Smallest poles of Igusa's and topological zeta functions and solutions of polynomial congruences

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Contents

Introduction

(0.1) Number theory and geometry are historically the first fields of mathematical interest and have always fascinated a lot of people. My research of the last few years led to new properties in these two domains.

(0.2) One of the basis concepts of this thesis is polynomial congruences. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial over the integers in n variables. Put $x = (x_1, \ldots, x_n)$. We want to study the number of solutions of $f(x) \equiv 0 \bmod m$ in $(\mathbb{Z}/m\mathbb{Z})^n$ for an arbitrary positive integer m. The Chinese remainder theorem reduces this problem to the case that m is a power of a prime. Let p be a fixed prime and let M_i, $i \in \mathbb{Z}_{\geq 0}$, be the number of solutions of the congruence $f(x) \equiv 0 \bmod p^i$ in $(\mathbb{Z}/p^i\mathbb{Z})^n$. If f has no singular point in $\mathbb{Z}_p^n$, then the behaviour of the M_i is well understood for large i because of Hensels lemma. Here $\mathbb{Z}_p$ is the ring of p-adic integers. The behaviour of the M_i is very complicated if f has a singular point in $\mathbb{Z}_p^n$. Igusa's p-adic zeta function allows us to study the singular case.

(0.3) For z in the field of p-adic numbers $\mathbb{Q}_p$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z, $|z| = p^{-\operatorname{ord} z}$ the absolute value of z and $\operatorname{ac} z = zp^{-\operatorname{ord} z}$ the angular component of z. Let χ be a character of $\mathbb{Z}_p^\times$, i.e., a group homomorphism $\chi : \mathbb{Z}_p^\times, . \to \mathbb{C}^\times, .$ with finite image. Let f be a polynomial over $\mathbb{Z}$ or more generally over $\mathbb{Z}_p$ or $\mathbb{Q}_p$. Let $X = \mathbb{Z}_p^n$. Igusa's p-adic zeta function $Z_{f,\chi}(s)$ of f and χ is defined by the p-adic integral

$$Z_{f,\chi}(s) := \int_X \chi(\operatorname{ac} f(x)) |f(x)|^s \, |dx|$$

for complex numbers s satisfying $\operatorname{Re}(s) > 0$, and by meromorphic continuation on the other part of $\mathbb{C}$.

Igusa's p-adic zeta function can be defined in a more general context. Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R, π a fixed uniformizing parameter for R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z, $|z| = q^{-\operatorname{ord} z}$ the absolute value of z and $\operatorname{ac} z = z\pi^{-\operatorname{ord} z}$ the angular component of z. Let χ be a character of $R^\times$. Let X be an open and compact subset of K^n and let f be a K-analytic function on X. Now Igusa's p-adic zeta function $Z_{f,\chi}(s)$ of f and χ is defined in the same way as before. Note that we have in our special case above that $K = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, $P = p\mathbb{Z}_p$, $\pi = p$, $q = p$, $X = \mathbb{Z}_p^n$ and $f \in \mathbb{Q}_p[x_1, \ldots, x_n]$.

Let f be a polynomial over R or, more generally, an arbitrary rigid K-analytic function on $X = R^n$ defined over R, i.e., a K-analytic function on R^n which is given by a power series over R which converges on R^n. Igusa's p-adic zeta function of such an f has an important connection with congruences. For $i \in \mathbb{Z}_{\geq 0}$ and $u \in R/P^i$, let $M_i(u)$ be the number of solutions of $f(x) \equiv u \bmod P^i$ in $(R/P^i)^n$. Put $M_i := M_i(0)$. Let e be the conductor of χ, i.e., the smallest $a \in \mathbb{Z}_{>0}$ such that χ is trivial on $1 + P^a$. Then the $M_{i+e}(\pi^i u)$, $u \in (R/P^e)^\times$, describe $Z_{f,\chi}(s)$ through the relation

$$Z_{f,\chi}(s) = \sum_{i=0}^{\infty} \sum_{u \in (R/P^e)^\times} \chi(u) M_{i+e}(\pi^i u) q^{-n(i+e)} q^{-is}.$$

If χ is the trivial character, all the M_i's describe and are described by $Z_{f,\chi}(s)$ through the relation

$$Z_{f,\chi}(s) = P(q^{-s}) - \frac{P(q^{-s}) - 1}{q^{-s}},$$

where the Poincaré series $P(t)$ of f is defined by

$$P(t) = \sum_{i=0}^{\infty} M_i(q^{-n}t)^i.$$

Using this connection, we can obtain properties of Igusa's p-adic zeta function from results on the number of solutions of congruences and vice versa.

The second method in our study of $Z_{f,\chi}(s)$ uses the existence of an embedded resolution of singularities of f, which is a composition of blowing-ups. We will calculate the defining integral of Igusa's p-adic zeta function on such an embedded resolution.

(0.4) The study of Igusa's p-adic zeta function by using an embedded resolution of singularities led to the introduction of a new geometric invariant.

Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^n$ which satisfies $f(0) = 0$ and which is not identically zero. Let $g : V \to U \subset \mathbb{C}^n$ be an embedded resolution of singularities of a representative of $f^{-1}\{0\}$. We denote by E_i, $i \in T$, the irreducible components of $g^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of $f \circ g$ and $g^*(dx_1 \wedge \cdots \wedge dx_n)$ along E_i. The (N_i, ν_i), $i \in T$, are called the numerical data of the embedded resolution. For $I \subset T$ denote also $E_I := \cap_{i \in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$. Note that V is the disjoint union of the $\overset{\circ}{E}_I$, $I \subset T$. Let $d \in \mathbb{Z}_{>0}$.

The local topological zeta function of f and d is defined as

$$Z_f^{(d)}(s) := \sum_{\substack{I \subset T \\ \forall i \in I \,:\, d | N_i}} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

Here s is a complex variable and $\chi(\cdot)$ denotes the topological Euler-Poincaré characteristic. The candidate poles of this rational function are the $-\nu_i/N_i$, with $i \in T$ satisfying $d \mid N_i$. It is striking that most candidate poles are actually not poles.

In our research, we will always take an embedded resolution g which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups g_i with an appropriate centre. The exceptional variety of g_i and the strict transforms of this variety are denoted by E_i. We will use the same notation if we have an embedded resolution in the context of Igusa's p-adic zeta function.

(0.5) The first chapter is an introduction to these zeta functions. It is written for people who are not familiar with this subject, but who have a mathematical background. The other chapters contain my research. They are all self contained, except Chapter 4. Chapters 2, 3 and 5 correspond to respectively [SV], [Se1] and [Se2].

(0.6) Chapter 2 contains our study of the local topological zeta function. We will see that it is easy to prove that for every $n \in \mathbb{Z}_{\geq 2}$, all the poles of a local topological zeta function are in the interval $[-n+1, 0[$.

If $n = 2$ and d is fixed, we determine all values less then $-1/2$ which occur as a pole of a local topological zeta function. They are all of the

form $-1/2-1/i$ with $i \in \mathbb{Z}_{>1}$, and all these values are the pole of a local topological zeta function with $d=1$. Moreover, if the multiplicity of f at the origin $0 \in \mathbb{C}^2$ is at least four, we will see that -1 is the only possible pole less than $-1/2$. Our result implies the well known fact that the log canonical threshold of f is never in $]5/6, 1[$.

If $n=3$ and $d=1$, we determine all values less then -1 which occur as a pole of a local topological zeta function. They are $-1-1/i$ with $i \in \mathbb{Z}_{>1}$. In particular, there are no poles less than $-3/2$. Moreover, if the multiplicity of f is at least three, we will see that there are no poles less than -1. The last part will be the most difficult because we will have to prove that some candidate poles are not poles.

(0.7) In Chapter 3 we study Igusa's p-adic zeta function by using the method of embedded resolution of singularities. We obtain results which are analogous to the ones of Chapter 2, but now for the real parts of the poles of Igusa's p-adic zeta function.

We will have to deal with an extra difficulty in the proof of the fact that some candidate poles of expected order one are not poles.
For the local topological zeta function, we could use the relations between the numerical data of Veys. If s_0 is a candidate pole of $Z_f^{(d)}(s)$ of expected order one and if E_i is an exceptional variety with candidate pole s_0, one can write down easily the contribution of E_i to the residue of $Z_f^{(d)}(s)$ at the candidate pole s_0. The relations of Veys allow you to prove that this contribution is zero for some intersection configurations on E_i at the stage when it is created.
For Igusa's p-adic zeta function, we also want to be able to calculate the contribution of an exceptional variety E_i with a candidate pole s_0 of expected order one to the residue of $Z_{f,\chi}(s)$ at s_0 by using only our information when E_i is created. Let s_0 be a candidate pole of $Z_{f,\chi}(s)$ of expected order one and let E_i be an exceptional variety with candidate pole s_0. Then it is generally known that the contribution of E_i to the residue of $Z_{f,\chi}(s)$ at s_0 is equal to the principal value integral of the Poincaré residue of $\chi(\mathrm{ac}\, f \circ g)|f \circ g|^{s_0} g^* dx$ on E_i. We deduce this formula again without using the terminology of principal value integrals and Poincaré residues. Our deduction is elementary. We will push forward this formula to the stage when E_i is created. In the specialized terminology, our result is that the contribution of E_i to the residue of $Z_{f,\chi}(s)$ at s_0 is equal to the principal value integral of the Poincaré residue of

$\chi(\mathrm{ac}\, f \circ g_1 \circ \cdots \circ g_i)|f \circ g_1 \circ \cdots \circ g_i|^{s_0} (g_1 \circ \cdots \circ g_i)^* dx$ on E_i.

Chapter 4 contains the calculations which are needed to obtain our result for Igusa's p-adic zeta functions with (non-trivial) character and $n = 3$.

(0.8) Chapter 5 contains results in general dimension n. Let f be a polynomial over R or a rigid K-analytic function on R^n defined over R. As in (0.3), let $M_i(u)$ be the number of solutions of $f(x) \equiv u \bmod P^i$ in $(R/P^i)^n$ and put $M_i := M_i(0)$. Let l be the smallest real part of a pole of Igusa's p-adic zeta function of f with trivial character. We prove that there exists an integer a which is independent of i such that M_i is an integer multiple of $q^{\lceil (n+l)i-a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$. We explained this result also in Chapter 3.

Together with our results of Chapter 3 in dimension two and three, we obtain some nice divisibility properties. The statements of these properties are so easy that we tried to find an elementary proof, and with success. It generalized easily to arbitrary dimension and to the more general class of numbers $M_i(u)$. More precisely, we will deduce that $M_i(u)$ is divisible by $q^{\lceil (n/2)(i-1) \rceil}$ for all $i \in \mathbb{Z}_{>0}$ and all $u \in R/P^i$.

We will also prove the converse of the result two paragraphs ago: if there exists an integer a such that M_i is an integer multiple of $q^{\lceil (n+l')i-a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$, then $l' \leq l$. This statement has an analogue if we are dealing with a character.

Let χ be an arbitrary character of $R^\times$. The last two paragraphs imply that $Z_{f,\chi}(s)$ has no poles with real part less than $-n/2$. This is the main result of the last chapter. We will see that it holds for an arbitrary K-analytic function on an open and compact subset of K^n, and not only for rigid K-analytic functions on R^n which are defined over R. Moreover, this result implies that a local topological zeta function has no poles less than $-n/2$. We also give an example which shows that this bound is optimal.

Chapter 1

Preliminaries on zeta functions

Abstract

We give an introduction to three zeta functions: Igusa's p-adic zeta function, the topological zeta function and the motivic zeta function. The first one belongs to the domain of number theory, the others are geometric invariants. We also try to explain the connection between these zeta functions.

1.1 Igusa's p-adic zeta function

(1.1.1) Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R, π a fixed uniformizing parameter for R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z and $|z| = q^{-\operatorname{ord} z}$ the absolute value of z. Let $f \in R[x_1, \ldots, x_n]$ and put $x = (x_1, \ldots, x_n)$.

(1.1.2) Let M_i, $i \in \mathbb{Z}_{\geq 0}$, be the number of solutions of $f(x) \equiv 0 \bmod P^i$ in $(R/P^i)^n$. Note that $M_0 = 1$. If $K = \mathbb{Q}_p$ and $f \in \mathbb{Z}[x_1, \ldots, x_n]$ or, more generally, $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$, it is just the number of solutions of $f(x) \equiv 0 \bmod p^i$ in $\{0, \ldots, p^i - 1\}^n$.

The *Poincaré series* $P(t)$ of f is the formal power series

$$P(t) = \sum_{i=0}^{\infty} M_i (q^{-n}t)^i.$$

If we consider t as a complex variable, it is absolutely convergent on the open unit disc $\{t \mid |t| < 1\} \subset \mathbb{C}$ because $M_i \leq q^{ni}$.

(1.1.3) A subset X of K^n is called open if for every $x \in X$, there exists an $i \in \mathbb{Z}_{\geq 0}$ such that $x + (P^i)^n \subset X$. In this way, K^n becomes a topological space. For $n = 1$, it is the topology on K induced by the absolute value, and for $n > 1$, it is the product topology on $K^n = K \times \cdots \times K$.

With this topology $K^n, +$ is a locally compact group. Therefore, there exists a unique translation invariant measure $\mu = |dx|$ on the Borel-σ-algebra of K^n for which $\mu(R^n) = 1$. This measure is called the Haar measure on K^n. Integration of a complex valued function on K^n with respect to the Haar measure is called p-adic integration.

Let $i \in \mathbb{Z}_{>0}$. Then R^n is the disjoint union of q^{ni} cosets of $(P^i)^n$. Because the Haar measure is translation invariant, the measure of each of these cosets is the same. Hence, we obtain $\mu((P^i)^n) = q^{-ni}$ because $\mu(R^n) = 1$ and because μ is additive.

(1.1.4) *Igusa's p-adic zeta function* $Z_f(s)$ of f is defined by a p-adic integral for complex numbers s satisfying $\mathrm{Re}(s) > 0$:

$$Z_f(s) = \int_{R^n} |f(x)|^s \, |dx|.$$

The integrand is the sth power of the absolute value of $f(x)$. We will see that $Z_f(s)$ is analytic and that it extends to a meromorphic function on $\mathbb{C}$, but until there $Z_f(s)$ is just a complex function on the right hand side of the complex plane.

(1.1.5) Because

$$\begin{aligned}
&\{x \in R^n \mid \operatorname{ord} f(x) = i\} \\
&\quad = \{x \in R^n \mid f(x) \equiv 0 \bmod P^i\} \setminus \{x \in R^n \mid f(x) \equiv 0 \bmod P^{i+1}\} \\
&\quad = \left(\bigcup_{\substack{a + (P^i)^n \in (R/P^i)^n \\ f(a) \equiv 0 \bmod P^i}} a + (P^i)^n \right) \setminus \left(\bigcup_{\substack{a + (P^{i+1})^n \in (R/P^{i+1})^n \\ f(a) \equiv 0 \bmod P^{i+1}}} a + (P^{i+1})^n \right),
\end{aligned}$$

where every union is a union of disjoint sets, we obtain that

$$\mu(\{x \in R^n \mid \operatorname{ord} f(x) = i\}) = M_i q^{-ni} - M_{i+1} q^{-n(i+1)}.$$

Moreover, it follows from this calculation that $|f(x)|$ is a locally constant function outside the zero locus of f. Consequently, $|f(x)|^s$ is a locally constant function, and thus continuous, outside the zero locus of f for every complex number s. One checks easily that $|f(x)|^s$ is continuous everywhere if $\mathrm{Re}(s) > 0$.

Since the defining integral of Igusa's p-adic zeta function is an integral of a continuous function on a compact set, it is meaningful. Actually, this is also clear by looking at the following explicit calculation of $Z_f(s)$:

$$\begin{aligned} Z_f(s) &= \int_{R^n} |f(x)|^s \, |dx| \\ &= \sum_{i=0}^{\infty} \mu(\{x \in R^n \mid \mathrm{ord}\, f(x) = i\}) q^{-is} \\ &= \sum_{i=0}^{\infty} (M_i q^{-ni} - M_{i+1} q^{-n(i+1)}) q^{-is} \\ &= \sum_{i=0}^{\infty} M_i q^{-ni} q^{-is} - \frac{\sum_{i=0}^{\infty} M_{i+1} q^{-n(i+1)} q^{-(i+1)s}}{q^{-s}} \\ &= P(q^{-s}) - \frac{P(q^{-s}) - 1}{q^{-s}}. \end{aligned} \tag{1.1}$$

Note that q^{-s} is an analytic function of s which maps a point of the right hand side of the complex plane to a point of the open unit disc. Moreover, $P(t)$ is an analytic function on the open unit disc. Therefore, we obtain that $Z_f(s)$ is an analytic function on the right hand side of the complex plane.

It is clear from equation (1.1) that $Z_f(s)$ depends only on q^{-s}, i.e., $Z_f(s_1) = Z_f(s_2)$ if $q^{-s_1} = q^{-s_2}$. One can observe this fact also directly by looking at the defining integral. We have that $q^{-s_1} = q^{-s_2}$ if and only if $s_1 = s_2 + 2k\pi\sqrt{-1}/\log q$ for some $k \in \mathbb{Z}$. Consequently, $Z_f(s)$ is periodic with period $2\pi\sqrt{-1}/\log q$.

Let t be another complex variable. Because the map $\varphi : \{s \mid \mathrm{Re}(s) > 0\} \to \{t \mid 0 < |t| < 1\} : s \mapsto t = q^{-s}$ is surjective and because $Z_f(s)$ depends only on q^{-s}, there exists a unique function $Z_f(t)$ on $\{t \mid 0 < |t| < 1\}$ such that $Z_f(s) = Z_f(t) \circ \varphi$. It follows from equation (1.1) that

$$Z_f(t) = P(t) - \frac{P(t) - 1}{t}.$$

Consequently, also $Z_f(t)$ is an analytic function. Because the constant term of $P(t)$ is 1, we can extend $Z_f(t)$ to an analytic function on $\{t \mid |t| < 1\}$. If we solve the previous equation to $P(t)$, we obtain

$$P(t) = \frac{1 - tZ_f(t)}{1 - t}.$$

(1.1.6) Fix $a = (a_1, \ldots, a_n) \in R^n$. Let M_i' be the number of solutions of $f(x) \equiv 0 \bmod P^i$ in $(R/P^i)^n$ which are equal to a modulo $P^n = P \times \cdots \times P$. Put $Q(t) = \sum_{i=1}^{\infty} M_i'(q^{-n}t)^i$. One obtains analogously as in (1.1.5) that

$$\int_{a+P^n} |f(x)|^s \, |dx| = Q(t) - \frac{Q(t) - q^{-n}t}{t},$$

where of course $t = q^{-s}$.

Our next goal is to calculate this integral under the conditions $f(a) \equiv 0 \bmod P$ and $(\partial f/\partial x_j)(a) \not\equiv 0 \bmod P$ for at least one j. We will do this by calculating the numbers M_i'. In the next paragraph, we will show that $M_i' = q^{(i-1)(n-1)}$. This is actually Hensel's lemma in several variables.

Fix a $j \in \{1, \ldots, n\}$ for which $(\partial f/\partial x_j)(a) \not\equiv 0 \bmod P$. Fix also $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n \in R$ satisfying $a_l \equiv b_l \bmod P$ for every $l \in \{1, \ldots, j-1, j+1, \ldots, n\}$. We want to count the number of solutions $y \in R/P^{i-1}$ of

$$g(y) := f(b_1, \ldots, b_{j-1}, a_j + \pi y, b_{j+1}, \ldots, b_n) \equiv 0 \bmod P^i.$$

The constant term of g as a polynomial in y is $f(b_1, \ldots, b_{j-1}, a_j, b_{j+1}, \ldots, b_n)$ and this is congruent to $f(a)$ and thus to 0 mod P. The coefficient of y in g is equal to $\pi(\partial f/\partial x_j)(b_1, \ldots, b_{j-1}, a_j, b_{j+1}, \ldots, b_n)$. This is 0 mod P and different from 0 mod P^2 because it is $\pi(\partial f/\partial x_j)(a) \bmod P^2$. The coefficient of every y^k is of course in P^k. Put $h(y) = g(y)/\pi$. We divide the previous congruence by π, so we obtain that we have to count the number of solutions $y \in R/P^{i-1}$ of

$$h(y) \equiv 0 \bmod P^{i-1}.$$

Note that h is a polynomial in which the coefficient of y is not in P and in which the coefficients in terms of higher degree are in P. First, we determine the solutions of $h(y) \equiv 0 \bmod P$ in R/P. Because this is actually a linear congruence, it has one solution $c_1 + P$. Next, we

determine the solutions of $h(y) \equiv 0 \bmod P^2$ in R/P^2. Therefore, we have to determine the solutions of $h(c_1 + \pi y_2) \equiv 0 \bmod P^2$ in R/P, which is actually a linear congruence after we have divided by π. So we obtain that there is one solution $c_2 + P^2$ of $h(y) \equiv 0 \bmod P^2$. Thereafter, we determine the solutions of $h(y) \equiv 0 \bmod P^3$ in R/P^3. Therefore, we have to determine the solutions of $h(c_2 + \pi^2 y_3) \equiv 0 \bmod P^3$ in R/P, which is actually a linear congruence after we have divided by π^2. If we continue in this way, we obtain finally that $h(y) \equiv 0 \bmod P^{i-1}$, and thus also $g(y) \equiv 0 \bmod P^i$, has one solution in R/P^{i-1}. Because we have q^{i-1} choices modulo P^i for every b_l, with $l \in \{1, \ldots, j-1, j+1, \ldots, n\}$, we obtain that $M_i' = q^{(i-1)(n-1)}$.

Because we know the M_i', we are able to calculate $Q(t)$:

$$\begin{aligned} Q(t) &= \sum_{i=1}^{\infty} M_i' q^{-ni} t^i \\ &= q^{-n+1} \sum_{i=1}^{\infty} q^{-i} t^i \\ &= q^{-n+1} \left(\frac{1}{1 - q^{-1}t} - 1 \right) \\ &= \frac{q^{-n}t}{1 - q^{-1}t}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{a+P^n} |f(x)|^s \, |dx| &= Q(t) - \frac{Q(t) - q^{-n}t}{t} \\ &= \frac{q^{-n}t}{1 - q^{-1}t} - \frac{q^{-n-1}t}{1 - q^{-1}t} \\ &= q^{-n} \frac{q-1}{q} \frac{t}{1 - q^{-1}t}. \end{aligned}$$

(1.1.7) We have obtained everything to write down *the p-adic stationary phase formula* [Ig7, Theorem 10.2.1]. It is actually the trivial equality

$$\int_{R^n} |f(x)|^s \, |dx| = \sum_{a+P^n \in (R/P)^n} \int_{a+P^n} |f(x)|^s \, |dx|,$$

in which we replace the integrals that we know by now. These are

$$\int_{a+P^n} |f(x)|^s \, |dx| = q^{-n}$$

if $f(a) \not\equiv 0 \bmod P$ and

$$\int_{a+P^n} |f(x)|^s \, |dx| = q^{-n} \frac{q-1}{q} \frac{t}{1-q^{-1}t}$$

if $f(a) \equiv 0 \bmod P$ and $(\partial f/\partial x_j)(a) \not\equiv 0 \bmod P$ for at least one $j \in \{1, \ldots, n\}$. If $f(a) \equiv 0 \bmod P$ and $(\partial f/\partial x_j)(a) \equiv 0 \bmod P$ for all j, then the integral stays unchanged in the formula.

The reduction of f modulo P, which is denoted by $\overline{f}$, is obtained by reducing all the coefficients of f modulo P and is thus a polynomial over the finite field $\mathbb{F}_q$. Note that $f(a) \equiv 0 \bmod P$ if and only if $\overline{f}(\overline{a}) = 0$ in $\mathbb{F}_q$ and that $(\partial f/\partial x_j)(a) \equiv 0 \bmod P$ if and only if $(\partial \overline{f}/\partial x_j)(\overline{a}) = 0$ in $\mathbb{F}_q$. Here, $\overline{a} \in \mathbb{F}_q^n$ is obtained by reducing every component of a modulo P. Note also that a singular point of a polynomial is by definition a zero of the polynomial and of all its partial derivatives of the first order. For example, every point is a singular point of the zero polynomial.

The p-adic stationary phase formula can sometimes be used to calculate Igusa's p-adic zeta function. I illustrate this with the polynomial $f = x_1x_2 + x_3^2$. We first calculate the number of solutions of $\overline{f} = x_1x_2 + x_3^2 = 0$ over the finite field $\mathbb{F}_q$. If $x_2 \neq 0$, this is a linear equation in x_1 for every $x_3 \in \mathbb{F}_q$. Such an equation has one solution. There are thus $q(q-1)$ solutions for which $x_2 \neq 0$. If $x_2 = 0$, then $x_3 = 0$ and x_1 is an arbitrary element of $\mathbb{F}_q$. There are thus q solutions for which $x_2 = 0$. We obtain that $x_1x_2 + x_3^2 = 0$ has $q(q-1) + q = q^2$ solutions over $\mathbb{F}_q$. There is only one singular point of $x_1x_2 + x_3^2 = 0$ over $\mathbb{F}_q$. Indeed, the system of equations $\overline{f} = x_1x_2 + x_3^2 = 0$, $\partial\overline{f}/\partial x_1 = x_2 = 0$, $\partial\overline{f}/\partial x_2 = x_1 = 0$ and $\partial\overline{f}/\partial x_3 = 2x_3 = 0$ only has $(0,0,0)$ as solution over $\mathbb{F}_q$. Note that this is also the case if we work in characteristic 2. The p-adic stationary phase formula states that

$$\begin{aligned} \int_{R^3} |x_1x_2 + x_3^2|^s \, |dx| &= (q^3 - q^2)q^{-3} + (q^2-1)q^{-3} \frac{q-1}{q} \frac{t}{1-q^{-1}t} \\ &\quad + \int_{P^3} |x_1x_2 + x_3^2|^s \, |dx|. \end{aligned}$$

If we apply the coordinate transformation $R^3 \to P^3 : (x_1, x_2, x_3) \mapsto (\pi x_1, \pi x_2, \pi x_3)$ to the last integral, we obtain

$$\begin{aligned} \int_{P^3} |x_1x_2 + x_3^2|^s \, |dx| &= \int_{R^3} |\pi x_1 \pi x_2 + \pi^2 x_3^2|^s \, q^{-3} |dx| \\ &= q^{-3}t^2 \int_{R^3} |x_1x_2 + x_3^2|^s \, |dx|. \end{aligned}$$

The integral we want to calculate appears in the last expression. Consequently, we get a functional equation. Solving this gives us

$$\int_{R^3} |x_1x_2 + x_3^2|^s \, |dx| = \frac{q-1}{q} \frac{1-q^{-3}t}{(1-q^{-1}t)(1-q^{-3}t^2)}.$$

In more difficult examples, one applies the p-adic stationary phase formula several times until one gets a system of equations in several undetermined functions from which Igusa's p-adic zeta function can be solved. It is an open problem whether this is always possible.

(1.1.8) Now I explain how Hoornaert obtained in her Ph.D. thesis (see also [DH]) a formula for Igusa's p-adic zeta function of polynomials which are non-degenerated over $\mathbb{F}_q$ with respect to their Newton polyhedron. This is in some sense a very large class of polynomials and consequently an interesting class for checking conjectures. It was also useful in my research: at some points it made me clear what I could expect and what certainly not. Indeed, the formula is in terms of the Newton polyhedron of f and is therefore very interesting to construct examples, which can be calculated explicitly by using the formula or the computer program of Hoornaert and Loots [HL] based on it.

Let $f(x) = \sum_{\omega \in \mathbb{Z}_{\geq 0}^n} a_\omega x^\omega = \sum_{\omega \in \mathbb{Z}_{\geq 0}^n} a_\omega x_1^{\omega_1} \cdots x_n^{\omega_n}$ be a non-zero polynomial over R with $f(0) = 0$. Here $\mathbb{Z}_{\geq 0}^n$ is the abbreviated notation for $(\mathbb{Z}_{\geq 0})^n$. Let $\mathrm{supp}(f) = \{\omega \in \mathbb{Z}_{\geq 0}^n \mid a_\omega \neq 0\}$. The Newton polyhedron $\Gamma(f)$ of f is defined as the convex hull in $\mathbb{R}_{\geq 0}^n$ of the set $\cup_{\omega \in \mathrm{supp}(f)} \omega + \mathbb{R}_{\geq 0}^n$. The intersection of $\Gamma(f)$ with a supporting hyperplane is called a face of $\Gamma(f)$ and a facet is a face of dimension $n-1$. If τ is a face of $\Gamma(f)$, we define $f_\tau(x) = \sum_{\omega \in \tau} a_\omega x^\omega$. We say that f is non-degenerated over $\mathbb{F}_q$ if not any of the polynomials $\overline{f}$ and $\overline{f_\tau}$, with τ a face of $\Gamma(f)$, has a singularity in $(\mathbb{F}_q^\times)^n$.

Let τ be a face of $\Gamma(f)$. Let $\gamma_1, \ldots, \gamma_i$ be the facets of $\Gamma(f)$ which contain τ. Let $v_1, \ldots, v_i$ be the vectors in $\mathbb{Z}_{\geq 0}^n \setminus \{(0, \ldots, 0)\}$ of minimal length which are perpendicular to respectively $\gamma_1, \ldots, \gamma_i$. Then $\Delta_\tau = \{\lambda_1 v_1 + \cdots + \lambda_i v_i \mid \lambda_i \in \mathbb{R}_{>0}\}$ is called the cone associated to τ. One proves that $\mathbb{R}_{\geq 0}^n \setminus \{(0, \ldots, 0)\}$ is the disjoint union of the Δ_τ.

For $x = (x_1, \ldots, x_n) \in R^n$, we denote $(\mathrm{ord}\, x_1, \ldots, \mathrm{ord}\, x_n)$ simply by $\mathrm{ord}\, x$. We have

$$\begin{aligned} Z_f(s) &= \int_{R^n} |f(x)|^s \, |dx| \\ &= \sum_{k \in \mathbb{Z}^n_{\geq 0}} \int_{\substack{x \in R^n \\ \operatorname{ord} x = k}} |f(x)|^s \, |dx| \\ &= \int_{(R^\times)^n} |f(x)|^s \, |dx| + \sum_{\tau} \sum_{k \in \mathbb{Z}^n_{\geq 0} \cap \Delta_\tau} \int_{\substack{x \in R^n \\ \operatorname{ord} x = k}} |f(x)|^s \, |dx|. \end{aligned}$$

Let $M_{\Gamma(f)}$ (resp. M_τ) be the number of solutions of $\overline{f}$ (resp. $\overline{f_\tau}$) in $(\mathbb{F}_q^\times)^n$.

Because f is non-degenerated over $\mathbb{F}_q$ we have that $\overline{f}$ has no singularities in $(\mathbb{F}_q^\times)^n$, and consequently

$$\begin{aligned} \int_{(R^\times)^n} |f(x)|^s \, |dx| &= ((q-1)^n - M_{\Gamma(f)})q^{-n} + M_{\Gamma(f)} q^{-n} \frac{q-1}{q} \frac{t}{1-q^{-1}t} \\ &= \left(\frac{q-1}{q}\right)^n - M_{\Gamma(f)} q^{-n} \frac{1-t}{1-q^{-1}t}. \end{aligned}$$

Let τ be a face of $\Gamma(f)$. Let $k \in \mathbb{Z}^n_{\geq 0} \cap \Delta_\tau$. One proves that the function $\theta : \Gamma(f) \to \mathbb{R} : x = (x_1, \ldots, x_n) \mapsto k \cdot x = k_1 x_1 + \cdots + k_n x_n$ reaches it minimum exactly on the points of τ. This minimum is denoted by $m(k)$. Put $\sigma(k) = k_1 + \cdots + k_n$. We apply the coordinate transformation $\varphi : (u_1, \ldots, u_n) \mapsto (\pi^{k_1} u_1, \ldots, \pi^{k_n} u_n)$. Then $\varphi^* |dx| = q^{-\sigma(k)} |du|$ and $\varphi^{-1}(\{x \in R^n \mid \operatorname{ord} x = k\}) = (R^\times)^n$. Because $\varphi^*(x^\omega) = \pi^{k \cdot \omega} u^\omega$ and because θ reaches its minimum $m(k)$ exactly on the points of τ, we can write $\varphi^*(f(x)) = \pi^{m(k)}(f_\tau(u) + \pi f_{\tau,k}(u))$, where $f_{\tau,k}$ is a polynomial over R. Because f is non-degenerated over $\mathbb{F}_q$ we have that $\overline{f_\tau}$, and thus also $\overline{f_\tau + \pi f_{\tau,k}}$, has no singularities in $(\mathbb{F}_q^\times)^n$. Therefore,

$$\begin{aligned} \int_{\substack{x \in R^n \\ \operatorname{ord} x = k}} |f(x)|^s \, |dx| &= \int_{(R^\times)^n} |\pi^{m(k)}(f_\tau(u) + \pi f_{\tau,k}(u))|^s \, q^{-\sigma(k)} |du| \\ &= q^{-\sigma(k)} t^{m(k)} \int_{(R^\times)^n} |f_\tau(u) + \pi f_{\tau,k}(u)|^s \, |du| \\ &= q^{-\sigma(k)} t^{m(k)} \left[\left(\frac{q-1}{q}\right)^n - M_\tau q^{-n} \frac{1-t}{1-q^{-1}t} \right]. \end{aligned}$$

The part between the brackets does not depend on the choice of $k \in \mathbb{Z}^n_{\geq 0} \cap \Delta_\tau$. We have thus that

$$\sum_{k \in \mathbb{Z}^n_{\geq 0} \cap \Delta_\tau} \int_{\substack{x \in R^n \\ \operatorname{ord} x = k}} |f(x)|^s \, |dx|$$

is equal to

$$\left(\sum_{k\in\mathbb{Z}^n_{\geq 0}\cap\Delta_\tau} q^{-\sigma(k)}t^{m(k)}\right)\left[\left(\frac{q-1}{q}\right)^n - M_\tau q^{-n}\frac{1-t}{1-q^{-1}t}\right].$$

One can prove that

$$\sum_{k\in\mathbb{Z}^n_{\geq 0}\cap\Delta_\tau} q^{-\sigma(k)}t^{m(k)} = \frac{A(t)}{(1-q^{-\sigma(v_1)}t^{m(v_1)})\cdots(1-q^{-\sigma(v_i)}t^{m(v_i)})},$$

where $A(t)$ is a polynomial which can be calculated explicitly and where $v_1,\ldots,v_i$ are the vectors in $\mathbb{Z}^n_{\geq 0}\setminus\{(0,\ldots,0)\}$ of minimal length which are perpendicular to the facets that contain τ.

The formula we obtained has important consequences. If f is non-degenerated over $\mathbb{F}_q$, it follows from the formula that $Z_f(s)$ has a meromorphic continuation to $\mathbb{C}$. Moreover, we can read the candidate poles of this meromorphic continuation from the Newton polyhedron of f. The real part of a candidate pole is -1 or of the form $-\sigma(v)/m(v)$ with v a vector which is perpendicular to a facet τ of $\Gamma(f)$. One checks directly that $-\sigma(v)/m(v) = -1/t_0$, where t_0 is the intersection of the supporting hyperplane of τ with the diagonal $\{(t,\ldots,t) \mid t\in\mathbb{R}\}$ of $\mathbb{R}^n$.

(1.1.9) All Igusa's p-adic zeta functions we have met so far are rational functions of t. This is true in general. We now give the idea behind the proof. For more details, see [Ig7] or Chapter 3.

To calculate integrals, one often tries to find a coordinate transformation such that the integral can be calculated easily in the new coordinates. Here, the appropriate coordinates lie on an embedded resolution of (f,dx), which always exists by Hironaka's theorem [Hi]. Hironaka actually proved that there exists a composition of blowing-ups which is an embedded resolution of (f,dx).

Let $g: Y\to R^n$ be an embedded resolution of (f,dx). Note that Y is a K-analytic manifold. Let b be an arbitrary point of Y. Then there exist analytic coordinates $y=(y_1,\ldots,y_n)$ on a neighbourhood V of b, non-vanishing K-analytic functions ε,η on V and integers $N_i\geq 0$, $\nu_i\geq 1$ for every $i\in\{1,\ldots,n\}$ such that on V

$$f\circ g = \varepsilon\prod_{i=1}^n y_i^{N_i} \qquad \text{and} \qquad g^*dx = \eta\prod_{i=1}^n y_i^{\nu_i-1}dy.$$

Because $|\varepsilon|$ and $|\eta|$ are locally constant functions, we can find a neighbourhood of b on which the integral

$$\int_Y |f \circ g|^s \, |g^* dx|$$

is of the form

$$|\varepsilon(b)|^s |\eta(b)| \int_{(P^k)^n} \prod_{i=1}^{n} |y_i|^{N_i s + \nu_i - 1} \, |dy|.$$

Note that $Z_f(s)$ is a finite sum of such integrals because Y is compact. Because the variables are separated, the integral is the product of integrals in one variable which can be calculated without a problem:

$$\int_{P^k} |y_i|^{N_i s + \nu_i - 1} \, |dy_i| = \frac{q-1}{q} \frac{q^{-k(N_i s + \nu_i)}}{1 - q^{-N_i s - \nu_i}}.$$

We have proved that $Z_f(t)$ is a rational function of t of the form

$$Z_f(t) = \frac{A(t)}{\prod_{j \in J} (1 - q^{-\nu_j} t^{N_j})},$$

where $A(t)$ is a polynomial. Indeed, a possible power of t in the denominator can be cancelled because we saw in (1.1.5) that $Z_f(t)$ is an analytic function on the open unit disc. Because $P(t) = (1 - tZ_f(t))/(1-t)$, we obtain that also $P(t)$ is a rational function of t.

Since $Z_f(t)$ is a rational function of t, we have that $Z_f(s)$ has a meromorphic continuation to $\mathbb{C}$. From now on, Igusa's p-adic zeta function $Z_f(s)$ of f is the meromorphic continuation of the function we considered before. Note that some facts of (1.1.5) such as the periodicity of $Z_f(s)$ and the relation between $Z_f(t)$ and $P(t)$ extend to this larger context. The candidate poles of $Z_f(s)$ are of the form

$$-\frac{\nu_j}{N_j} + \frac{2k\pi\sqrt{-1}}{N_j \log q},$$

with $k \in \mathbb{Z}$ and $j \in J$. Some of these candidate poles are poles and others aren't because they cancel with $A(q^{-s})$.

(1.1.10) Let us now formulate a theorem of Denef [De1]. Let F be a number field. Let $\mathcal{O}_F$ be the ring of integers in F. Let $f \in \mathcal{O}_F[x_1, \ldots, x_n]$.

We consider f as a polynomial function on $X = (F^{\mathrm{alg\,cl}})^n$. Let $g : Y \subset X \times \mathbb{P}^m(F^{\mathrm{alg\,cl}}) \to X$ be an algebraic embedded resolution of f which is defined over F.

