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# The topological asymptotic analysis for the Maxwell equations and applications

Bessem Samet

► **To cite this version:**

Bessem Samet. The topological asymptotic analysis for the Maxwell equations and applications. Mathematics [math]. Université Paul Sabatier - Toulouse III, 2004. English. NNT: . tel-00005827

**HAL Id: tel-00005827**

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Submitted on 8 Apr 2004

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# L'analyse asymptotique topologique pour les équations de Maxwell et applications

Bessem SAMET

5 avril 2004



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# Chapitre 1

## Synthèse des travaux

### Sommaire

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## 1.1 Introduction

Le but de l'optimisation de forme est de trouver "la meilleure" forme géométrique d'un objet donné par rapport à certains critères physiques. Cet objet peut être une structure mécanique (un pylone), une forme immergée dans un fluide (un avion), un dispositif électromagnétique (une antenne),... Les critères à optimiser peuvent être respectivement la solidité, la trainée, l'énergie transmise,... Ces exemples ont tous en commun que le critère à optimiser fait intervenir la solution d'une équation aux dérivées partielles posée dans un domaine, qui peut être l'espace occupé par l'objet ou le complémentaire de cet espace. Schématiquement, un problème d'optimisation de forme peut être exprimé de la manière suivante : trouver  $\Omega^* \in \mathcal{O}_{ad}$  tel que

$$J(\Omega^*, u_{\Omega^*}) = \min_{\Omega \in \mathcal{O}_{ad}} J(\Omega, u_{\Omega}), \quad (1.1)$$

où  $\mathcal{O}_{ad}$  est une classe de domaines admissibles,  $u_{\Omega}$  est la solution d'un problème d'état relatif à l'ouvert variable  $\Omega$  et  $J$  est le critère à minimiser.

En dehors des méthodes stochastiques comme les algorithmes génétiques [23, 24, 25, 26, 37] qui restent d'un coût de calculs élevé, les techniques usuelles d'optimisation requièrent le calcul de la dérivée de la fonction coût. Il apparaît donc important de pouvoir disposer de la dérivée du critère qu'on souhaite minimiser. Et c'est là que les difficultés commencent ! En effet, pour des raisons évidentes, on a l'habitude de ne définir une notion de différentiabilité que dans des espaces vectoriels normés. Or l'ensemble des domaines de  $\mathbb{R}^n$  n'est pas muni d'une telle structure d'espace vectoriel. Plusieurs possibilités ont été étudiées :

- En optimisation de forme classique [11, 32, 18, 19, 22, 29, 36, 39], chaque domaine  $\Omega \in \mathcal{O}_{ad}$  est écrit sous la forme  $\Omega = F(\Omega_0)$ , où  $\Omega_0$  est un domaine de référence et  $F$  est une fonction de transport. Par exemple, dans [11], l'ensemble des domaines admissibles est donné par

$$\mathcal{O}_{ad} = \{\Omega_t = \mathcal{F}(\Omega_0, t), t \geq 0\}.$$

Ici, la fonction  $\mathcal{F}$  est construite à partir d'un champ de déplacement  $V$  par

$$\begin{cases} \frac{d}{dt} \mathcal{X}_X(t) = V(\mathcal{X}_X(t), t), \\ \mathcal{X}_X(0) = X \in \Omega_0, \end{cases}$$

avec  $\mathcal{F}(X, t) = \mathcal{X}_X(t)$ . La fonction coût est alors redéfinie par

$$j(t) = J(\Omega_t, u_{\Omega_t}), \quad \forall t \geq 0.$$

Nous pouvons alors parfaitement considérer les quotients différentiels du type  $\frac{j(t) - j(0)}{t}$  et si ce quotient possède une limite quand  $t \rightarrow 0$ , nous l'appelons dérivée de  $J$  en  $\Omega$  suivant la direction  $V$ . Cette méthode présente les limitations suivantes :

1. Une déformation infinitésimale de la forme se traduit par un petit déplacement de la frontière du domaine. Ainsi, les domaines admissibles ont tous la même topologie. Par exemple, si  $\Omega_0$  est simplement connexe, alors tous les domaines  $F(\Omega_0)$  obtenus par itérations successives seront simplement connexes.
  2. Cette approche implique des modifications du modèle éléments finis au cours du processus d'optimisation, ce qui peut être contraignant.
- Pour surmonter les difficultés liées aux problèmes de non-existence de la forme optimale, la méthode de relaxation a été proposée. Elle consiste en fait à élargir l'espace des formes admissibles. Par exemple, en s'appuyant sur la théorie de l'homogénéisation, G. Allaire [1] considère comme formes admissibles les matériaux composites obtenus par un mélange fin de matériau et de vide. Une telle structure composite est déterminée par deux fonctions :  $\theta(x)$ , la densité volumique locale de matériau, prenant ses valeurs entre 0 et 1, et  $A^*(x)$ , le tenseur de Hooke correspondant à sa microstructure. Pour plus de détails sur la théorie de l'homogénéisation, le lecteur peut se référer à [1, 10, 33, 41]. Une des propriétés les plus intéressantes de cette méthode est qu'elle permet de capturer sur un maillage fixe une forme optimale dont la topologie n'est pas fixée ni dictée par le choix des conditions initiales. Il faut noter cependant que plusieurs difficultés liées à l'utilisation de cette méthode subsistent :
1. La solution homogénéisée n'est pas classique : c'est une distribution de matériau composite. Des méthodes de pénalisation doivent être appliquées pour retrouver un domaine admissible.
  2. Cette approche est limitée au cadre strict de l'élasticité linéaire. Son application en électromagnétisme reste problématique.
- En suivant les idées de Osher et Sethian [34], G. Allaire, F. Jouve et A.-M. Toader [2] ont introduit une nouvelle méthode basée sur la combinaison des techniques d'optimisation de forme classique et la méthode Eulerienne des courbes de niveaux (level-set method). Dans cette approche, toutes les formes admissibles  $\Omega$  sont contenues dans un domaine fixe  $D$ . La frontière de  $\Omega$  est paramétrée par une fonction  $\psi$ , appelée fonction courbe de niveaux. Cette fonction est définie sur le domaine fixe  $D$  par

$$\begin{cases} \psi(x) = 0 \Leftrightarrow x \in \partial\Omega, \\ \psi(x) < 0 \Leftrightarrow x \in \Omega, \\ \psi(x) > 0 \Leftrightarrow x \in D \setminus \bar{\Omega}. \end{cases} \quad (1.2)$$

A chaque itération, la fonction  $\psi$  est calculée en résolvant une équation d'Hamilton-Jacobi, posée dans le domaine  $D$  et faisant intervenir la dérivée de forme classique. Cette méthode présente les avantages suivants :

1. La forme est capturée sur un maillage fixe.
2. Durant le processus d'optimisation, elle permet de réduire le nombre de trous, donc de changer de topologie.

Toutefois, cette approche souffre des inconvénients suivants :

1. Il n'est pas permis de créer des trous.
  2. Comme dans le cas de l'optimisation de forme classique, la forme optimale dépend fortement du choix de la géométrie initiale.
- La fonction level-set  $\psi$  est soumise à une contrainte de positivité à l'extérieur du domaine et une contrainte de négativité à l'intérieur. Nous proposons une méthode permettant à  $\psi$  d'être positif à l'intérieur du domaine et de créer des trous. La fonction  $\psi$  que nous proposons a une propriété fondamentale : sa positivité est une condition nécessaire et même suffisante d'optimalité. Il s'agit de la fonction gradient topologique. Plus précisément, soit  $\Omega$  un domaine borné de  $\mathbb{R}^N$  ( $N=2$  ou  $3$ ) et  $\mathcal{J}(\Omega) = J(\Omega, u_\Omega)$  est le critère à minimiser. Nous notons  $B(x, \varepsilon)$ ,  $x \in \Omega$ , la boule de rayon  $\varepsilon > 0$ ,  $B(x, \varepsilon) = \{y \in \mathbb{R}^N, \|y - x\| < \varepsilon\}$  et  $\overline{B(x, \varepsilon)}$  désigne la fermeture de  $B(x, \varepsilon)$ . Dans la plus part des cas, la variation  $\mathcal{J}(\Omega \setminus \overline{B(x, \varepsilon)}) - \mathcal{J}(\Omega)$  admet un développement asymptotique (par rapport à  $\varepsilon$ ) qui s'écrit sous la forme

$$\mathcal{J}(\Omega \setminus \overline{B(x, \varepsilon)}) - \mathcal{J}(\Omega) = f(\varepsilon)g(x) + o(f(\varepsilon)), \quad (1.3)$$

où  $f(\varepsilon)$  est une fonction positive qui tend vers 0 avec  $\varepsilon$ . Nous appelons l'expression (1.3) l'asymptotique topologique. La fonction  $g$  est appelée gradient topologique. A chaque itération, un certain pourcentage de matière est enlevé ou inséré (selon la nature du problème) aux endroits où  $g$  est la plus négative. Ainsi, la forme optimale  $\Omega_{opt}$  est caractérisée par  $g(x) \geq 0, \forall x \in \Omega_{opt}$ .

Cette dernière méthode nécessite donc de connaître le développement asymptotique de la variation  $\mathcal{J}(\Omega \setminus \overline{B(x, \varepsilon)}) - \mathcal{J}(\Omega)$ . Le premier travail sur ce sujet est celui de Eschenauer, Kobelev et Schumacher [16], dans lequel ils ont étudié la minimisation de la compliance en élasticité linéaire avec une condition de Neumann sur le bord du trou. Leur idée consiste à faire bouger le bord de la perforation en utilisant les outils de l'optimisation de forme classique. Cette approche a été appelée : "the bubble method". Toujours dans le cadre de l'élasticité linéaire, J. Sokolowski et A. Zochowski [40] ont généralisé cette approche à une certaine classe de fonctions coûts. Dans [30], M. Masmoudi a introduit deux outils mathématiques pour calculer l'expression de l'asymptotique topologique : une adaptation de la méthode d'état adjoint [11] et une technique de troncature de domaine. Il a considéré pour la première fois une condition de Dirichlet sur le bord du trou (circulaire). Dans [17], S. Garreau, Ph. Guillaume et M. Masmoudi ont étudié le cas d'un trou de forme quelconque dans le cadre de l'élasticité linéaire. Dans [20, 21], Ph.

Guillaume et K. Sididris ont étudié les problèmes de Poisson et de Stokes avec une condition de Dirichlet sur le bord d'un trou de forme quelconque. Sur le plan numérique, cette approche a été appliquée avec succès à l'élasticité linéaire [12, 17], en pétrophysique [15], à la mécanique des fluides [38], en électromagnétisme [28, 35], etc. Une des propriétés les plus intéressantes de cette méthode est que la simple observation du gradient topologique  $g$ , calculé à la première itération, donne une bonne idée de la forme à obtenir.

Tous les cas cités ci-dessus concernent des opérateurs dont le symbole est homogène. Dans cette thèse, les thèmes suivants sont abordés :

- Etude d'opérateurs dont le symbole est non-homogène.
- Insertion d'un objet diélectrique dans le domaine. Dans cette partie, nous généralisons les travaux de M.S. Vogelius et H. Ammari. L'utilisation d'une approche adjointe nous permet d'obtenir des développements asymptotiques pour une large classe de fonctions coûts.
- Création d'un trou sur le bord du domaine.

Après la synthèse effectuée ci-dessous, le document est constitué de cinq articles publiés ou soumis à publication.

## 1.2 Equation de Helmholtz : condition de Dirichlet sur le bord d'un trou [Chap2, Chap3]

### 1.2.1 Position du problème

Soit  $\Omega$  un domaine borné de  $\mathbb{R}^N$  ( $N = 2$  ou  $3$ ) et de frontière  $\Gamma$  assez régulière. Etant donné  $h \in H^{-\frac{1}{2}}(\Gamma)$ , nous supposons qu'il existe un unique  $u_\Omega \in H^1(\Omega)$  vérifiant

$$\begin{cases} \Delta u_\Omega + k^2 u_\Omega = 0 & \text{dans } \Omega, \\ \partial_n u_\Omega = h & \text{sur } \Gamma, \end{cases} \quad (1.4)$$

où  $\partial_n$  désigne la dérivée normale. La condition aux limites sur le bord extérieur n'intervient pas dans l'analyse asymptotique.

Soit  $\omega$  un domaine borné de  $\mathbb{R}^N$  contenant l'origine et  $x_0 \in \Omega$ . Pour tout  $\varepsilon > 0$  assez petit, nous posons  $\omega_\varepsilon = x_0 + \varepsilon\omega$  et  $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$ . La possibilité de faire un changement de coordonnées nous permet de supposer que  $x_0 = 0$ . Nous posons  $u_{\Omega_\varepsilon} \in H^1(\Omega_\varepsilon)$  la solution du problème perturbé

$$\begin{cases} \Delta u_{\Omega_\varepsilon} + k^2 u_{\Omega_\varepsilon} = 0 & \text{dans } \Omega_\varepsilon, \\ u_{\Omega_\varepsilon} = 0 & \text{sur } \partial\omega_\varepsilon, \\ \partial_n u_{\Omega_\varepsilon} = h & \text{sur } \Gamma. \end{cases} \quad (1.5)$$

Pour simplifier la présentation dans ce chapitre introductif, les espaces fonctionnels considérés sont tous des  $\mathbb{R}$ -espaces vectoriels.

Nous considérons une application  $J$  différentiable de  $H^1(\mathcal{O})$  dans  $\mathbb{R}$ , où  $\mathcal{O}$  est une partie voisine de  $\Gamma$ . Nous posons

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}), \forall \varepsilon \geq 0. \quad (1.6)$$

Pour simplifier, nous écrivons  $j(\varepsilon) = J(u_{\Omega_\varepsilon})$ . L'hypothèse faite sur la fonction coût  $J$  s'explique par le fait que dans les problèmes électromagnétiques les critères telle que la matrice  $S$  sont définis sur les accès d'un guide d'onde ou à l'infini pour les obstacles bornés. Dans le chapitre 4 par exemple, nous avons étudié une large classe de fonctions coûts.

Notre but est d'étudier le comportement asymptotique de la différence  $j(\varepsilon) - j(0)$  lorsque  $\varepsilon$  tend vers 0. Nous utilisons deux outils mathématiques : la méthode adjointe généralisée et la troncature de domaine.

## 1.2.2 Outils mathématiques

### La méthode adjointe généralisée

Nous rappelons que cette méthode a été introduite par M. Masmoudi dans [30]. Nous la généralisons dans [Chap2, Chap3] au cas d'un champ complexe et d'un problème non coercif.

Soit  $\mathcal{V}$  un espace de Hilbert fixe,  $\ell$  une forme linéaire continue sur  $\mathcal{V}$  et  $(a_\varepsilon)_{\varepsilon \geq 0}$  est une famille de formes bilinéaires continues sur  $\mathcal{V}$ . Nous supposons que l'hypothèse suivante soit satisfaite.

**Hypothèse 1** *Pour tout  $\varepsilon \geq 0$ , la forme bilinéaire  $a_\varepsilon$  est uniformément coercive : il existe une constante  $\alpha > 0$  (indépendante de  $\varepsilon$ ) telle que*

$$a_\varepsilon(u, u) \geq \alpha \|u\|^2, \quad \forall u \in \mathcal{V}. \quad (1.7)$$

*Il existe  $\delta_a$  une forme bilinéaire continue sur  $\mathcal{V}$  et une fonction  $f(\varepsilon)$  définie sur  $\mathbb{R}_+^*$  telles que*

$$\lim_{\varepsilon \rightarrow 0^+} f(\varepsilon) = 0, \quad (1.8)$$

$$\|a_\varepsilon - a_0 - f(\varepsilon)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} = o(f(\varepsilon)), \quad (1.9)$$

où  $\mathcal{L}_2(\mathcal{V})$  désigne l'espace des formes bilinéaires continues sur  $\mathcal{V}$ .

Soit  $u_\varepsilon$  la solution du problème : trouver  $u_\varepsilon \in \mathcal{V}$  tel que

$$a_\varepsilon(u_\varepsilon, v) = \ell(v), \quad \forall v \in \mathcal{V}. \quad (1.10)$$

En utilisant l'inégalité (1.7) pour  $u = u_\varepsilon - u_0$ , pour tout  $\varepsilon \geq 0$  la fonction  $u_\varepsilon$  est la solution de l'équation (1.10), les égalités (1.8) et (1.9) et la continuité de la forme bilinéaire  $\delta_a$ , nous obtenons le lemme suivant.

**Lemme 1** *Nous avons*

$$\|u_\varepsilon - u_0\| = O(f(\varepsilon)).$$

Nous posons

$$j(\varepsilon) = J(u_\varepsilon), \quad \forall \varepsilon \geq 0, \quad (1.11)$$

où  $J$  est une fonction différentiable de  $\mathcal{V}$  dans  $\mathbb{R}$ . Pour tout  $\varepsilon \geq 0$ , nous définissons le lagrangien par

$$\mathcal{L}_\varepsilon(u, v) = J(u) + a_\varepsilon(u, v) - \ell(v), \quad \forall u, v \in \mathcal{V}. \quad (1.12)$$

Le résultat classique qui dit que la dérivée de la fonction coût est égale à la dérivée partielle du lagrangien est remplacée par

$$J(u_\varepsilon) - J(u_0) = \mathcal{L}_\varepsilon(u_\varepsilon, v) - \mathcal{L}_0(u_0, v), \quad \forall v \in \mathcal{V}. \quad (1.13)$$

En utilisant (1.13), la différentiabilité de la fonction  $J$ , le lemme 1, l'hypothèse 1 et la continuité de la forme bilinéaire  $\delta_a$ , nous obtenons le résultat suivant.

**Théorème 1** *La variation  $j(\varepsilon) - j(0)$  admet le développement asymptotique suivant :*

$$j(\varepsilon) - j(0) = f(\varepsilon)\delta_a(u_0, p_0) + o(f(\varepsilon)),$$

où  $u_0$  est la solution du problème directe (1.10) pour  $\varepsilon = 0$  et  $p_0$  est la solution du problème adjoint : trouver  $p_0 \in \mathcal{V}$  tel que

$$a_0(w, p_0) = -DJ(u_0).w, \quad \forall w \in \mathcal{V}. \quad (1.14)$$

Revenons maintenant au problème perturbé (1.5) et à la fonction coût  $j(\varepsilon) = J(u_{\Omega_\varepsilon})$ . Pour avoir un développement asymptotique de notre critère, nous ne pouvons pas appliquer directement le théorème 1 car  $u_{\Omega_\varepsilon} \in H^1(\Omega_\varepsilon)$ , espace fonctionnel qui dépend du paramètre  $\varepsilon$ . Pour cela, nous avons besoin de considérer la technique proposée dans le paragraphe suivant.

### La technique de troncature de domaine

Cette technique fournit le cadre mathématique nécessaire pour l'application de la méthode adjointe généralisée. Nous fixons  $R > 0$ . Nous utilisons les notations suivantes :

- $\Omega_R = \Omega \setminus \overline{B(0, R)}$ ,
- $\Gamma_R = \partial B(0, R)$ ,
- $D_\varepsilon = B(0, R) \setminus \overline{\omega_\varepsilon}$ .

Pour tout  $\varepsilon \geq 0$ , nous définissons l'opérateur *Dirichlet-to-Neumann*  $T_\varepsilon$  par

$$\begin{aligned} T_\varepsilon : H^{1/2}(\Gamma_R) &\longrightarrow H^{-1/2}(\Gamma_R) \\ \varphi &\longmapsto T_\varepsilon \varphi = \nabla u_\varepsilon^\varphi \cdot n|_{\Gamma_R}, \end{aligned} \quad (1.15)$$

où  $n|_{\Gamma_R}$  est la normale sortante sur  $\Gamma_R$  et  $u_\varepsilon^\varphi$  est la solution du problème

$$\begin{cases} \Delta u_\varepsilon^\varphi + k^2 u_\varepsilon^\varphi = 0 & \text{dans } D_\varepsilon, \\ u_\varepsilon^\varphi = \varphi & \text{sur } \Gamma_R, \\ u_\varepsilon^\varphi = 0 & \text{sur } \partial\omega_\varepsilon. \end{cases} \quad (1.16)$$

Pour tout  $\varepsilon \geq 0$ , nous posons  $u_\varepsilon$  la solution du problème tronqué

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 & \text{dans } \Omega_R, \\ \partial_n u_\varepsilon + T_\varepsilon u_\varepsilon = 0 & \text{sur } \Gamma_R, \\ \partial_n u_\varepsilon = h & \text{sur } \Gamma. \end{cases} \quad (1.17)$$

Nous avons alors le résultat suivant.

**Proposition 1** *La restriction à  $\Omega_R$  de  $u_{\Omega_\varepsilon}$  solution de (1.5) est la solution  $u_\varepsilon$  du problème (1.17).*

Nous rappelons que notre fonction coût est de la forme  $j(\varepsilon) = J(u_{\Omega_\varepsilon|\mathcal{O}})$ , où  $\mathcal{O}$  est une partie voisine de  $\Gamma$ . Ainsi, en utilisant la proposition 1, nous pouvons écrire

$$j(\varepsilon) = J(u_\varepsilon), \quad \forall \varepsilon \geq 0.$$

La formulation variationnelle de (1.17) peut alors s'écrire : trouver  $u_\varepsilon \in H^1(\Omega_R)$  tel que

$$a_\varepsilon(u_\varepsilon, v) = \ell(v), \quad \forall v \in H^1(\Omega_R),$$

où la forme bilinéaire  $a_\varepsilon$  et la forme linéaire  $\ell$  sont données par

$$\begin{aligned} a_\varepsilon(u, v) &= \int_{\Omega_R} \nabla u \cdot \nabla v \, dx - k^2 \int_{\Omega_R} uv \, dx + \int_{\Gamma_R} T_\varepsilon uv \, d\sigma(x), \\ \ell(v) &= \int_{\Gamma} hv \, d\sigma(x). \end{aligned}$$

Ainsi, la variation  $(a_\varepsilon - a_0)$  s'écrit

$$(a_\varepsilon - a_0)(u, v) = \int_{\Gamma_R} (T_\varepsilon - T_0)uv \, d\sigma(x), \quad \forall u, v \in H^1(\Omega_R),$$

et le problème se ramène à trouver une expression asymptotique de la variation  $T_\varepsilon - T_0$ .

### 1.2.3 Calcul asymptotique de $T_\varepsilon - T_0$

#### Première méthode : développement en séries de Fourier

Dans le cas où  $\omega$  est la boule unité, nous pouvons avoir une expression explicite de l'opérateur  $T_\varepsilon$  sous la forme d'une série de Fourier. Par exemple, en dimension  $N = 2$ , nous avons pour tout  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$

$$T_\varepsilon \varphi = k \sum_{n \in \mathbb{Z}} \frac{J'_n(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y'_n(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \varphi_n e^{in\theta},$$

où  $(\varphi_n)$  sont les coefficients de Fourier de  $\varphi$ ,  $(J_n)$  et  $(Y_n)$  sont respectivement les fonctions de Bessel de premier et de deuxième espèce.

**Deuxième méthode : approximation par la solution d'une équation intégrale**

Dans le cas où  $\omega$  est de forme quelconque, nous proposons la technique suivante pour obtenir un développement asymptotique de  $T_\varepsilon - T_0$ . L'idée principale consiste à approcher  $u_\varepsilon^\varphi - u_0^\varphi$  par la solution d'un problème extérieur à  $\omega_\varepsilon$ , où seulement la partie principale de l'opérateur non-homogène est considérée. Plus précisément, pour étudier la variation  $(T_\varepsilon - T_0)\varphi$  avec  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ , nous regardons d'abord le comportement asymptotique de  $u_\varepsilon^\varphi - u_0^\varphi$ , ce qui est naturel vu que

$$(T_\varepsilon - T_0)\varphi = \nabla(u_\varepsilon^\varphi - u_0^\varphi) \cdot n|_{\Gamma_R}.$$

Nous rappelons que pour  $\varepsilon \geq 0$  et  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ ,  $u_\varepsilon^\varphi$  est la solution du problème (1.16) et nous prenons  $N = 3$ . La variation  $u_\varepsilon^\varphi - u_0^\varphi$  est solution de

$$\begin{cases} \Delta(u_\varepsilon^\varphi - u_0^\varphi) + k^2(u_\varepsilon^\varphi - u_0^\varphi) = 0 & \text{dans } D_\varepsilon, \\ u_\varepsilon^\varphi - u_0^\varphi = -u_0^\varphi & \text{sur } \partial\omega_\varepsilon, \\ u_\varepsilon^\varphi - u_0^\varphi = 0 & \text{sur } \Gamma_R. \end{cases} \quad (1.18)$$

Nous approchons  $u_\varepsilon^\varphi - u_0^\varphi$  par  $u_{\varepsilon,\varphi}$  solution de

$$\begin{cases} \Delta u_{\varepsilon,\varphi} + k^2 u_{\varepsilon,\varphi} = 0 & \text{dans } D_\varepsilon, \\ u_{\varepsilon,\varphi} = -u_0^\varphi(0) & \text{sur } \partial\omega_\varepsilon, \\ u_{\varepsilon,\varphi} = 0 & \text{sur } \Gamma_R. \end{cases} \quad (1.19)$$

Cette première approche se justifie facilement par l'utilisation d'un développement de Taylor de  $u_0^\varphi$ . Nous approchons ensuite  $u_{\varepsilon,\varphi}$  par  $v_\varepsilon^\varphi$ , avec  $v_\varepsilon^\varphi(x) = v_\omega^\varphi\left(\frac{x}{\varepsilon}\right)$  et  $v_\omega^\varphi$  est la solution du problème extérieur

$$\begin{cases} -\Delta v_\omega^\varphi = 0 & \text{dans } \mathbb{R}^3 \setminus \bar{\omega}, \\ v_\omega^\varphi = 0 & \text{à } \infty, \\ v_\omega^\varphi = -u_0^\varphi(0) & \text{sur } \partial\omega. \end{cases} \quad (1.20)$$

Nous exprimons  $v_\varepsilon^\varphi$  sous la forme

$$v_\varepsilon^\varphi(x) = \varepsilon \left( \int_{\partial\omega} p_\omega(x) d\sigma(x) \right) E(x) + O(\varepsilon^2), \quad (1.21)$$

où  $E$  est la solution élémentaire du Laplacien et  $p_\omega \in H^{-\frac{1}{2}}(\partial\omega)$  est la solution de l'équation intégrale

$$\int_{\partial\omega} E(y-x) p_\omega(x) d\gamma(x) = -u_0^\varphi(0), \quad \forall y \in \partial\omega.$$

En posant  $P_\omega^\varphi(x) = \left( \int_{\partial\omega} p_\omega(x) d\sigma(x) \right) E(x)$ , nous pouvons écrire

$$(u_\varepsilon^\varphi - u_0^\varphi)(x) = \varepsilon P_\omega^\varphi(x) + R(\varepsilon). \quad (1.22)$$



Si nous introduisons l'opérateur  $\delta T$  défini par

$$\delta T\varphi = \nabla P_\omega^\varphi \cdot n|_{\Gamma_R} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma_R), \quad (1.23)$$

nous obtenons

$$\|T_\varepsilon - T_0 - \varepsilon\delta T\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon).$$

Ce résultat est non exploitable, la méthode adjointe généralisée exige  $o(\varepsilon)$  et non  $O(\varepsilon)$ . C'est ici qu'intervient le fait que le symbole de l'opérateur est non-homogène. Pour cela, nous corrigeons l'approximation (1.22) par la prise en compte du terme diagonal, en utilisant un terme correctif  $Q_\omega^\varphi$ , solution de

$$\begin{cases} \Delta Q_\omega^\varphi + k^2 Q_\omega^\varphi = k^2 P_\omega^\varphi & \text{dans } D_0, \\ Q_\omega^\varphi = P_\omega^\varphi|_{\Gamma_R} & \text{sur } \Gamma_R. \end{cases} \quad (1.24)$$

En posant  $\delta T$  l'opérateur défini par

$$\delta T\varphi = \nabla(P_\omega^\varphi - Q_\omega^\varphi) \cdot n|_{\Gamma_R} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma_R), \quad (1.25)$$

nous obtenons le résultat désiré

$$\|T_\varepsilon - T_0 - \varepsilon\delta T\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = o(\varepsilon). \quad (1.26)$$

### Comparaison entre les deux méthodes

La première méthode ne peut être utilisée que dans le cas d'un trou ayant une forme particulière. Par exemple, dans le chapitre 2, nous l'avons utilisée pour un trou circulaire. L'avantage de cette méthode est qu'elle nous permet d'avoir une expression explicite de la variation  $T_\varepsilon - T_0$ , ce qui nous donne une idée précise sur le terme dominant. Cependant, lorsque le problème est non-homogène, l'utilisation de cette méthode pose beaucoup de difficultés : le terme général de la série fait intervenir des fonctions spéciales (les fonctions de Bessel dans notre cas), l'étude du comportement du terme général lorsque  $n \rightarrow \infty$  et  $\varepsilon \rightarrow 0$ , convergence uniforme des séries, ... Néanmoins, lorsque l'on connaît mal les propriétés de l'opérateur que l'on étudie, cette méthode peut être utilisée comme première approche. Sinon, la deuxième méthode nous permet d'étudier un trou de forme quelconque, elle est donc plus générale et elle permet d'éviter beaucoup de calculs.

#### 1.2.4 Les principaux résultats

Dans le chapitre 2, nous étudions l'équation de Helmholtz avec une condition de Dirichlet sur le bord d'un trou circulaire. Nous obtenons

$$j(\varepsilon) - j(0) = -\frac{2\pi}{\log(\varepsilon)} \Re(u_\Omega(x) \overline{p_\Omega(x)}) + o\left(\frac{1}{\log(\varepsilon)}\right), \quad (1.27)$$

où  $\Re$  désigne la partie réelle d'un nombre complexe,  $u_\Omega$  et  $p_\Omega$  sont respectivement l'état direct et l'état adjoint. Notons que le terme  $k^2$  n'apparaît pas dans l'expression du gradient topologique.

Dans le chapitre 3, nous étudions le cas d'un trou de forme quelconque. Nous montrons qu'en dimension  $N = 2$ , la formule (1.27) est indépendante de la forme du trou, ce qui n'est pas le cas lorsque  $N = 3$ . Dans cette situation, l'asymptotique topologique est donnée par

$$j(\varepsilon) - j(0) = \varepsilon \Re \left( A_\omega(u_\Omega(x)) \overline{p_\Omega(x)} \right) + o(\varepsilon), \quad (1.28)$$

où  $A_\omega(u_\Omega(x))$  est une quantité qui dépend de la forme de  $\omega$ . Dans le cas d'un trou sphérique, nous obtenons

$$j(\varepsilon) - j(0) = 4\pi\varepsilon \Re(u_\Omega(x) \overline{p_\Omega(x)}) + o(\varepsilon). \quad (1.29)$$

## 1.2.5 Les applications numériques

### Optimisation d'un guide d'onde

Dans le chapitre 2, nous utilisons le gradient topologique pour l'optimisation d'un guide d'onde fonctionnant en plan H. Nous présentons trois tests numériques. Dans le premier test, nous utilisons une méthode itérative. A chaque itération, nous choisissons un seuil au-dessus duquel nous insérons du métal. Dans le deuxième cas, la forme est choisie à partir de l'allure du gradient topologique calculé à l'état initial. Le troisième test consiste à ajouter un point de métal (de la taille d'une maille) là où le gradient topologique est le plus négatif, et d'itérer en calculant à chaque fois un nouveau gradient. Dans les trois cas étudiés, nous obtenons des résultats satisfaisants.

### Détection d'objets métalliques enfouis

La partie numérique présentée dans le chapitre 3 concerne la détection d'objets métalliques enfouis dans un sol, à partir des mesures de champ effectuées au dessus du sol. Trois tests numériques sont étudiés :

- Le sol correspond à un sable très sec. Cinq objets métalliques sont présents dans ce sol. Le gradient topologique est calculé dans le sol, mais en l'absence des objets métalliques. Une seule itération est suffisante pour avoir une bonne idée sur la position des objets.
- Les mesures de références sont effectuées sur un sol bosselé, et dont la permittivité relative est variable. Le gradient topologique est calculé sur un sol parfaitement plat et de permittivité constante. Une seule itération, et les cinq objets sont localisés, avec des images légèrement déformées par rapport au résultat précédent.
- Une méthode itérative est utilisée. A chaque itération, un point de métal est inséré là où le gradient est le plus négatif. le résultat obtenu est satisfaisant.

### 1.3 Insertion d'une inhomogénéité dans le domaine [Chap4, Chap5]

Dans ces deux articles, nous étudions l'asymptotique topologique lorsque nous insérons une petite inhomogénéité dans le domaine. Plusieurs études ont été réalisées sur ce sujet. Voir par exemple les travaux de H. Ammari, S. Moskow, H. Kang, D. Volkov, C. Alves, A. Friedman, M.S. Vogelius et A. Khelifi [3, 4, 5, 6, 7, 8, 13, 27]. Dans ces travaux, des développements asymptotiques de la solution ont été considérés. Dans le chapitre 4, nous utilisons une méthode adjointe pour obtenir l'expression de l'asymptotique topologique pour une large classe de fonctions coûts. Comme exemple modèle, nous considérons l'équation de Helmholtz en dimension  $N$  ( $N = 2$  ou  $3$ ). Dans le chapitre 5, nous étudions les équations de Maxwell en dimension  $N = 3$ .

#### 1.3.1 Position du problème : équation de Helmholtz

Soit  $\Omega$  un domaine borné de  $\mathbb{R}^N$  et de frontière  $\Gamma$  assez régulière. Nous supposons que  $\Omega$  contient une inhomogénéité  $\omega_\varepsilon = \varepsilon\omega$ , où  $\varepsilon$  est un petit paramètre et  $\omega$  est un domaine borné de  $\mathbb{R}^N$  contenant l'origine. Soit  $\alpha_\varepsilon$  la fonction constante par morceaux, définie par

$$\alpha_\varepsilon = \begin{cases} \alpha_0 & \text{dans } \Omega \setminus \overline{\omega_\varepsilon}, \\ \alpha_1 & \text{dans } \omega_\varepsilon, \end{cases}$$

où  $\alpha_0$  et  $\alpha_1$  sont des constantes. Une autre fonction  $\beta_\varepsilon$  est définie d'une manière analogue. Pour  $\varepsilon \geq 0$ , nous posons  $u_\varepsilon$  la solution dans  $H^1(\Omega)$  du problème

$$\begin{cases} \nabla \cdot (\alpha_\varepsilon \nabla u_\varepsilon) + \beta_\varepsilon u_\varepsilon = 0 & \text{dans } \Omega, \\ \partial_n u_\varepsilon = h & \text{sur } \Gamma. \end{cases} \quad (1.30)$$

La condition aux limites  $\partial_n u_\varepsilon = h$  n'intervient pas dans le calcul asymptotique et elle peut être remplacée par une autre condition. Nous nous intéressons à la détermination d'un développement asymptotique de la variation  $j(\varepsilon) - j(0)$ , où la fonction coût  $j$  est définie par

$$j(\varepsilon) = J_\varepsilon(u_\varepsilon), \quad \forall \varepsilon \geq 0. \quad (1.31)$$

#### 1.3.2 Formulation variationnelle

Pour tout  $\varepsilon \geq 0$ ,  $u_\varepsilon$  est la solution dans  $H^1(\Omega)$  de l'équation variationnelle

$$a_\varepsilon(u_\varepsilon, v) = \ell(v), \quad \forall v \in H^1(\Omega), \quad (1.32)$$

où la forme bilinéaire  $a_\varepsilon$  et la forme linéaire  $\ell$  sont données par

$$a_\varepsilon(u, v) = \int_\Omega \alpha_\varepsilon \nabla u \cdot \nabla v \, dx - \int_\Omega \beta_\varepsilon uv \, dx, \quad (1.33)$$

$$\ell(v) = \alpha_0 \int_\Gamma hv \, d\sigma(x). \quad (1.34)$$

Dans le paragraphe suivant, nous présentons deux méthodes différentes pour calculer l'asymptotique topologique dans cette situation.

### 1.3.3 Différentes possibilités

#### Première possibilité : utilisation de la méthode adjointe généralisée

Dans cette situation, nous pouvons aussi appliquer la méthode adjointe généralisée. Mais, il ne faut surtout pas tomber dans le piège. Bien que nous travaillons dans un espace fonctionnel fixe :  $H^1(\Omega)$ , nous ne pouvons pas appliquer directement la méthode adjointe généralisée. En effet, dans cette situation, la variation de la forme bilinéaire est donnée par

$$(a_\varepsilon - a_0)(u, v) = (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \nabla u \cdot \nabla v \, dx - (\beta_1 - \beta_0) \int_{\omega_\varepsilon} uv \, dx, \quad \forall u, v \in H^1(\Omega).$$

Si nous écrivons que  $\delta_a(u_0, p_0) = (\alpha_1 - \alpha_0) \nabla u_0(0) \cdot \nabla p_0(0) - (\beta_1 - \beta_0) u_0(0) p_0(0)$ , où  $u_0$  et  $p_0$  sont respectivement l'état direct et l'état adjoint, nous ne pouvons pas en déduire la variation de  $j(\varepsilon) - j(0)$  vu que l'application  $\delta_a$  n'est pas continue dans  $H^1(\Omega)$ . Le résultat ainsi obtenu est faux. Ainsi, comme dans le cas d'une perforation, nous devons utiliser la technique de troncature de domaine, malgré le fait que l'espace fonctionnel soit fixe.

#### Deuxième possibilité : une méthode adjointe pour les inhomogénéités

Nous supposons que les hypothèses suivantes sont satisfaites.

##### Hypothèse 2

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= J_\varepsilon(u_0) + DJ_\varepsilon(u_0) \cdot (u_\varepsilon - u_0) + f(\varepsilon) \delta J_1 + o(f(\varepsilon)), \\ J_\varepsilon(u_0) &= J_0(u_0) + f(\varepsilon) \delta J_2 + o(f(\varepsilon)), \end{aligned}$$

où  $\delta J_1$  et  $\delta J_2$  sont deux nombres réels qui dépendent de la fonction coût.

##### Hypothèse 3 Il existe $\delta_a \in \mathbb{R}$ tel que

$$(a_\varepsilon - a_0)(u_0, p_\varepsilon) = f(\varepsilon) \delta_a + o(f(\varepsilon)),$$

où  $p_\varepsilon$  est la solution de

$$a_\varepsilon(w, p_\varepsilon) = -DJ_\varepsilon(u_0) \cdot w, \quad \forall w \in H^1(\Omega).$$

Nous notons que l'hypothèse 2 est vérifiée pour une large classe de fonctions coûts. Nous obtenons alors le théorème suivant.

