# Harmonic analysis of Banach space valued functions in the study of parabolic evolution equations

Pierre Portal

# Analyse harmonique des fonctions à valeurs dans un espace de Banach pour l'étude des équations d'évolution paraboliques

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**Abstract**

This work is motivated by the study of parabolic evolution equations and, in particular, of their regularity in a $L_p$ sense. Such questions lead to the investigation of singular integral operators with operator valued kernels acting on a Banach space. We are interested in boundedness results for such operators and their applications to evolution equations. Our focus is on the relationship between these results and the geometry of the underlying Banach space. We study various problems, both in discrete and continuous time, and relate their behavior to the R-boundedness of certain sets of bounded linear operators acting on a UMD space (for $L_p$ regularity with $1 < p < \infty$) or to the existence of a complemented copy of $c_0$ or $\ell_1$ (for $\ell_\infty$ and $\ell_1$ regularity).

Ce travail est motivé par l'étude des équations paraboliques et en particulier de leur régularité $L_p$. On est amené à considérer des opérateurs intégraux dont le noyau est une fonction à valeur dans un espace d'opérateurs agissant sur un espace de Banach. Les questions concernent alors le caractère borné de tels opérateurs intégraux et l'application de tels resultats à l'étude des équations d'évolution. Plus particulièrement on s'intéresse au rôle de la géométrie de l'espace de Banach sous-jacent dans ce type de résultats. Ce travail est une étude de différents problèmes abstraits, en temps discret et continu, où la régularité est liée au caractère R-borné de certains ensembles d'opérateurs linéaires agissant sur un espace de Banach UMD (régularité $L_p$ pour $1 < p < \infty$) ou à l'existence de copies complémentées de $c_0$ (resp. de $\ell_1$) dans l'espace de Banach sous-jacent (respectivement pour la régularité au sens de $\ell_\infty$ et de $\ell_1$).

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# Contents

"die Wissenschaft unter der Optik des Künstlers zu sehen, die Kunst aber unter der des Lebens..."
Friedrich Nietzsche in "Die Geburt der Tragödie". [1]

[1] "to look at scientific enquiry from the perspective of the artist, but to look at art from the perspective of life. . ." Friedrich Nietzsche in "The Birth of Tragedy".

# Introduction (français)

Ce travail concerne les équations d'évolution déterministes, c'est à dire les modèles mathématiques de l'évolution d'un système scientifique (e.g. biologique, physique, économique) dans le temps, en supposant que cette évolution est prédeterminée. De tels modèles utilisent des représentations mathématiques du temps et du système considéré. Pour le temps, on considère deux principaux objets : des intervalles de $\mathbb{R}$ pour les problèmes continus et des suites de $\mathbb{R}$ pour les problèmes discrets (d'autres approches existent cependant, voir la Section 3.2). Pour le système, on utilise un espace d'états qui, dans ce travail, est toujours un espace de Banach. Il est important de noter que, dans la plupart des cas, cet espace est de dimension infinie. Ce fait a d'importantes conséquences. Du point de vue philosophique, d'une part, il implique que, bien que les équations soient déterministes, les systèmes qu'elles représentent ne sont pas forcément prédictibles. Il se peut par exemple que les états ne puissent être interprétés que de manière probabiliste, comme c'est le cas en mécanique quantique. Du point de vue mathématique, d'autre part, les espaces de Banach de dimension infinie possèdent une très riche classification isomorphique linéaire. Il est donc naturel de se demander comment le choix de l'espace des états, par rapport à cette classification, influence le comportement des équations d'évolution. En schématisant largement, c'est là le sujet du présent travail.

Plus précisément, on approche les équations d'évolution par la théorie des semigroupes d'opérateurs linéaires. Ceci signifie que (en temps continu) l'on considère un espace de Banach $X$ et un ensemble $\{T_t\ ;\ t \in \mathbb{R}_+\} \subset B(X)$ (où $B(X)$ désigne l'ensemble des opérateurs linéaires agissant sur $X$) tel que

$$\begin{cases} T_t T_s = T_{t+s} \quad \forall (t,s) \in \mathbb{R}_+^2, \\ T_0 = I. \end{cases} \tag{1}$$

L'équation (1) signifie, bien sur, que $\{T_t\ ;\ t \in \mathbb{R}_+\}$ est un semigroupe. On peut aussi interpréter ce fait (voir l'épilogue de [EN00]) comme une expression mathématique du déterminisme scientifique puisque il énonce que, si le système est redémarré pour $s$ unités de temps après avoir évolué pendant $t$ unités de temps, il atteindra le même état que si il avait évolué pendant $t+s$ unités de temps. Un tel semigroupe (de la forme $T_t = e^{tA}$ où $Ax = \lim\limits_{h\to 0} \frac{T(h)x-x}{h}$ pour tout $x$ tel que cette limite existe) donne une solution faible $u(t) = T_t x + \int_0^t T_{t-s} f(s) ds$ du problème de Cauchy

$$(CP)_{\mathbb{R}_+,A} \qquad \left\{ \begin{array}{rcl} u'(t) - Au(t) & = & f(t) \quad \forall t \in \mathbb{R}_+, \\ u(0) & = & x. \end{array} \right.$$

Ceci siginifie que l'étude du problème $(CP)_{\mathbb{R}_+,A}$ (qui peut être une expression abstraite d'une équation aux dérivées partielles si $X$ est un espace de fonctions et $A$ un opérateur différentiel) peut se faire via l'étude du semigroupe $\{T_t\ ;\ t \in \mathbb{R}_+\}$ (voir [EN00]). De plus ceci suggère l'étude de fonctions de la variable de temps à valeurs dans l'espace de Banach $X$ et, en particulier, des fonctions de la forme

$$\begin{array}{rccl} O_x : & \mathbb{R}_+ & \to & X \\ & t & \mapsto & T_t x \end{array}$$

et

$$\begin{array}{rccl} I_f : & \mathbb{R}_+ & \to & X \\ & t & \mapsto & \int\limits_0^t T_{t-s} f(s) ds. \end{array}$$

Dans le deuxième cas, on reconnait un opérateur intégral invariant par translation avec un noyau opératoriel. Ceci suggère donc l'utilisation des espaces de Bochner $L_p(\mathbb{R}_+; X)$ et l'extension des résultats classiques de l'analyse harmonique des fonctions de $L_p(\mathbb{R}_+, \mathbb{C})$ à ces espaces de Bochner. Ce travail s'intéresse à ces extensions et à leur rôle dans l'étude des équations d'évolution et se concentre plus particulièrement sur les relations entre de tels résultats et la théorie des espaces de Banach.

La thèse comprend deux parties. La première s'intitule "Preliminaries" et contient des définitions et des résultats qui illustrent cette approche des problèmes d'évolution. La deuxième partie s'intitule "Contributions" et présente les problèmes particuliers que cette thèse considère. La présentation se veut autosuffisante en ce sens que les résultats et définitions utilisés dans la deuxième partie sont rappelés dans la première. Cette dernière comprend trois chapitres. Le premier concerne les espaces de Banach et les opérateurs entre espaces de Banach. Il commence par rappeler des propriétés géométriques nécessaires à l'extension des résultats d'analyse harmonique (espace UMD : Section 1.1), puis s'intéresse aux résulats sur l'existence de suites particulières et donc d'orbites particulières pour les semigroupes (espaces contenant $c_0$ : Section 1.2) et à l'existence de décomposition de Schauder particulières et donc de semigroupes particuliers (Sections 1.3 et 1.4). Ensuite on présente une notion cruciale sur le caractère aléatoirement borné de certains sous-ensembles de $B(X)$ (Section 1.5) qui s'avère être un élément clef dans l'extension de résultats hilbertiens au cadre des espaces UMD. Finalement la notion d'analyticité pour un semigroupe est présentée, à la fois en temps continu et en temps discret (respectivement Section 1.6 et Section 1.7). Ceci est bien sur très important puisque ce travail ne considère que les équations d'évolution paraboliques, c'est à dire les équations impliquant des générateurs de semigroupes analytiques.

Le deuxième chapitre est un résumé des résultats d'analyse harmonique pour les fonctions de $L_p(\mathbb{R}_+; X)$. Il commence avec le travail crucial de Bourgain dans les années 80 (Section 2.2) puis présente les principaux résultats récents tels que le théorème de multiplicateur de Weis.

Dans le troisième chapitre on aborde finalement les équations d'évolution et, en particulier, la question de leur régularité maximale. On considère donc, disons pour un problème continu, la relation entre la régularité (au sens d'un espace de fonctions donné) du terme inhomogène $f$ et de la solution $u$. Ceci est expliqué en détail dans la Section 3.1. La Section 3.2 présente ensuite rapidement la théorie des échelles de temps avant que ne soit introduite, dans différents sens du terme, la notion de régularité maximale (Sections 3.4, 3.5 et 3.6). Ce chapitre se termine ensuite avec un survol des problèmes concrets pour lesquels la régularité maximale a déjà été utilisée.

La seconde partie contient elle aussi trois chapitres. Le quatrième s'intéresse à la régularité maximale des équations d'évolution discrètes (i.e. avec $\mathbb{Z}_+$ comme notion de temps). On y montre tout d'abord que, en dehors des espaces de Hilbert, il existe des problèmes paraboliques qui n'ont pas la régularité maximale au sens de $\ell_2$. On considère ensuite la relation entre la régularité $\ell_1$ et $\ell_\infty$ des problèmes et l'existence d'une copie complémentée de $\ell_1$ (resp. de $c_0$) dans l'espace de Banach sous-jacent. Finalement, on présente d'autres relations du même type ainsi qu'une comparaison des différentes notions de régularité discrète (au sens de $\ell_1$, $\ell_p$ $(1 < p < \infty)$ et $\ell_\infty$ ). Les résultats de ce chapitre ont été publiés dans [Po03] (*Semigroup Forum*).

Le cinquième chapitre s'intéresse aussi aux problèmes discrets mais dans le cas où le pas de discrétisation n'est pas forcément constant (i.e. en considérant une suite croissante de $\mathbb{R}$ pour le temps). Pour ces problèmes on introduit les notions adéquates (notamment l'analyticité de certaines familles d'évolution) et on caractérise la régularité maximale $\ell_p$ $(1 < p < \infty)$ dans le cas où l'échelle de temps est basée sur un échantillon (i.e. quand la suite des pas de discrétisation est périodique). Finalement on obtient un résulat plus général par un argument de perturbation. Les résultats de ce chapitre ont été prépubliés dans [Po04] et soumis pour publication.

Le dernier chapitre s'intéresse à un problème général de l'analyse harmonique des fonctions de $L_p(\mathbb{R}_+; X)$ motivé par l'étude des problèmes d'évolution non autonome (comme par exemple les problèmes considérés dans le chapitre précédent). On y considère le caractére $L_p$ borné d'opérateurs pseudo-différentiels de la forme

$$T_a f(x) = \int_{\mathbb{R}^n} e^{ix.\xi} a(x,\xi) \widehat{f}(\xi) d\xi$$

où $\widehat{f}$ représente la transformée de Fourier de $f$ et $a(x,\xi) \in B(X)$ pour tout $(x,\xi) \in$

$\mathbb{R}^n \times \mathbb{R}^n$. On montre que $T_a$ se prolonge en un opérateur borné sur $L_p(\mathbb{R}^n; X)$ si l'on suppose que $X$ est UMD et que $a(x, \xi)$ vérifie certaines conditions de type Mihlin en $\xi$ et de régularité Hölderienne en $x$. En application on considère la question de la régularité maximale $L_p$ pour des équations non autonomes en temps continu. Ce chapitre donne une nouvelle preuve d'un résultat non publié de Željko Štrkalj et fait partie du travail en collaboration [PoS04]. L'article est en préparation.

# Introduction (english)

This work is motivated by deterministic evolution equations, i.e. mathematical models of the evolution of a scientific (e.g. biological, physical, economical) system with time, in the case where this evolution is thought to be predetermined. Such equations involve a mathematical representation of time and of the system under consideration. For time we use two main objects : intervals of $\mathbb{R}$ for continuous problems and sequences in $\mathbb{R}$ for discrete ones (although more general approaches can be investigated as it can be seen in Section 3.2). For the system we use a state space which, in our work, is always a Banach space. It is particularly important to remark that, in most cases, this space is infinite dimensional. A philosophical consequence of this fact is that, although the equations are deterministic, the systems they represent are not necessarily predictable. It may be, for instance, that the states can only be interpreted in a probabilistic way, as it is the case, for instance, in quantum mechanics. But this infinite dimensional nature of the state space has also an important mathematical consequence. Indeed, the classification of infinite dimensional Banach spaces, up to linear continuous isomorphisms, is very rich. It is therefore natural to wonder how the choice of the state space, from the point of view of Banach space theory, influences the behavior of the evolution equation. Put in a very general way, this is the purpose of the present work. More precisely we approach evolution equations with the theory of semigroups of linear operators. This means that (when time is assumed to be continuous) we consider a Banach (state) space $X$ and a set $\{T_t \; ; \; t \in \mathbb{R}_+\} \subset B(X)$ (where $B(X)$ denotes the set of bounded linear operator acting on $X$) such that

$$\begin{cases} T_t T_s = T_{t+s} \quad \forall (t,s) \in \mathbb{R}_+^2, \\ T_0 = I. \end{cases} \tag{2}$$

Equation 2 is, of course, the semigroup law. Note that it can also be seen (see the Epilogue of [EN00]) as the law of determinism for it states that the system, if restarted after $t$ units of time for $s$ units of time, will reach the state it would have reached after $t+s$ units of time if not restarted. Such a semigroup (of the form $T_t = e^{tA}$ where $Ax = \underset{h \to 0}{lim} \frac{T(h)x - x}{h}$ for each $x$ where this limit exists) gives the mild solution $u(t) = T_t x + \int_0^t T_{t-s} f(s) ds$ of the Cauchy problem

$$(CP)_{\mathbb{R}_+,A} \qquad \begin{cases} u'(t) - Au(t) &= f(t) \quad \forall t \in \mathbb{R}_+, \\ u(0) &= x. \end{cases}$$

This means that the study of $(CP)_{\mathbb{R}_+,A}$ (which can be seen as an abstract expression of a partial differential equation by letting $X$ be a function space and $A$ be differential operator) can be done through the study of the semigroup $\{T_t \ ; \ t \in \mathbb{R}_+\}$ (see [EN00]). Moreover this suggests to study the functions of the time variable with values in the Banach space $X$ and in particular the maps of the form

$$\begin{array}{rrcl} O_x : & \mathbb{R}_+ & \to & X \\ & t & \mapsto & T_t x \end{array} \quad \text{and} \quad \begin{array}{rrcl} I_f : & \mathbb{R}_+ & \to & X \\ & t & \mapsto & \int_0^t T_{t-s} f(s) ds. \end{array}$$

Considering the latter maps, one can recognize a translation invariant singular integral operator with an operator-valued kernel. This, in turn, suggests the use of Bochner spaces $L_p(\mathbb{R}_+; X)$ and the extension of harmonic analytic results for functions in $L_p(\mathbb{R}_+; \mathbb{C})$ to such spaces. This work is concerned with such extensions and their role in the study of evolution equations. But, most of all, it investigates the interplays between these issues and the theory of Banach spaces.

The thesis consists of two parts. The first one is called "Preliminaries" and gives various definitions and results which illustrate the questions and features of this approach to evolution equations. The second part is called "Contributions" and presents the particular problems I have been interested in during my PhD. The presentation is meant to be self-contained in the sense that the results and definitions used in the second part are recalled in the first one. The latter consists in three chapters.

It starts with Banach spaces and linear operators acting on them. This first chapter contains results on the geometry of Banach spaces related to the extension of harmonic analytic results (UMD spaces in Section 1.1), the existence of particular sequences and thus of particular orbits of semigroups (spaces containing $c_0$ in Section 1.2) and the existence of particular Schauder decompositions and thus of particular semigroups (Sections 1.3 and 1.4). Then it presents a crucial notion of boundedness for sets in $B(X)$ which turns out to be the key element in the extension of results from Hilbert to UMD spaces (Section 1.5). Finally the notion of analyticity for a semigroup (both in the continuous and discrete sense) is presented (Sections 1.6 and 1.7). This is especially important since we consider only parabolic evolution equations, i.e. equations involving generators of analytic semigroups.

Chapter 2 consists in an overview of the harmonic analytic theory for functions in $L_p(\mathbb{R}^n; X)$. It starts with the crucial work of Bourgain in the 80's (Section 2.2) and then presents the main recent results such as Weis' multiplier theorem.

In Chapter 3 we finally turn to evolution equations and, in particular, to the question of maximal regularity. This means that we consider, for a problem (say in continuous time) $(CP)_{\mathbb{R}_+,A}$, the relationship between the regularity (in the sense of a function space $Y$) of the inhomogeneous term $f$ and the solution $u$. This is explained in details in Section 3.1. Section 3.2 then presents briefly the time scale theory which allows to consider various notions of time. After this short digression, maximal regularity in different senses is considered (Sections 3.4, 3.5 and 3.6). This part is then concluded with a quick overview of the concrete problems to which maximal regularity has already been applied.

The second part also consists in three chapters. Chapter 4 deals with maximal regularity of discrete time evolution equations (i.e. with time being $\mathbb{Z}_+$). It is shown that, except in Hilbert spaces, not all parabolic evolution equations enjoy maximal regularity in the $\ell_2$ sense. Then it focuses on maximal regularity in the $\ell_1$ and the $\ell_\infty$ sense and considers the relationships between these properties and the fact that the Banach space contains or not a copy of $c_0$ (resp. a complemented copy of $\ell_1$). In the final section (4.5) more of these relationships are presented along with a comparison between maximal regularity in the $\ell_1$, $\ell_p$ $(1 < p < \infty)$ and $\ell_\infty$ sense. The results from this chapter can also be found in the article [Po03] from *Semigroup Forum*.

Chapter 5 is also concerned with discrete problems but in the case where the step size is not necessarily constant (i.e. with time being any increasing sequence of real numbers). For these problems we introduce the corresponding semigroup notions (especially analyticity) and then characterize maximal regularity in the $\ell_p$ sense $(1 < p < \infty)$ for the sample based case (i.e. when the sequence of step sizes is periodic). Finally more general cases are treated via a perturbation argument. The results from this chapter can be found in the preprint [Po04] from the *Université de Franche-Comté* which has been submitted for publication.

The last chapter deals with a general problem in the harmonic analysis of functions in $L_p(\mathbb{R}^n; X)$ which is motivated by the study of non autonomous evolution equations (such as continuous analogues of the equations considered in Chapter 5). We investigate the $L_p$ boundedness of pseudo-differential operators of the form

$$T_a f(x) = \int_{\mathbb{R}^n} e^{ix.\xi} a(x,\xi) \widehat{f}(\xi) d\xi$$

where $\widehat{f}$ denotes the Fourier transform of $f$ and $a(x,\xi) \in B(X)$ for each $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. We show that $T_a$ extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ provided $X$ is UMD and $a(x,\xi)$ satisfies some conditions of Mihlin's type in $\xi$ and of Hölder's type in $x$. As an application we consider the question of $L_p$-maximal regularity for non autonomous evolution equations in continuous time. This chapter gives a new proof of an unpublished result by Željko Štrkalj and is part of the joint work [PoS04]. The paper is in preparation.

# Part I

# Preliminaries

# Chapter 1

# Notions from the Banach spaces and operators theories

## 1.1 UMD spaces

Let $X$ be a Banach space and $B(X)$ denote the set of bounded linear operators acting on $X$. In order to extend harmonic analytic results from $L_p(\mathbb{R}^n;\mathbb{C})$ $(1 < p < \infty)$ to Bochner's spaces $L_p(\mathbb{R}^n;X)$ $(1 < p < \infty)$ one has to assume an adequate property of $X$. This is the purpose of the following definition.

**Definition 1.1.1**
*For $\varepsilon \in (0,1)$ and $1 < p < \infty$ let us define the operators $H_\varepsilon \in B(L_p(\mathbb{R};X))$ by*

$$H_\varepsilon = \frac{1}{\pi} \int\limits_{\varepsilon<|s|<\frac{1}{\varepsilon}} \frac{f(t-s)}{s} ds.$$

*$X$ is said to be a* UMD space *if $\underset{\varepsilon\to 0}{lim} H_\varepsilon f$ exists in $L_p(\mathbb{R};X)$ for every $f \in L_p(\mathbb{R};X)$. The operator $H = \underset{\varepsilon\to 0}{lim} H_\varepsilon$ is then called the* Hilbert transform.

This definition is natural for harmonic analytic applications and gives that $\mathbb{C}$ is a UMD space (this is a classical result due to Riesz, see for instance [Gr04]) However, much of the interest of the UMD class comes from the fact that it can be characterized by properties of a somewhat different nature.

**Theorem 1.1.2 (Bourgain 1983, Burkholder 1981,1983)**
*Let $X$ be a Banach space. The following assertions are equivalent.*

*(i) $X$ is a UMD space.*

*(ii) There exists $C > 0$ such that, for every probability space $(\Omega, \Sigma, \mu)$, every martingale $(f_k)_{k\in\mathbb{Z}_+} \subset L_p(\Omega; X)$ and every $n \in \mathbb{N}$,*

$$\sup_{\varepsilon_k=\pm 1} \| \sum_{k=0}^{n} \varepsilon_k (f_{k+1} - f_k) \|_{L_p(\Omega;X)} \leq C \| \sum_{k=0}^{n} (f_{k+1} - f_k) \|_{L_p(\Omega;X)}.$$

*(iii) There exists a biconvex function $\zeta : X \times X \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and for every $(x, y) \in X^2$ such that $\|x\| = \|y\| = 1$ we have*

$$\zeta(x, y) \leq \|x + y\|.$$

**Remark 1.1.3**

- *(ii) is in fact the original definition of a UMD space, which stands for "Unconditional Martingale Differences".*
- *$(i) \Longrightarrow (ii)$ was proved by Bourgain in [B83].*
- *$(ii) \Longrightarrow (i)$ was proved by Burkholder in [Bu83].*
- *$(iii) \Longleftrightarrow (ii)$ was proved by Burkholder in [Bu81].*
- *One could add other characterizations to Theorem 1.1.2. Namely one could replace the boundedness of the Hilbert transform on $L_p(\mathbb{R}; X)$ by analogue statements on $L_p(\mathbb{T}; X)$ or by the boundedness of the Riesz projection and one could also replace (ii) by analogue statements involving martingale transforms (see [Bu85] and [Rf86] for further information).*

Besides this interesting ubiquitous nature, UMD spaces possess important Banach spaces theoretical properties (see [Rf86], [Pi74] and [Pi75]).

**Theorem 1.1.4 (Pisier 1974)**
*UMD spaces are super-reflexive but not every super-reflexive space is UMD.*

Remark that Bourgain even found (in [B83]) a super-reflexive Banach lattice which is not UMD.

**Proposition 1.1.5**
*Let $X$ be a Banach space. Then the following hold.*

*(a) $X$ is UMD if and only if $X^*$ is UMD.*

*(b) If $X$ is UMD then for each measured space $\Omega$ with a $\sigma$ finite measure $\mu$, the space $L_p(\Omega; X)$ is UMD.*

*(c) If $X$ is UMD and $Y$ is a closed linear subspace of $X$ then $Y$ and $X/Y$ are UMD spaces.*

The UMD class therefore contains many classical Banach spaces, which makes it especially interesting in applications. For instance we have the following examples.

**Example 1.1.6**

- *Finite dimensional spaces, $\ell_p$ spaces and $L_p$ spaces ($1 < p < \infty$) are UMD spaces.*

- *For $1 < p < \infty$ and any von Neuman algebra $\mathcal{M}$, non commutative $L_p(\mathcal{M})$ spaces (such as the Schatten classes) are also UMD (see [PX97] and the survey [PX03]).*

Further information can be found in the survey articles [Rf86] by Rubio de Francia and [Bu85] by Burkholder.

## 1.2 Spaces containing $c_0$ : results by C. Bessaga and A. Pelczyński

Whereas the UMD property is crucial for the $L_p$ ($1 < p < \infty$) theory of evolution equations, the fact that the underlying Banach space contains or not a copy of $c_0$ is of importance for the $L_\infty$ theory (respectively complemented copies of $\ell_1$ play a role in the $L_1$ theory). For this reason we present here a characterization of this property, due to Bessaga and Pelczyński in [BP58], which will be of great use in Chapter 4. We start by recalling some definitions.

**Definition 1.2.1**
*Let $X$ be a Banach space and $(x_k)_{k\in\mathbb{N}} \subset X$. The series $\sum_{k=1}^{\infty} x_k$ is said to be* unconditionally convergent (u.c.) *if the series $\sum_{k=1}^{\infty} \varepsilon_k x_k$ converges for each choice of signs $\varepsilon_k = \pm 1$. It is said to be* weakly unconditionally convergent (w.u.c.) *if there exists a constant $C > 0$ such that*

$$sup_{n\in\mathbb{N}} sup_{\varepsilon_k=\pm 1} \| \sum_{k=1}^{n} \varepsilon_k x_k \| \leq C.$$

**Theorem 1.2.2 (Bessaga,Pelczyński 1958)**
*Let $X$ be a Banach space. The following assertions are equivalent.*

*(i) There exists in the space $X$ a w.u.c. series which is not u.c. ..*

*(ii) There exists a w.u.c. series $\sum_{k=1}^{\infty} x_k \in X$ such that $inf_{k\in\mathbb{N}} \|x_k\| > 0$.*

*(iii) $X$ contains a subspace isomorphic to $c_0$.*

In applications we use this fundamental theorem together with the following duality result.

**Theorem 1.2.3 (Bessaga,Pelczyński 1958)**
*Let $X$ be a Banach space such that $X^*$ contains a subspace isomorphic to $c_0$. Then $X$ contains a complemented subspace isomorphic to $\ell_1$.*

**Remark 1.2.4**
*In the above result the role of $c_0$ and $\ell_1$ can not be interchanged. For instance if $X = \ell_1$ then $\ell_1 \subset X^*$ but $c_0 \not\subset X$. However Johnson and Rosenthal have shown that if $X$ is separable and $X^*$ contains $\ell_1$ then $c_0$ is isomorphic to a quotient of $X$ (see Proposition 2.e.9 in volume 1 of [LT77]).*

We end this section with a particular case of a result of Sobczyk (in [So41]) which is also very useful.

**Theorem 1.2.5 (Sobczyk 1941)**
*Let $X$ be a separable Banach space with a subspace $Y$ isomorphic to $c_0$. Then $Y$ is a complemented subspace of $X$.*

All the results from this section can be found in [BP58] (Sobczyk's result is reproved using bases theory). Further information can be found in Section 2.e of [LT77] (volume 1).

## 1.3 Hilbert spaces among spaces with an unconditional basis

It is a common problem to decide whether a result, which is known to hold in Hilbert spaces, extends to more general Banach spaces. When it is not the case one may want to relate the result to characterizations of Hilbert spaces in Banach spaces theory. In this section we present such a characterization, which is used in [KL00] (and also in [Po03]) and which is a consequence of deep results by Lindenstrauss and Tzafriri and by Pelczyński. We start with some definitions.

**Definition 1.3.1**
*Let $X$ be a Banach space.*
$(x_k)_{k\in\mathbb{N}} \subset X$ *is called a* Schauder basis *if, for every $x \in X$, there exists a unique sequence $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $x = \sum\limits_{k=1}^{\infty} a_k x_k$.*
*It is called an* unconditional basis *if the series converges unconditionally.*
*A sequence of non-zero vectors $(u_j)_{j\in\mathbb{N}} \subset X$ of the form $u_j = \sum\limits_{n=p_j+1}^{p_{j+1}} a_n x_n$ for some $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and an increasing sequence of integers $(p_j)_{j\in\mathbb{N}}$ is called a* block basis of

$(x_k)_{k\in\mathbb{N}}$.
*Two bases* $(x_j)_{j\in\mathbb{N}}$ *and* $(y_j)_{j\in\mathbb{N}}$ *are called* equivalent *provided a series* $\sum_{j=1}^{\infty} a_j x_j$ *(for* $(a_j)_{j\in\mathbb{N}} \subset \mathbb{C}$*) converges if and only if* $\sum_{j=1}^{\infty} a_j y_j$ *converges.*

It is interesting to remark that many classical Banach spaces have an unconditional basis. It is, for instance, the case of finite dimensional spaces, $\ell_p$ spaces and $L_p$ spaces ($1 < p < \infty$). It is, however, not the case of $L_1$ (see [P61]) or of some non commutative $L_p$ spaces, even when $1 < p < \infty$ (see [PX03]).

