^*(v)$ has the distribution of X in Lemma 7.10, The conclusion of Lemma 7.10 now gives (7.45).

So for fixed K and arbitrary $v \in [n]$

$$P(|p(\mathcal{N}^*(v)) - c^*(v) - \frac{1}{2}\mathcal{G}(v)| > \varepsilon \sigma(\mathbf{p}), \frac{1}{2}\mathcal{G}(v) \leq (K+2\varepsilon)\sigma(\mathbf{p})) \leq \exp(-\delta/\sigma(\mathbf{p})) = o(1/n).$$

Using Boole's inequality gives

$$P\left(\frac{1}{\sigma(\mathbf{p})}|p(\mathcal{N}^*(v)) - c^*(v) - \frac{1}{2}\mathcal{G}(v)| > \varepsilon \text{ for some } v \text{ with } \frac{1}{2\sigma(\mathbf{p})}\mathcal{G}(v) \leq K+2\varepsilon\right) = o(1).$$

By (7.42) we may replace $p(\mathcal{N}^*(v)) - c^*(v)$ by $G_I^{\mathbf{p}}(e(v))$ in the previous expression. We now use a slightly fussy truncation procedure. Imposing an extra constraint,

$$P\left(\frac{1}{\sigma(\mathbf{p})} \max_v G_I^{\mathbf{p}}(e(v)) \leq K, \frac{1}{\sigma(\mathbf{p})}|G_I^{\mathbf{p}}(e(v)) - \frac{1}{2}\mathcal{G}(v)| > \varepsilon \text{ for some } v \text{ with } \frac{1}{2\sigma(\mathbf{p})}\mathcal{G}(v) \leq K+2\varepsilon\right) = o(1). \quad (7.46)$$

We claim that we can remove the restriction on v to get

$$P\left(\frac{1}{\sigma(\mathbf{p})} \max_v G_I^{\mathbf{p}}(e(v)) \leq K, \frac{1}{\sigma(\mathbf{p})} |G_I^{\mathbf{p}}(e(v)) - \frac{1}{2}\mathcal{G}(v)| > \varepsilon \text{ for some } v\right) = o(1). \quad (7.47)$$

Indeed, if v has parent v' then $\mathcal{G}(v) - \mathcal{G}(v') \leq \max_{i \notin [I]} p_i = o(\sigma(\mathbf{p}))$. So if there exists a v with $\frac{1}{2\sigma(\mathbf{p})}\mathcal{G}(v) > K + 2\varepsilon$ then (for large n) there is an ancestor w with $K + \varepsilon < \frac{1}{2\sigma(\mathbf{p})}\mathcal{G}(w) < K + 2\varepsilon$. But if the first event in (7.46) occurs, one obviously cannot have $\sigma(\mathbf{p})^{-1} |G_I^{\mathbf{p}}(e(w)) - \frac{1}{2}\mathcal{G}(w)| \leq \varepsilon$ by definition of w . Thus the probability in (7.47) is bounded by twice the probability in (7.46). This establishes (7.47). Since Proposition 7.2 implies $\frac{1}{\sigma(\mathbf{p})} \max_v G_I^{\mathbf{p}}(e(v))$ is tight as $n \rightarrow \infty$, (7.47) implies

$$\max_v \frac{1}{\sigma(\mathbf{p})} |G_I^{\mathbf{p}}(e(v)) - \frac{1}{2}\mathcal{G}(v)| \xrightarrow{p} 0. \quad (7.48)$$

Now let us show that the sequence $(\sigma(\mathbf{p}) \max_{v \in [n]} \text{ht}(v), n \geq 1)$ is tight. Fix $\varepsilon > 0$ and let $K > 0$ such that

$$P\left(\frac{1}{\sigma(\mathbf{p})} \max_v G_I^{\mathbf{p}}(e(v)) > K\right) < \varepsilon/2,$$

Then

$$P\left(\sigma(\mathbf{p}) \max_v \text{ht}(v) > K + 1\right) \leq \varepsilon/2 + P\left(\sigma(\mathbf{p}) \max_v \text{ht}(v) > K + 1, \frac{1}{\sigma(\mathbf{p})} \max_v G_I^{\mathbf{p}}(e(v)) < K\right),$$

but by the same kind of argument as above, if $\sigma(\mathbf{p}) \max_v \text{ht}(v) > K$, for n large there must exist some w with $K + 1/2 < \sigma(\mathbf{p}) \text{ht}(w) < K + 1$. By Lemma 7.9 we then have also $K + 1/2 < \sigma(\mathbf{p})^{-1}\mathcal{G}(v) < K + 1$ with high probability, so (7.48) implies that the right-hand side in the last expression is $< \varepsilon/2$ for n large. This being proved, Lemma 7.9 rewrites as $\max_v |\sigma(\mathbf{p})^{-1}\mathcal{G}(v) - \sigma(\mathbf{p})\theta_0^2 \text{ht}(v)| = o(1)$ in probability, which together with (7.48) establishes the proposition. \square

7.7 Miscellaneous comments

1. In principle Corollary 7.2 gives a criterion for boundedness of \mathcal{T}^θ , but one would prefer to have a condition directly in terms of θ . Here are some steps in that direction. From [66, Theorem 1.1], the process $X^{\text{br},\theta}$ may be put in the form $X_s^{\text{br},\theta} = X_s^1 + X_s^2$, $s \geq 0$, where X^1 is a Lévy process on $[0, \infty)$ and X^2 has exchangeable increments on $[0, 1]$ and in a certain sense behaves less wildly than X^1 . Precisely, X^1 has no drift, its Gaussian part is θ_0 and its Lévy measure is $\Lambda(dx) = \sum_{i \geq 1} \delta_{\theta_i}(dx)$, where $\delta_y(dx)$ is the Dirac mass at y . On the other hand, X^2 can be put in the form

$$X_s^2 = -X_1^1 s + \sum_{i \geq 1} \tau_i (1\{s \leq V_i\} - s)$$

for some square-summable random family (τ_i) and a sequence V_i of independent r.v.'s with uniform law (notice that X^1 and X^2 are by no means independent). Then, writing $\kappa_{X^{\text{br},\theta}} = \inf\{c > 0 : \sum_{i \geq 1} \theta_i^c < \infty\}$ and $\kappa_{X^2} = \inf\{c > 0 : \sum_{i \geq 1} \tau_i^c < \infty\}$ we have that

$$\kappa_{X^2} \leq \frac{\kappa_{X^{\text{br},\theta}}}{1 + \frac{1}{2}\kappa_{X^{\text{br},\theta}}}, \quad (7.49)$$

which is what we mean by “behaving less wildly”. It is therefore reasonable that the problem on the finiteness of the integral $\int_1^\infty ds/X_s^\theta$, which is a problem dealing with the behavior at the left of the overall minimum of $X^{\text{br},\theta}$, should be replaced by a problem on the Lévy process X^1 as soon as one can show that the overall minimum of $X^{\text{br},\theta}$ is actually attained at a local minimum of X^1 , and such that locally X^2 is negligible compared to X^1 at this time. Since X^1 has no negative jumps, the time-reversed process has no positive jumps, and such questions are addressed in Bertoin [18] and Millar [83]. Pushing the intuition one step further, by analogy with the standard criterion for non-extinction of continuous-state branching processes and the analogy of ICRT’s and Lévy trees mentioned above, we conjecture that $\int_0^\infty \Psi^{-1}(\lambda)d\lambda < \infty$ is equivalent to the boundedness of \mathcal{T}^θ , where Ψ is the Laplace exponent of X^1 :

$$\Psi(\lambda) = \theta_0 \lambda^2 / 2 + \sum_{i \geq 1} (\exp(-\lambda \theta_i) - 1 + \lambda \theta_i).$$

2. As we mentioned before, a natural guess would be that the exploration process of \mathcal{T}^θ in the general case $\theta_0 > 0$ is $\frac{2}{\theta_0} Y^\theta$. It is more difficult to get an intuition of what the exploration process of \mathcal{T}^θ should be in the cases when $\theta_0 = 0$, when the Brownian part of X^θ vanishes. By the general theory of continuum random trees, it should be easy to prove that compactness of the tree is enough to obtain the existence of an exploration process for \mathcal{T}^θ , which is the weak limit of $2(\theta_0^n)^{-2} Y^{\theta^n}$ for some $\theta^n \in \Theta \rightarrow \theta$ pointwise with $\theta_0^n > 0$ for every n . But this would not tell much about the look of this process. Another way would be to try to generalize local time methods used in [49], but these do not seem to adapt so easily to bridges with exchangeable increments instead of Lévy processes.