**Théorème 2** *La variation  $J_\varepsilon(u_\varepsilon) - J_0(u_0)$  admet le développement asymptotique suivant :*

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = f(\varepsilon) \delta j + o(f(\varepsilon)),$$

où  $\delta j = \delta_a + \delta J$  et  $\delta J = \delta J_1 + \delta J_2$ .

Fonction coût $j(\varepsilon)$	$\delta J$
$J(u_\varepsilon _\Gamma)$	0
$\int_\Omega \alpha_\varepsilon  u_\varepsilon - u_d ^2 dx$	$(\alpha_1 - \alpha_0) \omega   u_0(0) - u_d(0) ^2$
$\frac{1}{2} \int_\Omega \alpha_\varepsilon  \nabla(u_\varepsilon - u_d) ^2 dx$	$\frac{1}{2}(\alpha_1 - \alpha_0)(\nabla u_0(0)^T \mathcal{M}_\omega \overline{\nabla u_0(0)} -  \omega   \nabla u_0(0) ^2 +  \omega   \nabla u_0(0) - \nabla u_d(0) ^2)$

TAB. 1.1 – Fonctions coûts particulières (inhomogénéité).

### Comparaison des méthodes

Nous avons présenté deux méthodes différentes pour traiter un problème d'inhomogénéité. La première exige une troncature du domaine qui rend l'étude d'une fonction coût générale un peu compliquée (voir par exemple [20]). La deuxième évite les problèmes de la troncature et elle nous permet d'étudier plus facilement une large classe de fonctions coûts. Pour toutes ces raisons, nous utilisons la deuxième méthode pour obtenir l'expression de l'asymptotique topologique.

#### 1.3.4 Les principaux résultats

Dans le chapitre 4, nous étudions l'équation de Helmholtz en dimension  $N$ , posée dans un domaine contenant une inhomogénéité  $\omega_\varepsilon = \varepsilon\omega$ . Lorsque  $\omega$  est de forme quelconque, l'expression de l'asymptotique topologique est donnée par

$$j(\varepsilon) - j(0) = \varepsilon^N \Re \left\{ (\alpha_1 - \alpha_0) \nabla u_0(0)^T \mathcal{M}_\omega \overline{\nabla p_0} - (\beta_1 - \beta_0) |\omega| u_0(0) \overline{p_0(0)} + \delta J \right\} + o(\varepsilon^N),$$

où  $\mathcal{M}_\omega$  est une matrice qui dépend de la forme de  $\omega$ . Dans le cas particulier où  $\omega$  est la boule unité, nous obtenons

$$j(\varepsilon) - j(0) = \varepsilon^N \Re \left\{ \frac{N\alpha_0(\alpha_1 - \alpha_0)}{(N-1)\alpha_0 + \alpha_1} |\omega| \nabla u_0(0) \cdot \overline{\nabla p_0(0)} - (\beta_1 - \beta_0) |\omega| u_0(0) \overline{p_0(0)} + \delta J \right\} + o(\varepsilon^N).$$

Notons que le terme  $\delta J$  qui apparaît dans ces formules dépend de la fonction coût étudiée (voir Tableau 1.1). Nous obtenons aussi l'expression de l'asymptotique topologique dans le cas où  $\omega$  est une ellipse de grand axe  $a$  et de petit axe  $b$ . Dans cette situation, la matrice  $\mathcal{M}_\omega$  est donnée par

$$\mathcal{M}_\omega = \pi ab \begin{pmatrix} \frac{\alpha_0(1+a) + \alpha_1(b-1)}{\alpha_0 a + \alpha_1 b} & 0 \\ 0 & \frac{\alpha_0(1+b) + \alpha_1(a-1)}{\alpha_0 b + \alpha_1 a} \end{pmatrix}.$$

Il est intéressant de voir (formellement) que si  $\alpha_1 \rightarrow 0$ ,  $\beta_1 \rightarrow 0$ ,  $\alpha_0 = 1$  et  $\beta_0 = k^2$ , nous obtenons des formules d'asymptotique topologique pour l'équation de Helmholtz avec une condition de Neumann sur le bord d'un trou. Par exemple, dans le cas d'un trou sphérique, nous obtenons

$$j(\varepsilon) - j(0) = \varepsilon^N \Re \left\{ \frac{-N}{N-1} |\omega| \nabla u_0(0) \cdot \overline{\nabla p_0(0)} + |\omega| k^2 u_0(0) \overline{p_0(0)} + \delta J \right\} + o(\varepsilon^N).$$

Dans le cas d'un trou elliptique, la matrice  $\mathcal{M}_\omega$  est donnée par

$$\mathcal{M}_\omega = \begin{pmatrix} \pi(a+1)b & 0 \\ 0 & \pi(b+1)a \end{pmatrix}.$$

En faisant tendre  $b$  vers 0, nous retrouvons la formule dans le cas d'une fissure [9] :  $\omega = [-a, a] \times \{0\}$ . Elle est donnée par

$$j(\varepsilon) - j(0) = \varepsilon^2 \Re \left( -\pi a (\nabla u_0(0) \cdot \mathbf{n}) (\overline{\nabla p_0(0) \cdot \mathbf{n}}) + \delta J \right) + o(\varepsilon^2),$$

où  $\mathbf{n}$  est un vecteur normal à la fissure.

Dans le chapitre 5, nous étudions les équations de Maxwell en dimension  $N = 3$ . Plus précisément, pour tout  $\varepsilon \geq 0$ , nous considérons  $H_\varepsilon$  la solution dans  $H(\text{rot}, \Omega)$  de

$$\nabla \times (\alpha_\varepsilon \nabla \times H_\varepsilon) + \beta_\varepsilon H_\varepsilon = 0 \text{ dans } \Omega,$$

avec une condition aux limites sur  $\Gamma$ . Dans le cas d'une inhomogénéité de forme quelconque, nous obtenons

$$j(\varepsilon) - j(0) = \varepsilon^3 \Re \left\{ (\alpha_1 - \alpha_0) \nabla \times H_0(0) \cdot \overline{\mathcal{M}_\omega \left( \frac{\alpha_1}{\alpha_0} \right) \nabla \times p_0(0)} + \beta_0 \left( 1 - \frac{\beta_0}{\beta_1} \right) \right. \\ \left. H_0(0) \cdot \overline{\mathcal{M}_\omega \left( \frac{\beta_0}{\beta_1} \right) p_0(0)} \right\} + o(\varepsilon^3),$$

où  $\mathcal{M}_\omega$  est une matrice de polarisation qui dépend de la forme de  $\omega$ . Dans le cas d'une inhomogénéité sphérique, nous obtenons

$$j(\varepsilon) - j(0) = 4\pi\varepsilon^3 \Re \left\{ \frac{\alpha_0(\alpha_1 - \alpha_0)}{\alpha_0 + 2\alpha_1} \nabla \times H_0(0) \cdot \overline{\nabla \times p_0(0)} + \frac{\beta_0(\beta_1 - \beta_0)}{\beta_1 + 2\beta_0} H_0(0) \cdot \overline{p_0(0)} \right\} \\ + o(\varepsilon^3).$$

Lorsque  $\alpha_1 \rightarrow 0$ ,  $\beta_1 \rightarrow 0$ ,  $\alpha_0 = 1$  et  $\beta_0 = -k^2$ , nous obtenons (formellement) la formule dans le cas d'une bille métallique. Elle est donnée par

$$j(\varepsilon) - j(0) = 2\pi\varepsilon^3 \Re \left\{ -2\nabla \times H_0(0) \cdot \overline{\nabla \times p_0(0)} + k^2 H_0(0) \cdot \overline{p_0(0)} \right\} + o(\varepsilon^3).$$

### 1.3.5 Les applications numériques

Le chapitre 5 contient diverses applications numériques en électromagnétisme (six tests numériques). Les deux premiers tests ont été déjà présentés dans le chapitre 3. Le troisième, concerne la détection d'objets métalliques enfouis dans un sol à forte permittivité. Pour pouvoir traiter ce problème, on peut procéder en deux étapes :

- reconstruction du profil du sol en utilisant le gradient topologique par rapport à l'insertion de billes diélectriques,
- calcul du gradient topologique en prenant en compte le profil du sol.

Les quatrième et cinquième tests concernent la détection de la forme d'objets métalliques en espace libre. le sixième est dans le même contexte, mais ici, le gradient topologique est calculé par rapport à l'insertion de paillettes de métal.

## 1.4 Insertion d'un trou sur le bord du domaine [Chap6]

Dans cet article, nous étudions le cas d'un trou situé sur le bord du domaine. Plus précisément, nous considérons le problème suivant. Soit  $\Omega$  un domaine borné du plan. Une partie  $\Gamma_0$  du bord est définie par deux segments formant un angle de sommet  $O$  (l'origine) et de mesure  $\lambda\pi$ ,  $0 < \lambda \leq 2$ . Nous notons  $u_\Omega$  la solution du problème de Laplace posé dans le domaine  $\Omega$ , vérifiant  $u = 0$  sur  $\Gamma_0$  et une condition aux limites sur  $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma_0}$ . Pour  $\varepsilon > 0$  (assez petit), nous considérons le domaine perturbé  $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ , où  $S_\varepsilon$  est le secteur défini par  $S_\varepsilon = \{(r, \theta); 0 \leq r < \varepsilon, 0 \leq \theta \leq \lambda\pi\}$ . Notre but est de donner une expression asymptotique de la variation  $J(u_{\Omega_\varepsilon}) - J(u_\Omega)$ , où  $u_{\Omega_\varepsilon}$  est la solution du problème de Laplace posé dans le domaine perturbé avec une condition de Dirichlet imposée sur l'arc de cercle joignant les deux segments du secteur  $S_\varepsilon$ . La méthode adjointe généralisée et la technique de troncature sont proposées pour résoudre ce problème. Dans le cas où  $\lambda^{-1} \in \mathbb{N}^*$ , nous obtenons

$$J(u_{\Omega_\varepsilon}) - J(u_\Omega) = \pi \left[ \left( \frac{1}{\lambda} \right)! \right]^{-2} \varepsilon^{2/\lambda} \frac{\partial^{1/\lambda} u_\Omega}{\partial x^{1/\lambda}}(0) \frac{\partial^{1/\lambda} p_\Omega}{\partial x^{1/\lambda}}(0) + o(\varepsilon^{2/\lambda}),$$

où  $p_\Omega$  est l'état adjoint. Nous remarquons que l'expression du gradient topologique dépend de l'angle de singularité : plus l'angle est petit et plus des dérivées d'ordre élevé de l'état direct et de l'état adjoint apparaissent.

Nous avons présenté ci-dessus les principaux thèmes abordé dans les cinq articles qui constituent cette thèse. Cependant, du travail reste à faire dans cette direction. En particulier, les points suivants mériteraient d'être étudiés :

- Dans le chapitre 5, nous avons montré l'intérêt d'une perturbation filaire pour identifier une structure formée d'arêtes. L'expression du gra-

dient topologique utilisée, qui a donné d'excellents résultats, n'a pas encore été justifiée.

- Dans le chapitre 4, les formules asymptotiques obtenues ont de nombreuses applications potentielles en électromagnétisme notamment en imagerie.
- Nous n'avons pas encore étudié des problèmes d'évolution comme par exemple l'équation des ondes.





# Bibliographie

- [1] G. Allaire. *Shape optimization by the homogenization method*, Springer, Applied Mathematical Sciences Vol. 146 (2002).
- [2] G. Allaire, F. Jouve, A.-M. Toader. *A level-set method for shape optimization*, C. R. Acad. Sci. Paris, Série I, 334, pp. 1125-1130 (2002).
- [3] C. Alves, H. Ammari. *Boundary integral formulas for the reconstruction of imperfections of small diameter in an elastic medium*, SIAM J. Appl. Math. 62, 94-106 (2001).
- [4] H. Ammari. *An inverse initial boundary value problem for the wave equation in the presence of imperfections of small volume*, SIAM J. Control Optim. 41, 1194-1211 (2003).
- [5] H. Ammari, H. Kang. *A new method for reconstructing electromagnetic inhomogeneities of small volume*, Inverse problems 19, 63-71 (2003).
- [6] H. Ammari, H. Kang. *Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities*, submitted to J. Math. Anal. Appl.
- [7] H. Ammari, S. Moskow. *Asymptotic expansions for eigenvalues in the presence of small inhomogeneities*, Math. Meth. Appl. Sci. 26, 67-75 (2003).
- [8] H. Ammari, D. Volkov. *Asymptotic formulas for perturbations in the eigenfrequencies of the full Maxwell equations due to the presence of imperfections of small diameter*, Asympt. Anal. 30, 331-350 (2002).
- [9] S. Amstutz. *The topological asymptotic for the Helmholtz equation with respect to the creation of a hole and a crack with a Neumann condition*, article soumis dans SIAM J. Control Optim.
- [10] M. Bendsoe, N. Kikuchi. *Generating Optimal Topologies in Structural Design Using a Homogenization Method*, Comp. Meth. Appl. Mech. Eng., 71, 197-224 (1988).
- [11] J. Céa. *Conception optimale ou identification de forme, calcul rapide de la dérivée directionnelle de la fonction coût*, RAIRO Modél. Math. Anal. Numér., 20 pp. 371-402 (1986).

- [12] J. C ea, S. Garreau, Ph. Guillaume, M. Masmoudi. *The shape and topological optimizations connection*, In Fourth World Congress on Computational Mechanics (1998).
- [13] D.J. Cedio-Fengya, S. Moskov and M.S. Vogelius. *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction*, Inverse problems 14, 553-595 (1988).
- [14] G. Cheng, N. Olhoff. *An investigation concerning optimal design of solid elastic plates*, Int. J. Solids Struct. 16, pp. 305-323 (1981).
- [15] P.-F. EDOA. *Inversion de formes g ologiques en ing nierie de gisements en utilisant les donn es de production*, Th ese, Universit  Paul Sabatier (1999).
- [16] H. A. Eschenauer, V. V. Kobelev, A. Schumacher. *Bubble method for topology and shape optimization of structures*, Structural Optimization, 8 : 42-51 (1994).
- [17] S. Garreau, Ph. Guillaume, M. Masmoudi. *The topological asymptotic for PDE systems : the elasticity case*, SIAM J. Control Optim, 39, 1756-1778, (2001).
- [18] Ph. Guillaume. *D riv es d'ordre sup rieur en conception optimale de forme*, Th ese pr sent e   l'universit  Paul Sabatier, (1994).
- [19] Ph. Guillaume, M. Masmoudi. *D riv es d'ordre sup rieur en optimisation de domaines*, C.R. Acad. Sci. Paris, 315, s rie I : 859-862 (1992).
- [20] Ph. Guillaume, K. Sididris. *The topological asymptotic expansion for the Dirichlet problem*, SIAM J. Control Optim., 41 no 4, 1042-1072 (2002).
- [21] Ph. Guillaume, K. Sididris. *Topological sensitivity and shape optimization for the Stokes equations*, accepted in SIAM J. Control Optim.
- [22] E.J. Haug, K.K. Choi, V. Koskov. *Design Sensitivity Analysis of Structural Systems*, Academic Press (1986).
- [23] A.B. Kahng, B. R. Moon. *Toward more powerful recombinations*, In L. J. Eshelman, editor, Proceedings of the 6<sup>th</sup> International Conference on Genetic Algorithms, pages 96-103. Morgan Kaufmann (1995).
- [24] C. Kane, F. Jouve, M. Schoenauer. *Structural topology optimization in linear and nonlinear elasticity using genetical algorithms*, In 21 st ASME Design Automatic Conference, Boston MA (1995). ASME Press.
- [25] C. Kane, M. Schoenauer. *A drum shape optimization by genetic algorithms*, In Complex Systems - Mechanism of Adaptation. IOS Press and Ohmsha (1994).

- [26] C. Kane, M. Schoenauer. *Genetic operators for two-dimensional shape optimization*, volume 1063 of Lecture Notes in Computer Science. Springer-Verlag (1995).
- [27] A. Khelifi. *Electromagnetic scattering from small dielectric inhomogeneities*, PHD Thesis, Ecole Polytechnique, February 2002.
- [28] P. Mader. *Conception de composants antenne par gradient topologique*, Thèse, Université Paul Sabatier (2002).
- [29] M. Masmoudi. *Outils pour la conception optimale de formes*, Thèse d'état, Nice (1987).
- [30] M. Masmoudi. *The Topological Asymptotic*, Computational Methods for Control Applications, R. Glowinski, H. Kawarada and J. Periaux eds., GAKUTO Internat. Ser. Math. Sci. Appl. Vol. 16, pp. 53-72 (2001).
- [31] F. Murat. *Contre-exemples pour divers problèmes où le contrôle intervient dans les coefficients*, Ann. Mat. Pura Appl., 112, 49-68 (1977).
- [32] F. Murat, J. Simon. *Sur le contrôle par un domaine géométrique*, Thèse d'état, Paris (1976).
- [33] F. Murat, L. Tartar. *Calcul des variations et Homogénéisation*, Les méthodes de l'homogénéisation Théorie et Applications en Physique, Coll. Dir. Etudes et Recherches EDF, Eyrolles, 319-369 (1985).
- [34] S. Osher, J.A. Sethian. *Front propagation with curvature dependent speed : algorithms based on Hamilton-Jacobi formulations*, J. Comp. Phys. 78, 12-49 (1988).
- [35] J. Pommier. *L'asymptotique topologique en électromagnétisme*, Thèse, Université Paul Sabatier (2002).
- [36] B. Rousselet. *Quelques résultats en optimisation de domaines*, Thèse d'état, Nice (1982).
- [37] M. Schoenauer, L. Kallel, F. Jouve. *Mechanics inclusions identification by evolutionary computation*, Revue européenne des éléments finis, 5(5-6) : 619-648 (1996).
- [38] K. Sididris. *Sensibilité topologique en optimisation de forme*, Thèse, Université Paul Sabatier (2001).
- [39] J. Simon. *Domain variation for drag in Stokes flow*, Control theory of distributed systems and applications, Proceedings of IFIP Conference in Shanghai, éditeur Li Xunjing (1990).
- [40] J. Sokolowski, A. Zochowski. *On the topological derivative in shape optimization*, SIAM J. Control Optim. 37, no. 4, 1251-1272 (1999).
- [41] K. Suzuki, N. Kikuchi. *A homogeneization method for shape and topology optimization*, Comp. Meth. Appl. Mech. Eng., 93, 291-318 (1991).



# Chapitre 2

## The topological asymptotic for the Helmholtz equation

### Sommaire

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Ce chapitre est constitué de l'article : *The topological asymptotic for the Helmholtz equation*, SIAM J. Control Optim. Vol. 42, no. 5, pp. 1523-1544 (2003). C'est un travail réalisé en collaboration avec Samuel Amstutz et Mohamed Masmoudi.

## 2.1 Introduction

Classical shape optimization methods are based on the perturbation of the boundary of the initial shape. The initial and the final shape have the same topology. The aim of topological optimization is to find an optimal shape without any *a priori* assumption about the topology of the structure. Many important contributions in this field are concerned with structural mechanics and in particular the minimization of the compliance (external work) subject to a volume constraint. In view of the fact that the optimal structure has generally a large number of small holes, most authors [3, 5, 15] have considered composite material optimization. Using the homogenization theory G. Allaire *and al.* [3] exhibit a class of laminated materials with an explicit expression for the optimal material at any point of the structure. The range of application of this approach is quite restricted. For this reason, global optimization techniques like genetic algorithms and simulated annealing are used in order to solve more general problems [26]. Unfortunately, these methods are very slow.

The topological gradient has been introduced by A. Schumacher [27] to minimize a cost function  $j(\Omega) = J(\Omega, u_\Omega)$  where  $u_\Omega$  is the solution to a partial differential equation defined in the domain  $\Omega$ . The idea is to create a spherical hole  $B(x, \varepsilon)$  of radius  $\varepsilon$  around a point  $x$  in  $\Omega$ . Generally, an asymptotic expansion of the function  $j$  can be obtained in the following form :

$$j(\Omega \setminus \overline{B(x, \varepsilon)}) - j(\Omega) = f(\varepsilon)g(x) + o(f(\varepsilon)). \quad (2.1)$$

The function  $f(\varepsilon)$  is positive and tends to zero with  $\varepsilon$ . We call this expansion the topological asymptotic. To minimize the criterion, we have to create holes where  $g$  is negative. The optimality condition  $g \geq 0$  in  $\Omega$  is exactly what G. Buttazzo and G. Dal Maso [6] have obtained for the Laplace equation, using a relaxed formulation. The topological gradient  $g(x)$  has been computed by A. Schumacher [27] in the case of compliance minimization with Neumann condition on the boundary of the hole. In the same context, J. Sokolowski [25] gave some mathematical justifications in the plane stress case, and generalized it to various cost functions. A topological sensitivity framework using an adaptation of the adjoint method and a truncation technique has been introduced in [16] in the case of an homogeneous Dirichlet condition imposed on the boundary of a circular hole. The fundamental property of the adjoint technique is to provide the variation of a function with respect to a parameter by using a solution  $u_\Omega$  and an adjoint state  $p_\Omega$  which do not depend on

the chosen parameter. From the numerical view point, only two systems have to be solved for obtaining  $g(x)$  for all  $x \in \Omega$ . This observation leads to very efficient numerical algorithms. In [10, 11, 12], the topological sensitivity has been obtained in the contexts of linear elasticity, the Poisson equation and the Stokes problem with general shape functions and arbitrary shaped holes. These publications are concerned with PDE's operators whose symbols are homogeneous polynomials.

In this paper, we are interested in the differential operator

$$P = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} + k^2,$$

whose symbol is not homogenous. First, an adaptation of the adjoint method to the topological context is proposed in section 2.2 for the operator  $P$ . Next, a wave guide problem, the truncation method and the explicit expression of the topological asymptotic are presented in Section 2.3. Finally, an optimization algorithm and some applications of the topological gradient to wave guides optimization are given in section 2.4. This work is done in collaboration with Alcatel Space Industries.

## 2.2 A generalized adjoint method

In this section, the adjoint method is adapted to topological optimization. Let  $\mathcal{V}$  be a fixed complex Hilbert space. For  $\varepsilon \geq 0$ , let  $a_\varepsilon(\cdot, \cdot)$  be a sesquilinear and continuous form on  $\mathcal{V}$  and  $l_\varepsilon$  be a semilinear and continuous form on  $\mathcal{V}$ . We consider the following assumptions.

**Hypothesis 1** *There exists a sesquilinear and continuous form  $\delta_a$ , a semilinear and continuous form  $\delta_l$ , and a real function  $f(\varepsilon) > 0$  defined on  $\mathbb{R}_+^*$  such that*

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0, \quad (2.2)$$

$$\|a_\varepsilon - a_0 - f(\varepsilon)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} = o(f(\varepsilon)), \quad (2.3)$$

$$\|l_\varepsilon - l_0 - f(\varepsilon)\delta_l\|_{\mathcal{L}(\mathcal{V})} = o(f(\varepsilon)), \quad (2.4)$$

where  $\mathcal{L}(\mathcal{V})$  (respectively  $\mathcal{L}_2(\mathcal{V})$ ) denotes the space of continuous and semilinear (respectively sesquilinear) forms on  $\mathcal{V}$ .

**Hypothesis 2** *There exists a constant  $\alpha > 0$  such that*

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_0(u, v)|}{\|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}} \geq \alpha.$$

We say that  $a_0$  satisfies the inf-sup condition.



According to (2.3), there exists a constant  $\beta > 0$  (independent of  $\varepsilon$ ) such that

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_\varepsilon(u, v)|}{\|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}} \geq \beta \quad \forall \varepsilon \geq 0.$$

For  $\varepsilon \geq 0$ , we suppose that the following problem has one solution : find  $u_\varepsilon \in \mathcal{V}$  such that

$$a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v) \quad \forall v \in \mathcal{V}. \quad (2.5)$$

According to Hypothesis 2, this solution is unique. We have the following lemma.

**Lemma 1** *If Hypotheses 1 and 2 are satisfied then*

$$\|u_\varepsilon - u_0\|_{\mathcal{V}} = O(f(\varepsilon)).$$

*Proof.* It follows from Hypothesis 2 that there exists  $v_\varepsilon \in \mathcal{V}, v_\varepsilon \neq 0$  such that

$$\beta \|u_\varepsilon - u_0\|_{\mathcal{V}} \|v_\varepsilon\|_{\mathcal{V}} \leq |a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon)|,$$

which implies

$$\begin{aligned} \beta \|u_\varepsilon - u_0\|_{\mathcal{V}} \|v_\varepsilon\|_{\mathcal{V}} &\leq |a_\varepsilon(u_0, v_\varepsilon) - l_\varepsilon(v_\varepsilon)| \\ &= |a_\varepsilon(u_0, v_\varepsilon) - (l_\varepsilon - l_0 - f(\varepsilon)\delta_l)(v_\varepsilon) - l_0(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\ &= |(a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon)) - (l_\varepsilon - l_0 - f(\varepsilon)\delta_l)(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\ &\leq |a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon) - f(\varepsilon)\delta_a(u_0, v_\varepsilon)| + |l_\varepsilon(v_\varepsilon) - l_0(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\ &\quad + f(\varepsilon)(|\delta_a(u_0, v_\varepsilon)| + |\delta_l(v_\varepsilon)|). \end{aligned}$$

Using Hypothesis 1, we obtain

$$\beta \|u_\varepsilon - u_0\|_{\mathcal{V}} \|v_\varepsilon\|_{\mathcal{V}} \leq (o(f(\varepsilon)) + f(\varepsilon)(\|\delta_a\|_{\mathcal{L}_2(\mathcal{V})}\|u_0\|_{\mathcal{V}} + \|\delta_l\|_{\mathcal{L}(\mathcal{V})})) \|v_\varepsilon\|_{\mathcal{V}}.$$

□

Consider now a cost function  $j(\varepsilon) = J(u_\varepsilon)$ , where the functional  $J$  satisfies

$$J(u + h) = J(u) + \Re(L_u(h)) + o(\|h\|_{\mathcal{V}}) \quad \forall u, h \in \mathcal{V}. \quad (2.6)$$

Here,  $L_u$  is a linear and continuous form on  $\mathcal{V}$ . We suppose that the problem : find  $p_0 \in \mathcal{V}$  such that

$$a_0(v, p_0) = -L_{u_0}(v) \quad \forall v \in \mathcal{V}, \quad (2.7)$$

has a unique solution  $p_0$ , called the adjoint state. For  $\varepsilon \geq 0$ , we define the lagrangian operator  $\mathcal{L}_\varepsilon$  by

$$\mathcal{L}_\varepsilon(u, v) = J(u) + a_\varepsilon(u, v) - l_\varepsilon(v) \quad \forall u, v \in \mathcal{V}.$$

The next theorem gives the asymptotic expansion of  $j(\varepsilon)$ .

**Theorem 1** *If Hypotheses 1 and 2 are satisfied, then*

$$j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta_{\mathcal{L}}(u_0, p_0)) + o(f(\varepsilon)), \quad (2.8)$$

where  $u_0$  is the solution to Equation (2.5) with  $\varepsilon = 0$ ,  $p_0$  is the adjoint state, solution to Problem (2.7) and

$$\delta_{\mathcal{L}}(u, v) = \delta_a(u, v) - \delta_l(v), \quad \forall u, v \in \mathcal{V}.$$

*Proof.* We have that

$$j(\varepsilon) = \mathcal{L}_\varepsilon(u_\varepsilon, v) \quad \forall \varepsilon \geq 0, \forall v \in \mathcal{V}.$$

Next, choosing  $v = p_0$ , we obtain

$$\begin{aligned} j(\varepsilon) - j(0) &= \mathcal{L}_\varepsilon(u_\varepsilon, p_0) - \mathcal{L}_0(u_0, p_0) \\ &= J(u_\varepsilon) - J(u_0) + a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0) + l_0(p_0) - l_\varepsilon(p_0) \\ &= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0)) - \Re(l_\varepsilon(p_0) - l_0(p_0)) \\ &= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0) + a_0(u_\varepsilon - u_0, p_0)) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Using (2.6), we have that

$$J(u_\varepsilon) - J(u_0) = \Re(L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}).$$

Hence,

$$\begin{aligned} j(\varepsilon) - j(0) &= \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + \Re(a_0(u_\varepsilon - u_0, p_0) + L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Using that  $p_0$  is the adjoint solution, we obtain

$$\begin{aligned} j(\varepsilon) - j(0) &= \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)) \\ &= \Re((a_\varepsilon - a_0)(u_0, p_0)) + \Re((a_\varepsilon - a_0)(u_\varepsilon - u_0, p_0)) + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

It follows from Hypothesis 1 that

$$\begin{aligned} j(\varepsilon) - j(0) &= f(\varepsilon)\Re(\delta_a(u_0, p_0)) + o(f(\varepsilon)) + f(\varepsilon)\Re(\delta_a(u_\varepsilon - u_0, p_0)) + o(f(\varepsilon))\|u_\varepsilon - u_0\|_{\mathcal{V}} \\ &\quad + o(\|u_\varepsilon - u_0\|_{\mathcal{V}}) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Finally, from Lemma 1 and the hypothesis  $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$ , we have

$$j(\varepsilon) = j(0) + f(\varepsilon)\Re(\delta_a(u_0, p_0) - \delta_l(p_0)) + o(f(\varepsilon)),$$

since  $\delta_a$  is continuous by assumption. □

### 2.3 A wave guide problem

In this section, we study a problem of a wave guide as a component of a spatial antenna feeding system. Because the wave guide  $\mathcal{O}$  has a uniform thickness,  $\mathcal{O} = \Omega \times ]a, b[$ ,  $\Omega \subset \mathbb{R}^2$  and the electric field has a vertical polarization (normal to  $\Omega$ ), the 3D problem can be reduced to a 2D problem in  $\Omega$ , called the H-plane model. We assume that  $\Omega$  is a domain of  $\mathbb{R}^2$  with a regular boundary  $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_N$ ,  $N \in \mathbb{N}^*$ . We denote by  $u_\Omega$  the normal component to  $\Omega$  of the electric field. It is a solution to the Helmholtz problem :

$$\begin{cases} \Delta u_\Omega + k^2 u_\Omega = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_0, \\ \partial_n u_\Omega - iku_\Omega = h_j & \text{on } \Gamma_j, j = 1, 2, \dots, N, \end{cases} \quad (2.9)$$

where  $\partial_n u_\Omega$  is the normal derivative of  $u_\Omega$ ,  $k \in \{k \in \mathbb{C}^* / \Im(k) \geq 0\}$  and  $h_j \in H_{00}^{\frac{1}{2}}(\Gamma_j)'$  for all  $j \in \{1, 2, \dots, N\}$ . The first boundary condition means that  $\Gamma_0$  is a perfect metallic surface. When  $h_j = 0$ , the last equation is an approximate absorbing boundary condition (the normal incident plane waves are completely absorbed). When  $h_j \neq 0$ , it is a transmission condition. We prove in Appendix 2.5.1 that Problem (2.9) has one and only one solution in the Hilbert space

$$\mathcal{V}_\Omega = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_0\}. \quad (2.10)$$

Here and in the following, all the Sobolev spaces involve complex-valued functions.

For a given  $x \in \Omega$ , let us consider the perforated open set  $\Omega_\varepsilon = \Omega \setminus \overline{B(x, \varepsilon)}$ , where  $x$  is a point of  $\Omega$ ,  $B(x, \varepsilon)$  is the ball of center  $x$  and of radius  $\varepsilon$  (see Figure 2.1). We assume that  $\varepsilon > 0$  is small enough, and we denote  $\Sigma_\varepsilon = \partial B(x, \varepsilon)$ . Our aim is to get the sensitivity analysis of  $u_{\Omega_\varepsilon}$ , being the

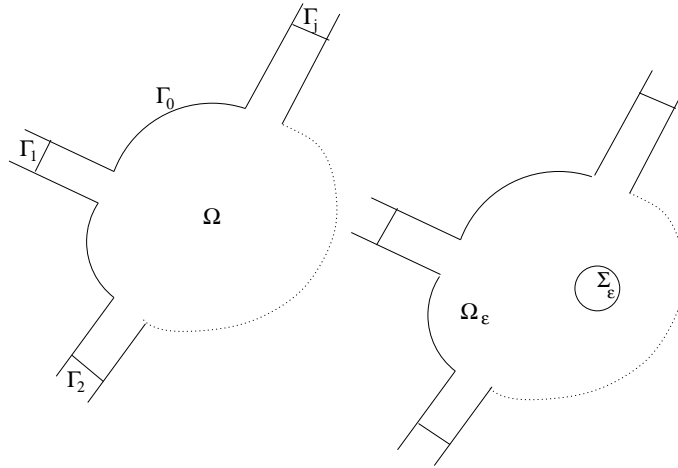


FIG. 2.1 – The initial domain and the same domain after the perforation.

unique solution (see Appendix 2.5.1) to :

$$\begin{cases} \Delta u_{\Omega_\varepsilon} + k^2 u_{\Omega_\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\ u_{\Omega_\varepsilon} = 0 & \text{on } \Gamma_0, \\ u_{\Omega_\varepsilon} = 0 & \text{on } \Sigma_\varepsilon, \\ \partial_n u_{\Omega_\varepsilon} - iku_{\Omega_\varepsilon} = h_j & \text{on } \Gamma_j, j = 1, 2, \dots, N, \end{cases} \quad (2.11)$$

with respect to  $\varepsilon$  at  $\varepsilon = 0$ . The solution of Problem (2.11) is defined on the variable open set  $\Omega_\varepsilon$ , thus it belongs to a functional space which depends on  $\varepsilon$ . Hence, if we want to derive the asymptotic expansion of a function of the form

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}), \quad (2.12)$$

we cannot apply directly the tools of Section 2.2, which require a fixed functional space. In classical shape optimization, this requirement can be satisfied with the help of a domain parameterization technique [13, 20, 17]. This technique involves a fixed domain and a bi-Lipshitz map between this domain and the modified one. In the topology optimization context, such a map does not exist between  $\Omega$  and  $\Omega_\varepsilon$ . However, a functional space independent of  $\varepsilon$  can be constructed by using a domain truncation technique.

### 2.3.1 The domain truncation

Let  $R > \varepsilon$  be such that the ball  $B(x, R)$  is included in  $\Omega$ . The boundary of  $B(x, R)$  is denoted by  $\Sigma_R$ . The truncated domain  $\Omega \setminus \overline{B(x, R)}$  is denoted by  $\Omega_R$  and  $D_\varepsilon$  denotes the corona  $B(x, R) \setminus \overline{B(x, \varepsilon)}$  (see Figure 2.2).

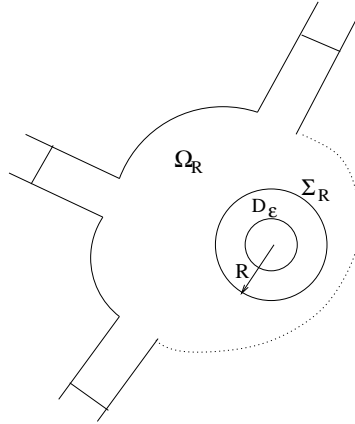


FIG. 2.2 – The truncated domain.

For a  $\Psi \in H^{\frac{1}{2}}(\Sigma_R)$ , we consider  $u_\Psi^\varepsilon$  the solution to the problem

$$\begin{cases} \Delta u_\Psi^\varepsilon + k^2 u_\Psi^\varepsilon = 0 & \text{in } D_\varepsilon, \\ u_\Psi^\varepsilon = \Psi & \text{on } \Sigma_R, \\ u_\Psi^\varepsilon = 0 & \text{on } \Sigma_\varepsilon, \end{cases} \quad (2.13)$$

and the *Dirichlet-to-Neumann* operator

$$\begin{aligned} T^\varepsilon : H^{1/2}(\Sigma_R) &\longrightarrow H^{-1/2}(\Sigma_R) \\ \Psi &\longmapsto T^\varepsilon \Psi = \nabla u_\Psi^\varepsilon \cdot n|_{\Sigma_R}, \end{aligned}$$

where  $n|_{\Sigma_R}$  denotes the outward normal to the boundary  $\Sigma_R$ . Using the Poincaré inequality, we obtain that, for  $\varepsilon < R < (\sqrt{2}|k|)^{-1}$ , the problem (2.13) is coercive. Hence it has one and only one solution.

We consider the truncated problem : find  $u_\varepsilon$  such that

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 & \text{in } \Omega_R, \\ u_\varepsilon = 0 & \text{on } \Gamma_0, \\ \partial_n u_\varepsilon + T^\varepsilon u_\varepsilon = 0 & \text{on } \Sigma_R, \\ \partial_n u_\varepsilon - ik u_\varepsilon = h_j & \text{on } \Gamma_j, j = 1, 2, \dots, N. \end{cases} \quad (2.14)$$

The variational formulation associated to Problem (2.14) is the following : find  $u_\varepsilon \in \mathcal{V}_R$  such that

$$a_\varepsilon(u_\varepsilon, v) = l(v) \quad \forall v \in \mathcal{V}_R, \quad (2.15)$$

where the functional space  $\mathcal{V}_R$ , the sesquilinear form  $a_\varepsilon$ , and the semilinear form  $l$  are defined by

$$\mathcal{V}_R = \{u \in H^1(\Omega_R), u = 0 \text{ on } \Gamma_0\}, \quad (2.16)$$

$$a_\varepsilon(u, v) = \int_{\Omega_R} \nabla u \cdot \overline{\nabla v} \, dx - k^2 \int_{\Omega_R} u \overline{v} \, dx + \int_{\Sigma_R} (T^\varepsilon u) \overline{v} \, d\gamma(x) \quad (2.17)$$

$$-ik \sum_{j=1}^N \int_{\Gamma_j} u \overline{v} \, d\gamma(x), \quad (2.18)$$

$$l(v) = \sum_{j=1}^N \int_{\Gamma_j} h_j \overline{v} \, d\gamma(x). \quad (2.19)$$

Here,  $\nabla u \cdot \overline{\nabla v} = \sum_{i=1}^2 \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_i}$  and  $d\gamma(x)$  is the Lebesgue measure on the boundary. The following result is standard in PDE theory.

**Proposition 2** *The problem (2.14) has one and only one solution in  $\mathcal{V}_R$  which is the restriction to  $\Omega_R$  of the solution to (2.11).*

*Proof.* Existence : Applying the definition of  $T^\varepsilon$ , we prove that the restriction to  $\Omega_R$  of the solution to (2.11) is a solution to (2.14).

Uniqueness : Any solution  $u$  to Problem (2.14) can be extended in  $\Omega_\varepsilon$  to the solution to Problem (2.11) : we use the solution  $u_\Psi^\varepsilon$  to (2.13) with  $\Psi = u|_{\Sigma_R}$ .  $\square$

We have now at our disposal the fixed Hilbert space  $\mathcal{V}_R$  required by Section 2.2. We assume that the function  $J$  is defined in a neighbor part of  $\Gamma$ . Then we have

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}) = J(u_\varepsilon) \quad \forall \varepsilon \geq 0. \quad (2.20)$$

### 2.3.2 Variation of the sesquilinear form

The variation of the sesquilinear form  $a_\varepsilon - a_0$  reads

$$a_\varepsilon(u, v) - a_0(u, v) = \int_{\Sigma_R} ((T^\varepsilon - T^0)u) \bar{v} d\gamma(x). \quad (2.21)$$

Hence, the problem reduces to the computation of  $(T^\varepsilon - T^0)\Psi$  for  $\Psi = u|_{\Sigma_R}$ . We have the following proposition.