**Theorem 1.3.2 (Lindenstrauss, Tzafriri 1971)**
*Let* $(x_k)_{k\in\mathbb{N}} \subset X$ *be a normalized unconditional basis of a Banach space* $X$*. Assume that for every permutation* $\pi$ *and for every block basis* $(u_j)_{j\in\mathbb{N}}$ *of* $(x_{\pi(k)})_{k\in\mathbb{N}}$*, the closed linear span of* $(u_j)_{j\in\mathbb{N}}$ *is complemented. Then* $(x_k)_{k\in\mathbb{N}}$ *is equivalent to the canonical basis of* $c_0$ *or* $\ell_p$ *for some* $1 \leq p < \infty$*.*

**Theorem 1.3.3 (Pelczyński 1960)**
*For* $p \in (1,2) \cup (2,\infty)$*, the spaces* $\ell_p$ *have two non equivalent unconditional bases.*

The following corollary is then a direct consequence.

**Corollary 1.3.4**
*Let* $X$ *be a Banach space with an unconditional basis. Assume that* $X \not\simeq c_0$ *and* $X \not\simeq \ell_1$*. Moreover, assume that for each normalized unconditional basis* $(x_k)_{k\in\mathbb{N}} \subset X$*, each permutation* $\pi$ *and each block basis* $(u_j)_{j\in\mathbb{N}}$ *of* $(x_{\pi(k)})_{k\in\mathbb{N}}$*, the closed linear span of* $(u_j)_{j\in\mathbb{N}}$ *is complemented.*
*Then* $X \simeq \ell_2$*.*

Further information can be found in Section 2.a. of [LT77].
This concludes our preliminaries on the geometry of Banach spaces. We now focus our attention on operator theoretical notions.

## 1.4 Multipliers on a Schauder decomposition

In order to relate properties of linear operators (and ultimately of evolution equations) to the geometry of the underlying Banach space one can consider special operators, which properties are closely related to properties of the space. Such operators are defined as follows.

**Definition 1.4.1**
*A normalized sequence* $(\Delta_k)_{k\in\mathbb{Z}} \subset B(X)$ *is called* a Schauder decomposition *of* $X$ *if the following hold.*

*(i)* $k \neq l \implies \Delta_k \Delta_l = 0$.

*(ii)* $\forall x \in X \quad \sum\limits_{k \in \mathbb{Z}} \Delta_k x = x$.

*It is called* unconditional *if the series converges unconditionally.*
*Given a sequence* $(\lambda_n)_{n \in \mathbb{Z}}$ *and a Schauder decomposition* $(\Delta_n)_{n \in \mathbb{Z}} \subset B(X)$*, the linear operator* $T$ *defined by*

$$T\Big(\sum_{k=-n}^{n} \Delta_k x\Big) = \sum_{k=-n}^{n} \lambda_k \Delta_k x$$

*is called a* multiplier *on* $(\Delta_n)_{n \in \mathbb{Z}}$ *associated to* $(\lambda_n)_{n \in \mathbb{Z}}$.

It is then a simple but useful fact that

$$\left\{ \begin{array}{l} (\lambda_n)_{n \in \mathbb{Z}} \in \ell_\infty \\ (\Delta_n)_{n \in \mathbb{Z}} \,\text{unconditional} \end{array} \right. \implies T \in B(X).$$

This property characterizes unconditional decompositions in the sense that, whenever the decomposition is conditional, there exists a bounded sequence $(\lambda_n)_{n \in \mathbb{Z}} \subset \mathbb{C}$ such that the corresponding multiplier is unbounded. For a conditional decomposition it is therefore unclear which conditions should be imposed on $(\lambda_n)_{n \in \mathbb{Z}}$ in order to define a bounded operator $T$. Nevertheless we have that

$$(\lambda_{n+1} - \lambda_n)_{n \in \mathbb{Z}} \in \ell_1 \implies T \in B(X).$$

Moreover we have the following lemma which can essentially be found in [Ve93] (see also [La98]).

**Lemma 1.4.2**
*Let* $X$ *be a Banach space,* $(\Delta_n)_{n \in \mathbb{N}} \subset B(X)$ *be a Schauder decomposition of* $X$ *and* $(\lambda_n)_{n \in \mathbb{N}}$ *be a sequence of scalars. Let us define*

$$\begin{aligned} D(T) = \{x \in X \; ; \; \sum_{k \in \mathbb{N}} \lambda_k \Delta_k x \,\text{converges in}\, X\} \\ \text{and}\, \forall x \in D(T) \; Tx = \sum_{k \in \mathbb{N}} \lambda_k \Delta_k x \end{aligned} \tag{1.1}$$

*Then :*

*(i)* $T$ *is closed and densely defined.*

*(ii)* *If* $\lambda_n \neq 0 \, \forall n \in \mathbb{N}$ *then* $T$ *is injective and has dense range.*

This provides us with an interesting way to construct (counter)examples. The idea goes back, at least, to Baillon (in [Ba80]) and will be of importance in Chapter 4. Further information on Schauder decompositions can be found in Section 1.g of [LT77]. For multipliers important references are the article [CPSW00] and the PhD thesis [Wi00].

## 1.5 R-boundedness of a family of bounded linear operators

The notion of a multiplier on an unconditional Schauder decomposition can be extended by considering $(T_k)_{k\in\mathbb{N}} \subset B(X)$ such that $T_k$ commutes with $\Delta_k$ for each $k$ and the operator defined by

$$T(\sum_{k=1}^{n} \Delta_k x) = \sum_{k=1}^{n} T_k \Delta_k x \quad \forall x \in X \quad \forall n \in \mathbb{N}.$$

In Hilbert spaces, if we assume uniform boundedness of the operators $T_k$ we obtain that $T \in B(X)$. But this result is not clear in general Banach spaces. However we can obtain an analogue result if we replace uniform boundedness by a stronger assumption. This will be the following notion of R-boundedness.

**Definition 1.5.1**
*Let $X$ be a Banach space. $\Psi \subset B(X)$ is called* R-bounded *if*
$\exists C > 0, \quad \forall n \in \mathbb{N}, \quad \forall T_1, ..., T_n \in \Psi, \quad \forall x_1, ..., x_n \in X$

$$\int_0^1 \| \sum_{j=1}^{n} \varepsilon_j(t) T_j x_j \| dt \leq C \int_0^1 \| \sum_{j=1}^{n} \varepsilon_j(t) x_j \| dt \tag{1.2}$$

*where $(\varepsilon_j)_{j\in\mathbb{N}}$ is the usual sequence of Rademacher functions on $[0,1]$ (i.e. $\varepsilon_j(t) = sign(sin(2^j \pi t))$).*

**Notation 1.5.2**
*For a R-bounded set $\Psi \subset B(X)$, we denote by $\mathcal{R}(\Psi)$ the smallest constant $C$ such that 1.2 holds.*

R-boundedness stands for Riesz-boundedness, Rademacher-boundedness or Randomized-boundedness depending on the authors. Given that a similar notion can be defined using Gaussian variables instead of Rademacher variables (which turns out to be equivalent in spaces with cotype), we choose to call this notion Rademacher-boundedness. It appears implicitly in the work [B83] of Bourgain and explicitly in the article [BG94] of Berkson and Gillespie. It turns out to be a crucial notion in Banach space valued harmonic analysis (see Chapter 2) but, from an abstract point of view, the role of this notion is to a great extent summarized in the following proposition (see [CPSW00] and [Wi00]).

**Proposition 1.5.3 (Clément, De Pagter, Sukochev, Witvliet 2000)**
*Let $X$ be a Banach space and $(\Delta_k)_{k\in\mathbb{Z}} \subset B(X)$ be an unconditional decomposition*

*of $X$. Consider a R-bounded family $(T_k)_{k\in\mathbb{N}} \subset B(X)$ such that $T_k$ commutes with $\Delta_k$ for each $k$ and the operator defined by*

$$T(\sum_{k=1}^{n} \Delta_k x) = \sum_{k=1}^{n} T_k \Delta_k x \quad \forall x \in X \quad \forall n \in \mathbb{N}.$$

*Then $T$ defines a bounded linear operator on $X$.*

The application to the harmonic analysis of functions in $L_p(\mathbb{R}; X)$ then follows from the existence of special unconditional decompositions of the spaces $L_p(\mathbb{R}; X)$ (see Section 2.1). Moreover R-boundedness generalizes the notion of square functions estimates in $L_q$ spaces. More precisely consider $\Psi \subset B(L_q)$ for some $1 < q < \infty$. Then by Khintchine-Kahane's inequalities and Fubini's theorem, $\Psi$ is R-bounded if and only if
$\exists C > 0, \quad \forall n \in \mathbb{N}, \quad \forall T_1, ..., T_n \in \Psi, \quad \forall x_1, ..., x_n \in L_q$

$$\|(\sum_{j=1}^{n} |T_j x_j|^2)^{\frac{1}{2}}\|_q \leq C \|(\sum_{j=1}^{n} |x_j|^2)^{\frac{1}{2}}\|_q.$$

In applications one also needs some stability results for R-boundedness. They can be found, for instance, in [CPSW00] and are summarized in the following lemma.

**Lemma 1.5.4**
*Let $X$ be a Banach space and $\Psi \subset B(X)$ be a R-bounded family. Then the absolute convex hull and the strong closure of $\Psi$ are also R-bounded.*

Nice surveys on R-boundedness can be found in the PhD [Wi00] and the master thesis [Hy00]. More informations and applications can be found in [BG94], [CPSW00], [W01] and [KW01].

## 1.6 Analytic semigroups of bounded linear operators

Since evolution equations are our main motivation, the families of operators that we consider are often semigroups. Moreover, in the applications we are interested in (parabolic evolution equations), these semigroups are even analytic. In this section we recall what it means along with some classical properties.

**Definition 1.6.1**
*Let $\delta \in (0, \frac{\pi}{2}]$ and $\Sigma_\delta = \{\lambda \in \mathbb{C} \ ; \ |arg(\lambda)| < \delta\} \cup \{0\}$. A family of operators $(T_z)_{z\in\Sigma_\delta}$ is called an* analytic semigroup *(of angle $\delta$) if*

*(i) $T(0) = I$ and $T(z + z') = T(z)T(z')$ for all $z, z' \in \Sigma_\delta$.*

*(ii) The map* $z \mapsto T(z)$ *is analytic in* $\Sigma_\delta$*.*

*(iii)* $\underset{z\in\Sigma_{\delta'}, z\to 0}{lim} T(z)x = x$ *for all* $x \in X$ *and* $0 < \delta' < \delta$*.*

*If in addition* $\{\|T(z)\| \ ; \ z \in \Sigma_{\delta'}\}$ *is bounded for each* $0 < \delta' < \delta$*, then* $(T_z)_{z\in\Sigma_\delta}$ *is called a* bounded analytic semigroup.

It is also important to recall that such semigroups are exactly those which generators are sectorial in the following sense.

**Definition 1.6.2**
*A closed linear operator* $(A, D(A))$ *with dense domain acting on a Banach space* $X$ *is called* sectorial *(of angle* $\theta$*) if there exists* $\theta \in (0, \frac{\pi}{2}]$ *such that*

*(i)* $\Sigma_{\frac{\pi}{2}+\theta}\backslash\{0\} \subset \rho(A)$*.*

*(ii) For each* $\varepsilon \in (0, \theta)$ *there exists* $M_\varepsilon > 0$ *such that*

$$\|\lambda R(\lambda, A)\| \leq M_\varepsilon \quad \forall\lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\theta-\varepsilon}.$$

We have then the following characterization (see Theorem 4.6 in [EN00]).

**Theorem 1.6.3**
*For a closed linear operator* $(A, D(A))$ *acting on a Banach space* $X$ *the following assertions are equivalent.*

*(a)* $A$ *generates a bounded analytic semigroup* $(T_z)_{z\in\Sigma_\delta}$*.*

*(b) There exists* $\theta \in (0, \frac{\pi}{2})$ *such that the operators* $e^{\pm i\theta}A$ *generate bounded strongly continuous semigroups.*

*(c)* $A$ *generates a bounded strongly continuous semigroup* $(T_t)_{t\in\mathbb{R}_+}$ *such that* $Ran(T(t)) \subset D(A)$ *for all* $t > 0$ *and*

$$\underset{t>0}{sup}\|tAT(t)\| < \infty.$$

*(d)* $A$ *generates a bounded strongly continuous semigroup* $(T_t)_{t\in\mathbb{R}_+}$ *and there exists a constant* $C > 0$ *such that*

$$\|sR(r + is, A)\| \leq C \quad \forall r > 0 \quad \forall s \in \mathbb{R}.$$

*(e)* $A$ *is sectorial.*

This result is of course of great importance since it is the starting point of useful theories developed for analytic semigroups. For instance property (c) suggests the use of operator-valued singular integral operators (see Chapter 3) whereas property (e) allows a functional calculus approach. The latter is unfortunately outside the scope of this work. The interested reader could look, for instance, at [Ha03] and [KW01] for further information.

For our purposes it is also important to remark that one can construct easily analytic semigroups using multipliers on a Schauder decomposition.

**Remark 1.6.4**
*If $(\lambda_n)_{n\in\mathbb{N}}$ is a monotone sequence of non positive real numbers then the operator $T$ defined by (1.1) generates an analytic semigroup (see [BC91] or [La98]).*

Besides these Banach space theoretical examples it is also noticeable that many classical differential operators do generate analytic semigroup. The main examples are given by elliptic operators on various functions spaces ($L_p$ for $1 < p \leq \infty, C^\alpha$ for $0 \leq \alpha \leq 1$) and under various smoothness conditions on the coefficients and the boundary.

Informations on these concrete examples can be found in [Lu95] and in Chapter 8 of [Ar04]. For general informations on semigroups our main reference is [EN00] where Section 2.4.a is devoted to analytic ones. The recent survey [Ar04] also contains a chapter on analytic semigroups.

## 1.7 Power-bounded operators and Ritt's condition

Semigroups of bounded linear operators $(T(t))_{t\in\mathbb{R}_+}$ are important objects in the study of continuous time evolution equations. But we are also interested in discrete time evolution equation which suggest the use of discrete semigroups $(T^n)_{n\in\mathbb{Z}_+}$ where $T \in B(X)$ and $\mathbb{Z}_+$ denotes the set of non negative integers. We say that such a semigroup is *bounded* if $T$ is power-bounded (i.e. $\exists C > 0\ \forall n \in \mathbb{Z}_+\ \|T^n\| \leq C$). It is, however, harder to define an analogue of the notion of analyticity. The following definition was nevertheless introduced by Coulhon and Saloff-Coste in [CS90] and turns out to be the right one.

**Definition 1.7.1**
*Let $X$ be a Banach space. A discrete time semigroup $(T^n)_{n\in\mathbb{Z}_+} \subset B(X)$ is said to be* analytic *if*

$$\exists C > 0\ \|n(T^{n+1} - T^n)\| \leq C\ \forall n \in \mathbb{Z}_+.$$

The interest of this definition is clear when one consider the following result independently obtained by Nagy,Zemanek in [NZ99] and Lyubich in [Ly99].

**Theorem 1.7.2 (Nagy,Zemanek / Lyubich 1999)**
*Let $X$ be a Banach space and $T \in B(X)$. The following assertions are equivalent.*

*(i)* $\exists C > 0 \ \|T^n\| \leq C$ *and* $\|n(T^{n+1} - T^n)\| \leq C \ \forall n \in \mathbb{Z}_+$.

*(ii)* $\exists K > 0 \ \|(\lambda - 1)R(\lambda, T)\| \leq K \ \forall |\lambda| > 1$.

The condition (ii) is called Ritt's condition and is studied in operator theory in relation with the growth of the powers of an operator (see [Ne01]). The question of the relationship between boundedness and analyticity of discrete semigroups has also been deeply studied. The main results in this direction are the following (see [Es83], [KT86], [KMSOT04]).

**Theorem 1.7.3 (Esterle/Katznelson, Tzafriri 1986)**
*Let $X$ be a Banach space, $T \in B(X)$ be power-bounded and $\sigma(T) \cap \mathbb{T} = \{1\}$. Then*

$$\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.$$

**Theorem 1.7.4**
**(Kalton, Montgomery-Smith, Oleszkiewicz, Tomilov 2004)**
*Let $X$ be a Banach space, $T \in B(X)$ and assume that*

$$\limsup_{n \to \infty} \|n(T^{n+1} - T^n)\| < \frac{1}{e}.$$

*Then $T$ is power-bounded.*

Remark that this improves a former result by Esterle in [Es83] where the constant $\frac{1}{96}$ was used instead of $\frac{1}{e}$. In the above result $\frac{1}{e}$ is sharp (see [KMSOT04]).

**Theorem 1.7.5**
**(Kalton, Montgomery-Smith, Oleszkiewicz, Tomilov 2004)**
*On any infinite dimensional Banach space there exists an analytic semigroup wich is not bounded.*

These theorems are, of course, of great importance. Our focus, however, is on bounded analytic discrete time semigroups and our use of these operator theoretical results only involves Theorem 1.7.2 as a (recent) discrete analogue to the well known Theorem 1.6.3.

Further information can be found in [Ne01], [NZ99], [Ly99] and [KMSOT04].

# Chapter 2

# Results in vector-valued harmonic analysis

## 2.1 Introduction

In this chapter we consider some harmonic analytic results for functions in $L_p(\mathbb{R}^n; X)$ where $1 < p < \infty$ and $X$ is a Banach space. More precisely we are interested in singular integral operators initially defined on the Schwartz class $\mathcal{S}(\mathbb{R}^n; X)$ by

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy \quad \forall x \in \mathbb{R}^n \tag{2.1}$$

with $k(x,y) \in B(X)$ $\forall (x,y) \in \mathbb{R}^{2n}$. We want to obtain conditions on $k$ for $T$ to extend to a bounded operator on $L_p(\mathbb{R}^n; X)$ $(1 < p < \infty)$. This question has of course also a pseudodifferential formulation. Namely we want to know whether the operator initially defined on $\mathcal{S}(\mathbb{R}^n; X)$ by

$$Tf(x) = \int_{\mathbb{R}^n} e^{ix.\xi} a(x,\xi)\widehat{f}(\xi)d\xi \quad \forall x \in \mathbb{R}^n \tag{2.2}$$

with $a(x,\xi) \in B(X)$ $\forall (x,\xi) \in \mathbb{R}^{2n}$ extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ $(1 < p < \infty)$. As a particular but highly interesting case we consider translation invariant operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y)f(y)dy \quad \forall x \in \mathbb{R}^n \tag{2.3}$$

or equivalently

$$Tf(x) = \int_{\mathbb{R}^n} e^{ix.\xi} m(\xi)\widehat{f}(\xi)d\xi \quad \forall x \in \mathbb{R}^n \tag{2.4}$$

where the functions $k$ and $m$ take values in $B(X)$. This means that we are interested in vector-valued extensions of $L_p$ boundedness results for singular integral operators, Fourier multipliers and pseudodifferential operators. The first results in this direction were obtained in the 60's by J.Schwartz in [Sc61] and Benedek, Calderón and Panzone in [BCP62]. On the one hand, following the "real method" of Calderón and Zygmund, Benedek, Calderón and Panzone showed the following analogue of a classical result by Hörmander.

**Theorem 2.1.1 (Benedek, Calderón, Panzone 1962)**
*Let $X$ be a Banach space and $T$ be defined by 2.3. Assume, moreover that there exists $C > 0$ such that*

$$\int\limits_{|y|>2|x|} |k(x) - k(x-y)|dy \leq C \quad \forall x \in \mathbb{R}^n.$$

*Then the following assertions are equivalent.*

*(i) $T$ extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ for some $1 < p < \infty$.*

*(ii) $T$ extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ for all $1 < p < \infty$.*

This result, however, is not as helpful as in the scalar valued case, since Plancherel's theorem does hold on $L_2(\mathbb{R}^n; X)$ only if $X$ is a Hilbert space. On the other hand, J. Schwartz obtained in [Sc61] the following extension to Calderón-Zygmund's theorem in the case where $X = L_q$ for some $1 < q < \infty$.

**Theorem 2.1.2 (J. Schwartz, 1961)**
*Let $k : R^n\backslash\{0\} \to \mathbb{R}$ be of the form $k(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n}$ where $\Omega : \mathbb{R}^n\backslash \to \mathbb{R}$ is a smooth function, homogeneous of order 0 and whose surface integral over the surface of the unit sphere is zero. Then $T$ defined by 2.3 extends to a bounded operator on $L_p(\mathbb{R}^n; L_q)$ for all $1 < p, q < \infty$.*

In the same paper he also gave the following hilbertian extension of Marcinkiewicz's multiplier theorem.

**Theorem 2.1.3 (J. Schwartz, 1961)**
*Let $H$ be a Hilbert space, $1 < p < \infty$, $\chi_k$ be the characteristic function of $[2^k, 2^{k+1}]$ and $m : \mathbb{R}\backslash\{0\} \to B(H)$ be a bounded function such that*

$$\sup_{k\in\mathbb{Z}} Var(m\chi_k) < \infty.$$

*Then the Fourier multiplier defined by 2.4 extends to a bounded operator on $L_p(\mathbb{R}; H)$.*

But, although these results were natural extensions of the scalar-valued ones, they did not contain the ideas needed to extend the theory to a larger class of Banach spaces. In the 80's a main step towards such a generalization was then reached by Bourgain. From his papers [B83] and [B86] it turns out that one has to assume that $X$ is a UMD Banach space (see Section 1.1). Indeed the UMD class appears to be the class of spaces to which Littlewood-Paley's decomposition can be extended.

Further information, including a nice historical perspective, can be found in the introductory chapter of the PhD thesis [Hy03]. More on the extension of Calderón-Zygmund's theory can be found in [RRT86].

## 2.2 Vector-valued Littlewood-Paley decomposition

In the scalar-valued case, many results rely on the dyadic decomposition of Littlewood and Paley. Let us recall (see for instance [S93] for details) that it is obtained by considering *dyadic partitions of unity in* $\mathcal{S}(\mathbb{R}^n,\mathbb{R})$, i.e. sequences $(\phi_k)_{k\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n,\mathbb{R})$ defined by

$$\phi_k(x) = \psi(2^{-k}|x|)$$

for some non negative function $\psi \in \mathcal{S}(\mathbb{R},\mathbb{R})$ supported in $[\frac{1}{2},2]$ and such that $\sum\limits_{k\in\mathbb{Z}} \psi(2^{-k}x) = 1 \quad \forall x \in \mathbb{R}^*$. The decomposition is then given by a sequence $(\Delta_k)_{k\in\mathbb{Z}} \in B(L_p(\mathbb{R}^n;X))$ defined by

$$\begin{array}{rccc} \Delta_k: & L_p(\mathbb{R}^n;X) & \to & L_p(\mathbb{R}^n;X), \\ & f & \mapsto & f * \check{\phi}_k. \end{array} \tag{2.5}$$

This is indeed a decomposition in the sense that $f = \sum\limits_{k\in Z} \Delta_k f \ \forall f \in L_p(\mathbb{R}^n;X)$. Although this is not a Schauder decomposition since $\Delta_k(X) \cap \Delta_{k+1}(X) \neq \{0\} \ \forall k \in \mathbb{Z}$ it plays the same role in applications. It is in fact close to an unconditional decomposition in the following sense.

**Theorem 2.2.1 (Littlewood,Paley)**
*Let $1 < p < \infty$, $f \in L_p(\mathbb{R}^n)$ and $(\Delta_k)_{k\in\mathbb{Z}} \in B(L_p(\mathbb{R}^n;\mathbb{C}))$ be defined as above. Then there exists $C > 0$ such that*

$$\frac{1}{C}\|f\|_p \leq \|(\sum_{k\in\mathbb{Z}} |\Delta_k f|^2)^{\frac{1}{2}}\|_p \leq C\|f\|_p.$$

In [B86], Bourgain generalized this result to UMD spaces. It is particularly interesting to note that his proof uses both the boundedness of the Hilbert transform (assumption 1 of Theorem 1.1.2) and the unconditional martingale differences property (assumption 2 of Theorem 1.1.2). In the following Theorem $(\varepsilon_k)_{k\in\mathbb{Z}}$ denotes a sequence of Rademacher functions (see Section 1.5).

**Theorem 2.2.2 (Bourgain 1986)**
*Let $X$ be a UMD Banach space, $1 < p < \infty$ and $(\Delta_k)_{k\in\mathbb{Z}} \in B(L_p(\mathbb{R}^n;X))$ be defined by 2.5. Then there exists $C > 0$ such that for all $f \in L_p(\mathbb{R}^n;X)$*

$$\frac{1}{C}\|f\|_{L_p(\mathbb{R}^n;X)} \leq \int_0^1 \|\sum_{k\in\mathbb{Z}} \varepsilon_k(t)\Delta_k f\|_{L_p(\mathbb{R}^n;X)} \leq C\|f\|_{L_p(\mathbb{R}^n;X)}.$$

¿From this result Bourgain was able to deduce the following generalization of Marcinkievicz's multiplier theorem. The highest dimensional case is due to Zimmermann in [Z89].

**Theorem 2.2.3 (Bourgain (n=1) 1986, Zimmermann 1989)**
*Let $X$ be a UMD Banach space, $1 < p < \infty$ and $m : \mathbb{R}^n\backslash\{0\} \to \mathbb{C}$ be a bounded function such that $\sup_{j\in\mathbb{Z}} Var(\phi_j m) < \infty$ Then the Fourier multiplier defined by 2.4 extends to a bounded operator on $L_p(\mathbb{R}^n;X)$.*

Further information can be found in [B86] and in [GW03-2] (where these results are given as steps towards fully vector-valued ones as presented in Section 2.3). For a nice historical point of view one can also read the introduction of [Hy03].

## 2.3 Vector-valued Fourier multipliers

In the applications to evolution equations, the need for vector-valued harmonic analytic results was first focused on Fourier multipliers (see [Am97],[W01]), i.e. on operators of the form 2.4. Although Bourgain's extension of Marcinkiewicz's mutliplier theorem provides many insights on the general case it lacks the possibility of having an operator-valued symbol $m$. This can be overcome in vector-valued Besov spaces (see [Am97], [GW03-1]) but not in the case of $L_p(\mathbb{R}^n;X)$ spaces. To treat this case one has to assume that $X$ is UMD in order to apply Bourgain's results but one also has to assume that some quantities are R-bounded to fully exploit Theorem 2.2.2. This leads to the following generalization of Marcinkiewicz's theorem.

**Theorem 2.3.1 (Weis 2001)**
*Let $X$ and $Y$ be UMD Banach spaces and $1 < p < \infty$. Consider a function $m : \mathbb{R}^* \to B(X,Y)$ of the form*

$$m(\xi) = \sum_{j\in\mathbb{Z}} \widetilde{m}(\xi)\phi_j(\xi)M_j$$

*where $(\phi_j)_{j\in\mathbb{Z}}$ is a dyadic partition of unity in $\mathcal{S}(\mathbb{R}^n,\mathbb{R})$, $\{M_j\ ;\ j \in \mathbb{Z}\}$ is R-bounded and $\sup_{j\in\mathbb{Z}} Var(\phi_j\widetilde{m}) < \infty$. Then the operator defined by 2.4 extends to a bounded operator from $L_p(\mathbb{R};X)$ to $L_p(\mathbb{R};Y)$.*

Then, approximation arguments allow to show the following generalization of Mihlin's theorem.

**Theorem 2.3.2 (Weis 2001)**
*Let $X$ and $Y$ be UMD Banach spaces and $1 < p < \infty$. Consider a differentiable function $m : \mathbb{R}^* \to B(X, Y)$ such that the following hold.*

*(i)* $\{m(t)\ ;\ t \in \mathbb{R}^*\}$ *is R-bounded.*

*(ii)* $\{tm'(t)\ ;\ t \in \mathbb{R}^*\}$ *is R-bounded.*

*Then the operator defined by 2.4 extends to a bounded operator from $L_p(\mathbb{R}; X)$ to $L_p(\mathbb{R}; Y)$.*

This result also holds in the periodic case (i.e. on $L_p(\mathbb{T}; X)$) by a result of Arendt and Bu in [AB02] and in the discrete case (i.e. on $\ell_p(X)$) by a result of Blunck in [Bl01-1].