A p-adic completion of F is a completion with respect to a prime ideal of $\mathcal{O}_F$. If K is an extension of finite degree of a p-adic completion, then the embedded resolution g induces an embedded resolution $g_K : Y_K \subset K^n \times \mathbb{P}^m(K) \to K^n$ of (f, dx) by using the same equations for Y_K and by using the same rule for g_K.

For almost all p-adic completions of F we have the following. Let K be an extension of finite degree of this p-adic completion. Let R be the valuation ring of K, P the maximal ideal of R and $\overline{K} \cong \mathbb{F}_q$ the residue field R/P. Let $g_K : Y_K \to K^n$ be the embedded resolution induced by g. Let E_i, $i \in T_K$, be the K-irreducible components of $g_K^{-1}(f^{-1}\{0\})$. Let N_i be the multiplicity of E_i in the divisor of $f \circ g_K$ on Y_K and let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $g_K^* dx$ on Y_K. Let $\overline{Y_K}$ (resp. $\overline{E_i}$, $i \in T$) denote the reduction modulo P of Y_K (resp. E_i). Then

$$Z_f(s) = q^{-n} \sum_{I \subset T_K} C_I \prod_{i \in I} \frac{q-1}{q^{\nu_i + sN_i} - 1},$$

where $C_I = \mathrm{card}\{a \in \overline{Y_K}(\overline{K}) \mid a \in \overline{E_i} \text{ if and only if } i \in I\}$. Remark that $Z_f(s)$ depends on K.

Denef actually proves that this holds for a p-adic completion with respect to the prime ideal λ if $g : Y \to X$ has good reduction modulo λ. Remark also that the formula can be formulated directly in terms of $g : Y \to X$ by using the F-irreducible components of $g^{-1}(f^{-1}\{0\})$ and by using reduction modulo λ.

(1.1.11) We will also consider some generalizations of the previous definition of Igusa's p-adic zeta function.

Let $f \in K[x_1, \ldots, x_n]$. Let χ be a character of $R^\times$, i.e., a group homomorphism $\chi : R^\times, . \to \mathbb{C}^\times, .$ with finite image. Put formally $\chi(0) = 0$. For $z \in K$, let $\mathrm{ac}\, z = z\pi^{-\mathrm{ord}\, z}$ be the angular component of z. Igusa's p-adic zeta function $Z_{f,\chi}(s)$ of f and χ is defined by the p-adic integral

$$Z_{f,\chi}(s) = \int_{R^n} \chi(\mathrm{ac}\, f(x)) |f(x)|^s \, |dx|$$

for $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 0$, and on the other part of the complex plane by meromorphic continuation.

Moreover, if f is an arbitrary K-analytic function on an open and compact subset X of K^n, then the meromorphic continuation of

$$Z_{f,\chi}(s) = \int_X \chi(\mathrm{ac}\, f(x))|f(x)|^s\, |dx|,$$

where $\mathrm{Re}(s) > 0$, is also called Igusa's p-adic zeta function of f and χ.

(1.1.12) The monodromy conjecture is one of the most important reasons for the study of the poles of Igusa's p-adic zeta function. For a polynomial f over a number field, this conjecture of Igusa asserts a connection between the poles of Igusa's p-adic zeta function and the topology of the map $f : (\mathbb{C}^n, f^{-1}\{0\}) \to (\mathbb{C}, 0)$. Moreover, the striking fact that most candidate poles of $Z_{f,\chi}(s)$ are actually not poles would be elucidated if this monodromy conjecture is true. We now state the conjecture.

Let F be a number field. Let $f \in F[x_1, \ldots, x_n]$. For almost all p-adic completions K of F we have the following for any character χ of $R^\times$. If s is a pole of $Z_{f,\chi}(s)$, then $\exp(2\pi\sqrt{-1}\,\mathrm{Re}(s))$ is an eigenvalue of the local monodromy of f at some complex point of $f^{-1}\{0\}$.

(1.1.13) Finally, I want to enumerate some different methods which can be used to study Igusa's p-adic zeta function. Some of them can be used to obtain a general statement, others study a special class of functions f.

1. One can use the relation between $Z_{f,\chi}(s)$ and the number of solutions of congruences. Results for the number of solutions of congruences can have implications for $Z_{f,\chi}(s)$.
2. One can use an embedded resolution of (f, dx). Especially embedded resolutions which are a composition of blowing-ups with well chosen centra can be very useful.
3. One can use the formula of Denef to obtain results for polynomials with good reduction modulo P.
4. One can use the formula of Hoornaert to obtain results for polynomials which are non-degenerated over $\mathbb{F}_q$ with respect to their Newton polyhedron.
5. One can study the class of polynomials which are a relative invariant of a prehomogeneous vector space.
6. One can use p-adic cell decomposition. Denef used this method for example to obtain another proof of the rationality of $Z_{f,\chi}(t)$.

1.2 Topological and motivic zeta function

(1.2.1) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function (resp. $f \in \mathbb{C}[x_1, \ldots, x_n]$). We will always assume that f is different from 0. Let $g : Y \to X \subset \mathbb{C}^n$ (resp. $g : Y \to \mathbb{C}^n$) be an embedded resolution of singularities of a representative $f : X \to \mathbb{C}$ (resp. of $f : \mathbb{C}^n \to \mathbb{C}$). We denote by E_i, $i \in T$, the irreducible components of $g^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of $f \circ g$ and g^*dx along E_i. The (N_i, ν_i), $i \in T$, are called the numerical data of the embedded resolution. For $I \subset T$, we denote $E_I := \cap_{i \in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$. Note that Y is the disjoint union of the $\overset{\circ}{E}_I$ with $I \subset T$.

(1.2.2) To the germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of a holomorphic function we associate the local topological zeta function

$$Z_{\mathrm{top},0,f}(s) := \sum_{I \subset T} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

To a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ we associate the global topological zeta function

$$Z_{\mathrm{top},gl,f}(s) := \sum_{I \subset T} \chi(\overset{\circ}{E}_I) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

In both cases, s is a complex variable and $\chi(\cdot)$ denotes the topological Euler-Poincaré characteristic. Of course, one has to prove that the expressions on the right hand side do not depend on the chosen embedded resolution. This remarkable fact was first proved as follows by Denef and Loeser in [DL1]. For a polynomial f over the ring of integers $\mathcal{O}_F$ of a number field F, take a completion K of F such that the formula in (1.1.10) holds for all extensions of this completion. Using p-adic interpolation, we can consider the global topological zeta function as the limit for $e \to 0$ of Igusa's p-adic zeta functions of f over certain non-ramified extensions of degree e over K. One has to use a cohomological interpretation of the numbers C_I, which is provided by Grothendieck's trace formula. To treat the other cases, one has to do more.

Note that one can also define a twisted version of the local and global topological zeta function for every $d \in \mathbb{Z}_{\geq 2}$. The definition is exactly the same as above, except for the fact that the sum runs only over subsets I of T for which $d \mid N_i$ for every $i \in I$.

(1.2.3) We are interested in the poles of these zeta functions. The candidate poles are the $-\nu_i/N_i$, $i \in T$. Some of them are poles, others are not. We are going to discuss the case $n = 2$. Let E_i, $i \in T$, be an exceptional curve. We have thus $E_i \cong \mathbb{P}^1$. Let E_j, $j \in J$, be the irreducible components of $g^{-1}(f^{-1}\{0\})$ which are different from E_i and which intersect E_i. Suppose that $-\nu_j/N_j \neq -\nu_i/N_i$, which is equivalent to $\alpha_j := \nu_j - (\nu_i/N_i)N_j \neq 0$, for every $j \in J$. Actually, we assume thus that the sum of the terms in the definition of the topological zeta function for which $i \in I$ has a candidate pole $-\nu_i/N_i$ of expected order one. The contribution $\mathcal{R}$ of E_i to the residue at the candidate pole $-\nu_i/N_i$ is by definition the residue of this sum at $-\nu_i/N_i$. Note that $\mathcal{R} = 0$ if and only if this sum has not a pole in $-\nu_i/N_i$.

Everything we write down from now on is for the local topological zeta function $Z_{\mathrm{top},0,f}(s)$ associated to the germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ of a holomorphic function. One obtains easily that

$$\mathcal{R} = \frac{1}{N_i}\left(\chi(\overset{\circ}{E}_{\{i\}}) + \sum_{j \in J} \alpha_j^{-1}\right). \tag{1.2}$$

We will prove that this is equal to zero if J contains one or two elements. Therefore, we need a relation between the α_j, $j \in J$.

We give here a deduction of Veys which uses intersection theory. Note that we may assume that f is a polynomial because there exists a degree from which the terms do not influence the embedded resolution and the numerical data. Because $\mathrm{div}(f \circ g) = \sum_{j \in T} N_j E_j$ is a principal divisor, we obtain that $(\sum_{j \in T} N_j E_j) \cdot E_i = 0$. Because $E_j \cdot E_i = 1$ if $j \in J$ and $E_j \cdot E_i = 0$ if $j \notin J \cup \{i\}$, we get

$$\sum_{j \in J} N_j = -N_i E_i^2. \tag{1.3}$$

Here, E_i^2 is the self intersection of E_i on Y. Because $\mathrm{div}(g^* dx) = \sum_{j \in T}(\nu_j - 1)E_j$ is the canonical divisor, we obtain from the adjunction formula that $(\sum_{j \in T}(\nu_j - 1)E_j) \cdot E_i = -2 - E_i^2$. Hence, we get

$$\sum_{j \in J}(\nu_j - 1) + 2 = -\nu_i E_i^2. \tag{1.4}$$

Relation (1.4) minus ν_i/N_i times relation (1.3) gives us a relation between

the α_j:

$$\sum_{j\in J}(\alpha_j - 1) + 2 = 0. \tag{1.5}$$

We are now able to prove that $\mathcal{R} = 0$ if J contains one or two elements. If $J = \{1\}$ contains one element, then it follows from relation (1.5) that $\alpha_1 = -1$. We obtain from (1.2) that

$$\mathcal{R} = \frac{1}{N_i}\left(1 + \frac{1}{-1}\right) = 0.$$

If $J = \{1, 2\}$ contains two elements, then it follows from (1.5) that $\alpha_1 + \alpha_2 = 0$. We obtain from (1.2) that

$$\mathcal{R} = \frac{1}{N_i}\left(0 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) = \frac{1}{N_i}\left(\frac{\alpha_2 + \alpha_1}{\alpha_1\alpha_2}\right) = 0.$$

We have treated the easy part of the following theorem. The other part is more difficult and is proved in [Ve5].

Theorem. *We have that s_0 is a pole of $Z_{\mathrm{top},0,f}(s)$ if and only if $s_0 = -\nu_i/N_i$ for some exceptional curve E_i intersecting at least three times other components, or $s_0 = -1/N_j$ for some irreducible component E_j of the strict transform of $f^{-1}\{0\}$.*

(1.2.4) The topological zeta function involves the topological Euler-Poincaré characteristic. This invariant is quite rough: sometimes one wants to detect more. Now we introduce the universal Euler characteristic, which is the finest invariant with the following properties of the topological Euler-Poincaré characteristic: $\chi(X) = \chi(Y)$ if X is isomorphic to Y and $\chi(X \setminus Y) = \chi(X) - \chi(Y)$ if Y is a closed subvariety of X. This universal Euler characteristic is the class of an algebraic variety in the Grothendieck ring of algebraic varieties over $\mathbb{C}$. We first recall this notion, which we will give relative to an arbitrary field of characteristic zero.

Let k be a field of characteristic zero. An algebraic variety over k is in this paragraph a reduced separated scheme of finite type over k. An algebraic variety is thus not necessarily irreducible. Let G be the free abelian group generated by the symbols $[V]$, where V is an algebraic variety over k. Let N be its subgroup generated by $\{[V] - [V'] \mid V \text{ is isomorphic to } V' \text{ over } k\} \cup \{[V] - [V \setminus W] - [W] \mid W \text{ is a closed } k\text{-subvariety of } V\}$. The Grothendieck ring $K_0(\mathrm{Var}_k)$ of algebraic varieties over k is the quotient group G/N. Its ring structure is determined

by $[V]\cdot[W] := [V \times W]$. We set $\mathbb{L} := [\mathbb{A}^1_k]$ and we denote by $\mathcal{M}_k$ the localization of $K_0(\mathrm{Var}_k)$ with respect to $\mathbb{L}$.

(1.2.5) If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is the germ of a holomorphic function, the local motivic zeta function of f is the following element of $\mathcal{M}_{\mathbb{C}}[[t]]$:

$$Z_{\mathrm{mot},0,f}(t) = \mathbb{L}^{-n} \sum_{I \subset T} [\overset{\circ}{E}_I \cap g^{-1}\{0\}] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{-\nu_i} t^{N_i}}{1-\mathbb{L}^{-\nu_i} t^{N_i}}.$$

If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial over $\mathbb{C}$, the global motivic zeta function of f is the following element of $\mathcal{M}_{\mathbb{C}}[[t]]$:

$$Z_{\mathrm{mot},gl,f}(t) = \mathbb{L}^{-n} \sum_{I \subset T} [\overset{\circ}{E}_I] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{-\nu_i} t^{N_i}}{1-\mathbb{L}^{-\nu_i} t^{N_i}}.$$

These motivic zeta functions are defined intrinsically by Denef and Loeser in [DL2]. In this paper, they obtained the previous expressions which are in terms of an embedded resolution.

Note that $1-\mathbb{L}^{-\nu_i} t^{N_i}$ is a unit in $\mathcal{M}_{\mathbb{C}}[[t]]$ and that its inverse is $1/(1-\mathbb{L}^{-\nu_i} t^{N_i}) = 1 + \mathbb{L}^{-\nu_i} t^{N_i} + \mathbb{L}^{-2\nu_i} t^{2N_i} + \cdots$. We can consider these motivic zeta functions as an element of the localization of the polynomial ring $\mathcal{M}_{\mathbb{C}}[t]$ with respect to the set $\{1-\mathbb{L}^{-\nu_i} t^{N_i} \mid i \in T\}$ or to the set $\{1-\mathbb{L}^{-a} t^b \mid a, b \in \mathbb{Z}_{>0}\}$. These localizations are subrings of $\mathcal{M}_{\mathbb{C}}[[t]]$.

(1.2.6) The topological zeta functions are specializations of the motivic zeta functions. This gives us another proof of the fact that the expressions in the definitions of the topological zeta functions are independent of the chosen embedded resolution.

We now explain how $Z_{\mathrm{top},gl,f}(s)$ can be seen as a specialization of $Z_{\mathrm{mot},gl,f}(t)$. First we specialize $Z_{\mathrm{mot},gl,f}(t)$ to the Hodge zeta function $Z_{\mathrm{Hod},gl,f}(t)$:

$$Z_{\mathrm{Hod},gl,f}(t) \quad = \quad (uv)^{-n} \sum_{I \subset T} H(\overset{\circ}{E}_I) \prod_{i \in I} \frac{(uv-1)(uv)^{-\nu_i} t^{N_i}}{1-(uv)^{-\nu_i} t^{N_i}}.$$

Here, $H(V) \in \mathbb{Z}[u, v]$ represents the Hodge polynomial of the algebraic variety V. This polynomial codes important information of the mixed Hodge theory of V. It is an Euler characteristic at an intermediate level: the elements of the Grothendieck ring specialize to Hodge polynomials

and one can specialize these further to the topological Euler-Poincaré characteristic by putting $u = v = 1$.

Let $s \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned} Z_{\mathrm{Hod},gl,f}((uv)^{-s}) &= (uv)^{-n} \sum_{I \subset T} H(\overset{\circ}{E}_I) \prod_{i \in I} \frac{(uv-1)(uv)^{-\nu_i - sN_i}}{1-(uv)^{-\nu_i - sN_i}} \\ &= (uv)^{-n} \sum_{I \subset T} H(\overset{\circ}{E}_I) \prod_{i \in I} \frac{uv-1}{(uv)^{\nu_i + sN_i}-1} \\ &= (uv)^{-n} \sum_{I \subset T} H(\overset{\circ}{E}_I) \prod_{i \in I} \frac{1}{1+uv+\cdots+(uv)^{\nu_i + sN_i - 1}} \end{aligned}$$

We apply the topological Euler-Poincaré characteristic to this expression. So we have to replace $H(\overset{\circ}{E}_I)$ by $\chi(\overset{\circ}{E}_I)$ and $H(\mathbb{L}) = uv$ by 1. We obtain the function

$$\mathbb{Z}_{\geq 0} \to \mathbb{C} : s \mapsto \sum_{I \subset T} \chi(\overset{\circ}{E}_I) \prod_{i \in I} \frac{1}{\nu_i + sN_i}. \tag{1.6}$$

This function is independent of the embedded resolution and can be extended uniquely to a rational function on $\mathbb{C}$, which is

$$Z_{\mathrm{top},gl,f}(s) = \sum_{I \subset T} \chi(\overset{\circ}{E}_I) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

We used Hodge polynomials to deduce that (1.6) is independent of the embedded resolution. I try to clarify why this is crucial by giving a slightly different deduction. We want to consider $Z_{\mathrm{mot},gl,f}(\mathbb{L}^{-s})$, but we can not give sense to this in $\mathcal{M}_{\mathbb{C}}$. We deal with this problem as follows. Suppose that we have two embedded resolutions. Then

$$\mathbb{L}^{-n} \sum_{I \subset T} [\overset{\circ}{E}_I] \prod_{i \in I} \frac{(\mathbb{L}-1)\mathbb{L}^{-\nu_i} t^{N_i}}{1-\mathbb{L}^{-\nu_i} t^{N_i}} = \mathbb{L}^{-n} \sum_{I' \subset T'} [\overset{\circ}{E}_{I'}] \prod_{j \in I'} \frac{(\mathbb{L}-1)\mathbb{L}^{-\nu_j} t^{N_j}}{1-\mathbb{L}^{-\nu_j} t^{N_j}}.$$

We bring the denominators to the other side, so we get a polynomial identity in which we make the substitution $t = \mathbb{L}^{-s}$ for $s \in \mathbb{Z}_{\geq 0}$. If we apply now the topological Euler-Poincaré characteristic, we get the trivial identity $0 = 0$ because $(\mathbb{L}-1)^{|T|+|T'|}$ is a factor of each side and because $\chi(\mathbb{L}-1) = 0$. We have to cancel $(\mathbb{L}-1)^{|T|+|T'|}$ before we apply the topological Euler-Poincaré characteristic, but this is a problem because

we do not know that $\mathbb{L}-1$ is not a zero divisor in $\mathcal{M}_{\mathbb{C}}$. However, if we specialize first to the Hodge polynomials, we can cancel $H(\mathbb{L}-1) = uv-1$ because $\mathbb{Z}[u,v]_{uv}$ is an integral domain. Thereafter, we apply the topological Euler-Poincaré characteristic and we bring every factor to its original side to obtain (1.6).

(1.2.7) If we work relative to a number field, there exists a connection between the motivic zeta function and Igusa's p-adic zeta function.

Let F be a number field. Let $f \in \mathcal{O}_F[x_1,\ldots,x_n]$ be a non-zero polynomial. Let $g: Y \to \mathbb{C}^n$ be an embedded resolution of $f: \mathbb{C}^n \to \mathbb{C}$ which is defined over F. We denote by E_i, $i \in T$, the F-irreducible components of $g^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of $f\circ g$ and g^*dx along E_i. For $I \subset T$, we denote $E_I := \cap_{i\in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j\notin I} E_j)$. The global motivic zeta function $Z_{\mathrm{mot},gl,f}(t)$ is in this context an element of $\mathcal{M}_F[[t]]$, and is given by the same expression as in (1.2.4).

Let λ be a prime ideal of $\mathcal{O}_F$ such that the statement in (1.1.10) holds for the p-adic completion of F with respect to λ. Let K be an extension of finite degree of this p-adic completion. Let R be the valuation ring of K and let P be the maximal ideal of R. Let q be the number of elements of the residue field $\mathcal{O}_F/\lambda$ and let $e \in \mathbb{Z}_{\geq 1}$ be such that $R/P \cong \mathbb{F}_{q^e}$. For an algebraic variety X over F, let $\overline{X}$ be the reduction modulo λ and let $\overline{X}(\mathbb{F}_{q^e})$ be the $\mathbb{F}_{q^e}$-rational points of $\overline{X}$. Because $\mathrm{card}(\overline{X}(\mathbb{F}_{q^e})) = \mathrm{card}(\overline{Y}(\mathbb{F}_{q^e}))$ if $X \cong_F Y$ and $\mathrm{card}(\overline{X\setminus Y}(\mathbb{F}_{q^e})) = \mathrm{card}(\overline{X}(\mathbb{F}_{q^e})) - \mathrm{card}(\overline{Y}(\mathbb{F}_{q^e}))$ if Y is a closed F-subvariety of X, we can define $\mathrm{card}(\alpha(\mathbb{F}_{q^e}))$ for $\alpha \in K_0(\mathrm{Var}_F)$ in an obvious way. We have that $\mathrm{card}((\alpha\beta)(\mathbb{F}_{q^e})) = \mathrm{card}(\alpha(\mathbb{F}_{q^e}))\mathrm{card}(\beta(\mathbb{F}_{q^e}))$ for $\alpha,\beta \in K_0(\mathrm{Var}_F)$ and that it induces a notion of cardinality on $\mathcal{M}_F$. Because of (1.1.10), we obtain Igusa's p-adic zeta function by specialization.

(1.2.8) The weak factorization conjecture is proved recently (see [AKMW] or [Wł]) and can be used to obtain the independence of the embedded resolution of the global topological and the global motivic zeta function. The weak factorization theorem states the following. Let $f: X \dashrightarrow Y$ be a proper birational map between non-singular algebraic varieties X and Y over an algebraically closed field of characteristic zero, which is an isomorphism over an open set U. Then f can be factored as

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{n-1}}{\dashrightarrow} X_n = Y,$$

where each X_i is a non-singular algebraic variety and f_i is a blowing-up or blowing-down at a smooth centre which is an isomorphism over U. Moreover, if $X \setminus U$ and $Y \setminus U$ are divisors with normal crossings, then each $D_i := X_i \setminus U$ is a divisor with normal crossings and f_i is a blowing-up or blowing-down at a smooth centre which has normal crossings with the components of D_i.

(1.2.9) *Example.* Let $f = x^5z^2 + x^8y^2 + x^{11}$. We used the computer program of Hoornaert and Loots [HL] to compute Igusa's p-adic zeta function of f and the global topological zeta function of f.

Let $K = \mathbb{Q}_p$. If $p \equiv 3 \bmod 4$, we obtained that $-1/2$ is a pole of $Z_f(s)$. If $p \equiv 1 \bmod 4$ we obtained that $-1/2$ is not a pole of $Z_f(s)$. In both cases, there are poles with real part $-1/2$ and with imaginary part between 0 and $2\pi\sqrt{-1}/\log p$.

Because $-1/2$ is a pole of infinitely many p-adic fields $\mathbb{Q}_p$, we have that $Z_{\mathrm{mot},gl,f}(t)$ has a pole in $-1/2$. However, we obtained that $Z_{\mathrm{top},gl,f}(s)$ has not a pole in $-1/2$. For the definition of a pole of the motivic zeta function, see [RV2].

1.3 Historical note

The study of these zeta functions has a rich history. Igusa's p-adic zeta function was introduced in 1965 by Weil [We] because he wanted to prove the convergence of the Eisenstein-Siegel series. The basis properties of this zeta function were first studied by Igusa ([Ig1], [Ig2], [Ig3]). He also used $Z_f(s)$ to solve a conjecture of Borevich and Shafarevich. They introduced the Poincaré series $P(t)$ in the 1950's and they conjectured the rationality of it [BS]. Igusa proved the rationality of $P(t)$ first in [Ig2] and in [Ig3] he gave the proof of this fact which we have seen before and which is much easier. See [De2] for more information and for further developments on Igusa's p-adic zeta function.

In the early 1990's Denef and Loeser introduced the topological zeta function [DL1]. They obtained this invariant as a limit of Igusa's p-adic zeta functions. In 1998 they defined the motivic zeta function [DL2]. This invariant specializes to the topological zeta function and is defined intrinsically.

Chapter 2

The topological zeta function for curves and surfaces

Abstract

We study the local topological zeta function associated to a complex function that is holomorphic at the origin of $\mathbb{C}^2$ (respectively $\mathbb{C}^3$). We determine all possible poles less than $-1/2$ (respectively -1). On $\mathbb{C}^2$ our result is a generalization of the fact that the log canonical threshold is never in $]5/6, 1[$. Similar statements are true for the motivic zeta function.

2.1 Introduction

(2.1.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^n$ which satisfies $f(0) = 0$ and which is not identically zero. Let $g : V \to U \subset \mathbb{C}^n$ be an embedded resolution of a representative of $f^{-1}\{0\}$. We denote by E_i, $i \in T$, the irreducible components of $g^{-1}(f^{-1}\{0\})$, and by N_i and $\nu_i - 1$ the multiplicities of $f \circ g$ and $g^*(dx_1 \wedge \cdots \wedge dx_n)$ along E_i. The (N_i, ν_i), $i \in T$, are called the numerical data of the resolution (V, g). For $I \subset T$ denote also $E_I := \cap_{i \in I} E_i$ and $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$.

This chapter corresponds to [SV].

The set of germs of holomorphic functions on a neighbourhood of $0 \in \mathbb{C}^n$ will be denoted by $\mathcal{O}_n$.

(2.1.2) To f one associates the local topological zeta function

$$Z_f(s) = Z_{\mathrm{top},0,f}(s) := \sum_{I \subset T} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

Here s is a complex variable and $\chi(\cdot)$ denotes the topological Euler-Poincaré characteristic. The remarkable fact that $Z_f(s)$ does not depend on the chosen resolution was first proved in [DL1] by expressing it as a limit of Igusa's p-adic zeta functions.

(2.1.3) The log canonical threshold $c_0(f)$ of f at $0 \in \mathbb{C}^n$ is by definition the supremum of the set $\{c \in \mathbb{Q} \mid \text{the pair } (\mathbb{C}^n, c \text{ div } f) \text{ is log canonical in a neighbourhood of } 0\}$. We can describe it (see [Ko2, Proposition 8.5]) in terms of the embedded resolution (V, g) as $c_0(f) = \min\{\nu_i/N_i \mid i \in T\}$. In particular, this minimum is independent of the chosen resolution. Consequently, $-c_0(f)$ is the largest candidate pole of $Z_f(s)$. The log canonical threshold has already been studied in various papers of Alexeev, Ein, Kollár, Kuwata, Mustaţă, Prokhorov, Reid, Shokurov and others; in particular, the sets

$$\mathcal{T}_n := \{c_0(f) \mid f \in \mathcal{O}_n\},$$

with $n \in \mathbb{Z}_{>0}$, are the subject of interesting conjectures.

It is natural to investigate whether more quotients $-\nu_i/N_i$, $i \in T$, yield invariants of the germ of f at 0. Of course, the whole set $\{-\nu_i/N_i \mid i \in T\}$ depends on the chosen resolution (for $n = 2$, however, one could consider such a set associated to the minimal resolution), but its subset consisting of the poles of $Z_f(s)$ is an invariant of f. Philosophically, these poles are induced by 'important' components E_i, which occur in every resolution. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}_n$ by

$$\mathcal{P}_n := \{s_0 \mid \exists f \in \mathcal{O}_n \,:\, Z_f(s) \textit{ has a pole in } s_0\}.$$

The case $n = 1$ is trivial: $\mathcal{T}_1 = \{1/i \mid i \in \mathbb{Z}_{>0}\}$ and $\mathcal{P}_1 = \{-1/i \mid i \in \mathbb{Z}_{>0}\}$.

(2.1.4) When $n = 2$, it is known that $\mathcal{T}_2 \cap]5/6, 1[= \emptyset$ (see [Ku1] or [Re]). Because it follows from [Ve5] that $-c_0(f)$ is a pole (and thus the largest pole) of $Z_f(s)$, the statement $\mathcal{P}_2 \cap]-1, -5/6[= \emptyset$ would be a remarkable

generalization. It is in fact not hard to prove this generalization. In this article, we will prove more:

$$\begin{aligned} \mathcal{P}_2 \cap]-\infty, -1/2[&= \{-1/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \\ &= \{-1, -5/6, -3/4, -7/10, \ldots\}. \end{aligned} \tag{2.1}$$

We will also see that -1 is the only possible pole less than $-1/2$ if f has multiplicity at least four.

(2.1.5) Kollár proved in [Ko1] that $\mathcal{T}_3 \cap]41/42, 1[= \emptyset$. It turns out that there is no analogous result for $\mathcal{P}_3$. In fact, we will give examples of zeta functions with poles in $]-1, -41/42[$ which are, moreover, arbitrarily near to -1. On the other hand, we prove the analogue of (2.1), which appears to be

$$\mathcal{P}_3 \cap]-\infty, -1[= \{-1 - 1/i \mid i \in \mathbb{Z}_{>1}\}. \tag{2.2}$$

In particular, there are no poles less than $-3/2$. Moreover, if the multiplicity of f is at least three, we will see that there are no poles less than -1. The last part will be the most difficult because we will have to prove that some candidate poles are not poles.

(2.1.6) In general, we expect for $n \geq 2$ that

$$\mathcal{P}_n \cap]-\infty, -(n-1)/2[= \{-(n-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\}.$$

We will see that for $n \geq 2$, it is easy to prove that

$$\mathcal{P}_n \cap]-\infty, -n+1[= \emptyset.$$

2.2 Curves

(2.2.1) We will determine $\mathcal{P}_2 \cap]-\infty, -1/2[$. Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^2$ which satisfies $f(0) = 0$ and which is not identically zero. Let (V, g) be the *minimal* embedded resolution of $f^{-1}\{0\}$. Write $g = g_1 \circ \cdots \circ g_t$ as a composition of blowing-ups g_i, $i \in T_e := \{1, \ldots, t\}$. The exceptional curve of g_i and also the strict transforms of this curve are denoted by E_i. The irreducible components of $f^{-1}\{0\}$ and their strict transforms are denoted by E_j, $j \in T_s$.

(2.2.2) The dual (minimal) embedded resolution graph of $f^{-1}\{0\}$ is obtained as follows. One associates a vertex to each exceptional curve in the minimal embedded resolution (represented by a dot), and to each branch of the strict transform of $f^{-1}\{0\}$ (represented by a circle). One also associates to each intersection an edge, connecting the corresponding vertices. The fact that E_i has numerical data (N_i, ν_i) is denoted by $E_i(N_i, \nu_i)$.

(2.2.3) Let E_i be an exceptional curve and let E_j, $j \in J$, be the components that intersect E_i in V. Set $\alpha_j = \nu_j - (\nu_i/N_i)N_j$ for $j \in J$. Then we have the relation

$$\sum_{j\in J}(\alpha_j - 1) + 2 = 0, \tag{2.3}$$

which was first proved by Loeser in [Lo1], and later more conceptually by the second author in [Ve2].
Suppose that $\alpha_j \neq 0$, which is equivalent to $-\nu_i/N_i \neq -\nu_j/N_j$, for all $j \in J$. Then one computes easily that the contribution $\mathcal{R}$ of E_i to the residue of $Z_f(s)$ at the candidate pole $-\nu_i/N_i$ is

$$\frac{1}{N_i}\left(\chi(\overset{\circ}{E}_{\{i\}}) + \sum_{j\in J}\alpha_j^{-1}\right). \tag{2.4}$$

From (2.3) and (2.4) it follows that $\mathcal{R} = 0$ if J contains one or two elements. This is the easy part of the following theorem. The other part is more difficult and is proved in [Ve5].

(2.2.4) Theorem. *We have that s_0 is a pole of $Z_f(s)$ if and only if $s_0 = -\nu_i/N_i$ for some exceptional curve E_i intersecting at least three times other components, or $s_0 = -1/N_j$ for some irreducible component E_j of the strict transform of $f^{-1}\{0\}$.*

The following lemma is obtained by elementary calculations.

(2.2.5) Lemma. *Suppose that we have blown up k times but we do not yet have an embedded resolution. Let P be a point of the strict transform of $f^{-1}\{0\}$ with multiplicity μ in which we do not yet have normal crossings. Let g_{k+1} be the blowing-up at P.*

(a) Suppose that two exceptional curves E_i and E_j contain P. Then the new candidate pole $-\nu_{k+1}/N_{k+1} = -(\nu_i+\nu_j)/(N_i+N_j+\mu)$ is larger than $\min\{-\nu_i/N_i, -\nu_j/N_j\}$.

(b) Suppose that exactly one exceptional curve E_i contains P and that $\mu \geq 2$. Then E_{k+1} has numerical data $(N_i+\mu, \nu_i+1)$ and $-(\nu_i+1)/(N_i+\mu)$ is between $-1/\mu$ and $-\nu_i/N_i$.

(c) Suppose that exactly one exceptional curve E_i contains P and that $\mu = 1$. Note that the two curves are tangent at P because we do not have normal crossings at P. Let g_{k+2} be the blowing-up at $E_i \cap E_{k+1}$. Because the strict transform of $f^{-1}\{0\}$ does not intersect E_{k+1} after this blowing-up, we do not have to blow up at a point of E_{k+1} anymore. Because E_{k+1} is intersected once, it follows from (2.2.3) that the contribution of E_{k+1} to the residue at the candidate pole $-\nu_{k+1}/N_{k+1}$ is zero. The numerical data of E_{k+2} are $(2N_i+2, 2\nu_i+1)$, and $-(2\nu_i+1)/(2N_i+2)$ is between $-1/2$ and $-\nu_i/N_i$.

(2.2.6) Suppose that after some blowing-ups, we do not have normal crossings at a point P. Suppose also that the candidate poles associated to the exceptional curves through P are all larger than or equal to $-1/2$. Then it follows from the above lemma that the components above P in the final resolution do not give a contribution to a pole less than $-1/2$.

Corollary. *Zeta functions of singularities of multiplicity at least four do not have a pole in $]-\infty, -1/2[\backslash\{-1\}$.*

Indeed, every exceptional curve in the minimal embedded resolution of $f^{-1}\{0\}$ lies above a point of E_1 (considered in the stage when it is created), which has a candidate pole larger than or equal to $-1/2$.

(2.2.7) To deal with multiplicity two and three, we will study an 'easier' element of $\mathcal{O}_2$ with the same zeta function. We mention two methods to obtain a simple function with this property.

METHOD 1. (See [Ku2]) Let $f \in \mathcal{O}_n$ have multiplicity d and let f_d be the homogeneous part of degree d in the Taylor series of f. Let $N \in \mathbb{Z}_{>d}$. Take a maximal set $\mathcal{V}$ of homogeneous polynomials of degree larger than d and at most N which are linearly independent in the quotient vector space $\mathcal{O}_n/(\partial f_d/\partial x_1, \ldots, \partial f_d/\partial x_n)$. Then f is holomorphically equivalent to $f_d + \sum_{u_i \in \mathcal{V}} a_i u_i + \varphi$, for some $a_i \in \mathbb{C}$ and some $\varphi \in \mathcal{O}_n$ which satisfies $\mathrm{mult}(\varphi) > N$.

Remark. (i) The similar statement in [Ku2, Lemma 3.2] is not correct; some homogeneity condition is needed.

(ii) Let $f^{-1}\{0\}$ have an isolated singularity at the origin and suppose that we have an embedded resolution which is an isomorphism outside this singularity. Then φ does not influence the embedded resolution and the numerical data if N is big enough. Consequently, to calculate the local topological zeta function, we can omit φ if we take N big enough. Note that when $n = 2$, $f^{-1}\{0\}$ has an isolated singularity at 0 if and only if f is reduced and $\text{mult}(f) \geq 2$.

METHOD 2. *(Weierstrass Preparation Theorem)* If $f(z_1, \ldots, z_{n-1}, w) = f(z, w) \in \mathcal{O}_n$ is not identically zero on the w-axis, then f can be written uniquely as $f = (w^e + a_1(z)w^{e-1} + \cdots + a_e(z))h$, where $a_i(z) \in \mathcal{O}_{n-1}$ satisfies $a_i(0) = 0$ and $h \in \mathcal{O}_n$ satisfies $h(0) \neq 0$.
Because $h(0) \neq 0$, the resolutions and the local topological zeta functions of f and $w^e + a_1(z)w^{e-1} + \cdots + a_e(z)$ are the same. After an appropriate coordinate transformation, the desired form will appear. For example, the coordinate transformation $(z, w) \mapsto (z, w - a_1(z)/e)$ cancels the term $a_1(z)w^{e-1}$.

(2.2.8) *Example.* Let $f \in \mathcal{O}_2$ have multiplicity three and let $f_3 = y^3 + xy^2 = y^2(y + x)$. First we illustrate method 1. Because $\partial f_3/\partial x = y^2$ and $\partial f_3/\partial y = 3y^2 + 2xy$, we get $(\partial f_3/\partial x, \partial f_3/\partial y) = (y^2, xy)$. Therefore, we set $\mathcal{V} = \{x^4, x^5, \ldots, x^N\}$, and we obtain that f is holomorphically equivalent to a function of the form $y^3 + xy^2 + a_4x^4 + \cdots + a_Nx^N + g(x, y)$, with $\text{mult}(g(x, y)) > N$. If all a_i can be taken zero for every N, then f is holomorphically equivalent to $y^3 + xy^2$. If there exists an a_i different from zero, which is the reduced case, the form above will allow us to calculate the local topological zeta function of f.

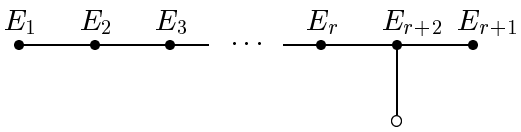
Now we illustrate method 2. By the Weierstrass Preparation Theorem, we may work with a function of the form $y^3 + a_1(x)y^2 + a_2(x)y + a_3(x)$, with $\text{mult}(a_1(x)) = 1$, $\text{mult}(a_2(x)) \geq 3$ and $\text{mult}(a_3(x)) \geq 4$. One can check that there exists a coordinate transformation $(x, y) \mapsto (x, y - k(x))$ such that the function becomes of the form $y^3 + b_1(x)y^2 + b_3(x)$, with $\text{mult}(b_1(x)) = 1$ and $\text{mult}(b_3(x)) \geq 4$. After another coordinate transformation, we get the form $y^3 + xy^2 + g(x)$, with $\text{mult}(g(x)) \geq 4$.

(2.2.9) Theorem. *We have*

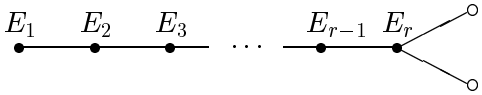
$$\mathcal{P}_2 \cap \left]-\infty, -\frac{1}{2}\right[= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}$$

and every local topological zeta function has at most one pole in the interval $]-1, -1/2]$.

