UNIVERSITÉ
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La suite spectrale de Leray-Serre en cohomologie de Floer des variétés symplectiques compactes à bord de type contact

The Leray-Serre spectral sequence for Floer cohomology of compact symplectic manifolds with contact type boundary

Auteur : Alexandru Oancea

Directeur de thèse : Claude Viterbo

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SUMMARY

– Motivation : Computation of Floer (co)homology
Existence of closed characteristics
Lagrangian embeddings
Symplectic capacities
Symplectic hyperbolicity

– (Extended) def. of Floer cohomology
Asymptotically linear Hamiltonians

– Generic spectral sequence theorem for Floer cohomology

– The Künneth formula (trivial fibration)

– LS spectral sequence
1. classical setting, Morse cohomology
2. symplectic fibrations, Floer cohomology

CLOSED CHARACTERISTICS

Definition : (M, ω) compact symplectic manifold
Contact type boundary (CTB) :
vector field X,
pointing outwards,
defined near ∂M,
$L_X \omega = \omega$.
Restricted CTB : X defined globally.

Notations :
- X Liouville v. field, φ flow of X,
- $\lambda = \iota_X \omega$ Liouville form near ∂M,
- R Reeb v. field : $\lambda(R) = 1$, $R \in \ker \omega|_{T\partial M}$
- X_H Hamiltonian vector field, defined for $H : \mathbb{S}^1 \times M \longrightarrow \mathbb{R}$ by $\omega(X_H, \cdot) = -dH$.

Trivialisation :

$$\Big(\partial M \times [1-\delta,\, 1],\, d(S\lambda)\Big) \longrightarrow \Big(\mathcal{V}(\partial M),\, \omega\Big) \ ,$$

$$(p,\, S) \longmapsto \varphi_{\ln S}(p) \ .$$

CLOSED CHARACTERISTICS

Remark : 1-1 correspondence between

1-periodic orbits of a hamiltonian $h(S)$

and

closed characteristics of period $h'(S)$ w.r.t Reeb vector field.

Symplectic completion :

$$\widehat{M} = (M, \omega) \bigcup \Big(\partial M \times [1, \infty[, d(S\lambda)\Big) \ .$$

Computation is a tool for applications : Morphism

$$FH^*(M) \xrightarrow{c^*} H^{n+*}(M, \partial M) \ .$$

If c^* is not bijective $\Rightarrow$ existence of closed characteristics.

CONSTRUCTION OF FLOER COHOMOLOGY

Assumptions : $\langle c_1, \pi_2 \rangle = 0$, $\langle \omega, \pi_2 \rangle = 0$.

Action functional for $H : \mathbb{S}^1 \times \widehat{M} \longrightarrow \mathbb{R}$:

$$A_H : C^\infty_{\text{contr}}(\mathbb{S}^1, \widehat{M}) \longrightarrow \mathbb{R}$$

$$A_H(\gamma) = -\int_{D^2} \bar{\gamma}^* \omega - \int_{\mathbb{S}^1} H(t, \gamma(t)) dt$$

Key :

$$\boxed{\text{Crit}(A_H) \longleftrightarrow \text{1-per. orbits of } X_H}$$

Compatible almost complex structure :

$$\omega(\cdot, J\cdot) \text{ Riemannian metric}$$

The Floer complex is based on the 1-periodic orbits and the differential counts L^2-gradient trajectories of A_H.

Remark : Analogue for Morse functions on finite dimensional manifolds : Morse (co)homology.

CONSTRUCTION OF FLOER COHOMOLOGY

$$FC^*_{[a,\,b]}(H,\,J) = \bigoplus_{\substack{i_{CZ}(x)=*\\ A_H(x)\in[a,\,b]\\ x\in\mathcal{P}(H)}} k\langle x\rangle,\ \ k \text{ field}$$

$$d\ \ : FC^*_{[a,\,b]}(H,\,J) \longrightarrow FC^{*+1}_{[a,\,b]}(H,\,J)$$

$$\langle x\rangle \longmapsto \sum \#\mathcal{M}(y,\,x)/\mathbb{R}\cdot\langle y\rangle$$

$$\mathcal{M}(y,\,x) = \{u\ \ : \mathbb{R}\times\mathbb{S}^1 \longrightarrow \widehat{M}\ \ : \bar{\partial}_J u = \nabla H\}$$

$$\lim_{s\to-\infty} u(s,\cdot) = y,\ \ \lim_{s\to+\infty} u(s,\cdot) = x$$

$$FH^*_{[a,\,b]}(H,J) = H^*(FC^*_{[a,\,b]}(H,\,J),\,d)$$

$$FH^*_{[a,\,b]}(M) = \varprojlim FH^*_{[a,\,b]}(H)$$

$$FH^*(M) = \varprojlim FH^*_{[a,\,b]}(M),\ \ a<0,\ \ b\to\infty$$

Key question : What class of hamiltonians ?
Link with convexity.