**Theorem 2.3.3 (Blunck 2001)**
*Let $X$ be a UMD Banach space and $1 < p < \infty$. Let $I = (-\pi, 0) \cup (0, \pi)$ and $M : I \to B(X)$ be a differentiable function such that the set*

$$\{M(t)\ ;\ t \in I\} \cup \{tM'(t)\ ;\ t \in I\}$$

*is R-bounded. Then the operator $T_M$, initially defined for a finitely supported sequence $(f_k)_{k\in\mathbb{Z}_+}$ by*

$$T_M f(m) = \int_I e^{imt} M(t)\Big(\sum_{k\in\mathbb{Z}_+} f_k e^{-ikt}\Big)dt$$

*extends to a bounded operator on $\ell_p(X)$.*

One should also remark that assumption (i) in Theorem 2.3.2 can not be avoided. This is partially shown in [W01] and, in fact, the following stronger statement holds (see [CP00]).

**Proposition 2.3.4 (Clément, Prüss 2000)**
*Let $X$ and $Y$ be Banach spaces and $m : \mathbb{R}^* \to B(X, Y)$ be such that the operator defined by 2.4 extends to a bounded operator on $L_2(\mathbb{R}; X)$.*
*Then $\{m(\xi)\ ;\ \xi$ is a Lebesgue point of $m\}$ is R-bounded.*

In [Bl01-1] this was generalized to locally compact abelian groups in the following way.

**Proposition 2.3.5 (Blunck 2001)**
*Let $X$ be a Banach space, $1 < p < \infty$ and $G$ be a LCA group with Haar measure $\mu$. Let the dual group $(\widehat{G}, \widehat{\mu})$ be equipped with a translation invariant metric such that*

$$\sup_{n\in\mathbb{N}} \widehat{\mu}(B_{\widehat{G}}(e,n^{-1}))^{-1} \|\mathcal{F}^{-1}(\chi_{B_{\widehat{G}}(e,n^{-1})})\|_{L_p(G)} \|\mathcal{F}^{-1}(\chi_{B_{\widehat{G}}(e,n^{-1})})\|_{L_{p'}(G)} < \infty,$$

*where $\mathcal{F}$ denotes the Fourier transform and $B_{\widehat{G}}(e,n^{-1})$ denotes the ball in $\widehat{G}$ centered at the identity $e$ and of radius $\frac{1}{n}$. Consider a function $m \in L_{1,loc}(\widehat{G}, B(X))$ such that $f \mapsto \mathcal{F}^{-1}m\mathcal{F}f$ defines a bounded operator on $L_p(G;X)$. Then the set*

$$\{m(\xi) \; ; \; \xi \textit{ is a Lebesgue point of} m\}$$

*is R-bounded.*

Following the ideas of Zimmermann in [Z89] a higher dimensional version of Theorem 2.3.2 was then obtained independantly by Štrkalj and Weis in [SW00] and Haller, Heck and Noll in [HHN02]. These results have recently been improved by Girardi and Weis in [GW03-2]. They obtained a generalization of classical Fourier multiplier theorems under Hörmander and Mihlin types of condition and with an optimal order of the derivatives involved. It is interesting to remark that this order depends on the Fourier type of $X$ (let us recall that a Banach space $X$ is said to have Fourier type $p$ if the Fourier transform defines a bounded linear operator from $L_p(\mathbb{R};X)$ to $L_{p'}(\mathbb{R};X)$). As a corollary of a general abstract result they, in fact, obtain the following (where $[a]$ denotes the integer part of $a$).

**Theorem 2.3.6 (Girardi, Weis 2004)**
*Let $X$ and $Y$ be UMD spaces with Fourier type $p \in (1,2]$, $l = [\frac{n}{p}] + 1$ and $m : \mathbb{R}^n\backslash\{0\} \to B(X,Y)$ be a measurable function whose distributional derivatives are represented by measurable functions. Assume that one of the following hold.*

1. $\exists C > 0 \quad \forall \alpha \in \mathbb{Z}_+^n \; |\alpha|_1 \le l \quad \mathcal{R}(\{|\xi|^{|\alpha|_1} D^\alpha m(\xi) \; ; \; \xi \in \mathbb{R}^n\backslash\{0\}\})$.
2. $\exists C > 0 \quad \max_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|_1 \le l}} \int_{\frac{1}{2}\le|\xi|\le 2} \left(\mathcal{R}(\{|2^k\xi|^{|\alpha|_1} D^\alpha m(2^k\xi) \; ; \; k \in \mathbb{Z}\})\right)^p d\xi \le C$.

*Then the operator defined by 2.4 extends to a bounded operator from $L_q(\mathbb{R}^n;X)$ to $L_q(\mathbb{R}^n;Y)$ for all $1 < q < \infty$.*

In the theorem above, condition 1 is of course of Mihlin type whereas condition 2 is of Hörmander type. But, in [Hy04-2], Hytönen proves that, in fact, these conditions can be weaken into a "Hörmander $\cap$ Mihlin" one. This is especially surprising since it also improves on the classical scalar-valued results.

**Theorem 2.3.7 (Hytönen 2004)**
*Let $X$ and $Y$ be UMD spaces with Fourier type $p \in (1,2]$, $l = [\frac{n}{p}] + 1$ and $m : \mathbb{R}^n \backslash \{0\} \to B(X,Y)$ be a measurable function whose distributional derivatives are represented by measurable functions. Assume that the set*

$$\{|\xi|^\alpha D^\alpha m(\xi) \ ; \ \xi \in \mathbb{R}^n \backslash \{0\}\})$$

*is R-bounded for all $\alpha \in \mathbb{Z}_+^n$ such that $|\alpha|_\infty \leq 1$ and $|\alpha|_1 \leq [\frac{n}{p}] + 1$. Then the operator defined by 2.4 extends to a bounded operator from $L_q(\mathbb{R}^n; X)$ to $L_q(\mathbb{R}^n; Y)$ for all $1 < q < \infty$.*

Moreover [Hy04-2] also contains analogue results for multipliers on vector-valued Besov and $H^1$ spaces.

Further information can be found in [GW03-2], [Hy03] and [Hy04-2]. Since the periodic case contains the main ideas and less technicalities, the article [AB02] is also very useful. See also Chapter 6 from the recent survey [Ar04].

## 2.4 Vector-valued singular integral operators

We now turn to operators of the form 2.1 and, in particular, of the form 2.3 in the translation-invariant case. As mentionned in Section 2.1, a first result in that direction (for convolution integrals) was obtained in 1962 by Benedek, Calderón and Panzone in [BCP62]. Later on, a Calderón-Zygmund theory for operator-valued kernels was developed, in particular in the article [RRT86] by Rubio de Francia, Ruiz and Torrea. In this work they generalized Theorem 2.1.1 to the non translation-invariant setting in the following way.

**Definition 2.4.1**
*Let $X$ and $Y$ be Banach spaces. A kernel $K : \mathbb{R}^{2n} \to B(X,Y)$ satisfies* (D) *if there exists $(c_k)_{k\in\mathbb{N}} \in \ell_1$ such that*

$$\int\limits_{S_k(y,z)} \|K(x,y) - K(x,z)\| dx \leq \frac{c_k}{|S_k(y,z)|}$$

*for all $k \in \mathbb{N}$ and all $(y,z) \in \mathbb{R}^{2n}$ where*

$$S_k(y,z) = \{x \in \mathbb{R}^n \ ; \ 2^k|y-z| \leq |x-z| \leq 2^{k+1}|y-z|\}.$$

**Theorem 2.4.2 (Rubio de Francia, Ruiz, Torrea 1986)**
*Let $X$ and $Y$ be Banach spaces, $1 < q < \infty$ and $k : \mathbb{R}^{2n} \to B(X,Y)$ be a kernel such that the operator $T$ defined by 2.1 extends to a bounded operator from $L_q(\mathbb{R}^n; X)$ to $L_q(\mathbb{R}^n; Y)$. If $k$ satisfies (D) then the following hold.*

*(i) For all $1 < p \leq q$ the operator $T$ extends to a bounded operator from $L_q(\mathbb{R}^n; X)$ to $L_q(\mathbb{R}^n; Y)$.*

*(ii) The operator $T$ extends to a bounded operator from $L_1(\mathbb{R}^n; X)$ to weak $L_1(\mathbb{R}^n; Y)$.*

*(iii) For all $1 < p \leq q$ the operator $T$ extends to a bounded operator from $H_1(\mathbb{R}^n; X)$ to $L_p(\mathbb{R}^n; Y)$.*

A dual result result is also given. Moreover they obtained, in the convolution case, the following version of a classical theorem by Hörmander (see [H60]) which generalizes Theorem 2.1.1.

**Theorem 2.4.3 (Benedek, Calderón, Panzone 1962)**
*Let $X$ and $Y$ be Banach spaces and $T$ be defined by 2.3. Assume, moreover that there exists $C > 0$ such that*

$$\int\limits_{|y|>2|x|} \|(k(x) - k(x-y))a\|_Y dy \leq C\|a\|_X \quad \forall a \in X \quad \forall x \in \mathbb{R}^n.$$

*Then the following assertions are equivalent.*

*(i) $T$ extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ for some $1 < p < \infty$.*

*(ii) $T$ extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ for all $1 < p < \infty$.*

[RRT86] also contains various applications of these results, in particular to maximal operators. However, for our purpose (namely $L_p$ boundedness of such integral operators), Theorem 2.4.2 (and its dual version) and Theorem 2.4.3 are not fully satisfying since they require a priori estimates which are hard to obtain outside Hilbert spaces. For operators of the convolution type 2.3 this motivated a recent work of Hytönen and Weis who obtained results in UMD-valued $L_p$ spaces (in [HW04]) and in vector-valued Besov spaces (in [HW02]). We state their results in the $L_p$ setting, starting with an extension of Theorem 2.4.2.

**Theorem 2.4.4 (Hytönen, Weis 2004)**
*Let $X$ and $Y$ be Banach spaces and $k \in \mathcal{S}'(\mathbb{R}^n; B(X,Y))$ be a kernel whose Fourier transform is strongly measurable, essentially bounded and such that, for every $x \in X$, $k(.)x$ agrees, away from the origin, with a locally integrable $Y$-valued function. Assume the following*

*(i) $\{\widehat{k}(\xi)\ ;\ \xi \in \mathbb{R}^n\}$ is R-bounded.*

*(ii) There exists $C > 0$ such that*

$$\int\limits_{|t|>2|s|} \mathcal{R}(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t))\ ;\ j \in \mathbb{Z}\})\omega(t)dt \leq C\omega(s) \quad \forall s \in \mathbb{R}^n$$

*where $\omega(t) = log(2 + |t|)$.*

*Then the operator defined by 2.1 extends to a bounded operator from $L_q(\mathbb{R}^n; X)$ to $L_q(\mathbb{R}^n; Y)$ for all $1 < q < \infty$.*

In this result it is particularily interesting to point out the growth factor $log(2+|t|)$. This is a consequence of the fact that, although the group of translations on $L_p(\mathbb{R}^n)$ is not R-bounded (for $p \neq 2$), one does have the following estimates (see [B86]).

**Proposition 2.4.5 (Bourgain 1986)**
*Let $X$ be a UMD space and $(f_j)_{j\in\mathbb{Z}} \subset L_p(\mathbb{R}^n; X)$ be a finitely non-zero sequence such that $supp\widehat{f_j} \subset \overline{B}(0, 2^j)$. Let $(h_j)_{j\in\mathbb{Z}} \subset \mathbb{R}^n$ be a sequence, lying on a line through the origin and such that $|h_j| < K2^j$ for some constant $K > 0$. Then there exists $C > 0$ such that*

$$\int_0^1 \| \sum_{j=1}^{n} \varepsilon_j(t) f_j(. - h_j)\|_{L_p(\mathbb{R}^n;X)} dt \leq C log(2+K) \int_0^1 \| \sum_{j=1}^{n} \varepsilon_j(t) f_j \|_{L_p(\mathbb{R}^n;X)} dt$$

*where $(\varepsilon_j)_{j\in\mathbb{N}}$ is a sequence of Rademacher functions on $[0, 1]$.*

Moreover Hytönen and Weis also proved the following more concrete result.

**Theorem 2.4.6 (Hytönen, Weis 2004)**
*Let $X$ and $Y$ be UMD Banach spaces and $k \in C^1(\mathbb{R}^n\backslash\{0\}; B(X, Y))$ be an odd kernel such that the following hold.*

*(i) $\{|t|^n k(t)\ ;\ t \in \mathbb{R}^n\backslash\{0\}\}$ is R-bounded.*

*(ii) $\{|t|^{n+1}\nabla k(t)\ ;\ t \in \mathbb{R}^n\backslash\{0\}\}$ is R-bounded.*

*Then the operator defined by 2.1 extends to a bounded operator from $L_q(\mathbb{R}^n; X)$ to $L_q(\mathbb{R}^n; Y)$ for all $1 < q < \infty$.*

In this result, assuming that the kernel is odd may be unreasonable in applications. However, Hytönen and Weis also give (in [HW04]) other conditions that can replace the oddness assumption and that are considerably less restrictive (although more technical).

Further information can be found in [Hy03] and [HW04].

## 2.5 Vector-valued pseudo-differential operators

As observed in Sections 2.3 and 2.4, translation-invariant operators of the form 2.3 and their Fourier multiplier counterpart (of the form 2.4) are now fairly well understood. In the general case of singular intergal operators of the form 2.1 and of pseudodifferential operators of the form 2.2 the situation is, however, not as good.

So far, Theorem 2.4.2 is the only available result for general singular integrals. But, as explained in Section 2.4, it requires a priori estimates which are difficult to obtain. For pseudo-differential operators, however, some results have been obtained. In Hilbert spaces, first of all, the following result is due to Hieber and Monniaux in [HM00].

**Theorem 2.5.1 (Hieber, Monniaux 2000)**
*Let $H$ be a Hilbert space and $a \in L_\infty(\mathbb{R}^2, B(H))$. Assume that $\xi \mapsto a(x,\xi)$ admits a holomorphic extension $z \mapsto a(x,z)$ to $\Sigma_\theta$ and $-\Sigma_\theta$ for some $\theta \in (0, \frac{\pi}{2})$ such that $\sup\limits_{z\in\Sigma_\theta\cup(-\Sigma_\theta)} \sup\limits_{x\in\mathbb{R}} \|a(x,z)\|_{B(H)} < \infty$. Then the operator defined by 2.2 extends to a bounded operator on $L_2(\mathbb{R}; H)$.*

The case of general UMD Banach spaces was then considered by Štrkalj in his Phd thesis [St00] where the following result was obtained.

**Theorem 2.5.2 (Štrkalj 2000)**
*Let $X$ be a UMD Banach space, $1 < p < \infty$, $n \in \mathbb{N}$, $\delta \in [0,1)$ and $r \in (0,1)$. Consider $a : \mathbb{R}^n \times \mathbb{R}^n \to B(X))$ such that $\forall \alpha \in \mathbb{Z}_+^n \quad \exists C_\alpha > 0$*

$$\left\{ \begin{array}{c} \mathcal{R}(\{(1+|\xi|)^{|\alpha|}\partial_\xi^\alpha a(x,\xi) \ : \ \xi \in \mathbb{R}^n\}) \leq C_\alpha \quad \forall x \in \mathbb{R}^n \\ \|\partial_\xi^\alpha a(x,\xi) - \partial_\xi^\alpha a(y,\xi)\| \leq C_\alpha |x-y|^r (1+|\xi|)^{\delta r - |\alpha|} \quad \forall (x,y) \in \mathbb{R}^{2n} \end{array} \right. \tag{2.6}$$

*Then $T_a$ defined by 2.2 extends to a bounded operator on $L_p(\mathbb{R}^n, X)$.*

This gives a natural extension of Weis' multiplier theorem 2.3.2 to a class of non regular pseudo-differential operators (i.e. with just Hölder regularity in the first variable). Unfortunately Štrkalj stopped doing research after his PhD and never published this result. A different, hopefully simpler, proof is presented in the joint work [PoS04] and is the subject of Chapter 6.

Further information can be found in Chapter 6 and in [HM00], [St00], [PoS04].

# Chapter 3

# Maximal regularity of parabolic evolution equations

## 3.1 Introduction

We now come to our initial motivation, namely parabolic evolution equations. We therefore consider a Banach (state) space $X$ and a set $\mathcal{T} \subset \mathbb{R}_+$ (such that $0 \in \mathcal{T}$) which represents time and is called a *time scale*. This notion comes from the theory of dynamic equations on time scales, which includes difference equations ($\mathcal{T}$ being a sequence) and differential equations ($\mathcal{T}$ being an interval), but also provides a wide generalization of these models of time (see Section 3.2 for more informations on this theory). In this introduction we restrict our attention to the case where $\mathcal{T}$ is either an interval or $\mathbb{Z}_+$. Acting on $X$ we then consider an operator $(A, D(A))$ which is assumed to generate an analytic semigroup in the appropriate sense. This means that in the case where $\mathcal{T}$ is an interval, then $(A, D(A))$ is sectorial of angle $\phi < \frac{\pi}{2}$ (see Section 1.6) and that, in the case where $\mathcal{T} = \mathbb{Z}_+$, $A$ is a bounded operator such that $((I-A)^n)_{n\in\mathbb{Z}_+}$ is a discrete time bounded analytic semigroup (see Section 1.7). Given the time scale $\mathcal{T}$, the state space $X$ and the operator $A$ acting on $X$ we consider a (input) function $f : \mathcal{T} \to X$ and the Cauchy problem

$$(CP)_{\mathcal{T},A} \qquad \left\{ \begin{array}{rcll} u^{\Delta}(t) - Au(t) & = & f(t) & \forall t \in \mathcal{T}, \\ u(0) & = & 0, & \end{array} \right.$$

where, for all $t \in \mathcal{T}$, $u^{\Delta}(t) = u'(t)$ if $\mathcal{T}$ is an interval and $u^{\Delta}(t) = u(t+1) - u(t)$ if $\mathcal{T} = \mathbb{Z}_+$. We are then interested in the relationship between the regularity of $f$ and the regularity of the mild solution of $(CP)_{\mathcal{T},A}$ (where the mild solution is given by $u(t) = \int\limits_0^t e^{(t-s)A} f(s)ds$ if $\mathcal{T}$ is an interval and $u(n+1) = \sum\limits_{j=0}^{n} T^{n-j} f_j$, for $T = I - A$, if $\mathcal{T} = \mathbb{Z}_+$). By "regularity" we mean that the function belongs to a given subspace $Y(X)$ of the space $\mathcal{F}(\mathcal{T}; X)$ of functions from $\mathcal{T}$ to $X$ (e.g. $Y = L_2(\mathbb{R}_+), Y = \ell_2$). More precisely we consider the following maximal regularity property.

**Definition 3.1.1**
*Let $X$ be a Banach space, $\mathcal{T} \subset \mathbb{R}_+$ be a time scale, $(A, D(A))$ be a linear operator acting on $X$ and $Y(X) \subset \mathcal{F}(\mathcal{T}; X)$. Then $(CP)_{\mathcal{T},A}$ is said to have $Y$-*maximal regularity *if, for each $f \in Y(X)$, the mild solution $u$ is such that $u^\Delta \in Y(X)$. It is said to have* strong $Y$-maximal regularity *if, for each $f \in Y(X)$, the mild solution $u$ belongs to $Y(X)$ and is such that $u^\Delta \in Y(X)$.*

Of course if $A = 0$ then $(CP)_{\mathcal{T},A}$ has $Y$-maximal regularity for all Banach spaces, all time scales $\mathcal{T} \subset \mathbb{R}_+$ and all $Y(X) \subset \mathcal{F}(\mathcal{T}; X)$. The question we are interested in is then the following.

**Question 3.1.2**
*Given a Banach space $X$, a time scale $\mathcal{T}$ and a notion of regularity defined by $Y(X) \subset \mathcal{F}(\mathcal{T}; X)$, what should be assumed on $A$ in order to obtain that $(CP)_{\mathcal{T},A}$ has $Y$-maximal regularity (resp. strong $Y$-maximal regularity) ?*

And, more specifically, what are the relationships between these conditions on $A$ and the geometrical properties of $X$ ?
Before we address this question, which is, to a great extent, the main question of this thesis, let us recall that, for the continuous time scales (i.e $\mathcal{T}$ being an interval), the study of maximal regularity usually divides into two branches. The first one is concerned with the case $Y = L_p$ ($p \in [1, \infty]$) whereas the second one is concerned with the case $Y = C^\alpha$ ($\alpha \in (0, 1)$). Since $L_p$ spaces are more natural from an harmonic analytic point of view we are mostly interested in maximal regularity in this sense. $C^\alpha$ maximal regularity, which is an important theory, deeply based on interpolation spaces, is therefore not considered in this work.

Further information can be found in the book [Am95] of Amann ($L_p$ case) and in the book [Lu95] of Lunardi ($C^\alpha$ case). More references, in various settings, are given in the next sections.

## 3.2 Introduction to calculus on time scales

Before going further with maximal regularity we present briefly the basics of the time scale theory. The latter was introduced by Hilger (in his PhD thesis [Hilg88]) in order to unify discrete and continuous calculus but turned out to provide not only a unification but also an extension of the concepts of time used in dynamic equations. A time scale $\mathcal{T}$ is an arbitrary closed non empty subset of the real line. The theory consists in a calculus for functions from $\mathcal{T}$ to $\mathbb{R}$ and results on dynamic equations on $\mathcal{T}$. The basics definitions are as follows and can be found in the book [BP01].

**Definition 3.2.1**
*Let $\mathcal{T}$ be a time scale. For $t \in \mathcal{T}$ we define the* forward jump operator $\sigma : \mathcal{T} \to \mathcal{T}$ *by*

$$\sigma(t) = inf\{s \in \mathcal{T}\ ;\ s > t\}$$

*with the convention that $inf\varnothing = sup\mathcal{T}$. The function $\mu : \mathcal{T} \to \mathbb{R}_+$ defined by $\mu(t) = \sigma(t) - t$ is then called the* graininess function.

We can now define the notion of derivative on a time scale (called delta or Hilger derivative).

**Definition 3.2.2**
*Let $\mathcal{T}$ be a time scale, $X$ be a Banach space, and consider $f : \mathcal{T} \to X$. We say that $f$ is* delta differentiable *at a point $t \in \mathcal{T}$ provided there exists a unique point $f^{\Delta}(t) \in X$ with the property that, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $\forall s \in (t-\delta, t+\delta) \cap \mathcal{T}$*

$$\|(f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\| \leq \varepsilon|\sigma(t) - s|.$$

*$f^{\Delta}(t)$ is then called the* delta derivative *(or Hilger derviative) of $f$ at $t$.*

Of course, delta-differentiation corresponds to classical differentiation if $\mathcal{T}$ is an interval and we have $f^{\Delta}(t) = f(t+1) - f(t)$ if $\mathcal{T} = \mathbb{Z}_+$, but one can also obtain many results in intermediate cases (see [BP01]). Integration in both the Riemann and the Lebesgue sense can be defined too and the integral of $f$ between $t$ and $s$ is denoted by $\int\limits_t^s f(\xi)\Delta\xi$. Since we use it in only very special cases we will not recall its construction here. We refer the reader to the chapter 5 "Riemann and Lebesgue Integration" in the book [BP03] for a detailed construction of both integrals. In this work we will be interested in the cases where $\mathcal{T}$ is either an interval (continuous time) or a sequence (discrete time scale). In the continuous setting, the time scale calculus coincides with the classical calculus. In the discrete time scale setting (i.e. $\mathcal{T} = \{t_k\ ;\ k \in \mathbb{Z}_+$ for an increasing sequence $(t_k)_{k\in\mathbb{Z}_+}$), we have that every function $f : \mathcal{T} \to X$ (where $X$ is a Banach space) is differentiable at each point $t$ with derivative

$$f^{\Delta}(t_k) = \frac{f(t_{k+1}) - f(t_k)}{\mu(t_k)}.$$

Moreover $\int\limits_{t_k}^{t_l} f(t)\Delta t = \sum\limits_{j=k}^{l-1} \mu(t_j)f(t_j)$. These time scales are particularily interesting since they correspond to discrete schemes with non constant step sizes (the step size being precisely the graininess). $\ell_p$ maximal regularity for evolution equations on such time scales is the subject of Chapter 5.

Further information on the time scale theory can be found in the books [BP01] and [BP03].

## 3.3 $L_p$ maximal regularity $(1 < p < \infty)$

For continuous time evolution equations (i.e $\mathcal{T}$ being an interval), maximal regularity in the $L_p$ sense has been deeply studied. In this case, some important basic observations should be made. Their proofs can be found, for instance, in Dore's survey article [Do00].

**Proposition 3.3.1**
*Let $X$ be a Banach space, $1 < p < \infty$, $T \in \mathbb{R}^*$ and consider $(CP)_{[0,T],A}$ as in Section 3.1. Then the following hold.*

*(i) If $(CP)_{[0,T],A}$ has $L_p$-maximal regularity, then $A$ generates an analytic semigroup.*

*(ii) If $(CP)_{\mathbb{R}_+,A}$ has $L_p$-maximal regularity then $(CP)_{[0,T_0],A}$ has $L_p$-maximal regularity for each $T_0 \in \mathbb{R}_+$.*

*(iii) If $(CP)_{[0,T],A}$ has $L_p$-maximal regularity and $A$ is invertible then $(CP)_{\mathbb{R}_+,A}$ has $L_p$-maximal regularity.*

*(iv) If $(CP)_{[0,T],A}$ has $L_p$-maximal regularity for some $1 < p < \infty$ then it has $L_q$-maximal regularity for all $1 < q < \infty$.*

In the above Proposition (i) means that $L_p$-maximal regularity can only occur in parabolic problems. (ii) and (iii) gives an interesting idea of the difference between the finite and infinite time situation. For finite interval one can always assume that $A$ is invertible (rescaling the semigroup) and hence $L_p$ maximal regularity and strong $L_p$ maximal regularity are equivalent. But, on an infinite interval, it seems that $L_p$-maximal regularity is more interesting than its strong counterpart since the latter would require invertibility of $A$ and some natural differential operators may not be invertible (the Laplacian for instance). We therefore focus our attention on the weak notion. Let us also mention that there exists a Banach space $X$ and an operator $A$ such that $(CP)_{[0,T],A}$ has $L_p$-maximal regularity for each $T \in \mathbb{R}_+$ but $(CP)_{\mathbb{R}_+,A}$ does not have $L_p$-maximal regularity. The counterexample is due to Le Merdy in [LeM99]. Finally, let us point out that the independance in $p$ given in (iv) is a consequence of the harmonic analytic Theorem 2.1.1.
We now turn to the main question, namely what are the sufficient conditions which can ensure $L_p$-maximal regularity. The first result in this direction is due to De Simon in [DeS64].

**Theorem 3.3.2** *(***De Simon 1964***)*
*Let $H$ be a Hilbert space, $1 < p < \infty$ and $(A, D(A))$ be the generator of an analytic semigroup. Then $(CP)_{\mathbb{R}_+,A}$ has $L_p$-maximal regularity.*

In the case $X = L_q$ for some $1 < q < \infty$, the following results were then obtained.

- $(CP)_{[0,T],A}$ has $L_p$-maximal regularity if $A$ generates a contraction semigroup on $L_q$ for all $1 \leq q < \infty$ which is analytic on $L_q$ for $1 < q < \infty$ (Lamberton in [Lam87]).

- $(CP)_{\mathbb{R}_+,A}$ has $L_p$-maximal regularity if $A$ generates an analytic positive semigroup of contractions on $L_q$ (Weis in [W00]).

- $(CP)_{[0,T],A}$ has $L_p$-maximal regularity if the kernel corresponding to the semigroup $(e^{tA})_{t\in\mathbb{R}_+}$ satisfies Poisson estimates (Hieber and Prüss in [HP97] and Coulhon and Duong in [CD00]).

For more abstract spaces the following results were obtained with a different method (namely the "sum of operators" method) by Da Prato and Grisvard first and then by Dore and Venni.

- Let $(A, D(A))$ be the generator of an analytic semigroup on a Banach space $\widetilde{X}$ and let $X = (\widetilde{X}, D(A))_{\theta,r}$ for some $\theta \in (0,1)$. Then $(CP)_{[0,T],A}$ has $L_p$-maximal regularity on $X$ (Da Prato and Grisvard in [DaPG75]).

- Let $X$ be a UMD space and $A$ have bounded imaginary powers. Then $(CP)_{[0,T],A}$ has $L_p$-maximal regularity on $X$ (Dore and Venni in [DV87]).

