Réduction et intégration symbolique des systèmes d'équations différentielles non-linéaires

Gérard Eichenmüller

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Gérard Eichenmüller

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Reduction et Intégration symbolique des systèmes d'équations différentielles non-linéaires

Formal reduction and integration of systems of nonlinear differential equations

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Composition du Jury :

L. BRENIG	Université Libre de Bruxelles	(président et rapporteur)
G. CHEN	Université de Lille	(rapporteur)
J. DELLA DORA	Institut National Polytechnique de Grenoble	(examinateur)
J. THOMANN	Université Louis Pasteur de Strassbourg	(examinateur)
E. TOURNIER	Université Joseph Fourier	(examinateur)

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Contents

Introduction

Dynamical systems are present everywhere in science. Many models of natural processes yield dynamical systems. A dynamical system is a system whose state changes with time. There exist two main types of dynamical systems: discrete dynamical systems that are represented by difference equations and continuous dynamical systems that are represented by differential equations.

The state of a system can be described by a number of variables that we will centralize in the n-dimensional vector $X = (x_1, \ldots, x_n)$. The variable t denotes time. We are interested in continuous dynamical systems that can be described by a system of autonomous differential equations

$$\dot{X} = F(X) \tag{1}$$

where $F = (f_1, \ldots, f_n)$ is a vector. The components $f_i \in C^{\geq 1}(M, \mathbb{C}^n)$ are defined on the open convex subset M. The fact that we consider systems implies that n is greater than 1. In particular the cases $n = 2$ and $n = 3$ will be studied very closely.

The solutions of dynamical systems are given by their flow that is denoted by Φ. The velocity of the flow is given by the vector field F. In general the flow Φ is approximated by numerical algorithms. But those methods are not very precise in the neighbourhood of singularities and they do not allow the study of systems containing parameters. On the other hand most differential equations (even simple ones) have no explicit solution. Therefore in this thesis we will employ another approach proposed by H. Poincaré to solve this dilemma. The explicit analytic study of a differential equation is replaced by qualitative studies.

To study the qualitative behaviour of dynamical systems means to classify them into equivalence classes of similar behaviour. This classification is realized by local diffeomorphisms. That means that those studies are strictly local. They are only valid in the neighbourhood of a point or another object. In two dimensions and for some higher dimensional problems the results of those considerations can be used to approximate algebraic solutions.

Dynamical systems described by equations as (1) often arise from modeling problems in science. The variable X denotes involved physical quantities that change with time. Those changes are described by a system of differential equations.

Example 1 (Planar Pendulum) *Newton's third law*

$$F = m\,a$$

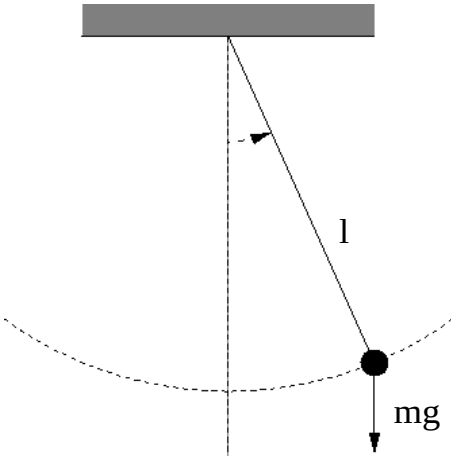


Figure 1: Planar pendulum, see example 1.

describes the behaviour of many physical systems. In absence of friction we find the relations

$$F = -m\,g\,\sin x$$
$$a = \frac{\partial^2}{\partial t^2}(l\,x)$$

for a planar pendulum (see figure 1). The variable m denotes the mass of the pendulum , l its length and g the gravitational constant. As the acceleration a is the second derivative of displacement x this is a second order differential equation

$$\ddot{x} + \frac{g}{l}\sin x = 0$$

that can be rewritten as a first order system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} sin(x) \end{cases}$$

in the variables x and the new variable y.

This thesis is split into three parts. The first part introduces some main tools that will be used in the algorithms for the reduction of two- and higher-dimensional systems. Those algorithms are described in the second part. The third part of this thesis deals with the implementation of the algorithms and the programming aspects. We will also give some examples for the use of the Maple package that has been implemented by the author.

We will give a more precise overview for each chapter of this thesis.

First part: The integration of dynamical systems

Formal solutions for dynamical systems

This chapter introduces some basic notations and definitions for dynamical systems as they can be found in many references on dynamical systems (see for example the works

from K. Alligood, T. Sauer and J. Yorke [39], F. Verhulst [26], J. Hale and H. Korcak [33], S. Chow and J. Hale [14], J. Guckenheimer and P. Holmes [36], D. Arrowsmith and C. Place [17] and J. Hubbard and B. West [34]).

Systems of differential equations and vector fields can be used to represent the same dynamics. Therefore the theory of dynamical systems largely uses the notations of vector fields. Their solutions are called global or local flows. In general global flows can't be calculated. That is why the computation of local flows and local studies of vector fields or differential equations is the main intention in the theory of dynamical systems.

If the linear part of a system exists and if the linearized vector field is equivalent or conjugated to the initial one a lot of qualitative attributes of vector fields can be derived from the linearized one. Therefore definitions for equivalence of vector fields using diffeomorphisms play a very important role in qualitaive studies of vector fields and differential equations.

Transformations that are no equivalence transformations, so called time transformations, are also frequently used to find solution curves for differential equations. The transformations do not yield equivalent vector fields and we can no longer retransform solutions of time transformed equations to solution of the initial differential equations. Therefore the curves obtained from an integration of the new system can no longer be considered as equivalent to the flow of the initial one. That is why we introduce the notation of solution curves that are parametrizations of the flows of the initial system.

In general normal form constructions that are introduced in chapter 3 yield diverging series. For this reason all calculated solutions might also be divergent. That is why all considerations are purely formal. We work in the ring of formal power series. The implemented algorithms work with truncations that are polynomials as formal power series can not be handeled in computations.

The Newton diagram

Many transformations applied to a vector field can be interpreted geometrically and the geometric aspects of vector fields can be used to find transformations that reduce and simplify the considered vector field.

The most important tool for the geometric interpretation of transformation is the Newton diagram or the Newton polygon. The Newton polygon is also used directly to find solutions for differential equations (see for example works from J. Della Dora and F. Jung [19], F. Beringer and F. Jung [6] and J. Cano [37]). For the algorithms proposed here however the Newton diagram and the support of a considered system will be used to calculate matrices that define power transformations.

Power transformations are a very powerful tool for handling systems of differential equations. They can be used to reduce singularities of systems having a nilpotent linear part as we will see in chapter 4 or to integrate systems that are in normal form (see chapter 7 and 9). Power transformation manipulate the exponents of the concerned system. The geometric interpretation of this manipulation is very simple as it induces an affine transformation on the support of the concerned system. Some results concerning power transformations are given in works from A. Bruno ([9] and [1]).

To validate the use of power transformations we have to prove that power transformations are diffeomorphisms. But some power transformations are not injective on the whole

definition set. Therefore the definition set is limited such that the transformations are injective on this set. Their surjectivity is guranteed by a construction that makes them "piecewise surjective".

In section 4 and 8 we will handle systems that also have negative exponents. They can be reduced by power transformations as their support lies within a cone. Therefore the intention of geometric manipulations of vector field often consists in manipulating those cones.

Normal forms

As mentioned before some of the most important properties of a vector field can be deduced from its linear part if it exists.

In certain cases there exist diffeomorphisms that transform a considered vector field into a linear one. Though in general it is not possible to linearize a given vector field entirely we can find diffeomorphisms that reduce the vector field to a "simpler" vector field. However the definition of "simpler" is not unique. The complexity of a vector field usually depends on the number and the properties of the nonlinear terms. A "simpler" vector field usually has fewer nonlinear terms or its nonlinear terms have special properties. When the system can no longer be reduced is said to be in normal form.

There exist many different approaches to normal forms. We use the Poincaré-Dulac normal form as it can be integrated for two-dimensional systems (chapter 7). The Poincaré-Dulac normal form is mainly due to H. Poincaré [47], H. Dulac [21] and G. Birkhoff [7]. It can be calculated using the adjoint representation method that is due to G. Iooss [35] or more efficiently using Lie theory as it has been proposed by K. T. Chen [40] or A. Deprit [3].

Recently a lot of work has been done in normal form theory. The first step in calculating the Poincaré-Dulac normal form is the calculation of the Jordan form of the matrix representing its linear part. This is far from being trivial (see I. Gil [29] and M. Griesbrecht [31]). But it can be avoided by using the Frobenius form of a matrix (see works from G. Chen [13]). R. Cushman and A. Sanders [16] propose an algorithm that can be used for the calculation of vector fields with nilpotent but non-vanishing linear part. The case of normal forms for Lotka-Voltera systems have been studied by S. Louies and L. Brenig [50]. Algorithms that use Carleman-linearizations to compute normal forms have been proposed by L. Stolovitch [56] and G. Chen [13].

Resolution of singularities by blowing-up

The Poincaré-Dulac normal form theorem can no longer be applied if the linear part of the considered vector field is nilpotent. In this case blowing-up is used. Blowing-up involves changes of coordinates (polar coordinates or power transformations) which expand or blow-up the singularity of the vector field into a set on which a finite number of simpler singularities occur.

Directional blowing-up was first introduced for plane algebraic curves by O. Zariski [66] and for two-dimensional differential equation by A. Seidenberg [53]. Since then many others have worked on this subject. For our constructions we will use quasihomogeneous blowing-up that is given by power transformations and that have first been used by A.

Bruno [9] and more recently by M. Brunella and M. Miari [8]. They use unimodular matrices and work mainly with two-dimensional problems. We extend blowing-up construction to power transformations defined by any invertible matrix as unimodular matrices are not sufficient for problems appearing in dimension 3. It will be shown in chapter 2 that those matrices define diffeomorphisms in a subset of the concerned neighbourhood and that their use is therefore allowed.

Second part: The Algorithms

Classification

The proposed algorithms handle the reduction of dynamical systems and the computation of solution curves for several cases. The case of a simple or regular point, the case of an elementary singular point (non-nilpotent linear part) and the case of a nonelementary singular point (nilpotent linear part) are treated separately. The first step in any calculation is to classify a given system to allow to handle it with the appropriate methods.

Regular points

In the neighbourhood of a regular point any system can be reduced to a system with a very simple form that is a kind of normal form. It can easily be integrated. The problem of calculating this normal form can be reduced to the problem of calculating the flow of a system of differential equations near a simple point. The change of coordinates is computed using Taylor series. This method is described in many references treating Lie theory and vector fields (W. Groebner [62], W. Groebner and H.Knapp [63] and P. Olver[46]).

Two-dimensional elementary singular points

A singular point is called elementary singular point if the considered system has a non-nilpotent Jacobian matrix there. Then the Poincaré-Dulac theorem can be applied to calculate the normal form of the considered system. The two dimensional Poincaré-Dulac normal form is integrable as it has been shown by A. Bruno [9].

A special case is represented by systems with purely imaginary eigenvalues. They yield periodic solutions that are best represented in polar coordinates. This also allows us to obtain real solution curves if the initial system is real.

Two-dimensional nonelementary singular points

In the case of a nonelementary singular point there exist two methods to find solution curves. Blowing-ups that are introduced in chapter 4 reduce the complexity of the considered singularity. They yield several new systems instead of only one initial system. The new systems are treated recursively be applying the entire algorithm (starting with the classification). Another method that is to use time transformations to compute non-nilpotent systems that have their support in a cone. An appropiate power transformation reduces this system to a non-nilpotent system with integer exponents. Solution curves for this system can easily be calculated via normal forms as in the case of an elementary singular point.

Both methods are controlled by the Newton diagram. The edges are used to compute the matrices defining the power transformations for the blowing-ups and the vertices define the time transformations.

These methods yield many solutions. Therefore a central point of this part of the algorithm is the use of sectors. The sectors define the domains of the concerned neighbourhood where the calculated solution curves are valid. They allow a very efficient handling of the solutions.

Three and higher dimensional elementary singular points

The algorithms described in this chapter have so far only been treated very superficially by A. Bruno [1]. Using these works as a starting point we propose a more complete study of the case of three- and higher-dimensional elementary singular points.

In a first step the Poincaré-Dulac normal form is calculated. This normal form can be reduced to a system from which we can split a system of lower dimension. The power transformation used for the reduction has to verify very strict conditions. Problems can occur if some of those vectors have negative coordinates. These problems arise from the higher dimension of the resonant plane and its position in the space of exponents. We propose a classification of three-dimensional normal forms that allows the reduction and integration of any three-dimensional normal form.

The virtual Newton diagram allows to generalize the results obtained from the intense study of three-dimensional normal forms to higher dimensional systems.

Three and higher dimensional nonelementary singular points

Three and higher dimensional systems with nonelementary singular points can also be treated by blowing-ups. However there still remain many problems as it has been shown for 3 dimensions by X. Gomez-Mont and I. Luengo [30]. Three dimensional systems can not always be reduced entirely by a finite number of successive blowing-ups. Nevertheless we give some examples that use 3 dimensional blowing-ups. For these examples the correctness of the constructions have been proved by F. Cano and D. Cerveau [10]. We propose a construction of blowing-ups that is controlled by the Newton diagram or the virtual Newton diagram and that considers a definition of sectors that is different to the definition given by A. Bruno [9]. It is strongly connected to the cones that contain the support of the initial system.

Third part: Implementation

The algorithms described in the previous part have been implemented in the FRIDAY [1] MAPLE package.

The FRIDAY package

This chapter gives a description of the package and a large number of examples for its use. The FRIDAY package is organized in modules according to the classification of dynamical

[1] FRIDAY stands for **F**ormal **R**eduction and **I**ntegration of **D**ynamical **A**utonomous **Sy**stems

systems. However only a few procedures are visible to the user. The main procedure can be used to integrate any two-dimensional and a large number of three-dimensional dynamical systems. Besides this procedure the modules that compute normal forms can be used separately.

In this thesis we will show how tools as normal forms and power transformations can be used for the formal reduction and integration of dynamical systems. For two-dimensional systems and three-dimensional systems with elementary singular points the proposed algorithms have been implemented and tested. However there still remain theoretical and practical problems especially in the field of three- and higher dimensional systems with nonelementary singular points.

Introduction

Les systèmes dynamiques sont présents partout dans la science. Ils proviennent de nombreux modèles simulant des phenomènes naturels. Il y a deux types principaux de systèmes dynamiques: les systèmes discrets qui sont représentés par des équations aux différences et les systèmes continus qui sont caractérisés par des systèmes d'équations différentielles.

L'état d'un système peut être décrit par un nombre de variables qui sont réunies dans le vecteur $X = (x_1, \ldots, x_n)$. La variable t dénote le temps. Dans cette thèse nous nous intéressons aux systèmes dynamiques continus qui peuvent être représentés par un système d'équations différentielles autonomes. Ces systèmes seront notés

$$\dot{X} = F(X) \tag{2}$$

où $F = (f_1, \ldots, f_n)$ est un vecteur de dimension n. Le fait que nous étudions des systèmes implique que $n > 1$. Nous allons en particulier étudier les cas $n = 2$ et $n = 3$.

Les solutions des systèmes dynamiques sont données par leur flot Φ. La vitesse du flot est définie par le champ de vecteurs F. En général le flot Φ est approché par des algorithmes numériques. Mais ces méthodes ne sont pas très précises dans le voisinage des singularités et elles ne permettent pas l'utilisation de paramètres. Par contre la plupart des équations différentielles (mêmes les plus simples) n'ont pas de solutions explicites. Dans cette thèse nous allons utiliser une autre approche, proposé par Poincaré. L'étude explicite analytique d'une équation différentielle est remplacée par une analyse qualitative.

Etudier le comportement qualitatif des systèmes dynamiques signifie les répartir dans des classes d'équivalence représentant des systèmes ayant le même comportement. Cette classification est réalisée à l'aide de difféomorphismes locaux. Cela veut dire que l'analyse qualitative utilisant cette approche fournit des résultats qui ne sont valables que localement dans le voisinage d'un point. Dans le cas de systèmes dynamiques de dimension deux cette analyse peut être utilisée pour approcher des solutions algébriques.

Des systèmes dynamiques de la forme (2) sont souvent issus de la modélisation de problèmes scientifiques. La variable X représente les quantités physiques concernées qui changent au cours du temps t. Ces changements sont caracterisés par un système d'équations différentielles.

Exemple 1 (Pendule) *Le comportement de nombreux modèles en mécanique est décrit par la troisième loi universelle de la mécanique*

$$F = m\,a\ .$$

En l'absence de friction, les équations

$$F = -m\,g\,\sin x$$
$$a = \frac{\partial^2}{\partial t^2}(l\,x)$$

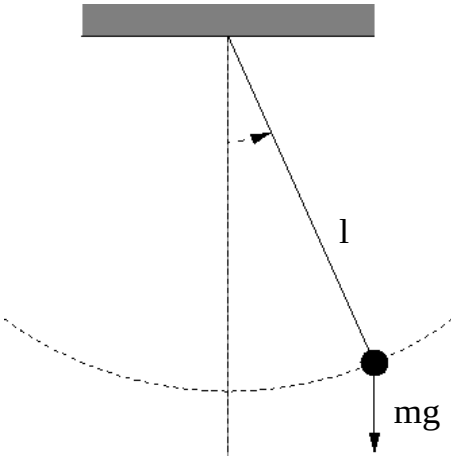


Figure 2: Le pendule de l'exemple 1.

définissent le comportement d'un pendule (voir figure 2). La variable m représente la masse du pendule, l sa longueur et g est la constante gravitationelle. Nous pouvons décrire ces relations par une équation différentielle d'ordre deux

$$\ddot{x} + \frac{g}{l} \sin x = 0$$

car l'accélération a est égale à la deuxième dérivée du déplacement x. Ceci définit un système d'ordre un en deux dimensions

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} sin(x) \end{cases}$$

que nous obtenons en introduisant la variable y.

Dans cette thèse nous avons montré comment des outils comme les formes normales et les transformation quasi-monomiales peuvent être utilisés pour la réduction et l'intégration formelle des systèmes dynamiques. Pour les systèmes en dimension deux et les systèmes non nilpotents en dimension trois les algorithmes proposés ont éte implantés et testés. Néanmoins, de nombreux problèmes théoriques et pratiques restent à résoudre, surtout dans le domaine des systèmes nilpotents.

Resumé par chapitre

Cette thèse est divisée en trois parties. Dans la première partie nous allons introduire des outils essentiels qui seront utilisés pour la réduction des systèmes considérés. Dans la deuxième partie nous allons décrire les aspects algorithmiques de ces réductions. L'implantation en MAPLE de ces algorithmes est le sujet de la troisième partie. Nous allons décrire les aspects de programmation et quelques exemples qui illustrent l'utilisation du logiciel implanté. Nous détaillons ci-dessous le plan de chaque chapitre.

Première partie: Intégration des systèmes dynamiques

Les solutions formelles des systèmes dynamiques

Dans ce chapitre nous introduisons les notions et les définitions de base que nous utiliserons constamment par la suite. Elles peuvent également être trouvées dans de nombreuses références sur les systèmes dynamiques (voir par exemple les travaux de K. Alligood, T. Sauer et J. Yorke [39], F. Verhulst [26], J. Hale et H. Korcak [33], S. Chow et J. Hale [14], J. Guckenheimer et P. Holmes [36], D. Arrowsmith et C. Place [17] et J. Hubbard et B. West [34]).

Les systèmes dynamiques et les champs de vecteurs peuvent être utilisés pour représenter la même dynamique. Par conséquent la théorie des systèmes dynamiques fait largement appel à la notion de champs de vecteurs. En général les flots globaux des champs de vecteurs ne peuvent pas être calculés. Pour cette raison le calcul des flots locaux et l'étude locale sont les buts principaux de la théorie des systèmes dynamiques. Si la partie linéaire d'un champs de vecteurs existe dans le voisinage d'un point singulier celle-ci peut être équivalente ou conjugé au champ de vecteurs non linéaire associé. Cela signifie que de nombreuses caractéristiques du champ de vecteurs non linéaire peuvent être déduits du champ linéarisé.

Pour cette raison la notion d'équivalence de champs de vecteurs est très importante dans l'analyse qualitative des champs de vecteurs et des équations différentielles.

Nous allons souvent utiliser des changements de temps. Ces transformations ne sont pas des transformations d'équivalence. Par conséquent les champs de vecteurs issus d'un changement de temps ne sont pas équivalents au champ de vecteurs initial. Les flots calculés pour des systèmes transformés par un changement de temps ne peuvent en général pas être transformés en des solutions du système initial. Pour cette raison nous introduisons la notion de courbes de solutions. Ces courbes représentent des paramétrisations du flot du système initial.

Pourtant, dans certains cas, il existe une relation d'équivalence entre le système de

départ et le système transformé par un changement de temps. Dans le cas des systèmes hamiltoniens les deux systèmes ont les mêmes intégrales premières. Si nous considérons uniquement les systèmes en deux dimensions, les deux systèmes peuvent être considérés comme provenant de la même équation différentielle scalaire. Ces relations peuvent également être utilisées pour vérifier les résultats calculés.

Le calcul des formes normales que nous allons introduire au chapitre 3 donne souvent des séries divergentes. C'est pourquoi nous nous placerons souvent dans l'anneau des séries formelles. Toutes les considérations seront alors purement formelles.

Le diagramme de Newton

De nombreuses transformations que nous allons utiliser pour la réduction des champs de vecteurs peuvent être interprétées géométriquement. De plus, les aspects géométriques des champs de vecteurs peuvent être utilisés pour trouver des transformations qui réduisent le champ de vecteurs comme on le souhaite.

L'outil le plus important pour l'analyse géométrique est le diagramme ou le polygone de Newton. Il peut aussi être utilisé directement pour calculer des solutions d'équations différentielles (voir par exemple J. Della Dora and F. Jung [19], F. Beringer and F. Jung [6] and J. Cano [37]). Néanmoins, pour les algorithmes que nous allons proposer ici le diagramme de Newton et le support d'un système vont être utilisés pour définir des tranformations quasi-monomiales. Ces transformations sont un outil très puissant pour manipuler des systèmes dynamiques. Nous allons les utiliser pour réduire les singularités des systèmes nilpotents (voir chapitre 4) et pour intégrer des systèmes qui sont sous forme normale (voir les chapitres 7 et 9). Les transformations quasi-monomiales manipulent les exposants des systèmes concernés. L'interprétation géométrique de ces manipulations peut être décrite par l'effet d'une transformation affine sur les exposants. Quelques résultats sur les transformations quasi-monomiales peuvent être trouvés dans les travaux de A. Bruno ([9] et [1]).

Pour valider l'utilisation des transformations quasi-monomiales nous devons prouver que ce sont des difféomorphismes. Néanmoins certaines transformations ne sont pas injectives sur l'ensemble du domaine de définition. Pour cette raison nous allons limiter ce domaine de sorte que la transformation devienne injective. La surjectivité des transformations quasi-monomiales est assurée par une construction qui les rend surjectives par morceaux.

Dans les chapitres 8 et 9 nous allons travailler sur des systèmes avec des exposants négatifs. Ces systèmes ne peuvent être manipulés que parce que leur support est inclus dans un cône convexe. Pour cette raison la manipulation des champs de vecteurs est souvent fortement liée à la manipulations de cônes dans l'espace des exposants.

Les formes normales

Comme nous venons de le mentionner, de nombreuses caractéristiques d'un champ de vecteurs peuvent être déduites de sa partie linéaire si celle-ci existe. Cependant, dans certains cas le champ de vecteurs linéarisé n'est pas conjugé au champs de vecteurs initial car il n'existe aucun difféomorphisme permettant de linéariser le champs de vecteurs. Même si cette linéarisation n'est pas possible nous pouvons trouver des difféomorphismes qui simplifient le champ de vecteurs considéré. Mais la définition de ce que "simple" veut

dire n'est pas unique. Nous dirons qu'un champ de vecteurs est "simple" lorsque sa partie linéaire peut être décrite par une matrice sous forme de Jordan et que sa partie non-linéaire ne contient que des termes résonnants. Un champs de vecteurs simplifié est appelé forme normale du champ de vecteur de départ.

Il existent de nombreuses approches aux formes normales. Nous allons utiliser la forme normale de Poincaré-Dulac car elle peut être intégrée en dimension deux (voir chapitre 7). Cette forme normale est due à H. Poincaré [47], H. Dulac [21] et G. Birkhoff [7]. Elle peut être calculée en utilisant la méthode de la représentation adjointe due à G. Iooss [35] ou plus efficacement en utilisant la théorie des transformations de Lie (voir K. T. Chen [40] et A. Deprit [3]). Récemment, beaucoup de rapports ont été publiés sur le sujet des formes normales. La première étape du calcul de la forme normale de Poincaré-Dulac est le calcul de la forme de Jordan de la matrice représentant la partie linéaire. A cause de la complexité de la représentation des nombres algébriques ceci n'est pas un problème trivial. La forme de Jordan peut être calculée à partir de la forme de Frobenius d'une matrice (voir I. Gil [29] et M. Griesbrecht [31]). Un algorithme de calcul de formes normales pour des systèmes avec une partie linéaire nilpotente a été proposé par R. Cushman et A. Sanders [16]. Les formes normales peuvent aussi être calculées en utilisant les linéarisations de Carleman (L. Stolovitch [56] et G. Chen [13]).

Résolution de singularités par éclatements

Nous ne pouvons plus appliquer le théorème de Poincaré-Dulac si la partie linéaire du système concerné est nilpotente. Dans ce cas nous allons utiliser des éclatements. Ces éclatements sont définis par des changements de variables qui déploient ou éclatent la singularité. Ce procédé nous donne un nombre fini de singularités plus simples. Des changements de variables qui permettent d'éclater une singularité sont par exemple l'introduction de coordonnées polaires ou certaines transformations quasi-monomiales.

Les éclatemets directionnels ont été introduits par O. Zariski [66] pour les courbes algébriques et par A. Seidenberg [53] pour les systèmes d'équations différentielles de dimension deux. Depuis, de nombreux travaux ont été réalisés sur ce sujet. Pour nos constructions nous allons utiliser les éclatements quasi-homogènes qui sont définis par des transformations quasi-momomiales et qui ont été introduits par A. Bruno [9]. Plus récemment, M.Brunella et M. Miari [8] ont travaillé dans ce domaine. Ils utilisent des transformations définies par des matrices unimodulaires et travaillent essentiellement en dimension 2. Nous étendons la construction des éclatements à l'utilisation de toute matrice inversible. Ceci est nécessaire car l'utilisation des matrices unimodulaires n'est pas appropriée à de nombreux problèmes en dimension supérieure. Nous pouvons utiliser ces matrices uniquement grâce aux résultats du chapitre 2 où nous démontrons que ces matrices définissent des difféomorphismes.

Deuxième partie: Les algorithmes

La classification

Les algorithmes que nous allons proposer réduisent et intègrent des systèmes dynamiques pour différents cas. Ces cas doivent être traités séparément. Donc, la première étape de

tous les calculs effectués est la classification des systèmes concernés. Nous allons distinguer le cas d'un point régulier, le cas d'un point singulier élémentaire et le cas d'un point singulier non élémentaire, ce qui nous permet de traiter les systèmes considerés avec des méthodes convenables.

Les points réguliers

Dans le voisinage d'un point régulier nous pouvons réduire chaque système à une forme très simple que nous appellerons "forme normale". Elle peut facilement être intégrée. Nous allons ramener le problème du calcul de cette forme normale au problème du calcul du flot d'un champ de vecteurs au voisinage d'un point régulier. Le changement de variables qui permet de mettre le système initial sous forme normale peut être calculé par des séries de Taylor. Cette méthode peut être trouvée dans de nombreuses références sur la théorie des transformations de Lie (par exemple dans W. Groebner [62], W. Groebner et H.Knapp [63] et P. Olver[46]).

Les points singuliers élémentaires en dimension deux

Nous appelons un point singulier "point singulier élémentaire" si la matrice jacobienne du système considéré est non nilpotente. Dans ce cas nous pouvons appliquer le théorème de Poincaré-Dulac pour calculer la forme normale de ce système. En deux dimensions toute forme normale peut être intégrée. Les systèmes réels dont la matrice jacobienne a des valeurs propres imaginaires pures représentent un cas particulier. Les solutions de ces systèmes sont des solutions périodiques. La meilleure façon de les représenter est d'introduire des coordonnées polaires. Ceci nous permet également d'obtenir des solutions réelles à condition que le système de départ soit réel.

Les points singuliers non élémentaires en dimension deux

Dans le cas d'un point singulier non élémentaire, nous allons utiliser des éclatements et des changements de temps pour réduire les systèmes concernés. Les calculs sont organisés d'après les faces du diagramme de Newton.

Pour chaque sommet un changement de temps permet d'obtenir un système non nilpotent dont le support est inclus dans un cône convexe. Comme les exposants du système obtenu sont négatifs nous devons appliquer une transformation quasi-monomiale qui nous fournit un système dont les exposants sont des entiers positifs. Ce système peut être traité comme un système au voisinage d'un point singulier élémentaire. Pour chaque arête du diagramme de Newton nous utilisons les éclatements, déjà introduit au chapitre 4. Ces éclatements nous fournissent plusieurs systèmes au lieu d'un seul système de départ. Ceux-ci peuvent être traités récursivement.

Ces méthodes fournissent de nombreuses solutions. Un point central de l'algorithme est donc de définir des domaines du voisinage étudié où ces solutions sont valables. Nous allons appeler ces domaines "secteurs" et les calculer de façon à ce qu'ils recouvrent entièrement un voisinage du point étudié.

Les points singuliers élémentaires en dimension n

Les algorithmes que nous décrivons dans ce chapitre n'ont jusqu'a présent été étudiés que d'une manière superficielle par A. Bruno [9]. Nous utilisons ses travaux comme point de départ pour proposer une étude plus complète. Dans une première étape la forme normale de Poincaré-Dulac du système concerné est calculée. Celle-ci peut être transformée en un système que nous pouvons diviser en un système de dimension réduite et un système intégrable. Néanmoins, la transformation quasi-monomiale utilisée dans ce but doit vérifier des conditions très strictes.

Pour les systèmes de dimension trois, nous proposons une classification des formes normales qui permet la réduction et l'intégration de toute forme normale. Les résultats obtenus grâce à une étude approfondie des systèmes en dimension trois peuvent être généralisés aux systèmes de dimension supérieure.

Cependant, si la dimension du système considéré est supérieure à trois, de nombreux problèmes peuvent apparaître à cause de la dimension croissante du plan résonnant. Si la dimension du plan résonnant dépasse deux, le cône contenant les exposants de la forme normale peut être défini par un nombre de vecteurs trop important et ces vecteurs peuvent avoir des coordonnées négatives. Pour résoudre ces problèmes, nous proposons une construction complémentaire, le diagramme de Newton virtuel. Elle permet de construire un ensemble de cônes qui possèdent une structure plus régulière. Cet ensemble de cônes définit des éclatements qui permettent d'obtenir des systèmes de dimension réduite et de couvrir entièrement le voisinage étudié par des secteurs.

