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Said Bellaassali

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Université de Bourgogne  
Laboratoire d'Analyse  
Appliquée et Optimisation

# Thèse

présentée par

**Said Bellaassali**

en vue d'obtenir le titre de

DOCTEUR DE L'UNIVERSITÉ DE BOURGOGNE

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## *Contributions à l'optimisation multicritère*

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*à Fabienne et à mes parents*



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## Introduction générale

L'analyse et l'optimisation différentiable se basent surtout sur le théorème de Fermat qui atteste que le gradient d'une fonction s'annule en tout point où elle atteint son minimum. Le passage à l'optimisation non différentiable tout en cherchant à obtenir les mêmes résultats, a suscité l'idée du sous-gradient qui remplacera le gradient. L'ensemble des sous-gradients nous donne la notion de sous-différentiel. À cette motivation vient s'adjoindre celle de travailler sans convexité, ce qu'on appelle l'analyse non régulière.

Parmi les sous-différentiels connus, citons le sous-différentiel généralisé de Clarke, le sous-différentiel géométrique, le sous-différentiel approché, le sous-différentiel proximal et les sous-différentiels Fréchet et Fréchet limite. Ce travail n'utilisera que le sous-différentiel Fréchet limite et le sous-différentiel approché pour plusieurs raisons : notamment car ces deux sous-différentiels sont « plus petits » que celui de Clarke, ils contiennent moins d'erreurs et sont intéressants pour les richesses de leurs calculs.

On utilisera le concept de préférence qui est apparu pour la première fois en économie. Plusieurs auteurs définissaient la préférence par une fonction utilité. Debreu [28] a montré qu'en dimension finie, une préférence  $\prec$  est définie par une fonction utilité si et seulement si pour tout  $x$

$$\{y : x \prec y\} \quad \text{et} \quad \{y : y \prec x\} \quad \text{sont fermés.} \quad (1)$$

Ce théorème assure l'existence de la fonction utilité mais ne donne cependant pas la méthode pour l'avoir. De plus il existe des préférences qui ne peuvent pas être définies par des fonctions utilités comme l'ordre lexicographique par exemple.

Le travail est constitué de quatre chapitres. Nous présentons dans le premier chapitre les outils d'analyse non lisses accompagnés du calcul sous-différentiel indispensable pour la compréhension de cette thèse.

Le deuxième chapitre porte sur l'existence des multiplicateurs de Lagrange des problèmes d'optimisation vectorielle non lisse de la forme

$$(P) \begin{cases} \min f(x) \\ \text{sous contraintes} \\ x \in C \text{ et } g(x) \in D \end{cases}$$

où  $f: X \mapsto Z$  et  $g: X \mapsto Y$  sont des fonctions,  $X$ ,  $Y$  et  $Z$  sont des espaces de Banach et  $C \subset X$  et  $D \subset Y$  sont des ensembles fermés. On travaille ici avec une préférence générale  $\prec$ .

On dit qu'un élément  $x$  de  $X$  est admissible pour le problème  $(P)$  s'il vérifie  $x \in C$  et  $g(x) \in D$ . On dit que  $\bar{x}$  est une solution du problème  $(P)$  si  $\bar{x}$  est un point admissible et s'il n'existe aucun  $x$  admissible pour le problème  $(P)$  tel que  $f(x) \prec f(\bar{x})$ .

Question : comment peut-on établir des conditions nécessaires d'optimalité (multiplicateurs de Lagrange) du problème  $(P)$  avec une préférence générale ?

On rappelle que le problème  $(P)$  peut être étudié en dimension finie ou infinie.

Les conditions nécessaires d'optimalité du problème  $(P)$  peuvent être établies en utilisant des conditions de qualification. On peut en rappeler ici quelques unes comme celles de Guignard, Slater, Mangasarian-Fromovitz, Robinson, Zowe et Kurcyusz. Plusieurs auteurs ont utilisé des conditions de qualification pour assurer la non-vacuité et le caractère borné de l'ensemble des multiplicateurs de Lagrange du type Kuhn-Tucker [30], [33], [51], [78] et [103], ou pour étudier les propriétés de la fonction marginale associée à un programme non linéaire [34], [81] et [91], ou tout simplement pour obtenir des règles de calcul sous-différentiel.

D'autres ont utilisé les fonctions localement lipschitziennes [47], [60] et [70], les fonctions fortement compactement lipschitziennes [30] et [35] et le principe variationnel d'Ekeland [29] en supposant que  $D$  est un cône convexe fermé d'intérieur non vide ou que  $D = D_1 \times \{0\}$  avec  $\{0\} \subset \mathbb{R}^n$  et  $D_1$  est un cône convexe fermé d'intérieur non vide.

En utilisant le sous-différentiel approché, l'auteur de [52] a obtenu les multiplicateurs de Lagrange du type Fritz-John et Kuhn-Tucker pour le problème  $(P)$  et ceci en travaillant dans des espaces de Banach et en supposant que  $D$  est « épi-lipschitz like » [11] et [12] et dans [59] en supposant que  $D$  est compactement épi-lipschitzien.

Dans le deuxième chapitre, on donnera d'abord la définition de la régularité d'une préférence  $\prec$ . En utilisant la propriété d'Aubin pour les fonctions multivoques et son rapport avec la propriété de calme [82], la notion des fonctions compactement épi-lipschitziennes de Borwein et Strojwas [12]-[18], on montrera l'existence des multiplicateurs de Lagrange du type Karush-Kuhn-Tucker exprimés en terme du sous-différentiel approché. On obtiendra ainsi :

**Théorème 0.0.1** *Soit  $\bar{x}$  une solution du problème  $(P)$ . Supposons que la*

préférence  $\prec$  est régulière en  $f(\bar{x})$  et que la fonction multivoque  $M : Y \mapsto X$ , définie par  $M(y) = \{x \in C : y \in -g(x) + D\}$ , est calme en  $(0, \bar{x})$ . Alors il existe  $z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x}))), f(\bar{x}))$ , avec  $z^* \neq 0$ , et  $y^* \in N(D, g(\bar{x}))$  tels que

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Ici  $\tilde{N}(cl(\mathcal{L}(f(\bar{x}))))$  est un « élargissement » du cône normal approché. Sous les mêmes hypothèses que le Théorème 0.0.1, on déduira l'existence des multiplicateurs de Lagrange pour le problème (P) avec quelques exemples de préférences :

- Pareto généralisé : on considère  $K \subset Z$  un cône convexe avec  $K^0$  localement compact, et on définit la préférence  $\prec$  par :

$$z \prec z' \text{ si et seulement si } z - z' \in K \text{ et } z \neq z'.$$

On obtiendra l'existence de  $z^* \in K^0$ , avec  $z^* \neq 0$ , et  $y^* \in N(D, g(\bar{x}))$  tels que

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x})$$

où  $K^0 = \{z^* \in Z^* : \langle z^*, z \rangle \leq 0 \quad \forall z \in K\}$ .

- Une préférence définie par une fonction utilité  $u$  : on suppose que l'épigraphe de  $u$  est compactement épi-lipschitzien en  $(\bar{z}, u(\bar{z}))$  et que  $0 \notin \partial u(\bar{z})$  où  $\bar{z} = f(\bar{x})$ . On obtiendra l'existence de  $y^* \in N(D, g(\bar{x}))$  et  $z^* \in \partial^\infty u(f(\bar{x})) \cup [\cup_{\lambda > 0} \lambda \partial u(f(\bar{x}))]$ , avec  $z^* \neq 0$ , tels que

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Pour le cas non calme on montrera l'existence des multiplicateurs de Lagrange du type Fritz-John. On obtiendra le théorème suivant :

**Théorème 0.0.2** *Soit  $\bar{x}$  une solution du problème (P). Supposons que la préférence  $\prec$  est régulière en  $f(\bar{x})$  et que  $D$  est compactement épi-lipschitzien en  $g(\bar{x})$ . Alors il existe  $z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x}))), f(\bar{x}))$ , et  $y^* \in N(D, g(\bar{x}))$ , avec  $(z^*, y^*) \neq 0$ , tels que*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

En utilisant respectivement le Théorème 0.0.1 et le Théorème 0.0.2, on obtiendra respectivement les conditions nécessaires d'optimalité des problèmes :

- $\min_{x \in X} F(x)$  où  $F : X \rightarrow Y$  est une fonction multivoque à graphe fermé.

Si on suppose que  $\bar{x}$  est le minimum (relatif à  $\bar{y}$  où  $(\bar{x}, \bar{y}) \in GrF$ ) par rapport à une préférence générale  $\prec$  et que cette préférence est régulière en  $\bar{y}$ , alors il existe  $z^* \in \tilde{N}(cl\mathcal{L}(\bar{y}), \bar{y})$  avec  $z^* \neq 0$ , tel que  $0 \in D^*F(\bar{x}, \bar{y})(z^*)$ .

Ici  $D^*F(x, y)$  désigne la codérivée de  $F$  au point  $(x, y) \in GrF$ .

On s'intéressera ensuite à l'étude des conditions nécessaires d'optimalité pour un problème multiobjectif gouverné par une inclusion différentielle, ceci en travaillant toujours avec une préférence générale. On aura besoin de quelques résultats sur le problème généralisé de Bolza. Le troisième chapitre est ainsi consacré à l'existence d'une nouvelle inclusion d'Euler-Lagrange des problèmes suivants :

### Le problème de Bolza

$$\min_{x \in W^{1,1}([a,b])} \{g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt\}. \quad (P_B)$$

### Le problème de Bolza avec contrôle

$$\min_{(x,u)} \{g(x(a), x(b)) + \int_a^b h(t, x(t), u(t)) dt\} \quad (R_1)$$

sous contraintes

$$\dot{x}(t) = f(t, x(t), u(t)) \quad p.p \quad \text{et} \quad u(t) \in U(t).$$

### Le problème isopérimétrique ( $I_S$ )

$$\min \{g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt\} \quad (I_s)$$

sous contraintes

$$\int_a^b f(t, x(t), \dot{x}(t)) dt \in K, \quad \dot{x}(t) \in U(t) \quad p.p \quad t \in [a, b]$$

où  $K \subset \mathbb{R}^m$  est un ensemble fermé,  $U$  est une fonction multivoque de  $[a, b]$  dans  $\mathbb{R}^n$  et  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  et  $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  sont des fonctions. Remarquons que pour les problèmes ( $P_B$ ) et ( $R_1$ ) on a  $m = n$ .

Pour établir les conditions nécessaires d'optimalité du problème ( $R_1$ ), plusieurs travaux ([21], [45], [46], [67]-[71], [76], [83], etc) ont utilisé des résultats d'analyse non lisse et ont prouvé des inclusions d'Euler-Lagrange de la forme

$$\dot{p}(t) \in \partial_x^c [h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle](z(t), v(t)) \quad p.p \quad t \in [a, b] \quad (2)$$

ou

$$\dot{p}(t) \in \text{co}\partial_x[h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle](z(t), v(t)) \quad p.p \ t \in [a, b] \quad (3)$$

où  $\partial_x^c$  et  $\partial_x$  sont respectivement le sous-différentiel partiel de Clarke et le sous-différentiel partiel Fréchet limite.

En utilisant la notion de calme qui est plus générale que la condition de la normalité, l'auteur de [56] a obtenu l'inclusion d'Euler-Lagrange suivante :

$$\dot{p}(t) \in \text{co}\{q : (q, 0) \in \partial[h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle + \psi_{U(t)}(\cdot)](z(t), v(t))\} \quad p.p \quad (4)$$

ainsi que l'inclusion Hamiltonienne :

$$\dot{p}(t) \in \text{co}\{q : (-q, \dot{z}(t), v(t)) \in \partial H(t, z(t), p(t), 0)\} \quad p.p \ t \in [a, b] \quad (5)$$

où  $\psi_C$  est la fonction caractéristique de l'ensemble  $C$  et  $H$  est l'Hamiltonien défini par

$$H(t, x, p, q) = \sup_{u \in U(t)} \{\langle p, f(t, x, u) \rangle + \langle q, u \rangle - h(t, x, u)\}$$

Notre but dans le troisième chapitre est d'établir une nouvelle inclusion d'Euler-Lagrange du type (4) sans supposer ni la notion de calme, ni la différentiabilité, ni la convexité du lagrangien. On travaillera avec le sous-différentiel Fréchet limite.

On commencera par reformuler  $(P_B)$  en un problème d'inclusion différentielle. En utilisant le Théorème 1 de [45], on obtiendra :

**Théorème 0.0.3** *Soit  $z$  une solution locale (dans  $W^{1,1}$ ) du problème  $(P_B)$ . Supposons que  $h$  est épi-mesurable en  $t$  et épi-lipschitzienne en  $z$ . Alors il existe un arc  $p$  et  $\lambda \in \{0, 1\}$ , avec  $(\lambda, p) \neq 0$ , tels que*

$$\dot{p}(t) \in \text{co}\{q : (q, p(t), -\lambda) \in N(\text{epi } h(t, \cdot); ((z(t), \dot{z}(t)), h(t, z(t), \dot{z}(t))))\} \quad p.p \ t \in [a, b] \quad (6)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), l(z(a), z(b)))) \quad (7)$$

$$\langle p(t), \dot{z}(t) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \max_{v \in \mathbb{R}^n} \{\langle p(t), v \rangle - \lambda h(t, z(t), v)\} \quad p.p \ t \in [a, b]. \quad (8)$$



On déduira le résultat de Vinter-Zheng [95] concernant le problème

$$\min_{x \in W^{1,1}} \left\{ g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt \right\} \quad (Q_1)$$

sous contraintes

$$\dot{x}(t) \in F(t, x(t)) \quad p.p., \quad (x(a), x(b)) \in C \quad (9)$$

où  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  et  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  sont des fonctions  $F : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  est une fonction multivoque à valeurs non bornées et  $C \subset \mathbb{R}^n \times \mathbb{R}^n$  est un ensemble fermé.

**Corollaire 0.0.1** *Supposons que  $z$  est une solution locale de  $(Q_1)$  et qu'il existe  $\varepsilon > 0$ , deux fonctions intégrables  $k, k_h : [a, b] \rightarrow \mathbf{R}$  et une constante  $k_g > 0$  tels qu'on ait*

$$(H_g) \text{ Pour tout } (x, y), (x', y') \in (z(a), z(b)) + \varepsilon(\mathbb{B} \times \mathbb{B})$$

$$|g(x, y) - g(x', y')| \leq k_g \|(x, y) - (x', y')\|.$$

$(H_F)$   $F(t, x)$  est mesurable en  $t$ , a valeurs fermées et pour tout  $x, x' \in z(t) + \varepsilon\mathbb{B}$

$$F(t, x') \subset F(t, x) + k(t)\|x' - x\|\mathbb{B}.$$

$(H_h)$   $h$  is épi-mesurable en  $t$  et pour tout  $(x, v), (x', v') \in (z(t) + \varepsilon\mathbb{B}) \times \mathbb{R}^n$

$$|h(t, x, v) - h(t, x', v')| \leq k_h(t)\|(x, v) - (x', v')\|.$$

$(H_k)$   $kk_h$  est intégrable .

On a alors l'existence d'un arc  $p$  et d'un scalaire  $\lambda \in \{0, 1\}$ , avec  $(p, \lambda) \neq 0$ , tels que

$$\begin{aligned} \dot{p}(t) \in \text{co}\{q : (q, p(t)) \in \lambda \partial h(t, z(t), \dot{z}(t)) \\ + N(\text{Gr}F(t, \cdot), (z(t), \dot{z}(t)))\} \quad p.p \quad t \in [a, b] \end{aligned}$$

$$(p(a), -p(b)) \in \lambda \partial g(z(a), z(b)) + N(C, (z(a), z(b)))$$

$$\langle p(t), \dot{z}(t) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \max_{v \in F(t, z(t))} \{ \langle p(t), v \rangle - \lambda h(t, z(t), v) \} \quad p.p \quad t \in [a, b].$$

On peut également en déduire le résultat de Ioffe-Rockafellar [46] :

**Corollaire 0.0.2** *Soit  $z$  une solution locale du problème de Bolza ( $P_B$ ). Supposons que :*

i)  $g$  est s.c.i

ii)  $h$  est mesurable en  $t$  et  $h(t, \cdot)$  est s.c.i

iii)  $h$  est à valeurs finies et pour tout  $N > 0$ , il existe  $\varepsilon_N > 0$  et  $k_N \in L^1$  tels que pour tout  $x, x' \in z(t) + \varepsilon_N B$  et  $v \in \dot{z}(t) + NB$

$$| h(t, x', v) - h(t, x, v) | \leq k_N(t) \|x' - x\|.$$

On a alors l'existence d'un arc  $p$  satisfaisant les expressions (6), (7) et (8) du Théorème 0.0.3 avec  $\lambda = 1$ .

En appliquant toujours le Théorème 0.0.3, on dérivera le principe du maximum du problème ( $R_1$ ).

**Théorème 0.0.4** *Supposons que  $(z, v)$  est une solution locale de ( $R_1$ ) et qu'il existe  $\varepsilon > 0$  et une fonction intégrable  $k : [a, b] \rightarrow \mathbb{R}$  tels que pour presque tout  $t \in [a, b]$  et pour tout  $z_1, z_2 \in z(t) + \varepsilon B$ ,  $u \in U(t)$  on a*

$$\|f(t, z_1, u) - f(t, z_2, u)\| \leq k(t) \|z_1 - z_2\|$$

$$| h(t, z_1, u) - h(t, z_2, u) | \leq k(t) \|z_1 - z_2\|.$$

Alors il existe un arc  $p$  et  $\lambda \in \{0, 1\}$  tels que  $(p, \lambda) \neq (0, 0)$  et

$$\dot{p}(t) \in \text{cod}_x \{ \lambda h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle \} (z(t), v(t)) \quad p.p \quad t \in [a, b] \quad (10)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), g(z(a), z(b)))) \quad (11)$$

$$\langle p(t), f(t, z(t), v(t)) \rangle - \lambda h(t, z(t), v(t)) = \max_{u \in U(t)} \{ \langle p(t), f(t, z(t), u) \rangle - \lambda h(t, z(t), u) \} \quad p.p \quad t \in [a, b]. \quad (12)$$

Si on suppose de plus que  $f(t, \cdot)$  et  $h(t, \cdot)$  sont localement lipschitziennes en  $(z(t), v(t))$ , alors l'inclusion d'Euler-Lagrange (10) peut être remplacée par la condition suivante :

$$\dot{p}(t) \in \text{co} \{ q : (q, 0) \in \partial [ \lambda h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle ] (z(t), v(t)) + \{0\} \times N(U(t), v(t)) \} \quad p.p \quad t \in [a, b]. \quad (13)$$

Maintenant, en utilisant le principe du maximum (Théorème 0.0.4), on obtiendra les conditions nécessaires d'optimalité des problèmes isopérimétriques ( $I_S$ ).

**Théorème 0.0.5** *Soit  $z$  une solution locale du problème ( $I_S$ ). Supposons que*

- a)  $f(t, x, u)$  est mesurable en  $t$  et continue en  $u$ ,
- b)  $h(t, \cdot)$  est semi-continue inférieurement et mesurable en  $t$ ,
- c)  $g$  est semi-continue inférieurement,
- d)  $U$  est mesurable et à valeurs fermées.

*Supposons qu'il existe  $\varepsilon > 0$  et une fonction intégrable  $k : [a, b] \rightarrow \mathbb{R}$  tels que pour presque tout  $t \in [a, b]$  et pour tous  $z_1, z_2 \in z(t) + \varepsilon B$  et  $u \in U(t)$  on a*

$$\|f(t, z_1, u) - f(t, z_2, u)\| \leq k(t)\|z_1 - z_2\|$$

$$|h(t, z_1, u) - h(t, z_2, u)| \leq k(t)\|z_1 - z_2\|.$$

*Alors il existe un arc  $p$ ,  $\lambda \in \{0, 1\}$  et un vecteur  $\gamma \in N(K; \int_a^b f(t, z(t), \dot{z}(t)) dt)$  tels que  $(p, \gamma, \lambda) \neq 0$  et*

$$\dot{p}(t) \in \text{co}\partial_x \{\lambda h(t, \cdot) - \langle \gamma, f(t, \cdot) \rangle\}(z(t), \dot{z}(t)) \quad p.p \ t \in [a, b] \quad (14)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), g(z(a), z(b)))) \quad (15)$$

$$\begin{aligned} \langle p(t), \dot{z}(t) \rangle + \langle \gamma, f(t, z(t), \dot{z}(t)) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \\ \max_{v \in U(t)} \{ \langle p(t), v \rangle + \langle \gamma, f(t, z(t), v) \rangle - \lambda h(t, z(t), v) \} \\ p.p \ t \in [a, b]. \quad (16) \end{aligned}$$

*Si on suppose de plus que  $f(t, \cdot)$  et  $h(t, \cdot)$  sont localement lipschitziennes en  $(z(t), \dot{z}(t))$  alors l'inclusion d'Euler-Lagrange (14) peut être remplacée par :*

$$\dot{p}(t) \in \text{co}\{q : (q, p(t)) \in \partial[\lambda h(t, \cdot) - \langle \gamma, f(t, \cdot) \rangle](z(t), v(t)) + \{0\} \times N(U(t), \dot{z}(t))\} \quad p.p \ t \in [a, b]. \quad (17)$$

On conclura le troisième chapitre en appliquant les résultats trouvés à des problèmes économiques du type :

$$\min_{(x, c)} \{g(x(b)) + \int_a^b h(t, x(t), c(t)) dt\}; \quad (P_e)$$

tels que

$$\dot{x}(t) = f(t, x(t), c(t)) \quad p.p \ t \in [a, b];$$

$$\begin{aligned} c(t) &\geq 0 \quad p.p \quad t \in [a, b]; \\ x(t) &> 0 \quad \forall t \in [a, b]; \\ x(a) &= \alpha; \end{aligned}$$

où  $\alpha$  est le capital initial,

$x(t) = (x_1(t), \dots, x_n(t))$  le taux du capital à l'instant  $t$ ,

et  $c(t) = (c_1(t), \dots, c_n(t))$  est la consommation à l'instant  $t$ .

En utilisant le Théorème 0.0.4, on obtiendra :

**Théorème 0.0.6** *Soit  $(z, v)$  une solution locale de  $(P_e)$ . Supposons que l'on ait l'un des deux cas suivants :*

*soit les consommations  $v_i$ ,  $i = 1, \dots, n$ , sont positives, soit les facteurs  $\varphi_i$ ,  $i = 1, \dots, n$ , sont positifs presque par tout sur  $[a, b]$ .*

*Supposons également qu'il existe  $\varepsilon > 0$  et des fonctions intégrables  $k_i : [a, b] \rightarrow \mathbb{R}$ , avec  $\max(\frac{k_i}{|\varphi_i|})$  intégrable, tels que pour presque tout  $t \in [a, b]$ , pour tout  $z', z'' \in z(t) + \varepsilon B$ , pour tout  $c \in \mathbb{R}_+^n$  et pour tout  $i = 1, \dots, n$  on a*

$$\|\varrho_i(t, z') - \varrho_i(t, z'')\| \leq k_i(t) \|z' - z''\|.$$

*Alors il existe un arc  $p$  et  $\lambda \in \{0, 1\}$  tels que  $(p, \lambda) \neq 0$  et*

$$\dot{p}(t) \in \text{co}\partial \left\{ \sum_{i=1}^n \frac{p_i(t) \varrho_i(t, \cdot)}{\varphi_i(t)} \right\} (z(t)) \quad p.p \quad t \in [a, b];$$

$$(-p(b), -\lambda) \in N(\text{epi } g, (z(b), g(z(b)))) \quad (18)$$

$$\sum_{i=1}^n \frac{p_i(t) v_i(t)}{\varphi_i(t)} - \lambda u(t, v(t)) =$$

$$\max_{c \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^n \frac{p_i(t) c_i}{\varphi_i(t)} - \lambda u(t, c) \right\} \quad p.p \quad t \in [a, b]. \quad (19)$$

*Si on suppose de plus que  $g$  est localement lipschitzienne en  $z(b)$  alors  $\lambda = 1$ .*

Dans le même contexte (économique) on trouvera les conditions nécessaires d'optimalité du modèle général de Ramsey

$$\text{minimiser } \int_a^b u(t, c_1(t), \dots, c_n(t)) dt \quad (R_m)$$

sur toutes les paires  $(x, c)$  vérifiant

$$\dot{x}(t) = c(t) - f(t, x(t), c(t)) \quad p.p \ t \in [a, b];$$

$$c(t) \geq 0 \quad p.p \ t \in [a, b];$$

$$x(t) > 0 \quad \forall t \in [a, b];$$

$$x(a) = \alpha;$$

où  $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$  est la fonction définie par

$$f(t, x, c) = (\varrho_1(t, x, c_1), \dots, \varrho_n(t, x, c_n))$$

et où  $\alpha$  est le capital initial.

Remarquer que  $(R_m)$  est un cas particulier de  $(P_e)$ . En effet, on suppose que tout les produits fabriqués en même période doivent être consommés ou investis sans détérioration ou dépréciation c'est à dire

$$I_i(t) = \dot{x}_i(t), \quad p.p \ t \in [a, b].$$

On suppose aussi que les productions  $y_i(t)$  sont présentées par les fonctions  $\varrho_i(t, x_1(t), \dots, x_n(t))$ . On obtient ainsi pour tout  $i = 1, \dots, n$ ,

$$\dot{x}_i(t) = c_i(t) - \varrho_i(t, x_1(t), \dots, x_n(t)).$$

En utilisant le Théorème 0.0.6, on déduira :

**Théorème 0.0.7** *Soit  $(z, v)$  une solution locale de  $(R_m)$ . Supposons qu'il existe  $\varepsilon > 0$  et une fonction intégrable  $k : [a, b] \rightarrow \mathbb{R}$  tels que pour presque tout  $t \in [a, b]$ , pour tout  $z', z'' \in z(t) + \varepsilon B$ ,  $c \in \mathbb{R}_+^n$  et pour tout  $i = 1, \dots, n$  on a*

$$\|\varrho_i(t, z') - \varrho_i(t, z'')\| \leq k(t) \|z' - z''\|.$$

Alors il existe un arc  $p$  tel que :

$$\dot{p}(t) \in \text{co}\partial \left\{ \sum_{i=1}^n p_i(t) \varrho_i(t, \cdot) \right\} (z(t)) \quad p.p \ t \in [a, b] \quad (20)$$

$$p(b) = 0 \quad (21)$$

$$\langle p(t), v(t) \rangle - u(t, v(t)) = \max_{c \in \mathbb{R}_+^n} \{ \langle p(t), c \rangle - u(t, c) \} \quad (22)$$

$$p.p \ t \in [a, b].$$

On peut également appliquer nos résultats à des problèmes chimiques du type

$$\begin{aligned} \min_{(x,\theta,\gamma)} \sum_{i=1}^n g_i(x_i(T)) & \quad (P_c) \\ \text{sous contraintes} & \\ \dot{x}(t) = f(t, x(t), \theta(t), \gamma(t)); & \\ (\theta(t), \gamma(t)) \in \Pi; & \\ x(0) = x^0 & \end{aligned}$$

où  $x_1(t), \dots, x_n(t)$  sont les concentrations à l'instant  $t$  de  $n$  substances où  $n$  réactions chimiques ont lieu ;

$x_i^0$  est la concentration de la substance  $i$  ;

$\theta(t)$  est la température et  $\gamma(t)$  est la pression dans le réacteur à l'instant  $t$  ;

$\Pi = \{(\theta, \gamma) \in \mathbb{R} \times \mathbb{R} : \theta_0 \leq \theta \leq \theta_1 \text{ et } \gamma_0 \leq \gamma \leq \gamma_1\}$ . On obtiendra :

**Théorème 0.0.8** *Supposons que  $(z, \bar{\theta}, \bar{\gamma})$  est une solution locale de  $(P_c)$  et qu'il existe  $\varepsilon > 0$  et une fonction intégrable  $k : [a, b] \rightarrow \mathbb{R}$  tels que pour presque tout  $t \in [0, T]$ , pour tous  $z', z'' \in z(t) + \varepsilon B$ ,  $(\theta, \gamma) \in \Pi$  et pour tout  $i \in \{1, \dots, n\}$  on a*

$$\|f_i(t, z', \theta, \gamma) - f_i(t, z'', \theta, \gamma)\| \leq k(t) \|z' - z''\|.$$

Alors il existe un arc  $p$  et  $\lambda \in \{0, 1\}$  tels que  $(p, \lambda) \neq 0$  et

$$\begin{aligned} \dot{p}(t) \in \text{cod}_x \left\{ - \sum_{i=1}^n \langle p_i(t), f_i(t, \cdot) \rangle (z(t), \bar{\theta}(t), \bar{\gamma}(t)) \right. & \quad (23) \\ \left. p.p t \in [0, T] \right. & \end{aligned}$$

$$(-p_i(T), -\lambda) \in N(\text{epi } g_i; (z_i(T), g_i(z_i(T)))) , i = 1, \dots, n \quad (24)$$

$$\begin{aligned} \sum_{i=1}^n \langle p_i(t), f_i(t, z(t), \bar{\theta}(t), \bar{\gamma}(t)) \rangle = & \\ \max_{(v', v'') \in \Pi(t)} \left\{ \sum_{i=1}^n \langle p_i(t), f_i(t, z(t), v', v'') \rangle \right\} p.p t \in [0, T]. & \end{aligned}$$

Revenons maintenant au Théorème 0.0.3. Si on prend  $g$  localement lipschitzienne en  $(z(a), z(b))$  on obtiendra  $\lambda = 1$ . Cette conclusion sera un outil parmi d'autres, pour obtenir des conditions nécessaires d'optimalité des problèmes multiobjectifs gouvernés par une inclusion différentielle. Ceci sera l'objectif majeur du quatrième chapitre. Dans ce dernier, on s'intéressera aux conditions nécessaires d'optimalité du problème suivant

$$\begin{aligned} \min f(x(a), x(b)) & \qquad \qquad \qquad (P_m) \\ (x(a), x(b)) & \in S \\ \dot{x}(t) & \in F(t, x(t)) \quad p.p \quad t \in [a, b] \end{aligned}$$

où  $f: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^m$  est une application,  $S \subset \mathbb{R}^n \times \mathbb{R}^n$  est un ensemble non vide et fermé et  $F: [a, b] \times \mathbb{R}^n \mapsto \mathbb{R}^n$  est une fonction multivoque à valeurs fermées et mesurable en la première variable de l'intervalle  $[a, b]$ .

Ce genre de problème est utilisable surtout en économie [28], en génie chimique (polymérisation) [8] - [9] et en contrôle multiobjectif du design [96] et [19].

Plusieurs travaux antérieurs ont utilisé des préférences déterminées par des cônes (Pareto faible et Pareto optimum faible), des préférences définies par des fonctions utilités ou le concept d'équilibre de Nash.

Il y a plusieurs approches et des résultats variés concernant des conditions nécessaires d'optimalité du problème  $(P_m)$ . Plusieurs auteurs se sont intéressés aux solutions au sens de Pareto et ses généralisations (voir [13], [22], [26], [70], [88] et [97]-[99]), d'autres ont affiné les conditions nécessaires d'optimalité pour des problèmes du genre  $(P_m)$  mais à valeurs réelles (voir [45], [56], [67]-[69], [89] et [95]) ou les conditions hamiltoniennes (voir [24], [31], [32], [73]-[76], [101]). Les résultats sont exprimés en terme de dérivées généralisées y compris celles de Clarke [22]. La majorité de ces résultats sont obtenus en supposant que l'inclusion différentielle est lipschitzienne, intégrablement sous-lipschitzienne, bornée ou non bornée.