**Proposition 3** *The solution  $u_\Psi^\varepsilon$  to Problem (2.13) and the operator  $T^\varepsilon$  are given by the explicit expressions :*

$$u_\Psi^\varepsilon(r, \theta) = \sum_{n \in \mathbb{Z}} \frac{J_n(kr)Y_n(k\varepsilon) - J_n(k\varepsilon)Y_n(kr)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n e^{in\theta}$$

and

$$T^\varepsilon \psi = k \sum_{n \in \mathbb{Z}} \frac{J'_n(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y'_n(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n e^{in\theta}, \quad (2.22)$$

where  $(r, \theta)$  are the polar coordinates in  $\mathbb{R}^2$ ,  $(\Psi_n)$  are the Fourier coefficients of  $\Psi$ ,  $(J_n)$  and  $(Y_n)$  are respectively the Bessel functions of the first and the second kind.

*Proof.* We have in polar coordinates :

$$u_\Psi^\varepsilon(r, \theta) = \sum_{n \in \mathbb{Z}} c_n(r) e^{in\theta},$$

where  $c_n(r)$  satisfies the differential equation :

$$\frac{d^2 c_n}{dr^2} + \frac{1}{r} \frac{dc_n}{dr} + (k^2 - \frac{n^2}{r^2}) c_n(r) = 0 \quad \forall n \in \mathbb{Z},$$

and thus  $c_n$  is a linear combination of  $J_n$  and  $Y_n$  Bessel functions :

$$c_n(r) = a_n J_n(kr) + b_n Y_n(kr) \quad \forall n \in \mathbb{Z}.$$

Using the boundary conditions, we obtain

$$a_n = \frac{Y_n(k\varepsilon)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n, \quad b_n = \frac{-J_n(k\varepsilon)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n.$$

□

In particular, for  $\varepsilon = 0$  we have :

**Proposition 4** *The solution  $u_\Psi^0$  and the operator  $T^0$  are given by the explicit expressions*

$$u_\Psi^0(r, \theta) = \sum_{n \in \mathbb{Z}} \frac{J_n(kr)}{J_n(kR)} \psi_n e^{in\theta}$$

and

$$T^0\psi = k \sum_{n \in \mathbb{Z}} \frac{J'_n(kR)}{J_n(kR)} \psi_n e^{in\theta}, \quad (2.23)$$

where  $u_\psi^0$  is the solution to (2.13) for  $\varepsilon = 0$ .

For  $\Psi \in H^s(\Sigma_R)$ , let

$$\|\psi\|_{s, \Sigma_R}^2 = \sum_{n \in \mathbb{Z}} |\psi_n|^2 (1 + |n|)^{2s} \quad (2.24)$$

be the norm of  $\Psi$  in this space. The so defined norm is equivalent to the usual norm of  $H^s(\Sigma_R)$ . We introduce the operator :

$$\begin{aligned} \delta_T : H^{1/2}(\Sigma_R) &\longrightarrow H^{-1/2}(\Sigma_R) \\ \Psi &\longmapsto \delta_T \Psi = \frac{1}{RJ_0^2(kR)} \Psi_0. \end{aligned}$$

We have the following lemma.

**Lemma 2** *We have that*

$$\|T^\varepsilon - T^0 - \frac{-1}{\log(\varepsilon)} \delta_T\|_{\mathcal{L}(H^{1/2}(\Sigma_R); H^{-1/2}(\Sigma_R))} = o\left(\frac{-1}{\log(\varepsilon)}\right).$$

*Proof.* Let  $\Psi \in H^{\frac{1}{2}}(\Sigma_R)$ . Using the series (2.22) and (2.23), we obtain

$$\begin{aligned} (T^\varepsilon - T^0)\psi &= k \sum_{n \in \mathbb{Z}} \frac{J'_n(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y'_n(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n e^{in\theta} - k \sum_{n \in \mathbb{Z}} \frac{J'_n(kR)}{J_n(kR)} \psi_n e^{in\theta} \\ &= k \sum_{n \in \mathbb{Z}^*} \frac{J'_n(kR)Y_n(k\varepsilon) - J_n(k\varepsilon)Y'_n(kR)}{J_n(kR)Y_n(k\varepsilon) - Y_n(kR)J_n(k\varepsilon)} \psi_n e^{in\theta} - k \sum_{n \in \mathbb{Z}^*} \frac{J'_n(kR)}{J_n(kR)} \psi_n e^{in\theta} \\ &\quad - k \frac{Y'_0(kR)J_0(kR) - Y_0(kR)J'_0(kR)}{J_0^2(kR)} \frac{J_0(k\varepsilon)J_0(kR)}{J_0(kR)Y_0(k\varepsilon) - Y_0(kR)J_0(k\varepsilon)} \psi_0. \end{aligned}$$

We have that [1]

$$\begin{aligned} \frac{Y'_0(kR)J_0(kR) - Y_0(kR)J'_0(kR)}{J_0^2(kR)} &= \frac{W\{J_0(kR), Y_0(kR)\}}{J_0^2(kR)} \\ &= \frac{2}{\pi kR} \frac{1}{J_0^2(kR)}, \end{aligned}$$

where  $W$  is the Wronskian. Then

$$\begin{aligned} (T^\varepsilon - T^0)\psi &= k \sum_{n \in \mathbb{Z}^*} \frac{J_n(k\varepsilon)Y_n(kR)}{Y_n(k\varepsilon)J_n(kR) - Y_n(kR)J_n(k\varepsilon)} \left( \frac{J'_n(kR)}{J_n(kR)} - \frac{Y'_n(kR)}{Y_n(kR)} \right) \psi_n e^{in\theta} \\ &\quad - \frac{2}{\pi} \frac{J_0(k\varepsilon)J_0(kR)}{J_0(kR)Y_0(k\varepsilon) - Y_0(kR)J_0(k\varepsilon)} \frac{1}{RJ_0^2(kR)} \psi_0. \end{aligned} \quad (2.25)$$

We have the following formula [1] :

$$Y_0(k\varepsilon) = \frac{2}{\pi} \left( \log\left(\frac{k\varepsilon}{2}\right) + \gamma \right) J_0(k\varepsilon) + \varepsilon\alpha(\varepsilon), \quad (2.26)$$

where  $\gamma$  denotes Euler's constant and  $\alpha(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ . We insert (2.26) into (2.25) :

$$(T^\varepsilon - T^0)\psi = \varepsilon R_\varepsilon \Psi + \frac{-1}{\log(\varepsilon)} \left( 1 + \frac{M}{\log(\varepsilon)} + \varepsilon\theta(\varepsilon) \right)^{-1} \delta_T \Psi,$$

where  $M$  is a constant independent of  $\varepsilon$ ,  $\theta(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$  and

$$R_\varepsilon \psi = \sum_{n \in \mathbb{Z}^*} \frac{k}{\varepsilon} \frac{J_n(k\varepsilon) Y_n(kR)}{Y_n(k\varepsilon) J_n(kR) - Y_n(kR) J_n(k\varepsilon)} \left( \frac{J'_n(kR)}{J_n(kR)} - \frac{Y'_n(kR)}{Y_n(kR)} \right) \psi_n e^{in\theta}.$$

Then

$$\left( T^\varepsilon - T^0 - \frac{-1}{\log(\varepsilon)} \delta_T \right) \psi = \varepsilon R_\varepsilon \psi + O(1) \left( \frac{-1}{\log(\varepsilon)} \right)^2 \frac{1}{R J_0^2(kR)} \psi_0.$$

Using (2.24), we have

$$\begin{aligned} \| R_\varepsilon \psi \|_{-\frac{1}{2}; \Sigma_R}^2 &= \sum_{n \in \mathbb{Z}^*} \frac{|k|^2}{\varepsilon^2} \left| \frac{J_n(k\varepsilon) Y_n(kR)}{Y_n(k\varepsilon) J_n(kR) - Y_n(kR) J_n(k\varepsilon)} \right|^2 \\ &\quad \cdot \left| \frac{J'_n(kR)}{J_n(kR)(1+|n|)} - \frac{Y'_n(kR)}{Y_n(kR)(1+|n|)} \right|^2 (1+|n|) |\psi_n|^2. \end{aligned}$$

Let us prove that there exists a constant  $c > 0$  (independent of  $\Psi$  and  $\varepsilon$ ) such that for all  $0 < \varepsilon < \varepsilon_0 < R$ ,

$$\| R_\varepsilon \psi \|_{-\frac{1}{2}; \Sigma_R} \leq c \| \psi \|_{\frac{1}{2}; \Sigma_R}.$$

We have [1]

$$\frac{1}{1+|n|} \frac{J'_n(kR)}{J_n(kR)} = -\frac{1}{1+|n|} \frac{J_{n+1}(kR)}{J_n(kR)} + \frac{n}{1+|n|} \frac{1}{kR}$$

and for  $n \rightarrow \infty$

$$J_n(z) \sim (2\pi n)^{-\frac{1}{2}} \left( \frac{ez}{2n} \right)^n.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{1+|n|} \frac{J_{n+1}(kR)}{J_n(kR)} = 0$$



and

$$\left| \frac{1}{1+|n|} \frac{J'_n(kR)}{J_n(kR)} \right| \leq c \quad \forall n \in \mathbb{Z}^*.$$

Here and in what follows,  $c$  is a positive constant independent of the data (e.g., of  $\varepsilon$  and  $n$ ). Similarly, we have

$$\left| \frac{1}{1+|n|} \frac{Y'_n(kR)}{Y_n(kR)} \right| \leq c \quad \forall n \in \mathbb{Z}^*.$$

Hence,

$$\left| \frac{J'_n(kR)}{J_n(kR)(1+|n|)} - \frac{Y'_n(kR)}{Y_n(kR)(1+|n|)} \right| \leq c \quad \forall n \in \mathbb{Z}^*.$$

We denote

$$f_n(\varepsilon) = \frac{1}{\varepsilon} \left| \frac{J_n(k\varepsilon)Y_n(kR)}{Y_n(k\varepsilon)J_n(kR) - Y_n(kR)J_n(k\varepsilon)} \right|.$$

We have also

$$f_n(\varepsilon) = \left| \frac{\varepsilon J_n(kR)Y_n(k\varepsilon)}{J_n(k\varepsilon)Y_n(kR)} - \varepsilon \right|^{-1}.$$

We show in Appendix 2.5.3 that there exist  $n_0$  and  $\varepsilon_0$  such that

$$\left| \frac{J_n(kR)}{J_n(k\varepsilon)} \right| \geq c \left( \frac{R}{\varepsilon} \right)^{n-1} \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0 \quad (2.27)$$

and

$$\left| \frac{Y_n(k\varepsilon)}{Y_n(kR)} \right| \geq c \left( \frac{R}{\varepsilon} \right)^n \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0. \quad (2.28)$$

Using (2.27) and (2.28), we obtain

$$\left| \frac{\varepsilon Y_n(k\varepsilon)J_n(kR)}{J_n(k\varepsilon)Y_n(kR)} \right| \geq c \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0$$

and

$$f_n(\varepsilon) \leq c \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0.$$

For  $p \in \{1, 2, \dots, n_0 - 1\}$ , we have  $f_p(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ . Then

$$f_n(\varepsilon) \leq c \quad \forall n \in \mathbb{Z}^*, \forall \varepsilon < \varepsilon_0.$$

Hence

$$\|R_\varepsilon \psi\|_{-\frac{1}{2}, \Sigma_R} \leq c \|\psi\|_{\frac{1}{2}, \Sigma_R} \quad \forall \psi \in H^{\frac{1}{2}}(\Sigma_R).$$

This completes the proof. □

From this lemma we obtain

**Proposition 5** *Let  $\delta_a$  be the sesquilinear and continuous form defined on  $\mathcal{V}_R$  by*

$$\delta_a(u, v) = \frac{u^{mean} \overline{v^{mean}}}{J_0(kR) J_0(kR)},$$

where  $u^{mean}$  and  $v^{mean}$  denote respectively the mean values of  $u$  and  $v$  on  $\Sigma_R$ . We have

$$\left| a_\varepsilon(u, p) - a_0(u, p) - \frac{-2\pi}{\log(\varepsilon)} \delta_a(u, p) \right| = o\left(\frac{-1}{\log(\varepsilon)}\right) \|u\|_{\mathcal{V}_R} \|p\|_{\mathcal{V}_R} \quad \forall u, p \in \mathcal{V}_R.$$

### 2.3.3 The asymptotic expansion

We prove in Appendix 2.5.2 that the sesquilinear form  $a_0$  satisfies Hypothesis 2 (inf-sup condition).

The adjoint problem is : find  $p_\Omega \in \mathcal{V}_\Omega$  such that

$$\int_{\Omega} (\nabla v \cdot \overline{\nabla p_\Omega} - k^2 v \overline{p_\Omega}) dx - ik \sum_{j=1}^N \int_{\Gamma_j} v \overline{p_\Omega} d\gamma(x) = -L_{u_\Omega}(v) \quad \forall v \in \mathcal{V}_\Omega. \quad (2.29)$$

This problem has one and only one solution (see Appendix 2.5.1). If  $L_{u_\Omega} \in H_{00}^{\frac{1}{2}}(\Gamma_m)'$ ,  $m \in \{1, 2, \dots, N\}$ , the strong formulation of Problem (2.29) is :

$$\begin{cases} \Delta p_\Omega + \bar{k}^2 p_\Omega = 0 & \text{in } \Omega, \\ p_\Omega = 0 & \text{on } \Gamma_0, \\ \partial_n p_\Omega + i\bar{k} p_\Omega = -L_{u_\Omega} & \text{on } \Gamma_m, \\ \partial_n p_\Omega + i\bar{k} p_\Omega = 0 & \text{on } \Gamma_j, j \in \{1, 2, \dots, N\} \setminus \{m\}. \end{cases} \quad (2.30)$$

Hence, all the assumptions of Section 2.2 are satisfied and we can apply the adjoint method. Then we have the following theorem.

**Theorem 2** *The function  $j$  has the following asymptotic expansion*

$$j(\varepsilon) - j(0) = \frac{-2\pi}{\log(\varepsilon)} \Re(u_\Omega(x) \overline{p_\Omega}(x)) + o\left(\frac{-1}{\log(\varepsilon)}\right).$$

*Proof.* Using Theorem 1, we obtain

$$j(\varepsilon) - j(0) = \frac{-2\pi}{\log(\varepsilon)} \Re(\delta_a(u_0, p_0)) + o\left(\frac{-1}{\log(\varepsilon)}\right),$$

where  $u_0$  is the solution to (2.15) for  $\varepsilon = 0$  and  $p_0$  is the solution to the adjoint problem

$$a_0(v, p_0) = -L_{u_0}(v) \quad \forall v \in \mathcal{V}_R. \quad (2.31)$$

As observed in Proposition 2,  $u_0$  is the restriction to  $\Omega_R$  of  $u_\Omega$ . Let us prove that the same property holds for  $p_0$  and  $p_\Omega$ . For  $v \in \mathcal{V}_\Omega$ , we denote by  $p_R$  and  $v_R$  the restriction of  $p_\Omega$  and  $v$  to  $\Omega_R$ . On the one hand, we have

$$\begin{aligned}
& \int_{\Omega} (\nabla v \cdot \overline{\nabla p_\Omega} - k^2 v \overline{p_\Omega}) dx - ik \sum_{j=1}^N \int_{\Gamma_j} v \overline{p_\Omega} d\gamma(x) \\
&= \int_{\Omega_R} (\nabla v_R \cdot \overline{\nabla p_R} - k^2 v_R \overline{p_R}) dx - ik \sum_{j=1}^N \int_{\Gamma_j} v_R \overline{p_R} d\gamma(x) + \int_{D_0} (\nabla v \cdot \overline{\nabla p_\Omega} - k^2 v \overline{p_\Omega}) dx \\
&= \int_{\Omega_R} (\nabla v_R \cdot \overline{\nabla p_R} - k^2 v_R \overline{p_R}) dx - ik \sum_{j=1}^N \int_{\Gamma_j} v_R \overline{p_R} d\gamma(x) + \int_{\Sigma_R} (T^0 v_R) \overline{p_R} d\gamma(x) \\
&= a_0(v_R, p_R). \tag{2.32}
\end{aligned}$$

On the other hand, due to the fact that  $J$  is defined in a neighbor part of  $\Gamma$ , we have that  $J(u) = J(u_R)$  for all  $u \in \mathcal{V}_\Omega$ . Hence

$$L_{u_\Omega}(v) = L_{u_0}(v_R). \tag{2.33}$$

Then, gathering (2.32), (2.29) and (2.33), we obtain

$$a_0(v_R, p_R) = -L_{u_0}(v_R) \quad \forall v_R \in \mathcal{V}_R,$$

which proves that  $p_R$  is the solution to (2.31). Then  $p_0$  is the restriction to  $\Omega_R$  of  $p_\Omega$ . It remains to prove that  $\delta_a(u_\Omega|_{\Omega_R}, p_\Omega|_{\Omega_R}) = u_\Omega(x) \cdot p_\Omega(x)$ . Using that  $u_\Omega$  is the solution to Helmholtz equation in the ball  $B(x, R)$ , we obtain

$$u_\Omega(x) = \frac{u_\Omega|_{\Sigma_R}^{mean}}{J_0(kR)}.$$

Similarly, we have

$$\overline{p_\Omega}(x) = \frac{\overline{p_\Omega|_{\Sigma_R}^{mean}}}{J_0(kR)}.$$

Hence

$$\begin{aligned}
\delta_a(u_0, p_0) &= \delta_a(u_\Omega|_{\Omega_R}, p_\Omega|_{\Omega_R}) \\
&= u_\Omega(x) \overline{p_\Omega(x)}.
\end{aligned}$$

This completes the proof. □

Then the topological gradient is

$$g = \Re(u_\Omega \overline{p_\Omega}).$$

## 2.4 Numerical results

### 2.4.1 T-shaped waveguide

We use the topological gradient to design a H-plane T-shaped waveguide. The geometric constraints are shown on Figure 2.3.(a). The input  $\Gamma_1$  is excited by the TE10 mode (see the second boundary condition of (2.34)) : the excitation is given by

$$u_e(y) = \cos\left(\frac{\pi y}{d}\right) \forall y \in \Gamma_1.$$

We follow the two ideas [22] :

- the initial guess is the free space
- instead of minimizing the reflected energy, we maximize the transmitted energy on  $\Gamma_2$  and  $\Gamma_3$ .

At the beginning, only the input and output channels have metallic boundaries. In order to use the finite element method, the design domain is delimited by a fictitious boundary  $\Gamma_4$  on which an absorbing condition is imposed (see Figure 2.3.(b)). The problem is modeled as follows

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \partial_n u - ik' u = \partial_n u_e - ik' u_e & \text{on } \Gamma_1, \\ \partial_n u - ik' u = 0 & \text{on } \Gamma_2, \Gamma_3, \\ \partial_n u - ik u = 0 & \text{on } \Gamma_4, \end{cases} \quad (2.34)$$

where  $k^2 = k'^2 + \frac{\pi^2}{d^2}$ ,  $d$  being the length of  $\Gamma_1$ . The perfect conduction on the metallic boundary leads to the first boundary condition  $u = 0$  on  $\Gamma_0$ . The third boundary condition prevent reflections on  $\Gamma_2, \Gamma_3$ . The last equation is an approximate absorbing boundary condition. Here and in the following, we take  $k = 10$ .

The cost function to maximize is

$$J(u) = |S_{12}(u)|^2 + |S_{13}(u)|^2,$$

where  $S_{1j}(u)$  is given by

$$S_{1j}(u) = \int_{\Gamma_j} u|_{\Gamma_j} \cos\left(\frac{\pi x}{d}\right) dx, \quad j \in \{2, 3\}.$$

The adjoint state is solution to

$$\begin{cases} \Delta \bar{p} + k^2 \bar{p} = 0 & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \Gamma_0, \\ \partial_n \bar{p} - ik' \bar{p} = 0 & \text{on } \Gamma_1, \\ \partial_n \bar{p} - ik' \bar{p} = -2\overline{S_{12}(u)} \cos\left(\frac{\pi x}{d}\right) & \text{on } \Gamma_2, \\ \partial_n \bar{p} - ik' \bar{p} = -2\overline{S_{13}(u)} \cos\left(\frac{\pi x}{d}\right) & \text{on } \Gamma_3, \\ \partial_n \bar{p} - ik \bar{p} = 0 & \text{on } \Gamma_4. \end{cases} \quad (2.35)$$

Then the topological gradient is  $g = \Re(u\bar{p})$  (see Figure 2.4.(b)). We are interested in the relative loss of energy

$$P(u) = \frac{E_e - (E_2 + E_3)(u)}{E_e},$$

where  $E_e$  is the entering energy and  $E_j(u)$  is the outgoing energy through  $\Gamma_j$ ,  $j \in \{2, 3\}$ .

We present here the topological optimization procedure. The underlying idea is the following : in the  $\ell$ -th step of the process, if  $\bar{x}$  is such that the topological gradient is higher than a certain value  $t_\ell$ , we insert at this point a Dirichlet node (metal). The constant  $t_\ell$  is chosen by the user, which allows him to take into account other constraints, as for example the feasibility. The process is stopped when the topological gradient is everywhere negative in the design domain or when the shape suits the designer. The algorithm is the following :

- Initialization : choose the initial domain  $\Omega_0$ , and set  $\ell = 0$ . The domain  $\Omega_0$  is meshed and it is identified with the set of the nodes :  $\Omega_0 = \{x_k, k \in \{1, 2, \dots, n\}\}$ . The grid is fixed during the process.
- Repeat
  1. compute  $u_\ell, p_\ell$  the direct and adjoint solutions in the domain  $\Omega_\ell$ ,
  2. compute the topological gradient  $g_\ell = \Re(u_\ell \bar{p}_\ell)$ ,
  3. set  $\Omega_{\ell+1} = \Omega_\ell \setminus \{x_k, g_\ell(x_k) \geq t_{\ell+1}\}$ ,
  4.  $\ell \leftarrow \ell + 1$ .

Figure 2.4 shows the isovalues of  $|u|$  and the topological gradient for the initial geometry. In this case, 94.4% of the energy is lost. After two iterations, the loss is reduced to 2.02% (see Figure 2.5) and the topological gradient is everywhere negative. The last step consists in smoothing the boundary of the domain by inserting some metal where  $|u|$  is close to zero. The loss of energy of this waveguide is equal to 1.5% (see Figure 2.6). The convergence history is given by Figure 2.7.

## 2.4.2 L-shaped waveguide

Here, we use the topological gradient like a decision help system to build a junction between two rectangular waveguides. The initial geometry and the design domain are given by Figure 2.8. The cost function to maximize is

$$J(u) = |S_{12}(u)|^2.$$

Figure 2.9.(a) shows the isovalues of  $|u|$  for the initial geometry. In this case, 95.43% of the energy is lost. We observe that the topological gradient is high on a quarter of circle where we decide to put metal (see Figure 2.9.(b)). The loss of energy of the obtained waveguide is now equal to 0.34% (see Figure 2.10).

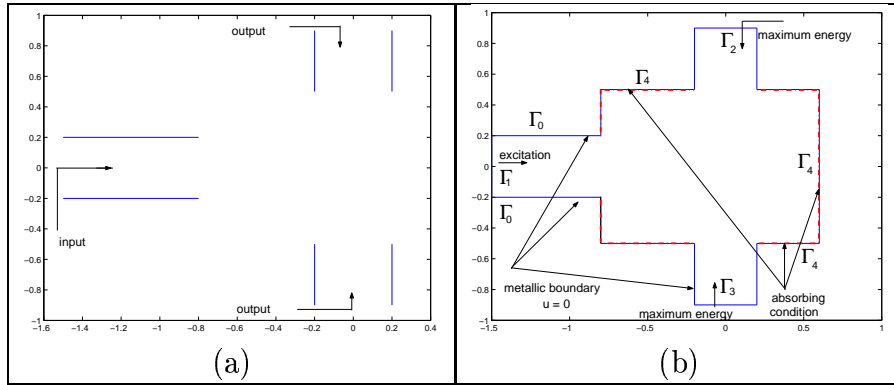


FIG. 2.3 – The initial geometry (a) and the design domain (b).

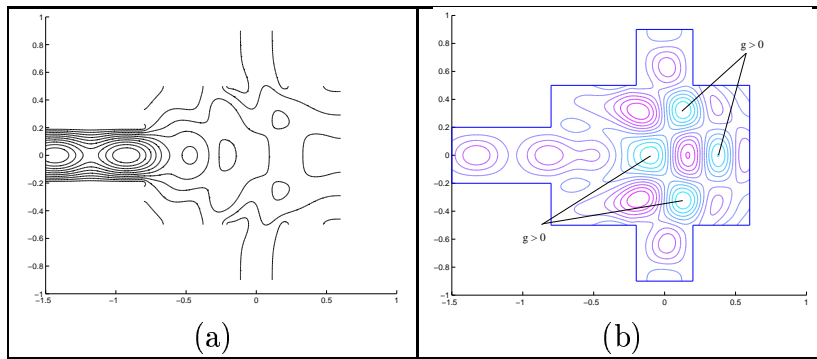


FIG. 2.4 – Modulus of the electric field (a) and topological gradient (b).

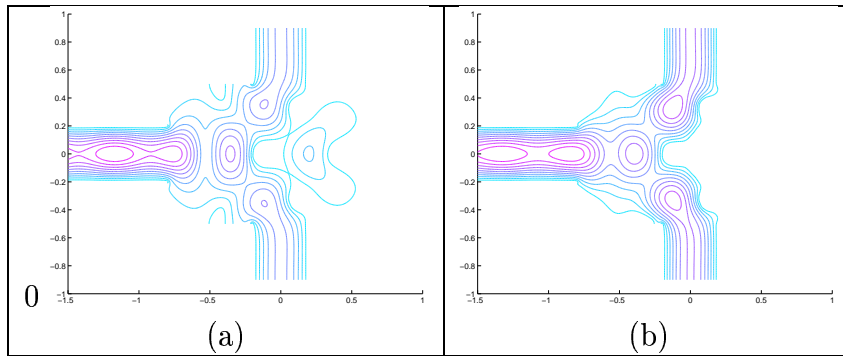


FIG. 2.5 – Modulus of the electric fields obtained after a first iteration (a) and after two iterations (b).

### 2.4.3 U-shaped waveguide

Here, the initial guess is a metallic cavity. The geometry of the waveguide is shown in Figure 2.11. The cost function to maximize is

$$J(u) = |S_{12}(u)|^2.$$

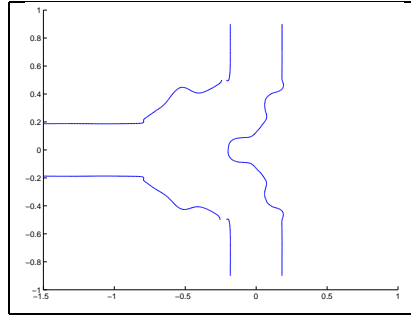


FIG. 2.6 – Final geometry.

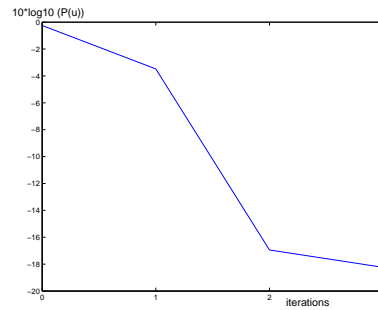


FIG. 2.7 – Convergence history.

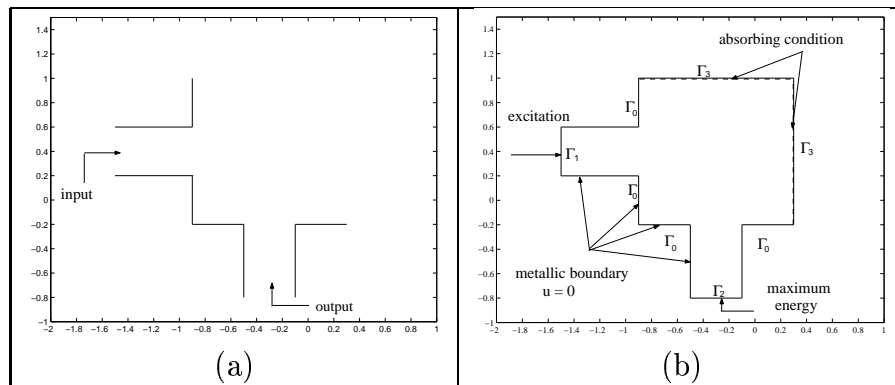


FIG. 2.8 – The initial geometry (a) and the design domain (b).

Figure 2.12.(a) shows the isovalues of  $|u|$  for the initial geometry. In this case, 88.45% of the energy is reflected. There are three local maximas of the topological gradient (see Figure 2.12.(b)). At each local maxima, we introduce a pointwise Dirichlet condition (a metallic plot). The new energy distribution is shown in Figure 2.13.(a). The loss of energy is now equal to 39.19%. A new analysis is performed : after the introduction of another metallic plot, we obtain the design of Figure 2.13.(b). The objective is fulfilled, the loss

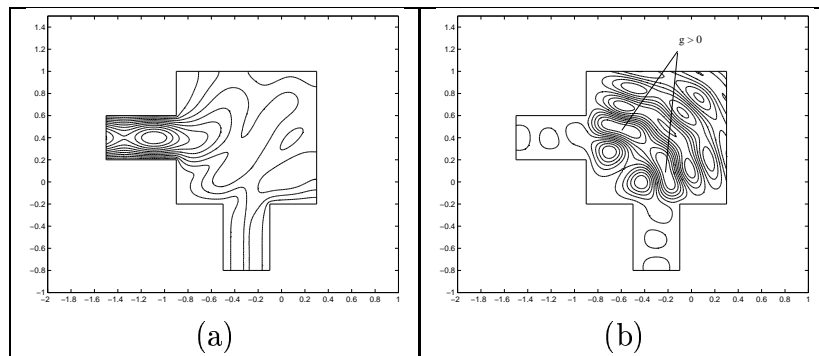


FIG. 2.9 – Modulus of the electric field (a) and topological gradient (b).

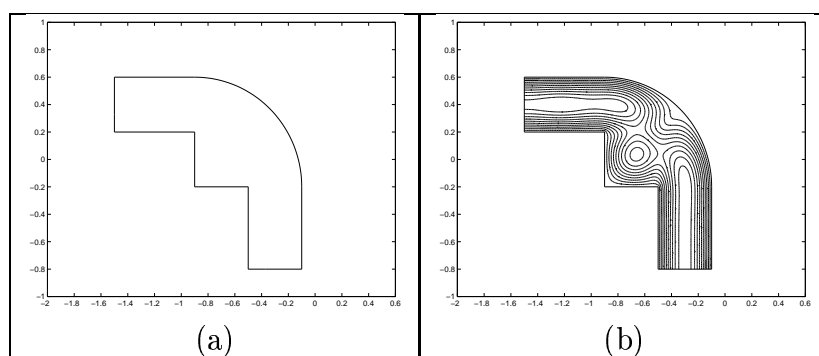


FIG. 2.10 – Final geometry (a) and modulus of the electric field (b).

of energy is equal to 0.7%. For feasibility reasons, we decide not to insert additional plots.

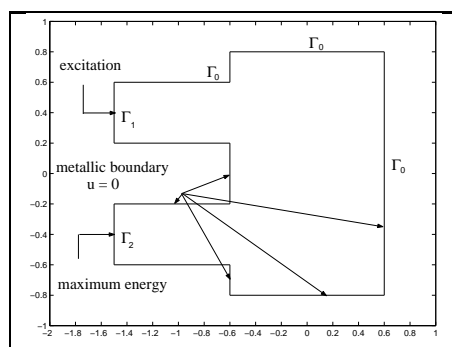


FIG. 2.11 – Geometry of the initial guide.



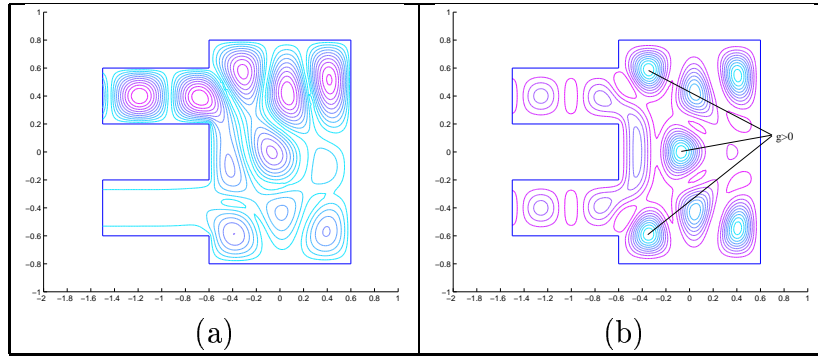


FIG. 2.12 – Modulus of the electric field (a) and topological gradient (b).

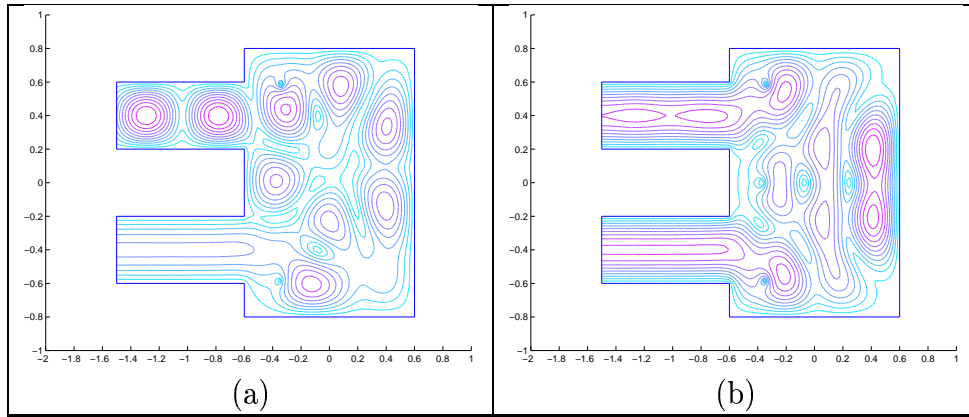


FIG. 2.13 – Modulus of the electric fields obtained after a first iteration (a) and after two iterations (b).

## 2.5 Appendix

### 2.5.1 Existence and uniqueness of solution

Here we establish the existence and uniqueness of solution to problem (2.9). Replacing  $\Omega$  with  $\Omega_\varepsilon$ , the argumentation would be the same for Problem (2.11). Without any loss of generality, we suppose here that  $N = 1$ . The variational form of problem (2.9) is : find  $u \in \mathcal{V}_\Omega$  satisfying

$$a(u, v) = l(v) \quad \forall v \in \mathcal{V}_\Omega, \quad (2.36)$$

where the functional space  $\mathcal{V}_\Omega$ , the sesquilinear form  $a$  and the semilinear form  $l$  are defined by

$$\begin{aligned}\mathcal{V}_\Omega &= \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\}, \\ a(u, v) &= \int_\Omega (\nabla u \cdot \overline{\nabla v} - k^2 u \bar{v}) dx - ik \int_{\Gamma_1} u \bar{v} d\gamma(x), \\ l(v) &= \int_{\Gamma_1} g \bar{v} d\gamma(x).\end{aligned}$$

We split  $a$  in the following form :

$$a(u, v) = b(u, v) + c(u, v), \quad (2.37)$$

where  $b$  and  $c$  are defined by

$$b(u, v) = \int_\Omega (\nabla u \cdot \overline{\nabla v} + u \bar{v}) dx, \quad (2.38)$$

$$c(u, v) = -(1 + k^2) \int_\Omega u \bar{v} dx - ik \int_{\Gamma_1} u \bar{v} d\gamma(x). \quad (2.39)$$

We recall the following result which is a consequence of the Lax-Milgram theorem.

**Lemma 3** *For all  $f \in \mathcal{V}'_\Omega$ , there exists a unique  $u_f \in \mathcal{V}_\Omega$  such that*

$$b(u_f, v) = \langle f, v \rangle_{\mathcal{V}'_\Omega, \mathcal{V}_\Omega}.$$

*The operator  $f \mapsto u_f$  is continuous from  $\mathcal{V}'_\Omega$  to  $\mathcal{V}_\Omega$ .*

We define

$$\begin{aligned}\mathcal{C} : \mathcal{V}_\Omega &\longrightarrow \mathcal{V}_\Omega \\ u &\longmapsto \mathcal{C}u\end{aligned}$$

such that

$$b(\mathcal{C}u, v) + c(u, v) = 0 \quad \forall v \in \mathcal{V}_\Omega. \quad (2.40)$$

We have the following lemma.

**Lemma 4** *The operator  $\mathcal{C}$  is compact.*

*Proof.* By Lemma 3, it suffices to prove that the operator

$$u \longmapsto c(u, \cdot)$$

from  $\mathcal{V}_\Omega$  to  $\mathcal{V}'_\Omega$  is compact. Let  $(u_i)$  be a sequence bounded in  $\mathcal{V}_\Omega$ . The following imbeddings :  $\mathcal{V}_\Omega \rightarrow L^2(\Omega)$  and  $H^{\frac{1}{2}}(\Gamma_1) \rightarrow L^2(\Gamma_1)$  are compact, then there exists a subsequence always denoted by  $(u_i)$  such that

$$u_i \rightharpoonup w_1 \text{ in } L^2(\Omega)$$

and

$$\gamma_0 u_i \rightarrow w_2 \text{ in } L^2(\Gamma_1).$$

Then,

$$c(u_i, \cdot) \rightarrow l_{w_1}^{w_2} \text{ in } \mathcal{V}'_{\Omega},$$

where  $l_{w_1}^{w_2}$  is defined by

$$\langle l_{w_1}^{w_2}, v \rangle_{\mathcal{V}'_{\Omega}, \mathcal{V}_{\Omega}} = -(1 + k^2) \int_{\Omega} w_1 \bar{v} dx - ik \int_{\Gamma_1} w_2 \bar{v} d\gamma(x) \quad \forall v \in \mathcal{V}_{\Omega}.$$

Hence the operator  $\mathcal{C}$  is compact. □

Using (2.40), Problem (2.36) can be written as follows : find  $u \in \mathcal{V}_{\Omega}$  such that

$$b((I - \mathcal{C})u, v) = l(v) \quad \forall v \in \mathcal{V}_{\Omega}. \quad (2.41)$$

We have the following lemma.