Les points singuliers non élémentaires en dimension n

Les éclatements permettent également de traiter des systèmes nilpotents de dimension trois et supérieure. Néanmoins certains systèmes ne peuvent être entièrement réduits comme l'ont démontré X. Gomez-Mont et I. Luengo [30]. Seuls quelques cas précis somme les systèmes non dicritiques peuvent être réduits par cette méthode (voir F. Cano et D. Cerveau [10]). Pour ces systèmes nous proposons une construction qui utilise des éclatements controlées par le diagramme de Newton et son extension, le diagramme de Newton virtuel. Cette méthode permet de définir des secteurs qui couvrent entièrement le voisinage concerné. La définition des secteurs obtenus par cette méthode est différente de celle proposée par A. Bruno. Sa construction est fortement liée à la manipulation de cônes dans l'espace des exposants.

Troisième partie: Le logiciel

Les algorithmes décrits dans la partie précédente ont été implantés en MAPLE dans le package FRIDAY[2].

Le logiciel FRIDAY

Dans ce chapitre nous donnons une description du package FRIDAY et de nombreux exemples pour son utilisation. Ce package est organisé en modules correspondant à la

[2]FRIDAY est un acronyme pour **F**ormal **R**eduction and **I**ntegration of **D**ynamical **A**utonomous S**y**stems

classification des systèmes dynamiques. Cependant, l'utilisateur n'a accès qu'à certaines procédures de contrôle.

La procédure principale peut être utilisée pour l'intégration d'un système quelconque en dimension deux et pour de nombreux systèmes en trois dimensions. A part cette procédure ce sont surtout les modules calculant les formes normales qui peuvent être utilisés séparément.

Dans cette thèse nous avons montré comment des outils comme les formes normales et les transformations quasi-monomiales peuvent être utilisés pour la réduction et l'intégration formelle des systèmes dynamiques. Pour les systèmes en dimension deux et les systèmes non nilpotents en dimension trois les algorithmes proposés ont éte implantés et testés. Néanmoins, de nombreux problèmes théoriques et pratiques restent à résoudre, surtout dans le domaine des systèmes nilpotents.

Chapter 1

Formal Solutions for Dynamical Systems

This chapter introduces some definitions that are basic for the developement of the theory in the following chapters. Consider a system of autonomous differential equations of the form

$$\frac{\partial X}{\partial t} = F(X) \tag{1.1}$$

where $F = (f_1, f_2, \ldots, f_n)$, $f_i \in C^{\geq 1}(M, \mathbb{C}^n)$ and $X = (x_1, x_2, \ldots, x_n)$ are vectors of dimension n. The system is called autonomous because the right hand side of equation (1.1) does not depend on the independent variable t that usually stands for time. We are looking for solutions represented by the dependent variable $X(t)$.

The results and notations presented in this chapter have been subject of many publications. For example in the works from K. Alligood, T. Sauer and J. Yorke [39], F. Verhulst [26], J. Hale and H. Korcak [33], S. Chow and J. Hale [14], J. Guckenheimer and P. Holmes [36], D. Arrowsmith and C. Place [17] and J. Hubbard and B. West [34].

1.1 Flows, Vector Fields and Differential Equations

Let M be a convex open subset of $\mathbb{C}^n$ or $\mathbb{R}^n$.

Definition 1 (global flow) *A global flow on M is a continuously differentiable function $\phi : \mathbb{R} \times M \to M$ such that $\forall X \in M$*

1. $\phi(0, X) = X$
2. $\phi(t, \phi(s, X)) = \phi(t + s, X)$, $\forall t, s \in \mathbb{R}$.

The flow is called global because it is defined for all $t \in \mathbb{R}$. It can be related to differential equations by the definition of vector fields.

Definition 2 (vector field) *A vector field associated to a flow ϕ is a function $F : M \to \mathbb{R}^n$, $F \in C^0$, defined on the open subset M, that associates a vector in $\mathbb{R}^n$ to any point in M such that $\forall X \in M$:*

$$F(X) = \frac{d\phi}{dt}(t, X)|_{t=0} = lim_{\epsilon \to 0} \left\{ \frac{\phi(\epsilon, X) - \phi(0, X)}{\epsilon} \right\}.$$

$X(t) = \phi(t, X_0)$ is a solution of the initial value problem $\dot{X} = F(X),\ X(0) = X_0$. The existence and the uniqueness of a local flow representing a solution is guaranteed by the following theorem.

Theorem 1 (existence and uniqueness) *Let M be an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ and $F : M \to \mathbb{R}^n$ or $\mathbb{C}^n$ be a continuously differentiable map and let $X_0 \in M$. Then there is some constant $c > 0$ and a unique solution $X(t) = \phi(t, X_0) : (-c, c) \to M$ of the initial value problem*

$$\dot{X} = F(X),\ X(0) = X_0\,.$$

From definition 2 it follows that every flow corresponds to autonomous differential equation (1.1). The opposite is not true because in general the solutions of (1.1) can not be extended indefinitely in time. But for every autonomous differential equation there can be found local flows defined on a subset of $\mathbb{R} \times M$.

Definition 3 (local flow) *Let A be an open subset of $\mathbb{R} \times M$. A local flow is a continuously differentiable function $\phi : A \to M$ such that*

1. $\{0\} \times M \subset A \subset \mathbb{R} \times M$
2. $\forall X \in M : A \cap (\mathbb{R} \times \{X\})$ *is convex*
3. $\phi(0, X) = X$ *and* $\forall t, s \in \mathbb{R},\ \forall X \in M : \phi(t, \phi(s, X)) = \phi(s + t, X)$ *if this expression makes sense.*

As it can be deduced from above flows, vector fields and autonomous differential equations can be used to represent the same dynamics.

1.2 Linearization of vector fields

Solutions of differential equations or vector fields are particulary interesting in the neighbourhood of isolated singular points as the behaviour of the solutions can be quite complex there.

Definition 4 (singular point) *A point $X_0 \in M$ is called singular point of a vector field F if $F(X_0) = 0$. It is an isolated singular point if X_0 has a non-empty neighbourhood Ω such that $F(X) \neq 0\ \forall X \in \Omega - X_0$.*

A singular point is often called equilibrium point or singularity whereas all other points are called simple, regular or ordinary points.

Any system (1.1) can be linearized in a point X_0. That means that instead of studying the behaviour of the nonlinear system we can study the linear system

$$\frac{\partial X}{\partial t} = DF(X_0)X \tag{1.2}$$

where $DF(X_0)$ denotes the jacobian matrix of F in X_0. As linear systems are well known this is the simplest way to obtain information about a differential equation (1.1).

The point X_0 is called a nonelementary singular point if X_0 is a singular point and all eigenvalues of $DF(X_0)$ are zero. In this case the matrix $DF(X_0)$ and the linear system (1.2) are called nilpotent. If X_0 is a singular point but $DF(X_0)$ is non-nilpotent we call X_0 an elementary singular point of the system F.

For the calculation of solutions of differential equations an important question arises. The question is if the linearized system (1.2) and the corresponding non-linear system (1.1) have locally the same flow structure. Under precise conditions the Hartman-Grobman theorem gives a positive answer to this question.

Theorem 2 (The Hartman-Grobman theorem) *If none of the eigenvalues of the matrix $DF(X_0)$ has a zero real part then the vector field F is topologically conjugated to the linearized vector field $DF(X_0)X$ in a neighbourhood of X_0.*

The theorem gives no result for vector fields if the matrix $DF(X_0)$ has zero eigenvalues. In those cases certain nonlinear parts of the vector field have a determining role.

If a nonlinear system is topologically conjugated to a linear one there exists a homeomorphism that linearizes the nonlinear system. But homeomorphisms are not necessarily smooth and do therefore not preserve very well the qualitative behaviour of a concerned system. For this reason we have to use a stronger definition of conjugacy and equivalence that is based on diffeomorphisms.

1.3 Equivalence of vector fields

To study the qualitative behaviour of vector fields or systems of differential equations means to classify them into equivalence classes of similar behaviour and to describe the characteristics of those classes. This classification is done via diffeomorphisms. Systems that can be transformed into each other by diffeomorphisms are called equivalent.

Let M and V be open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 5 (diffeomorphisms) *Let M and V be open sets on E. A map $H : M \to V$ is a C^k-diffeomorphism if*

1. *H is of class C^k and*
2. *H is invertible and its inverse is also of class C^k.*

C^k-diffeomorphisms can be used to define equivalence or conjugacy relations.

Definition 6 (C^k equivalence of vector fields) *Let F and $\tilde{F}$ be two vector fields defined respectively on M and V, that are two open subsets of E. ϕ and ψ denote their local flows.*

F and $\tilde{F}$ are called locally C^k-equivalent if there exists a C^k-diffeomorphism $H : V \to M$ which takes the flow ψ of $\tilde{F}$ to the flow ϕ of F and preserves the orientation of the flows.

If in addition to this, the parametrization of the flows is preserved the systems are called C^k-conjugated.

Applying a diffeomorphism $H : X \mapsto H(X)$ to a differential equation (1.1) yields

$$DH(X)\,\dot{X} = F(H(X))$$

and a new vector field

$$\tilde{F} = DH^{-1}\,(F \circ H) \tag{1.3}$$

that is conjugated to the vector field F. Equation (1.3) defines the action of a diffeomorphism H on a vector field F. $\tilde{F}$ is often denoted $H * F$ and the operator $*$ is called pull-back. A diffeomorphism H such that the relation (1.3) is verified can be found if the vector fields F and $\tilde{F}$ are conjugated.

Example 2 (Translations) *As it is much simpler to study vector fields in the neighbourhood of the origin than in the neighbourhood of a point X_0 we will often use diffeomorphisms given by a translation*

$$H(X) = X + X_0\,.$$

H transforms any vector field F into the conjugated vector field

$$H * F = DH^{-1}(F \circ H)\,.$$

The vector field $\tilde{F}$ is similar to F except that X_0 has been transformed to the origin as $DH = id$.

Remark 1 *In the Hartman-Grobman theorem the notation of topological conjugacy is used. Topologically conjugated means C^0-conjugated and refers to C^0-diffeomorphisms that are homeomorphisms.*

Transformations defined by diffeomorphisms are the main tool for the reduction of vector fields. All normalizing transformations and power transformations that are used in the following are diffeomorphisms.

1.4 Time transformations

A dynamical system describes the one-parameter evolution of several dynamic variables. The parameter in which they evolve is called time t. At certain steps of the proposed algorithms we will apply transformations to t. These transformations are called time transformations. In contrast to previously introduced transformations by C^k-diffeomorphisms time transformations do not preserve the orientation of flows. The resulting vector fields are therefore not equivalent to the initial vector fields.

For some vector fields there exists a factorization

$$F(X) = h(X)\tilde{F}(X)$$

with the factor $h(X)$. It is much simpler to study the vector field $\tilde{F}$ that results from a division of F by h instead of studying F. $\tilde{F}$ can be seen as a result of an application of a

time transformation to F. If we substitute $h(X)\partial t$ by $\partial\tilde{t}$ the differential equation (1.1) is transformed to a new equation

$$\frac{\partial X}{\partial \tilde{t}} = \tilde{F}(X)\,. \tag{1.4}$$

The change of variables $\partial\tilde{t} = h(X)\partial t$ is equivalent to a change of variables

$$\tilde{t} = \gamma(t)$$

where γ is the solution of the differential equation

$$\frac{\partial \tilde{t}}{\partial t} = h(X)\,. \tag{1.5}$$

The solution $X(t)$ of the initial system can easily be calculated from the solutions $X(\tilde{t})$ of equation (1.4) if γ is known.

Example 3 (Time transformations) *Consider the 2 dimensional system*

$$\begin{cases} \frac{dx}{dt} = x^2 \\ \frac{dy}{dt} = xy \end{cases} \tag{1.6}$$

that can be transformed to the system

$$\begin{cases} \frac{dx}{d\tilde{t}} = x \\ \frac{dy}{d\tilde{t}} = y \end{cases} \tag{1.7}$$

by the time change given by $\partial\tilde{t} = h\partial t$ with $h = x$. The solutions of (1.7) are given by $X(\tilde{t}) = (a\,e^{\tilde{t}}, b\,e^{\tilde{t}})$. Solving (1.5) yields

$$\gamma(t) = -ln(-a(t+c))$$

and the solution of the system (1.6) is

$$\left(\frac{-1}{t+c}, \frac{-b}{a}\frac{1}{t+c}\right).$$

Observe that the behaviour of the solutions $(x(t), y(t))$ for $t \to \infty, -c, -\infty$ is different from the behaviour of $(x(\tilde{t}), y(\tilde{t}))$ for $\tilde{t} \to \infty, -c, -\infty$. As a consequence the deduction of some qualitative characteristics such as stability of the solutions of F from the behaviour of $X(\tilde{t})$ is impossible if γ is not known.

The time change $\tilde{t} = \gamma(t)$ alters the parametrization of the solution curves and it is easy to see that the orientation of the flow is not always preserved. F and $\tilde{F}$ are therefore neither equivalent nor conjugated. The only common characteristic between the solutions of F and $\tilde{F}$ is that they are different parametrizations of the same curve.

However we can define an equivalence relation between the vector field F and the vector field $\tilde{F}$ in the case of Hamiltonian vector fields and if we treat two-dimensional systems. Those equivalence relations are important for the verification of computed solution curves.

For two dimensional systems calculating solutions $X(t)$ for

$$\frac{\partial X}{\partial t} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \tag{1.8}$$

can be reduced to the problem of finding parametrized solutions for the scalar differential equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}. \tag{1.9}$$

This equation can be parametrized as in equation (1.8) or as

$$\frac{\partial X}{\partial \tilde{t}} = \begin{pmatrix} \frac{f(x,y)}{h(x,y)} \\ \frac{g(x,y)}{h(x,y)} \end{pmatrix} \tag{1.10}$$

which is equivalent to the application of a time change with $\partial\tilde{t} = h(x,y)\partial t$ to equation (1.8). We can therefore say that the systems (1.8) and (1.10) are equivalent in the sense that their solutions are both parametrized solution curves for the scalar equation (1.9).

Another equivalence relation between F and $\tilde{F}$ can be given if we consider Hamiltonian systems of differential equations. A Hamiltonian system is characterized by its energy function H. The dependent variables are given by $X = (Y, Z)$ where Y and Z are vectors. A Hamiltonian system is given by

$$\begin{cases} \dot{Y} = \frac{\partial H}{\partial Z} \\ \dot{Z} = -\frac{\partial H}{\partial Y} \end{cases}. \tag{1.11}$$

A major property of Hamiltonian systems is that they own first integrals.

Definition 7 (first integral) *let U be open and nonempty. A real valued map*

$$\Phi : U \to \mathbb{R};\ X \mapsto \Phi(X),\ \Phi \in C^1$$

that is not constant on any open subset of $\mathbb{R}^n$ is called a first integral of a differential equation (1.1) if the function Φ is constant along any solution $X(t)$ with initial value $X(0) = X_0$.

$$\Phi(X(t)) = \Phi(X_0) \tag{1.12}$$

for all t for which $X(t)$ is defined.

It is obvious that the Energy function H is a first integral of the Hamiltonian system (1.11) as

$$\frac{\partial H(Y(t), Z(t))}{\partial t} = 0$$

for any $(Y(t), Z(t))$ solving equation (1.11).

Applying a time change to a Hamiltonian vector field yields a vector field that is equivalent to the initial one in the sense that it has the same first integrals. This is due to the fact that the condition (1.12) does not depend on the parametrization of the curve $X(t)$. Most systems of differential equations however do not possess first integrals. For this reason this conclusion can not be generalized to non-Hamiltonian systems.

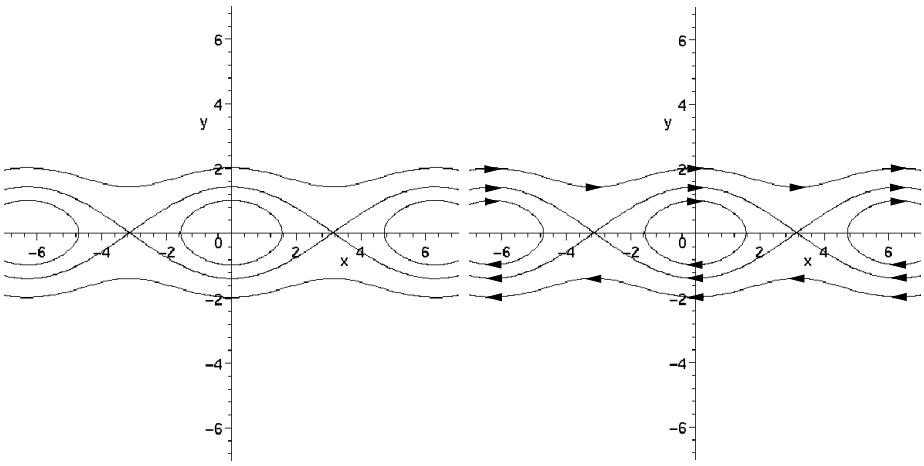


Figure 1.1: The level curves of H and the solution curves of the dynamical system (1.13) are identic but the level curves have no direction. See example 4.

Example 4 (pendulum) *The pendulum equation is either given by the second order scalar equation*

$$\ddot{x} + \frac{g}{l} sin(x) = 0$$

or by the first order system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} sin(x) \end{cases} . \tag{1.13}$$

The points $(k\pi, 0)$ with $k \in \mathbb{Z}$ are singularities of (1.13). In chapter 7 we will approximate solution curves in the neighbourhood or these singularities.

The total energy of the system (kinetic plus potential energy) is given by

$$H(x, y) = (1/2)ml^2y^2 + mgl(1 - cos(x)) .$$

H is constant along the solutions $(x(t), y(t))$ of the system and along any parametrization $(x(\tilde{t}), y(\tilde{t}))$ with $\tilde{t} = \gamma(t)$ of those curves. In contrast to the real solutions $(x(t), y(t))$ it makes no sense to give a direction to the curves $(x(\tilde{t}), y(\tilde{t}))$, that denote the lines where the total energy of the system is constant, as the sense of parametrization might have been inversed by the time transformation. The curves $(x(\tilde{t}), y(\tilde{t}))$ are called level curves of H. Level curves and the solutions of equation (1.13) are sketched in figure 1.1 for $g/l = 1/2$.

Time transformations will be extensively used in the following. Therefore all calculated solution curves can only be interpreted as level curves of the energy function if we treat Hamiltonian systems or parametrized solutions of the associated scalar differential equations if we deal with 2-dimensional systems. Those properties are used to verify computed solution curves. That also means that for 3 dimensional systems the verification of calculated results is much more difficult.

1.5 Convergence and Formal Solutions

Normal form calculations, that will be introduced in chapter 3, often yield diverging series. Therefore the ring of convergent power series $k\{X\}$ is not sufficient for our calculations. The following calculations and considerations are purely formal. We work in the ring of formal power series $k[[X]]$, that extends the ring of polynomials $k[X]$, as it also admits infinite sums without presuming that they are converging.

The aspects of convergence will not be considered here though conditions for the convergence of normal form transformations have been given by H. Poincaré [47], H. Dulac [21], C. Siegel [54], A. Bruno and S. Walcher [2] and others.

We will mainly work with systems of first order differential equation of dimension 2 and 3. First order means that only the first derivative occurs in the equation. However higher order differential equations can always be transformed to a system of order one. As we will see with some restrictions the algorithm can also be used for problems that admit parameters. The systems of dimension two are very well known due to a large number of publications in this domain. Three dimensional problems have so far not been studied extensively using the approach proposed here. So far there didn't exist any programms for integrating a large number of 3 dimensional systems of differential equations.

Chapter 2

The Newton diagram

In this chapter power transformations are introduced. Those transformations act on the exponents of the monomials of a given system

$$\dot{X} = F(X) \ . \tag{2.1}$$

Their effect on the exponents of a given system can be interpreted geometrically for a better illustration of the action of those transformations. Further those geometric aspects can be used to find power transformations that manipulate the exponents of a system in an appropriate way.

Therefore some geometric notations as the Newton diagram and the support of a system are needed. In chapter 4 we will use the Newton diagram to calculate matrices that define power transformations as it has been done by A.Bruno [9] and more recently by M. Brunella and M. Miari [8].

Section 2.2 is closely related to the work of A. Bruno [1] who states theorem 3. In his approach he concentrates on the use of unimodular matrices. However unimodular matrices are not sufficient to solve all problems concerning the integration of two- and higher-dimensional systems as it will be shown in the chapters 8 and 9. For this reason we extend the definition of power transformations to the use of any invertible matrix as they have also been studied by L. Brenig and A. Goriely [23]. As some of those transformations are not injective and therefore no diffeomorphism we will introduce some complementary methods to make them bijective.

Another important role in the geometric study of differential equations is played by cones as some systems can have their support within a cone. That is why in section 2.4 we introduce some basic notations about cones that can for example be found in A. Goldman and A. Tucker [4].

2.1 The support and the Newton diagram

Power transformations are applied to a system in order to integrate it in the case of an elementary singular point in section 7.1 or in order to simplify it by blowing-ups that are the subject of chapter 4. To study the geometrical aspects of power transformations we will work in the space of exponents (which is subset of $\mathbb{Z}^n$) where the support and the

Newton diagram for any dynamical system are defined. To simplify the representation the notations

$$X = (x_1, x_2, \ldots, x_n)\,,$$

$$Q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n\,,$$

$$X^Q = (x_1^{q_1}, x_2^{q_2}, \ldots, x_n^{q_n})\,,$$

$$A = (a_{ij}) \in \mathcal{M}_n(\mathbb{Z})\,,$$

$$X^A = \begin{pmatrix} x_1^{a_{11}} x_2^{a_{12}} \ldots x_n^{a_{1n}} \\ \ldots \\ x_1^{a_{n1}} x_2^{a_{n2}} \ldots x_n^{a_{nn}} \end{pmatrix},$$

$$F(X) = \begin{pmatrix} x_1 \sum_{Q \in \mathcal{N}_1} \alpha_{Q1} X^Q \\ \ldots \\ x_n \sum_{Q \in \mathcal{N}_n} \alpha_{Qn} X^Q \end{pmatrix},\ \alpha_{Qi} \in \mathbb{R}\ or\ \mathbb{C}$$

where the sets $\mathcal{N}_i$ are defined as

$$\mathcal{N}_i = \{Q \in \mathbb{N}^{i-1} \times \mathbb{N} \cup \{-1\} \times \mathbb{N}^{n-i} : \sum q_i \geq 0\}$$

will be used. Further the set $\mathcal{N}$ defined as

$$\mathcal{N} = \bigcup_{i=1}^{n} \mathcal{N}_i$$

will frequently appear. Based on the above notations the support and the Newton diagram for F are defined as follows :

Definition 8 (Support) *The set*

$$supp(F) := \bigcup_{i=1}^{n} \{Q \in \mathcal{N}_i : \alpha_{Qi} \neq 0\} \subset \mathbb{Z}^n$$

is called the support of the system (2.1).

The definition of the support allows to construct the set

$$\Omega = \bigcup_{Q \in supp(F)} \{Q + P : P \in \mathbb{R}_+^n\}$$

that is used to define the Newton diagram.

Definition 9 (Newton diagram) *The lower left part of the convex hull of the set Ω is called the Newton diagram of F. It does not contain horizontal or vertical faces. It is denoted by $\Gamma(F)$.*

$\Gamma(F)$ consists of a finite number of j-dimensional faces that are denoted by $\Gamma_i^{(j)}$. The faces $\Gamma_i^{(1)}$ and $\Gamma_i^{(0)}$ are called edges and vertices of the Newton diagram. The faces of the Newton diagram will be used to compute matrices defining quasihomogeneous blowing-ups in chapter 4.

The Newton polygon which is almost identical to the Newton diagram can also be used in a different way. It can be applied for the direct calculation of solutions of algebraic or differential equations (see for example J. Della Dora and F. Jung [19], F. Beringer and F. Jung [6] and J. Cano [37]).

Example 5 *Consider the system*

$$\dot{X} = \begin{pmatrix} -x^4 + yx^3 \\ \frac{13}{9}y^2x^6 - x^2y^2 + xy^3 \end{pmatrix}$$

which has a nonelementary singularity in the origin. Its support consists of the points

$$supp(F) = \{(3,0),(2,1),(6,1),(1,2)\}$$

and its Newton diagram is shown in figure 2.1.

2.2 Power transformations

Power transformations are defined as

$$\begin{cases} k^n \to k^n, \ k = \mathbb{R} \text{ or } k = \mathbb{C} \\ X \mapsto X^{A^T}. \end{cases} \tag{2.2}$$

A^T denotes the transposed of an invertible matrix $A \in Gl_n(\mathbb{Z})$ with integer coefficients. We use those transformations as coordinate change $X = \tilde{X}^{A^T}$.

The effect of the coordinate change on the exponents of the system is described by the following theorem.

Theorem 3 *A change of coordinates $X = \tilde{X}^{A^T}$ applied to a system (2.1) induces an affine transformation $\tilde{Q} = AQ$ on the points $Q \in supp(F)$.*

Proof 1 *To prove this theorem we will study the system*

$$\frac{\partial \log X}{\partial t} = G(X) = (\frac{f_1}{x_1}, \ldots, \frac{f_n}{x_n})$$

that is equivalent to (2.1). Under the change of coordinates $X = \tilde{X}^{A^T}$ *we obtain*

$$\begin{aligned}
\frac{\partial \log \tilde{X}^{A^T}}{\partial t} &= \begin{pmatrix} \sum \frac{\partial log \tilde{X}^{(a_{11},a_{21},\ldots,a_{n1})}}{\partial \tilde{x}_i} \frac{\partial \tilde{x}_i}{\partial t} \\ \cdots \\ \sum \frac{\partial log \tilde{X}^{(a_{1n},a_{2n},\ldots,a_{nn})}}{\partial \tilde{x}_i} \frac{\partial \tilde{x}_i}{\partial t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\tilde{X}^{(a_{11},\ldots,a_{n1})}} \left(a_{11} X^{(a_{11}-1,a_{21},\ldots,a_{n1})} \frac{\partial \tilde{x}_1}{\partial t} + \ldots \right) \\ \cdots \\ \frac{1}{\tilde{X}^{(a_{1n},\ldots,a_{nn})}} \left(a_{1n} X^{(a_{1n}-1,a_{2n},\ldots,a_{nn})} \frac{\partial \tilde{x}_1}{\partial t} + \ldots \right) \end{pmatrix} \\
&= \begin{pmatrix} (\frac{a_{11}}{\tilde{x}_1}, \ldots, \frac{a_{n1}}{\tilde{x}_n}) \frac{\partial \tilde{X}}{\partial t} \\ \vdots \\ (\frac{a_{1n}}{\tilde{x}_1}, \ldots, \frac{a_{nn}}{\tilde{x}_n}) \frac{\partial \tilde{X}}{\partial t} \end{pmatrix} \\
&= \begin{pmatrix} (a_{11}, a_{21}, \ldots, a_{n1}) \frac{\partial log \tilde{X}}{\partial t} \\ \cdots \\ (a_{1n}, a_{2n}, \ldots, a_{nn}) \frac{\partial log \tilde{X}}{\partial t} \end{pmatrix} \\
&= A^T \frac{\partial log \tilde{X}}{\partial t}
\end{aligned}$$

and the new system

$$\frac{\partial \log \tilde{X}}{\partial t} = A^{-T} G(\tilde{X}^{A^T}).$$

A simple calculation shows that each monomial X^Q *in* $G(X)$ *is transformed to a monomial* $\tilde{X}^{AQ}$ *belonging to* $G(\tilde{X}^{A^T})$.

Theorem 3 is used to calculate appropriate matrices for power transformations. Finding a matrix A that handles the support or the Newton diagram in a suitable way makes sure that exponents of the system transformed by the power transformation $X = \tilde{X}^{A^T}$ have the required caracteristics.

2.3 Power transformations as diffeomorphisms

The application of power transformations in order to transform a vector field implies that the coordinate change must be a diffeomorphism. However most power transformations are not injective in any neighbourhood U of the origin in $\mathbb{R}^n$ or $\mathbb{C}^n$ but we will show that they are in some regions of U. If the power transformations are defined on those regions instead of U their use is allowed. The methods employed to find those regions are different in $\mathbb{C}^n$ and $\mathbb{R}^n$. For this reason both cases are treated separately.

For the following considerations we work in the set $U' = U - \bigcup_i \{X | x_i = 0\}$ where none of the components of X is zero. The sets $\{X | x_i = 0\}$ will be considered in chapter 4. They play the role of exceptional divisors for a special kind of power transformations that are called blowing-ups.

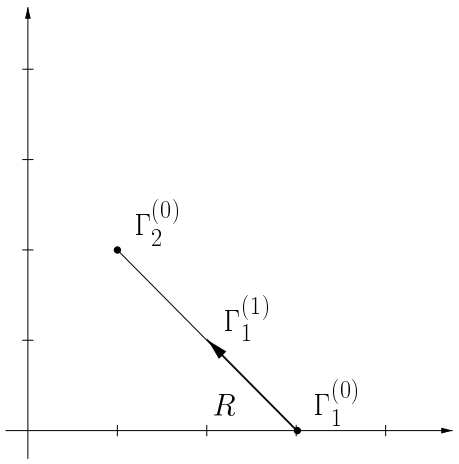


Figure 2.1: The Newton diagram for the system treated in example 5.

2.3.1 Power transformations in $\mathbb{R}^n$

In a first step it will be shown that X^{A^T} is injective within the set $\{X \in U' : x_1, \ldots, x_n > 0\}$. Suppose that $X_0^{A^T} = X_1^{A^T}$. This can also be written as a system of equations

$$\begin{aligned}
x_{01}^{a_{11}} x_{02}^{a_{21}} \ldots x_{0n}^{a_{n1}} &= x_{11}^{a_{11}} x_{12}^{a_{21}} \ldots x_{1n}^{a_{n1}} \\
x_{01}^{a_{12}} x_{02}^{a_{22}} \ldots x_{0n}^{a_{n2}} &= x_{11}^{a_{12}} x_{12}^{a_{22}} \ldots x_{1n}^{a_{n2}} \\
&\ldots \\
x_{01}^{a_{1n}} x_{02}^{a_{2n}} \ldots x_{0n}^{a_{nn}} &= x_{11}^{a_{1n}} x_{1n}^{a_{2n}} \ldots x_{1n}^{a_{nn}} .
\end{aligned}$$

Exponentiating the k-th lines with a_{11} and dividing them by the first line exponentiated by a_{1k} leads to a new equation system

$$\begin{aligned}
x_{01}^{a_{11}} x_{02}^{a_{21}} \ldots x_{0n}^{a_{n1}} &= x_{11}^{a_{11}} x_{12}^{a_{21}} \ldots x_{1n}^{a_{n1}} \\
1 \cdot x_{02}^{\tilde{a}_{22}} \ldots x_{0n}^{\tilde{a}_{n2}} &= 1 \cdot x_{12}^{\tilde{a}_{22}} \ldots x_{1n}^{\tilde{a}_{n2}} \\
&\ldots \\
1 \cdot x_{02}^{\tilde{a}_{2n}} \ldots x_{0n}^{\tilde{a}_{nn}} &= 1 \cdot x_{1n}^{\tilde{a}_{2n}} \ldots x_{1n}^{\tilde{a}_{nn}} .
\end{aligned}$$

Further exponentiations and divisions (that are allowed as $x_{1i}, x_{0i} \neq 0$ for all $i \in \{1, \ldots, n\}$) leads to

$$X_0^R = X_1^R$$

where R is an upper triangular matrix. Because $x_{0i} > 0$ and $x_{1i} > 0$ for all $i \in \{1, \ldots, n\}$ we can conclude that $X_0 = X_1$.