Chapter 8

Brownian Bridge Asymptotics for Random \mathbf{p} -mappings

Contents

8.1	Introduction	219
8.2	Background	221
8.2.1	Mappings	221
8.2.2	Coding trees and mappings by marked walks	222
8.2.3	The convergence theorem	223
8.2.4	\mathbf{p} -trees, \mathbf{p} -mappings and the Joyal bijection	224
8.2.5	Weak convergence of random tree walks	226
8.3	Proof of Theorem 8.1	227
8.3.1	Representation of the mapping walk with \mathbf{p} -trees	227
8.3.2	A transformation on paths	228
8.3.3	Pushing forward tree walks to mapping walks	229
8.3.4	\mathbf{J} transforms B^{ex} to B^{br}	231
8.3.5	Completing the proof of Theorem 8.1	232
8.4	Final comments	233

8.1 Introduction

A *mapping* $m : [n] \rightarrow [n]$ is just a function, identified with its digraph $\mathcal{D}(m) = \{(i, m(i)), i \in [n]\}$. Exact and asymptotic properties of random mappings have been studied extensively in the combinatorial literature since the 1960s [60, 72]. Aldous and Pitman [9] introduced the method of associating a *mapping-walk* with a mapping, and showed that (for a uniform random mapping) rescaled mapping-walks converge in law to reflecting Brownian bridge. The underlying idea – that to rooted trees one can associate tree-walks in such a way that random tree-walks have tractable stochastic structure – has been developed in many

directions over the last 15 years, and this paper, together with a companion paper [14], takes another look at invariance principles for random mappings with better tools.

As is well known, the digraph $\mathcal{D}(m)$ decomposes into trees attached to cycles. The argument of [9] was that the walk-segments corresponding to different cycles, considered separately, converge to Brownian excursions, and that the process of combining these walk-segments into the mapping-walk turned out (by calculation) to be the same as the way that excursions of reflecting Brownian bridge are combined. That proof (and its reinterpretation by Biane [33]) made the result seem rather coincidental. In this paper we give a conceptually straightforward argument which both proves convergence and more directly identifies the limit. The argument is based on the *Joyal bijection* J between doubly-rooted trees and mappings. Being a bijection it takes uniform law to uniform law; less obviously, it takes the natural \mathbf{p} -tree model of random trees to the natural \mathbf{p} -mapping model of random mappings. Theorem 8.1 will show that under a natural hypothesis, mapping-walks associated with random \mathbf{p} -mappings converge weakly to reflecting Brownian bridge. We can outline the proof in four sentences.

- It is known that rescaled walks associated with random \mathbf{p} -trees converge in law to Brownian excursion, under the natural hypothesis (8.4) on (\mathbf{p}^n) (section 8.2.5).
- There is an transformation $\mathbf{J} : D[0, 1] \rightarrow D[0, 1]$ which “lifts” the trees→mappings Joyal bijection to the associated walks (section 8.3.3).
- \mathbf{J} has appropriate continuity properties (section 8.3.2).
- \mathbf{J} takes Brownian excursion to reflecting Brownian bridge (section 8.3.4).

Filling in the details is not difficult, and indeed it takes longer in section 8.2 to describe the background material (tree walks, mapping walks, the Joyal bijection in its probabilistic form, its interpretation for walks) than to describe the new arguments in section 8.3. One unusual aspect is that to handle the natural class (8.4) of \mathbf{p} -mappings, we need to use a certain $*$ -topology on $D[0, 1]$ which is weaker than the usual Skorokhod topology (in brief, it permits upward spikes of vanishing width but non-vanishing height).

A companion paper [14] uses a quite different approach to study a range of models for random trees or mappings, based on spanning subgraphs of random vertices. We will quote from there the general result (Theorem 8.2(b)) that rescaled random \mathbf{p} -tree walks converge in the $*$ -topology to Brownian excursion, but our treatment of random mappings will be essentially self-contained. We were motivated in part by a recent paper of O’Cinneide and Pokrovskii [85], who gave a more classically-framed study of random \mathbf{p} -trees (under the same hypothesis (8.4)) from the viewpoint of limit distributions for a few explicit statistics. See [9, 14, 15, 13] for various explicit limit distributions derived from the Brownian bridge.

When (8.4) fails the asymptotics of \mathbf{p} -trees and \mathbf{p} -mappings are quite different: Brownian excursion and reflecting Brownian bridge are replaced by certain jump processes with infinite-dimensional parametrization. Technicalities become much more intricate in this case, but the general method of using the operator \mathbf{J} will still work. We will treat this in a sequel [8].

The recent lecture notes of Pitman [91] provide a broad general survey of this field of probabilistic combinatorics and stochastic processes.

8.2 Background

8.2.1 Mappings

Let S be a finite set. For any mapping $m : S \rightarrow S$, write $\mathcal{D}(m)$ for the mapping digraph whose edges are $s \rightarrow m(s)$, and write $\mathcal{C}(m)$ for the set of cyclic points of m (i.e. the points that are mapped to themselves by some iterate of m).

Let $\mathcal{T}_c(m)$ be the tree component of the mapping digraph with root $c \in \mathcal{C}(m)$. The tree components are bundled by the disjoint cycles $\mathcal{C}_j(m) \subseteq \mathcal{C}(m)$ to form the basins of attraction of the mapping, say

$$\mathcal{B}_j(m) := \bigcup_{c \in \mathcal{C}_j(m)} \mathcal{T}_c(m) \supseteq \mathcal{C}_j(m) \quad \text{with} \quad \bigcup_j \mathcal{B}_j(m) = S \quad \text{and} \quad \bigcup_j \mathcal{C}_j(m) = \mathcal{C}(m) \quad (8.1)$$

where all three unions are disjoint unions, and the $\mathcal{B}_j(m)$ and $\mathcal{C}_j(m)$ are indexed by $j = 1, 2, \dots$ in such a way that these sets are non-empty iff $j \leq k$, the number of cycles of the digraph, which is also the number of basins of the digraph. The choice of ordering will be important later, but first we define the random mappings we will consider.

From now on, suppose that $S = \{1, 2, \dots, n\} =: [n]$. Consider a probability law \mathbf{p} on $[n]$, and assume that $p_i > 0$ for each i . A random mapping M is called a \mathbf{p} -mapping if for every $m \in [n]^{[n]}$,

$$P(M = m) = \prod_{x \in [n]} p_{m(x)}. \quad (8.2)$$

In other words, each point of $[n]$ is mapped independently of the others to a point of $[n]$ chosen according to the probability law \mathbf{p} .

We now define an order on the basins of attraction and cycles of a \mathbf{p} -mapping which will be relevant to our study. Consider a random sample (X_2, X_3, \dots) of i.i.d. points of $[n]$ with common law \mathbf{p} , independent of M (our unusual choice of index set $\{2, 3, \dots\}$ will become clear in section 8.2.4). Then order the basins of M in their order of appearance in the \mathbf{p} -sample. More precisely, since $p_i > 0$ for every $i \in [n]$, we have that $\{X_2, X_3, \dots\} = [n]$ a.s., so the following procedure a.s. terminates:

- Let $\mathcal{B}_1(M)$ be the basin of M containing X_2 and $\mathcal{C}_1(M)$ be the cycle included in $\mathcal{B}_1(M)$. Define $\tau_1 = 2$.
- Suppose $(\tau_i)_{1 \leq i \leq j}$ and the non-empty $(\mathcal{B}_i(M))_{1 \leq i \leq j}$ and $(\mathcal{C}_i(M))_{1 \leq i \leq j}$ are given. As long as $\cup_{1 \leq i \leq j} \mathcal{B}_i(M) \neq [n]$, let $\tau_{j+1} = \inf\{k : X_k \notin \cup_{1 \leq i \leq j} \mathcal{B}_i(M)\}$ and $\mathcal{B}_{j+1}(M)$ be the basin containing $X_{\tau_{j+1}}$.