Mais la question principale dans cette partie est : comment peut-on avoir les conditions nécessaires d'optimalité du problème  $(P_m)$  en travaillant avec une préférence générale et avec les hypothèses les plus faibles possibles ?

Zhu, dans son papier [101], a utilisé le récent progrès de l'analyse non lisse et en particulier, le calcul des sous-différentiels des fonctions semi-continues inférieurement (voir [14], [15], [25] et [46]), les méthodes pour prouver le principe extrémal (voir [63], [71] et [75]), les conditions Hamiltoniennes nécessaires d'optimalité qui généralisent les conditions classiques établies pour des problèmes de contrôle optimal en supposant que l'inclusion différentielle est uniformément

lipschitzienne, à valeurs bornées et convexes. Ces résultats sont exprimés en terme du sous-différentiel généralisé de Clarke qui est, comme on l'a dit au début de l'introduction, plus large que le sous-différentiel Fréchet limite. Dans cette troisième partie, on donnera une définition de la régularité d'une préférence, similaire à celle utilisée dans la première partie, différente de celle utilisée par Zhu [101]. On donnera un contre-exemple qui montre que la troisième condition de régularité (utilisée par Zhu) n'est pas valide pour une préférence définie par une fonction utilité.

On supposera que l'inclusion différentielle est sous-lipschitzienne (notion utilisé par Loewen et Rockafellar [67]) en la solution. En se basant sur la notion de la semi-normalité et le principe variationnel d'Ekeland on obtiendra les conditions nécessaires d'optimalité pour le problème  $(P_m)$  avec une préférence générale.

**Théorème 0.0.9** *Soit  $z$  une solution locale de  $(P_m)$ . Supposons que  $F$  est sous-lipschitzienne en  $z$  et que la préférence  $\prec$  est régulière en  $f(z(a), z(b))$ . Alors il existe un arc  $p$ ,  $\lambda \geq 0$  et  $w \in \tilde{N}(cl\mathcal{L}(f(z(a), z(b))), f(z(a), z(b)))$ , avec  $|\omega| = 1$  et  $(\lambda, p) \neq 0$ , tels que*

$$\dot{p}(t) \in co\{q : (q, p(t)) \in N(GrF(t, \cdot); (z(t), \dot{z}(t)))\} \quad p.p t \in [a, b]; \quad (25)$$

$$(p(a), -p(b)) \in \lambda\partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \quad (26)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \quad p.p t \in [a, b]. \quad (27)$$

*Si de plus que  $F$  est à valeurs convexes, alors (25) peut être remplacée par la condition hamiltonienne*

$$\dot{p}(t) \in co \{q : (-q, \dot{z}(t)) \in \partial H(t, (z(t), p(t)))\} \quad p.p t \in [a, b]. \quad (28)$$

Sous les mêmes hypothèses que le Théorème 0.0.9, on obtiendra les conditions nécessaires d'optimalité du problème  $(P_m)$  :

- lorsque  $f$  est à valeurs réelles, on obtiendra le résultat de Ioffe [45] :  
il existe un arc  $p$ ,  $\lambda \geq 0$  tels que  $(\lambda, p) \neq 0$  et

$$\dot{p}(t) \in co\{q : (q, p(t)) \in N(GrF(t, \cdot); (z(t), \dot{z}(t)))\} \quad p.p t \in [a, b]; \quad (29)$$

$$(p(a), -p(b)) \in \lambda\partial f(z(a), z(b)) + N(S; (z(a), z(b))); \quad (30)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \quad p.p t \in [a, b]. \quad (31)$$



- Lorsque la préférence est associée à un cône convexe  $K$  :

il existe un arc  $p$ ,  $\lambda \geq 0$  et  $\omega \in K^0$  avec

$|\omega| = 1$ , tels que  $(\lambda, p) \neq 0$  et

$$\dot{p}(t) \in \text{co}\{q : (q, p(t)) \in N(\text{Gr}F(t, \cdot); (z(t), \dot{z}(t)))\} p.p t \in [a, b]; \quad (32)$$

$$(p(a), -p(b)) \in \lambda \partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \quad (33)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) p.p t \in [a, b]. \quad (34)$$

- Lorsque la préférence est déterminée par une fonction utilité  $u$  vérifiant

$0 \notin \partial u(f(z(a), z(b)))$  :

il existe un arc  $p$ ,  $\lambda \geq 0$  et

$$\omega \in \partial^\infty u(f(z(a), z(b))) \cup \left( \bigcup_{a>0} a \partial u(f(z(a), z(b))) \right)$$

avec  $|\omega| = 1$ , tels que  $(\lambda, p) \neq 0$  et

$$\dot{p}(t) \in \text{co}\{q : (q, p(t)) \in N(\text{Gr}F(t, \cdot); (z(t), \dot{z}(t)))\} p.p t \in [a, b]; \quad (35)$$

$$(p(a), -p(b)) \in \lambda \partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \quad (36)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) p.p t \in [a, b]. \quad (37)$$

Nous aimerions prévenir le lecteur qu'une partie de cette thèse, parce qu'elle correspond à des publications à paraître, a été rédigée en anglais.

# Chapitre 1

## Quelques concepts d'analyse non lisse

### 1.1 Introduction

Le concept de la différentiabilité joue un rôle crucial dans l'étude des problèmes d'optimisation. Les origines de l'analyse non-lisse remontent au début des années 70 quand les théoriciens du contrôle et de la programmation non-linéaire ont essayé d'établir des conditions nécessaires d'optimalité pour des problèmes avec des données non-lisses ou avec des fonctions non-lisses (comme le maximum de plusieurs fonctions lisses), qui surgissent même dans beaucoup de problèmes avec des données lisses. Les exemples suivants [16] illustrent comment de telles fonctions non-lisses peuvent apparaître dans des problèmes avec des données lisses.

**Exemple 1** *Souvent on souhaite travailler avec le maximum de deux ou plusieurs fonctions.*

*Soit  $f(x) = \max(f_1(x), f_2(x))$ . Pour des fonctions simples et lisses définies sur  $\mathbb{R}$ ,  $f_1(x) = x$  et  $f_2(x) = -x$  on obtient  $f(x) = |x|$ , qui n'est pas une fonction lisse.*

**Exemple 2** *Considérons le problème d'optimisation suivant, avec des contraintes simples*

$$\min_{g(x)=a} f(x)$$

*où  $a \in \mathbb{R}$  est un paramètre. Dans la pratique, il est important de savoir comment le modèle répond à la perturbation  $a$ . Pour ceci on considère la*

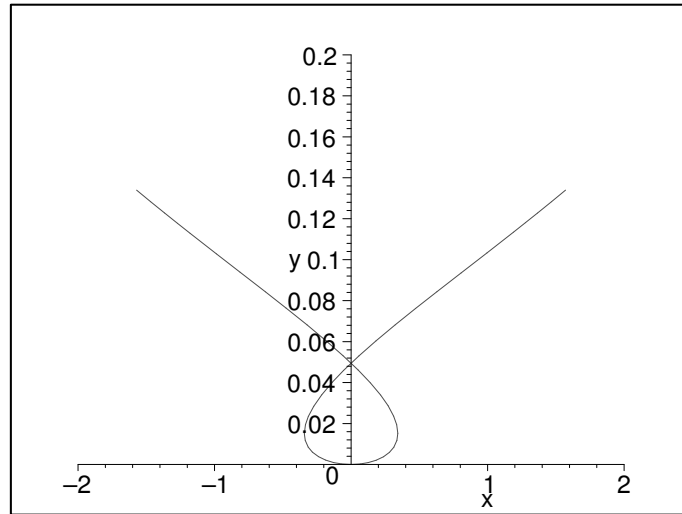


FIG. 1.1 – la courbe de  $(g(x), f(x))$  pour  $x \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ .

*valeur optimale*

$$v(a) = \inf\{f(x) : g(x) = a\}$$

comme fonction de  $a$ . D'une façon concrète, supposons que  $f(x) = 1 - \cos(x)$ ,  $g(x) = \sin(6x) - 3x$  et  $a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  ce qui correspond à  $x \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ . Les courbes de  $(g(x), f(x))$  et de  $v$  sont présentées respectivement par les figures FIG. 1 et FIG. 2. La fonction  $v$  n'est pas lisse, elle n'est même pas continue.

Dans le but de résoudre ce genre de problèmes, divers concepts de dérivées généralisées ont été proposés pour remplacer la dérivée. Le but était de définir une dérivée généralisée pour chaque point dans le domaine d'une fonction appartenant à une classe particulière telles les fonctions localement lipschitziennes.

La première dérivée généralisée était introduite par Clarke [20]. Plusieurs autres concepts de dérivées généralisées fréquemment utilisés sont les codérivées introduites par Mordukhovich [71], [72] et [76], les sous-différentiels approchés introduits par Ioffe [43] et [44], les dérivées contingentes [4] et [79] et les B-dérivées de Treiman [92] et [93], etc ...

Même si les dérivées généralisées sont très utiles dans l'étude des problèmes non-lisses, leurs définitions sont compliquées et elles sont souvent difficiles à calculer, ce qui a poussé à penser aux concepts géométriques.

Le concept géométrique des vecteurs perpendiculaires à un ensemble était utilisé par Clarke [20], tandis que Hirriart-Urruty [39] était le premier à

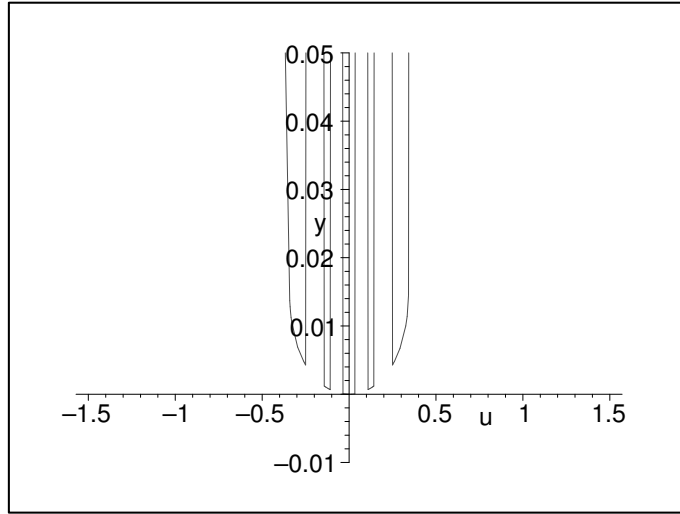


FIG. 1.2 – la courbe de  $v$  pour  $a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .

montrer comment obtenir une formule explicite pour le cône tangent convexe correspondant. La construction de dérivées généralisées possibles à valeurs non convexes a été développée par Mordukhovich [71] et cela en utilisant ces vecteurs normaux. Le sous-différentiel proximal était introduit par Mordukhovich [77] mais il a été présenté explicitement par Rockafellar [85] où les caractérisations du sous-gradient généralisé de Clarke et le cône normal de Clarke ont été donnés. Toujours Mordukhovich a introduit [72] la très subtile et importante notion de sous-différentiel approché que Kruger [62] a développé. Treiman [94] a exprimé le sous-différentiel de Clarke en termes de  $\varepsilon$ -sous-différentiel de Fréchet. Ioffe a donné dans [40] certaines propriétés en dimension finie du sous-différentiel approché et a étendu la notion du sous-différentiel approché à des fonctions définies sur un espace localement convexe quelconque [41]. Ces caractérisations ont été plus tard prolongées à autre dérivées généralisées, sous-différentiels et cônes normaux.

Quand une fonction  $f$  atteint un minimum en un point  $x$ , on a alors  $0 \in \partial f(x)$  et ceci pour toute notion de sous-différentiel. Toutefois on conçoit bien que cette notion n'est vraiment intéressante que si l'on peut exprimer ou estimer le sous-différentiel d'une somme ou d'une composition, c'est à dire si l'on peut établir de bonnes règles de calcul sous-différentiel pour cette notion.

On a mené ce travail en utilisant seulement les sous-différentiels approché et Fréchet limite, le premier étant utilisé pour des fonctions définies sur un

espace de Banach et le deuxième pour celles qui sont définies sur  $\mathbb{R}^n$ .

On désigne par  $S$  un sous-ensemble fermé de  $\mathbb{R}^n$ ,  $\mathbb{B}$  la boule unité,  $\|\cdot\|$  la norme dans  $\mathbb{R}^n$  et  $\text{co}$  l'enveloppe convexe.  $d(\cdot, S)$  est la fonction distance de l'ensemble  $S$  où

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

$x \xrightarrow{f} x_0$  et  $x \xrightarrow{S} x_0$  signifient  $x \rightarrow x_0$  avec  $f(x) \rightarrow f(x_0)$  et  $x \rightarrow x_0$  avec  $x \in S$ , respectivement. Soit  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ , rappelons que l'épigraphe de  $f$  ainsi que son domaine de définition sont

$$\text{epi } f = \{(x, r) : f(x) \leq r\} \subset X \times \mathbb{R}$$

$$\text{dom } f = \{x : f(x) < \infty\}.$$

Rappelons aussi que la fonction indicatrice de  $S$  est définie par :

$$\Psi(S, x) = \begin{cases} 0 & \text{si } x \in S, \\ +\infty & \text{sinon.} \end{cases}$$

## 1.2 Sous-différentiel Fréchet limite

Soit  $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  une fonction semi-continue inférieurement,  $\bar{x} \in \text{dom } f$  et  $\varepsilon > 0$ . Le  $\varepsilon$ -sous-différentiel Fréchet de  $f$  en  $\bar{x}$ , qu'on note  $\partial_F^\varepsilon f(\bar{x})$  est défini par

$$\partial_F^\varepsilon f(\bar{x}) = \left\{ x^* : \liminf_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x}) - \langle x^*, h \rangle}{\|h\|} \geq -\varepsilon \right\}.$$

Lorsqu'on prend  $f = \Psi(S, \cdot)$ , on obtient le cône  $\varepsilon$ -normal Fréchet à  $S$  en  $\bar{x} \in S$  de la manière suivante :

$$\hat{N}_\varepsilon(S, \bar{x}) = \partial_F^\varepsilon \Psi(S, \bar{x}) = \left\{ x^* \in \mathbb{R}^n : \liminf_{x \xrightarrow{S} \bar{x}} \frac{\langle -x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq -\varepsilon \right\}.$$

Ce sous-différentiel ne possède pas de règles de calcul exactes. Par exemple le sous-différentiel de Clarke ( $\partial_c$ ) vérifie la règle de calcul de la somme suivante :

$$\partial_c(f + g)(\bar{x}) \subset \partial_c f(\bar{x}) + \partial_c g(\bar{x})$$

pourvu que l'une des fonctions soit localement lipschitzienne en  $\bar{x}$ . Malheureusement cette propriété n'est plus valable pour le  $\varepsilon$ -sous-différentiel Fréchet.

Pour s'en convaincre, il suffit de prendre  $f(x) = |x|$ ,  $g(x) = -|x|$ ,  $0 < \varepsilon < 1$  et  $\bar{x} = 0$ . Alors  $\partial_F^\varepsilon(f+g)(0) = \{0\}$  mais  $\partial_F^\varepsilon f(0) + \partial_F^\varepsilon g(0) = \emptyset$  car  $\partial_F^\varepsilon g(0) = \emptyset$ . Il faut noter que ce sous-différentiel possède toujours des règles de calcul sous-différentiel « flous ». Pour avoir des règles de calcul exactes, les auteurs précédemment cités ont pris les limites supérieures (au sens ensemblistes) des  $\varepsilon$ -sous-différentiels Fréchet, c'est à dire, l'ensemble

$$\partial_F f(\bar{x}) = \limsup_{\substack{x \rightarrow \bar{x} \\ \varepsilon \rightarrow 0}} \partial_F^\varepsilon f(x)$$

appelé le sous-différentiel Fréchet limite de  $f$  en  $\bar{x}$ .

Autrement dit

$$\partial_F f(\bar{x}) = \left\{ x^* : x^* = \lim_{n \rightarrow \infty} x_n^*, x_n^* \in \partial_F^{\varepsilon_n} f(x_n), x_n \xrightarrow[n \rightarrow \infty]{f} \bar{x} \text{ et } \varepsilon_n \downarrow 0 \right\}.$$

Lorsque  $f = \Psi(S, \cdot)$ , on obtient la notion de cône normal Fréchet limite de la manière suivante :

$$N_F(S; \bar{x}) := \limsup_{\substack{S \\ x \rightarrow \bar{x} \\ \varepsilon \rightarrow 0^+}} \hat{N}_\varepsilon(S, x).$$

La notion du sous-différentiel Fréchet limite a été développée indépendamment par Ioffe, Kruger et Mordukhovich dans les années 80, est efficace pour dériver les conditions d'optimalité, stable par construction et satisfait remarquablement les règles de calcul.

Comme les autres notions de sous-différentiels, le sous-différentiel Fréchet limite a aussi une caractérisation géométrique. Kruger [62] a donné la caractérisation géométrique du sous-différentiel Fréchet limite de la façon suivante :

$$\partial_F f(\bar{x}) = \{x^* \in \mathbb{R}^n : (x^*, -1) \in N_F(\text{epi } f; (\bar{x}, f(\bar{x})))\}.$$

Le sous-différentiel singulier Fréchet limite de  $f$  en  $\bar{x}$  est également défini par

$$\partial_F^\infty f(\bar{x}) = \{x^* \in \mathbb{R}^n : (x^*, 0) \in N_F(\text{epi } f; (\bar{x}, f(\bar{x})))\}.$$

Rappelons que si  $f$  est localement lipshitzienne en  $\bar{x}$  alors  $\partial_F^\infty f(\bar{x}) = 0$ . Rappelons maintenant quelques propriétés de calcul du sous-différentiel Fréchet.

**Théorème 1.2.1** (la somme)[77]

Soit  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  deux fonctions semi-continues inférieurement en  $\bar{x} \in \text{dom } f_1 \cap \text{dom } f_2$  telles que

$$\partial_F^\infty f_1(\bar{x}) \cap (-\partial_F^\infty f_2(\bar{x})) = \{0\}. \tag{1.1}$$

On a alors

$$\partial_F(f_1 + f_2)(\bar{x}) \subset \partial_F f_1(\bar{x}) + \partial_F f_2(\bar{x}).$$

**Remarque 1.2.1** Si  $f_1$  et  $f_2$  sont localement lipschitziennes en  $x$ , la condition (1.1) est triviale.

### 1.2.1 Lien avec le sous-différentiel de Mordukhovich

Le sous-différentiel de Mordukhovich [77] qu'on définit ci-après est utilisé pour les fonctions définies dans des espaces de dimensions finies.

On définit l'ensemble des projections de  $x$  sur  $S$  (au sens de Mordukhovich) par

$$P(x, S) = \{y \in S : \|x - y\| = d(x, S)\}.$$

L'ensemble

$$N_m(S, \bar{x}) = \limsup_{x \rightarrow \bar{x}} [\text{cône}(x - P(x, S))]$$

est un cône fermé (non nécessairement convexe), appelé le cône normal de Mordukhovich à  $S$  en  $\bar{x} \in S$ .

Soit  $f$  une fonction définie de  $\mathbb{R}^n$  dans  $\mathbb{R} \cup \{\infty\}$  et  $\bar{x} \in \text{dom } f$ . Le sous-différentiel de Mordukhovich de  $f$  au point  $\bar{x}$  est l'ensemble :

$$\partial_m f(\bar{x}) = \{x^* \in \mathbb{R}^n : (x^*, -1) \in N_m(\text{epi } f, (\bar{x}, f(\bar{x})))\}.$$

Le sous-différentiel singulier de Mordukhovich de  $f$  en  $\bar{x}$ , qu'on note  $\partial_m^\infty f(x)$ , est défini par

$$\partial_m^\infty f(\bar{x}) = \{x^* \in \mathbb{R}^n : (x^*, 0) \in N_m(\text{epi } f, (\bar{x}, f(\bar{x})))\}.$$

Voyons maintenant la relation entre le cône normal Fréchet limite et celui de Mordukhovich.

**Proposition 1.2.1** On a  $N_F(S, \bar{x}) = N_m(S, \bar{x})$ .

**Preuve :** Soit  $\zeta \in N_F(S, \bar{x})$ , il existe  $\zeta_n \rightarrow \zeta$ ,  $y_n \xrightarrow{S} \bar{x}$ ,  $\varepsilon_n \rightarrow 0$  et  $\beta_n \rightarrow 0$  tels que pour tout  $x \in B(y_n, 2\beta_n) \cap S$  on a

$$-\langle \zeta_n, x - y_n \rangle + \varepsilon_n \|x - y_n\| \geq 0 \quad (1.2)$$

On peut supposer que  $\|\zeta\| = \|\zeta_n\| = 1$ . Posons  $x_n = y_n + \beta_n \zeta_n$  et choisissons  $u_n \in S$  tel que

$$d(x_n, S) = \|u_n - x_n\|.$$

On a

$$\|x_n - u_n\| \leq \|x_n - y_n\| \leq \beta_n$$

par suite

$$\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \leq 2\beta_n.$$

En remplaçant  $x$  par  $u_n$  dans (1.2), on obtient

$$\langle \zeta_n, u_n - y_n \rangle \leq \varepsilon_n \|u_n - y_n\|. \quad (1.3)$$

Comme  $\|x_n - u_n\| \leq \beta_n$  on a  $\|x_n - u_n\|^2 \leq \beta_n^2$  et donc  $\|y_n - u_n\|^2 \leq 2\beta_n \langle \zeta_n, u_n - y_n \rangle$  et donc en utilisant (1.3), il résulte que

$$\|y_n - u_n\| \leq 2\beta_n \varepsilon_n.$$

Ainsi si on pose  $\zeta'_n = \frac{x_n - u_n}{\beta_n}$ , on obtient  $\|\zeta_n - \zeta'_n\| \leq 2\varepsilon_n$  et  $\zeta'_n \in \text{cône}(x_n - P(x_n, S))$ . Par conséquent  $\zeta \in N_m(S, \bar{x})$ .

Inversement, soit  $\zeta \in N_m(S, \bar{x})$ , il existe  $\zeta_n \rightarrow \zeta$ ,  $x_n \rightarrow \bar{x}$ ,  $y_n \in S$  et un nombre  $t_n > 0$  tels que

$$\zeta_n = t_n(x_n - y_n) \text{ et } d(x_n, S) = \|x_n - y_n\|.$$

Comme  $\frac{\|\zeta_n\|}{\|x_n - y_n\|} \rightarrow +\infty$  alors  $t_n \rightarrow +\infty$  Pour tout  $x \in S$ , on a

$$\begin{aligned} \|x_n - y_n\|^2 &\leq \|x - y_n\|^2 \\ &= \|x_n - y_n\|^2 + \|x - y_n\|^2 - 2\langle x_n - y_n, x - y_n \rangle \\ &= \|x_n - y_n\|^2 + \|x - y_n\|^2 - \frac{2}{t_n} \langle \zeta_n, x - y_n \rangle \end{aligned} \quad (1.4)$$

donc on a

$$\langle \zeta_n, x - y_n \rangle \leq \frac{t_n}{2} \|x - y_n\|^2.$$

Posons  $\beta_n = \frac{1}{t_n^2}$  et  $\varepsilon_n = \frac{1}{2t_n}$ . Alors pour tout  $x \in B(y_n, \beta_n) \cap S$ , on a

$$-\langle \zeta_n, x - y_n \rangle + \varepsilon_n \|x - y_n\| \geq 0.$$

D'où  $\zeta \in N_F(S, \bar{x})$ . □

Il découle de la proposition [?] :



**Proposition 1.2.2** *Soit  $f$  une fonction semi-continue inférieurement, on a*

$$\partial_F f(\bar{x}) = \partial_m f(\bar{x}).$$

Le sous-différentiel de Mordukhovich, comme on a pu remarquer, est défini en utilisant la notion des projections. Un autre sous-différentiel est défini en utilisant la notion des projections, il s'agit du sous-différentiel proximal. C'est le but du prochain paragraphe, dans lequel on donne la définition du sous-différentiel proximal et on établit la relation entre les deux sous-différentiels (proximal et Fréchet).

### 1.2.2 Lien avec le sous-différentiel proximal

On dit que le vecteur  $x^* \in \mathbb{R}^n$  est normal proximal à  $S$  en  $\bar{x}$ , s'ils existent  $u \notin S$  et  $\lambda > 0$  tels que  $x^* = \lambda(u - \bar{x})$  et  $d(u, S) = \|u - \bar{x}\|$ .

D'une autre manière équivalente,  $x^*$  est normal proximal à  $S$  en  $\bar{x}$  si et seulement s'il existe  $t > 0$  tel que

$$d(\bar{x} + tx^*, S) = t \|x^*\|.$$

Soit  $N_p(S, \bar{x})$  l'ensemble des normaux proximaux à  $S$  en  $\bar{x}$ , c'est à dire l'ensemble des éléments de  $\mathbb{R}^n$  dont la projection sur  $S$  est  $\bar{x}$ . C'est évident que c'est un cône.

On a aussi la caractérisation suivante :

**Proposition 1.2.3** [25]  *$x^* \in N_p(S, \bar{x})$  si et seulement s'il existe  $\sigma \geq 0$  tel que pour tout  $x \in S$  on a*

$$\langle x^*, x - \bar{x} \rangle \leq \sigma \|x - \bar{x}\|^2.$$

**Preuve :** Soit  $x^* \in N_p(S, \bar{x})$ , il existe  $t > 0$  tel que  $d(\bar{x} + tx^*, S) = t \|x^*\|$ . Soit  $x \in S$  et posons  $u = x + tx^*$ , considérons  $v \in S$  tel que  $d(u, S) = \|u - v\|$ . On a

$$\begin{aligned} t^2 \|x^*\|^2 &\leq \|\bar{x} + tx^* - x\|^2 \\ &= \langle x - \bar{x} - tx^*, x - \bar{x} - tx^* \rangle \\ &= \|x - \bar{x}\|^2 + t^2 \|x^*\|^2 - 2t \langle x^*, x - \bar{x} \rangle. \end{aligned}$$

En posant  $\sigma = \frac{1}{2t}$ , on obtient

$$\langle x^*, x - \bar{x} \rangle \leq \sigma \|x - \bar{x}\|^2.$$

Voyons maintenant la condition suffisante. Posons  $\sigma = \frac{1}{2t}$  et considérons  $s \in S$  tel que

$$d(\bar{x} + tx^*, S) = \| \bar{x} + tx^* - s \| .$$

On a par hypothèse

$$0 \geq \| s - \bar{x} \|^2 - 2t \langle x^*, s - \bar{x} \rangle$$

donc

$$t^2 \| x^* \|^2 \leq \| s - \bar{x} \|^2 + t^2 \| x^* \|^2 - 2t \langle x^*, s - \bar{x} \rangle$$

et ainsi

$$t \| x^* \| \leq \| \bar{x} + tx^* - s \|$$

et par suite  $x^* \in N_p(S, \bar{x})$ . □

Cette caractérisation du cône normal proximal est intéressante lorsque  $S$  est convexe.

**Proposition 1.2.4** *Si  $S$  est convexe, alors  $x^* \in N_p(S, \bar{x})$  si et seulement si  $\langle x^*, x - \bar{x} \rangle \leq 0$  pour tout  $x \in S$ .*

**Preuve :** La condition suffisante est triviale. Voyons maintenant la condition nécessaire. Soit  $x^* \in N_p(S, \bar{x})$ , il existe  $t_0 > 0$  tel que

$$d(\bar{x} + t_0 x^*, S) = t_0 \| x^* \| .$$

Soit  $t \in [0, 1]$  et  $x \in S$ . Posons  $x' = \bar{x} + t(x - \bar{x}) = tx + (1 - t)\bar{x}$ .  $S$  est convexe,  $x' \in S$ . On a

$$\langle x^*, x' - \bar{x} \rangle \leq \sigma t^2 \| x - \bar{x} \|$$

et donc

$$\langle x^*, x - \bar{x} \rangle \leq \sigma t \| x - \bar{x} \| .$$

Comme  $t$  est arbitraire, on obtient

$$\langle x^*, x - \bar{x} \rangle \leq 0.$$

□

**Proposition 1.2.5**  *$N_p(S, x) \neq \{0\}$  pour tout  $x$  appartenant à un ensemble dense de  $\partial S$  (où  $\partial S$  est le bord de  $S$ ).*

**Preuve :** Soit  $x \in \partial S$ , il faut donc montrer qu'il existe une suite  $(s_n)_n$  de points de  $\partial S$  qui converge vers  $x$  et telle que

$$\forall n \in \mathbb{N}, \quad N_p(S, s_n) \neq \{0\}.$$

Comme  $x$  est sur le bord de  $S$ , il existe une première suite  $(x_n)_n$  n'appartenant pas à  $S$  telle que  $x_n \rightarrow x$ . Soit  $s_n \in S$  tel que  $d(x_n, S) = \|x_n - s_n\|$ . Par construction  $x_n - s_n \in N_p(S, s_n)$  et  $s_n \rightarrow x$ .  $\square$

Donnons maintenant la définition du sous-différentiel proximal. Soit  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  une fonction semi-continue inférieurement. On dit que  $x^* \in \mathbb{R}^n$  est un sous-gradient proximal de  $f$  en  $\bar{x}$  si

$$(x^*, -1) \in N_p(\text{epi } f, (\bar{x}, f(\bar{x}))).$$

L'ensemble des sous-gradients proximaux de  $f$  en  $\bar{x}$  est appelé le sous-différentiel proximal de  $f$  en  $\bar{x}$  noté  $\partial_p f(\bar{x})$ .

Donnons maintenant une caractérisation analytique du sous-différentiel proximal.

**Théorème 1.2.2** [25]  $x^* \in \partial_p f(\bar{x})$  si et seulement s'il existe  $\sigma \geq 0$  tel que

$$f(x) - f(\bar{x}) + \sigma \|x - \bar{x}\|^2 \geq \langle x^*, \bar{x} - x \rangle$$

pour tout  $y$  dans un voisinage de  $\bar{x}$ .

A partir de la notion du cône normal proximal à  $S$  en  $x \in S$ , on peut définir le cône normal proximal limite à  $S$  en  $x$  de la manière suivante :

$$N_L(S, \bar{x}) = \{x^* : x^* = \lim_{n \rightarrow \infty} x_n^* \text{ où } x_n^* \in N_p(S, x_n) \text{ et } x_n \xrightarrow{S} \bar{x}\}.$$

On peut définir maintenant le sous-différentiel proximal limite de  $f$  en  $\bar{x}$  par

$$\partial_L f(\bar{x}) = \{x^* : x^* = \lim_{n \rightarrow \infty} x_n^* \text{ où } x_n^* \in \partial_p f(x_n) \text{ et } x_n \xrightarrow{f} \bar{x}\}.$$

Ce n'est pas difficile de voir que

- \*  $x^* \in \partial_L f(\bar{x})$  si et seulement si  $(x^*, -1) \in N_L(\text{epi } f; (\bar{x}, f(\bar{x})))$ .
- \* Si  $S$  est convexe alors pour tout  $x \in S$  on a  $N_p(S, \bar{x}) = N_L(S, \bar{x})$ .

On a un lien entre le cône normal proximal limite et le cône normal Fréchet.

**Proposition 1.2.6**

$$N_F(S, \bar{x}) = N_L(S, \bar{x}).$$

La preuve de la proposition est triviale.

De la proposition 1.2.6 et des caractérisations géométriques, on déduit :

$$\partial_F f(\bar{x}) = \partial_L f(\bar{x}).$$

Voyons maintenant le rapport entre le cône normal proximal et le cône tangent de Bouligand.