Proof. (a) Suppose that $\mathrm{mult}(f)$, the multiplicity of f at the origin of $\mathbb{C}^2$, is equal to two. Then f is holomorphically equivalent to y^2 or $y^2 + x^k$ for some $k \in \mathbb{Z}_{>1}$. If it is y^2, the only pole of $Z_f(s)$ is $-1/2$. If $k = 2$, the only pole of $Z_f(s)$ is -1. If k is odd, write $k = 2r + 1$. After r blowing-ups, the strict transform of $f^{-1}\{0\}$ is non-singular and tangent to E_r. The numerical data of E_i, $i = 1, \ldots, r$, are $(2i, i+1)$. To get the minimal embedded resolution, we now blow up twice. The dual resolution graph is



and the numerical data are $E_1(2,2)$, $E_2(4,3)$, $E_3(6,4)$, $\ldots$, $E_r(2r, r+1)$, $E_{r+1}(2r+1, r+2)$ and $E_{r+2}(4r+2, 2r+3)$.
If k is even and larger than 2, write $k = 2r$. By easy calculations we obtain the dual resolution graph



and the numerical data $E_1(2,2)$, $E_2(4,3)$, $E_3(6,4)$, $\ldots$, $E_{r-1}(2r-2, r)$ and $E_r(2r, r+1)$.
Because $-(2r+3)/(4r+2) = -1/2 - 1/(2r+1)$ and $-(r+1)/(2r) = -1/2 - 1/(2r)$, it follows from (2.2.4) that

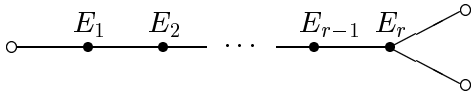
$$\begin{aligned} &\{s_0 \mid \exists f \in \mathcal{O}_2 : \mathrm{mult}(f) = 2 \text{ and } Z_f(s) \text{ has a pole in } s_0\} \\ &\quad = \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\} \cup \left\{ -\frac{1}{2} \right\}. \end{aligned}$$

Note that Newton polyhedra could also be used to deal with (a), see [DL1].

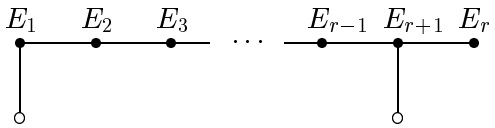
(b) Suppose that $\mathrm{mult}(f) = 3$. Up to an affine coordinate transformation, there are three cases for f_3.

(b.1) Case $f_3 = xy(x+y)$. After one blowing-up we get an embedded resolution. The poles of $Z_f(s)$ are -1 and $-2/3 = -1/2 - 1/6$.

(b.2) Case $f_3 = y^2(y+x)$. According to example 2.2.8, we may suppose that $f = y^3 + xy^2 + g(x)$, where $g(x)$ is a holomorphic function in the variable x of multiplicity $k \geq 4$. If $g(x) = 0$, the poles of $Z_f(s)$ are -1 and $-1/2$. Now consider the case when k is odd. Write $k = 2r+1$. After r blowing-ups we get an embedded resolution with dual resolution graph



and with numerical data $E_1(3,2)$, $E_2(5,3)$, $\ldots$, $E_{r-1}(2r-1,r)$ and $E_r(2r+1,r+1)$.
If k is even, write $k = 2r$. After $r+1$ blowing-ups we get



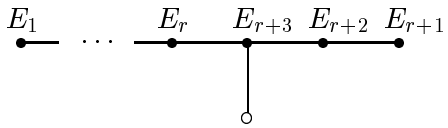
and $E_1(3,2)$, $E_2(5,3)$, $E_3(7,4)$, $\ldots$, $E_{r-1}(2r-1,r)$, $E_r(2r,r+1)$ and $E_{r+1}(4r,2r+1)$.
The poles appearing in (b.2) are in the desired set because $-(r+1)/(2r+1) = -1/2 - 1/(4r+2)$ and $-(2r+1)/(4r) = -1/2 - 1/(4r)$.

(b.3) Case $f_3 = y^3$. We may suppose that f is of the form

$$y^3 + a_4x^4 + b_3yx^3 + a_5x^5 + b_4yx^4 + a_6x^6 + b_5yx^5 + \cdots,$$

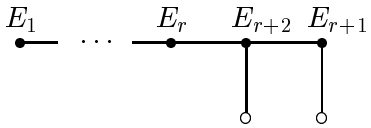
where $a_i, b_i \in \mathbb{C}$. If $f = f_3 = y^3$ then the only pole of $Z_f(s)$ is $-1/3$. Otherwise there is an integer $r \geq 1$ such that after blowing up r times and always taking the charts determined by $g_i(x,y) = (x, xy)$, we get $(g_1 \circ \cdots \circ g_r)^* dx \wedge dy = x^r dx \wedge dy$ and $f \circ g_1 \circ \cdots \circ g_r = x^{3r}(y^3 + a_{3r+1}x + b_{2r+1}yx + a_{3r+2}x^2 + b_{2r+2}yx^2 + a_{3r+3}x^3 + \cdots)$, with $a_{3r+1}, b_{2r+1}, a_{3r+2}, b_{2r+2}$ and a_{3r+3} not all zero. The equation of E_r in this chart is $x = 0$ and the numerical data of E_r are $(3r, r+1)$. The zero locus of $y^3 + a_{3r+1}x + b_{2r+1}yx + a_{3r+2}x^2 + b_{2r+2}yx^2 + a_{3r+3}x^3 + \cdots$ is the strict transform of $f^{-1}\{0\}$. Note that it intersects only E_r at this stage.

(b.3.i) If $a_{3r+1} \neq 0$, we obtain after blowing up three more times the dual resolution graph



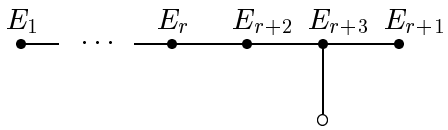
and the numerical date $E_r(3r, r+1)$, $E_{r+1}(3r+1, r+2)$, $E_{r+2}(6r+2, 2r+3)$ and $E_{r+3}(9r+3, 3r+4)$. The pole $-(3r+4)/(9r+3)$ is in the interval $]-\infty, -1/2]$ if and only if $r = 1$, and in this case the pole is equal to $-1/2 - 1/12$.

(b.3.ii) If $a_{3r+1} = 0$ and $b_{2r+1} \neq 0$, calculations give us



and the numerical date $E_r(3r, r+1)$, $E_{r+1}(3r+2, r+2)$ and $E_{r+2}(6r+3, 2r+3)$. The pole $-(2r+3)/(6r+3)$ is in the interval $]-\infty, -1/2]$ if and only if $r = 1$, and in this case the pole is equal to $-1/2 - 1/18$.

(b.3.iii) If $a_{3r+1} = b_{2r+1} = 0$ and $a_{3r+2} \neq 0$, we get the dual resolution graph



and the numerical data $E_r(3r, r+1)$, $E_{r+1}(3r+2, r+2)$, $E_{r+2}(6r+3, 2r+3)$ and $E_{r+3}(9r+6, 3r+5)$. The pole $-(3r+5)/(9r+6)$ is in the interval $]-\infty, -1/2]$ if and only if $r = 1$, and in this case the pole is equal to $-1/2 - 1/30$.

(b.3.iv) The last case is $a_{3r+1} = b_{2r+1} = a_{3r+2} = 0$ and ($b_{2r+2} \neq 0$ or $a_{3r+3} \neq 0$).
If $y^3 + b_{2r+2}yx^2 + a_{3r+3}x^3$ is a product of three distinct linear factors, we get an embedded resolution after one blowing-up. The numerical data of E_{r+1} are $(3r+3, r+2)$ and $-(r+2)/(3r+3) \notin]-\infty, -1/2[$.
If $y^3 + b_{2r+2}yx^2 + a_{3r+3}x^3$ is not a product of three distinct linear factors, then it is equal to $y^3 + xy^2$ after an affine coordinate transformation that does not change the equation $x = 0$ of E_r. Let g_{r+1} be the blowing-up

at the origin of the chart we consider. The strict transform of $f^{-1}\{0\}$ only intersects the exceptional curve E_{r+1}, which has numerical data $(3r+3, r+2)$. Because $-(r+2)/(3r+3) \geq -1/2$ for all r, it follows from (2.2.4) and (2.2.6) that $Z_f(s)$ has no pole in $]-\infty, -1/2[$ different from -1.

(c) Suppose that $\mathrm{mult}(f) \geq 4$. We explained in (2.2.6) that $Z_f(s)$ has no pole in $]-\infty, -1/2[$ different from -1. $\square$

(2.2.10) We now present a similar result for the following generalized zeta functions [DL1]. The case $d = 2$ is used in the next section. To $f \in \mathcal{O}_n$ and $d \in \mathbb{Z}_{>0}$ one associates the local topological zeta function

$$Z_f^{(d)}(s) = Z_{\mathrm{top},0,f}^{(d)}(s) := \sum_{\substack{I \subset T \\ \forall i \in I \,:\, d \mid N_i}} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i}.$$

For $n, d \in \mathbb{Z}_{>0}$, we set

$$\mathcal{P}_n^{(d)} := \{s_0 \mid \exists f \in \mathcal{O}_n \,:\, Z_f^{(d)}(s) \text{ has a pole in } s_0\}.$$

Consequently, $Z_f(s) = Z_f^{(1)}(s)$ and $\mathcal{P}_n = \mathcal{P}_n^{(1)}$.

(2.2.11) Let E_i be an exceptional curve and let E_j, $j \in J$, be the components that intersect E_i in V. Then

$$\sum_{j \in J} N_j \equiv 0 \pmod{N_i}, \tag{2.5}$$

see, e.g., [Lo1] or [Ve3]. Fix $d \in \mathbb{Z}_{>0}$ and suppose that $d \mid N_i$. Let $J_d \subset J$ be the subset of indices j satisfying $d \mid N_j$. Suppose that $\alpha_j := \nu_j - (\nu_i/N_i)N_j$ is different from 0 for all $j \in J_d$. Then the contribution $\mathcal{R}$ of E_i to the residue of $Z_f^{(d)}(s)$ at the candidate pole $-\nu_i/N_i$ is

$$\frac{1}{N_i}\left(\chi(\overset{\circ}{E}_{\{i\}}) + \sum_{j \in J_d} \alpha_j^{-1}\right). \tag{2.6}$$

This contribution is zero if J contains one or two indices. Indeed, if J contains one element, relation (2.5) implies that $J = J_d$. Therefore, the contribution $\mathcal{R}$ is the same as in the case $d = 1$ and by (2.2.3) we get

$\mathcal{R} = 0$. If J contains two elements, relation (2.5) implies that $J_d = J$ or $J_d = \emptyset$. If $J_d = J$, we obtain $\mathcal{R} = 0$ analogously as in the previous case. If $J_d = \emptyset$, we get $\mathcal{R} = 0$ because the Euler-Poincaré characteristic of a projective line minus two points is zero.

(2.2.12) Theorem. *Let $f \in \mathcal{O}_2$ and $d \in \mathbb{Z}_{>1}$. Then*

$$\mathcal{P}_2^{(d)} \cap \left]-\infty, -\frac{1}{2}\right[= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>2} \text{ and } d|\mathrm{lcm}(2,i) \right\}.$$

Proof. We use the notation and the calculations of (2.2.9). First we consider the candidate pole $-\nu_j/N_j$, $j \in T_s$. Because E_j has numerical data $(N_j, 1)$, the candidate pole $-\nu_j/N_j$ is less than $-1/2$ if and only if $N_j = 1$, and in this case $d \nmid N_j$.

(a) Suppose that $\mathrm{mult}(f) = 2$. So f is holomorphically equivalent to (i) y^2 or (ii) $y^2 + x^k$ for some $k \in \mathbb{Z}_{>1}$. In case (i) with $d = 2$ the only pole of $Z_f^{(d)}(s)$ is $-1/2$. In case (i) with $d > 2$ and in case (ii) for $k = 2$, we obtain that $Z_f^{(d)}(s)$ is identically zero.
If $k = 2r + 1$, we obtain from (2.2.11) and the consideration above about E_j, $j \in T_s$, that only the candidate pole associated to E_{r+2} can be a pole. So we have to compute the residue at the candidate pole $-1/2-1/(2r+1)$. If $d \nmid N_{r+2}$, then $\mathcal{R} = 0$. If $d|N_r$, $d|N_{r+1}$ and $d|N_{r+2}$, then $d = 1$ and we have a contradiction. If $d|N_r$, $d|N_{r+2}$ and $d \nmid N_{r+1}$ (which is equivalent to $d = 2$), then the contribution $\mathcal{R}$ is $r/(2r+1)$ and this is not zero because $r \geq 1$. If $d|N_{r+1}$, $d|N_{r+2}$ and $d \nmid N_r$, then $\mathcal{R} = 1/(4r+2)$. If $d|N_{r+2}$, $d \nmid N_r$ and $d \nmid N_{r+1}$, then $\mathcal{R} = -1/(4r+2)$. We conclude that $-1/2 - 1/(2r+1)$ is a pole of $Z_f^{(d)}(s)$ if and only if $d|4r+2$.
If $k = 2r$, $r \geq 2$, only the candidate pole associated to E_r can be a pole. If $d \nmid N_r$, then $\mathcal{R} = 0$. If $d|N_r$ and $d|N_{r-1}$, then $\mathcal{R} = (r-1)/(2r) \neq 0$. If $d|N_r$ and $d \nmid N_{r-1}$, then $\mathcal{R} = -1/(2r)$. Consequently, $-1/2 - 1/(2r)$ is a pole of $Z_f^{(d)}(s)$ if and only if $d|2r$.
Remark that we have proved that

$$\begin{aligned} &\{s_0 \mid \exists f \in \mathcal{O}_2 : \mathrm{mult}(f) = 2 \text{ and } Z_f^{(d)}(s) \text{ has a pole in } s_0\} \setminus \{-1/2\} \\ &\qquad = \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>2} \text{ and } d \mid \mathrm{lcm}(2,i) \right\}. \end{aligned}$$

(b) Suppose that $\mathrm{mult}(f) = 3$.

(b.1) Case $f_3 = xy(x+y)$. We get that $-1/2 - 1/6$ is a pole if and only if $d|3$ (which is equivalent to $d = 3$). This is consistent with the claim because $3 \mid \mathrm{lcm}(2,6)$.

(b.2) Case $f_3 = y^2(x+y)$. If $g(x) = 0$, then $Z_f^{(d)}(s)$ is identically zero (for every $d \geq 2$).
If $k = 2r+1$, only the candidate pole associated to E_r can be a pole. If $d|N_r$, then $d \nmid N_{r-1}$ because $d > 1$ and because N_r and N_{r-1} are odd numbers with difference 2. Consequently, if $d|2r+1$, then $\mathcal{R} = -1/(2r+1) \neq 0$.
If $k = 2r$, the only candidate pole which can be a pole is $-\nu_{r+1}/N_{r+1}$. If $d|N_{r+1}$ and $d|N_{r-1}$, then $d = 1$, which is a contradiction. So we have to consider two cases. If $d|N_{r+1}$, $d \nmid N_r$ and $d \nmid N_{r-1}$, then $\mathcal{R} = -1/(4r)$. If $d|N_{r+1}$, $d|N_r$ and $d \nmid N_{r-1}$, then $\mathcal{R} = 1/(4r)$. We obtain that $-1/2 - 1/(4r)$ is a pole if and only if $d|4r$.

(b.3) Case $f_3 = y^3$. We get analogously that if we have a pole, it is of the desired form. Remark that we only have to consider the case $r = 1$ in (b.3.i), (b.3.ii) and (b.3.iii).

(c) Suppose that $\mathrm{mult}(f) \geq 4$. As before we get that $Z_f^{(d)}(s)$ has no pole less than $-1/2$. $\square$

2.3 Surfaces

In this section, we prove the following theorem.

(2.3.0) Theorem. *We have*

$$\mathcal{P}_3 \cap]-\infty, -1[= \left\{ -1 - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}.$$

Moreover, if $f \in \mathcal{O}_3$ has multiplicity three or more, then $Z_f(s)$ has no pole less than -1.

Remark. (i) It is a priori not obvious that the smallest value of $\mathcal{P}_3$ is $-3/2$. This is in contrast to the fact that it easily follows from Lemma 2.2.5 that the smallest value of $\mathcal{P}_2$ is -1.

(ii) In (2.3.3.9) we give functions $f_k \in \mathcal{O}_3$ of arbitrary multiplicity such that $Z_{f_k}(s)$ has a pole in s_k, where $(s_k)_k$ is a sequence of real numbers larger than -1 and converging to -1. In particular, $\mathcal{P}_3 \cap]-1, -41/42[\neq \emptyset$, which is in contrast to $\mathcal{T}_3 \cap]41/42, 1[= \emptyset$.

2.3.1 On candidate poles which are not poles

(2.3.1.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^3$ which satisfies $f(0) = 0$ and which is not identically zero. Let Y be the zero set of f. Fix an embedded resolution $g : X_t \to X_0 \subset \mathbb{C}^3$ for Y which is an isomorphism outside the singular locus of Y and which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : X_i \to X_{i-1}$ with irreducible non-singular centre D_{i-1} and exceptional variety $E_i^{(0)}$ satisfying for $i = 0, \ldots, t-1$:

(a) the codimension of D_i in X_i is at least two;

(b) D_i is a subset of the strict transform of Y under $g_1 \circ \cdots \circ g_i$;

(c) the union of the exceptional varieties of $g_1 \circ \cdots \circ g_i$ has only normal crossings with D_i, i.e., for all $P \in D_i$, there are three surface germs through P which are in normal crossings such that each exceptional surface germ through P is one of them and such that the germ of D_i at P is the intersection of some of them;

(d) the origin 0 of $\mathbb{C}^3$ is an element of $(g_1 \circ \cdots \circ g_i)D_i$; and

(e) D_i contains a point in which $(g_1 \circ \cdots \circ g_i)^{-1}Y$ has not normal crossings.

Note that such a resolution always exists by Hironaka's theorem [Hi].

(2.3.1.2) Fix an exceptional variety $E_i^{(0)}$. The strict transform E_i of $E_i^{(0)}$ in X_t is obtained by a finite succession of blowing-ups h_j, $j \in T_e := \{1, \ldots, m\}$,

$$E_i^{(0)} \xleftarrow{h_1} E_i^{(1)} \xleftarrow{h_2} \cdots E_i^{(j-1)} \xleftarrow{h_j} E_i^{(j)} \cdots \xleftarrow{h_{m-1}} E_i^{(m-1)} \xleftarrow{h_m} E_i^{(m)} = E_i$$

with centre $P_{j-1} \in E_i^{(j-1)}$ and exceptional curve $C_j^{(j)}$. The irreducible components of the intersection of $E_i^{(0)}$ with irreducible components of $(g_1 \circ \cdots \circ g_i)^{-1}Y$ different from $E_i^{(0)}$ are denoted by $C_j^{(0)}$, $j \in T_s$. The strict transform of $C_j^{(k)}$ in $E_i^{(l)}$ is denoted (whenever this makes sense) by $C_j^{(l)}$ and we set $C_j = C_j^{(m)}$. Note that $h := h_1 \circ \cdots \circ h_m$ is an embedded resolution of $\cup_{j \in T_s} C_j^{(0)}$. For each $j \in T := T_s \cup T_e$ the curve C_j is an irreducible component of the intersection of E_i with exactly one other component of $g^{-1}Y$. Let this component have numerical data (N_k, ν_k) and set $\alpha_j = \nu_k - (\nu_i/N_i)N_k$.

(2.3.1.3) Suppose that $E_i^{(0)} \subset (g_1 \circ \cdots \circ g_i)^{-1}\{0\}$ and that $\alpha_j \neq 0$ for every $j \in T$. The contribution $\mathcal{R}$ of E_i to the residue of $Z_f(s)$ at the candidate pole $-\nu_i/N_i$ is

$$\frac{1}{N_i}\left(\sum_{I\subset T} \chi(\overset{\circ}{C}_I)\prod_{j\in I}\alpha_j^{-1}\right), \tag{2.7}$$

where $\overset{\circ}{C}_I$ denotes the subset $(\cap_{j\in I}C_j)\setminus(\cup_{j\notin I}C_j)$ of E_i. Note that $\overset{\circ}{C}_\emptyset = E_i \setminus (\cup_{j\in T}C_j)$. We now state some relations between the α_j, which will allow us to prove that this contribution is identically zero (i.e. zero for any value of the alphas) for many intersection configurations on $E_i^{(0)}$.

(2.3.1.4) To the creation of $E_i^{(0)} \subset (g_1 \circ \cdots \circ g_i)^{-1}\{0\}$ in the resolution process we associate the relation

$$\sum_{j\in T_s} d_j(\alpha_j - 1) + 3 - \dim D_{i-1} = 0, \tag{2.8}$$

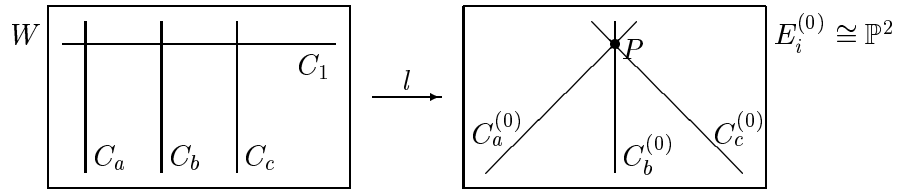
where d_j, $j \in T_s$, is the degree of the intersection cycle $C_j^{(0)} \cdot F$ on F for a general fibre F of $g_i|_{E_i^{(0)}} : E_i^{(0)} \to D_{i-1}$ over a point of D_{i-1}. In particular, when D_{i-1} is a point, we have that $E_i^{(0)} \cong \mathbb{P}^2$ and that d_j is just the degree of the curve $C_j^{(0)}$. To the blowing-up h_j we associate the relation

$$\alpha_j = \sum_{k\in T_s\cup\{1,\ldots,j-1\}} \mu_k(\alpha_k - 1) + 2, \tag{2.9}$$

where μ_k, $k \in T_s \cup \{1,\ldots,j-1\}$, is the multiplicity of P_{j-1} on $C_k^{(j-1)}$. See [Ve2] for more general statements in arbitrary dimensions and for proofs.

(2.3.1.5) Now we proceed in the same way as in [Ve4] for Igusa's p-adic zeta function. One can easily verify that the number (2.7) does not change when we do an extra blowing-up h_{m+1} at a point $P_m \in E_i^{(m)}$ and when we associate to the new exceptional curve a number α using (2.9). Because of this observation, one can compute $\mathcal{R}$ if one has the curves $C_j^{(0)}$, $j \in T_s$, on $E_i^{(0)}$ together with the associated values α_j as follows. Compute the *minimal* embedded resolution of $\cup_{j\in T_s}C_j^{(0)}$ and compute the alpha associated to an exceptional curve using (2.9). By putting these data in (2.7), we get $\mathcal{R}$.

(2.3.1.6) *Example.* Suppose that $E_i^{(0)}$ is the exceptional variety of a blowing-up at a point and suppose that the intersection configuration on $E_i^{(0)}$ consists of three projective lines $C_j^{(0)}$, $j \in T_s := \{a, b, c\}$, all passing through the same point P. Suppose that $\alpha_j \neq 0$ for all $j \in T$. The minimal embedded resolution $l : W \to E_i^{(0)}$ is the blowing-up at P. By abuse of notation, we denote the exceptional curve by C_1 and the strict transform of $C_j^{(0)}$, $j \in T_s$, by C_j.



By relations (2.8) and (2.9) we have $\alpha_a + \alpha_b + \alpha_c = 0$ and $\alpha_1 = \alpha_a + \alpha_b + \alpha_c - 1 = -1$, respectively. Now we can calculate the contribution $\mathcal{R}$ of the strict transform of $E_i^{(0)}$ in X_t to the residue of $Z_f(s)$ at the candidate pole $-\nu_i/N_i$:

$$\begin{aligned}\mathcal{R} &= \frac{1}{N_i}\left(\sum_{I\subset T}\chi(\overset{\circ}{C}_I)\prod_{j\in I}\alpha_j^{-1}\right)\\ &= \frac{1}{N_i}\left(-1-\frac{1}{\alpha_1}+\frac{1}{\alpha_a}+\frac{1}{\alpha_b}+\frac{1}{\alpha_c}+\frac{1}{\alpha_1\alpha_a}+\frac{1}{\alpha_1\alpha_b}+\frac{1}{\alpha_1\alpha_c}\right)\\ &= 0.\end{aligned}$$

We stress that $\mathcal{R}$ is zero for any possible values of α_a, α_b and α_c.

2.3.2 Multiplicity two

(2.3.2.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^n$ which satisfies $f(0) = 0$, and let F be the germ of the holomorphic function $f + x_{n+1}^2$ on a neighbourhood of the origin 0 in $\mathbb{C}^{n+1}$. Then the following equality is obtained in [ACLM], see also the Thom-Sebastiani principle in [DL3]:

$$Z_F(s) = \frac{1}{2s+1} + \frac{s(2s+3)}{2(s+1)(2s+1)}Z_f\left(s+\frac{1}{2}\right) - \frac{3s}{2(s+1)}Z_f^{(2)}\left(s+\frac{1}{2}\right).$$

(2.3.2.2) Proposition. *The set*

$$\{s_0 \mid \exists f \in \mathcal{O}_3 : \text{mult}(f) = 2 \text{ and } Z_f(s) \text{ has a pole in } s_0\} \cap]-\infty, -1[$$

is equal to

$$\left\{ -1 - \frac{1}{i} \middle| \, i \in \mathbb{Z}_{>1} \right\}.$$

Proof. Let f be an element of $\mathcal{O}_3$ with multiplicity two. Up to an affine coordinate transformation, the part of degree two in the Taylor series of f is equal to x^2, x^2+y^2 or $x^2+y^2+z^2$. Using (2.2.7), we may suppose that f is of the form $x^2 + g(y,z)$ with $g(y,z) \in \mathcal{O}_2$. The formula in (2.3.2.1) and the result for curves imply that every pole of $Z_f(s)$ less than -1 is of the form $-1-1/i$, $i \in \mathbb{Z}_{>1}$. For the other inclusion, we remark that the poles of the local topological zeta function associated to $x^2+y^2+z^i$, $i \geq 2$, are $-1-1/i$ and -1. $\square$

(2.3.2.3) Our next goal is to give a sequence of poles larger than -1 and converging to -1. Keeping in mind the formula in (2.3.2.1), we try to find functions $f_k \in \mathcal{O}_2$ such that $Z_{f_k}(s)$ has a pole in s_k, where $(s_k)_k$ is a sequence of real numbers larger than $-1/2$ and converging to $-1/2$. Set $f_k = x^3y^2 + x^k$ for $k \geq 5$.
We obtain the following equalities after some calculations:

$$\begin{aligned}
Z_{f_{2r+4}}(s) &= \frac{3s^2+2rs+8s+2r+3}{(4rs+8s+2r+3)(3s+1)(s+1)}, \\
Z^{(2)}_{f_{2r+4}}(s) &= \frac{1}{4rs+8s+2r+3}, \\
Z_{f_{2r+3}}(s) &= \frac{3s^2-rs-2s-r-1}{(2rs+3s+r+1)(3s+1)(s+1)}, \\
Z^{(2)}_{f_{2r+3}}(s) &= 0.
\end{aligned}$$

Now we use the formula in (2.3.2.1) to calculate the local topological zeta function of $F_k := f_k + z^2$. We obtain for even and odd k that

$$Z_{F_k}(s) = \frac{(6k-6)s^2 + (15k-5)s + 10k - 5}{(6s+5)(s+1)(2ks+2k-1)}.$$

Finally, we make the substitution $s = -(2k-1)/(2k)$ in the numerator in order to check that this value, which converges to -1 if k goes to infinity,

is a pole. We obtain

$$\frac{(k-1)(k-3)(2k-1)}{2k^2}.$$

This value never becomes zero because $k \geq 5$. Therefore, $-(2k-1)/(2k)$ is always a pole of $Z_{F_k}(s)$.

Remark. In particular, we obtain that $\mathcal{P}_3 \cap]-1, -41/42[\neq \emptyset$, which is in contrast to $\mathcal{T}_3 \cap]41/42, 1[= \emptyset$.

2.3.3 Multiplicity larger than two

(2.3.3.1) Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^3$ which satisfies $f(0) = 0$ and which is not identically zero. Let Y be the zero set of f. Fix an embedded resolution g for Y which is a composition of blowing-ups $g_{ij} : X_i \to X_j$ with irreducible non-singular centre D_j and exceptional surface E_i as in (2.3.1.1). Denote the irreducible components of Y by E_i, $i \in T_s$. The strict transform of a variety E_i by a succession of blowing-ups will be denoted in the same way. The numerical data of E_i are (N_i, ν_i).

(2.3.3.2) The following table gives the numerical data of E_i. In the columns, the dimension of D_j is kept fixed. In the rows, the number of exceptional surfaces through D_j is kept fixed. So E_k, E_l and E_m represent exceptional surfaces that contain D_j. The multiplicity of D_j on the strict transform of Y is denoted by μ_{D_j}.

	D_j is a point P	D_j is a curve L
/	$(\mu_P, 3)$	$(\mu_L, 2)$
E_k	$(N_k + \mu_P, \nu_k + 2)$	$(N_k + \mu_L, \nu_k + 1)$
E_k and E_l	$(N_k + N_l + \mu_P, \nu_k + \nu_l + 1)$	$(N_k + N_l + \mu_L, \nu_k + \nu_l)$
E_k, E_l and E_m	$(N_k + N_l + N_m + \mu_P, \nu_k + \nu_l + \nu_m)$	/

(2.3.3.3) Lemma. *Suppose that* $\mathrm{mult}(f) \geq 3$. *If there is no exceptional surface through* D_j, *then* $-\nu_i/N_i \geq -1$.

Proof. The case that the centre D_j is a point P through which no exceptional surface passes can only occur in the first blowing-up because of condition (d) in (2.3.1.1) and because the inverse image of 0 in X_j is contained in the union of the exceptional surfaces in X_j. Since $\mathrm{mult}(f) \geq 3$,

we have in this case $-\nu_i/N_i = -3/\mu_P = -3/\mathrm{mult}(f) \geq -1$.
If the centre D_j is a curve L contained in no exceptional surface, then $\mu_L \geq 2$ because our embedded resolution is an isomorphism outside the singular locus of Y. Consequently, we get in this case $-\nu_i/N_i = -2/\mu_L \geq -1$. $\square$

(2.3.3.4) Suppose that D_j is contained in at least one exceptional surface and that the candidate poles associated to the exceptional surfaces that pass through D_j are larger than or equal to -1. Then the table in (2.3.3.2) implies that also $-\nu_i/N_i \geq -1$, unless D_j is a non-singular point P of the strict transform of Y through which only one exceptional surface E_0 passes and $-\nu_0/N_0 = -1$. Suppose that we are in this situation. Denote the unique irreducible component of the strict transform of Y which passes through P by E_a. Consider now a small enough neighbourhood Z_0 of P on which E_a is non-singular such that, if we restrict the blowing-ups g_{ij} to the inverse image of Z_0, we get an embedded resolution $h = h_1 \circ \cdots \circ h_s$ for the germ of $E_a \cup E_0$ at P which is a composition of blowing-ups $h_i : Z_i \to Z_{i-1}$, $i \in \{1, \ldots, s\}$, with irreducible non-singular centre $D'_{i-1} := D_{i-1} \cap Z_{i-1}$ and exceptional surface $E'_i := E_i \cap Z_i$ satisfying for $i = 0, \ldots, s-1$:

(a) the codimension of D'_i in Z_i is at least two;
(b) D'_i is a subset of $E'_a := E_a \cap Z_i$;
(c) $\cup_{l \in \{0,1,\ldots,i\}} E'_l$ has only normal crossings with D'_i, where $E'_0 := E_0 \cap Z_0$;
(d) the image of D'_i under $h_1 \circ \cdots \circ h_i$ contains P; and
(e) if $D_i = D'_i$, then D_i contains a point where there are not normal crossings.

Note that it can happen that g_{ij} is an isomorphism on the inverse image of Z_0. Because we did not specify the indices in (2.3.3.1), we were able to get a nice notation here. Note also that $D_i = D'_i$ if D_i is a point. From now on, we study the resolution $h : Z_s \to Z_0$ for the germ of $E_a \cup E_0$ at P.

(2.3.3.5) Lemma. *If $D_i = D'_i$, then D_i is a subset of E'_0.*

Proof. Note that D_i has to lie in an exceptional surface because E'_a is non-singular and because an embedded resolution is an isomorphism outside the singular locus of Y.

First we consider the case that $D_i = D_i'$ is a point contained in exceptional surfaces different from E_0' and in the surface E_a'. The union of these surfaces has normal crossings at D_i because E_a', considered as a subset of Z_0, is non-singular. This is in contradiction with (e). Note that it can thus not happen that E_a' and three exceptional surfaces different from E_0' have a point in common.

The case that $D_i = D_i'$ is a curve contained in exactly two exceptional surfaces different from E_0' and in the surface E_a' cannot occur because E_a' is a non-singular subset of Z_0 and therefore these three surfaces should have normal crossings.

Finally we study the case that $D_i = D_i'$ is a curve contained in one exceptional surface E_j' different from E_0' and in E_a'. Condition (c) implies that every point of D_i is contained in at most one exceptional surface different from E_j'. Moreover, such an exceptional surface has to be transversal to D_i. This implies that there are normal crossings at every point of D_i, which is in contradiction with (e). Therefore, this case cannot occur. □

(2.3.3.6) Lemma. *Suppose that* $\mathrm{mult}(f) \geq 3$. *Then we have* $\nu_i \leq N_i + 1$ *for every exceptional surface* E_i, $i \in \{1, \ldots, s\}$. *Moreover,* $\nu_i = N_i + 1$ *if and only if* D_{i-1} *is a point and the numerical data of every exceptional surface* E_j *different from* E_0 *and through* D_{i-1} *satisfy* $\nu_j = N_j + 1$.

Proof. The proof is by induction on i. Since $\nu_0 = N_0$, we have that $\nu_1 = N_1 + 1$. Suppose now that $\nu_j \leq N_j + 1$ for every exceptional surface E_j through D_{i-1}.

Case 1: D_{i-1} is a point. We obtain from (2.3.3.5) that D_{i-1} is a subset of E_0'. Because $\nu_0 = N_0$ and because every other exceptional surface E_j through D_{i-1} satisfies $\nu_j \leq N_j + 1$, the table of (2.3.3.2) gives us that $\nu_i \leq N_i + 1$.

Case 2: D_{i-1} is a curve. If $D_{i-1} \neq D_{i-1}'$, then $D_{i-1}' \not\subset (h_1 \circ \cdots \circ h_{i-1})^{-1}P$ and therefore we get as in the beginning of (2.3.3.4) that $-\nu_i/N_i \geq -1$. If $D_{i-1} = D_{i-1}'$, one computes from (2.3.3.2) and the previous lemma that $-\nu_i/N_i \geq -1$.

We have now proved the first part of the lemma. Using this first part and the table of (2.3.3.2), we get the second part. □

(2.3.3.7) Lemma. *If* $\mathrm{mult}(f) \geq 3$ *and if the numerical data of* E_i *satisfy* $\nu_i = N_i + 1$, *then* $-\nu_i/N_i \neq -\nu_j/N_j$ *for every exceptional surface*

E_j that intersects E_i at some stage of the resolution process.

Proof. Let E_j be an exceptional surface that intersects E_i at some stage of the resolution process. If E_j is created before E_i, then E_j contains the point D_{i-1}. Otherwise, E_j is created by a blowing-up at a point of E_i or by a blowing-up along a curve.

If E_j is created by a blowing-up along a curve, then $-\nu_j/N_j \geq -1$ and, consequently, $-\nu_i/N_i \neq -\nu_j/N_j$. Now we consider the case that E_j contains the point D_{i-1}. There is no problem if $\nu_j \leq N_j$. Consequently, suppose that $\nu_j = N_j + 1$. From the table in (2.3.3.2), we get $N_j < N_i$. Therefore, $-\nu_i/N_i = -(N_i+1)/N_i > -(N_j+1)/N_j = -\nu_j/N_j$. The case that E_j is created by a blowing-up at a point of E_i is treated analogously. □

(2.3.3.8) Proposition. *If* $\mathrm{mult}(f) \geq 3$, *then no pole of* $Z_f(s)$ *is less than* -1.

Proof. Suppose that $\mathrm{mult}(f) \geq 3$.

We have only to consider exceptional surfaces with a candidate pole less than -1. Recall from (2.3.3.6) that $-\nu_i/N_i < -1$ if and only if D_{i-1} is a point and all exceptional surfaces through the point D_{i-1} different from E_0 have a candidate pole less than -1. We will determine all possible intersection configurations on such surfaces just after their creation.

If $-\nu_i/N_i \geq -1$ and $-\nu_{i+1}/N_{i+1} < -1$, then the blowing-ups along D_{i-1} and D_i commute with each other. Therefore, we may assume that there is a k (larger than zero because $-\nu_1/N_1 < -1$) such that $-\nu_i/N_i < -1$ for $1 \leq i \leq k$ and $-\nu_i/N_i \geq -1$ for $k < i \leq s$.

The intersection configuration on E_1 consists of one projective line, which is the intersection with E_0 and E_a. The points of Z_1 in which we do not have normal crossings and which lie above P are those on this projective line. This implies the following statement for $i = 2$:

> If Q is a point of Z_{i-1}, $i \in \{2, \ldots, k\}$, in which we do not have normal crossings and which lies above P (so consequently Q is a point of E_0, of one or two other exceptional surfaces and of E_a), then there exists an exceptional surface E_l through Q with the property $E_0 \cap E_l = E_a \cap E_l$. (∗)

We prove this statement by induction on i. Suppose that it is true for $i = j \in \{2, \ldots, k-1\}$. We give the proof for $i = j + 1$. The statement

follows from the induction hypothesis for points not on E_j, because a blowing-up is an isomorphism outside the exceptional surface. Therefore, we prove it for points on E_j. By the induction hypothesis applied to the point D_{j-1}, we obtain that there exists an exceptional surface E_l through D_{j-1} such that $E_0 \cap E_l = E_a \cap E_l$ in Z_{j-1}. But then $E_0 \cap E_l = E_a \cap E_l$ in Z_j, which solves the problem for the point $E_0 \cap E_l \cap E_j$. There are other points on E_j in which we do not have normal crossings if and only if E_a is tangent to E_0 in D_{j-1}. In this case, the points in which we do not have normal crossings are the points of $E_0 \cap E_j$. Because $E_0 \cap E_j = E_a \cap E_j$, we are done.

Because the centre of a blowing-up satisfies the conditions of the statement, we obtain that the possible intersection configurations are the following configurations of lines in $\mathbb{P}^2$: (i) one line; (ii) two lines; (iii) three lines through one point; (iv) three lines in general position; and (v) three lines through one point and a fourth line not through that point.