For the purpose of defining a useful marked random walk in the next section, shall also introduce an order on all the cyclic points, as follows. With the above notations, let $c_j \in \mathcal{C}_j(M)$ be the cyclic point which is the root of the subtree of the digraph of M that contains X_{τ_j} . Then within $\mathcal{C}_j(M)$ the vertices are ordered as follows:

$$M(c_j), M^2(c_j), \dots, M^{|\mathcal{C}_j(M)|-1}(c_j), c_j.$$

Together with the order on basins, this induces an order on all cyclic points.

Call this order (on basins, cycles, or cyclic points) the **p-biased random order**.

8.2.2 Coding trees and mappings by marked walks

Let \mathbf{T}_n^o be the set of ordered rooted trees on n vertices. By ordered, we mean that the sons of each vertex of the tree, if any, are ordered (i.e. we are given a map from the set of children into $\{1, 2, 3, \dots\}$). Consider some tree T in \mathbf{T}_n^o . Denote by $H_i(T)$ the height of vertex i in this tree (*height* = number of edges between i and the root). Suppose that each vertex i has a weight $w_i > 0$, to be interpreted as the duration of time that the walk spends at each vertex. Then one can define the *height process* of the tree as follows. First put the vertices in *depth-first order* (the root is first, and coming after a certain vertex is either its first child, or (if it has no children) its next brother, or (if he has no brother either), the next brother of its parent, and so on). This order can be written as a permutation σ : we say that $\sigma(i)$ is the label of the i -th vertex. For $s \leq \sum_{i=1}^n w_{\sigma(i)}$ set

$$H_s^T = H_{\sigma(i)}(T) \quad \text{if } \sum_{j=1}^{i-1} w_{\sigma(j)} \leq s < \sum_{j=1}^i w_{\sigma(j)},$$

and $H_{\sum_{i=1}^n w_{\sigma(i)}}^T = H_{\sigma(n)}(T)$ (so the process is right-continuous). This also induces a map $s \mapsto s^T$ from $[0, \sum_i w_i]$ to $[n]$, where $s^T = \sigma(i)$ whenever $\sum_{j=1}^{i-1} w_{\sigma(j)} \leq s < \sum_{j=1}^i w_{\sigma(j)}$, and $1^T = \sigma(n)$. With this notation, $H_s^T = H_{s^T}(T)$. We say that s is a time at which the vertex s^T is *visited* by the height process of T .

Now consider a \mathbf{p} -mapping M on $[n]$ with the assumptions above on \mathbf{p} . Given the choice of a particular order on the cyclic points, say (c_1, \dots, c_K) , one can construct the “height process” associated with the \mathbf{p} -mapping, as follows: in the digraph of M , delete the edges between cyclic points and consider the tree components $\mathcal{T}_{c_1}, \mathcal{T}_{c_2}, \dots, \mathcal{T}_{c_K}$ of the resulting random forest, with respective roots c_1, c_2, \dots, c_K . The tree components are unordered trees, but we can make them into ordered trees by putting each set of children of the vertices of the \mathcal{T}_{c_i} 's into uniform random order. This induces a depth-first order on each \mathcal{T}_{c_i} . Let $H^{\mathcal{T}_{c_i}}$ be the height process of \mathcal{T}_{c_i} (where the weight w_x of a point x is its \mathbf{p} -value p_x). Now define the *mapping walk* $(H_s^M, 0 \leq s \leq 1)$ to be the concatenation of these tree-walks, in the order dictated by the order on the cyclic points. That is, for $0 \leq s \leq 1$ set

$$H_s^M = H_{s - \sum_{j < i} p(\mathcal{T}_{c_j})}^{\mathcal{T}_{c_i}} \quad \text{if } \sum_{j < i} p(\mathcal{T}_{c_j}) \leq s < \sum_{j \leq i} p(\mathcal{T}_{c_j}), \quad (8.3)$$

and $H_1^M = H_{1^-}^M$. As for the trees, we denote by s^M the point that is visited by H^M at time s . Also, for $x \in [n]$ let $[g^M(x), d^M(x))$ be the interval where x is visited by the walk associated to M . Several features of the mapping M are coded within this walk, such as the number of cyclic points (which is the number of points x such that $H_{g^M(x)}^M = 0$), and the shapes of the trees planted on the cyclic points, which can be deduced from the excursions of the walk away from 0.

Now suppose that c_1, \dots, c_K , and the basins $\mathcal{B}_1(M), \mathcal{B}_2(M), \dots$ are in the \mathbf{p} -biased random order. Put a mark Z_i at the time when the i -th non-empty basin of M has been entirely visited. This has to be a time when H^M is 0 (this is the time when the walk visits the first cyclic point of the next basin), unless Z_i is the time when the last basin has been visited, and then one has $Z_i = 1$. The marks $0 = Z_0, Z_1, Z_2, \dots$ determine the visits of each basin, i.e. the portion of H^M between Z_{j-1} and Z_j is the mapping walk corresponding to the j -th basin of the mapping. In particular $p(\mathcal{B}_j) = Z_j - Z_{j-1}$.

Last, we denote by ℓ_s^M the number of cyclic points that are before s^M in depth-first order. Precisely,

$$\ell_s^M = \sum_{j \leq i} \mathbb{1}_{\{H_{\sum_{k=1}^j p_{\sigma(k)}}^M = 0\}} \quad \text{if } \sum_{j < i} p_{\sigma(j)} \leq s < \sum_{j \leq i} p_{\sigma(j)},$$

and $\ell_1^M = \ell_{1-}^M = |\mathcal{C}(M)|$.

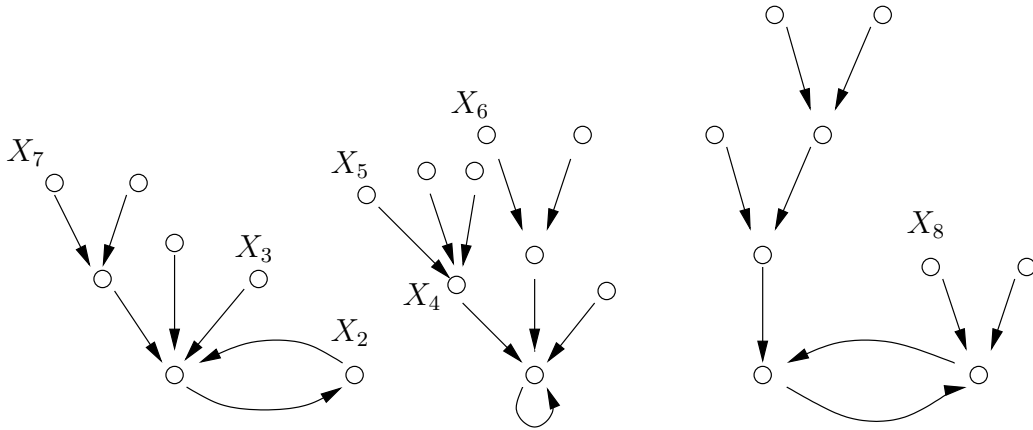


Figure 8.1: A mapping pattern digraph and a \mathbf{p} sample run until it has visited the three basins

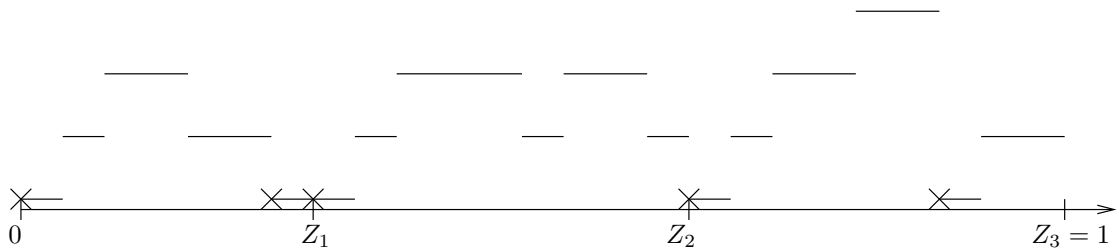


Figure 8.2: The corresponding marked walk, where crosses indicate visits to cyclic points

Remark. (a) Because the walk can visit two cyclic points consecutively, some information about the mapping pattern (i.e. the digraph with unlabeled vertices) is lost in $(H^M, (Z_1, Z_2, \dots))$. But when we are also given $((g^M(i), d^M(i)))_{i \in [n]}$, which is a partition of $[0, 1]$, we can recover the mapping pattern.