This result can be generalized to any of the 2^n quadrants

$$\{X \in U' : x_{i_1}, \ldots, x_{i_l} > 0,\ x_{i_{l+1}}, \ldots, x_{i_n} < 0\},\ i_1, \ldots, i_n \in \{1, \ldots, n\},\ i_k = i_j \Leftrightarrow k = j$$

of U' but it is not necessarily true for the whole set U'. In this case a subset of U' can be found such that X^{A^T} is injective within this subset. This subset will be denoted by $\hat{U}$.

To determine $\hat{U}$ the neighbourhood is split into 2^n quadrants and a set of those quadrants is chosen such that X^{A^T} is injective within this set. This is closer illustrated in the following algorithm :

Algorithm 1
compute_$\hat{U}(A)$
input: the matrix A
output: *results* that is a list caracterizing the region of injectivity of X^{A^T} and *image* that is a list that allows to define a piecewise surjective transformation
begin
list = list of all $e = (e_1, \ldots, e_n),\ e_i \in \{0, 1\}$
results = empty list
image = empty list
for $i = 1, \ldots, 2^n$ **do**
$a[i] := (list[i])^{A^T}$
for $j = 1, \ldots, 2^n$ **do**
if $a[i] = list[j]$ *and* $j \notin$ *image* **then**
add i to results
add j to image
return*(results,image)*
end.

If a change of coordinates characterized by a power transformation is used for our problems the complete initial neighbourhood has to be covered by the image of $X^{A^T} : \hat{U} \to \mathbb{R}^n$. This is true if the transformation is surjective.

In $\mathbb{R}^n$ power transformations are usually not surjective if $U' \neq \hat{U}$. This problem can be solved by using X^{A^T} and the transformations defined by

$$X \mapsto \begin{pmatrix} (-1)^{\alpha_{j1}} X^{A_1} \\ \vdots \\ (-1)^{\alpha_{jn}} X^{A_n} \end{pmatrix} \tag{2.3}$$

with vectors $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ such that $\alpha_{ji} \in \{1, 0\}$. The vectors A_j denote the row vectors of A^T. Choosing appropriate transformations (2.3) defined on the same set $\hat{U}$ as X^{A^T} yields a set of transformations that is "piecewise" surjective. The vectors α_j can be deduced from the list *image* computed by algorithm 1.

Example 6 *Consider the transformation* X^{A^T} *with*

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 0 & 0 & 1 \\ 4 & -2 & -2 \end{pmatrix}$$

and $\det A = 4$. *The list* result *returned by algorithm 1 contains 2 entries that allows to construct a set*

$$\hat{U} = \{X : y > 0,\ z > 0\}\,.$$

that contains only two quadrants. $X^{A^T} : \hat{U} \to \mathbb{R}^n$ *is injective.*

The list image *allows to conclude that the image of the power transformation* X^{A^T} *is the set* $\{X : x > 0,\ y > 0\}$*. That means we also have to use the 3 transformations*

$$X \mapsto \begin{pmatrix} (-1)^{\alpha_{j1}} X^{(-2,0,4)} \\ (-1)^{\alpha_{j2}} X^{(2,0,-2)} \\ (-1)^{\alpha_{j3}} X^{(1,1,-2)} \end{pmatrix}$$

with

$$\alpha_j = (\alpha_{j1}, \alpha_{j2}, \alpha_{j3}) \in \{(1,0,0), (0,1,0), (1,1,0)\}$$

that are all defined on $\hat{U}$*, to construct a piecewise surjective transformation.*

2.3.2 Power transformations in $\mathbb{C}^n$

Handling the case of transformations defined in $\mathbb{C}^n$ is less complicated as $\hat{U}$ does not consist of a set of quadrants and any injective transformation is surjective.

Suppose that X^{A^T} is not injective. That means there exist $X_0 \neq X_1$ such that

$$X_0^{A^T} = X_1^{A^T}. \tag{2.4}$$

X_j can be written in trigonometric form

$$X_j = \begin{pmatrix} r_{j1} e^{i\alpha_{j1}} \\ \vdots \\ r_{jn} e^{i\alpha_{jn}} \end{pmatrix}$$

with $r_{jk} > 0$ and $\alpha_{jk} \in [0, 2\pi[$. With $R_j = (r_{j1}, \ldots, r_{jn})$, $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn})$ and $k = (k_1, \ldots, k_n)$, $k_j \in \mathbb{Z}$ the condition (2.4) holds if the $2n$ conditions

$$\begin{aligned} R_0^{A^T} &= R_1^{A^T} \\ A^T \alpha_0 &= A^T \alpha_1 + k\, 2\pi \end{aligned} \tag{2.5}$$

are verified. That means that the arguments modulo 2π have to be equal. As $r_{jk} > 0$ the first n conditions are verified only if $R_0 = R_1$ so we can focus on the remaining conditions on the arguments. The second condition in equation (2.5) can be multiplied by $A^{-T} = \frac{1}{detA} A^*$ where A^* denotes the adjoint matrix to A^T. All entries of A^T are integer. That yields

$$\alpha_0 - \alpha_1 = \frac{2\pi}{detA} A^* k. \tag{2.6}$$

Any power transformation $X^{A^T} : \hat{U} \to \mathbb{C}^n$ is injective for $\hat{U} = U_i'$ with

$$U_i' = \{\tilde{X} \in U' : arg(\tilde{x}_i) < \frac{2\pi}{detA}\}$$

for $i \in \{1, \ldots n\}$.

On U_i' the inverse to X^{A^T} exists. It is given by $X^{A^{-T}}$.

When applying a change of coordinates characterized by a power transformation all of the initial coordinates has to be covered by the image of X^{A^T}. This is true if the transformation is surjective.

We will show that for any $X \in U'$ there exists a $\tilde{X} \in U_i'$ such that $X = \tilde{X}^{A^T}$. Writing X in trigonometric form

$$X = \begin{pmatrix} r_1 e^{i\alpha_1} \\ \vdots \\ r_n e^{i\alpha_n} \end{pmatrix}$$

and applying the inverse transformation $\tilde{X} = X^{A^{-T}}$ yields

$$\tilde{X} = R^{A^{-T}} e^{iA^{-T}\alpha} .$$

If $\tilde{X} \in U_i'$ the $2n$ conditions

$$\begin{array}{rcl} (R^{A^{-T}})_i & > & 0 \\ (A^{-T}\alpha)_j \, mod \, 2\pi & \in & [0, 2\pi[\forall j \neq i \\ (A^{-T}\alpha)_i \, mod \, 2\pi & \in & [0, \frac{2\pi}{detA}[\end{array}$$

have to be verified. $(.)_j$ denotes the j-th component of a vector. The first two conditions are always true so only the last condition has to be proved.

Without loss of generality let $i = 1$. Suppose that $(A^{-T}\alpha)_1 \notin [0, \frac{2\pi}{detA}[$. Equation (2.6) will be used to find another $\tilde{X} \in U'$ such that $X = \tilde{X}^{A^T}$. For any $l \in \mathbb{Z}$ there exists a k' such that $(A^* k')_1 = l$ and an $l \in Z$ such that

$$\left((A^{-T}\alpha)_1 + \frac{2\pi}{detA} l \right) \, mod \, 2\pi \; \in [0, \frac{2\pi}{detA}[\; .$$

Defining

$$\beta = A^{-T}\alpha + \frac{2\pi}{detA} A^* k'$$

we can easily verify that $A^T\beta = \alpha$. It has been proved that the transformation X^{A^T} is bijective on the set U'.

Example 7 *The power transformation from example 6 is bijective if it is defined as* $X^{A^T} : U_1' \to U'$ *with*

$$U_1' = \{X : arg(x) \, mod \, 2\pi \in [0, \frac{\pi}{2}[\} \; .$$

Remark 2 *Power transformations defined by unimodular matrices are bijective. For this reason their use for defining power transformations is advantageous in many cases.*

For the power transformations used in the following it can be assumed that they are bijective on U' or that they have been defined on a subset $\hat{U}$ of U' such that they are bijective. This section has shown that with some additional constructions any power transformation is a diffeomorphism and can therefore be used to transform vector fields.

2.4 Cones

In the following a certain type of vector fields will play an important role: vector fields having their support within a cone V in the space of exponents. Those systems are often refered to as class $\mathcal{V}$-systems. Time- and power transformation that are used to transform a considered system (2.1) to an equivalent system can also be interpreted as manipulations of the cone V that contains $supp(F)$. Further cones can be used to define transformations that manipulate a given system in an appropriate way.

A set $V \subset \mathbb{R}^n$ is called a cone if along with the point P it contains any point αP with $\alpha \geq 0$, $\alpha \in \mathbb{R}$. Any cone can be defined by a set of vectors $Q_0, Q_1, \ldots, Q_m$ as the set

$$V = \{Q : Q = Q_0 + \sum_{i=1}^{m} \alpha_i Q_i,\ \alpha_i \geq 0,\ \alpha_i \in \mathbb{R}\}\ . \tag{2.7}$$

A cone is called convex if it is a convex set. All cones considered in the following are convex cones. A cone V is called degenerate if it contains an entire line αP with $\alpha \in \mathbb{R}$.

Consider a time transformation defined by

$$d\tilde{t} = X^{Q_0} dt$$

that transforms the initial system (2.1) to a system

$$\frac{\partial X}{\partial \tilde{t}} = \frac{1}{X^{Q_0}} F(X)\ .$$

It is obvious that each point $Q \in supp(F)$ is translated to the point $\tilde{Q} = Q - Q_0$. With the points of the support of F the time transformation also tranlates the Newton diagram of F or any cone V.

The effect of power transformations on cones is more complex. Consider a convex degenerate or non-degenerate cone V that is defined as in equation (2.7) by the vector Q_0 and the linearly independent vectors $Q_1, \ldots, Q_m \in \mathbb{Z}^n$ where of course $m \leq n$. A change of coordinates $X = \tilde{X}^{A^T}$ can be defined via the inverse of the matrix A. Using the vectors $Q_1, \ldots, Q_m$ as the first row vectors of A^{-1} yields the matrix

$$A^{-1} = (Q_1 | \ldots | Q_m | \ldots)$$

that is completed to a $n \times n$ matrix such that A^{-1} is invertible.

According to theorem 3 the coordinate change $X = \tilde{X}^{A^T}$ transforms all vectors Q_i, $i = 1, \ldots, m$ to

$$\tilde{Q}_i = AQ_i = det A\, e_i\ .$$

Therefore the cone V has been transformed to a cone

$$\tilde{V} = \{Q : Q = AQ_0 + \sum \tilde{\alpha}_i e_i, \tilde{\alpha}_i \geq 0, \tilde{\alpha}_i \in \mathbb{R}\}$$

that is defined by the vectors $e_1, \ldots, e_m$ and AQ_0.

However, the matrix A^{-1} might not be unimodular and A might therefore have fractional coefficients. For this reason instead of the inverse matrix the adjoint matrix

$$A^* = det A\, A^{-1} = (Q_1 | \ldots | Q_m | \ldots)$$

is used to define the power transformation. The matrix A is not uniquely defined by A^* and can therefore be chosen such that it has only integer coefficients. The matrix A is computed in the following way: The adjoint matrix B^* of any matrix B is defined as $B^* = detB\, B^{-1}$. Now define $A = \frac{k}{detB}B$ with $k \in \mathbb{N}$ such that $\forall i,j : a_{ij} \in \mathbb{Z}$ and $gcd(a_{11}, \ldots, a_{nn}) = 1$ for the coefficients a_{ij}, $i,j = 1, \ldots, n$ of the matrix A.

A is the matrix used for the power transformation and B^* is its adjoint as $detA = k$ yields

$$A^* = k\, A^{-1} = det\, B\, B^{-1} = B^* \ .$$

So for $A^* = B^*$ the matrix A is the appropriate matrix for the power transformation.

Chapter 3

Normal Forms

The theory of normal forms is due to H. Poincaré [47] who introduced qualitative methods in the study of solutions of ordinary differential equations. To study the qualitative behaviour of a systems of differential equations

$$\dot{X} = F(X) \tag{3.1}$$

means to classify them locally into equivalence classes of similar behaviour. The classification is performed by formal diffeomorphisms. For systems with non-vanishing linear part the corresponding classes can be represented by a set of elements that are said to be in normal form. These elements are the "simplest" elements of their class. In general these representative elements are not unique and their choice depend on the definition of what "simplest" means.

In the following chapters the Poincare-Dulac normal form is used. Here "simplest" means that the matrix characterizing the linear part of the system is in Jordan form and the nonlinear part of the system contains as few terms as possible.

The computation of the Jordan form is a difficult problem for higher-dimensional matrices. However it represents the first step in the computation of the Poincaré-Dulac normal form. The reduction of the non-linear terms is performed step by step for terms of increasing degree. The computations yield a normal form and the formal diffeomorphism that is used to normalize the considered system (3.1). There exist many approaches for the computation of normal forms. In this chapter we will focus on the construction of the Poincaré-Dulac normal form using Lie transformations and the matrix representation method.

Normal forms have been the subject of many publications. The basic results of H. Poincaré [47] have been extended by H. Dulac [21] and G. Birkhoff [7]. Lie theory has been introduced to normal form theory by K.T. Chen [40] and W. Groebner [62]. The concerning algorithms have been optimized by A. Deprit [3]. Recently normal form theory has rapidly developped since it is essential in bifurcation theory. See for example the works of A. Bruno [9], J. Guckenheimer and P. Holmes [36], S. Walcher [65], S. Ushiki [58], F. Takens [25], Shui-Nee Chow, Chengzhi Li and Duo Wang [15], G. Gaeta [27], S. Louies and L. Brenig [50], L. Vallier [60], G. Iooss and M. Adelmeyer [35] and F. Zinoun [67].

There exist other algorithms for normal form computations that are however not considered here. For example the computation of normal forms by Carleman linearizations (J.

Della Dora and L. Stolovitch [20] and G. Chen [13]) and the computation of normal forms for systems with nilpotent linear parts (R. Cushman and J. Sanders [16]). The problem of computing the Jordan form can be avoided by using the Frobenius form of a matrix (G. Chen [13]).

The components of the map F in equation (3.1) are considered to be formal power series. Therefore the only possible singularity is $X = 0$. As a consequence we will suppose that $F(0) = 0$. All transformations are considered to be formal power series.

3.1 The Poincaré-Dulac normal form

The basic theory of normal forms is due to H. Poincaré [47]. He stated that systems of the form (3.1) are formally equivalent to their linearized system if the eigenvalues of the concerned system are non-resonant. In this context resonance is defined as a relation between the points $Q \in \mathcal{N}$ that can appear in the support of the vector field F and the eigenvalues of the matrix $DF(0)$. The set $\mathcal{N}$ defines the set of all points that can appear in the support of a vector field F. It has already been defined in section 2.1.

Definition 10 (resonances) *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $DF(0)$. They verify a resonance condition if*

$$\exists Q = (q_1, \ldots, q_n) \in \mathcal{N} : \ \langle Q, \Lambda \rangle = \sum_{i=1}^{n} q_i \lambda_i = 0 \tag{3.2}$$

where $\Lambda = (\lambda_1, \ldots, \lambda_n)$. They verify a resonance condition of order k if the condition (3.2) holds and if $|Q| = \sum q_i = k - 1$.

Now the Poincaré theorem can be formulated as follows:

Theorem 4 (Poincaré theorem) *If the eigenvalues of the matrix $DF(0)$ are non-resonant the system (3.1) is formally equivalent to its linear part. That means that the nonlinear system (3.1) can be reduced to a linear system*

$$\dot{\tilde{X}} = DF(0)\, \tilde{X}$$

by a formal change of coordinates $X = H(\tilde{X})$.

The Poincaré theorem has been extended by H. Dulac [21] to systems whose matrices $DF(0)$ have resonant eigenvalues. He states that any system (3.1) can be reduced to a system $\dot{X} = \tilde{F}(X)$ where the matrix $D\tilde{F}(0)$ is in Jordan form and the nonlinear part contains only resonant terms.

Theorem 5 (Poincaré-Dulac theorem) *The differential equation (3.1) can be reduced to a system*

$$\frac{\partial \tilde{X}}{\partial t} = \tilde{F}(\tilde{X}) = J\tilde{X} + W(\tilde{X}) \tag{3.3}$$

by a formal change of coordinates $X = H(\tilde{X})$. In equation (3.3) the matrix $D\tilde{F}(0) = J$ is in Jordan form and $\tilde{F}$ contains only resonant terms. That means that the resonance condition

$$\forall Q \in supp(\tilde{F}) : \ \langle Q, \Lambda \rangle = 0$$

holds for all exponents of the normal form.

The transformation $X = H(\tilde{X})$ is called normalizing transformation. It can be decomposed into

$$H(\tilde{X}) = P\tilde{X} + V(\tilde{X})$$

where $P\tilde{X}$ denotes its linear part and $V(\tilde{X})$ its nonlinear part. The matrix P is the transition matrix that transforms the matrix $DF(0)$ into Jordan form

$$J = P^{-1}DF(0)P \ .$$

The new system (3.3) is called Poincaré-Dulac normal form of the initial system (3.1). The particular structure of its support is used for further reductions or for the integration of normal forms. The points of $supp(\tilde{F})$ lie on the so called resonant plane.

Definition 11 (resonant plane) *The set of points*

$$M = \{Q \in \mathcal{N} : \langle Q, \Lambda \rangle = 0\}$$

is called the resonant plane for a normalized system (3.3). It is a subset of the space of exponents and represents all points that can appear in the support of a normal form.

It is obvious that resonances occur for all $Q \in \mathcal{N}$ if $\lambda_1 = \ldots = \lambda_n = 0$. For this reason the Poincaré-Dulac normal form only yields a reduction for systems (3.1) with non-nilpotent linear part. However there exist normal form constructions for nilpotent systems with non-vanishing linear part (see R. Cushman and J. Sanders [16]).

Efficient algorithms for the computations of the Poincaré-Dulac normal form will be introduced in the following. These algorithms use Poincaré- or Lie-transformations. However the Poincaré and the Poincaré-Dulac theorem are the basis for all reductions and integrations concerning normal forms that are used in the following chapters.

3.2 The Jordan form

The first step in calculating the Poincaré-Dulac normal form is the calculation of the Jordan form of the matrix $DF(0)$ that represents the linear part of the system (3.1). Let

$$J = P^{-1}DF(0)\,P$$

be the Jordan form of the matrix $DF(0)$. Then the linear change of coordinates $X = P\tilde{X}$ yields a system

$$\frac{\partial \tilde{X}}{\partial t} = F(\tilde{X})$$

where the matrix $D\tilde{F}(0) = J$ is in Jordan form. However due to the problems of representation of algebraic numbers the computations of the Jordan form for $n \times n$ matrices with $n > 2$ is difficult. In this case the computations can be performed by algorithms that are based on works from I. Gil [29], P. Ozello [44] and M. Griesbrecht [31].

3.3 Computation of normal forms by Poincaré Transformations

To understand the basic idea of normal form computations we will study the effect of so called Poincaré transformations to a given system (3.1). Poincaré transformations have the form

$$X = \tilde{X} + H_k(\tilde{X}) \tag{3.4}$$

where $H_k \in \mathcal{H}_n^k$. $\mathcal{H}_n^k$ denotes the product of n copies of the space of homogeneous polynomials of degree k with n variables $x_1, \ldots, x_n$. The effect of Poincaré transformations on the initial system (3.1) is computed step by step for increasing degree $k \geq 2$. To study this effect the Taylor expansion

$$\dot{X} = AX + F_2(X) + F_3(X) + \ldots \tag{3.5}$$

of system (3.1) with $F_k \in \mathcal{H}_k^n$ is considered. Applying a Poincaré-transformation (3.4) to the vector field (3.5) yields the so called homological equation for the terms of degree k of the resulting normalized system. Calculating normal forms can be reduced to the problem of finding solutions for the homological equation. This problem can be solved for example by using the matrix representation method.

3.3.1 The homological equation

The effect of Poincaré transformations on a system (3.1) or (3.5) can be computed straightforward. The resulting equation that is called homological equation is basic to all normal form theory. Introducing the Poincaré transformation (3.4) into equation (3.1) yields the new system

$$\frac{\partial \tilde{X}}{\partial t} = (Id + DH_k(\tilde{X}))^{-1} F(\tilde{X} + H_k(\tilde{X})) = \tilde{F}(\tilde{X}) \tag{3.6}$$

that has been computed according to $H * F = (DH)^{-1}F(H)$. This expression can be simplified by introducing

$$(Id + DH_k(\tilde{X}))^{-1} = \sum_{i=0}^{\infty} (-DH_k(\tilde{X}))^i = Id - DH_k(\tilde{X}) + O(\tilde{X}^k) \; .$$

Now the system (3.6) can be written as

$$\frac{\partial \tilde{X}}{\partial t} = (Id - DH_k(\tilde{X}) + O(\tilde{X}^k))^{-1} F(\tilde{X} + H_k(\tilde{X})).$$

Ordering all terms according to their degree yields

$$\begin{aligned} \tfrac{\partial \tilde{X}}{\partial t} = \; & A\tilde{X} + F_2(\tilde{X}) + \ldots + F_{k-1}(\tilde{X}) \\ & + (F_k(\tilde{X}) + AH_k(\tilde{X}) - DH_k(\tilde{X})A\tilde{X}) \\ & + O(\tilde{X}^{k+1}). \end{aligned} \tag{3.7}$$

This equation can be decomposed into three parts. The terms of degree lower than k remain unchanged in equation (3.7). The terms of degree higher than k are changed but

they are not considered at this step of the algorithm. The term $\tilde{F}_k(\tilde{X})$ of the normal form is obtained by the relation

$$\tilde{F}_k(\tilde{X}) = F_k(\tilde{X}) + AH_k(\tilde{X}) - DH_k(\tilde{X})A\tilde{X}. \tag{3.8}$$

The task consists in finding an appropriate H_k such that $\tilde{F}_k$ is "as simple as possible". For the Poincaré-Dulac normal form that means to find $H_k(\tilde{X})$ such that a maximum of terms in equation (3.8) vanish. For this purpose we try to solve the equation

$$F_k(\tilde{X}) = DH_k(\tilde{X})A\tilde{X} - AH_k(\tilde{X}) \tag{3.9}$$

that is called the "homological equation". An equivalent formulation

$$F_k(\tilde{X}) = L_A^k(H)$$

can be given by introducing the linear operator

$$L_A^k : \begin{cases} \mathcal{H}_n^k \to \mathcal{H}_n^k \\ H_k(\tilde{X}) \mapsto L_A^k(H_k) = DH_k(\tilde{X})A\tilde{X} - AH_k(\tilde{X}). \end{cases}$$

that is called the homological operator. The subscript means that the linear operator L_A^k only makes use of the informations available from the linear part A of the system and k refers to the degree of the polynomials in H_k. Theoretically the problem of using Poincaré-transformations for calculating normal forms reduces to calculating the inverse of the operator $L_A^k(H)$.

If the eigenvalues of $L_A^k(H)$ do not contain zero the operator $L_A^k(H)$ is invertible and equation (3.9) can be solved. If $L_A^k(H)$ is not invertible the space $\mathcal{H}_n^k$ is split into

$$\mathcal{H}_n^k = R_n^k(A) \oplus C_n^k(A)$$

where $R_n^k(A)$ denotes the range of $L_A^k(H)$ and $C_n^k(A)$ a complementary space. The terms belonging to $C_n^k(A)$ can not be removed. In the case of the Poincaré-Dulac normal form those terms are the resonant terms.

The normal forms are not unique since $C_n^k(A)$ is not uniqueley determined. Several methods can be used for finding the complementary subspaces C^k for a given matrix A. We will use the matrix representation method.

3.3.2 The matrix representation of the homological operator

A possible method for computing the subspace $C_n^k(A)$ is the matrix representation method. In liteature this method can be found for example in Shui-Nee Chow, Chengzhi Li and Duo Wang [15] and L. Vallier [60]. The linear operator L_A^k can be represented by a matrix L in a suitable basis. The matrix L has a structure that can be derived from the structure of the matrix A. The space $Ker(L)$, that is associated to $C_n^k(A)$ can easily be computed for the Poincaré-Dulac normal form as the matrix A is in Jordan form.

The operator L_A^k can be represented by a matrix with respect to a basis of $\mathcal{H}_n^k$. A basis for $\mathcal{H}_n^k$ can be given by the basis elements $X^Q e_i$ with $|Q| = k$ and $i = 1, \ldots, n$. The basis elements are ordered in the lexicographic order

$$X^Q e_j < X^P e_i \;\Leftrightarrow\; (i, q_1, \ldots, q_n) < (j, p_1, \ldots, p_n)\ . \tag{3.10}$$

In equation (3.10) the relation $(i, q_1, \ldots, q_n) < (j, p_1, \ldots, p_n)$ holds only if $i < j$ or if $i = j$ and the first components q_i and p_i with $q_i \neq p_i$ verify $q_i < p_i$. The structure of the matrix L depends on the structure of A.

Lemma 1 *If A is a diagonal matrix then L is also diagonal. If A is lower (upper) triangular then L is lower (upper) block triangular. The element l_{ii} is $\langle Q, \Lambda \rangle - \lambda_j$ where i means the i-th element in the lexicographical ordering and it corresponds to the basis element $e_j X^Q$.*

The maps H_k and $F_k(\tilde{X}) - \tilde{F}_k(\tilde{X})$ can be represented by the vectors h and f with respect to the basis of $\mathcal{H}_k^n$. Equation (3.8) can be written as a linear system

$$Lh = f \ .$$

If A is in Jordan form the range of L and a complementary subspace can be easily read off the matrix L because the range of L is spanned by the columns of the matrix. We can easily see that L has zero eigenvalues if the eigenvalues of A verify resonance conditions of order k. That means that resonant monomials can not be reduced by the operator L_A^k.

The terms in the space C_k^n are the terms of degree k that remain in the normal form. They can not be removed in equation (3.8). However the Poincaré transformation (3.4) also affects terms of higher degree than k. To perform the next step of the computation of the normal form (for $k+1$) these terms need to be known. They are calculated using equation (3.7). However these computations are not very efficient. A better approach for this problem is the use of Lie transformation methods.

3.4 Computation of normal forms by Lie transformation

A main problem in calculating normal forms using Poincaré transformations is that the calculated transformation $X = \tilde{X} + H_k(\tilde{X})$ does not only affect terms of degree k but it also changes terms of higher degree. In the previous section the application of Poincaré transformation to the initial system was computed according to equation (3.7). However a much more efficient way to perform these computations is to use Lie transformations.

The introduction of Lie theory to the theory of normal forms is due to W. Groebner [62] and K.T. Chen [40]. Since then this subject has been developped by many others. See for example the works of P. Olver [46], G. Chen [13] and K. Meyer [43]. A. Deprit [3], S. Chow and J. Hale [14] and L. Vallier [60] have optimized the organization of the computations.

3.4.1 Definitions

The elementary operators used in Lie theory are the Lie derivative and the Lie bracket.

Definition 12 (Lie derivative) *Let $F = (f_1, \ldots, f_n)$ be a vector field defined on an open subset M of $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $g : M \to \mathbb{R}[[X]]$ or $\mathbb{C}[[X]]$ be a function on M. The operator*

$\mathcal{L}_F(.)$ *with*

$$\mathcal{L}_F(g) = \sum_{i=1}^{n} f_i \frac{\partial g}{\partial x_i}$$

is called the Lie derivative.

$\mathcal{L}_F(g)$ can be interpreted as the derivative of the function g in direction of the vector field F.

Definition 13 (Lie bracket) *Let F and G be two vector fields defined on open subsets. The vector field*

$$[F, G] = \frac{\partial G}{\partial X} F(X) - \frac{\partial F}{\partial X} G(X) \tag{3.11}$$

is called the Lie bracket of F and G.

In equation (3.11) the expression $\frac{\partial F}{\partial X}$ denotes the Jacobian matrix of F. The Lie bracket is used to define the adjoint operator.

Definition 14 (adjoint operator) *The operator*

$$ad_F = [F, .]$$

that associates a vector field $[F, G]$ to any vector field G, is called the adjoint operator associated to a vector field F.

The main idea of Lie transformation theory is to introduce a new parameter ϵ and to consider the transformation $H(\tilde{X}, \epsilon)$ either as the flow of a vector field $G(H)$ or as solution of the associated differential equation

$$\frac{\partial H(\tilde{X}, \epsilon)}{\partial \epsilon} = G(H(\tilde{X}, \epsilon) \ .$$

The normalizing transformation introduced in section 3.1 is obtained from the flow $H(\tilde{X}, \epsilon)$ for $\epsilon = 1$. It transforms the initial system (3.1) to Poincaré-Dulac normal form. The vector field $H * F$ is not computed as the action of the transformation H on F but as the action of the vector field G on F.

The task consists in transforming F to normal form by the action of the vector field G. Having calculated $H * F$ and G, the transformation H is computed as the flow of the non-singular vector field G.

3.4.2 Action on a vector field

Consider the vector field F defined on an open subset M of $\mathbb{C}^n$ or $\mathbb{R}^n$ and the vector field G defined on an open subset $\bar{M}$ of $\mathcal{H}_n$

$$\begin{aligned} F &: X \in M \mapsto \mathbb{C}^n \ or \mathbb{R}^n \\ G &: H \in \bar{M} \mapsto \mathcal{H}_n \ . \end{aligned}$$

Introducing the parameter ϵ the normalizing transfomation H can be written as

$$H(\tilde{X}, \epsilon) = \epsilon \tilde{X} + \epsilon^2 H_2(\tilde{X}) + \epsilon^3 H_3(\tilde{X}) + \dots \ .$$

H can be considered as the flow of the non-singular vector field $G(H)$ or the solution of the associated differential equation

$$\frac{\partial H(\tilde{X},\epsilon)}{\partial \epsilon} = G(H(\tilde{X},\epsilon)) \ . \tag{3.12}$$

Now the action of the vector field G on the vector field F is given by the following theorem:

Theorem 6 *The action of a vector field G on a vector field F is given by the formal Taylor serie*

$$H * F = F + \epsilon[G,F] + \frac{\epsilon^2}{2\,!}[G,[G,F]] + \ldots = e^{\epsilon\, ad_G}(F) \tag{3.13}$$

where $H * F$ denotes the transformed vector field.