For all these configurations, we can calculate as in (2.3.1.6) that the contribution to the residue is 0. The second author has done this already in [Ve4] for Igusa's p-adic zeta function. The point is that $(*)$ excludes the configuration consisting of four lines in general position, for which this contribution is not zero. Note also that we need here that the alphas are not zero, a fact we proved in (2.3.3.7). $\square$

(2.3.3.9) In (2.3.2.3), we found functions $f_k \in \mathcal{O}_3$ of multiplicity two such that $Z_{f_k}(s)$ has a pole in s_k, where $(s_k)_k$ is a sequence of real numbers larger than -1 and converging to -1. Here we construct for *every* $n \geq 0$ functions $f_k \in \mathcal{O}_3$ of multiplicity $n+2$ with this property. We use the formula obtained by Denef and Loeser in [DL1, Théorème 5.3], which expresses the local topological zeta function of a non-degenerated polynomial in terms of its Newton polyhedron. Fix $n \geq 0$ and set $f_k = x^n z^2 + x^{3+n} y^2 + x^k$ for $k \geq n+4$. Then

$$Z_{f_k}(s) = \frac{\begin{array}{c}(-2n^2-6n)s^3 + (n^2+3kn-4n+6k-6)s^2 \\ +(-4n^2+4kn-7n+15k-5)s - 10n + 10k - 5\end{array}}{(6s+2ns+5)(s+1)(2ks+2k-2n-1)(ns+1)}.$$

Consequently, $-(2k-2n-1)/(2k)$ is a pole if and only if it is not a zero of the numerator. So we make the substitution $s = -(2k-2n-1)/(2k)$ in the numerator and obtain

$$\frac{(k-1-2n)(k-n-3)(2k-2n-1)(2n^2-2kn+n+2k)}{4k^3}.$$

Because $k \geq n+4$, this is zero if and only if $k = 1+2n$. Thus we have found for any multiplicity larger than one a sequence with the desired property.

2.4 Arbitrary dimension

(2.4.0) Theorem. *Let $n \geq 2$. Then we have*

$$\mathcal{P}_n \cap]-\infty, -n+1[= \emptyset.$$

Proof. Let f be the germ of a holomorphic function on a neighbourhood of the origin 0 in $\mathbb{C}^n$ which satisfies $f(0) = 0$ and which is not identically zero. Let Y be the zero set of f. Fix an embedded resolution $g : X_t \to X_0 \subset \mathbb{C}^n$ for Y which is an isomorphism outside the singular locus of Y and which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : X_i \to X_{i-1}$ with irreducible non-singular centre D_{i-1} and exceptional variety E_i satisfying for $i = 0, \ldots, t-1$:

(a) the codimension of D_i in X_i is at least two;
(b) D_i is a subset of the strict transform of Y under $g_1 \circ \cdots \circ g_i$;
(c) the union of the exceptional varieties of $g_1 \circ \cdots \circ g_i$ has only normal crossings with D_i;
(d) the origin 0 of $\mathbb{C}^n$ is an element of $(g_1 \circ \cdots \circ g_i)D_i$; and
(e) D_i contains a point in which $(g_1 \circ \cdots \circ g_i)^{-1}Y$ has not normal crossings.

Note that such a resolution always exists by Hironaka's theorem [Hi].

We prove that all candidate poles are larger than or equal to $-n+1$. Let $i \in \{1, \ldots, t\}$. Suppose that $-\nu_j/N_j \geq -n+1$ for every $j \in \{1, \ldots, i-1\}$. We prove that $-\nu_i/N_i \geq -n+1$.

Case 1. If the centre D_{i-1} of the blowing-up g_i is not contained in an exceptional variety, then the multiplicity $\mu_{D_{i-1}}$ of D_{i-1} on the strict transform of Y is larger than or equal to two because g is an isomorphism outside the singular locus. So $N_i \geq 2$ and $\nu_i \leq n$ and consequently, $-\nu_i/N_i \geq -n/2 \geq -n+1$.

Case 2. Suppose that there are $r \geq 1$ exceptional varieties with numerical data $(N_{1,i}, \nu_{1,i}), \ldots, (N_{r,i}, \nu_{r,i})$ which contain the k-dimensional centre D_{i-1}. Then

$$-\frac{\nu_i}{N_i} = -\frac{\nu_{1,i} + \cdots + \nu_{r,i} + n - k - r}{N_{1,i} + \cdots + N_{r,i} + \mu_{D_{i-1}}} \geq -n+1$$

because $-\nu_{1,i}/N_{1,i} \geq -n+1, \ldots, -\nu_{r,i}/N_{r,i} \geq -n+1$ and $-(n-k-r)/\mu_{D_{i-1}} \geq -n+1$. $\square$

2.5 Other zeta functions

(2.5.1) Denef and Loeser in [DL2] associate to a polynomial its motivic zeta function, which is a much finer invariant than its topological zeta function. Instead of the usual topological Euler-Poincaré characteristic, it involves the so-called universal Euler characteristic of an algebraic variety, i.e. its class in the Grothendieck ring.

We recall this notion. The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex algebraic varieties is the free abelian group generated by the symbols $[V]$, where V is a variety, subject to the relations $[V] = [V']$, if V is isomorphic to V', and $[V] = [V \setminus W] + [W]$, if W is closed in V. Its ring structure is given by $[V] \cdot [W] := [V \times W]$. We set $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}]$ and denote by $\mathcal{M}_{\mathbb{C}}$ the localization of $K_0(\mathrm{Var}_{\mathbb{C}})$ with respect to $\mathbb{L}$.

(2.5.2) In [DL2] the motivic zeta function is more generally defined for a regular function f on a smooth algebraic variety X, with respect to a subvariety W of X; we refer the reader to [DL2, section 2] for this definition. One can easily verify that the construction is still valid for a germ f of a holomorphic function at $0 \in \mathbb{C}^n$ when $W = \{0\}$; we denote this (local) motivic zeta function by $Z_{\mathrm{mot},0,f}(s)$. Then, with the notation of (2.1.1), the formula of [DL2, Theorem 2.2.1] yields that

$$Z_{\mathrm{mot},0,f}(s) = \mathbb{L}^{-n} \sum_{I \subset T} [\overset{\circ}{E}_I \cap g^{-1}\{0\}] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i + sN_i} - 1}.$$

Here $\mathbb{L}^{-s}$ should be considered as a variable, and this expression lives in a localization of the polynomial ring $\mathcal{M}_{\mathbb{C}}[\mathbb{L}^{-s}]$.

(2.5.3) The motivic zeta function $Z_{\mathrm{mot},0,f}(s)$ specializes to $Z_{\mathrm{top},0,f}(s)$ [DL2, subsection 2.3], but also to various 'intermediate level' zeta functions. An important one uses Hodge polynomials. Recall that the Hodge polynomial of a complex algebraic variety V is

$$H(V) = H(V,u,v) := \sum_{p,q} \left(\sum_{i \geq 0} (-1)^i h^{p,q}\left(H^i_c(V,\mathbb{C})\right) \right) u^p v^q \in \mathbb{Z}[u,v],$$

where $h^{p,q}\left(H_c^i(V,\mathbb{C})\right)$ is the rank of the (p,q)-Hodge component of the ith cohomology group with compact support of V. The zeta function of f on this level is

$$Z_{\mathrm{Hod},0,f}(s) = (uv)^{-n} \sum_{I \subset T} H\left(\overset{\circ}{E}_I \cap g^{-1}\{0\}\right) \prod_{i \in I} \frac{uv-1}{(uv)^{\nu_i + sN_i} - 1};$$

here $(uv)^{-s}$ is a variable, and this zeta function lives, e.g., in the field of rational functions in $(uv)^{-s}$ over $\mathbb{Q}(u,v)$.

(2.5.4) As in [RV2] we define the poles of $Z_{\mathrm{Hod},0,f}(s)$ to be the real numbers s_0 such that $(uv)^{-s_0}$ is a pole of $Z_{\mathrm{Hod},0,f}(s)$, considered as rational function in $(uv)^{-s}$. Then we have the following.

Theorems 2.2.9 and 2.3.0 are still valid with $Z_f(s) = Z_{\mathrm{top},0,f}(s)$ replaced by $Z_{\mathrm{Hod},0,f}(s)$ and $\mathcal{P}_n = \{s_0 \mid \exists f \in \mathcal{O}_n \ : \ Z_{\mathrm{Hod},0,f}(s)$ has a pole in $s_0\}$. The proofs are the same as before; they essentially just use the 'geometry' of a resolution.

A good definition of poles of $Z_{\mathrm{mot},0,f}(s)$ is not immediately clear, due to the fact that $\mathcal{M}_{\mathbb{C}}$ could have zero divisors (at present this is an open question). Using the definition of [RV2] for real poles, Theorems 2.2.9 and 2.3.0 are also valid for $Z_{\mathrm{mot},0,f}(s)$.

(2.5.5) One could and should also wonder whether there are analogous results for Igusa's p-adic zeta function. This problem is studied in the following chapter.

Chapter 3

Igusa's p-adic zeta function

Abstract

Let K be a p-adic field. We explore Igusa's p-adic zeta function, which is associated to a K-analytic function on an open and compact subset of K^n. First we deduce a formula for an important coefficient in the MacLaurin series of this meromorphic function at a candidate pole. Afterwards we use this formula to determine all values less than $-1/2$ for $n = 2$ and less than -1 for $n = 3$ which occur as the real part of a pole.

3.1 Introduction

(3.1.1) Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z and $|z| = q^{-\operatorname{ord} z}$ the absolute value of z.

(3.1.2) Let f be a K-analytic function on an open and compact subset X of K^n and put $x = (x_1, \ldots, x_n)$. Igusa's p-adic zeta function of f is defined by

$$Z_f(s) = \int_X |f(x)|^s \, |dx|$$

for $s \in \mathbb{C}$, $\operatorname{Re}(s) \geq 0$, where $|dx|$ denotes the Haar measure on K^n, so

This chapter corresponds to [Se1].

normalized that R^n has measure 1. Igusa proved that it is a rational function of q^{-s}, so that it extends to a meromorphic function $Z_f(s)$ on $\mathbb{C}$ which is also called Igusa's p-adic zeta function of f.

(3.1.3) This zeta function has an interesting connection with number theory. Let f be a K-analytic function on R^n defined by a power series over R which is convergent on the whole of R^n. Let M_i be the number of solutions of $f(x) \equiv 0 \bmod P^i$ in $(R/P^i)^n$. All the M_i's are described by $Z_f(s)$ through the relation

$$Z_f(s) = (1 - q^s)P(q^{-s}) + q^s,$$

where the Poincaré series $P(t)$ of f is defined by

$$P(t) = \sum_{i=0}^{\infty} M_i (q^{-n}t)^i.$$

Remark that $P(t)$ is a rational function of t because $Z_f(s)$ is a rational function of q^{-s}.

(3.1.4) The poles of $Z_f(s)$ are an interesting object of study because they are related to the monodromy conjecture (1.1.12) and because they determine the asymptotic behaviour of the M_i. The poles with largest real part give the largest contribution to the M_i. In this paper we are concerned with the smallest real part l of a pole of $Z_f(s)$. A non-trivial consequence of the fact that the M_i are integers is that l is larger than or equal to $-n$. Our main results are stated in the next paragraph and sharpen this bound by using a completely different method. This better bound has number theoretic consequences because the knowledge of l gives us interesting information about the M_i: there exists an $a \in \mathbb{Z}$ such that M_i is divisible by $q^{\lceil (n+l)i - a \rceil}$ for all i (for which $(n+l)i - a \geq 0$). This is proved in Section 3.5. Remark that a is independent of i and that the number in the exponent is the smallest integer larger than or equal to $(n+l)i - a$.

Let F_n^K denote the set of all K-analytic functions defined on an arbitrary open and compact subset of K^n. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}_n^K$ by

$$\mathcal{P}_n^K := \{s_0 \mid \exists f \in F_n^K \,:\, Z_f(s) \textit{ has a pole with real part } s_0\}.$$

In this article, we will prove for $n = 2$ that

$$\begin{aligned} \mathcal{P}_2^K \cap]-\infty, -1/2[&= \{-1/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \\ &= \{-1, -5/6, -3/4, -7/10, \ldots\}. \end{aligned}$$

Moreover, if f has not a singular point of multiplicity two or three, we will see that there are no poles with real part smaller than $-1/2$ and different from -1.
For $n = 3$, we prove that

$$\mathcal{P}_3^K \cap]-\infty, -1[= \{-1 - 1/i \mid i \in \mathbb{Z}_{>1}\}.$$

In particular, there are no poles with real part smaller than $-3/2$. Moreover, if f has not a singular point of multiplicity two, we will see that there are no poles with real part smaller than -1. We will have to prove that some candidate poles of expected order one are not poles. The geometric part of this proof is analogous to the one in Section 2.3 for the topological zeta function. We will prove that some contributions to the residue of $Z_f(s)$ at a candidate pole of expected order one are zero. In Section 3.2 we develop a method to calculate the contribution of an exceptional variety E_i with a candidate pole s_0 of expected order one to the residue of $Z_f(s)$ at s_0 by using only our information when E_i is created. In general, we expect that $\mathcal{P}_n^K \cap]-\infty, -(n-1)/2[= \{-(n-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\}$. One can easily show that

$$\mathcal{P}_n^K \cap]-\infty, -n+1[= \emptyset$$

if $n \geq 2$. The proof is analogous to the one in Section 2.4 for the local topological zeta function.

(3.1.5) Let $f \in K[x_1, x_2]$. Consider f as a polynomial over $K^{\text{alg cl}}$. Suppose that the minimal embedded resolution g of $f^{-1}\{0\} \subset (K^{\text{alg cl}})^2$ is defined over K, i.e., all irreducible components of $g^{-1}(f^{-1}\{0\})$ over $K^{\text{alg cl}}$ and all points in the intersection of two such components are defined over K. Then it is generally known that an exceptional curve which is intersected once or twice does not contribute to the residues of its candidate poles with expected order 1. Because $K^{\text{alg cl}} \cong \mathbb{C}$, we can use the calculations in Section 2.2 to conclude that the real part of a pole of $Z_f(s)$ is of the form $-1/2 - 1/i$, $i \in \mathbb{Z}_{>1}$, if it is smaller than $-1/2$.

Let $f \in K[x_1, x_2, x_3]$. Consider f again as a polynomial over $K^{\mathrm{alg\,cl}} \cong \mathbb{C}$. Suppose that f has not a singular point of multiplicity two. Suppose that there exists an embedded resolution g of $f^{-1}\{0\} \subset (K^{\mathrm{alg\,cl}})^3 \simeq \mathbb{C}^3$ for which the induced embedded resolution of the germ at each point P of $\mathbb{C}^3$ satisfies the conditions in (2.3.1.1), which is defined over K and which has good reduction modulo P (see [De1, Section 2]). Then the vanishing results in [Ve4] based on the relations on Veys and the calculations in Section 2.3 imply that $Z_f(s)$ has no pole with real part smaller than -1.

Consequently, starting from Chapter 2, it is rather easy to deal with polynomials which allow an appropriate embedded resolution. However it is very difficult to verify the existence of such an embedded resolution for a concrete function f. In a lot of cases there does not exist an embedded resolution which is defined over K, and if it exists, the condition of good reduction modulo P is very hard to check. This gives us a strong motivation to study the general case. In this article there are no constraints on f: we will not require that g is defined over K and that g has good reduction modulo P.

3.2 The tool for our vanishing results

(3.2.1) Let K be a p-adic field. Let $x = (x_1, \ldots, x_n)$ be the coordinates of K^n. Let f be a K-analytic function on an open and compact subset X of K^n. An embedded resolution $g : Y \to X$ of (f, dx) consists of a K-analytic manifold Y, a proper K-analytic map g and a finite set $\{E_i \mid i \in T\}$ of closed submanifolds of Y of codimension one with a pair of positive integers (N_i, ν_i), called numerical data, assigned to each E_i such that

1. the union of the E_i is equal to $g^{-1}(f^{-1}\{0\})$,
2. the restriction $Y \setminus g^{-1}(f^{-1}\{0\}) \to X \setminus f^{-1}\{0\}$ is a K-bianalytic map,
3. for every point b of Y, if $E_1, \ldots, E_k$ are all the E_i that contain b, there exists a chart $(V, y = (y_1, \ldots, y_n))$ around b with y_i, $1 \leq i \leq k$, an equation of E_i on V such that

$$f \circ g = \varepsilon \prod_{i=1}^{k} y_i^{N_i} \qquad \text{and} \qquad g^* dx = \eta \prod_{i=1}^{k} y_i^{\nu_i - 1} dy$$

on V for non-vanishing K-analytic functions ε and η on V.

Let Y be a n-dimensional K-analytic manifold, ω a K-analytic differential n-form on Y and h a K-analytic function on an open subset U of Y. We say that a chart (V, y) on Y is a good chart for (h, ω) if $V \subset U$, $h = \varepsilon \prod_{i=1}^{k} y_i^{N_i}$ on V and $\omega = \eta \prod_{i=1}^{k} y_i^{\nu_i - 1} dy$ on V for $k \in \{0, \ldots, n\}$, $N_i \in \mathbb{Z}_{>0}$, $\nu_i \in \mathbb{Z}_{>0}$ and non-vanishing K-analytic functions ε, η on V. We say that (h, ω) has normal crossings at a point $P \in U$ if there exists a good chart for (h, ω) around P. So when we say normal crossings, we mean normal crossings over K.

Let f be a K-analytic function on an open and compact subset X of K^n. Let Y be an n-dimensional K-analytic manifold and $g : Y \to X$ a K-analytic map which is a composition of blowing-ups along K-analytic closed submanifolds which are contained in the zero locus of the pullback of f. In this situation we have the following. If $y = (y_1, \ldots, y_n)$ is a system of local parameters at $P \in Y$ such that $f \circ g = \varepsilon \prod_{i=1}^{k} y_i^{N_i}$ for $k \in \{0, \ldots, n\}$, $N_i \in \mathbb{Z}_{>0}$ and ε a unit in the local ring at P, then $g^* dx = \eta \prod_{i=1}^{k} y_i^{\nu_i - 1} dy$ for $\nu_i \in \mathbb{Z}_{>0}$ and η a unit in the local ring at P. Consequently we will talk in this context about an embedded resolution of f instead of (f, dx) and about normal crossings of $f \circ g$ at P instead of $(f \circ g, g^* dx)$. Although the condition on $f \circ g$ does not imply the condition on $g^* dx$ globally, we will talk about a good chart for $f \circ g$ instead of $(f \circ g, g^* dx)$ in this context.

(3.2.2) Let $g : Y \to X$ be an embedded resolution of (f, dx). We study Igusa's p-adic zeta function $Z_f(s)$ by calculating the integral on the resolution Y:

$$\begin{aligned} Z_f(s) &= \int_X |f(x)|^s |dx| \\ &= \int_Y |f \circ g|^s |g^* dx|. \end{aligned}$$

Because $|\varepsilon|$ and $|\eta|$ are locally constant functions on each chart and because Y is a compact K-analytic manifold, we can choose a finite set J of good charts (V, y) for $(f \circ g, g^* dx)$ such that $|\varepsilon|$ and $|\eta|$ are constant on each chart, the V's form a partition of Y and for each chart (V, y) we have $y(V) = P^j := P^{j_1} \times \cdots \times P^{j_n}$ for some $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$ depending on (V, y). Remark that we may even require that $j_1 = \cdots = j_n$ and that this value does not depend on the chart, but we will not do this.

Because Y is a finite disjoint union of the V's, we obtain

$$Z_f(s) = \sum_{(V,y)\in J} \int_V |(f\circ g)(y)|^s |g^* dx|.$$

We can calculate these integrals very explicitly because we know in the one variable case that

$$\int_{P^j} |x|^{\alpha-1} |dx| = \frac{q-1}{q}\frac{q^{-j\alpha}}{1-q^{-\alpha}}$$

for a complex number α with $\mathrm{Re}(\alpha) > 0$ (the integral is not defined if $\mathrm{Re}(\alpha) \leq 0$) and because $|\varepsilon|$ and $|\eta|$ are constant on each chart:

$$\begin{aligned}
\int_V |(f\circ g)(y)|^s |g^* dx| &= \int_V |\varepsilon|^s |\eta| \prod_{i=1}^k |y_i|^{N_i s+\nu_i-1} |dy| \\
&= \int_{P^j} |\varepsilon|^s |\eta| \prod_{i=1}^k |y_i|^{N_i s+\nu_i-1} |dy| \\
&= |\varepsilon|^s |\eta| \prod_{i=1}^k \int_{P^{j_i}} |y_i|^{N_i s+\nu_i-1} |dy_i| \prod_{i=k+1}^n \int_{P^{j_i}} |dy_i| \\
&= |\varepsilon|^s |\eta| q^{-\sum_{i=k+1}^n j_i} \left(\frac{q-1}{q}\right)^k \prod_{i=1}^k \frac{q^{-j_i(N_i s+\nu_i)}}{1-q^{-N_i s-\nu_i}}.
\end{aligned}$$

This shows that $Z_f(s)$ is a rational function of q^{-s}. Moreover, these calculations imply that the integral

$$\int_V |(f\circ g)(y)|^s |g^* dx|$$

is defined if and only if $\mathrm{Re}(s) > \max\{-\nu_i/N_i \mid 1 \leq i \leq k\}$. Consequently, the integral

$$\int_X |f(x)|^s |dx|$$

is defined if and only if $\mathrm{Re}(s) > \max\{-\nu_i/N_i \mid i \in T\}$. We obtain also from this calculation that every pole of $Z_f(s)$ is of the form

$$-\frac{\nu_i}{N_i} + \frac{2k\pi\sqrt{-1}}{N_i \log q},$$

with $k \in \mathbb{Z}$ and $i \in T$. These values are called the candidate poles of $Z_f(s)$. If $i \in T$ is fixed, the values $-\nu_i/N_i + (2k\pi\sqrt{-1})/(N_i \log q)$, $k \in \mathbb{Z}$, are called the candidate poles of $Z_f(s)$ associated to E_i.

Let s_0 be a candidate pole of $Z_f(s)$. Because the poles of $1/(1 - q^{-N_i s - \nu_i})$ have order one, we define the expected order $m = m(s_0)$ of s_0 as the highest number of E_i's with candidate pole s_0 and with non-empty intersection. The order of s_0 is of course less than or equal to m. It is less than m if and only if b_{-m}, which is defined by the MacLaurin series

$$\frac{b_{-m}}{(s-s_0)^m} + \frac{b_{-m+1}}{(s-s_0)^{m-1}} + \cdots + b_0 + b_1(s-s_0) + \cdots$$

of $Z_f(s)$ at s_0, is equal to zero. Remark that a candidate pole of expected order one is a pole if and only if $b_{-1} \neq 0$.

(3.2.3) Fix a uniformizing parameter π for R. For $z \in K$ let $\operatorname{ac} z := z\pi^{-\operatorname{ord} z}$ be the angular component of z. Let χ be a character of $R^\times$, i.e., a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. Put formally $\chi(0) = 0$. The generalized Igusa's p-adic zeta function of f is now defined by

$$Z_{f,\chi}(s) = \int_X \chi(\operatorname{ac} f(x)) |f(x)|^s \, |dx|$$

for $s \in \mathbb{C}$, $\operatorname{Re}(s) \geq 0$.

In the one variable case we have now that

$$\int_{P^j} \chi(\operatorname{ac} x) |x|^{\alpha - 1} \, |dx| = \begin{cases} \frac{q-1}{q} \frac{q^{-j\alpha}}{1-q^{-\alpha}} & \text{if } \chi = 1 \\ 0 & \text{if } \chi \neq 1 \end{cases}$$

for a complex number α with $\operatorname{Re}(\alpha) > 0$. The integral is not defined if $\operatorname{Re}(\alpha) \leq 0$.

Because $\chi(\operatorname{ac} \varepsilon)$ is a locally constant function on each chart, we can choose a finite set J of good charts (V, y) for $(f \circ g, g^* dx)$ such that $|\varepsilon|$, $|\eta|$ and $\chi(\operatorname{ac} \varepsilon)$ are constant on each chart, the V's form a partition of Y and for each chart (V, y) we have $y(V) = P^j$ for some $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$. Analogously as in (3.2.2) we obtain

$$\begin{aligned} Z_{f,\chi}(s) &= \sum_{(V,y) \in J} \int_V \chi(\operatorname{ac}(f \circ g)(y)) |(f \circ g)(y)|^s \, |g^* dx| \\ &= \sum_{(V,y) \in J} \chi(\operatorname{ac} \varepsilon) |\varepsilon|^s |\eta| q^{-\sum_{i=k+1}^n j_i} \prod_{i=1}^k \int_{P^{j_i}} \chi^{N_i}(\operatorname{ac} y_i) |y_i|^{N_i s + \nu_i - 1} \, |dy_i|. \end{aligned}$$

This implies that $Z_{f,\chi}(s)$ is again a rational function of q^{-s}, so that it extends to a meromorphic function $Z_{f,\chi}(s)$ on $\mathbb{C}$. The integral

$$\int_X \chi(\mathrm{ac}\, f(x))|f(x)|^s\,|dx|$$

is defined if and only if $\mathrm{Re}(s) > \max\{-\nu_i/N_i \mid i \in T\}$. We obtain that every pole of $Z_{f,\chi}(s)$ is of the form $-\nu_i/N_i + (2k\pi\sqrt{-1})/(N_i \log q)$, with $k \in \mathbb{Z}$ and $i \in T$ such that $\chi^{N_i} = 1$. These values are called the candidate poles of $Z_{f,\chi}(s)$. Obviously, we associate only candidate poles to E_i if $\chi^{N_i} = 1$. The expected order $m = m(s_0)$ of a candidate pole s_0 of $Z_{f,\chi}(s)$ is the highest number of E_i's with candidate pole s_0 (thus also with $\chi^{N_i} = 1$) and with non-empty intersection.

Remark that everything we have done up till now is well known. More details can be found for example in [Ig7].

(3.2.4) Let X be an open and compact subset of K^n. Let $\xi, f, f_1, \ldots, f_l$ be K-analytic functions on X. Let a_i, b_i, $1 \leq i \leq l$, be non-negative integers. We associate to these data the zeta function

$$Z(s_1, \ldots, s_l) = \int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_1 + b_1} \ldots |f_l|^{a_l s_l + b_l}|dx|,$$

which is defined on a set U that contains all points $(s_1, \ldots, s_l) \in \mathbb{C}^l$ with $\mathrm{Re}(s_i) \geq -b_i/a_i$ if f_i vanishes on X and s_i arbitrary if f_i does not vanish on X. Loeser [Lo2] already studied this zeta function. By looking at an embedded resolution of $\xi f f_1 \ldots f_l$, one proves analogously as in (3.2.2) that $Z(s_1, \ldots, s_l)$ is a rational function of $q^{-s_1}, \ldots, q^{-s_l}$. Consequently, it extends to a meromorphic function on $\mathbb{C}^l$, which we also denote by $Z(s_1, \ldots, s_l)$. As before, we can also obtain an explicit description of U, which turns out to be an open subset of $\mathbb{C}^l$.

The meromorphic continuation of a function h will be denoted by $[h]^{\mathrm{mc}}$ and the evaluation of this meromorphic continuation in the point $s = s_0$ of the domain will be denoted by $[h]^{\mathrm{mc}}_{s=s_0}$.

In our study of Igusa's p-adic zeta function, we will have to deal with expressions of the form

$$\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s + b_1} \ldots |f_l|^{a_l s + b_l}|dx|\right]^{\mathrm{mc}}_{s=s_0}.$$

The zeta function in more complex variables can be used to modify this expression. If $U \cap \{(s_1, \ldots, s_l) \in \mathbb{C}^l \mid s_1 = s_0\} \neq \emptyset$, then

$$\begin{aligned}
&\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s+b_1}|f_2|^{a_2 s+b_2}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}_{s=s_0} \\
&= \left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_1+b_1}|f_2|^{a_2 s_2+b_2}\ldots|f_l|^{a_l s_l+b_l}|dx|\right]^{\mathrm{mc}}_{s_1=\cdots=s_l=s_0} \\
&= \left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_0+b_1}|f_2|^{a_2 s+b_2}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}_{s=s_0}.
\end{aligned}$$

We explain the first equality. The composition of the map

$$A : \mathbb{C} \to \mathbb{C}^l : s \mapsto (s, \ldots, s)$$

with the meromorphic function $Z : \mathbb{C}^l \to \mathbb{C}$ which sends $(s_1, \ldots, s_l) \in U$ to

$$\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_1+b_1}\ldots|f_l|^{a_l s_l+b_l}|dx|$$

is a meromorphic function on $\mathbb{C}$ which is equal to the meromorphic function

$$\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s+b_1}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}$$

because they agree on an open subset of $\mathbb{C}$. Consequently, the first equality is nothing more than $(Z \circ A)(s_0) = Z(A(s_0))$. For the second equality, we have to use the map

$$B : \mathbb{C} \to \mathbb{C}^l : s \mapsto (s_0, s, \ldots, s).$$

(3.2.5) Let f be a K-analytic function on an open and compact subset X of K^n and let $g : Y \to X$ be an embedded resolution of (f, dx) as in (3.2.1). For $I \subset T$ denote $E_I = \cap_{i \in I} E_i$. Let χ be a character of $R^\times$. Let s_0 be a candidate pole of $Z_{f,\chi}(s)$ and let m be its expected order. Let E_I, $I \in S$, be all the non-empty intersections of m varieties E_i, $i \in T$, with candidate pole s_0 (and thus also with $\chi^{N_i} = 1$). Fix $I \in S$ and suppose for the ease of notation that $I = \{1, \ldots, m\}$. Let W_1 and W_2 be open and compact subsets of Y which satisfy $E_I \cap W_1 = E_I \cap W_2 \neq \emptyset$ and which do not meet any E_K, $K \in S \setminus \{I\}$. Then the contribution of W_1 to b_{-m} and the contribution of W_2 to b_{-m} are the same because they are both equal to the contribution of $W_1 \cap W_2$ to b_{-m}. Consequently we can

speak of the contribution of $E_I \cap W_1 = E_I \cap W_2$ to b_{-m}. In particular, the contribution of E_I to b_{-m} is well defined.

Consider a set J of disjoint compact charts (V, y) that intersect E_I, that cover E_I and that are disjoint with all E_K, $K \in S \setminus \{I\}$. This set J is necessarily finite and the contribution of E_I to b_{-m} is the sum over J of the contributions

$$\lim_{s \to s_0} (s - s_0)^m \left[\int_V \chi(\mathrm{ac}\,(f \circ g)(y)) |(f \circ g)(y)|^s |g^* dx| \right]^{\mathrm{mc}}$$

of V to b_{-m}.

We introduce some notation. Let (V, y) be a chart. We have that $\overline{y} = (y_{m+1}, \ldots, y_n)$ determines a chart on the closed submanifold $\overline{V}$ of V defined by $y_1 = \cdots = y_m = 0$. Denote $dy_{m+1} \wedge \cdots \wedge dy_n$ by $d\overline{y}$. It is a volume form on $\overline{V}$. If $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$, then we denote $P^{j_{m+1}} \times \cdots \times P^{j_n}$ by $\overline{P^j}$.

Suppose that (V, y) is a compact chart such that $E_1, \ldots, E_m$ have equations $y_1 = 0, \ldots, y_m = 0$ respectively, and such that

$$f \circ g = \alpha \prod_{i=1}^m y_i^{N_i} \qquad \text{and} \qquad g^* dx = \beta \prod_{i=1}^m y_i^{\nu_i - 1} dy$$

on V, for K-analytic functions α and β on V with $|\alpha|$, $|\beta|$ and $\chi(\mathrm{ac}\,\alpha)$ *independent of* $y_1, \ldots, y_m$. Remark that a good chart (V, y) for $(f \circ g, g^* dx)$ in which $|\varepsilon|$, $|\eta|$ and $\chi(\mathrm{ac}\,\varepsilon)$ are constant satisfies this condition for $\alpha = \varepsilon \prod_{i=m+1}^k y_i^{N_i}$ and $\beta = \eta \prod_{i=m+1}^k y_i^{\nu_i - 1}$. Remark also that $\overline{V} = V \cap E_I$. Suppose also that $y(V)$ is of the form P^j with $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$. Then

$$\begin{aligned}
&\lim_{s \to s_0} (s - s_0)^m \left[\int_V \chi(\mathrm{ac}\,(f \circ g)(y)) |(f \circ g)(y)|^s |g^* dx| \right]^{\mathrm{mc}} \\
&= \lim_{s \to s_0} (s - s_0)^m \left[\int_{P^j} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| \prod_{i=1}^m \chi^{N_i}(\mathrm{ac}\,y_i) |y_i|^{N_i s + \nu_i - 1} |dy| \right]^{\mathrm{mc}} \\
&= \left(\prod_{i=1}^m \lim_{s \to s_0} (s - s_0) \left[\int_{P^{j_i}} |y_i|^{N_i s + \nu_i - 1} |dy_i| \right]^{\mathrm{mc}} \right) \left[\int_{\overline{P^j}} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| |d\overline{y}| \right]_{s = s_0}^{\mathrm{mc}} \\
&= \left(\prod_{i=1}^m \frac{q-1}{q N_i \log q} \right) \left[\int_{\overline{V}} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| |d\overline{y}| \right]_{s = s_0}^{\mathrm{mc}}.
\end{aligned}$$

We have derived that the last expression is the contribution of V to b_{-m}. Consequently, the only aspect of the chart (V, y) it depends on is

$\overline{V}$. In the next section we will see that we do not have to require that $|\alpha|$, $|\beta|$ and $\chi(\mathrm{ac}\,\alpha)$ are independent of $y_1, \ldots, y_m$ and that we are in an embedded resolution to have this independence.

(3.2.6) Suppose that $g : Y = Y_t \to X = Y_0$ is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$. Suppose that each g_i is a blowing-up along a K-analytic closed submanifold of codimension larger than one which has only normal crossings with the union of the exceptional varieties of $g_1 \circ \cdots \circ g_{i-1}$. Let $I = \{1, \ldots, m\} \in S$ as in (3.2.5). Let $r \in \{0, \ldots, t\}$. Suppose that E_I already exists in Y_r and that the E_i, $i \in I$, intersect transversally in Y_r. Remark that the last condition is satisfied if all the E_i, $i \in I$, are exceptional. We will write $E_I \subset Y_r$ if we want to stress that we consider E_I as a subset of Y_r.

We call a chart (V, y) a good chart for $E_I \subset Y_r$ if (V, y) is a chart on Y_r such that V intersects E_I and such that $y_1 = 0, \ldots, y_m = 0$ are the equations of respectively $E_1, \ldots, E_m$ on V.

Let (V, y) be a good compact chart for $E_I \subset Y_r$. Then we have

$$f \circ g_1 \circ \cdots \circ g_r = \alpha \prod_{i=1}^{m} y_i^{N_i} \quad \text{and} \quad (g_1 \circ \cdots \circ g_r)^* dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1} dy$$

on V, for K-analytic functions α and β on V.

We will now prove that the only aspect of the chart (V, y) that

$$\left[\int_{\overline{V}} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| |d\overline{y}| \right]_{s=s_0}^{\mathrm{mc}} \tag{3.1}$$

depends on is $\overline{V}$.

Let (W, z) be another chart on Y_r such that $\overline{V} = \overline{W}$ and such that $z_1 = 0, \ldots, z_m = 0$ are the equations of respectively $E_1, \ldots, E_m$ on W. We may suppose that $V = W$ because we can restrict them both to $V \cap W$. For every $i \in \{1, \ldots, m\}$ there exists a non-vanishing K-analytic function f_i on V such that $y_i = f_i z_i$ because y_i and z_i are equations of the same E_i. Thus

$$\begin{aligned} f \circ g_1 \circ \cdots \circ g_r &= \alpha \prod_{i=1}^{m} (f_i z_i)^{N_i} \\ &= \alpha \left(\prod_{i=1}^{m} f_i^{N_i} \right) \prod_{i=1}^{m} z_i^{N_i} \end{aligned}$$

and

$$\begin{aligned}(g_1 \circ \cdots \circ g_r)^* dx &= \beta \prod_{i=1}^{m} (f_i z_i)^{\nu_i - 1} \det\left(\frac{\partial y}{\partial z}\right) dz \\ &= \beta \left(\prod_{i=1}^{m} f_i^{\nu_i - 1}\right) \det\left(\frac{\partial y}{\partial z}\right) \prod_{i=1}^{m} z_i^{\nu_i - 1} dz.\end{aligned}$$

We have to prove that (3.1) is equal to

$$\left[\int_{\overline{V}} \chi(\operatorname{ac}\alpha)|\alpha|^s |\beta| \left(\prod_{i=1}^{m} \chi^{N_i}(\operatorname{ac} f_i)|f_i|^{N_i s + \nu_i - 1}\right) \left|\det\left(\frac{\partial y}{\partial z}\right)\right| |d\overline{z}|\right]_{s=s_0}^{\mathrm{mc}} . \quad (3.2)$$

In (3.1) we have that $|d\overline{y}| = |\det(\partial\overline{y}/\partial\overline{z})||d\overline{z}|$. Recall that $\chi^{N_i} = 1$ for $i \in \{1, \ldots, m\}$. Because f_i, $i \in I$, is a non-vanishing function, we may replace each $|f_i|^{N_i s + \nu_i - 1}$ in (3.2) by $|f_i|^{N_i s_0 + \nu_i - 1}$ according to (3.2.4), and this is equal to $|f_i|^{-1}$ because $N_i s_0 + \nu_i = 0$ for $i \in I$. Consequently, we have to prove that

$$\begin{aligned}&\left[\int_{\overline{V}} \chi(\operatorname{ac}\alpha)|\alpha|^s |\beta| \left|\det\left(\frac{\partial \overline{y}}{\partial \overline{z}}\right)\right| |d\overline{z}|\right]_{s=s_0}^{\mathrm{mc}} \\ &= \left[\int_{\overline{V}} \chi(\operatorname{ac}\alpha)|\alpha|^s |\beta| \prod_{i=1}^{m} |f_i|^{-1} \left|\det\left(\frac{\partial y}{\partial z}\right)\right| |d\overline{z}|\right]_{s=s_0}^{\mathrm{mc}} .\end{aligned}$$

Because $(\partial y/\partial z)$ is equal to

$$\begin{pmatrix} f_1 + z_1\frac{\partial f_1}{\partial z_1} & \cdots & z_1\frac{\partial f_1}{\partial z_m} & z_1\frac{\partial f_1}{\partial z_{m+1}} & \cdots & z_1\frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_m\frac{\partial f_m}{\partial z_1} & \cdots & f_m + z_m\frac{\partial f_m}{\partial z_m} & z_m\frac{\partial f_m}{\partial z_{m+1}} & \cdots & z_m\frac{\partial f_m}{\partial z_n} \\ \frac{\partial y_{m+1}}{\partial z_1} & \cdots & \frac{\partial y_{m+1}}{\partial z_m} & \frac{\partial y_{m+1}}{\partial z_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial z_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial z_1} & \cdots & \frac{\partial y_n}{\partial z_m} & \frac{\partial y_n}{\partial z_{m+1}} & \cdots & \frac{\partial y_n}{\partial z_n} \end{pmatrix},$$

we obtain that

$$\left[\det\left(\frac{\partial y}{\partial z}\right)\right]_{z_1 = \cdots = z_m = 0} = \left[\left(\prod_{i=1}^{m} f_i\right) \det\left(\frac{\partial \overline{y}}{\partial \overline{z}}\right)\right]_{z_1 = \cdots = z_m = 0} .$$

Consequently we have proved our statement.