**Définition 1** *Un vecteur  $v$  est dit tangent à l'ensemble  $S$  en un point  $\bar{x}$  si*

$$\liminf_{t \downarrow 0} \frac{d(\bar{x} + tv, S)}{t} = 0.$$

*L'ensemble de ces vecteurs qu'on note  $T_B(S, \bar{x})$  est un cône appelé le cône tangent de Bouligand à  $S$  en  $x$ .*

Autrement dit un vecteur  $h \in T_B(S, x)$  si et seulement s'il existe deux suites  $t_n \rightarrow 0^+$  et  $h_n \rightarrow h$  telles que pour  $n$  assez grand,  $\bar{x} + t_n h_n \in S$ . Ceci est équivalent en termes ensemblistes à

$$T_B(S, x) = \limsup_{t \rightarrow 0^+} t^{-1}(S - x).$$

Nous pouvons d'ores et déjà relier le cône de Bouligand au cône normal proximal au plûtôt à son polaire  $N_p(S, x)^\circ = \{x^* : \langle x^*, y \rangle \leq 0, \forall y \in N_p(S, x)\}$ .

**Proposition 1.2.7** [25] *Pour tout  $x \in S$ , on a  $T_B(S, x) \subset N_p(S, x)^\circ$ .*

La preuve est laissé au lecteur.

Rappelons que le cône normal proximal à  $S$  en  $x$  est toujours convexe alors que le cône de Bouligand n'est pas forcément convexe.

Nous introduisons maintenant la notion du sous-différentiel approché dans  $\mathbb{R}^n$ .

### 1.2.3 Lien avec le sous-différentiel approché de Ioffe

Commençons tout d'abord par rappeler la définition du sous-différentiel de Dini.

Soient  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  et  $\bar{x} \in \text{dom } f$ . On pose

$$d^- f(\bar{x}, h) = \liminf_{\substack{u \rightarrow h \\ t \downarrow 0}} t^{-1}(f(\bar{x} + tu) - f(\bar{x}))$$

la dérivée directionnelle (inférieure) de Dini de  $f$  en  $\bar{x}$  dans la direction  $h$ .  
Le  $\varepsilon$ -sous-différentiel de Dini de  $f$  en  $\bar{x}$  est l'ensemble

$$\partial_\varepsilon^- f(\bar{x}) = \{x^* \in \mathbb{R}^n : \langle x^*, h \rangle \leq d^- f(\bar{x}; h) + \varepsilon \|h\|, \forall h \in \mathbb{R}^n\} \text{ si } \bar{x} \in \text{dom } f$$

et

$$\partial_\varepsilon^- f(\bar{x}) = \emptyset \text{ si } \bar{x} \notin \text{dom } f.$$

Pour  $\varepsilon = 0$  on écrit  $\partial^- f(\bar{x})$ .

**Définition 2** L'ensemble  $\partial_a f(x) = \limsup_{\substack{f \\ x \rightarrow \bar{x}}} \partial^- f(x)$  est appelé le sous-différentiel approché de  $f$  en  $\bar{x}$ .

**Remarque 1.2.2** Le sous-différentiel approché est semi-continue supérieurement dans le sens suivant :

$$\partial_a f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \partial_a f(x).$$

Soit  $\bar{x} \in S$ , l'ensemble

$$N_a(S, \bar{x}) = \partial_a \Psi(S, \bar{x}) = \limsup_{x \xrightarrow{S} \bar{x}} \partial^- \Psi(S, x) \quad (1.5)$$

est appelé le cône normal approché de  $S$  en  $\bar{x}$ .

Voyons maintenant le rapport entre le cône normal approché à  $S$  en  $\bar{x}$  et celui de Bouligand. Remarquons d'abord que pour  $h \in \mathbb{R}^n$  on a  $d^- \Psi(S, \bar{x}, h) = 0$  si et seulement s'il existe  $u_n \rightarrow h$  et  $t_n \downarrow 0$  tels que  $\bar{x} + t_n u_n \in S$ . Ainsi on a

$$d^- \Psi(S, \bar{x}, h) = \Psi(T_B(S, \bar{x}), h). \quad (1.6)$$

En utilisant l'équation (1.6), on montre facilement la proposition suivante :

**Proposition 1.2.8**

$$T_B(S, \bar{x})^\circ = \partial^- \Psi(S, \bar{x}).$$

Donc d'après la Proposition 1.2.8 et (1.5), on a la caractérisation suivante :

$$N_a(S, \bar{x}) = \limsup_{\substack{f \\ x \rightarrow \bar{x}}} T_B(S, x)^\circ.$$

Voyons maintenant la relation entre le cône normal approché et le cône normal Fréchet limite.

**Proposition 1.2.9**

$$N_F(S, \bar{x}) = N_a(S, \bar{x}).$$

Pour établir la preuve de la Proposition 1.2.9, on aura besoin du lemme suivant :

**Lemme 1.2.1** [40] Si  $0 \in \partial^- f(\bar{x})$  alors pour tout  $\varepsilon > 0$ , la fonction

$$x \rightarrow f(x) + \varepsilon \|x - \bar{x}\|$$

atteint un minimum local strict en  $\bar{x}$ .

**Preuve :** Supposons qu'il existe  $\varepsilon > 0$  et une suite  $x_n \rightarrow \bar{x}$  tels que

$$f(x_n) + \varepsilon \|x_n - \bar{x}\| < f(\bar{x}).$$

Posons  $t_n = \|x_n - \bar{x}\|$ ,  $u_n = \frac{x_n - \bar{x}}{t_n}$  et supposons que la suite  $u_n \rightarrow h$  avec  $\|h\| = 1$ . On conclut que

$$\begin{aligned} d^- f(\bar{x}, h) &\leq \liminf_{n \rightarrow \infty} t_n^{-1} (f(\bar{x} + t_n u_n) - f(\bar{x})) \\ &= \liminf_{n \rightarrow \infty} t_n^{-1} (f(x_n) - f(\bar{x})) \\ &\leq \liminf_{n \rightarrow \infty} t_n^{-1} (-t_n^{-1} \varepsilon) \\ &= -\varepsilon \end{aligned} \tag{1.7}$$

ce qui contredit le fait que  $0 \in \partial^- f(\bar{x})$ . □

**Preuve de la Proposition 1.2.9 :** L'inclusion  $N_F(S, \bar{x}) \subset N_a(S, \bar{x})$  est évidente. Réciproquement, soit  $\zeta \in N_a(S, \bar{x})$ , il existe  $x_n \xrightarrow{S} \bar{x}$  et  $\zeta_n \rightarrow \zeta$  tels que

$$\zeta_n \in \partial^- \Psi(S, x_n).$$

Donc

$$0 \in \partial^- (\Psi(S, \cdot) - \langle \zeta_n, \cdot \rangle)(x_n)$$

et donc d'après le Lemme 1.2.1, il existe  $\varepsilon_n \rightarrow 0$  et  $\beta_n \rightarrow 0$  tels que pour tout  $u \in S$  vérifiant  $\|u - x_n\| < \beta_n$ , on a

$$-\langle \zeta_n, u - x_n \rangle + \varepsilon_n \|u - x_n\| > 0$$

et donc  $\zeta_n \in \hat{N}_\varepsilon(S, x_n)$  ce qui entraîne que  $\zeta \in N_F(S, \bar{x})$ . □

De la proposition 1.2.9, on conclut :

**Proposition 1.2.10**

$$\partial_F f(\bar{x}) = \partial_a f(\bar{x}).$$

On peut également donner la définition du sous-différentiel singulier approché

$$\partial_a^\infty f(\bar{x}) = \{x^* : (x^*, 0) \in N_a(\text{epi } f, (\bar{x}, f(\bar{x})))\}.$$

Rockafellar dans [85] a démontré que

$$\partial_a^\infty f(\bar{x}) = \limsup_{\substack{f \\ x \rightarrow \bar{x} \\ t \downarrow 0}} t \partial_p f(x).$$

### 1.3 Le sous-différentiel approché de Ioffe en dimension infinie

Après avoir défini le sous-différentiel approché en dimension finie, considérons maintenant un espace de Banach  $X$  et donnons quelques résultats relatifs au sous-différentiel approché en dimension infinie.

On désigne par  $Y$  un autre espace de Banach,  $X'$  et  $Y'$  les duals topologiques respectifs de  $X$  et  $Y$ ,  $B_X$  et  $B_{X'}$  les boules unités de  $X$  et  $X'$  respectivement,  $\sigma$  les topologies  $\sigma(X, X')$  et  $\sigma(Y, Y')$ ,  $\sigma'$  la topologie faible étoile  $X'$  ou  $Y'$  et  $\langle \cdot, \cdot \rangle$  le crochet de dualité entre  $X$  et  $X'$ . Soient  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  et  $x \in \text{dom } f$ . Comme en dimension finie, pour  $h \in X$  on considère la dérivée directionnelle (inférieure) de Dini faible de  $f$  en  $x$  définie par

$$d^- f(x, h) = \liminf_{\substack{u \rightarrow h \\ t \downarrow 0}} t^{-1}(f(x + tu) - f(x)),$$

et le  $\varepsilon$ -sous-différentiel de Dini de  $f$  en  $x$

$$\partial_\varepsilon^- f(x) = \{x^* \in X' : \langle x^*, h \rangle \leq d^- f(x; h) + \varepsilon \|h\|, \forall h \in X\} \text{ si } x \in \text{dom } f$$

et

$$\partial_\varepsilon^- f(x) = \emptyset \text{ si } x \notin \text{dom } f.$$

Pour  $\varepsilon = 0$  on écrit, comme en dimension finie,  $\partial^- f(x)$ .

Notons par  $\mathcal{F}(X)$  la collection des sous-espaces de dimensions finies de  $X$ . Le sous-différentiel approché de  $f$  en  $x_0 \in \text{dom } f$  est défini par les expressions suivantes (voir Ioffe [41] et [43])

$$\partial_A f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x)$$

où

$$\limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \{x^* \in X' : x^* = \sigma' - \lim x_i^*, x_i^* \in \partial^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\},$$

c'est l'ensemble de  $\sigma'$ -limites de telles suites généralisées.

Ici

$$f_{x+L}(x) = \begin{cases} f(x) & \text{si } x \in x + L, \\ +\infty & \text{sinon.} \end{cases}$$

Comme en dimension finie, il est facile de voir que la fonction multivoque  $x \rightarrow \partial_A f(x)$  est semi-continue supérieurement dans le sens suivant

$$\partial_A f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial_A f(x).$$

Dans [41] et [43] Ioffe a montré que si  $S$  est un sous-ensemble fermé de  $X$  et  $x_0 \in S$  alors

$$\partial_A d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- d_{x+L}(x, S) \cap (1 + \varepsilon)B_{X'}.$$

le cône normal à  $S$  en  $x_0 \in S$  est l'ensemble

$$N_a(S, x_0) = \partial_A \Psi(S, x_0).$$

Pour  $x_0 \notin S$  on pose  $N(S, x_0) = \emptyset$ . Lorsque l'ensemble  $S$  est compactement épi-lipschitzien en  $x_0$  on a [49]

$$N_A(S, x_0) = \mathbb{R}_+ \partial_A d(x_0, S).$$

Rappelons qu'un ensemble compactement épi-lipschitzien en  $x_0 \in S$  (c.e.l) ([12], [17] et [18]) s'il existe  $\gamma > 0$  et  $H \subset X$  compact en norme tels que

$$S \cap (x_0 + \gamma B_X) + t\gamma B_X \subset S - tH, \quad \text{pour tout } t \in ]0, \gamma[.$$

On peut exprimer le sous-différentiel approché de  $f$  en  $x$  en fonction du cône normal approché :

$$\partial_A f(x) = \{x^* \in X' : (x^*, -1) \in N_A(\text{epi } f, (x, f(x)))\}.$$

On peut définir également le sous-différentiel singulier approché de  $f$  en  $x$  de la manière suivante :

$$\partial_A^\infty f(x) = \{x^* : (x^*, 0) \in N_A(\text{epi } f, (x, f(x)))\}.$$



**Théorème 1.3.1** [41](La somme)

Soit  $f : X \rightarrow \mathbb{R}$  une fonction s.c.i au voisinage de  $x_0$  et  $g : X \rightarrow \mathbb{R}$  une fonction lipschitzienne au voisinage de  $x_0$ . On a alors

$$\partial_A(f_1 + f_2) \subset \partial_A f_1(x) + \partial_A f_2(x).$$

Le sous-différentiel approché est utilisé dans la première partie de ce travail. On aura besoin de la définition suivante :

**Définition 3** [90] La fonction  $g : X \mapsto Y$  est dite fortement compactement lipschitzienne (f.c.l) en un point  $x_0$  s'il existe une fonction multivoque  $R : X \mapsto 2^{Comp(Y)}$ , où  $Comp(Y)$  est l'ensemble de tous les sous-ensembles de  $Y$  compacts en norme, et une fonction  $r : X \times X \rightarrow \mathbb{R}_+$  vérifiant

(i)  $\lim_{\substack{x \rightarrow x_0 \\ h \rightarrow 0}} r(x, h) = 0,$

(ii) il existe  $\alpha > 0$  tel que

$$t^{-1}[g(x + th) - g(x)] \in R(h) + \|h\|r(x, th)B_Y$$

pour tout  $x \in x_0 + \alpha B_X, h \in \alpha B_X$  et  $t \in ]0, \alpha[$ ,

(iii)  $R(0) = \{0\}$  et  $R$  est semi-continue supérieurement.

On peut montrer que toute fonction f.c.l est localement lipschitzienne [90]. En dimension finie les deux notions coïncident.

Ioffe [I2] a développé des règles de calcul sous-différentiel approché de la composition de deux fonctions. En utilisant la définition précédente, Jourani et Thibault [57] ont développé le théorème suivant :

**Théorème 1.3.2** (La composition)[57]

Soit  $g : X \rightarrow Y$  une fonction f.c.l en  $x_0$  et  $f : Y \rightarrow \mathbb{R}$  une fonction localement lipschitzienne en  $g(x_0)$ . Alors  $f \circ g$  est une fonction localement lipschitzienne en  $x_0$  et

$$\partial_A(f \circ g)(x_0) \subset \bigcup_{y^* \in \partial_A f(g(x_0))} \partial_A(y^* \circ g)(x_0).$$

on aura besoin de la proposition suivante qui est une conséquence directe de la Proposition 2.3 de [59].

**Proposition 1.3.1** Soit  $g : X \rightarrow Y$  une fonction f.c.l en  $x_0$ ,  $(y_i^*)$  une suite généralisée bornée de  $Y'$ , convergent faible étoile vers zéro dans  $Y'$  et  $(x_i)$  une suite généralisée convergent en norme vers  $x_0$  dans  $X$ . Si  $x_i^* \in \partial(y_i^* \circ g)(x_i)$ , alors  $(x_i^*)$  converge faible étoile vers zéro dans  $X'$ .

Dans la première partie de ce travail, on aura besoin de la propriété d'Aubin et la propriété de calme. Soit  $M : Y \mapsto X$  une fonction multivoque, on dit que  $M$  a la propriété d'Aubin en  $(\bar{y}, \bar{x})$  élément du graphe de  $M$ , s'il existe des voisinages  $\mathcal{Y}$  et  $\mathcal{X}$  de  $\bar{y}$  et  $\bar{x}$  respectivement et  $K > 0$  tels que

$$d(x, M(y)) \leq K\|y - y'\| \quad \forall y, y' \in \mathcal{Y} \quad \forall x \in M(y') \cap \mathcal{X}.$$

En fixant les paramètres  $y$  en  $\bar{y}$  dans la définition de la propriété d'Aubin, on trouve que  $M$  est calme en  $(\bar{y}, \bar{x})$  ([82]) :

$$d(x, M(\bar{y})) \leq K\|y - \bar{y}\| \quad \forall y \in \mathcal{Y} \quad \forall x \in M(y) \cap \mathcal{X}.$$

La propriété d'Aubin implique calme mais la réciproque n'est pas vraie (prendre  $M(y) = \{x : x^2 \geq y\}$ ).

**Théorème 1.3.3** [59] *Soient  $D \subset Y$  et  $C \subset X$  deux ensembles fermés et  $g : X \rightarrow Y$  une fonction f.c.l en  $\bar{x} \in C \cap g^{-1}(D)$ . Supposons que  $D$  est compactement épi-lipschitzien en  $g(x_0)$  et que la condition de régularité suivante est vérifiée en  $\bar{x}$*

$$[y^* \in \partial_A d(g(\bar{x}), D) \quad \text{et} \quad 0 \in \partial_A (y^* \circ g + d(\cdot, C))(\bar{x})] \implies y^* = 0.$$

Alors la fonction multivoque  $M : Y \mapsto X$  définie par

$$M(y) = \{x \in C : y \in -g(x) + D\}$$

a la propriété d'Aubin au point  $(0, \bar{x})$  et donc calme en ce point, ou d'une façon équivalente, il existe  $a \geq 0$  et  $r > 0$  tels que

$$d(x, C \cap g^{-1}(D)) \leq a[d(g(x), D) + d(x, C)]$$

pour tout  $x \in \bar{x} + rB_X$ .

Pour la démonstration, voir Jourani et Thibault [59].

**Remarque 1.3.1** *Les nouvelles conditions « bornées » pour la propriété calme des fonctions multivoques dans des espaces de dimensions finies sont découvertes dans [37]. Des conditions similaires ont été présentées dans [36] pour le cas d'un système convexe et des espaces de dimensions infinies.*

En utilisant la définition du sous-différentiel approché et de la propriété de calme on obtient :

**Proposition 1.3.2** *Supposons que  $g$  est f.c.l en  $\bar{x} \in C \cap g^{-1}(D)$  et que la fonction multivoque  $M$  du Théorème 1.3.3 est calme en  $(0, \bar{x})$ . Alors*

$$\partial_A d(\cdot, M(0))(\bar{x}) \subset \bigcup_{y^* \in N(D, g(\bar{x}))} \partial_A(y^* \circ g)(\bar{x}) + \mathbb{R}_+ \partial_A d(\bar{x}, C).$$

Il nous reste à conclure ce chapitre en donnant quelques résultats concernant les fonctions multivoques en dimensions finies et qui nous seront utiles pour la suite.

## 1.4 Fonctions multivoques

Dans ce paragraphe, on considère  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  une fonction multivoque. Rappelons que le graphe de  $F$  est défini par  $\text{Gr } F = \{(x, y) : y \in F(x)\}$ .

**Définition 4** *On dit que  $F$  est semi-continue supérieurement (s.c.s) sur un sous-ensemble non vide  $V$  de  $\mathbb{R}^n$  si pour tout  $\varepsilon > 0$  et pour tout  $u \in V$  il existe un voisinage  $U$  de  $u$  dans  $V$  tel que*

$$F(x) \subset F(u) + \varepsilon \mathbb{B}, \quad \forall x \in U.$$

**Lemme 1.4.1** *Supposons que  $F$  est définie sur un sous-ensemble  $V$  non vide de  $\mathbb{R}^n$  et que :*

- i)  $\text{Gr } F$  est fermé ;
- ii) il existe un compact  $K$  de  $\mathbb{R}^m$  tel que

$$F(x) \subset K, \quad \forall x \in V ;$$

Alors  $F$  est s.c.s sur  $V$ .

A l'aide du Lemme 1.4.1, on peut établir le lemme suivant :

**Lemme 1.4.2** *Supposons que la fonction  $f : (x_0, y_0) + r\mathbb{B} \mapsto \mathbb{R}$  est lipschitzienne de constante  $K$ . Définissons la fonction multivoque*

$\Gamma : (x_0, y_0) + r\mathbb{B} \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$  *par :*

$$\Gamma(x, y, p, s) = \text{co}\{q : (q, p) \in \partial_F f(x, y) + s\mathbb{B}\}.$$

Alors pour tout  $\lambda \in ]0, 1[$ , tout  $(x, y, s) \in (x_0, y_0, 0) + \lambda r\mathbb{B}$  et tout  $p \in \mathbb{R}^n$ , avec  $\Gamma(x, y, p, s) \neq \emptyset$ ,  $\Gamma$  est s.c.s en  $(x, y, p, s)$ .

**Démonstration du Lemme 1.4.2** Remarquer que la fonction multivoque  $\Gamma$ , vérifie les conditions *ii)* et *iii)* du Lemme 1.4.1. D'autre part, il est clair que  $\Gamma$  est de graphe fermé.

**Définition 5** On dit que  $F$  est lipschitzienne sur un ensemble  $C$  de  $\mathbb{R}^n$  s'il existe une constante  $k > 0$  telle que

$$F(x) \subset F(u) + k |x - u| \mathbb{B} \quad \forall \quad x, u \in C.$$

Rappelons qu'on peut définir  $D^*F(x_0, y_0) : \mathbb{R}^m \rightarrow \mathbb{R}^n$  la codérivée de  $F$  au point  $(x_0, y_0)$  de son graphe  $Gr F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\}$  par

$$D^*F(x_0, y_0)(y^*) = \{x^* : (x^*, -y^*) \in N_a(Gr F; (x_0, y_0))\}.$$

**Définition 6** On dit que  $F$  est pseudo-lipschitzienne, ([2], [3], [87]), en  $(x_0, y_0) \in GrF$  s'il existe  $k > 0$  et  $r > 0$  tels que pour tout  $x, u \in x_0 + r\mathbb{B}$

$$F(x) \cap (y_0 + r\mathbb{B}) \subset F(u) + k|x - u|\mathbb{B}.$$

**Lemme 1.4.3** On suppose que  $F$  est pseudo-lipschitzienne de constante  $k$  en  $(x_0, y_0) \in GrF$ . Alors pour tout  $y^* \in \mathbb{R}^n$ , avec  $D^*F(x_0, y_0)(y^*) \neq \emptyset$ , on a

$$\sup \{|x^*| : x^* \in D^*F(x_0, y_0)(y^*)\} \leq k|y^*|.$$

Si on suppose de plus que la fonction multivoque  $F$  est à valeurs fermées, alors pour tout  $(x, y) \in (x_0 + \frac{r}{12}\mathbb{B}) \times (y_0 + \frac{r}{12}\mathbb{B})$ , avec  $(x, y) \notin GrF$  et pour tout  $(x^*, y^*) \in \partial d(\cdot; F(\cdot))(x, y)$ , on a

$$|y^*| = 1, \text{ et } |x^*| \leq K|y^*|.$$

Pour la démonstration voir Chapitre 2.

On termine ce paragraphe par le lemme de compacité suivant :

**Lemme 1.4.4** [22] Soient  $\varepsilon$  un nombre strictement positif et  $\Gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$  une fonction multivoque tels que pour presque tout  $t \in [a, b]$ ,  $\Gamma(t, \cdot)$  est à valeurs non vides, compactes et convexes autour de  $(z(t), \dot{z}(t), p, s)$ , où  $s \in [0, \varepsilon]$  et  $\Gamma(z(t), \dot{z}(t), p, s) \neq \emptyset$ . On considère les suites  $(z_k)$ ,  $(p_k)$  de  $W^{1,1}$ ,  $(\phi_k)$  de  $L^1([a, b], ]0, +\infty[)$ ,  $(\alpha_k)$  et  $(s_k)$  de  $\mathbb{R}_+$ , telles que :

$$z_k \rightarrow z \text{ dans } W^{1,1}, \alpha_k \rightarrow 0, s_k \rightarrow 0, \text{ et } \phi_k \rightarrow \phi \text{ dans } L^1([a, b], ]0, +\infty[),$$

où  $\phi$  est une fonction intégrable. On suppose que

i) pour tout  $(x, y, p, s)$  de l'intérieur de l'ensemble

$$\{(x', y', p', s') : t \in [a, b], x' \in z(t) + \varepsilon \mathbb{B}, y' \in \dot{z}(t) + \varepsilon \mathbb{B}, \\ s' \in [0, \varepsilon], \Gamma(t, x', y', p', s') \neq \emptyset\},$$

la fonction multivoque  $t' \mapsto \Gamma(t', x, y, p, s)$  est mesurable ;

ii) pour tout  $k$ ,  $|\dot{p}_k(t)| \leq \phi_k(t)$  pour presque tout  $t \in [a, b]$  ;

iii) pour tout  $k$ ,  $\dot{p}_k(t) \in \Gamma(t, z_k(t), \dot{z}_k(t), p_k(t), s_k) + \alpha_k \mathbb{B}$  p.p.  $t \in [a, b]$  ;

iv) Pour presque tout  $t \in [a, b]$ , pour tout  $p \in \mathbb{R}^n$  où  $\Gamma(t, z(t), \dot{z}(t), p, 0) \neq \emptyset$ , la fonction multivoque  $(x', y', p', s') \mapsto \Gamma(t, x', y', p', s')$  est semi-continue supérieurement en  $(z(t), \dot{z}(t), p, 0)$  ;

v) La suite  $(p_k(a))$  est bornée ;

vi) il existe une fonction intégrable  $\psi$  telle que

$$\sup_{\{(p', s') : s' \in [0, \varepsilon], \Gamma(t, z(t), \dot{z}(t), p', s') \neq \emptyset\}} \max_{y \in \Gamma(t, z(t), \dot{z}(t), p', s')} |y| \leq \psi(t) \text{ p.p.}$$

Alors il existe une sous-suite de  $(p_k)$ , convergeant uniformément vers un arc  $p$  satisfaisant :

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \text{ p.p. } t \in [a, b].$$

# Chapitre 2

## Lagrange multipliers for multiobjective programs with a general preference

**Abstract.** We consider a nonsmooth multiobjective optimization problems related to a general preference in Banach spaces. We introduce Calmness definition in Banach spaces involving approximate subdifferential and using compactly epi-Lipschitzian notion. We derive Lagrange multipliers of Karush-Kuhn-Tucker type and Fritz-John type. Examples of useful preferences are given.

**Key words.** Nonsmooth analysis, calmness, Lagrange multipliers, preference, multiobjective optimization, scalar optimization.

### 2.1 Introduction

In this paper we shall obtain existence of Lagrange multipliers for multiobjective optimization problems in Banach spaces with a general preference.

The problem that we consider here is of the form

$$(P) \begin{cases} \min f(x) \\ \text{subject to} \\ x \in C \text{ and } g(x) \in D \end{cases}$$

where  $f: X \mapsto Z$  and  $g: X \mapsto Y$  are mappings,  $X$ ,  $Y$  and  $Z$  are Banach

spaces and  $C \subset X$  and  $D \subset Y$  are closed sets.

Historically, the concept of preference appeared in the value theory in economics. Many authors in the early studies often defined the preference by an utility function, i.e, given a preference whether its always possible to find an utility function that can determine the preference.

In [28] the author proved, in finite dimension, that a preference  $\prec$  determined by a continuous utility function if and only if for any  $x$  the sets

$$\{y : x \prec y\} \quad \text{and} \quad \{y : y \prec x\} \quad \text{are closed.} \quad (2.1)$$

This theorem is not general and besides this it's an existence theorem (i.e dont provide methods for determining utility function) and there are some useful preference that dont check (2.1).

Always in a historical context, when Kuhn and Tucker proved the Kuhn-Tucker theorem in 1950, which gives necessary conditions for the existence of an optimal solution to a nonlinear programming problem, they launched the theory of nonlinear programming. Karush derived a result that was comparable to the Kuhn-Tucker theorem. Like Karush, Fritz-John looked at the finite-dimensional case and formulated a result that later was acknowledged as a version of Kuhn-Tucker theorem.

For nonsmooth multiobjective optimization problem, necessary optimality conditions can be established under various qualification conditions (constraint qualification conditions). Among well known qualification conditions are the Guignard qualification, Slater qualification, linear objective qualification, Mangasarian-Fromovitz qualification, Robinson qualification, Zowe qualification and Kurcyusz qualification. It has been shown that the last three ones imply the regularity of the corresponding set-valued mappings expressing feasibility.

Many works are devoted to the qualification conditions to ensuring the non-vacuity and boundedness of Lagrange multipliers sets of Kuhn-Tucker type ([30], [33], [51], [78] and [103]), or to study differential stability of a marginal function in nonlinear programs ([34], [81] and [91]) and or to obtain subdifferential calculus rules.

Problem ( $P$ ) is studied in finite or infinite dimensional spaces. For deriving necessary conditions, some authors involved locally Lipschitz functions ([47],[60], [70] and [100]) others consider strongly compactly Lipschitzian functions ([30] and [35]), using Ekeland's variational principle [29] and assuming that  $D$  is a closed convex cone with nonempty interior or  $D = D_1 \times \{0\}$

with  $\{0\} \subset \mathbb{R}^n$  and  $D_1$  a closed convex cone with nonempty interior.

Involving the approximate subdifferential, the authors in [52] derived Lagrange multipliers of Fritz-John type and Kuhn-Tucker type for problem  $(P)$  in Banach spaces by assuming that  $D$  is epi-Lipschitz like [11] and [12] and in [59] by assuming that  $D$  is compactly epi-Lipschitzian.

In this paper we give a regularity definition of preferences. We use calmness qualification (i.e. calmness of set-valued mappings [82]) and compactly epi-Lipschitzian notion of Borwein and Strojwas [12], [17] and [18], to show the existence of Karush-Kuhn-Tucker Lagrange multipliers in terms of the approximate subdifferential for multiobjective optimization problem with a general preference, and from this result we derive some examples. The noncalmness case is treated and gives Fritz-John Lagrange multipliers of problem  $(P)$ . Our result is applied to produce necessary optimality conditions for problems of type

$$\min_{x \in X} F(x)$$

where  $F : X \rightarrow Y$  is a set-valued mapping with closed graph.

The paper is organized as follows. Section 2 contains the key definitions and some useful results. Section 3 shows the existence of Karush-Kuhn-Tucker Lagrange multipliers for problem  $(P)$ . Some examples of preferences are given. Section 4 is devoted to Fritz-John Lagrange multipliers of problem  $(P)$ . Section 5 is concerned with Lagrange multipliers for single-objective programs. Section 6 is concerned with differentiable case.

## 2.2 Approximate subdifferentials and preliminaries

Throughout we shall assume that  $X, Y$  and  $Z$  are Banach spaces,  $X^*, Y^*$  and  $Z^*$  are their topological duals and  $\langle \cdot, \cdot \rangle$  is the pairing between the spaces. We denote by  $B_X, B_{X^*}, \dots$  the closed unit balls of  $X, X^*, \dots$ . By  $d(\cdot, S)$  we denote the usual distance function to the set  $S$

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$



We write  $x \xrightarrow{f} x_0$  and  $x \xrightarrow{S} x_0$  to express  $x \rightarrow x_0$  with  $f(x) \rightarrow f(x_0)$  and  $x \rightarrow x_0$  with  $x \in S$ , respectively.

If  $f$  is an extended-real-valued function on  $X$ , we write for any subset  $S$  of  $X$

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

The function

$$d^- f(x, h) = \liminf_{\substack{u \rightarrow h \\ \varepsilon \downarrow 0}} t^{-1}(f(x + tu) - f(x))$$

is the lower Dini directional derivative of  $f$  at  $x$  and the Dini  $\varepsilon$ -subdifferential of  $f$  at  $x$  is the set

$$\partial_\varepsilon^- f(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq d^- f(x; h) + \varepsilon \|h\|, \forall h \in X\}$$

for  $x \in \text{Dom} f$  and  $\partial_\varepsilon^- f(x) = \emptyset$  if  $x \notin \text{Dom} f$ , where  $\text{Dom} f$  denotes the effective domain of  $f$ . For  $\varepsilon = 0$  we write  $\partial^- f(x)$ .