¿From these particular results one can wonder which class of Banach spaces should be considered. It turns out that the UMD class is especially interesting. Namely, Coulhon and Lamberton have proved in [CL84] that if $A$ generates the Poisson semigroup on $X = L_2(Y)$ (i.e. $e^{tA} = P_t \otimes I_Y$ where $Y$ is a Banach space) then $(CP)_{[0,T],A}$ has $L_p$-maximal regularity on $X$ if and only if $X$ is UMD. By that time the conjecture was then that, in UMD spaces, $(CP)_{[0,T],A}$ would have $L_p$-maximal regularity if and only if $A$ generates an analytic semigroup. It turns out to be wrong. In fact, De Simon's result is, in some sense, a characterization of Hilbert spaces. This surprising result was obtained by Kalton and Lancien in [KL00] (some extensions are also given in [KL01]). The theorem is the following.

**Theorem 3.3.3 (Kalton, Lancien 2000)**
*Let $X$ be a Banach space with an unconditional basis. Assume that, for each analytic semigroup $(e^{tA})_{t\in\mathbb{R}_+}$, the Cauchy problem $(CP)_{[0,T],A}$ has $L_p$-maximal regularity. Then $X$ is isomorphic to $\ell_2$.*

Shortly afterwards the right analogue of De Simon's theorem was found by Weis (see [W01]). The result is based on Theorem 2.3.2. Indeed the operator defined by 2.1 with $k(t) = e^{tA}\chi_{\mathbb{R}_+} \quad \forall t \in \mathbb{R}$ extends to a bounded operator on $L_p(\mathbb{R}; X)$ if and only if the Fourier multiplier $T$ defined by 2.4 with $m(\xi) = i\xi R(i\xi; A) \quad \forall \xi \in \mathbb{R}^*$ extends to a bounded operator on $L_p(\mathbb{R}; X)$. This gives the following result.

**Theorem 3.3.4 (Weis 2001)**
*Let $X$ be a UMD space, $1 < p < \infty$ and $(A, D(A))$ the generator of a bounded analytic semigroup on $X$. Then the following are equivalent :*

*(i)* $(CP)_{\mathbb{R}_+;A}$ *has $L_p$-maximal regularity,*

*(ii)* $\{itR(it, A)\,;\, t \in \mathbb{R}^*\}$ *is R-bounded,*

*(iii)* $\{e^{tA}\,,\, tAe^{tA}\,;\, t > 0\}$ *is R-bounded.*

An operator $A$ satisfying (ii) is then called *R-sectorial* and the semigroup generated by $A$ is called *R-analytic*.

**Remark 3.3.5**
*This result solves the problem of $L_p$-maximal regularity using Banach space valued harmonic analyis but it also suggests that R-boundedness has to be used to generalize results on the sum of operators such as the famous Dore-Venni theorem. This is indeed the case and has been achieved by Kalton and Weis in [KW01].*

Further information can be found in the book of Amann [Am95], the survey articles of Dore [Do00], [D093], the sixth chapter of the recent survey [Ar04] and the articles of Weis [W00] and [W01].

## 3.4 $L_p$ maximal regularity ($p \in \{1, \infty\}$)

In the preceeding section we have seen that, in the case $1 < p < \infty$, $L_p$-maximal regularity is closely related to harmonic analytic questions in Banach spaces. To extend this approach to the case $p = 1$ it is therefore natural to consider $H^1$-maximal regularity (in the sense of the real-variable $H^1$-space) instead of $L_1$-maximal regularity. This was done by Hytönen in [Hy04-1] where the following result was proved.

**Theorem 3.4.1 (Hytönen 2004)**
*Let $X$ be a UMD space and $(A, D(A))$ the generator of a bounded analytic semigroup on $X$. Then the following are equivalent :*

*(i)* $(CP)_{\mathbb{R}_+;A}$ *has $H_1$-maximal regularity,*

*(ii)* $\{itR(it, A)\,;\, t \in \mathbb{R}^*\}$ *is R-bounded,*

*(iii)* $\{e^{tA}\,,\, tAe^{tA}\,;\, t > 0\}$ *is R-bounded.*

But considering $L_1$ (or $L_\infty$) maximal regularity is also possible and turns out to be related to other geometrical properties of the underlying Banach space than the UMD property required in the harmonic analytic approach. The main result in this direction was obtained by Baillon in [Ba80] where the following was shown.

**Theorem 3.4.2 (Baillon 1980)**
*Let $X$ be a separable Banach space and $(A, D(A))$ be the generator of an analytic semigroup. Then the following conditions are equivalent.*

*(i) $X$ does not contain a copy of $c_0$.*

*(ii) If $(CP)_{[0,T];A}$ has $L_\infty$-maximal regularity then $A$ is bounded.*

Later on, a dual version was obtained by Guerre-Delabriere in [GD95].

**Theorem 3.4.3 (Guerre-Delabriere 1995)**
*Let $X$ be a Banach space and $(A, D(A))$ be the generator of an analytic semigroup. Then the following conditions are equivalent.*

*(i) $X$ does not contain a complemented copy of $\ell_1$.*

*(ii) If $(CP)_{[0,T];A}$ has $L_1$-maximal regularity then $A$ is bounded.*

These two results, which are based on the work of Bessaga and Pelczyński (see Section 1.2) are the motivation of an analogue study in the discrete case which is part of Chapter 4.

Further information can be found in [Hy04-1], [Ba80] and [GD95].

## 3.5 $\ell_p$-maximal regularity $(1 < p < \infty)$

Because of its interest in applications (see Section 3.6), maximal regularity, in various settings, is a common question in the study of continuous time evolution equations. But, as explained in Section 3.1, it is also a natural question on different time scales and in particular for difference equations. Moreover, discrete analogues of the basic results on continuous time analytic semigroups have been obtained recently (see Section 1.7) and allow to approach $(CP)_{\mathbb{Z}_+;A}$ in a way which is similar to the semigroup approach to $(CP)_{\mathbb{R}_+;A}$. This was first done by Blunck in [Bl01-1]. In this work $(CP)_{\mathbb{Z}_+;A}$ is considered and the question of $\ell_p$-maximal regularity $(1 < p < \infty)$ is treated with harmonic analytic techniques. To do so, Blunck gives a discrete analogue of Weis' multiplier Theorem 2.3.2 together with Proposition 2.3.5 which is a converse to the multiplier theorem. As in the continuous case, $\ell_p$-maximal regularity is equivalent to the boundedness, in $\ell_p(X)$, of a "singular integral" operator given by

$$K_A f(m) = \sum_{n=0}^{m} k_A(n) f(m-n),$$

where $k_A = \begin{cases} A(I-A)^n & \text{if } n \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases}$
We can then express this operator as a Fourier multiplier and obtain that $(CP)_{\mathbb{Z}_+;A}$ has $\ell_p$-maximal regularity if and only if the operator $T_A$, defined by

$$T_A f(m) = \int_{\mathbb{T}} \lambda^m (\lambda - 1) R(\lambda, I - A)(\sum_{k \in \mathbb{Z}_+} f_k \lambda^{-k}) d\lambda,$$

is bounded on $\ell_p(X)$. This leads to the following result.

**Theorem 3.5.1 (Blunck 2001)**
*Let $X$ be a UMD space, $1 < p < \infty$ and $(T^n)_{n \in \mathbb{Z}_+}$ be a bounded analytic discrete time semigroup. Then the following are equivalent :*

*(i)* $(CP)_{\mathbb{Z}_+;I-T}$ *has $\ell_p$-maximal regularity.*

*(ii)* $\{T^n \, , \, n(T^{n+1} - T^n) \, ; \, n \in \mathbb{Z}_+\}$ *is R-bounded.*

*(iii)* $\{(\lambda - 1)R(\lambda, T) \, ; \, |\lambda| = 1 \, , \, \lambda \neq 1\}$ *is R-bounded.*

*(iv)* $\{e^{t(I-T)} \, , \, t(I-T)e^{t(I-T)} \, ; \, t > 0\}$ *is R-bounded.*

*(v)* $(CP)_{\mathbb{R}_+;(I-T)}$ *has $L_p$-maximal regularity,*

In particular this gives the following concrete condition.

**Theorem 3.5.2 (Blunck 2001)**
*Let $1 < p, q < \infty$ and $T \in B(\ell_q)$ be a subpositive contraction (i.e. there exists a positive operator $S$ such that $|Tf| \leq S|f|$ for all $f \in \ell_q$) such that $(T^n)_{n \in \mathbb{Z}_+}$ is a bounded analytic discrete time semigroup. Then $(CP)_{\mathbb{Z}_+;I-T}$ has $\ell_p$-maximal regularity.*

**Remark 3.5.3**
*More "concrete" cases, namely when $T$ is an integral operator satisfying certain Poisson bounds, are given in [Bl01-2].*

**Remark 3.5.4**
*Just as in the continuous case, $\ell_p$-maximal regularity for $1 < p < \infty$ turns out to be independant of $p$ (see [Bl01-1]).*

**Remark 3.5.5**
*Without looking at the proof, Theorem 3.5.1 may be somewhat misleading. Indeed, one should keep in mind that the equivalence between (i) and (v) is obtained via Weis' characterization and not directly.*

This work gives a fairly complete treatment of the case $1 < p < \infty$. However, one may wonder, given that Kalton-Lancien's counterexamples in the continuous case (see Section 3.3) are unbounded operators, whether or not R-boundedness is needed in Theorem 3.5.1. Also, one may consider the question of $\ell_1$ and $\ell_\infty$-maximal regularity, especially in the light of the results presented in Section 3.4. This is the motivation of Chapter 4.

Further information can be found in [Bl01-1] and [Po03].

## 3.6 Applications of maximal regularity

In the continuous time setting maximal regularity, both in the $L_p$ and the $C^\alpha$ sense, has been used in various applications to solve problems which are more complex than the simple Cauchy problem $(CP)_{\mathbb{R}_+;A}$. A precise exposition of these applications is clearly far beyond the scope of this section. We therefore restrict ourselves to a brief presentation of some examples which illustrate the wide range of problems in which maximal regularity has been used.

### Non autonomous parabolic equations

In the study of problems of the form

$$\left\{ \begin{array}{rcl} u'(t) - A(t)u(t) & = & f(t) \quad \forall t \in [0,T], \\ u(0) & = & x, \end{array} \right.$$

where $(A(t), D)_{t\in[0,T]}$ is a family of generators of analytic semigroups, maximal regularity of $(CP)_{[0,T],A(t_0)}$ for each $t_0 \in [0,T]$ is used to obtain a solution (with a desired regularity) through perturbation arguments. This is done for instance in Chapter 6 of [Lu95], under the assumption that $A(.) \in C^\alpha([0,T]; B(D,X))$ for some $\alpha \in (0,1)$, where a strict solution is obtained provided $f \in C^\alpha([0,T]; X), x \in D, A(0)x + f(0) \in \overline{D}$. Such a result is also obtained in [PS04] where a solution in $W^{1,p}([0,T]; X) \cap L_p([0,T]; D)$ is found under the assumption that $A(.) \in C([0,T]; B(D,X))$, $f \in L_p([0,T]; X)$ and $x \in (X,D)_{1-\frac{1}{p},p}$. More results of this type can be found in [Am95].

### Nonlinear equations

Maximal regularity is also used in the study of problems of the form

$$(QLCP) \qquad \left\{ \begin{array}{rcl} u'(t) - Au(t) & = & g(t,u(t)) + f(t) \quad \forall t \in [0,T], \\ u(0) & = & x, \end{array} \right.$$

where $g$ and $f$ are Lipschitz continuous functions and $A$ is the generator of an analytic semigroup. Here one considers an operator $\Gamma$ defined by $\Gamma u = v$ where $v$ is the solution of

$$(LCP) \qquad \begin{cases} v'(t) - Av(t) &= g(t, u(t)) + f(t) \quad \forall t \in [0, T], \\ u(0) &= x. \end{cases}$$

Using maximal regularity of (LCP) , one can then obtain a local solution (i.e. a solution defined in $[0, T_0]$ for a sufficiently small $T_0$ ) of (QLCP) as a fixed point of $\Gamma$. Such approaches can be found, for instance, in [Am95], [Lu95] and [CL93].

## Integrodifferential and delay equations

Consider problems of the form

$$\begin{cases} u'(t) - Au(t) &= \int\limits_0^t k(t,s)Au(s)ds \quad \forall t \in [0, T], \\ u(0) &= x. \end{cases}$$

where $k$ is a real valued function and of the form

$$\begin{cases} u'(t) - Au(t) &= Au(t-r) \quad \forall t \in [0, T], \\ u_{|[-r,0]} &= \phi_0. \end{cases}$$

with $r > 0$ and $\phi_0 : [-r, 0] \to D(A)$. For such problems, maximal regularity has been used, also through a fixed point argument, by considering the integral term (resp. the delay term) as a perturbation of $u' - Au$. These techniques are presented in the books of Prüss [Pr93] and Lunardi [Lu95].

## More applications

Besides the main applications presented above, maximal regularity has also been used for second order problems (see [CGD98]) and to prove the uniqueness of the solution of a Navier-Stokes equation (see [Mo99]).

Further information can be found in the books [Am95] by Amann, [Lu95] by Lunardi and [Pr93] by Prüss.

# Part II

# Contributions

# Chapter 4

# On discrete maximal regularity

## 4.1 Introduction

In this chapter we are interested in the question of $\ell_p$-maximal regularity ($1 \leq p \leq \infty$) for discrete time parabolic evolution equations. We therefore consider a complex Banach space $X$, the time scale $\mathcal{T} = \mathbb{Z}_+$ and an operator $A \in B(X)$ such that $((I-A)^n)_{n\in\mathbb{Z}_+}$ is a bounded analytic semigroup (see Section 1.7). In the case $1 < p < \infty$, the problem has already been considered by Blunck in [Bl01-1] (see Section 3.5). In the latter paper, $\ell_p$-maximal regularity ($1 < p < \infty$) is characterized by the R-analyticity of the discrete time semigroup $((I-A)^n)_{n\in\mathbb{Z}_+}$. It is, however, unclear, whether this notion differs from the simple analyticity of $((I-A)^n)_{n\in\mathbb{Z}_+}$. This leads us to a first question.

**Question 4.1.1**
*Given a Banach space $X$, does there exist $T \in B(X)$ such that*

$$\{T^n \; ; \; n \in \mathbb{Z}_+\} \cup \{n(T^{n+1} - T^n) \; ; \; n \in \mathbb{Z}_+\}$$

*is bounded but not R-bounded ?*

This is, of course, a discrete version of the famous "problem of $L_p$-maximal regularity" in the continuous setting, which was solved by Kalton and Lancien in [KL00] (see Theorem 3.3.3). Their counterexamples are unbounded and, unfortunately, can not directly be adapted. Given Blunck's Theorem 3.5.1, Question 4.1.1 can therefore be reformulated as follows : given a Banach space $X$, does there exist a bounded operator $A \in B(X)$ which generates a (continuous) bounded analytic semigroup which is not R-analytic ? In Section 4.2, we show that such a counterexample can be found in any Banach space with an unconditional basis, which is not isomorphic to a Hilbert space.

We then turn to $\ell_p$-maximal regularity in the case where $p \in \{1, \infty\}$. Given the results of Baillon (Theorem 3.4.2) and Guerre-Delabrière (Theorem 3.4.3) in the continuous setting we are interested in the following question.

**Question 4.1.2**
*Given a Banach space $X$, does the fact that $c_0 \not\subset X$ (resp. $\ell_1 \not\subset_c X$) impose a condition on the operators $A \in B(X)$ such that $(CP)_{\mathbb{Z}_+,A}$ has $\ell_\infty$ (resp. $\ell_1$) maximal regularity ?*

This question seems to be especially natural since, in the continuous setting, such a condition exists and states that $A$ has to be bounded, and since boundedness is not a restriction in the discrete setting. It turns out that such a condition also exists in the discrete setting and states that $\{nA(I-A)^n \ ; \ n \in \mathbb{Z}_+\}$ has to be bounded from below. This is shown is Section 4.3 for the case $p = \infty$ and in Section 4.4 for the case $p = 1$.

Finally we consider the relationships between these notions of maximal regularity. The last section is therefore devoted to the following question.

**Question 4.1.3**
*Given a Banach space $X$, are the notions of $\ell_1$, $\ell_p$ $(1 < p < \infty)$, and $\ell_\infty$ maximal regularity comparable ?*

We show the following.

- If $X$ is UMD then $\ell_p$-maximal regularity $(1 < p < \infty)$ does not imply $\ell_1$ (resp. $\ell_\infty$) maximal regularity.
- If $c_0 \subset_c X$ (resp. $\ell_1 \subset_c X$) then $\ell_\infty$ (resp. $\ell_1$) maximal regularity does not imply $\ell_1$ (resp. $\ell_\infty$) maximal regularity.
- If $c_0 \not\subset_c X$ (resp. $\ell_1 \not\subset_c X$) and if $T^n$ tends to zero in the strong operator topology (resp. $(T^*)^n x^*$ tends to zero in the $w^*$-topology for each $x^* \in X^*$ ) then $\ell_\infty$ (resp. $\ell_1$) maximal regularity implies $\ell_p$-maximal regularity $(1 < p < \infty)$.

In fact we obtain a much stronger result than the latter statement. Indeed we show that, under the above hypothesis, $\ell_\infty$ (resp. $\ell_1$ )maximal regularity can only occur in trivial cases (namely when the spectral radius $r(I - A)$ is strictly smaller than 1).

## 4.2 Counterexamples to $\ell_2$-discrete maximal regularity

### 4.2.1 preliminaries

First we consider the following definitions (the first one is already used in [Hy00]).

**Definition 4.2.1**
*In a Banach space $X$, a sequence $(T_n)_{n\in\mathbb{N}} \subset B(X)$ is called R-bounded relative to a sequence $(X_n)_{n\in\mathbb{N}}$ of closed subspaces of $X$ if*
$\exists C > 0 \quad \forall n \in \mathbb{N} \quad \forall x_k \in X_k \; 1 \leq k \leq n$

$$\int_0^1 \|\sum_{j=1}^n \varepsilon_j(t) T_j x_j\| dt \leq C \int_0^1 \|\sum_{j=1}^n \varepsilon_j(t) x_j\| dt.$$

**Definition 4.2.2**
*A Banach space $X$ is said to have the discrete maximal regularity property (DMRP) if for every discrete-time bounded analytic semigroup the associated discrete Cauchy problem has $\ell_2$-discrete maximal regularity .*

**Definition 4.2.3**
*A discrete-time analytic semigroup is called* R-analytic *(resp. $R^*$-analytic) if the set $\{T^n, n(T^{n+1} - T^n) \; ; \; n \in \mathbb{N}\}$ (resp. $\{T^{*n}, n(T^{*n+1} - T^{*n}) \; ; \; n \in \mathbb{N}\}$) is R-bounded on $X$ (resp. on $X^*$).*

**Definition 4.2.4**
*A Banach space $X$ is said to have the $(AR)$ property (resp. the $(AR)^*$ property) if every discrete-time bounded analytic semigroup is R-analytic (resp. $R^*$-analytic).*

The results of Sönke Blunck ([Bl01-1]) show that the R-analyticity of a discrete-time bounded analytic semigroup is a necessary, and in (UMD) spaces sufficient, condition for $(DCP)_T$ to have $\ell_2$-discrete maximal regularity. We show that (DMRP) implies that $X$ has both properties $(AR)$ and $(AR)^*$. We then deduce some properties of Schauder decompositions in such spaces. We recall that a Schauder decomposition of a Banach space $X$ is a sequence of closed subspaces of $(X_n)_{n\in\mathbb{N}}$ such that every element $x \in X$ has a unique representation of the form $x = \sum\limits_{n\in\mathbb{N}} x_n$ with $x_n \in X_n$ for all $n$ (see [LT77] pages 47-52 and Section 1.4 for further information). Finally we use the results of N.Kalton and G.Lancien (see [KL00] and Theorem 3.3.3) to obtain that Hilbert spaces are the only spaces with this property among spaces with an unconditional basis. More precisely we prove the following Theorem.

**Theorem 4.2.5**
*Let $X$ be a Banach space with (DMRP) and $(E_n, P_n)_{n\in\mathbb{N}}$ be a Schauder decomposition of $X$. Then $\{P_{2n} \; , \; n \in \mathbb{N}\}$ is R-bounded relative to $(span(E_{2n-1}, E_{2n}))_{n\in\mathbb{N}}$ and $\{P^*_{2n} \; , \; n \in \mathbb{N}\}$ is R-bounded relative to $(span(P^*_{2n-1}(X^*), P^*_{2n}(X^*)))_{n\in\mathbb{N}}$.*

Once Theorem 4.2.5 is proved, the proofs of Corollary 3.2 and Theorem 3.3 in [KL00] lead to the following result.

**Theorem 4.2.6**
*Let $X$ be a Banach space with (DMRP) and an unconditional basis, then $X$ is isomorphic to a Hilbert space.*

We recall that these proofs are based on two facts. First it is impossible to have the conclusion of Theorem 4.2.5 in $c_0$ and $\ell_1$. Secondly, this conclusion implies that every block basis of every permutation of the unconditional basis of $X$ spans a complemented subspace. The conclusion then follows from Corollary 1.3.4 which is itself based on Theorem 1.3.2 by Lindenstrauss and Tzafriri and Theorem 1.3.3 by Pelczyński. We now turn to the proof of Theorem 4.2.5.

### 4.2.2 proof of the main result

We need the following lemma.

**Lemma 4.2.7**
*Let $X$ be a Banach space and $(T^n)_{n\in\mathbb{N}}$ be a discrete-time bounded analytic semigroup . $(DCP)_T$ has then $\ell_2$-discrete maximal regularity on $X$ if and only if $(DCP)_{T^*}$ has $\ell_2$-discrete maximal regularity on $X^*$.*

*Proof :*
We consider the following.

$$\begin{array}{rcl} k_T \; : \; \mathbb{Z} & \to & B(X), \\ n & \mapsto & \begin{cases} (T^{n+1} - T^n) & if\ n \in \mathbb{N}, \\ 0 & otherwise. \end{cases} \end{array}$$

As remarked in [Bl01-1] (Remark 2.2), $(DCP)_T$ has $\ell_2$-discrete maximal regularity if and only if the operator

$$\begin{array}{rcl} K_T \; : \; \ell_1(\mathbb{Z}, X) & \to & \ell_\infty(\mathbb{Z}, X), \\ (f_n)_{n\in\mathbb{Z}} & \mapsto & (\sum\limits_{j\in\mathbb{Z}} k_T(n-j) f_j)_{n\in\mathbb{Z}}, \end{array}$$

extends to a bounded operator on $\ell_2(\mathbb{Z}, X)$. We are going to show that it is the case if and only if $K_{T^*}$ extends to a bounded operator on $\ell_2(\mathbb{Z}, X^*)$. We need the following isometry.

$$\begin{array}{rcl} J \; : \; \ell_2(\mathbb{Z}, X) & \to & \ell_2(\mathbb{Z}, X), \\ (f_n)_{n\in\mathbb{Z}} & \mapsto & (f_{-n})_{n\in\mathbb{Z}}. \end{array}$$

For $f \in \ell_2(\mathbb{Z}, X)$ and $g^* \in \ell_2(\mathbb{Z}, X^*)$ two sequences with finite support, we then have the following.

$$\begin{aligned} < K_{T^*}(g^*), f > &= \sum_{n\in\mathbb{Z}} < \sum_{j\in\mathbb{Z}} k_{T^*}(n-j) g^*_j, f_n >= \sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} < k_{T^*}(-n+j) g^*_{-j}, f_{-n} >, \\ &= \sum_{n\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} < g^*_{-j}, k_T(j-n) f_{-n} >, \\ &=< J^* g^*, K_T(J(f)) > . \end{aligned} \tag{4.1}$$

This gives the desired result by density of finitely supported sequences. $\square$

**Corollary 4.2.8**
*Let $X$ be a Banach space with (DMRP). $X$ has then both properties $(AR)$ and $(AR)^*$.*

According to this corollary, the only thing which remains to prove is the following.

**Proposition 4.2.9**
*Let $X$ be a Banach space with $(AR)$ (resp. $(AR)^*$) and $(E_n, P_n)_{n\in\mathbb{N}}$ be a Schauder decomposition of $X$. $\{P_{2n}\ ,\ n \in \mathbb{N}\}$ is then $R$-bounded relative to $(span(E_{2n-1}, E_{2n}))_{n\in\mathbb{N}}$ (resp. $\{P^*_{2n}\ ,\ n \in \mathbb{N}\}$ $R$-bounded relative to $(span(P^*_{2n-1}(X^*), P^*_{2n}(X^*)))_{n\in\mathbb{N}}$ ) .*

We need another lemma.

**Lemma 4.2.10**
*A multiplier on a Schauder decomposition associated to a monotone increasing sequence of real numbers belonging to $[0, 1]$ defines a discrete-time bounded analytic semigroup .*

*Proof :*
In view of Theorem 2.3 in [Bl01-1], it is equivalent to show that $I - T$ is the negative generator of a bounded analytic semigroup or to show that $I - T$ is a sectorial operator with angle less than $\frac{\pi}{2}$. But this is a classical fact (see Remark 1.6.4 and [BC91]). $\square$

*Proof of proposition 4.2.9 :*
Let $T_a$ and $T_b$ be the multipliers on $(E_n)_{n\in\mathbb{N}}$ associated respectively to the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defined by

$$a_{2n-1} = b_{2n-1} = b_{2n} = 1 - 2^{-(n-1)} \quad a_{2n} = 1 - 2^{-n}.$$

These operators are power-bounded and analytic by Lemma 4.2.10.
We obtain that for all $n$ and for all $u_n \in span(E_{2n}, E_{2n+1})$

$$i2^{-n}(1 + i2^{-n} - T_b)^{-1}u_n - i2^{-n}(1 + i2^{-n} - T_a)^{-1}u_n = \frac{1}{(i + 2)(i + 1)} P_{2n}u_n.$$

From [Bl01-1] and the fact that $X$ has $(DMRP)$ we obtain that the following sets are $R-bounded.$

$$\{(\lambda - 1)R(\lambda, T_a)\ ;\ \lambda \in 1 + i\mathbb{R}\ ,\ \lambda \neq 1\},$$
$$\{(\lambda - 1)R(\lambda, T_b)\ ;\ \lambda \in 1 + i\mathbb{R}\ ,\ \lambda \neq 1\}.$$

Thus we have that the sets

$$\{i2^{-n}(1+i2^{-n}-T_a)^{-1}\ ;\ n\in\mathbb{N}\},$$
$$\{i2^{-n}(1+i2^{-n}-T_b)^{-1}\ ;\ n\in\mathbb{N}\},$$

are $R-bounded$ and then that the family of projections $(P_{2n})_{n\in\mathbb{N}}$ is $R-bounded$ relative to $(span(E_{2n-1},E_{2n}))_{n\in\mathbb{N}}$ . The same proof with $T_a^*$ and $T_b^*$ gives us that property $(AR)^*$ implies that $\{P_{2n}^*\ ,\ n\in\mathbb{N}\}$ is R-bounded relative to $(span(P_{2n-1}^*(X^*),P_{2n}^*(X^*)))_{n\in\mathbb{N}}$.
□

## 4.3 The Banach space $c_0$ and $\ell_\infty$-discrete maximal regularity

In the continuous time setting (see [Ba80] and Theorem 3.4.2) the existence of semigroups with unbounded generators such that the associated Cauchy problem has $L_\infty$-maximal regularity is equivalent to the condition $c_0\subset X$ in separable Banach spaces. Looking for such a result in the discrete time setting, we need to replace the assumption "having an unbounded generator" by the equivalent condition for analytic semigroups (see [Hi50]) : $\limsup_{t\to 0}\|tAT(t)\|\geq\frac{1}{e}$. This condition can be expressed in the discrete setting in the following way.

$$\exists C>0\quad:\quad\forall n\in\mathbb{N}\quad\|n(T^{n+1}-T^n)\|\geq C.\tag{4.2}$$

We obtain the following results.