(b) The height process of a tree is a particular instance of a “tree walk”, i.e. a walk associated with a tree. The fact that the walk spends time p_i at vertex i is important; but other walks with this property might also be usable.

8.2.3 The convergence theorem

At this point we can state precisely the result of this paper, Theorem 8.1. For a probability law \mathbf{p} on $[n]$ write

$$c(\mathbf{p}) := \sqrt{\sum_i p_i^2}.$$

For a sequence $(\mathbf{p}^{(n)})$ of probability laws on $[n]$, introduce the *uniform asymptotic negligibility* condition

$$\frac{\max_i p_i^{(n)}}{c(\mathbf{p}^{(n)})} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (8.4)$$

This turns out to be natural because of the *birthday tree* construction of \mathbf{p} -trees [39], or the direct study of iterates of random mappings [14]. It is easy to check that (8.4) implies $c(\mathbf{p}^{(n)}) \rightarrow 0$.

Let M_n be a $\mathbf{p}^{(n)}$ -mapping, and let $(H^{M_n}, (Z_1^n, Z_2^n, \dots))$ be the associated marked random walk. Let $B^{|br|}$ be standard reflected Brownian bridge on $[0, 1]$, and let $(L_s, 0 \leq s \leq 1)$ be its local time at 0, normalized as **half** the density of the occupation measure at 0 of $B^{|br|}$. Define the random points (D_1, D_2, \dots) as follows: take U_1 uniform on $[0, 1]$ independent of $B^{|br|}$, and let $D_1 = \inf\{s \geq U_1 : B_s^{|br|} = 0\}$. Then conditionally on D_1 take U_2 uniform on $[D_1, 1]$ independent of $(B_s^{|br|}, D_1 \leq s \leq 1)$, and let $D_2 = \inf\{s \geq U_2 : B_s^{|br|} = 0\}$, and so on.

Theorem 8.1 :

Suppose $(\mathbf{p}^{(n)})$ satisfies (8.4).

(i) There is convergence in law

$$c(\mathbf{p}^{(n)})H^{M_n} \rightarrow 2B^{|br|} \quad (8.5)$$

with respect to the $*$ -topology on $D[0, 1]$ defined in section 8.2.5. If $\mathbf{p}^{(n)}$ is uniform on $[n]$ then we can use the usual Skorokhod topology on $D[0, 1]$.

(ii) Jointly with the convergence in (i), the marks (Z_1^n, Z_2^n, \dots) converge in law to the sequence (D_1, D_2, \dots) .

(iii) Jointly with the above convergences we have the limit in law (for the uniform topology)

$$(c(\mathbf{p}^{(n)})\ell_s^{M_n}, 0 \leq s \leq 1) \rightarrow (L_s, 0 \leq s \leq 1). \quad (8.6)$$

This immediately yields

Corollary 8.1 :

The following convergence in law holds jointly with (8.6) in Theorem 8.1 :

$$(p^{(n)}(\mathcal{B}_j(M_n)), c(\mathbf{p}^{(n)})|\mathcal{C}_j(M_n)|)_{j \geq 1} \xrightarrow[n \rightarrow \infty]{(d)} (D_j - D_{j-1}, L_{D_j} - L_{D_{j-1}})_{j \geq 1}. \quad (8.7)$$

For uniform $\mathbf{p}^{(n)}$ we have $c(\mathbf{p}^{(n)}) = n^{-1/2}$ and these results rederive the results of [9]. For $(\mathbf{p}^{(n)})$ satisfying (8.4), these results imply results proved by other methods in [14] while adding assertion (iii) which cannot be proved by those methods.

8.2.4 \mathbf{p} -trees, \mathbf{p} -mappings and the Joyal bijection

Let \mathbf{T}_n be the set of unordered rooted labeled trees on $[n]$. We define a random object, the **\mathbf{p} -tree**, as a random rooted unordered labeled tree whose law is given by

$$P(\mathcal{T} = T) = \prod_{x \in [n]} p_x^{|T_x|}, \quad (8.8)$$

where T_x is the set of children of vertex x . It is not obvious that the normalizing factor on the right hand side of (8.8) is 1, that is, that this formula indeed defines a probability

law. This known fact [90] can be seen as a consequence of our following discussion.

As shown by Joyal [64] and reviewed in Pitman [90] one can define a bijection J between $\mathbf{T}_n \times [n]$ and $[n]^{[n]}$ which pushes forward the law of the \mathbf{p} -tree, together with an independent \mathbf{p} -vertex X_1 , to the law of the \mathbf{p} -mapping. This bijection maps the *spine* of the tree, that is, the vertices of the path from the root X_0 to the distinguished vertex X_1 , to the cyclic points of the mapping. As a deterministic bijection it would involve an arbitrary matching of two sets of some cardinality $K!$, but for our probabilistic uses it is more convenient to have the matching implemented by an explicit rule based on external randomization, as follows.

Let (\mathcal{T}, X_1) denote a \mathbf{p} -tree \mathcal{T} , rooted at some vertex X_0 , together with an independent \mathbf{p} -point X_1 . Let $X_0 = c_1, c_2, \dots, c_K = X_1$ be the path from the root to X_1 , which we call the *spine* of the tree. Delete the edges between the vertices of the spine, obtaining K trees $\mathcal{T}_{c_1}, \dots, \mathcal{T}_{c_K}$. Recall that (X_2, X_3, \dots) is an independent random \mathbf{p} -sample. As before the following construction a.s. terminates:

- Let $\tau_1 = 2$ and $\mathcal{T}_{c_{k_1}}$ be the tree containing X_2 . Then bind the trees $\mathcal{T}_{c_1}, \dots, \mathcal{T}_{c_{k_1}}$ together by putting edges $c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_{k_1} \rightarrow c_1$. Let $\mathcal{C}_1 = \{c_1, \dots, c_{k_1}\}$ and $\mathcal{B}_1 = \cup_{1 \leq i \leq k_1} \mathcal{T}_{c_i}$.

- Knowing $(\tau_i)_{1 \leq i \leq j}, (k_i)_{1 \leq i \leq j}, (\mathcal{C}_i)_{1 \leq i \leq j}$ and $(\mathcal{B}_i)_{1 \leq i \leq j}$ whose union is not $[n]$, let $\tau_{j+1} = \inf\{k : X_k \notin \cup_{1 \leq i \leq j} \mathcal{B}_i\}$. Then let $\mathcal{T}_{c_{k_{j+1}}}$ be the tree containing $X_{\tau_{j+1}}$, bind together the trees $\mathcal{T}_{c_{k_j+1}}, \dots, \mathcal{T}_{c_{k_{j+1}}}$ by putting edges $c_{k_j+1} \rightarrow c_{k_j+2} \rightarrow \dots \rightarrow c_{k_{j+1}} \rightarrow c_{k_j+1}$. Let $\mathcal{C}_{j+1} = \{c_{k_j+1}, \dots, c_{k_{j+1}}\}$ and $\mathcal{B}_{j+1} = \cup_{k_j+1 \leq i \leq k_{j+1}} \mathcal{T}_{c_i}$.

- When this ends (i.e. all the tree is examined), write $J(\mathcal{T}, X_1)$ for the mapping whose basins are $\mathcal{B}_1, \mathcal{B}_2, \dots$, and whose digraph is given by the following edges within each basin: within each tree \mathcal{T}_c for $c \in \mathcal{C} = \cup \mathcal{C}_i$, the edges are pointing towards the root c , and the cyclic points are pointing to each other according to the binding of trees described above.

Proposition 8.1 :

The mapping $J(\mathcal{T}, X_1)$ is a \mathbf{p} -mapping, and its basins and cyclic points are in \mathbf{p} -biased order.