By  $\mathcal{F}(X)$  we denote the collection of finite dimensional subspaces of  $X$ . The approximate subdifferential of  $f$  at  $x_0 \in \text{Dom} f$  is defined by the following expressions (see Ioffe [41] and [43])

$$\partial f(x_0) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{f} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f_{x+L}(x)$$

where

$$\limsup_{x \xrightarrow{f} x_0} \partial^- f_{x+L}(x) = \{x^* \in X^* : x^* = w^* - \lim x_i^*, x_i^* \in \partial^- f_{x_i+L}(x_i), x_i \xrightarrow{f} x_0\},$$

that is, the set of  $w^*$ -limits of all such nets.

It is easily seen that the set-valued mapping  $x \rightarrow \partial f(x)$  is upper semicontinuous in the following sense

$$\partial f(x_0) = \limsup_{x \xrightarrow{f} x_0} \partial f(x)$$

and in [41] and [43] Ioffe has shown that when  $S$  is a closed subset of  $X$  and  $x_0 \in S$

$$\partial d(x_0, S) = \bigcap_{L \in \mathcal{F}(X)} \limsup_{\substack{x \xrightarrow{S} x_0 \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- d_{x+L}(x, S) \cap (1 + \varepsilon)B_{X^*}.$$

The normal cone  $N(S, x_0)$  to  $S$  at  $x_0 \in S$  is defined by

$$N(S, x_0) = \bigcup_{\lambda > 0} \lambda \partial d(x_0, S).$$

For  $x_0 \notin S$  we set  $N(S, x_0) = \emptyset$ .

In the sequel we shall need the following class of mappings between Banach spaces.

**Definition 1** [90]. *A mapping  $g : X \mapsto Y$  is said to be strongly compactly Lipschitzian (s.c.L.) at a point  $x_0$  if there exist a set-valued mapping  $R : X \mapsto 2^{Comp(Y)}$ , where  $Comp(Y)$  denotes the set of all norm compact subsets of  $Y$ , and a function  $r : X \times X \rightarrow \mathbb{R}_+$  satisfying*

(i)  $\lim_{\substack{x \rightarrow x_0 \\ h \rightarrow 0}} r(x, h) = 0,$

(ii) *there exists  $\alpha > 0$  such that*

$$t^{-1}[g(x + th) - g(x)] \in R(h) + \|h\|r(x, th)B_Y$$

for all  $x \in x_0 + \alpha B_X, h \in \alpha B_X$  and  $t \in ]0, \alpha[$ ,

(iii)  $R(0) = \{0\}$  and  $R$  is upper semicontinuous.

It can be shown [90] that every s.c.L. mapping is locally Lipschitzian. In finite dimensions the concepts coincide.

Recently we have developed in [57] a chain rule for this class of mappings. Let us note that this chain rule has been obtained before by Ioffe in [43] for maps with compact prederivatives.

**Theorem 2.2.1** [57]. *Let  $g : X \rightarrow Y$  be a s.c.L. mapping at  $x_0$  and let  $f : Y \rightarrow \mathbb{R}$  be a locally Lipschitz function at  $g(x_0)$ . Then  $f \circ g$  is locally Lipschitz at  $x_0$  and*

$$\partial(f \circ g)(x_0) \subset \bigcup_{y^* \in \partial f(g(x_0))} \partial(y^* \circ g)(x_0).$$

To complete this section we note the following property of s.c.L. mappings which is a direct consequence of Proposition 2.3 in [59].

**Proposition 2.2.1** *Let  $g : X \rightarrow Y$  be s.c.L. at  $x_0$  and let  $(y_i^*)$  any bounded net of  $Y^*$  which  $w^*$ -converges to zero in  $Y^*$  and let  $(x_i)$  be a net norm-converging to  $x_0$  in  $X$ . If  $x_i^* \in \partial(y_i^* \circ g)(x_i)$ , then  $(x_i^*)$   $w^*$ -converges to zero in  $X^*$ .*

## 2.3 Karush-Kuhn-Tucker Lagrange multipliers

We start by recalling some of the prominent Lipschitz properties formulated for set-valued mappings. Let  $M : Y \mapsto X$  be a set-valued mapping.  $M$  is said to have the *Aubin property* at some  $(\bar{y}, \bar{x})$  in the graph of  $M$ , if there exist neighborhoods  $\mathcal{Y}$  and  $\mathcal{X}$  of  $\bar{y}$  and  $\bar{x}$  as well as some  $K > 0$  such that

$$d(x, M(y)) \leq K\|y - y'\| \quad \forall y, y' \in \mathcal{Y} \quad \forall x \in M(y') \cap \mathcal{X}.$$

Fixing one of the  $y$ -parameters as  $\bar{y}$  in the definition of the Aubin property, yields the calmness ([82]) of  $M$  at  $(\bar{y}, \bar{x})$  :

$$d(x, M(\bar{y})) \leq K\|y - \bar{y}\| \quad \forall y \in \mathcal{Y} \quad \forall x \in M(y) \cap \mathcal{X}.$$

Obviously, the Aubin property implies calmness whereas the converse is not true (e.g.  $M(y) = \{x : x^2 \geq y\}$ ).

Before giving sufficient conditions for calmness for a special class of set-valued mappings, let us recall the following notion by Borwein and Strojwas [12], [17] and [18]. A set  $S \subset X$  is said to be *compactly epi-Lipschitzian* at  $x_0 \in S$  if there exist  $\gamma > 0$  and a norm compact set  $H \subset X$  such that

$$S \cap (x_0 + \gamma B_X) + t\gamma B_X \subset S - tH, \quad \text{for all } t \in ]0, \gamma[.$$

**Theorem 2.3.1** *Let  $D \subset Y$  and  $C \subset X$  be two closed subsets and  $g : X \rightarrow Y$  be a s.c.L. mapping at  $\bar{x} \in C \cap g^{-1}(D)$ . Suppose that  $D$  is compactly epi-Lipschitz at  $g(x_0)$ . Suppose also that the following regularity condition holds at  $\bar{x}$*

$$[y^* \in \partial d(g(\bar{x}), D) \quad \text{and} \quad 0 \in \partial(y^* \circ g + d(\cdot, C))(\bar{x})] \implies y^* = 0.$$

*Then the set-valued mapping  $M : Y \mapsto X$  defined by*

$$M(y) = \{x \in C : y \in -g(x) + D\}$$

*has the Aubin property at the point  $(0, \bar{x})$  and hence it is calm at this point or equivalently for some real numbers  $a \geq 0$  and  $r > 0$*

$$d(x, C \cap g^{-1}(D)) \leq a[d(g(x), D) + d(x, C)]$$

*for all  $x \in \bar{x} + rB_X$ .*

**Proof.** See Jourani and Thibault [59]. □

Note that new “boundary” conditions for calmness for this class of set-valued mappings in finite dimension have been discovered in [37]. Similar conditions have been presented in [36] for convex systems in infinite dimensional situation.

Using the definition of the approximate subdifferential we can easily get

**Proposition 2.3.1** *Suppose that  $g$  is s.c.L. at  $\bar{x} \in C \cap g^{-1}(D)$  and that the set-valued mapping  $M$  in Theorem 2.3.1 is calm at  $(0, \bar{x})$ . Then*

$$\partial d(\cdot, M(0)) \subset \bigcup_{y^* \in N(D, g(\bar{x}))} \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Our aim in this section is to show how to use calmness to obtain the existence of Karush-Kuhn-Tucker Lagrange multipliers for multiobjective optimization problems with a general preference. The problem that we will consider is of the form

$$(P) \begin{cases} \min f(x) \\ \text{subject to} \\ x \in C \text{ and } g(x) \in D \end{cases}$$

where  $f : X \mapsto Z$  and  $g : X \mapsto Y$  are mappings and  $C \subset X$  and  $D \subset Y$  are closed sets.

Let  $\prec$  be a nonreflexive preference for vector in  $Z$ . Let  $\bar{x}$  be a feasible point for  $(P)$ . We say  $\bar{x}$  is a solution to problem  $(P)$  provided that there exists no other feasible point  $x$  for  $(P)$  such that  $f(x) \prec f(\bar{x})$ .

For any  $z \in Z$  we denote  $\mathcal{L}(z) := \{z' \in Z : z' \prec z\}$ .

Then  $\bar{x}$  is a solution to  $(P)$  if and only if  $f(C \cap g^{-1}(D)) \cap \mathcal{L}(f(\bar{x})) = \emptyset$ .

We need the following regularity assumptions on the preference.

**Definition 2** *We say that a preference  $\prec$  is regular at  $z \in Z$  provided that*

(D<sub>1</sub>) *for any  $u \in Z$  near  $z$ ,  $u \in \text{cl}\mathcal{L}(u)$ ;*

(D<sub>2</sub>) *for any  $u$  and  $w$  near  $z$  satisfying  $u \prec z$ ,  $w \in \text{cl}\mathcal{L}(u)$  we have  $w \prec v$ ;*

(D<sub>3</sub>) *there exists a locally compact cone  $K^*$  in  $Z^*$  such that for any nets  $z_i, z'_i \rightarrow z$  in  $Z$*

$$N(\text{cl}\mathcal{L}(z_i), z'_i) \subset K^*.$$

Recall that a set  $K^*$  in  $Z^*$  is *weak-star locally compact* if every point of  $K^*$  lies in a weak-star open set  $V$  such that  $\text{cl}^*(V) \cap K$  is weak-star compact.

The first important property of these cones has been established by Loewen in [65] in a reflexive Banach space (but the proof works in any Banach space). He showed that if  $(z_i^*)$  is a net in a locally compact cone  $K^*$  then

$(z_i^*)$  weak-star converges to 0 iff it converges in norm to 0.

Note that our assertion  $(D_3)$  always holds in finite dimensional situation. Our definition of regularity differs from that introduced by Zhu [101]. Indeed the definition in [101] was given in finite dimensional spaces and assumed in addition to  $(D_1)$  and  $(D_2)$  the following one : for any sequences  $z_n, z'_n \rightarrow z$

$$\limsup_{n \rightarrow \infty} N(\text{cl}\mathcal{L}(z_n), z'_n) \subset N(\text{cl}\mathcal{L}(\bar{z}), \bar{z}).$$

Unfortunately, contrary to what it is indicated in [101], the Zhu's regularity does not hold for a preference determined by an utility function.

**Example 1** [7] *Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  be an utility function defined by  $u(x, y) = |x| - |y|$ . Then  $N(\mathcal{L}(0), 0) = \{(x, y) \in \mathbb{R}^2 : |x| = |y|\}$  while  $\cup_{\lambda > 0} \lambda \partial u(0, 0) = \cup_{\lambda > 0} \lambda([-1, 1] \times \{-1, 1\})$ . So that the preference is not regular in the sense by Zhu [101], but it is regular in the sense of Definition 2.*

For this reason we introduce the cone

$$\tilde{N}(\text{cl}\mathcal{L}(z), z) := w^* - \limsup_{z'', z' \rightarrow z} N(\text{cl}\mathcal{L}(z''), z').$$

Note that for the preference in Example 1 we have

$$\tilde{N}(\text{cl}\mathcal{L}(0), 0) = \cup_{\lambda \geq 0} \lambda([-1, 1] \times \{-1, 1\}).$$

Throughout the rest of this section, we make the following standing assumption :  $f$  and  $g$  are s.c.L. at  $\bar{x}$ . Our main result of this section is the following.

**Theorem 2.3.2** *Let  $\bar{x}$  be a solution to problem  $(P)$ . Suppose the preference  $\prec$  is regular at  $f(\bar{x})$  and the set-valued mapping  $M : Y \mapsto X$ , defined by  $M(y) = \{x \in C : y \in -g(x) + D\}$ , is calm at  $(0, \bar{x})$ . Then there exist  $z^* \in \tilde{N}(\text{cl}(\mathcal{L}(f(\bar{x})), f(\bar{x})))$ , with  $z^* \neq 0$ , and  $y^* \in N(D, g(\bar{x}))$  such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

**Proof.** Let  $(\theta_k)$  be a sequence in  $Z$  such that

$$\theta_k \prec f(\bar{x}) \quad \text{and} \quad \|\theta_k - f(\bar{x})\| < \frac{1}{k^2}.$$

Define the sets  $\Theta := cl(\mathcal{L}(\theta_k))$  and  $\Gamma = C \cap g^{-1}(D)$  and the function

$$h(x, \theta) = \begin{cases} \|f(x) - \theta\| & \text{if } x \in B(\bar{x}, s_1), \\ +\infty & \text{otherwise.} \end{cases}$$

where  $s_1$  is such that  $f$  and  $g$  are Lipschitzian on  $B(\bar{x}, s_1)$  with constant  $k_f = k_g$ . Because of  $(D_1)$ ,  $(\bar{x}, \theta_k) \in \Gamma \times \Theta$  and hence

$$h(\bar{x}, \theta_k) \leq \inf_{(x, \theta) \in \Gamma \times \Theta} h(x, \theta) + \frac{1}{k^2}.$$

So, since  $\Gamma$  and  $\Theta$  are closed and  $h$  is lower semicontinuous, Ekeland's variational principle produces  $(x_k, \gamma_k) \in \Gamma \times \Theta$  such that

$$\|x_k - \bar{x}\| + \|\theta_k - \gamma_k\| < \frac{1}{k}$$

$$h(x_k, \gamma_k) \leq h(x, \theta) + \frac{1}{k}(\|x - x_k\| + \|\theta - \gamma_k\|) \quad \forall (x, \theta) \in \Gamma \times \Theta.$$

As  $h$  is locally Lipschitzian around  $(x_k, \gamma_k)$  we get for  $(x, \theta)$  near  $(x_k, \gamma_k)$

$$\begin{aligned} h(x_k, \gamma_k) &\leq h(x, \theta) + \frac{1}{k}(\|x - x_k\| + \|\theta - \gamma_k\|) \\ &\quad + (k_f + 2)[d(x, \Gamma) + d(\theta, \Theta)] \end{aligned}$$

and hence

$$(0, 0) \in \partial h(x_k, \gamma_k) + (k_f + 2)[\partial d(x_k, \Gamma) \times \partial d(\gamma_k, \Theta)].$$

As  $\bar{x}$  is a local solution to problem  $(P)$ , then by  $(D_2)$ , and the choice of  $\theta_k$  one has  $\gamma_k \neq f(x_k)$ . Since  $f$  is s.c.L., Theorem 2.2.1 implies that

$$\partial h(x_k, \gamma_k) \subset \bigcup_{\|z^*\|=1} (\partial(z^* \circ f)(x_k) \times \{-z^*\}).$$

Thus there are  $z_k^* \in Z^*$  with  $\|z_k^*\| = 1$  and  $(a_k^*, b_k^*) \in \frac{1}{k}\mathbb{B}$  such that

$$a_k^* \in \partial(z_k^* \circ f)(x_k) + (k_f + 2)\partial d(x_k, \Gamma)$$

$$b_k^* + z_k^* \in (k_f + 2)\partial d(\gamma_k, cl\mathcal{L}(\theta_k)).$$

Now, extracting subnet if necessary, we may suppose, using  $(D_3)$  and Proposition 2.2.1,  $z_k^* \rightarrow z^*$ , with  $z^* \neq 0$ ,

$$0 \in \partial(z^* \circ f)(\bar{x}) + (k_f + 2)\partial d(\bar{x}, \Gamma)$$

$$z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x}))), f(\bar{x})).$$

So the proof is terminated by applying Proposition 2.3.1.  $\square$

In the remainder of this section, we will examine few examples.

**Example 2** (*Generalized Pareto*) Let  $K \subset Z$  be a convex cone with  $K^0$  locally compact ( $K^0 = \{z^* \in Z^* : \langle z^*, z \rangle \leq 0 \quad \forall z \in K\}$ ). We now define the preference by  $z \prec z'$  if and only if  $z - z' \in K$  and  $z \neq z'$ . In this case  $\mathcal{L}(z) = \{z' \in Z : z' \neq z, z' - z \in K\}$ . Multiobjective programs with this preference are called generalized Pareto optimization problems. When  $Z = \mathbb{R}^m$  and  $K = \mathbb{R}_-^m$  (resp.  $K = \text{int } \mathbb{R}_-^m$ ) we get Pareto (resp. weak Pareto) optimization problems. It is easy to check that preference defined in this way satisfies assumptions  $(D_1) - (D_3)$  in Definition 2.

Thus we have

**Corollary 2.3.1** Under assumptions of Theorem 2.3.2, there exist  $z^* \in K^0$ , with  $z^* \neq 0$ , and  $y^* \in N(D, g(\bar{x}))$  such that

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

Our next example considers a preference determined by an utility function.

**Example 3** (*A preference defined by an utility function*) Let  $u : Z \mapsto \mathbb{R} \cup \{+\infty\}$  be a continuous utility function that determines the preference, i.e.,  $z \prec z'$  if and only if  $u(z) < u(z')$ . We need additional assumptions to ensure the regularity of the preference which we summarize in the following proposition.

**Proposition 2.3.2** Let  $u$  be a continuous utility function determining the preference  $\prec$ . Suppose that the epigraph  $\text{epi}u$  of  $u$  is compactly epi-Lipschitzian at  $(\bar{z}, u(\bar{z}))$  and

$$0 \notin \partial u(\bar{z}). \tag{2.2}$$

Then the preference is regular at  $\bar{z}$  and

$$\tilde{N}(cl\mathcal{L}(\bar{z}), \bar{z}) = w^* - \limsup_{z \rightarrow \bar{z}} N(cl\mathcal{L}(z), z) = \partial^\infty u(\bar{z}) \bigcup_{\lambda > 0} (\bigcup_{\lambda > 0} \lambda \partial u(\bar{z})).$$

**Proof.** Conditions  $(D_1)$  and  $(D_2)$  follow from the continuity of  $u$ . Since  $\text{epi}u$  is CEL at  $(\bar{z}, u(\bar{z}))$ , condition  $(D_3)$  follows from [59]. Relation (2.2) guarantees the existence of two real numbers  $a > 0$  and  $\alpha > 0$  such that

$$d(z, \{v \in Z : u(z) \leq r\}) \leq a \max(0, u(z) - r) \quad (2.3)$$

for all  $z \in B(\bar{z}, \alpha)$  and  $r \in B(u(\bar{z}), \alpha)$ . Let  $z^* \in \tilde{N}(cl\mathcal{L}(\bar{z}), \bar{z})$ , with  $z^* \neq 0$ . Then there are nets  $z_i, z'_i \rightarrow \bar{z}$ ,  $(\lambda_i) \subset ]0, +\infty[$  and  $(z_i^*)$  such that

$$\lambda_i z_i^* \rightarrow z^*, \quad z'_i \in cl\mathcal{L}(z_i), \quad z_i^* \in \partial d(cl\mathcal{L}(z_i), z'_i).$$

Thus for each collection  $(L)$  of finite dimensional subspaces of  $Z$  there are nets  $z_{ij} \rightarrow z_i$ ,  $z_{ij}^* \rightarrow z_i^*$  and  $r_j, s_j \rightarrow 0^+$  such that

$$z_{ij} \in cl\mathcal{L}(z_i), \quad \|z_{ij}^*\| \leq 1 + s_j$$

and

$$d(z, cl\mathcal{L}(z_i)) + (2 + s_j)d(z, z_{ij} + L) - \langle z_{ij}^*, z - z_{ij} \rangle + s_j \|z - z_{ij}\| \geq 0$$

for all  $z \in B(z_{ij}, r_j)$ . Since  $z^* \neq 0$  we obtain  $u(z_i) = u(z_{ij})$ . Invoking (2.3) we get

$$a \max(0, u(z) - u(z_i)) + (2 + s_j)d(z, z_{ij} + L) - \langle z_{ij}^*, z - z_{ij} \rangle + s_j \|z - z_{ij}\| \geq 0$$

for  $z$  near  $z_{ij}$ . Thus  $z_i^* \in \partial[\max(0, u(\cdot) - u(z_i))](z_i)$  and hence  $z_i^* \in \partial^\infty u(z_i)$  or there exists  $\beta_i \in [0, a]$  such that  $z_i^* \in \beta_i \partial u(z_i)$ . Thus

$$\lambda_i z_i^* \in \partial^\infty u(z_i) \cup \lambda_i \beta_i \partial u(z_i).$$

Having in mind relation (2.2) we get the boundedness of  $(\lambda_i \beta_i)$  and

$$z^* \in \partial^\infty u(\bar{z}) \bigcup_{\lambda > 0} (\bigcup_{\lambda > 0} \lambda \partial u(\bar{z})).$$

Conversely let  $z^* \in \partial u(\bar{z})$ , with  $z^* \neq 0$ . As  $\text{epi}u$  is CEL at  $(\bar{x}, u(\bar{x}))$ , it follows ([49]) that  $(z^*, -1) \in \bigcup_{\lambda > 0} \lambda \partial d((\bar{z}, u(\bar{z})), \text{epi}u)$ . Then there exist  $(w^*, r^*) \in$



$\partial d((\bar{z}, u(\bar{z})), \text{epi}u)$  and  $\lambda > 0$  such that  $(z^*, -1) = \lambda(w^*, r^*)$ . For each collection  $(L)$  of finite dimensional subspaces of  $Z$  there exist nets  $z_i \rightarrow z$ ,  $r_i^* \rightarrow r^*$ ,  $z_i^* \rightarrow w^*$  and  $r_i, s_i \rightarrow 0^+$  such that

$$d((z, r), \text{epi}u) + (2 + s_i)d(z, z_i + L) - \langle z_i^*, z - z_i \rangle - r_i^*(r - u(z_i)) + s_i[\|z - z_i\| + |r - u(z_i)|] \geq 0$$

for all  $z \in B(z_i, r_i)$  and  $r \in B(u(z_i), r_i)$ . Thus

$$(2 + s_i)d(z, z_i + L) - \langle z_i^*, z - z_i \rangle + s_i\|z - z_i\| \geq 0$$

for all  $z \in \text{cl}\mathcal{L}(z_i) \cap B(z_i, r_i)$ . Thus

$$z_i^* \in (4 + s_i)\partial d(z_i, \text{cl}\mathcal{L}(z_i)) + s_i B_{Z^*} + L^\perp$$

and hence  $z^* \in \tilde{N}(\text{cl}\mathcal{L}(\bar{z}), \bar{z})$  and the proof is complete.  $\square$

Using this proposition we have the following corollary of Theorem 2.3.2.

**Corollary 2.3.2** *Let  $\prec$  be a preference determined by an utility function  $u$ . Suppose that  $u$  satisfies the conditions of Proposition 2.3.2 with  $\bar{z} = f(\bar{x})$ . Then under the assumptions of Theorem 2.3.2 there exist  $y^* \in N(D, g(\bar{x}))$  and  $z^* \in \partial^\infty u(f(\bar{x})) \bigcup_{\lambda > 0} [\bigcup \lambda \partial u(f(\bar{x}))]$ , with  $z^* \neq 0$ , such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

From Theorem 2.3.2 we can obtain the following corollary, on necessary optimality conditions for set-valued optimization problem

$$\min_{x \in X} F(x) \tag{2.4}$$

where  $F : X \rightarrow Y$  is a set-valued mapping with closed graph.

Let  $(\bar{x}, \bar{y}) \in \text{Gr}F$ . The point  $\bar{x}$  said to be a minimum (with respect to  $\bar{y}$ ) to the problem (2.4) related to a general preference  $\prec$  if and only if

$$F(X) \cap \mathcal{L}(\bar{y}) = \emptyset.$$

Let us recall that the set-valued mapping  $D^*F(x, y) : Y \rightarrow X$  defined by :

$$D^*F(x, y)(y^*) = \{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N(\text{Gr}F; (x, y))\}$$

is called the coderivative of  $F$  at the point  $(x, y) \in \text{Gr}F$ .

**Corollary 2.3.3** *Let  $\bar{x}$  be a minimum (with respect to  $\bar{y}$ ) to the problem (2.4) related to a general preference  $\prec$ . Suppose that  $\prec$  is regular at  $\bar{y}$ . Then there exists  $z^* \in \tilde{N}(cl\mathcal{L}(\bar{y}), \bar{y})$  with  $z^* \neq 0$ , such that  $0 \in D^*F(\bar{x}, \bar{y})(z^*)$ .*

**Proof.** Let  $f : X \times Y \rightarrow Y$  be a mapping defined by  $f(x, y) = y$ . Consider the following multiobjective optimization problem

$$\min_{(x,y) \in GrF} f(x, y). \quad (2.5)$$

Then  $\bar{x}$  is a minimum (with respect to  $\bar{y}$ ) to the problem (2.4), if and only if  $(\bar{x}, \bar{y})$  is a solution of problem (2.5). Using Theorem 2.3.2, there exist  $z^* \in \tilde{N}(cl\mathcal{L}(\bar{y}), \bar{y})$  with  $z^* \neq 0$ , such that

$$0 \in \partial(z^* \circ f)(\bar{x}, \bar{y}) + N(GrF, (\bar{x}, \bar{y}))$$

which gives  $0 \in D^*F(\bar{x}, \bar{y})(z^*)$ . □

In [53] the author got the same result, but for the case of Pareto optimum.

## 2.4 Fritz-John Lagrange multipliers

In this section, we consider the case where the set-valued mapping  $M$  in Theorem 2.3.2 is not necessarily calm.

**Theorem 2.4.1** *Let  $\bar{x}$  be a solution to problem (P). Suppose the preference  $\prec$  is regular at  $f(\bar{x})$  and that  $D$  is CEL at  $g(\bar{x})$ . Then there exist  $z^* \in \tilde{N}(cl(\mathcal{L}(f(\bar{x})), f(\bar{x})))$ , and  $y^* \in N(D, g(\bar{x}))$ , with  $(z^*, y^*) \neq 0$ , such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

**Proof.** Use Theorems 2.3.1 and 2.3.2. □

We have to note that the conclusion of this theorem holds under assumptions of the previous corollaries without calmness. Namely :

**Corollary 2.4.1** *Let the assumptions of Corollary 2.3.2 hold without calmness assumption. Then there exist  $y^* \in N(D, g(\bar{x}))$  and  $z^* \in \partial^\infty u(f(\bar{x})) \bigcup [\bigcup_{\lambda > 0} \lambda \partial u(f(\bar{x}))]$ , with  $(y^*, z^*) \neq 0$ , such that*

$$0 \in \partial(z^* \circ f)(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

## 2.5 Lagrange multipliers for single-objective programs

In this section we consider problem  $(P)$  with  $Z = \mathbb{R}$ . Our aim here is to give necessary optimality conditions under general assumptions.

**Theorem 2.5.1** *Let  $\bar{x}$  be a solution to problem  $(P)$  with  $Z = \mathbb{R}$  and  $\mathcal{L}(r) = r + \mathbb{R}_+$ . Suppose that*

- 1)  $f$  is lower semicontinuous on  $X$  and  $g$  is s.c.L. at  $\bar{x}$ ;
- 2) the set-valued mapping  $M : Y \mapsto X$  defined by  $M(y) = \{x \in C : y \in -g(x) + D\}$  is calm at  $(0, \bar{x})$ ;
- 3) either
  - i)  $\text{epi} f$  is CEL at  $(\bar{x}, f(\bar{x}))$  or
  - ii)  $C$  and  $D$  are CEL at  $\bar{x}$  and  $g(\bar{x})$  respectively and for all  $y^* \in \partial d(D, g(\bar{x}))$ , with  $y^* \neq 0$ , we have

$$0 \notin \partial(y^* \circ g)(\bar{x}) + \partial d(C, \bar{x});$$

- 4)  $\partial^\infty f(\bar{x}) \cap (-N(C \cap g^{-1}(D))) = \{0\}$ .

Then there exists  $y^* \in N(D, g(\bar{x}))$  such that

$$0 \in \partial f(\bar{x}) + \partial(y^* \circ g)(\bar{x}) + N(C, \bar{x}).$$

**Proof.** The case of 4) – i) is established in Jourani [48]. So it suffices to consider case 4) – ii). Since  $\bar{x}$  is a solution to  $(P)$ , then  $0 \in \partial(f + \Psi_{C \cap g^{-1}(D)})(\bar{x})$ , where  $\Psi_H$  denotes the indicator function of the set  $H$ . It follows from [50] that the set  $C \cap g^{-1}(D)$  is CEL at  $\bar{x}$ . Now using 5) we get ([48])  $0 \in \partial f(\bar{x}) + N(C \cap g^{-1}(D), \bar{x})$ . The proof is terminated by applying Proposition 2.3.1.  $\square$

## 2.6 The differentiable case

In this section, we consider programs with differentiable data. A mapping  $h : X \mapsto Y$  is said to be strictly differentiable at  $\bar{x}$  if

$$\lim_{x, y \rightarrow \bar{x}} \frac{h(x) - h(y) - Dh(\bar{x})(x - y)}{\|x - y\|} = 0.$$

To simplify we assume in our problem (P) that  $C = X$ . Thus in the differentiable case our previous results may be expressed in a simple way.

**Corollary 2.6.1** *Suppose in addition to the assumptions of Theorem 2.3.2 (resp. Theorem 2.4.1) that  $f$  and  $g$  are strictly differentiable at  $\bar{x}$ . Then there exist  $z^* \in \tilde{N}(\text{cl}(\mathcal{L}(f(\bar{x})), f(\bar{x}))$  and  $y^* \in N(D, g(\bar{x}))$ , with  $z^* \neq 0$  (resp.  $(z^*, y^*) \neq 0$ ), such that  $z^* \circ Df(\bar{x}) + y^* \circ Dg(\bar{x}) = 0$ .*

**Corollary 2.6.2** *In addition to the assumptions of Corollary 2.3.2 (resp. Corollary 2.4.1) we suppose that  $f$  and  $g$  are strictly differentiable at  $\bar{x}$ . Then there exist  $y^* \in N(D, g(\bar{x}))$  and  $z^* \in \partial^\infty u(f(\bar{x})) \bigcup [\bigcup_{\lambda > 0} \lambda \partial u(f(\bar{x}))]$ , with  $z^* \neq 0$  (resp.  $(z^*, y^*) \neq 0$ ), such that  $z^* \circ Df(\bar{x}) + y^* \circ Dg(\bar{x}) = 0$ .*

Note that if we suppose in Corollary 2.6.2 that  $u$  is strictly differentiable at  $f(\bar{x})$  then there exist  $y^* \in N(D, g(\bar{x}))$  and  $\lambda \geq 0$  with  $\lambda \neq 0$  (resp.  $(\lambda, y^*) \neq 0$ ), such that

$$\lambda \nabla u(f(\bar{x})) \circ Df(\bar{x}) + y^* \circ Dg(\bar{x}) = 0.$$



## Chapitre 3

# New Euler-Lagrange inclusion with applications to general isoperimetric problems and to Ramsey model

**Abstract.** The aim of the paper is to establish a new Euler-Lagrange inclusion in nonsmooth optimal control problems in terms of a small subdifferential. In the obtained inclusion, the subdifferential is taken simultaneously in state and control. Our approach lies in reducing the optimal control problem into a generalized Bolza one for which we give a necessary optimality conditions under minimal assumptions. Our results are used to obtain new necessary optimality conditions for generalized isoperimetric problems and to study a general economic problems including Ramsey model of economic growth with multiple consumptions and productions.

### 3.1 Introduction

In this paper we shall state the maximum principle with a new Euler-Lagrange inclusion and shall use it to obtain new necessary optimality conditions for general isoperimetric problems and for some economic problems including the generalized Ramsey model of economic growth.

The optimal control problem that we consider is of the form

$$\min\{g(x(a), x(b)) + \int_a^b h(t, x(t), u(t)) dt\}$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad a.e. t \in [a, b], \quad (x(a), x(b)) \in S.$$

This problem does not assume convexity nor differentiability. There are different approaches and various results in this area (see [21], [45], [46], [67], [68], [69], [71], [76], [83] etc ). These works, which use one or another tool of nonsmooth analysis, contain Euler-Lagrange inclusion of the form

$$\dot{p}(t) \in \partial_x^c[h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle](z(t), v(t)) \quad a.e. t \in [a, b] \quad (3.1)$$

or the form

$$\dot{p}(t) \in \text{co}\partial_x[h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle](z(t), v(t)) \quad a.e. t \in [a, b] \quad (3.2)$$

where  $\partial_x^c$  and  $\partial_x$  stand for the partial Clarke's subdifferential and the partial limiting Fréchet subdifferential.