**Proposition 4.3.1**
*There exists a bounded operator $T$ acting on $c_0$ and verifying the condition*

$$\exists C>0\quad:\quad\forall n\in\mathbb{N}\quad\|n(T^{n+1}-T^n)\|\geq C,$$

*such that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity.*

*Proof :*Let $T$ be the multiplier on the canonical basis $(e_k)_{k\in\mathbb{N}}$ associated to the sequence $(1-\frac{1}{k})_{k\in\mathbb{N}}$. This operator is power-bounded and analytic by Lemma 4.2.10. First we prove that $T$ satisfies condition (4.2). We have that

$$\|(T^{n+1}-T^n)e_n\|=\frac{1}{n}|1-\frac{1}{n}|^n.$$

This proves that $\|n(T^{n+1}-T^n)\|\geq\frac{e}{2}$ for $n$ sufficiently large, which gives us (4.2). Now we have to prove that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity. Let $(f_n)_{n\in\mathbb{N}}$

be a sequence belonging to $\ell_\infty(c_0)$. Let $n$ be an integer and denote by $(P_k)_{k\in\mathbb{N}}$ the coordinate functionals of the canonical basis, we have that

$$\begin{aligned}\|\sum_{j=0}^{n}(T^{j+1}-T^j)f_{n-j}\| &= \sup_{k\in\mathbb{N}} |\sum_{j=0}^{n}\frac{1}{k}(1-\frac{1}{k})^j P_k(f_{n-j})|, \\ &\leq \sup_{k\in\mathbb{N}} \sum_{j=0}^{n}\frac{1}{k}|1-\frac{1}{k}|^j \|(f_n)_{n\in\mathbb{N}}\|_{\ell_\infty(c_0)}, \\ &\leq \sup_{k\in\mathbb{N}} (1-|1-\frac{1}{k}|^{n+1})\|(f_n)_{n\in\mathbb{N}}\|_{\ell_\infty(c_0)}, \\ &\leq \|(f_n)_{n\in\mathbb{N}}\|_{\ell_\infty(c_0)}.\end{aligned}$$

This proves that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity. $\square$

**Remark 4.3.2**
*There is an interesting relationship between this example and the one constructed by Giovanni Dore in section 8 of [Do00]. The later is a multiplier on the canonical basis of the space c of converging sequences associated to the sequence $(-k)_{k\in\mathbb{N}}$ such that the corresponding Cauchy problem has $L_\infty$-maximal regularity (and therefore $L_2$-maximal regularity by section 7 of [Do00]). The operator $T$ defined in the above proposition is therefore such that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity and $(CP)_{(I-T)^{-1}}$ has $L_\infty$-maximal regularity. It is remarkable since, in UMD spaces, it follows from Sönke Blunck's result (in [Bl01-1]) that $(DCP)_T$ has $\ell_2$-discrete maximal regularity if and only if $(CP)_{(I-T)^{-1}}$ has $L_2$-maximal regularity. This fact has been pointed out to me by the anonymous referee. I would like to thank him/her for that and for many other valuable comments.*

Let us now state the main result.

**Theorem 4.3.3**
*Let $X$ be a separable Banach space. The following assertions are equivalent.*

*(i)* $c_0 \subset X$.

*(ii) There exists a bounded operator $T$ acting on $X$ and verifying the condition*

$$\exists C > 0 \quad : \quad \forall n \in \mathbb{N} \quad \|n(T^{n+1}-T^n)\| \geq C,$$

*such that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity.*

*Proof :*
Since every separable Banach space containing $c_0$ contains it as a complemented subspace (see [So41] and Theorem 1.2.5), Proposition 4.3.1 gives us that (i) implies

(ii). Indeed one defines an operator on $X$ equal to the operator of proposition 4.3.1 on the complemented subspace isomorphic to $c_0$ and equal to the identity on its complement. Now consider a bounded operator $T$ verifying the conditions of (ii) and define the following sequences.

$$\begin{aligned}(y_n)_{n\in\mathbb{N}} \subset X \; ; \; \forall n \in \mathbb{N} \quad \|y_n\| = 1 \; and \; \|n(T^{n+1} - T^n)y_n\| > \frac{C}{2},\\ (u_n)_{n\in\mathbb{N}} \subset \mathbb{N} \; ; \; \begin{cases} u_n &= 2^n u_{n-1} \; \forall n > 0, \\ u_0 &= 1, \end{cases}\\ (x_n)_{n\in\mathbb{N}} \subset X \; ; \; \forall n \in \mathbb{N} \quad x_n = u_n(T^{u_n+1} - T^{u_n})y_{u_n}.\end{aligned} \tag{4.3}$$

We thus have that $\forall n \in \mathbb{N} \quad \|x_n\| > \frac{C}{2}$. Using a result of C.Bessaga and A.Pełczyński (see [BP58] and Theorem 1.2.2), we obtain the result if this sequence $(x_n)_{n\in\mathbb{N}}$ also verifies

$$\exists M > 0 \quad \forall n \in \mathbb{N} \quad \sup_{\varepsilon_k = \pm 1} \|\sum_{k=0}^{n} \varepsilon_k x_k\| \leq M. \tag{4.4}$$

We consider the following sequence

$$f_{u_n-j} = \begin{cases} \varepsilon_{i+1} T^{u_{i+1}-j} y_{u_{i+1}} & if \quad u_i \leq j < u_{i+1} \quad (i = 0, .., n-2), \\ 0 & otherwise. \end{cases}$$

Since $T$ is power-bounded this sequence belongs to $\ell_\infty(X)$. We thus obtain

$$\begin{aligned}\sum_{j=0}^{u_n} (T^{j+1} - T^j) f_{u_n-j} &= \sum_{i=0}^{n-2} \sum_{k=u_i}^{u_{i+1}-1} \varepsilon_{i+1}(T^{u_{i+1}+1} - T^{u_{i+1}}) y_{u_{i+1}}, \\ &= \sum_{i=0}^{n-2} \varepsilon_{i+1}(u_{i+1} - u_i)(T^{u_{i+1}+1} - T^{u_{i+1}}) y_{u_{i+1}}, \\ &= \sum_{i=1}^{n-1} \varepsilon_i (u_i - u_{i-1})(T^{u_i+1} - T^{u_i}) y_{u_i}.\end{aligned}$$

This leads to

$$\forall n \geq 2 \quad \|\sum_{i=1}^{n-1} \varepsilon_i (u_i - u_{i-1})(T^{u_i+1} - T^{u_i}) y_{u_i}\| \leq C_{DMR} \|(f_j)_{j\in\mathbb{N}}\|_{\ell_\infty(X)}, \tag{4.5}$$

where $C_{DMR}$ is the constant of $\ell_\infty$-discrete maximal regularity.

Moreover we have the following.

$$\begin{aligned}
&\|\sum_{i=0}^{n}\varepsilon_i(T^{u_i+1}-T^{u_i})y_{u_i}-\sum_{i=1}^{n-1}\varepsilon_i(u_i-u_{i-1})(T^{u_i+1}-T^{u_i})y_{u_i}\|,\\
\leq\ &\|(T^2-T)y_1\|+\|\sum_{i=1}^{n-1}\varepsilon_i\frac{u_{i-1}}{u_i}u_i(T^{u_i+1}-T^{u_i})y_{u_i}\|+\|(T^{u_{n+1}}-T^{u_n})y_{u_n}\|,\\
&\leq 2K_T+\sum_{i=1}^{n-1}\frac{K_T}{2^i}\leq 3K_T,
\end{aligned}$$

where $K_T$ is the constant of analyticity of $T$.
Using (4.5) we obtain

$$\forall n\geq 2 \quad \|\sum_{i=1}^{n-1}\varepsilon_i u_i(T^{u_i+1}-T^{u_i})y_{u_i}\|\leq C_{DMR}\|(f_j)_{j\in\mathbb{N}}\|_{\ell_\infty(X)}+3K_T,$$

which gives (4.4). □

**Remark 4.3.4**
*It should be noticed that the separability of $X$ was only used to prove that (i) implies (ii) and that the converse assertion remains valid if we drop this assertion.*

In this theorem the fact that (ii) implies (i) remains valid if we weaken condition 4.2 by replacing $\mathbb{N}$ by an unbounded subset $E$ of $\mathbb{N}$ (just choose $(u_n)_{n\in\mathbb{N}}\subset E$ such that $u_n\geq 2^n u_{n-1}\ \forall n\in\mathbb{N}$).
We therefore obtain the following corollary.

**Corollary 4.3.5**
*Let $X$ be a Banach space such that $c_0\not\subset X$ and $(T^n)_{n\in\mathbb{N}}$ be a discrete-time bounded analytic semigroup such that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity . We then have that*

$$\|n(T^{n+1}-T^n)\| \underset{n\to\infty}{\to} 0.$$

## 4.4 The Banach space $\ell_1$ and $\ell_1$-discrete maximal regularity

In the continuous setting, Sylvie Guerre Delabriere (see [GD95] and Theorem 3.4.3) proved the following dual result to Baillon's theorem : $X$ contains $\ell_1$ as a complemented subspace (which will be denoted $\ell_1\subset_c X$) if and only if there exists a bounded analytic semigroup with an unbounded generator such that $(CP)_A$ has $L_1$-maximal regularity. In a similar way, we are going to prove the following.

**Theorem 4.4.1**
*Let $X$ be a Banach space. The following assertions are equivalent.*

*(i)* $\ell_1 \subset_c X$.

*(ii) There exists a bounded operator $T$ acting on $X$ and verifying the condition*

$$\exists C > 0 \quad : \quad \forall n \in \mathbb{N} \quad \|n(T^{n+1} - T^n)\| \geq C,$$

*such that $(DCP)_T$ has $\ell_1$-discrete maximal regularity.*

To prove this theorem, we need the following lemma on the duality between $\ell_\infty$-discrete maximal regularity and $\ell_1$-discrete maximal regularity.

**Lemma 4.4.2**
*Let $X$ be a Banach space and $(T^n)_{n\in\mathbb{N}}$ be a discrete-time bounded analytic semigroup . $(DCP)_T$ has then $\ell_1$-discrete maximal regularity on $X$ if and only if $(DCP)_{T^*}$ has $\ell_\infty$-discrete maximal regularity on $X^*$.*

*Proof :*
We first assume that $(DCP)_T$ has $\ell_1$-discrete maximal regularity and write $S = T^*$. As it is done in Lemma 4.2.7, we define

$$\begin{array}{rcl} k_T \ : \ \mathbb{Z} & \rightarrow & B(X), \\ n & \mapsto & \begin{cases} (T^{n+1} - T^n) & if\ n \in \mathbb{N}, \\ 0 & otherwise. \end{cases} \end{array}$$

The problem in this case is that the translates of sequences supported in $\mathbb{Z}_+$ are not dense in $\ell_\infty(\mathbb{Z}, X^*)$. It is therefore not possible to apply the reasoning of Lemma 4.2.7.
However, since $(DCP)_T$ has $\ell_1$-discrete maximal regularity, we can also define the following bounded operator

$$\begin{array}{rcl} K_T \ : \ \ell_1(X) & \rightarrow & \ell_1(X), \\ (f_n)_{n\in\mathbb{N}} & \mapsto & (\sum_{j=0}^{\infty} k_T(n-j)f_j)_{n\in\mathbb{N}}. \end{array}$$

Therefore we have that $K_T^*$ belongs to $B(\ell_1(X)^*)$ and then that $K_T^*$ belongs to $B(\ell_\infty(X^*))$. Now let $N$ be an integer,$h^*$ be an element of $\ell_\infty(X^*)$ and $f$ be an element of $\ell_1(X)$. We define the following sequence $g^* \in \ell_\infty(X^*)$ .

$$g_n^* = \begin{cases} 0 & if\ n < N, \\ h_{2N-m}^* & if\ N \leq n \leq 2N, \\ 0 & otherwise. \end{cases}$$

We have that

$$
\begin{aligned}
< K_T^*(g^*), f > &= \sum_{n\in\mathbb{N}} < g_n^*, \sum_{j\in\mathbb{N}} k_T(n-j) f_j >, \\
&= \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} < k_T(n-j)^* g_n^*, f_j > .
\end{aligned}
$$

Now denote by $k_{T^*}^{\vee}$ the following operator

$$
\begin{array}{rcl}
k_{T^*}^{\vee} \ : \ \mathbb{Z} & \to & B(X^*), \\
n & \mapsto & \begin{cases} (S^{-n+1} - S^{-n}) & if\ n \leq 0, \\ 0 & otherwise. \end{cases}
\end{array}
$$

We obtain that $< K_T^*(g^*), f >= \sum\limits_{j\in\mathbb{N}} < \sum\limits_{n\in\mathbb{N}} k_{T^*}^{\vee}(j-n) g_n^*, f_j >$.
Therefore we have that $K_T^*(g^*) = (\sum\limits_{m\geq j} k_{T^*}^{\vee}(j-m) g_m^*)_{j\in\mathbb{N}}$. But, since $K_T^*$ belongs to $B(\ell_\infty(X^*))$ we have that

$$
\| \sum_{m\geq N} k_{T^*}^{\vee}(N-m) g_m^* \| \leq \|K_T^*\| \|g^*\|.
$$

And then

$$
\begin{aligned}
\| \sum_{m=0}^{N} (S^{m+1} - S^m) h_{N-m}^* \| &= \| \sum_{m=0}^{N} (S^{m+1} - S^m) g_{N+m}^* \|, \\
&= \| \sum_{m=-N}^{0} k_{T^*}^{\vee}(m) g_{N-m}^* \|, \\
&\leq \|K_T^*\| \|g^*\| \leq \|K_T^*\| \|h^*\|.
\end{aligned}
$$

which proves that $(DCP)_S$ has $\ell_\infty$-discrete maximal regularity.
Now we turn to the other direction and assume that $(DCP)_{T^*}$ has $\ell_\infty$-discrete maximal regularity on $X^*$. In this case we define $K_{T^*}$ as in Lemma 4.2.7. Since the translates of sequences supported in $\mathbb{Z}_+$ are dense in $c_0(\mathbb{Z}, X^*)$ and $(DCP)_{T^*}$ has $\ell_\infty$-discrete maximal regularity on $X^*$ we obtain that $K_{T^*}$ extends to a bounded operator on $c_0(\mathbb{Z}, X^*)$. The result then follows from the same idendity for finitely supported sequences as in (4.1) of Lemma 4.2.7 and the fact the $c_0(\mathbb{Z}, X^*)$ norms $\ell_1(\mathbb{Z}, X)$. $\square$

**Proposition 4.4.3**
*There exists a bounded operator $T$ acting on $\ell_1$ and verifying the condition (4.2)*

$$
\exists C > 0 \quad : \quad \forall n \in \mathbb{N} \quad \|n(T^{n+1} - T^n)\| \geq C,
$$

*such that $(DCP)_T$ has $\ell_1$-discrete maximal regularity.*

*Proof :*
Let $T$ be the multiplier on the canonical basis of $\ell_1$ ($(e_k)_{k\in\mathbb{N}}$) associated to the sequence $(1-\frac{1}{k})_{k\in\mathbb{N}}$. Since $T = S^*$ where $S$ is the multiplier associated to the same sequence but acting on the canonical basis of $c_0$ it is clear, in view of the proof proposition 4.3.1, that this operator is power-bounded, analytic and satisfy condition (4.2). Now we have to prove that $(DCP)_T$ has $\ell_1$-discrete maximal regularity.
Let $(f_n^*)_{n\in\mathbb{N}}$ be a sequence belonging to $\ell_1(\ell_1)$. Since $\ell_1(\ell_1) = c_0(c_0)^*$ we have the following (denoting by $(\lambda_{n,j})_{(n,j)\in\mathbb{N}^2}$ some adapted sequence of complex numbers of modulus 1)

$$\begin{aligned}
\sum_{n=0}^{\infty}\|\sum_{j=0}^{n}(T^{n-j+1}-T^{n-j})f_j^*\| &= \sup_{g\in B_{c_0(c_0)}}\sum_{n=0}^{\infty}\sum_{j=0}^{n} < (T^{n-j+1}-T^{n-j})f_j^*, \lambda_{n,j}.g_n >,\\
&= \sup_{g\in B_{c_0(c_0)}}\sum_{j=0}^{\infty}\sum_{n=j}^{\infty} < f_j^*, (S^{n-j+1}-S^{n-j})\lambda_{n,j}.g_n >,\\
&= \sup_{g\in B_{c_0(c_0)}}\sum_{j=0}^{\infty} < f_j^*, \sum_{m=0}^{\infty}(S^{m+1}-S^{m})\lambda_{m+j,j}.g_{m+j} >,\\
&\leq \sup_{g\in B_{c_0(c_0)}}\sum_{j=0}^{\infty}\|f_j^*\|.\sup_{k\in\mathbb{N}}\sum_{m=0}^{\infty}\frac{1}{k}(1-\frac{1}{k})^m.\|g\|_{c_0(c_0)},\\
&\leq \|f^*\|_{\ell_1(\ell_1)}.
\end{aligned}$$

which proves the proposition. □
Now we can finish the proof.

*Proof of Theorem 4.4.1:*

Proposition 4.4.3 gives us that (i) implies (ii).
Now suppose (ii). $T^*$ is then an operator on $X^*$ satisfying (4.2). Moreover, Lemma 4.4.2 implies that $(DCP)_{T^*}$ has $\ell_\infty$-discrete maximal regularity.
So, by Theorem 4.3.3 and Remark 4.3.4 we obtain that $c_0 \subset X^*$.
But, by a result of C.Bessaga and A.Pelczyński (see [BP58] and Theorem 1.2.3), this implies that $X$ contains $\ell_1$ as a complemented subspace. □
In view of this result and of Corollary 4.3.5 we also obtain the following corollary.

**Corollary 4.4.4**
*Let $X$ be a Banach space such that $\ell_1 \not\subset_c X$ and $(T^n)_{n\in\mathbb{N}}$ be a discrete-time bounded analytic semigroup such that $(DCP)_T$ has $\ell_1$-discrete maximal regularity . We then have that*

$$\|n(T^{n+1}-T^n)\| \underset{n\to\infty}{\longrightarrow} 0.$$

## 4.5 Discrete maximal regularities

With the preceding results in mind, we would like to compare these three notions of discrete maximal regularity. First we apply the preceeding results to show some differences between them in particular Banach spaces. We then show that, in some sense, $\ell_2$-discrete maximal regularity is a weaker notion than $\ell_\infty$ (resp. $\ell_1$)-discrete maximal regularity in Banach spaces that do not contain $c_0$ (resp. a complemented copy of $\ell_1$).

**Proposition 4.5.1**

*a) Let $X$ be a $(UMD)$ Banach space with an unconditional basis. There exists $T \in B(X)$ such that $(DCP)_T$ has $\ell_2$-discrete maximal regularity but neither $\ell_1$ nor $\ell_\infty$-discrete maximal regularity .*

*b) There exists $T \in B(c_0)$ such that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity but not $\ell_1$-discrete maximal regularity .*

*c) There exists $T \in B(\ell_1)$ such that $(DCP)_T$ has $\ell_1$-discrete maximal regularity but not $\ell_\infty$-discrete maximal regularity .*

*Proof :*In all the three cases we look at multipliers $T$ associated to the sequence $(1-\frac{1}{k})_{k\in\mathbb{N}}$ acting on the unconditional basis (the canonical one in the $c_0$ and the $\ell_1$ case ) $(e_n)_{n\in\mathbb{N}}$. Such an operator is power-bounded and analytic by Lemma 4.2.10. Moreover, we have that
$\|(T^{n+1}-T^n)e_n\| = \dfrac{1}{n}|1-\dfrac{1}{n}|^n$ which gives us (4.2).
*a) the $(UMD)$ case*
By Theorem 1.1 of [Bl01-1] it suffices to show that $I-T$ has $L_2$-maximal regularity. But, since $I-T$ is a multiplier on an unconditional basis it admits a bounded $H^\infty$ calculus ($f(I-T)$ for functions in $H^\infty$ of some sector are multipliers associated to the bounded sequence $(f(1-\frac{1}{n}))_{n\in\mathbb{N}}$ and unconditionality leads the boundedness of $f(I-T)$). Therefore $I-T$ has $L_2$-maximal regularity by Dore-Venni's theorem (see the survey on maximal regularity [D093]). Now, since $X$ is reflexive and thus does not contain an isomorphic copy of $c_0$ we obtain that $(DCP)_T$ does not have $\ell_\infty$-discrete maximal regularity by Theorem 4.3.3 and since $X$ also does not contain a complemented copy of $\ell_1$ we obtain that $(DCP)_T$ does not have $\ell_1$-discrete maximal regularity by Theorem 4.4.1.
*b) the $c_0$ case*
Since $c_0$ does not contain $\ell_1$ we obtain that $(DCP)_T$ does not have $\ell_1$-discrete maximal regularity by Theorem 4.4.1 and that it has $\ell_\infty$-discrete maximal regularity by Proposition 4.3.1.
*c) the $\ell_1$ case*
Since $\ell_1$ does not contain $c_0$ we obtain that $(DCP)_T$ does not have $\ell_\infty$-discrete maximal regularity by Theorem 4.3.3 and that it has $\ell_1$-discrete maximal regularity by

proposition 4.4.3. $\square$

Now using Corollary (4.3.5) we can deduce a spectral property for operators associated with $\ell_\infty$ (resp. $\ell_1$) regular discrete Cauchy problems in Banach spaces that do not contain $c_0$ (resp. a complemented copy of $\ell_1$). I thank Sönke Blunck and Gilles Lancien for suggesting the following corollary.

**Corollary 4.5.2**
*Let $X$ be a Banach space such that $c_0 \not\subset X$, and $(T^n)_{n\in\mathbb{N}}$ be a discrete-time bounded analytic semigroup such that $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity .*
*Then there exists $0 \leq s_0 < 1$ such that*

$$\sigma(T) \subset B(0, s_0) \cup \{1\},$$

*where $B(0, s_0)$ denotes the closed disc of center $0$ and radius $s_0$.*

*Proof* :Since $\|n(T^{n+1}-T^n)\|$ tends to 0 as $n$ tends to infinity we have that $\underset{\lambda\in\sigma(T)}{sup}\, n|1-\lambda||\lambda|^n$ must also tend to 0. But this also implies that

$$\underset{\lambda\in\sigma(T)}{sup}\, n(1-|\lambda|)|\lambda|^n \underset{n\to\infty}{\to} 0. \tag{4.6}$$

Now consider the function of one real variable defined by $f_n(s) = n(1-s)s^n \quad \forall s \in [0,1]$. A closer look at this function shows that it is increasing from 0 to $1 - \frac{1}{n+1}$ and decreasing afterwards. Moreover $\{f_n(1-\frac{1}{n})\,;\, n \in \mathbb{N}\}$ is bounded from below. We then obtain that

$$\exists C > 0 \quad \forall n \geq 2 \quad \forall s \in [1-\frac{1}{n}, 1-\frac{1}{n+1}] \quad f_n(s) \geq C. \tag{4.7}$$

But using (4.6) and (4.7) we obtain that

$$\exists n_0 \in \mathbb{N} \quad \forall \lambda \in \mathbb{D}\ |\lambda| \neq 1 \quad |\lambda| \geq 1 - \frac{1}{n_0} \implies \lambda \notin \sigma(T).$$

Now the proof is completed using the fact that the analyticity of $T$ already implies that $\sigma(T) \cap \mathbb{T} \subset \{1\}$ (see [Bl01-1]). $\square$

With the same proof we also obtain the following.

**Corollary 4.5.3**
*Let $X$ be a Banach space such that $\ell_1 \not\subset_c X$, and $(T^n)_{n\in\mathbb{N}}$ be a discrete-time bounded analytic semigroup such that $(DCP)_T$ has $\ell_1$-discrete maximal regularity .*
*Then there exists $0 \leq s_0 < 1$ such that*

$$\sigma(T) \subset B(0, s_0) \cup \{1\},$$

*where $B(0, s_0)$ denotes the disc of center $0$ and radius $s_0$.*

Finally we are going to show that, with an additional assumption, the situation of the preceeding corollaries can only happen in trivial cases. This comes from the following theorem whose idea is due to Nigel Kalton.

**Theorem 4.5.4**
*Let $X$ be a Banach space and $(T^n)_{n\in\mathbb{Z}_+}$ be a discrete-time bounded analytic semigroup such that*

*(i) $T^n$ tends to zero in the strong operator topology,*

*(ii) $(DCP)_T$ has $\ell_\infty$-discrete maximal regularity .*

*Then* $\exists C_1 > 0\ \exists C_2 > 0\ \exists a \in \mathbb{N}\ \forall x \in X$

$$C_1 \sup_{k\in\mathbb{Z}_+} \|(T^{a^{k+1}} - T^{a^k})x\| \leq \|x\| \leq C_2 \left( \sup_{k\in\mathbb{Z}_+} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| \right).$$

*Proof :*
Let us first remark that for any increasing sequence $(n_k)_{k\in\mathbb{Z}_+} \subset \mathbb{Z}_+$ and any $(x_k)_{k\in\mathbb{Z}_+} \in \ell_\infty(X)$ we have that

$$\|\sum_{k=0}^{\infty}(T^{n_{k+1}} - T^{n_k})x_k\| \leq C_{DMR} \sup_{k\in\mathbb{Z}_+} \|x_k\|,$$

where $C_{DMR}$ denotes the maximal regularity constant. Moreover we remark that (i) implies that, for all $a \in \mathbb{N}$,

$$\|x\| \leq \|\sum_{k\in\mathbb{Z}_+} (T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\|.$$

We then obtain the following, where $\preceq$ denotes inequalities up to a constant which does not depend on $x$.

$$\begin{aligned}
&\| \sum_{k\in\mathbb{Z}_+} (T^{a^{k+1}} - T^{a^k})x\| \\
&\quad \preceq \| \sum_{\substack{(k,j)\in\mathbb{Z}_+^2 \\ |k-j|\leq 3}} (T^{a^{k+1}} - T^{a^k})(T^{a^{j+1}} - T^{a^j})x\| + \| \sum_{k\in\mathbb{Z}_+} \sum_{m=3}^{\infty} (T^{a^{k+1}} - T^{a^k})(T^{a^{k+1+m}} - T^{a^{k+m}})x\| \\
&\qquad + \| \sum_{k\in\mathbb{Z}_+} (T^{a^{k+1}} - T^{a^k})(I-T)x\| \\
&\quad \preceq \underset{k\in\mathbb{Z}_+}{sup} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| + \| \sum_{k\in\mathbb{Z}_+} \sum_{m=3}^{\infty} (T^{a^{k+1}} - T^{a^k})(T^{a^{k+1+m}} - T^{a^{k+m}})x\| \\
&\quad \preceq \underset{k\in\mathbb{Z}_+}{sup} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| \\
&\qquad + \sum_{m=3}^{\infty} \| \sum_{k\in\mathbb{Z}_+} (T^{a^{2k+1}+a^{2k+m-1}} - T^{a^{2k}+a^{2k+m-1}})(T^{a^{2k+1+m}-a^{2k+m-1}} - T^{a^{2k+m}-a^{2k+m-1}})x\| \\
&\qquad + \sum_{m=3}^{\infty} \| \sum_{k\in\mathbb{Z}_+} (T^{a^{2k+2}+a^{2k+m}} - T^{a^{2k+1}+a^{2k+m}})(T^{a^{2k+2+m}-a^{2k+m}} - T^{a^{2k+m+1}-a^{2k+m}})x\| \\
&\quad \preceq \underset{k\in\mathbb{Z}_+}{sup} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| \\
&\qquad + \sum_{m=3}^{\infty} \underset{k\in\mathbb{Z}_+}{sup} \|(T^{a^{k+1}+a^{k+m-1}} - T^{a^k+a^{k+m-1}})x\| \quad \text{(using maximal regularity)} \\
&\quad \preceq \underset{k\in\mathbb{Z}_+}{sup} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| \\
&+ \quad \sum_{m=3}^{\infty} \underset{k\in\mathbb{Z}_+}{sup} \left( log(\frac{a^{k+1}+a^{k+m-1}}{a^k+a^{k+m-1}}) \right) \|x\| \quad \text{(using the analyticity of } (T^n)_{n\in\mathbb{Z}_+}) \\
&\quad \preceq \underset{k\in\mathbb{Z}_+}{sup} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| + \sum_{m=3}^{\infty} a^{2-m}\|x\|.
\end{aligned}$$

This gives the result by picking $a$ large enough (the other inequality of course follows from the boundedness of $(T^n)_{n\in\mathbb{Z}_+}$). □

Now using the same theorem due to C.Bessaga and A.Pelczyński (see [BP58] and Theorem 1.2.2) as in Theorem 4.3.3 we can show the following.