**Lemma 5** *For  $k \in \{k \in \mathbb{C}^* / \Im(k) \geq 0\}$ , the following problem : find  $u \in \mathcal{V}_{\Omega}$  such that*

$$a(u, v) = 0 \quad \forall v \in \mathcal{V}_{\Omega}, \quad (2.42)$$

*has no non-trivial solution.*

*Proof.* Let  $u$  be a solution to Problem (2.42). For  $v = u$ , we have

$$a(u, u) = 0.$$

Then,

$$\int_{\Omega} |\nabla u|^2 dx - k^2 \int_{\Omega} |u|^2 dx - ik \int_{\Gamma_1} |u|^2 d\gamma(x) = 0. \quad (2.43)$$

By writing  $k = k_1 + ik_2$ , where  $(k_1, k_2) \in \mathbb{R}^2$  and using (2.43), we obtain

$$\int_{\Omega} |\nabla u|^2 dx - (k_1^2 - k_2^2) \int_{\Omega} |u|^2 dx + k_2 \int_{\Gamma_1} |u|^2 d\gamma(x) = 0, \quad (2.44)$$

and

$$k_1 \int_{\Gamma_1} |u|^2 d\gamma(x) + 2k_1 k_2 \int_{\Omega} |u|^2 dx = 0. \quad (2.45)$$

Two cases can arise :

– first case :  $k_2 > 0$ . If  $k_1 = 0$ , using (2.44) we obtain

$$\int_{\Omega} |\nabla u|^2 dx + k_2^2 \int_{\Omega} |u|^2 dx + k_2 \int_{\Gamma_1} |u|^2 d\gamma(x) = 0.$$

Then,  $u = 0$  in  $\Omega$ . If  $k_1 \neq 0$ , using (2.45) we obtain

$$\int_{\Gamma_1} |u|^2 d\gamma(x) + 2k_2 \int_{\Omega} |u|^2 dx = 0.$$

Then,  $u = 0$  in  $\Omega$ .

– second case :  $k_2 = 0$  and  $k_1 \neq 0$ . Using (2.45), we obtain

$$u = 0 \quad \text{on } \Gamma_1.$$

Let  $\tilde{\Omega}$  be a regular domain containing  $\Omega$  and so that  $\Gamma_0 \subset \partial\tilde{\Omega}$ . Extending  $u$  by zero in  $\tilde{\Omega} \setminus \Omega$ . We obtain a function  $\tilde{u}$  that satisfies

$$\Delta\tilde{u} + k^2\tilde{u} = 0 \quad \text{in } \mathcal{D}'(\tilde{\Omega}).$$

This extension is analytic, it is equal to zero in an open subset of a connex domain, thus  $\tilde{u} = 0$  in  $\tilde{\Omega}$ .

This completes the proof. □

By Lemma 5, Lemma 4 and using the Fredholm alternative, we obtain the following result.

**Theorem 3** *For  $k \in \{k \in \mathbb{C}^* / \Im(k) \geq 0\}$ , problem (2.36) has one and only one solution.*

## 2.5.2 The inf-sup condition

Our aim is to prove that the sesquilinear form  $a_0$  defined by (2.17) for  $\varepsilon = 0$ , satisfies the inf-sup condition (see Hypothesis 2). We have the following lemma.

**Lemma 6** *The sesquilinear form  $a$  defined in (2.36) satisfies the inf-sup condition.*

*Proof.* Let  $u \in \mathcal{V}_\Omega$ . We set  $v = (I - \mathcal{C})u$ , where  $\mathcal{C}$  is the operator defined by (2.40). According to (2.40), we have

$$\begin{aligned} a(u, v) &= b(v, v) \\ &= \|(I - \mathcal{C})u\|_{\mathcal{V}_\Omega} \|v\|_{\mathcal{V}_\Omega} \\ &\geq \alpha \|u\|_{\mathcal{V}_\Omega} \|v\|_{\mathcal{V}_\Omega}, \end{aligned}$$

where  $\alpha = \|(I - \mathcal{C})^{-1}\|_{\mathcal{L}(\mathcal{V}_\Omega, \mathcal{V}_\Omega)}^{-1}$ . Thus the sesquilinear form  $a$  satisfies the inf-sup condition. □

We have the following result.

**Proposition 6** *The sesquilinear form  $a_0$  satisfies the inf-sup condition.*

*Proof.* We have

$$a_0(u, v) = \int_{\Omega_R} (\nabla u \cdot \overline{\nabla v} - k^2 u \bar{v}) dx + \int_{\Sigma_R} (T^0 u) \bar{v} d\gamma(x) - ik \int_{\Gamma_1} u \bar{v} d\gamma(x) \quad \forall u, v \in \mathcal{V}_R.$$

For all  $u \in \mathcal{V}_R$  we set

$$\tilde{u} = \begin{cases} u & \text{in } \Omega_R \\ u_\psi^0 & \text{in } B(x, R), \end{cases}$$

where  $\psi = u|_{\Sigma_R}$  and  $u_\psi^0$  is the solution to

$$\begin{cases} \Delta u_\psi^0 + k^2 u_\psi^0 = 0 & \text{in } B(x, R) \\ u_\psi^0 = \psi & \text{on } \Sigma_R. \end{cases}$$

It can easily be proved that

$$a_0(u, v|_{\Omega_R}) = a(\tilde{u}, v) \quad \forall u \in \mathcal{V}_R, \forall v \in \mathcal{V}_\Omega.$$

According to Lemma 6, there exists  $v \in \mathcal{V}_\Omega, v \neq 0$ , such that

$$\begin{aligned} a_0(u, v|_{\Omega_R}) = a(\tilde{u}, v) &\geq \alpha \|\tilde{u}\|_{\mathcal{V}_\Omega} \|v\|_{\mathcal{V}_\Omega} \\ &\geq \alpha \|u\|_{\mathcal{V}_R} \|v|_{\Omega_R}\|_{\mathcal{V}_R}. \end{aligned}$$

This completes the proof. □

### 2.5.3 Some useful inequalities

We have the following proposition.

**Proposition 7** *There exists  $c > 0$  such that*

$$\left| \varepsilon \frac{J_n(kR)}{J_n(k\varepsilon)} \right| \geq c \left( \frac{R}{\varepsilon} \right)^{n-1} \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0.$$

*Proof.* The Bessel function  $J_n(z)$  is defined by

$$J_n(z) = \left( \frac{1}{2} z \right)^n \sum_{p=0}^{+\infty} \frac{(-\frac{1}{4} z^2)^p}{p! \Gamma(n+p+1)}.$$

Then we have

$$\begin{aligned}
\varepsilon \frac{J_n(kR)}{J_n(k\varepsilon)} &= \varepsilon \left(\frac{R}{\varepsilon}\right)^n \frac{\sum_{p=0}^{+\infty} \frac{(-\frac{1}{4}k^2 R^2)^p}{p! \Gamma(n+p+1)}}{\sum_{p=0}^{+\infty} \frac{(-\frac{1}{4}k^2 \varepsilon^2)^p}{p! \Gamma(n+p+1)}} \\
&= \varepsilon \left(\frac{R}{\varepsilon}\right)^n \frac{(\Gamma(n+1))^{-1} + \sum_{p=1}^{+\infty} \frac{(-\frac{1}{4}k^2 R^2)^p}{p! \Gamma(n+p+1)}}{(\Gamma(n+1))^{-1} + \sum_{p=1}^{+\infty} \frac{(-\frac{1}{4}k^2 \varepsilon^2)^p}{p! \Gamma(n+p+1)}} \\
&= \varepsilon \left(\frac{R}{\varepsilon}\right)^n \frac{1 + \sum_{p=1}^{+\infty} \frac{n!}{p!(n+p)!} (-\frac{1}{4}k^2 R^2)^p}{1 + \sum_{p=1}^{+\infty} \frac{n!}{p!(n+p)!} (-\frac{1}{4}k^2 \varepsilon^2)^p} \\
&= \left(\frac{R}{\varepsilon}\right)^{n-1} u_n(\varepsilon),
\end{aligned}$$

where  $u_n(\varepsilon)$  is defined by

$$u_n(\varepsilon) = \frac{R + \sum_{p=1}^{+\infty} \frac{Rn!}{p!(n+p)!} (-\frac{1}{4}k^2 R^2)^p}{1 + \sum_{p=1}^{+\infty} \frac{n!}{p!(n+p)!} (-\frac{1}{4}k^2 \varepsilon^2)^p}.$$

It is easy to see that the series which intervene in the expression of  $u_n(\varepsilon)$  converge normally with respect to  $(n, \varepsilon)$ . Hence, we have

$$\lim_{(n, \varepsilon) \rightarrow (\infty, 0)} u_n(\varepsilon) = R.$$

Using the limit definition, there exists  $c > 0$  such that

$$|u_n(\varepsilon)| \geq c \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0.$$

This completes the proof. □

By the same techniques we obtain the following result.

**Proposition 8** *There exists  $c > 0$  such that*

$$\left| \frac{Y_n(k\varepsilon)}{Y_n(kR)} \right| \geq c \left(\frac{R}{\varepsilon}\right)^n \quad \forall n \geq n_0, \forall \varepsilon < \varepsilon_0.$$



# Bibliographie

- [1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover 1964.
- [2] G. ALLAIRE AND R. KOHN, *Optimal bounds on the effective behavior of a mixture of two well-order elastic materials*, Quartely of applied Mathematics, LI(4) :643-674, 1993.
- [3] G. ALLAIRE AND R. KOHN, *Optimal design for minimum weight and compliance in plane stress using extremal microstructures*, *European Journal of Mechanics, A/Solids*, 12(6) :839-878, 1993.
- [4] M. BECKER, *Optimisation topologique de structure en variables discrètes*, Université de Liège, 1996.
- [5] M. BENDSOE, *Optimal topology design of continuum structure : an introduction*, Technical report, Department of mathematics, Technical University of Denmark, DK2800 Lyngby, Denmark, september 1996.
- [6] G. BUTTAZZO AND G. DAL MASO, *Shape optimization for Dirichlet problems : Relaxed Formulation and optimality conditions*, Appl. Math. Optim. 23, 17-49, 1991.
- [7] J. CÉA *Conception optimale ou identification de forme, calcul rapide de la dérivée directionnelle de la fonction coût*, M.A.A.N., 20(3) :371-402, 1986.
- [8] J. CÉA, A.GIOAN AND J. MICHEL, *Quelques résultats sur l'identification de domaines*, CALCOLO, 1973.
- [9] R. DAUTRAY AND J.-L.LIONS, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Masson, collection CEA, 1987.
- [10] S. GARREAU, P. GUILLAUME AND M. MASMOUDI, *The topological sensitivity for linear isotropic elasticity*, European Conference on Computational Mechanics (ECCM99), 1999, rapport MIP 99.45.
- [11] P. GUILLAUME, K. SIDIDRIS, *The topological asymptotic expansion for the Dirichlet problem*, accepted in SIAM journal on Control and Optimization.
- [12] P. GUILLAUME, K. SIDIDRIS, *Topological sensitivity and shape optimization for the Stokes equations*, rapport MIP no. 01-24, 2001.



- [13] P. GUILLAUME AND M. MASMOUDI, *Computation of high order derivatives in optimal shape design*, Numerische Mathematik 67, 231-250, 1994.
- [14] P. GUILLAUME, *Dérivées d'ordre supérieur en conception optimale de forme*, Thèse, Université Paul Sabatier, 1994.
- [15] J. JACOBSEN, N. OLHOFF, AND E. RONHOLT, *Generalized shape optimization of three-dimensional structures using materials with optimum microstructures*, Technical report, Institute of Mechanical Engineering, Aalborg University, DK-9920 Aalborg, Denmark, 1996.
- [16] M. MASMOUDI, *The Topological Asymptotic*, in Computational Methods for Control Applications, ed. H. Kawarada and J. Periaux, International Series GAKUTO, 2002.
- [17] M. MASMOUDI, *Outils pour la conception optimale de formes*, Thèse d'état, Université de Nice, 1987.
- [18] M. MASMOUDI, *Numerical Solution for Exterior Problems*, Numerische Mathematik. 51, 87-101, Springer-Verlag 1987.
- [19] M. MASMOUDI, *Résolution numérique de problèmes extérieurs*, Thèse présentée à l'université de Nice, 1979.
- [20] F. MURAT AND J. SIMON, *Sur le contrôle par un domaine géométrique*, thèse d'état, Paris, 1976.
- [21] F. MURAT AND L. TARTAR, *Calcul des variations et homogénéisation*, In Les méthodes de l'homogénéisation : Théorie et Applications en Physique, pages 319-369. Eyrolles, 1985.
- [22] P. MADER, *Optimisation topologique pour la conception de composants guidés*, Thèse, Université Paul Sabatier, 2002.
- [23] J-C. NÉDÉLEC, *Acoustic and Electromagnetic Equations*, Springer, 2000.
- [24] P.A. RAVIART AND J.M. THOMAS, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson 1988.
- [25] J. SOKOLOWSKI AND A. ZOCHOWSKI, *On the topological derivative in shape optimization*, Technical report, INRIA, 1997.
- [26] M. SHOENAUER, L. KALLEL AND F. JOUVE, *Mechanics inclusions identification by evolutionary computation*, Revue européenne des éléments finis, 5(5-6) :619-648, 1996.
- [27] A. SCHUMACHER, *Topologieoptimierung von Bauteilstrukturen unter Verwendung von Lophpositionierungskriterien*, Thesis, Universität-Gesamthochschule Siegen, 1995.
- [28] CH. VASSALLO, *Théorie des guides d'ondes électromagnétiques*, Collection Technique et scientifique des Télécommunications, Eyrolles 1985.

# Chapitre 3

## The topological asymptotic for the Helmholtz equation II

### Sommaire

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Ce chapitre est constitué de l'article : *The topological asymptotic for the Helmholtz equation with Dirichlet condition on the boundary of an arbitrary shaped hole*, accepté pour publication dans SIAM J. Control Optim. C'est un travail réalisé en collaboration avec Julien Pommier.

### 3.1 Introduction

The same numerical methods are generally used in shape inversion and optimal shape design. There are mainly two categories of shape inversion or shape optimization methods. In the first category we deform continuously the boundary of the object to optimize in order to decrease a given cost function [5, 20, 24, 27, 30]. The final shape has the same topology as the initial shape given by the designer. Therefore, to reach the optimal geometry we need *a priori* knowledge of its topology. However the topology of the optimal shape is often the main unknown in object detection problems. For example, the knowledge of the number and the locations of buried mines is more important than their accurate shapes. The second category of algorithms allow topology modifications. Many important contributions in this field are concerned with structural mechanics and in particular the optimization of the compliance (external work) subject to a volume constraint [4, 16]. In view of the fact that the optimal structure has generally a large number of small holes, most authors [1, 3, 14] have considered composite material optimization. Using the homogenization theory G. Allaire *et al.* [1] exhibit a class of laminated materials with an explicit expression for the optimal material at any point of the structure. In this case, the optimal solution is not a classical design : it is a distribution of composite materials. Then penalization methods must be applied in order to retrieve a realistic shape. For all these reasons global optimization methods are used to solve more general problems [15, 25]. Unfortunately these methods are quite slow.

More recently, Eschenauer and Olhoff [7], Schumacher [26], C ea *et al.* [6], Garreau *et al.* [8] and Sokolowski *et al.* [28, 29, 21] presented a method to obtain the optimal topology by calculating the so-called topological gradient (or topological derivative). This gradient is a function defined in the domain of interest where, at each point, it gives the sensitivity of the cost function when a small hole is created at that point. This approach seems to be general and efficient. To present the basic idea, we consider  $\Omega$  a domain of  $\mathbb{R}^n$ ,  $n=2$  or  $3$  and  $j(\Omega) = J(u_\Omega)$  a cost function to be minimized, where  $u_\Omega$  is the solution to a given PDE problem defined in  $\Omega$ . For  $\varepsilon > 0$ , let  $\Omega_\varepsilon = \Omega \setminus \overline{x_0 + \varepsilon\omega}$  be the subset obtained by removing a small part  $\overline{x_0 + \varepsilon\omega}$  from  $\Omega$ , where  $x_0 \in \Omega$  and  $\omega \subset \mathbb{R}^n$  is a fixed open and bounded subset containing the origin. We can generally prove that the variation of the criterion is given by the asymptotic

expansion :

$$j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon)g(x_0) + o(f(\varepsilon)), \quad (3.1)$$

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0, \quad f(\varepsilon) > 0. \quad (3.2)$$

This expansion is called the topological asymptotic. To minimize the criterion, we have to create holes where  $g$  (called the topological gradient) is negative.

In this paper, using the adjoint method and the domain truncation technique introduced in [17], we compute the topological asymptotic expansion for the Helmholtz equation in two and three space dimensions with Dirichlet condition on the boundary of an arbitrary shaped hole. The originality of this work is that the symbol of the Helmholtz operator is non-homogeneous. The basic idea is to say that the leading term of the topological asymptotic expansion is given by the principal part of the operator on the case of a Dirichlet condition on the boundary of the hole. Our work generalizes the contribution of Ph. Guillaume *et al.* [9] for the Poisson equation and it is easily applicable to other problems of which the symbol of the operator is non-homogeneous, as for example, the Quasi-stokes problem and the elastic waves problem. In the numerical part, We present some applications that illustrate the ability of the topological sensitivity approach to solve inverse scattering problems.

As a background to our work, we cite the contributions of A. M. Il'in [11, 12, 13] for the construction of asymptotic expansions of solutions to boundary value problems in domain with small holes in the case of second order scalar equations by the use of the method of matched asymptotic expansions. Various spectral problems in domains with small holes are investigated by V. G. Maz'ya, S. A. Nazarov, B. A. Plamenevskij and Shin Osava [23, 18, 22]. In [31], M. S. Vogelius *et al.* provided a rigorous derivation for solutions to the time-harmonic Maxwell's equations of a TE (transverse electric) nature, in the presence of a finite number of diametrically small inhomogeneities. Based on layer potential techniques, H. Ammari *et al.* [2] provided a rigorous derivation of complete asymptotic expansions for solutions to the Helmholtz equation in two and three dimensions, in the presence of small inhomogeneities in the domain. In our work, we derive asymptotic expansions not for solutions but for a given cost function.

The generalized adjoint method is recalled in Section 3.2. Next, the formulation of the Helmholtz problem is presented in Section 3.3 and its truncated version is described in Section 3.4. Section 3.5 presents the main results whose proofs are given in Section 3.6. Finally, numerical examples illustrate in Section 3.7 the abilities of the topological sensitivity to solve inverse scattering problems.

### 3.2 A generalized adjoint method

In this section, the generalized adjoint method introduced in [17, 8] is slightly modified. The first modification is due to the fact that the cost function is defined in a  $\mathbb{C}$ -Hilbert space and it takes values in  $\mathbb{R}$ , then it is not differentiable. For this reason, the differentiability property is replaced by the formulation (3.7). The second modification is due to the fact that the sesquilinear form associated to our problem is not coercive. For this reason, the coercivity property is replaced by the inf-sup condition (see Hypothesis 4).

Let  $\mathcal{V}$  be a fixed complex Hilbert space. For  $\varepsilon \geq 0$ , let  $a_\varepsilon(\cdot, \cdot)$  be a sesquilinear and continuous form on  $\mathcal{V}$  and  $l_\varepsilon$  be a semilinear and continuous form on  $\mathcal{V}$ . We consider the following assumptions.

**Hypothesis 3** *There exists a sesquilinear and continuous form  $\delta a$ , a semilinear and continuous form  $\delta_l$ , and a real function  $f(\varepsilon) > 0$  defined on  $\mathbb{R}_+^*$  such that*

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0, \quad (3.3)$$

$$\|a_\varepsilon - a_0 - f(\varepsilon)\delta a\|_{\mathcal{L}_2(\mathcal{V})} = o(f(\varepsilon)), \quad (3.4)$$

$$\|l_\varepsilon - l_0 - f(\varepsilon)\delta_l\|_{\mathcal{L}(\mathcal{V})} = o(f(\varepsilon)), \quad (3.5)$$

where  $\mathcal{L}(\mathcal{V})$  (respectively  $\mathcal{L}_2(\mathcal{V})$ ) denotes the space of continuous and semilinear (respectively sesquilinear) forms on  $\mathcal{V}$ .

**Hypothesis 4** *There exists a constant  $\alpha > 0$  such that*

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_0(u, v)|}{\|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}} \geq \alpha.$$

We say that  $a_0$  satisfies the inf-sup condition.

According to (3.4), there exists a constant  $\beta > 0$  independent of  $\varepsilon$  such that

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_\varepsilon(u, v)|}{\|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}} \geq \beta.$$

For  $\varepsilon \geq 0$ , let  $u_\varepsilon$  be the solution to the problem : find  $u_\varepsilon \in \mathcal{V}$  such that

$$a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v) \quad \forall v \in \mathcal{V}. \quad (3.6)$$

We have the following lemma.

**Lemma 7** *If Hypotheses 3 and 4 are satisfied then*

$$\|u_\varepsilon - u_0\|_{\mathcal{V}} = O(f(\varepsilon)).$$

*Proof.* It follows from Hypothesis 4 that there exists  $v_\varepsilon \in \mathcal{V}, v_\varepsilon \neq 0$  such that

$$\beta \|u_\varepsilon - u_0\|_{\mathcal{V}} \|v_\varepsilon\|_{\mathcal{V}} \leq |a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon)|,$$

which implies

$$\begin{aligned} \beta \|u_\varepsilon - u_0\|_{\mathcal{V}} \|v_\varepsilon\|_{\mathcal{V}} &\leq |a_\varepsilon(u_0, v_\varepsilon) - l_\varepsilon(v_\varepsilon)| \\ &= |a_\varepsilon(u_0, v_\varepsilon) - (l_\varepsilon - l_0 - f(\varepsilon)\delta_l)(v_\varepsilon) - l_0(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\ &= |(a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon)) - (l_\varepsilon - l_0 - f(\varepsilon)\delta_l)(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\ &\leq |a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon) - f(\varepsilon)\delta a(u_0, v_\varepsilon)| + |l_\varepsilon(v_\varepsilon) - l_0(v_\varepsilon) - f(\varepsilon)\delta_l(v_\varepsilon)| \\ &\quad + f(\varepsilon)(|\delta a(u_0, v_\varepsilon)| + |\delta_l(v_\varepsilon)|). \end{aligned}$$

Using Hypothesis 3 we obtain

$$\beta \|u_\varepsilon - u_0\|_{\mathcal{V}} \|v_\varepsilon\|_{\mathcal{V}} \leq (o(f(\varepsilon)) + f(\varepsilon)(\|\delta a\|_{\mathcal{L}_2(\mathcal{V})}\|u_0\|_{\mathcal{V}} + \|\delta_l\|_{\mathcal{L}(\mathcal{V})})) \|v_\varepsilon\|_{\mathcal{V}}.$$

□

Consider now a cost function  $j(\varepsilon) = J(u_\varepsilon)$ , where the functional  $J$  satisfies

$$J(u + h) = J(u) + \Re(L_u(h)) + o(\|h\|) \quad \forall u, h \in \mathcal{V}, \quad (3.7)$$

where  $L_u$  is a linear and continuous form on  $\mathcal{V}$ .

For  $\varepsilon \geq 0$ , we define the Lagrangian operator  $\mathcal{L}_\varepsilon$  by

$$\mathcal{L}_\varepsilon(u, v) = J(u) + a_\varepsilon(u, v) - l_\varepsilon(v) \quad \forall u, v \in \mathcal{V}.$$

The next theorem gives the asymptotic expansion of  $j(\varepsilon)$ .

**Theorem 4** *If Hypotheses 3 and 4 are satisfied, then*

$$j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta_{\mathcal{L}}(u_0, p_0)) + o(f(\varepsilon)), \quad (3.8)$$

where  $u_0$  is the solution to Equation (3.6) with  $\varepsilon = 0$ ,  $p_0$  is the solution to the adjoint problem : find  $p_0 \in \mathcal{V}$  such that

$$a_0(v, p_0) = -L_{u_0}(v) \quad \forall v \in \mathcal{V} \quad (3.9)$$

and

$$\delta_{\mathcal{L}}(u, v) = \delta a(u, v) - \delta_l(v), \quad \forall u, v \in \mathcal{V}.$$

*Proof.* We have that

$$j(\varepsilon) = \mathcal{L}_\varepsilon(u_\varepsilon, v) \quad \forall \varepsilon \geq 0, \forall v \in \mathcal{V}.$$

Next, choosing  $v = p_0$ , we obtain

$$\begin{aligned} j(\varepsilon) - j(0) &= \mathcal{L}_\varepsilon(u_\varepsilon, p_0) - \mathcal{L}_0(u_0, p_0) \\ &= J(u_\varepsilon) - J(u_0) + a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0) + l_0(p_0) - l_\varepsilon(p_0) \\ &= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0)) - \Re(l_\varepsilon(p_0) - l_0(p_0)) \\ &= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0) + a_0(u_\varepsilon - u_0, p_0)) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Using (3.7), we have that

$$J(u_\varepsilon) - J(u_0) = \Re(L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|).$$

Hence,

$$\begin{aligned} j(\varepsilon) - j(0) &= \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + \Re(a_0(u_\varepsilon - u_0, p_0) + L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Using that  $p_0$  is the adjoint solution, we obtain

$$\begin{aligned} j(\varepsilon) - j(0) &= \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + o(\|u_\varepsilon - u_0\|) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)) \\ &= \Re((a_\varepsilon - a_0)(u_0, p_0)) + \Re((a_\varepsilon - a_0)(u_\varepsilon - u_0, p_0)) + o(\|u_\varepsilon - u_0\|) \\ &\quad - \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

It follows from Hypothesis 3 that

$$\begin{aligned} j(\varepsilon) - j(0) &= f(\varepsilon)\Re(\delta a(u_0, p_0)) + o(f(\varepsilon)) + f(\varepsilon)\Re(\delta a(u_\varepsilon - u_0, p_0)) + o(f(\varepsilon))\|u_\varepsilon - u_0\| \\ &\quad + o(\|u_\varepsilon - u_0\|) - f(\varepsilon)\Re(\delta_l(p_0)). \end{aligned}$$

Finally, from Lemma 7 and the hypothesis  $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$ , we have

$$j(\varepsilon) = j(0) + f(\varepsilon)\Re(\delta a(u_0, p_0) - \delta_l(p_0)) + o(f(\varepsilon)),$$

since  $\delta_a$  is continuous by assumption. □

### 3.3 The Helmholtz problem in a domain with a small hole

Let  $\Omega$  be an open and bounded subset of  $\mathbb{R}^n$  with boundary  $\Gamma = \Gamma_0 \cup \Gamma_1$ ,  $n = 2$  or  $3$ . The Helmholtz problem is

$$\begin{cases} \Delta u_\Omega + k^2 u_\Omega = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \Gamma_0, \\ \frac{\partial u_\Omega}{\partial n} = \Lambda u_\Omega + \Theta & \text{on } \Gamma_1, \end{cases} \quad (3.10)$$

where  $k \in \mathbb{R}^*$ ,  $\Theta \in H_{00}^{\frac{1}{2}}(\Gamma_1)'$  and  $\Lambda \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma_1), H_{00}^{\frac{1}{2}}(\Gamma_1)')$ .

We define

$$\begin{cases} \mathcal{V}(\Omega) &= \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\} \\ a(\Omega, u, v) &= \int_\Omega \nabla u \cdot \nabla \bar{v} \, dx - k^2 \int_\Omega u \bar{v} \, dx - \langle \Lambda u, \bar{v} \rangle \\ \ell(v) &= \langle \Theta, \bar{v} \rangle, \end{cases} \quad (3.11)$$

where  $\langle \cdot, \cdot \rangle$  is the duality product between  $H_{00}^{\frac{1}{2}}(\Gamma_1)'$  and  $H_{00}^{\frac{1}{2}}(\Gamma_1)$ . The variational formulation associated to problem (3.10) is the following : find  $u_\Omega \in \mathcal{V}(\Omega)$  such that

$$a(\Omega, u_\Omega, v) = \ell(v) \quad \forall v \in \mathcal{V}(\Omega). \quad (3.12)$$

We consider the following assumption.

**Hypothesis 5** *The operator  $\Lambda$  is split into  $\Lambda_0 + \Lambda_1$  with*

–  $\Lambda_1 \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma_1), H_{00}^{\frac{1}{2}}(\Gamma_1)'),$  and satisfies

$$\Re \langle \Lambda_1 \psi, \bar{\psi} \rangle \leq 0 \quad \forall \psi \in H_{00}^{\frac{1}{2}}(\Gamma_1), \quad (3.13)$$

–  $\Lambda_2 \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma_1), H_{00}^{\frac{1}{2}}(\Gamma_1)).$

We assume the following property of uniqueness

**Hypothesis 6** *We have*

$$a(\Omega, u, v) = 0 \quad \forall v \in \mathcal{V}(\Omega) \Rightarrow u = 0, \quad (3.14)$$

$$a(\Omega, u, v) = 0 \quad \forall u \in \mathcal{V}(\Omega) \Rightarrow v = 0. \quad (3.15)$$

From the Lax-Milgram theorem and the fact that the imbeddings  $\mathcal{V}_\Omega \rightarrow L^2(\Omega)$  and  $H_{00}^{\frac{1}{2}}(\Gamma_1) \rightarrow L^2(\Gamma_1)$  are compact, and due to the Fredholm alternative, we obtain (see *e.g.* [10] for a detailed argumentation) the following result.

**Proposition 9** *If hypotheses 5 and 6 are satisfied, we have that*

1. *Problem (3.12) has one and only one solution*
2. *The sesquilinear form  $a(\Omega, \cdot, \cdot)$  satisfies the inf-sup condition : there exists a constant  $a > 0$  such that*

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_\Omega(u, v)|}{\|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)}} \geq a. \quad (3.16)$$

For a given  $x_0 \in \Omega$ , consider the modified open subset  $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$ ,  $\omega_\varepsilon = x_0 + \varepsilon\omega$ , where  $\omega$  is a fixed open and bounded subset of  $\mathbb{R}^n$  containing the origin ( $\omega_\varepsilon = \emptyset$  if  $\varepsilon = 0$ ), whose boundary  $\partial\omega$  is connected and piecewise of class  $\mathcal{C}^1$ . The modified solution  $u_{\Omega_\varepsilon}$  satisfies

$$\begin{cases} \Delta u_{\Omega_\varepsilon} + k^2 u_{\Omega_\varepsilon} &= 0 & \text{in } \Omega_\varepsilon, \\ u_{\Omega_\varepsilon} &= 0 & \text{on } \Gamma_0, \\ u_{\Omega_\varepsilon} &= 0 & \text{on } \partial\omega_\varepsilon, \\ \frac{\partial u_{\Omega_\varepsilon}}{\partial n} &= \Lambda u_{\Omega_\varepsilon} + \Theta & \text{on } \Gamma_1. \end{cases} \quad (3.17)$$



The function  $u_{\Omega_\varepsilon}$  is defined on the variable open set  $\Omega_\varepsilon$ , thus it belongs to a functional space which depends on  $\varepsilon$ . Hence, if we want to derive the asymptotic expansion of a function of the form

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}), \tag{3.18}$$

we cannot apply directly the tools of section 3.2, which require a fixed functional space. For this reason, we use the domain truncation method introduced in [17] to avoid this complication.

### 3.4 The truncation method

Let  $R > 0$  be such that the closed ball  $\overline{B(x_0, R)}$  is included in  $\Omega$ . It is supposed throughout this paper that  $\varepsilon$  remains small enough so that  $\overline{\omega_\varepsilon} \subset B(x_0, R)$ . The truncated open subset is defined by

$$\Omega_R = \Omega \setminus \overline{B(x_0, R)}. \tag{3.19}$$

The open subset  $B(x_0, R) \setminus \overline{\omega_\varepsilon}$  is denoted by  $D_\varepsilon$  (see Figure 3). For  $\varphi \in$

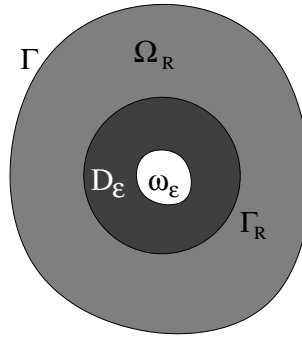


FIG. 3.1 – The truncated domain.

$H^{\frac{1}{2}}(\Gamma_R)$  and  $\varepsilon > 0$ , let  $u_\varepsilon^\varphi$  be the solution to the problem : find  $u_\varepsilon^\varphi$  such that

$$\begin{cases} \Delta u_\varepsilon^\varphi + k^2 u_\varepsilon^\varphi = 0 & \text{in } D_\varepsilon, \\ u_\varepsilon^\varphi = 0 & \text{on } \partial\omega_\varepsilon, \\ u_\varepsilon^\varphi = \varphi & \text{on } \Gamma_R, \end{cases} \tag{3.20}$$

where  $\Gamma_R$  is the boundary of the ball  $B(x_0, R)$ . For  $\varepsilon = 0$ ,  $u_0^\varphi$  is the solution to

$$\begin{cases} \Delta u_0^\varphi + k^2 u_0^\varphi = 0 & \text{in } B(x_0, R), \\ u_0^\varphi = \varphi & \text{on } \Gamma_R. \end{cases} \tag{3.21}$$

Using the Poincaré inequality, it can easily be seen that for  $R < \frac{1}{\sqrt{2}|k|}$ , problem (3.20) has one and only one solution.

For  $\varepsilon \geq 0$ , the Dirichlet-to-Neumann operator  $T_\varepsilon$  is defined by

$$\begin{aligned} T_\varepsilon : H^{1/2}(\Gamma_R) &\longrightarrow H^{-1/2}(\Gamma_R) \\ \varphi &\longmapsto T_\varepsilon \varphi = \nabla u_\varepsilon^\varphi \cdot n|_{\Gamma_R}, \end{aligned}$$

where the normal  $n|_{\Gamma_R}$  is chosen outward to  $D_\varepsilon$  on  $\Gamma_R$  and  $\partial\omega_\varepsilon$ .

Finally, we define for  $\varepsilon \geq 0$  the solution  $u_\varepsilon$  to the truncated problem

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon &= 0 & \text{in } \Omega_R, \\ u_\varepsilon &= 0 & \text{on } \Gamma_0, \\ \frac{\partial u_\varepsilon}{\partial n} &= \Lambda u_\varepsilon + \Theta & \text{on } \Gamma_1, \\ \frac{\partial u_\varepsilon}{\partial n} - T_\varepsilon u_\varepsilon|_{\Gamma_R} &= 0 & \text{on } \Gamma_R. \end{cases} \quad (3.22)$$

The variational formulation associated to (3.22) is : find  $u_\varepsilon \in \mathcal{V}_R$  such that

$$a_\varepsilon(u_\varepsilon, v) = \ell(v) \quad \forall v \in \mathcal{V}_R, \quad (3.23)$$

where the functional space  $\mathcal{V}_R$  and the sesquilinear form  $a_\varepsilon$  are defined by

$$\mathcal{V}_R = \{v \in H^1(\Omega_R); v|_{\Gamma_0} = 0\}, \quad (3.24)$$

$$a_\varepsilon(u, v) = \int_{\Omega_R} \nabla u \cdot \nabla \bar{v} \, dx - k^2 \int_{\Omega_R} u \cdot \bar{v} \, dx - \langle \Lambda u, \bar{v} \rangle + \int_{\Gamma_R} T_\varepsilon u \bar{v} \, dx \quad (3.25)$$

Here,  $\int_{\Gamma_R}$  denotes the duality product between  $H^{1/2}(\Gamma_R)$  and  $H^{-1/2}(\Gamma_R)$ . The following result is standard in PDE theory.

**Proposition 10** *Problems (3.17) and (3.22) have a unique solution. Moreover, the restriction to  $\Omega_R$  of the solution  $u_{\Omega_\varepsilon}$  to problem (3.17) is the solution  $u_\varepsilon$  to problem (3.22).*

We have now at our disposal the fixed Hilbert space  $\mathcal{V}_R$  required by Section 3.2. We assume that the following hypothesis holds.

**Hypothesis 7** *The function  $J$  introduced in (3.18) is defined in a neighbor part of  $\Gamma$  and it satisfies*

$$J(u + h) = J(u) + \Re(L_u(h)) + o(\|h\|) \quad \forall u, h \in \mathcal{V}_R,$$

where  $L_u$  is a linear and continuous form on  $\mathcal{V}_R$ .

Then, we obtain that

$$j(\varepsilon) = J(u_{\Omega_\varepsilon}) = J(u_\varepsilon) \quad \forall \varepsilon \geq 0. \quad (3.26)$$

**Remark 1** *We can also consider more general cost function (see e.g. [30]), the truncation method does not restrict the choice of the function. In the numerical part of this work, only measurements on the boundary of the domain are used. For this reason and to simplify the presentation, we considered the previous assumption about the cost function.*

Let  $v_\Omega$  be the solution to the adjoint problem

$$a(\Omega, w, v_\Omega) = -L_{u_\Omega}(w), \quad \forall w \in \mathcal{V}(\Omega), \quad (3.27)$$

where the functional space  $\mathcal{V}(\Omega)$  and the sesquilinear form  $a(\Omega, \cdot, \cdot)$  are defined in (3.11). It has been shown in Proposition 10 that  $u_0$  is the restriction to  $\Omega_R$  of  $u_\Omega$ . Similarly,  $v_0$ , the solution to

$$a_0(w, v_0) = -L_{u_0}(w), \quad \forall w \in \mathcal{V}_R, \quad (3.28)$$

is the restriction to  $\Omega_R$  of  $v_\Omega$ .

### 3.5 The main results

This section contains the main results of this paper. All the proofs are reported in Section 3.6. Henceforth we have to distinguish the cases  $n = 2$  and  $n = 3$ . This is due to the fact that the fundamental solutions to the Laplace equation in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  have an essentially different asymptotic expansion at infinity, and problem (3.29) has generally no solution if  $n = 2$ .

#### 3.5.1 The three dimensional case

Possibly changing the co-ordinate system, we can suppose for convenience that  $x_0 = 0$ . In order to derive the topological sensitivity of the function  $j$ , we introduce two auxiliary problems.