In [56], the second author obtained the following new Euler-Lagrange inclusion where the limiting Fréchet subdifferential is taken simultaneously in the state and the control

$$\dot{p}(t) \in \text{co}\{q : (q, 0) \in \partial[h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle + \psi_{U(t)}(\cdot)](z(t), v(t))\} \quad a.e. \quad (3.3)$$

and the Hamiltonian inclusion

$$\dot{p}(t) \in \text{co}\{q : (-q, \dot{z}(t), v(t)) \in \partial H(t, z(t), p(t), 0)\} \quad a.e. t \in [a, b] \quad (3.4)$$

where  $\psi_C$  denotes the indicator function of the set  $C$  and  $H$  is the Hamiltonian defined by

$$H(t, x, p, q) = \sup_{u \in U(t)} \{\langle p, f(t, x, u) \rangle + \langle q, u \rangle - h(t, x, u)\}.$$

The two last inclusions was established under a calmness assumption which is more general than normality condition.

Our aim in this paper is to prove a new Euler-Lagrange inclusion of type (3.3) without calmness assumption and to apply the result to the following generalized isoperimetric problem

$$\min\{g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt\} \quad (I_S)$$

subject to

$$\int_a^b f(t, x(t), \dot{x}(t)) dt \in K, \quad (x(a), x(b)) \in S, \quad \dot{x}(t) \in U(t) \text{ a.e. } t \in [a, b]$$

and to some economic problems including the following generalized Ramsey model of economic growth

$$\min \int_a^b u(c(t)) dt \tag{R_m}$$

subject to

$$\dot{x}(t) = \varphi(t, x(t)) - c(t), \quad c(t) \in U(t) \text{ a.e. } t \in [a, b], \quad x(t) \in D.$$

Our approach is based on necessary optimality conditions for the generalized Bolza problem

$$\min \{g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt\} \tag{P_B}$$

that we establish here under minimal assumptions. These conditions allow us to derive results like those by Ioffe-Rockafellar [46] and Vinter-Zheng [95].

The paper is organized as follows. Section 2 contains the key definitions of normals, subgradients and coderivatives used in the sequel. Note that our attention is confined to generalized differential constructions of variational analysis related to the Fréchet differentiability and subdifferentiability in finite dimensional spaces. In section 3, we state and establish necessary optimality conditions for the generalized Bolza problem ( $P_B$ ) and as a consequence we will obtain the extend Euler-Lagrange condition for nonconvex variational problems [95] and to derive the result by Ioffe-Rockafellar [46]. In section 4, we use these conditions to derive the maximum principle with a new Euler-Lagrange inclusion to special classes of problems. In section 5, we show how to obtain a new first order necessary conditions for the generalized isoperimetric problem ( $I_S$ ) from the maximum principle. In section 6, we shall illustrate how the maximum principle is used to study a general economic problems and we take up the generalized Ramsey model of economic growth with multiple consumptions and productions.

### 3.2 Background

The domain, which we use here, is typically one of the absolutely continuous functions  $x : [a, b] \rightarrow \mathbb{R}^n$  for which  $\|\dot{x}\|$  is integrable on  $[a, b]$ ,  $\dot{x}$  denotes



the derivative almost every where of  $x$ . This domain is nothing but the Sobolev space  $W^{1,1}$  ( $:=W^{1,1}([a, b], \mathbb{R}^n)$ ), which we endow it with the norm

$$\|x\| = \|x(a)\| + \int_a^b \|\dot{x}(t)\| dt.$$

Here and throughout the paper, we will use  $\|\cdot\|$  to denote both the euclidian norm of  $\mathbb{R}^n$  and the norm of  $W^{1,1}$  and  $B$  to denote the open unit ball of  $\mathbb{R}^n$ .

Now we state basic tools of generalized differentiation that are more appropriate for our main purpose. Details may be found in [71] and [76].

Let  $C$  be a closed subset of  $\mathbb{R}^n$  containing some point  $c$ . The  $\varepsilon$ -normal cone to  $C$  at  $c$  is the set

$$\hat{N}_\varepsilon(C, c) := \{\zeta \in \mathbb{R}^n : \liminf_{x \in C \rightarrow c} \frac{\langle -\zeta, x - c \rangle}{\|x - c\|} \geq -\varepsilon\}.$$

The normal cone to  $C$  at  $c$  is the set

$$N(C, c) := \limsup_{\substack{x \in C \rightarrow c \\ \varepsilon \rightarrow 0^+}} \hat{N}_\varepsilon(C, c)$$

Now let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous (l.s.c) function and let  $c \in \mathbb{R}^n$  such that  $f(c) < \infty$ . The limiting Fréchet subdifferential of  $f$  at  $c$  is the set

$$\partial f(c) = \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N(\text{epi } f; (c, f(c)))\}$$

where  $\text{epi } f$  denotes the epigraph of  $f$ . We have the following analytic characterization of  $\partial f(c)$  :

$$\partial f(c) = \limsup_{\substack{x \rightarrow c \\ f(x) \rightarrow f(c) \\ \varepsilon \rightarrow 0^+}} \partial_\varepsilon f(x)$$

where

$$\partial_\varepsilon f(x) = \{x^* \in X^* : \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq -\varepsilon\}.$$

Next we consider the multivalued mapping  $G$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  of closed graph

$$\text{Gr}G = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in G(x)\}.$$

The multivalued mapping  $D^*G(x, y) : \mathbb{R}^m \rightarrow \mathbb{R}^n$  defined by :

$$D^*G(x, y)(y^*) = \{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N(\text{Gr}G; (x, y))\}$$

is called the coderivative of  $G$  at the point  $(x, y) \in \text{Gr}G$ .

### 3.3 Necessary optimality conditions for the generalized Bolza problem

In this section we derive necessary optimality conditions for the following Bolza problem

$$\min\{g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt\} \quad (P_B)$$

where  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  are functions such that for almost all  $t \in [a, b]$ , the functions  $g$  and  $h(t, \cdot)$  are l.s.c.

**Definition 3** [21] *The function  $h$  is epi-Lipschitzian at an arc  $z$  if there exist an integrable function  $k : [a, b] \rightarrow \mathbb{R}$  and positive  $\varepsilon$  satisfying the following conditions : for almost all  $t \in [a, b]$  given two points  $z_1$  and  $z_2$  in  $z(t) + \varepsilon B$  and  $u_1 \in \mathbb{R}^n$  such that  $h(t, z_1, u_1)$  is finite, there exist a point  $u_2 \in \mathbb{R}^n$  and  $\delta \geq 0$  such that  $h(t, z_2, u_2)$  is finite and*

$$\|u_1 - u_2\| + |h(t, z_1, u_1) - h(t, z_2, u_2) - \delta| \leq k(t)\|z_1 - z_2\|.$$

This is equivalent to saying that the multivalued mapping

$$E(t, s) = \{(u, r) \in \mathbb{R}^n \times \mathbb{R} : h(t, s, u) \leq r\}$$

is Lipschitzian in  $s$  on  $z(t) + \varepsilon B$  (i.e. for all  $s, s' \in z(t) + \varepsilon B$  we have  $E(t, s') \subset E(t, s) + k(t) |s' - s| B$ ), (see the Appendix 1).

It is easy to see that  $h$  is epi-Lipshitzian at an arc  $z$  iff there exists an integrable function  $k : [a, b] \rightarrow \mathbb{R}$  and a positive  $\varepsilon$  satisfying for almost all  $t \in [a, b]$  given two points  $z_1$  and  $z_2$  in  $z(t) + \varepsilon B$  and  $u_1 \in \mathbb{R}^n$  such that  $h(t, z_1, u_1)$  is finite, there exists a point  $u_2 \in \mathbb{R}^n$  such that  $h(t, z_2, u_2)$  is finite and

$$\|u_1 - u_2\| + h(t, z_2, u_2) - h(t, z_1, u_1) \leq k(t)\|z_1 - z_2\|.$$

**Definition 4** [21] *The function  $h$  is epi-measurable (in  $t$ ) if for each  $x \in \mathbb{R}^n$ , the multivalued mapping  $E(t, x) = \text{epih}(t, x, \cdot)$  is Lebesgue measurable in  $t$ .*

Now we can state the main result of this section.

**Theorem 3.3.1** *Let  $z$  be a local solution (in  $W^{1,1}$ ) to the problem  $(P_B)$ . Suppose that  $h$  is epi-measurable in  $t$  and epi-Lipschitzian in  $z$ . Then there exist an arc  $p$  and  $\lambda \in \{0, 1\}$ , with  $(\lambda, p) \neq 0$ , such that*

$$\dot{p}(t) \in \text{co}\{q : (q, p(t), -\lambda) \in N(\text{epi } h(t, \cdot); ((z(t), \dot{z}(t)), h(t, z(t), \dot{z}(t))))\} \text{ a.e. } t \in [a, b] \quad (3.5)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), l(z(a), z(b)))) \quad (3.6)$$

$$\langle p(t), \dot{z}(t) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \max_{v \in \mathbb{R}^n} \{\langle p(t), v \rangle - \lambda h(t, z(t), v)\} \text{ a.e. } t \in [a, b]. \quad (3.7)$$

Proof. We begin by reducing our problem to an optimal control problem governed by a differential inclusion. Indeed, let  $F : [a, b] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  be a multivalued mapping defined by

$$F(t, x, \alpha, r) = \{(v, s, 0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : h(t, x, v) \leq s\}.$$

Consider the set  $S$  and the function  $\varphi : (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R})^2 \rightarrow \mathbb{R}$  defined by

$$S = \{(s_1, s_2, s_3, \gamma_1, \gamma_2, \gamma_3) \in (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R})^2 : g(s_1, \gamma_1) \leq \gamma_3; s_2 = 0\}$$

$$\varphi(x_1, \alpha_1, r_1, x_2, \alpha_2, r_2) = \alpha_2 + r_2.$$

Set  $\bar{r}(t) = g(z(a), z(b))$ ,  $\bar{\alpha}(t) = \int_a^t h(s, z(s), \dot{z}(s)) dt$ , and consider the problem

$$\min_{(x, \alpha, r)} \varphi(x(a), \alpha(a), r(a), x(b), \alpha(b), r(b)) \quad (P'_B)$$

subject to

$$(\dot{x}(t), \dot{\alpha}(t), \dot{r}(t)) \in F(t, x(t), \alpha(t), r(t)) \text{ a.e. } t \in [a, b]$$

$$(x(a), \alpha(a), r(a), x(b), \alpha(b), r(b)) \in S.$$

Then

- $z$  is a local solution to  $(P_B)$  iff  $(z, \bar{\alpha}, \bar{r})$  is a local solution of  $(P'_B)$ .
- $h$  is epi-Lipschitzian at  $z$  iff  $F$  is locally Lipschitzian at  $(z, \bar{\alpha}, \bar{r})$ .
- $h$  is epi-measurable in  $t$  iff  $F$  is measurable in  $t$ .

The lower semicontinuity of  $L(t, \cdot)$  and  $l$  implies that  $F$  is a closed-valued and  $S$  is closed. By Theorem 1 in [45] there exist an arc  $(p, \theta, u)$  and  $\lambda \in \{0, 1\}$  with  $(p, \theta, u, \lambda) \neq 0$ , such that

$$\begin{aligned} (\dot{p}(t), \dot{\theta}(t), \dot{u}(t)) \in \text{co}D^*F(t, (z(t), \bar{\alpha}(t), \bar{r}(t)), \\ (\dot{z}(t), \dot{\bar{\alpha}}(t), \dot{\bar{r}}(t)))(-p(t), -\theta(t), -u(t)) \text{ a.e. } t \in [a, b] \end{aligned} \quad (3.8)$$

$$\begin{aligned} ((p(a), \theta(a), u(a)); (-p(b), -\theta(b), -u(b))) \in \\ \lambda \partial \varphi((z(a), \bar{\alpha}(a), \bar{r}(a)); (z(b), \bar{\alpha}(b), \bar{r}(b))) + \\ N(S, ((z(a), \bar{\alpha}(a), \bar{r}(a)); (z(b), \bar{\alpha}(b), \bar{r}(b)))) \end{aligned} \quad (3.9)$$

$$\begin{aligned} \langle (p(t), \theta(t), u(t)), (\dot{z}(t), \dot{\bar{\alpha}}(t), \dot{\bar{r}}(t)) \rangle \\ = \max_{(v, \alpha, r) \in F(t, z(t), \bar{\alpha}(t), \bar{r}(t))} \langle (p(t), \theta(t), u(t)), (v, \alpha, r) \rangle \quad a.e. \ t \in [a, b]. \end{aligned} \quad (3.10)$$

From (3.9) we have

$$\begin{aligned} ((p(a), \theta(a), u(a)); (-p(b), -\theta(b) - \lambda, -u(b) - \lambda)) \in \\ N(S, ((z(a), \bar{\alpha}(a), \bar{r}(a)); (z(b), \bar{\alpha}(b), \bar{r}(b)))) \end{aligned} \quad (3.11)$$

and hence

$$(p(a), -p(b), -u(b) - \lambda) \in N(\text{epi } g; ((z(a), z(b)), g(z(a), z(b))))$$

$$u(a) = 0 \quad \text{and} \quad -\theta(b) - \lambda = 0.$$

As  $GrF(t, \cdot) = \{(x, \alpha, r, v, s, \gamma) : (x, v, s) \in \text{epih}(t, \cdot), \gamma = 0, \alpha, r \in \mathbb{R}\}$  we deduce from (3.8)

$$\begin{aligned} \dot{p}(t) \in \text{co}\{q_1 : (q_1, p(t), \theta(t)) \in N(\text{epih}(t, \cdot); ((z(t), \dot{z}(t)); h(t, z(t), \dot{z}(t))))\} \quad a.e. \\ \dot{\theta}(t) = 0, \quad \dot{u}(t) = 0. \end{aligned}$$

So that  $\theta(t) = -\lambda$  and  $u(t) = 0$  whence (3.5). In the other hand, relation (3.9) yields  $(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), l(z(a), z(b))))$  which gives (3.6). From (3.10), we get

$$\begin{aligned} \langle p(t), \dot{z}(t) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \max_{v \in \mathbb{R}^n} \{ \langle p(t), v \rangle - \lambda h(t, z(t), v) \} \\ a.e. \ t \in [a, b]. \end{aligned}$$

□

**Corollary 3.3.1** *suppose in addition to the assumptions of Theorem 3.3.1 that  $g$  is locally Lipschitzian at  $(z(a), z(b))$  then Theorem 3.3.1 holds with  $\lambda > 0$  ( we can assume that  $\lambda = 1$  )*

**Proof** (See the Appendix).

Now we state the result by Vinter-Zheng [95] on necessary optimality conditions for a variational problem governed by nonconvex unbounded differential inclusions. The problem is of the form

$$\begin{aligned} \min_{x \in W^{1,1}} \{ & g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt \} & (Q_1) \\ & \text{subject to} \\ & \dot{x}(t) \in F(t, x(t)), \quad (x(a), x(b)) \in C \end{aligned}$$

where  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  are functions  $F : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a multivalued mapping and  $C \subset \mathbb{R}^n \times \mathbb{R}^n$  is a closed set.

In order to state necessary conditions for this problem, we need to put some assumptions. We suppose that for an arc  $z$  there exist  $\varepsilon > 0$ , integrable functions  $k, k_h : [a, b] \rightarrow \mathbf{R}$  and a constant  $k_g > 0$  such that

$$(H_g) \text{ For all } (x, y), (x', y') \in (z(a), z(b)) + \varepsilon(B \times B)$$

$$|g(x, y) - g(x', y')| \leq k_g \|(x, y) - (x', y')\|.$$

$$(H_F) F(t, x) \text{ is measurable in } t, \text{ closed-valued and for all } x, x' \in z(t) + \varepsilon B$$

$$F(t, x') \subset F(t, x) + k(t)\|x' - x\|B.$$

$$(H_h) h \text{ is epi-measurable in } t \text{ and for all } (x, v), (x', v') \in (z(t) + \varepsilon B) \times \mathbb{R}^n$$

$$|h(t, x, v) - h(t, x', v')| \leq k_h(t)\|(x, v) - (x', v')\|.$$

$$(H_k) k k_h \text{ is integrable .}$$

**Corollary 3.3.2** [95] *Let  $z$  be a local solution to the problem  $(Q_1)$ . Suppose that  $(H_g)$ ,  $(H_F)$ ,  $(H_h)$  and  $(H_k)$  are satisfied. Then there exist an arc  $p$  and  $\lambda \in \{0, 1\}$ , with  $(p, \lambda) \neq 0$ , such that*

$$\begin{aligned} \dot{p}(t) \in \text{co}\{ & q : (q, p(t)) \in \lambda \partial h(t, z(t), \dot{z}(t)) \\ & + N(\text{Gr}F(t, \cdot), (z(t), \dot{z}(t))) \} \text{ a.e. } t \in [a, b] \end{aligned}$$

$$(p(a), -p(b)) \in \lambda \partial g(z(a), z(b)) + N(C, (z(a), z(b)))$$

$$\begin{aligned} \langle p(t), \dot{z}(t) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \\ \max_{v \in F(t, z(t))} \{ \langle p(t), v \rangle - \lambda h(t, z(t), v) \} \text{ a.e. } t \in [a, b]. \end{aligned}$$

**Proof.** We reformulate the problem  $(Q_1)$  into the generalized Bolza problem  $(P_B)$  by setting

$$L(t, x, y) = h(t, x, y) + \psi_{GrF(t, \cdot)}(x, y)$$

$$l(x, y) = g(x, y) + \psi_C(x, y)$$

and let  $(Q'_1)$  be the following Bolza problem

$$\min_{x \in W^{1,1}} \{ l(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t)) dt \}.$$

Note that  $z$  is a local solution to  $(Q_1)$  iff  $z$  is a local solution to  $(Q'_1)$ . It is not difficult to see that all the hypotheses of Theorem 3.3.1 are satisfied. Thus there exist an arc  $p$  and  $\lambda \in \{0, 1\}$  such that  $(p, \lambda) \neq 0$  and

$$\begin{aligned} \dot{p}(t) \in \text{co} \{ q : (q, p(t), -\lambda) \in N(\text{epi } L(t, \cdot); ((z(t), \dot{z}(t)), L(t, z(t), \dot{z}(t)))) \} \\ \text{a.e. } t \in [a, b] \end{aligned} \quad (3.12)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } l; ((z(a), z(b)), l(z(a), z(b)))) \quad (3.13)$$

$$\begin{aligned} \langle p(t), \dot{z}(t) \rangle - \lambda L(t, z(t), \dot{z}(t)) = \\ \max_{v \in \mathbb{R}^n} \{ \langle p(t), v \rangle - \lambda L(t, z(t), v) \} \text{ a.e. } t \in [a, b]. \end{aligned} \quad (3.14)$$

From (3.12) we have

$$\begin{aligned} \dot{p}(t) \in \text{co} \{ q : (q, p(t), -\lambda) \in N(\text{epi } h(t, \cdot) \cap (GrF(t, \cdot) \times \mathbb{R}); \\ ((z(t), \dot{z}(t)), h(t, z(t), \dot{z}(t)))) \} \text{ a.e. } t \in [a, b]. \end{aligned}$$

Since  $h(t, \cdot)$  is locally Lipschitzian at  $(z(t), \dot{z}(t))$

$$\begin{aligned} \dot{p}(t) \in \text{co} \{ q : (q, p(t), -\lambda) \in N(\text{epi } h(t, \cdot); ((z(t), \dot{z}(t)), h(t, z(t), \dot{z}(t)))) \\ + N(GrF(t, \cdot); (z(t), \dot{z}(t))) \times \{0\} \} \text{ a.e. } t \in [a, b]. \end{aligned}$$

and the proof is complete by using (3.13) and (3.14).  $\square$

Now we may derive the result by Ioffe-Rockafellar [46] from our Theorem 3.3.1.

**Corollary 3.3.3** [46] *Let  $z$  be a local solution to the Bolza problem  $(P_B)$  where we assume that*

*i)  $g$  is l.s.c*

*ii)  $h$  is measurable in  $t$  and  $h(t, \cdot)$  is l.s.c*

*iii)  $h$  is finite-valued and for all  $N > 0$  there exist  $\varepsilon_N > 0$  and  $k_N \in L^1$  such that for all  $x, x' \in z(t) + \varepsilon_N B$  and  $v \in \dot{z}(t) + NB$*

$$|h(t, x', v) - h(t, x, v)| \leq k_N(t) \|x' - x\|.$$

*Then there exists an arc  $p$  satisfying assertions (3.5), (3.6) and (3.7) of Theorem 3.3.1 with  $\lambda = 1$ .*

**Proof.** Let  $N$  be an integer and consider the function  $L_N$  defined by

$$h_N(t, x, y) = \begin{cases} h(t, x, y) & \text{if } y \in \dot{z}(t) + NB \\ +\infty & \text{otherwise.} \end{cases}$$

We consider the following Bolza problem

$$\min_{x \in W^{1,1}} \{g(x(a), x(b)) + \int_a^b h_N(t, x(t), \dot{x}(t)) dt\}. \quad (P_N)$$

Since  $z$  is a local solution to the problem  $(P_B)$ , then it is a local solution to  $(P_N)$ . Note that  $h_N$  is epi-measurable in  $t$  and epi-Lipschitzian at  $z$  and that  $g$  is l.s.c. Then from Theorem 3.3.1, there exist an arc  $p_N$  and  $\lambda_N \in \{0, 1\}$  such that  $(p_N, \lambda_N) \neq 0$  and

$$\begin{aligned} \dot{p}_N(t) \in \text{co}\{q : (q, p_N(t), -\lambda_N) \in N(\text{epi } h_N(t, \cdot); ((z(t), \dot{z}(t)), \\ , h_N(t, z(t), \dot{z}(t))))\} \text{ a.e. } t \in [a, b] \end{aligned} \quad (1_N)$$

$$(p_N(a), -p_N(b), -\lambda_N) \in N(\text{epi } g; ((z(a), z(b)), g(z(a), z(b)))) \quad (2_N)$$

$$\begin{aligned} \langle p_N(t), \dot{z}(t) \rangle - \lambda_N h_N(t, z(t), \dot{z}(t)) = \max_{v \in \mathbb{R}^n} \{\langle p_N(t), v \rangle - \lambda_N h_N(t, z(t), v)\} \\ \text{a.e. } t \in [a, b]. \end{aligned} \quad (3_N)$$

We claim that  $\lambda_N = 1$ . Indeed suppose  $\lambda_N = 0$ . Then, since  $h_N$  is finite-valued on  $\dot{z}(t) + NB$ , assertion 3<sub>N</sub>) yields

$$\langle p_N(t), \dot{z}(t) \rangle = \max_{v \in \dot{z}(t) + NB} \langle p_N(t), v \rangle \text{ a.e. } t \in [a, b]$$

and hence  $p_N(t) = 0$  a.e.  $t \in [a, b]$ , or equivalently  $p_N = 0$  (because of the continuity of  $p_N$ ). This contradicts the fact that  $(p_N, \lambda_N) \neq 0$ . So that

relations  $(1_N)$ ,  $(2_N)$  and  $(3_N)$  hold with  $\lambda_N = 1$ . Moreover in  $1_N$ ) the function  $h_N$  can be replaced by  $h$ .

First we show that for almost all  $t \in [a, b]$  the sequence  $(p_N(t))$  is bounded. Indeed, consider the sets

$$I_N = \{t \in [a, b] : \langle p_N(t), \dot{z}(t) \rangle - h(t, z(t), \dot{z}(t)) = \max_{v \in \dot{z}(t) + NB} \{ \langle p_N(t), v \rangle - h(t, z(t), v) \}$$

and  $I = \bigcap_N I_N$ . Suppose that there exists  $t_0 \in I$  such that  $(p_N(t_0))_N$  is not bounded. Then, extracting subsequence, we may suppose that  $\|p_N(t_0)\| \rightarrow +\infty$ . Since the sequence  $(\frac{p_N(t_0)}{\|p_N(t_0)\|})_N$  is bounded, we can assume that it converges to  $w \in \mathbb{R}^n$ , with  $\|w\| = 1$ . As

$$\langle \frac{p_N(t_0)}{\|p_N(t_0)\|}, \dot{z}(t_0) \rangle \geq \langle \frac{p_N(t_0)}{\|p_N(t_0)\|}, v \rangle \quad \forall v \in \dot{z}(t_0) + B$$

and by passing to the limit we get

$$\langle w, \dot{z}(t_0) \rangle \geq \langle w, v \rangle \quad \forall v \in \dot{z}(t_0) + B$$

or equivalently  $w = 0$ . This contradiction implies that  $(p_N(t))_N$  is bounded for all  $t \in I$ . Now from the assumption *iii*) and the assertion  $(1_N)$ , it follows that

$$\|\dot{p}_N(t)\| \leq k_1(t) \text{ a.e. } t \in [a, b]$$

and hence for all  $t \in I$  we have

$$\|p_N(a)\| - \|p_N(t)\| \leq \|p_N(a) - p_N(t)\| \leq \int_a^t k_1(s) ds$$

so that

$$\|p_N(a)\| \leq \|p_N(t)\| + \int_a^t k_1(s) ds.$$

As  $(p_N(t))_N$  is bounded for all  $t \in I$  then  $(p_N(a))_N$  is also bounded.

Consider the multivalued mapping  $\Gamma$  defined by

$$\Gamma(t, \theta) = \text{co}\{q : (q, \theta) \in \partial h(t, z(t), \dot{z}(t))\}.$$

Then  $\Gamma$  is integrably bounded, nonempty, compact and convex-valued, and measurable in  $t$ . Moreover for almost all  $t$ , the graph of  $\Gamma(t, \cdot)$  is closed and hence  $\Gamma(t, \cdot)$  is upper semicontinuous. Then, by Theorem 3.1.7 in [22], there



exists a subsequence of  $\{p_N\}_N$  converging uniformly to an arc  $p$  which is a trajectory for  $\Gamma$ , and by passing to the limit (for this subsequence) in  $2_N$  and  $3_N$ ) we obtain (3.6) and (3.7) of Theorem 3.3.1 with  $\lambda = 1$  and the proof is terminated.  $\square$

**Remak 3.3.1** *Note that in [46], the authors assumed that, in addition to the assumptions of Corollary 3.3.3,*

$$h(t, z(t), v) \geq -k(t) \forall v \in z(t) + NB \text{ a.e. } t \in [a, b].$$

### 3.4 The maximum principle

Our goal in this section is to give a simple proof of the maximum principle using Theorem 3.3.1 and to obtain a new Euler-Lagrange inclusion for optimal control problem.

To every integrable map  $u(t)$  taking values in a given set  $U(t)$  of  $\mathbb{R}^m$  we associate the solution  $x$  to the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)).$$

We have to minimize over all such pairs  $(x, u)$  the functional

$$g(x(a), x(b)) + \int_a^b h(t, x(t), u(t)) dt \quad (R_1)$$

where  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ , and  $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  are mappings.

We shall assume the following assumptions :

- a)  $f(t, x, u)$  is measurable in  $t$  and continuous in  $u$
- b)  $h(t, \cdot)$  is l.s.c and measurable in  $t$
- c)  $g$  is l.s.c
- d)  $U$  is measurable and closed-valued .

**Theorem 3.4.1** *Suppose  $(z, v)$  solves locally the optimal control problem  $(R_1)$  and that there exist  $\varepsilon > 0$  and an integrable function  $k : [a, b] \rightarrow \mathbb{R}$  such that for almost all  $t \in [a, b]$  and for all  $z_1, z_2 \in z(t) + \varepsilon B$ ,  $u \in U(t)$  we have*

$$\begin{aligned} \|f(t, z_1, u) - f(t, z_2, u)\| &\leq k(t)\|z_1 - z_2\| \\ |h(t, z_1, u) - h(t, z_2, u)| &\leq k(t)\|z_1 - z_2\|. \end{aligned}$$

Then there exist an arc  $p$  and  $\lambda \in \{0, 1\}$  such that  $(p, \lambda) \neq (0, 0)$  and

$$\dot{p}(t) \in \text{co}\partial_x \{\lambda h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle\}(z(t), v(t)) \text{ a.e. } t \in [a, b] \quad (3.15)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), g(z(a), z(b)))) \quad (3.16)$$

$$\begin{aligned} \langle p(t), f(t, z(t), v(t)) \rangle - \lambda h(t, z(t), v(t)) = \\ \max_{u \in U(t)} \{\langle p(t), f(t, z(t), u) \rangle - \lambda h(t, z(t), u)\} \text{ a.e. } t \in [a, b]. \end{aligned} \quad (3.17)$$

If in addition  $f(t, \cdot)$  and  $h(t, \cdot)$  are locally Lipschitzian around  $(z(t), v(t))$  then the Euler-Lagrange inclusion may be replaced by the following one

$$\begin{aligned} \dot{p}(t) \in \text{co}\{q : (q, 0) \in \partial[\lambda h(t, \cdot) - \langle p(t), f(t, \cdot) \rangle](z(t), v(t)) + \\ \{0\} \times N(U(t), v(t))\} \text{ a.e. } t \in [a, b]. \end{aligned} \quad (3.18)$$

**Proof.** We reframe problem  $(R_1)$  to a Bolza problem. Let  $l : (\mathbb{R}^n \times \mathbb{R}^m)^2 \longrightarrow \mathbb{R} \cup \{+\infty\}$  and  $L : [a, b] \times (\mathbb{R}^n \times \mathbb{R}^m)^2 \longrightarrow \mathbb{R} \cup \{+\infty\}$  be functions defined by

$$\begin{aligned} l(s, s', u, r) &= g(s, u) \\ L(t, s, s', u, u') &= \begin{cases} h(t, s, u') & \text{if } u = f(t, s, u') \text{ and } u' \in U(t) \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

and set

$$\omega(t) = \int_a^t v(s) ds.$$

Consider the following Bolza problem (denoted by  $(P'_B)$ )

$$\min \{l(x(a), u(a), x(b), u(b)) + \int_a^b L(t, x(t), u(t), \dot{x}(t), \dot{u}(t)) dt\}$$

over all  $(x, u) \in W^{1,1}([a, b], \mathbb{R}^n) \times W^{1,1}([a, b], \mathbb{R}^m)$ . It is easy to check that  $(z, v)$  is a local solution to  $(R_1)$  implies  $(z, w)$  is a local solution to  $(P'_B)$ . Notice that  $L(t, \cdot)$  and  $l$  are l.s.c,  $L$  is epi-Lipschitzian at  $(z, w)$  and epi-measurable in  $t$ . Therefore the assumptions of Theorem 3.3.1 are fulfilled. Then, by this theorem, there exist an arc  $(p, r)$  and  $\lambda \in \{0, 1\}$  such that  $(p, r, \lambda) \neq 0$  and

$$\begin{aligned} (\dot{p}(t), \dot{r}(t)) \in \text{co}\{(q, q') : (q, q', p(t), r(t), -\lambda) \in \\ N(\text{epi } L(t, \cdot); ((z(t), w(t), \dot{z}(t), \dot{w}(t)); L(t, z(t), w(t), \dot{z}(t), \dot{w}(t))))\} \\ \text{a.e. } t \in [a, b] \end{aligned} \quad (3.19)$$

$$(p(a), r(a), -p(b), -r(b), -\lambda) \in N(\text{epi } l; ((z(a), w(a), z(b), w(b)); l(z(a), w(a), z(b), w(b)))) \quad (3.20)$$

$$\begin{aligned} \langle (p(t), r(t)), (\dot{z}(t), \dot{w}(t)) \rangle - \lambda L(t, z(t), w(t), \dot{z}(t), \dot{w}(t)) = \\ \max_{(\theta, v') \in \mathbb{R}^n \times \mathbb{R}^m} \{ \langle (p(t), r(t)), (\theta, v') \rangle - \lambda L(t, z(t), w(t), \theta, v') \} \\ \text{a.e. } t \in [a, b]. \quad (3.21) \end{aligned}$$

Now it suffices to use the definition of the limiting Fréchet subdifferential and the definitions of  $l$  and  $L$  to get the conclusion of the theorem.  $\square$

### 3.5 Necessary optimality conditions for general isoperimetric problems

In this section we give new necessary optimality conditions for general isoperimetric problems using Theorem 3.4.1.