**Corollary 4.5.5**

*Let $X$ be a Banach space and let $(T^n)_{n\in\mathbb{Z}_+}$ be a discrete-time bounded analytic semigroup such that* *(i)* $T^n$ *tends to zero in the strong operator topology,*

*(ii)* $(DCP)_T$ *has* $\ell_\infty$*-discrete maximal regularity .*

*(iii)* $c_0 \not\subset X$.

*Then* $\|T^n\| \underset{n\to\infty}{\to} 0$.

*Proof :*
We first claim that

$$\exists N \in \mathbb{N} \quad \exists C > 0 \quad \forall x \in X \quad \|x\| \leq C \left( \underset{0\leq k\leq N}{max} \|(T^{a^{k+1}} - T^{a^k})x\| + \|(I-T)x\| \right).$$

To prove this we assume the contrary and pick an increasing sequence of positive integers $(p_n)_{n\in\mathbb{Z}_+}$ and a sequence $(x_n)_{n\in\mathbb{Z}_+}$ in the unit ball of $X$ such that for some $\delta > 0$ we have, for all $n \in \mathbb{N}$, that

$$\|T^{a^{p_n+1}} - T^{a^{p_n}} x_n\| \geq \delta.$$

Now let $y_n = T^{a^{p_n+1}} - T^{a^{p_n}} x_n$. We have that $\|y_n\| \geq \delta \quad \forall n \in \mathbb{Z}_+$. Moreover $\ell_\infty$-maximal regularity leads to the following.

$$\underset{M\in\mathbb{N}}{sup}\ \underset{\varepsilon_n=\pm1}{sup} \|\sum_{n=0}^{M} \varepsilon_n y_n\| \leq C.$$

Therefore, by Theorem 1.2.2 of Bessaga and Pelczyński, we obtain that $c_0 \subset X$, which is a contradiction. This means that

$$\exists C > 0 \quad \forall x \in X \quad \forall n \in \mathbb{N} \quad \|T^n x\| \leq C \left( \underset{0\leq k\leq N}{max} \|(T^{a^{k+1}+n} - T^{a^k+n})x\| + \|(T^{n+1} - T^n)x\| \right).$$

The results then follows from the analyticity of $(T^n)_{n\in\mathbb{Z}_+}$. □

It should be noticed that the assumption (iii) in the above statement can not be removed. Indeed, the multiplier $T$, acting on the canonical basis of $c_0$ and associated to the sequence $(1 - \frac{1}{2^n})_{n\in\mathbb{N}}$ is such that $(DCP)_T$ has $\ell_\infty$-maximal regularity (for the same reason as the example considered in Proposition 4.3.1). Moreover, given $\varepsilon > 0$ and $x \in c_0$, there exists $N \in \mathbb{N}$ such that

$$\forall k \in \mathbb{Z}_+ \quad \|T^k x\| \leq \underset{n\leq N}{max} (1 - \frac{1}{2^n})^k \|x\| + \varepsilon,$$

which gives that $T^k$ tends to zero in the strong operator topology. But $T^k$ can not tend to zero in norm since the spectrum of $T$ contains $\{1\}$.

We also obtain the following dual results.

**Proposition 4.5.6**
*Let $X$ be a Banach space and $(T^n)_{n\in\mathbb{Z}_+}$ be a discrete-time bounded analytic semigroup . Let us denote by $S$ the adjoint operator of $T$ acting on $X^*$ and assume that*

*(i) For each $x^* \in X^*$, $S^n x^*$ tends to zero in the $w^*$-topology,*

*(ii) $(DCP)_T$ has $\ell_1$-discrete maximal regularity .*

*Then the following hold.*

1. $\exists C_1 > 0\ \exists C_2 > 0\ \forall x \in X$

$$C_1 \sum_{j=0}^{\infty} \|(T^{j+1} - T^j)x\| \le \|x\| \le C_2 \sum_{j=0}^{\infty} \|(T^{j+1} - T^j)x\|.$$

2. $\exists C_3 > 0\ \exists C_4 > 0\ \exists a \in \mathbb{N}\ \forall x^* \in X^*$

$$C_3 \sup_{k\in\mathbb{Z}_+} \|(S^{a^{k+1}} - S^{a^k})x^*\| \le \|x^*\| \le C_4 \left( \sup_{k\in\mathbb{Z}_+} \|(S^{a^{k+1}} - S^{a^k})x^*\| + \|(I-S)x^*\| \right).$$

*Proof :*
Let $x \in X$. For $n \in \mathbb{Z}_+$, pick $x_n^* \in X^*$ such that

$$\|(T^{n+1} - T^n)x\| =< (T^{n+1} - T^n)x, x_n^* > .$$

Since $(CP)_{Z_+,I-S}$ has $\ell_\infty$-maximal regularity we have that there exists $C > 0$ such that for each $N \in \mathbb{N}$

$$\sum_{k=0}^{N} \|(T^{k+1} - T^k)x\| \le \| \sum_{k=0}^{N} (S^{k+1} - S^k)x_n^*\| \|x\| \le C\|x\|.$$

To prove the other direction we let $x^*$ be a norm 1 functional and use (i) to obtain that there exists $C > 0$ such that

$$< x, x^* >=< x, \sum_{j=0}^{\infty} (S^j - S^{j+1})x^* >\le C \sum_{j=0}^{\infty} \|(T^{j+1} - T^j)x\|.$$

This proves 1. The proof of 2 is then the same as the proof of Theorem 4.5.4. □

**Corollary 4.5.7**
*Let $X$ be a Banach space and $(T^n)_{n\in\mathbb{Z}_+}$ be a discrete-time bounded analytic semigroup . Let us denote by $S$ the adjoint operator of $T$ acting on $X^*$ and assume that*

*(i) For each $x^* \in X^*$, $S^n x^*$ tends to zero in the $w^*$-topology,*

*(ii) $(DCP)_T$ has $\ell_1$-discrete maximal regularity .*

*(iii) $c_0 \not\subset X^*$.*

*Then $\|T^n\| \underset{n\to\infty}{\rightarrow} 0$.*

*Proof :*
The proof is the same as in Corollary 4.5.5. Thanks to Proposition 4.5.6 and to (iii) we obtain that $\exists C > 0\ \exists a \in \mathbb{N}\ \ \exists N \in \mathbb{N}\ \forall x^* \in X^*$

$$\|x^*\| \le C\left(\max_{0\le k\le N}\|(S^{a^{k+1}} - S^{a^k})x^*\| + \|(I-S)x^*\|\right).$$

But this implies that $\|S^n\|$ and thus $\|T^n\|$ tends to zero. □

Since $(DCP)_T$ clearly has $\ell_2$-discrete maximal regularity if $\|T^n\|$ tends to zero, we finally obtain the following.

**Corollary 4.5.8**
*Let $X$ be a Banach space such that $c_0 \not\subset X$ (resp. $\ell_1 \not\subset_c X$). Let $(T^n)_{n\in\mathbb{Z}_+}$ be a discrete-time bounded analytic semigroup , denote by $S$ the adjoint operator of $T$ acting on $X^*$ and assume that*

*(i) $T^n$ tends to zero in the strong operator topology (resp. for each $x^* \in X^*$, $S^n x^*$ tends to zero in the $w^*$-topology),*

*(ii) $(DCP)_T$ has $\ell_\infty$(resp. $\ell_1$)-discrete maximal regularity .*

*Then $(DCP)_T$ has $\ell_2$-discrete maximal regularity .*

# Chapter 5

# Maximal regularity of evolution equations on discrete time scales

## 5.1 Introduction

In this chapter we are interested in the question of $\ell_p$-maximal regularity ($1 < p < \infty$) for evolution equations on discrete time scales. This generalizes the problem treated in Chapter 4 by replacing the time scale $\mathbb{Z}_+$ by a general increasing sequence $\mathcal{T} = \{t_k \ ; \ k \in \mathbb{Z}_+\} \subset \mathbb{R}_+$. The corresponding Cauchy problem $(CP)_{\mathcal{T},A}$ may therefore be seen as a discretized one where the discretization scheme has a variable step size. In fact the sequence of step sizes $(\mu(t_k))_{k\in Z_+}$ (where $\mu(t_k) = t_{k+1} - t_k$) is nothing but the graininess sequence of $\mathcal{T}$. The question we consider is then the following.

**Question 5.1.1**
*Given a Banach space $X$ and a time scale $\mathcal{T} = \{t_k \ ; \ k \in \mathbb{Z}_+\} \subset \mathbb{R}_+$ (where $(t_k)_{k\in\mathbb{Z}_+}$ is increasing), which necessary and/or sufficient conditions should be imposed on an operator $A \in B(X)$ for $(CP)_{\mathcal{T},A}$ to have $\ell_p$-maximal regularity ($1 < p < \infty$) ?*

This means, of course, that we are looking for an analogue of Weis' Theorem 3.3.4 and of Blunck's Theorem 3.5.1 , and hence that we think about these conditions as an appropriate analogue to R-analyticity. The problem is that the solution of $(CP)_{\mathcal{T},A}$ is not simply given by a translation invariant singular integral operator as in the cases $\mathcal{T} = \mathbb{R}_+$ or $\mathcal{T} = \mathbb{Z}_+$. In fact $(CP)_{\mathcal{T},A}$ can be seen as the following non autonomous problem

$$\begin{cases} u(t_{n+1}) - u(t_n) + \mu(t_n)Au(t_n) &= \mu(t_n)f(t_n) \quad \forall n \in \mathbb{Z}_+, \\ u(t_0) &= 0. \end{cases}$$

This suggests the use of evolution families instead of semigroups and the expression of the solution with a singular integral operator with a non symmetric kernel (namely we have $u(t_{n+1}) = \sum_{k=0}^{n} \mu(t_k)(\prod_{j=k}^{n} I - \mu(t_j)A)f(t_k) \quad \forall n \in \mathbb{Z}_+)$.

In Section 5.2, the adequate definitions are given, especially the notion of analyticity for such an evolution family.

We then turn, in Section 5.3, to a special case, namely the case of sample based time scales (i.e. time scales such that $(\mu(t_k))_{k\in\mathbb{Z}_+}$ is periodic). This particular setting has many interesting properties. First it allows us to prove the necessity of the analyticity of the evolution family. Then it makes it possible to relate the properties of the evolution family to properties of a discrete semigroup $(T^n)_{n\in\mathbb{Z}_+}$. And finally it allows to rewrite the problem $(CP)_{\mathcal{T},A}$ on the Banach space $X$ as a problem $(CP)_{\mathbb{Z}_+,A\otimes I_r}$ on a product space $X^r$ where $A \otimes I_r$ denotes the diagonal operator with all diagonal entries equal to $A$. This, in turn, makes it possible to use Blunck's multiplier Theorem 2.3.3, although with more complicated symbols as in the original result of Blunck (see [Bl01-1]). This leads our Theorem 5.3.7 which gives the desired answer to Question 5.1.1.

The last section is then devoted to a perturbation result which allows to consider discrete time scales which are not sample based but just "asymptotically sample based". More precisely we replace $\mathcal{T}$ by a time scale $\widetilde{\mathcal{T}} = \{\widetilde{t_k} \ ; \ k \in Z_+\}$ where $\mu(\widetilde{t_k}) = \mu(t_k) + \varepsilon_k$ and give an admissibility condition on the perturbation sequence $(\varepsilon_k)_{k\in Z_+}$ in order to preserve $\ell_p$-maximal regularity. Although this condition (see Definition 5.4.1) depends both on the initial time scale $\mathcal{T}$ and on the operator $A$, it allows any perturbation $(\varepsilon_k)_{k\in Z_+} \in \ell_1$ (see Example 5.4.2).

## 5.2 Cauchy problems on discrete time scales

We consider a Banach space $X$ and a time scale $\mathcal{T} = \{t_k; k \in \mathbb{Z}_+\} \subset \mathbb{R}_+$ (where $t_k$ is increasing). In this setting one considers the Cauchy problem

$$(CP)_{\mathcal{T},A} \qquad \left\{ \begin{array}{rcl} u^{\Delta}(t) - Au(t) & = & f(t) \quad \forall t \in \mathcal{T}, \\ u(t_0) & = & 0, \end{array} \right.$$

where $u^{\Delta}$ is defined in the sense of the time scale calculus. Let us recall that, in this sense, we can define the derivative of a function on $\mathcal{T}$ as a sequence $(u^{\Delta}(t_n))_{n\in\mathbb{N}}$ with $u^{\Delta}(t_n) = \frac{u(t_{n+1})-u(t_n)}{\mu_{n+1}}$ (where $(\mu_k)_{k\in\mathbb{N}}$ is the *graininess function* defined by $\mu_k = t_k - t_{k-1}$ for $k \geq 1$) and the integral between $t_0$ and $t_n$ of a sequence $(u(t_n))_{n\in\mathbb{N}}$ as $\sum_{k=0}^{n-1} \mu_{k+1}u(t_k)$ (see [BP01] and Section 3.2 for more informations on time scale calculus). We therefore define an $\ell_p(\mathcal{T}; X)$ norm for functions on the time scale :

$$\|f\|_{\ell_p(\mathcal{T};X)} = (\sum_{k=0}^{\infty} \mu_{k+1}\|f(t_k)\|^p)^{\frac{1}{p}}.$$

The corresponding notion of maximal regularity is then as follows.

**Definition 5.2.1**
*We say that* $(CP)_{\mathcal{T},A}$ *has* $\ell_p$-maximal regularity *if there exists a constant* $C > 0$ *such that for every input function* $f \in \ell_p(\mathcal{T}; X)$

$$\|u^{\Delta}\|_{\ell_p(\mathcal{T};X)} \leq C\|f\|_{\ell_p(\mathcal{T};X)}.$$

We now relate $\ell_p$-maximal regularity to properties of an evolution family generated by $A$.

**Definition 5.2.2**
*The family* $E_{A,\mathcal{T}} = (e_{A,\mathcal{T}}(t_k, t_n))_{(k,n)\in\mathbb{Z}_+^2, n\geq k}$ *where*

$$e_{A,\mathcal{T}}(t_k, t_n) \;=\; \begin{cases} \prod_{j=k}^{n-1}(I - \mu_{j+1}A) & \text{if } k < n \\ I & \text{if } k = n \end{cases}$$

*is called the evolution family associated to* $(CP)_{A,\mathcal{T}}$.

Moreover, one defines

$$\begin{array}{rccc} \mathcal{E}_{A,\mathcal{T}}: & \mathcal{T} & \to & B(X) \\ & t_k & \mapsto & e_{A,\mathcal{T}}(t_k, t_0) \end{array}$$

that gives a functional counterpart to $e_{A,\mathcal{T}}$ and satisfies the differential equation

$$\mathcal{E}^{\Delta}_{A,\mathcal{T}}(t_k) = -A\mathcal{E}_{A,\mathcal{T}}(t_k) \;\forall k \in \mathbb{N}.$$

This allows to define the following analogue to classical semigroups properties.

**Definition 5.2.3**
*The associated evolution family* $E_{A,\mathcal{T}}$ *is called* bounded *(resp.* R-bounded*) if it is a bounded (resp.* R-bounded*) set. It is called* analytic *(resp.* R-analytic*) if the set* $\{(t_k - t_0)A\mathcal{E}_{A,\mathcal{T}}(t_{k+1})\ ;\ k \in \mathbb{Z}_+\}$ *is bounded (resp.* R-bounded*).*

**Remark 5.2.4**
*If the time scale is* $\mathbb{Z}_+$ *we recover the corresponding notions for discrete time semigroups. In this case* $\mathcal{E}_{A,\mathcal{T}}(t_k) = (I - A)^k$*, the semigroup is bounded (resp. R-bounded) if and only if* $((I - A)^n)_{n\in\mathbb{Z}_+}$ *is and it is analytic (resp. R-analytic) if and only if* $(A((I - A)^{n+1} - (I - A)^n))_{n\in\mathbb{Z}_+}$ *is bounded (resp. R-bounded).*

Let us now point out that the solution of $(CP)_{A,\mathcal{T}}$ can be expressed using the associated evolution family :

$$u(t_n) = \sum_{k=0}^{n-1} \mu_{k+1}.e_{A,\mathcal{T}}(t_k, t_n)f(t_k) \quad \forall n \in \mathbb{N}.$$

Moreover, assuming that $E_{A,\mathcal{T}}$ is bounded and defining an operator $K$ with

$$K(f)(t_k) = \begin{cases} A\sum\limits_{k=0}^{n-1} \mu_{k+1}.e_{A,\mathcal{T}}(t_k,t_n)f(t_k) & \text{if } k>0, \\ 0 & \text{if } k=0, \end{cases}$$

one obtains that $(CP)_{A,\mathcal{T}}$ has $\ell_p$-maximal regularity if and only if $K$ extends to a bounded operator on $\ell_p(\mathcal{T},X)$. Under an additional assumption on the time scale we now exhibit relationships between the boundedness of $K$ and R-analyticity of $E_{A,\mathcal{T}}$.

## 5.3 Maximal regularity on sample based time scales

**Definition 5.3.1**
*A time scale $\mathcal{T} = \{t_k; k \in \mathbb{N}\} \subset \mathbb{R}_+$ is called* sample based *with period $r$ if the graininess sequence is r-periodic.*

On such a time scale we now have the following necessary condition for $\ell_p$-maximal regularity. The proof is essentially the same as in both the discrete and the continuous case.

**Proposition 5.3.2**
*Let $X$ be a Banach space and $\mathcal{T} = \{t_k; k \in \mathbb{Z}_+\}$ be a sample based time scale with period $r$. Let $1 < p < \infty$ and assume that $E_{A,\mathcal{T}}$ is bounded and that $(CP)_{A,\mathcal{T}}$ has $\ell_p$-maximal regularity. Then $E_{A,\mathcal{T}}$ is analytic.*

*Proof :*
Let $x \in X$, $N \in \mathbb{N}$ and define $f(t_k) = \begin{cases} \mathcal{E}_{A,\mathcal{T}}(t_{k+1})x & \text{if} \quad k<N \\ 0 & \text{otherwise} \end{cases}$.
We have that there exists $M > 0$ such that $\|e_{A,\mathcal{T}}(t_k,t_n)\| \le M \quad \forall k \ge n$. Therefore,

$$\|f\|^p_{\ell_p(\mathcal{T};X)} \le M^p(\sum_{k=0}^{N} \mu_{k+1})\|x\|^p. \tag{5.1}$$

Since $\mathcal{T}$ is sample based we remark that there exists $C > 0$ such that

(i) $\frac{N}{C} \le \sum\limits_{k=0}^{N} \mu_{k+1} \le CN$,

(ii) $\frac{N^{1+p}}{C} \le \sum\limits_{k=0}^{N} \mu_{k+1}(\sum\limits_{j=0}^{k-1} \mu_{j+1})^p \le CN^{1+p}$.

We thus obtain the following, where $\succeq$ denotes inequalities up to a constant which does not depend on $N$.

$$\|K(f)\|^p_{\ell_p(\mathcal{T};X)} = \sum_{l=0}^{\infty} \mu_{l+1} \| \sum_{k=0}^{l-1} \mu_{k+1} A e_{A,\mathcal{T}}(t_{k+1}, t_l) f(t_k)\|^p,$$
$$\geq \sum_{l=0}^{N} \mu_{l+1} \| \sum_{k=0}^{l-1} \mu_{k+1} A e_{A,\mathcal{T}}(t_0, t_l) x\|^p \geq M^{-p} \sum_{l=0}^{N} \mu_{l+1} \| \sum_{k=0}^{l-1} \mu_{k+1} A \mathcal{E}_{A,\mathcal{T}}(t_N) x\|^p,$$
$$= M^{-p} \sum_{l=0}^{N} \mu_{l+1} (\sum_{k=0}^{l-1} \mu_{k+1})^p \|A\mathcal{E}_{A,\mathcal{T}}(t_N)x\|^p \succeq N^{1+p} \|A\mathcal{E}_{A,\mathcal{T}}(t_N)x\|^p,$$
$$\succeq N \|(t_{N-1} - t_0) A\mathcal{E}_{A,\mathcal{T}}(t_N)x\|^p.$$

Combining the later with (5.1) leads to $\|(t_{N-1} - t_0)A\mathcal{E}_{A,\mathcal{T}}(t_N)x\| \leq C'\|x\|$ for some constant $C' > 0$. $\square$

**Remark 5.3.3**
*One can notice that the assumption on the boundedness of $E_{A,\mathcal{T}}$, which is usual in the study of maximal regularity, is not a consequence of analyticity. Even in the case where $\mathcal{T} = \mathbb{Z}_+$ it has been shown by N.Kalton, S.Montgomery-Smith, K.Oleszkiewicz and Y.Tomilov ([KMSOT04]) that, on any infinite dimensional Banach space, there exists a discrete analytic semigroup that is not bounded (see Theorems 1.7.4 and 1.7.5)*

We now assume that $E_{A,\mathcal{T}}$ is bounded and analytic and define

$$T = \mathcal{E}_{A,\mathcal{T}}(t_r).$$

This is of course a discrete analogue of the *monodromy operator* used in the study of periodic nonautonomous Cauchy problem in the continuous case. The boundedness of $E_{A,\mathcal{T}}$ implies that $(T^n)_{n\in\mathbb{Z}_+}$ defines a bounded discrete time semigroup. We show that it is actually analytic and that it is R-analytic if and only if $E_{A,\mathcal{T}}$ is. This is a consequence of the following lemma.

**Lemma 5.3.4**
*Let $\mathcal{T}$ be a sample based time scale with period $r$ and $E_{A,\mathcal{T}}$ be an analytic evolution family. Then $T = \mathcal{E}_{A,\mathcal{T}}(t_r)$ is such that there exists a bounded invertible operator $B \in B(X)$ such that*

$$I - T = A.B$$

*Proof :*
There exists a polynomial $P$, of order less than $r$, such that $B = P(A)$. It is thus

enough to show that $0 \notin \sigma(P(A))$. We first remark that $P(0) = \sum_{j=1}^{r} \mu_j \neq 0$. Using the spectral mapping theorem it is then sufficient to show that

$$\forall \lambda \in \sigma(A) \quad P(\lambda) = 0 \implies \lambda = 0.$$

Let us therefore consider $\lambda_0 \in \sigma(A)$ such that $1 = \prod_{j=1}^{r} (1 - \mu_j \lambda_0)$. Then

$$\|(t_{rn} - t_0) A\mathcal{E}(t_{rn})\| \geq |\lambda_0| (t_{rn} - t_0) \quad \forall n \in \mathbb{N}.$$

But, because of the analyticity of $e_{A,\mathcal{T}}$, this implies that $\lambda_0 = 0$. □
We thus obtain the desired result as a corollary.

**Corollary 5.3.5**
*In the setting of the above lemma we have that $(T^n)_{n\in\mathbb{Z}_+}$ is an analytic discrete time semigroup and that it is R-analytic if and only if $E_{A,\mathcal{T}}$ is.*

*Proof :*
This follows directly from the fact that there exists $C > 0$ such that for any $N \in \mathbb{N}$ and any $(x_n)_{n=1}^{N} \subset X$,

$$\begin{aligned} \frac{1}{C} \int_0^1 \| \sum_{n=1}^{N} \varepsilon_n(t)(t_{rn} - t_0) A\mathcal{E}(t_{rn}) x_n \| dt &\leq \int_0^1 \| \sum_{n=1}^{N} \varepsilon_n(t) n(T - I) T^n x_n \| dt \\ &\leq C \int_0^1 \| \sum_{n=1}^{N} \varepsilon_n(t)(t_{rn} - t_0) A\mathcal{E}(t_{rn}) x_n \| dt \end{aligned}$$

where $(\varepsilon_j)_{j\in\mathbb{N}}$ is the usual sequence of Rademacher functions on $[0,1]$. □
The last step to allow the use of Fourier multipliers techniques is now to restate the problem in $\ell_p(X^r)$ (where $X^r$ is equipped, for instance, with the $\ell_p^r$ norm). Denoting by $\left((u_n^{(1)}, ..., u_n^{(r)})\right)_{n\in\mathbb{Z}}$ an element of $\ell_p(X^r)$, we define the operators

$$\mathcal{D}\left((u_n^{(1)}, ..., u_n^{(r)})\right)_{n\in\mathbb{Z}} = \left( (\frac{u_n^{(2)} - u_n^{(1)}}{\mu_1}, ..., \frac{u_n^{(r)} - u_n^{(r-1)}}{\mu_{r-1}}, \frac{u_{n+1}^{(1)} - u_n^{(r)}}{\mu_r}) \right)_{n\in\mathbb{Z}}$$

and

$$\mathcal{A}\left((u_n^{(1)}, ..., u_n^{(r)})\right)_{n\in\mathbb{Z}} = \left((Au_n^{(1)}, ..., Au_n^{(r)})\right)_{n\in\mathbb{Z}}.$$

By considering sequences $(u^{(j)})_{n\in\mathbb{Z}}$ (for $j = 1, ..., r$) with $u_n^{(j)} = \begin{cases} u(t_{j-1+rn}) & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$ for a function $u$ of the time scale one obtains that the $\ell_p$-maximal regularity of $(CP)_{\mathcal{T},A}$ is equivalent to the existence of a constant $C > 0$ such that

$$\|\mathcal{D}y\| \leq C\|(\mathcal{D}+\mathcal{A})y\| \quad \forall y \in \ell_p(X^r) \tag{5.2}$$

$\mathcal{D}$ is now our differentiation operator which turns out to be a Fourier multiplier. Let us recall that a Fourier multiplier is an operator of the form $\mathcal{F}_M f = \mathcal{F}^{-1}(M.\mathcal{F}f)$ where $M$ is a map from $(-\pi,\pi)$ to $B(X)$, $\mathcal{F}$ denotes the Fourier transform for the group $\mathbb{Z}$ (thus with the dual group $\mathbb{T}$) and the operator is initially defined for finitely supported sequences. A direct computation of the Fourier transform shows that $\mathcal{D} = \mathcal{F}_M$ where

$$M(t) = \begin{pmatrix} -\frac{I}{\mu_1} & \frac{I}{\mu_1} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{I}{\mu_2} & \frac{I}{\mu_2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\frac{I}{\mu_{r-1}} & \frac{I}{\mu_{r-1}} \\ \frac{e^{-it}}{\mu_r}I & 0 & \cdots & \cdots & 0 & -\frac{I}{\mu_r} \end{pmatrix}$$

It should be noticed that in the discrete and the continuous settings, the differentiation operator is a Fourier multiplier with an essentially scalar symbol (i.e. a symbol of the form $m(t)I$ where $m$ is scalar valued), whereas in this setting the symbol is operator valued. The problem can now be expressed as an operator matrix valued Fourier multiplier question. To do so remark that $\mathcal{M}(t)+\mathcal{A}$ is invertible provided its determinant

$$\Delta(t,A) = (-1)^{r+1}\left(\prod_{k=1}^{r}\frac{1}{\mu_k}\right)(e^{-it}-T)$$

is. But since $E_{A,\mathcal{T}}$ is analytic $T^{n+1}-T^n$ tends to zero by Corollary (5.3.5). Therefore $\sigma(T)\cap\mathbb{T}\subset\{1\}$. $\Delta(t,A)$ is thus invertible for all $t\in(-\pi,0)\cup(0,\pi)$ which allows us to define

$$\begin{array}{rccc} \mathcal{N}: & (-\pi,0)\cup(0,\pi) & \to & B(X) \\ & t & \mapsto & \mathcal{M}(t).(\mathcal{M}(t)+\mathcal{A})^{-1} \end{array}.$$

Since inequality (5.2) holds if and only if $\mathcal{F}_{\mathcal{N}}$ extends to a bounded operator on $\ell_p(X^r)$ we have the following lemma.

**Lemma 5.3.6**
*Let $X$ be a Banach space, $\mathcal{T}=\{t_k; k\in\mathbb{Z}_+\}\subset\mathbb{R}_+$ be a sample based time scale, $1<p<\infty$ and $E_{A,\mathcal{T}}$ be an associated bounded analytic evolution family. Then $(CP)_{\mathcal{T},A}$ has $\ell_p$-maximal regularity if and only if $\mathcal{F}_{\mathcal{N}}$ extends to a bounded operator on $\ell_p(X^r)$.*

Using Blunck's multiplier Theorem 2.3.3 we have that, if $X$ is an UMD space, and if the sets

$$\{\mathcal{N}(t); t\in(-\pi,0)\cup(0,\pi)\} \text{ and } \{(e^{-it}-1)(e^{-it}+1)\mathcal{N}'(t); t\in(-\pi,0)\cup(0,\pi)\}$$

are R-bounded, then $\mathcal{F}_{\mathcal{N}}$ defines a bounded multiplier on $\ell_p(X^r)$. Reciprocally the later implies that the set

$$\{\mathcal{N}(t); t \in (-\pi, 0) \cup (0, \pi)\}$$

is R-bounded. This leads our result.