// Fix m , a particular mapping on $[n]$. We condition on the \mathbf{p} -sample (X_2, X_3, \dots) . Then it is not difficult to see that there exists a unique (T, y) such that $J(T, y) = m$. This tree is obtained as follows: take the first cyclic point c of m to which X_2 is mapped by some iterate of m . If it is not the unique cyclic point of the basin of m in which X_2 has fallen, we delete the edge between the previous cyclic point (i.e. the cyclic point c' such that $m(c') = c$) and c . We then write $c_1 = c', c_2 = m(c'), c_3 = m^2(c'), \dots, c_{k_1} = c$. We reverse the edges between these cyclic points, i.e. we put directed edges $c_{k_1} \rightarrow \dots \rightarrow c_2 \rightarrow c_1$. Then we do the same with the next basin discovered by (X_2, X_3, \dots) , and, with obvious notations, we put an edge $c_{k_1+1} \rightarrow c_{k_1}$. We then call y the *top* of the spine of the tree T thus built, so that y is the root of the tree in which the point of the \mathbf{p} -sample (X_2, X_3, \dots) that has “discovered” the last basin of m has fallen. In fact, what we have done here is the way to invert the map J .

Now, the probability that (\mathcal{T}, X_1) , the \mathbf{p} -tree with an independent \mathbf{p} -vertex, is equal to (T, y) , is easily seen to be equal to $\prod_{x \in [n]} p_x^{|m^{-1}(x)|}$. Indeed, for each vertex x of T except y , the number of edges pointing to x is the

same as in the mapping digraph, and for y there is one ingoing edge missing, but this is compensated by our choice of $X_1 = y$ which has probability p_y .

Moreover, the probability does not depend on the values of X_2, X_3, \dots . So we can uncondition on (X_2, X_3, \dots) and then the fact that the basins of $J(\mathcal{T}, X_1)$ are in \mathbf{p} -biased random order is obvious. //

Remark. As hinted before, there are different ways of implementing the Joyal bijection in a probabilistic context. In the Brownian bridge limit setting of Theorem 8.1, these lead to different recursive decompositions of Brownian bridge, discussed in detail in [15].

In Fig. 8.3, we draw a tree with a spine (\bullet vertices) and we run a \mathbf{p} sample on it. The crosses indicate the edges that must be removed to form the mapping digraph, which is the same as in Fig. 8.1.

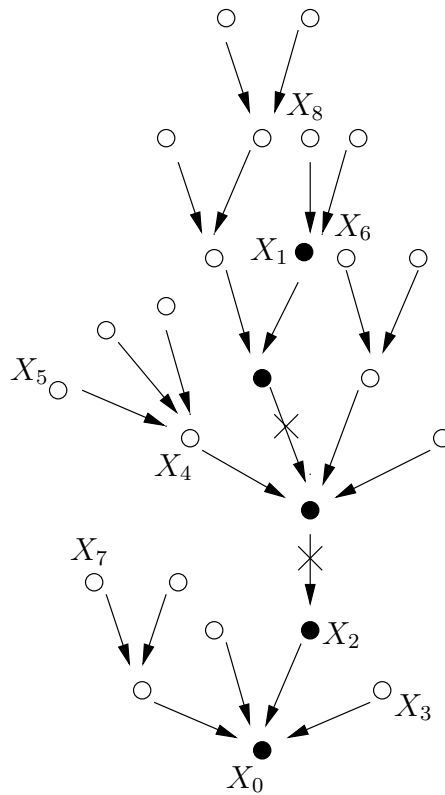


Figure 8.3: A tree and a \mathbf{p} -sample giving the mapping of Fig. 8.1 by the Joyal map

8.2.5 Weak convergence of random tree walks

Let \mathcal{T}_n be a random $\mathbf{p}^{(n)}$ -tree and let $H^{(n)} = H^{\mathcal{T}_n}$ be the associated height process from section 8.2.2. Let B^{ex} be standard Brownian excursion. We quote the following theorem: part (a) is from [5] (see [78] for recent variations) and part (b) is [14] Theorem 4.

Theorem 8.2 :

(a) If $\mathbf{p}^{(n)}$ is uniform on $[n]$ then

$$n^{-1/2}H^{(n)} \rightarrow 2B^{ex} \text{ in law}$$

with respect to the usual Skorokhod topology on $D[0, 1]$.

(b) If the sequence $(\mathbf{p}^{(n)})$ satisfies the uniform asymptotic negligibility condition (8.4) then

$$c(\mathbf{p}^{(n)})H^{(n)} \rightarrow 2B^{ex} \text{ in law}$$

with respect to the $*$ -topology on $D[0, 1]$ described below.

Examples show [14] that Skorokhod convergence does not hold in the complete generality of (8.4). In [7] we have sufficient conditions on $(\mathbf{p}^{(n)})$ for Skorokhod convergence, but we do not have a conjecture for the precise necessary and sufficient conditions.

Here are the properties of the $*$ -topology that we need (stated slightly differently than in [14]). Write $\xrightarrow{\text{unif}}$ for uniform convergence on $[0, 1]$.

Lemma 8.1 :

Let $f^n \in D[0, 1]$ and $f^\infty \in C[0, 1]$. Then $f^n \rightarrow^* f^\infty$ if and only if there exist functions $g^n, h^n \in D[0, 1]$ such that

$$f^n = g^n + h^n$$

$$g^n \xrightarrow{\text{unif}} f^\infty$$

$$h^n \geq 0$$

$$\text{Leb}\{x : h^n(x) > 0\} \rightarrow 0.$$

8.3 Proof of Theorem 8.1

8.3.1 Representation of the mapping walk with \mathbf{p} -trees

As in section 8.2.4, let (\mathcal{T}, X_1) denote a \mathbf{p} -tree \mathcal{T} , rooted at some vertex X_0 , together with an independent \mathbf{p} -point X_1 . Recall the definition of the mapping $J(\mathcal{T}, X_1)$ defined in terms of (\mathcal{T}, X_1) and a \mathbf{p} -sample (X_2, X_3, \dots) . We are now going to use Proposition 8.1 to construct the \mathbf{p} -mapping walk H^M , for $M = J(\mathcal{T}, X_1)$, from (\mathcal{T}, X_1) . Recall that $\mathcal{T}_{c_1}, \dots, \mathcal{T}_{c_K}$ are the subtrees of \mathcal{T} obtained when the edges between the vertices of the spine are deleted, and rooted at these vertices. To each of these we can associate the height processes $H^{\mathcal{T}_{c_i}}$ (with weights on vertices being the \mathbf{p} -values). If we now concatenate these walks together, just as in (8.3), it should be clear from Proposition 8.1 that the resulting process is the walk H^M associated to the \mathbf{p} -mapping $M = J(\mathcal{T}, X_1)$, with the order on basins induced by the Joyal map. With this interpretation, the mapping walk is thus what we call the height process of the \mathbf{p} -tree *above the spine*.

Next, we need to incorporate the time-marks of the mapping walk. Recall that these time-marks give the successive intervals $[Z_i, Z_{i+1})$ of exploration of the j -th basin. By Proposition 8.1, the order on basins is determined by the visits of a \mathbf{p} -sample of components of the \mathbf{p} -tree. So it should be clear that we may obtain the marks as follows (this has to be understood as a conditional form of the recursive constructions above). Let $Z_0 = 0$. Recall the notation $[g^M(i), d^M(i))$ for the interval during which the walk H^M visits the point i .

- Take U_2 uniform $(0, 1)$ independent of the \mathbf{p} -tree. Then let $Z_1 = \inf\{g^M(i) : g^M(i) > U_2, H_{g^M(i)}^M = 0\} \wedge 1$. If $Z_1 = 1$ we are done.
- Given $(Z_i)_{0 \leq i \leq j}$ with $Z_j < 1$, let U_{j+2} be uniform on $(Z_j, 1)$ independent of the tree,

and $Z_{j+1} = \inf\{g^M(i) : g^M(i) > U_{j+2}, H_{g^M(i)}^M = 0\} \wedge 1$. If this is 1 we are done.