The first problem, called the “exterior problem”, is formulated in  $\mathbb{R}^3 \setminus \bar{\omega}$  and consists to find  $v_\omega$ , solution to

$$\begin{cases} -\Delta v_\omega = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega}, \\ v_\omega = 0 & \text{at } \infty, \\ v_\omega = u_\Omega(x_0) & \text{on } \partial\omega, \end{cases} \quad (3.29)$$

where  $u_\Omega$  is the solution to the direct problem (3.10). Here, one can remark that just the principal part of the Helmholtz operator is used, which described by the Laplace equation. The function  $v_\omega$  can be expressed by a single layer potential on  $\partial\omega$ . Let

$$E(y) = \frac{1}{4\pi r} \quad (3.30)$$

with  $r = ||y||$ . It is a fundamental solution for the Laplace equation in  $\mathbb{R}^3$ . Then, the function  $v_\omega$  reads

$$v_\omega(y) = \int_{\partial\omega} E(y-x)p_\omega(x) d\gamma(x), \quad y \in \mathbb{R}^3 \setminus \bar{\omega}, \quad (3.31)$$

where  $p_\omega \in H^{-\frac{1}{2}}(\partial\omega)$  is the solution to boundary integral equation

$$\int_{\partial\omega} E(y-x)p_\omega(x) d\gamma(x) = u_\Omega(x_0), \quad \forall y \in \partial\omega. \quad (3.32)$$

For  $x$  bounded and large  $r = \|y\|$ , we have

$$E(y - x) = E(y) + O\left(\frac{1}{r^2}\right) \quad (3.33)$$

and the asymptotic expansion at infinity of the function  $v_\omega$  is given by

$$v_\omega(y) = P_\omega(y) + W_\omega(y), \quad (3.34)$$

$$P_\omega(y) = A_\omega(u_\Omega(x_0)) E(y), \quad (3.35)$$

$$A_\omega(u_\Omega(x_0)) = \int_{\partial\omega} p_\omega(x) d\gamma(x), \quad (3.36)$$

$$W_\omega(y) = O\left(\frac{1}{r^2}\right). \quad (3.37)$$

Notice that  $P_\omega \in L_{loc}^m$  for all  $m < 3$ . Clearly, the function  $\alpha \mapsto A_\omega(\alpha)$  is linear on  $\mathbb{R}$ , and the number  $A_\omega(\alpha)$  depends on the shape of  $\omega$ .

The second problem, which we call ‘‘interior problem’’, is formulated in  $D_0 = B(x_0, R)$  and consists to find  $Q_\omega^1$  solution to

$$\begin{cases} \Delta Q_\omega^1 + k^2 Q_\omega^1 = 0 & \text{in } D_0, \\ Q_\omega^1 = P_\omega|_{\Gamma_R} & \text{on } \Gamma_R. \end{cases} \quad (3.38)$$

Here, the idea is to consider an interior and exterior problems that give a good ‘‘first order approximation’’ of  $(u_\varepsilon^\varphi - u_0^\varphi)|_{D_\varepsilon}$ ,  $\varphi = u_\Omega|_{\Gamma_R}$ , in the form  $f(\varepsilon)(Q_\omega^1 - P_\omega)$ , in a way which will be stated precisely in Section 3.6. But, the given formulation (3.38) of the interior problem, which is the ‘‘natural’’ choice, is not sufficient to get the behavior needed by the adjoint technique described in Section 3.2. More precisely, in this case one can construct the sesquilinear form  $\delta a$  but there is no positive function  $f(\varepsilon)$  such that  $\|a_\varepsilon - a_0 - f(\varepsilon)\delta a\|_{\mathcal{L}_2(\mathcal{V}_R)} = o(f(\varepsilon))$ . Indeed, one can observe through the proof of Proposition 11 that the behavior of  $\|a_\varepsilon - a_0 - f(\varepsilon)\delta a\|_{\mathcal{L}_2(\mathcal{V}_R)}$  is not of order  $o(\varepsilon)$ , but only of order  $O(\varepsilon)$ . This due to the approximation used on the exterior problem (3.29), where just the principal part of the operator is considered. For this reason, a new term  $Q_\omega^2$  is used in order to correct the error caused by this approximation. We construct  $Q_\omega^2$  as solution to

$$\begin{cases} \Delta Q_\omega^2 + k^2 Q_\omega^2 = k^2 P_\omega & \text{in } D_0, \\ Q_\omega^2 = 0 & \text{on } \Gamma_R. \end{cases} \quad (3.39)$$

Setting  $Q_\omega = Q_\omega^1 + Q_\omega^2$ , then  $Q_\omega$  is solution to

$$\begin{cases} \Delta Q_\omega + k^2 Q_\omega = k^2 P_\omega & \text{in } D_0, \\ Q_\omega = P_\omega|_{\Gamma_R} & \text{on } \Gamma_R. \end{cases} \quad (3.40)$$

Using the corrected interior problem (3.40), one can derive the good approximation of  $(u_\varepsilon^\varphi - u_0^\varphi)|_{D_\varepsilon}$ . The main result is the following, which will be proved in Section 3.6.

**Theorem 5** *Let  $j(\varepsilon) = J(u_{\Omega_\varepsilon})$  be a cost function satisfying Hypothesis 7. Then, the topological asymptotic expansion is given by*

$$j(\varepsilon) - j(0) = \varepsilon \Re \left( A_\omega(u_\Omega(x_0)) \overline{v_\Omega(x_0)} \right) + o(\varepsilon), \quad (3.41)$$

where  $u_\Omega$  is the direct state, solution to (3.10) and  $v_\Omega$  is the adjoint state, solution to (3.27).

Then, the topological gradient is given by

$$g(x) = \Re \left( A_\omega(u_\Omega(x)) \overline{v_\Omega(x)} \right), \quad \forall x \in \Omega$$

and only two systems must be solved in order to compute  $g(x)$  for all  $x \in \Omega$ .

When  $\omega$  is the unit ball  $B(0, 1)$ , then  $v_\omega(y)$ ,  $P_\omega(y)$  and  $W_\omega(y)$  can be computed explicitly :

$$v_\omega(y) = \frac{u_\Omega(x_0)}{r} = P_\omega(y), \quad W_\omega(y) = 0, \quad 0 \neq y \in \mathbb{R}^3. \quad (3.42)$$

Then it follows from (3.30) (3.35) that

$$A_\omega(u_\Omega(x_0)) = 4\pi u_\Omega(x_0). \quad (3.43)$$

Then, we have the following result.

**Corollary 1** *Under the assumptions of Theorem 5 and when  $\omega$  is the unit ball  $B(0, 1)$ , the topological asymptotic expansion is given by*

$$j(\varepsilon) - j(0) = 4\pi\varepsilon \Re \left( u_\Omega(x_0) \overline{v_\Omega(x_0)} \right) + o(\varepsilon). \quad (3.44)$$

### 3.5.2 The two dimensional case

In this paragraph, we intend to derive the asymptotic expansion of the function  $j$  in two dimensional case. The technical used is similar to that of the three dimensional case. We use the principal part of the Helmholtz operator to derive the topological sensitivity expression. Next, we briefly describe the transposition of the previous results to the two dimensional case. As before,  $u_\Omega$  and the adjoint state  $v_\Omega$  are respectively the solutions to (3.10) and (3.27).

The exterior problem must now be defined differently than in (3.29). It consists to find  $v_\omega$ , solution to

$$\begin{cases} -\Delta v_\omega & = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega}, \\ v_\omega(y) / \log r & = u_\Omega(x_0) & \text{at } \infty, \\ v_\omega & = 0 & \text{on } \partial\omega. \end{cases} \quad (3.45)$$

A fundamental solution for the Laplace equation in  $\mathbb{R}^2$  is given by

$$E(y) = -\frac{1}{2\pi} \log r. \quad (3.46)$$

The function  $v_\omega$  has the form

$$v_\omega(y) = u_\Omega(x_0) \log \|y\| + P_\omega + W_\omega(y), \quad (3.47)$$

where  $P_\omega$  is constant and  $W_\omega(y) = o(1)$  at infinity [9]. In the next proposition (where  $\omega$  is not supposed to be a ball), one can observe that in the two dimensional case the topological sensitivity does not depend on the shape of the hole  $\omega$ , in contrast to the three dimensional case.

**Theorem 6** *The assumptions are the same as in Theorem 5. The function  $j$  has the following asymptotic expansion*

$$j(\varepsilon) = j(0) - \frac{2\pi}{\log \varepsilon} \Re \left( u_\Omega(x_0) \overline{v_\Omega(x_0)} \right) + o \left( \frac{1}{\log \varepsilon} \right). \quad (3.48)$$

The proof uses the same tools as for the three dimensional case (see Section 3.6) and will not be repeated for the two dimensional case.

## 3.6 Proofs

This section consists in the proof of Theorem 5. The variation of the sesquilinear form  $a_\varepsilon$  reads

$$a_\varepsilon(u, v) - a_0(u, v) = \int_{\Gamma_R} (T_\varepsilon - T_0) u \bar{v} \, d\gamma(x). \quad (3.49)$$

Hence, the problem reduces to the analysis of  $(T_\varepsilon - T_0)\varphi$  for  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ . More precisely, it will be shown that there exists an operator  $\delta T \in \mathcal{L} \left( H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R) \right)$  such that

$$\|T_\varepsilon - T_0 - \varepsilon \delta T\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}). \quad (3.50)$$

Consequently, defining  $\delta a$  by

$$\delta a(u, v) = \int_{\Gamma_R} \delta T u \bar{v} \, d\gamma(x) \quad \forall u, v \in \mathcal{V}_R \quad (3.51)$$

will yield straightforwardly

$$\|a_\varepsilon - a_0 - \varepsilon \delta a\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}). \quad (3.52)$$

First we need some definitions and preliminary lemmas.

### 3.6.1 Definitions

For convenience, the following norms and semi-norms are chosen for the functional spaces which will be used.

- For a bounded and open subset  $\mathcal{O} \subset \mathbb{R}^3$  and  $m \geq 0$ , the sobolev space  $H^m(\mathcal{O})$  is equipped with the norm defined by

$$\|u\|_{m,\mathcal{O}}^2 = \sum_{j=0}^m |u|_{j,\mathcal{O}}^2,$$

where the semi-norms  $|u|_{j,\mathcal{O}}$  are given by

$$|u|_{j,\mathcal{O}}^2 = \sum_{|\alpha|=j} \int_{\mathcal{O}} |\partial_{\alpha} u|^2 dx. \quad (3.53)$$

- For a given  $\varepsilon > 0$ , the space  $H^{\frac{1}{2}}(\Gamma_{R/\varepsilon})$  is equipped with the following norm :

$$\|u\|_{\frac{1}{2},\Gamma_{R/\varepsilon}} = \inf\{\|v\|_{1,C(R/2\varepsilon,R/\varepsilon)}; v|_{\Gamma_{R/\varepsilon}} = u\},$$

where  $C(r, r') = \{x \in \mathbb{R}^3; r < \|x\| < r'\}$ .

- The dual space  $H^{-\frac{1}{2}}(\Gamma_{R/\varepsilon})$  is equipped with the natural norm

$$\|w\|_{-\frac{1}{2},\Gamma_{R/\varepsilon}} = \sup\{|\langle w, v \rangle_{-\frac{1}{2},\frac{1}{2}}; v \in H^{\frac{1}{2}}(\Gamma_{R/\varepsilon}); \|v\|_{\frac{1}{2},\Gamma_{R/\varepsilon}} = 1\},$$

where  $\langle, \rangle_{-\frac{1}{2},\frac{1}{2}}$  is the duality product between  $H^{\frac{1}{2}}(\Gamma_{R/\varepsilon})$  and  $H^{-\frac{1}{2}}(\Gamma_{R/\varepsilon})$ .

### 3.6.2 Preliminary lemmas

Recall that  $x_0 = 0$ . We will use extensively the following change of variable : for a given function  $u$  defined on a subset  $\mathcal{O}$ , the function  $\tilde{u}$  is defined on  $\tilde{\mathcal{O}} = \mathcal{O}/\varepsilon$  by

$$\tilde{u}(y) = u(x), \quad y = \frac{x}{\varepsilon}.$$

**Lemma 8** *We have that*

$$|u|_{1,\mathcal{O}} = \varepsilon^{1/2} |\tilde{u}|_{1,\tilde{\mathcal{O}}} \quad (3.54)$$

$$\|u\|_{0,\mathcal{O}} = \varepsilon^{3/2} \|\tilde{u}\|_{0,\tilde{\mathcal{O}}}. \quad (3.55)$$

*proof.* Due to  $\nabla u(x) = \nabla \tilde{u}(y)/\varepsilon$  and to definition (3.53), we have

$$|u|_{1,\mathcal{O}}^2 = \int_{\mathcal{O}} |\nabla u|^2 dx = \frac{1}{\varepsilon^2} \int_{\tilde{\mathcal{O}}} |\nabla \tilde{u}|^2 \varepsilon^3 dy.$$

Similarly, we have

$$\|u\|_{0,\mathcal{O}} = \varepsilon^{3/2} \|\tilde{u}\|_{0,\tilde{\mathcal{O}}}.$$

□

**Lemma 9** [9] For  $\varphi \in H^{\frac{1}{2}}(\partial\omega)$  let  $v$  be the solution to the problem

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega}, \\ v = 0 & \text{at } \infty, \\ v = \varphi & \text{on } \partial\omega. \end{cases} \quad (3.56)$$

The function  $v$  is split into

$$\begin{aligned} v(y) &= V(y) + W(y) \\ V(y) &= E(y) \int_{\partial\omega} p(x) d\gamma(x), \end{aligned}$$

where  $E(y) = \frac{1}{4\pi\|y\|}$  and  $p \in H^{-\frac{1}{2}}(\partial\omega)$  is the unique solution to

$$\int_{\partial\omega} E(y-x)p(x) d\gamma(x) = \varphi(y) \quad \forall y \in \partial\omega. \quad (3.57)$$

There exists a constant  $c > 0$  (independent of  $\varphi$  and  $\varepsilon$ ) such that

$$\begin{aligned} \|V\|_{0,C(R/2\varepsilon,R/\varepsilon)} &\leq c\varepsilon^{-1/2}\|\varphi\|_{\frac{1}{2},\partial\omega} \\ |V|_{1,C(R/2\varepsilon,R/\varepsilon)} &\leq c\varepsilon^{1/2}\|\varphi\|_{\frac{1}{2},\partial\omega} \\ \|V\|_{0,D_\varepsilon/\varepsilon} &\leq c\varepsilon^{-1/2}\|\varphi\|_{\frac{1}{2},\partial\omega} \\ |V|_{1,D_\varepsilon/\varepsilon} &\leq c\|\varphi\|_{\frac{1}{2},\partial\omega} \\ \|W\|_{0,C(R/2\varepsilon,R/\varepsilon)} &\leq c\varepsilon^{1/2}\|\varphi\|_{\frac{1}{2},\partial\omega} \\ |W|_{1,C(R/2\varepsilon,R/\varepsilon)} &\leq c\varepsilon^{3/2}\|\varphi\|_{\frac{1}{2},\partial\omega} \\ \|W\|_{0,D_\varepsilon/\varepsilon} &\leq c\|\varphi\|_{\frac{1}{2},\partial\omega}. \end{aligned}$$

**Lemma 10** We assume that  $R < \frac{1}{\sqrt{2}|k|}$ . For a given  $\varepsilon > 0$ ,  $f_\varepsilon \in L^2(D_\varepsilon)$  and  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ , let  $v_\varepsilon$  be the solution to

$$\begin{cases} \Delta v_\varepsilon + k^2 v_\varepsilon = f_\varepsilon & \text{dans } D_\varepsilon, \\ v_\varepsilon = 0 & \text{sur } \partial\omega_\varepsilon, \\ v_\varepsilon = \varphi & \text{sur } \Gamma_R. \end{cases} \quad (3.58)$$

There exists a constant  $C(R, k) > 0$  (independent of  $\varphi$  and  $\varepsilon$ ) such that

$$\|v_\varepsilon\|_{1,D_\varepsilon} \leq C(R, k) \left( \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon} \right). \quad (3.59)$$

*Proof.* Let  $\mathcal{R}\varphi$  be the lifting of  $\varphi$  in the space  $H^1(C(R/2, R))$  such that  $\mathcal{R}\varphi|_{\Gamma_{R/2}} = 0$ . We extend  $\mathcal{R}\varphi$  by zero to the domain  $D_\varepsilon$ . We denote by  $\widetilde{\mathcal{R}\varphi}$  this extension. It belongs to  $H^1(D_\varepsilon)$ . We introduce

$$u_\varepsilon = \widetilde{\mathcal{R}\varphi} - v_\varepsilon, \quad (3.60)$$

$$g_\varepsilon = -f_\varepsilon + \Delta \widetilde{\mathcal{R}\varphi} + k^2 \widetilde{\mathcal{R}\varphi}. \quad (3.61)$$



The function  $g_\varepsilon$  belongs to the space  $H^{-1}(D_\varepsilon)$  and the new unknown  $u_\varepsilon$  is the solution to

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = g_\varepsilon & \text{in } D_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma_R. \end{cases} \quad (3.62)$$

Using the Poincaré inequality and the elliptic regularity, we obtain

$$\|u_\varepsilon\|_{1,D_\varepsilon} \leq \left( \frac{1 + 2R^2}{1 - 2k^2 R^2} \right) \|g_\varepsilon\|_{-1,D_\varepsilon}. \quad (3.63)$$

Finally, the result follows from (3.60), (3.61), (3.63) and the continuity of the lifting  $\mathcal{R}$  □

Here, and in the sequel, we assume that  $R < \frac{1}{\sqrt{2}|k|}$ .

**Lemma 11** For  $\varepsilon > 0$  and  $\psi \in H^1(D_0)$ , let  $X_\varepsilon$  the solution to the problem

$$\begin{cases} \Delta X_\varepsilon + k^2 X_\varepsilon = 0 & \text{in } D_\varepsilon, \\ X_\varepsilon = \psi & \text{on } \partial\omega_\varepsilon, \\ X_\varepsilon = 0 & \text{on } \Gamma_R. \end{cases} \quad (3.64)$$

There exists a constant  $c > 0$  (independent of  $\varphi$  and  $\varepsilon$ ), such that for all  $\varepsilon > 0$ ,

$$|X_\varepsilon|_{1,C(R/2,R)} \leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} \quad (3.65)$$

$$\|X_\varepsilon\|_{0,D_\varepsilon} \leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} \quad (3.66)$$

$$|X_\varepsilon|_{1,D_\varepsilon} \leq c\varepsilon^{1/2} \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.67)$$

*Proof.* Let  $\tilde{v}_\varepsilon$  be the solution to the exterior problem

$$\begin{cases} -\Delta \tilde{v}_\varepsilon = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega}, \\ \tilde{v}_\varepsilon = 0 & \text{at } \infty, \\ \tilde{v}_\varepsilon = \psi(\varepsilon y) & \text{on } \partial\omega. \end{cases} \quad (3.68)$$

The function  $X_\varepsilon$  can be written

$$X_\varepsilon = v_\varepsilon - w_\varepsilon,$$

where  $v_\varepsilon(x) = \tilde{v}_\varepsilon\left(\frac{x}{\varepsilon}\right)$ . The function  $w_\varepsilon$  itself is the solution to

$$\begin{cases} \Delta w_\varepsilon + k^2 w_\varepsilon = k^2 v_\varepsilon & \text{in } D_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon, \\ w_\varepsilon = v_\varepsilon & \text{on } \Gamma_R. \end{cases} \quad (3.69)$$

It follows from Lemma 10 that there exists a constant  $c > 0$  such that

$$\|w_\varepsilon\|_{1,D_\varepsilon} \leq c \left( \|v_\varepsilon|_{\Gamma_R}\|_{\frac{1}{2},\Gamma_R} + k^2 \|v_\varepsilon\|_{0,D_\varepsilon} \right). \quad (3.70)$$

It follows from Lemma 8 and Lemma 9 that

$$\|v_\varepsilon|_{\Gamma_R}\|_{\frac{1}{2},\Gamma_R} \leq c \|v_\varepsilon\|_{1,C(R/2,R)} \quad (3.71)$$

$$\leq c \left( \|v_\varepsilon\|_{0,C(R/2,R)} + |v_\varepsilon|_{1,C(R/2,R)} \right) \quad (3.72)$$

$$= c \left( \varepsilon^{3/2} \|\tilde{v}_\varepsilon\|_{0,C(R/2\varepsilon,R/\varepsilon)} + \varepsilon^{1/2} |\tilde{v}_\varepsilon|_{1,C(R/2\varepsilon,R/\varepsilon)} \right) \quad (3.73)$$

$$\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.74)$$

We have that

$$\|v_\varepsilon\|_{0,D_\varepsilon} = \varepsilon^{3/2} \|\tilde{v}_\varepsilon\|_{0,D_\varepsilon/\varepsilon} \quad (3.75)$$

$$\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.76)$$

From (3.70), (3.74) and (3.76), we obtain that

$$\|w_\varepsilon\|_{1,D_\varepsilon} \leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.77)$$

Then, we have

$$\bullet |X_\varepsilon|_{1,C(R/2,R)} = |v_\varepsilon - w_\varepsilon|_{1,C(R/2,R)} \quad (3.78)$$

$$\leq |v_\varepsilon|_{1,C(R/2,R)} + |w_\varepsilon|_{1,C(R/2,R)} \quad (3.79)$$

$$\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} + \|w_\varepsilon\|_{1,D_\varepsilon} \quad (3.80)$$

$$\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.81)$$

$$\bullet \|X_\varepsilon\|_{0,D_\varepsilon} \leq \|v_\varepsilon\|_{0,D_\varepsilon} + \|w_\varepsilon\|_{1,D_\varepsilon} \quad (3.82)$$

$$\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.83)$$

$$\bullet |X_\varepsilon|_{1,D_\varepsilon} \leq |v_\varepsilon|_{1,D_\varepsilon} + |w_\varepsilon|_{1,D_\varepsilon} \quad (3.84)$$

$$\leq \varepsilon^{1/2} |\tilde{v}_\varepsilon|_{1,D_\varepsilon/\varepsilon} + \|w_\varepsilon\|_{1,D_\varepsilon} \quad (3.85)$$

$$\leq c\varepsilon^{1/2} \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} + c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} \quad (3.86)$$

$$\leq c\varepsilon^{1/2} \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}. \quad (3.87)$$

This completes the proof. □

Lemmas 10 and 11 are summarized in the following Lemma.

**Lemma 12** For  $\varepsilon > 0$ ,  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ ,  $\psi \in H^1(D_0)$  and  $f_\varepsilon \in L^2(D_\varepsilon)$ , let  $v_\varepsilon$  be the solution to the problem

$$\begin{cases} \Delta v_\varepsilon + k^2 v_\varepsilon = f_\varepsilon & \text{in } D_\varepsilon, \\ v_\varepsilon = \psi & \text{on } \partial\omega_\varepsilon, \\ v_\varepsilon = \varphi & \text{on } \Gamma_R. \end{cases} \quad (3.88)$$

There exists a constant  $c > 0$  (independent of  $\varphi$ ,  $\psi$ ,  $f_\varepsilon$  and  $\varepsilon$ ) such that for all  $\varepsilon > 0$ ,

$$|v_\varepsilon|_{1,C(R/2,R)} \leq c \left( \varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} + \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon} \right) \quad (3.89)$$

$$\|v_\varepsilon\|_{0,D_\varepsilon} \leq c \left( \varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} + \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon} \right) \quad (3.90)$$

$$|v_\varepsilon|_{1,D_\varepsilon} \leq c \left( \varepsilon^{1/2} \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} + \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon} \right). \quad (3.91)$$

**Lemma 13** Let  $u$  belongs to the space  $H^1(C(R/2, R))$  and satisfies  $\Delta u + k^2 u = 0$  in  $C(R/2, R)$ ,  $u|_{\Gamma_R} = 0$ . Then, there exists a constant  $c > 0$  (independent of  $u$ ) such that

$$\|\nabla u \cdot n|_{\Gamma_R}\|_{-\frac{1}{2},\Gamma_R} \leq c |u|_{1,C(R/2,R)}. \quad (3.92)$$

*Proof.* Let  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ . We define  $v$  the solution to the problem

$$\begin{cases} \Delta v = 0 & \text{in } C(R/2, R), \\ v = 0 & \text{on } \Gamma_{R/2}, \\ v = \varphi & \text{on } \Gamma_R. \end{cases}$$

Using the Green formula, we obtain

$$\int_{\Gamma_R} \nabla u \cdot n|_{\Gamma_R} \bar{\varphi} \, d\gamma(x) = \int_{C(R/2,R)} \nabla u \cdot \nabla \bar{v} \, dx - k^2 \int_{C(R/2,R)} u \bar{v} \, dx.$$

Then, we have

$$\begin{aligned} \left| \int_{\Gamma_R} \nabla u \cdot n|_{\Gamma_R} \bar{\varphi} \, d\gamma(x) \right| &\leq |u|_{1,C(R/2,R)} \|v\|_{1,C(R/2,R)} + k^2 \|u\|_{0,C(R/2,R)} \|v\|_{1,C(R/2,R)} \\ &\leq |u|_{1,C(R/2,R)} \|\varphi\|_{\frac{1}{2},\Gamma_R} + ck^2 |u|_{1,C(R/2,R)} \|\varphi\|_{\frac{1}{2},\Gamma_R} \\ &\leq c |u|_{1,C(R/2,R)} \|\varphi\|_{\frac{1}{2},\Gamma_R}. \end{aligned}$$

This completes the proof.  $\square$

### 3.6.3 Variation of the sesquilinear form

The variation of the sesquilinear form  $a_\varepsilon$  reads

$$a_\varepsilon(u, v) - a_0(u, v) = \int_{\Gamma_R} (T_\varepsilon - T_0)u\bar{v} \, d\gamma(x).$$

For  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ , recall that  $u_\varepsilon^\varphi$  is the solution to (3.20) or (3.21) if  $\varepsilon = 0$ . Let  $v_\omega^\varphi$  be the solution to the problem

$$\begin{cases} \Delta v_\omega^\varphi = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\omega}, \\ v_\omega^\varphi = 0 & \text{at } \infty, \\ v_\omega^\varphi = u_0^\varphi(x_0) & \text{on } \partial\omega. \end{cases} \quad (3.93)$$

Like in (3.34)-(3.35), Let  $P_\omega^\varphi(y) = A_\omega(u_0^\varphi(x_0))E(y)$  be the dominant part of  $v_\omega^\varphi$ , and let  $Q_\omega^\varphi$  be the solution to the associated interior problem

$$\begin{cases} \Delta Q_\omega^\varphi + k^2 Q_\omega^\varphi = k^2 P_\omega^\varphi & \text{in } D_0, \\ Q_\omega^\varphi = P_\omega^\varphi|_{\Gamma_R} & \text{on } \Gamma_R. \end{cases} \quad (3.94)$$

The linear operator  $\delta T$  (independent of  $\varepsilon$ ) is defined as follows :

$$\begin{aligned} \delta T : H^{1/2}(\Gamma_R) &\longrightarrow H^{-1/2}(\Gamma_R), \\ \varphi &\longmapsto \delta T\varphi = \nabla(Q_\omega^\varphi - P_\omega^\varphi) \cdot n|_{\Gamma_R}. \end{aligned} \quad (3.95)$$

**Proposition 11** *The operator  $T_\varepsilon$  admits the following asymptotic expansion*

$$\|T_\varepsilon - T_0 - \varepsilon\delta T\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}).$$

*Proof.* Let  $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$ . For simplicity we drop the subscripts  $(\cdot)^\varphi$ . For  $y = x/\varepsilon$ , we have

$$v_\omega(y) = P_\omega(y) + W_\omega(y),$$

with  $P_\omega\left(\frac{x}{\varepsilon}\right) = \varepsilon P_\omega(x)$  and  $W_\omega(y) = O\left(\frac{1}{\|y\|^2}\right)$ . Let

$$\psi_\varepsilon(x) = (T_\varepsilon - T_0 - \varepsilon\delta T)\varphi(x).$$

We have

$$\begin{aligned} \psi_\varepsilon(x) &= (\nabla u_\varepsilon - \nabla u_0 - \varepsilon(\nabla Q_\omega - \nabla P_\omega)) \cdot n|_{\Gamma_R} \\ &= \nabla\left(w_\varepsilon(x) - W_\omega\left(\frac{x}{\varepsilon}\right)\right) \cdot n|_{\Gamma_R}, \end{aligned}$$

where  $w_\varepsilon$  is defined by

$$w_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon Q_\omega(x) + v_\omega\left(\frac{x}{\varepsilon}\right).$$

The function  $w_\varepsilon$  is solution to

$$\begin{cases} \Delta w_\varepsilon + k^2 w_\varepsilon = k^2 W_\omega(x/\varepsilon) & \text{in } D_\varepsilon, \\ w_\varepsilon = W_\omega(x/\varepsilon) & \text{on } \Gamma_R, \\ w_\varepsilon = -u_0(x) + u_0(0) - \varepsilon Q_\omega(x) & \text{on } \partial\omega_\varepsilon. \end{cases} \quad (3.96)$$

In order to apply Lemma 12, we have to estimate the right-hand side terms.

- in  $D_\varepsilon$ , we have

$$\|W_\omega(x/\varepsilon)\|_{0,D_\varepsilon} = \varepsilon^{3/2} \|W_\omega(y)\|_{0,D_\varepsilon/\varepsilon}.$$

Using Lemma 9, we obtain

$$\begin{aligned} \|W_\omega(y)\|_{0,D_\varepsilon/\varepsilon} &\leq c \|u_0(x_0)\|_{\frac{1}{2},\partial\omega} \\ &\leq c |u_0(x_0)| \\ &\leq c \|\varphi\|_{\frac{1}{2},\Gamma_R}. \end{aligned}$$

Then, we have

$$\|W_\omega(x/\varepsilon)\|_{0,D_\varepsilon} \leq c \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R}.$$

- On  $\Gamma_R$ , using Lemma (8), Lemma (9) and the elliptic regularity, we obtain

$$\begin{aligned} \|W_\omega(x/\varepsilon)\|_{\frac{1}{2},\Gamma_R} &\leq c \|W_\omega(x/\varepsilon)\|_{1,C(R/2,R)} \\ &\leq c (\|W_\omega(x/\varepsilon)\|_{0,C(R/2,R)} + |W_\omega(x/\varepsilon)|_{1,C(R/2,R)}) \\ &= c (\varepsilon^{3/2} \|W_\omega(y)\|_{0,C(R/2\varepsilon,R/\varepsilon)} + \varepsilon^{1/2} |W_\omega(y)|_{1,C(R/2\varepsilon,R/\varepsilon)}) \\ &\leq c \varepsilon^2 \|u_0(x_0)\|_{\frac{1}{2},\partial\omega} \\ &\leq c \varepsilon^2 |u_0(x_0)| \\ &\leq c \varepsilon^2 \|\varphi\|_{\frac{1}{2},\Gamma_R}. \end{aligned}$$

- On  $\partial\omega_\varepsilon$ , putting

$$\theta_\varepsilon(x) = \frac{-u_0(x) + u_0(x_0) - \varepsilon Q_\omega(x)}{\varepsilon},$$

we have for small  $\varepsilon$

$$\begin{aligned} \|\theta_\varepsilon(\varepsilon y)\|_{\frac{1}{2},\partial\omega} &\leq c \|\theta_\varepsilon(\varepsilon y)\|_{1,\omega} \\ &= c \left\| \frac{u_0(\varepsilon y) - u_0(x_0)}{\varepsilon} + Q_\omega(\varepsilon y) \right\|_{1,\omega} \\ &\leq c (\|u_0\|_{C^2(B(0,R/2))} + \|Q_\omega\|_{C^1(B(0,R/2))}) \\ &\leq c \|\varphi\|_{\frac{1}{2},\Gamma_R}. \end{aligned}$$

We can now apply Lemma 12, which gives

$$\begin{aligned} |w_\varepsilon|_{1,C(R/2,R)} &\leq c \left( \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R} + \varepsilon^2 \|\varphi\|_{\frac{1}{2},\Gamma_R} + \varepsilon \|\varepsilon \theta_\varepsilon(\varepsilon y)\|_{\frac{1}{2},\partial\omega} \right) \\ &\leq c \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R}. \end{aligned}$$

Finally, it follows from Lemma (8) and Lemma (13) that

$$\begin{aligned}
\|\psi\|_{-\frac{1}{2},\Gamma_R} &= \|\nabla(w_\varepsilon - W_\omega(x/\varepsilon)) \cdot n|_{\Gamma_R}\|_{-\frac{1}{2},\Gamma_R} \\
&\leq c \left( |w_\varepsilon|_{1,C(R/2,R)} + |W_\omega(x/\varepsilon)|_{1,C(R/2,R)} \right) \\
&= c \left( |w_\varepsilon|_{1,C(R/2,R)} + \varepsilon^{1/2} |W_\omega(y)|_{1,C(R/2\varepsilon,R/\varepsilon)} \right) \\
&\leq c \left( \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R} + \varepsilon^2 \|\varphi\|_{\frac{1}{2},\Gamma_R} \right) \\
&\leq c\varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R}.
\end{aligned}$$

Hence,

$$\|T_\varepsilon - T_0 - \varepsilon\delta T\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}).$$

□

The asymptotic expansion of the sesquilinear form  $a_\varepsilon$  follows now straightforwardly.

**Proposition 12** *Let*

$$\delta a(u, v) = \int_{\Gamma_R} \delta T u \bar{v} \, d\gamma(x) \quad u, v \in \mathcal{V}_R.$$

*Then the asymptotic expansion of the sesquilinear form  $a_\varepsilon$  is given by*

$$\|a_\varepsilon - a_0 - \varepsilon\delta a\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}).$$

### 3.6.4 Proof of Theorem 5

The proof of this theorem is done in two steps. Firstly, we prove that Hypothesis 4 is satisfied. More precisely, we prove that the sesquilinear form  $a_0$  satisfies the inf-sup condition. Secondly, we apply Theorem 1 to compute the topological asymptotic expansion.

#### The first step : the inf-sup condition

For all  $u \in \mathcal{V}_R$  we set

$$\tilde{u} = \begin{cases} u & \text{in } \Omega_R, \\ u_0^\varphi & \text{in } B(x_0, R), \end{cases}$$

where  $\varphi = u|_{\Gamma_R}$  and  $u_0^\varphi$  is the solution to

$$\begin{cases} \Delta u_0^\varphi + k^2 u_0^\varphi = 0 & \text{in } B(x_0, R), \\ u_0^\varphi = \varphi & \text{on } \Gamma_R. \end{cases}$$

It can easily be proved that

$$a_0(u, v|_{\Omega_R}) = a(\Omega, \tilde{u}, v) \quad \forall u \in \mathcal{V}_R, \forall v \in \mathcal{V}(\Omega),$$

where the functional space  $\mathcal{V}(\Omega)$  and the sesquilinear form  $a(\Omega, \cdot, \cdot)$  are defined by (3.11). From Proposition 9, the sesquilinear form  $a(\Omega, \cdot, \cdot)$  satisfies the inf-sup condition. As a consequence, there exists  $v \in \mathcal{V}(\Omega), v \neq 0$ , such that

$$\begin{aligned} a_0(u, v|_{\Omega_R}) = a(\Omega, \tilde{u}, v) &\geq a\|\tilde{u}\|_{\mathcal{V}(\Omega)}\|v\|_{\mathcal{V}(\Omega)} \\ &\geq a\|u\|_{\mathcal{V}_R}\|v|_{\Omega_R}\|_{\mathcal{V}_R}. \end{aligned}$$

Then  $a_0$  satisfies the inf-sup condition and Hypothesis 4 is satisfied.

**Applying Theorem 4**

All the hypotheses of section 3.2 are satisfied and we can apply Theorem 4. We obtain the following asymptotic formula :

$$\begin{aligned} j(\varepsilon) - j(0) &= \varepsilon\Re(\delta a(u_\Omega, v_\Omega)) + o(\varepsilon) \\ &= \varepsilon\Re\left(\int_{\Gamma_R} \nabla(Q_\omega^\varphi - P_\omega^\varphi) \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x)\right) + o(\varepsilon), \end{aligned}$$

where  $\varphi = u_\Omega|_{\Gamma_R} = u_0|_{\Gamma_R}$ . Thanks to Green's formula and (3.94), we obtain that

$$\begin{aligned} \int_{\Gamma_R} \nabla(Q_\omega^\varphi - P_\omega^\varphi) \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x) &= k^2 \int_{D_0} P_\omega \overline{v_\Omega} \, dx + \int_{\Gamma_R} \nabla \overline{v_\Omega} \cdot n|_{\Gamma_R} P_\omega \, d\gamma(x) \\ &\quad - \int_{\Gamma_R} \nabla P_\omega \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x). \end{aligned} \tag{3.97}$$

It can be shown that

$$\begin{aligned} \int_{\Gamma_R} \nabla \overline{v_\Omega} \cdot n|_{\Gamma_R} P_\omega \, d\gamma(x) - \int_{\Gamma_R} \nabla P_\omega \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x) &= A_\omega(u_\Omega(x_0)) \langle -\Delta E, \overline{v_\Omega} \psi \rangle_{\mathcal{D}'(D_0), \mathcal{D}(D_0)} \\ &\quad - k^2 \int_{D_0} P_\omega \overline{v_\Omega} \, dx \\ &= A_\omega(u_\Omega(x_0)) \langle \delta, \overline{v_\Omega} \psi \rangle_{\mathcal{D}'(D_0), \mathcal{D}(D_0)} \\ &\quad - k^2 \int_{D_0} P_\omega \overline{v_\Omega} \, dx \\ &= A_\omega(u_\Omega(x_0)) \overline{v_\Omega(x_0)} - k^2 \int_{D_0} P_\omega \overline{v_\Omega} \, dx, \end{aligned}$$

where  $\psi \in \mathcal{D}(D_0)$  satisfies  $\psi(x_0) = 1$ . We insert this expression into (3.97), and we obtain the desired result.

**3.7 Numerical results : Buried objects detection**

We will consider here a simple problem of detection of metallic object buried in a soil. The aim is to find the number and the positions of metallic

objects (supposedly infinite in the  $\vec{e}_z$  direction), using scattered field measurements from a mono-static antenna horizontally translated above the soil. This is a rough model of the facilities described in [19]. The 2D Helmholtz equation is solved with time-domain finite differences (FDTD), the frequency-domain solution being obtained with a Fourier transform. The antenna is roughly approximated by a single source point, which will be translated at various locations above the soil. At each point of the mesh, the topological sensitivity will be computed.

Let  $\mathcal{X} = \{x_i\}_{i=1..n_x}$  be the set of the successive locations of the source (and sensors, since the antenna is supposed to be mono-static), and  $\mathcal{F} = \{f_i\}_{i=1..n_f}$  the set of measurement frequencies. Let  $\varepsilon_s$  be the soil permittivity. The set of metallic objects buried in the soil is denoted by  $\Omega$ .