**Theorem 5.3.7**
*Let $X$ be a UMD Banach space, $\mathcal{T} = \{t_k; k \in \mathbb{Z}_+\} \subset \mathbb{R}_+$ be a sample based time scale, $1 < p < \infty$ and $E_{A,\mathcal{T}}$ be a bounded analytic associated evolution family. Then the following assertions are equivalent.*

*(i)* $E_{A,\mathcal{T}}$ *is R-analytic.*

*(ii)* $(CP)_{A,\mathcal{T}}$ *has* $\ell_p$*-maximal regularity.*

*(iii)* $(T^n)_{n\in\mathbb{Z}_+}$ *is a R-analytic discrete time semigroup.*

*(iv)* $(CP)_{I-T,\mathbb{Z}_+}$ *has* $\ell_p$*-maximal regularity.*

*Proof :*
$(iii) \iff (iv)$ is due to Sönke Blunck (see Theorem 3.5.1).
$(iii) \iff (i)$ is given by Corollary 5.3.5.
$(iii) \implies (ii)$ as remarked above, we only have to show that the sets

$$\{\mathcal{N}(t); t \in (-\pi, 0) \cup (0, \pi)\} \text{ and } \{(e^{-it} - 1)(e^{-it} + 1)\mathcal{N}'(t); t \in (-\pi, 0) \cup (0, \pi)\}$$

are R-bounded. Moreover we have that

$$\mathcal{N}(t) = I - (\mathcal{A})(\mathcal{M}(t) + \mathcal{A})^{-1} \text{ and } \mathcal{N}'(t) = \frac{-ie^{-it}}{\mu_r}(I - \mathcal{N}(t))E_{r,1}(\mathcal{M}(t) + \mathcal{A})^{-1}$$

where $(E_{i,j})_{1\le i,j\le r}$ denotes the canonical basis of the space of $r \times r$ matrices. It is therefore sufficient to show that the sets

$$\{\mathcal{A}(\mathcal{M}(t) + \mathcal{A})^{-1}; t \in (-\pi, 0) \cup (0, \pi)\}$$
$$\text{and } \{(e^{-it} - 1)(\mathcal{M}(t) + \mathcal{A})^{-1}; t \in (-\pi, 0) \cup (0, \pi)\}$$

are R-bounded (finite sum and products of R-bounded families are R-bounded). Since all entries of $\mathcal{M}(t)+\mathcal{A}$ are commuting we have that $(\mathcal{M}(t)+\mathcal{A})^{-1} = (\Delta(t, A)^{-1}\otimes I_r)B(t, A)$ where $\Delta(t, A)$ is the determinant of $\mathcal{M}(t)+\mathcal{A}$ , $B(t, A)$ is the transpose of its comatrix and, for $C \in B(X)$, $C \otimes I_r$ denotes the diagonal operator matrix with diagonal entries equal to $C$. Remarking that $B(t, A)$ has entries which are polynomial of order less than $r$ in $e^{-it}$ and $A$ we obtain that $\{B(t, A)\ ;\ t \in (-\pi, 0)\cup(0, \pi)\}$ is R-bounded. It therefore suffices to show that the sets

$$\{AR(e^{it}, T); t \in (-\pi, 0) \cup (0, \pi)\} \text{ and } \{(e^{it} - 1)R(e^{it}, T); t \in (-\pi, 0) \cup (0, \pi)\}$$

are R-bounded. But it follows from Lemma 5.3.4 that the first of these sets is R-bounded if and only if $\{(T-I)R(e^{it},T); t \in (-\pi,0)\cup(0,\pi)\}$ is R-bounded. The result now follows from Theorem 3.5.1 of [Bl01-1] that shows that the later is equivalent to (iii).
$(ii) \implies (iii)$ as remarked above, (ii) implies that the set $\{\mathcal{M}(t)(\mathcal{M}(t)+\mathcal{A})^{-1}; t \in (-\pi,0)\cup(0,\pi)\}$ is R-bounded. But, as seen is the proof of $(iii) \implies (ii)$ this also implies that the set

$$\{(AR(e^{-it},T)\otimes I_r)B(t,A); t \in (-\pi,0)\cup(0,\pi)\}$$

is R-bounded and therefore that, for any $1 \le i,j \le r$ the set

$$\{(AR(e^{-it},T)\otimes I_r).E_{i,j}B(t,A)E_{j,i}; t \in (-\pi,0)\cup(0,\pi)\}$$

is R-bounded. Now since the entry on the first row and the r-th column of $B(t,A)$ is equal to $\prod_{k=1}^{r}\frac{1}{\mu_k}$ we obtain that the set $\{AR(e^{-it},T); t \in (-\pi,0)\cup(0,\pi)\}$ is R-bounded. Lemma 5.3.4 finally gives that $\{(I-T)R(e^{-it},T); t \in (-\pi,0)\cup(0,\pi)\}$ is R-bounded which implies (iii) by Theorem 1.1 of [Bl01-1]. □

## 5.4 A perturbation result

In this section we obtain maximal regularity of a Cauchy problem on a time scale that is not necessarily sample based but that is asymptotically close to a sample based time scale on which maximal regularity holds. We consider a Banach space $X$, a sample based time scale (of period $r$) $\mathcal{T} = \{t_j,\ j \in \mathbb{Z}_+\}$ (let us recall that $\mu_j = t_j - t_{j-1}\ \forall j \in \mathbb{N}$), a bounded operator $A \in B(X)$, a sequence $(\varepsilon_j)_{j\in\mathbb{Z}_+} \subset \mathbb{R}_+$, and a perturbed time scale $\widetilde{\mathcal{T}} = \{\widetilde{t}_j,\ j \in \mathbb{Z}_+\}$ such that

$$\begin{cases} \widetilde{t}_0 = t_0, \\ \widetilde{\mu}_{j+1} = \widetilde{t}_{j+1} - \widetilde{t}_j = \mu_{j+1} + \varepsilon_j. \end{cases}$$

**Definition 5.4.1**
*In the above setting, the sequence $(\varepsilon_j)_{j\in\mathbb{Z}_+}$ is called an* admissible perturbation *of $(CP)_{\mathcal{T},A}$ if there exists an open set $\Omega \subset \mathbb{C}$ satisfying*

*(i)* $\sigma(A) \subset \Omega$,

*(ii)* $\frac{1}{\mu_j} \notin \Omega \quad \forall j = 1,...,r$,

*such that for $n \ge k \ge 0$ the functions*

$$\begin{array}{rccc} g_{n,k}: & \Omega & \to & \mathbb{C}, \\ & \lambda & \mapsto & \dfrac{\prod_{j=k}^{n}\left(1-\frac{\varepsilon_j\lambda}{1-\mu_{j+1}\lambda}\right)-1}{\lambda}, \end{array}$$

*define a sequence in $H^\infty(\Omega)$ such that $\exists C > 0\ \exists \alpha \in [0,1)$*

$$\|g_{n,k}\|_\infty \leq C|n-k|^\alpha.$$

**Example 5.4.2**
*If $(\varepsilon_j)_{j\in\mathbb{Z}_+} \in \ell_1$ then it is an admissible perturbation of $(CP)_{\mathcal{T},A}$ for any sample based time scale $\mathcal{T}$ and any operator $A \in B(X)$ such that $T = \prod_{j=1}^{r}(I - \mu_j A)$ is invertible.*

This follows from the fact that for $\lambda \neq 0$,

$$|g_{n,k}(\lambda)| \leq \frac{exp(\sum_{j=k}^{n} |\frac{\varepsilon_j \lambda}{1-\mu_{j+1}\lambda}|) + 1}{|\lambda|},$$

and that $|g_{n,k}(0)| = |\sum_{j=k}^{n} \varepsilon_j|$.

**Theorem 5.4.3**
*Let $X$ be a Banach space, $\mathcal{T} = \{t_k; k \in \mathbb{N}\} \subset \mathbb{R}_+$ a sample based time scale (of period $r$), $1 < p < \infty$ and $E_{A,\mathcal{T}}$ an associated bounded evolution family. Consider a perturbation $(\varepsilon_j)_{j\in\mathbb{Z}_+}$ of $(CP)_{\mathcal{T},A}$ and the corresponding perturbed time scale $\widetilde{\mathcal{T}} = \{\widetilde{t}_j\ ;\ j \in \mathbb{Z}_+\} \subset \mathbb{R}_+$ and assume the following.*

*(i)* $(CP)_{\mathcal{T},A}$ *has $\ell_p$-maximal regularity,*

*(ii)* $\exists C > 0\ \forall j \in \mathbb{N}\ \frac{1}{C} \leq \widetilde{\mu}_j \leq C$,

*(iii)* $(\varepsilon_j)_{j\in\mathbb{Z}_+}$ *is an admissible perturbation of* $(CP)_{\mathcal{T},A}$.

*Then $(CP)_{A,\widetilde{\mathcal{T}}}$ has $\ell_p$-maximal regularity.*

*Proof :*
Let $(f(\widetilde{t}_k))_{k\in\mathbb{Z}_+} \in \ell_p(\widetilde{\mathcal{T}}; X)$. We estimate $\widetilde{K}f$ where

$$\widetilde{K}(f)(\widetilde{t}_k) = \begin{cases} A\sum_{k=0}^{n-1} \widetilde{\mu}_{k+1}.e_{\widetilde{\mathcal{T}},A}(\widetilde{t}_k, \widetilde{t}_n) f(\widetilde{t}_k) & \text{if}\, n > 0, \\ 0 & \text{if}\, n = 0. \end{cases}$$

To do so let $N \in \mathbb{N}$ and let us denote by $\preceq$ inequalities up to a multiplicative constant independent of $n$ and $N$. We have

$$\begin{aligned}
\mathcal{I} &= \sum_{n=1}^{N} (\mu_{n+1} + \varepsilon_n) \| A \sum_{k=0}^{n-1} (\mu_{k+1} + \varepsilon_k) (\prod_{j=k}^{n-1} I - (\mu_{j+1} + \varepsilon_j) A) f(\widetilde{t}_k) \|^p, \\
&\preceq \sum_{n=1}^{N} \| A \sum_{k=0}^{n-1} \mu_{k+1} (\prod_{j=k}^{n-1} I - \mu_{j+1} A) (\frac{\mu_{k+1} + \varepsilon_k}{\mu_{k+1}}) f(\widetilde{t}_k) \|^p, \\
&+ \sum_{n=1}^{N} \left( \sum_{k=0}^{n-1} \| A (\prod_{j=k}^{n-1} I - (\mu_{j+1} + \varepsilon_j) A) - (\prod_{j=k}^{n-1} I - \mu_{j+1} A) \| \| f(\widetilde{t}_k) \| \right)^p .
\end{aligned}$$

Now using (i) and Dunford calculus we obtain

$$\mathcal{I} \preceq \|f\|^p_{\ell_p(\widetilde{\mathcal{T}};X)} + \sum_{n=1}^{N} \left( \sum_{k=0}^{n-1} \| A^2 (\prod_{j=k}^{n-1} I - \mu_{j+1} A) g_{n-1,k}(A) \| \| f(\widetilde{t}_k) \| \right)^p .$$

Lemma 5.3.4 and (iii) then imply that

$$\mathcal{I} \preceq \|f\|^p_{\ell_p(\widetilde{\mathcal{T}};X)} + \sum_{n=1}^{N} \left( \sum_{k=0}^{n-1} \| (I-T)^2 T^m \| |n-k|^\alpha \| f(\widetilde{t}_k) \| \right)^p .$$

where $m$ is the entire part of $\frac{n-k}{r}$. Since $(T^n)_{n \in \mathbb{Z}_+}$ is a discrete analytic semigroup by Proposition 5.3.2 and Corollary 5.3.5 we have that

$$\| (I-T)^2 T^m \| \preceq \frac{1}{|n-k|^2}.$$

The result follows therefore from Young's inequalities. $\square$

# Chapter 6

# $L_p$ boundedness of pseudodifferential operators with operator valued symbols and application

## 6.1 Introduction

This chapter is motivated by the question of $L_p$-maximal regularity for non autonomous Cauchy problems of the form

$$(NACP)\left\{\begin{array}{rcll} u'(t)-A(t)u(t) &=& f(t) & \forall t\in[0,T], \\ u(0) &=& 0, & \end{array}\right.$$

where $(A(t),D(A(t)))_{t\in[0,T]}$ is a family of generators of analytic semigroups. It has been shown in [HM00] that this problem is related to the boundedness, in $L_p(\mathbb{R}^n;X)$, of pseudodifferential operators defined on the Schwartz class by

$$T_a f(x)=\int_{\mathbb{R}^n} e^{ix.\xi}a(x,\xi)\widehat{f}(\xi)d\xi, \tag{6.1}$$

where $a:\mathbb{R}^n\times\mathbb{R}^n\to B(X)$ is an operator valued symbol. As recalled in Section 2.5, Hieber and Monniaux have proved in [HM00] the boundedness of a large class of such operators on $L_p(\mathbb{R}^n;H)$ where $H$ is a Hilbert space. On the other hand, Štrkalj proved boundedness results on Banach space valued Bessel and Besov function spaces in his PhD thesis [St00]. In this chapter we consider the following question.

**Question 6.1.1**
*Given a UMD Banach space, for which classes of symbols $a:\mathbb{R}^n\times\mathbb{R}^n\to B(X)$ do the operator $T_a$ defined by 6.1 extend to bounded operators on $L_p(\mathbb{R}^n;X)$ for $1<p<\infty$ ?*

This was already answered in [St00]. However the proof was rather involved and never published (except in Željko Štrkalj's thesis). We present here a different and hopefully more direct approach to this question which can also be found in the joint work [PoS04] with Željko Štrkalj.

In Section 6.2 we introduce two classes of symbols which naturally generalize classical classes of scalar valued symbols and state the corresponding boundedness results. The first result deals with regular symbols whereas the second one considers the general "non regular" case.

The theorem in the regular case is then proved in Section 6.3. Following the work of Coifman and Meyer (see [CM78]) we prove that the problem reduces to the study of some elementary symbols. $L_p$ boundedness for such symbols is then proved using an operator valued analogue of the technique of almost orthogonality.

The non regular case is then treated in Section 6.4 using a method due to Nagase (see [Na77]) which allows to reduce this general case to the one treated in Section 6.3.

Finally we give, in Section 6.5, an application of this result to the study of maximal regularity for non autonomous evolution equations in continuous time.

## 6.2 Main results

Let $X$ be a Banach space. We consider the following classes of symbols.

**Definition 6.2.1**
*Let $0 \leq \delta < 1$, $n \in \mathbb{N}$ and $m \in \mathbb{N}$.*
*Consider $a \in C^m(\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}, B(X))$. We say that*

$$a \in \mathcal{S}^0_{1,\delta}(m, X)$$

*if the following hold.*

*(a)* $\forall \alpha \in (\mathbb{Z}_+)^n\ |\alpha| \leq m \quad \exists C_\alpha > 0$

$$\mathcal{R}(\{(1+|\xi|)^{|\alpha|}\partial_\xi^\alpha a(x,\xi)\,,\,\xi \in \mathbb{R}^n\}) \leq C_\alpha \quad \forall x \in \mathbb{R}^n, \tag{6.2}$$

*(b)* $\forall \alpha \in (\mathbb{Z}_+)^n\ 0 < |\alpha| \leq m \quad \forall \beta \in (\mathbb{Z}_+)^n\ |\beta| \leq m\ \exists C_{\alpha,\beta} > 0$ *such that* $\forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$

$$\|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\| \leq C_{\alpha,\beta}(1+|\xi|)^{\delta|\beta|-|\alpha|}. \tag{6.3}$$

**Definition 6.2.2**
*Let $0 \leq \delta < 1$, $0 < r < 1$, $n \in \mathbb{N}$ and $m \in \mathbb{N}$.*
*Consider a symbol $a : \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\} \to B(X)$ of class $C^m$ in the second variable. We say that*

$$a \in \mathcal{S}^0_{1,\delta}(r, m, X)$$

*if the following hold.*

*(a)* $\forall \alpha \in (\mathbb{Z}_+)^n \; |\alpha| \leq m \quad \exists C_\alpha > 0$

$$\mathcal{R}(\{(1+|\xi|)^{|\alpha|}\partial_\xi^\alpha a(x+ty,\xi)\,,\ \xi \in \mathbb{R}^n\,,\ t \in \mathbb{R}\}) \leq C_\alpha \quad \forall (x,y) \in \mathbb{R}^{2n}, \tag{6.4}$$

*(b)* $\forall \alpha \in (\mathbb{Z}_+)^n \; 0 < |\alpha| \leq m \quad \exists C_\alpha > 0 \quad \forall (x,y) \in \mathbb{R}^{2n} \quad \forall \xi \in \mathbb{R}^n$

$$\|\partial_\xi^\alpha a(x,\xi) - \partial_\xi^\alpha a(y,\xi)\| \leq C_\alpha |x-y|^r (1+|\xi|)^{\delta r - |\alpha|}. \tag{6.5}$$

Our main result is then the following.

**Theorem 6.2.3**
*Let $X$ be a UMD space , $1 < p < \infty$, $a \in \mathcal{S}^0_{1,\delta}(r,m,X)$ and $m > max(\frac{2n}{1-\delta}, 2n+4)$, then $T_a$ defined by (6.1) extends to a bounded operator on $L_p(\mathbb{R}^n;X)$.*

As a first step we prove the following "regular" version.

**Theorem 6.2.4**
*Let $X$ be a UMD space , $1 < p < \infty$, $a \in \mathcal{S}^0_{1,\delta}(m,X)$, and $m > max(\frac{2n}{1-\delta}, 2n+4)$, then $T_a$ defined by (6.1) extends to a bounded operator on $L_p(\mathbb{R}^n;X)$.*

## 6.3 Proof of Theorem 6.2.4

We start by proving a similar result for "elementary" symbols. Then we use an operator valued version of the Coifman-Meyer decomposition to reduce the case $a \in \mathcal{S}^0_{1,\delta}(m,X)$ to the case of "elementary" symbols. This is based on the Littlewood-Paley decomposition. Hence we consider *dyadic partitions of unity in* $\mathcal{S}(\mathbb{R}^n,\mathbb{R})$, i.e. sequences $(\phi_k)_{k\in\mathbb{Z}_+} \subset \mathcal{S}(\mathbb{R}^n,\mathbb{R})$ defined by

$$\phi_k(x) = \begin{cases} \psi(2^{-k}|x|) & \text{if } k \neq 0, \\ 1 - \sum\limits_{k\in\mathbb{N}} \psi(2^{-k}|x|) & \text{if } k = 0, \end{cases}$$

for some non negative function $\psi \in \mathcal{S}(\mathbb{R},\mathbb{R})$ supported in $[\frac{1}{2},2]$ and such that $\sum\limits_{k\in\mathbb{Z}} \psi(2^{-k}x) = 1 \quad \forall x \in \mathbb{R}^*$. Given such a partition we consider operators

$$\begin{array}{cccc} D_k: & L_p(\mathbb{R}^n;X) & \to & L_p(\mathbb{R}^n;X) \\ & f & \mapsto & f * \check{\phi}_k \end{array}$$

Remark that we have $D_k D_j = 0$ whenever $|j-k| > 1$. We recall the following results of Bourgain.

**Theorem 6.3.1 (Bourgain 1986)**
*Let $X$ be a UMD Banach space, $1 < p < \infty$ and $(D_k)_{k\in\mathbb{Z}_+} \in B(L_p(\mathbb{R}^n; X))$ be defined as above. Then there exists $C > 0$ such that for all $f \in L_p(\mathbb{R}^n; X)$*

$$\frac{1}{C}\|f\|_{L_p(\mathbb{R}^n;X)} \leq \int_0^1 \|\sum_{k\in\mathbb{Z}_+} \varepsilon_k(t) D_k f\|_{L_p(\mathbb{R}^n;X)} dt \leq C\|f\|_{L_p(\mathbb{R}^n;X)}.$$

**Theorem 6.3.2 (Bourgain (n=1) 1986, Zimmermann 1989)**
*Let $X$ be a UMD Banach space, $1 < p < \infty$ and $m : \mathbb{R}^n\backslash\{0\} \to \mathbb{C}$ be a bounded function such that*

$$sup\{|\xi|^{|\alpha|}\|D^\alpha m(\xi)\| \ ; \ \xi \in \mathbb{R}^n\backslash\{0\} \ \alpha \in (\mathbb{Z}_+)^n \ |\alpha| \leq n\} < \infty.$$

*Then the Fourier multiplier, initially defined on $\mathcal{S}(\mathbb{R}^n, X)$ by $T_a f(x) = \int_{\mathbb{R}^n} e^{ix.\xi} m(\xi)\widehat{f}(\xi)d\xi$, extends to a bounded operator on $L_p(\mathbb{R}^n; X)$.*

We also need the following corollaries.

**Corollary 6.3.3 (Kalton, Weis 2001)**
*Let $X$ be a UMD Banach space and $1 < p < \infty$. Then $(D_k)_{k\in\mathbb{Z}_+} \in B(L_p(\mathbb{R}^n; X))$, defined as above, is a R-bounded family.*

*Proof :*
By Theorem 6.3.2 we have that

$$\sup_{n\in\mathbb{N}} \sup_{\varepsilon_k=\pm1} \|\sum_{k=1}^{n} \varepsilon_k D_k\| < \infty.$$

The result therefore follows from Theorem 3.3 in [KW01]. □

**Corollary 6.3.4**
*Let $X$ be a UMD Banach space and $1 < p < \infty$. Then $\exists C > 0 \quad \forall N \in \mathbb{N} \quad \forall (f_0, ..., f_N) \in L_p(\mathbb{R}^n; X)^N$*

$$\|\sum_{k=0}^{N} D_k f_k\|_{L_p(\mathbb{R}^n;X)} \leq C \int_0^1 \|\sum_{k=0}^{N} \varepsilon_k(t) D_k f_k\|_{L_p(\mathbb{R}^n;X)} dt.$$

*Proof :*
Denoting by $\preceq$ inequalities up to a constant which does not depend on $N$ and

$(f_0, ..., f_N)$ we have

$$\begin{aligned}
\| \sum_{k=0}^{N} D_k f_k \|_{L_p(\mathbb{R}^n;X)} &\preceq \int_0^1 \| \sum_{j \in \mathbb{Z}_+} \varepsilon_j(t) D_j \sum_{k=0}^{N} D_k f_k \|_{L_p(\mathbb{R}^n;X)} dt \quad \text{(by Theorem 6.3.1)} \\
&\preceq \int_0^1 (\| \sum_{k=1}^{N} \varepsilon_{k-1}(t) D_{k-1} D_k f_k \|_{L_p(\mathbb{R}^n;X)} + \| \sum_{k=0}^{N} \varepsilon_k(t) D_k D_k f_k \|_{L_p(\mathbb{R}^n;X)} \\
&+ \| \sum_{k=0}^{N} \varepsilon_{k+1}(t) D_{k+1} D_k f_k \|_{L_p(\mathbb{R}^n;X)}) dt \\
&\preceq \int_0^1 \| \sum_{k=0}^{N} \varepsilon_k(t) D_k f_k \|_{L_p(\mathbb{R}^n;X)} \quad \text{(by Corollary 6.3.3)}
\end{aligned}$$

□

**Proposition 6.3.5**
*Let $X$ be a UMD Banach space, $1 < p < \infty$, $n \in \mathbb{N}$, $\delta \in (0,1)$ and $m$ the smallest integer such that $m > max(\frac{n}{1-\delta}, n+2)$. Consider*

$$a(x,\xi) = \sum_{j \in \mathbb{Z}_+} \psi_j(\xi) a_j(x) \quad \forall x \in \mathbb{R}^n \quad \forall \xi \in \mathbb{R}^n$$

*where*

*(i)* $\exists C > 0 \quad \mathcal{R}(\{a_j(x)\, ;\, j \in \mathbb{Z}_+\}) \leq C \quad \forall x \in \mathbb{R}^n.$

*(ii)* $\forall j \in \mathbb{Z}_+ \quad a_j \in C^{2m}(\mathbb{R}^n, B(X))$ *and*

$$\exists C_\alpha > 0 \quad \|\partial^\alpha a_j(x)\| \leq C_\alpha 2^{j\delta|\alpha|} \quad \forall x \in \mathbb{R}^n$$

*(iii)* $(\psi_j)_{j \in \mathbb{Z}_+} \subset \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ *is such that*

   *(a)* $\exists C > 0 \quad |\partial_\xi^\alpha \psi_j(\xi)| \leq C 2^{j|\alpha|} \quad \forall \xi \in \mathbb{R}^n \quad \forall \alpha \in \mathbb{Z}_+^n |\alpha| \leq n+2,$

   *(b)* $supp(\psi_0) \subset \{\xi \in \mathbb{R}^n\ ;\ |\xi| \leq 1\}$ *and,* $\forall j \in \mathbb{N}$, $supp(\psi_j) \subset \{\xi \in \mathbb{R}^n\ ;\ 2^{j-2} \leq |\xi| \leq 2^{j+2}\}$.

*Then $T_a$ defined by (6.1) extends to a bounded operator on $L_p(\mathbb{R}^n;X)$.*

*Proof :*
For $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}, X)$ let us define

$$T_j f(x) = \int_{\mathbb{R}^n} e^{ix.\xi} \psi_j(\xi) a_j(x) \widehat{f}(\xi) d\xi \quad \forall x \in \mathbb{R}^n.$$

We have that

$$T_a f = \sum_{j\in\mathbb{Z}_+} T_j f = \sum_{k\in\mathbb{Z}_+}\sum_{j\in\mathbb{Z}_+} D_k T_j f \quad \forall f\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\},X).$$

Moreover, by Corollary 6.3.4 there exists a constant $C>0$ such that, letting $D_{-j} = 0\ \forall j\in\mathbb{N}$, the following holds.

$$\begin{aligned}\|\sum_{j\in\mathbb{Z}_+}\sum_{k\in\mathbb{Z}_+} D_k T_j f\| &\leq \sum_{k=-4}^{4}\|\sum_{j\in\mathbb{Z}_+} D_{j+k}T_j f\| + \left(\sum_{j\in\mathbb{Z}_+}\sum_{\substack{k\in\mathbb{Z}_+\\ |k-j|>4}}\|D_k T_j\|\right)\|f\|,\\ &\leq C\sum_{k=-4}^{4}\int_0^1\|\sum_{j\in\mathbb{Z}_+}\varepsilon_j(t)D_{j+k}T_j f\|dt + \left(\sum_{j\in\mathbb{Z}_+}\sum_{\substack{k\in\mathbb{Z}_+\\ |k-j|>4}}\|D_k T_j\|\right)\|f\|,\end{aligned}$$

where $(\varepsilon_j)_{j\in\mathbb{Z}_+}$ is a sequence of Rademacher functions. It therefore suffices to show

1. $\exists C>0 \quad \forall N\in\mathbb{N} \quad \int_0^1\|\sum_{j\in\mathbb{Z}_+}\varepsilon_j(t)D_{k(j)}T_j f\|_{L_p(\mathbb{R}^n;X)}dt \leq C\|f\|_{L_p(\mathbb{R}^n;X)}$, where, given an integer $l$ such that $|l|\leq 4$, $k:\mathbb{Z}_+\to\mathbb{Z}_+$ is of the form $k(m) = m+l \quad \forall m\in\mathbb{Z}$.
2. $\exists C'>0 \quad \sum_{\substack{(k,j)\in\mathbb{Z}_+^2\\ |k-j|>4}}\|D_k T_j\|_{B(L_p(\mathbb{R}^n;X))}\leq C'$.