The problems that we consider here are of the form

$$\min \left\{ g(x(a), x(b)) + \int_a^b h(t, x(t), \dot{x}(t)) dt \right\} \quad (I_s)$$

subject to

$$\int_a^b f(t, x(t), \dot{x}(t)) dt \in K, \quad \dot{x}(t) \in U(t) \text{ a.e. } t \in [a, b].$$

where  $K \subset \mathbb{R}^m$  is a closed set,  $U$  is a multivalued mapping from  $[a, b]$  to  $\mathbb{R}^n$  and  $g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  are functions.

We shall assume the following assumptions :

- a)  $f(t, x, u)$  is measurable in  $t$  and continuous in  $u$
- b)  $h(t, \cdot)$  is l.s.c and measurable in  $t$
- c)  $g$  is l.s.c
- d)  $U$  is measurable and closed-valued .

**Theorem 3.5.1** *Let  $z$  solves locally the problem  $(I_s)$ . Suppose that assumptions a), b), c) and d) are satisfied and that there exist  $\varepsilon > 0$  and an integrable function  $k : [a, b] \rightarrow \mathbb{R}$  such that for almost all  $t \in [a, b]$  and for all  $z_1, z_2 \in z(t) + \varepsilon B$ ,  $u \in U(t)$  we have*

$$\|f(t, z_1, u) - f(t, z_2, u)\| \leq k(t) \|z_1 - z_2\|$$

$$|h(t, z_1, u) - h(t, z_2, u)| \leq k(t) \|z_1 - z_2\|.$$

Then there exist an arc  $p$ ,  $\lambda \in \{0, 1\}$  and a vector  $\gamma \in N(K; \int_a^b f(t, z(t), \dot{z}(t)) dt)$  such that  $(p, \gamma, \lambda) \neq 0$  and

$$\dot{p}(t) \in \text{co}\partial_x \{ \lambda h(t, \cdot) - \langle \gamma, f(t, \cdot) \rangle \} (z(t), \dot{z}(t)) \text{ a.e. } t \in [a, b] \quad (3.22)$$

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g; ((z(a), z(b)), g(z(a), z(b)))) \quad (3.23)$$

$$\begin{aligned} \langle p(t), \dot{z}(t) \rangle + \langle \gamma, f(t, z(t), \dot{z}(t)) \rangle - \lambda h(t, z(t), \dot{z}(t)) = \\ \max_{v \in U(t)} \{ \langle p(t), v \rangle + \langle \gamma, f(t, z(t), v) \rangle - \lambda h(t, z(t), v) \} \\ \text{a.e. } t \in [a, b]. \end{aligned} \quad (3.24)$$

If in addition  $f(t, \cdot)$  and  $h(t, \cdot)$  are locally Lipschitzian around  $(z(t), \dot{z}(t))$  then the Euler-Lagrange inclusion (3.22) may be replaced by :

$$\begin{aligned} \dot{p}(t) \in \text{co}\{q : (q, p(t)) \in \partial[\lambda h(t, \cdot) - \langle \gamma, f(t, \cdot) \rangle](z(t), v(t)) + \\ \{0\} \times N(U(t), \dot{z}(t))\} \text{ a.e. } t \in [a, b]. \end{aligned} \quad (3.25)$$

**Proof.** We reframe the problem  $(I_s)$  to one of  $\mathbb{R}^{n+m}$  by the following definitions  $((x, v)$  will represent a point in  $\mathbb{R}^{n+m}$ ). Set  $\Delta = \{(v, v') \in \mathbb{R}^m \times \mathbb{R}^m : v' - v \in K\}$

$$l(x, v, x', v') = g(x, x') + \psi_\Delta(v, v')$$

$$L(t, x, v, w) = h(t, x, w)$$

$$J(t, x, v, w) = (w, f(t, x, w)).$$

Consider the following optimal control problem, which we denote  $(I'_s)$ , minimize over the arcs  $(x, v, w)$  the functional

$$l(x(a), v(a), x(b), v(b)) + \int_a^b L(t, x(t), v(t), w(t)) dt$$

subject to

$$(\dot{x}(t), \dot{v}(t)) = J(t, x(t), v(t), w(t)), \quad w(t) \in U(t) \text{ a.e. } t \in [a, b].$$

Note that  $z$  is a local solution to the problem  $(I_s)$  iff  $(z, \bar{v}, u)$  is a local solution to the problem  $(I'_s)$ , where  $\bar{v}(t) = \bar{v}(a) + \int_a^t f(s, z(s), \dot{z}(s)) ds$  and  $u(t) = \dot{z}(t)$ .

It is easy to check that :

- $J$  is measurable in  $t$  and continuous in  $w$  ;
- $L$  is measurable in  $t$  and  $L(t, \cdot)$  is l.s.c. ;
- $l$  is l.s.c.

Note that for almost all  $t \in [a, b]$ , for all  $z_1, z_2 \in z(t) + \varepsilon B$ ,  $v_1, v_2 \in \mathbb{R}^m$  and  $w \in U(t)$  we have

$$\|J(t, z_1, v_1, w) - J(t, z_2, v_2, w)\| \leq k(t)(\|z_1 - z_2\| + \|v_1 - v_2\|)$$

$$|L(t, z_1, v_1, w) - L(t, z_2, v_2, w)| \leq k(t)(\|z_1 - z_2\| + \|v_1 - v_2\|).$$

We are now in position to apply Theorem 3.4.1. So there exist an arc  $p$ , an arc  $q$  and  $\lambda \in \{0, 1\}$ , with  $(p, q, \lambda) \neq 0$ , such that

$$(\dot{p}(t), \dot{q}(t)) \in \text{co}\partial_{(x,v)}\{\lambda L(t, \cdot) - \langle (p(t), q(t)), J(t, \cdot) \rangle\}(z(t), \bar{v}(t), u(t)) \quad a.e. t \in [a, b] \quad (3.26)$$

$$(p(a), q(a), -p(b), -q(b), -\lambda) \in N(\text{epi } l; ((z(a), \bar{v}(a), z(b), \bar{v}(b)), l(z(a), \bar{v}(a), z(b), \bar{v}(b)))) \quad (3.27)$$

$$\begin{aligned} & \langle (p(t), q(t)), J(t, z(t), \bar{v}(t), u(t)) \rangle - \lambda L(t, z(t), \bar{v}(t), u(t)) = \\ & \max_{v \in U(t)} \{ \langle (p(t), q(t)), J(t, z(t), \bar{v}(t), v) \rangle - \lambda L(t, z(t), \bar{v}(t), v) \} \\ & \quad a.e. t \in [a, b]. \quad (3.28) \end{aligned}$$

From (3.26) we have  $\dot{q}(t) = 0$  and hence  $q$  is a constant mapping. We set  $q = \gamma$ . Then

$$\dot{p}(t) \in \text{co}\partial_x \{ \lambda h(t, \cdot) - \langle \gamma, f(t, \cdot) \rangle \}(z(t), \dot{z}(t)), \quad a.e. t \in [a, b].$$

From (3.27) and the formula

$$N(\Delta, (v, v')) \subset \{(v^*, -v^*) : v^* \in N(K, v - v')\}$$

we get

$$(p(a), -p(b), -\lambda) \in N(\text{epi } g, ((z(a), z(b)), g(z(a), z(b))))$$

and

$$\gamma \in N(K, \int_a^b f(s, z(s), \dot{z}(s)) ds).$$

If in addition  $f(t, \cdot)$  and  $h(t, \cdot)$  are locally Lipschitzian around  $(z(t), \dot{z}(t))$  then  $J(t, \cdot)$  and  $L(t, \cdot)$  are locally Lipschitzian around  $(z(t), \bar{v}(t), u(t))$ , and Theorem 3.4.1 yields

$$\begin{aligned} (\dot{p}(t), 0) \in \text{co} \{ (q_1, q_2) : (q_1, q_2, 0) \in \\ \partial[\lambda L(t, \cdot) - \langle (p(t), \gamma), J(t, \cdot) \rangle](z(t), \bar{v}(t), u(t))] + \{0\} \times N(U(t), u(t)) \} \\ \text{a.e. } t \in [a, b] \end{aligned}$$

and (3.25) follows from a simple computation of the subdifferential.  $\square$

### 3.6 Application to some economic problems

In this section we shall illustrate how the maximum principle is used to study general economic problems. We consider here an economy in which  $n$ -goods are produced with the aid of capitals  $x_1(t), \dots, x_n(t)$  which may depend on time  $t$  and in which the total outputs  $y_1(t), \dots, y_n(t)$  are either consumed or invested with possible deterioration or depreciation. Thus if  $I_1(t), \dots, I_n(t)$  denote the rates of investment of the  $n$ -commodities and if  $c_1(t), \dots, c_n(t)$  are the corresponding consumptions, we have

$$y_i(t) = c_i(t) + I_i(t) \quad i = 1, \dots, n$$

We assume that the production  $y_i(t)$  depend on the capitals  $x_1(t), \dots, x_n(t)$  at time  $t$ , and that  $y_i(t)$  is a known function  $\varrho_i(t, x_1(t), \dots, x_n(t))$ . We suppose also that the investment  $I_i(t)$  is proportional to the variation of the capital  $x_i(t)$ , that is

$$I_i(t) = \varphi_i(t)\dot{x}_i(t)$$

where  $\varphi_i(t)$  can be considered as a factor of deterioration or depreciation. Thus for each commodity  $i$  we have

$$\varrho_i(t, x_1(t), \dots, x_n(t)) = c_i(t) + \varphi_i(t)\dot{x}_i(t) \text{ a.e. } t \in [a, b].$$

Another point of view is to assume that investment  $I_i(t)$  is used both to augment the stock of capital  $x_i(t)$ , and to replace depreciated capital, that is,

$$I_i(t) = \psi_i(t)x_i(t) + \varphi_i(t)\dot{x}_i(t), \text{ a.e. } t \in [a, b].$$

But in this case, it is sufficient to replace  $\varrho_i(t, x(t))$  by  $\varrho_i(t, x(t)) + \psi_i(t)x_i(t)$ . Since the economic objective of any planing concerns the standard of living, we assume that an utility function  $-u(t, c_1(t), \dots, c_n(t))$  is known which

measures the instantaneous well-being of the economy and that the value of the end capital  $-g(x_1(b), \dots, x_n(b))$  is given. As any planing we should try to increase the consumption and the capital, i.e., to maximize the functional

$$\int_a^b -u(t, c_1(t), \dots, c_n(t)) dt - g(x_1(b), \dots, x_n(b))$$

or to minimize the functional

$$\int_a^b u(t, c_1(t), \dots, c_n(t)) dt + g(x_1(b), \dots, x_n(b))$$

where  $\int_a^b -u(t, c_1(t), \dots, c_n(t)) dt$  is the global utility function. We formulate this problem as a control problem. Let  $x(t) = (x_1(t), \dots, x_n(t))$  denotes the rate of capitals at time  $t$ , let  $c(t) = (c_1(t), \dots, c_n(t))$  denotes consumptions at time  $t$ . Define new functionals  $f_i : [a, b] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$f_i(t, x, c) = \frac{c_i}{\varphi_i(t)} - \frac{\varrho_i(t, x)}{\varphi_i(t)}$$

$f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$f(t, x, c) = (f_1(t, x, c_1), \dots, f_n(t, x, c_n))$$

and  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$h(t, x, c) = u(t, c).$$

Our economic problem is equivalent to the following optimal control problem

$$\min_{(x, c)} \left\{ g(x(b)) + \int_a^b h(t, x(t), c(t)) dt \right\} \quad (P_\epsilon)$$

subject to

$$\dot{x}(t) = f(t, x(t), c(t)) \quad a.e. \quad t \in [a, b]$$

$$c(t) \geq 0 \quad a.e. \quad t \in [a, b]$$

$$x(t) > 0 \quad \forall t \in [a, b]$$

$$x(a) = \alpha$$

where  $\alpha$  is the initial rate of capital. In order to study our problem we impose assumptions similar to those in Theorem 3.4.1. Namely :

- $\varrho_i(t, x)$  is measurable in  $t$  for all  $i = 1, \dots, n$ .
- For all  $i = 1, \dots, n$ ,  $\varphi_i$  is measurable and  $\varphi_i(t)$  not take 0 for all  $t \in [a, b]$ .
- $u(t, \cdot)$  is l.s.c and measurable in  $t$ .
- $g$  is l.s.c.

**Theorem 3.6.1** *Let  $(z, v)$  solves locally the control problem  $(P_e)$ . Suppose that either the consumptions  $v_i$ ,  $i = 1, \dots, n$ , are positive or the factors  $\varphi_i$ ,  $i = 1, \dots, n$ , are positive almost every where on  $[a, b]$ . Suppose also that there exist  $\varepsilon > 0$  and integrable functions  $k_i : [a, b] \rightarrow \mathbb{R}$ , with  $\max(\frac{k_i}{|\varphi_i|})$  integrable, such that for almost all  $t \in [a, b]$  and for all  $z', z'' \in z(t) + \varepsilon B$ , and for all  $c \in \mathbb{R}_+^n$  and for all  $i = 1, \dots, n$  we have*

$$\|\varrho_i(t, z') - \varrho_i(t, z'')\| \leq k_i(t)\|z' - z''\|.$$

Then there exist an arc  $p$  and  $\lambda \in \{0, 1\}$  such that  $(p, \lambda) \neq 0$  and

$$\begin{aligned} \dot{p}(t) \in \text{co}\partial \left\{ \sum_{i=1}^n \frac{p_i(t)\varrho_i(t, \cdot)}{\varphi_i(t)} \right\} (z(t)) \quad \text{a.e. } t \in [a, b] \\ (-p(b), -\lambda) \in N(\text{epi } g, (z(b), g(z(b)))) \end{aligned} \quad (3.29)$$

$$\begin{aligned} \sum_{i=1}^n \frac{p_i(t)v_i(t)}{\varphi_i(t)} - \lambda u(t, v(t)) = \\ \max_{c \in \mathbb{R}_+^n} \left\{ \sum_{i=1}^n \frac{p_i(t)c_i}{\varphi_i(t)} - \lambda u(t, c) \right\} \quad \text{a.e. } t \in [a, b]. \end{aligned} \quad (3.30)$$

If in addition  $g$  is locally Lipschitzian around  $z(b)$  then  $\lambda$  may be chosen equal to 1.

**Proof.** Notice that  $(z, v)$  is a local solution of  $(P_e)$  iff  $(z, v)$  is a local solution to the following problem

$$\min_{(x, c)} \left\{ g(x(b)) + \int_a^b h(t, x(t), c(t)) dt \right\} \quad (P'_e)$$



subject to

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), c(t)) \text{ a.e. } t \in [a, b] \\ c(t) &\geq 0 \text{ a.e. } t \in [a, b] \\ x(a) &= \alpha. \end{aligned}$$

Indeed, it suffices to see that if  $z_i(t) > 0$  for all  $i \in \{1, \dots, n\}$  and  $t \in [a, b]$ , then for  $\beta = \min_i \min_t z_i(t)$ , we have  $\beta > 0$  and for each  $0 < r \leq \frac{\beta}{2}$  and each  $x \in B(z, r)$  we get  $x_i(t) > \frac{\beta}{2}$ . It remains now to apply Theorem 3.4.1. If  $g$  is locally Lipschitzian around  $z(b)$ , then  $\lambda = 1$ . Indeed, if  $\lambda = 0$ , we obtain, by using (3.29), that  $p(b) = 0$ . From (3.30) we have for all  $i = 1, \dots, n$

$$\frac{p_i(t)v_i(t)}{\varphi_i(t)} = \max_{c_i \in \mathbb{R}_+} \frac{p_i(t)c_i}{\varphi_i(t)} \text{ a.e.}$$

or equivalently  $p_i(t)v_i(t) = 0$  and  $\frac{p_i(t)}{\varphi_i(t)} \leq 0$  a.e.. So if  $v_i$ ,  $i = 1, \dots, n$ , are positive almost every where on  $[a, b]$  then  $p = 0$ . Now if  $\varphi_i$ ,  $i = 1, \dots, n$ , are positive almost every where on  $[a, b]$  then  $p_i(t) \leq 0$  for all  $i$  and  $t$ . Formula(3.29) implies

$$\|\dot{p}(t)\| \leq k(t)\|p(t)\|$$

where  $k(t) = \max_i \frac{k_i(t)}{|\varphi_i(t)|}$  and hence

$$\sum_{i=1}^n |\dot{p}_i(t)| \leq nk(t) \sum_{i=1}^n |p_i(t)|.$$

So that

$$\sum_{i=1}^n \dot{p}_i(t) + nk(t) \sum_{i=1}^n p_i(t) \leq 0.$$

Let

$$v(t) = \sum_{i=1}^n p_i(t) \exp \left( n \int_a^t k(s) ds \right).$$

Then we have

$$\dot{v}(t) = \sum_{i=1}^n (\dot{p}_i(t) + nk(t)p_i(t)) \exp n \int_a^t k(s) ds.$$

Thus for all  $t \in [a, b]$

$$\int_t^b \dot{v}(s) ds \leq 0$$

and hence  $v(b) - v(t) \leq 0$  and  $v(t) \geq 0$ . So that  $p_i(t) \geq 0$  for all  $t \in [a, b]$ . Finally we have  $p_i(t) = 0$  for all  $t \in [a, b]$  which is in contradiction with  $(p, \lambda) \neq 0$ .  $\square$

Now we apply the previous economic problem to the Ramsey Model of economic growth. In this model it is assumed that all the commodities produced in a given period must either be consumed or invested without deterioration or depreciation, i.e.,

$$I_i(t) = \dot{x}_i(t), \text{ a.e. } t \in [a, b].$$

We assume that productions  $y_i(t)$  are known functions  $\varrho_i(t, x_1(t), \dots, x_n(t))$ . Thus

$$\dot{x}_i(t) = c_i(t) - \varrho_i(t, x_1(t), \dots, x_n(t)).$$

The objective of any planing is to maximize the utility functional

$$\int_a^b -u(t, c_1(t), \dots, c_n(t)) dt$$

or to minimize the functional

$$\int_a^b u(t, c_1(t), \dots, c_n(t)) dt \tag{R_m}$$

over all pairs  $(x, c)$  satisfying

$$\begin{aligned} \dot{x}(t) &= c(t) - f(t, x(t), c(t)) \quad \text{a.e. } t \in [a, b] \\ c(t) &\geq 0 \quad \text{a.e. } t \in [a, b] \\ x(t) &> 0 \quad \forall t \in [a, b] \\ x(a) &= \alpha \end{aligned}$$

where  $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$  is the functional defined by

$$f(t, x, c) = (\varrho_1(t, x, c_1), \dots, \varrho_n(t, x, c_n))$$

and  $\alpha$  is the initial rate of capital.

We assume assumptions similar to those imposed in Theorem 3.6.1, i.e.,

- $\varrho_i(t, x)$  is measurable in  $t$  for all  $i = 1, \dots, n$ ;
- $u(t, \cdot)$  is l.s.c and measurable in  $t$ .

**Theorem 3.6.2** *Suppose  $(z, v)$  solves locally the control problem  $(R_m)$  and that there exist  $\varepsilon > 0$  and an integrable function  $k : [a, b] \rightarrow \mathbb{R}$  such that almost all  $t \in [a, b]$  and for all  $z', z'' \in z(t) + \varepsilon B$ ,  $c \in \mathbb{R}_+^n$  and for all  $i = 1, \dots, n$  we have*

$$\|\varrho_i(t, z') - \varrho_i(t, z'')\| \leq k(t)\|z' - z''\|.$$

Then there exists an arc  $p$  such that

$$\dot{p}(t) \in \text{co}\partial \left\{ \sum_{i=1}^n p_i(t) \varrho_i(t, \cdot) \right\} (z(t)) \quad \text{a.e. } t \in [a, b] \quad (3.31)$$

$$p(b) = 0 \quad (3.32)$$

$$\langle p(t), v(t) \rangle - u(t, v(t)) = \max_{c \in \mathbb{R}_+^n} \{ \langle p(t), c \rangle - u(t, c) \} \quad (3.33)$$

$$\text{a.e. } t \in [a, b].$$

### 3.7 Application to a chemical problem

Our result may also be applied in Chemical Engineering. Indeed, let  $x_1(t), \dots, x_n(t)$  denote the concentrations at time  $t$  of  $n$  substances in a reactor in which  $n$  simultaneous chemical reactions are taking place. Let the rates of the reactions be governed by a system of differential equations

$$\begin{cases} \dot{x}_i(t) = f_i(t, x_1(t), \dots, x_n(t), \theta(t), \gamma(t)) & i = 1, \dots, n \\ x_i(0) = x_i^0 \end{cases} \quad (3.34)$$

where  $x_i^0$  is the initial concentration of the substance  $i$ ,  $\theta(t)$  is the temperature and  $\gamma(t)$  is the pressure in the reactor at time  $t$ . We can control the temperature and the pressure at each instant of time, subject to the constraints

$$\theta_0 \leq \theta(t) \leq \theta_1$$

$$\gamma_0 \leq \gamma(t) \leq \gamma_1$$

where  $\theta_0, \theta_1, \gamma_0, \gamma_1$  are constants. These represent the minimum and the maximum attainable temperature and pressure.

Note that the functional  $f_i$  describing the evolution is not assumed to be smooth. We let the reaction proceed for a time  $T$ . The concentrations at

this time are  $x_1(T), \dots, x_n(T)$ . Associated with each product is an economic function, or price function  $-g_i(x_i(T))$ .

The value of the end product is

$$\sum_{i=1}^n -g_i(x_i(T)).$$

The problem here is to chose piecewise continuous functions  $\theta$  and  $\gamma$  on the intervalle  $[0, T]$  then (3.34) is satisfied and so that  $\sum_{i=1}^n -g_i(x_i(T))$  is maximized or equivalently  $\sum_{i=1}^n g_i(x_i(T))$  is minimized, i.e.,

$$\min \sum_{i=1}^n g_i(x_i(T)) \quad (P_c)$$

over all pairs  $(x, \theta, \gamma)$  satisfying

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), \theta(t), \gamma(t)) \\ (\theta(t), \gamma(t)) &\in \Pi \\ x(0) &= x^0 \end{aligned}$$

where  $\Pi = \{(\theta, \gamma) \in \mathbb{R} \times \mathbb{R} : \theta_0 \leq \theta \leq \theta_1 \text{ and } \gamma_0 \leq \gamma \leq \gamma_1\}$ . and  $f$  is the function defined by

$$f(t, x, \theta, \gamma) = (f_1(t, x, \theta, \gamma), \dots, f_n(t, x, \theta, \gamma)).$$

We assume that :

For all  $i \in \{1, \dots, n\}$ ,  $f_i(t, x, \theta, \gamma)$  is measurable in  $t$  and continuous in  $(\theta, \gamma)$  and  $g_i$  is l.s.c .

**Theorem 3.7.1** *Suppose  $(z, \bar{\theta}, \bar{\gamma})$  solves locally the control problem  $(P_c)$  and that there exist  $\varepsilon > 0$  and an integrable function  $k : [a, b] \rightarrow \mathbb{R}$  such that for almost all  $t \in [0, T]$  and for all  $z', z'' \in z(t) + \varepsilon B$  ,  $(\theta, \gamma) \in \Pi$  and for all  $i \in \{1, \dots, n\}$  we have*

$$\|f_i(t, z', \theta, \gamma) - f_i(t, z'', \theta, \gamma)\| \leq k(t)\|z' - z''\|.$$

*Then there exist an arc  $p$  and  $\lambda \in \{0, 1\}$  such that  $(p, \lambda) \neq 0$  and*

$$\dot{p}(t) \in \text{co}\partial_x \left\{ - \sum_{i=1}^n \langle p_i(t), f_i(t, \cdot) \rangle \right\} (z(t), \bar{\theta}(t), \bar{\gamma}(t)) \quad (3.35)$$

*a.e  $t \in [0, T]$*

$$(-p_i(T), -\lambda) \in N(\text{epi } g_i; (z_i(T), g_i(z_i(T)))) , i = 1, \dots, n \quad (3.36)$$

$$\begin{aligned} \sum_{i=1}^n \langle p_i(t), f_i(t, z(t), \bar{\theta}(t), \bar{\gamma}(t)) \rangle = \\ \max_{(v', v'') \in \Pi(t)} \left\{ \sum_{i=1}^n \langle p_i(t), f_i(t, z(t), v', v'') \rangle \right\} \text{ a.e } t \in [0, T]. \end{aligned}$$

### 3.8 Appendix 1

#### • Proof of Corollary 3.3.1

We need the following Lemma to proof Corollary 3.3.1.

**Lemma 3.8.1** *let  $h : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be epi-Lipschitzian in  $z(t)$  and  $(q, p, \alpha) \in N(\text{epih}(t, \cdot); (z(t), \dot{z}(t)); h(t, z(t), \dot{z}(t)))$ , then*

$$\| q \| \leq k(t)(\| p \| + \| \alpha \|).$$

**Proof of Lemma 3.8.1**  $(q, p, \alpha) \in N(\text{epih}(t, \cdot); (z(t), \dot{z}(t)); L(t, z(t), \dot{z}(t)))$  then using the definition of normal cone limit Fréchet, there exist a sequences  $\{(q_k, p_k, \alpha_k)\}_k$ ,  $\{(x_k, y_k, \gamma_k)\}_k$ ,  $(\epsilon_k)_k$  and  $(\beta_k)_k$  such that

$$\begin{aligned} (q_k, p_k, \alpha_k) &\longrightarrow (q, p, \alpha) \\ (x_k, y_k, \gamma_k) &\xrightarrow{\text{epi } h} (z(t), \dot{z}(t), h(t, z(t), \dot{z}(t))) \\ \epsilon_k &\longrightarrow 0 \\ \beta_k &\longrightarrow 0^+ \end{aligned}$$

and

$$\begin{aligned} - \langle q_k, x - x_k \rangle - \langle p_k, y - y_k \rangle - \alpha_k(\gamma - \gamma_k) \\ + \epsilon_k(\| x - x_k \| + \| y - y_k \| + | \gamma - \gamma_k |) \geq 0 \\ \forall (x, y, \gamma) \in B(x_k, \beta_k) \times B(y_k, \beta_k) \times B(\gamma_k, \beta_k) \cap \text{epi } h(t, \cdot). \quad (3.37) \end{aligned}$$

Take  $x = x_k$ ,  $y = y_k$  and  $x' \in B(x_k, \frac{\beta_k}{k(t)})$ . Since  $h$  is epi-Lipschitzian at  $z$ , then there exist  $y'_k \in \mathbb{R}^n$  and  $\delta_k \geq 0$  such that  $h(t, x', y'_k)$  is finite and

$$\| y_k - y'_k \| + | h(t, x', y'_k) + \delta_k - h(t, x_k, y_k) | \leq k(t) \| x_k - x' \|$$

so that  $\| y_k - y'_k \| \leq k(t) \| x_k - x' \|$  and  $\| y_k - y'_k \| \leq \beta_k$ . We have also  $h(t, x', y'_k) \leq h(t, x_k, y_k) + k(t) \| x_k - x' \|$ . As  $(x_k, y_k, \gamma_k) \in \text{epi } h(t, \cdot)$  then  $h(t, x_k, y_k) \leq \gamma_k$  and  $h(t, x', y'_k) \leq \gamma_k + k(t) \| x_k - x' \|$ . Set  $\gamma = \gamma_k + k(t) \| x_k - x' \|$ , we have  $(x', y'_k, \gamma) \in \text{epi } h(t, \cdot)$  et  $\gamma \in B(\gamma_k, \beta_k)$  from (3.37) we have

$$\begin{aligned} - \langle q_k, x' - x_k \rangle - \langle p_k, y'_k - y_k \rangle - \alpha_k(\gamma - \gamma_k) \\ + \epsilon_k(\| x' - x_k \| \\ + \| y'_k - y_k \| + | \gamma - \gamma_k |) \geq 0 \end{aligned}$$

then

$$\begin{aligned} \langle q_k, x' - x_k \rangle &\leq \| p_k \| \| y_k - y'_k \| + \| \alpha_k \| \| \gamma_k - \gamma \| + \epsilon_k(\| x' - x_k \| \\ &\quad + \| y'_k - y_k \| + | \gamma - \gamma_k |) \\ &\leq k(t) \| p_k \| \| x_k - x' \| + k(t) \| \alpha_k \| \| x_k - x' \| + \epsilon_k(\| x' - x_k \| \\ &\quad + k(t) \| x' - x_k \| + k(t) \| x' - x_k \|). \end{aligned}$$

Set  $m = \| x' - x_k \|$  then

$$\frac{\langle q_k, m \rangle}{\| m \|} \leq k(t)(\| p_k \| + \| \alpha_k \|) + \epsilon_k(1 + 2k(t))$$

so that

$$\sup_m \frac{\langle q_k, m \rangle}{\| m \|} \leq k(t)(\| p_k \| + \| \alpha_k \|) + \epsilon_k(1 + 2k(t))$$

and

$$\| q_k \| \leq k(t)(\| p_k \| + \| \alpha_k \|) + \epsilon_k(1 + 2k(t)).$$

Finally we have

$$\| q \| \leq k(t)(\| p \| + \| \alpha \|).$$

□

As a consequence of Lemma 3.8.1, if we suppose in Theorem 3.3.1 that  $\lambda = 0$ , then from (3.5) we have  $\dot{p}(t) = \sum_{i=1}^{n+1} \lambda_i q_i$ , where  $\lambda_i \geq 0$ ,  $\sum_{i=1}^{n+1} \lambda_i = 1$  and

$$(q_i, p(t), 0) \in N(\text{epi } h(t, \cdot); (z(t), \dot{z}(t)); h(t, z(t), \dot{z}(t))) \quad \text{a.e } t \in [a, b].$$

From Lemma 3.8.1 we have  $\| q_i \| \leq k(t) \| p(t) \|$ , then

$$\| \dot{p}(t) \| \leq k(t) \| p(t) \| .$$

**proof of Corollary 3.3.1** Suppose that  $\lambda = 0$ , then from Theorem 3.3.1 we have

$$(p(a), -p(b), 0) \in N(\text{epil}; (z(a), z(b)); l(z(a), z(b)))$$

then  $(p(a), -p(b)) \in \partial^\infty l(z(a), z(b))$ . As  $g$  is locally Lipschitzian at  $(z(a), z(b))$  then  $p(a) = p(b) = 0$ .

Let  $v(t) = \int_a^t \| \dot{p}(s) \| ds$ , we have  $\| \dot{p}(t) \| \leq k(t) \| p(t) \|$  then  $\dot{v}(t) \leq k(t) \| p(t) \|$ . Set  $f(t) = \int_a^t k(s) ds$ , note that  $\| p(t) \| \leq \| v(t) \|$  then  $(v(t)e^{-f(t)})' \leq 0$  a.e  $t \in [a, b]$  and  $\int_a^t (v(s)e^{-f(s)})' ds \leq 0$ .