Proof 2 Equation (3.13) represents the Taylor serie of the vector field $H * F$. Therefore to prove theorem 6 it is sufficient to show that

$$\begin{aligned} H * F|_{\epsilon=0} &= F \\ \tfrac{\partial H*F}{\partial \epsilon}\big|_{\epsilon=0} &= [G,F] \\ \tfrac{\partial^2 H*F}{\partial \epsilon^2}\Big|_{\epsilon=0} &= [G,[G,F]] \\ &\ldots \end{aligned} \tag{3.14}$$

The first equation in (3.14) follows immediately from the definition of $H(\tilde{X},\epsilon)$. Now we will prove the second equation.

Applying the transformation $X = H(\tilde{X},\epsilon)$ to the initial equation (3.1) yields

$$DH\,\frac{\partial \tilde{X}}{\partial \epsilon} = F(H(\tilde{X},\epsilon)) \tag{3.15}$$

and the new system

$$\frac{\partial \tilde{X}}{\partial \epsilon} = (DH)^{-1} F(H(\tilde{X},\epsilon)) \tag{3.16}$$

that is also denoted by $H * F$. Deriving equation (3.15) yields

$$\frac{\partial DH}{\partial \epsilon}\, H * F + DH \frac{\partial H * F}{\partial \epsilon} = DF \frac{\partial H(\tilde{X},\epsilon)}{\partial \epsilon} \tag{3.17}$$

that can be simplified by the properties

$$\begin{aligned} \tfrac{\partial DH}{\partial \epsilon} &= D\tfrac{\partial H}{\partial \epsilon} \\ &= D(G(H)) \\ &= DG(H)\,DH \\ H * F &= (DH)^{-1} F(H) \\ \tfrac{\partial H}{\partial \epsilon} &= G(H) \ . \end{aligned}$$

Introducing these results into equation (3.17) yields

$$DG(H)F(H) + DH\frac{\partial H * F}{\partial \epsilon} = DF(H)\,G(H)\ .$$

With the definition of the Lie bracket this can be written as

$$\frac{\partial H * F}{\partial \epsilon} = (DH)^{-1}[G, F]\ .$$

For $\epsilon = 0$ this yields the second term in the Taylor serie for $H * F$. The terms of higher degree can be computed in a similar way (see also G. Chen [13]).

Now $H(\tilde{X}, \epsilon)$ can be computed as the flow of the vector field G. The normalizing transformation $H(\tilde{X})$ is obtained for $\epsilon = 1$.

3.4.3 The flow of a non-singular vector field

Let $G(H)$ be a non-singular vector field. The flow H of the vector field $G(H)$ can be computed very efficiently by computing the Taylor serie of H.

The derivations of $H(\tilde{X}, \epsilon)$ are given by

$$\begin{array}{rcl} \frac{\partial H(\tilde{X},\epsilon)}{\partial \epsilon} & = & G(H(\tilde{X}, \epsilon)) \\ \frac{\partial^2 H(\tilde{X},\epsilon)}{\partial \epsilon^2} & = & D\,G(H)\,G(H(\tilde{X}, \epsilon)) \\ \cdots & & \end{array}$$

Computing the Taylor serie for $H(\tilde{X}, \epsilon)$ around $\epsilon = 0$ yields

$$H(\tilde{X}, \epsilon) = H(\tilde{X}, 0) + \epsilon\ \frac{\partial H(\tilde{X}, \epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} + \frac{\epsilon^2}{2\,!}\ \frac{\partial^2 H(\tilde{X}, \epsilon)}{\partial \epsilon^2}\bigg|_{\epsilon=0} + \ldots$$

or

$$H(\tilde{X}, \epsilon) = H(\tilde{X}, 0) + \epsilon\ G(H(\tilde{X}, \epsilon))\Big|_{\epsilon=0} + \frac{\epsilon^2}{2\,!}\ D\,G(H)\,G(H(\tilde{X}, \epsilon))\Big|_{\epsilon=0} + \ldots\ .$$

Introducing the notation of Lie derivatives this can be written as

$$H(\tilde{X}, \epsilon) = H_0(\tilde{X}) + \epsilon\ \mathcal{L}_G(H_0(\tilde{X})) + \frac{\epsilon^2}{2\,!}\mathcal{L}_G^2(H_0(\tilde{X})) + \ldots = e^{\epsilon \mathcal{L}_G}(H_0(\tilde{X})) \qquad (3.18)$$

with $H_0(\tilde{X}) = H(\tilde{X}, 0)$. Considering that H is the normalizing transformation defined in section 3.1 and that the linear part has Jordan form we can consider that $H_0 = \tilde{X}$. Equation (3.18) is called the Lie serie of the vector field G.

The property (3.18) is used to compute the transformation H for the vector field $G(H)$ that is used to normalize the initial vector field F.

3.4.4 Organization of the computations

The computations of the action on a vector field (theorem 6) and of the flow of a non-singular vector field (equation (3.18)) can be performed very efficiently when they are organized in so called Lie triangles. This scheme has been introduced by A. Deprit [3]. Therefore the scalar parameter $\epsilon \in k$ is introduced. Now the vector fields F, G and $\tilde{F} = H * F$ and the transformation H are written as

$$F(X,\epsilon) = \sum_{m=0}^{\infty} F_{m+1}(X)\frac{\epsilon^m}{m!}$$

$$G(\tilde{X},\epsilon) = \sum_{m=0}^{\infty} G_{m+1}(X)\frac{\epsilon^m}{m!}$$

$$\tilde{F}(\tilde{X},\epsilon) = \sum_{m=0}^{\infty} \tilde{F}_{m+1}(\tilde{X})\frac{\epsilon^m}{m!}$$

$$H(\tilde{X},\epsilon) = \sum_{m=0}^{\infty} H_{m+1}(\tilde{X})\frac{\epsilon^m}{m!}\ .$$

where F_k, G_k, $\tilde{F}_k$, $H_k \in \mathcal{H}_k^n$. The calculations are performed according to the following theorems

Theorem 7 *If the sequence* $F_i^{(m)}(X),\ i = 1,2,\ldots,\ m = 1,2,\ldots,i-1$ *is defined by the recursive relations*

$$\begin{array}{l} F_i^{(0)} = F_i, \quad i = 1,2,\ldots \\ F_i^{(m)} = F_i^{(m-1)} + \sum_{1\le j\le i-m} \binom{i-1}{j-1} [G_{j+1}, F_{i-j}^{(m-1)}] \\ for\ i = 1,2,\ldots\ and\ m = 1,2,\ldots,i-1 \end{array}$$

then

$$\tilde{F}_i = F_i^{(i-1)}, \quad for\ i = 1,2,\ldots.$$

This allows to compute the normal form $\tilde{F}$ and the vector field G. It yields an equation that is equivalent to the homological equation (3.9). This equation can be solved by using the matrix representation method.

The transformation H can be computed according to the following theorem.

Theorem 8 *If we define the sequence*

$$\begin{array}{l} G_i^{(0)} = G_i, \quad i = 1,2,\ldots \\ G_i^{(m)} = G_i^{(m-1)} + \sum_{1\le j\le i-m} \binom{i-1}{j-1} \mathcal{L}_{G_{j+1}}(G_{i-j}^{(m-1)}) \\ for\ i = 1,2,\ldots\ and\ m = 1,2,\ldots,i-1 \end{array}$$

then

$$H_i = G_i^{(i-1)}, \quad for\ i = 1,2,\ldots.$$

These theorems allow to organize the computations in Lie triangles :

$$\begin{array}{ccccccc} F_0^{(0)} & & & & & & \\ F_1^{(0)} & F_1^{(1)} & & & & & \\ F_2^{(0)} & F_2^{(1)} & F_2^{(2)} & & & & \\ F_3^{(0)} & F_3^{(1)} & F_3^{(2)} & F_3^{(3)} & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{array} \qquad \begin{array}{cccccc} H_0^{(0)} & & & & & \\ H_1^{(0)} & H_1^{(1)} & & & & \\ H_2^{(0)} & H_2^{(1)} & H_2^{(2)} & & & \\ H_3^{(0)} & H_3^{(1)} & H_3^{(2)} & H_3^{(3)} & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

These Lie triangles can be computed very efficiently.

3.5 Example

The following example illustrates all steps of the proposed algorithm.

Example 8 (Pendulum) *Consider the dynamical system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} sin(x) \end{cases}$$

already introduced in the examples 1 and 4. Developping $sin(x)$ into its Taylor serie around $x = 0$ yields

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l}(x - \frac{1}{3}x^3 + \frac{1}{120}x^5 + O(x^7)) \ . \end{cases} \tag{3.19}$$

The linear part of the system (3.19) represented by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix}$$

is transformed to its jordan form by a change of variables

$$X = \begin{pmatrix} \frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{y} \\ \frac{i}{2}\sqrt{\frac{g}{l}}\tilde{x} - \frac{i}{2}\sqrt{\frac{g}{l}}\tilde{y} \end{pmatrix} \ .$$

This yields the new system

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} i\sqrt{\frac{g}{l}} & 0 \\ 0 & -i\sqrt{\frac{g}{l}} \end{pmatrix} \tilde{X} + \begin{pmatrix} -\frac{i}{48}\sqrt{\frac{g}{l}}\tilde{x}^3 - \frac{i}{16}\sqrt{\frac{g}{l}}\tilde{x}^2\tilde{y} - \frac{i}{16}\sqrt{\frac{g}{l}}\tilde{x}\tilde{y}^2 - \frac{i}{48}\sqrt{\frac{g}{l}}\tilde{y}^3 + O(\tilde{X}^4) \\ \frac{i}{48}\sqrt{\frac{g}{l}}\tilde{x}^3 + \frac{i}{16}\sqrt{\frac{g}{l}}\tilde{x}^2\tilde{y} + \frac{i}{16}\sqrt{\frac{g}{l}}\tilde{x}\tilde{y}^2 + \frac{i}{48}\sqrt{\frac{g}{l}}\tilde{y}^3 + O(\tilde{X}^4) \end{pmatrix} .$$

This system can be transformed to its normal form

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} i\sqrt{\frac{g}{l}} & 0 \\ 0 & -i\sqrt{\frac{g}{l}} \end{pmatrix} \hat{X} + \begin{pmatrix} -\frac{i}{16}\sqrt{\frac{g}{l}}\hat{x}^2\hat{y} + O(\hat{X}^4) \\ \frac{i}{16}\sqrt{\frac{g}{l}}\hat{x}\hat{y}^2 + O(\hat{X}^4) \end{pmatrix} \tag{3.20}$$

by a change of variables

$$\tilde{X} = \hat{X} + \begin{pmatrix} -\frac{1}{96}\hat{x}^3 + \frac{1}{32}\hat{x}\hat{y}^2 + \frac{1}{192}\hat{y}^3 \\ \frac{1}{192}\hat{x}^3 + \frac{1}{32}\hat{x}^2\hat{y} - \frac{1}{96}\hat{y}^3 \end{pmatrix}$$

The system (3.20) contains only resonant nonlinear terms. This is the Poincaré-Dulac normal form of the initial system (3.19).

Calculating with parameters in the linear part of the system in example 8 is only possible because in the resonance equation the parameters vanish.

Chapter 4

Resolution of singularities by blowing-up

Blowing-up is one of the most frequently used methods to reduce vector fields. It is mostly used for the reduction of nilpotent vector fields but in chapter 9 it is shown that blowing-ups can also be used for the reduction of non-nilpotent systems of differential equations.

The idea is to apply a change of coordinates that expands or "blows-up" the singularity of a vector field F or of the associated system of differential equations

$$\dot{X} = F(X) \ . \tag{4.1}$$

In section 4.3 it will be shown that F can no longer be supposed to be given by formal power series. Therefore we will presume that the components f_i of F are real or complex analytic power series in the variables $x_1, \ldots, x_n$.

The local study of the initial system in the singular point $X = 0$ is replaced by a study of the transformed system in the blown-up singularity. The concerned singularity $X = 0$ is called the center and the blown-up singularity is refered to as the exceptional divisor of the blowing-up.

The simplest blowing-up, that illustrates very well all aspects of this method, is the introduction of polar coordinates. However for systems defined by power series the use of directional and quasi-homogeneous directional blowing-ups is more appropriate. Those blowing-ups are represented by power transformations that were already introduced in chapter 2.

Directional blowing-up has first been introduced for the desingularization of plane algebraic curves by O. Zariski [66]. Those results were extended to two-dimensional systems of differential equations by A. Seidenberg [53] and A. van den Essen [61]. Quasihomogeneous directional blowing-up, that is controlled by the Newton diagram, was introduced by A. Bruno [9]. Like M. Brunella and M. Miari [8] he uses only unimodular matrices to construct the power transformations that define blowing-ups for two-dimensional systems. In the works of F. Dumortier [32], [22] and M. Pelletier [45] the matrices for the used quasihomogeneous blowing-ups are constructed using a different approach.

However we will see that the use of the matrices defining the blowing-ups can be generalized. This is necessary as unimodular matrices do not allow to treat all appearing problems. Therefore the definition of blowing-ups is extended to the use of any invertible

matrix. The used matrices are computed via their adjoint matrix. As those matrices do not always define injective power transformation this generalization is only possible due to the results from section 2.3.

The definitions of blowing-ups are given for n-dimensional systems. However blowing-ups of systems with $n > 2$ might yield difficulties. For some three-dimensional systems there might not exist a finite chain of successive blowing-up that entirely reduces the concerned system. This has been shown by J. Jouanolou [38] and X. Gómez-Mont and I. Luengo [30]. Three-dimensional vector fields can only be desingularized by blowing-ups for some particular cases as it was shown for nondicritical systems by F. Cano and D. Cerveau [10].

4.1 Two-dimensional polar blowing-up

The simplest possible blowing-up is the introduction of polar coordinates to the differential equation (4.1). In general polar blowing-up is not used for systems given by power series. However it is considered here as it illustrates well all aspects of blowing-up. We will consider a two-dimensional vector field $F = (f_1, f_2)$ where f_1 and f_2 are given by real analytic power series in the variables x_1 and x_2 ($f_1, f_2 \in \mathbb{R}\{x_1, x_2\}$).

Definition 15 (Polar blowing-up in $\mathbb{R}^2$) *The map*

$$\Phi : \begin{cases} \mathbb{R} \times [0, 2\pi[\to \mathbb{R}^2 \\ (r, \varphi) \mapsto (r \sin\varphi, r\cos\varphi) \end{cases}$$

is called a polar blowing-up in $\mathbb{R}^2$.

A considered system is blown-up by applying the change of coordinates $X = \Phi(\tilde{X})$. This yields a new system $\tilde{F} = \Phi * F$. The point $X = 0$ is "represented" by the set

$$\{(r, \varphi) : r = 0,\ \varphi \in [0, 2\pi[\} \ . \tag{4.2}$$

That means that Φ^{-1} is not defined for $X = 0$. Nevertheless the set (4.2) is denoted by $\Phi^{-1}(0)$. It will be called the exceptional divisor of the blowing-up. The point $X = 0$ is refered to as the center of the blowing-up. In contrast to directional blowing-up, that will be studied in the following section, the exceptional divisor of polar blowing-up is finite as $\varphi \in [0, 2\pi[$.

The system resulting from the blowing-up can be computed according to

$$\Phi * F = (D\Phi)^{-1} F(\Phi) \ .$$

This yields the system

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \sin\varphi & r\cos\varphi \\ \cos\varphi & -r\sin\varphi \end{pmatrix}^{-1} \begin{pmatrix} f_1(r\sin\varphi, r\cos\varphi) \\ f_2(r\sin\varphi, r\cos\varphi) \end{pmatrix}.$$

that can also be written as

$$\begin{cases} \dot{r} &=& \sin\varphi f_1(r\sin\varphi, r\cos\varphi) + \cos\varphi f_2(r\sin\varphi, r\cos\varphi) &=& r^k \tilde{f}_1(r, \varphi) \\ \dot{\varphi} &=& \frac{\cos\varphi}{r} f_1(r\sin\varphi, r\cos\varphi) - \frac{\sin\varphi}{r} f_2(r\sin\varphi, r\cos\varphi) &=& r^{k-1}\tilde{f}_2(r, \varphi) \end{cases}$$

with $\tilde{f}_1, \tilde{f}_2 \in \mathbb{R}[r, \sin\varphi, r\cos\varphi]$. Applying a time transformation $d\tilde{t} = r^{k-1}dt$ yields the new system

$$\begin{cases} \frac{\partial r}{\partial \tilde{t}} = r\tilde{f}_1(r,\varphi) \\ \frac{\partial \varphi}{\partial \tilde{t}} = \tilde{f}_2(r,\varphi) \ . \end{cases} \tag{4.3}$$

The local study of the initial system near $X = 0$ is replaced by the examination of the system (4.3) near the exceptional divisor $\Phi^{-1}(0)$. For further studies the new singularities on $\Phi^{-1}(0)$ are of particular interest. Those singularities exist as the studied systems are analytic. Two possible cases have to be considered for equation (4.3).

- $\tilde{f}_2(0,\varphi) \not\equiv 0$.
  The singularities on $\Phi^{-1}(0)$ are given by the set

  $$S = \{(0,\varphi) : \tilde{f}_2(0,\varphi) = 0\} \ .$$

  All other points on $\Phi^{-1}(0)$ are regular points. Here the solution curves are parallel to the exceptional divisor as $\frac{\partial r}{\partial \tilde{t}} = 0$. For the initial coordinates that means that only the solution curves computed for the singularities in S might pass through $X = 0$.

  Further applying the time transformation

  $$d\hat{t} = \tilde{f}_2(r,\varphi)d\tilde{t}$$

  yields the new system

  $$\begin{cases} \frac{\partial r}{\partial \hat{t}} = r\frac{\tilde{f}_1(r,\varphi)}{\tilde{f}_2(r,\varphi)} \\ \frac{\partial \varphi}{\partial \hat{t}} = 1 \ . \end{cases}$$

  that has the solution $(r,\varphi) = (0,\hat{t})$. Therefore the exceptional divisor is a solution curve for (4.3).

- $\tilde{f}_2(0,\varphi) \equiv 0$.
  Another time transformation $d\tilde{t} = rd\hat{t}$ yields the new system

  $$\begin{cases} \frac{\partial r}{\partial \hat{t}} = \tilde{f}_1(r,\varphi) \\ \frac{\partial \varphi}{\partial \hat{t}} = \frac{1}{r}\tilde{f}_2(r,\varphi) \ . \end{cases}$$

  The singularities of the new system are given by the set

  $$S = \{(0,\varphi) : \tilde{f}_1(0,\varphi) = 0 \ and \ \left.\frac{1}{r}\tilde{f}_2(r,\varphi)\right|_{r=0} = 0\} \ .$$

  All points $(0,\varphi) \notin S$ are regular points.

  For any point $(0,\phi_0) \notin S$ with $\tilde{f}_1(0,\phi_0) = 0$ there exits a solution curve that is tangent to the exceptional divisor. Those solution curves are called tangencies.

  As for all other points $(0,\phi_0) \notin S$ on the exceptional divisor $\frac{\partial r}{\partial \tilde{t}} \neq 0$, there exists a solution curve for equation (4.3) passing through this point. For the initial system (4.1) that means that there is an infinite number of solution curves passing through the singularity in $X = 0$.

In both cases the points in the set S can be studied by translating them to the origin and by applying either another blowing-up or by computing normal forms.

Remark 3 *The case $\tilde{f}_2(0,\varphi) \not\equiv 0$ is called the noncritical case. The case $\tilde{f}_2(0,\varphi) \equiv 0$ is refered to as the dicritical case. The name "dicritical" is due to the problems in the dicritical case for higher dimensional problems.*

Remark 4 *The transformation Φ is a diffeomorphism on the set $\mathbb{R} \times [0, 2\pi[- \Phi^{-1}(0)$. Therefore the initial vector field F and the vector field $\Phi * F$ are conjugated.*

Example 9 *Consider the system given by*

$$\dot{X} = \begin{pmatrix} -{x_1}^4 + {x_1}^3\, x_2 \\ \frac{13}{9}\, {x_1}^6\, {x_2}^2 - {x_1}^2\, {x_2}^2 + x_1\, {x_2}^3 \end{pmatrix} .$$

Applying a polar blowing-up and a time transformation with $d\tilde{t} = r^3 dt$ yields a new system

$$\begin{cases} \frac{\partial r}{\partial \tilde{t}} &= -r\sin(\varphi) + 2\, r \sin(\varphi)\cos(\varphi)^2 + r\cos(\varphi) + \ldots \\ \frac{\partial \varphi}{\partial \tilde{t}} &= -\cos(\varphi) + \cos(\varphi)^3 + 2\sin(\varphi)\cos(\varphi)^2 - 2\cos(\varphi)^4 \sin(\varphi) + \ldots \end{cases} \tag{4.4}$$

that verifies the conditions for the noncritical case. For $r = 0$ the system (4.4) has the form

$$\begin{cases} \frac{\partial r}{\partial t} &= 0 \\ \frac{\partial \varphi}{\partial t} &= -\cos(\varphi) + \cos(\varphi)^3 + 2\sin(\varphi)\cos(\varphi)^2 - 2\cos(\varphi)^4 \sin(\varphi) \end{cases}$$

that allows to compute its singularities on the exceptional divisor. As

$$-\cos(\varphi) + \cos(\varphi)^3 + 2\sin(\varphi)\cos(\varphi)^2 - 2\cos(\varphi)^4 \sin(\varphi) = \cos(\varphi)\sin(\varphi)^2 \left(\sin(2\,\varphi) - 1\right)$$

the singularities on the exceptional divisor $\Phi^{-1}(0)$ are given by the set

$$S = \{(0,\, 0),\ (0,\, \frac{1}{4}\,\pi),\ (0,\, \frac{1}{2}\,\pi),\ (0,\, \pi),\ (0,\, \frac{5}{4}\,\pi),\ (0,\, \frac{3}{2}\,\pi)\} .$$

The components of the systems resulting form polar blowing-up are analytic series in r, $\sin\varphi$ and $\cos\varphi$. As all further reductions are defined for power series, Taylor series expansions have to be used to allow further computations. Therefore directional blowing-up is more appropriate for the use with systems of the form (4.1).

4.2 Quasi-homogeneous directional blowing up

Among the most frequently used blowing-up constructions for systems (4.1) given by power series are quasihomogeneous directional blowing-ups. They are based on the fact that the qualitative properties of vector fields near the singularity in $X = 0$ are mainly determined by the quasihomogeneous parts of the concerned vector field. Those parts are computed using the Newton diagram. The degree of quasi-homogenity defines the coefficients of the matrix used for the blowing-up. That means that quasihomogeneous blowing-up is controlled by the Newton diagram.

4.2.1 Quasi-homogeneous vector fields

Quasi-homogeneous blowing-ups have been studied by F. Dumortier [32] and M. Pelletier [45]. They are defined by the type of quasi-homogenity of a quasi-homogeneous vector field.

Definition 16 (quasi-homogeneous functions and vector fields) *A function f defined on $\mathbb{R}$ or $\mathbb{C}$ is called quasi-homogeneous of type $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and degree k with $gcd(\alpha_1, \ldots, \alpha_n) = 1$ if*

$$f(r^{\alpha_1} x_1, \ldots, r^{\alpha_n} x_n) = r^k f(x_1, \ldots, x_n)$$

A vector field $F = (f_1, \ldots, f_n)$ is called quasi-homogeneous of type α and degree k if any f_j, $j = 1 \ldots n$ is quasi-homogeneous of type α and degree $k + \alpha_j$.

The quasi-homogeneous parts of a vector field with the lowest degree are given by the monomials associated to the faces of the Newton diagram. Consider the vector field

$$F_i^{(j)} = \begin{pmatrix} x_1 \sum_{Q \in \Gamma_i^{(j)}} a_{1Q} X^Q \\ \cdots \\ x_n \sum_{Q \in \Gamma_i^{(j)}} a_{nQ} X^Q \end{pmatrix}$$

containing all monomials associated to the points of a face $\Gamma_i^{(j)}$ of the Newton diagram. Choose α such that the vector α is orthogonal to the face $\Gamma_i^{(j)}$ and such that $\alpha_1, \ldots, \alpha_n \geq 0$. Due to the definition of the Newton diagram this vector exists for all faces $\Gamma_i^{(j)}$. Then all $Q \in \Gamma_i^{(j)}$ are lying on the hyperplane $\{Q : \langle \alpha, Q \rangle = k\}$. $F_i^{(j)}$ is quasihomogeneous of type α and degree k as

$$F_i^{(j)}(r^{\alpha_1} x_1, \ldots, r^{\alpha_n} x_n) = \begin{pmatrix} r^{\alpha_1} x_1 \sum_Q r^{\langle \alpha, Q \rangle} a_{1Q} X^Q \\ \cdots \\ r^{\alpha_n} x_n \sum_Q r^{\langle \alpha, Q \rangle} a_{nQ} X^Q \end{pmatrix} .$$

The vector field $F_i^{(j)}$ is called the quasihomogeneous part of F relative to the face $\Gamma_i^{(j)}$. Each vector field $F_i^{(0)}$ is quasihomogeneous of type α for any vector α. The vector α with $\alpha_1, \ldots, \alpha_n \geq 0$ is used to define quasi-homogeneous blowing-ups.

Quasihomogenous directional blowing-up is defined by a power transformation

$$\Psi_m : X \mapsto X^{A_m^T} \tag{4.5}$$

where the matrix A_m is defined as

$$A_m = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ \alpha_1 & \ldots & \alpha_m & \ldots & \alpha_n \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} .$$

The index m refers to the m-th row vector in A_m that is identical to the vector α. A considered system (4.1) is transformed to the conjugated blown-up system $\Psi_m * F$ by the change of coordinates $X = \Psi_m(\tilde{X})$.

Quasi-homogeneous blowing-ups of the form (4.5) are not necessarily diffeomorphisms. In this case the transformed system is not conjugated to the initial one. Therefore the additional constructions from section 2.3 are used to define a diffeomorphism corresponding to transformation (4.5).

Quasi-homogeneous blowing-up is defined by the vector α that is computed using the Newton diagram. Therefore the effects of the blowing-up Ψ_m on the support and on the Newton diagram of F are of particular interest.

4.2.2 The effect on the Newton diagram

All power transformations act on the exponents of the considered system and equivalently on the support of F and its Newton diagram. Consider the $n-1$ dimensional face $\Gamma_i^{(n-1)}$. According to theorem 3 in section 2.2 the effect of the power transfomation (4.5) on the points $Q \in supp(F)$ can be computed as

$$\tilde{Q} = AQ = \begin{pmatrix} q_1 \\ \vdots \\ \langle \alpha, Q \rangle \\ \vdots \\ q_n \end{pmatrix} .$$

That means that all points on the face $\Gamma_i^{(n-1)}$ are transformed to points with identical m-coordinate $k = \langle \alpha, Q_0 \rangle$ with $Q_0 \in \Gamma_i^{(n-1)}$. As $\Gamma_i^{(n-1)}$ lies on the lower left part of the convex hull and as $\alpha_1, \ldots, \alpha_n \geq 0$ it can be stated that all points $Q \in supp(F) - \Gamma_i^{(n-1)}$ are transformed to points AQ whose m-coordinate is greater than k.

This result can be interpreted geometrically. The face $\Gamma_i^{(n-1)}$ has been straightened up. That means that it has been transformed to a face that is parallel to the hyperplane

$$\{\tilde{Q} | \tilde{q}_m = 0\} . \tag{4.6}$$

This face can be translated to the set (4.6) by a time transformation

$$d\tilde{t} = \tilde{x}_m^k dt .$$

As all points $Q \in supp(F) - \Gamma_i^{(n-1)}$ are transformed to points AQ whose m-coordinate is greater than k the exponents of the system resulting from the blowing-up and the time transformation are positive.

Now consider the $n-2$ dimensional face $\Gamma_i^{(n-2)} \subset \Gamma_i^{(n-1)}$. This face can be seen as the intersection of the face $\Gamma_i^{(n-1)}$ and another face of dimension $n-1$. This face is either a $n-1$ dimensional face of the Newton diagram or a face of the convex hull of $supp(F)$. Let β be a normal vector β of this face that verifies $\beta \in \mathbb{N}^n$, $gcd(\beta_1, \ldots, \beta_n) = 1$. The vectors α and β are not colinear and all points $Q \in \Gamma_j^{(n-2)}$ lie on a hyperplane $\langle Q, \beta \rangle = \tilde{k}$. Further, for all $Q \in supp(F)$ the condition $\langle Q, \beta \rangle \geq \tilde{k}$ holds.

For this reason after having applied the transformation (4.5) another quasihomogeneous blowing-up Ψ_l can be applied to straighten the face $\Gamma_i^{(n-2)}$ such that it is transformed to a set of points that is parallel to the set

$$\{Q | q_m = 0 \; and \; q_l = 0\} \; .$$

The composition of those two blowing-ups is given by the power transformation $X = \tilde{X}^{A^T}$ with

$$A = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ \alpha_1 & \dots & \alpha_m & \dots & \dots & \dots & \alpha_n \\ & & & \ddots & & & \\ \beta_1 & \dots & \dots & \dots & \beta_l & \dots & \beta_n \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} .$$

As any face $\Gamma_i^{(j)}$ of the Newton diagram can be seen as an intersection of $n-j$ faces of dimension $n-1$ the $n-j$ normal vectors of these faces are orthogonal to $\Gamma_i^{(j)}$. Therefore $n-j$ elementary quasihomogeneous blowing-ups of the form (4.5) can be used to straighten the concerned face. These elementary blowing-ups can be joined to a single power transformation $\Psi : X \mapsto X^{A^T}$ where the normal vectors of the intersecting faces define the row vectors of the matrix A. This defines a blowing-up for each face $\Gamma_i^{(j)}$ of the Newton diagram. Ordering the quasihomogeneous blowing-ups ($m = 1, \; l = 2, \; \dots$) yields that the face $\Gamma_i^{(j)}$ is transformed to a face that is parallel to the set

$$\{\tilde{Q} : \tilde{q}_1 = 0, \; \tilde{q}_2 = 0, \; \dots, \; \tilde{q}_{n-j} = 0\} \; . \tag{4.7}$$

For all $\tilde{Q} \in supp(\Psi * F)$ the condition

$$\tilde{q}_1 \geq \tilde{q}_{01}, \; \dots, \; \tilde{q}_{n-j} \geq \tilde{q}_{0n-j}$$

with $\tilde{Q}_0 = (\tilde{q}_{01}, \dots, \tilde{q}_{0n-j}) = AQ_0, \; Q_0 \in \Gamma_i^{(j)}$ holds. Therefore a time transformation

$$d\tilde{t} = \tilde{x}_1^{\tilde{q}_{01}} \dots \tilde{x}_{n-j}^{\tilde{q}_{0n-j}} dt \tag{4.8}$$

translates the straightened face to the set (4.7). The exponents of the system resulting from the blowing-up Ψ and the time transformation (4.8) are positive.