(3.2.7) *Data and notations.* Let f be a K-analytic function on an open and compact subset X of K^n. Let χ be a character of $R^\times$. Let $g : Y = Y_t \to X = Y_0$ be an embedded resolution of (f, dx) as in (3.2.1) which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$. Suppose that each g_i is a blowing-up along a K-analytic closed submanifold of codimension larger than one which has only normal crossings with the union of the exceptional varieties of $g_1 \circ \cdots \circ g_{i-1}$. Let s_0 be a candidate pole of $Z_{f,\chi}(s)$ and let m be its expected order. Let b_{-m} be defined by the MacLaurin series

$$\frac{b_{-m}}{(s-s_0)^m} + \frac{b_{-m+1}}{(s-s_0)^{m-1}} + \cdots + b_0 + b_1(s-s_0) + \cdots$$

of $Z_{f,\chi}(s)$ at s_0. For $I \subset T$, denote the intersection in Y of the $E_i \subset Y$, $i \in I$, by E_I. Let E_I, $I \in S$, be all the non-empty intersections in Y of m varieties E_i, $i \in T$, with candidate pole s_0 (and thus also with $\chi^{N_i} = 1$). Fix $I \in S$ and suppose for the ease of notation that $I = \{1, \ldots, m\}$. Let $r \in \{0, \ldots, t\}$. Suppose that E_I already exists in Y_r and that the E_i, $i \in I$, intersect transversally in Y_r.

Definition. The contribution of an open and compact subset U of $E_I \subset Y_r$ to b_{-m} is the contribution of the strict transform of U in Y to b_{-m}.

Remark. The contribution of E_I to b_{-m} is not necessarily equal to the contribution of a 'very small' neighbourhood of $E_I \subset Y_r$ to b_{-m}, because it can happen that an E_K, $K \in S \setminus \{I\}$, lies above $E_I \subset Y_r$.

Data and notations. Let (V, y) be a good compact chart for $E_I \subset Y_r$, i.e., (V, y) is a chart on Y_r such that V is compact and intersects E_I and such that $y_1 = 0, \ldots, y_m = 0$ are the equations of respectively $E_1, \ldots, E_m$ on V. Write

$$f \circ g_1 \circ \cdots \circ g_r = \alpha \prod_{i=1}^{m} y_i^{N_i} \quad \text{and} \quad (g_1 \circ \cdots \circ g_r)^* dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1} dy$$

on V, for K-analytic functions α and β on V. We have that $\overline{y} = (y_{m+1}, \ldots, y_n)$ determines coordinates on the closed submanifold $\overline{V} = V \cap E_I$ which is defined by $y_1 = \cdots = y_m = 0$. Consider the volume form $d\overline{y} = dy_{m+1} \wedge \cdots \wedge dy_n$ on $\overline{V}$.

Proposition. *The contribution of $\overline{V}$ to b_{-m} is equal to*

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q} \right) \left[\int_{\overline{V}} \chi(\operatorname{ac} \alpha) |\alpha|^s |\beta| |d\overline{y}| \right]_{s=s_0}^{\mathrm{mc}} .$$

Notation. For the ease of notation, the first factor of this contribution will be denoted by κ:

$$\kappa := \prod_{i=1}^{m} \frac{q-1}{qN_i \log q}.$$

Proof. We proved in (3.2.5) that the proposition is true if we are working on the embedded resolution, thus if $r = t$. Note that $g_{r+1} \circ \cdots \circ g_t$ is a composition of a finite number of blowing-ups. Therefore, it is enough to prove the formula on Y_r under the assumption that it is true on Y_{r+1}. So we want to prove that the contribution of $\overline{V}$ to b_{-m} is equal to

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right) \left[\int_{\overline{V}} \chi(\operatorname{ac} \alpha)|\alpha|^s |\beta||d\overline{y}|\right]_{s=s_0}^{\mathrm{mc}}. \tag{3.3}$$

We have that (3.3) is equal to

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right)\left[\int_{g_{r+1}^*(\overline{V})} \chi(\operatorname{ac} g_{r+1}^*\alpha)|g_{r+1}^*\alpha|^s |g_{r+1}^*\beta||g_{r+1}^* d\overline{y}|\right]_{s=s_0}^{\mathrm{mc}}, \tag{3.4}$$

where $g_{r+1}^*(\overline{V})$ denotes the strict transform of $\overline{V}$ under g_{r+1}. On the other hand, the contribution of $\overline{V}$ to b_{-m} is equal to the contribution of $g_{r+1}^*(\overline{V})$ to b_{-m}. Consequently, we have to prove that the contribution of $g_{r+1}^*(\overline{V})$ to b_{-m} is equal to (3.4).

Because the centre of g_{r+1} does not contain E_I and has only normal crossings with $E_1 \cup \cdots \cup E_m$, we may suppose that g_{r+1} is the blowing-up along $y_1 = \cdots = y_a = y_{m+1} = \cdots = y_b = 0$, where $0 \leq a \leq m < b \leq n$. The transformation

$$(z_1, \ldots, z_n) \mapsto (z_1 z_b, \ldots, z_a z_b, z_{a+1}, \ldots, z_m, z_{m+1} z_b, \ldots, z_{b-1} z_b, z_b, z_{b+1}, \ldots, z_n)$$

determines coordinates z on an open subset O of Y_{r+1}. If we permute $y_{m+1}, \ldots, y_b$, it represents another transformation which determines coordinates on another open subset of Y_{r+1}; the open subsets we obtain in this way cover $g_{r+1}^*(\overline{V})$.

Let W be an open and compact subset of O that intersects $E_I \subset Y_{r+1}$. Then (W, z) is a good chart for $E_I \subset Y_{r+1}$. It is enough to prove that the contribution of $\overline{W}$ to b_{-m} is equal to κ times

$$\left[\int_{\overline{W}} \chi(\operatorname{ac} g_{r+1}^*\alpha)|g_{r+1}^*\alpha|^s |g_{r+1}^*\beta||g_{r+1}^* d\overline{y}|\right]_{s=s_0}^{\mathrm{mc}}.$$

Because

$$f \circ g_1 \circ \cdots \circ g_{r+1} = g_{r+1}^* \alpha \left(\prod_{i=1}^{m} z_i^{N_i} \right) z_b^{\sum_{i=1}^{a} N_i}$$

and

$$\begin{aligned} (g_1 \circ \cdots \circ g_{r+1})^* dx &= g_{r+1}^* \beta \left(\prod_{i=1}^{m} z_i^{\nu_i - 1} \right) z_b^{\sum_{i=1}^{a} (\nu_i - 1)} z_b^{b-m-1+a} dz \\ &= g_{r+1}^* \beta \left(\prod_{i=1}^{m} z_i^{\nu_i - 1} \right) z_b^{\sum_{i=1}^{a} \nu_i} z_b^{b-m-1} dz, \end{aligned}$$

it follows from our assumption that the contribution of $\overline{W}$ to b_{-m} is equal to κ times

$$\left[\int_{\overline{W}} \chi(\mathrm{ac}\, g_{r+1}^* \alpha) |g_{r+1}^* \alpha|^s |g_{r+1}^* \beta| |z_b|^{\sum_{i=1}^{a} (N_i s + \nu_i)} |z_b|^{b-m-1} |d\overline{z}| \right]_{s=s_0}^{\mathrm{mc}}.$$

We can use (3.2.4) to simplify this expression because we know that $N_i s_0 + \nu_i = 0$ for $i \in \{1, \ldots, a\}$ and that $b - m - 1 \geq 0$. Because moreover $g_{r+1}^* d\overline{y} = z_b^{b-m-1} d\overline{z}$, we obtain that the contribution of $\overline{W}$ to b_{-m} is equal to κ times

$$\left[\int_{\overline{W}} \chi(\mathrm{ac}\, g_{r+1}^* \alpha) |g_{r+1}^* \alpha|^s |g_{r+1}^* \beta| |g_{r+1}^* d\overline{y}| \right]_{s=s_0}^{\mathrm{mc}}.$$

This had to be proved. □

Alternative proof. Let F be the closed subset of $E_I \subset Y_t$ that contains the points at which $\pi = g_t \circ \cdots \circ g_{r+1} : Y_t \to Y_r$ is not K-bianalytic. Let $(B_n)_{n \in \mathbb{Z}_{\geq 0}}$ be a decreasing sequence of open and compact subsets of $E_I \subset Y_t$ such that $\cap_{n \in \mathbb{Z}_{\geq 0}} B_n = F$. Note that $\pi(F)$ is a closed subset of Y_r and that $\cap_{n \in \mathbb{Z}_{\geq 0}} \pi(B_n) = \pi(F)$. Note also that for every neighbourhood O of F (respectively $\pi(F)$), there exists a positive integer n such that $B_n \subset O$ (respectively $\pi(B_n) \subset O$).

Let J be a set of good compact charts (W, z) for $E_I \subset Y_t$ such that the $\overline{W}$'s form a partition of $\pi^{-1}(\overline{V})$. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Then

$$\begin{aligned} \int_{\overline{V}} \chi(\mathrm{ac}\, \alpha) |\alpha|^s |\beta| |d\overline{y}| &= \lim_{n \to \infty} \int_{\overline{V} \setminus \pi(B_n)} \chi(\mathrm{ac}\, \alpha) |\alpha|^s |\beta| |d\overline{y}| \\ &= \lim_{n \to \infty} \sum_{(W,z) \in J} \int_{\overline{W} \setminus B_n} \chi(\mathrm{ac}\, \alpha_z) |\alpha_z|^s |\beta_z| |d\overline{z}| \\ &= \sum_{(W,z) \in J} \int_{\overline{W}} \chi(\mathrm{ac}\, \alpha_z) |\alpha_z|^s |\beta_z| |d\overline{z}|. \end{aligned}$$

The first equality holds because

$$\lim_{n\to\infty}\int_{\overline{V}\cap\pi(B_n)}\chi(\mathrm{ac}\,\alpha)|\alpha|^s|\beta||d\overline{y}|=0.$$

Indeed, the measure of $\overline{V}\cap\pi(B_n)$ for $|d\overline{y}|$ decreases to zero if $n\to\infty$ and the real and the complex part of the integrand are bounded for complex numbers s satisfying $\mathrm{Re}(s)>0$. The last equality is obtained by using the same argument. We have written α_z and β_z to stress that these functions depend on the chart. For the second equality, we have to use that the $\overline{W}$'s form a partition of $\pi^{-1}(\overline{V})$, that $\pi: Y_t\to Y_r$ is a K-bianalytic map on a neighbourhood of $\overline{W}\setminus B_n$ in Y_t for every $(W,z)\in J$ and that the contribution is independent of the chosen coordinates, a fact we explained in (3.2.6).

Finally, we evaluate the meromorphic continuation in $s=s_0$ and we obtain:

$$\left[\int_{\overline{V}}\chi(\mathrm{ac}\,\alpha)|\alpha|^s|\beta||d\overline{y}|\right]^{\mathrm{mc}}_{s=s_0}=\sum_{(W,z)\in J}\left[\int_{\overline{W}}\chi(\mathrm{ac}\,\alpha_z)|\alpha_z|^s|\beta_z||d\overline{z}|\right]^{\mathrm{mc}}_{s=s_0}.$$

This had to be proved. □

(3.2.8) Let T_t be the set of all $j\in T\setminus I$ for which E_j intersects E_I in Y_t. Let F_j, $j\in T_t$, be the intersection of E_j and E_I in Y_t. We have that F_j has codimension one in $E_I\subset Y_t$. The set of all j, $j\in T_t$, for which $(g_{r+1}\circ\cdots\circ g_t)(F_j)$ has also codimension one in $E_I\subset Y_r$ will be denoted by T_r. For $j\in T_r$ we denote $(g_{r+1}\circ\cdots\circ g_t)(F_j)$ also by F_j and we put $\alpha_j=N_js_0+\nu_j$.

Let (V,y) be a good chart for E_I on Y_r on which F_j, $j\in T_r$, is given by $y_1=\cdots=y_m=y_{m+1}=0$. Write

$$\alpha(0,\ldots,0,y_{m+1},\ldots,y_n) \;=\; y_{m+1}^{N_{j,r}}h_1$$

and

$$\beta(0,\ldots,0,y_{m+1},\ldots,y_n) \;=\; y_{m+1}^{\nu_{j,r}-1}h_2$$

with h_1 and h_2 not divisible by y_{m+1}. We denote $N_{j,r}s_0+\nu_{j,r}$ by $\alpha_{j,r}$.

We deduce now the relations that will be used later. In this paragraph we suppose that $m=1$ and that $E_I=E_r$ is created by the blowing-up at

a point P of Y_{r-1}. Suppose that there exists a chart (V, y) centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part a homogeneous polynomial for which every irreducible factor over $K^{\text{alg cl}}$ is defined over K and for which the zero locus in $\mathbb{P}^{n-1}$ of every irreducible factor (over $K^{\text{alg cl}}$) contains a non-singular point defined over K. Remark that these conditions are satisfied if the lowest degree part is a product of linear factors defined over K. Write $f \circ g_1 \circ \cdots \circ g_{r-1} = e\left(\prod_{j \in T_r} f_j^{N_{j,r}}\right) + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho\left(\prod_{j \in T_r} f_j^{\nu_{j,r}-1}\right) dy$, where f_j is the equation of $F_j \subset E_r$ in the homogeneous coordinates $(y_1 : \cdots : y_n)$ on $E_r \subset Y_r$, $e \in K^\times$, θ is a power series with multiplicity larger than the degree of the homogeneous polynomial $\prod_{j \in T_j} f_j^{N_{j,r}}$ and ρ is a K-analytic function which does not vanish at P. Because the multiplicity of $f \circ g_1 \circ \cdots \circ g_{r-1}$ at P is equal to N_r, we obtain the first relation:

$$\sum_{j \in T_r} (\deg F_j) N_{j,r} = N_r. \qquad \text{(Relation 1)}$$

Our second relation will involve the $\alpha_{j,r}$, $j \in T_r$. There will appear differential forms with rational exponents in the calculations. One can make sense to this by considering them as an element of a tensor power of the module of rational differential forms (see [Ja2]), but we will not give details here. Let $i \in \{1, \ldots, n\}$. We look at the chart $(O, z = (z_1, \ldots, z_n))$ on Y_r for which $g_r(z_1, \ldots, z_n) = (z_1 z_i, \ldots, z_{i-1} z_i, z_i, z_{i+1} z_i, \ldots, z_n z_i)$. We have then that

$$f \circ g_1 \circ \cdots \circ g_r = z_i^{N_r} \left(e \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{N_{j,r}} + z_i \frac{\theta \circ g_r}{z_i^{N_r+1}} \right)$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx = z_i^{\nu_r - 1} (\rho \circ g_r) \left(\prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\nu_{j,r}-1} \right) dz.$$

Consequently the Poincaré residue of $(f \circ g_1 \circ \cdots \circ g_r)^{-\nu_r/N_r} (g_1 \circ \cdots g_r)^* dx$ on $E_I \subset Y_r$ (see [Ja2]) is equal to

$$e^{-\nu_r/N_r} \rho(P) \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\alpha_{j,r}-1} d\overline{z},$$

so that the canonical divisor of E_r is $\sum_{j\in T_r}(\alpha_{j,r}-1)F_j$. Because we know that the degree of the canonical divisor on $E_r \cong \mathbb{P}^{n-1}$ is $-n$, we obtain the second relation:

$$\sum_{j\in T_r}(\deg F_j)(\alpha_{j,r}-1) = -n. \qquad \text{(Relation 2)}$$

Remark that the condition on the lowest degree part of $f\circ g_1\circ\cdots\circ g_{r-1}$ has to be satisfied because otherwise some terms on the left hand side are missing. We need the two relations which we just derived in Section 3.3. In the next paragraph we will deduce that $\alpha_{j,r}=\alpha_j$ and that $N_{j,r}\equiv N_j \pmod{N_r}$ so that we obtain

$$\sum_{j\in T_r}(\deg F_j)N_j \equiv 0 \pmod{N_r}$$

and

$$\sum_{j\in T_r}(\deg F_j)(\alpha_j-1) = -n.$$

One can find these relations in a more general form in [Ve2], [Ve3] and [Ve6].

We prove that $\alpha_{j,r}=\alpha_j$ for $j\in T_r$. Because $g_{r+1}\circ\cdots\circ g_t$ is a composition of a finite number of blowing-ups, it is enough to prove that $\alpha_{j,r}=\alpha_{j,r+1}$. If the centre of g_{r+1} does not contain F_j, then $N_{j,r}=N_{j,r+1}$ and $\nu_{j,r}=\nu_{j,r+1}$ so that we are done. If the centre of g_{r+1} contains F_j, we may suppose that g_{r+1} is the blowing-up along $y_1=\cdots=y_a=y_{m+1}=0$, where $0<a\leq m$. The relevant chart is determined by the transformation

$$(z_1,\ldots,z_n)\mapsto(z_1z_{m+1},\ldots,z_az_{m+1},z_{a+1},\ldots,z_m,z_{m+1},\ldots,z_n).$$

Because

$$f\circ g_1\circ\cdots\circ g_{r+1} = g_{r+1}^*\alpha\left(\prod_{i=1}^m z_i^{N_i}\right)z_{m+1}^{\sum_{i=1}^a N_i}$$

and

$$(g_1\circ\cdots\circ g_{r+1})^*dx = g_{r+1}^*\beta\left(\prod_{i=1}^m z_i^{\nu_i-1}\right)z_{m+1}^{\sum_{i=1}^a \nu_i}dz,$$

we have to prove that $N_{j,r}s_0+\nu_{j,r} = (N_{j,r}+\sum_{i=1}^a N_i)s_0+(\nu_{j,r}+\sum_{i=1}^a\nu_i)$. This follows from the fact that $N_is_0+\nu_i=0$ for $i\in\{1,\ldots,a\}$. Remark that it follows also from these calculations that

$$N_{j,r}\equiv N_j \bmod \gcd(N_1,\ldots,N_m).$$

(3.2.9) *Example.* We give an illustration which is easy and well known. Let $f = x_1^2 + x_2^2$. Let $X = \mathbb{Z}_p \times \mathbb{Z}_p$. We want to determine the poles of Igusa's p-adic zeta function associated to f. Notice that -1 is a square in $\mathbb{Q}_p$ if and only if -1 is a square $\mathbb{Z}/(p)$ and $p \neq 2$.

If -1 is a square in $\mathbb{Q}_p$, then (f, dx) has already normal crossings. We obtain a good chart for (f, dx) by applying the coordinate transformation $(y_1, y_2) \mapsto ((y_1 + y_2)/2, (y_1 - y_2)/(2a))$, where a denotes a square root of -1. Because $|a| = 1$ we obtain

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} |x_1^2 + x_2^2|^s |dx_1 \wedge dx_2| = \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |y_1 y_2|^s |dy_1 \wedge dy_2|.$$

Consequently, the only candidate poles of $Z_f(s)$ are the values $-1 + (2k\pi\sqrt{-1})/(\log p)$, $k \in \mathbb{Z}$. All these candidate poles are poles because $b_{-2} = ((p-1)/(p \log p))^2$ for each candidate pole.

If -1 is not a square in $\mathbb{Q}_p$, then (f, dx) has not normal crossings at the origin. We obtain an embedded resolution after one blowing-up g. Remark that the zero locus of f contains only the origin and that the zero locus of $f \circ g$ is equal to the exceptional curve E of g. We will use the two charts on the blowing-up determined by $(y_1, y_2) \mapsto (y_1 y_2, y_2)$ and $(z_1, z_2) \mapsto (z_1, z_1 z_2)$. The sets $\{(y_1, y_2) \mid y_1 \in \mathbb{Z}_p\,,\ y_2 = 0\}$ and $\{(z_1, z_2) \mid z_1 = 0\,,\ z_2 \in p\mathbb{Z}_p\}$ form a partition of E. The candidate poles of $Z_f(s)$ are $s_k = -1 + (2k\pi\sqrt{-1})/(2 \log p)$, $k \in \mathbb{Z}$, and each b_{-1} is equal to

$$\left(\frac{p-1}{2p \log p}\right) \left(\left[\int_{\mathbb{Z}_p} |y_1^2 + 1|^s |dy_1| \right]_{s=s_k}^{\mathrm{mc}} + \left[\int_{p\mathbb{Z}_p} |1 + z_2^2|^s |dz_2| \right]_{s=s_k}^{\mathrm{mc}} \right).$$

If $p \neq 2$, we have that $|1 + x^2| = 1$ for every $x \in \mathbb{Z}_p$, so that $b_{-1} = (p^2 - 1)/(2p^2 \log p)$. If $p = 2$, we have that $|1 + x^2| = 1$ for every $x \in 2\mathbb{Z}_2$ and $|1 + x^2| = 1/2$ for every $x \in 1 + 2\mathbb{Z}_2$, so that $b_{-1} = 1/(2 \log 2)$ if k is even and $b_{-1} = 0$ if k is odd.

Remark that Igusa's p-adic zeta function of $x_1^2 + x_2^2$ can be calculated completely elementarily in all the cases.

3.3 The vanishing results

3.3.1 Curves

Let X be an open and compact subset of K^2. Let f be a K-analytic function on X. Let $g : Y \to X$ be an embedded resolution of f. Write $g = g_1 \circ \cdots \circ g_t : Y = Y_t \to X = Y_0$ as a composition of blowing-ups $g_i : Y_i \to Y_{i-1}$, $i \in \{1, \ldots, t\}$. The exceptional curve of g_i and also the strict transforms of this curve are denoted by E_i. Let χ be a character of $R^\times$.

Proposition. *Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V, y = (y_1, y_2))$ centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part a (non-constant) monomial. Then the contribution of E_r to the residue b_{-1} of $Z_{f,\chi}(s)$ at s_0 is zero.*

Remark. This proposition is essentially well known. Our proof differs slightly from the ones in [Ig5] and [Lo1] because we will calculate the contribution of E_r to b_{-1} just after the creation of E_r instead of on the embedded resolution. We incorporate this proof here because the same technique will be used in the proof of the more difficult result of Section 3.3.2.

Proof. We may suppose that (V, y) is a chart centred at P such that $f \circ g_1 \circ \cdots \circ g_{r-1} = e y_1^k y_2^l + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{c-1} y_2^{d-1} dy$ with $k, l \in \mathbb{Z}_{\geq 0}$, $c, d \in \mathbb{Z}_{>0}$, $e \in K^\times$ and ρ, θ K-analytic functions satisfying $\rho(0,0) \neq 0$ and $\mathrm{mult}(\theta) > k + l$. We consider here the case that k and l are both not zero. The case that k or l is zero can be treated analogously.

We look at the chart $(O, z = (z_1, z_2))$ on Y_r for which $g_r(z_1, z_2) = (z_1, z_1 z_2)$. Then

$$f \circ g_1 \circ \cdots \circ g_r \quad = \quad z_1^{k+l} \left(e z_2^l + z_1 \frac{\theta(z_1, z_1 z_2)}{z_1^{k+l+1}} \right)$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx \quad = \quad \rho(z_1, z_1 z_2) z_1^{c+d-1} z_2^{d-1} dz.$$

Remark that the equation of E_r is $z_1 = 0$, that $N_r = k + l$ and that $\nu_r = c + d$. Using the notation of (3.2.8), let $T_r = \{1, 2\}$ and let F_1 be

the origin of this chart. The contribution to b_{-1} of an open and compact subset A of E_r which is contained in O is equal to

$$\left(\frac{q-1}{qN_r \log q}\right)\left[\int_A \chi(\mathrm{ac}\, e)\chi^l(\mathrm{ac}\, z_2)|e|^s|\rho(0,0)||z_2|^{ls+d-1}|dz_2|\right]_{s=s_0}^{\mathrm{mc}}.$$

Let $(O', z' = (z_1', z_2'))$ be the chart on Y_r for which $g_r(z_1', z_2') = (z_1'z_2', z_2')$. The origin of this chart is the point F_2. Analogously as before, we obtain that the contribution to b_{-1} of an open and compact subset B of E_r which is contained in O' is equal to

$$\left(\frac{q-1}{qN_r \log q}\right)\left[\int_B \chi(\mathrm{ac}\, e)\chi^k(\mathrm{ac}\, z_1')|e|^s|\rho(0,0)||z_1'|^{ks+c-1}|dz_1'|\right]_{s=s_0}^{\mathrm{mc}}.$$

Because $\chi^{N_r} = 1$ (otherwise there are no candidate poles associated to E_r) and because $k+l = N_r$, we have that $\chi^k = 1$ if and only if $\chi^l = 1$.

Case 1: $\chi^k = \chi^l = 1$. Then the contribution of E_r to b_{-1} is equal to

$$\frac{\chi(\mathrm{ac}\, e)|e|^{s_0}|\rho(0,0)|(q-1)}{qN_r \log q}$$

times

$$\begin{aligned}
&\left[\int_R |z_2|^{ls+d-1}|dz_2|\right]_{s=s_0}^{\mathrm{mc}} + \left[\int_P |z_1'|^{ks+c-1}|dz_1'|\right]_{s=s_0}^{\mathrm{mc}} \\
&= \frac{q-1}{q}\frac{1}{1-q^{-\alpha_1}} + \frac{q-1}{q}\frac{q^{-\alpha_2}}{1-q^{-\alpha_2}} \\
&= \left(\frac{q-1}{q}\right)\left(\frac{1-q^{-\alpha_2}+q^{-\alpha_2}-q^{-\alpha_1-\alpha_2}}{(1-q^{-\alpha_1})(1-q^{-\alpha_2})}\right) \\
&= 0.
\end{aligned}$$

The last equality follows from $\alpha_1 + \alpha_2 = 0$, which is Relation 2 of (3.2.8).

Case 2: $\chi^k \neq 1$ and $\chi^l \neq 1$. Then the contribution of E_r to b_{-1} is equal to zero because both terms in the sum

$$\left[\int_R \chi^l(\mathrm{ac}\, z_2)|z_2|^{ls+d-1}|dz_2|\right]_{s=s_0}^{\mathrm{mc}} + \left[\int_P \chi^k(\mathrm{ac}\, z_1')|z_1'|^{ks+c-1}|dz_1'|\right]_{s=s_0}^{\mathrm{mc}}$$

are equal to zero. □

3.3.2 Surfaces

Let X be an open and compact subset of K^3. Let f be a K-analytic function on X. Let $g : Y = Y_t \to X = Y_0$ be an embedded resolution of f which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$ with centre a K-analytic closed submanifold which has only normal crossings with the union of the exceptional surfaces in Y_{i-1} and with exceptional surface E_i.

Proposition. *Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V, y = (y_1, y_2, y_3))$ centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part of the form $e y_1^k y_2^l y_3^m (y_1 + y_2)^n$, with $e \in K^\times$ and $k, l, m, n \in \mathbb{Z}_{\geq 0}$. Then the contribution of E_r to the residue b_{-1} of $Z_f(s)$ at s_0 is zero.*

Proof. We may suppose that $f \circ g_1 \circ \cdots \circ g_{r-1} = e y_1^k y_2^l y_3^m (y_1 + y_2)^n + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{a-1} y_2^{b-1} y_3^{c-1} (y_1 + y_2)^{d-1} dy$ with $a, b, c, d \in \mathbb{Z}_{>0}$ and ρ, θ K-analytic functions satisfying $\rho(0,0) \neq 0$ and $\mathrm{mult}(\theta) > k+l+m+n$. Remark that at least one of the numbers a, b, d is equal to 1. We consider here the case that k, l, m and n are all different from zero. The other cases are treated analogously. Let $T_r = \{1, 2, 3, 4\}$ and suppose that F_i, $i \in \{1, 2, 3\}$, is given by $y_i = 0$ and that F_4 is given by $y_1 + y_2 = 0$ in the homogeneous coordinates $(y_1 : y_2 : y_3)$ on $E_r \subset Y_r$.

We look at the chart $(O, z = (z_1, z_2, z_3))$ on Y_r for which $g_r(z_1, z_2, z_3) = (z_1 z_3, z_2 z_3, z_3)$. Then

$$f \circ g_1 \circ \cdots \circ g_r \;=\; z_3^{k+l+m+n} \left(e z_1^k z_2^l (z_1 + z_2)^n + z_3 \frac{\theta(z_1 z_3, z_2 z_3, z_3)}{z_3^{k+l+m+n+1}} \right)$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx \;=\; \rho(z_1 z_3, z_2 z_3, z_3) z_1^{a-1} z_2^{b-1} z_3^{a+b+c+d-2} (z_1 + z_2)^{d-1} dz.$$

Remark that the equation of E_r is $z_3 = 0$, that $N_r = k + l + m + n$ and that $\nu_r = a+b+c+d-1$. The contribution to b_{-1} of an open and compact subset A of E_r which is contained in O is equal to $\kappa = (q-1)/(qN_r \log q)$ times

$$\left[\int_A |e|^s |\rho(0,0,0)| |z_1|^{ks+a-1} |z_2|^{ls+b-1} |z_1 + z_2|^{ns+d-1} |dz_1 \wedge dz_2| \right]_{s=s_0}^{\mathrm{mc}}.$$

Let $(O', z' = (z'_1, z'_2, z'_3))$ be the chart on Y_r for which $g_r(z'_1, z'_2, z'_3) = (z'_1 z'_2, z'_2, z'_2 z'_3)$. Analogously as before, we obtain that the contribution to b_{-1} of an open and compact subset B of E_r which is contained in O' is equal to κ times

$$\left[\int_B |e|^s |\rho(0,0,0)| |z'_1|^{ks+a-1} |z'_3|^{ms+c-1} |z'_1+1|^{ns+d-1} |dz'_1 \wedge dz'_3|\right]_{s=s_0}^{\mathrm{mc}}.$$

Let $(O'', z'' = (z''_1, z''_2, z''_3))$ be the chart on Y_r for which $g_r(z''_1, z''_2, z''_3) = (z''_1, z''_1 z''_2, z''_1 z''_3)$. Analogously as before, we obtain that the contribution to b_{-1} of an open and compact subset C of E_r which is contained in O'' is equal to κ times

$$\left[\int_C |e|^s |\rho(0,0,0)| |z''_2|^{ls+b-1} |z''_3|^{ms+c-1} |1+z''_2|^{ns+d-1} |dz''_2 \wedge dz''_3|\right]_{s=s_0}^{\mathrm{mc}}.$$

Now we take $A = P \times P$, $B = P \times R$ and $C = R \times R$. Because these sets form a partition of E_r, the contribution to b_{-1} of E_r is the sum of the three contributions above.

We have to prove that the contribution to b_{-1} of E_r is equal to zero, so we have to prove that

$$\begin{aligned} &\left[\int_A |z_1|^{ks+a-1} |z_2|^{ls+b-1} |z_1+z_2|^{ns+d-1} |dz_1 \wedge dz_2|\right]_{s=s_0}^{\mathrm{mc}} \\ &+ \left[\int_B |z'_1|^{ks+a-1} |z'_3|^{ms+c-1} |z'_1+1|^{ns+d-1} |dz'_1 \wedge dz'_3|\right]_{s=s_0}^{\mathrm{mc}} \qquad (*) \\ &+ \left[\int_C |z''_2|^{ls+b-1} |z''_3|^{ms+c-1} |1+z''_2|^{ns+d-1} |dz''_2 \wedge dz''_3|\right]_{s=s_0}^{\mathrm{mc}} \end{aligned}$$

is equal to zero.

To calculate the first term in $(*)$, we partition A into

$$\begin{aligned} A_1 &= \{(z_1, z_2) \in P \times P \mid \operatorname{ord} z_1 > \operatorname{ord} z_2\} \\ &= \bigsqcup_{i \in \mathbb{Z}_{>0}} \{(z_1, z_2) \mid \operatorname{ord} z_1 > \operatorname{ord} z_2 = i\} \\ A_2 &= \{(z_1, z_2) \in P \times P \mid \operatorname{ord} z_1 < \operatorname{ord} z_2\} \\ &= \bigsqcup_{i \in \mathbb{Z}_{>0}} \{(z_1, z_2) \mid i = \operatorname{ord} z_1 < \operatorname{ord} z_2\} \\ A_3 &= \{(z_1, z_2) \in P \times P \mid \operatorname{ord} z_1 = \operatorname{ord} z_2\} \\ &= \bigsqcup_{i \in \mathbb{Z}_{>0}} \{(z_1, z_2) \mid \operatorname{ord} z_1 = \operatorname{ord} z_2 = i\} \end{aligned}$$

The contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\left[\sum_{i=1}^{\infty}\int_{P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}|z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_2|\right)|dz_1|\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left[\sum_{i=1}^{\infty}\frac{q-1}{q}q^{-i}q^{-i(ls+b-1)}q^{-i(ns+d-1)}\int_{P^{i+1}}|z_1|^{ks+a-1}|dz_1|\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left[\sum_{i=1}^{\infty}\frac{q-1}{q}q^{-i}q^{-i(ls+b-1)}q^{-i(ns+d-1)}\frac{q-1}{q}\frac{q^{-i(ks+a)}}{q^{ks+a}-1}\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left[\left(\frac{q-1}{q}\right)^2\frac{1}{q^{ks+a}-1}\sum_{i=1}^{\infty}q^{-i(ks+a+ls+b+ns+d-1)}\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left[\left(\frac{q-1}{q}\right)^2\frac{1}{(q^{ks+a}-1)(q^{ks+a+ls+b+ns+d-1}-1)}\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left(\frac{q-1}{q}\right)^2\frac{1}{(q^{\alpha_1}-1)(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)}. \qquad (3.5)
\end{aligned}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to

$$\left(\frac{q-1}{q}\right)^2\frac{1}{(q^{\alpha_2}-1)(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)}. \qquad (3.6)$$

The contribution of A_3 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\left[\sum_{i=1}^{\infty}\int_{(P^i\setminus P^{i+1})^2}|z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_1\wedge dz_2|\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left[\sum_{i=1}^{\infty}q^{-i(ks+a-1)}q^{-i(ls+b-1)}\int_{(P^i\setminus P^{i+1})^2}|z_1+z_2|^{ns+d-1}|dz_1\wedge dz_2|\right]_{s=s_0}^{\mathrm{mc}}\\
&\quad=\left[\sum_{i=1}^{\infty}q^{-i(ks+a+ls+b-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{-z_2+P^{i+1}}|z_1+z_2|^{ns+d-1}|dz_1|\right.\right.\\
&\qquad\left.\left.+\int_{(P^i\setminus P^{i+1})\setminus(-z_2+P^{i+1})}|z_1+z_2|^{ns+d-1}|dz_1|\right)|dz_2|\right]_{s=s_0}^{\mathrm{mc}}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b-2)} \right. \\
&\qquad \left. \int_{P^i \setminus P^{i+1}} \frac{q-1}{q}\frac{q^{-i(ns+d)}}{q^{ns+d}-1} + \frac{q-2}{q} q^{-i} q^{-i(ns+d-1)} |dz_2| \right]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b-2)} \right. \\
&\qquad \left. \frac{q-1}{q} q^{-i} \left(\frac{q-1}{q}\frac{q^{-i(ns+d)}}{q^{ns+d}-1} + \frac{q-2}{q} q^{-i(ns+d)} \right) \right]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\left(\frac{q-1}{q}\right)^2 \frac{1}{q^{ns+d}-1} \sum_{i=1}^{\infty} q^{-i(ks+a+ls+b+ns+d-1)} \right. \\
&\qquad \left. + \frac{q-1}{q}\frac{q-2}{q} \sum_{i=1}^{\infty} q^{-i(ks+a+ls+b+ns+d-1)} \right]_{s=s_0}^{\mathrm{mc}} \\
&= \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_4}-1)(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)} \qquad (3.7) \\
&\qquad + \left(\frac{q-1}{q}\right)\left(\frac{q-2}{q}\right) \frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}. \qquad (3.8)
\end{aligned}
$$

The second term of $(*)$ is equal to

$$
\begin{aligned}
&\left[\int_P |z_1|^{ks+a-1}|z_1+1|^{ns+d-1}|dz_1| \int_R |z_3|^{ms+c-1}|dz_3|\right]_{s=s_0}^{\mathrm{mc}} \\
&\qquad = \left[\int_P |z_1|^{ks+a-1}|dz_1| \int_R |z_3|^{ms+c-1}|dz_3|\right]_{s=s_0}^{\mathrm{mc}} \\
&\qquad = \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_1}-1)(1-q^{-\alpha_3})}. \qquad (3.9)
\end{aligned}
$$

The third term of $(*)$ is equal to

$$
\begin{aligned}
&\left[\int_R |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2| \int_R |z_3|^{ms+c-1}|dz_3|\right]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\left(\int_{R\setminus(P\cup -1+P)} |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2| \right.\right. \\
&\qquad \left.\left. + \int_P |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2| + \int_{-1+P} |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2|\right)\right.
\end{aligned}
$$

$$\left(\int_R |z_3|^{ms+c-1}|dz_3|\right)\Big]_{s=s_0}^{\mathrm{mc}}$$

$$= \left[\left(1-\frac{2}{q}+\int_P |z_2|^{ls+b-1}|dz_2| + \int_{-1+P}|1+z_2|^{ns+d-1}|dz_2|\right)\right.$$
$$\left.\left(\int_R |z_3|^{ms+c-1}|dz_3|\right)\right]_{s=s_0}^{\mathrm{mc}}$$

$$= \left(1-\frac{2}{q}+\frac{q-1}{q}\frac{1}{q^{\alpha_2}-1}+\frac{q-1}{q}\frac{1}{q^{\alpha_4}-1}\right)\left(\frac{q-1}{q}\frac{1}{1-q^{-\alpha_3}}\right)$$

$$= \left(\frac{q-1}{q}\right)\left(\frac{q-2}{q}\right)\frac{1}{1-q^{-\alpha_3}} \tag{3.10}$$

$$+\left(\frac{q-1}{q}\right)^2\frac{1}{(q^{\alpha_2}-1)(1-q^{-\alpha_3})} \tag{3.11}$$

$$+\left(\frac{q-1}{q}\right)^2\frac{1}{(q^{\alpha_4}-1)(1-q^{-\alpha_3})}. \tag{3.12}$$

Relation 2 of (3.2.8) is $\alpha_1+\alpha_2+\alpha_3+\alpha_4-1=0$, so that we obtain

$$\begin{aligned}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}+\frac{1}{1-q^{-\alpha_3}} &= \frac{1-q^{-\alpha_3}+q^{\alpha_1+\alpha_2+\alpha_4-1}-1}{(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)(1-q^{-\alpha_3})}\\ &= \frac{q^{\alpha_1+\alpha_2+\alpha_3+\alpha_4-1}-1}{(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)(q^{\alpha_3}-1)}\\ &= 0,\end{aligned}$$

and consequently (3.5) + (3.9) = 0. Analogously, we obtain that (3.6) + (3.11) = (3.7) + (3.12) = (3.8) + (3.10) = 0. Consequently, the contribution of E_r to b_{-1} is equal to zero. □

Remark. (i) If k, l, m and n are not all different from zero, then the same calculations are valid. Now, $T_r \subset \{1,2,3,4\}$ and F_i, $i \in T_r$, is given by the same equation as before. If $i \in \{1,2,3,4\}\backslash T_r$, we have to put formally $\alpha_i = 1$.