Thus we can study mapping-walks directly in terms of trees, as summarized in

Proposition 8.2 :

Let \mathcal{T} be a \mathbf{p} -tree and X_1 a \mathbf{p} -sample. The marked height process above the spine, $(H, (Z_1, \dots))$, has the law of the marked walk of the \mathbf{p} -mapping $J(\mathcal{T}, X_1)$, with basins in \mathbf{p} -biased random order.

8.3.2 A transformation on paths

Motivated by the discrete transformation (height process \rightarrow height process above spine) above, we introduce a transformation $\mathbf{J}^u : D[0, 1] \rightarrow D[0, 1]$. Fix $0 \leq u \leq 1$. Consider $f = (f_t) \in D[0, 1]$. Define the *pre- and post- infimum process of f before and after u* , written $\underline{f}(u)$, as follows:

$$\underline{f}_s(u) = \begin{cases} \inf_{t \in [s, u]} f_t & \text{for } s < u \\ \inf_{t \in [u, s]} f_t & \text{for } s \geq u. \end{cases}$$

An “excursion” of f above $\underline{f}(u)$ is a portion of path of $f - \underline{f}(u)$ on a constancy interval of $\underline{f}(u)$. Each of these excursions has a starting time g which is at some height $h = f_g = \underline{f}_g(u)$, and if two or more of these excursions have the same starting height, we stick them together in the order induced by $(0, 1)$, so that each height specifies at most one “generalized” excursion of f above \underline{f} . Write $\varepsilon_1(\cdot), \varepsilon_2(\cdot), \dots$ for these generalized excursions, ranked for example in decreasing order of lifetimes l_1, l_2, \dots , with some arbitrary convention if repeats appear in this sequence. Let h_i be the height of the starting point of excursion ε_i . We now concatenate these excursions in increasing order of starting height. That is, for $s \in [0, 1)$, let $h = h_i$ be the unique height such that $\sum_{j: h_j < h_i} l_j \leq s < \sum_{j: h_j \leq h_i} l_j$ and define

$$(\mathbf{J}^u(f))_s = \varepsilon_i \left(s - \sum_{j: h_j < h_i} l_j \right).$$

If the sum s_0 of lengths of constancy intervals of \underline{f} , that is $\sum_j l_j$, equals 1, then $\mathbf{J}^u(f)$ is defined for all $0 \leq s \leq 1$; otherwise we just define $\mathbf{J}^u(f)$ to equal 0 on $s_0 < s \leq 1$. We call $\mathbf{J}^u(f)$ the *process f reflected above $\underline{f}(u)$* .

Lemma 8.2 :

Let $f^n \in D[0, 1]$ and $f^\infty \in C[0, 1]$. Suppose that, for each $0 \leq u \leq 1$, the lengths of intervals of constancy of $\underline{f}^\infty(u)$ sum to 1, and suppose that the different (non-generalized) excursions of \underline{f}^∞ above $\underline{f}^\infty(u)$ start at different heights.

(a) If $f^n \xrightarrow{\text{unif}} f^\infty$ then $\mathbf{J}^u(f^n) \xrightarrow{\text{unif}} \mathbf{J}^u(f^\infty)$.

(b) If $f^n \rightarrow^* f^\infty$ and U has uniform $(0, 1)$ law then $\mathbf{J}^U(f^n) \rightarrow^* \mathbf{J}^U(f^\infty)$ in probability.

// We outline the argument, omitting some details. Fix u . Consider an interval of constancy of $\underline{f}^\infty(u)$, say $[a_k, b_k]$. From the hypotheses on f^∞ we have $f(s) > f(a_k)$ on $a_k < s < b_k$. Consider the case $f^n \xrightarrow{\text{unif}} f^\infty$. Then for large n there must be intervals of constancy of $\underline{f}^n(u)$, say $[a_k^n, b_k^n]$, such that $a_k^n \rightarrow a_k, b_k^n \rightarrow b_k, f^n(a_k^n) \rightarrow f^\infty(a_k)$. This implies

$$(f^n(a_k^n + s) - f^n(a_k^n), 0 \leq s \leq b_k^n - a_k^n) \xrightarrow{\text{unif}} (f^\infty(a_k + s) - f^\infty(a_k), 0 \leq s \leq b_k - a_k).$$

Since $\sum_k (b_k^n - a_k^n) \rightarrow \sum_k (b_k - a_k) = 1$, we easily see that in the case $u = 1$ we have $\mathbf{J}^u(f^n) \xrightarrow{\text{unif}} \mathbf{J}^u(f^\infty)$. For general u , apply the argument above separately to $[0, u]$ and $[u, 1]$, and check that the operation of "concatenation of excursions in order of starting height" is continuous; again we deduce $\mathbf{J}^u(f^n) \xrightarrow{\text{unif}} \mathbf{J}^u(f^\infty)$.

Now consider the case $f^n \rightarrow^* f^\infty$. Recall the Lemma 8.1 decomposition $f^n = g^n + h^n$. By passing to a subsequence we may assume that for almost all $0 \leq u \leq 1$

$$h^n(u) = 0 \text{ ultimately .} \tag{8.9}$$

Fix such a u ; it is enough to show $\mathbf{J}^u(f^n) \rightarrow^* \mathbf{J}^u(f^\infty)$. The previous case implies that $\mathbf{J}^u(g^n) \xrightarrow{\text{unif}} \mathbf{J}^u(f^\infty)$. Consider, as in the previous argument, an interval of constancy $[a_k^n, b_k^n]$ of g^n converging to an interval of constancy of $[a_k, b_k]$ of f^∞ . Since $f^n = g^n + h^n$ with $h^n \geq 0$, for n large enough there is a corresponding interval of constancy of f^n which contains the interval $[\tilde{a}_k^n, \tilde{b}_k^n]$ defined by

$$\tilde{a}_k^n = \inf\{a \geq a_k^n : h^n(a) = 0\}, \quad \tilde{b}_k^n = \sup\{b \leq b_k^n : h^n(b) = 0\}.$$

Use (8.9) to see that $\tilde{b}_k^n - \tilde{a}_k^n \rightarrow b_k - a_k$. We now see that the analog of $\mathbf{J}^u(g^n)$ using only excursions over $\cup_k [\tilde{a}_k^n, \tilde{b}_k^n]$ will converge uniformly to $\mathbf{J}^u(f^\infty)$. After adding the contribution of h^n over these intervals, we will still have $*$ -convergence; and the contribution to $\mathbf{J}^u(f^n)$ from the complement of $\cup_k [\tilde{a}_k^n, \tilde{b}_k^n]$ is asymptotically negligible for $*$ -convergence. //

8.3.3 Pushing forward tree walks to mapping walks

Let \mathcal{T} be a \mathbf{p} -tree on $[n]$, and put the children of each vertex in uniform random order. Let U be uniform on $(0, 1)$, independent of \mathcal{T} , and let X_1 be the vertex visited by the height process $H^{\mathcal{T}}$ at time U . The fact that the height process spends time p_x at vertex x implies that X_1 is a \mathbf{p} -sample. By Proposition 8.1, $M = J(\mathcal{T}, X_1)$ is a random \mathbf{p} -mapping with basins in \mathbf{p} -biased random order. Let H^M be the associated marked random walk, constructed as in section 8.3.1, which by Proposition 8.2 is the height process of \mathcal{T} above the spine.

So to get H^M from $H^{\mathcal{T}}$ we have to extract from $H^{\mathcal{T}}$ the height processes of the subtrees rooted on the spine. This will be done by applying the transformation \mathbf{J} to a slightly modified version of $H^{\mathcal{T}}$.

Write $c_1, c_2, \dots, c_K = X_1$ for the vertices of the spine of \mathcal{T} in order of height, and as before write $[g(c_i), d(c_i))$ for the interval in which the height process $H^{\mathcal{T}}$ "visits" c_i . Now we consider the process

$$K_s = \begin{cases} H_{g(c_i)}^{\mathcal{T}} + 1 & \text{if } s \in (g(c_i), d(c_i)), \text{ for some } i \\ H_s^{\mathcal{T}} & \text{else.} \end{cases} \tag{8.10}$$

In other words, we "lift" the heights of the spine vertices by 1, but we use a small artifact here: at the point $g(c_i)$, the process stays at the value $H_{g(c_i)}^{\mathcal{T}}$, and the process is not càdlàg

in general. Now reflect this process K above $\underline{K}(U)$ to obtain the process $\mathbf{J}^U(K)$.