We associate to  $\Omega$  a set of “measurements”  $\mathcal{M}(\Omega)$  : at each couple  $(x_i, f_j) \in \mathcal{X} \times \mathcal{F}$ , we first define the field  $u_{x_i, f_j}^\Omega$ , solution of

$$\begin{cases} \Delta u + k_j^2 u & = s_{x_i} & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ u & = 0 & \text{on } \partial\Omega, \\ \lim_{r \rightarrow \infty} \sqrt{r}(\partial_r u - iku) & = 0, \end{cases} \quad (3.98)$$

where  $s_{x_i}$  represents a source point centered at  $x_i$ , and where

$$\begin{aligned} k_j^2(x) &= \varepsilon(x)\mu\omega_j^2, \\ w_j &= 2\pi f_j, \\ \varepsilon(x) &= \begin{cases} \varepsilon_0 & \text{if } x \geq 0, \\ \varepsilon_s & \text{if } x < 0. \end{cases} \end{aligned}$$

Then the “measurements” are  $\mathcal{M}(\Omega) = \{m_{x_i, f_j}(\Omega)\}$ , in our numerical tests,  $m_{x_i, f_j}(\Omega)$  is the value of the scattered field at the point  $x_i$ .

We call *reference measurements*  $\widetilde{\mathcal{M}} = \{\widetilde{m}_{x_i, f_j}\}$  the values which are obtained with the real objects in the soil. Ideally these would have been real measurement, but in the following numerical results, we only consider synthetic data obtained via FDTD.

The cost function, which expresses the adequacy between the measurements obtained for a distribution of metallic objects  $\Omega$  and the reference data, is :

$$j(\Omega) = \|\mathcal{M} - \widetilde{\mathcal{M}}\|^2 = \sum_{i,j} j_{x_i, f_j}(\Omega), \quad (3.99)$$

where

$$j_{x_i, f_j}(\Omega) = |m_{x_i, f_j}(\Omega) - \widetilde{m}_{x_i, f_j}|^2. \quad (3.100)$$

Applying the expression of the topological asymptotic (see Proposition 6), one has

$$j(\Omega \setminus \overline{B(x, \varepsilon)}) - j(\Omega) = \sum_{i,j} -\frac{2\pi}{\log \varepsilon} \Re \left( u_{x_i, f_j}^\Omega(x) \overline{v_{x_i, f_j}^\Omega(x)} \right) + o\left(\frac{1}{\log \varepsilon}\right), \quad (3.101)$$



where  $v_{x_i, f_j}^\Omega$  is the adjoint state associated to the couple  $(x_i, f_j)$ .

The first example (see Fig. 3.2) shows the topological sensitivity computed on an “ideal” case : there is no noise on the data, and the reference soil is a flat and homogeneous dry sand soil. One can see that the top of the five objects is clearly identified by the negative values of the topological sensitivity . This topological sensitivity can be obtained very quickly since it is evaluated on a empty flat soil, which is invariant by translation : all direct states and adjoint states are just horizontal translations of a “canonical” solution. The computational cost is only 10 seconds on a 300MHz personal computer.

The second example (see Fig. 3.3) is a little bit more realistic : the data is artificially noised since the reference data  $\widetilde{\mathcal{M}}$  was obtained on a non-flat inhomogeneous soil, while the topological sensitivity was still computed on a flat homogeneous soil. One can observe that, although the objects are still located correctly, the image (see Fig. 3.3(b)) is a little bit distorted.

The third example shows that using an iterative process might give good results, and the expense of some computational cost. In this example, the basic iterative algorithm just inserts a metal point at the point where the topological sensitivity is the most negative. Then the topological sensitivity is reevaluated, taking into account the metal points inserted at previous iterations, etc.. the figure 3.4 shows the objects and the metal points that were inserted at iteration 10 and iteration 55.

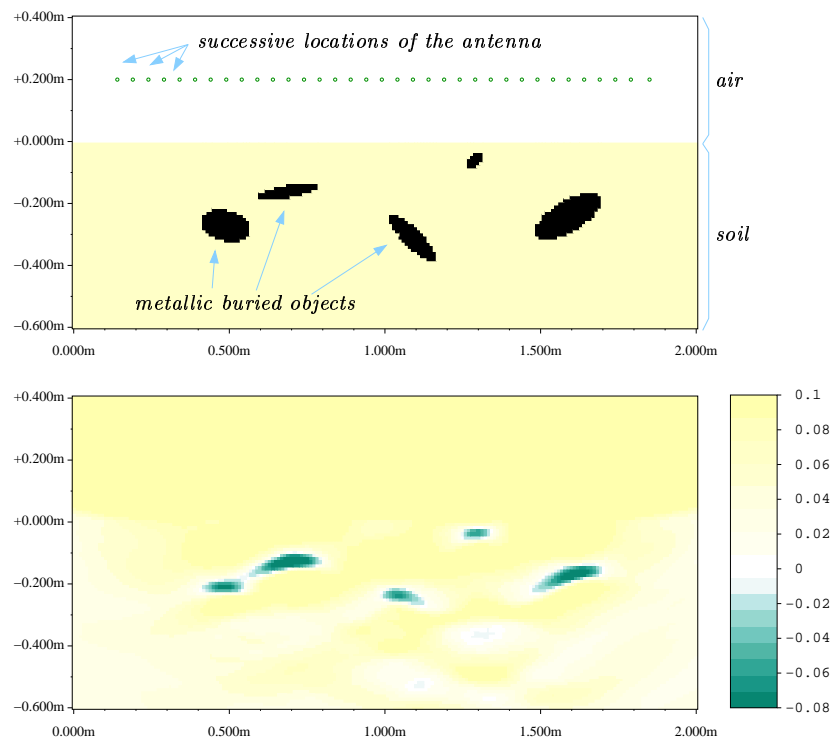


FIG. 3.2 – Repartition of metallic objects in the soil, and the corresponding topological sensitivity computed on an empty flat soil (dry soil, flat surface  $\varepsilon_r = 2.3, 20$  Frequencies ranging from 400MHz to 2GHz).

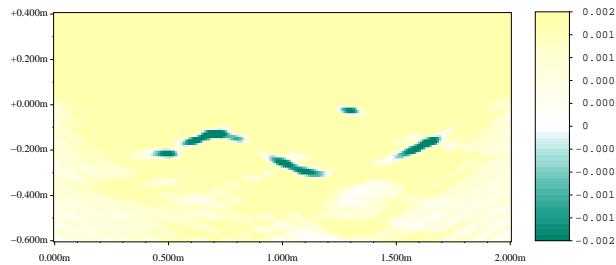
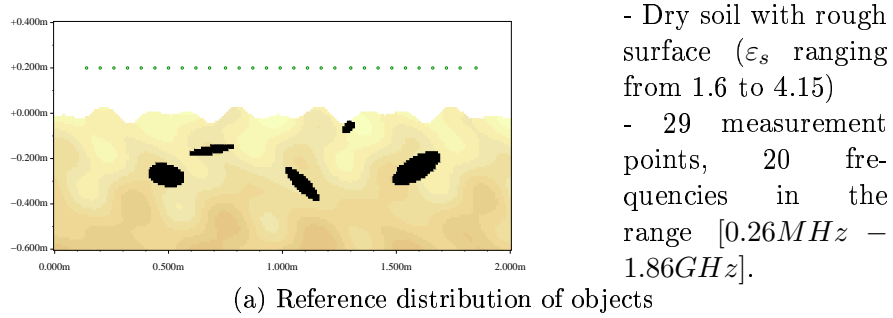


FIG. 3.3 – Topological sensitivity with noisy data.

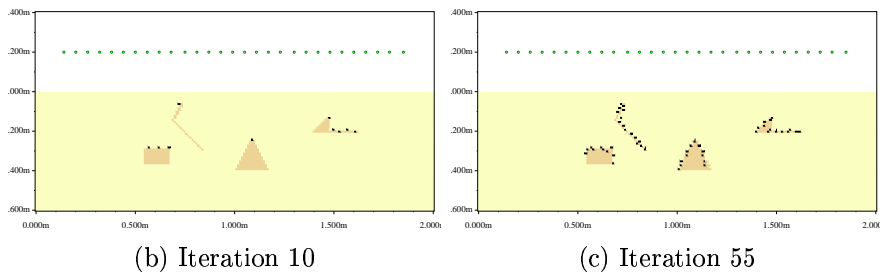
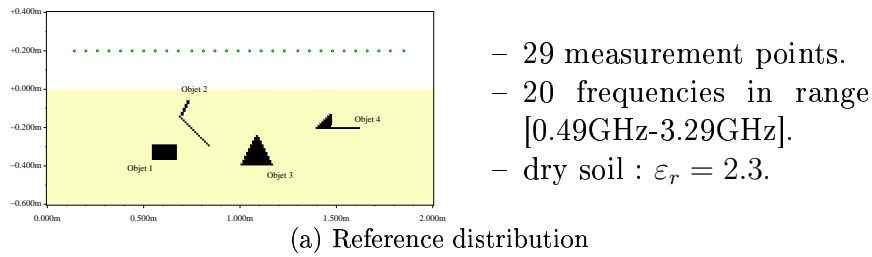


FIG. 3.4 – Iterations

# Bibliographie

- [1] G. ALLAIRE, R. KOHN, *Optimal design for minimum weight and compliance in plane stress using extremal microstructures*, *European Journal of Mechanics, A/Solids*, 12(6) :839-878, 1993.
- [2] H. AMMARI, H. KANG, *Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities*, submitted to *J. Math. Anal. Appl.*
- [3] M. BENDSOE, *Optimal topology design of continuum structure : an introduction*, Technical report, Departement of mathematics, Technical University of Denmark, DK2800 Lyngby, Denmark, september 1996.
- [4] E. BONNETIER, C. CONCA, *Approximation of Young measures by functions and application to a problem of optimal design for plates with variable thickness*, *Proc. Royal Soc. Edinburgh* **124A**, pp. 399-422, 1994.
- [5] J. CÉA, *Conception optimale ou identification de forme, calcul rapide de la dérivée directionnelle de la fonction coût*, *M.A.A.N.*, 20(3) :371-402, 1986.
- [6] J. CÉA, S. GARREAU, PH. GUILLAUME, M. MASMOUDI, *Shape and Topological Optimizations Connection*, Research Report, UFR MIG, Université Paul Sabatier, Toulouse, France, 1998. Also published in *Comput. Methods Appl. Mech. Engrg.* 188, 713-726, 2000.
- [7] H. A. ESCHENAUER, N. OLSHOFF, *Topology optimization of continuum structures*, a review, *Appl. Mech. Rev.* 54, 331-390, 2001.
- [8] S. GARREAU, PH. GUILLAUME, M. MASMOUDI, *The topological asymptotic for PDE systems : the elasticity case*, *SIAM J. Control Optim.* 39 (2001) 1756-1778.
- [9] PH. GUILLAUME, K. SIDIDRIS, *The topological asymptotic expansion for the Dirichlet problem*, accepted in *SIAM Journal on Control and Optimization*.
- [10] F. IHLENBURG AND I. BABUSKA, *Finite element solution of the Helmholtz equation with high wave number partII : the h-p version of the FEM*, *Siam J. Numer. Anal.* Vol. 34, No. 1, pp. 315-358, February 1997.

- [11] A. M. IL'IN, *A boundary value problem for the elliptic equation of second order in a domain with a narrow slit. I. The two-dimensional case*, Mat. sb. 99 (141)(1976) 514-537. (Eng transl. in Math USSR Sbornik **28** (1976)).
- [12] A. M. IL'IN, *Study of the asymptotic behavior of the solution of an elliptic boundary value problem in a domain with a small hole*, Trudy Sem. Petrovsk. **6**(1981) 57-82 (Russian).
- [13] A. M. IL'IN, *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, Translations of Mathematical Monographs, Vol. 102, AMS 1992.
- [14] J. JACOBSEN, N. OLHOFF, AND E. RONHOLT, *Generalized shape optimization of three-dimensionnal structures using materials with optimum microstructures*, Technical report, Institute of Mechanical Engineering, Aalborg University, DK-9920 Aalborg, Denmark, 1996.
- [15] C. KANE, M. SCHOENAUER, *Optimization topologique de formes par algorithmes génétiques*, Revue Française de Mécanique, pages 4 :237-246, 1997.
- [16] R. V. KOHN, M. S. VOGELIUS, *Thin plates with varying thicknesses and their relation to structural optimization*, in Homogenization and Effective Moduli of Materials and Media, IMA vol. 1, eds. J.L. Ericksen et al., Springer, 1986.
- [17] M. MASMOUDI, *The topological asymptotic*, in Computational Methods for Control Applications, ed. H. Kawarada and J. Periaux, International Series GAKUTO, 2002.
- [18] V. G. MAZ'YA, S. A. NAZAROV, B. A. PLAMENEVSKIJ, *Asymptotic expansions of the eigenvalues of boundary value problems for the Laplace operator in domains with small holes*, Izv. Akad. Nauk SSSR. Ser. Mat. **48**(2)(1984) 347-371. (English transl. in Math. USSR Izvestiya. **24**(1985) 321-345).
- [19] P. MILLOT, J.C. BUREAU, P. BORDERIES, E. BACHELIER, C. PICHOT, E. LEBRUSQ, E. LEBRUSQ, E. GUILIANTON, AND J.Y. DAUVIGNAC, *Experimental study of near surface radar imaging of buried objects with adaptive focussed synthetic aperture processing*, in subsurface Sensing Technologies and Applications II, Proceedings of SPIE Vol. 4129, pages 515-523. CamNyugen, 2000.
- [20] F. MURAT, S. SIMON, *Etudes de problèmes d'optimal design*, Lecture Notes in Computer Science 41, 54-62, Springer Verlag, Berlin, 1976.
- [21] S. A. NAZAROV, J. SOKOLOWSKI, *Asymptotic analysis of shape functionals*, Rapport de recherche de l'INRIA, RR-4633, 2002.
- [22] S. A. NAZAROV, *Asymptotic expansions of eigenvalues*, Leningrad Univ. 1987 (Russian).

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- [23] S. OSAVA, *Singular Hadamard's variation of domains and eigenvalues of Laplacian*, Parts 1, 2 Proc. Japan Acad Sci **56** (1980)306-310, **57**(1981) 242-246.
- [24] O. PIRONNEAU, *Optimal shape design for elliptic systems*, Springer-Verlag, New York, 1984.
- [25] M. SCHOENAUER, L. KALLEL, F. JOUVE, *Mechanics inclusions identification by evolutionary computation*, Revue européenne des éléments finis, 5(5-6), pp. 619-648, 1996.
- [26] A. SCHUMACHER, *Topologieoptimierung von Bauteilstrukturen unter Verwendung von Lochpositionierungskriterien*, Doctoral Thesis, FOMMAS-Report Nr T09-01.96, Univ of Siegen, 1996.
- [27] J. SIMON, *Differentiation with respect to the domain in boundary value problems*, Num. Funct. Anal. Optimz., 2, 649-687, 1980.
- [28] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *On topological derivative in shape optimization*, SIAM J. Control Optim. 37, 1251-1272, 1999.
- [29] J. SOKOŁOWSKI, A. ŻOCHOWSKI, *Topological derivatives for elliptic problems*, Inverse Problems 15, 123-134, 1999.
- [30] J. SOKOŁOWSKI, J.P. ZOLESIO, *Introduction to shape optimization : shape sensitivity analysis*, Springer Series in Computational Mathematics, Vol. 10, Springer, Berlin, 1992.
- [31] M. S. VOGELIUS, D. VOLKOV, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter*, Math. Model. Numer. Anal. 34, pp. 723-748, 2000.



# Chapitre 4

## Sensitivity analysis with respect to the insertion of small inhomogeneities

### Sommaire

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Ce chapitre est une version détaillée de la note “*Sensitivity analysis with respect to the insertion of small inhomogeneities*” soumise aux Comptes Rendus de l’Académie des Sciences. C’est un travail réalisé en collaboration avec Samuel Amstutz.

## 4.1 Introduction

The aim of the topological sensitivity analysis is to obtain an asymptotic expansion of a cost function with respect to the creation of a small hole in the domain. The principle is the following. We consider a cost function  $\mathcal{J}(\Omega) = J(\Omega, u_\Omega)$  where  $u_\Omega$  is the solution to a partial differential equation defined in the domain  $\Omega \subset \mathbb{R}^d$  ( $d=2$  or  $3$ ), a point  $x_0 \in \Omega$  and a fixed domain  $\overline{B} \subset \mathbb{R}^d$ , containing the origin. We search an asymptotic expansion of  $\mathcal{J}(\Omega \setminus (x_0 + \varepsilon B))$  when  $\varepsilon$  tends to 0. In most cases, it is of the form

$$\mathcal{J}(\Omega \setminus \overline{(x_0 + \varepsilon B)}) - \mathcal{J}(\Omega) = f(\varepsilon)G(x_0) + o(f(\varepsilon)). \quad (4.1)$$

Here,  $f(\varepsilon)$  is an explicit positive function going to zero with  $\varepsilon$  and the function  $G$ , called the topological gradient (or sensitivity), is very easy to compute. We call (4.1) the topological asymptotic expansion. Hence, to minimize the criterion  $\mathcal{J}$  we just have to create small holes at some points  $\tilde{x}$  where the topological gradient is negative. For more details about this approach we refer the reader to S. Garreau et al. [10], Ph. Guillaume and K. Sididris [11], M. Masmoudi [12], A. Schumacher [15], A.A. Novotny et al. [13] and J. Sokolowski and A. Żochowski [16]. In all these works, the authors are interested in only one way of perturbing the domain : the insertion of a hole.

Another situation, firstly considered by D.J. Cedio-Fengya, S. Moskov and M.S. Vogelius [9], consists in perturbing the domain by the insertion of small inhomogeneities with constitutive parameters different from those of the background medium. In this publication, the authors are interested in the identification of conductivity imperfections of small diameter by the use of boundary measurements. Other references can be found in [4, 3, 5, 2, 17, 6, 1]. In all these publications, only asymptotic formulas of solutions are presented.

In the present work, we combine these two approaches. More precisely, we compute the topological asymptotic with respect to the insertion of small inhomogeneities in the domain. As a model example, we consider solutions to the Helmholtz equation in two and three dimensions. An adjoint method is used to obtain the expressions of  $f(\varepsilon)$  and  $G(x_0)$  (see Formula (4.1)) for a large class of shape functions. This method requires an asymptotic expansion of the solution. It is given in this case in [17].

The paper is organized as follows. In Section 4.2 we present the adjoint method. The Helmholtz problem in the presence of small inhomogeneities and the adjoint equation are presented in Section 4.3. In Section 4.4 we compute the expression of the topological asymptotic. Some particular cases of cost

functions and of shape of inserted objects are presented in Section 4.5. The case of metallic objects is retrieved formally [7].

## 4.2 The adjoint method

Let  $\mathcal{V}$  be a complex Hilbert space. For all  $\varepsilon \geq 0$ , let  $a_\varepsilon$  be a sesquilinear and continuous form on  $\mathcal{V}$  and  $\ell_\varepsilon$  be a semilinear and continuous form on  $\mathcal{V}$ . We assume that for all  $\varepsilon \geq 0$ , The following problem :

$$\begin{cases} u_\varepsilon \in \mathcal{V} \\ a_\varepsilon(u_\varepsilon, v) = \ell_\varepsilon(v) \quad \forall v \in \mathcal{V} \end{cases} \quad (4.2)$$

has one and only one solution. Consider now a cost function  $j(\varepsilon) = J_\varepsilon(u_\varepsilon) \in \mathbb{R}$ ,  $\varepsilon \geq 0$ . Suppose that the following hypotheses hold.

**Hypothesis 8** *There exist a linear and continuous form on  $\mathcal{V}$ , denoted by  $L_\varepsilon$ , a function  $f(\varepsilon) > 0$  and two real numbers  $\delta J_1$  and  $\delta J_2$  such that*

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_0) + \Re L_\varepsilon(u_\varepsilon - u_0) + f(\varepsilon)\delta J_1 + o(f(\varepsilon)), \quad (4.3)$$

$$J_\varepsilon(u_0) = J_0(u_0) + f(\varepsilon)\delta J_2 + o(f(\varepsilon)), \quad (4.4)$$

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0. \quad (4.5)$$

**Hypothesis 9** *There exist two complex numbers  $\delta a$  and  $\delta \ell$  such that*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta a + o(f(\varepsilon)), \quad (4.6)$$

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = f(\varepsilon)\delta \ell + o(f(\varepsilon)), \quad (4.7)$$

where  $v_\varepsilon$  is the solution to

$$a_\varepsilon(w, v_\varepsilon) = -L_\varepsilon(w) \quad \forall w \in \mathcal{V}. \quad (4.8)$$

It is supposed that for all  $\varepsilon \geq 0$ , Problem (4.8) has one and only one solution. We call  $v_0$  the adjoint state. We have the following result.

**Theorem 7** *The variation of the cost function  $j$  with respect to  $\varepsilon$  is given by*

$$j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta j) + o(f(\varepsilon)),$$

where  $\delta j = \delta a - \delta \ell + \delta J$  and  $\delta J = \delta J_1 + \delta J_2$ .

*Proof.* We can write that

$$j(\varepsilon) - j(0) = [J_\varepsilon(u_\varepsilon) - J_0(u_0)] + \Re[a_\varepsilon(u_\varepsilon, v_\varepsilon) - a_0(u_0, v_\varepsilon)] - \Re[\ell_\varepsilon(v_\varepsilon) - \ell_0(v_\varepsilon)].$$

Using Hypothesis 9, we obtain that

$$\begin{aligned} j(\varepsilon) - j(0) &= [J_\varepsilon(u_\varepsilon) - J_0(u_0)] + \Re a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + f(\varepsilon)\Re(\delta a - \delta \ell) + o(f(\varepsilon)) \\ &= [J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0)] + [J_\varepsilon(u_0) - J_0(u_0)] + \Re a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) \\ &\quad + f(\varepsilon)\Re(\delta a - \delta \ell) + o(f(\varepsilon)). \end{aligned}$$

It follows from Hypothesis 8 that

$$j(\varepsilon) - j(0) = \Re[a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + L_\varepsilon(u_\varepsilon - u_0)] + f(\varepsilon)\Re(\delta J_1 + \delta J_2 + \delta a - \delta \ell) + o(f(\varepsilon)).$$

Using Equation (4.8), we obtain the desired result. □

## 4.3 Problem formulation

### 4.3.1 The Helmholtz problem

Let  $\Omega$  be a bounded, smooth subdomain of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ . For simplicity we take  $\partial\Omega$  to be  $C^\infty$ , but this condition could be considerably weakened. We suppose that  $\Omega$  contains a small inhomogeneity  $\omega_\varepsilon$  of the form  $\omega_\varepsilon = \varepsilon B$ , where  $B \subset \mathbb{R}^d$  is a bounded, smooth ( $C^\infty$ ) domain containing  $0$  (the origin) and  $\varepsilon$  is the order of magnitude of the diameter of the inhomogeneity. Let  $u_\varepsilon$  be the solution to the Helmholtz problem :

$$\begin{cases} \nabla \cdot (\alpha_\varepsilon \nabla u_\varepsilon) + \beta_\varepsilon u_\varepsilon = 0 \text{ in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial n} = \Lambda u_\varepsilon + \sigma \text{ on } \partial\Omega. \end{cases} \quad (4.9)$$

Here,  $\Lambda \in \mathcal{L}(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))$ ,  $\sigma \in H^{-\frac{1}{2}}(\partial\Omega)$ ,  $\mathbf{n}$  denotes the outward unit normal to  $\partial\Omega$  and  $\alpha_\varepsilon$  is a piecewise constant function defined by

$$\alpha_\varepsilon(x) = \begin{cases} \alpha_0 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \\ \alpha_1 & \text{if } x \in \omega_\varepsilon, \end{cases} \quad (4.10)$$

where  $\alpha_0$  and  $\alpha_1$  are positive constants. The piecewise constant function  $\beta_\varepsilon$  is defined analogously.

The variational formulation associated to (4.9) is :

$$\begin{cases} u_\varepsilon \in H^1(\Omega) \\ a_\varepsilon(u_\varepsilon, v) = \ell(v) \quad \forall v \in H^1(\Omega), \end{cases} \quad (4.11)$$

where for all  $u, v$  in  $H^1(\Omega)$ ,

$$a_\varepsilon(u, v) = \int_\Omega \alpha_\varepsilon \nabla u \cdot \overline{\nabla v} \, dx - \int_\Omega \beta_\varepsilon u \bar{v} \, dx - \alpha_0 \int_{\partial\Omega} (\Lambda u) \bar{v} \, ds(x), \quad (4.12)$$

$$\ell(v) = \alpha_0 \int_{\partial\Omega} \sigma \bar{v} \, ds(x). \quad (4.13)$$

We assume that the following hypotheses hold.

**Hypothesis 10** *The operator  $\Lambda$  is splitted in  $\Lambda_0 + \Lambda_1$  with*

–  $\Lambda_0 \in \mathcal{L}(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))$  and satisfies

$$\Re \int_{\partial\Omega} (\Lambda_0 \varphi) \overline{\varphi} \, ds(x) \leq 0 \quad \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega).$$

–  $\Lambda_1 \in \mathcal{L}(H^{\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega))$ .

**Hypothesis 11** For all  $\varepsilon \geq 0$ , we have

$$a_\varepsilon(u, v) = 0 \quad \forall v \in H^1(\Omega) \implies u = 0, \quad (4.14)$$

$$a_\varepsilon(u, v) = 0 \quad \forall u \in H^1(\Omega) \implies v = 0. \quad (4.15)$$

We have the following result that is proved in Section 4.6.

**Proposition 13** *If Hypotheses 10 and 11 are satisfied, then for all  $\varepsilon \geq 0$  Problem (4.11) has one and only one solution.*

### 4.3.2 The cost function and the adjoint problem

We consider a cost function  $j(\varepsilon) = J_\varepsilon(u_\varepsilon) \in \mathbb{R}$ ,  $\varepsilon \geq 0$ . We assume that  $J_\varepsilon$  satisfies Hypothesis 8 with  $f(\varepsilon) = \varepsilon^d$ ,  $\mathcal{V} = H^1(\Omega)$  and the linear form  $L_0$  satisfies the following hypothesis.

**Hypothesis 12** *The distribution  $L_0$  is of regularity  $H^2$  in a neighborhood of the origin.*

For all  $\varepsilon \geq 0$ , we define  $v_\varepsilon$ , solution to the following problem :

$$\begin{cases} v_\varepsilon \in H^1(\Omega) \\ a_\varepsilon(w, v_\varepsilon) = -L_\varepsilon(w) \quad \forall w \in H^1(\Omega). \end{cases} \quad (4.16)$$

We have the following result (see Section 4.6).

**Proposition 14** *If Hypotheses 10 and 11 are satisfied, Problem (4.16) has one and only one solution.*

## 4.4 The main result

### 4.4.1 Variation of the sesquilinear form

Here, our aim is to check that Hypothesis 9 is satisfied and to compute  $\delta a$ ,  $\delta \ell$  and  $f(\varepsilon)$ . The sesquilinear form  $\ell$  defined in (4.13) is independent of  $\varepsilon$ . Then, we have that

$$\delta \ell = 0. \quad (4.17)$$

Using (4.12) and (4.10), we obtain that

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \nabla u_0 \cdot \overline{\nabla v_\varepsilon} \, dx - (\beta_1 - \beta_0) \int_{\omega_\varepsilon} u_0 \cdot \overline{v_\varepsilon} \, dx. \quad (4.18)$$

Denoting  $w_\varepsilon = v_\varepsilon - v_0$  and using (4.18), we obtain

$$\begin{aligned} (a_\varepsilon - a_0)(u_0, v_\varepsilon) &= (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \nabla u_0 \cdot \overline{\nabla w_\varepsilon} \, dx + (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \nabla u_0 \cdot \overline{\nabla v_0} \, dx \\ &\quad - (\beta_1 - \beta_0) \int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx - (\beta_1 - \beta_0) \int_{\omega_\varepsilon} u_0 \cdot \overline{v_0} \, dx. \end{aligned}$$

We obtain by using the Green formula :

$$\begin{aligned} (a_\varepsilon - a_0)(u_0, v_\varepsilon) &= (\alpha_1 - \alpha_0) \left[ \int_{\partial\omega_\varepsilon} \frac{\partial u_0}{\partial n} \overline{w_\varepsilon} \, ds + \int_{\omega_\varepsilon} \nabla u_0 \cdot \overline{\nabla v_0} \, dx \right] \\ &\quad - (\beta_1 - \beta_0) \int_{\omega_\varepsilon} u_0 \cdot \overline{v_0} \, dx + \left( \frac{\alpha_1}{\alpha_0} \beta_0 - \beta_1 \right) \int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx. \end{aligned} \quad (4.19)$$

**Lemma 14** *We have the following estimate.*

$$\int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx = o(\varepsilon^d).$$

*Proof.* We have that

$$\left| \int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx \right| \leq \|u_0\|_{L^\infty(\omega_\varepsilon)} \int_{\omega_\varepsilon} |w_\varepsilon| \, dx.$$

From the Hölder inequality, we obtain that for all  $p, q \in [1, +\infty[$  satisfying  $1/p + 1/q = 1$ ,

$$\left| \int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx \right| \leq c\varepsilon^{d/p} \|w_\varepsilon\|_{L^q(\omega_\varepsilon)}.$$

We choose  $p = 3/2$  and  $q = 3$ . It follows from the Sobolev imbedding theorem that  $H^1(\Omega) \subset L^q(\Omega)$  with a continuous imbedding. Hence,

$$\left| \int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx \right| \leq c\varepsilon^{d/p} \|w_\varepsilon\|_{H^1(\Omega)}.$$

It is proved in [17] that  $\|w_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^{d/2})$  (the energy estimate). Using this result, we obtain that

$$\left| \int_{\omega_\varepsilon} u_0 \cdot \overline{w_\varepsilon} \, dx \right| \leq c\varepsilon^{7d/6}.$$

□

**Lemma 15** *We have the following estimate.*

$$\int_{\omega_\varepsilon} u_0 \cdot \overline{v_0} \, dx = \varepsilon^d |B| u_0(0) \cdot \overline{v_0(0)} + o(\varepsilon^d).$$

*Proof.* We can write

$$\int_{\omega_\varepsilon} u_0 \cdot \overline{v_0} \, dx = \varepsilon^d |B| u_0(0) \cdot \overline{v_0(0)} + \int_{\omega_\varepsilon} [u_0 \cdot \overline{v_0} - u_0(0) \cdot \overline{v_0(0)}] \, dx.$$

We obtain by a change of variable that

$$\int_{\omega_\varepsilon} [u_0 \cdot \overline{v_0} - u_0(0) \cdot \overline{v_0(0)}] \, dx = \varepsilon^d \int_B [u_0(\varepsilon y) \cdot \overline{v_0(\varepsilon y)} - u_0(0) \cdot \overline{v_0(0)}] \, dy.$$

By the Taylor expansion, we obtain

$$\int_B [u_0(\varepsilon y) \cdot \overline{v_0(\varepsilon y)} - u_0(0) \cdot \overline{v_0(0)}] \, dy = O(\varepsilon).$$

Hence,

$$\int_{\omega_\varepsilon} u_0 \cdot \overline{v_0} \, dx = \varepsilon^d |B| u_0(0) \cdot \overline{v_0(0)} + O(\varepsilon^{d+1}).$$

□

In a similar manner, we prove the following estimate.

**Lemma 16**

$$\int_{\omega_\varepsilon} \nabla u_0 \cdot \overline{\nabla v_0} \, dx = \varepsilon^d |B| \nabla u_0(0) \cdot \overline{\nabla v_0(0)} + o(\varepsilon^d).$$

Now we introduce the function  $\Phi$  solution to

$$\begin{cases} \Delta \Phi = 0 \text{ in } B \text{ and } \mathbb{R}^d \setminus \overline{B}, \\ \Phi \text{ is continuous across } \partial B, \\ \frac{\alpha_0}{\alpha_1} \left( \frac{\partial \Phi}{\partial n} \right)^+ - \left( \frac{\partial \Phi}{\partial n} \right)^- = -\mathbf{n}, \\ \lim_{|y| \rightarrow \infty} |\Phi(y)| = 0. \end{cases} \quad (4.20)$$

Here,  $\mathbf{n}$  denotes the outward unit normal to  $\partial B$ ; superscript  $+$  and  $-$  indicate the limiting values as we approach  $\partial B$  from outside  $B$ , and from inside  $B$ , respectively. We have the following result.

**Lemma 17** *We have the following estimate.*

$$\int_{\partial \omega_\varepsilon} \frac{\partial u_0}{\partial n} \overline{w_\varepsilon} \, ds = \varepsilon^d \left( \frac{\alpha_0}{\alpha_1} - 1 \right) \nabla u_0(0)^T \left[ \int_{\partial B} \mathbf{n} \otimes \Phi(y) \, ds(y) \right] \overline{\nabla v_0(0)} + o(\varepsilon^d),$$

where  $\otimes$  denotes the tensorial product between two vectors;  $U \otimes V = (U_i V_j)_{1 \leq i, j \leq d} \quad \forall U, V \in \mathbb{R}^d$ .

*Proof.* We can write

$$\int_{\partial\omega_\varepsilon} \frac{\partial u_0}{\partial n} \overline{w_\varepsilon} ds = \int_{\partial\omega_\varepsilon} \nabla u_0(0) \cdot \mathbf{n} \overline{w_\varepsilon} ds + \int_{\partial\omega_\varepsilon} [\nabla u_0(x) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon} ds. \quad (4.21)$$

By a change of variable we obtain

$$\int_{\partial\omega_\varepsilon} [\nabla u_0(x) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon} ds(x) = \varepsilon^{d-1} \int_{\partial B} [\nabla u_0(\varepsilon y) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon(\varepsilon y)} ds(y).$$

We have

$$\left| \int_{\partial B} [\nabla u_0(\varepsilon y) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon(\varepsilon y)} ds(y) \right| \leq c \|(\nabla u_0(\varepsilon y) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n})\|_{-\frac{1}{2}, \partial B} \|w_\varepsilon(\varepsilon y)\|_{\frac{1}{2}, \partial B}.$$

A Taylor expansion, the trace theorem and a change of variable yield

$$\left| \int_{\partial B} [\nabla u_0(\varepsilon y) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon(\varepsilon y)} ds(y) \right| \leq c\varepsilon \left( \frac{1}{\varepsilon^{d/2}} \|w_\varepsilon\|_{L^2(\omega_\varepsilon)} + \varepsilon^{1-d/2} |w_\varepsilon|_{1, \omega_\varepsilon} \right).$$

Yet, using the Hölder inequality, we obtain that for all  $p, q \in [1, +\infty[$  with  $1/p + 1/q = 1$ ,

$$\|w_\varepsilon\|_{L^2(\omega_\varepsilon)} \leq \varepsilon^{d/2q} \|w_\varepsilon\|_{L^{2p}(\omega_\varepsilon)}.$$

We choose  $p = 3$  and  $q = 3/2$ . Using that  $\|w_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^{d/2})$  and the Sobolev imbedding theorem, we obtain

$$\left| \int_{\partial B} [\nabla u_0(\varepsilon y) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon(\varepsilon y)} ds(y) \right| \leq c\varepsilon(\varepsilon^{d/3} + \varepsilon).$$

Then,

$$\int_{\partial\omega_\varepsilon} [\nabla u_0(x) \cdot \mathbf{n} - \nabla u_0(0) \cdot \mathbf{n}] \overline{w_\varepsilon} ds(x) = o(\varepsilon^d). \quad (4.22)$$

In [17], it is proved that for all  $y \in \partial B$ ,

$$\overline{w_\varepsilon(\varepsilon y)} = \varepsilon \left( \frac{\alpha_0}{\alpha_1} - 1 \right) \Phi(y) \cdot \overline{\nabla v_0(0)} + R_\varepsilon(y), \quad (4.23)$$

where

$$\|R_\varepsilon(y)\|_{C(\partial\omega)} = \begin{cases} O(\varepsilon^2 \ln \varepsilon) & \text{if } d = 2, \\ O(\varepsilon^{\frac{5}{2}}) & \text{if } d = 3. \end{cases} \quad (4.24)$$

Using a change of variable, (4.23) and (4.24), we obtain that

$$\int_{\partial\omega_\varepsilon} \nabla u_0(0) \cdot \mathbf{n} \overline{w_\varepsilon} ds = \varepsilon^d \left( \frac{\alpha_0}{\alpha_1} - 1 \right) \nabla u_0(0)^T \left[ \int_{\partial B} \mathbf{n} \otimes \Phi(y) ds(y) \right] \overline{\nabla v_0(0)} + o(\varepsilon^d). \quad (4.25)$$

From (4.21), (4.22) and (4.25) we obtain the desired result.  $\square$

It follows from (4.19), Lemmas 14, 15, 16 and 17 that the following result holds.

**Proposition 15** *We have*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = \varepsilon^d (\alpha_1 - \alpha_0) \nabla u_0(0)^T \left[ \left( \frac{\alpha_0}{\alpha_1} - 1 \right) \int_{\partial B} \mathbf{n} \otimes \Phi(y) ds(y) \right. \\ \left. + |B|I] \overline{\nabla v_0(0)} + \varepsilon^d (\beta_0 - \beta_1) |B| u_0(0) \overline{v_0(0)} + o(\varepsilon^d). \right.$$

Then, Hypothesis 9 is satisfied with  $f(\varepsilon) = \varepsilon^d$  and

$$\delta a = (\alpha_1 - \alpha_0) \nabla u_0(0)^T \left[ \left( \frac{\alpha_0}{\alpha_1} - 1 \right) \int_{\partial B} \mathbf{n} \otimes \Phi(y) ds(y) + |B|I \right] \overline{\nabla v_0(0)} \\ + \varepsilon^d (\beta_0 - \beta_1) |B| u_0(0) \overline{v_0(0)}.$$

#### 4.4.2 The topological asymptotic expansion

All the assumptions of the adjoint method are satisfied and we can apply Theorem 7 to compute the topological asymptotic expansion. We obtain the following result.

**Theorem 8** *The cost function  $j$  has the following asymptotic expansion :*

$$j(\varepsilon) - j(0) = \varepsilon^d \Re \left\{ (\alpha_1 - \alpha_0) \nabla u_0(0)^T \left[ \left( \frac{\alpha_0}{\alpha_1} - 1 \right) \int_{\partial B} \mathbf{n} \otimes \Phi(y) ds(y) + |B|I \right] \overline{\nabla v_0(0)} \right. \\ \left. - (\beta_1 - \beta_0) |B| u_0(0) \overline{v_0(0)} + \delta J \right\} + o(\varepsilon^d).$$

## 4.5 Particular cases

### 4.5.1 Preliminary lemmas

**Lemma 18** [7] *We denote  $\Omega_R = \Omega \setminus \overline{B(0, R)}$ ,  $R$  being a fixed radius such that  $B(0, R) \subset \Omega$ . We have that*

$$\|u_\varepsilon - u_0\|_{H^1(\Omega_R)} = O(\varepsilon^d).$$

**Lemma 19** *We have that*

$$\int_{\Omega} \alpha_\varepsilon |u_\varepsilon - u_0|^2 dx = o(\varepsilon^d).$$

*Proof.* To simplify the presentation, we make the proof only for  $d = 3$ . We have that

$$\int_{\Omega} \alpha_\varepsilon |u_\varepsilon - u_0|^2 dx = \alpha_0 \int_{\Omega \setminus \overline{\omega_\varepsilon}} |u_\varepsilon - u_0|^2 dx + \alpha_1 \int_{\omega_\varepsilon} |u_\varepsilon - u_0|^2 dx \\ = E_1(\varepsilon) + E_2(\varepsilon),$$



where  $E_1(\varepsilon) = \alpha_0 \int_{\Omega \setminus \overline{\omega_\varepsilon}} |u_\varepsilon - u_0|^2 dx$  and  $E_2(\varepsilon) = \alpha_1 \int_{\omega_\varepsilon} |u_\varepsilon - u_0|^2 dx$ .