However quasihomogeneous blowing-ups defined that way are not uniquely defined as in a face $\Gamma_i^{(j)}$ with $j < n-2$, any number of $n-1$ dimensional faces of the Newton diagram can intersect. Each of those $n-1$ dimensional faces can be used to define the matrix A by its normal vector. Therefore quasihomogeneous blowing-ups defined that way do not allow adequate control on the cones of the Newton diagram. That induces that the sectors that will be introduced in section 8 can not be controlled sufficiently. A better method to construct matrices that perform well directed manipulations on the Newton diagram and its cones is the construction of A via its adjoint matrix.

4.2.3 Construction of blowing-ups via adjoint matrices

The construction of the matrix A, that defines a quasi-homogeneous blowing-up, via its adjoint matrix is based on theorem 3 in section 2.2 and on the results in section 2.4. It yields a blowing-up that has the same effect on the Newton diagram as the blowing-ups constructed in section 4.2.2.

Consider the linearly independent vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$. They form the columns of the matrix

$$A^* = (v_1 | \ldots | v_n) \ .$$

According to the results from section 2.4 the matrix A^* can be used to compute a matrix A that defines a power transformation $X = \tilde{X}^{A^T}$. According to theorem 3 in section 2.2 the effect of this transformation on the vectors $v_k,\ k = 1, \ldots, n$ is given by

$$\tilde{v}_k = Av_k = detA\, e_k \ .$$

If there exists a point Q_0 such that any point $Q \in supp(F)$ can be written as

$$Q = Q_0 + \sum_{k=1}^{n} \gamma_k v_k \ with\ \gamma_k \geq 0 \tag{4.9}$$

those points are transformed to the points

$$\tilde{Q} = AQ_0 + \sum_{k=1}^{n} \gamma_k e_k \ .$$

The condition (4.9) is equivalent to the condition that $supp(F)$ has to be contained in the set $Q_0 + V$ where V is the convex cone spanned by the vectors $v_1, \ldots, v_n$. A time transformation

$$d\tilde{X} = \tilde{X}^{AQ_0} dt$$

translates the point AQ_0 to the origin and yields a new system with positive support.

These properties can be used to construct a quasihomogeneous blowing-up with the same effects as the blowing up Ψ defined in section 4.2.2.

Consider the face $\Gamma_i^{(j)}$ and let $Q_0 = \Gamma_i^{(0)} \in \Gamma_i^{(j)}$ be a vertex on the face $\Gamma_i^{(j)}$. There exist j linearly independent vectors

$$v_k = Q_0 - Q_k,\ Q_k \in \Gamma_i^{(j)},\ k = n - j - 1, \ldots, n$$

on the face $\Gamma_i^{(j)}$ such that all $Q \in \Gamma_i^{(j)}$ can be represented as

$$Q = Q_0 + \sum_{k=n-j-1}^{n} \gamma_k v_k,\ \gamma_k \in \mathbb{R}^+ \ .$$

With the vectors $v_1, \ldots, v_{n-j}$ they can be completed to a set of vectors that define the convex cone

$$V = \sum_{k=n-j+1}^{n} \mathbb{R}^+ v_k + \sum_{k=1}^{n-j} \mathbb{R}^+ v_k$$

such that all $Q \in supp(F)$ lie within the set $Q_0 + V$.

The cone V can be computed using for example the virtual Newton diagram that will be introduced in section 9.4 and chapter 10.

The blowing-up defined by $X = \tilde{X}^{A^T}$ with

$$A^* = \det A A^{-1} = (v_1 | \dots | v_n)$$

has the same effects as the blowing up Ψ from section 4.2.2. That means that the face $\Gamma_i^{(j)}$ is straightened and an appropriate time transformation yields a system with integer exponents.

The choice of the vectors $v_1, \dots, v_n$ and therefore the construction of V is not unique. Further the construction of the matrix A^* is not limited to the above choices of Q_0 and V. Any cone V spanned by n vectors and any point Q_0 such that $supp(F) \subset Q_0 + V$ can be used for the above construction if $n - j$ vectors in the set of vectors defining V lie on $\Gamma_i^{(j)}$.

However the construction of quasihomogeneous blowing-ups by adjoint matrices is in certain cases easier to handle than the quasihomogeneous blowing-ups defined in section 4.2.2. The main advantage is that the sector definition, that will be introduced in chapter 8 and 10, is much simpler to control as the inverse matrix of A is known.

It is obvious that not all power transformations define blowing-ups. Therefore a central point in the use of power transformations is to determine whether they define blowing-ups or not and to consider this in the construction of the power transformations.

4.2.4 The exceptional divisor

A power transformation defines a blowing-up if the center of the blowing-up can be represented by an object of higher dimension in the new coordinates. That means that the exceptional divisor has to contain more than a single point. Consider the transformation $\Psi : X \to X^{A^T}$ where A is an invertible matrix. The exceptional divisor $\Psi^{-1}(0)$ can be computed by considering the inverse of the transformation $X = \tilde{X}^{A^T}$ where it is defined. The transformation $X = \tilde{X}^{A^T}$ can be inverted on the set

$$U' = K^n - \bigcup_k \{X \in K^n | x_k = 0\}, \ K = \mathbb{R} \ or \ K = \mathbb{C} \, .$$

If A is defined by its adjoint matrix A^* with the column vectors $v_1, \dots, v_n$ this yields the inverse transformation

$$\tilde{X} = X^{A^{-T}} = \begin{pmatrix} X^{\frac{1}{det\, A} v_1} \\ \vdots \\ X^{\frac{1}{det\, A} v_n} \end{pmatrix} \, . \tag{4.10}$$

The exceptional divisor is defined as the set of values that the expression

$$\lim_{X \to 0} \tilde{X} \ = \lim_{X \to 0} \begin{pmatrix} X^{\frac{1}{det\, A} v_1} \\ \vdots \\ X^{\frac{1}{det\, A} v_n} \end{pmatrix} \tag{4.11}$$

can take. To evaluate this expression the point $X = 0$ is approached on so called curves of class $\mathcal{W}$. Those curves have been introduced by A. Bruno [9]. They are defined as

$$\mathcal{F} : \begin{cases} x_1(t) = t^{\alpha_1}(c_1 + O(1/t)) \\ \dots \\ x_n(t) = t^{\alpha_n}(c_n + O(1/t)) \end{cases} .$$

with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, $\alpha_1, \dots, \alpha_n \leq 0$ and $c_1, \dots, c_n \neq 0$. Evaluating expression (4.11) on the curve $\mathcal{F}$ yields

$$\lim_{X \to 0,\, X \in \Gamma_i^{(j)}} \tilde{X} = \lim_{t \to \infty} \begin{pmatrix} \tilde{c}_1\, t^{\frac{1}{det\, A}\langle v_1, \alpha \rangle} \\ \vdots \\ \tilde{c}_n\, t^{\frac{1}{det\, A}\langle v_n, \alpha \rangle} \end{pmatrix} \tag{4.12}$$

with $\tilde{c}_k = \frac{1}{\det A} \sum_l v_{kl} c_l$ for $v_k = (v_{k1}, \dots, v_{kn})$. Each line in equation 4.12 can be evaluated separately. For the line k this yields

$$\tilde{c}_k\, t^{\langle v_k, \alpha \rangle} \to \begin{cases} 0 & if\ \langle v_k, \alpha \rangle < 0 \\ \tilde{c}_k & if\ \langle v_k, \alpha \rangle = 0 \\ \infty & if\ \langle v_k, \alpha \rangle > 0 \end{cases}$$

as $\det A > 0$. It is obvious that $\tilde{c}_k\, t^{\langle v_k, \alpha \rangle} \neq 0$ only if the vector v_k has at least one negative coefficient. In this case there exists an α with $\langle \alpha, v_k \rangle = 0$. That means that $\tilde{x}_k$ can take any value if $X \to 0$. As a consequence the transformation $X = \tilde{X}^{A^T}$ is a blowing-up.

Any power transformation $X = \tilde{X}^{A^T}$ is a blowing-up if the matrix A^* has negative entries.

From the construction of the vectors $v_{n-j+1}, \dots, v_n$ it follows that they lie on a hyperplane $\langle ., \alpha \rangle = 0$ with $\alpha \in \mathbb{Z}^n_-$. As those vectors are used for the construction of the matrix A^* the power transformation $X = \tilde{X}^{A^T}$ with A and A^* defined as in section (4.2.3) is a blowing-up. The components $\tilde{x}_{n-j+1}, \dots, \tilde{x}_n$ can take any value and the set

$$S = \{\tilde{X} : \tilde{x}_1 = 0, \dots, \tilde{x}_{n-j} = 0\} \tag{4.13}$$

is part of the exceptional divisor. We are interested in the singularities of the new system $\Psi * F$ on the set (4.13). It is obvious that those singularities are given by the singularities of

$$\Psi * F_i^{(j)}(0, \dots, 0, \tilde{x}_{n-j+1}, \dots, \tilde{x}_n) = \begin{pmatrix} \tilde{x}_1 \tilde{f}_1(0, \dots, 0, \tilde{x}_{n-j+1}, \dots, \tilde{x}_n) \\ \vdots \\ \tilde{x}_n \tilde{f}_n(0, \dots, 0, \tilde{x}_{n-j+1}, \dots, \tilde{x}_n) \end{pmatrix} . \tag{4.14}$$

We have to distinguish two cases:

- If the condition

  $$\exists k \in \{n - j + 1, \dots, n\} : \tilde{f}_k(0, \dots, 0, \tilde{x}_{n-j+1}, \dots, \tilde{x}_n) \not\equiv 0$$

  holds, the singularities are given by the points $\tilde{X}_0 = (0, \dots, 0, *, \dots, *)$ that verify

  $$\tilde{f}_{n-j+1}(\tilde{X}_0) = 0, \dots, \tilde{f}_n(\tilde{X}_0) = 0 .$$

All other points $\tilde{X}_0$ are regular points. As for those points the condition

$$\Psi * F(\tilde{X}_0) = (0, \ldots, 0, *, \ldots, *)$$

holds, the solution curves $\tilde{X}(t)$ in the neighbourhood of those points verify $\tilde{x}_1(t), \ldots, \tilde{x}_{n-j}(t) = const$. The curves $\tilde{X}(t)$ near $\tilde{X}_0$ are all parallel to the exceptional divisor. The $n-j$ dimensional system

$$\left\{ \begin{array}{rcl} \frac{\partial \tilde{x}_{n-j+1}}{\partial t} & = & \tilde{x}_{n-j+1}\tilde{f}_{n-j+1}(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \\ & \ldots & \\ \frac{\partial \tilde{x}_n}{\partial t} & = & \tilde{x}_n\tilde{f}_n(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \end{array} \right.$$

yields solution curves on the exceptional divisor. As they are all transformed to the point $X = 0$ they are not interesting for the computation of solution curves $X(t)$.

This case is called the noncritical case.

- If the condition

$$\tilde{f}_{n-j+1}(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \equiv \ldots \equiv \tilde{f}_n(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \equiv 0$$

holds, equation (4.14) is identical to zero. The entire set S is a non-isolated singularity. However there exists an index k such that the time transformation

$$d\bar{t} = \tilde{x}_k d\tilde{t}$$

yields a new system with at least one non-vanishing component on the set (4.13). The singularities of the resulting system are identical to the singularities of

$$\frac{\Psi * F_i^{(j)}(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n)}{\tilde{x}_k} = \left(\begin{array}{c} \frac{\tilde{x}_1}{\tilde{x}_k}\tilde{f}_1(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \\ \vdots \\ \tilde{f}_k(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \\ \vdots \\ \frac{\tilde{x}_n}{\tilde{x}_k}\tilde{f}_n(0, \ldots, 0, \tilde{x}_{n-j+1}, \ldots, \tilde{x}_n) \end{array} \right) .$$

In general this system will have negative exponents. If this is the case a further study with the methods introduced here is not possible.

This case is called the dicritical case.

In this section it has been shown how quasi-homogeneous blowing-up can be used to reduce vector fields. However in the dicritical case a further reduction is not always possible. In general a single blowing-up might not be sufficient to reduce a considered vector field entirely. This is due to the fact that the first blowing-up might still yield nonelementary singularities. In this case further blowing-ups are needed. The composition of several blowing-ups is called successive blowing-up.

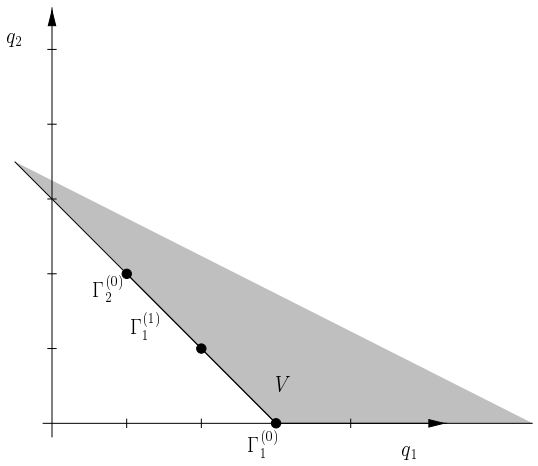


Figure 4.1: This figure shows the cone V constructed to define the blowing-up applied in the case of an edge $\Gamma_1^{(1)}$ in example 10.

4.3 Successive blowing-up

The singularities of a system resulting from a single blowing-up might still be nonelementary. In this case a single blowing-up is not sufficient to reduce a nonelementary singular point entirey. The concerned singularity is translated to the origin and another blowing-up is applied. This yields a chain of compositions of a blowing-up, a time transformation and a translation. This chain is called successive blowing-up.

An important question arises in this context. Is it possible to reduce any nonelementary singular point by a finite successive blowing-up ?

This is true for two-dimensional vector fields if the vector field is analytic in the neighbourhood of an isolated singularity. This has been shown for example by A. van den Essen [61], F. Cano [24] and F. Dumortier [32].

For three-dimensional problems this is not true for any case. This has been shown by J.P. Jouanolou [38] and X. Gómez-Mont and I. Luengo [30]. However some particular cases can be solved. It has been shown by F. Cano and D. Cerveau [10] that any non-dicritical system can be reduced by a finite number of blowing-ups. Non-dicritical means that none of all admissible blowing-ups yield a dicritical case.

Example 10 (Blowing-up of a two-dimensional vector field) *Consider the two-dimensional nilpotent system of differential equations*

$$\dot{X} = \begin{pmatrix} -x_1^4 + x_2 x_1^3 \\ \frac{13}{9} x_2^2 x_1^6 - x_1^2 x_2^2 + x_1 x_2^3 \end{pmatrix} . \tag{4.15}$$

It's Newton diagram is drawn in figure 4.1. It contains a single edge $\Gamma_1^{(1)} = \binom{-1}{1}$. *The blowing-up corresponding to that edge can be constructed using* $Q_0 = \Gamma_1^{(0)}$ *and the matrix*

$$A^* = (v_1|v_2)$$

with $v_2 = (-1, 1)$. *The matrix* A^* *can be completed by any vector* $v_1 = (v_{11}, v_{12})$ *with*

$$0 \geq \frac{v_{12}}{v_{11}} > -1, \ v_{11} > 0 \ .$$

Choosing $v_1 = (1, 0)$ *yields the matrices*

$$A^* = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ .$$

The power transformation $X = \tilde{X}^{A^T}$ *yields the system*

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \begin{pmatrix} -\tilde{x}_1^4 + \tilde{x}_1^4 \tilde{x}_2 \\ \tilde{x}_1^3 \tilde{x}_2 - 2\tilde{x}_1^3 \tilde{x}_2^2 + x_1^3 x_2^3 + \frac{13}{9} x_1^7 x_2^2 \end{pmatrix}.$$

The points $\Gamma_1^{(0)}$ *and* $\Gamma_2^{(0)}$ *have been transformed to the point* $(3, 0)$ *and* $(3, 2)$ *respectively. Therefore a time transformation*

$$d\tilde{t} = \tilde{x}_1^3 dt$$

transforms the straightened edge to the $\tilde{q}_2$*-axis. The resulting system*

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \begin{pmatrix} -\tilde{x}_1 + \tilde{x}_1 \tilde{x}_2 \\ \tilde{x}_2 - 2\tilde{x}_2^2 + \tilde{x}_2^3 + \frac{13}{9} \tilde{x}_1^4 \tilde{x}_2^2 \end{pmatrix} \tag{4.16}$$

verifies the noncritical case. Therefore the singularities of (4.16) are given by the set

$$S = \{\tilde{X} : \tilde{x}_1 = 0, \ \tilde{x}_2(1 - 2\tilde{x}_2 + \tilde{x}_2^2) = 0\} = \{(0, 0), \ (0, 1)\} \ .$$

In $\tilde{X} = (0, 0)$ *the system (4.16) is non-nilpotent. The singularity in* $\tilde{X} = (0, 1)$ *however is nilpotent. It can be reduced by a further blowing-up.*

This chapter has shown how power transformations and quasihomogeneous blowing-up can be used for the desingularization of nilpotent vector fields. They are controlled by the Newton diagram.

Chapter 5

Classification

In the previous chapters a number of methods that can efficiently be used to reduce vector fields were introduced. These methods are the basis of the algorithms introduced in the following. This chapter will show how those methods can be linked together to integrate any 2-dimensional and a large number of 3- and higher-dimensional system of autonomous differential equations.

Consider a system of differential equations given by

$$\dot{X} = F(X) \tag{5.1}$$

where the components of F are given as real or complex analytic power series in the variable X. The intention of all algorithms proposed in the following is to reduce this system such that formal solution curves $X(t)$ can be calculated in the neighbourhood

$$U = \{X : ||X - X_0||_\infty \leq \epsilon\}$$

of a point X_0. We will presume that the point of interest X_0 has been translated to the origin to simplify all considerations. (For more details about translations see example 2). The algorithm is split into three main parts according to the classification of the point $X_0 = 0$ that has already been mentioned in section 1.2:

- The origin is a regular point if $F(0) \neq 0$. The flow of a vector field in the neighbourhood of a regular point has a very simple structure. According to the flow box theorem there exists a change of coordinates $X = H(\tilde{X})$ that transforms the system (5.1) into a new system of the form

  $$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

  The integration of this system is obvious. The change of coordinates can be calculated as a Taylor serie.

  The flow box theorem can be found for example in M. Hirsch and S. Smale [42] or J. Hale and H. Korcak [33].

- The origin is an elementary singular point if $F(0) = 0$ and the Jacobian matrix $DF(0)$ is non-nilpotent. In this case the Poincaré-Dulac theorem introduced in chapter 3 can be applied. The resulting normal forms can always be integrated in two dimensions. For 3- and higher-dimensional problems, if a direct integration is not possible, an appropriate power transformation can reduce the normal form to a system of lower dimension. These systems are treated recursively by applying the entire algorithm again.

- The origin is a nonelementary singular point if $F(0) = 0$ and $DF(0)$ is nilpotent. In this case there exist two possibilities to reduce the considered system. They are controlled by the Newton diagram that was introduced in chapter 2. For any of its vertices there exists a time transformation that yields a non-nilpotent system that has its support within a cone. Applying a power transformation this system can be reduced to a form that allows to apply reduction and integration algorithms for elementary singular points. All other faces of the Newton diagram can be used to define matrices for blowing-ups that where introduced in chapter 4. Those blowing-ups yield new systems that have either regular points, elementary singular points or nonelementary singular points. Those systems can be treated by applying the entire algorithm recurively. Blowing-ups are very well known for 2-dimensional problems (see for example A. van den Essen [61], F. Cano [24] and F. Dumortier [32]). For higher dimensional systems there still remain many unsolved problems as it has been shown by J.P. Jouanolou [38] and X. Gómez-Mont and I. Luengo [30].

An overwiev on the splitting of the algorithms and how the different parts are linked is given in figure 5.1.

The different cases are the subject of the following chapters. Chapter 6 deals with n-dimensional regular points. Chapter 7 describes the integration of systems with 2-dimensional elementary singular points. Systems with 2-dimensional nonelementary singular points are treated in chapter 8. In chapter 9 we show how 3 dimensional normal forms can be integrated and where the limits of the proposed algorithms are. The propositions obtained for 3-dimensional systems can partly be extended to higher-dimensional systems. Chapter 8 gives some examples for 3- and higher-dimensional blowing-ups.

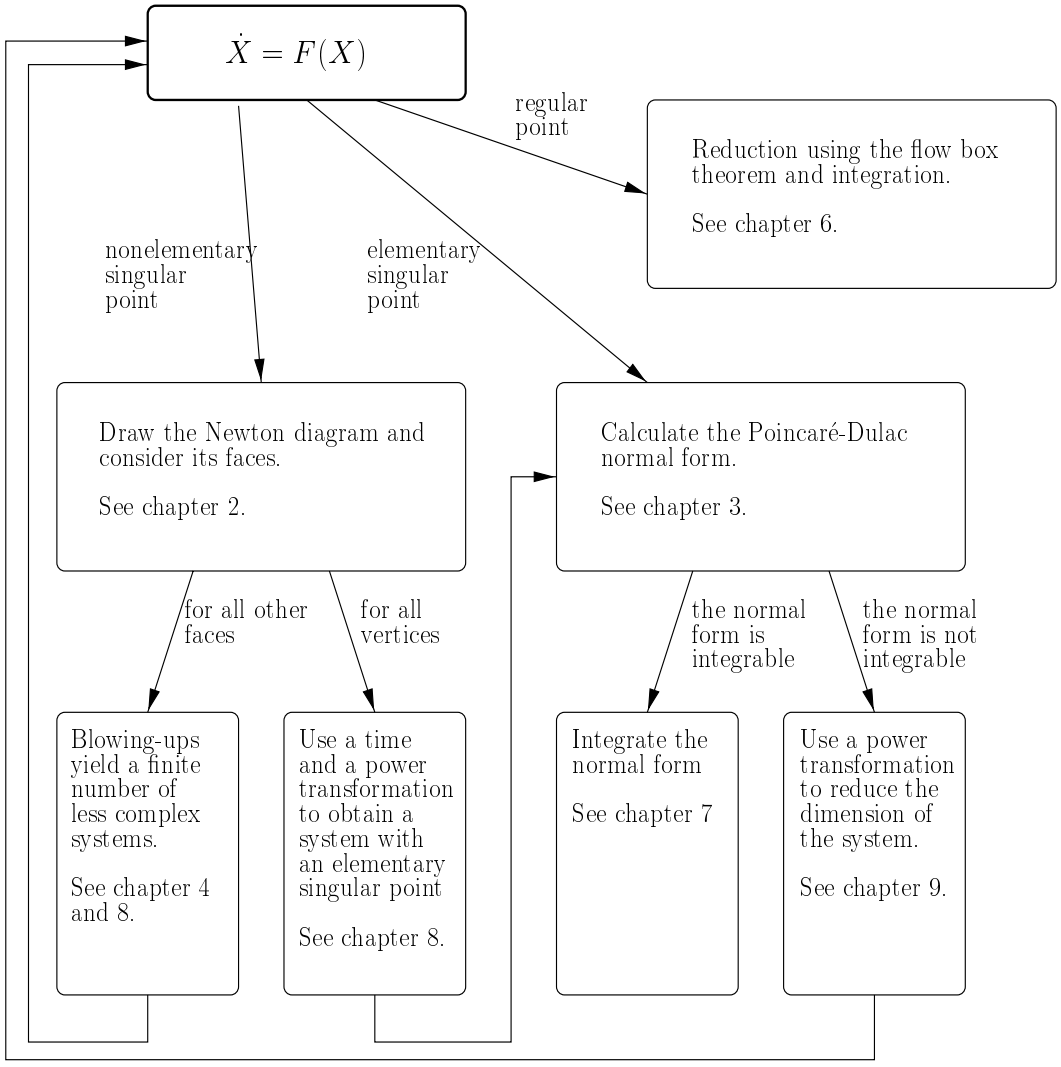


Figure 5.1: An overwiew on the entire algorithm and its composites.

Chapter 6

Regular Points

The simplest examples of dynamical systems are systems in the neighbourhood of a regular point $X_0 = 0$. In this case the constant part of the considered system

$$\dot{X} = F(X) \tag{6.1}$$

is different form zero. The flow or the normal form of a regular system (6.1), where the components of F are given as real or complex, analytic or formal power series, has a very simple structure as it is stated in the flow box theorem.

Theorem 9 (Flow Box Theorem) *In a sufficient small neighbourhood of a regular point of (6.1) there exists a differentiable, analytic change of variables*

$$X = H(\tilde{X}),\ H = (h_1, \ldots, h_n),\ h_i \in k\{X\},\ K = \mathbb{R} \text{ or } \mathbb{C} \tag{6.2}$$

that transforms the initial system to

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.3}$$

in the new coordinates.

A proof of this theorem can be found in works from M. Hirsch and S. Smale [42] and J. Hale and H. Kocak [33]. Theorem 9 guarantees the existence of a box-like neighbourhood of any regular point such that the flow of the system enters the box at one end and flows out at the other. The flow of a system in the neighbourhood of a simple point and a corresponding flow box are sketched in figure 6.1.

The computation of the change of variables H can be reduced to the problem of calculating the flow of a regular vector field. An algorithm that solves this problem was introduced in section 3.4.3. The method is based on Lie theory and is very efficient. It allows to compute the flow $X(t, X_0)$ for any initial value $X_0 = X(0, X_0)$ if X_0 is a regular point.

The solution of the reduced system (6.3) is

$$\tilde{X}(t) = \begin{pmatrix} t + c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} .$$

Transformed to the initial coordinates the solutions have the form

$$X(t) = H(\tilde{X}(t)) .$$

They must coincide with the flow $X(t, X_0)$ calculated using the algorithm from section 3.4.3. Therefore H can be obtained from substituting

$$\begin{aligned} t &= \tilde{x}_1, \\ x_{01} &= 0, \\ x_{02} &= \tilde{x}_2, \\ &\dots \\ x_{0n} &= \tilde{x}_n \end{aligned}$$

in $X(t, X_0)$ where $X_0 = (x_{01}, \dots, x_{0n})$.

Example 11 *The system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} sin(x) \end{cases} \tag{6.4}$$

that issued from the pendulum model in example 1 can be developped into a Taylor serie around the regular point $(\pi/2, 0)$. *This yields a system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} + \frac{1}{2}\frac{g}{l}(x - \frac{1}{2}\pi)^2 + O((x - \frac{1}{2}\pi)^4) \end{cases}$$

that is transformed to

$$\begin{cases} \frac{\partial \tilde{x}}{\partial t} = 1 \\ \\ \frac{\partial \tilde{y}}{\partial t} = 0 \end{cases}$$

by the transformation

$$X = \begin{pmatrix} \frac{\pi}{2} + \tilde{y} - \frac{1}{2}\frac{g}{l}x^2 + \dots \\ -\frac{g}{l}x + \dots \end{pmatrix} .$$

The solution curves $\tilde{X} = (t, c)$ *for the reduced system are transformed to the initial coordinates. The obtained curves* $X(t, c)$ *are solution curves for the initial system (6.4). They are sketched for* $g/l = 1/2$ *in figure 6.1.*

The calculation of solutions of dynamical systems (6.1) in the neighbourhood of a regular point is not very interesting for itself. Nevertheless many blowing-ups of nilpotent vector fields yield systems with regular points. Therefore solution curves of nilpotent vector fields can only be calculated if the case of a regular point can be solved.

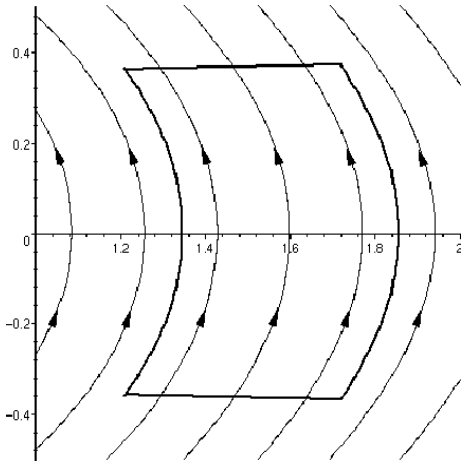


Figure 6.1: Solution curves in the neighbourhood of a regular point for the system treated in example 11. The lines flow through a so called "flow box".

Chapter 7

Two dimensional elementary singular points

In chapter 3 non-nilpotent singular systems of differential equations

$$\dot{X} = F(X) \tag{7.1}$$

where transformed to their Poincaré-Dulac normal form. In two dimensions this yields systems of the form

$$\begin{cases} \frac{\partial x}{\partial t} = x \sum_{\{Q \in \mathcal{N}_1 : \langle Q, \Lambda \rangle = 0\}} a_Q x^{q_1} y^{q_2} \\ \frac{\partial y}{\partial t} = y \sum_{\{Q \in \mathcal{N}_2 : \langle Q, \Lambda \rangle = 0\}} b_Q x^{q_1} y^{q_2} \end{cases} \tag{7.2}$$

that have their support on the resonant plane

$$M = \{Q \in \mathcal{N} : \langle Q, \Lambda \rangle = 0\}$$

with $\Lambda = (\lambda_1, \lambda_2)$. A possible resonant plane is sketched in figure 7.1. In this chapter it will be shown that 2-dimensional normal forms can always be integrated. The results from section 7.1 are based on works from A. Bruno [9] and the results from section 7.2 can partly be found in works from S. Chow, C. Li and D. Wang [15].

7.1 Integration

To integrate the Poincaré-Dulac normal form (7.2) a central point of interest are the possible solution of the resonance equations

$$\langle Q, \Lambda \rangle = 0 \tag{7.3}$$

Let $\lambda := \frac{\lambda_1}{\lambda_2}$, $\lambda_2 \neq 0$ without loss of generality. This allows to classify the possible solutions of equation (7.3)

1. $\lambda \in \mathbb{C} - \mathbb{Q}$
   No resonance is possible, the normal form is a linear differential equation

$$\begin{cases} \frac{\partial x}{\partial t} = \lambda_1 x \\ \frac{\partial y}{\partial t} = \lambda_2 y. \end{cases}$$

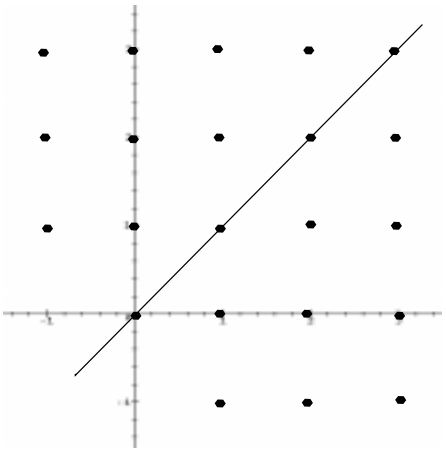


Figure 7.1: This figure shows the resonant plane for the normal form in example 12. All points in $N_1 \cup N_2$ that intersect the line $\{Q \in \mathbb{R} :< Q, \Lambda >= 0\}$ are elements of the resonant plane M and can be in the support of the normal form.