Lemma 8.3 :

$$\mathbf{J}^U(K)|_s = (H_s^M - 1)^+, \quad 0 \leq s \leq 1. \quad (8.11)$$

// Suppose that the height process H^T of the tree is currently visiting a spine vertex, say c_i , which is not the top of the spine. Write h for its height ($h = H^T(c_i) = K_{g(c_i)} = K_{g(c_{i+})} - 1$). Then c_i has some children, one of them being c_{i+1} . Now we want to recover the height process of the subtree \mathcal{T}_{c_i} rooted at c_i when we delete the edges between the vertices of the spine. First, during the time interval $(g(c_i), g(c_{i+1}))$, the height process of \mathcal{T}_n visits c_i and the vertices of \mathcal{T}_{c_i} that are located to the left of the spine (i.e. the descendants of the children of c_i located before c_{i+1}), if any. Then the process examines all the descendants of c_{i+1} , hence staying at heights greater than $h + 1$, and after that visits the children of c_i that are to the right of the spine, if any, starting say at time $g'_i > U$.

Hence, $K_{g(c_i)} = h$, $K_s \geq h + 1$ for $s \in (g(c_i), g(c_{i+1}))$ and $K_s > h + 1$ for $s \in (g(c_{i+1}), U)$. So $(g(c_i), g(c_{i+1}))$ is an excursion interval of K above \underline{K} , for an excursion starting at height $h + 1$. This excursion is easily seen as being $(H^{\mathcal{T}_{c_i}} - 1)^+$ restricted to the vertices that are to the left hand side of the spine, where $H^{\mathcal{T}_{c_i}}$ is the height process of \mathcal{T}_{c_i} .

Then $K_{g'_i} = h + 1$, so g'_i is the starting time of an excursion of K above $\underline{K}(U)$, with starting height $h + 1$, and this excursion is now $(H^{\mathcal{T}_{c_i}} - 1)^+$ restricted to the vertices that are to the right hand side of the spine. The analysis is easier if $c_i = X_1$ is the top of the spine, in which case there is no child of c_i at the left or right-hand side of the spine. From the description in 8.3.1 this gives the result. //

Note that our "artifact" was designed to give an exact equality in Lemma 8.3. Removing the artifact to make processes càdlàg can only change the processes involved by ± 1 , which will not affect our subsequent asymptotic arguments.

Figure 8.4 shows the height process of the tree of Fig. 8.3, with U such that the spine is the same. We also draw the process \underline{K} . As noted before, the unmarked walk associated to the image of the last tree by the Joyal map depends only on the spine, and so this walk is that of Fig. 8.2. The next figure depicts the process $\mathbf{J}^U(K)$.

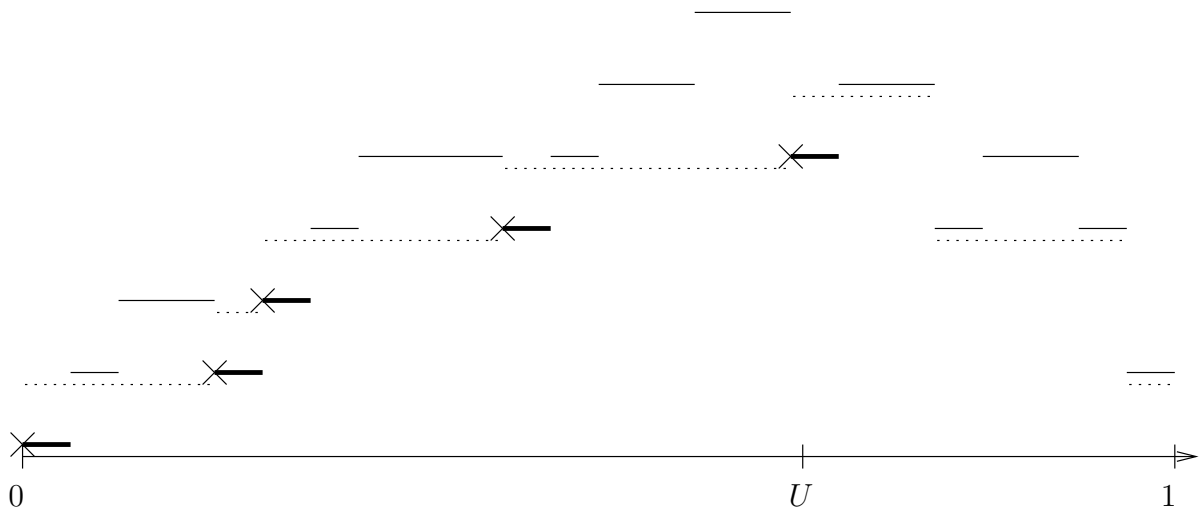


Figure 8.4: The process H^T and the process \underline{K} (dashed), where crosses and thick lines represent visits to vertices of the spine

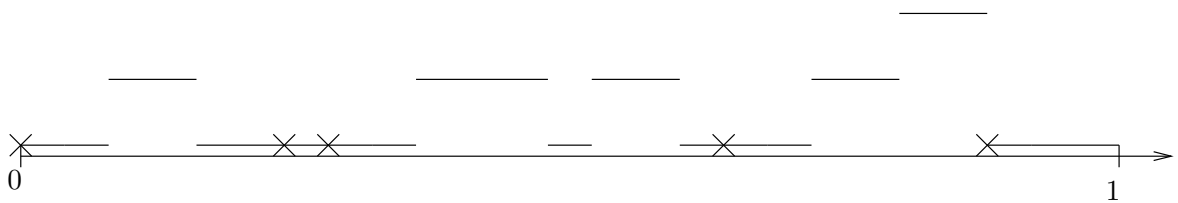


Figure 8.5: The process $\mathbf{J}^U(K)$ (compare with Fig. 8.2)

8.3.4 J transforms B^{ex} to $B^{|br|}$

Lemma 8.4 :

Let B^{ex} be standard Brownian excursion, and let U be uniform independent on $[0, 1]$. Then $\mathbf{J}^U(B^{ex})$ is distributed as $B^{|br|}$, reflecting Brownian bridge on $[0, 1]$.

// By [94], the reflecting Brownian bridge is obtained from the family of its excursions by concatenating them in exchangeable random order. Precisely, let $(\varepsilon_1, \varepsilon_2, \dots)$ be the excursions of $B^{|br|}$ away from 0, ranked by decreasing order of their durations $l_1 \geq l_2 \geq \dots > 0$, and let \preceq be a random order (\prec is then the associated strict order) on \mathbb{N} independent of the excursions, such that for every k , each one of the $k!$ possible strict orderings on the set $[k]$ are equally likely. Then the process

$$X_s = \varepsilon_i \left(s - \sum_{j \prec i} l_j \right) \quad \text{for} \quad \sum_{j \prec i} l_j \leq s \leq \sum_{j \preceq i} l_j$$

has the same law as $B^{|br|}$.