(ii) Suppose that we are in the same situation as in the proposition and let χ be an arbitrary character of $R^\times$. Then one can show that the contribution of E_r to the residue b_{-1} of $Z_{f,\chi}(s)$ at s_0 is zero. The proof consists of very long calculations involving character sums. This will appear in Chapter 4.

3.4 Determination of the smallest poles

The main ideas and results of this section have the same flavour as those in Chapter 2, where the local topological zeta function is studied. However it is here sometimes more complicated because the field K is not algebraically closed.

3.4.1 Curves

(3.4.1.1) In this section we will determine $\mathcal{P}_2^K \cap]-\infty, -1/2[$. Let f be a K-analytic function on an open and compact subset of K^2 and let g be the *minimal* embedded resolution of f. The poles of $Z_f(s)$ with real part less than $-1/2$ and different from -1 are only associated to exceptional curves. Consequently, these poles are completely determined by the germs of f at the points where f has not normal crossings. It is thus sufficient to study the germs of K-analytic functions at the origin, which will be identified with the convergent power series. The set of all convergent power series in the variables x and y is classically denoted by $K\ll x, y\gg$.

(3.4.1.2) Let $f \in K\ll x, y\gg$. Let $g : Y \to X$ be the *minimal* embedded resolution of a representative of f. Write $g = g_1 \circ \cdots \circ g_t : Y = Y_t \to X = Y_0$ as a composition of blowing-ups $g_i : Y_i \to Y_{i-1}$, $i \in \{1, \ldots, t\}$. The exceptional curve of g_i and also the strict transforms of this curve are denoted by E_i. Let T be as in (3.2.1) and obviously we suppose that $\{1, \ldots, t\} \subset T$.

Let $k \in \{1, \ldots, t\}$. Let $P \in Y_k$ be a point on an exceptional curve, i.e., a point which is mapped to the origin under the map $g_1 \circ \cdots \circ g_k$. The strict transform of f around P is defined as the germ at P of the K-analytic function $f \circ g_1 \circ \cdots \circ g_k$ divided by the highest possible powers of local equations of exceptional curves through P. Remark that the strict transform of f around P is defined modulo the germ of a K-analytic function which does not vanish at P as a factor.

We call a complex number 'a candidate pole of $Z_f(s)$' if it is a candidate pole associated to an E_i, $i \in T$, satisfying $0 \in g(E_i)$. A candidate pole of $Z_f(s)$ is called a pole of $Z_f(s)$ if there exists an arbitrarily small neighbourhood of 0 for which it is a pole.

The following lemma is trivial.

(3.4.1.3) Lemma. *Suppose that we have blown up k times but we do not yet have an embedded resolution. Let P be a point at which $f \circ g_1 \circ \cdots \circ g_k$ has not normal crossings. Let μ be the multiplicity in P of the strict transform of f around P and let g_{k+1} be the blowing-up at P.*

(a) Suppose that two exceptional curves E_i and E_j contain P. Then $-\nu_{k+1}/N_{k+1}$ is equal to $-(\nu_i+\nu_j)/(N_i+N_j+\mu)$ and this is larger than $\min\{-\nu_i/N_i, -\nu_j/N_j\}$.

(b) Suppose that exactly one exceptional curve E_i contains P and that $\mu \geq 2$. Then E_{k+1} has numerical data $(N_i+\mu, \nu_i+1)$ and $-(\nu_i+1)/(N_i+\mu)$ is between $-1/\mu$ and $-\nu_i/N_i$.

(c) Suppose that exactly one exceptional curve E_i contains P and that $\mu = 1$. Note that the two curves are tangent at P because we do not have normal crossings at P. Let g_{k+2} be the blowing-up at $E_i \cap E_{k+1}$. Remark that we do not have to blow up at a point of E_{k+1} anymore. The numerical data of E_{k+2} are $(2N_i+2, 2\nu_i+1)$, and $-(2\nu_i+1)/(2N_i+2)$ is between $-1/2$ and $-\nu_i/N_i$. Let s_0 be a candidate pole associated to E_{k+1}. Because s_0 is not a candidate pole associated to E_{k+2}, which is a consequence of $-\nu_{k+1}/N_{k+1} \neq -\nu_{k+2}/N_{k+2}$, the contribution of E_{k+1} to the coefficient b_{-2} in the MacLaurin series of $Z_f(s)$ at s_0 is zero. It follows from the proposition in Section 3.3.1 that E_{k+1} does not give a contribution to the residue b_{-1} of $Z_f(s)$ at s_0.

(3.4.1.4) Suppose that after some blowing-ups, the pullback of f has not normal crossings at a point P. Suppose also that the real parts of the candidate poles associated to the exceptional curves through P are all larger than or equal to $-1/2$. Then it follows from the above lemma that the components above P in the final resolution do not give a contribution to a candidate pole with real part less than $-1/2$.

Corollary. *Zeta functions of convergent power series of multiplicity at least four do not have a pole with real part in $]-\infty, -1/2[\backslash\{-1\}$.*

Indeed, every exceptional curve in the minimal embedded resolution of f lies above a point of E_1 (considered in the stage when it is created), which has a candidate pole with real part larger than or equal to $-1/2$.

(3.4.1.5) To deal with multiplicity two and three, we will study an 'easier' element of $K \ll x, y \gg$. We will use the following theorem (see [Ig7, Theorem 2.3.1]).

Weierstrass Preparation Theorem.
If $f(z_1,\ldots,z_{n-1},w)=f(z,w)\in K\ll z,w\gg$ is not identically zero on the w-axis, then f can be written uniquely as $f=(w^e+a_1(z)w^{e-1}+\cdots+a_e(z))h$, where $a_i(z)\in K\ll z\gg$ satisfies $a_i(0)=0$ and $h\in K\ll z,w\gg$ satisfies $h(0)\neq 0$.

Because $h(0)\neq 0$ implies that $|h|$ is constant on a neighbourhood of 0, we have that Igusa's p-adic zeta functions of f and $w^e+a_1(z)w^{e-1}+\cdots+a_e(z)$ have the same poles. After an appropriate coordinate transformation, the desired form will appear. For example, the coordinate transformation $(z,w)\mapsto(z,w-a_1(z)/e)$ cancels the term $a_1(z)w^{e-1}$.

(3.4.1.6) *Example.* Let $f\in K\ll x,y\gg$ have multiplicity three and let $f_3=y^3+xy^2=y^2(y+x)$ be the homogeneous part of f of degree 3. By the Weierstrass Preparation Theorem, we may work with a function of the form $y^3+a_1(x)y^2+a_2(x)y+a_3(x)$, with $\mathrm{mult}(a_1(x))=1$, $\mathrm{mult}(a_2(x))\geq 3$ and $\mathrm{mult}(a_3(x))\geq 4$. One can check that there exists a coordinate transformation $(x,y)\mapsto(x,y-k(x))$ such that the function becomes of the form $y^3+b_1(x)y^2+b_3(x)$, with $\mathrm{mult}(b_1(x))=1$ and $\mathrm{mult}(b_3(x))\geq 4$. After another coordinate transformation, we get the form $y^3+xy^2+g(x)$, with $\mathrm{mult}(g(x))\geq 4$.

(3.4.1.7) Theorem. *We have*

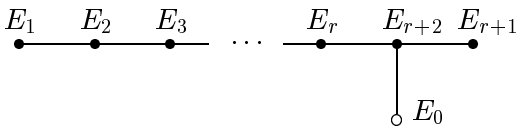
$$\mathcal{P}_2^K\cap\left]-\infty,-\frac{1}{2}\right[=\left\{-\frac{1}{2}-\frac{1}{i}\,\middle|\, i\in\mathbb{Z}_{>1}\right\}$$

and every Igusa's p-adic zeta function has at most one pole in $]-1,-1/2]$. *Moreover, if* $f\in K\ll x,y\gg$ *has multiplicity at least four, then* $Z_f(s)$ *has no pole with real part in* $]-\infty,-1/2[\setminus\{-1\}$.

Proof. Because the calculations are analogous to the calculations in (2.2.9) for the local topological zeta function, we do not treat all the cases in this paper.

(a) Suppose that f is an element of $K\ll x_1,x_2\gg$ with multiplicity two. When we apply the ideas of (3.4.1.5), we see that it is enough to consider x_1^2 and $x_1^2+ax_2^l$, with $l\in\mathbb{Z}_{>1}$ and $a\in K^\times$. If $f=x_1^2$, the candidate poles of $Z_f(s)$ are $-1/2+(k\pi\sqrt{-1})/(\log p)$, $k\in\mathbb{Z}$. If $l=2$, the calculations are analogous as in (3.2.9). If l is odd, write $l=2r+1$. After r blowing-ups, the strict transform of $f^{-1}\{0\}$ is non-singular and

tangent to E_r. The numerical data of E_i, $i = 1, \ldots, r$, are $(2i, i+1)$. To get the minimal embedded resolution, we now blow up twice. Let E_0 be the strict transform of $f^{-1}\{0\}$. Remark that $T = \{0, 1, \ldots, r\}$. The dual resolution graph is



and the numerical data are $E_1(2,2)$, $E_2(4,3)$, $E_3(6,4)$, $\ldots$, $E_r(2r, r+1)$, $E_{r+1}(2r+1, r+2)$ and $E_{r+2}(4r+2, 2r+3)$. It follows from Section 3.3.1 that the candidate poles associated to $E_1, \ldots, E_{r+1}$ are not poles. The other candidate poles have real part -1 or $-(2r+3)/(4r+2) = -1/2 - 1/(2r+1)$. We calculate the residue of $Z_f(s)$ at the candidate pole $s_0 = -1/2 - 1/(2r+1)$. Because

$$\begin{aligned}
&\left[\int_{aR} |y_1|^{(2r+1)s+r+1} |y_1 + a|^s |dy_1|\right]_{s=s_0}^{\mathrm{mc}} \\
&= |a|^{-1/(2r+1)} \left[\int_R |y|^{(2r+1)s+r+1} |y+1|^s |dy|\right]_{s=s_0}^{\mathrm{mc}} \\
&= |a|^{-1/(2r+1)} \left[\int_{R\setminus(-1+P)} |y|^{(2r+1)s+r+1} |dy| + \int_{-1+P} |y+1|^s |dy|\right]_{s=s_0}^{\mathrm{mc}} \\
&= |a|^{-1/(2r+1)} \left(\frac{q-2}{q} + \frac{q-1}{q}\frac{1}{q^{\alpha_{r+1}}-1} + \frac{q-1}{q}\frac{1}{q^{\alpha_0}-1}\right)
\end{aligned}$$

and

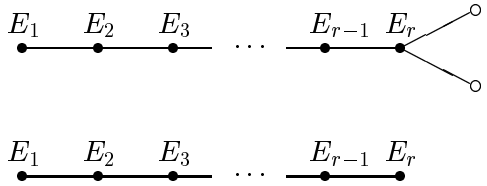
$$\begin{aligned}
&\left[\int_{\frac{1}{a}P} |y_2|^{2rs+r} |1 + a y_2|^s |dy_2|\right]_{s=s_0}^{\mathrm{mc}} \\
&= |a|^{-1/(2r+1)} \left[\int_P |y|^{2rs+r} |1+y|^s |dy|\right]_{s=s_0}^{\mathrm{mc}} \\
&= |a|^{-1/(2r+1)} \left[\int_P |y|^{2rs+r} |dy|\right]_{s=s_0}^{\mathrm{mc}} \\
&= |a|^{-1/(2r+1)} \frac{q-1}{q}\frac{1}{q^{\alpha_r}-1}
\end{aligned}$$

the residue of $Z_f(s)$ at the candidate pole $s_0 = -1/2 - 1/(2r+1)$ is

$$|a|^{-1/(2r+1)} \left(\frac{q-2}{q} + \frac{q-1}{q}\frac{1}{q^{\alpha_{r+1}}-1} + \frac{q-1}{q}\frac{1}{q^{\alpha_0}-1} + \frac{q-1}{q}\frac{1}{q^{\alpha_r}-1}\right)$$

multiplied by the factor κ which is different from zero (see (3.2.7)). Because $\alpha_{r+1} = (2r+1)s_0+r+2 = 1/2 > 0$, $\alpha_0 = s_0+1 = 1/2-1/(2r+1) > 0$ and $\alpha_r = 2rs_0 + r + 1 = 1/(2r+1) > 0$, we have that the last three terms of this expression are strictly positive. Consequently the whole expression is strictly positive and thus different from zero, so that $-1/2 - 1/(2r+1)$ is a pole of $Z_f(s)$.

If l is even and larger than 2, write $l = 2r$. We have to blow up r times to obtain an embedded resolution. We have $E_1(2,2)$, $E_2(4,3)$, $E_3(6,4)$, ..., $E_{r-1}(2r-2,r)$, $E_r(2r,r+1)$. We obtain the first dual resolution graph if $-a$ is a square in K. Otherwise, we obtain the second dual resolution graph.



It follows from Section 3.3.1 that the candidate poles associated to $E_1, \ldots, E_{r-1}$ are not poles. The other candidate poles have real part -1 or $-(r+1)/(2r) = -1/2 - 1/(2r)$ in the first case and $-(r+1)/(2r) = -1/2 - 1/(2r)$ in the second case. Now we prove that $-1/2 - 1/(2r)$ is an element of $\mathcal{P}_2^K$. Suppose first that $p \neq 2$. Then there exists an element a of K with norm 1 for which $-a$ is not a square in K. For such an a, the residue of $Z_f(s)$ at $s_0 = -1/2 - 1/(2r)$ is the non-zero factor κ times

$$\frac{q-1}{q}\frac{1}{q^{\alpha_{r-1}}-1} + 1.$$

Suppose now that $p = 2$. Remark that every element of the residue field is a square in this case. Let $b \in R^\times$. If $b' \in b + P$, then $b'^2 - b^2 \in P^2$. Consequently, there exists an $a \in -b^2 + P$ such that $|a + x^2| = 1/q$ for all $x \in b + P$. For such an a, the residue of $Z_f(s)$ at $s_0 = -1/2 - 1/(2r)$ is the non-zero factor κ times

$$\frac{q-1}{q}\frac{1}{q^{\alpha_{r-1}}-1} + \frac{q-1}{q} + \frac{1}{q}\left(\frac{1}{q}\right)^{s_0}.$$

Because $\alpha_{r-1} = (2r-2)s_0 + r = 1/r > 0$, we obtain in the two cases that this residue is strictly positive, which implies that $-1/2 - 1/(2r)$ is a pole.

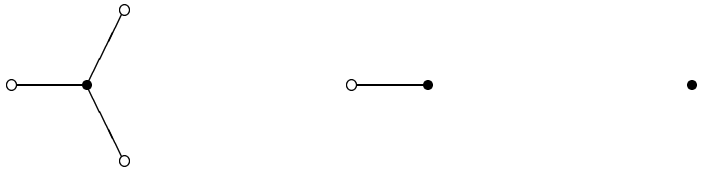
Our conclusion of part (a) is thus that the set $\{s_0 \mid \exists f \in K\ll x_1, x_2\gg :$ mult$(f) = 2$ and $Z_f(s)$ has a pole with real part $s_0\}$ is equal to

$$\left\{ -\frac{1}{2} - \frac{1}{i} \middle| \, i \in \mathbb{Z}_{>1} \right\} \cup \left\{ -\frac{1}{2} \right\}.$$

Remark that Newton polyhedra could also be used to deal with (a) if $p \neq 2$, see [DH].

(b) Suppose that f is an element of $K\ll x_1, x_2\gg$ with multiplicity three. Up to an affine coordinate transformation, there are three cases for f_3.

We consider the case that f_3 is a product of three different linear factors over $K^{\text{alg cl}}$. Then we obtain an embedded resolution after one blowing-up. There are three possibilities for the dual resolution graph, depending on whether f_3 splits into linear factors over K, f_3 is a product of a linear factor and an irreducible factor of degree 2 over K or f_3 is irreducible over K. The dual resolution graphs are respectively



The equations of $f_3 \circ g$ in the charts determined by $(y_1, y_2) \mapsto (y_1, y_1y_2)$ and $(z_1, z_2) \mapsto (z_1z_2, z_2)$ are respectively of the form $y_1^3h_1$ and $z_2^3h_2$. In the last case for example, we have that h_1 and h_2 are non-vanishing on the exceptional curve.
The real parts of the candidate poles of $Z_f(s)$ are -1 and $-2/3 = -1/2 - 1/6$ in the first two cases and $-2/3 = -1/2 - 1/6$ in the last case.

The other cases are treated in (2.2.9) for the topological zeta function and are very similar for Igusa's p-adic zeta function.

(c) Suppose that f is an element of $K\ll x_1, x_2\gg$ with multiplicity at least four. We explained in (3.4.1.4) that $Z_f(s)$ has no pole with real part in $]-\infty, -1/2[\backslash\{-1\}$. $\square$

(3.4.1.8) Let χ be a character of $R^\times$. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}^K_{n,\chi}$ by

$$\mathcal{P}^K_{n,\chi} := \{s_0 \mid \exists f \in F^K_n \ : \ Z_{f,\chi}(s) \text{ has a pole with real part } s_0\}.$$

Theorem. *We have*

$$\mathcal{P}^K_{2,\chi} \cap \left]-\infty, -\frac{1}{2}\right[\subset \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}$$

and every Igusa's p-adic zeta function has at most one pole in $]-1, -1/2]$.

Proof. In the proof of the previous theorem we needed only the proposition in Section 3.3.1 to obtain that some candidate poles were not poles. Now we work with a character and we can use the same proposition to prove that these candidate poles are not poles. □

3.4.2 Surfaces

In this section, we prove the following theorem.

(3.4.2.0) Theorem. *We have*

$$\mathcal{P}^K_3 \cap]-\infty, -1[= \left\{ -1 - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}.$$

Moreover, if $f \in K\ll x, y, z\gg$ *has multiplicity at least three, then* $Z_f(s)$ *has no pole with real part less than* -1.

Remark. (i) It is a priori not obvious that the smallest value of $\mathcal{P}^K_3$ is $-3/2$. This is in contrast with the fact that it easily follows from Lemma 3.4.1.3 that the smallest value of $\mathcal{P}^K_2$ is -1.
(ii) Let χ be a character of $R^\times$. Then one proves analogously as below that an element of $\mathcal{P}^K_{3,\chi}$ less than -1 is of the form $-1 - 1/i$, $i \in \mathbb{Z}_{>1}$. Using remark (ii) in Section 3.3.2, the arguments below will also imply that $Z_{f,\chi}(s)$ has no pole with real part less than -1 if $f \in K\ll x, y, z\gg$ has multiplicity at least three.

3.4.2.1 Multiplicity two

(3.4.2.1.1) Let $f(x)$, $x = (x_1, \ldots, x_n)$, be a K-analytic function on an open and compact subset X of K^n. Let $g(y)$, $y = (y_1, \ldots, y_m)$, be a K-analytic function on an open and compact subset Y of K^m. Then

$f(x)+g(y)$ is a K-analytic function on the open and compact subset $X\times Y$ of K^{n+m}. Put $A(s,\rho):=q^{s+1}-1$ if ρ is the trivial character of $R^\times$ and $A(s,\rho):=1$ if ρ is another character of $R^\times$.

Fix a character χ of $R^\times$. Suppose that the only critical value of f and g is zero. Then the poles of $A(s,\chi)Z_{f+g,\chi}(s)$ are of the form s_1+s_2 with s_1 a pole of $A(s,\chi')Z_{f,\chi'}(s)$ and s_2 a pole of $A(s,\chi'')Z_{g,\chi''}(s)$ for some characters χ' and χ'' of $R^\times$ satisfying $\chi'\chi''=\chi$ (see [Ig4] or [De2, (5.1)]).

(3.4.2.1.2) Proposition. *The set* $\{s_0 \mid \exists f\in K\ll x,y,z\gg : mult(f)=2$ *and* $Z_f(s)$ *has a pole in* $s_0\}\cap]-\infty,-1[$ *is equal to*

$$\left\{-1-\frac{1}{i}\;\middle|\; i\in\mathbb{Z}_{>1}\right\}.$$

Proof. Let f be an element of $K\ll x,y,z\gg$ with multiplicity two. Up to an affine coordinate transformation, the part of degree two of f is equal to $ax^2+by^2+cz^2$, with $a,b,c\in K$ and $a\neq 0$. Using (3.4.1.5), we may suppose that f is of the form $x^2+g(y,z)$ with $g(y,z)\in K\ll y,z\gg$. The statement in (3.4.2.1.1) and the result for curves imply that every pole of $Z_f(s)$ less than -1 is of the form $-1-1/i$, $i\in\mathbb{Z}_{>1}$.

Now we prove the other inclusion. Using the p-adic stationary phase formula (1.1.7), we obtain that Igusa's p-adic zeta function of $xy+z^i$, $i\geq 2$, is equal to

$$\left(\frac{q-1}{q}\right)\left(\frac{1-q^{-s-3}+(q-1)(q^{-2s-4}+q^{-3s-5}+\cdots+q^{-(i-1)s-(i+1)})}{(1-q^{-s-1})(1-q^{-is-(i+1)})}\right).$$

The real poles of this zeta function are -1 and $-1-1/i$. □

3.4.2.2 Multiplicity larger than two

(3.4.2.2.1) Let f be an element of $K\ll x,y,z\gg$. Fix a (small enough) neighbourhood X of $0\in K^3$ on which f is convergent and an embedded resolution $g: Y\to X$ of f which is a K-bianalytic map at the points where f has normal crossings and which is a composition of blowing-ups $g_{ij}: X_i\to X_j$ with centre a K-analytic closed submanifold D_j and with exceptional surface E_i satisfying:

(a) the codimension of D_j in X_j is at least two;

(b) D_j is a subset of the zero locus of the strict transform of f on each chart (the strict transform of f is not defined globally);

(c) the union of the exceptional varieties in X_j has only normal crossings with D_j, i.e., for all $P \in D_j$, there are three surface germs through P which are in normal crossings such that each exceptional surface germ through P is one of them and such that the germ of D_j at P is the intersection of some of them;

(d) the image of D_j in $X \subset K^3$ contains the origin of K^3; and

(e) D_j contains a point in which the pullback of f has not normal crossings.

Remark that such a resolution always exists by Hironaka's theorem [Hi].

(3.4.2.2.2) The following table gives the numerical data of E_i. In the columns, the dimension of D_j is kept fixed. In the rows, the number of exceptional surfaces through D_j is kept fixed. So E_k, E_l and E_m represent exceptional surfaces that contain D_j. The multiplicity of the strict transform of f in D_j is denoted by μ_{D_j}.

	D_j is a point P	D_j is a curve L
/	$(\mu_P, 3)$	$(\mu_L, 2)$
E_k	$(N_k + \mu_P, \nu_k + 2)$	$(N_k + \mu_L, \nu_k + 1)$
E_k and E_l	$(N_k + N_l + \mu_P, \nu_k + \nu_l + 1)$	$(N_k + N_l + \mu_L, \nu_k + \nu_l)$
E_k, E_l and E_m	$(N_k + N_l + N_m + \mu_P, \nu_k + \nu_l + \nu_m)$	/

(3.4.2.2.3) Lemma. *Suppose that* $\mathrm{mult}(f) \geq 3$. *If there is no exceptional surface through* D_j, *then* $-\nu_i/N_i \geq -1$.

Proof. The analogous statement for the local topological zeta function is treated in (2.3.3.3). The proof of the lemma is a trivial adaptation of the proof there. □

(3.4.2.2.4) Suppose that D_j is contained in at least one exceptional surface and that the real parts of the candidate poles associated to the exceptional surfaces that pass through D_j are larger than or equal to -1. Then the table in (3.4.2.2.2) implies that also $-\nu_i/N_i \geq -1$, unless D_j is a regular point P of the strict transform of f around P through which only one exceptional surface E_0 passes and $-\nu_0/N_0 = -1$. Suppose that we are in this situation. Let Z_0 be a (small enough) neighbourhood of P

such that, if we restrict the blowing-ups g_{ij} to the inverse image of Z_0, we get an embedded resolution $h = h_1 \circ \cdots \circ h_s$ of the pullback of f which is a composition of blowing-ups $h_i : Z_i \to Z_{i-1}$, $i \in \{1, \ldots, s\}$, with centre $D'_{i-1} := D_{i-1} \cap Z_{i-1}$ and exceptional surface $E'_i := E_i \cap Z_i$ for which P is in the image of D'_{i-1} under $h_1 \circ \cdots \circ h_{i-1}$.

Remark that it can happen that g_{ij} is a K-bianalytic map on the inverse image of Z_0. Because we did not specify the indices in (3.4.2.2.1), we were able to get a nice notation here. From now on, we study the resolution $h : Z_s \to Z_0$ of the pullback of f.

Lemma. *(a) If $D_i = D'_i$, then D_i is a subset of $E'_0 := E_0 \cap Z_0$.*

(b) Suppose that $\mathrm{mult}(f) \geq 3$. *Then we have $\nu_i \leq N_i + 1$ for every exceptional surface E_i, $i \in \{1, \ldots, s\}$. Moreover, $\nu_i = N_i + 1$ if and only if D_{i-1} is a point and the numerical data of every exceptional surface E_j different from E_0 and through D_{i-1} satisfy $\nu_j = N_j + 1$.*

(c) If $\mathrm{mult}(f) \geq 3$ *and if the numerical data of E_i satisfy $\nu_i = N_i + 1$, then $-\nu_i/N_i \neq -\nu_j/N_j$ for every exceptional surface E_j that intersects E_i at some stage of the resolution process.*

Proof. See (2.3.3.5),(2.3.3.6) and (2.3.3.7). □

Proposition. *If* $\mathrm{mult}(f) \geq 3$, *then $Z_f(s)$ has no pole with real part less than -1.*

Proof. The proof is analogous to the one in (2.3.3.8). Now we have to use the proposition in Section 3.3.2. □

3.5 Consequences for the M_i

Suppose that f is a K-analytic function on R^n defined by a power series over R which is convergent on the whole of R^n. Let l be the smallest real part of a pole of $Z_f(s)$ and let M_i be the number of solutions of $f(x) \equiv 0 \bmod P^i$ in $(R/P^i)^n$.

Proposition. *There exists an integer a which is independent of i such that M_i is an integer multiple of $q^{\lceil (n+l)i - a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$.*

Remark. (i) The number $\lceil (n+l)i - a \rceil$ is the smallest integer larger than or equal to $(n+l)i - a$, which rises ($n + l > 0$) linearly as a function of i with a slope depending on l.

(ii) The statement is trivial if $(n+l)i-a \leq 0$ because the M_i are integers. If $(n+l)i-a > 0$, which is the case for i large enough, it claims that M_i is divisible by $q^{\lceil (n+l)i-a \rceil}$.

Proof. Put $t = q^{-s}$. It follows from (3.2.2) that we can write

$$Z_f(t) = \frac{A(t)}{\prod_{j \in T} \left(1 - q^{-\nu_j} t^{N_j}\right)},$$

where $A(t)$ is a polynomial with coefficients in the set $S := \{z/q^i \mid z \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 0}\}$. By using the division algorithm for polynomials we can write

$$Z_f(t) = \frac{B(t)}{\prod_{j \in K} \left(1 - q^{-\nu_j} t^{N_j}\right)},$$

where $B(t)$ is a polynomial with coefficients in S and where $K := \{j \in T \mid -\nu_j/N_j \geq l\}$.

The Poincaré series $P(t)$ of f is defined by

$$P(t) = \sum_{i=0}^{\infty} M_i \frac{t^i}{q^{ni}}.$$

and can be obtained from $Z_f(t)$ by the relation

$$P(t) = \frac{1 - tZ_f(t)}{1-t}.$$

It easily follows from the defining integral of Igusa's p-adic zeta function that $Z_f(t=1) = 1$. Consequently, $1 - tZ_f(t)$ is divisible by $1-t$ and $P(t)$ can be written as

$$P(t) = \frac{C(t)}{\prod_{j \in K} \left(1 - q^{-\nu_j} t^{N_j}\right)},$$

where $C(t)$ is a polynomial with coefficients in S.

We will say that a formal power series in t has the divisibility property if the coefficient of t^i/q^{ni} is an integer multiple of $q^{\lceil (n+l)i \rceil}$ for every i.

For $j \in K$, the series

$$\frac{1}{1 - q^{-\nu_j} t^{N_j}} = \sum_{i=0}^{\infty} q^{-i\nu_j} t^{iN_j} = \sum_{i=0}^{\infty} q^{i(nN_j - \nu_j)} \frac{t^{iN_j}}{q^{niN_j}}$$

has the divisibility property because $nN_j - \nu_j$ is an integer larger than or equal to $N_j(n+l)$. Let a be an integer such that the polynomial $D(t) := q^a C(t)$ has the divisibility property. Note that $C(t) = q^{-a} D(t)$.

One can easily check that the product of a finite number of power series with the divisibility property also has the divisibility property. This implies that $P(t)$ is a power series with the divisibility property, multiplied by q^{-a}. Hence M_i is an integer multiple of $q^{\lceil (n+l)i \rceil - a} = q^{\lceil (n+l)i - a \rceil}$ for all i. $\square$

Chapter 4

Vanishing results with character

Abstract

In the previous chapter we obtained a vanishing result for a candidate pole of Igusa's p-adic zeta function associated to a K-analytic function in three variables and with trivial character. Here we do the calculations which are needed to obtain this result if we are dealing with a non-trivial character. I consider character sums in the first section because they appear in a natural way.

4.1 Character sums

(4.1.1) Let $G,.$ be a finite group. Let $\mathbb{C}^\times,.$ be the multiplicative group of the field of complex numbers. A character χ of G is a group homomorphism $\chi : G \to \mathbb{C}^\times$. Note that $\chi(x)$ is a $(|G|)$th root of unity for every $x \in G$.

Let R be a discrete valuation ring, abbreviated DVR. Let P be the maximal ideal of R and suppose that the residue field R/P is isomorphic to $\mathbb{F}_q$. Let π be a fixed uniformizing parameter for R. A character χ of the group $R^\times,.$ is a group homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. The conductor $e_\chi = e$ of χ is the smallest $u \in \mathbb{Z}_{>0}$ such that χ is trivial on $1 + P^u$.

Valuation rings of p-adic fields are the DVRs which are interesting for our purposes. Other interesting DVRs with finite residue field are the rings of formal power series $\mathbb{F}_q[[t]]$.

For every $u \in \mathbb{Z}_{>0}$, there is a natural one to one correspondence between characters $\chi : (R/P^u)^\times \to \mathbb{C}^\times$ and characters $\chi : R^\times \to \mathbb{C}^\times$ with conductor less than or equal to u.

Let $u \in \mathbb{Z}_{>0}$ and let $L \subset R$ be a union of cosets of P^u. By abuse of notation, we will consider L sometimes as a subset of R/P^u. We will write $L \subset R/P^u$ if we want to stress this. If all elements of $L \subset R/P^u$ are units, we will also write $L \subset (R/P^u)^\times$.

The characters of a finite group $G, .$ (and of $R^\times, .$) can be multiplied in an obvious way. The set of characters becomes a group for this operation. The identity of this group is the constant map on 1, and this character is called the trivial character.

We now give a lot of propositions on character sums which will be used in Section 4.2. In [IR, Chapter 8], character sums of $\mathbb{F}_p^\times$ are treated. We will use similar techniques in our proofs.

(4.1.2) Proposition. *Let χ be a non-trivial character of a finite group G. Then $\sum_{x\in G} \chi(x) = 0$.*

Proof. Fix $a \in G$ such that $\chi(a) \neq 1$. Then

$$\chi(a) \sum_{x\in G} \chi(x) = \sum_{x\in G} \chi(ax) = \sum_{x\in G} \chi(x).$$

The last equality is a consequence of the fact that ax runs over all element of G if x does. Our statement follows because $\chi(a) \neq 1$. □

The previous proposition is well known. Now comes the serious work.

(4.1.3) Proposition. *Let R be a DVR and let χ be a non-trivial character of $R^\times$ with conductor e. Then*

$$\begin{aligned} \sum_{x\in(R/P^e)^\times} \chi(x) &= 0, \\ \sum_{\{(x_1,x_2)|x_1,x_2,x_1+x_2\in(R/P^e)^\times\}} \chi(x_1+x_2) &= 0. \end{aligned}$$

Proof. (1) This is Proposition 4.1.2 for $G = (R/P^e)^\times$.

(2) Every element of $(R/P^e)^\times$ can be written as $x_1 + x_2$, with $x_1, x_2 \in (R/P^e)^\times$, in exactly $(q-2)q^{e-1}$ ways. Consequently,

$$\begin{aligned}\sum_{\{(x_1,x_2)|x_1,x_2,x_1+x_2\in(R/P^e)^\times\}} \chi(x_1+x_2) &= (q-2)q^{e-1}\sum_{t\in(R/P^e)^\times}\chi(t)\\ &= 0.\end{aligned}$$

□

(4.1.4) Proposition. *Let R be a DVR and let χ be a non-trivial character of $R^\times$ with conductor $e \geq 2$. Let $a \in R^\times$, let $i \in \{1, \ldots, e-1\}$ and let $j \geq i$. Then*

$$\begin{aligned}\sum_{x\in a+P^i\subset(R/P^e)^\times}\chi(x) &= 0,\\ \sum_{x\in 1+P^i\subset(R/P^e)^\times}\chi(\pi^j a+x) &= 0,\\ \sum_{x\in(R/P^e)^\times}\chi(\pi^j a+x) &= 0.\end{aligned}$$

Proof. (1) Because

$$a + P^i \to 1 + P^i : x \mapsto a^{-1}x$$

is a bijection, we obtain

$$\begin{aligned}\sum_{x\in a+P^i\subset(R/P^e)^\times}\chi(x) &= \sum_{x\in a+P^i\subset(R/P^e)^\times}\chi(a)\chi(a^{-1}x)\\ &= \chi(a)\sum_{t\in 1+P^i\subset(R/P^e)^\times}\chi(t)\\ &= 0.\end{aligned}$$

The last equality follows from Proposition 4.1.2 because $1 + P^i$ is a subgroup of $(R/P^e)^\times$ on which χ is non-trivial.

(2) Because

$$1 + P^i \to 1 + P^i : x \mapsto \pi^j a + x$$

is a bijection, we obtain

$$\begin{aligned}\sum_{x\in 1+P^i\subset(R/P^e)^\times}\chi(\pi^j a+x) &= \sum_{t\in 1+P^i\subset(R/P^e)^\times}\chi(t)\\ &= 0.\end{aligned}$$

(3) The proof of the last equality is analogous to (2). □

(4.1.5) Proposition. *Let R be a DVR and let χ be a non-trivial character of $R^\times$ with conductor e. Let $a\in R^\times$ and let $i\in\{1,\dots,e-1\}$. Then*

$$\begin{aligned}\sum_{x\in(R/P^e)^\times}\chi(x)\chi^{-1}(x+\pi^i a) &= \begin{cases}0 & \text{if } i\in\{1,\dots,e-2\}\\ -q^{e-1} & \text{if } i=e-1\end{cases},\\ \sum_{x\in(R/P^e)^\times\setminus(-a+P)}\chi(x)\chi^{-1}(x+a) &= \begin{cases}0 & \text{if } e>1\\ -1 & \text{if } e=1\end{cases},\\ \sum_{x\in(R/P^e)^\times}\chi(x)\chi^{-1}(\pi^i x+a) &= 0.\end{aligned}$$

Proof. (1) In this proof all calculations in R are modulo P^e. Because $\chi(x)\chi^{-1}(x+\pi^i a)=\chi(x/(x+\pi^i a))$, we study the values $x/(x+\pi^i a)$ if x runs over $(R/P^e)^\times$. We have that $x/(x+\pi^i a)=t$ if and only if $x(1-t)=\pi^i a t$, and such a t is of the form $t=1+\pi^i b$ for some $b\in R^\times$. Moreover the $x\in(R/P^e)^\times$ which satisfy this equation for such a fixed t are exactly the elements which are equal to $-ab^{-1}t$ modulo P^{e-i}. We thus have q^i values of $x\in(R/P^e)^\times$ for such a fixed t. Consequently

$$\begin{aligned}\sum_{x\in(R/P^e)^\times}\chi(x)\chi^{-1}(x+\pi^i a) &= \sum_{x\in(R/P^e)^\times}\chi\left(\frac{x}{x+\pi^i a}\right)\\ &= q^i\sum_{t\in 1+(P^i\setminus P^{i+1})\subset(R/P^e)^\times}\chi(t)\\ &= q^i\left(\sum_{t\in 1+P^i}\chi(t)-\sum_{t\in 1+P^{i+1}}\chi(t)\right)\\ &= \begin{cases}0 & \text{if } i\in\{1,\dots,e-2\}\\ -q^{e-1} & \text{if } i=e-1\end{cases}.\end{aligned}$$

The last equality follows from Proposition 4.1.2 because $1+P^i$ and $1+P^{i+1}$ are subgroups of $(R/P^e)^\times$.

(2) One verifies easily that the map

$$(R/P^e)^\times \setminus (-a+P) \to (R/P^e)^\times \setminus (1+P) : x \mapsto \frac{x}{x+a}$$

is a bijection. Consequently

$$\begin{aligned}\sum_{x\in(R/P^e)^\times\setminus(-a+P)} \chi(x)\chi^{-1}(x+a) &= \sum_{x\in(R/P^e)^\times\setminus(-a+P)} \chi\left(\frac{x}{x+a}\right)\\ &= \sum_{t\in(R/P^e)^\times\setminus(1+P)} \chi(t)\\ &= \begin{cases} 0 & \text{if } e>1\\ -1 & \text{if } e=1\end{cases}.\end{aligned}$$

(3) One verifies easily that the map

$$(R/P^e)^\times \to (R/P^e)^\times : x \mapsto \frac{x}{\pi^i x+a}$$

is a bijection. Consequently

$$\begin{aligned}\sum_{x\in(R/P^e)^\times} \chi(x)\chi^{-1}(\pi^i x+a) &= \sum_{x\in(R/P^e)^\times} \chi\left(\frac{x}{\pi^i x+a}\right)\\ &= \sum_{t\in(R/P^e)^\times} \chi(t)\\ &= 0.\end{aligned}$$

□

From now on, e is always the maximum of the conductors of the characters which are involved.