Its integration

$$\begin{aligned} x &= c_1 e^{\lambda_1 t} \\ y &= c_2 e^{\lambda_2 t} \end{aligned}$$

is obvious.

2. $\lambda \in \mathbb{Q},\ \lambda > 0$

   - If $\lambda = m,\ m \in \mathbb{N},\ \lambda \neq 1$ only one vector $Q = \binom{m}{-1}$ can solve the resonance equation (7.3). With $\lambda = m$ ($\lambda_1 = m\lambda_2$) the normal form is

   $$\begin{cases} \frac{\partial x}{\partial t} = m\lambda_2 x + a_{(-1,m)} y^m \\ \frac{\partial y}{\partial t} = \lambda_2 y. \end{cases}$$

   This can be integrated to

   $$\begin{aligned} x &= (a_{(-1,m)} c_2^m t + c_1) e^{\lambda_2 m t} \\ y &= c_2 e^{\lambda_2 t} \end{aligned}$$

   using the method of the variation of the constant.

   - If $\lambda = \frac{1}{m},\ m \in \mathbb{N},\ \lambda \neq 1$ we get

   $$\begin{cases} \frac{\partial x}{\partial t} = \lambda_1 x \\ \frac{\partial y}{\partial t} = m\lambda_1 y + b_{(m,-1)} x^m \end{cases}$$

   which can be integrated in a same way as the previous case.

   - If $\lambda = 1$, σ might be different from zero but no nonlinear monomials will appear. This case is similar to the previous one ($\lambda = m,\ m \in \mathbb{N}$) with m=1 and $a_{(-1,1)} = \sigma = 1$). If $\sigma = 0$ the resulting system is linear.

- If $\lambda \notin \{\frac{1}{m}, m | m \in \mathbb{N}\}$ the obtained system is linear.

3. $\lambda \in \mathbb{Q},\ \lambda = 0 \ \Leftrightarrow\ \lambda_1 = 0$.
All vectors $Q = \binom{q_1}{0}$ with $q_1 \in \mathbb{N}$ verify the condition (7.3). The normalized system (7.2) has the form
$$\begin{cases} \frac{\partial x}{\partial t} = x \sum_k a_{(k,0)} x^k =: xf(x) \\ \frac{\partial y}{\partial t} = \lambda_2 y + y \sum_k b_{(k,0)} x^k =: yg(x) \end{cases} \tag{7.4}$$
f and g depend only on x. If $f \equiv 0$ a time transformation
$$d\tilde{t} = yg(x)\ dt$$
yields the system
$$\begin{cases} \frac{\partial x}{\partial \tilde{t}} = 0 \\ \frac{\partial y}{\partial \tilde{t}} = 1 \end{cases}$$
that can be integrated to
$$\begin{aligned} x(\tilde{t}) &= c_1 \\ y(\tilde{t}) &= \tilde{t} + c_2 \end{aligned}$$
with
$$\tilde{t} = c_2 e^{g(c_1)t}\ .$$
If $f \not\equiv 0$ the corresponding scalar differential equation
$$\frac{\partial log y}{\partial x} = \frac{g(x)}{xf(x)}$$
is formally integrable as the right hand side does not depend on y. The result is
$$log y = \int^x \frac{g(s)ds}{sf(s)} + c_1.$$
This can be parametrized in the following way
$$\begin{aligned} x &= \tilde{t} \\ y &= exp(\textstyle\int^{\tilde{t}} \frac{g(s)ds}{sf(s)} + c_1). \end{aligned}$$
Note that in this case time transformations $\tilde{t} = \gamma(t)$ with unknown map γ are used. $\gamma(t)$ is the solution of the scalar differential equation
$$\frac{d\tilde{t}}{dt} = xf(x)\ . \tag{7.5}$$
Therefore to transform the calculated curves $X(\tilde{t})$ to solutions $X(t)$ of the initial system the solution of equation (7.5) has to be known.

4. $\lambda \in \mathbb{Q},\ \lambda < 0$
If $\lambda \in \mathbb{Q}^-$ there exists a vector $Q = \binom{q_1}{q_2}$ with $q_1, q_2 \in \mathbb{N}$ and $gcd(q_1, q_2) = 1$ that solves the equation (7.3). This is also true for any vector $kQ, k \in \mathbb{N}$. Figure 7.1 shows the set $N = N_1 \cup N_2$ of the points that can be in the support of the initial

system $supp(F)$ and the line defined by the resonance condition. The points on this line can be in the support of the resulting normal form

$$\begin{cases} \frac{\partial x}{\partial t} = \lambda_1 x + x \sum_k a_{(kq_1,kq_2)} (x^{q_1} y^{q_2})^k \\ \frac{\partial y}{\partial t} = \lambda_2 y + y \sum_k b_{(kq_1,kq_2)} (x^{q_1} y^{q_2})^k. \end{cases} \tag{7.6}$$

This system has two particular solutions

$$\begin{array}{l} (x,y) = (\tilde{t},0) \ with \ \tilde{t} = ce^{\lambda_1 t} \\ (x,y) = (0,\tilde{t}) \ with \ \tilde{t} = ce^{\lambda_2 t} \ . \end{array} \tag{7.7}$$

Further the normal form (7.6) can be brought to a system of the form (7.4), that is treated by the previous case, by a suitable change of coordinates $X = \tilde{X}^{A^T}$. According to theorem 3 in section 2.2 the matrix A is computed such that

$$AQ = \begin{pmatrix} a & -b \\ -q_2 & q_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} detA \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.8}$$

and $a, b \geq 0$. The resulting system can be integrated as in the previous case ($\lambda = 0$).

The transformation $X = \tilde{X}^{A^T}$ where A verifies equation (7.8) is not a blowing-up as the inverse of A does not have any negative matrix entries. As a consequence a local integration of the resulting system of the form (7.4) is sufficient to find solution curves for system (7.6).

However it can be deduced from the power transformation that the particular solutions (7.7) can not be obtained from the integration of the reduced system.

The solution curve $(x,y) = (\tilde{x}^a \tilde{y}^{-q_2}, \tilde{x}^{-b} \tilde{y}^{q_1}) = (\tilde{t}, 0)$ can only be obtained for $\tilde{y} = 0$ but

$$lim_{\tilde{y} \to 0} x = lim_{\tilde{y} \to 0} \tilde{x}^a \tilde{y}^{-q_2} < \infty$$

only if $q_2 = 0$. $q_2 = 0$ is impossible for this case.

For this reason the solutions (7.7) have to be added separately to the set of calculated solutions.

Example 12 *This example computes solution curves for the pendulum equation introduced in example 1 in the neighbourhood of the elementary singularity* $X_0 = (\pi, 0)$.

In a first step the pendulum equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l} sin(x) \end{cases}$$

is translated to move the point of interest to the origin. Developping the translated system into a Taylor serie around 0 *we obtain the system*

$$\dot{X} = \begin{pmatrix} y \\ \frac{g}{l}(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)) \end{pmatrix}$$

that we can treat with normal form algorithms. The Poincaré-Dulac normal form is

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} -\sqrt{\frac{g}{l}} & 0 \\ 0 & \sqrt{\frac{g}{l}} \end{pmatrix} \tilde{X} + \begin{pmatrix} \frac{1}{16}\sqrt{\frac{g}{l}} \tilde{x}^2 \tilde{y} + O(\tilde{X}^4) \\ -\frac{1}{16}\sqrt{\frac{g}{l}} \tilde{x} \tilde{y}^2 + O(\tilde{X}^4) \end{pmatrix} . \tag{7.9}$$

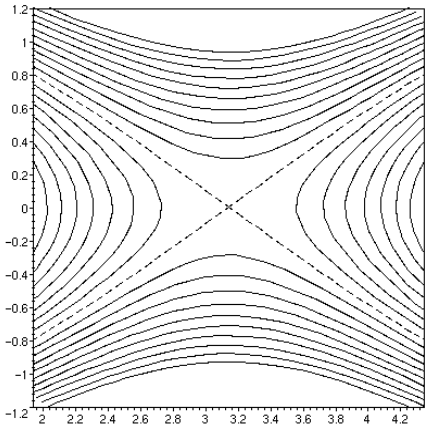


Figure 7.2: Solution curves calculated for the pendulum equation in the neighbourhood of the singular point $(\pi, 0)$ in example 12. The curves obtained from the particular solutions (7.7) are dashed.

Resonances occur for all $Q = k\binom{1}{1}$ *for* $k \in \mathbb{N}$. *Applying a power transformation* $\tilde{X} = \bar{X}^{A^T}$ *with*

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

yields the new system

$$\frac{\partial \bar{X}}{\partial t} = \begin{pmatrix} 0 \\ \frac{1}{16}\sqrt{\frac{g}{l}}\bar{x}\bar{y} - \sqrt{\frac{g}{l}}y \end{pmatrix}$$

This system has been truncated to order 3. It can be integrated to $\bar{X}(\tilde{t}) = (c, \tilde{t})$ *according to case 3 in section 7.1. The corresponding solution curves and the particular solution curves* $\tilde{X} = (\tilde{t}, 0)$ *and* $\tilde{X} = (0, \tilde{t})$ *of system (7.9) have been retransformed to the initial coordinates and sketched for* $g/l = 1/2$ *in figure 7.2 without considering problems of convergence. (The dashed lines represent the solution curves obtained for* $\tilde{X} = (\tilde{t}, 0)$ *and* $\tilde{X} = (0, \tilde{t})$*).*

7.2 Systems with real coefficients

A system (7.1) that has real coefficients can have a linear part with complex eigenvalues λ_1 and $\lambda_2 = \bar{\lambda}_1$. In this case the previously described method will transform the system into its complex Jordan canonical form and therefore introduce complex coefficients and yield complex solutions. In this section it will be shown that it is possible to calculate only the real solutions for real systems even if the computations use complex numbers.

There are two cases to consider. The case where the eigenvalues are purely imaginary $\lambda_1 = -\lambda_2$ and the case when they have a non-vanishing real part.

- $Re(\lambda_1) \neq 0$
  In this case we know that the resonance condition (7.3) has no solutions for $Q \in$

$N_1 \cup N_2$. The Poincaré-Dulac normal form is linear. According to Sternberg [55] the system (7.1) can be linearized by a normalizing algorithm even without transforming the linear part of the system into its Jordan canonical form. To calculate this so called rational or A-normal form we can use the algorithms described in chapter 3. The integration of the resulting linear system $\dot{\tilde{X}} = DF(0)\tilde{X}$ is obvious.

- $\lambda_1 = -\lambda_2$
  In this case the linear part of F has the form

$$DF(0)X = \begin{pmatrix} 0 & -\alpha \\ \beta & 0 \end{pmatrix} X,\ \alpha, \beta > 0,\ \alpha, \beta \in \mathbb{R}\ .$$

This linear part is transformed to its Jordan canonical form

$$J\tilde{X} = \begin{pmatrix} i\sqrt{\alpha\beta} & 0 \\ 0 & -i\sqrt{\alpha\beta} \end{pmatrix} \tilde{X}$$

by a linear transformation

$$X = P\tilde{X}\ with\ P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2}\sqrt{\frac{\beta}{\alpha}} & \frac{i}{2}\sqrt{\frac{\beta}{\alpha}} \end{pmatrix} \tag{7.10}$$

The resulting system has the particular structure

$$\frac{\partial \tilde{X}}{\partial t} = P^{-1}F(P\tilde{X}) = \begin{pmatrix} \tilde{x}\sum a_Q \tilde{X}^Q \\ \tilde{y}\sum b_Q \tilde{X}^Q \end{pmatrix}$$

where the coefficients verify $a_Q = \bar{b}_Q$ for all Q with $q_1 = q_2$. This structure is preserved by the normal form construction. Applying the inverse of transformation (7.10) to the resulting Poincaré-Dulac normal form

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} \hat{x}\sum_j \hat{a}_{(j,j)}\hat{X}^{(j,j)} \\ \hat{y}\sum_j \overline{\hat{a}_{(j,j)}}\hat{X}^{(j,j)} \end{pmatrix}$$

yields a new system

$$\frac{\partial \tilde{\tilde{X}}}{\partial t} = \begin{pmatrix} \sum_j (\tilde{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\tilde{y}}^2)^j \left(\mathrm{Re}(\hat{a}_{(j,j)})\tilde{\tilde{x}} - \mathrm{Im}(\hat{a}_{(j,j)})\sqrt{\frac{\alpha}{\beta}}\tilde{\tilde{y}}\right) \\ \sum_j (\tilde{\tilde{x}}^2 + \frac{\alpha}{\beta}\tilde{\tilde{y}}^2)^j \left(\mathrm{Im}(\hat{a}_{(j,j)})\sqrt{\frac{\alpha}{\beta}}\tilde{\tilde{x}} + \mathrm{Re}(\hat{a}_{(j,j)})\tilde{\tilde{y}}\right) \end{pmatrix}$$

that is transformed to

$$\begin{cases} \dot{r} = c_1 r + c_3 r^3 + \dots \\ \dot{\theta} = d_0 + d_2 r^2 + d_4 r^4 + \dots \end{cases}$$

by the introduction of polar coordinates $\tilde{\tilde{X}} = \binom{r\sqrt{\alpha}sin\theta}{r\sqrt{\beta}cos\theta}$. This new system can be integrated using previously introduced methods.

Example 13 (Pendulum) *In example 8 the normal form of the system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{l}(x - \frac{1}{3}x^3 + \frac{1}{120}x^5 + O(x^7)) \end{cases}.$$

was computed. This system issued from modelling a planar pendulum in example 1. The normal form

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} i\sqrt{\frac{g}{l}} & 0 \\ 0 & -i\sqrt{\frac{g}{l}} \end{pmatrix} \hat{X} + \begin{pmatrix} -\frac{i}{16}\sqrt{\frac{g}{l}}\hat{x}^2\hat{y} + O(\hat{X}^4) \\ \frac{i}{16}\sqrt{\frac{g}{l}}\hat{x}\hat{y}^2 + O(\hat{X}^4) \end{pmatrix} \tag{7.11}$$

is transformed to the new system

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix} \tilde{X} + \begin{pmatrix} -\frac{1}{16}\tilde{x}^2\tilde{y} - \frac{1}{16}\frac{l}{g}\tilde{y}^3 + O(\tilde{X}^4) \\ \frac{1}{16}\tilde{x}^3 + \frac{1}{16}\frac{l}{g}\tilde{x}\tilde{y}^2 + O(\tilde{X}^4) \end{pmatrix}$$

by a transformation

$$\hat{X} = \begin{pmatrix} \tilde{x} - i\sqrt{\frac{l}{g}}\tilde{y} \\ \tilde{x} + i\sqrt{\frac{l}{g}}\tilde{y} \end{pmatrix}$$

that is the inverse to the transformation that was used to obtain a system with a linear part in Jordan form. The introduction of polar coordinates

$$\tilde{X} = \begin{pmatrix} r\sin(\alpha) \\ r\sqrt{\frac{g}{l}}\cos(\alpha) \end{pmatrix}$$

yields a new truncated system

$$\begin{cases} \frac{\partial r}{\partial t} &= 0 \\ \frac{\partial \alpha}{\partial t} &= \sqrt{\frac{g}{l}}(1 - \frac{r^2}{16}) \end{cases}$$

that can be integrated to $(r(\tilde{t}), \alpha(\tilde{t})) = (c, \tilde{t})$ *with* $\tilde{t} = \sqrt{\frac{g}{l}}(1 - \frac{c_1^2}{16})t + c_2$ *according to case 3 in section 7.1. These curves can be transformed to the initial coordinates. They are sketched together with the exact solutions in figure 7.3 without considering eventual problems of convergence.*

This chapter has shown that all 2 dimensional systems can be integrated if they are in Poincaré-Dulac normal form. However, the obtained results are in general no solutions of the considered systems but parametrizations of the solution curves.

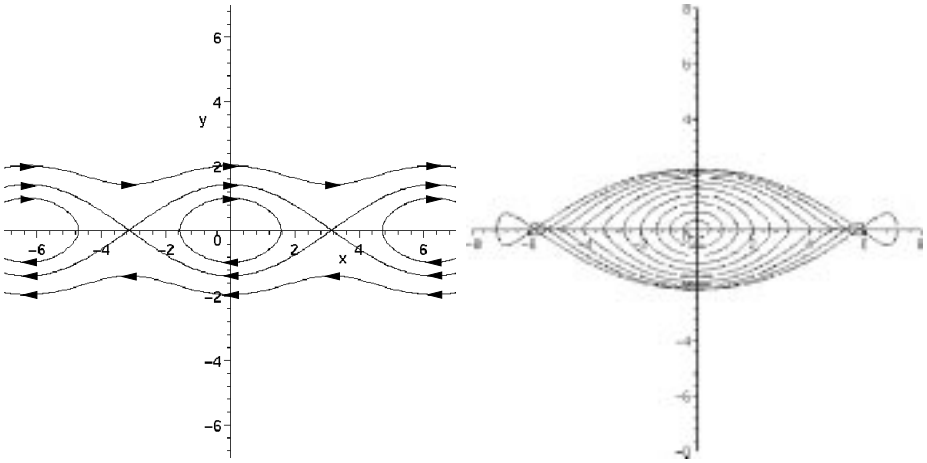


Figure 7.3: This figure shows the exact solutions and the solution curves calculated in example 13 for the planar pendulum for $g/l = 1/2$. We can see that the curves coincide well in the neighbourhood of $(0, 0)$.

Chapter 8

Two dimensional nonelementary singular points

In the case of a nonelementary singular point the application of the Poincaré-Dulac normal form to a system

$$\dot{X} = F(X) \tag{8.1}$$

with $F \in \mathbb{C}^2\{X\}$ or $\mathbb{R}^2\{X\}$ fails because $\lambda_1 = \lambda_2 = 0$ and resonances occur for any exponent. For this reasons other techniques, that are controlled by the Newton diagram of F, are employed. For two-dimensional systems the Newton diagram yields a set of vertices $\Gamma_j^{(0)}$ with $j = 1, \ldots, k$ and a set of edges $\Gamma_j^{(1)}$ with $j = 1, \ldots, k-1$. For each of those faces there exist algorithms to calculate solution curves.

For each vertex a time transformation and a power transformation transform the initial system (8.1) to a non-nilpotent system. This system can be integrated using normal form constructions and the methods introduced in chapter 7. The edges of the Newton diagram are used to define blowing-ups that yield a finite number of "less complicated" new systems. Those systems are treated by applying the entire algorithm recursively.

However the solutions computed by the algorithms associated to the faces of the Newton diagram are only valid on parts of the initial neighbourhood. Those parts are called sectors. Those sectors are defined such that they cover the entire concerned neighbourhood though they might sometimes intersect each other. That means that solution curves are computed for any point of the concerned neighbourhood.

Blowing-ups of two-dimensional systems have been subject of many publications. This chapter is mainly based on works from F. Dumortier [32], M. Pelletier [45] and A. Bruno [9]. Though we give a different explanation for the appearance of sectors the sector notation introduced by A. Bruno is largely used in this chapter.

8.1 The Vertices

Each vertice $Q := \Gamma_j^{(0)}$ of the Newton diagram can be used to define a time transformation

$$d\tilde{t} = X^Q dt. \tag{8.2}$$

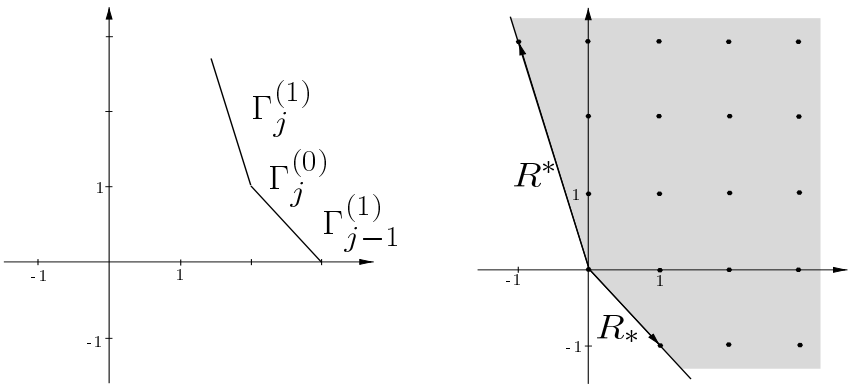


Figure 8.1: The time transformation (8.2) translates the considered vertex to the origin. The new system (8.3) has its support within a convex cone V that contains the first quadrant.

that translates the point Q to the origin in the space of exponents (see figure 8.1). This yields a new system

$$\frac{\partial X}{\partial \tilde{t}} = \frac{1}{X^Q} F(X) \tag{8.3}$$

that has a non-zero linear part. This system and the initial one in equation (8.1) are equivalent in the sense that the solution curves of both systems are parametrized solutions of the same scalar differential equation. If the Newton diagram of the system consists of more than one vertex the support of the new system has negative exponents and it lies within a convex cone

$$V = \mathbb{R}^+ \, R^* + \mathbb{R}^+ \, R_*$$

(see figure 8.1). The cone V contains the first quadrant and it is defined by the vectors R^* and R_*. For non-extremal vertices those vectors lie on the edges $\Gamma_j^{(1)}$ and $\Gamma_{j-1}^{(1)}$. R^* and R_* lead away from the origin and they verify

$$gcd(s,t) = 1, \; gcd(u,v) = 1 \; with \; R_* = \begin{pmatrix} s \\ t \end{pmatrix}, \; R^* = \begin{pmatrix} u \\ v \end{pmatrix} .$$

For the extremal vertex $\Gamma_1^{(0)}$ the vector R_* is defined as $R_* = (1,0)$ and for the extremal vertex $\Gamma_k^{(0)}$ the vector R^* is defined as $R^* = (0,1)$.

Systems having their support within a convex cone are called class $\mathcal{V}$ systems.

8.1.1 The Poincaré-Dulac Normal Form for Class $\mathcal{V}$ Systems

Class $\mathcal{V}$ systems have been introduced by A. Bruno [9]. They denote systems that have their support within a convex cone V. In the case of a vertex this cone is defined by the vectors R^* and R_*.

A class $\mathcal{V}$ system (8.3) can either be transformed to normal form directly via the definition of the generalized Poincaré-Dulac normal form, as it is done by A. Bruno [9], or it can be transformed to a system with positive integer exponents by a power transformation

$$X = \tilde{X}^{A^T} . \tag{8.4}$$

Theorem 10 *The change of coordinates* $X = \tilde{X}^{A^T}$ *described by the matrix*

$$A := \begin{pmatrix} v & -u \\ -t & s \end{pmatrix}$$

transforms a class $\mathcal{V}$ *system (8.3) to a new system with integer exponents that has its support within the first quadrant* ($R_* = \binom{s}{t}$ *and* $R^* = \binom{u}{v}$, $s, v > 0$, $t, u \leq 0$).

Proof 3 *All points* Q *that might appear in the support of the class* $\mathcal{V}$ *system (8.3) can be written as*

$$Q = \alpha R^* + \beta R_*$$

with $\alpha, \beta, \in \mathbb{R}^+$. *According to theorem 3 in chapter 2 the vectors* R^* *and* R_* *are transformed to the vectors*

$$\begin{pmatrix} 0 \\ detA \end{pmatrix} \text{ and } \begin{pmatrix} detA \\ 0 \end{pmatrix}$$

respectively. For this reason Q *is transformed to the point*

$$\tilde{Q} = \alpha \begin{pmatrix} 0 \\ detA \end{pmatrix} + \beta \begin{pmatrix} detA \\ 0 \end{pmatrix}$$

that has positive coordinates. The coordinates of $\tilde{Q}$ *are integer as* A *has only integer coefficients.*

According to the results from section 4.2.3 the matrix A can also be computed via its adjoint matrix. For the matrix

$$A^* = (R_* | R^*)$$

we obtain the matrix A form theorem 10.

The new system is transformed to its Poincaré-Dulac normal form and integrated with the algorithms introduced in chapter 7.

The advantage of this method compared to the computation of generalized normal forms for class $\mathcal{V}$ systems is that the classical normal form algorithms from chapter 3 can be used. They are more efficient than algorithms computing normal form for class $\mathcal{V}$ systems.

Remark 5 *If the employed power transformation is not bijective the constructions introduced in chapter 2 can be used to define injective and piecewise surjective transformation.*

8.1.2 The Sectors

A very important characteristic of the solution curves calculated by the algorithms associated to a vertex of the Newton diagram is that they are not valid in the entire concerned neighbourhood

$$U = \{X : |x| \leq \epsilon, \ |y| \leq \epsilon\} \ .$$

This results from the fact that the transformation (8.4) is a blowing-up. The origin $X = 0$ is transformed to the exceptional divisor of the blowing-up and the neighbourhood U of the origin is transformed to the set $\tilde{U}$ that denotes a neighbourhood of the exceptional

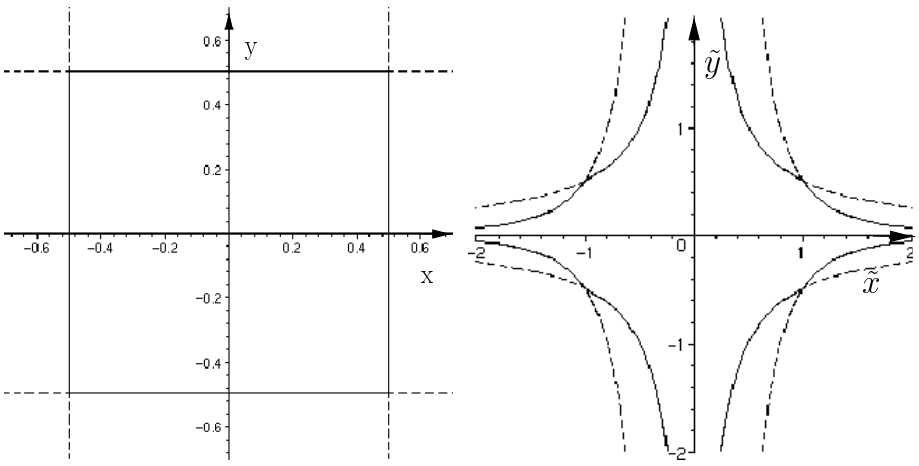


Figure 8.2: The neighbourhood U of the singularity $X = 0$ is transformed into a neighbourhood $\tilde{U}$ of the exceptional divisor.

divisor. This is shown in figure 8.2. The solution curves for the set $\tilde{U}$ can no longer be found with local methods as $\tilde{U}$ is not a neighbourhood of a point.

The algorithms from section 8.1 yield solution curves that are only locally valid in a neighbourhood of $\tilde{X} = 0$. We will presume that this neighbourhood can be denoted as

$$\tilde{U}_j^{(0)} = \{\tilde{X} : |\tilde{x}| \leq \delta_j,\ |\tilde{y}| \leq \delta_j\}$$

with δ_j sufficiently small.

Transforming $\tilde{U}_j^{(0)}$ to the initial coordinates yields the definition sets

$$U_j^{(0)} = \{X : |X|^{R_*} \leq \delta_j^{|detA|},\ |X|^{R^*} \leq \delta_j^{|detA|}\}$$

for the curves computed for the vertices $\Gamma_j^{(0)}$. The sets $U_j^{(0)}$ are limited by the curves

$$\begin{cases} |x| = \delta_j^{-\frac{detA}{s}} & if\ t = 0 \\ |y| = \delta_j^{\frac{detA}{t}} |x|^{-\frac{s}{t}} & if\ t \neq 0 \\ |x| = \delta_j^{-\frac{detA}{u}} & if\ v = 0 \\ |y| = \delta_j^{\frac{detA}{v}} |x|^{-\frac{u}{v}} & if\ v \neq 0 \end{cases}$$

The sets $U_j^{(0)}$ are called sectors. An example for a sector obtained in example 14 is sketched in figure 8.4.

An important point in the definition of the set $\tilde{U}_j^{(0)}$ is the choice of the parameter δ_j. As it has been shown in chapter 4 the blowing-up might yield new singularities on the exceptional divisor besides the one in $\tilde{X} = 0$. We have to consider all those singularities. The singularities different from $\tilde{X} = 0$ will be studied in the algorihms associated to the edges of the Newton diagram. Therefore the set $\tilde{U}_j^{(0)}$ must be chosen such that it does not

contain any further singularities besides $\tilde{X} = 0$. This is guaranteed if the δ_j are chosen sufficiently small.

To simplify the representation of the sectors we choose

$$\epsilon = min\left\{1, min_{j=1}^{k}\{\delta_j^{|detA|}\}\right\}$$

and work with the neighbourhoods

$$\tilde{U}_j^{(0)} = \{X : |\tilde{x}| \leq \epsilon^{\frac{1}{|detA|}},\ |\tilde{y}| \leq \epsilon^{\frac{1}{|detA|}}\}$$

in the coordinates $\tilde{X}$ and with the the sectors

$$U_j^{(0)} = X : |X|^{R_*} \leq \epsilon, |X|^{R^*} \leq \epsilon, |X|_\infty \leq \epsilon\}$$

in the initial coordinates. The parts of U that are not covered by the sectors associated to the vertices of the Newton diagram are associated to the edges $\Gamma_j^{(1)}$ and treated separately. They are denoted by

$$U_j^{(1)} = \{X : \epsilon \leq |X|^R \leq \epsilon^{-1},\ |X|_\infty \leq \epsilon\}\ .$$

The vector R is defined as lying on the edge $\Gamma_j^{(1)}$ and verifying the properties

$$r_1 < 0,\ r_2 > 0, gcd(r_1, r_2) = 1$$

with $R = (r_1, r_2)$. R is identical to the vector R^* for the vertex $\Gamma_j^{(0)}$.

The following example illustrates how solution curves in a sector associated to a vertex of the Newton diagram are calculated. The used power transformation is not bijective. This makes additional constructions necessary.