ASYMPTOTICALLY LINEAR HAMILTONIANS

Definition : Standard almost complex structure J on $\partial M \times [1, \infty[$:

$$
\begin{aligned}
J_{(p,\,S)}(\frac{\partial}{\partial S}) &= \frac{1}{S} R(p)\ , \\
J_{(p,\,S)}(R(p)) &= -S \frac{\partial}{\partial S}\ , \\
J_{(p,\,S)}|_{\xi} &= J_0\ ,
\end{aligned}
$$

where $\xi = \ker \lambda$, J_0 compatible with $\omega|_{\xi}$.

Definition : Asymptotically linear Hamiltonian $H(t, p, S)\ : \mathbb{S}^1 \times \partial M \times [1, \infty[\longrightarrow \mathbb{R}$ verifies :

There is a function $f : \partial M \longrightarrow \mathbb{R}^+$ s.t.

$$|X_H(t, p, S) - X_F(p, S)| = o(\sqrt{S})\ ,$$

where $F(p, S) = S f(p)$.

Key fact : Compactness theorems are valid in the class of asymptotically linear Ham.

GENERIC THEOREM

Theorem : Let $\widehat{F} \hookrightarrow \widehat{E} \to B$ be a fibration with $\widehat{F}$, $\widehat{E}$ symplectic completions of CTB manifolds. Under "suitable" assumptions, there is a spectral sequence $E_r^{*,*} \Rightarrow FH^*(\widehat{E})$ with :

- $E_2^{p,q} \simeq FH^p(B\ ; \mathcal{FH}^q(\widehat{F}))$

- Morphism of spectral sequences to classical Leray-Serre spectral sequence

$K_r^{*,*} \Rightarrow H^*(E, \partial E)$, $K_2^{p,q} \simeq H^p(B, \mathcal{H}^q(F, \partial F))$,

induced at E_2 level by

$c^*\ : FH^q(\widehat{F}) \longrightarrow H^{k+q}(F, \partial F)$, $\dim F = 2k$.

Proved cases :

- trivial fibration $E = B \times F$, where B, F have restricted CTB (Künneth formula).
- negative line bundle over compact symplectic base B.
- Reduced to conjectured energy estimate for general symplectic fibration over compact symplectic base.

KÜNNETH FORMULA

ASSUMPTIONS : $(\widehat{M}, \omega_1)$, $(\widehat{N}, \omega_2)$

symplectic completions of restricted CTB mfd,

symplectically aspherical : $\langle c_1, \pi_2 \rangle = 0$,

$\langle \omega_i, \pi_2 \rangle = 0$, $i = 1, 2$.

Symplectic form on $\widehat{M} \times \widehat{N}$: $\omega_1 \oplus \omega_2$.

Liouville v. field : $X_1 + X_2$.

Cohomology : coefficients in a field.

Commutative diagram :

$$\begin{tikzcd}
\bigoplus_{r+s=k} FH^r(M) \otimes FH^s(N) \arrow[r, "\sim"] \arrow[d, "c^* \otimes c^*"'] & FH^k(M \times N) \arrow[d, "c^*"] \\
\bigoplus_{r+s=k} H^{m+r}(M, \partial M) \otimes H^{n+s}(N, \partial N) \arrow[r, "\sim"] & H^{m+n+k}(M \times N, \partial(M \times N))
\end{tikzcd}$$

SPECTRAL SEQUENCES

Spectral sequence associated to filtered differential graded complex (C, d) : bigraded differential complex

$$(E_r^{p,q}, d_r)_{r \geq 0}$$

satisfying $E_{r+1} = H(E_r)$

Bidegree of differential is $(r, -r+1)$.

$$E_0^{p,q} = C_p^{p+q}/C_{p+1}^{p,q}$$

Roughly : $E_r^{p,q}$ kills elements which are in $d(C_{p-r+1})$.

$p + q =$ total degree

$p =$ degree of filtration

$r =$ depth of cohomological relations already exploited

Coefficients in a field : $H^k(C) \simeq \bigoplus_{p+q=k} E_\infty^{p,q}$

SYMPLECTIC FIBRATIONS

Assumptions : Fibration $\widehat{F} \hookrightarrow \widehat{E} \xrightarrow{\pi} B$

- (B, β) closed symplectic

- $\Omega \in \Omega^2(\widehat{E})$, $\Omega|_{\text{fiber}}$ symplectic, $d\Omega = 0$

- $d\Theta = \Omega$, $d\Lambda = \pi^*\beta$ outside compact set

- $\forall\ 0 < \epsilon \leq \epsilon_0$, $\Omega_\epsilon = \pi^*\beta + \epsilon\Omega$ symplectic, $\Theta_\epsilon = \Lambda + \epsilon\Theta$ Liouville form

Fundamental example : $\mathcal{L} \longrightarrow B$ hermitian line bundle, $c_1(\mathcal{L}) = -[\beta]$. Put $\Lambda = \theta^\nabla$, $\Omega = |\cdot|^2\theta^\nabla$, where θ^∇ angular form for hermitian connection with curvature $-2i\pi\beta$.

Fundamental objects :

- Horizontal distribution $H = (\ker \pi_*)^{\perp_{\Omega_\epsilon}}$ - tangent (or not) to ∂E ;

- Liouville vector field - preserving (or not) the horizontal distribution

Corollary of existence of spectral sequence for negative line bundles :

$$FH^*(\mathcal{L}) = 0 \ .$$

!!! This is a positive result!!!