(4.1.6) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi \neq \rho^{-1}$ and $\psi \neq \rho^{-1}$. Let $a \in R^\times$ and let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned}\sum_{x\in(R/P^e)^\times} \psi(x)\rho(x+\pi^i a) &= 0 \quad \text{if} \quad e_\rho \leq e_{\psi\rho}+i-1,\\ \sum_{x\in(R/P^e)^\times} \chi(x)\rho(\pi^i x+a) &= 0 \quad \text{if} \quad e_\rho \leq e_\chi+i-1.\end{aligned}$$

Proof. (1) Let $v \in 1 + P^{e_{\psi\rho}-1}$ such that $(\psi\rho)(v) \neq 1$. Then

$$\begin{aligned}\sum_{x\in(R/P^e)^\times} \psi(x)\rho(x+\pi^i a) &= \sum_{x\in(R/P^e)^\times} \psi(vx)\rho(vx+\pi^i a)\\ &= \sum_{x\in(R/P^e)^\times} \psi(vx)\rho(vx+\pi^i va)\\ &= (\psi\rho)(v) \sum_{x\in(R/P^e)^\times} \psi(x)\rho(x+\pi^i a).\end{aligned}$$

Because $(\psi\rho)(v) \neq 1$, we get our statement. In the first equality, we used a translation in the group $(R/P^e)^\times$. For the second equality, we used that $e_\rho \leq e_{\psi\rho} + i - 1$ and that $v \in 1 + P^{e_{\psi\rho}-1}$.

(2) Let $v \in 1 + P^{e_\chi - 1}$ such that $\chi(v) \neq 0$. Then

$$\begin{aligned}\sum_{x\in(R/P^e)^\times} \chi(x)\rho(\pi^i x+a) &= \sum_{x\in(R/P^e)^\times} \chi(vx)\rho(\pi^i vx+a)\\ &= \sum_{x\in(R/P^e)^\times} \chi(vx)\rho(\pi^i x+a)\\ &= \chi(v) \sum_{x\in(R/P^e)^\times} \chi(x)\rho(\pi^i x+a).\end{aligned}$$

Because $\chi(v) \neq 1$, we get our statement. In the first equality, we used a translation in the group $(R/P^e)^\times$. For the second equality, we used that $e_\rho \leq e_\chi + i - 1$ and that $v \in 1 + P^{e_\chi - 1}$. □

(4.1.7) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi\psi\rho = 1$. Note that the largest two values of e_χ, e_ψ, e_ρ are equal. Let $a, x_1, x_2 \in R^\times$ and let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned}\sum_{x_2\in(R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2+\pi^i x_1) &= \sum_{x\in(R/P^e)^\times} \chi(x)\rho(a+\pi^i x)\\ &= \sum_{x_1\in(R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2+\pi^i x_1),\end{aligned}$$

and this is equal to 0 if $e_\rho \leq e_\chi + i - 1$.

Proof. Fix $v \in (R/P^e)^\times$. Then

$$\begin{aligned}
&\sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) \\
&\quad = (\chi\psi\rho)(v) \sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) \\
&\quad = \sum_{x_2 \in (R/P^e)^\times} \chi(vx_1)\psi(vx_2)\rho(avx_2 + \pi^i v x_1) \\
&\quad = \sum_{x_2 \in (R/P^e)^\times} \chi(vx_1)\psi(x_2)\rho(ax_2 + \pi^i v x_1).
\end{aligned}$$

Because vx_1 takes all values of $(R/P^e)^\times$ if v runs over $(R/P^e)^\times$, we obtain that

$$\sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1)$$

is independent of $x_1 \in (R/P^e)^\times$. In the first equality, we put $x_1 = 1$:

$$\begin{aligned}
\sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) &= \sum_{x_2 \in (R/P^e)^\times} \psi(x_2)\rho(ax_2 + \pi^i) \\
&= \sum_{x \in (R/P^e)^\times} \psi(x^{-1})\rho(ax^{-1} + \pi^i) \\
&= \sum_{x \in (R/P^e)^\times} (\psi\rho)^{-1}(x)\rho(a + \pi^i x) \\
&= \sum_{x \in (R/P^e)^\times} \chi(x)\rho(a + \pi^i x).
\end{aligned}$$

This is the first equality we had to prove. Analogously as before, we obtain that

$$\sum_{x_1 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1)$$

is independent of x_2. If we put $x_2 = 1$, we obtain the second equality. We can use either of the equalities of (4.1.6) to prove that it is equal to 0 under the condition $e_\rho \leq e_\chi + i - 1$. □

(4.1.8) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi\psi\rho = 1$. Then*

$$\frac{1}{q^e} \sum_{x_1, x_2, x_1 + x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(x_1 + x_2) = \frac{q-1}{q} \sum_{x \in (R/P^e)^\times \setminus (-1+P)} \psi(x)\rho(1+x).$$

Proof. The map $\{(x_1, x_2) \mid x_1, x_2 \in (R/P^e)^\times \text{ and } x_2 \notin -1 + P\} \to \{(x_1, x_2) \mid x_1, x_2, x_1+x_2 \in (R/P^e)^\times\} : (x_1, x_2) \mapsto (x_1, x_1x_2)$ is a bijection. Therefore

$$\begin{aligned}
&\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(x_1+x_2) \\
&\quad = \sum_{x_1,x_2\in(R/P^e)^\times;x_2\notin -1+P} \chi(x_1)\psi(x_1x_2)\rho(x_1+x_1x_2) \\
&\quad = \sum_{x_1,x_2\in(R/P^e)^\times;x_2\notin -1+P} \psi(x_2)\rho(1+x_2) \\
&\quad = (q-1)q^{e-1} \sum_{x\in(R/P^e)^\times\setminus(-1+P)} \psi(x)\rho(1+x). \qquad \square
\end{aligned}$$

(4.1.9) Proposition. *Let R be a DVR and let χ, ρ be non-trivial characters of $R^\times$ such that $\chi\rho \neq 1$. Let $a \in R^\times$. Let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned}
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_1+\pi^i x_2) &= 0, \\
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_2+\pi^i x_1) &= 0, \\
\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(x_1)\rho(x_2) &= 0.
\end{aligned}$$

Proof. Let $v \in (R/P^e)^\times$ such that $(\chi\rho)(v) \neq 0$. The map $(x_1, x_2) \mapsto (vx_1, vx_2)$ is a bijection of $\{(x_1, x_2) \mid x_1, x_2 \in (R/P^e)^\times\}$ and also of $\{(x_1, x_2) \mid x_1, x_2, x_1 + x_2 \in (R/P^e)^\times\}$. Therefore

$$\begin{aligned}
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_1+\pi^i x_2) &= \sum_{x_1,x_2\in(R/P^e)^\times} \chi(vx_1)\rho(avx_1+\pi^i vx_2) \\
&= (\chi\rho)(v) \sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_1+\pi^i x_2),
\end{aligned}$$

$$\begin{aligned}
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_2+\pi^i x_1) &= \sum_{x_1,x_2\in(R/P^e)^\times} \chi(vx_1)\rho(avx_2+\pi^i vx_1) \\
&= (\chi\rho)(v) \sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_2+\pi^i x_1)
\end{aligned}$$

and

$$\begin{aligned}\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times}\chi(x_1)\rho(x_2) &= \sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times}\chi(vx_1)\rho(vx_2)\\ &= (\chi\rho)(v)\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times}\chi(x_1)\rho(x_2).\end{aligned}$$

Because $(\chi\rho)(v)\neq 1$, we obtain our statements. □

(4.1.10) Proposition. *Let R be a DVR and let χ,ψ,ρ be non-trivial characters of $R^\times$ such that $\chi\psi\rho\neq 1$. Let $a\in R^\times$. Let $i\in\mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned}\sum_{x_1,x_2\in(R/P^e)^\times}\chi(x_1)\psi(x_2)\rho(ax_2+\pi^i x_1) &= 0,\\ \sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times}\chi(x_1)\psi(x_2)\rho(x_1+x_2) &= 0.\end{aligned}$$

Proof. We obtain these equalities analogously as in (4.1.9). Now we have to take $v\in(R/P^e)^\times$ such that $(\chi\psi\rho)(v)\neq 1$. □

4.2 The vanishing result

Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R, π a fixed uniformizing parameter for R and q the cardinality of the residue field R/P. For $z\in K$, let $\operatorname{ord} z\in\mathbb{Z}\cup\{+\infty\}$ denote the valuation of z, $|z|=q^{-\operatorname{ord} z}$ the absolute value of z and $\operatorname{ac} z=z\pi^{-\operatorname{ord} z}$ the angular component of z.

Let X be an open and compact subset of K^3. Let f be a K-analytic function on X. Let $g:Y=Y_t\to X=Y_0$ be an embedded resolution of f which is a composition $g_1\circ\cdots\circ g_t$ of blowing-ups $g_i:Y_i\to Y_{i-1}$ with centre a K-analytic closed submanifold which has only normal crossings with the union of the exceptional surfaces in Y_{i-1} and with exceptional surface E_i. Let χ be a character of $R^\times$.

Proposition. *Let $r\in\{1,\ldots,t\}$ and let $P\in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V,y=(y_1,y_2,y_3))$ centred at P on which $f\circ g_1\circ\cdots\circ g_{r-1}$ is given by a power*

series with lowest degree part of the form $ey_1^k y_2^l y_3^m (y_1+y_2)^n$, *with* $e \in K^\times$ *and* $k,l,m,n \in \mathbb{Z}_{\geq 0}$. *Then the contribution of* E_r *to the residue* b_{-1} *of* $Z_{f,\chi}(s)$ *at* s_0 *is zero.*

Proof. Put $A = P \times P$, $B = P \times R$ and $C = R \times R$. We obtain analogously as in Section 3.3.2 that we have to prove that

$$\begin{aligned}
&\int_A \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1 \wedge dz_2| \\
&\quad + \int_B \chi^k(\mathrm{ac}\, z_1)\chi^m(\mathrm{ac}\, z_3)\chi^n(\mathrm{ac}\, z_1+1)|z_1|^{\alpha_1-1}|z_3|^{\alpha_3-1}|z_1+1|^{\alpha_4-1}|dz_1 \wedge dz_3| \qquad (*) \\
&\quad + \int_C \chi^l(\mathrm{ac}\, z_2)\chi^m(\mathrm{ac}\, z_3)\chi^n(\mathrm{ac}\, 1+z_2)|z_2|^{\alpha_2-1}|z_3|^{\alpha_3-1}|1+z_2|^{\alpha_4-1}|dz_2 \wedge dz_3|
\end{aligned}$$

is equal to zero. The definition of the α's and the relation between them are in (3.2.8). Note that we have omitted the brackets in for example $\mathrm{ac}(z_1+z_2)$. To simplify our notation, we have also put $s = s_0$ in the integrand. This is not exact because the integrals do not have to converge. We actually have to calculate these integrals for complex numbers s satisfying $\mathrm{Re}(s) > 0$, and we have to evaluate the meromorphic continuation in $s = s_0$. This can be done in mind while reading the calculations. The expression above is equal to

$$\begin{aligned}
&\int_A \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1 \wedge dz_2| \\
&\quad + \int_P \chi^k(\mathrm{ac}\, z_1)\chi^n(\mathrm{ac}\, z_1+1)|z_1|^{\alpha_1-1}|z_1+1|^{\alpha_4-1}|dz_1| \int_R \chi^m(\mathrm{ac}\, z_3)|z_3|^{\alpha_3-1}|dz_3| \\
&\quad + \int_R \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|z_2|^{\alpha_2-1}|1+z_2|^{\alpha_4-1}|dz_2| \int_R \chi^m(\mathrm{ac}\, z_3)|z_3|^{\alpha_3-1}|dz_3|.
\end{aligned}$$

Note that

$$H := \int_R \chi^m(\mathrm{ac}\, z_3)|z_3|^{\alpha_3-1}\,|dz_3| = \begin{cases} \frac{q-1}{q}\frac{1}{1-q^{-\alpha_3}} & \text{if } \chi^m = 1 \\ 0 & \text{if } \chi^m \neq 1. \end{cases}$$

To calculate the first term in $(*)$, we partition A into

$$\begin{aligned}
A_1 &= \{(z_1,z_2) \in P \times P \mid \mathrm{ord}\, z_1 > \mathrm{ord}\, z_2\} = \bigsqcup_{i \in \mathbb{Z}_{>0}} \{(z_1,z_2) \mid \mathrm{ord}\, z_1 > \mathrm{ord}\, z_2 = i\} \\
A_2 &= \{(z_1,z_2) \in P \times P \mid \mathrm{ord}\, z_1 < \mathrm{ord}\, z_2\} = \bigsqcup_{i \in \mathbb{Z}_{>0}} \{(z_1,z_2) \mid i = \mathrm{ord}\, z_1 < \mathrm{ord}\, z_2\} \\
A_3 &= \{(z_1,z_2) \in P \times P \mid \mathrm{ord}\, z_1 = \mathrm{ord}\, z_2\} = \bigsqcup_{i \in \mathbb{Z}_{>0}} \{(z_1,z_2) \mid \mathrm{ord}\, z_1 = \mathrm{ord}\, z_2 = i\}
\end{aligned}$$

To calculate the third term in $(*)$, we partition C into $C_1 = (R \setminus (P \cup -1+P)) \times R$, $C_2 = P \times R$ and $C_3 = (-1+P) \times R$.

We have that N_r is a multiple of the order of χ because s_0 is a candidate pole of $Z_{f,\chi}(s)$ associated to E_r. On the other hand, Relation 1 in (3.2.8) says that $N_r = k+l+m+n$. Consequently, $1 = \chi^{N_r} = \chi^{k+l+m+n}$.

Let e be the maximum of the conductors of χ^k, χ^l, χ^m and χ^n.

Case 1. $\chi^k = \chi^l = \chi^m = \chi^n = 1$

The calculations for this case are already done in Section 3.3.2.

Case 2. $\chi^k = \chi^m = 1$ and χ^l, $\chi^n \neq 1$

Note that $\chi^l = \chi^{-n}$ and that $e = e_{\chi^l} = e_{\chi^n}$.
The contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i \setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)} \int_{P^{i+1}} |z_1|^{\alpha_1-1} \left(\int_{P^i \setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|dz_2| \right) |dz_1| \\
&= -\frac{q-1}{q^2} q^{-(e-1)\alpha_1} \frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1} + \left(\frac{q-1}{q}\right)^2 \frac{q^{-e\alpha_1}}{(1-q^{-\alpha_1})(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)}.
\end{aligned}$$

For the last equality, note that by Proposition 4.1.5

$$\begin{aligned}
\int_{P^i \setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|dz_2| &= \frac{1}{q^{i+e}} \sum_{z_2 \in (R/P^e)^\times} \chi^l(z_2)\chi^n(\pi^{\mathrm{ord}\, z_1 - i}\mathrm{ac}(z_1)+z_2) \\
&= \begin{cases} 0 & \text{if } \mathrm{ord}\, z_1 - i \in \{1, \ldots, e-2\} \\ -q^{-i-1} & \text{if } \mathrm{ord}\, z_1 - i = e-1 \\ (q-1)q^{-i-1} & \text{if } \mathrm{ord}\, z_1 - i \geq e. \end{cases}
\end{aligned}$$

The contribution of A_2 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i \setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1| \right) |dz_2| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_4-2)} \int_{P^{i+1}} \chi^l(\mathrm{ac}\, z_2)|z_2|^{\alpha_2-1} \left(\int_{P^i \setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|dz_1| \right) |dz_2| \\
&= 0.
\end{aligned}$$

For the last equality, note that by Proposition 4.1.4

$$\begin{aligned}\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|dz_1| &= \frac{1}{q^{i+e}} \sum_{z_1\in (R/P^e)^\times} \chi^n(z_1+\pi^{\mathrm{ord}\, z_2-i}\mathrm{ac}\, z_2)\\ &= 0.\end{aligned}$$

The contribution of A_3 to the first term in $(*)$ is equal to

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2|\\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\chi^l(\mathrm{ac}\, z_2)\int_{-z_2+P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_1|\right.\\
&\qquad \left.+\chi^l(\mathrm{ac}\, z_2)\int_{(P^i\setminus P^{i+1})\setminus(-z_2+P^{i+1})} \chi^n(\mathrm{ac}\, z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2|\\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\chi^l(\mathrm{ac}\, z_2)\int_{P^{i+1}} \chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz|\right.\\
&\qquad \left.+\chi^l(\mathrm{ac}\, z_2)q^{-i(\alpha_4-1)}\int_{(P^i\setminus P^{i+1})\setminus(z_2+P^{i+1})} \chi^n(\mathrm{ac}\, z)|dz|\right)|dz_2|\\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\left(-\frac{\chi^l(\mathrm{ac}\, z_2)}{q^{i+e}}\sum_{z\in \mathrm{ac}\, z_2+P} \chi^n(z)\right)|dz_2|\\
&= \begin{cases} 0 & \text{if } e>1\\ -\frac{q-1}{q^2}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1} & \text{if } e=1.\end{cases}\end{aligned}$$

For the third equality, note that

$$\int_{P^{i+1}} \chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz| = 0$$

and that for $z_2\in P^i\setminus P^{i+1}$

$$\begin{aligned}&\int_{(P^i\setminus P^{i+1})\setminus(z_2+P^{i+1})} \chi^n(\mathrm{ac}\, z)|dz|\\
&\qquad = \int_{(P^i\setminus P^{i+1})} \chi^n(\mathrm{ac}\, z)|dz| - \int_{z_2+P^{i+1}} \chi^n(\mathrm{ac}\, z)|dz|\\
&\qquad = \frac{1}{q^{i+e}}\sum_{z\in(R/P^e)^\times}\chi^n(z) - \frac{1}{q^{i+e}}\sum_{z\in \mathrm{ac}\, z_2+P\subset(R/P^e)^\times}\chi^n(z)\\
&\qquad = -\frac{1}{q^{i+e}}\sum_{z\in \mathrm{ac}\, z_2+P\subset(R/P^e)^\times}\chi^n(z).\end{aligned}$$

The second term of $(*)$ is equal to

$$\begin{aligned}
&\int_P \chi^n(\mathrm{ac}\, z_1+1)|z_1|^{\alpha_1-1}|dz_1| \int_R |z_3|^{\alpha_3-1}|dz_3| \\
&= \left(\sum_{i=1}^{e-1} q^{-i(\alpha_1-1)} \int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+1)|dz_1| + \int_{P^e} |z_1|^{\alpha_1-1}|dz_1|\right) \int_R |z_3|^{\alpha_3-1}|dz_3| \\
&= -\frac{q-1}{q^2} q^{-(e-1)\alpha_1} \frac{1}{1-q^{-\alpha_3}} + \left(\frac{q-1}{q}\right)^2 \frac{q^{-e\alpha_1}}{(1-q^{-\alpha_1})(1-q^{-\alpha_3})}.
\end{aligned}$$

For the last equality, note that by Proposition 4.1.4

$$\begin{aligned}
\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+1)|dz_1| &= \int_{P^i} \chi^n(\mathrm{ac}\, z_1+1)|dz_1| - \int_{P^{i+1}} \chi^n(\mathrm{ac}\, z_1+1)|dz_1| \\
&= \frac{1}{q^e} \sum_{z\in 1+P^i\subset (R/P^e)^\times} \chi^n(z) - \frac{1}{q^e} \sum_{z\in 1+P^{i+1}\subset (R/P^e)^\times} \chi^n(z) \\
&= \begin{cases} 0 & \text{if } i\in\{1,\dots,e-2\} \\ -\frac{1}{q^e} & \text{if } i = e-1. \end{cases}
\end{aligned}$$

Using Proposition 4.1.5 we obtain that the contribution of C_1 to the third term in $(*)$ is equal to

$$\begin{aligned}
&\int_{R\setminus(P\cup -1+P)} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|dz_2| \int_R |z_3|^{\alpha_3-1}|dz_3| \\
&= \left(\frac{1}{q^e} \sum_{z_2\in (R/P^e)^\times\setminus(-1+P)} \chi^l(z_2)\chi^n(1+z_2)\right) \frac{q-1}{q} \frac{1}{1-q^{-\alpha_3}} \\
&= \begin{cases} 0 & \text{if } e>1 \\ -\frac{q-1}{q^2}\frac{1}{1-q^{-\alpha_3}} & \text{if } e=1. \end{cases}
\end{aligned}$$

Using Proposition 4.1.5 we obtain that the contribution of C_2 to the third term in $(*)$ is equal to H multiplied by

$$\begin{aligned}
&\int_P \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|z_2|^{\alpha_2-1}|dz_2| \\
&= \sum_{i=1}^{e-1} q^{-i(\alpha_2-1)} \int_{P^i\setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|dz_2| + \int_{P^e} \chi^l(\mathrm{ac}\, z_2)|z_2|^{\alpha_2-1}|dz_2| \\
&= \sum_{i=1}^{e-1} \frac{q^{-i(\alpha_2-1)}}{q^{i+e}} \sum_{z\in (R/P^e)^\times} \chi^l(z)\chi^n(1+\pi^i z) \\
&= 0.
\end{aligned}$$

Using Proposition 4.1.5 we obtain that the contribution of C_3 to the third term in $(*)$ is equal to H multiplied by

$$\begin{aligned}
&\int_{-1+P} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|1+z_2|^{\alpha_4-1}|dz_2| \\
&= \int_P \chi^l(\mathrm{ac}\ -1+z_2)\chi^n(\mathrm{ac}\, z_2)|z_2|^{\alpha_4-1}|dz_2| \\
&= \sum_{i=1}^{e-1} q^{-i(\alpha_4-1)} \int_{P^i\setminus P^{i+1}} \chi^l(\mathrm{ac}\ -1+z_2)\chi^n(\mathrm{ac}\, z_2)|dz_2| \\
&\qquad +\chi^l(-1)\int_{P^e} \chi^n(\mathrm{ac}\, z_2)|z_2|^{\alpha_4-1}|dz_2| \\
&= \sum_{i=1}^{e-1} \frac{q^{-i(\alpha_4-1)}}{q^{i+e}} \sum_{z\in(R/P^e)^\times} \chi^l(-1+\pi^i z)\chi^n(z) \\
&= 0.
\end{aligned}$$

The contribution of A_1 cancels with the one of B. The contribution of A_3 cancels with the one of C_1. Consequently, the contribution of E_r to b_{-1} is equal to zero in this case.

Case 3. $\chi^k = \chi^l = 1$ and χ^m, $\chi^n \neq 1$

Using Proposition 4.1.4 we obtain that the contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)} \int_{P^{i+1}} |z_1|^{\alpha_1-1} \left(\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)} \int_{P^{i+1}} |z_1|^{\alpha_1-1} \left(\frac{1}{q^{i+e}} \sum_{z_2\in(R/P^e)^\times} \chi^n(\pi^{\mathrm{ord}\, z_1-i}\mathrm{ac}(z_1)+z_2) \right) |dz_1| \\
&= 0.
\end{aligned}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.

Using Proposition 4.1.3 we obtain that the contribution of A_3 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^n(\mathrm{ac}\,z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2| \\
&= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{-z_2+P^{i+1}}\chi^n(\mathrm{ac}\,z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_1|\right. \\
&\qquad \left.+\int_{(P^i\setminus P^{i+1})\setminus(-z_2+P^{i+1})}\chi^n(\mathrm{ac}\,z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2| \\
&= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\int_{(P^i\setminus P^{i+1})\setminus(-z_2+P^{i+1})}\chi^n(\mathrm{ac}\,z_1+z_2)|dz_1||dz_2| \\
&= \sum_{i=1}^{\infty}\frac{q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}}{q^{2(i+e)}}\sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times}\chi^n(z_1+z_2) \\
&= 0.
\end{aligned}$$

The second and the third term in $(*)$ are both equal to zero. Indeed, we have that $H=0$ since $\chi^m\neq 1$.

Case 4. $\chi^m=1$ and χ^k, χ^l, $\chi^n\neq 1$

We may suppose that $e_{\chi^n}\leq e_{\chi^k}$ and that $e_{\chi^n}\leq e_{\chi^l}$. Note that $e_{\chi^k}=e_{\chi^l}$, and e is by definition this value.
The contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty}\int_{P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1| \\
&= \sum_{i=1}^{\infty}q^{-i(\alpha_2+\alpha_4-2)}\int_{P^{i+1}}\chi^k(\mathrm{ac}z_1)|z_1|^{\alpha_1-1}\left(\int_{P^i\setminus P^{i+1}}\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|dz_2|\right)|dz_1| \\
&= 0.
\end{aligned}$$

For the last equality, note that by Proposition 4.1.6

$$\begin{aligned}
\int_{P^i\setminus P^{i+1}}\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|dz_2| &= \frac{1}{q^{i+e}}\sum_{z_2\in(R/P^e)^\times}\chi^l(z_2)\chi^n(\pi^{\mathrm{ord}\,z_1-i}\mathrm{ac}(z_1)+z_2) \\
&= 0.
\end{aligned}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.

The contribution

$$\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$

of A_3 to the first term in $(*)$ is the sum of two parts.

Part 1. Using Proposition 4.1.7 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{-z_1+P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,-z_1)\int_{P^{i+e}}\chi^n(\mathrm{ac}\,z_2)|z_2|^{\alpha_4-1}|dz_2|\right.\\
&\qquad\left.+\sum_{j=1}^{e-1}\int_{P^{i+j}\setminus P^{i+j+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,-z_1+z_2)\chi^n(\mathrm{ac}\,z_2)|z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\\
&\qquad\int_{P^i\setminus P^{i+1}}\sum_{j=1}^{e-1}\frac{q^{-(i+j)(\alpha_4-1)}}{q^{i+j+e}}\sum_{z_2\in(R/P^e)^\times}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}(-z_1)+\pi^j z_2)\chi^n(z_2)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-2)}\int_{P^i\setminus P^{i+1}}\sum_{j=1}^{e-1}\frac{q^{-j\alpha_4}}{q^e}\sum_{z_2\in(R/P^e)^\times}\chi^l(-1+\pi^j z_2)\chi^n(z_2)|dz_1|\\
&\quad=\frac{q-1}{q}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}\sum_{j=1}^{e-1}\frac{q^{-j\alpha_4}}{q^e}\sum_{z_2\in(R/P^e)^\times}\chi^l(-1+\pi^j z_2)\chi^n(z_2).
\end{aligned}$$

Part 2.

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\left(\int_{(P^i\setminus P^{i+1})\setminus(-z_1+P^{i+1})}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|dz_2|\right)|dz_1|\\
&\quad=\frac{1}{q^{2e}}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}\sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times}\chi^k(z_1)\chi^l(z_2)\chi^n(z_1+z_2).
\end{aligned}$$

Using Proposition 4.1.6 we obtain that the second term in $(*)$ is equal to H multiplied by

$$\begin{aligned}
&\int_P\chi^k(\mathrm{ac}\,z_1)\chi^n(\mathrm{ac}\,z_1+1)|z_1|^{\alpha_1-1}|dz_1|\\
&\quad=\sum_{i=1}^{e-1} q^{-i(\alpha_1-1)}\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^n(\mathrm{ac}\,z_1+1)|dz_1|+\int_{P^e}\chi^k(\mathrm{ac}\,z_1)|z_1|^{\alpha_1-1}|dz_1|\\
&\quad=\sum_{i=1}^{e-1}\frac{q^{-i(\alpha_1-1)}}{q^{i+e}}\sum_{z\in(R/P^e)^\times}\chi^k(z)\chi^n(\pi^i z+1)\\
&\quad=0.
\end{aligned}$$

The contribution of C_1 to the third term in $(*)$ is equal to

$$\begin{aligned}&\int_{R\setminus(P\cup -1+P)} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|dz_2| \int_R |z_3|^{\alpha_3-1}|dz_3| \\ &= \left(\frac{1}{q^e}\sum_{z_2\in(R/P^e)^\times\setminus(-1+P)} \chi^l(z_2)\chi^n(1+z_2)\right)\frac{q-1}{q}\frac{1}{1-q^{-\alpha_3}}.\end{aligned}$$

The contribution of C_2 to the third term in $(*)$ is 0. The calculation is the same as the calculation of the second term.
The contribution of C_3 to the third term in $(*)$ is equal to H multiplied by

$$\begin{aligned}&\int_{-1+P} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|1+z_2|^{\alpha_4-1}|dz_1| \\ &= \int_P \chi^l(\mathrm{ac}\, -1+z)\chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz| \\ &= \sum_{j=1}^{e-1} q^{-j(\alpha_4-1)}\int_{P^j\setminus P^{j+1}} \chi^l(\mathrm{ac}\, -1+z)\chi^n(\mathrm{ac}\, z)|dz| + \chi^l(-1)\int_{P^e}\chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz| \\ &= \sum_{j=1}^{e-1}\frac{q^{-j\alpha_4}}{q^e}\sum_{z\in(R/P^e)^\times}\chi^l(-1+\pi^j z)\chi^n(z)\end{aligned}$$

The contribution of C_3 cancels with the one of the first part of A_3. Using Proposition 4.1.8 we obtain that the contribution of C_1 cancels with the one of the second part of A_3.

Case 5. $\chi^n = 1$ and χ^k, χ^l, $\chi^m \neq 1$

The contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1| \\ &= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)}\int_{P^{i+1}}\chi^k(\mathrm{ac}\, z_1)|z_1|^{\alpha_1-1}\left(\int_{P^i\setminus P^{i+1}}\chi^l(\mathrm{ac}\, z_2)|dz_2|\right)|dz_1| \\ &= 0.\end{aligned}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.
The contribution

$$\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$

of A_3 to the first term in $(*)$ is the sum of two parts.
Part 1. Using Proposition 4.1.9 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \int_{P^i\setminus P^{i+1}} \left(\int_{-z_1+P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|z_1+z_2|^{\alpha_4-1}|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \int_{P^i\setminus P^{i+1}} \left(\int_{P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, -z_1+z_2)|z_2|^{\alpha_4-1}|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \sum_{j=1}^{\infty} q^{-(i+j)(\alpha_4-1)} \int_{P^i\setminus P^{i+1}} \int_{P^{i+j}\setminus P^{i+j+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, -z_1+z_2)|dz_2||dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)} \sum_{j=1}^{\infty} \frac{q^{-j(\alpha_4-1)}}{q^{2i+j+2e}} \sum_{z_1,z_2\in(R/P^e)^\times} \chi^k(z_1)\chi^l(-z_1+\pi^j z_2) \\
&= 0.
\end{aligned}$$

Part 2. Using Proposition 4.1.9 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)} \int_{P^i\setminus P^{i+1}} \left(\int_{(P^i\setminus P^{i+1})\setminus(-z_1+P^{i+1})} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|dz_2| \right) |dz_1| \\
&= \frac{1}{q^{2e}} \frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1} \sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times} \chi^k(z_1)\chi^l(z_2) \\
&= 0.
\end{aligned}$$

The second and the third term in $(*)$ are both equal to zero. Indeed, we have that $H = 0$ since $\chi^m \neq 1$.

Case 6. χ^k, χ^l, χ^m, $\chi^n \neq 1$

Using Proposition 4.1.10 we obtain that the contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i\setminus P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)-(i+j)(\alpha_1-1)} \\
&\qquad \int_{P^{i+j}\setminus P^{i+j+1}} \int_{P^i\setminus P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|dz_2||dz_1| \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{q^{-i(\alpha_2+\alpha_4-2)-(i+j)(\alpha_1-1)}}{q^{2i+j+2e}} \sum_{z_1,z_2\in(R/P^e)^\times} \chi^k(z_1)\chi^l(z_2)\chi^n(\pi^j z_1+z_2) \\
&= 0.
\end{aligned}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.
The contribution

$$\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$

of A_3 to the first term in $(*)$ is the sum of two parts.
Part 1. Using Proposition 4.1.10 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{-z_1+P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{P^{i+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,-z_1+z_2)\chi^n(\mathrm{ac}\,z_2)|z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\sum_{j=1}^{\infty} q^{-(i+j)(\alpha_4-1)}\\
&\qquad\int_{P^i\setminus P^{i+1}}\int_{P^{i+j}\setminus P^{i+j+1}}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,-z_1+z_2)\chi^n(\mathrm{ac}\,z_2)|dz_2||dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\sum_{j=1}^{\infty}\frac{q^{-j(\alpha_4-1)}}{q^{2i+j+2e}}\sum_{z_1,z_2\in(R/P^e)^\times}\chi^k(z_1)\chi^l(-z_1+\pi^j z_2)\chi^n(z_2)\\
&\quad=0.
\end{aligned}$$

Part 2. Using Proposition 4.1.10 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\left(\int_{(P^i\setminus P^{i+1})\setminus(-z_1+P^{i+1})}\chi^k(\mathrm{ac}\,z_1)\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|dz_2|\right)|dz_1|\\
&\quad=\quad\frac{1}{q^{2e}}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}\sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times}\chi^k(z_1)\chi^l(z_2)\chi^n(z_1+z_2)\\
&\quad=\quad 0.
\end{aligned}$$

The second and the third term in $(*)$ are again both equal to zero.

We have treated all the cases by using the geometric symmetry of the problem, so our proof is finished. □

Chapter 5

Congruences and lower bound in arbitrary dimension

Abstract

Let K be a p-adic field, R the valuation ring of K, P the maximal ideal of R and q the cardinality of the residue field R/P. Let f be a polynomial over R in n variables and let χ be a character of $R^\times$. Let $M_i(u)$ be the number of solutions of $f = u$ in $(R/P^i)^n$ for $i \in \mathbb{Z}_{\geq 0}$ and $u \in R/P^i$. These numbers are related with Igusa's p-adic zeta function $Z_{f,\chi}(s)$ of f. We explain the connection between the $M_i(u)$ and the smallest real part of a pole of $Z_{f,\chi}(s)$. We also prove that $M_i(u)$ is divisible by $q^{\lceil (n/2)(i-1) \rceil}$, where the corners indicate that we have to round up. This will imply our main result: $Z_{f,\chi}(s)$ has no poles with real part less than $-n/2$. We will also consider arbitrary K-analytic functions f.

5.1 Introduction

(5.1.1) Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R, π a fixed

This chapter corresponds to [Se2].

uniformizing parameter for R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z, $|z| = q^{-\operatorname{ord} z}$ the absolute value of z and $\operatorname{ac} z = z\pi^{-\operatorname{ord} z}$ the angular component of z.

Let χ be a character of $R^\times$, i.e., a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. We formally put $\chi(0) = 0$. Let e be the conductor of χ, i.e., the smallest $a \in \mathbb{Z}_{>0}$ such that χ is trivial on $1 + P^a$.

(5.1.2) Let f be a K-analytic function on an open and compact subset X of K^n and put $x = (x_1, \ldots, x_n)$. Igusa's p-adic zeta function of f and χ is defined by

$$Z_{f,\chi}(s) = \int_X \chi(\operatorname{ac} f(x))|f(x)|^s \, |dx|$$

for $s \in \mathbb{C}$, $\operatorname{Re}(s) \geq 0$, where $|dx|$ denotes the Haar measure on K^n, so normalized that R^n has measure 1. Igusa proved that it is a rational function of q^{-s}, so that it extends to a meromorphic function $Z_{f,\chi}(s)$ on $\mathbb{C}$ which is also called Igusa's p-adic zeta function of f. We will write $Z_{f,\chi}(t)$ if we consider $Z_{f,\chi}(s)$ as a function in the variable $t = q^{-s}$. If χ is the trivial character, we will also write $Z_f(s)$ and $Z_f(t)$.

(5.1.3) A power series $f = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}_{\geq 0}^n} c_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$ over K is convergent in $(a_1, \ldots, a_n) \in K^n$ if and only if $|c_{i_1,\ldots,i_n} a_1^{i_1} \ldots a_n^{i_n}| \to 0$ if $i_1 + \cdots + i_n \to \infty$. If f is convergent at every $(a_1, \ldots, a_n) \in (P^k)^n$ for some $k \in \mathbb{Z}$, then f is called a convergent power series.

Since a K-analytic function is locally described by convergent power series, we only have to consider this type of functions in the study of Igusa's p-adic zeta function. By performing a dilatation of the form $(x_1, \ldots, x_n) \mapsto (\pi^k x_1, \ldots, \pi^k x_n)$, we may moreover suppose that f is rigid, i.e., convergent on R^n. The coefficients of a rigid K-analytic function f on R^n have the property that $|c_{i_1,\ldots,i_n}| \to 0$ if $i_1 + \cdots + i_n \to \infty$. Consequently, $|c_{i_1,\ldots,i_n}|$ is bounded and we can multiply f by an element of K to obtain a series over R. So we only have to study $Z_f(s)$ for rigid K-analytic functions f on R^n defined over R. See also [Ig7, Chapter 2].

(5.1.4) Let f be a rigid K-analytic function on R^n defined over R. Igusa's p-adic zeta function of such an f has an important connection with congruences. For $i \in \mathbb{Z}_{\geq 0}$ and $u \in R/P^i$, let $M_i(u)$ be the number of solutions

of $f(x) \equiv u \bmod P^i$ in $(R/P^i)^n$. Put $M_i := M_i(0)$.

The $M_{i+e}(\pi^i u)$, $u \in (R/P^e)^\times$, describe $Z_{f,\chi}(t)$ through the relation

$$Z_{f,\chi}(t) = \sum_{i=0}^{\infty} \sum_{u \in (R/P^e)^\times} \chi(u) M_{i+e}(\pi^i u) q^{-n(i+e)} t^i.$$

If χ is the trivial character, all the M_i's describe and are described by $Z_f(t)$ through the relation

$$Z_f(t) = P(t) - \frac{P(t)-1}{t},$$

where the Poincaré series $P(t)$ of f is defined by

$$P(t) = \sum_{i=0}^{\infty} M_i (q^{-n}t)^i.$$

Remark that $P(t)$ is a rational function because $Z_f(t)$ has this property.

(5.1.5) If $n = 2$ it is generally known that $Z_{f,\chi}(s)$ has no poles with real part less than -1. We determined in Chapter 3 all the values less than $-1/2$ which occur as the real part of a pole of some $Z_{f,\chi}(s)$ if $n = 2$, and all values less than -1 if $n = 3$. In particular, we proved that there are no poles with real part less than $-3/2$ if $n = 3$. In arbitrary dimension n, we saw that it is easy to prove that there are no poles with real part less than $-(n-1)$ and we conjectured that this bound can be sharpened to $-n/2$.

Let f be a rigid K-analytic function on R^n defined over R. In Section 3.5 we proved that there exists an integer a such that M_i is an integer multiple of $q^{\lceil (n/2)i - a \rceil}$ for all i if this conjecture is true in dimension n for the trivial character. Consequently, this divisibility property of the M_i is true for $n = 2$ and $n = 3$. The statement of this property is so easy that we tried to find an elementary proof, and with success. It generalized easily to arbitrary dimension and to the more general class of numbers $M_i(u)$. This is the subject of the second section. We deduce there that $M_i(u)$ is divisible by $q^{\lceil (n/2)(i-1) \rceil}$ for all $i \in \mathbb{Z}_{>0}$.