Let us start with 1. Let $f\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\},X)$. Denoting by $\preceq$ inequalities up to a constant which does not depend on $f$ and $N$ we obtain

$$\begin{aligned}&(\int_0^1\|\sum_{j=0}^{N}\varepsilon_j(t)D_{k(j)}T_j f\|_{L_p(\mathbb{R}^n;X)}dt)^p\\ &\quad\preceq\int_0^1\|\sum_{j=0}^{N}\varepsilon_j(t)D_{k(j)}T_j f\|^p_{L_p(\mathbb{R}^n X)}dt \quad\text{(by Kahane's inequalities)}\\ &\quad\preceq\int_0^1\|\sum_{j=0}^{N}\varepsilon_j(t)T_j f\|^p_{L_p(\mathbb{R}^n;X)}dt \quad\text{(by Corollary 6.3.3)}\\ &\quad\preceq\int_0^1\int_{\mathbb{R}^n}\|\sum_{j=0}^{N}\varepsilon_j(t)a_j(x)\int_{\mathbb{R}^n}e^{ix.\xi}\psi_j(\xi)\widehat{f}(\xi)d\xi\|_X^p dxdt\\ &\quad\preceq\int_0^1\int_{\mathbb{R}^n}\|\int_{\mathbb{R}^n}e^{ix.\xi}\sum_{j=0}^{N}\varepsilon_j(t)\psi_j(\xi)\widehat{f}(\xi)d\xi\|_X^p dxdt \quad\text{(by (i))}\\ &\quad\preceq\|f\|^p_{L_p(\mathbb{R}^n;X)} \quad\text{(by Theorem 6.3.2 and (iii)).}\end{aligned}$$

We now turn to 2. Let $k$ and $j$ be two integers such that $|k-j|>4$. First we remark that

$$\forall x \in \mathbb{R}^n \quad D_k T_j f(x) = \int_{R^n} K_{j,k}(x,y) f(y) dy$$

where $K_{j,k}(x,y) = \int_{\mathbb{R}^{3n}} \phi_k(\eta)\psi_j(\xi) e^{i\xi.(z-y)} e^{i\eta.(x-z)} a_j(z) dz d\eta d\xi$. Now we consider a function $\chi \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ with compact support satisfying $\chi(0) = 1$ and define, for $\varepsilon \in (0,1)$ the following kernels.

$$K_{j,k,\varepsilon}(x,y) = \int_{\mathbb{R}^{3n}} \phi_k(\eta)\psi_j(\xi) e^{i\xi.(z-y)} e^{i\eta.(x-z)} \chi(\varepsilon z) a_j(z) dz d\eta d\xi..$$

Integrating by parts in the $z$-variable and denoting by $\Delta_z$ the Laplace operator with respect to the variable $z$ we obtain

$$K_{j,k,\varepsilon}(x,y) = \int\limits_{\mathbb{R}^{3n}} \frac{e^{i\xi(z-y)} e^{i\eta(x-z)}}{(1+|\xi-\eta|^2)^m} \phi_k(\eta)\psi_j(\xi)(I-\Delta_z)^m \chi(\varepsilon z) a_j(z) dz d\eta d\xi$$

Now integrating by parts in the other variables we obtain the following, where $m'$ is such that $n < 2m' \leq n+2$.

$$K_{j,k,\varepsilon}(x,y) = \int\limits_{\mathbb{R}^{3n}} \frac{e^{i\xi(z-y)} e^{i\eta(x-z)}}{(1+|z-x|^2)^{m'}(1+|z-y|^2)^{m'}}$$
$$(I-\Delta_\eta)^{m'}(I-\Delta_\xi)^{m'} \left( \frac{\phi_k(\eta)\psi_j(\xi)}{(1+|\xi-\eta|^2)^m} \right) (I-\Delta_z)^m \chi(\varepsilon z) a_j(z) dz d\eta d\xi.$$

Using (ii) we obtain a constant $C>0$, independant of $\varepsilon$ such that

$$\|K_{j,k,\varepsilon}(x,y)\|_{B(X)} \leq C 2^{2j\delta m} \int\limits_{\mathbb{R}^{2n}} |(I-\Delta_\eta)^{m'}(I-\Delta_\xi)^{m'} \left( \frac{\phi_k(\eta)\psi_j(\xi)}{(1+|\xi-\eta|^2)^m} \right)| d\xi d\eta$$
$$\int\limits_{\mathbb{R}^n} \frac{1}{(1+|z-x|^2)^{m'}} \frac{1}{(1+|z-y|^2)^{m'}} dz.$$

Now using the fact that $supp(\phi_l) \subset \{s \in \mathbb{R}^n\, ;\, 2^{l-1} \leq |s| \leq 2^{l+1}\}$ and $|k-j|>4$ we remark that there exists a constant $C'>0$ such that

$$\int\limits_{\mathbb{R}^{2n}} |(I-\Delta_\eta)^{m'}(I-\Delta_\xi)^{m'} \left( \frac{\phi_k(\eta)\psi_j(\xi)}{(1+|\xi-\eta|^2)^m} \right)| d\xi d\eta$$
$$\leq C' 2^{max(j,k)(2n-2m)}.$$

Therefore, letting $\varepsilon$ tend to zero, we obtain a constant $\widetilde{C}>0$ such that

$$\|K_{j,k}(x,y)\| \leq \widetilde{C} 2^{2j\delta m} 2^{max(j,k)(2n-2m)} \int\limits_{\mathbb{R}^n} \frac{1}{(1+|z-x|^2)^{m'}} \frac{1}{(1+|z-y|^2)^{m'}} dz.$$

And thus $\exists C' > 0$ such that

$$\|D_k T_j\|_{B(L_p(\mathbb{R}^n;X))} \leq C' 2^{2j\delta m} 2^{max(j,k)(2n-2m)},$$

which concludes the proof since

$$\sum_{(j,k)\in\mathbb{Z}_+\times\mathbb{Z}_+} 2^{2j\delta m} 2^{max(j,k)(2n-2m)} \leq 2\sum_{j=0}^{\infty} j 4^{j(n-2(1-\delta)m)} < \infty.$$

$\square$

We now consider an analogue of the Coifman-Meyer decomposition.

**Proposition 6.3.6**
*Let $X$ be a Banach space, $n \in \mathbb{N}$, $m \in \mathbb{N}$ such that $m \geq 2n+4$ and $a \in \mathcal{S}^0_{1,\delta}(m, X)$. Then there exists a sequence $(c_j)_{j\in\mathbb{Z}^n} \subset \ell_1$, a sequence $(\psi_k)_{k\in\mathbb{Z}_+} \subset \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ and a sequence of functions from $\mathbb{R}^n$ to $B(X)$ $(a_{j,k})_{(j,k)\in\mathbb{Z}^n\times\mathbb{Z}_+}$ such that*

*(i)* $a(x,\xi) = \sum\limits_{(j,k)\in\mathbb{Z}^n\times\mathbb{Z}_+} c_j \psi_k(\xi) \frac{e^{2^{-k} ij.\xi}}{(1+|j|)^{n+2}} a_{j,k}(x) \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$,

*(ii)* $\exists C > 0 \quad \mathcal{R}(\{a_{j,k}(x)\,;\, k \in \mathbb{Z}_+\}) \leq C \quad \forall x \in \mathbb{R}^n \quad \forall j \in \mathbb{Z}$,

*(iii)* $\exists K > 0 \quad \forall \beta \in (\mathbb{Z}_+)^n \ |\beta| \leq m$

$$\|\partial^\beta_x a_{j,k}(x)\| \leq K 2^{k\delta|\beta|} \quad \forall x \in \mathbb{R}^n \quad \forall j \in \mathbb{Z}^n \quad \forall k \in \mathbb{Z}_+.$$

*Proof :*
Let $(\phi_k)_{k\in\mathbb{Z}_+}$ be a dyadic partition of unity in $\mathcal{S}(\mathbb{R}^n,\mathbb{R})$. For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$ define

$$b_k(x,\xi) = a(x, 2^k\xi)\phi_k(2^k\xi),$$

and consider the functions $B_k : \xi \mapsto \sum\limits_{j\in\mathbb{Z}^n} b_k(x, \xi - 2j\pi)$. Expanding those functions as Fourier series, and considering two functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ such that $supp(\psi_0) \subset \{\xi \in \mathbb{R}^n \,;\, |\xi| \leq 1\}$ and $supp(\psi_1) \subset \{\xi \in \mathbb{R}^n \,;\, 2^{-2} \leq |\xi| \leq 4\}$ we can write

$$b_0(x,\xi) = \sum_{j\in\mathbb{Z}^n} \frac{1}{(1+|j|^2)^{n+2}} e^{ij.\xi} \psi_0(\xi) a_{j,k}(x) \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$$

and, for all $k \in \mathbb{N}$,

$$b_k(x,\xi) = \sum_{j\in\mathbb{Z}^n} \frac{1}{(1+|j|^2)^{n+2}} e^{ij.\xi} \psi_1(\xi) a_{j,k}(x) \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$$

where

$$a_{j,k}(x) = \int\limits_{[-\pi,\pi]^n} e^{-ij.\xi}(I-\Delta_\xi)^{n+2} b_k(x,\xi)d\xi \quad \forall (j,k) \in \mathbb{Z}^n \times \mathbb{Z}_+ \quad \forall x \in \mathbb{R}^n. \quad (6.6)$$

Letting, for all $k \in \mathbb{N}, \psi_k(\xi) = \psi_1(2^{-k}\xi) \quad \forall \xi \in \mathbb{R}^n$, we have that, for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$,

$$a(x,\xi) = \sum_{k\in\mathbb{Z}_+} b_k(x, 2^{-k}\xi) = \sum_{(k,j)\in\mathbb{Z}_+\times\mathbb{Z}^n} \frac{(1+|j|)^{n+2}}{(1+|j|^2)^{n+2}} \psi_k(\xi) \frac{e^{2^{-k}ij.\xi}}{(1+|j|)^{n+2}} a_{j,k}(x).$$

This gives (i) with $c_j = \frac{(1+|j|)^{n+2}}{(1+|j|^2)^{n+2}} \quad \forall j \in \mathbb{Z}^n$.
Since $a \in \mathcal{S}^0_{1,\delta}(X)$ we also have that

$$\exists C > 0 \quad \mathcal{R}(\{(1+|2^k\xi|^2)^{n+2}\left((I-\Delta_\xi)^{n+2}a(x,.)\right)(2^k\xi)\,;\xi \in \mathbb{R}^n\,,\, k \in \mathbb{Z}_+\}) \leq C \quad \forall x \in \mathbb{R}^n,$$

and therefore that

$$\exists C' > 0 \quad \mathcal{R}(\{(I-\Delta_\xi)^{n+2} b_k(x,\xi)\,;\xi \in \mathbb{R}^n\,,\, k \in \mathbb{Z}_+\}) \leq C' \quad \forall x \in \mathbb{R}^n.$$

Given (6.6) this leads (ii).
Finally we observe that, since $a \in \mathcal{S}^0_{1,\delta}(X)$,
$\forall \beta \in (\mathbb{Z}_+)^n \; |\beta| \leq m \quad \exists C_\beta, C'_\beta > 0$ such that $\forall (k,j) \in \mathbb{Z}_+ \times \mathbb{Z}^n$

$$\begin{aligned} \|\partial_x^\beta a_{j,k}(x)\| &\leq C_\beta \int\limits_{[-\pi,\pi]^n} \|\partial_x^\beta (I-\Delta_\xi)^{n+2} b_k(x,\xi)\| d\xi \\ &\leq C'_\beta 2^{k\delta|\beta|}. \end{aligned}$$

This gives (iii). □

*Proof of Theorem (6.2.4) :*
Let $a \in \mathcal{S}^0_{1,\delta}(m, X)$ and $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}, X)$. By Proposition (6.3.6) we have

$$a(x,\xi) = \sum_{(j,k)\in\mathbb{Z}^n\times\mathbb{Z}_+} c_j \psi_k(\xi) \frac{e^{2^{-k}ij.\xi}}{(1+|j|)^{n+2}} a_{j,k}(x) \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}.$$

Let $\widetilde{\psi}_{k,j}(\xi) = \psi_k(\xi) \frac{e^{2^{-k}ij.\xi}}{(1+|j|)^{n+2}} \quad \forall \xi \in \mathbb{R}^n \quad \forall (k,j) \in \mathbb{Z}_+ \times \mathbb{Z}^n$ and $\widetilde{a}_j(x,\xi) = \sum\limits_{k\in\mathbb{Z}_+} \psi_k(\xi) a_{j,k}(x) \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\} \quad \forall j \in \mathbb{Z}^n$. By Proposition 6.3.5 the operators $T_{\widetilde{a}_j}$ are bounded on $L_p(\mathbb{R}^n; X)$ uniformly in $j \in \mathbb{Z}^n$. The result therefore follows from the fact that $(c_j)_{j\in\mathbb{Z}^n} \in \ell_1$ and that $T_a = \sum\limits_{j\in\mathbb{Z}^n} T_{\widetilde{a}_j}$. □

## 6.4 Proof of Theorem 6.2.3

To reduce this case to the one treated in Theorem 6.2.4 we use an analogue of a decomposition technique due to M.Nagase. Starting with a symbol $a \in \mathcal{S}^0_{1,\tau}(r, m, X)$ we show in this section that

$$a = b + r$$

with $b \in \mathcal{S}^0_{1,\delta}(m, X)$ ( for some $\delta \in (0, 1)$) and $r$ being such that $T_r$ defined by (6.1) extends to a bounded operator on $L_p(\mathbb{R}^n; X)$ $(1 < p < \infty)$. We start with a lemma giving a condition on a symbol $r$ for $T_r$ to be $L_p$ bounded.

**Lemma 6.4.1**
*Let $X$ be a Banach space, $1 < p < \infty$, $n \in \mathbb{N}$ and $r : \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\} \to B(X)$ be such that $\exists \rho \in (0, 1)$ $\forall \alpha \in (\mathbb{Z}_+)^n$ $|\alpha| \leq n + 1$ $\exists C_\alpha > 0$ such that*

$$(1 + |\xi|)^{|\alpha|+\rho} \|\partial^\alpha_\xi r(x, \xi)\| \leq C_\alpha.$$

*Then $T_r$ defined by (6.1) extends to a bounded operator on $L_p(\mathbb{R}^n; X)$.*

*Proof :*
This follows the proof of Theorem 2 from [Na77]. Let $\chi \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ be a compactly supported function satisfying $\chi(0) = 1$, $\varepsilon \in (0, 1)$ and define the following.

$$\begin{aligned}
r_\varepsilon(x, \xi) &= \chi(\varepsilon\xi) r(x, \xi)\\
\|r\|_\rho &= \sum_{|\alpha| \leq n+1} sup\{(1 + |\xi|)^{|\alpha|+\rho} \|\partial^\alpha_\xi r(x, \xi)\| \ ; \ \xi \in \mathbb{R}^n\}\\
\|r_\varepsilon\|_\rho &= \sum_{|\alpha| \leq n+1} sup\{(1 + |\xi|)^{|\alpha|+\rho} \|\partial^\alpha_\xi r_\varepsilon(x, \xi)\| \ ; \ \xi \in \mathbb{R}^n\}\\
K_\varepsilon(x, y) &= \int_{\mathbb{R}^n} e^{i(x-y)\xi} \chi(\varepsilon\xi) r(x, \xi) d\xi
\end{aligned}$$

Letting $\alpha \in (\mathbb{Z}_+)^n$, $|\alpha| \in \{n, n + 1\}$ and integrating by parts leads, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$

$$\|(x-y)^\alpha K_\varepsilon(x, y)\| = \|\int_{\mathbb{R}^n} e^{i(x-y)\xi} \partial^\alpha_\xi (\chi(\varepsilon\xi) r(x, \xi)) d\xi\| = \|\int_{\mathbb{R}^n} (e^{i(x-y)\xi} - 1) \partial^\alpha_\xi (\chi(\varepsilon\xi) r(x, \xi)) d\xi\|.$$

Moreover $|e^{i(x-y)\xi} - 1| \leq 2|x - y|^{\rho'} (1 + |\xi|)^{\rho'}$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ and all $\rho' \in (0, 1)$. Picking $\rho' \in (0, min(\rho, 1))$ we obtain that, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$, there exists $C > 0$ independant of $\varepsilon$ such that $\|(x - y)^\alpha K_\varepsilon(x, y)\| \leq C|x - y|^{\rho'-\rho}$. Therefore we have that, as $\varepsilon$ tends to 0, $K_\varepsilon$ tends to a function $K : \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\} \to B(X)$ such that

$$T_r f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \quad \forall x \in \mathbb{R}^n,$$

which satisfies

$$\|(1+|x-y|)|x-y|^n K(x,y)\| \leq C'|x-y|^{\rho'} \quad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$$

for some constant $C'$ independant of $\varepsilon$. Schur's lemma therefore gives the result. $\square$

We now turn to the decomposition itself.

**Proposition 6.4.2**
*Let $X$ be a Banach space, $(r,\tau) \in (0,1)^2$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $m \geq n+1$ and $a \in \mathcal{S}^0_{1,\tau}(r,m,X)$. Then there exists functions $b, r$, from $\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ to $B(X)$, such that*

*(i)* $\exists \delta \in (0,1)$ *such that* $b \in \mathcal{S}^0_{1,\delta}(m,X)$.

*(ii)* $\exists \rho \in (0,1) \quad \forall \alpha \in (\mathbb{Z}_+)^n \ |\alpha| \leq n+1 \quad \exists C_\alpha > 0$ *such that*

$$(1+|\xi|)^{|\alpha|+\rho} \|\partial_\xi^\alpha r(x,\xi)\| \leq C_\alpha$$

*Proof :*
Let $\delta \in (\tau, 1)$ and consider $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\int\limits_{\mathbb{R}^n} \phi(y)dy = 1$. Define, for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$,

$$b(x,\xi) = \int\limits_{\mathbb{R}^n} \phi((1+|\xi|)^\delta (x-y))(1+|\xi|)^{n\delta} a(y,\xi)dy \text{ and } r(x,\xi) = a(x,\xi) - b(x,\xi).$$

We have to show that $b$ satisfies (i) and that $r$ satisfies (ii).
Let us start with (i).
Let $\alpha \in (\mathbb{Z}_+)^n$, $0 < |\alpha| \leq m$, $\beta \in (\mathbb{Z}_+)^n$, $|\beta| \leq m$. We have $\forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$

$$\partial_x^\beta \partial_\xi^\alpha b(x,\xi) = \sum_{\alpha' \leq \alpha} \frac{\alpha!}{\alpha'!(\alpha-\alpha')!} \int\limits_{\mathbb{R}^n} \partial_\xi^{\alpha'} (\phi_{(\beta)}((1+|\xi|)^\delta (x-y))(1+|\xi|)^{\delta(n+|\beta|)}) \partial_\xi^{\alpha-\alpha'} a(y,\xi)dy$$

where $\phi_{(\beta)}(z) = \partial_z^\beta \phi(z)$. As in [Na77] we remark that, denoting by $\preceq$ inequalities up to a constant independant of $x, y$ and $\xi$, we have, for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$,

$$\begin{aligned} &|\partial_\xi^{\alpha'} (\phi_{(\beta)}((1+|\xi|)^\delta (x-y))(1+|\xi|)^{\delta(n+|\beta|)})| \\ &\preceq \sum_{\gamma \leq \alpha'} (1+|\xi|)^{-|\alpha'|} | \left((1+|\xi|)^\delta (x-y)\right)^\gamma || \phi_{(\beta+\gamma)}((1+|\xi|)^\delta (x-y))| (1+|\xi|)^{\delta(n+|\beta|)}. \end{aligned}$$

Therefore

$$\|\partial_x^\beta \partial_\xi^\alpha b(x,\xi)\| \preceq \sum_{\alpha' \leq \alpha} \sum_{\gamma \leq \alpha'} \int\limits_{\mathbb{R}^n} (1+|\xi|)^{-|\alpha|+\delta|\beta|} |z^\gamma \phi_{(\beta+\gamma)}(z)| dz \preceq (1+|\xi|)^{-|\alpha|+\delta|\beta|}$$

since $\phi$ has compact support.

Moreover, considering $N \in \mathbb{N}$, $(\varepsilon_j)_{j=1}^N$ a sequence of Rademacher functions, $(\xi_j)_{j=1}^N \subset \mathbb{R}^n$, $(x_j)_{j=1}^N \subset X$ and $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le m$, we have that

$$\int_0^1 \| \sum_{j=1}^N \varepsilon_j(t)(1+|\xi_j|)^{|\alpha|}\partial_\xi^\alpha b(x,\xi_j)x_j\|dt \preceq \sum_{\alpha'\le\alpha} \int_0^1 \| \sum_{j=1}^N \varepsilon_j(t)(1+|\xi_j|)^{|\alpha|}$$
$$\int_{\mathbb{R}^n} \partial_\xi^{\alpha'}(\phi_{(\beta)}((1+|\xi_j|)^\delta(x-y))(1+|\xi_j|)^{\delta(n+|\beta|)})\partial_\xi^{\alpha-\alpha'}a(y,\xi_j)dy\|dt$$

Now let us define a function $\Psi_{\alpha'} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned}&\partial_\xi^{\alpha'}(\phi_{(\beta)}((1+|\xi|)^\delta(x-y))(1+|\xi|)^{\delta(n+|\beta|)})\\ &\quad = \Psi_{\alpha'}(\xi)(1+|\xi|)^{\delta n}\left((1+|\xi|)^\delta(x-y)\right)^\gamma \phi_{(\gamma)}((1+|\xi|)^\delta(x-y))\end{aligned}$$

and remark that

$$|\Psi_{\alpha'}(\xi)| \preceq (1+|\xi|)^{-|\alpha'|}.$$

We then have that

$$\begin{aligned}&\int_0^1 \| \sum_{j=1}^N \varepsilon_j(t)(1+|\xi_j|)^{|\alpha|}\partial_\xi^\alpha b(x,\xi_j)x_j\|dt\\
&\preceq \sum_{\alpha'\le\alpha}\sum_{\gamma\le\alpha'} \int_0^1 \| \sum_{j=1}^N \varepsilon_j(t)(1+|\xi_j|)^{|\alpha|} \int_{\mathbb{R}^n} \Psi_{\alpha'}(\xi) z^\gamma \phi_{(\gamma)}(z)\partial_\xi^{\alpha-\alpha'}a(x-\frac{z}{(1+|\xi_j|)^\delta}\xi_j)x_j dz\|dt,\\
&\preceq \sum_{\alpha'\le\alpha}\sum_{\gamma\le\alpha'} \int_0^1 \int_{\mathbb{R}^n} |z^\gamma\phi_{(\gamma)}(z)| \| \sum_{j=1}^N \varepsilon_j(t)(1+|\xi_j|)^{|\alpha-\alpha'|}\partial_\xi^{\alpha-\alpha'}a(x-\frac{z}{(1+|\xi_j|)^\delta}\xi_j)x_j\|dtdz,\\
&\preceq \sum_{\alpha'\le\alpha}\sum_{\gamma\le\alpha'} \int_0^1 \int_{\mathbb{R}^n} |z^\gamma||\phi_{(\gamma)}(z)|dz \| \sum_{j=1}^N \varepsilon_j(t)x_j\|dt,\\
&\preceq \int_0^1 \| \sum_{j=1}^N \varepsilon_j(t)x_j\|dt,\end{aligned}$$

since $a \in \mathcal{S}^0_{1,\tau}(r,m,X)$.

We now turn to (ii).

Let $\alpha \in (\mathbb{Z}_+)^n$ , $|\alpha| \le n+1$. Pick $\rho \in (0,(\delta-\tau)r)$. ¿From the fact that $\int_{\mathbb{R}^n} \phi(z)dz = 1$ we have that $\forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$

$$r(x,\xi) = \int_{\mathbb{R}^n} \phi((1+|\xi|)^\delta(x-y))(1+|\xi|)^{n\delta}(a(x,\xi)-a(y,\xi))dy.$$

As before we then have that

$$\|\partial_\xi^\alpha r(x,\xi)\| \preceq \sum_{\alpha'\leq\alpha}\sum_{\gamma\leq\alpha'}(1+|\xi|)^{-|\alpha'|}$$
$$\int_{\mathbb{R}^n} |\left((1+|\xi|)^\delta y\right)^\gamma||\phi_{(\gamma)}((1+|\xi|)^\delta y)|(1+|\xi|)^{\delta n}\|\partial_\xi^{\alpha-\alpha'}(a(x-y),\xi)-a(x,\xi)\|dy,$$
$$\preceq \sum_{\alpha'\leq\alpha}\sum_{\gamma\leq\alpha'}\int_{\mathbb{R}^n} |\left((1+|\xi|)^\delta y\right)^\gamma||\phi_{(\gamma)}((1+|\xi|)^\delta y)|(1+|\xi|)^{\delta n}|y|^r(1+|\xi|)^{-|\alpha|+\tau r}dy,$$
$$\preceq \int_{\mathbb{R}^n} |z^\gamma\phi_{(\gamma)}(z)||z^r|(1+|\xi|)^{-|\alpha|+(\tau-\delta)r}dz \preceq (1+|\xi|)^{-|\alpha|-\rho}.$$

## 6.5 Application to maximal regularity of non autonomous evolution equations

We consider the following Cauchy problem, where $(A(t),D(A(t)))_{t\in[0,T]}$ is a family of generators of analytic semigroups.

$$(NACP)\begin{cases} u'(t)-A(t)u(t) &= f(t) \quad \forall t\in[0,T],\\ u(0) &= 0,\end{cases}$$

This problem is said to have $L_p$-maximal regularity ($1<p<\infty$) if, for each $f\in L_p([0,T];X)$, it has a unique solution $u\in W_0^{1,p}([0,T];X)$. We make the following assumptions.

$$(A) \quad \exists\phi\in(0,\frac{\pi}{2}) \quad \exists M>0 \quad \forall t\in[0,T] \quad \forall\lambda\notin\overline{\Sigma}_\phi,$$
$$\sigma(A(t))\subset\Sigma_\phi \quad \text{and} \quad \|(1+|\lambda|)R(\lambda,-A(t))\|\leq M.$$

$$(AT) \quad \exists K>0 \quad \exists 0\leq\alpha<\beta\leq 1 \quad \exists\phi\leq\psi<\frac{\pi}{2} \quad \forall(t,s)\in[0,T]^2 \quad \forall\lambda\notin\overline{\Sigma}_\psi,$$
$$\|A(t)R(\lambda,-A(t))(A(t)^{-1}-A(s)^{-1})\|\leq K\frac{|t-s|^\beta}{1+|\lambda|^{1-|\alpha|}}.$$

Since the article [AT87], where Acquistapace and Terreni introduced condition (AT), the problem (NACP) is known to have a unique classical solution as soon as (A) and (AT) are assumed (see [MR00] for a proof using evolution families) Moreover, Hieber and Monniaux have proved in [HM00] that (NACP) has $L_p$-maximal regularity if and only if the operator $T_a$ defined by 6.1 with

$$a(x,\xi)=\begin{cases} i\xi R(i\xi,A(0)) & \text{if } x<0,\\ i\xi R(i\xi,A(x)) & \text{if } x\in[0,T],\\ i\xi R(i\xi,A(T)) & \text{if } x>T,\end{cases} \tag{6.7}$$

extends to a bounded operator on $L_p(\mathbb{R};X)$. To obtain such a maximal regularity we therefore replace (A) by the following stronger hypothesis.

$$\begin{aligned} (RA) \quad & \exists \phi \in (0, \frac{\pi}{2}) \quad \exists M > 0 \quad \sigma(A(t)) \subset \Sigma_\phi \quad \forall t \in [0,T] \\ & \text{and} \quad \mathcal{R}(\{(1+|\lambda|)R(\lambda, -A(t))\ ;\ \lambda \notin \overline{\Sigma}_\phi\ ,\ t \in [0,T]\}) \leq M. \end{aligned}$$

We then obtain the following result.

**Corollary 6.5.1**
*Let $X$ be a UMD Banach space, $1 < p < \infty$ and $(A(t), D(A(t)))_{t\in[0,T]}$ be a family of operators satisfying (RA) and (AT). Then (NACP) has $L_p$-maximal regularity.*

*Proof :*
Given Theorem 6.2.3 we just have to show that the symbol $a$ defined by 6.7 belongs to $\mathcal{S}^0_{1,\delta}(m,r;X)$ for some $m > max(\frac{n}{1-\delta}, 2n+4)$ and some $(\delta, r) \in (0,1)$. Condition 6.4 follows from (RA). Moreover, for $(t,s) \in [0,T]^2$ and $\xi \in \mathbb{R}$, we have by (RA) and (AT) that

$$\begin{aligned} \|R(i\xi, A(t)) - R(i\xi, A(s))\| &\leq \|A(s)R(i\xi, A(s))\|\|A(t)R(i\xi, A(t))(A(t)^{-1} - A(s)^{-1})\|, \\ &\leq MK\frac{|t-s|^\beta}{(1+|\xi|)^{1-|\alpha|}}, \end{aligned}$$

which leads 6.5 with $r = \beta$ and $\delta = \frac{\alpha}{\beta}$. □

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