$v(a) = 0$  then  $v(t)e^{-f(t)} \leq 0$  which implies  $v(t) \leq 0$  and then  $v(t) = 0$ . Using the norm of  $p$  in  $W^{1,1}$ , we have  $\| p \| = 0$  which is in contradiction with  $(p, \lambda) \neq 0$ .  $\square$

•  **$L$  is epi-Lipschitz at  $z$  iff  $E(t, s)$  is Lipschitzian in  $s$  on  $z(t) + \varepsilon\mathbb{B}$**

$L$  is epi-Lipschitz at an arc  $z$  is equivalent to saying that for almost all  $t \in [a, b]$  the multivalued mapping

$$E(t, s) = \{(u, r) \in \mathbb{R}^n \times \mathbb{R} : L(t, s, u) \leq r\}$$

is Lipschitzian in  $s$  on  $z(t) + \varepsilon\mathbb{B}$  (i.e. for all  $s, s' \in z(t) + \varepsilon\mathbb{B}$  we have  $E(t, s') \subset E(t, s) + k(t) |s' - s| \mathbb{B}$ ). Indeed, let  $z_1, z_2 \in z(t) + \varepsilon\mathbb{B}$  and  $(u, r) \in E(t, z_1)$ .  $L$  is epi-Lipschitz at  $z$ , then there exist a point  $v \in \mathbb{R}^n$  and  $\delta \geq 0$  such that  $L(t, z_2, v)$  is finite and

$$|u - v| + |L(t, z_1, u) - L(t, z_2, v) - \delta| \leq k(t)|z_1 - z_2|.$$

so that

$$L(t, z_2, v) \leq L(t, z_1, u) + k(t)|z_1 - z_2|$$

and

$$|u - v| \leq k(t)|z_1 - z_2|.$$

Set  $s = L(t, z_1, u) + k(t)|z_1 - z_2|$ , then we have

$$|r - s| + |u - v| \leq 2k(t) |z_1 - z_2|$$

and thus  $E(t, s)$  is Lipschitzian in  $s$  on  $z(t) + \varepsilon\mathbb{B}$ .

reciprocally, let  $z_1, z_2 \in z(t) + \varepsilon\mathbb{B}$  and  $u \in \mathbb{R}^n$  such that  $L(t, z_1, u)$  is finite. We have  $(u, L(t, z_1, u)) \in E(t, z_1)$ , then there exist  $(v, r) \in E(t, z_2)$  such that

$$|L(t, z_1, u) - r| + |u - v| \leq k(t) |z_1 - z_2| .$$

Then we have  $|u - v| \leq k(t) |z_1 - z_2|$  and

$$L(t, z_2, v) - L(t, z_1, u) \leq r - L(t, z_1, u) \leq k(t) |z_1 - z_2|.$$

Set  $\delta = k(t) |z_1 - z_2| + L(t, z_1, u) - L(t, z_2, v)$ .  $\delta$  is positive and we have

$$\begin{aligned} |u - v| + |L(t, z_1, u) - L(t, z_2, v) - \delta| &= |u - v| + |-k(t) |z_1 - z_2|| \\ &\leq 2k(t) |z_1 - z_2|. \end{aligned}$$

Thus  $L$  is epi-Lipschitzian at  $z$ .

□





# Chapitre 4

## Necessary optimality conditions in multiobjective dynamic optimization <sup>1</sup>

**Abstract.** We consider a nonsmooth multiobjective optimal control problems related to a general preference. Both differential inclusion and endpoint constraints are involved. Necessary conditions and Hamiltonian necessary conditions expressed in terms of the limiting Fréchet subdifferential are developed. Examples of useful preferences are given.

**Key words.** Multiobjective optimal control, necessary conditions, Hamiltonian necessary conditions, preference, utility function, differential inclusions.

### 4.1 Introduction

This paper is mainly concerned with the following multiobjective dynamic optimization problem with the dynamic governed by a differential inclusion

$$\begin{aligned} \min f(x(a), x(b)) & \qquad (P) \\ (x(a), x(b)) & \in S \\ \dot{x}(t) & \in F(t, x(t)) \quad a.e. \quad t \in [a, b] \end{aligned}$$

where  $f: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^m$  is a mapping,  $S \subset \mathbb{R}^n \times \mathbb{R}^n$  is a closed nonempty set and  $F: [a, b] \times \mathbb{R}^n \mapsto \mathbb{R}^n$  is a closed-valued multivalued mapping which

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<sup>1</sup>to appear in SIAM Journal on Control and Optimization

is measurable in  $t \in [a, b]$ .

These problems naturally arise, for example, in economic (economic growth models) ([28] and references therein), in chemical engineering (polymerization processes) ([8] - [9] and references therein) and in multiobjective control design ([96], [19] and references therein). Problems considered use preferences determined by cones (Pareto and weak Pareto optimum), by utility function or use the concept of Nash equilibrium.

Our aim in this paper is to use a general preference including the previous ones in order to state necessary and Hamiltonian necessary conditions for multiobjective optimal control problems ( $P$ ).

The concept of preference appeared in the value theory in economics. Many authors, in the early studies, often define the preference by an utility function, i.e. given a preference whether its always possible to find an utility function that can determine the preference.

In [28] the author proved that a preference  $\prec$  can be determined by a continuous utility function if and only if for any  $x$  the sets

$$\{y : x \prec y\} \quad \text{and} \quad \{y : y \prec x\} \quad \text{are closed.} \quad (4.1)$$

This theorem is not general and besides this it is an existence theorem (i.e. does provide methods for determining a utility function) and there are some useful preference that does not satisfy (4.1) (like the preference determined by lexicographical order).

There are different approaches and various results on necessary conditions for ( $P$ ). Several researchs have been devoted to the weak Pareto solution and its generalization ( see [13], [22], [26], [70], [88], [97], [98] and references therein). Other research gets refinements of necessary optimality conditions for real-valued objective optimal control problems (see [45], [67]-[69], [95], [89] and [56]), or Hamiltonian necessary conditions (see [76], [31], [32], [24], [73], [74] and [101]).

These results are expressed in terms of various generalized derivatives including Clarke's generalized subgradient [22], limiting subgradient which is also known under other names : limiting subgradient set in [23], approximate subdifferential in Ioffe [40], subdifferential in Mordukhovich [74], subgradient set in the general sens in Rockafellar [86]. Most of these results are obtained for Lipschitz, integrably sub-Lipschitz, bounded or unbounded differential inclusions.

In [45], Ioffe used results of [86] and [46] to obtain a general necessary optimality conditions and Hamiltonian optimality conditions for single-objective optimal control problems.

In [101], Zhu used recent progress in nonsmooth analysis, in particular, calculus for smooth subdifferentials of lower semicontinuous functions ([14], [15], [25], [46]), the methods for proving the extremal principle ([63], [71], [75]) and techniques in handling the Hamiltonian for a differential inclusion, to prove Hamiltonian necessary conditions that extend the classical Hamiltonian necessary conditions for optimal control problems that had previously been derived for uniformly Lipschitz, bounded and convex-valued differential inclusions related to a general preference. The obtained conditions are expressed in terms of Clarke's generalized gradient which is larger than the limiting F chet subdifferential. The regularity conditions (A3) imposed in [101], which use the usual limiting normal cone, is too strong to include the preference defined by a utility function (see Example 6).

In this paper we propose a different approach. We introduce a definition of regularity modified from that introduced in [101]. To solve the problem of regularity of preference determined by a utility function, we define a larger limiting normal cone to replace the usual one in [101]. Under our regularity condition of the general preference and a sub-Lipschitz property of multi-valued mappings, introduced by Loewen and Rockafellar in [67], we obtain Euler-Lagrange necessary optimality conditions for multiobjective optimal control problems with nonconvex differential inclusion constraints in terms of the limiting Fr chet subdifferential. Necessary optimality conditions for the weak Pareto solution and its generalization can be derived and refined by using our necessary conditions.

Our main result extends the necessary optimality condition of Ioffe (Theorem 1 [45]) from a single objective optimal control of differential inclusion problem to multiobjective one. This is also an extension of the Hamiltonian necessary optimality conditions for convex differential inclusions obtained in [101].

The paper is organized as follows. Section 2 contains the key definitions normals subgradient and coderivatives used in the sequel. In Section 3 we state our main result, establish necessary optimality conditions for multiobjective control problems with some examples and discussions. Then we derive necessary conditions for these examples of preferences. In Section 4 we give a technical proof of the main result.

## 4.2 Background

Now we state basic tools of generalized differentiation that are more appropriate for our main purpose. Details may be found in [71]. Let  $C$  be a closed subset of  $\mathbb{R}^n$  containing some point  $c$ . The  $\varepsilon$ -normal cone to  $C$  at  $c$  is the set

$$\hat{N}_\varepsilon(C, c) := \{\zeta \in \mathbb{R}^n : \liminf_{x \in C \rightarrow c} \frac{\langle -\zeta, x - c \rangle}{\|x - c\|} \geq -\varepsilon\}.$$

The normal cone to  $C$  at  $c$  is the set

$$N(C; c) := \limsup_{\substack{x \in C \rightarrow c \\ \varepsilon \rightarrow 0^+}} \hat{N}_\varepsilon(C, c).$$

Now let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  be a lower semicontinuous (l.s.c) function and let  $c \in \mathbb{R}^n$  such that  $f(c) < \infty$ . The limiting Fréchet subdifferential of  $f$  at  $c$  is the set

$$\partial f(c) = \{\zeta \in \mathbb{R}^n : (\zeta, -1) \in N(\text{epi } f; (c, f(c)))\}$$

where  $\text{epi } f$  denotes the epigraph of  $f$ . We have the following analytic characterization of  $\partial f(c)$  :

$$\partial f(c) = \limsup_{\substack{x \rightarrow c \\ f(x) \rightarrow f(c) \\ \varepsilon \rightarrow 0^+}} \partial_\varepsilon f(x)$$

where

$$\partial_\varepsilon f(x) = \{x^* \in X^* : \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq -\varepsilon\}.$$

The singular subdifferential of  $f$  at  $c$  is the set

$$\partial^\infty f(c) = \{\zeta \in \mathbb{R}^n : (\zeta, 0) \in N(\text{epi } f; (c, f(c)))\}.$$

Next we consider a multivalued mapping  $F$  from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  of the closed graph

$$\text{Gr}F := \{(x, y) : y \in F(x)\}.$$

The multivalued mapping  $D^*F(x, y) : \mathbb{R}^m \mapsto \mathbb{R}^n$  defined by

$$D^*F(x, y)(y^*) := \{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N(\text{Gr}F; (x, y))\}$$

is called the coderivative of  $F$  at the point  $(x, y) \in \text{Gr}F$ .

The domain over which our study occurs is typically one of the functions  $W^{1,1}([a, b], \mathbb{R}^n)$  (abbreviated  $W^{1,1}$ ) consisting of all absolutely continuous functions  $x: [a, b] \mapsto \mathbb{R}^n$  for which  $|\dot{x}|$  is integrable on  $[a, b]$  ( $\dot{x}$  denotes the derivative (almost everywhere) of  $x$ ). An *arc* is a function in  $W^{1,1}$ . The space  $W^{1,1}$  is endowed with the norm

$$\|x\| = |x(a)| + \int_a^b |\dot{x}(t)| dt$$

where  $|\cdot|$  denotes the euclidean norm of  $\mathbb{R}^n$ . Here  $\mathbb{B}$  stands for the closed unit ball in  $\mathbb{R}^n$  and

$$B(z, r) = \{x \in W^{1,1}: \|x - z\| \leq r\}.$$

The distance function on  $W^{1,1}$ ,  $\mathbb{R}^n$  or  $\mathbb{R}^n \times \mathbb{R}^n$  will be denoted by  $d(\cdot, \cdot)$ .

The following lemma is needed.

**Lemma 4.2.1** *Let  $G$  be pseudo-Lipschitzian ([2], [87]) around  $(x_0, y_0) \in \text{Gr}G$  with modulus  $K$ , i.e., there exists  $r > 0$  such that for all  $x, u \in x_0 + r\mathbb{B}$*

$$G(x) \cap (y_0 + r\mathbb{B}) \subset G(u) + K|x - u|\mathbb{B}.$$

*Then for all  $y^* \in \mathbb{R}^n$ , with  $D^*G(x_0, y_0)(y^*) \neq \emptyset$ , one has*

$$\sup \{|x^*| : x^* \in D^*G(x_0, y_0)(y^*)\} \leq K|y^*|.$$

*If in addition  $G$  is closed-valued, then for all  $(x, y) \in (x_0 + \frac{r}{12}\mathbb{B}) \times (y_0 + \frac{r}{12}\mathbb{B})$ , with  $(x, y) \notin \text{Gr}G$ , and all  $(x^*, y^*) \in \partial d(\cdot; G(\cdot))(x, y)$  we have*

$$|y^*| = 1, \text{ and } |x^*| \leq K|y^*|.$$

**Proof.** It suffices to establish the second part, the first one follows from the definition of limiting Fréchet subdifferential. Let  $(x, y) \in (x_0 + \frac{r}{12}\mathbb{B}) \times (y_0 + \frac{r}{12}\mathbb{B})$ , with  $(x, y) \notin \text{Gr}G$ , and let  $(x^*, y^*) \in \partial d(\cdot; G(\cdot))(x, y)$ . Then there are sequences  $x_k \rightarrow x$ ,  $y_k \rightarrow y$ ,  $x_k^* \rightarrow x^*$ ,  $y_k^* \rightarrow y^*$ ,  $\varepsilon_k \rightarrow 0^+$  and  $r_k \rightarrow 0^+$  such that

$$d(v; G(u)) - d(y_k; G(x_k)) - \langle x_k^*, u - x_k \rangle - \langle y_k^*, v - y_k \rangle + \varepsilon_k[|u - x_k| + |v - y_k|] \geq 0$$

for all  $u \in x_k + r_k\mathbb{B}$  and  $v \in y_k + r_k\mathbb{B}$ . For each integer  $k$ , there exists  $v_k \in G(x_k)$  such that

$$d(y_k; G(x_k)) = |y_k - v_k|.$$

So

$$|y' - v| - |y_k - v_k| - \langle x_k^*, u - x_k \rangle - \langle y_k^*, v - y_k \rangle + \varepsilon_k[|u - x_k| + |v - y_k|] \geq 0$$

for all  $u \in x_k + r_k\mathbb{B}$ ,  $v \in y_k + r_k\mathbb{B}$  and  $y' \in G(u)$ .

Consider the function  $g$  defined by

$$g(u, y', v) = |y' - v| - \langle x_k^*, u - x_k \rangle - \langle y_k^*, v - y_k \rangle + \varepsilon_k[|u - x_k| + |v - y_k|].$$

Then

$$(0, 0, 0) \in \partial g(x_k, v_k, y_k) + N(\text{Gr}G; (x_k, v_k)) \times \{0\}.$$

As for  $k$  large enough  $y_k \neq v_k$  then

$$\partial g(x_k, v_k, y_k) \subset \{(0, v^*, -v^*) : |v^*| = 1\} + (-x_k^*, 0, -y_k^*) + \varepsilon_k\mathbb{B} \times \{0\} \times \varepsilon_k\mathbb{B}$$

and hence we obtain  $(u_k^*, v_k^*) \in N(\text{Gr}G; (x_k, v_k))$ , with  $|v_k^*| = 1$ , such that

$$|x_k^* - u_k^*| \leq \varepsilon_k \text{ and } |y_k^* - v_k^*| \leq \varepsilon_k.$$

Now since  $d(y_k; G(x_k)) = |y_k - v_k|$ , we get for  $k$  sufficiently large

$$|y_k - v_k| \leq \frac{r}{2}$$

and hence

$$|x_0 - x_k| + |y_0 - v_k| \leq \frac{5r}{6}.$$

Thus for all  $u, u' \in x_k + \frac{r}{6}\mathbb{B}$

$$G(u) \cap (v_k + \frac{r}{6}\mathbb{B}) \subset G(u') + K|u - u'|\mathbb{B}.$$

So the first part of the lemma ensures that

$$|u_k^*| \leq K|v_k^*|$$

and since  $u_k^* \rightarrow x^*$  and  $v_k^* \rightarrow y^*$  we get  $|x^*| \leq K|y^*|$  and the proof is complete.  $\square$

**Lemma 4.2.2** *Let  $G: V \mapsto \mathbb{R}^m$  be a multivalued mapping, where  $V$  is a nonempty set in  $\mathbb{R}^n$ . Suppose that :*

i) *GrG is closed and*

ii) there exists a compact set  $K$  in  $\mathbb{R}^m$  such that

$$G(x) \subset K, \quad \forall x \in V.$$

Then  $G$  is upper semi-continuous (u.s.c.) on  $V$ , that is, for all  $u \in V$  and all  $\varepsilon > 0$  there exists a neighborhood  $U$  of  $u$  in  $V$  such that

$$G(x) \subset G(u) + \varepsilon\mathbb{B}, \quad \forall x \in U.$$

With the help of the last lemma, we can prove the following one.

**Lemma 4.2.3** *Suppose that the mapping  $f: (x_0, y_0) + r\mathbb{B} \mapsto \mathbb{R}$  is Lipschitzian with constant  $K$ . Define the multivalued mapping  $\Gamma: (x_0, y_0) + r\mathbb{B} \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$  by*

$$\Gamma(t, x, y, p, s) = \text{co}\{q: (q, p) \in \partial f(x, y) + s\mathbb{B}\}.$$

Then for all  $\lambda \in ]0, 1[$ , all  $(x, y, s) \in (x_0, y_0, 0) + \lambda r\mathbb{B}$  and all  $p \in \mathbb{R}^n$ , with  $\Gamma(x, y, p, s) \neq \emptyset$ ,  $\Gamma$  is u.s.c. at  $(x, y, p, s)$  in the sense of Lemma 4.2.2.

**Proof.** Note that ii) of Lemma 4.2.2 is satisfied. It is not difficult to show that  $\Gamma$  is of closed graph and to apply Lemma 4.2.2.  $\square$

**Lemma 4.2.4** [22] *Let  $\varepsilon > 0$  and  $\Gamma: [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$  be a multivalued mapping such that for almost all  $t \in [a, b]$ ,  $\Gamma(t, \cdot)$  has nonempty, compact and convex values around  $(z(t), \dot{z}(t), p, s)$ , with  $s \in [0, \varepsilon]$  and  $\Gamma(z(t), \dot{z}(t), p, s) \neq \emptyset$ . For sequences  $(z_k)$  and  $(p_k)$  in  $W^{1,1}$ ,  $(\phi_k)$  in  $L^1([a, b], ]0, +\infty[)$ ,  $(\alpha_k)$  and  $(s_k)$  in  $\mathbb{R}_+$  with  $z_k \rightarrow z$  in  $W^{1,1}$ ,  $\phi_k \rightarrow \phi$  in  $L^1([a, b], ]0, +\infty[)$ , for some integrable function  $\phi$ ,  $\alpha_k \rightarrow 0$  and  $s_k \rightarrow 0$  we suppose that*

i) *For every  $(x, y, p, s)$  in the interior of the set*

$$\{(x', y', p', s') : t \in [a, b], x' \in z(t) + \varepsilon\mathbb{B}, y' \in \dot{z}(t) + \varepsilon\mathbb{B}, s' \in [0, \varepsilon], \Gamma(t, x', y', p', s') \neq \emptyset\}$$

*the multivalued mapping  $t' \mapsto \Gamma(t', x, y, p, s)$  is measurable.*

ii) *For all  $k$ ,  $|\dot{p}_k(t)| \leq \phi_k(t)$  for almost all  $t \in [a, b]$ .*

iii) *For all  $k$ ,  $\dot{p}_k(t) \in \Gamma(t, z_k(t), \dot{z}_k(t), p_k(t), s_k) + \alpha_k\mathbb{B}$  a.e.  $t \in [a, b]$ .*

iv) *For almost all  $t \in [a, b]$ , for every  $p \in \mathbb{R}^n$  with  $\Gamma(t, z(t), \dot{z}(t), p, 0) \neq \emptyset$ , the multivalued mapping  $(x', y', p', s') \mapsto \Gamma(t, x', y', p', s')$  is upper semicontinuous at  $(z(t), \dot{z}(t), p, 0)$ .*



v) The sequence  $(p_k(a))$  is bounded.

vi) There exists an integrable function  $\psi$  such that

$$\sup_{\{(p', s') : s' \in [0, \varepsilon], \Gamma(t, z(t), \dot{z}(t), p', s') \neq \emptyset\}} \max_{y \in \Gamma(t, z(t), \dot{z}(t), p', s')} |y| \leq \psi(t) \text{ a.e.}$$

Then there is a subsequence of  $(p_k)$  which converges uniformly to an arc  $p$  satisfying

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad \text{a.e. } t \in [a, b].$$

We conclude this section by recalling necessary optimality conditions for the following generalized problem of Bolza

$$\min \left\{ \ell(x(a), x(b)) + \int_a^b L(t, x(t), \dot{x}(t)) dt \right\} \quad (P_B)$$

where the functions  $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$  and  $\ell : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$  are such that for each  $t \in [a, b]$ , the functions  $L(t, \cdot, \cdot)$  and  $\ell$  are l.s.c. on  $\mathbb{R}^n \times \mathbb{R}^n$ .

The function  $L$  is *epi-Lipschitz* at an arc  $z$  if there exist an integrable function  $k : [a, b] \mapsto \mathbb{R}$  and a positive  $\varepsilon$  satisfying the following conditions : for almost all  $t \in [a, b]$ , given two points  $z_1$  and  $z_2$  within  $\varepsilon$  of  $z(t)$  and  $u_1 \in \mathbb{R}^n$  such that  $L(t, z_1, u_1)$  is finite, there exist a point  $u_2 \in \mathbb{R}^n$  and  $\delta \geq 0$  such that  $L(t, z_2, u_2)$  is finite and

$$|u_1 - u_2| + |L(t, z_1, u_1) - L(t, z_2, u_2) - \delta| \leq k(t)|z_1 - z_2|.$$

This is equivalent to saying that the multivalued mapping

$$E(t, s) = \{(u, r) \in \mathbb{R}^n \times \mathbb{R} : L(t, s, u) \leq r\}$$

is Lipschitzian in  $s$  on  $z(t) + \varepsilon\mathbb{B}$  (i.e. for all  $s, s' \in z(t) + \varepsilon\mathbb{B}$  we have  $E(t, s') \subset E(t, s) + k(t) |s' - s| \mathbb{B}$ ), (see the Appendix 1).

$L$  is said to be *epi-measurable* (in  $t$ ) if for each  $s \in \mathbb{R}^n$ , the multivalued mapping  $E(t, s)$  is Lebesgue measurable in  $t$ .

The notation  $\partial L$  will denote the limiting Fréchet subdifferential of the function  $L(t, \cdot, \cdot)$ .

Now we may state a variant of the necessary conditions for the generalized Bolza problem established in Jourani [56].

**Theorem 4.2.1** *Let  $z$  solves locally the generalized problem of Bolza ( $P_B$ ) (in  $W^{1,1}$ ). Suppose that  $L(t, z, u)$  is epi-measurable in  $t$ , and  $L(t, \cdot, \cdot)$  is epi-Lipschitzian at  $z$  and  $\ell$  is locally Lipschitzian around  $(z(a), z(b))$ . Then there exists an arc  $p$  such that one has :*

$$\begin{aligned} \dot{p}(t) &\in \text{co}\{q : (q, p(t)) \in \partial L(t, z(t), \dot{z}(t))\} \quad \text{a.e. } t \in [a, b] \\ (p(a), -p(b)) &\in \partial \ell(z(a), z(b)) \\ \langle p(t), \dot{z}(t) \rangle - L(t, z(t), \dot{z}(t)) &= \max\{\langle p(t), v \rangle - L(t, z(t), v) : v \in \mathbb{R}^n\}. \end{aligned}$$

### 4.3 The main result

**Definition 5**  *$F$  is said to be sub-Lipschitzian in the sense of Loewen- Rockafellar [67] at  $z$  if there exist  $\beta > 0$ ,  $\varepsilon > 0$  and a summable function  $k : [a, b] \mapsto \mathbb{R}$  such that for almost all  $t \in [a, b]$ , for all  $N > 0$ , for all  $x, x' \in z(t) + \varepsilon \mathbb{B}$  and  $y \in \dot{z}(t) + N \mathbb{B}$  one has*

$$d(y, F(t, x)) - d(y, F(t, x')) \leq (k(t) + \beta N)|x - x'|.$$

Let  $\prec$  be a (nonreflexive) preference for vectors in  $\mathbb{R}^m$ . We consider the following multiobjective optimization problem.

$$\begin{aligned} \min f(x(a), x(b)) & \tag{P} \\ (x(a), x(b)) &\in S \\ \dot{x}(t) &\in F(t, x(t)) \quad \text{a.e. } t \in [a, b] \end{aligned}$$

where  $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^m$  is a mapping,  $S \subset \mathbb{R}^n \times \mathbb{R}^n$  is a closed nonempty set and  $F : [a, b] \times \mathbb{R}^n \mapsto \mathbb{R}^n$  is a closed-valued multivalued mapping which is measurable in  $t \in [a, b]$ .

We say that an arc  $x \in W^{1,1}$  is a feasible trajectory for problem (P) if  $x$  satisfies  $(x(a), x(b)) \in S$  and  $\dot{x}(t) \in F(t, x(t))$  a.e.  $t \in [a, b]$ .

$z$  is a solution to (P) provided that it is feasible and there does not exists any feasible trajectory  $x$  of (P) such that  $f(x(a), x(b)) \prec f(z(a), z(b))$ . For all  $r \in \mathbb{R}^m$ , we denote

$$\mathcal{L}(r) := \{s \in \mathbb{R}^m : s \prec r\}.$$

We will need the following regularity assumptions on the preference modified from [101].

**Definition 6** We say that a preference  $\prec$  is regular at  $r \in \mathbb{R}^m$  provided that

(A<sub>1</sub>) for any  $s \in \mathbb{R}^m$ ,  $s \in \text{cl}\mathcal{L}(s)$ ;

(A<sub>2</sub>) for any  $r \prec s$ ,  $t \in \text{cl}\mathcal{L}(r)$  implies that  $t \prec s$ ;

**Remak 4.3.1** The preference determined by the Lexicographical order  $\prec$  is defined by  $r \prec s$  if there exists an integer  $q \in \{0, 1, \dots, m-1\}$  such that  $r_i = s_i$ ,  $i = 1, \dots, q$  and  $r_{q+1} < s_{q+1}$ . This preference is not regular. Indeed we consider in  $\mathbb{R}^3$  the vectors  $r = (1, 1, 3)$ ,  $s = (1, 1, 5)$  and  $t = (1, 1, 6)$ . We have  $r \prec s$  and  $t \in \text{cl}\mathcal{L}(r)$  but  $s \prec t$ , then (A<sub>2</sub>) is not hold and so that  $\prec$  is not regular at  $r$ .

Note that a preference determined by the lexicographical order does not correspond to any real utility function [28].

**Remak 4.3.2** Our definition of regularity is different from that given by Zhu in [101] where the following third condition is in force : for any sequences  $r_k, \theta_k \mapsto r$  in  $\mathbb{R}^m$

$$\limsup_{k \rightarrow +\infty} N(\text{cl}\mathcal{L}(r_k); \theta_k) \subset N(\text{cl}\mathcal{L}(r); r).$$

But with this condition, preferences defined by an utility function (e.g.  $u$ ) are not regular at any  $r \in \mathbb{R}^m$  even if

$$\lim_{s \rightarrow r} d(0, \partial u(s)) > 0.$$

For more details see Example 6.

We consider the following enlargement cone of the limiting Fréchet normal cone

$$\tilde{N}(\text{cl}\mathcal{L}(x), x) = \limsup_{y, x' \rightarrow x} N(\text{cl}\mathcal{L}(y); x')$$

Before stating our main result we recall that the Hamiltonian associated to  $F$  is defined by

$$H(t, x, y) = \sup_{v \in F(t, x)} \langle y, v \rangle.$$

**Theorem 4.3.1** Let  $z$  be a local solution to the multiobjective optimal control problem (P). Suppose that  $F$  is sub-Lipschitzian at  $z$  and that the preference  $\prec$  is regular at  $f(z(a), z(b))$ . Then there exist  $p \in W^{1,1}$ ,  $\lambda \geq 0$  and  $w \in \tilde{N}(\text{cl}\mathcal{L}(f(z(a), z(b))), f(z(a), z(b)))$ , with  $|\omega| = 1$ , such that  $(\lambda, p) \neq 0$  and

$$\dot{p}(t) \in \text{co}D^*F(t, z(t), \dot{z}(t))(-p(t)) \text{ a.e. } t \in [a, b]; \quad (4.2)$$

$$(p(a), -p(b)) \in \lambda\partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \quad (4.3)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \text{ a.e. } t \in [a, b]. \quad (4.4)$$

If in addition  $F$  is convex-valued, then (4.2) may be replaced by the following one

$$\dot{p}(t) \in \text{co} \{q : (-q, \dot{z}(t)) \in \partial H(t, (z(t), p(t)))\} \text{ a.e. } t \in [a, b]. \quad (4.5)$$

The aim of Theorem 4.3.1 is to extend the necessary optimality conditions of Ioffe (Theorem 1 [45]) from a single objective optimal control of differential inclusion problem to multiobjective one. By using the large class of sub-Lipschitz differential inclusion, Theorem 4.3.1 extends also the Hamiltonian necessary optimality conditions for convex-valued differential inclusions obtained in [101].

In the remainder of this section we now examine a few examples. The proof of Theorem 4.3.1 is postponed to the next section.

**Example 4** (*A generalized Pareto optimal*)

Let  $K$  be a pointed convex cone ( $K \cap (-K) = \{0\}$ ). We define the preference  $\prec$  by :  $r \prec s$  if and only if  $r - s \in K$  and  $r \neq s$ . Multiobjective optimal control problem with this preference is called generalized Pareto optimal control problem. Notice that if  $K = \mathbb{R}_-^m$  (resp.  $K = \text{int}\mathbb{R}_-^m$  where  $\mathbb{R}_-^m = \{(x_1, x_2, \dots, x_m) \in \mathbb{R}^m : x_i \leq 0 \text{ for all } i = 1, \dots, m\}$ ) we get Pareto (resp. weak Pareto) optimal control problems. This preference is regular at any  $r \in \mathbb{R}^m$ . Moreover, for any  $r \in \mathbb{R}^m$  we have  $\tilde{N}(\text{cl}\mathcal{L}(r), r) = K^0$  with  $K^0 = \{s \in \mathbb{R}^m : \langle s, q \rangle \leq 0 \forall q \in K\}$ .

**Corollary 4.3.1** *Let  $z$  be a local solution to the generalized Pareto multiobjective optimal control problem (P). Then there exist  $p \in W^{1,1}$ ,  $\lambda \geq 0$  and  $\omega \in K^0$  with  $|\omega| = 1$ , such that  $(\lambda, p) \neq 0$  and*

$$\dot{p}(t) \in \text{co}D^*F(t, z(t), \dot{z}(t))(-p(t)) \text{ a.e. } t \in [a, b]; \quad (4.6)$$

$$(p(a), -p(b)) \in \lambda\partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \quad (4.7)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \text{ a.e. } t \in [a, b]. \quad (4.8)$$

**Example 5** (*A preference determined by an utility function*)

Let  $u$  be a continuous function, we define the preference  $\prec$  determined by utility function  $u$  by :  $r \prec s$  if and only if  $u(r) < u(s)$ .