Example 14 *The Newton polygon of the system*

$$\begin{cases} \frac{\partial x}{\partial t} = x^2y + xy^4 \\ \frac{\partial y}{\partial t} = 2xy^2 + x^2y \end{cases}$$

is sketched in figure 8.3. It has a vertex $\Gamma_2^{(0)} = \binom{1}{1}$. *Applying the time transformation*

$$d\tilde{t} = X^{(1,1)}dt$$

yields a system

$$\begin{cases} \frac{\partial x}{\partial \tilde{t}} = x + x^{-1}y^3 \\ \frac{\partial y}{\partial \tilde{t}} = 2y + xy^{-1} \end{cases} \tag{8.5}$$

that has its support within a cone V *defined by the vectors* $R^* = \binom{-1}{3}$ *and* $R_* = \binom{1}{-1}$. *The cone is shown in figure 8.1. The transformation*

$$X = \tilde{X}^{A^T},\ A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

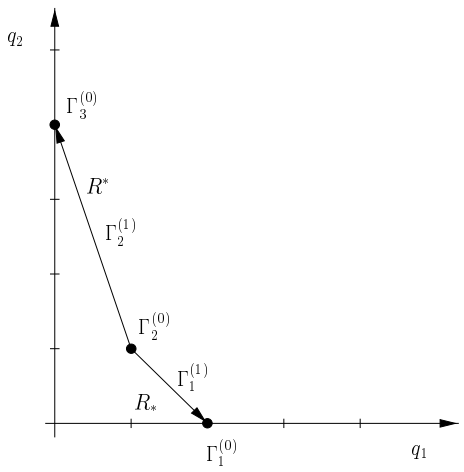


Figure 8.3: The Newton diagram for the system treated in example 14.

is not bijective. Therefore we use the two transformations

$$X = \tilde{X}^{A^T} \text{ and } X = \begin{pmatrix} \tilde{x}^3\tilde{y} \\ -\tilde{x}\tilde{y} \end{pmatrix} \tag{8.6}$$

defined on the set

$$\tilde{U}_2^{(0)} = \{\tilde{X} : |\tilde{x}| \leq \epsilon^{\frac{1}{2}},\ |\tilde{y}| \leq \epsilon^{\frac{1}{2}},\ \tilde{x} \geq 0\}$$

that consists of two quadrants of a neighbourhood of $\tilde{X} = 0$.

The transformations (8.6) transform system (8.5) to the two systems

$$\begin{cases} \frac{\partial \tilde{x}}{\partial \tilde{t}} = -\frac{1}{2}\tilde{x} - \frac{1}{2}\tilde{x}^3 + \frac{1}{2}\tilde{x}\tilde{y}^2 \\ \frac{\partial \tilde{y}}{\partial \tilde{t}} = \frac{5}{2}\tilde{y} + \frac{3}{2}\tilde{x}^2\tilde{y} - \frac{1}{2}\tilde{y}^3 \end{cases} \tag{8.7}$$

and

$$\begin{cases} \frac{\partial \tilde{x}}{\partial \tilde{t}} = -\frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{x}^3 - \frac{1}{2}\tilde{x}\tilde{y}^2 \\ \frac{\partial \tilde{y}}{\partial \tilde{t}} = \frac{5}{2}\tilde{y} - \frac{3}{2}\tilde{x}^2\tilde{y} + \frac{1}{2}\tilde{y}^3 \end{cases} . \tag{8.8}$$

The set $\tilde{U}_2^{(0)}$ *is transformed into the two sets*

$$U_{2,1}^{(0)} = \{X : |X|^{(-1,3)} \leq \epsilon,\ |X|^{(1,-1)} \leq \epsilon,\ xy > 0\}$$
$$U_{2,2}^{(0)} = \{X : |X|^{(-1,3)} \leq \epsilon,\ |X|^{(1,-1)} \leq \epsilon,\ xy < 0\}$$

That form the sector $U_2^{(0)}$. *To find solution curves valid for* $U_{2,1}^{(0)}$ *and* $U_{2,2}^{(0)}$ *the two systems (8.7) and (8.7) are treated like any system with an elementary singular point.*

Figure 8.4 shows $\tilde{U}_2^{(0)}$ *and the sector* $U_2^{(0)}$ *for* $\epsilon = \frac{1}{2}$.

For $\epsilon < 1$ the constructions for the vertices of the Newton diagram yield solution curves that do not entirely cover the initial neighbourhood U. Solutions for the remaining sectors are associated to the vertices of the Newton diagram. They are found using blowing-ups.

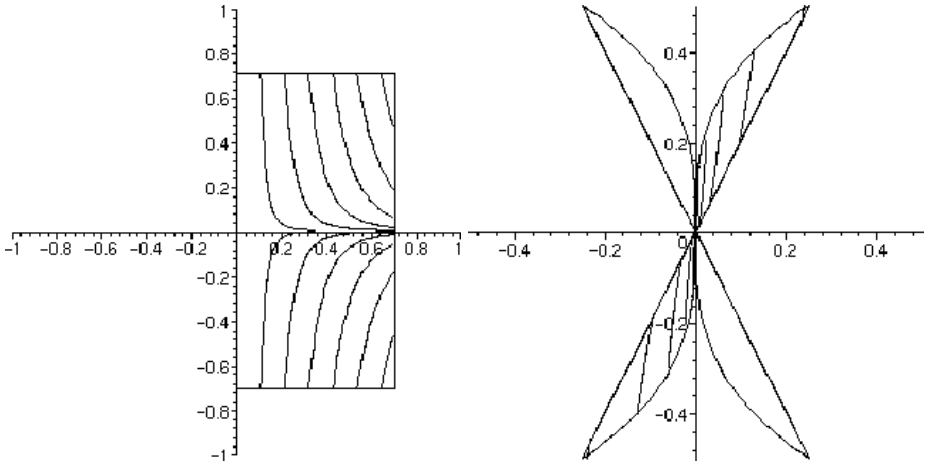


Figure 8.4: The set $\tilde{U}_2^{(0)}$ and the solution curves $\tilde{X}(t)$ otained for the system (8.7) are transformed to the initial coordinates. We obtain the set $U_{2,1}^{(0)}$ and solution curves $X(t)$ for the initial system that have been sketched without considering any problems of convergence (see section 1.5). See also example 14.

8.2 The Edges

The sectors

$$U_j^{(1)}(\epsilon) = \{X : \epsilon \leq |X|^R \leq \epsilon^{-1},\ ||X||_\infty \leq \epsilon\}$$

are associated to the edges $\Gamma_j^{(1)}$ of the Newton diagram. Within these sectors the initial system (8.1) is reduced by quasihomogenous blowing-ups that were introduced in section 4.2.3. These blowing-ups yield a finite number of new systems that have no or "less complex" singularities . The matrix defining the corresponding power transformation is defined by the vectors R verifying

$$r_1 < 0,\ r_2 > 0, gcd(r_1, r_2) = 1$$

and lying on the edges $\Gamma_j^{(1)}$ of the Newton diagram.

8.2.1 Reduction of the Singularity

The sector

$$U_j^{(1)}(\epsilon) = \{X : \epsilon \leq |X|^R \leq \epsilon^{-1},\ ||X||_\infty \leq \epsilon\}$$

is limited by the plane algebraic curves

$$|y| = \epsilon^{\frac{1}{r_2}} |x|^{-\frac{r_1}{r_2}}$$

and

$$|y| = \epsilon^{-\frac{1}{r_2}} |x|^{-\frac{r_1}{r_2}}$$

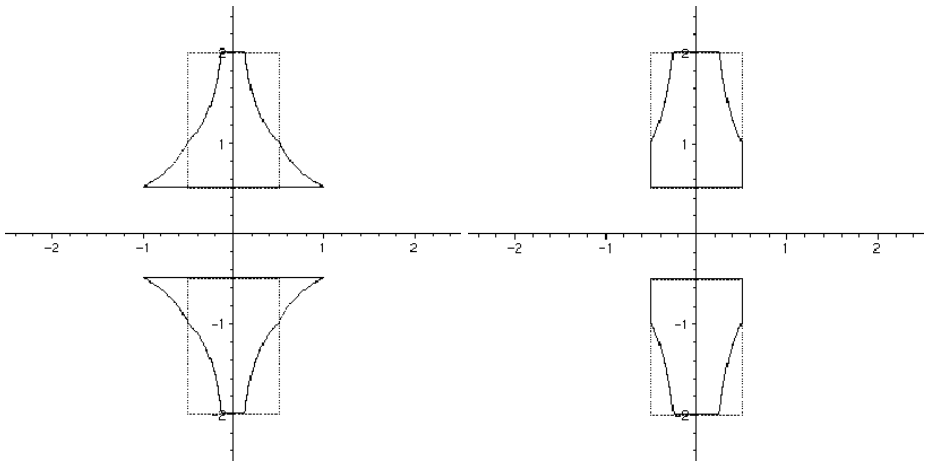


Figure 8.5: The figure shows that the set set $\tilde{U}_j^{(1)}$ does not entirely cover the set (8.11) if any unimodular matrix is used (left). If matrices of the form (8.9) or (8.10) are used, the set $\tilde{U}_j^{(1)}$, that is drawn with dashed lines, entirely contains the set (8.11) (right figure).

To study the behaviour of the system (8.1) within this sector an appropriate blowing-up can separate these curves. This blowing-up is defined by the change of coordinates

$$X = \tilde{X}^{A^T}$$

with the matrix

$$A = \begin{pmatrix} r_2 & -r_1 \\ c & d \end{pmatrix}$$

with $R = (r_1, r_2)$. The matrix coefficients c and d can be chosen such that $detA = 1$. This makes sure that the power transformation is a diffeomorphism (see section 2.3). However, we will also consider the matrices

$$A = \begin{pmatrix} r_2 & -r_1 \\ 0 & 1 \end{pmatrix} \tag{8.9}$$

if r_2 is odd and the matrix

$$A = \begin{pmatrix} r_2 & -r_1 \\ 1 & 0 \end{pmatrix} \tag{8.10}$$

if r_2 is even. Those matrices also define diffeomorphisms. Using those matrices the representation of the sectors $U_j^{(1)}$ is more complicated as in the case of unimodular matrices but the calculated solution curves are valid in the entire sector $U_j^{(1)}$. This is not the case if unimodular matrices are used (see figure 8.5).

Applying a blowing up $X = \tilde{X}^{A^T}$ to the initial system (8.1) yields a new system

$$\frac{\partial \tilde{X}}{\partial t} = \tilde{F}$$

that has the following properties :

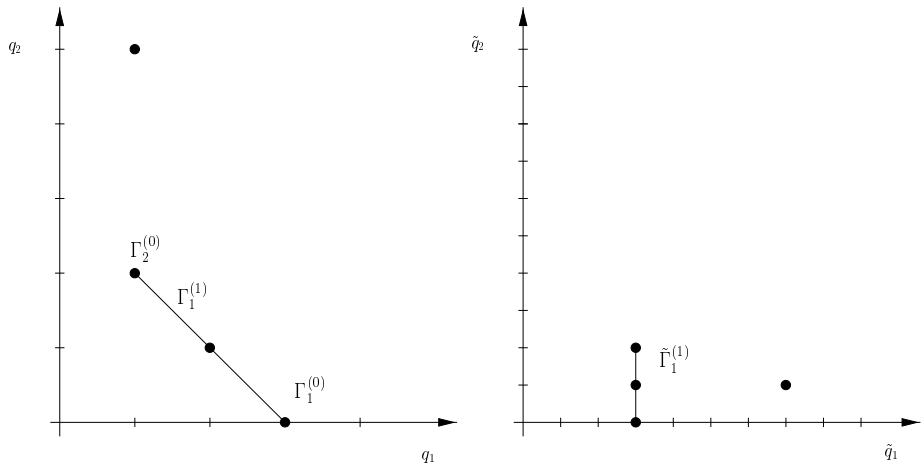


Figure 8.6: The figure shows the Newton diagram and the support of the system treated in example 15. After having applied a blowing up the edge $\Gamma_1^{(1)}$ has been straightened up to the vertical edge $\tilde{\Gamma}_1^{(1)}$.

- For the matrices (8.9) and (8.10) the origin is blown up to the exceptional divisor $\{0\} \times k$ where $k = \mathbb{C}$ or $\mathbb{R}$. In the case of unimodular matrices different from the matrices (8.9) and (8.10) the exceptional divisor is $k \times \{0\} \cup \{0\} \times k$.

- The sector $U_j^{(0)}$ has been transformed to the set

$$\{\tilde{X} : \epsilon^{\frac{1}{|detA|}} \leq |\tilde{y}| \leq \epsilon^{-\frac{1}{|detA|}},\ |\tilde{x}|^{r_2}|\tilde{y}^c| \leq \epsilon, |\tilde{x}|^{-r_1}|\tilde{y}^d| \leq \epsilon\} \tag{8.11}$$

  but we will work with the simplified expression

$$\tilde{U}_j^{(1)} = \{\tilde{X} : \epsilon^{\frac{1}{|detA|}} \leq |\tilde{y}| \leq \epsilon^{-\frac{1}{|detA|}},\ |\tilde{x}| \leq \epsilon^{\frac{1}{|detA|}}\} \tag{8.12}$$

  that contains the set (8.11) for the matrices (8.9) and (8.10).

  In the case of unimodular matrices the sector $\tilde{U}_j^{(1)}$ does in general not entirely contain the set (8.11). This is sketched in figure 8.5.

- The edge $\Gamma_j^{(1)}$ has been "straightened up". That means that the vector R has been transformed to a vertical vector

$$\tilde{R} = A\,R = \begin{pmatrix} 0 \\ detA \end{pmatrix}.$$

  The points $\Gamma_j^{(0)}$ and $\Gamma_{j+1}^{(0)}$ lying on the edge $\Gamma_j^{(1)}$ have been transformed to points with identical $\tilde{q}_1$-coordinates. This is illustrated in figure 8.6.

- As the support of the new system lies entirely on the left of the straightened edge the new system has the form

$$\begin{cases} \frac{\partial \tilde{x}}{\partial t} = \tilde{x}^{s+1}\tilde{f}(\tilde{x},\tilde{y}) \\ \frac{\partial \tilde{y}}{\partial t} = \tilde{y}\tilde{x}^{s}\tilde{g}(\tilde{x},\tilde{y}) \end{cases} \tag{8.13}$$

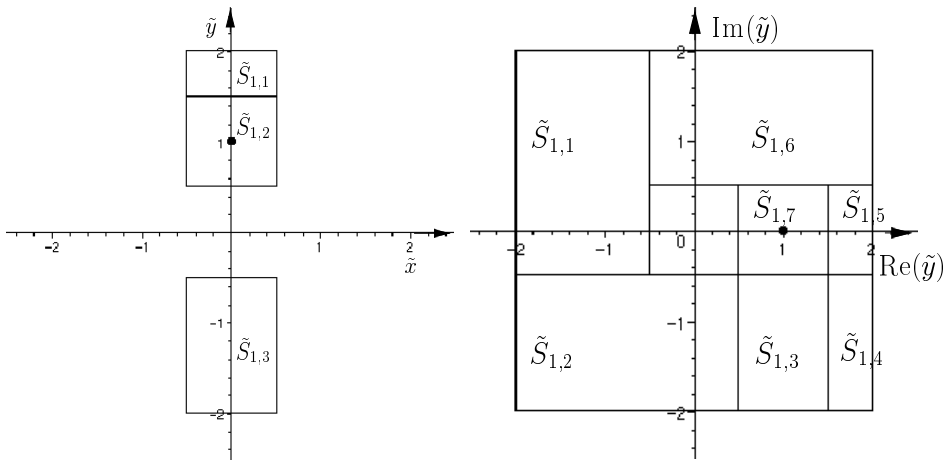


Figure 8.7: The set $\tilde{U}_1^{(1)}$ from example 15 is decomposed in subsectors for the real and the complex case. The only appearing singularity is the one in $\tilde{X} = (0, 1)$.

where s is the q_1-coordinate of the straightened edge.

The initial problem was to find solution curves for the initial system (8.1) in the neighbourhood U of the origin $X = 0$. Applying the blowing-up this has been reduced to a new problem. Now we must find solution curves of a new system (8.13) in the neighbourhood $\tilde{U}_j^{(1)}$ of the $\tilde{y}$-axis. This problem can be solved by splitting the set $\tilde{U}_j^{(1)}$ into so called subsectors and by defining a recursion.

8.2.2 Subsectors and Recursion

To solve the problem of finding solution curves within $\tilde{U}_j^{(1)}$ two different cases have to be considered. To distinguish those cases the truncated system

$$\dot{X} = \hat{F}_j = \begin{pmatrix} x \sum_{\{Q:Q\in\Gamma_j^{(1)}\}} a_Q X^Q \\ y \sum_{\{Q:Q\in\Gamma_j^{(1)}\}} b_Q X^Q \end{pmatrix} \tag{8.14}$$

is defined. This system contains only those monomials in (8.1) that have their support on the edge $\Gamma_j^{(1)}$. By the blowing up the truncated system (8.14) is transformed to the system

$$\begin{cases} \frac{d\tilde{x}}{dt} = \tilde{x}^{s+1}\hat{f}(\tilde{y}) \\ \frac{d\tilde{y}}{dt} = \tilde{x}^s\tilde{y}\hat{g}(\tilde{y}). \end{cases} \tag{8.15}$$

that has its support on the straightened edge $\tilde{\Gamma}_j^{(0)}$.

The transformed truncated system (8.15) is used to distinguish the two cases:

- $\hat{g}(\tilde{y}) \equiv 0$.
  Applying a time transformation $d\tilde{t} = \tilde{x}^{s+1}dt$ to system (8.13) yields

$$\begin{cases} \frac{\partial\tilde{x}}{\partial\tilde{t}} = \tilde{f}(\tilde{x}, \tilde{y}) \\ \frac{\partial\tilde{y}}{\partial t} = \frac{\tilde{y}}{\tilde{x}}\tilde{g}(\tilde{x}, \tilde{y}). \end{cases} \tag{8.16}$$

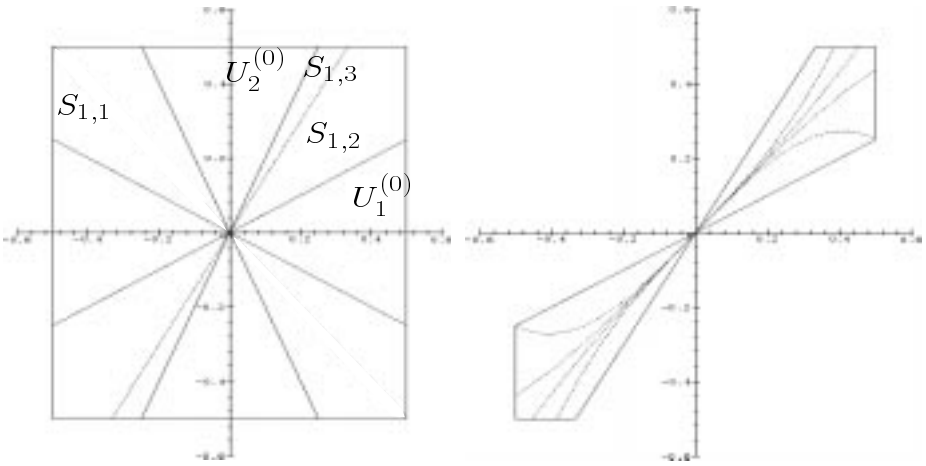


Figure 8.8: The division of the neighbourhood of the origin into sectors and subsectors in the first and the second level of recursion for the system treated in example 15 ($U_1^{(1)} = S_{1,1} \cup S_{1,2} \cup S_{1,3}$).

To calculate the new singular points on the exceptional divisor let $\tilde{x} = 0$. The resulting system

$$\begin{cases} \frac{d\tilde{x}}{dt} = \hat{f}(\tilde{y}) \\ \frac{d\tilde{y}}{dt} = \left(\frac{\tilde{y}}{\tilde{x}}\tilde{g}(\tilde{x},\tilde{y})\right)_{\tilde{x}=0}. \end{cases}$$

has the following properties:

1. The y-axis is no solution curve.
2. All points in $\{(0,\tilde{y}) : \tilde{f}(0,\tilde{y}) \neq 0\}$ are regular points and $\tilde{x}(t)$ is not constant there. Therefore there exists a solution curve passing through each of those points.
3. All points in $\{(0,\tilde{y}) : \tilde{f}(0,\tilde{y}) = 0, \left(\frac{\tilde{y}}{\tilde{x}}\tilde{g}(\tilde{x},\tilde{y})\right)_{\tilde{x}=0} \neq 0\}$ are regular points but here the solution curves are parallel to the $\tilde{y} - axis$ as $\tilde{x}(t)$ is constant. Those points are called tangencies.
4. All points in $\{(0,\tilde{y}) : \tilde{f}(0,\tilde{y}) = 0, \left(\frac{\tilde{y}}{\tilde{x}}\tilde{g}(\tilde{x},\tilde{y})\right)_{\tilde{x}=0} = 0\}$ are singular points. Those points are translated to the origin and the entire algorithm is applied recursively to the resulting systems.

This case is called the dicritical case as tangencies and singularities can occur.

- $\hat{g}(\tilde{y}) \not\equiv 0$.
  Applying the time transformation $d\tilde{t} = \tilde{x}^s dt$. (8.13) yields the system

$$\begin{cases} \frac{\partial\tilde{x}}{\partial\tilde{t}} = \tilde{x}\tilde{f}(\tilde{x},\tilde{y}) \\ \frac{\partial\tilde{y}}{\partial\tilde{t}} = \tilde{y}\tilde{g}(\tilde{x},\tilde{y}). \end{cases} \tag{8.17}$$

On the exceptional divisor, for $\tilde{x} = 0$ the system is

$$\begin{cases} \frac{\partial \tilde{x}}{\partial t} = 0 \\ \frac{\partial \tilde{y}}{\partial t} = \tilde{y}\hat{g}(\tilde{y}). \end{cases} \tag{8.18}$$

Some properties for the system (8.17) can now be given:

1. The y-axis ($\tilde{x}(t) = 0, \tilde{y}(t) = t$) is a solution curve of system (8.17).
2. All points $\{\tilde{X} \in \tilde{U}_j^{(0)} : \tilde{g}(0, \tilde{y}) \neq 0\}$ are regular points. From equation (8.18) shows that here the solution curves will be parallel to the y-axis.
3. All points in $\{\tilde{X} \in \tilde{U}_j^{(0)} : \tilde{g}(0, \tilde{y}) = 0\}$ are singular points. They might be elementary or not. Those points are translated to the origin and the entire algorithm is applied recursively to the resulting systems.

This case is called the noncritical case as tangencies do not occur.

The above classification has shown that in any case the resulting system can be solved in any point. A central point is the recursive application of the entire algorithm. Those recursions have to be defined such that they yield solution curves for the whole set $\tilde{U}_j^{(1)}$. Therefore the set $\tilde{U}_j^{(1)}$ is split into so called subsectors $\tilde{S}_{j,1}, \ldots, \tilde{S}_{j,k}$. Any of those subsectors contains either one singular point or only regular points. The subsectors are considered as neighbourhoods of the points $\tilde{X}_{j,1}, \ldots, \tilde{X}_{j,k}$. The point $\tilde{X}_{j,i}$ with $i = 1, \ldots, k$ is either a singular points or any regular point. A possible splitting of a set $\tilde{U}_j^{(1)}$ into subsectors in the case of real and complex variables is sketched in figure 8.7.

With the systems (8.16) or (8.17), the subsector $\tilde{S}_{j,i}$ and the point of interest $\tilde{X}_{j,i}$ the entire algorithm can be called again. The first step in the next level of recursion will be a translation of the point $\tilde{X}_{j,i}$ to the origin.

If the singularity has not been reduced by a first application of a blowing-up another one is applied. A. van den Essen [61] has proved that any isolated singularity can be reduced entirely by a finite number of blowing-ups. So this algorithm will come to an end after a finite number of steps.

Example 15 *The Newton diagram of the considered system*

$$\dot{X} = \begin{pmatrix} -x^4 + yx^3 \\ \frac{13}{9}y^2x^6 - x^2y^2 + xy^3 \end{pmatrix} \tag{8.19}$$

has only one edge $\Gamma_1^{(1)}$ *with* $R = \binom{-1}{1}$ *(see figure 8.6). For* $\epsilon = \frac{1}{2}$ *the sector*

$$U_1^{(1)} = \{X : \frac{1}{2} \leq |x|^{-1}|y| \leq 2,\ |x| \leq \frac{1}{2},\ |y| \leq \frac{1}{2}\}$$

is associated to this edge. The blowing-up $X = \tilde{X}^{A^T}$ *with*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

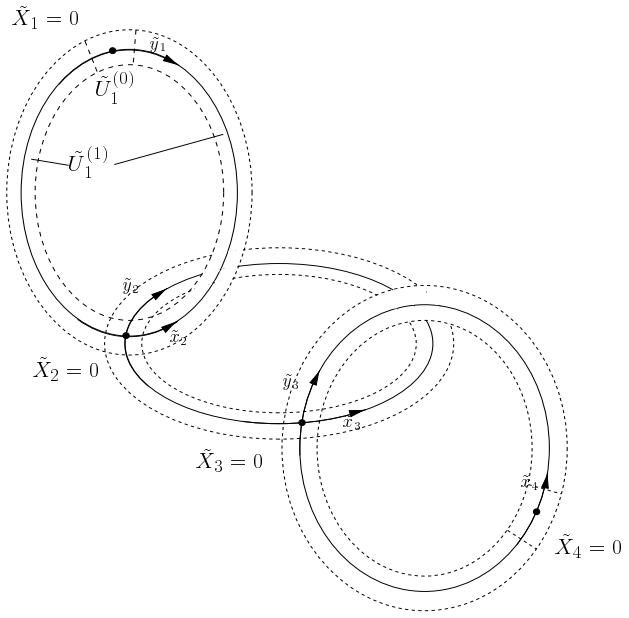


Figure 8.9: This figure shows the structure of the exceptional divisors of the blowing-ups used for the faces of the Newton diagram.

transforms $U_1^{(1)}$ *into*

$$\tilde{U}_1^{(1)} = \{\tilde{X} : \frac{1}{2} \leq |\tilde{y}| \leq 2,\ |\tilde{x}| \leq \frac{1}{2}\}$$

and yields the new system

$$\dot{\tilde{X}} = \begin{pmatrix} -\tilde{x}^4 + \tilde{x}^4\tilde{y} \\ \tilde{x}^3\tilde{y} - 2\tilde{x}^3\tilde{y}^2 + x^3y^3 + \frac{13}{9}x^7y^2 \end{pmatrix}.$$

That means a noncritical case. A new time transformation with $s = 3$. *yields a resulting system that has a nonelementary singular point in* $(0, 1)$. *For the real case the sector* $\tilde{U}_1^{(1)}$ *is divided into 3 subsectors as shown in figure 8.7. The subsectors* $\tilde{S}_{1,1}$ *and* $\tilde{S}_{1,3}$ *contain only simple points but the remaining subsector* $\tilde{S}_{1,2}$ *contains a nonelementary singular points. Therefore in the second level of recursion another blowing-up has to be performed.*

In the second level of recursion the solution curves for all sectors and subsectors of $S_{1,2}$ *can be calculated directly as the blown-up system has only simple points. Figure 8.8 shows the sectors and subsectors resulting for this example in the first and second level of recursion.*

This chapter has shown how blowing-ups and time transformations can be used to find solution curves for two-dimensional systems of differential equations in the neighbourhood of a nonelementary singular point. The computed solutions are valid only on parts of the neighbourhood that are called sectors.

For both cases, for the case of an edge and for the case of a vertex, blowing-ups are used to reduce the considered system. This fact imposes to study the structure of the blowing-ups and their exceptional divisors more closely. Consider the blowing-ups for the vertices $\Gamma_j^{(0}$ and $\Gamma_{j+1}^{(0}$ and the edge $\Gamma_j^{(1)}$. They are denoted by

$$X = \tilde{X}_j^{A_j^T},\ X = \tilde{X}_{j+1}^{A_{j+1}^T}\ and\ X = \hat{X}_j^{B_j^T}$$

respectively. As those power transformations are invertible anywhere except in their exceptional divisors we can construct the transformations

$$\tilde{X}_j = (\tilde{X}_{j+1}^{A_{j+1}^T})^{A_j^{-T}} \tag{8.20}$$

and

$$\tilde{X}_j = (\hat{X}_j^{B_j^T})^{A_j^{-T}}\ . \tag{8.21}$$

Those power transformations can be completed in the exceptional divisors of the initial power transformations. Computing the transformations (8.20) and (8.21) explicitely yields

$$\tilde{y}_j = \frac{1}{\tilde{x}_{j+1}^k},\ k > 0,\ k \in \mathbb{Q}$$

and

$$\tilde{y}_j = \frac{1}{\hat{y}_j^k},\ k > 0,\ k \in \mathbb{Q}$$

if the matrix (8.9) is used to define the blowing-up for the edge and

$$\tilde{y}_j = \hat{y}_j^k,\ k > 0,\ k \in \mathbb{Q}$$

if the matrix (8.10) is used. That means that a part of the exceptional divisor for all three considered blowing-ups is identical. However as

$$lim_{\tilde{x}_{j+1} \to 0}\ \tilde{y}_j = \infty$$

the point $\tilde{X}_j = (0, \infty)$ is identic to the point $\tilde{X}_{j+1} = (0, 0)$. To visualize this relation the lines $\tilde{x}_j = 0$ and $\tilde{y}_j = 0$ of the exceptional divisors can be displayed as circles as it has been done in figure 8.9. This figure shows the structure of the exceptional divisors computed for a Newton diagram that consists of four vertices and three edges. The neighbourhoods of the exceptional divisors are split into the sets $\tilde{U}_1^{(0)}, \ldots, \tilde{U}_3^{(1)}$ according to this figure.

Now, it is obvious that the power transformations applied for the reductions of class $\mathcal{V}$ systems could also be used for the case of an edge to define a recursive process. Nevertheless the case of an edge and the case of a vertex of the Newton diagram are treated separately for two reasons. The sector definition is simpler and the use of transformations that are no diffeomorphisms to define recursions would increase the cost of the algorithm. This is due to the additional constructions introduced in section 2.3 that have to be used in the case of non-injective power transformations.

Chapter 9

Three- and higher-dimensional elementary singular points

The case of a two-dimensional elementary singular point has been subject of chapter 7. There it was shown that any two-dimensional system can be integrated in the neighbourhood of an elementary singular point. In this chapter we are interested to know whether these results can be extended to higher dimensional systems.

Like in the case of two-dimensional systems the first step is the transformation of the considered system to Poincaré-Dulac normal form. For higher-dimensional systems the computation of the Jordan form might cause some problems that were already mentioned in section 3.2. However, here we will presume that the Jordan form can be computed and that the considered system

$$\dot{X} = F(X) \tag{9.1}$$

is given by a formal power serie and that it is in Poincaré-Dulac normal form. The resonant plane

$$M = \{Q \in \mathcal{N} : \langle Q, \Lambda \rangle = 0\}$$

of the normal form (9.1) has been defined in section 3.1. It plays a central role in the reduction and integration of the system (9.1). The set M is used to define the vector space

$$\overline{M} = \{P : P = \sum \alpha_i Q_i,\ \alpha_i \in \mathbb{R},\ Q_i \in M\}\ .$$

The methods used for the reduction of the normal form depend on m that denotes the maximum number of linearly independent vectors in $\overline{M}$. It is obvious that m is smaller than the dimension of the system.

For $m = 0$ the system (9.1) can be integrated directly as it is linear and the matrix $DF(0)$ is in Jordan form. For $m = 1$ the considered normal form can always be integrated. The integration methods are similar to those already used in section 7.1. If M contains two linearly independent vectors one or several power transformations can be used to reduce the normal form. The reduction yields systems that can be integrated via the integration of a m-dimensional nilpotent system. However, the employed power transformations have to verify very strict conditions.