We denote  $X_\varepsilon = u_\varepsilon - u_0$ . It is the solution to

$$\begin{cases} \Delta X_\varepsilon + \frac{\beta_0}{\alpha_0} X_\varepsilon = 0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon}, \\ X_\varepsilon = u_\varepsilon - u_0 & \text{on } \partial\omega_\varepsilon, \\ X_\varepsilon = u_\varepsilon - u_0 & \text{on } \partial\Omega. \end{cases}$$

In [14], it is proved that

$$\|X_\varepsilon\|_{L^2(\Omega \setminus \overline{\omega_\varepsilon})} \leq c \left( \varepsilon \|(u_\varepsilon - u_0)(\varepsilon y)\|_{1/2, \partial B} + \|u_\varepsilon - u_0\|_{1/2, \partial\Omega} \right).$$

Using Lemma 18 and the trace theorem, we obtain

$$\|X_\varepsilon\|_{L^2(\Omega \setminus \overline{\omega_\varepsilon})} \leq c\varepsilon \|(u_\varepsilon - u_0)(\varepsilon y)\|_{1/2, \partial B} + c\varepsilon^3.$$

Using (4.23), we obtain

$$\|X_\varepsilon\|_{L^2(\Omega \setminus \overline{\omega_\varepsilon})} \leq c\varepsilon(\varepsilon + \varepsilon^{5/2}) + c\varepsilon^3 \leq c\varepsilon^2.$$

Then,

$$E_1(\varepsilon) = O(\varepsilon^4). \tag{4.26}$$

For any positive numbers  $p$  and  $q$  such that  $1/p + 1/q = 1$ , the Hölder inequality brings

$$\|X_\varepsilon\|_{L^2(\omega_\varepsilon)} \leq c\varepsilon^{\frac{d}{2q}} \|X_\varepsilon\|_{L^{2p}(\omega_\varepsilon)}. \tag{4.27}$$

Next, the Sobolev imbedding theorem provides

$$\|X_\varepsilon\|_{L^{2p}(\Omega)} \leq c \|X_\varepsilon\|_{H^1(\Omega)}. \tag{4.28}$$

We choose  $p = 3$ . Then, using that  $\|X_\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^{d/2})$ , (4.27) and (4.28), we obtain

$$E_2(\varepsilon) = O(\varepsilon^5). \tag{4.29}$$

From (4.26) and (4.29) we obtain the desired result for  $d = 3$ . □

Using the Taylor expansion and a change of variable, we can prove the following result.

**Lemma 20** *We have*

$$\int_{\omega_\varepsilon} |u_0 - u_d|^2 dx = |B|\varepsilon^d |u_0(0) - u_d(0)|^2 + o(\varepsilon^d),$$

where  $u_d \in H^2(\Omega)$ .

**Lemma 21** *We have*

$$\begin{aligned} \int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_\varepsilon - u_0)} dx &= -\varepsilon^d \frac{(\alpha_1 - \alpha_0)^2}{\alpha_1} \nabla u_0(0)^T \left( \int_{\partial B} \mathbf{n} \otimes \Phi(y) ds(y) \right) \overline{\nabla u_0(0)} \\ &+ o(\varepsilon^d). \end{aligned}$$

*Proof.* We denote  $X_\varepsilon = u_\varepsilon - u_0$ . We have

$$\int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_\varepsilon - u_0)} dx = \int_{\Omega \setminus \overline{\omega_\varepsilon}} \alpha_0 \nabla X_\varepsilon \cdot \overline{\nabla X_\varepsilon} dx + \int_{\omega_\varepsilon} \alpha_1 \nabla X_\varepsilon \cdot \overline{\nabla X_\varepsilon} dx.$$

Using the Green formula we obtain

$$\begin{aligned} \int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_\varepsilon - u_0)} &= \alpha_0 \left[ - \int_{\Omega \setminus \overline{\omega_\varepsilon}} \Delta X_\varepsilon \cdot \overline{X_\varepsilon} - \int_{\partial \omega_\varepsilon} \left( \frac{\partial X_\varepsilon}{\partial n} \right)^+ \overline{X_\varepsilon} \right. \\ &\quad \left. + \int_{\partial \Omega} \frac{\partial X_\varepsilon}{\partial n} \overline{X_\varepsilon} \right] + \alpha_1 \left[ - \int_{\omega_\varepsilon} \Delta X_\varepsilon \cdot \overline{X_\varepsilon} + \int_{\partial \omega_\varepsilon} \left( \frac{\partial X_\varepsilon}{\partial n} \right)^- \overline{X_\varepsilon} \right]. \end{aligned}$$

Using Lemmas 18 and 19 we obtain

$$\int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_\varepsilon - u_0)} dx = \int_{\omega_\varepsilon} \left[ \alpha_\varepsilon \frac{\partial X_\varepsilon}{\partial n} \right] \overline{X_\varepsilon} ds + o(\varepsilon^d),$$

where

$$\left[ \alpha_\varepsilon \frac{\partial X_\varepsilon}{\partial n} \right] = \int_{\omega_\varepsilon} \left( \alpha_1 \left( \frac{\partial X_\varepsilon}{\partial n} \right)^- - \alpha_0 \left( \frac{\partial X_\varepsilon}{\partial n} \right)^+ \right) \overline{X_\varepsilon} ds.$$

Using that

$$\alpha_1 \left( \frac{\partial u_\varepsilon}{\partial n} \right)^- - \alpha_0 \left( \frac{\partial u_\varepsilon}{\partial n} \right)^+ = 0$$

We obtain

$$\int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_\varepsilon - u_0)} dx = (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} \frac{\partial u_0}{\partial n} \overline{X_\varepsilon} ds + o(\varepsilon^d).$$

From Lemma 17 we obtain the desired result.  $\square$

Using the Taylor expansion and a change of variable, we can prove the following result.

**Lemma 22** *We have*

$$\int_{\omega_\varepsilon} |\nabla(u_0 - u_d)|^2 dx = \varepsilon^d |B| |\nabla u_0(0) - \nabla u_d(0)|^2 + o(\varepsilon^d),$$

where  $u_d \in H^3(\Omega)$ .

## 4.5.2 Particular cost functions

### First example

We denote  $\Omega_R = \Omega \setminus \overline{B(0, R)}$ ,  $R$  being a fixed radius such that  $\overline{B(0, R)} \subset \Omega$ . We consider a cost function of the form :  $j(\varepsilon) = J_\varepsilon(u_\varepsilon) = J(u_\varepsilon|_{\Omega_R})$ . We

assume that there exists  $L$ , a linear and continuous form on  $H^1(\Omega_R)$  such that

$$J(u_{0|\Omega_R} + h) = J(u_{0|\Omega_R}) + \Re(L(h)) + o(\|h\|_{1,\Omega_R}) \quad \forall h \in H^1(\Omega_R).$$

Using Lemma 18, it is easy to prove that Hypotheses 9 and 12 are satisfied with

$$L_\varepsilon(u) = L(u|_{\Omega_R}) \quad \forall u \in H^1(\Omega), \forall \varepsilon \geq 0$$

and

$$\delta J_1 = \delta J_2 = 0.$$

### Second example

It consists in the cost function

$$j(\varepsilon) = J_\varepsilon(u_\varepsilon) = \int_{\Omega} \alpha_\varepsilon |u_\varepsilon - u_d|^2 dx,$$

where  $u_d \in H^2(\Omega)$ .

• *Expression of  $J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0)$  :* we have that

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = \int_{\Omega} \alpha_\varepsilon |u_\varepsilon - u_0|^2 dx + 2\Re \int_{\Omega} \alpha_\varepsilon (u_\varepsilon - u_0) \cdot \overline{(u_0 - u_d)} dx. \quad (4.30)$$

Using (4.30) and Lemma 19 we obtain that

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = \Re L_\varepsilon(u_\varepsilon - u_0) + \varepsilon^d \delta J_1 + o(\varepsilon^d),$$

where

$$L_\varepsilon(u) = 2 \int_{\Omega} \alpha_\varepsilon u \cdot \overline{(u_0 - u_d)} dx \quad \forall u \in H^1(\Omega),$$

and

$$\delta J_1 = 0.$$

• *Expression of  $J_\varepsilon(u_0) - J_0(u_0)$  :* we have that

$$\begin{aligned} J_\varepsilon(u_0) - J_0(u_0) &= \int_{\Omega} \alpha_\varepsilon |u_0 - u_d|^2 dx - \alpha_0 \int_{\Omega} |u_0 - u_d|^2 dx \\ &= (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} |u_0 - u_d|^2 dx. \end{aligned}$$

Using Lemma 20, we obtain that

$$J_\varepsilon(u_0) - J_0(u_0) = \varepsilon^d \delta J_2 + o(\varepsilon^d),$$

where

$$\delta J_2 = (\alpha_1 - \alpha_0) |B| |u_0(0) - u_d(0)|^2.$$

**Third example**

It consists in the cost function

$$j(\varepsilon) = J_\varepsilon(u_\varepsilon) = \frac{1}{2} \int_{\Omega} \alpha_\varepsilon |\nabla(u_\varepsilon - u_d)|^2 dx,$$

where  $u_d \in H^3(\Omega)$ .

• *Expression of  $J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0)$  :* we have that

$$\begin{aligned} J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) &= \frac{1}{2} \int_{\Omega} \alpha_\varepsilon |\nabla(u_\varepsilon - u_d)|^2 dx - \frac{1}{2} \int_{\Omega} \alpha_\varepsilon |\nabla(u_0 - u_d)|^2 dx \\ &= \frac{1}{2} \int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_\varepsilon - u_0)} dx + \Re \int_{\Omega} \alpha_\varepsilon \nabla(u_\varepsilon - u_0) \cdot \overline{\nabla(u_0 - u_d)} dx. \end{aligned}$$

Using Lemma 21 we obtain that

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = \Re L_\varepsilon(u_\varepsilon - u_0) + \varepsilon^d \delta J_1 + o(\varepsilon^d),$$

where

$$L_\varepsilon(u) = \int_{\Omega} \alpha_\varepsilon \nabla u \cdot \overline{\nabla(u_0 - u_d)} dx \quad \forall u \in H^1(\Omega)$$

and

$$\delta J_1 = -\frac{1}{2} \frac{(\alpha_1 - \alpha_0)^2}{\alpha_1} \nabla u_0(0)^T \left( \int_{\partial B} \mathbf{n} \otimes \Phi(y) ds(y) \right) \overline{\nabla u_0(0)}.$$

• *Expression of  $J_\varepsilon(u_0) - J_0(u_0)$  :* we have that

$$\begin{aligned} J_\varepsilon(u_0) - J_0(u_0) &= \frac{1}{2} \int_{\Omega} \alpha_\varepsilon |\nabla(u_0 - u_d)|^2 dx - \frac{1}{2} \int_{\Omega} \alpha_0 |\nabla(u_0 - u_d)|^2 dx \\ &= \frac{1}{2} (\alpha_1 - \alpha_0) \int_{\omega_\varepsilon} |\nabla(u_0 - u_d)|^2 dx. \end{aligned}$$

Using Lemma 22, we obtain that

$$J_\varepsilon(u_0) - J_0(u_0) = \varepsilon^d \delta J_2 + o(\varepsilon^d),$$

where

$$\delta J_2 = \frac{1}{2} (\alpha_1 - \alpha_0) |B| |\nabla u_0(0) - \nabla u_d(0)|^2.$$

**4.5.3 Particular shaped dielectric objects**

When  $B$  is the unit ball, we can explicitly determine  $\Phi$  the solution to Problem (4.20). In this special case we obtain

$$\Phi(y) = \frac{1}{(d-1) \frac{\alpha_0}{\alpha_1} + 1} \times \begin{cases} y & \text{in } B, \\ \frac{y}{|y|^d} & \text{in } \mathbb{R}^d \setminus \overline{B}. \end{cases} \quad (4.31)$$

Using Theorem 8 and (4.31) we obtain the following result.

**Corollary 2** (ball) *If  $B$  is the unit ball, the function  $j$  has the following asymptotic expansion :*

$$j(\varepsilon) - j(0) = \varepsilon^d \Re \left\{ \frac{d\alpha_0(\alpha_1 - \alpha_0)}{(d-1)\alpha_0 + \alpha_1} |B| \nabla u_0(0) \cdot \overline{\nabla v_0(0)} - (\beta_1 - \beta_0) |B| u_0(0) \overline{v_0(0)} + \delta J \right\} + o(\varepsilon^d).$$

Now we consider the case when  $B$  is an ellipse whose semi-major axis is of length  $a$ , and whose semi-minor axis is of length  $b$ . In this case we obtain the following result.

**Corollary 3** (ellipse) *If  $B$  is an ellipse whose semi-major axis is of length  $a$ , and whose semi-minor axis is of length  $b$  (2D problem), the function  $j$  has the following asymptotic expansion :*

$$j(\varepsilon) - j(0) = \varepsilon^2 \Re \left\{ \pi ab (\alpha_1 - \alpha_0) \nabla u_0(0)^T P \overline{\nabla v_0(0)} - (\beta_1 - \beta_0) \pi ab u_0(0) \overline{v_0(0)} + \delta J \right\} + o(\varepsilon^2),$$

where  $P$  is given by

$$\begin{pmatrix} \frac{\alpha_0(1+a) + \alpha_1(b-1)}{\alpha_0 a + \alpha_1 b} & 0 \\ 0 & \frac{\alpha_0(1+b) + \alpha_1(a-1)}{\alpha_0 b + \alpha_1 a} \end{pmatrix}.$$

For the expression of the function  $\Phi$  we refer the reader to [8].

#### 4.5.4 Metallic objects

Setting  $\alpha_1 \rightarrow 0$ ,  $\beta_1 \rightarrow 0$ ,  $\alpha_0 = 1$  and  $\beta_0 = k^2$ , we obtain from Corollary 2 and Corollary 3 the following results.

**Corollary 4** (ball) *If  $B$  is the unit ball,  $\omega_\varepsilon$  is a hole and  $\frac{\partial u_\varepsilon}{\partial n} = 0$  on  $\partial\omega_\varepsilon$ , we obtain*

$$j(\varepsilon) - j(0) = \varepsilon^d \Re \left\{ \frac{-d}{d-1} |B| \nabla u_0(0) \cdot \overline{\nabla v_0(0)} + |B| k^2 u_0(0) \overline{v_0(0)} + \delta J \right\} + o(\varepsilon^d).$$

**Corollary 5** (ellipse) *If  $B$  is an ellipse whose semi-major axis is of length  $a$ , and whose semi-minor axis is of length  $b$ ,  $\omega_\varepsilon$  is a hole and  $\frac{\partial u_\varepsilon}{\partial n} = 0$  on  $\partial\omega_\varepsilon$ , we obtain*

$$j(\varepsilon) - j(0) = \varepsilon^2 \Re \left\{ -\nabla u_0(0)^T \begin{pmatrix} \pi(a+1)b & 0 \\ 0 & \pi(b+1)a \end{pmatrix} \overline{\nabla v_0(0)} + \pi ab k^2 u_0(0) \overline{v_0(0)} + \delta J \right\} + o(\varepsilon^2).$$

Setting  $b \rightarrow 0$ , we obtain from Corollary 5 the following result.

**Corollary 6** (*straight crack*) *If  $B$  is the segment  $[-a, a] \times \{0\}$  and  $\frac{\partial u_\varepsilon}{\partial n} = 0$  on  $\partial\omega_\varepsilon$ , we obtain that*

$$j(\varepsilon) - j(0) = \varepsilon^2 \Re \left( -\pi a \frac{\partial u_0}{\partial n}(0) \overline{\frac{\partial v_0}{\partial n}(0)} + \delta J \right) + o(\varepsilon^2).$$

In Corollaries 4 and 6, in the case  $d = 2$ , we retrieve formulas previously proved in [7] for particular cost functions.

## 4.6 Well-posedness

In this section, our aim is to prove Proposition 13. For  $\varepsilon \geq 0$  we split  $a_\varepsilon$  in  $a_\varepsilon^0 + a_\varepsilon^1$  with

$$\begin{cases} a_\varepsilon^0(u, v) = \int_{\Omega} \alpha_\varepsilon (\nabla u \cdot \overline{\nabla v} + u \cdot \bar{v}) \, dx - \alpha_0 \int_{\partial\Omega} \Lambda_0 u \cdot \bar{v} \, ds(x), \\ a_\varepsilon^1(u, v) = - \int_{\Omega} (\beta_\varepsilon + \alpha_\varepsilon) u \cdot \bar{v} \, dx - \alpha_0 \int_{\partial\Omega} \Lambda_1 u \cdot \bar{v} \, ds(x). \end{cases}$$

We introduce the operator

$$\begin{aligned} \mathcal{A}_\varepsilon : H^1(\Omega) &\longrightarrow H^1(\Omega) \\ u &\longmapsto \mathcal{A}_\varepsilon u, \end{aligned}$$

such that

$$a_\varepsilon^0(\mathcal{A}_\varepsilon u, v) + a_\varepsilon^1(u, v) = 0 \quad \forall v \in H^1(\Omega). \quad (4.32)$$

It follows from Hypothesis 10 and the Lax-Milgram theorem that the operator  $\mathcal{A}_\varepsilon$  is well-defined. Using that the imbeddings  $H^1(\Omega) \rightarrow L^2(\Omega)$  and  $H^{\frac{1}{2}}(\partial\Omega) \rightarrow L^2(\partial\Omega)$  are compact, we obtain that the operator  $\mathcal{A}_\varepsilon$  is compact. From (4.32) we obtain

$$a_\varepsilon(u, v) = a_\varepsilon^0((I - \mathcal{A}_\varepsilon)u, v) \quad \forall u, v \in H^1(\Omega). \quad (4.33)$$

The Fredholm alternative and Hypothesis 11 imply that the operator  $I - \mathcal{A}_\varepsilon$  is invertible. Then, using the Lax-Milgram theorem, we obtain the desired result.

Proposition 14 can be proved by a similar manner.



# Bibliographie

- [1] C. Alves, H. Ammari. *Boundary integral formulas for the reconstruction of imperfections of small diameter in an elastic medium*, SIAM J. Appl. Math. 62 (2001), 94-106.
- [2] H. Ammari. *An inverse initial boundary value problem for the wave equation in the presence of imperfections of small volume*, SIAM J. Control Optim. 41 (2003), 1194-1211.
- [3] H. Ammari, D. Volkov. *Asymptotic formulas for perturbations in the eigenfrequencies of the full Maxwell equations due to the presence of imperfections of small diameter*, Asympt. Anal. 30 (2002), 331-350.
- [4] H. Ammari, S. Moskow. *Asymptotic expansions for eigenvalues in the presence of small inhomogeneities*, Math. Meth. Appl. Sci. 26 (2003), 67-75.
- [5] H. Ammari, H. Kang. *A new method for reconstructing electromagnetic inhomogeneities of small volume*, Inverse problems 19 (2003), 63-71.
- [6] H. Ammari, H. Kang. *Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities*, submitted to J. Math. Anal. Appl.
- [7] S. Amstutz. *The topological asymptotic for the Helmholtz equation : insertion of a hole, a crack and a dielectric object*, Rapport MIP no. 03-05.
- [8] M. Brühl, M. Hanke and M.S. Vogelius. *A direct impedance tomography algorithm for locating small inhomogeneities*, to appear in Numer. Math.
- [9] D.J. Cedio-Fengya, S. Moskow and M.S. Vogelius. *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction*, Inverse problems 14 (1998), 553-595.
- [10] S. Garreau, P. Guillaume and M. Masmoudi. *The topological sensitivity for linear isotropic elasticity*, European conference on computational Mechanics (ECCM99), 1999, rapport MIP no. 99.45.



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- [11] P. Guillaume, K. Sididris. *The topological asymptotic expansion for the Dirichlet problem*, accepted in SIAM journal on Control and Optimization.
  - [12] M. Masmoudi. *The Topological Asymptotic*, in Computational Methods for Control Applications, ed.H. Kawarada and J. Periaux, International Series GAKUTO, 2002.
  - [13] A.A. Novotny, R.A. Feijóo, E. Taroco and C. Padra. *Topological sensitivity analysis*, Comput. Methods Appl. Mech. Engrg. 192 (2003) 803-829.
  - [14] J. Pommier, B. Samet. *The topological asymptotic for the Helmholtz equation with Dirichlet condition on the boundary of an arbitrary shaped hole*, submitted to SIAM Journal on Control and Optimization.
  - [15] A. Schumacher. *Topologieoptimierung von Bauteilstrukturen unter Verwendung von Lophpositionierungskriterien*, Thesis, Universität-Gesamthochschule Siegen, 1995.
  - [16] J. Sokolowski, A. Żochowski. *On the topological derivative in shape optimization*, Technical report, INRIA, 1997.
  - [17] M.S. Vogelius, D. Volkov. *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter*, Math. Model. Numer. Anal. 34, pp. 723-748, 2000.

# Chapitre 5

## The topological asymptotic expansion for the Maxwell equations and some applications

### Sommaire

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Ce chapitre est constitué de l'article : *The topological asymptotic expansion for the Maxwell equations and some applications*, soumis dans ESAIM : Control, Optimization and Calculus of Variations. C'est un travail réalisé en collaboration avec Julien Pommier et Mohamed Masmoudi.

## 5.1 Introduction

Topological optimization methods become very attractive for industrial applications. It becomes possible to satisfy more challenging specifications of industrial products by allowing modifications of the topology of the initial design.

The most relevant topological optimization methods are based on the computation of a level set function :

- the material density in the case of the topological optimization via the homogenization theory. An optimal domain is defined by a threshold of the material density function [2, 3, 8],
- the built-in level set function in the level set method [24, 1],
- the topological gradient provided by the topological asymptotic expansion, which is the concern of this paper.

In the latter case, at convergence, the positivity of the topological gradient inside the final domain provides a necessary and sufficient optimality condition. We will present in this paper the topological asymptotic expansion for the Maxwell equations. Then we will show that our topological optimization method is very promising for solving shape inverse problems in electromagnetic applications.

In [2, 3, 8] the optimal shape is derived from the optimization of material properties. The range of application of this approach is quite restricted. The difficulties arise when we have to identify highly contrasted medias : electromagnetic identification of metallic objects in the free space or identification of an obstacle immersed in a fluid. In both cases the boundary condition that appears on the surface of the obstacle is of Dirichlet type. For these reasons global optimization methods are used to solve more general problems [16, 25]. The more recent level set method [24, 1] gives very promising results. Even if it belongs to the classical shape optimization methods, it allows the modification of the number of connected components of the domain. Unfortunately these methods are quite slow.

The topological asymptotic expansion seems to be general and efficient. To present the basic idea, we consider  $\Omega$  a domain of  $\mathbb{R}^n$  ( $n=2$  or  $3$ ) and  $j(\Omega) = J(u_\Omega)$  a cost function to be minimized, where  $u_\Omega$  is the solution to a given PDE problem defined in  $\Omega$ . For  $\varepsilon > 0$ , let  $\Omega_\varepsilon = \Omega \setminus \overline{x_0 + \varepsilon\omega}$  be the subset obtained by removing a small part  $\overline{x_0 + \varepsilon\omega}$  from  $\Omega$ , where  $x_0 \in \Omega$  and  $\omega \subset \mathbb{R}^n$  is a fixed domain containing the origin. We can generally prove that

the variation of the criterion is given by the asymptotic expansion :

$$j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon)g(x_0) + o(f(\varepsilon)), \quad (5.1)$$

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0, f(\varepsilon) > 0. \quad (5.2)$$

This expansion is called the topological asymptotic. To minimize the criterion  $j$  we just have to create infinitely small holes at some points  $\tilde{x}$  where the topological gradient (or sensitivity)  $g(\tilde{x})$  is negative.

The first definition of the topological gradient has been introduced by A. Schumacher *et al.* [26] under the name of *bubble method* in the context of compliance optimization for linear elasticity problems. In the same context, J. Sokolowski *et al.* [27] gave some mathematical justifications in the plane stress case and generalized it to various cost functions. In [18, 14], using an adaptation of the adjoint method [9] and a domain truncation technique, S. Garreau *et al.* presented a method to obtain the topological asymptotic expansion. For more details about this approach, we refer the reader to [28, 20, 21, 15, 23]. In all these works, only the insertion of a hole is considered to modify the domain.

In this paper, using an adjoint method, we derive the topological asymptotic expansion for the 3D Maxwell equations. Here, we consider two cases of domain perturbation : insertion of a dielectric object and the insertion of a metallic obstacle (a hole). The latter result is obtained by considering the limit of the topological asymptotic expansion when the permittivity goes to infinity and the permeability goes to zero. As a background to our work, we cite the contribution of H. Ammari *et al.* [6] for the study of solutions to the time-harmonic Maxwell equations in the presence of small inhomogeneities in the domain. Other contributions in this context can be found in [13, 5, 4, 10, 29, 30, 17]. In all these publications, only asymptotic formulas of solutions are given. Here, we derive asymptotic expansions not for solutions but for a given cost function.

In Section 5.2, we present the adjoint method. The main contribution of this paper is to make the topological gradient  $g$  easy to compute by using the adjoint state. The formulation of the Maxwell problem with a small inhomogeneity is presented in Section 5.3. In Section 5.4, we compute topological asymptotic expansions when we insert a dielectric object in the domain and when we create a spherical hole. Some numerical results are presented in Section 5.5, in the context of buried objects detection and shape inversion of 3D objects in free space from time-domain scattered field data.

## 5.2 The adjoint method

Let  $\mathcal{V}$  be a complex Hilbert space. For all  $\varepsilon \geq 0$ , let  $a_\varepsilon$  be a sesquilinear and continuous form on  $\mathcal{V}$  and  $\ell_\varepsilon$  be a semilinear and continuous form on  $\mathcal{V}$ .

We assume that for all  $\varepsilon \geq 0$ , The following problem :

$$\begin{cases} u_\varepsilon \in \mathcal{V}, \\ a_\varepsilon(u_\varepsilon, v) = \ell_\varepsilon(v) \quad \forall v \in \mathcal{V}, \end{cases} \quad (5.3)$$

has one and only one solution. Consider now a cost function  $j(\varepsilon) = J(u_\varepsilon) \in \mathbb{R}$ ,  $\varepsilon \geq 0$ . Suppose that the following hypotheses hold.

**Hypothesis 13** *There exist a linear and continuous form on  $\mathcal{V}$  denoted by  $L_0$  and a function  $f(\varepsilon) > 0$  such that*

$$J(u_\varepsilon) = J(u_0) + \Re L_0(u_\varepsilon - u_0) + o(f(\varepsilon)), \quad (5.4)$$

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0. \quad (5.5)$$

**Hypothesis 14** *There exist two complex numbers  $\delta a$  and  $\delta \ell$  such that*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta a + o(f(\varepsilon)), \quad (5.6)$$

$$(\ell_\varepsilon - \ell_0)(v_\varepsilon) = f(\varepsilon)\delta \ell + o(f(\varepsilon)), \quad (5.7)$$

where  $v_\varepsilon$  is the solution to

$$a_\varepsilon(w, v_\varepsilon) = -L_0(w) \quad \forall w \in \mathcal{V}. \quad (5.8)$$

It is supposed that for all  $\varepsilon \geq 0$ , Problem (5.8) has one and only one solution. We call  $v_0$  the adjoint state.

**Theorem 9** *The variation of the cost function  $j$  with respect to  $\varepsilon$  is given by*

$$j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta j) + o(f(\varepsilon)),$$

where  $\delta j = \delta a - \delta \ell$ .

*Proof.* We have that

$$\begin{aligned} j(\varepsilon) - j(0) &= (J(u_\varepsilon) - J(u_0)) + \Re(a_\varepsilon(u_\varepsilon, v_\varepsilon) - a_0(u_0, v_\varepsilon)) - \Re(\ell_\varepsilon(v_\varepsilon) - \ell_0(v_\varepsilon)) \\ &= (J(u_\varepsilon) - J(u_0)) + \Re(a_\varepsilon(u_0, v_\varepsilon) - a_0(u_0, v_\varepsilon)) + \Re a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) \\ &\quad - \Re(\ell_\varepsilon(v_\varepsilon) - \ell_0(v_\varepsilon)). \end{aligned}$$

Using Hypotheses 13 and 14, we obtain that

$$j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta a - \delta \ell) + \Re(a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + L_0(u_\varepsilon - u_0)) + o(f(\varepsilon)). \quad (5.9)$$

It follows from the definition of  $v_\varepsilon$  that

$$\Re(a_\varepsilon(u_\varepsilon - u_0, v_\varepsilon) + L_0(u_\varepsilon - u_0)) = 0 \quad \forall \varepsilon \geq 0. \quad (5.10)$$

Using (5.9) and (5.10), we obtain the desired result. □

### 5.3 The Maxwell problem

Let  $\Omega$  be a bounded open domain of  $\mathbb{R}^3$ , with a smooth boundary. For simplicity, we take  $\partial\Omega$  to be  $C^\infty$ , but this regularity condition could be considerably weakened. We suppose that  $\Omega$  contains a small inhomogeneity  $D_\varepsilon$  of the form  $D_\varepsilon = \varepsilon B$ , where  $B$  is a bounded, smooth ( $C^\infty$ ) domain containing 0 (the origin) and  $\varepsilon$  is the order of magnitude of the diameter of the inhomogeneity. The magnetic field in the presence of the inhomogeneity is denoted  $H_\varepsilon$ . It is the solution to :

$$\begin{cases} \nabla \times (\alpha_\varepsilon \nabla \times H_\varepsilon) + \beta_\varepsilon H_\varepsilon = 0 & \text{in } \Omega, \\ \alpha_\varepsilon (\nabla \times H_\varepsilon) \times \mathbf{n} = g & \text{on } \partial\Omega. \end{cases} \quad (5.11)$$

Here,  $\mathbf{n}$  denotes the outward unit normal to  $\partial\Omega$  and  $\alpha_\varepsilon$  is a piecewise constant function defined by

$$\alpha_\varepsilon(x) = \begin{cases} \alpha_0 & \text{if } x \in \Omega \setminus \overline{D_\varepsilon}, \\ \alpha_1 & \text{if } x \in D_\varepsilon, \end{cases} \quad (5.12)$$

where  $\alpha_0$  and  $\alpha_1$  are constants. If we allow the degenerate case  $\varepsilon = 0$ , then the function  $\alpha_0(x)$  equals the constant  $\alpha_0$ . The piecewise constant function  $\beta_\varepsilon$  is defined analogously. Problem (5.11) can be formulated as follows :

$$\begin{cases} \nabla \times (\alpha_0 \nabla \times H_\varepsilon) + \beta_0 H_\varepsilon = 0 & \text{in } \Omega \setminus \overline{D_\varepsilon}, \\ \nabla \times (\alpha_1 \nabla \times H_\varepsilon) + \beta_1 H_\varepsilon = 0 & \text{in } \varepsilon B, \\ H_\varepsilon \times \mathbf{n} \text{ is continuous across } \partial(\varepsilon B), \\ \alpha_0 (\nabla \times H_\varepsilon)^+ \times \mathbf{n} - \alpha_1 (\nabla \times H_\varepsilon)^- \times \mathbf{n} = 0 & \text{on } \partial(\varepsilon B), \\ \beta_0 H_\varepsilon^+ \cdot \mathbf{n} - \beta_1 H_\varepsilon^- \cdot \mathbf{n} = 0 & \text{on } \partial(\varepsilon B), \\ \alpha_0 (\nabla \times H_\varepsilon) \times \mathbf{n} = g & \text{on } \partial\Omega. \end{cases} \quad (5.13)$$

Here,  $\mathbf{n}$  denotes the outward unit normal to  $\partial(\varepsilon B)$  (and to  $\partial\Omega$ ); superscript + and - indicate the limiting values as we approach  $\partial(\varepsilon B)$  from outside  $\varepsilon B$ , and from inside  $\varepsilon B$ , respectively. The magnetic field,  $H_0$ , in the absence of any inhomogeneities, satisfies :

$$\begin{cases} \nabla \times (\alpha_0 \nabla \times H_0) + \beta_0 H_0 = 0 & \text{in } \Omega, \\ \alpha_0 (\nabla \times H_0) \times \mathbf{n} = g & \text{on } \partial\Omega. \end{cases} \quad (5.14)$$

In order to define the variational formulation of Problem (5.14), let  $a_0$  denotes the sesquilinear form

$$a_0(u, v) = \int_{\Omega} \alpha_0 \nabla \times u \cdot \overline{\nabla \times v} \, dx + \int_{\Omega} \beta_0 u \cdot \bar{v} \, dx, \quad (5.15)$$

defined on  $H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ , with  $H(\text{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \nabla \times u \in L^2(\Omega)^3\}$ . Let  $TH_{\text{div}}^{-\frac{1}{2}}(\partial\Omega)$  denotes the space of tangential vector fields on

$\partial\Omega$  that lie in  $H^{-\frac{1}{2}}(\partial\Omega)$  and whose surface divergences also lie in  $H^{-\frac{1}{2}}(\partial\Omega)$ . Suppose that :

$$g \in TH_{\text{div}}^{-\frac{1}{2}}(\partial\Omega).$$

Let  $\ell$  denotes the semilinear form

$$\ell(v) = \int_{\partial\Omega} g \cdot \bar{v} \, d\sigma_x, \quad (5.16)$$

defined on  $H(\text{curl}, \Omega)$ . It is well known that for such  $g$  the semilinear  $\ell$  is continuous on  $H(\text{curl}, \Omega)$  (the integral on  $\partial\Omega$  is to be interpreted as the duality pairing between the appropriate spaces of distributions and test functions). The variational formulation of Problem (5.14) is that  $H_0 \in H(\text{curl}, \Omega)$  and satisfies :

$$a_0(H_0, v) = \ell(v) \quad \forall v \in H(\text{curl}, \Omega). \quad (5.17)$$

We assume that the following hypothesis holds.

**Hypothesis 15** *The variational problem (5.17) has one and only one solution (for all  $g$ ).*

Similarly, Problem (5.11) has the weak formulation :

$$H_\varepsilon \in H(\text{curl}, \Omega) \text{ and } a_\varepsilon(H_\varepsilon, v) = \ell(v) \quad \forall v \in H(\text{curl}, \Omega), \quad (5.18)$$

where the sesquilinear form  $a_\varepsilon$  is given by

$$a_\varepsilon(u, v) = \int_{\Omega} \alpha_\varepsilon \nabla \times u \cdot \overline{\nabla \times v} \, dx + \int_{\Omega} \beta_\varepsilon u \cdot \bar{v} \, dx \quad (5.19)$$

for all  $u$  and  $v$  in  $H(\text{curl}, \Omega)$ . We have the following result [6].

**Proposition 16** *If Hypothesis 15 holds, then, there exists a constant  $\varepsilon_0 > 0$ , such that for all  $0 \leq \varepsilon < \varepsilon_0$ , Problem (5.18) has one and only one solution.*

We consider now a cost function  $j(\varepsilon) = J(H_\varepsilon) \in \mathbb{R}, \varepsilon \geq 0$ . We assume (for simplicity) that the following hypothesis holds.

**Hypothesis 16** *The cost function  $J$  is defined in a neighbor part of  $\partial\Omega$  and it satisfies Hypothesis 13 with  $f(\varepsilon) = \varepsilon^3$ .*

For all  $\varepsilon \geq 0$ , we define  $v_\varepsilon$  the solution to the following problem :

$$a_\varepsilon(w, v_\varepsilon) = -L_0(w) \quad \forall w \in H(\text{curl}, \Omega). \quad (5.20)$$

As in Proposition 16, if Hypothesis 15 holds and  $L_0$  is regular enough, then, for all  $\varepsilon \geq 0$  ( $\varepsilon$  is small enough), Problem (5.20) has one and only one solution [6].

## 5.4 Asymptotic expansions

### 5.4.1 Variation of the sesquilinear form : a preliminary version

Here, our aim is to check that Hypothesis 14 is satisfied and to calculate  $f(\varepsilon)$ ,  $\delta a$  and  $\delta \ell$ . The semilinear form  $\ell$  defined in (5.16) is independent of the parameter  $\varepsilon$ . Then, we have that

$$\delta \ell = 0. \quad (5.21)$$

Using (5.15) and (5.19), we obtain that

$$(a_\varepsilon - a_0)(H_0, v_\varepsilon) = \int_{\Omega} (\alpha_\varepsilon - \alpha_0) \nabla \times H_0 \cdot \overline{\nabla \times v_\varepsilon} dx + \int_{\Omega} (\beta_\varepsilon - \beta_0) H_0 \cdot \overline{v_\varepsilon} dx. \quad (5.22)$$

Using (5.12) and (5.22), we obtain that

$$(a_\varepsilon - a_0)(H_0, v_\varepsilon) = (\alpha_1 - \alpha_0) \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times v_\varepsilon} dx + (\beta_1 - \beta_0) \int_{D_\varepsilon} H_0 \cdot \overline{v_\varepsilon} dx.$$

Denoting  $w_\varepsilon = v_\varepsilon - v_0$ , we obtain that

$$\begin{aligned} (a_\varepsilon - a_0)(H_0, v_\varepsilon) &= (\alpha_1 - \alpha_0) \left( \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times w_\varepsilon} dx + \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times v_0} dx \right) \\ &\quad + (\beta_1 - \beta_0) \int_{D_\varepsilon} H_0 \cdot \overline{v_0} dx + (\beta_1 - \beta_0) \int_{D_\varepsilon} H_0 \cdot \overline{w_\varepsilon} dx. \end{aligned} \quad (5.23)$$

Now, we introduce the vector field  $h^*$  solution to the following problem :

$$\begin{cases} \Delta h^* = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B} \text{ and in } B, \\ \nabla \cdot h^* = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B} \text{ and in } B, \\ \alpha_0 (\nabla \times h^*)^+ \times \mathbf{n} - \alpha_1 (\nabla \times h^*)^- \times \mathbf{n} = (\alpha_1 - \alpha_0) (\nabla \times v_0)(0) \times \mathbf{n} \text{ on } \partial B, \\ \beta_0 (h^* \cdot \mathbf{n})^+ = \beta_1 (h^* \cdot \mathbf{n})^- \text{ on } \partial B \text{ and } h^* \times \mathbf{n} \text{ is continuous across } \partial B, \\ h^*(y) = O(|y|^{-1}) \text{ uniformly as } |y| \rightarrow \infty. \end{cases} \quad (5.24)$$

We have the following result.