By [30, Theorem 3.2], we already know that the excursions away from 0 of $\mathbf{J}^U(B^{ex})$ are those of a reflecting Brownian bridge. It thus remains to show that the different ordering of the excursions used to define the process $\mathbf{J}^U(B^{ex})$ is an independent exchangeable order. Now, by a conditioned form of Bismut's decomposition (see e.g. Biane [32]), conditionally on U and B_U^{ex} , the paths $(B_{U-s}^{ex}, 0 \leq s \leq U)$ and $(B_{s+U}^{ex}, 0 \leq s \leq 1 - U)$ are independent Brownian paths starting at B_U^{ex} , conditioned to first hit 0 at time U and $1 - U$ respectively, and killed at these times. Still conditionally on (U, B_U^{ex}) , consider the excursions $(\varepsilon_1^1, \varepsilon_2^1, \dots)$ of $(B_s^{ex}, 0 \leq s \leq U)$ above its future infimum process, ordered in decreasing lifetimes order, and their respective heights (h_1^1, h_2^1, \dots) . Let also $(\varepsilon_1^2, \varepsilon_2^2, \dots)$ be the excursions of $(B_{s+U}^{ex}, 0 \leq s \leq 1 - U)$ above its infimum process, also ordered in decreasing lifetimes order, and denote their respective heights by (h_1^2, h_2^2, \dots) . Then we have from [94, Proposition 6.2] that $(h_1^1/B_U^{ex}, h_2^1/B_U^{ex}, \dots)$ and $(h_1^2/B_U^{ex}, h_2^2/B_U^{ex}, \dots)$ are independent conditionally on B_U^{ex} , and are two sequences of i.i.d. uniform $[0, 1]$ r.v.'s. Hence, the concatenation of these two sequences is again a sequence of independent uniform $[0, 1]$ r.v.'s. This thus holds also unconditionally on (U, B_U^{ex}) . Now, by definition, the order of the excursions of $\mathbf{J}^U(B^{ex})$ is that induced by this concatenated family, meaning that the excursion ε_k^i appears before excursion $\varepsilon_{k'}^j$ if and only if $h_k^i < h_{k'}^j$ for $k, k' \geq 1, i, j \in \{1, 2\}$. The excursions are thus in exchangeable random order, and the claim follows. //

Notice also from [94] that for the reflected Brownian bridge $B^{|br|} = \mathbf{J}^U(B^{ex})$, one can extend the fact that $L_1 = 2B_U^{ex}$ to

$$L_s = 2h_s \quad (8.12)$$

if s is not a zero of $B^{|br|}$, where h_s is the height of the starting point of the excursion of B^{ex} that is matched to the excursion of $\mathbf{J}^U(B^{ex})$ straddling s , and L is then defined on all $[0, 1]$ by continuity.

8.3.5 Completing the proof of Theorem 8.1

// As in Proposition 8.1, we may take the $\mathbf{p}^{(n)}$ -mapping M_n in its representation $M_n = J(\mathcal{T}_n, X_{1,n})$, where \mathcal{T}_n is a $\mathbf{p}^{(n)}$ -tree and $X_{1,n}$ is a $\mathbf{p}^{(n)}$ sample from \mathcal{T}_n . By Theorem 8.2(b) and the Skorokhod representation Theorem, we may suppose that we have a.s. convergence of $c(\mathbf{p}^{(n)})H^{\mathcal{T}_n}$ to $2B^{ex}$. (Here and below, convergence is $*$ -convergence in general, and uniform convergence in the special case of uniform $\mathbf{p}^{(n)}$). As before, we may suppose that $X_{1,n}$ is the vertex that is visited at time U by $H^{\mathcal{T}_n}$ for an independent uniform U , and use the same U for every n . From the definition (8.10) of K^n we also have a.s. convergence of $c(\mathbf{p}^{(n)})K^n$ to $2B^{ex}$. Then by Lemmas 8.2 and 8.4, the process $c(\mathbf{p}^{(n)})\mathbf{J}^U(K^n)$ converges to $2B^{|br|}$. Hence, so does $c(\mathbf{p}^{(n)})H^{M_n}$ according to Lemma 8.3. This is assertion (i) of the Theorem.

For (ii), the assertion about the marks (Z_1^n, Z_2^n, \dots) follows easily by incorporating the representation of section 8.3.1 into the argument above and using the fact from 8.3.2 and 8.3.3 that the excursions of $c(\mathbf{p}^{(n)})H^{M_n}$

away from 0 converge to that of $2B^{|br|}$ (the only possible trouble is when a U_i falls on a zero of H^n , but this happens with probability going to 0).

To obtain (iii) we observe that the number of cyclic points visited in depth-first order before the vertex coded by $s \in [0, 1]$ is equal (except for an unimportant possible error of 1) to the starting height of some excursion of K^n above \underline{K}^n . Now suppose that s is not a zero of $B^{|br|}$, so that it is strictly included in the excursion interval of, say the k -th longest-lifetime excursion of $2B^{|br|}$ away from 0. Then for n sufficiently big, s also belongs to the excursion interval of the k -th longest-lifetime excursion of H^{M_n} away from 0, which corresponds to the k -th longest-lifetime excursion of K^n above \underline{K}^n . But this excursion's starting height, once multiplied by $c(\mathbf{p}^{(n)})$, converges to the starting height of the k -th longest-lifetime excursion of $2B^{ex}$ above $2\underline{B}^{ex}$. It now follows from the remark after the proof of Lemma 8.4 that this last height is equal to L_s . We can now conclude, since the limiting process L is continuous and increasing on $[0, 1]$, and since the lengths of excursions of $2B^{ex}$ above $2\underline{B}^{ex}$ sum to 1, that the convergence of $c(\mathbf{p}^{(n)})\ell^{M_n}$ to L holds uniformly and not only pointwise. //

8.4 Final comments

1. At the start of the proof of Lemma 8.4 we used the result from [30] that the excursions of $\mathbf{J}^U(B^{ex})$ away from 0 are those of a reflecting Brownian bridge. Here is a way to rederive that result. First, by the well-known formula for the entrance law of the Brownian excursion, one easily gets that the law of (U, B_U^{ex}) has the same law as $(T_{R/2}, R/2)$ given $T_R = 1$, where T is the first-passage subordinator associated with Brownian motion and R is an independent r.v. with Rayleigh distribution. By Bismut's decomposition, one deduces that the process Y defined by $Y_t = B_{U-t}^{ex}$ for $0 \leq t \leq U$ and $Y_t = B_t^{ex} - 2B_U^{ex}$ for $U \leq t \leq 1$ is, conditionally on B_U^{ex} but unconditionally on U , a first-passage bridge of the Brownian motion, i.e. a Brownian motion conditioned to first hit $-2B_U^{ex}$ at time 1. By [94], its associated reflected process above its infimum is a reflecting Brownian bridge conditioned to have local time $2B_U^{ex}$ at level 0, and we can uncondition on B_U^{ex} , since $2B_U^{ex}$ has the Rayleigh law, which is that of the local time at 0 of $B^{|br|}$.

2. Our work implicitly answers a question of Pitman [91]. Let

$$C_n^k = |\{i \in [n] : M_n^{k-1}(i) \notin \mathcal{C}(M_n), M_n^k(i) \in \mathcal{C}(M_n)\}|$$

be the number of vertices at distance k of the set of cyclic points of the uniform random mapping M_n (for the distance induced by the digraph of M_n). In particular, $C_n^0 = \ell_1^{M_n}$ with our previous notations. Drmota and Gittenberger [47] show that the process $(n^{-1/2}C_n^{\lfloor 2s\sqrt{n} \rfloor}, s \geq 0)$ converges in law to the process $(L_1^s(B^{|br|}), s \geq 0)$ of local times of $B^{|br|}$ (with our choice of normalization as half the occupation density). One of the question raised in Pitman [91] is whether this convergence holds jointly with the convergences of our main theorem. To show this is true, first note that from the tightness of each individual component, we get that the pair $(n^{-1/2}H^{M_n}, n^{-1/2}C_n^{\lfloor 2\sqrt{n} \cdot \cdot \rfloor})$ is tight (this is true because the limit in law of the process $n^{-1/2}H^{M_n}$ is continuous, see [61, p. 353, Corollary 3.33]). Call $(2B^{|br|}, L')$ its weak limit through some subsequence, and suppose that the

convergence in law holds a.s. by Skorokhod's representation theorem. If we prove that $L'(s) = L_1^s(B^{|br|})$ for every s , we will have shown that $(2B^{|br|}, L_1(B^{|br|}))$ is the only possible limit, hence that $(n^{-1/2}H^{M_n}, n^{-1/2}C_n^{[2\sqrt{n}s]})$ jointly converges to this limit. Now for every $s \geq 0$ one has that

$$\int_0^1 dt \mathbb{1}_{\{H_t^{M_n} \leq [2\sqrt{n}s]\}} = n^{-1} \sum_{k=0}^{[2\sqrt{n}s]} C_n^k \rightarrow 2 \int_0^s du L'(u),$$

whereas the left-hand term converges to $\int_0^1 dt \mathbb{1}_{\{B_t^{|br|} \leq s\}}$, which equals $2 \int_0^s du L_1^u(B^{|br|})$. Hence the result by identification.

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