**Lemma 4.3.1** *Let  $u$  be a continuous utility function determining the preference  $\prec$ . Suppose that*

$$0 \notin \partial u(r). \quad (4.9)$$

Then the preference  $\prec$  is regular at  $r$  and

$$\tilde{N}(cl\mathcal{L}(r), r) = \limsup_{r' \rightarrow r} N(cl\mathcal{L}(r'); r') = \partial^\infty u(r) \bigcup \left( \bigcup_{a>0} a\partial u(r) \right).$$

**Proof.** The proof of Lemma 4.3.1 is similar to that given in [101]. From (4.9),  $\mathcal{L}(r)$  is nonempty and from the continuity of  $u$ , it follows that  $\prec$  satisfies  $(A_1)$  and  $(A_2)$  in Definition 6 and thus  $\prec$  is regular. Now for  $r'$  sufficiently close to  $r$ ,  $cl\mathcal{L}(r') = \{s \in \mathbb{R}^m : u(s) - u(r') \leq 0\}$ . Then

$$\partial_\varepsilon u(r') \subset \hat{N}_\varepsilon(cl\mathcal{L}(r'), r').$$

By passing to the limits we have

$$\partial^\infty u(r) \bigcup \left( \bigcup_{a>0} a\partial u(r) \right) \subset \limsup_{r' \rightarrow r} N(cl\mathcal{L}(r'); r') \subset \tilde{N}(cl\mathcal{L}(r), r).$$

Conversely let  $\zeta \in \tilde{N}(cl\mathcal{L}(r), r)$  such that  $\zeta \neq 0$ . Then there are sequences  $\zeta_k \rightarrow \zeta$ ,  $r_k, r'_k \rightarrow r$  such that  $\zeta_k \in N(cl\mathcal{L}(r_k); r'_k)$ . By the definition of limiting Fréchet normal cone, we may assume that  $\zeta_k \in \hat{N}_{\varepsilon_k}(cl\mathcal{L}(r_k), r'_k)$ . We must have  $u(r_k) = u(r'_k)$ . Indeed,  $\hat{N}_{\varepsilon_k}(cl\mathcal{L}(r_k), r'_k) = \{0\}$  when  $u(r'_k) < u(r_k)$  and empty when  $u(r'_k) > u(r_k)$ . Then  $\hat{N}_{\varepsilon_k}(cl\mathcal{L}(r_k), r'_k) = \hat{N}_{\varepsilon_k}(cl\mathcal{L}(r_k), r_k)$ . From  $\hat{N}_{\varepsilon_k}(cl\mathcal{L}(r_k), r_k) = \hat{N}_{\varepsilon_k}(\{s : u(s) - u(r_k) \leq 0\}, r_k)$  and [16], there exist  $a_k > 0$  and  $\theta_k \in \partial_{\varepsilon_k} u(r)$  such that  $|a_k \theta_k - \zeta_k| < \frac{1}{k}$ . So that

$$\lim_{k \rightarrow \infty} a_k \theta_k = \zeta.$$

We claim that  $(a_k)$  is bounded. Indeed, suppose the contrary. Then  $(a_k)$  has a subsequence going to the infinity. But in this case  $(\theta_k)$  must have a subsequence converging to zero, and this contradicts (4.9). So  $(a_k)$  is bounded and we can assume that  $a_k \rightarrow a$ . If  $a \neq 0$  then  $\zeta \in a\partial u(r)$ . If  $a = 0$  then  $\zeta \in \partial^\infty u(r)$ , and the proof is complete.  $\square$

From Lemma 4.3.1 and Theorem 4.3.1 we have the following corollary.

**Corollary 4.3.2** *Let  $\prec$  be a preference determined by an utility function  $u$  and that  $z$  be a local solution to the multiobjective optimal control problem (P). Suppose that*

$$0 \notin \partial u(f(z(a), z(b))).$$

*Then there exist  $p \in W^{1,1}$ ,  $\lambda \geq 0$  and*

$$\omega \in \partial^\infty u(f(z(a), z(b))) \cup \left( \bigcup_{a>0} a \partial u(f(z(a), z(b))) \right)$$

*with  $|\omega| = 1$ , such that  $(\lambda, p) \neq 0$  and*

$$\dot{p}(t) \in \text{co}D^*F(t, z(t), \dot{z}(t))(-p(t)) \text{ a.e. } t \in [a, b]; \quad (4.10)$$

$$(p(a), -p(b)) \in \lambda \partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b))); \quad (4.11)$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \text{ a.e. } t \in [a, b]. \quad (4.12)$$

In [101], the author showed that, for a preference  $\prec$  defined by a continuous utility function  $u$ ,  $N(\text{cl}\mathcal{L}(r); r) = \partial^\infty u(r) \cup \left( \bigcup_{a>0} a \partial u(r) \right)$  provided that  $\lim_{s \rightarrow r} d(0, \partial u(s)) > 0$ . This could give him the regularity and the explicit shape of  $N(\text{cl}\mathcal{L}(r); r)$ . But there is a gap in the proof. The following example shows that Zhu's regularity does not hold.

**Example 6** *Consider the function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by*

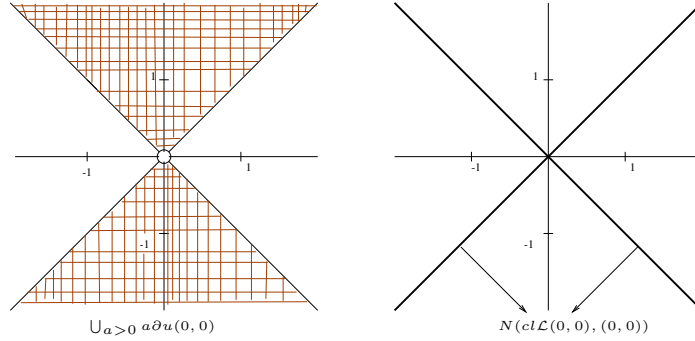
$$u(x, y) = |x| - |y|.$$

*Then  $u$  is Lipschitz continuous, and satisfies  $\partial u(0, 0) = [-1, 1] \times \{-1, 1\}$ . So that  $(0, 0) \notin \partial u(0, 0)$ ,  $\partial^\infty u(0, 0) = \{(0, 0)\}$  and*

$$N(\text{cl}\mathcal{L}(0, 0); (0, 0)) = \{(x, y) \in \mathbb{R}^2 : |y| = |x|\}.$$

*Then it is clear that*

$$N(\text{cl}\mathcal{L}(0, 0); (0, 0)) \neq \partial^\infty u(0, 0) \cup \left( \bigcup_{a>0} a \partial u(0, 0) \right).$$



## 4.4 Proof of Theorem 4.3.1

Since  $F$  is sub-Lipschitzian at  $z$  there exist  $\beta > 0$ ,  $\varepsilon > 0$  and a summable function  $k : [a, b] \mapsto \mathbb{R}$  such that for almost all  $t \in [a, b]$ , for all  $N > 0$ , for all  $x, x' \in z(t) + \varepsilon \mathbb{B}$  and  $y \in \dot{z}(t) + N\mathbb{B}$  one has

$$d(y, F(t, x)) - d(y, F(t, x')) \leq (k(t) + \beta N)|x - x'|.$$

Let  $G$  be the solution set of the system

$$\dot{x}(t) \in F(t, x(t)) \text{ a.e.}, (x(a), x(b)) \in S \quad (4.13)$$

Let  $\varepsilon$  as above. We say that the system (4.13) is semi-normal at  $z$  if there exist  $\alpha > 0$  and  $r > 0$  such that for all  $x \in B(z, r)$

$$d(x, G \cap B(z, \varepsilon)) \leq \alpha \left\{ d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t); F(t, x(t))) dt \right\} \quad (4.14)$$

Set  $G_\varepsilon = G \cap B(z, \varepsilon)$ .

We divide the proof into two parts and each part is divided into two steps.

**Part 1.** When system (4.13) is not semi-normal at  $z$ . The proof of this part is similar to that given in [45].

**Step 1.** (*Application of Ekeland's variational principle and Theorem 4.2.1*). Consider the function  $h$  defined by

$$h(x) = d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t); F(t, x(t))) dt.$$

Since  $F$  is sub-Lipschitzian at  $z$  then  $h$  is lower semicontinuous on the set  $B(z, \varepsilon)$  and  $G_\varepsilon$  is closed (see the Appendix). If system (4.13) is not semi-normal at  $z$ , then there is a sequence  $x_k \rightarrow z$  in  $W^{1,1}$  such that, for  $k$  large

enough

$$d(x_k, G_\varepsilon) > kh(x_k).$$

Set  $\varepsilon_k = \sqrt{h(x_k)} > 0$ ,  $\lambda_k = \min(\varepsilon_k, k\varepsilon_k^2)$  and  $s_k = \frac{\varepsilon_k^2}{\lambda_k}$ . Then  $\varepsilon_k \rightarrow 0^+$  and  $s_k \rightarrow 0^+$ . Therefore one has

$$h(x_k) \leq \inf_{x \in B(z, \varepsilon)} h(x) + \varepsilon_k^2.$$

By Ekeland variational principle we get  $z_k \in B(z, \varepsilon)$  satisfying

$$\|z_k - x_k\| < \lambda_k \tag{4.15}$$

$$h(z_k) \leq h(x) + s_k \|x - z_k\|, \quad \forall x \in B(z, \varepsilon) \tag{4.16}$$

Observe that for  $k$  sufficiently large  $\|z_k - z\| \leq \frac{\varepsilon}{2}$ . By the closedness of  $G_\varepsilon$  and relation (4.15)  $z_k \notin G$  and by (4.16)  $z_k$  is a local solution to the following Bolza problem

$$\min \left\{ \ell_k(x(a), x(b)) + \int_a^b L_k(t, x(t), \dot{x}(t)) dt \right\}$$

where

$$\ell_k(u, v) = d((u, v); S) + s_k |u - z_k(a)|$$

and

$$L_k(t, x, y) = \begin{cases} d(y; F(t, x)) + s_k |y - \dot{z}_k(t)| & \text{if } (x, y) \in A(t), \\ +\infty & \text{otherwise.} \end{cases}$$

where  $A(t) = (z(t) + \varepsilon\mathbb{B}) \times (\dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B})$  and  $N > 0$  is an arbitrary integer.

Since  $L_k(t, \cdot, \cdot)$  is l.s.c, epi-Lipschitzian at  $z_k$  (see the Appendix) and epi-measurable in  $t$  and  $\ell_k$  is locally Lipschitzian around  $(z_k(a), z_k(b))$  then Theorem 4.2.1 yields the existence of an arc  $p_k$  in  $W^{1,1}$  satisfying

$$\dot{p}_k(t) \in \text{co}\{q : (q, p_k(t)) \in \partial L_k(t, z_k(t), \dot{z}_k(t))\} \quad \text{a.e. } t \in [a, b] \tag{4.17}$$

$$(p_k(a), -p_k(b)) \in \partial \ell_k(z_k(a), z_k(b)) \tag{4.18}$$



$$\langle p_k(t), \dot{z}_k(t) \rangle - L_k(t, z_k(t), \dot{z}_k(t)) = \max_{v \in \mathbb{R}^n} \{ \langle p_k(t), v \rangle - L_k(t, z_k(t), v) \}. \quad (4.19)$$

From (4.17), (4.18) and (4.19) we have

$$(p_k(a), -p_k(b)) \in \partial d((z_k(a), z_k(b)); S) + s_k \mathbb{B} \times \{0\} \quad (4.20)$$

$$\dot{p}_k(t) \in \text{co} \{ q : (q, p_k(t)) \in \partial d(\cdot; F(t, \cdot))(z_k(t), \dot{z}_k(t)) + \{0\} \times s_k \mathbb{B} \} \quad a.e. \quad (4.21)$$

$$\begin{aligned} & \langle p_k(t), \dot{z}_k(t) \rangle - d(\dot{z}_k(t); F(t, z_k(t))) = \\ & \max_{v \in \dot{z}_k(t) + (N + |\dot{z}_k(t) - \dot{z}_k(t)|) \mathbb{B}} \{ \langle p_k(t), v \rangle - d(v; F(t, z_k(t))) - s_k |v - \dot{z}_k(t)| \} \quad a.e. \end{aligned}$$

**Step 2.** (*Application of Lemmas 4.2.1-4.2.4*).

By (4.20) there exists  $\zeta_k \in \partial d((z_k(a), z_k(b)); S)$  such that

$$(p_k(a), -p_k(b)) - \zeta_k \in s_k \mathbb{B} \times \{0\}. \quad (4.22)$$

Since  $z_k \notin G$ , we have either

$$|\zeta_k| = 1 \text{ if } (z_k(a), z_k(b)) \notin S \quad (4.23)$$

or (because of Lemma 4.2.1 and (4.21)), on a set of positive measure on which  $\dot{z}_k(t) \notin F(t, z_k(t))$  we have

$$1 - s_k \leq |p_k(t)| \leq 1 + s_k \quad (4.24)$$

It follows from (4.22)-(4.24) that

$$\frac{1}{\sqrt{2}} - s_k \leq \max_{t \in [a, b]} |p_k(t)| \leq 1 + s_k. \quad (4.25)$$

Now let  $\Gamma : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}^n$  be the multivalued mapping defined by

$$\Gamma(t, x, y, w, s) = \text{co} \{ q : (q, w) \in \partial d(\cdot; F(t, \cdot))(x, y) + \{0\} \times s \mathbb{B} \}.$$

Then

$$(p_k(a), -p_k(b)) - \zeta_k \in s_k \mathbb{B} \times \{0\} \quad (4.26)$$

$$\dot{p}_k(t) \in \Gamma(t, z_k(t), \dot{z}_k(t), p_k(t), s_k) \quad a.e. \quad (4.27)$$

$$\begin{aligned} \langle p_k(t), \dot{z}_k(t) \rangle - d(\dot{z}_k(t); F(t, z_k(t))) = \\ \max_{v \in \dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B}} \{ \langle p_k(t), v \rangle - d(v; F(t, z_k(t))) - s_k |v - \dot{z}_k(t)| \} \quad a.e. \end{aligned} \quad (4.28)$$

Extracting a subsequence if necessary we may suppose that  $\zeta_k \rightarrow \zeta$  for some  $\zeta$  in  $\partial d((z(a), z(b)); S)$  with

$$|\zeta| = 1 \text{ if } (z_k(a), z_k(b)) \notin S \text{ for infinite number of } k.$$

On the other hand, by Lemma 4.2.2, the multivalued mapping  $\Gamma(t, \cdot)$  is upper semicontinuous with compact convex values and by the definition of the limiting Fréchet subdifferential and the sub-Lipschitz condition we have (via Lemma 4.2.1 and (4.21)) for all  $k$

$$|\dot{p}_k(t)| \leq 1 + k(t) + \beta(1 + |\dot{z}(t) - \dot{z}_k(t)|) \quad a.e.$$

Note that  $\Gamma(t, x, y, w, s)$  is measurable in  $t$  (see the Appendix). By Lemma 4.2.4 there exists a subsequence of  $(p_k)$  converging uniformly to an arc  $p$  satisfying

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad a.e. \quad (4.29)$$

and hence we obtain, by passing to the limit in (4.26) and (4.28),

$$(p(a), -p(b)) \in \partial d((z(a), z(b)); S) \quad (4.30)$$

$$\langle p(t), \dot{z}(t) \rangle = \max_{v \in F(t, z(t)) \cap (\dot{z}(t) + N\mathbb{B})} \langle p(t), v \rangle \quad a.e. \quad (4.31)$$

Now because of (4.25) the pair  $(\zeta, p)$  must be nonzero. In fact we have

$$\frac{1}{\sqrt{2}} \leq \max_{t \in [a, b]} |p(t)| \leq 1. \quad (4.32)$$

As  $p$  depends on  $N$ , we obtain a sequence  $(p_N)$  satisfying (4.29)-(4.32) and

$$|\dot{p}_N(t)| \leq 1 + k(t) + \beta \quad a.e.$$

Again Lemma 4.2.4 produces a subsequence of  $(p_N)$  converging uniformly to some  $p$  which satisfies the following

$$\begin{aligned}
\dot{p}(t) &\in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad a.e. \\
(p(a), -p(b)) &\in \partial d((z(a), z(b)); S) \\
\langle p(t), \dot{z}(t) \rangle &= \max_{v \in F(t, z(t))} \langle p(t), v \rangle. \\
\frac{1}{\sqrt{2}} &\leq \max_{t \in [a, b]} |p(t)| \leq 1.
\end{aligned}$$

Finally we have

$$\dot{p}(t) \in \text{co}D^*F(t, z(t), \dot{z}(t))(-p(t)) \quad a.e. \quad t \in [a, b],$$

$$(p(a), -p(b)) \in N(S; (z(a), z(b))),$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \quad a.e. \quad t \in [a, b].$$

**Part 2.** When system (4.13) is semi-normal.

**Step 1.** (*Application of Ekeland's variational principle*). Let  $k$  be a positive integer and choose  $\theta_k \prec f(z(a), z(b))$  such that  $|\theta_k - f(z(a), z(b))| < \frac{1}{k^2}$  and define  $\Theta := \text{cl}\mathcal{L}(\theta_k)$ . Define the function

$$h(x, \theta) = \begin{cases} |f(x(a), x(b)) - \theta| & \text{if } x \in B(z, s_1), \\ +\infty & \text{otherwise.} \end{cases}$$

where  $s_1$  is such that  $f$  is Lipschitzian on  $(z(a), z(b)) + s_1\mathbb{B}$  with constant  $k_f$ . From  $(A_1)$  we have  $(z, \theta_k) \in G_\varepsilon \times \Theta$  and hence

$$h(z, \theta_k) \leq \inf_{(x, \theta) \in G_\varepsilon \times \Theta} h(x, \theta) + \frac{1}{k^2}.$$

Note that  $G_\varepsilon$  and  $\Theta$  are closed in  $W^{1,1}$  and  $\mathbb{R}^m$  respectively, and that  $h$  is lower semicontinuous on  $G_\varepsilon \times \Theta$ . Then by Ekeland variational principle there exists  $(z_k, \gamma_k) \in G_\varepsilon \times \Theta$  such that

$$\|z_k - z\| + |\gamma_k - \theta_k| \leq \frac{1}{k} \quad (4.33)$$

and

$$h(z_k, \gamma_k) \leq h(x, \theta) + \frac{1}{k} [\|z_k - x\| + |\gamma_k - \theta|], \quad \forall (x, \theta) \in G_\varepsilon \times \Theta. \quad (4.34)$$

From (4.34) one gets

$$h(z_k, \gamma_k) \leq h(x, \gamma_k) + \frac{1}{k} \|z_k - x\|, \forall x \in G_\varepsilon \quad (4.35)$$

and

$$h(z_k, \gamma_k) \leq h(z_k, \theta) + \frac{1}{k} |\gamma_k - \theta|, \forall \theta \in \Theta. \quad (4.36)$$

Since  $z$  is an optimal local solution to problem  $(P)$ , then, by  $(A_2)$  and the choice of  $\theta_k$ , one has  $\gamma_k \neq f(z_k(a), z_k(b))$ . Set  $w_k = \frac{f(z_k(a), z_k(b)) - \gamma_k}{|f(z_k(a), z_k(b)) - \gamma_k|}$ . Extracting subsequence we may assume that  $(w_k)$  converges to some  $w$ , with  $|w| = 1$ . So that, by (4.36) one has

$$w \in \limsup_{k \rightarrow +\infty} N(\text{cl}\mathcal{L}(\theta_k); \gamma_k)$$

and then

$$w \in \tilde{N}(\text{cl}\mathcal{L}(f(z(a), z(b))), f(z(a), z(b))).$$

Now from (4.35) and the seminormality of (4.13) there exist  $\alpha > 0$  and  $\min(s_1, r, \varepsilon) > s > 0$  (both not depending on  $k$ ) such that

$$h(z_k, \gamma_k) \leq h(x, \gamma_k) + \frac{1}{k} \|z_k - x\| + \alpha(k_f + 1) \left[ d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t), F(t, x(t))) dt \right]$$

for all  $x \in B(z, s)$  where  $r$  and  $\alpha$  are as in (4.14).

Define the functions

$$\ell_k(u, v) = |f(u, v) - \gamma_k| + \frac{1}{k} |u - z_k(a)| + \alpha(k_f + 1) d((u, v); S)$$

and

$$L_k(t, x, y) = \begin{cases} \alpha(k_f + 1) d(y; F(t, x)) + \frac{1}{k} |y - \dot{z}_k(t)| & \text{if } (x, y) \in A(t), \\ +\infty & \text{otherwise.} \end{cases}$$

where  $A(t) = (z(t) + s\mathbb{B}) \times (\dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B})$ . So that  $z_k$  is a local solution to the Bolza problem

$$\min \left\{ \ell_k(x(a), x(b)) + \int_a^b L_k(t, x(t), \dot{x}(t)) dt \right\}$$

**Step 2.** (*Application of Theorem 4.2.1 and Lemmas 4.2.1-4.2.4*). It is easy to check that  $\ell_k$  is l.s.c and locally-Lipschitzian around  $(z_k(a), z_k(b))$ ,  $L_k(t, \cdot, \cdot)$  is l.s.c,  $L_k$  is epi-measurable in  $t$  and epi-Lipschitzian at  $z_k$  (see the Appendix). Then by Theorem 4.2.1 there exist an arc  $p_k$  in  $W^{1,1}$  satisfying

$$\dot{p}_k(t) \in \text{co}\{q : (q, p_k(t)) \in \partial L_k(t, z_k(t), \dot{z}_k(t))\} \quad a.e. t \in [a, b] \quad (4.37)$$

$$(p_k(a), -p_k(b)) \in \partial \ell_k(z_k(a), z_k(b)) \quad (4.38)$$

$$\langle p_k(t), \dot{z}_k(t) \rangle - L_k(t, z_k(t), \dot{z}_k(t)) = \max_{v \in \mathbb{R}^n} \{\langle p_k(t), v \rangle - L_k(t, z_k(t), v)\}. \quad (4.39)$$

Consider the multivalued mapping defined by

$$\Gamma(t, x, y, w, s) = \text{co} \{q : (q, w) \in \alpha(k_f + 1)\partial d(\cdot; F(t, \cdot))(x, y) + \{0\} \times s\mathbb{B}\}.$$

From (4.37)-(4.39) we have

$$(p_k(a), -p_k(b)) \in \partial(|f(\cdot) - \gamma_k|)(z_k(a), z_k(b)) + N(S; (z_k(a), z_k(b))) + \frac{1}{k}\mathbb{B} \times \{0\} \quad (4.40)$$

$$\dot{p}_k(t) \in \Gamma(t, z_k(t), \dot{z}_k(t), p_k(t), \frac{1}{k}) \quad a.e. \quad (4.41)$$

$$\begin{aligned} \langle p_k(t), \dot{z}_k(t) \rangle - \alpha(k_f + 1)d(\dot{z}_k(t); F(t, z_k(t))) = \\ \max_{v \in \dot{z}_k(t) + (N + |\dot{z}_k(t) - \dot{z}_k(t)|)\mathbb{B}} \{\langle p_k(t), v \rangle - \alpha(k_f + 1)d(v; F(t, z_k(t))) - s_k|v - \dot{z}_k(t)|\} \quad a.e. \end{aligned} \quad (4.42)$$

By Lemma 4.2.2, the multivalued mapping  $\Gamma(t, \cdot)$  is upper semicontinuous with compact convex values and by the definition of the limiting Fréchet subdifferential and the sub-Lipschitz condition we have (via Lemma 4.2.1 and (4.41)) for all  $k$

$$|\dot{p}_k(t)| \leq \alpha(k_f + 1)(1 + k(t) + \beta(1 + |\dot{z}(t) - \dot{z}_k(t)|)) \quad a.e.$$

By Lemma 4.2.4 there exists a subsequence of  $(p_k)$  converging uniformly to an arc  $p$  satisfying

$$\dot{p}(t) \in \Gamma(t, z(t), \dot{z}(t), p(t), 0) \quad a.e. \quad (4.43)$$

Note that

$$\partial(|f(\cdot, \cdot) - \gamma_k|)(z_k(a), z_k(b)) \subset \partial(\langle w_k, f(\cdot, \cdot) \rangle)(z_k(a), z_k(b))$$

and hence, by passing to the limit in (4.40) and (4.42) and using the same argument as in Part1, step 2, we have

$$(p(a), -p(b)) \in \partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S; (z(a), z(b)))$$

$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \quad a.e.$$

Now if we assume that  $F$  is convex-valued then, by (4.29) and/or (4.43) and Rockafaller result [86], we obtain

$$\dot{p}(t) \in \text{co} \{q : (-q, \dot{z}(t)) \in \partial H(t, z(t), p(t))\} \quad a.e. \quad t \in [a, b].$$

Which completes the proof. □

## 4.5 Appendix 2

–  $h(x) = d((x(a), x(b)); S) + \int_a^b d(\dot{x}(t); F(t, x(t)))dt$  is **lower semicontinuous on**  $B(z, \varepsilon)$ .

Since  $F$  is sub-Lipschitzian at  $z$  then there exist  $\beta > 0$ ,  $\varepsilon > 0$  and a summable function  $k : [a, b] \mapsto \mathbb{R}$  such that for almost all  $t \in [a, b]$ , for all  $N > 0$ , for all  $x, x' \in z(t) + \varepsilon \mathbb{B}$  and  $y \in \dot{z}(t) + N \mathbb{B}$  one has

$$d(y, F(t, x)) - d(y, F(t, x')) \leq (k(t) + \beta N)|x - x'|.$$

Let  $x \in B(z, \varepsilon)$  and  $\varepsilon' > 0$  and set  $\delta < \frac{\varepsilon'}{1 + \int_a^b k(t) dt + \beta(\varepsilon + b - a)}$ .

Let  $x' \in B(z, \varepsilon)$  such that  $\|x - x'\| < \delta$  and set  $N = |\dot{x}'(t) - \dot{z}(t)| + 1$ .

We have

$$\begin{aligned}
& \left| \int_a^b d(\dot{x}(t), F(t, x(t))) dt - \int_a^b d(\dot{x}'(t), F(t, x'(t))) dt \right| \leq \int_a^b |\dot{x}(t) - \dot{x}'(t)| dt \\
& \quad + \left| \int_a^b d(\dot{x}'(t), F(t, x(t))) dt - \int_a^b d(\dot{x}'(t), F(t, x'(t))) dt \right| \\
& \leq \delta + \int_a^b (k(t) + \beta N) |x(t) - x'(t)| dt \\
& \leq \delta + \delta \left( \int_a^b k(t) dt + \beta(\varepsilon + b - a) \right) \leq \varepsilon'.
\end{aligned}$$

Thus  $h$  is l.s.c on  $B(z, \varepsilon)$ .

–  $G_\varepsilon$  is closed.

Let  $(x_n)$  be a subsequence in  $G_\varepsilon$  such that  $x_n \rightarrow x$  in  $W^{1,1}$ . Since  $S$  is closed  $(x(a), x(b)) \in S$ . Set  $N' = \|\dot{x}(t) - \dot{z}(t)\| + 1$ , since  $F$  is sub-Lipschitzian at  $z$  we have

$$d(\dot{x}(t), F(t, x(t))) \leq (k(t) + \beta N) |x(t) - x_n(t)| + d(\dot{x}(t), F(t, x_n(t))).$$

So that

$$\begin{aligned}
\int_a^b d(\dot{x}(t), F(t, x(t))) dt & \leq \|x - x_n\| \int_a^b (k(t) + \beta N) dt \\
& \quad + \int_a^b d(\dot{x}(t), F(t, x_n(t))) dt \\
& \leq \|x - x_n\| \left( \beta \|x - z\| + \beta(b - a) + \int_a^b k(t) dt \right) \\
& \quad + \|x - x_n\|.
\end{aligned}$$

Then  $d(\dot{x}(t), F(t, x(t))) = 0$  a.e and since  $F$  is closed-valued  $x \in G_\varepsilon$ .  $\square$

–  $L_k(t, \cdot, \cdot)$  is epi-Lipschitzian at  $z_k$ .

We have

$$L_k(t, x, y) = \begin{cases} \alpha(k_f + 1)d(y; F(t, x)) + \frac{1}{k}|y - \dot{z}_k(t)| & \text{if } (x, y) \in A(t), \\ +\infty & \text{otherwise} \end{cases}$$

where  $A(t) = (z(t) + s\mathbb{B}) \times (\dot{z}(t) + (N + |\dot{z}(t) - \dot{z}_k(t)|)\mathbb{B})$ .

For  $k$  large enough we can suppose that  $|z_k(t) - z(t)| < \frac{s}{2}$ . Let  $x_1, x_2 \in B(z_k(t), \frac{s}{2})$  and  $y \in \mathbb{R}^n$  such that  $L_k(t, x_1, y)$  is finite. Then

$$|x_1 - z(t)| \leq s \text{ and } |y - \dot{z}(t)| \leq N + |\dot{z}(t) - \dot{z}_k(t)|.$$

Since  $|x_2 - z(t)| < s$ ,  $L_k(t, x_2, y)$  is finite and using the fact that  $F$  is sub-Lipschitzian at  $z$  we get

$$\begin{aligned} L_k(t, x_2, y) - L_k(t, x_1, y) &= \alpha(k_f + 1)[d(y; F(t, x_2)) - d(y; F(t, x_1))] \\ &\leq \alpha(k_f + 1)(k(t) + \beta(N + |\dot{z}(t) - \dot{z}_k(t)|)) |x_1 - x_2|. \end{aligned}$$

Then  $L_k(t, \cdot, \cdot)$  is epi-Lipschitzian at  $z_k$ . □

–  $\Gamma(t, x, y, w, s)$  is measurable in  $t$ .

The measurability of the multivalued mapping  $\Gamma(t, x, y, w, s)$  in  $t$  follows from the two following Lemmas.

**Lemma 4.5.1** *Let  $G : [a, b] \rightarrow \mathbb{R}^n$  be a measurable multivalued mapping and let  $K$  be a compact set in  $\mathbb{R}^n$ . Then the multivalued mapping  $G(\cdot) + K$  is also measurable.*

**Proof.** It suffices to see that for any set  $A$  in  $\mathbb{R}^n$  we have

$$(G(\cdot) + K)^{-1}(A) = G^{-1}(A - K),$$

where  $G^{-1}(A) = \{t : G(t) \cap A \neq \emptyset\}$ . □

**Lemma 4.5.2** *Let  $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  be a lower semicontinuous function in  $(x, y)$  and measurable in  $(t, x, y)$ . Consider the multivalued mapping*

$$R(t, x, y, p) = \{q : (q, -p) \in \partial f(t, x, y) + \{0\} \times s\mathbb{B}\}.$$

*Then  $R$  and  $\bar{c}o R$  are measurable in  $t$ .*

**Proof.** It follows from Lemma 2 in [42] that the graph of the multivalued mapping  $t \rightarrow \partial f(t, x, y)$  is measurable. As this multivalued mapping is closed-valued, Theorem 8.1.4 in [4] implies that it is measurable in  $t$ . Now Lemma 4.5.1 asserts that the multivalued mapping

$$t \longrightarrow \partial f(t, x, y) + \{0\} \times s\mathbb{B}$$

is measurable in  $t$ . The measurability of  $t \rightarrow R(t, x, y, p)$  follows from the formula

$$(\partial f(\cdot, x, y) + \{0\} \times s\mathbb{B})^{-1}(A \times \{-p\}) = R^{-1}(\cdot, x, y, p)(A).$$

The measurability of  $\bar{c}o R$  follows from Theorem 8.2.2 in [4]. □





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