#### Lemma 23

$$\int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times w_\varepsilon} dx = \varepsilon^3 \nabla \times H_0(0) \cdot \overline{\int_B \nabla \times h^*(y) dy} + O(\varepsilon^4).$$

*Proof.* We have that

$$\begin{aligned} \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times w_\varepsilon} dx &= \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\left( \nabla_x \times \left( w_\varepsilon - \varepsilon h^* \left( \frac{x}{\varepsilon} \right) \right) \right)} dx \\ &\quad + \varepsilon \int_{D_\varepsilon} (\nabla \times H_0(x) - \nabla \times H_0(0)) \cdot \overline{\nabla_x \times h^* \left( \frac{x}{\varepsilon} \right)} dx \\ &\quad + \varepsilon \int_{D_\varepsilon} \nabla \times H_0(0) \cdot \overline{\nabla_x \times h^* \left( \frac{x}{\varepsilon} \right)} dx. \end{aligned}$$



We obtain by a change of variable that

$$\begin{aligned} \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times w_\varepsilon} \, dx &= \varepsilon^3 \nabla \times H_0(0) \cdot \overline{\int_B \nabla \times h^*(y) \, dy} \\ &+ \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\left( \nabla_x \times \left( w_\varepsilon - \varepsilon h^* \left( \frac{x}{\varepsilon} \right) \right) \right)} \, dx \\ &+ \varepsilon \int_{D_\varepsilon} (\nabla \times H_0(x) - \nabla \times H_0(0)) \cdot \overline{\nabla_x \times h^* \left( \frac{x}{\varepsilon} \right)} \, dx. \end{aligned} \quad (5.25)$$

Using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned} \left| \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\left( \nabla_x \times \left( w_\varepsilon - \varepsilon h^* \left( \frac{x}{\varepsilon} \right) \right) \right)} \, dx \right| &\leq \varepsilon^{3/2} |B|^{1/2} \|\nabla \times H_0\|_{L^\infty(\Omega)} \\ &\quad \left\| \nabla_x \times \left( w_\varepsilon - \varepsilon h^* \left( \frac{x}{\varepsilon} \right) \right) \right\|_{L^2(\Omega)}. \end{aligned}$$

In [6], it is proved that

$$\left\| \nabla_x \times \left( w_\varepsilon - \varepsilon h^* \left( \frac{x}{\varepsilon} \right) \right) \right\|_{L^2(\Omega)} = O(\varepsilon^{5/2}).$$

Then, we obtain

$$\left| \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\left( \nabla_x \times \left( w_\varepsilon - \varepsilon h^* \left( \frac{x}{\varepsilon} \right) \right) \right)} \, dx \right| = O(\varepsilon^4). \quad (5.26)$$

By a change of variable, we obtain that

$$\int_{D_\varepsilon} (\nabla \times H_0(x) - \nabla \times H_0(0)) \cdot \overline{\nabla_x \times h^* \left( \frac{x}{\varepsilon} \right)} \, dx = \varepsilon^2 \int_B (\nabla \times H_0(\varepsilon y) - \nabla \times H_0(0)) \cdot \overline{\nabla \times h^*(y)} \, dy. \quad (5.27)$$

By the Taylor expansion, we obtain that

$$\left| \int_B (\nabla \times H_0(\varepsilon y) - \nabla \times H_0(0)) \cdot \overline{\nabla \times h^*(y)} \, dy \right| = O(\varepsilon). \quad (5.28)$$

Using (5.27) and (5.28), we obtain

$$\left| \int_{D_\varepsilon} (\nabla \times H_0(x) - \nabla \times H_0(0)) \cdot \overline{\nabla_x \times h^* \left( \frac{x}{\varepsilon} \right)} \, dx \right| = O(\varepsilon^3). \quad (5.29)$$

Finally, using (5.25), (5.26) and (5.29), we obtain the desired result.  $\square$

We have the following result.

**Lemma 24**

$$\int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times v_0} \, dx = \varepsilon^3 |B| \nabla \times H_0(0) \cdot \overline{\nabla \times v_0(0)} + O(\varepsilon^4).$$

*Proof.* By a change of variable, we obtain that

$$\begin{aligned} \int_{D_\varepsilon} \nabla \times H_0 \cdot \overline{\nabla \times v_0} \, dx &= \varepsilon^3 |B| \nabla \times H_0(0) \cdot \overline{\nabla \times v_0(0)} \\ &\quad + \varepsilon^3 \int_B ((\nabla \times H_0 \cdot \overline{\nabla \times v_0})(\varepsilon y) - (\nabla \times H_0 \cdot \overline{\nabla \times v_0})(0)) \, dy. \end{aligned}$$

By the Taylor expansion, we obtain the desired result.  $\square$

By a similar manner, we can prove the following result.

**Lemma 25**

$$\int_{D_\varepsilon} H_0 \cdot \overline{v_0} \, dx = \varepsilon^3 |B| H_0(0) \cdot \overline{v_0(0)} + O(\varepsilon^4).$$

Let  $q^*$  be the unique (scalar) solution to :

$$\begin{cases} \Delta q^* = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B} \text{ and in } B, \\ q^* \text{ is continuous across } \partial B, \\ \beta_0(\nabla q^* \cdot \mathbf{n})^+ - \beta_1(\nabla q^* \cdot \mathbf{n})^- = (\beta_1 - \beta_0)v_0(0) \cdot \mathbf{n}, \\ \lim_{|y| \rightarrow \infty} q^*(y) = 0. \end{cases} \quad (5.30)$$

We have the following asymptotic expansion.

**Lemma 26**

$$\int_{D_\varepsilon} H_0 \cdot \overline{w_\varepsilon} \, dx = \varepsilon^3 H_0(0) \cdot \overline{\int_B \nabla q^*(y) \, dy} + O(\varepsilon^4).$$

*Proof.* We have

$$\begin{aligned} \int_{D_\varepsilon} H_0 \cdot \overline{w_\varepsilon} \, dx &= \int_{D_\varepsilon} H_0 \cdot \overline{\left(w_\varepsilon - \nabla_y q^* \left(\frac{x}{\varepsilon}\right)\right)} \, dx + \int_{D_\varepsilon} H_0(0) \cdot \overline{\nabla_y q^* \left(\frac{x}{\varepsilon}\right)} \, dx \\ &\quad + \int_{D_\varepsilon} (H_0(x) - H_0(0)) \cdot \overline{\nabla_y q^* \left(\frac{x}{\varepsilon}\right)} \, dx. \end{aligned} \quad (5.31)$$

Using the Cauchy-Schwartz inequality, we obtain

$$\left| \int_{D_\varepsilon} H_0 \cdot \overline{\left(w_\varepsilon - \nabla_y q^* \left(\frac{x}{\varepsilon}\right)\right)} \, dx \right| \leq |B|^{1/2} \varepsilon^{3/2} \|H_0\|_{L^\infty(\Omega)} \left\| w_\varepsilon - \nabla_y q^* \left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)}.$$

In [6], it is proved that

$$\left\| w_\varepsilon - \nabla_y q^* \left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = O(\varepsilon^{5/2}).$$

Then, we obtain

$$\left| \int_{D_\varepsilon} H_0 \cdot \overline{\left(w_\varepsilon - \nabla_y q^* \left(\frac{x}{\varepsilon}\right)\right)} \, dx \right| = O(\varepsilon^4). \quad (5.32)$$

By a change of variable, we obtain

$$\int_{D_\varepsilon} H_0(0) \cdot \overline{\nabla_y q^* \left( \frac{x}{\varepsilon} \right)} dx = \varepsilon^3 H_0(0) \cdot \overline{\int_B \nabla q^*(y) dy}. \quad (5.33)$$

Using the Taylor expansion, it is easy to see that

$$\left| \int_{D_\varepsilon} (H_0(x) - H_0(0)) \cdot \overline{\nabla_y q^* \left( \frac{x}{\varepsilon} \right)} dx \right| = O(\varepsilon^4). \quad (5.34)$$

From (5.31), (5.32), (5.33) and (5.34) we obtain the desired result. □

Then, it follows from (5.23) and Lemmas 23, 24, 25 and 26 that the asymptotic expansion of  $(a_\varepsilon - a_0)(H_0, v_\varepsilon)$  is given by the following result.

**Proposition 17** *We have*

$$\begin{aligned} (a_\varepsilon - a_0)(H_0, v_\varepsilon) &= \varepsilon^3 (\alpha_1 - \alpha_0) \nabla \times H_0(0) \cdot \overline{\left( \int_B \nabla \times h^*(y) dy + |B| \nabla \times v_0(0) \right)} \\ &\quad + \varepsilon^3 (\beta_1 - \beta_0) H_0(0) \cdot \overline{\left( |B| v_0(0) + \int_B \nabla q^*(y) dy \right)} + O(\varepsilon^4). \end{aligned}$$

This is our preliminary version of the asymptotic expansion of  $(a_\varepsilon - a_0)(H_0, v_\varepsilon)$ . In the following subsection, we show that this formula can be rewritten by the use of two polarization tensors.

### 5.4.2 Rewriting the formula presented in Proposition 17

By  $\Phi_j$ ,  $1 \leq j \leq 3$ , we denote the solution to :

$$\begin{cases} \Delta \Phi_j = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B} \text{ and in } B, \\ \Phi_j \text{ is continuous across } \partial B, \\ c(\nabla \Phi_j \cdot \mathbf{n})^+ - (\nabla \Phi_j \cdot \mathbf{n})^- = 0 \text{ on } \partial B, \\ \Phi_j(y) - y_j \rightarrow 0 \text{ as } |y| \rightarrow \infty, \end{cases} \quad (5.35)$$

where  $c > 0$ . The existence and uniqueness of  $\Phi_j$  can be established (in the real, as well as in the complex case) using single layer potentials with suitably chosen densities, see [11, 12]. We now define the polarization tensor  $M(c)$  by :

$$M_{ij}(c) = c^{-1} \int_B \frac{\partial}{\partial y_i} \Phi_j dy, \quad 1 \leq i, j \leq 3. \quad (5.36)$$

In [6], it is proved that

$$|B| v_0(0) + \int_B \nabla_y q^*(y) dy = \frac{\beta_0}{\beta_1} M \left( \frac{\beta_0}{\beta_1} \right) v_0(0) \quad (5.37)$$

and

$$|B|\nabla \times v_0(0) + \int_B \nabla_y \times h^*(y) dy = M \left( \frac{\alpha_1}{\alpha_0} \right) \nabla \times v_0(0). \quad (5.38)$$

Insertion of (5.37) and (5.38) into the formula presented in Proposition (17) yields to the following result.

**Proposition 18** *We have*

$$(a_\varepsilon - a_0)(H_0, v_\varepsilon) = \varepsilon^3 \left\{ (\alpha_1 - \alpha_0) \nabla \times H_0(0) \cdot \overline{M \left( \frac{\alpha_1}{\alpha_0} \right) \nabla \times v_0(0)} + \beta_0 \left( 1 - \frac{\beta_0}{\beta_1} \right) H_0(0) \cdot \overline{M \left( \frac{\beta_0}{\beta_1} \right) v_0(0)} \right\} + O(\varepsilon^4).$$

Then, Hypothesis 14 is satisfied with  $f(\varepsilon) = \varepsilon^3$  and

$$\delta a = \left\{ (\alpha_1 - \alpha_0) \nabla \times H_0(0) \cdot \overline{M \left( \frac{\alpha_1}{\alpha_0} \right) \nabla \times v_0(0)} + \beta_0 \left( 1 - \frac{\beta_0}{\beta_1} \right) H_0(0) \cdot \overline{M \left( \frac{\beta_0}{\beta_1} \right) v_0(0)} \right\}.$$

### 5.4.3 The topological asymptotic

All the assumptions of Section 5.2 are satisfied. Then, we can apply Theorem 9 and we obtain the principal result of this paper.

**Theorem 10** *If Hypotheses 2.8 and 16 are satisfied, the topological asymptotic expansion with respect the insertion of a small inhomogeneity is given by*

$$j(\varepsilon) - j(0) = \varepsilon^3 \Re \left\{ (\alpha_1 - \alpha_0) \nabla \times H_0(0) \cdot \overline{M \left( \frac{\alpha_1}{\alpha_0} \right) \nabla \times v_0(0)} + \beta_0 \left( 1 - \frac{\beta_0}{\beta_1} \right) H_0(0) \cdot \overline{M \left( \frac{\beta_0}{\beta_1} \right) v_0(0)} \right\} + o(\varepsilon^3).$$

#### Case of a spherical inhomogeneity

When  $B$  is the unit ball  $B(0, 1)$ , the polarization tensor  $M$  is given by [10] :

$$M_{ij}(c) = \frac{3|B|}{1+2c} \delta_{ij} \quad \forall c > 0, 1 \leq i, j \leq 3. \quad (5.39)$$

Insertion of (5.39) with  $c = \frac{\alpha_1}{\alpha_0}$  (and  $c = \frac{\beta_0}{\beta_1}$ ) into the topological asymptotic expansion given by Theorem 10 yields to the following result.

**Corollary 7** *Under the assumptions of Theorem 10 and when  $B$  is the unit ball, the topological asymptotic expansion is given by*

$$j(\varepsilon) - j(0) = 4\pi\varepsilon^3 \Re \left\{ \frac{\alpha_0(\alpha_1 - \alpha_0)}{\alpha_0 + 2\alpha_1} \nabla \times H_0(0) \cdot \overline{\nabla \times v_0(0)} + \frac{\beta_0(\beta_1 - \beta_0)}{\beta_1 + 2\beta_0} H_0(0) \cdot \overline{v_0(0)} \right\} + o(\varepsilon^3).$$

**Case of a spherical metallic object**

Setting  $\alpha_1 = \beta_1 = 0$ ,  $\alpha_0 = 1$  and  $\beta_0 = -k^2$  ( $k$  is the wave number), we obtain from Corollary 7 the topological asymptotic expansion with respect to the insertion of a spherical metallic object in the domain  $\Omega : D_\varepsilon$  is a spherical hole with the boundary condition  $(\nabla \times H_\varepsilon) \times \mathbf{n} = 0$  on  $\partial(\varepsilon B)$ .

**Corollary 8** *The topological asymptotic expansion with respect to the insertion of a spherical metallic object in the domain  $\Omega$  is given by*

$$j(\varepsilon) - j(0) = 2\pi\varepsilon^3 \Re \left\{ -2\nabla \times H_0(0) \cdot \overline{\nabla \times v_0(0)} + k^2 H_0(0) \cdot \overline{v_0(0)} \right\} + o(\varepsilon^3).$$

## 5.5 Some applications

### 5.5.1 Topological asymptotic expansions for electromagnetic problems

In the same manner we derive asymptotic expansions in different electromagnetic cases, when a small metallic obstacle is inserted in the domain. The obtained results are presented in the following table. In Table 5.1 we

2D, TM polarization $(E_z, H_x, H_y)$	$-\frac{2\pi}{\log(\varepsilon)} \Re(E_0(x) \overline{v_0(x)})$
2D, TE polarization $(E_x, E_y, H_z)$	$\pi\varepsilon^2 \Re(k^2 H_0(x) \cdot \overline{v_0(x)} - 2\nabla H_0(0) \cdot \overline{\nabla v_0(0)})$
3D Maxwell equations	see Corollary 8

TAB. 5.1 – Expressions of the topological asymptotic with respect to the insertion of a metallic object (a small hole).

denote by  $E_x, E_y, E_z$ , the components of the electrical field and by  $H_x, H_y, H_z$  the components of the magnetic field. In the 2D/TM polarization case, the electrical field has only one non null component  $E_z$ . It satisfies the 2D Helmholtz equation with Dirichlet condition on the boundary of the hole. Notice that the frequency does not appear in the 2D/TM expression. The full justification of the topological asymptotic expansion in this case is given in [23]. For the 2D/TE polarization, we consider the magnetic field  $H_z$  that satisfies the 2D Helmholtz equation with a Neumann condition on the boundary of the hole. The mathematical analysis of this case can be found in [7, 22].

### 5.5.2 Shape inversion from frequency domain data

We will consider here a simple problem of detection of metallic objects buried in the soil. The problem here is based on a 2D/TM model, hence we are looking for infinite metallic wires placed in the  $z$  direction. The aim is to find

the number and the locations of metallic objects using scattered field measurements from a mono-static antenna horizontally translated above the soil. This is a rough model of the problems described in [19]. The 2D Helmholtz equation is solved by the Finite Differences Time-Domain (FDTD) method. The frequency-domain solution being obtained with a Fourier transform. The antenna is roughly approximated by a punctual source, which will be translated at various locations above the soil.

Let  $\mathcal{X} = \{x_i\}_{i=1..n_x}$  be the set of the successive locations of the source (and sensors, since the antenna is supposed to be mono-static) and  $\mathcal{F} = \{f_i\}_{i=1..n_f}$  the set of measurement frequencies. Let  $\varepsilon_s$  be the soil permittivity and  $\sigma_s$  its conductivity. The area occupied by the buried objects is denoted by  $\Omega$ . We associate to  $\Omega$  its response  $\mathcal{M}(\Omega)$  : for each couple  $(x_i, f_j) \in \mathcal{X} \times \mathcal{F}$ , we consider the field  $E_{x_i, f_j}^\Omega$  the solution to

$$\begin{cases} \Delta E_{x_i, f_j}^\Omega + k_j^2 E_{x_i, f_j}^\Omega = s_{x_i} & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ E_{x_i, f_j}^\Omega = 0 & \text{on } \partial\Omega, \\ \lim_{r \rightarrow \infty} \sqrt{r} (\partial_r E_{x_i, f_j}^\Omega - ik E_{x_i, f_j}^\Omega) = 0, \end{cases} \quad (5.40)$$

where  $s_{x_i}$  represents a source point located at  $x_i$  and where

$$\begin{aligned} k_j^2(x) &= \varepsilon(x)\mu\omega_j^2 + i\omega_j\mu\sigma(x), \\ w_j &= 2\pi f_j, \\ (\varepsilon(x), \sigma(x)) &= \begin{cases} (\varepsilon_0, 0) & \text{if } x \geq 0, \\ (\varepsilon_s, \sigma_s) & \text{if } x < 0. \end{cases} \end{aligned}$$

Then the measurement are  $\mathcal{M}(\Omega) = (m_{x_i, f_j}(\Omega) := \langle d_{x_i}, E_{x_i, f_j}^\Omega \rangle)$ , where  $d_{x_i}$  is the measurement function (in our numerical tests,  $m_{x_i, f_j}(\Omega)$  is the value of the scattered field at the source point  $x_i$ ).

We call *reference measurements*  $\tilde{\mathcal{M}} = (\tilde{m}_{x_i, f_j})$  the measured data. For the moment, we only consider synthetic data obtained via FDTD.

We have to minimize the cost function that evaluates de discrepancy between measured data and the response obtained for a given distribution of metallic objects :

$$j(\Omega) = \|\mathcal{M} - \tilde{\mathcal{M}}\|^2 = \sum_{i,j} |m_{x_i, f_j}(\Omega) - \tilde{m}_{x_i, f_j}|^2. \quad (5.41)$$

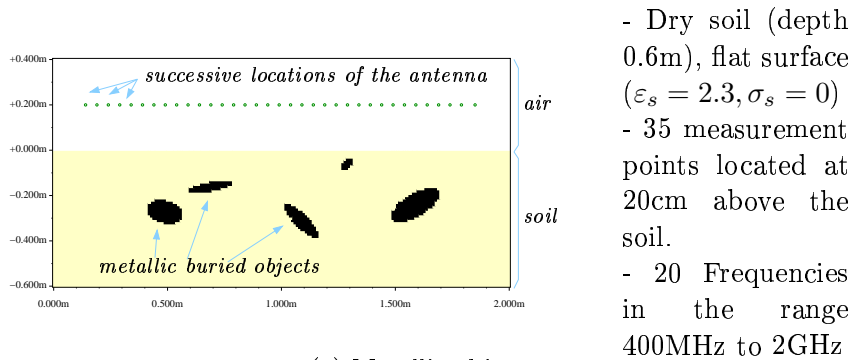
Since the cost function  $j$  involves many direct states  $E_{x_i, f_j}^\Omega$  (and therefore many adjoint states  $v_{x_i, f_j}^\Omega$ ), we shall adapt the result given in Table 5.1 to our context. Indeed, one can rewrite  $j$  as a sum of cost functions  $j_{x_i, f_j}$

$$\begin{aligned} j(\Omega) &= \sum_{i,j} j_{x_i, f_j}(\Omega), \\ j_{x_i, f_j}(\Omega) &= |m_{x_i, f_j}(\Omega) - \tilde{m}_{x_i, f_j}|^2. \end{aligned}$$

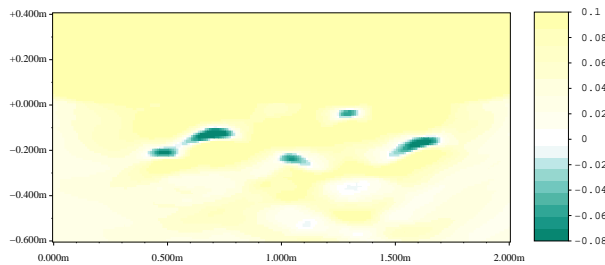
Then, applying the topological sensitivity for the 2D/TM Maxwell equations, one obtains

$$j(\Omega \setminus \overline{B(x, \varepsilon)}) - j(\Omega) = \sum_{i,j} -\frac{2\pi}{\log \varepsilon} \Re(E_{x_i, f_j}^\Omega(x) \cdot \overline{v_{x_i, f_j}^\Omega(x)}) + o\left(\frac{1}{\log \varepsilon}\right).$$

Since the function  $f(\varepsilon) = 1/\log \varepsilon$  is stiff, a “small” hole whose radius is 1/20th of the wavelength still causes an important variation of the cost function. Since only the linearized cost function is involved in the topological sensitivity computation (via the adjoint), the nonlinearity of the cost function would be an important source of error. Indeed, one can get a better estimation of the variation of the cost function. Instead of computing the sensitivities of the nonlinear cost functions  $j_{x_i, f_j}$ , we first compute the sensitivities of the linear functionals  $m_{x_i, f_j}$ . Then, we compute the effective variation of the nonlinear cost function for a given  $\varepsilon$ .



(a) Metallic objects



(b) Topological sensitivity

FIG. 5.1 – Repartition of metallic objects in the soil and the corresponding topological sensitivity computed on an empty flat soil.

The first example (Figure 5.1) shows the topological sensitivity computed in a simple case : the data is not noisy and the reference soil is a flat and homogeneous dry sand soil (low permittivity and null conductivity). One can see that the top of the five objects (Figure 5.1(a)) is clearly identified

by the negative region of the topological sensitivity (Figure 5.1(b)). This topological sensitivity can be obtained very quickly since it is evaluated on a flat soil without mines, which is invariant by translation : all direct states and adjoint states are just horizontal translations of a “canonical” solution. The computational cost is only 10 seconds on a 300MHz personal computer.

The second example (Figure 5.2) is a little bit more realistic : the data is noisy since the reference data  $\tilde{\mathcal{M}}$  was obtained on a non-flat inhomogeneous soil, while the topological sensitivity was still computed on a flat homogeneous soil. One can observe that, although the objects are still located correctly, the image (Figure 5.2(b)) is a little bit distorted.

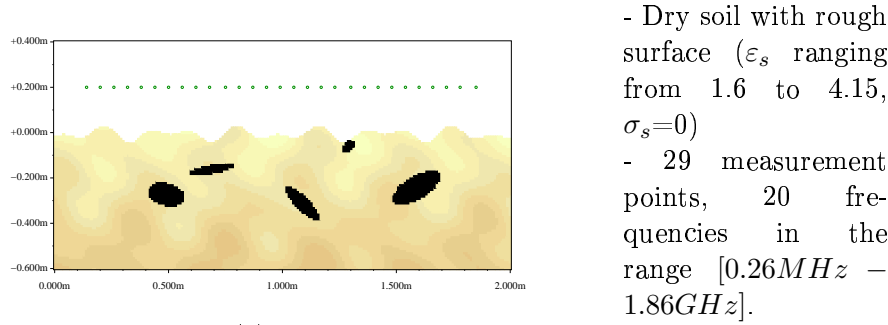
The third example (Figure 5.3) goes further in this direction : the soil is no more a dry sandy soil, but a wet ground (high permittivity and conductivity). Hence the waves quickly vanish in the soil and its surface is highly reflective : its roughness has a great influence on the measurements and computing a topological sensitivity without taking into account its roughness would give an irrelevant result. So we proceed in two steps : the first step recovers the shape of the ground surface, by computing a topological sensitivity in the free space (without soil and without mines) with respect to small balls of undergrass instead of small balls of metal (Figure 5.3(b)). One can see Figure 5.3(c) that the result does not seem to be affected by the influence of the two buried objects. But it is possible to recover these two buried objects if we take into account the detected soil profile in the computation (Figure 5.3(d)).

### 5.5.3 Shape inversion from time-domain data

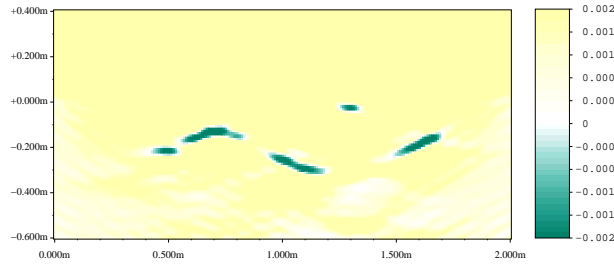
The next numerical tests follow a different approach. We now consider time-domain measurements in the context of 3D Maxwell equations. As it is indicated in Table 5.1, the topological sensitivities for 3D Maxwell equations and 2D transverse magnetic are very different. The main difference is the behavior of the function  $f(\varepsilon)$ , since  $\varepsilon^3$  goes much faster to zero with  $\varepsilon$  than  $-1/\log \varepsilon$ , we won't have to face the difficulties arising from a large variation of the solution when a relatively small (i.e.  $\simeq 1/20th$  of wavelength) metallic sphere is inserted. Using a time-domain solver, it is possible to keep the computational cost very low with respect to the size of the problems (the mesh size here is  $140 \times 140 \times 140$ ). The same FDTD code is used to compute the direct solution and the adjoint solution (which is computed backward, from the last time-step to the first time-step), and the topological sensitivity is just the integration with respect to time  $t$  of their product. Indeed, the expression for the topological sensitivity that was used in these experiments is simply

$$g(x) = \Re \int_0^T E(x, t) \cdot \overline{v(x, t)} dt,$$





(a) Reference distribution of objects



(b) Topological sensitivity on a flat empty homogeneous soil ( $\varepsilon_s = 2.3$ )

FIG. 5.2 – Topological sensitivity with noisy data.

where  $E(x, t)$  (resp.  $v(x, t)$ ) is the electric field solution of the direct (resp. adjoint) problem.

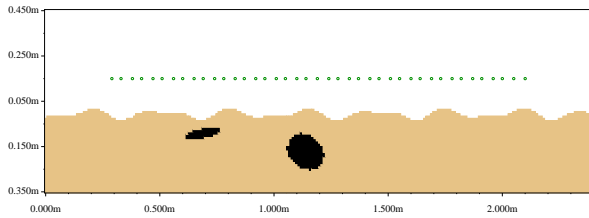
The first object, the metallic edges of a cube, is placed in free Space and is illuminated by 6 plane waves (whose time-domain signal is a gaussian distribution). The measurements are the values, for each time-step, of the tangential electric field along a “virtual surface” enclosing the cube. Figure 5.4 shows a surface of iso-values of the topological sensitivity, taking the free space as initial guess. Figure 5.5 shows the same experience with a more complex object.

Another major difference with the 2D/TM case is that the topological sensitivity expression depends on the shape of the holes. So one can try to see if the result is improved if the metallic obstacles are no more spherical but “oriented” : the expression for an infinitely small wire of direction  $d$  is

$$\Re \int_0^T (E(t) \cdot d) \overline{(v(t) \cdot d)} dt.$$

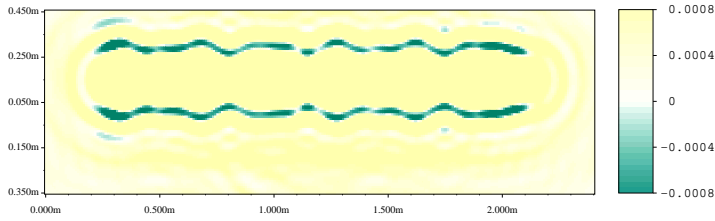
We can write that

$$\Re \int_0^T (E(t) \cdot d) \overline{(v(t) \cdot d)} dt = \left( \Re \int_0^T A(t) dt \right) d \cdot d,$$

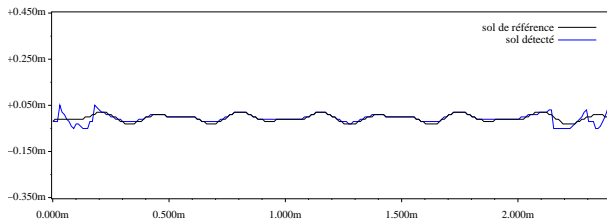


- 41 measurement points
- 30 frequencies in the range  $[0.33GHz - 1GHz]$
- wet soil :  $[\epsilon_s = 13, \sigma_s = 0.05]$ .

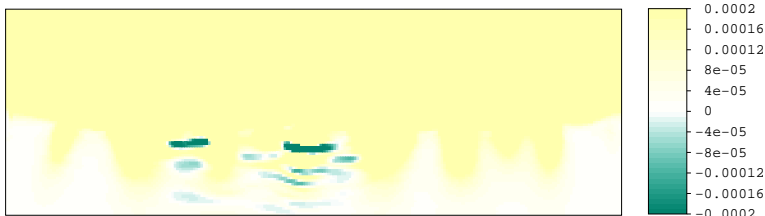
(a) Reference distribution



(b) Topological asymptotic computed on an empty initial guess (no soil at all), with respect to small balls of undergrass



(c) Detected soil surface



(d) Topological sensitivity taking in account the detected soil surface

FIG. 5.3 – Topological sensitivity in two steps.

where

$$A(t) = \frac{E(t) \otimes \overline{v(t)} + v(t) \otimes \overline{v(t)}}{2}.$$

Then at each point of the domain, the optimal orientation of the wire is given

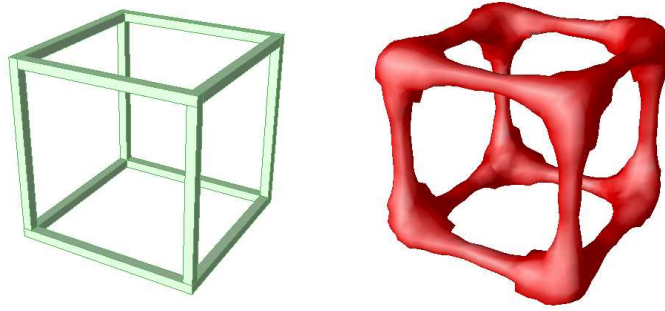


FIG. 5.4 – Metallic edges of a cube and an iso-values surface of the corresponding topological sensitivity (computed in the free space as initial guess).

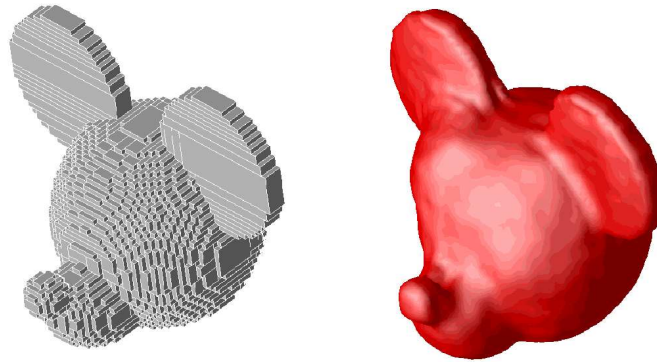


FIG. 5.5 – Topological sensitivity for a slightly more complex object.

by

$$\min_{d \in \mathbb{R}^3, \|d\|=1} \left( \Re \int_0^T A(t) dt \right) d.d.$$

Hence, we have two interesting informations :

- the variation of the cost function, for an optimally oriented wire, is given by the smallest eigenvalue of

$$\Re \int_0^T A(t) dt,$$

- the corresponding eigenvector gives the optimal orientation of the wire.

Applying this new topological sensitivity to the wired cube of the figure 5.4 we show a significant improvement of the result. Figure 5.6 shows an isosurface of the so obtained topological sensitivity.

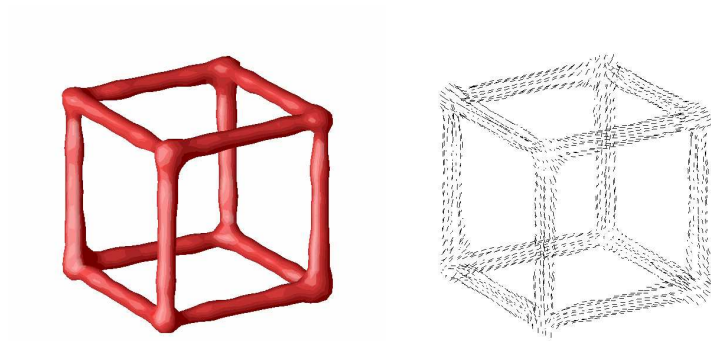


FIG. 5.6 – An isosurface of the topological sensitivity for optimally oriented infinitely small wires.



# Bibliographie

- [1] G. Allaire, F. Jouve, A.M. Toader, *A level-set method for shape optimization*, C. R. Acad. Sci, Série I, (334) : 1125-1130, 2002.
- [2] G. Allaire, R. Kohn, *Optimal design for minimum weight and compliance in plane stress using extremal microstructures*, European Journal of Mechanics, A/Solids, 12(6) :839-878, 1993.
- [3] G. Allaire, R. Kohn, *Optimal bounds on the effective behavior of a mixture of two well-order elastic materials*, Quartely of applied Mathematics, LI(4) : 643-674, 1993.
- [4] C. Alves, H. Ammari, *Boundary integral formulas for the reconstruction of imperfections of small diameter in an elastic medium*, SIAM J. Appl. Math. 62, 94-106, 2001.
- [5] H. Ammari, S. Moskow and M.S. Vogelius, *Boundary integral formulas for the reconstruction of electric and electromagnetic inhomogeneities of small volume*, ESAIM : COCV 9, 49-66, 2003.
- [6] H. Ammari, MS. Vogelius and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations*, J. Math. Pures Appl. 80, 8, 769-814, 2001
- [7] S. Amstutz, *The topological asymptotic for the Helmholtz equation : insertion of a hole, a crack and a dielectric object*, Rapport MIP no. 03-05, 2003.
- [8] M. Bendsoe, N. Kikuchi, *Generating Optimal Topologies in Structural Design Using a Homogenisation Method*, Comp. Meth. Appl. Mech. Eng. 71, 197-224, 1988.
- [9] J. Céa, *Conception optimale ou identification de forme, calcul rapide de la dérivée directionnelle de la fonction coût*, M.A.A.N., 20(3) :371-402, 1986.
- [10] D.J. Cedio-Fengya, S. Moskow and M. S. Vogelius, *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction*, Inverse problems 14, pp. 553-595, 1998.
- [11] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory*, Krieger Publ. CO., Malabar, Florida, 1992.

- [12] G.B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, 1976.
- [13] A. Friedman, M.S. Vogelius, *Identification of small inhomogeneities of extreme conductivity by boundary measurements : a theorem of continuous dependence*, Arch. Rational Mech. Anal. 105(4), pp. 299-326, 1989.
- [14] S. Garreau, P. Guillaume and M. Masmoudi, *The topological sensitivity for linear isotropic elasticity*, European conference on computational Mechanics (ECCM99), rapport MIP 99.45 1999.
- [15] P. Guillaume, K. Sididris, *The topological asymptotic expansion for the Dirichlet problem*, accepted in SIAM Journal on Control and Optimization.
- [16] C. Kane, M. Schoenauer, *Optimisation topologique de formes par algorithmes génétiques*, Revue Française de Mécanique, 4 :237-246, 1997.
- [17] A. Khelifi, *Electromagnetic scattering from small dielectric inhomogeneities*, PHD Thesis, Ecole Polytechnique, 2002.
- [18] M. Masmoudi, *The Topological Asymptotic*, in Computational Methods for Control Applications, ed.H. Kawarada and J. Periaux, International Series GAKUTO, 2002.
- [19] P. Millot, J. Bureau, P. Borderies, E. Bachelier, C. Pichot, E. LE Brusq, E. Guillanton and J. Dauvignac, *Experimental study of near surface radar imaging of buried objects with adaptive focussed synthetic aperture processing*, In Subsurface Sensing Technologies and Applications II, Proceedings of SPIE Vol. 4129, pages 515-523. Cam Nyugen, 2000.
- [20] S.A. Nazarov, J. Sokolowski, *Asymptotic analysis of shape functionals*, Rapport de recherche de l'INRIA, RR-4633, 2002.
- [21] A.A. Novotny, R.A. Fej3oo, C. Padra and E. Taroco, *Topological Sensitivity Analysis*, Comput. Methods Appl. Mech. Engrg. 192, 803-829, 2003.
- [22] B. Samet, S. Amstutz, *Sensitivity analysis with respect to the insertion of small inhomogeneities*, submitted to C.R. Acad. Sci. Paris, 2003.
- [23] B. Samet, S. Amstutz and M. Masmoudi, *The topological asymptotic for the Helmholtz equation*, accepted in SIAM Journal on Control and Optimization.
- [24] F. Santosa, *A level set approach for inverse problems involving obstacles*, ESAIM : Control, Optimization and Calculus of Variations, 1 :17-33, 1996.
- [25] M. Schoenauer, L. Kallel and F. Jouve, *Mechanics inclusions identification by evolutionary computation*, Revue européenne des éléments finis, 5(5-6), pp. 619-648, 1996.

- [26] A. Schumacher, V.V. Koboletv and H.A. Eschenauer, *Bubble method for topology and shape optimization of structures*, Journal of Structural Optimization, 8, 42-51, 1994.
- [27] J. Sokolowski, A. Żochowski, *On the topological derivative in shape optimization*, Technical report, INRIA, 1997.
- [28] J. Sokolowski, A. Żochowski, *A topological derivative for optimal control problems*, Control and Cybernetics **28**(3), 611-626, 1999.
- [29] M.S. Vogelius, D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter*, *M<sup>2</sup>AN* 34, 723-748, 2000.
- [30] D. Volkov, *An inverse problem for the time harmonic Maxwell equations*, PhD thesis, Rutgers University, New Brunswick, NJ, 2001.





## Chapitre 6

The topological asymptotic with  
respect to a singular boundary  
perturbation