(5.1.6) The poles of Igusa's p-adic zeta function are an interesting object of study for example because they are related to the monodromy conjecture [De2, (2.3.2)]. In the third section, we explain the connection

between the $M_i(u)$ and the smallest real part of a pole of Igusa's p-adic zeta function. Let l be the smallest real part of a pole of $Z_f(s)$. We proved in Section 3.5 that there exists an integer a which is independent of i such that M_i is an integer multiple of $q^{\lceil (n+l)i-a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$. We repeat this proof for completeness and we also prove the converse: if there exists an integer a such that M_i is an integer multiple of $q^{\lceil (n+l')i-a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$, then $l' \leq l$. The last statement has an analogue if we are dealing with a character. Together with (5.1.5), this will imply that $Z_{f,\chi}(s)$ has no pole with real part less than $-n/2$. Remark that this bound is optimal: $Z_f(s)$ has a pole in $-n/2$ if f is equal to $x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n$ for n even and $x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} + x_n^2$ for n odd, see [Ig7, Corollary 10.2.1].

5.2 A theorem on the number of solutions of congruences

(5.2.1) Let f be a rigid K-analytic function on R^n defined over R. Let $i \in \mathbb{Z}_{\geq 0}$ and $u \in R/P^i$. We denote by $M_i(f,u)$ or $M_i(u)$ the number of solutions of $f(x) \equiv u \bmod P^i$ in $(R/P^i)^n$. If $u = 0$, we will also write $M_i(f)$ or M_i.

(5.2.2) Let f be a rigid K-analytic function on R^n defined over R. Let $(b_1, \ldots, b_n) \in R^n$. Then $g(y_1, \ldots, y_n) := f(b_1 + y_1, \ldots, b_n + y_n)$ is a rigid K-analytic function on R^n defined over R. Consequently, $h(z_1, \ldots, z_n) := g(\pi z_1, \ldots, \pi z_n) = f(b_1 + \pi z_1, \ldots, b_n + \pi z_n)$ is a power series which is convergent on $\pi^{-1}R \supset R$ and the coefficient of a monomial of degree r in this power series is in P^r.

Remark also that the coefficients of a convergent power series are related with partial derivatives.

(5.2.3) Theorem. *Let $n \in \mathbb{Z}_{>1}$. Then we have that*

$$q^{\lceil (n/2)(i-1) \rceil} \mid M_i(f,u)$$

for all rigid K-analytic functions f on R^n defined over R, $i \in \mathbb{Z}_{>0}$ and $u \in R/P^i$.

Remark. The number $\lceil (n/2)(i-1) \rceil$ is the smallest integer larger than or equal to $(n/2)(i-1)$.

Proof. Note that we may suppose that u is zero, because u can be absorbed by f. So we have to prove that

$$q^{\lceil (n/2)(i-1) \rceil} \mid M_i(f)$$

for every rigid K-analytic function f on R^n defined over R and for every $i \in \mathbb{Z}_{>0}$.

The argument is by induction on i. For $i = 1$, the statement is trivial. Let $k \in \mathbb{Z}_{\geq 2}$. Suppose that the statement is true for $i = 1, \ldots, k-1$. We prove the statement for $i = k$. Let $(b_1, \ldots, b_n) \in R^n$. It is enough to prove that the number of solutions of

$$f(b_1 + \pi z_1, \ldots, b_n + \pi z_n) \equiv 0 \bmod P^k \tag{5.1}$$

in $(R/P^{k-1})^n$ is a multiple of $q^{\lceil (n/2)(k-1) \rceil}$. Put $h(z_1, \ldots, z_n) := f(b_1 + \pi z_1, \ldots, b_n + \pi z_n)$. Then h is a rigid K-analytic function on R^n which is defined over R. Moreover, the coefficients of the z_j, $j = 1, \ldots, n$, are in P and the coefficients in terms of higher degree are in P^2. We explained this in (5.2.2).

Case 1: Not all the coefficients in the linear part of h are in P^2. Then the number of solutions of (5.1) in $(R/P^{k-1})^n$ is equal to 0 or $q^{(n-1)(k-1)}$. This is actually Hensel's lemma. Because $(n-1)(k-1) \geq \lceil (n/2)(k-1) \rceil$, we are done.

Case 2: All the coefficients in the linear part of h are in P^2. Write $h(z_1, \ldots, z_n) = \pi^2 \tilde{h}(z_1, \ldots, z_n)$, where $\tilde{h}$ is a rigid K-analytic function on R^n defined over R. Equation (5.1) becomes

$$\tilde{h}(z_1, \ldots, z_n) \equiv 0 \bmod P^{k-2}. \tag{5.2}$$

We want to prove that the number of solutions of this congruence in $(R/P^{k-1})^n$ is a multiple of $q^{\lceil (n/2)(k-1) \rceil}$. If $k = 2$, the number of solutions of (5.2) in $(R/P)^n$ is q^n, so we are done because $n \geq \lceil n/2 \rceil$. If $k > 2$, the number of solutions of (5.2) in $(R/P^{k-1})^n$ is $q^n M_{k-2}(\tilde{h})$, which is a multiple of $q^n q^{\lceil (n/2)(k-3) \rceil} = q^{\lceil (n/2)(k-1) \rceil}$. Here we used the induction hypothesis for $\tilde{h}$ and $i = k-2$. $\square$

(5.2.4) *Remark.* The previous theorem can be proved without induction. We give an outline. Let $\mathcal{O} \subset (R/P^i)^n$ be the set of solutions of $f(x_1, \ldots, x_n) \equiv u \bmod P^i$ in $(R/P^i)^n$. We give a partition of $\mathcal{O}$ with the property that the number of elements of every subset in this partition is a multiple of $q^{\lceil (n/2)(i-1) \rceil}$.

Let r be $i/2$ if i is even and $(i-1)/2$ if i is odd. We associate a subset of $\mathcal{O}$ to every element $(b_1,\ldots,b_n) \in \mathcal{O}$. If $(\partial f/\partial x_j)(b_1,\ldots,b_n) \equiv 0 \bmod P^r$ for every $j \in \{1,\ldots,n\}$, then we associate to $(b_1,\ldots,b_n)$ the subset $\mathcal{O}_{(b_1,\ldots,b_n)} := (b_1,\ldots,b_n) + P^{i-r}$ of $\mathcal{O}$. Otherwise let k be the number in $\{0,\ldots,r-1\}$ such that $(\partial f/\partial x_j)(b_1,\ldots,b_n) \equiv 0 \bmod P^k$ for every $j \in \{1,\ldots,n\}$ and such that $(\partial f/\partial x_j)(b_1,\ldots,b_n) \not\equiv 0 \bmod P^{k+1}$ for some $j \in \{1,\ldots,n\}$. We associate to $(b_1,\ldots,b_n)$ in this case the set $\mathcal{O}_{(b_1,\ldots,b_n)} := \mathcal{O} \cap ((b_1,\ldots,b_n) + P^{k+1})$.

One can prove that the number of elements of every $\mathcal{O}_{(b_1,\ldots,b_n)}$ is a multiple of $q^{\lceil (n/2)(i-1) \rceil}$. Moreover, if $(b'_1,\ldots,b'_n) \in \mathcal{O}_{(b_1,\ldots,b_n)}$, then $\mathcal{O}_{(b'_1,\ldots,b'_n)} = \mathcal{O}_{(b_1,\ldots,b_n)}$. Consequently, the set $\{\mathcal{O}_{(b_1,\ldots,b_n)} \mid (b_1,\ldots,b_n) \in \mathcal{O}\}$ is a partition of $\mathcal{O}$.

5.3 The smallest poles of Igusa's zeta function and congruences

(5.3.1) Let f be a rigid K-analytic function on R^n which is defined over R. Let $S := \{z/q^i \mid z \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 0}\}$. It is generally known (see [Ig7] or Chapter 3) that Igusa's p-adic zeta function of f is a rational function in t of the form

$$Z_f(t) = \frac{A(t)}{\prod_{j\in J}(1-q^{-\nu_j}t^{N_j})},$$

where $A(t) \in S[t]$, where all ν_j and N_j are in $\mathbb{Z}_{>0}$ and where $A(t)$ is not divisible by any of the $1-q^{-\nu_j}t^{N_j}$. Remark that the real parts of the poles of $Z_f(s)$ are the $-\nu_j/N_j$, $j \in J$. Put $l := \min\{-\nu_j/N_j \mid j \in J\}$.

It follows from $P(t) = (1-tZ_f(t))/(1-t)$ and $Z_f(t=1) = 1$ that the Poincaré series $P(t) = \sum_{i=0}^{\infty} M_i q^{-ni} t^i$ of f can be written as

$$P(t) = \frac{B(t)}{\prod_{j\in J}(1-q^{-\nu_j}t^{N_j})},$$

where $B(t) \in S[t]$ is not divisible by any of the $1-q^{-\nu_j}t^{N_j}$.

In the next paragraphs, we will work in a more general context. By abuse of notation, we will use the symbols of this particular situation.

(5.3.2) Let $P(t)$ be an *arbitrary* rational function in t of the form

$$P(t) = \frac{B(t)}{\prod_{j\in J}(1-q^{-\nu_j}t^{N_j})},$$

where $B(t) \in S[t]$, where all ν_j and N_j are in $\mathbb{Z}_{>0}$ and where $B(t)$ is not divisible by any of the $1 - q^{-\nu_j} t^{N_j}$. Put $l := \min\{-\nu_j/N_j \mid j \in J\}$. Define the numbers M_i by the equality

$$P(t) = \sum_{i=0}^{\infty} M_i q^{-ni} t^i.$$

The following proposition is also in Section 3.5.

(5.3.3) Proposition. *There exists an integer a which is independent of i such that M_i is an integer multiple of $q^{\lceil (n+l)i - a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$.*

Remark. (i) The statement in the proposition is obviously equivalent to the following. If $l' \leq l$, then there exists an integer a which is independent of i such that M_i is an integer multiple of $q^{\lceil (n+l')i - a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$.
(ii) Suppose that we are in the situation of (5.3.1). Then $n + l > 0$, see Chapter 3 or (5.3.4), so that $(n+l)i - a$ rises linearly as a function of i with a slope depending on l. The statement is trivial if $(n+l)i - a \leq 0$ because the M_i are integers. If $(n+l)i - a > 0$, which is the case for i large enough, it claims that M_i is divisible by $q^{\lceil (n+l)i - a \rceil}$.

Proof. We will say that a formal power series in t has the divisibility property if the coefficient of t^i/q^{ni} is an integer multiple of $q^{\lceil (n+l)i \rceil}$ for every i.

For $j \in J$, the series

$$\frac{1}{1 - q^{-\nu_j} t^{N_j}} = \sum_{i=0}^{\infty} q^{-i\nu_j} t^{iN_j} = \sum_{i=0}^{\infty} q^{i(nN_j - \nu_j)} \frac{t^{iN_j}}{q^{niN_j}}$$

has the divisibility property because $nN_j - \nu_j$ is an integer larger than or equal to $N_j(n+l)$. Let a be an integer such that the polynomial $C(t) := q^a B(t)$ has the divisibility property. Remark that $B(t) = q^{-a} C(t)$.

One can easily check that the product of a finite number of power series with the divisibility property also has the divisibility property. This implies that $P(t)$ is a power series with the divisibility property, multiplied by q^{-a}. Hence M_i is an integer multiple of $q^{\lceil (n+l)i \rceil - a} = q^{\lceil (n+l)i - a \rceil}$ for all i. $\square$

(5.3.4) Proposition. *There exist an integer a which is independent of i and positive integers R and c such that M_{iR+c} is not an integer multiple of $q^{\lceil (n+l)(iR+c) + a \rceil}$ for i large enough.*

Consequences. (i) If there exists an integer a such that M_i is an integer multiple of $q^{\lceil (n+l')i-a \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$, then $l' \leq l$. This is the converse of Proposition 5.3.3.
(ii) Because we saw in the previous section that M_i is an integer multiple of $q^{\lceil (n/2)(i-1) \rceil}$ if we are in the situation of (5.3.1), we obtain already that $Z_f(s)$ has no poles with real part less than $-n/2$.

Proof. Put $J_1 = \{j \in J \mid -\nu_j/N_j = l\}$ and $J_2 = J \setminus J_1$. Let N be the lowest common multiple of the N_j, $j \in J_1$, and let ν be the lowest common multiple of the ν_j, $j \in J_1$. Remark that $\nu/N = \nu_j/N_j$ for all $j \in J_1$. Let m be the cardinality of J_1. Because $1 - q^{-\nu}t^N$ is a multiple of $1 - q^{-\nu_j}t^{N_j}$ for all $j \in J_1$, we can write

$$P(t) = \frac{D(t)}{(1 - q^{-\nu}t^N)^m \prod_{j \in J_2}(1 - q^{-\nu_j}t^{N_j})},$$

where $D(t) \in S[t]$. Using decomposition into partial fractions, we obtain that

$$P(t) = \frac{r_1}{(1 - q^{-\nu}t^N)^m} + \cdots + \frac{r_m}{1 - q^{-\nu}t^N} + \frac{E(t)}{\prod_{j \in J_2}(1 - q^{-\nu_j}t^{N_j})},$$

where $r_i \in S[t]$ with $\deg(r_i) < N$ and where $E(t) \in S[t]$.

Let $k \in \mathbb{Z}_{>0}$. Then

$$\frac{1}{(1 - q^{-\nu}t^N)^k} = \sum_{i=0}^{\infty} f_k(i) q^{-i\nu} t^{iN},$$

where $f_k : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is a polynomial function with rational coefficients of degree $k-1$. Indeed, because $\sum_{i=0}^{\infty} f_k(i) q^{-i\nu} t^{iN} = (\sum_{i=0}^{\infty} q^{-i\nu} t^{iN})$ $(\sum_{i=0}^{\infty} f_{k-1}(i) q^{-i\nu} t^{iN})$, we obtain that $f_k(n) = \sum_{i=0}^{n} f_{k-1}(i)$. Consequently, it follows by induction on k since $\sum_{i=0}^{n} g(i)$ is a polynomial function in n of degree r with rational coefficients if g is such a function of degree $r-1$.

There exists an integer d, an integer z which is not divisible by q and polynomial functions with integer coefficients $g_b : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, $b \in \{0, 1, \ldots, N-1\}$, such that

$$\begin{aligned} &\frac{r_1}{(1 - q^{-\nu}t^N)^m} + \frac{r_2}{(1 - q^{-\nu}t^N)^{m-1}} + \cdots + \frac{r_m}{1 - q^{-\nu}t^N} \\ &= \sum_{b=0}^{N-1} \sum_{i=0}^{\infty} \frac{g_b(i)}{z q^d} q^{-i\nu} t^{iN+b} \end{aligned}$$

$$= \sum_{b=0}^{N-1} \sum_{i=0}^{\infty} \frac{g_b(i)}{z} q^{(n+l)(iN+b)-d-bl} \frac{t^{iN+b}}{q^{n(iN+b)}} \tag{5.3}$$

and such that z is a divisor of $g_b(i)$ for every $i \in \mathbb{Z}_{\geq 0}$. Remark that $(n+l)(iN+b)-d-bl$ is an integer. Because $D(t)$ is not divisible by $(1-q^{-\nu}t^N)^m$, at least one of the polynomials g_b is different from the zero polynomial. Fix from now on a $b \in \{0,\ldots,N-1\}$ for which g_b is different from 0. Because g_b is a polynomial function, it has only a finite number of zeros. Let h be a positive integer which is not a zero of g_b. Let r be an integer such that $g_b(h)$ is not a multiple of q^r. Because g_b is a polynomial with integer coefficients, we obtain for every $i \in \mathbb{Z}_{\geq 0}$ that $g_b(iq^r+h)$ is not a multiple of q^r. We put $a = \llcorner r-d-bl \lrcorner$ and because $(iq^r+h)N+b = iq^rN+hN+b$, we put $R = q^rN$ and $c = hN+b$. With this notation, we have that the coefficient of $t^{iR+c}/q^{n(iR+c)}$ in (5.3) is not an integer multiple of $q^{(n+l)(iR+c)+r-d-bl} = q^{\lceil (n+l)(iR+c)+a \rceil}$ for every $i \in \mathbb{Z}_{\geq 0}$.

Now we consider the remaining part

$$\frac{E(t)}{\prod_{j\in J_2}(1-q^{-\nu_j}t^{N_j})} \tag{5.4}$$

of $P(t)$. We obtain from Proposition 5.3.3 that there exists an $l' > l$ and an integer a' such that the coefficient of t^i/q^{ni} in (5.4) is an integer multiple of $q^{\lceil (n+l')i-a' \rceil}$ for all $i \in \mathbb{Z}_{\geq 0}$. Consequently, the coefficient of t^i/q^{ni} is an integer multiple of $q^{\lceil (n+l)i+a \rceil}$ for i large enough.

Because M_{iR+c} is the sum of two numbers of which exactly one is an integer multiple of $q^{\lceil (n+l)(iR+c)+a \rceil}$ for i large enough, we obtain that M_{iR+c} is not an integer multiple of $q^{\lceil (n+l)(iR+c)+a \rceil}$ for i large enough. □

(5.3.5) Let χ be a character of $R^\times$ with conductor e. Suppose that the image of χ consists of the nth roots of unity. Let $\xi = \exp(2\pi\sqrt{-1}/n)$. The minimal polynomial of ξ over $\mathbb{Q}$ is the nth cyclotomic polynomial having degree the Euler number $\phi(n)$. Remark also that $\{1,\xi,\ldots,\xi^{\phi(n)-1}\}$ is a basis of $\mathbb{Q}(\xi)$ as a vector space over $\mathbb{Q}$.

Write

$$Z_{f,\chi}(t) = \sum_{i=0}^{\infty} \sum_{u\in(R/P^e)^\times} \chi(u) M_{i+e}(\pi^i u) q^{-n(i+e)} t^i$$

$$= \sum_{0 \le k < \phi(n)} \left(\sum_{i=0}^{\infty} \widetilde{M}_{i+e,k} q^{-n(i+e)} t^i \right) \xi^k, \tag{5.5}$$

where $\widetilde{M}_{i+e,k}$ is a linear combination of the $M_{i+e}(\pi^i u)$, $u \in (R/P^e)^\times$, with integer coefficients because the nth cyclotomic polynomial is monic and has integer coefficients. Since $M_{i+e}(\pi^i u)$ is an integer multiple of $q^{\lceil (n/2)(i+e-1) \rceil}$, this implies that $\widetilde{M}_{i+e,k}$ is also an integer multiple of $q^{\lceil (n/2)(i+e-1) \rceil}$.

On the other hand, it is generally known that $Z_{f,\chi}(t)$ can be written in the form

$$Z_{f,\chi}(t) = \sum_{0 \le k < \phi(n)} \frac{A_k(t)}{\prod_{j \in J_k} (1 - q^{-\nu_j} t^{N_j})} \xi^k, \tag{5.6}$$

where $A_k(t) \in S[t]$, where every ν_j and N_j is in $\mathbb{Z}_{>0}$ and where $A_k(t)$ is not divisible by $1 - q^{-\nu_j} t^{N_j}$ for every $j \in J_k$.

Because $\{1, \xi, \ldots, \xi^{\phi(n)-1}\}$ is a basis of $\mathbb{Q}(\xi)$ as a vector space over $\mathbb{Q}$, we obtain from (5.5) and (5.6) that

$$\frac{A_k(t)}{\prod_{j \in J_k} (1 - q^{-\nu_j} t^{N_j})} = \sum_{i=0}^{\infty} \widetilde{M}_{i+e,k} q^{-ne} q^{-ni} t^i$$

for every $k \in \{0, 1, \ldots, \phi(n) - 1)\}$. The consequence (i) of (5.3.4) can be applied to this equality. Because $\widetilde{M}_{i+e,k} q^{-ne}$ is an integer multiple of $q^{\lceil (n/2)(i+e-1) - ne \rceil}$, which is equal to $q^{\lceil (n-n/2)i - (n/2)(e+1) \rceil}$, we obtain that $Z_{f,\chi}(s)$ has no pole with real part less than $-n/2$.

We obtain the following theorem by using (5.1.3).

Theorem. *Let f be a K-analytic function on an open and compact subset of K^n. Let χ be a character of $R^\times$. Then we have that Igusa's p-adic zeta function $Z_{f,\chi}(s)$ of f has no poles with real part less than $-n/2$.*

Remark. To any $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $d \in \mathbb{Z}_{>0}$ Denef and Loeser associate the topological zeta function $Z^{(d)}_{\mathrm{top},f}(s)$, see [DL1] or [De2]. Because $Z^{(d)}_{\mathrm{top},f}(s)$ is a limit of Igusa's p-adic zeta functions, we obtain that $Z^{(d)}_{\mathrm{top},f}(s)$ has no poles less than $-n/2$. This is generally known for $n = 2$ and we proved this already for $n = 3$ and $d = 1$ in Section 2.3.

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Nederlandse Samenvatting

Getaltheorie en meetkunde zijn historisch gezien de eerste deelgebieden van de wiskunde die bestudeerd werden. Mijn onderzoek van de laatste jaren leidde tot nieuwe eigenschappen in deze twee domeinen.

N.1 Basisbegrippen en -eigenschappen

N.1.1 Igusa's p-adische zetafunctie

(N.1.1.1) Eén van de basisbegrippen van dit proefschrift is veeltermcongruenties. Zij $f \in \mathbb{Z}[x_1, \ldots, x_n]$ een veelterm over de gehele getallen in n veranderlijken. Noteer $x = (x_1, \ldots, x_n)$. We willen het aantal oplossingen bestuderen van de congruentie $f(x) \equiv 0 \bmod m$ in $(\mathbb{Z}/m\mathbb{Z})^n$ voor een willekeurig positief geheel getal m. De Chinese reststelling reduceert dit probleem tot het geval dat m een macht van een priemgetal is. Zij p een gefixeerd priemgetal en zij M_i, $i \in \mathbb{Z}_{\geq 0}$, het aantal oplossingen van de congruentie $f(x) \equiv 0 \bmod p^i$ in $(\mathbb{Z}/p^i\mathbb{Z})^n$. Het gedrag van de M_i's is door Hensels lemma gekend voor grote i als f geen singulier punt heeft in $\mathbb{Z}_p^n$. Hier stelt $\mathbb{Z}_p$ de ring van p-adische gehelen voor. Het gedrag van de M_i's is veel ingewikkelder als f een singulier punt heeft in $\mathbb{Z}_p^n$. Igusa's p-adische zetafunctie laat ons toe om het singuliere geval te bestuderen.

(N.1.1.2) Voor een p-adisch getal $z \in \mathbb{Q}_p$ noteren we de valuatie van z met $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$, de absolute waarde van z met $|z| = p^{-\operatorname{ord} z}$ en de angulaire component van z met $\operatorname{ac} z = zp^{-\operatorname{ord} z}$. Zij χ een karakter van $\mathbb{Z}_p^\times$, d.w.z. een groepsmorfisme $\chi : \mathbb{Z}_p^\times, . \to \mathbb{C}^\times, .$ met eindig beeld. We stellen formeel $\chi(0) = 0$. Zij f een veelterm over $\mathbb{Z}$ of algemener over $\mathbb{Z}_p$ of $\mathbb{Q}_p$. Zij $X = \mathbb{Z}_p^n$. Igusa's p-adische zetafunctie $Z_{f,\chi}(s)$ van f en χ

wordt gedefinieerd door de p-adische integraal

$$Z_{f,\chi}(s) := \int_X \chi(\operatorname{ac} f(x))|f(x)|^s \, |dx| \tag{N.1}$$

voor complexe getallen s met positief reëel deel, en door de meromorfe continuatie te nemen op de rest van $\mathbb{C}$. Hier is $|dx|$ de Haarmaat op $\mathbb{Q}_p^n$ waarvoor $\mathbb{Z}_p^n$ maat 1 heeft.

(N.1.1.3) Igusa's p-adische zetafunctie kan in een algemenere context gedefinieerd worden. Zij K een p-adisch veld, d.w.z. een uitbreiding van $\mathbb{Q}_p$ van eindige graad. Zij R de valuatiering van K, P het maximaal ideaal van R, π een gefixeerde uniformerende parameter voor R en q het aantal elementen van de quotiëntring R/P. Merk op dat dus $R/P \cong \mathbb{F}_q$. Voor $z \in K$, zij $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ de valuatie van z, $|z| = q^{-\operatorname{ord} z}$ de absolute waarde van z en $\operatorname{ac} z = z\pi^{-\operatorname{ord} z}$ de angulaire component van z. Zij χ een karakter van $R^\times$. Zij X een open en compact deel van K^n en zij f een K-analytische functie op X. Igusa's p-adische zetafunctie $Z_{f,\chi}(s)$ van f en χ wordt nu op dezelfde manier gedefinieerd als voordien. Merk op dat in het speciale geval van hierboven $K = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, $P = p\mathbb{Z}_p$, $\pi = p$, $q = p$, $X = \mathbb{Z}_p^n$ en $f \in \mathbb{Q}_p[x_1, \ldots, x_n]$.

Door gebruik te maken van een ingebedde resolutie van (f, dx) bewees Igusa dat de p-adische integraal in (N.1) een rationale functie is van q^{-s}. Bijgevolg kunnen we $Z_{f,\chi}(s)$, $\operatorname{Re}(s) > 0$, uitbreiden tot een meromorfe functie op gans $\mathbb{C}$, een feit dat we in de definitie van Igusa's p-adische zetafunctie gebruikt hebben. Als we $Z_{f,\chi}(s)$ beschouwen als functie in de variabele $t = q^{-s}$ zullen we $Z_{f,\chi}(t)$ schrijven. Als χ het triviale karakter is van $R^\times$ schrijven we ook $Z_f(s)$ en $Z_f(t)$.

(N.1.1.4) Zij f een veelterm over R of algemener, een strakke K-analytische functie op R^n gedefinieerd over R, d.w.z. een K-analytische functie op R^n die gegeven wordt door een convergente machtreeks op R^n met coëfficiënten in R. Igusa's p-adische zetafunctie van zo'n f heeft een belangrijk verband met congruenties. Voor $i \in \mathbb{Z}_{>0}$ en $u \in R/P^i$ beschouwen we het aantal oplossingen $M_i(u)$ van de congruentie $f(x) \equiv u \bmod P^i$ in $(R/P^i)^n$. Noteer $M_i := M_i(0)$. Zij e de conductor van χ, d.w.z. de kleinste $a \in \mathbb{Z}_{>0}$ waarvoor χ triviaal is op $1 + P^a$.

De $M_{i+e}(\pi^i u)$, $u \in (R/P^e)^\times$, beschrijven $Z_{f,\chi}(t)$ via de relatie

$$Z_{f,\chi}(t) = \sum_{i=0}^{\infty} \sum_{u \in (R/P^e)^\times} \chi(u) M_{i+e}(\pi^i u) q^{-n(i+e)} t^i.$$

Als χ het triviale karakter is, beschrijven de M_i's Igusa's p-adische zetafunctie $Z_f(t)$ ook via de relatie

$$Z_f(t) = P(t) - \frac{P(t)-1}{t}, \tag{N.2}$$

waarbij de Poincaré reeks $P(t)$ van f gedefinieerd wordt door

$$P(t) = \sum_{i=0}^{\infty} M_i (q^{-n} t)^i.$$

Omdat we relatie (N.2) kunnen oplossen naar $P(t)$ hebben we ook dat $Z_f(t)$ alle M_i's beschrijft.

N.1.2 De lokale topologische zetafunctie

(N.1.2.1) Zij f de kiem van een holomorfe functie op een omgeving van de oorsprong 0 in $\mathbb{C}^n$ waarvoor $f(0) = 0$. We veronderstellen steeds dat f niet identiek nul is. Zij $g : V \to U \subset \mathbb{C}^n$ een ingebedde resolutie van singulariteiten van een representant van $f^{-1}\{0\}$. We noteren de irreducibele componenten van $g^{-1}(f^{-1}\{0\})$ met E_i, $i \in T$, en de multipliciteiten van $f \circ g$ en $g^*(dx_1 \wedge \cdots \wedge dx_n)$ langsheen E_i met N_i en $\nu_i - 1$. De (N_i, ν_i), $i \in T$, worden de numerische data genoemd van de resolutie (V, g). Voor $I \subset T$ noteren we $E_I := \cap_{i \in I} E_i$ en $\overset{\circ}{E}_I := E_I \setminus (\cup_{j \notin I} E_j)$. Merk op dat V de disjuncte unie is van de $\overset{\circ}{E}_I$, $I \subset T$. Zij $d \in \mathbb{Z}_{\geq 1}$.

(N.1.2.2) Aan f en d associëren we de lokale topologische zetafunctie

$$Z_f^{(d)}(s) := \sum_{\substack{I \subset T \\ \forall i \in I \,:\, d | N_i}} \chi(\overset{\circ}{E}_I \cap g^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + s N_i}.$$

Hier is s een complexe variabele en $\chi(\cdot)$ stelt de topologische Euler-Poincaré karakteristiek voor. Het merkwaardige feit dat $Z_f^{(d)}(s)$ onafhankelijk is van de gekozen resolutie werd voor het eerst bewezen door Denef

en Loeser. Ze bekwamen de uitdrukking voor de topologische zetafunctie immers als limiet van Igusa's p-adische zetafuncties. Verder noteren we $Z_f(s) := Z_f^{(1)}(s)$.

De kandidaatpolen van de rationale functie $Z_f^{(d)}(s)$ zijn de $-\nu_i/N_i$, waarbij $i \in T$ voldoet aan $d \mid N_i$. Het is opvallend dat de meeste kandidaatpolen geen pool blijken te zijn.

N.2 Studie met behulp van ingebedde resoluties

N.2.1 De lokale topologische zetafunctie

(N.2.1.1) We bestuderen eerst het geval $n = 2$. Het is dan algemeen gekend dat de lokale topologische zetafunctie geen polen heeft kleiner dan -1. We bepalen voor elke gefixeerde d de waarden kleiner dan $-1/2$ die de pool zijn van een lokale topologische zetafunctie. Ze zijn allemaal van de vorm $-1/2 - 1/i$ met $i \in \mathbb{Z}_{>1}$, en al deze waarden zijn de pool van een lokale topologische zetafunctie met $d = 1$.

Als f multipliciteit vier of meer heeft in de oorsprong $0 \in \mathbb{C}^2$, argumenteren we dat -1 de enige mogelijke pool is kleiner dan $-1/2$. Om de multipliciteiten twee en drie te behandelen gebruiken we het Preparatietheorema van Weierstrass om een eenvoudigere functie te vinden met dezelfde lokale topologische zetafunctie.

Voor willekeurige n kan de logcanonieke drempel $c_0(f)$ van f beschreven worden in termen van een ingebedde resolutie van f door $c_0(f) = \min\{\nu_i/N_i \mid i \in T\}$. In het bijzonder is dit minimum dus onafhankelijk van de ingebedde resolutie van f. Als $n = 2$ is het gekend dat $-c_0(f)$ een pool is van $Z_f(s)$. Bijgevolg is ons resultaat een opmerkelijke veralgemening van het feit dat de logcanonieke drempel nooit in het interval $]5/6, 1[$ zit.

(N.2.1.2) Als $n = 3$ en $d = 1$ bepalen we alle waarden kleiner dan -1 die de pool zijn van een lokale topologische zetafunctie. Ze zijn $-1 - 1/i$ met $i \in \mathbb{Z}_{>1}$. Merk op dat er dus geen polen zijn kleiner dan $-3/2$.

Als f multipliciteit drie of meer heeft, bewijzen we bovendien dat er geen polen zijn kleiner dan -1. Dit is een lastig deel omdat we moeten argumenteren dat sommige kandidaatpolen geen pool zijn. Hiervoor kunnen we de relaties tussen de numerische data van Veys gebruiken. Als s_0 een kandidaatpool is van $Z_f^{(d)}(s)$ met verwachte orde één en als E_i een

exceptionele variëteit is met kandidaatpool s_0, dan kunnen we gemakkelijk de bijdrage van E_i tot het residu van $Z_f^{(d)}(s)$ in de kandidaatpool s_0 neerschrijven. De relaties van Veys laten je toe om te bewijzen dat deze bijdrage nul is voor sommige intersectieconfiguraties op E_i op het ogenblik dat E_i gecreëerd wordt. We bepalen alle mogelijke intersectieconfiguraties op exceptionele variëteiten met kandidaatpool kleiner dan -1 op het ogenblik dat ze gecreëerd worden. Voor deze intersectieconfiguraties is de bijdrage tot het residu nul.

N.2.2 Igusa's p-adische zetafunctie

(N.2.2.1) Voor Igusa's p-adische zetafuncties bekomen we analoge resultaten als voor de topologische zetafunctie, maar nu voor de reële delen van de polen. We krijgen hier wel te maken met een extra moeilijkheid in het bewijs dat sommige kandidaatpolen geen pool zijn. We willen de bijdrage berekenen van een exceptionele variëteit E_i met een kandidaatpool s_0 van verwachte orde 1 tot het residu van $Z_{f,\chi}(s)$ in s_0 door alleen informatie te gebruiken die gekend is als E_i gecreëerd wordt. Ons resultaat van volgende sectie geeft een methode om dit te doen.

(N.2.2.2) Zij f een K-analytische functie op een open en compact deel X van K^n. Zij $g : Y = Y_t \to X = Y_0$ een ingebedde resolutie van f die een samenstelling $g_1 \circ \cdots \circ g_t$ is van opblazingen $g_i : Y_i \to Y_{i-1}$. Veronderstel dat elke g_i een opblazing is langsheen een K-analytisch gesloten deelmanifold van Y_{i-1} dat gelegen is in de nulpuntsverzameling van $f \circ g_1 \circ \cdots \circ g_{i-1}$ en dat enkel transversale snijdingen heeft met de unie van de exceptionele variëteiten van $g_1 \circ \cdots \circ g_{i-1}$.

Veronderstel dat s_0 een kandidaatpool is van $Z_{f,\chi}(s)$ met verwachte orde m. Zij $r \in \{1, \ldots, t\}$. Zij $(V, y = (y_1, \ldots, y_n))$ een kaart op een open en compact deel V van Y_r. Veronderstel dat het zin heeft om over de strikt getransformeerde van $y_1 = \cdots = y_m = 0$ in Y te spreken en dat

$$f \circ g_1 \circ \cdots \circ g_r = \alpha \prod_{i=1}^{m} y_i^{N_i} \quad \text{en} \quad (g_1 \circ \cdots \circ g_r)^* dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1} dy$$

waarbij geldt:

1. α en β zijn K-analytische functies op V die niet nul worden op een $n - m$ dimensionaal deel van $y_1 = y_2 = \cdots = y_m = 0$,
2. voor elke $i \in \{1, \ldots, m\}$ is $\chi^{N_i} = 1$ en

3. voor elke $i \in \{1, \ldots, m\}$ bestaat er een $k_i \in \mathbb{Z}$ zodat $s_0 = -\nu_i/N_i + (2k_i\pi\sqrt{-1})/(N_i \log q)$.

Dan is de bijdrage van de strikt getransformeerde van $y_1 = \cdots = y_m = 0$ in Y tot de coëfficiënt b_{-m} in de MacLaurin reeks

$$\frac{b_{-m}}{(s-s_0)^m} + \frac{b_{-m+1}}{(s-s_0)^{m-1}} + \cdots + b_0 + b_1(s-s_0) + \cdots$$

van $Z_{f,\chi}(s)$ in s_0 gelijk aan

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right) \left[\int_{y_1=\cdots=y_m=0} \chi(\mathrm{ac}\,\alpha)|\alpha|^s |\beta||dy_{m+1} \wedge \cdots \wedge dy_n|\right]_{s=s_0}^{\mathrm{mc}}.$$

Hier stelt $[\,\cdot\,]_{s=s_0}^{\mathrm{mc}}$ de evaluatie voor in $s = s_0$ van de meromorfe continuatie van de functie tussen de vierkante haakjes. Het was reeds gekend dat de bijdrage tot het residu van $Z_{f,\chi}(s)$ in s_0 hieraan gelijk is als we op de ingebedde resolutie Y werken, dus als $r = t$. Voor andere r bekomen we ons resultaat door deze formule voort te duwen langsheen de opblazingen.

N.3 Studie met behulp van congruenties

(N.3.1) Zij f een veelterm over R of algemener een strakke K-analytische functie op R^n gedefinieerd over R. Zij l het kleinste reële deel van een pool van Igusa's p-adische zetafunctie van f met triviaal karakter. We bewijzen dat er een geheel getal a bestaat dat onafhankelijk is van i zodat M_i een geheel veelvoud is van $q^{\ulcorner (n+l)i-a \urcorner}$ voor alle $i \in \mathbb{Z}_{\geq 0}$.

(N.3.2) Als $n = 2$ heeft $Z_f(s)$ geen polen met reëel deel kleiner dan -1. Als $n = 3$ hebben we met behulp van ingebedde resoluties bewezen dat $Z_f(s)$ geen polen heeft met reëel deel kleiner dan $-3/2$. Tesamen met de vorige paragraaf geeft dit mooie deelbaarheidseigenschappen voor de M_i's als $n = 2$ en als $n = 3$. Omdat deze zo eenvoudig te formuleren zijn, gaan we op zoek naar een elementair bewijs, en we hebben succes. Dit bewijs veralgemeent gemakkelijk naar willekeurige dimensie en naar de algemenere klasse van getallen $M_i(u)$. Ons resultaat is dat $M_i(u)$ deelbaar is door $q^{\ulcorner (n/2)(i-1) \urcorner}$ voor alle $i \in \mathbb{Z}_{>0}$ en alle $u \in R/P^i$.

(N.3.3) We willen dit resultaat gebruiken om te bewijzen dat $Z_f(s)$ geen polen heeft met reëel deel kleiner dan $-n/2$. Hiervoor bewijzen we het

omgekeerde van het resultaat in (N.3.1): als er een geheel getal a bestaat zodat M_i een geheel veelvoud is van $q^{\ulcorner (n+l')i-a \urcorner}$ voor alle $i \in \mathbb{Z}_{\geq 0}$, dan geldt dat $l' \leq l$. Omdat we een analoog resultaat hebben wanneer we te maken hebben met een karakter χ zijn we in staat om te bewijzen dat $Z_{f,\chi}(s)$ geen polen heeft met reëel deel kleiner dan $-n/2$. Voorlopig hebben we deze ondergrens alleen geformuleerd voor strakke K-analytische functies op R^n gedefinieerd over R. We zullen echter argumenteren dat hieruit deze ondergrens ook volgt voor een willekeurige K-analytische functie op een open en compact deel van K^n. Merk op dat deze grens optimaal is: $Z_f(s)$ heeft een pool in $-n/2$ als f gelijk is aan $x_1x_2+x_3x_4+\cdots+x_{n-1}x_n$ voor n even en $x_1x_2+x_3x_4+\cdots+x_{n-2}x_{n-1}+x_n^2$ voor n oneven.

(N.3.4) Omdat de lokale topologische zetafunctie een limiet is van Igusa's p-adische zetafuncties verkrijgen we ook dat $Z_f^{(d)}(s)$ geen polen heeft kleiner dan $-n/2$.