The methods for reducing systems with elementary singular points are extensively studied for three-dimensional systems. A generalization to n-dimensional systems is pos-

sible but some further difficulties arise from the fact that a three- or higher-dimensional cone can be spanned by any number of vectors.

The general ideas for the algorithms described in this chapter have been studied by A.Bruno [9] and L. Brenig and A. Goriely [23]. In this chapter many aspects of those algorithms are studied more intensely and some further constructions are introduced. The sections 9.2 and 9.3.3 deal with real normal forms for systems with two complex conjugated, purely imaginary eigenvalues. They are based on works from S. Chow, C. Li and D. Wang [15].

9.1 Integration of n-dimensional normal forms for $m = 1$

If the resonant plane M of a normalized system (9.1) contains only one linearly independent vector ($m = 1$) the results from section 7.1 (case 3 and 4) can be generalized. Two cases have to be studied separately:

- $M \cap \mathbb{N}^n = \{0\}$
  Without loss of generality it can be presumed that

$$Q_0 = \begin{pmatrix} -1 \\ q_{02} \\ \vdots \\ q_{0n} \end{pmatrix}$$

  with $q_{0i} \in \mathbb{N}$ is the only vector different from $Q = 0$ that lies in M. The normalized system (9.1) has the form

$$\frac{\partial X}{\partial t} = DF(0)X + \begin{pmatrix} a_{1,Q_0} X^{(0,q_{02},\ldots,q_{0n})} \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

  It can be integrated using the method of the variation of the constant.

- $M \cap \mathbb{N}^n \neq \{0\}$
  There exists a vector $Q_0 \in M$ such that all other vectors $Q \in M$ can be written as kQ_0 with $k \in \mathbb{N}$. The normalized system (9.1) has the form

$$\dot{X} = DF(0)X + \begin{pmatrix} x_1 \sum_k a_{1k} X^{kQ_0} \\ \ldots \\ x_n \sum_k a_{nk} X^{kQ_0} \end{pmatrix} . \tag{9.2}$$

  Like in section 7.1 there exists a power transformation $X = \tilde{X}^{A^T}$ that transforms this system to a system

$$\frac{\partial \tilde{X}}{\partial t} = D\tilde{F}(0)\tilde{X} + \begin{pmatrix} \tilde{x}_1 \sum_k \tilde{a}_{1k} \tilde{x}_1^k \\ \ldots \\ \tilde{x}_n \sum_k \tilde{a}_{nk} \tilde{x}_1^k \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 g_1(\tilde{x}_1) \\ \ldots \\ \tilde{x}_n g_n(\tilde{x}_1) \end{pmatrix} \tag{9.3}$$

that can be integrated. If $g_1 \not\equiv 0$ the resulting equation (9.3) can be transformed to a new system of differential equations

$$\begin{cases} \frac{\partial \tilde{x}_1}{\partial \tilde{t}} = 1 \\ \frac{\partial log \tilde{x}_i}{\partial \tilde{t}} = \frac{g_i(\tilde{x}_1)}{\tilde{x}_1 g_1(\tilde{x}_1)}, \ i = 2, \ldots, n \end{cases} \tag{9.4}$$

by a time transformation

$$d\tilde{t} = \tilde{x}_1 g_1(\tilde{x}_1)\, dt$$

In equation (9.4) each line is integrable. The lines $2, \ldots, n$ can be integrated to

$$log \tilde{x}_i = \int^{\tilde{t}} \frac{g_i(s)}{s g_1(s)} ds + c_i \ .$$

$g_1 \equiv 0$ yields $\tilde{x}_1(t) = c_1$ for the first variable. As a consequence equation (9.3) has the form

$$\frac{\partial X}{\partial t} = \begin{pmatrix} 0 \\ \tilde{x}_2 k_2 \\ \vdots \\ \tilde{x}_n k_n \end{pmatrix}$$

with the constant terms $k_i = g_i(c_1)$, $i = 2, \ldots, n$. This can formally be integrated to

$$\tilde{X}(t) = (c_1, c_2 e^{k_2 t}, \ldots, c_n e^{k_n t}) \ .$$

The matrix A for the power transformation is best calculated via its adjoint matrix $A^* = detA\, A^{-1}$. Any matrix A with

$$A^* = (Q_0 | \ldots)$$

such that A^* has no negative entries can be used. The fact that A^* has no negative entries guarantees that the power transformation is no blowing-up.

Like in section 7.1 the particular solutions

$$\begin{aligned} X(\tilde{t}) &= (\tilde{t}, c_1, \ldots, c_{n-1}) \ with \ \tilde{t} = e^{\lambda_1 t}, \ c_1, \ldots, c_{n-1} \in \mathbb{R} \ or \ \mathbb{C} \\ &\ldots \\ X(\tilde{t}) &= (c_1, \ldots, c_{n-1}, \tilde{t}) \ with \ \tilde{t} = e^{\lambda_n t}, \ c_1, \ldots, c_{n-1} \in \mathbb{R} \ or \ \mathbb{C} \end{aligned}$$

of equation (9.2) have to be considered separately if they can not be obtained from solution curves calculated for the reduced system (9.3).

Example 16 *The normal form of the considered system*

$$\dot{X} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} X + \begin{pmatrix} x_1^2 x_2 x_3 \\ 0 \\ -x_1 x_2 x_3^2 \end{pmatrix}$$

has the form

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 1+\sqrt{2} & & \\ & 1-\sqrt{2} & \\ & & -2 \end{pmatrix} \tilde{X} + \begin{pmatrix} \frac{\sqrt{2}}{16} \tilde{x}_1^2 \tilde{x}_2 \tilde{x}_3 \\ -\frac{\sqrt{2}}{16} \tilde{x}_1 \tilde{x}_2^2 \tilde{x}_3 \\ 0 \end{pmatrix} \ .$$

The resonant plane

$$M = \left\{ k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : k \in \mathbb{N} \right\}$$

is transformed to the $\tilde{q}_1$-axis by the change of coordinates $\tilde{X} = \bar{X}^{A^T}$ with

$$A^* = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} .$$

This yields the new system

$$\frac{\partial \bar{X}}{\partial t} = \begin{pmatrix} 0 & & \\ & 1+\sqrt{2} & \\ & & 1-\sqrt{2} \end{pmatrix} \bar{X} + \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{16}\bar{x}_2\bar{x}_1 \\ -\frac{\sqrt{2}}{16}\bar{x}_3\bar{x}_1 \end{pmatrix}$$

that can be integrated to

$$\bar{X}(t) = \left(c_1, c_2 e^{(\sqrt{2}+1+\frac{\sqrt{2}}{16}c_1)t}, c_3 e^{(-\sqrt{2}+1-\frac{\sqrt{2}}{16}c_1)t} \right)$$

with $c_1, c_2, c_3 \in \mathbb{R}$ or $\mathbb{C}$.

9.2 Integration of real n-dimensional normal forms for $m = 1$

A particular normal form with $m = 1$ can issue for a system with real coefficients and two complex conjugated, purely imaginary eigenvalues. The two-dimensional case has been the subject of section 7.2.

The system $\dot{X} = F(X)$ that has real coefficients and the purely imaginary eigenvalues λ_1 and $\lambda_2 = -\lambda_1$ can be transformed to a system

$$\frac{\partial \tilde{X}}{\partial t} = \tilde{F}(\tilde{X})$$

with real coefficients where the linear part is in real Jordan form

$$D\tilde{F}(0) = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix} .$$

The block

$$B_1 = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

represents the eigenvalues λ_1 and λ_2 ($\alpha = \mathrm{Im}(\lambda_1)$). The other blocks are either Jordan blocks for real eigenvalues or real Jordan blocks for complex conjugated eigenvalues. The used linear transformation $X = T\tilde{X}$ is also real.

The next step is the application of a linear transformation

$$\tilde{X} = P\bar{X} \ with \ P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \\ -\frac{i}{2} & \frac{i}{2} & \\ & & \ddots \end{pmatrix}$$

that yields a new system with a linear part in Jordan form

$$D\bar{F}(0)\bar{X} = \begin{pmatrix} i\alpha & 0 & \\ 0 & -i\alpha & \\ & & \ddots \end{pmatrix} \bar{X} \ .$$

This system is transformed to its Poincaré-Dulac normal form

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} i\alpha & 0 & \\ 0 & -i\alpha & \\ & & \ddots \end{pmatrix} \hat{X} + \begin{pmatrix} \hat{x}_1 \sum a_{1k} \hat{x}_1^k \hat{x}_2^k \\ \hat{x}_2 \sum a_{2k} \hat{x}_1^k \hat{x}_2^k \\ \cdots \\ \hat{x}_n \sum a_{2n} \hat{x}_1^k \hat{x}_2^k \end{pmatrix}$$

that has the particular form $a_{1k} = \overline{a_{2k}}$. Applying the linear transformation $\hat{X} = P\tilde{\tilde{X}}$ to the normalized system yields a new system

$$\frac{\partial \tilde{\tilde{X}}}{\partial t} = \begin{pmatrix} 0 & \alpha & \\ -\alpha & 0 & \\ & & \ddots \end{pmatrix} \tilde{\tilde{X}} + \begin{pmatrix} \sum (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k (\mathrm{Re}(a_{1k}) \tilde{\tilde{x}}_1 - \mathrm{Im}(a_{1k}) \tilde{\tilde{x}}_2) \\ \sum (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k (\mathrm{Im}(a_{1k}) \tilde{\tilde{x}}_1 + \mathrm{Re}(a_{1k}) \tilde{\tilde{x}}_2) \\ \tilde{\tilde{x}}_3 \sum \tilde{\tilde{a}}_{3k} (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k \\ \cdots \\ \tilde{\tilde{x}}_n \sum \tilde{\tilde{a}}_{nk} (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k \end{pmatrix}$$

Introducing polar coordinates

$$\tilde{\tilde{X}} = \begin{pmatrix} r \sin \phi \\ r \cos \phi \\ \tilde{\tilde{x}}_3 \\ \vdots \\ \tilde{\tilde{x}}_n \end{pmatrix}$$

for the coordinates $\tilde{\tilde{x}}_1$ and $\tilde{\tilde{x}}_2$ yields the integrable system

$$\begin{cases} \dot{r} &= c_1 r + c_3 r^3 + \ldots \\ \dot{\phi} &= d_0 + d_2 r^2 + d_4 r^4 + \ldots \\ \frac{\partial \tilde{\tilde{x}}_3}{\partial t} &= \tilde{\tilde{x}}_3 (e_0 + e_2 r^2 + e_4 r^4 + \ldots) \\ \cdots \end{cases} .$$

Example 17 *Consider the system*

$$\dot{X} = \begin{pmatrix} 0 & 4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} X + \begin{pmatrix} 3\,{x_1}^2\,x_2 - {x_3}^3 \\ 0 \\ {x_1}^2\,x_2\,x_3 - x_1\,x_2 \end{pmatrix} .$$

Its linear part is in real Jordan form. Transforming it to Jordan form using the transformation $X = P\tilde{X}$ yields the new system

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} -4i & 0 & 0 \\ 0 & 4i & 0 \\ 0 & 0 & 2 \end{pmatrix} \tilde{X} + \begin{pmatrix} \frac{3}{8} I \tilde{x}_1^3 - \frac{3}{8} I \tilde{x}_1^2 \tilde{x}_2 + \frac{3}{8} I \tilde{x}_1 \tilde{x}_2^2 + \frac{3}{8} I \tilde{x}_2^3 - \tilde{x}_3^3 \\ \frac{3}{8} I \tilde{x}_1^3 - \frac{3}{8} I \tilde{x}_1^2 \tilde{x}_2 + \frac{3}{8} I \tilde{x}_1 \tilde{x}_2^2 + \frac{3}{8} I \tilde{x}_2^3 - \tilde{x}_3^3 \\ \frac{1}{4} I \tilde{x}_1^2 - \frac{1}{4} I \tilde{x}_2^2 + \dots \end{pmatrix} .$$

This system is transformed to Poincaré-Dulac normal form

$$\frac{\partial \bar{X}}{\partial t} = \begin{pmatrix} -4I & 0 & 0 \\ 0 & 4I & 0 \\ 0 & 0 & 2 \end{pmatrix} \bar{X} + \begin{pmatrix} -4\, I\, \bar{x}_1 - \frac{3}{8} I \bar{x}_1^2 \bar{x}_2 + \frac{27}{1024} I \bar{x}_1^3 \bar{x}_2^2 + \dots \\ 4\, I\, \bar{x}_2 + \frac{3}{8} I \bar{x}_1 \bar{x}_2^2 - \frac{27}{1024} I \bar{x}_1^2 \bar{x}_2^3 + \dots \\ \bar{x}_3 - \frac{75159}{2211825664} \bar{x}_1^4 \bar{x}_3 \bar{x}_2^4 + \dots \end{pmatrix}$$

and retransformed by the transformation $\bar{X} = P^{-1}\hat{X}$. This yields the new system

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} 0 & 4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \hat{X} + \begin{pmatrix} \frac{3}{8} \hat{x}_1^2 \hat{x}_2 + \frac{3}{8} \hat{x}_2^3 + \dots \\ -\frac{3}{8} \hat{x}_1^3 - \frac{3}{8} \hat{x}_1 \hat{x}_2^2 + \dots \\ -\frac{2245477}{1105912834} \hat{x}_1^4 \hat{x}_2^4 \hat{x}_3 + \dots \end{pmatrix} .$$

Introducing polar coordinates

$$\hat{X} = \begin{pmatrix} r \sin\phi \\ r \cos\phi \\ \hat{x}_3 \end{pmatrix}$$

yields a new integrable system

$$\begin{cases} \frac{\partial r}{\partial t} &= \frac{843792}{5399965} r^9 + \dots \\ \frac{\partial \phi}{\partial t} &= -4 - 6\, r^2 + \frac{27}{4}\, r^4 + \dots \\ \frac{\partial \hat{x}_3}{\partial t} &= 2\, \hat{x}_3 + \dots \ . \end{cases}$$

9.3 Reduction of three-dimensional normal forms for m=2

The reduction and integration of three dimensional normal forms is of particular interest as the dimension of the resonant plane does not exceed 2 and two-dimensional systems can be integrated by methods introduced previously in the chapters 5 to 8.

Three-dimensional normal forms have a resonant plane with 0,1 or 2 linearly independent vectors. For $m = 0$ the integration of the normal form is obvious. For $m = 1$ the algorithms from section 9.1 can be used to integrate the considered normal form. For $m = 2$ a power transformation $X = \tilde{X}^{A^T}$ can be used to reduced the normal form to a two-dimensional system. The choice of the matrix A is the main problem.

Systems that are treated in a particular way in section 9.3.3 are systems with real coefficients and two complex conjugated, purely imaginary eigenvalues. It will be shown that they can be reduced to systems that are also real.

9.3.1 The Choice of the Matrix A

If M contains 2 linearly independent vectors a power transformation can be used to simplify the normal form (9.1).

Suppose that the vectors Q_1, $Q_2 \in M$ form a basis for $\overline{M}$. Then the normal form (9.1) can also be written as

$$\dot{X} = \begin{pmatrix} x_1 \sum_{k\geq 0} \sum_{|R|=k} a_{1R} X^R \\ x_2 \sum_{k\geq 0} \sum_{|R|=k} a_{2R} X^R \\ x_3 \sum_{k\geq 0} \sum_{|R|=k} a_{3R} X^R \end{pmatrix}$$

with

$$R = \alpha_1 Q_1 + \alpha_2 Q_2$$

and

$$|R| = |r_1| + |r_2| + |r_3|$$

where $R = (r_1, r_2, r_3)$. $\alpha_1 \in \mathbb{Z}$ and $\alpha_2 \in \mathbb{Z}$ are chosen such that $R \in \mathcal{N}$. According to theorem 3 in section 2.2 applying a power transformation $X = \tilde{X}^{A^T}$ with

$$AQ_1 = \det A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; and \; AQ_2 = \det A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

yields a new system

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} \tilde{x}_1 \sum_{k\geq 0} \sum_{|\tilde{R}|=k} \tilde{a}_{1\tilde{R}} \tilde{X}^{\tilde{R}} \\ \tilde{x}_2 \sum_{k\geq 0} \sum_{|\tilde{R}|=k} \tilde{a}_{2\tilde{R}} \tilde{X}^{\tilde{R}} \\ \tilde{x}_3 \sum_{k\geq 0} \sum_{|\tilde{R}|=k} \tilde{a}_{3\tilde{R}} \tilde{X}^{\tilde{R}} \end{pmatrix} \tag{9.5}$$

with

$$\tilde{R} = \alpha_1 AQ_1 + \alpha_2 AQ_2, \; |\tilde{R}| = |\tilde{r}_1| + |\tilde{r}_2|) \; .$$

The system (9.5) can also be written in the form

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} \tilde{x}_1 g_1(\tilde{x}_1, \tilde{x}_2) \\ \tilde{x}_2 g_2(\tilde{x}_1, \tilde{x}_2) \\ \tilde{x}_3 g_3(\tilde{x}_1, \tilde{x}_2) \end{pmatrix} \; . \tag{9.6}$$

The first two lines only depend on the variables $\tilde{x}_1$ and $\tilde{x}_2$. The 2-dimensional system

$$\begin{cases} \frac{\partial \tilde{x}_1}{\partial t} = \tilde{x}_1 g(\tilde{x}_1, \tilde{x}_2) \\ \frac{\partial \tilde{x}_2}{\partial t} = \tilde{x}_2 g_2(\tilde{x}_1, \tilde{x}_2) \end{cases} \tag{9.7}$$

is formally integrated by previously introduced methods. If the system (9.7) is integrable the solution curves $\tilde{X}(t)$ can be obtained from the solution curves $(\tilde{x}_1(t), \tilde{x}_2(t))$ by formally integrating the scalar differential equations

$$\frac{\partial log \, \tilde{x}_3}{\partial t} = g_3(\tilde{x}_1(t), \tilde{x}_2(t))$$

that is equivalent to the remaining equation in system (9.6). The problem of finding solution curves for the initial normalized system (9.1) has been reduced to the problem of finding an appropriate power transformation and an appropriate matrix A. A is best constructed via its adjoint matrix

$$A^* = (Q_1 | Q_2 | \ldots)$$

where the vectors Q_1 and Q_2 define the first and the second row vectors. However some essential properties are required for the system (9.6). Those properties affect the choice of the vectors Q_1 and Q_2 and therefore the computaion of the matrix A.

- as it has been shown the vectors Q_1 and Q_2 have to form basis of M.
- the coefficients of the system (9.6) have to be integer and positive. Otherwise the resulting system can not be integrated by the previously introduced algorithms.
- the power transformation $X = \tilde{X}^{A^T}$ has to be a diffeomorphism. This is verified if the matrix A is unimodular. If A is an non-unimodular invertible matrix the constructions introduced in section 2.3 can be used to define a corresponding bijective diffeomorphism.
- if the power transformation $X = \tilde{X}^{A^T}$ is not a blowing-up, the solutions of the resulting systems are valid in the whole neighbourhood of $X = 0$.

  If the power transformation $X = \tilde{X}^{A^T}$ is a blowing-up, the reduction and integration yield solutions that are only valid in sectors As a single transformation is not sufficient to cover a neighbourhood of $X = 0$ with sectors, a serie of blowing-ups controlled by a Newton diagram are needed.

The following classification allows to construct appropriate matrices A for any three-dimensional normal form.

9.3.2 The Classification

In this section a classification of three-dimensional Poincaré-Dulac normal forms is given. This classification uses the eigenvalues of the linear part of the system and can easily be implemented. It allows an exact definition of the resonant plane M. Further it makes sure that all cases are considered and it allows the reduction of any three-dimensional normal form.

In the study of three-dimensional normal forms we will focus on systems with rational eigenvalues and show later that all other cases can be derived from these systems.

1. $\lambda_1 > 0,\ \lambda_2 < 0,\ \lambda_3 = 0$
   Let
   $$k_1 := min\{k \in \mathbb{Q} : \frac{\lambda_1}{k} \in \mathbb{Z}\ and\ \frac{\lambda_2}{k} \in \mathbb{Z}\}$$
   the set M can be written as $M = \{\frac{\alpha}{k_1}\begin{pmatrix} -\lambda_2 \\ \lambda_1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} :\ \alpha \in \mathbb{N},\ \beta \in \mathbb{N},\ \alpha+\beta \geq 0\}$
   The support lies on a plane ($m = 2$). All points in $supp(F)$ are in $\mathbb{N}^3$.

   Let $Q_1, Q_2 \in M \cap \mathbb{N}^3$ be the two vectors that span the cone V that includes entirely the support of the considered system. Then the power transformation $X = \tilde{X}^{A^T}$ defined by the matrix
   $$A^* = \left(Q_1|Q_2|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)$$
   with $a, b, c \in \mathbb{N}$ yields a system that verifies all conditions for the further integration of the corresponding two-dimensional system.

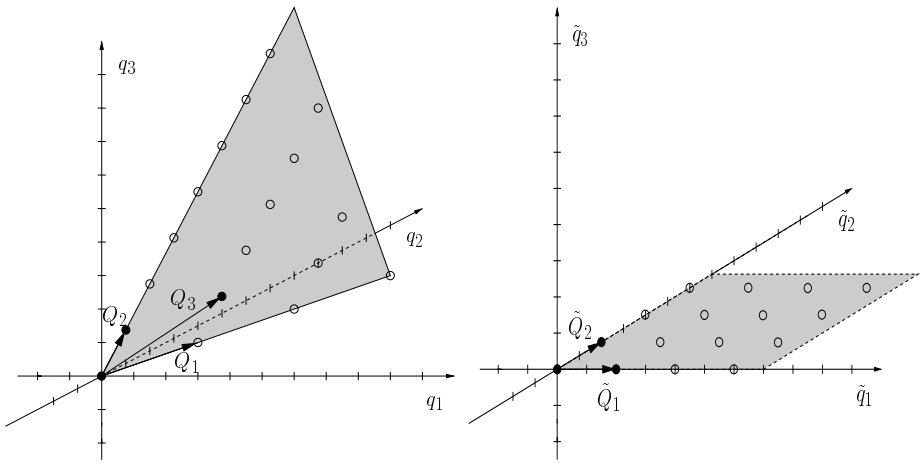


Figure 9.1: This picture shows the plane $< \Lambda, Q >= 0$, $supp(F)$ and the vectors that are choosen for the construction of the matrix A^* in example 18. The support of the new system lies in the $\tilde{q}_1\tilde{q}_2$-plane.

Proof 4 *Any point $Q \in supp(F)$ can be written as*

$$Q = \alpha_1 Q_1 + \alpha_2 Q_2$$

with $\alpha_1, \alpha_2 \geq 0$. Q is transformed to the point

$$\tilde{Q} = \alpha_1 \det A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \det A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

that has only positive coordinates. As A has only integer coefficients the exponents $\tilde{Q} = AQ$ of the new system are integer and positive. The first and the second line in the resulting system only depend on the variables $\tilde{x}_1$ and $\tilde{x}_2$.

As the matrix $A^* = \det A\, A^{-1}$ has no negative coefficients the transformation $X = \tilde{X}^{A^T}$ is not a blowing-up.

Example 18 *Consider the normalized system*

$$\dot{X} = \begin{pmatrix} 1 & & \\ & 3 & \\ & & -3 \end{pmatrix} X + \begin{pmatrix} x_1^4 x_2 x_3^2 \\ x_2^2 x_3 \\ 0 \end{pmatrix} .$$

As we can see in figure 9.1 the cone V that contains the set M is spanned by the vectors $Q_1 = (3, 0, 1)$ and $Q_2 = (0, 1, 1)$. They are used to build the matrix

$$A^* = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} .$$

The power transformation $X = \tilde{X}^{A^T}$ *yields the new system*

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tilde{X} + \begin{pmatrix} 3\tilde{x}_1^2\tilde{x}_2 \\ \tilde{x}_2^2 \\ 2\tilde{x}_1\tilde{x}_2\tilde{x}_3 \end{pmatrix} .$$

with positive, integer support. Using a cone that does not contain supp(F), for example the cone spanned by the vectors Q_1 *and* $Q_3 = (3, 1, 2)$*, yields a system with negative support.*

2. $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0$ or $\lambda_1 < 0,\ \lambda_2 < 0,\ \lambda_3 < 0$

   - $\exists k, l \in \mathbb{N} : \lambda_1 = k\lambda_2,\ \lambda_2 = l\lambda_3$
     $M = \{ \begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ -1 \\ l \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ l \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} : \alpha \leq k,\ \alpha \in \mathbb{N} \}$
     The resulting system can be written as
     $$\dot{X} = \begin{pmatrix} \lambda_1 & \sigma_1 & \\ & \lambda_2 & \sigma_2 \\ & & \lambda_3 \end{pmatrix} X + \begin{pmatrix} p_1(x_2, x_3) \\ p_2(x_3) \\ 0 \end{pmatrix}$$
     where p_1 is a power serie in x_2 and x_3, p_2 is a power serie in x_3 and $\sigma_i \in \{0, 1\}$. The system can be integrated directly using the separation of the constant.
   - $\exists k \in \mathbb{N} : \lambda_1 = k\lambda_2$, and there does not exist a $l \in \mathbb{N}$ such that $\lambda_2 = l\lambda_3$
     $M = \{ \begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$
     Like above the system is directly integrable.
   - There does not exist a $k, l \in \mathbb{N}$ such that $\lambda_1 = l\lambda_2,\ \lambda_2 = k\lambda_3$
     $M = \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$
     The system is diagonal and therefore integrable.

3. $\lambda_1 \neq 0,\ \lambda_2 = 0,\ \lambda_3 = 0$
   $M = \{ \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : \ (\alpha, \beta) \in \mathbb{N}^2 \cup \{(-1, k), (k, -1) : k \in \mathbb{N}, k \geq 1\} \}$
   The system has the form
   $$\dot{X} = \begin{pmatrix} \lambda_1 & & \\ & 0 & \\ & & 0 \end{pmatrix} X + \begin{pmatrix} x_1 f(x_2, x_3)) \\ g(x_2, x_3) \\ h(x_2, x_3) \end{pmatrix}$$
   A two-dimensional system in the variables x_2 and x_3 can be split directly. After having solved the two-dimensional system the third equation can be integrated.

4. $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 = 0$ or $\lambda_1 < 0,\ \lambda_2 < 0,\ \lambda_3 = 0$

- $\lambda_1 = \lambda_2$

$$M = \{\alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} : \ \alpha \in \mathbb{N}, \beta, \gamma \in \{0, 1\}\}$$

As and basis of M contains at least one vector with negative coordinates the construction of a power transformation that is not a blowing-up is not possible. For this reason blowing-ups controlled by the Newton diagram of the concerned system will be used.

The support of the considered system lies in the cone V defined by two vectors $Q_1 = (-1, 1, \alpha_1)$ and $Q_2 = (1, -1, 1 + \alpha_2)$. Then the Newton diagram consists of the three vertices

$$\begin{aligned} \Gamma_1^{(0)} &= (-1, 1, \alpha_1), \\ \Gamma_2^{(0)} &= (0, 0, 0), \\ \Gamma_3^{(0)} &= (1, -1, 1 + \alpha_2) \end{aligned}$$

and the edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ defined by the vectors

$$R_1 = (-1, 1, \alpha_1), \ R_2 = (1, -1, 1 + \alpha_2)$$

respectively. Like in the case of a two-dimensional nonelementary singular point the edges and vertices are considered separately.

The vertices $\Gamma_j^{(0)}$ are transformed to the point $Q = 0$ by a time transformation

$$d\tilde{t} = X^{\Gamma_j^{(0)}} dt \ .$$

Now, the power transformation defined by the matrix

$$A^* = \left(R_* | R^* | \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right)$$

where $a, b, c \geq 0$ is used to reduce the system. The vectors R_* and R^* are defined as the vectors lying on the edges adjoinig the vertex and leading away from the vertex. For the extremal vertices the remaining vector is defined as $Q_3 = (0, 0, 1) \in M$. As the vectors R_* and R^* form a cone that includes the support of F, the resulting system verifies all conditions required for a further reduction and local integration.

The solution curves computed for the resulting system are valid in a neighbourhood

$$\tilde{U}_j^{(0)} = \{\tilde{X} : |\tilde{X}|_\infty \leq \epsilon\}$$

of $\tilde{X} = 0$. Therefore the curves $X(t)$ are valid in a sector

$$U_j^{(0)} = \{X : |X|^{R_*} \leq \epsilon, \ |X|^{R^*} \leq \epsilon, \ |X|^{(a,b,c)} \leq \epsilon\}$$

in the initial coordinates. The remaining sectors are associated to the edges.

The blowing-up for the edges $\Gamma_j^{(1)}$ is defined by the matrix

$$A^* = \left(R_j |Q_3| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right)$$

with $a, b, c \in \mathbb{N}$. In the new coordinates the sector

$$U_j^{(0)} = \{X : \epsilon \leq |X|^{R_j} \leq \epsilon^{-1}, \ \ldots\}$$

is transformed to a neighbourhood of the exceptional divisor $\{k\}\times 0\times 0$. Solution curves on those sets can be computed like in section 8.2 by the definition of subsectors and recursions. The system reduced by the power transformation verifies all conditions required for the reduction to a two-dimensional system.

Example 19 *Consider the system*

$$\dot{X} = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ x_1\, x_3 \\ x_3^2 \end{pmatrix}$$

with the three vertices $\Gamma_1^{(0)} = (-1,\ 1,\ 0)$, $\Gamma_2^{(0)} = (0,\ 0,\ 0)$ *und* $\Gamma_3^{(0)} = (1,\ -1,\ 1)$. *The support and the resonant plane M are shown in figure 9.3.*
For the first vertex $\Gamma_1^{(0)}$ *the time transformation*

$$d\tilde{t} = X^{(-1,1,0)} dt$$

yields the system

$$\frac{\partial X}{\partial \tilde{t}} = \begin{pmatrix} {x_1}^2 x_2^{-1} + x_1 \\ x_1 + {x_1}^2\, x_3 x_2^{-1} \\ {x_3}^2\, x_1 x_2^{-1} \end{pmatrix}$$

A power transformation defined by the matrices

$$A^* = A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ \textit{and}\ A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

yields the new system

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \begin{pmatrix} -\tilde{x}_2\, \tilde{x}_1^3 + \tilde{x}_1 \\ \tilde{x}_2^2\, \tilde{x}_1 \\ \tilde{x}_1\, \tilde{x}_3 + \tilde{x}_3 \end{pmatrix} \tag{9.8}$$

that can be integrated by the algorithms introduced in section 9.3.1. Its solution curves are valid in the sector

$$U_1^{(0)} = \{X : |X|^{(1,-1,0)} \leq \epsilon,\ |x_3| \leq \epsilon,\ |x_1| \leq \epsilon\}\ .$$

Blowing-ups for the other vertices defined by the matrices

$$A_2^* = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ A_3^* = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

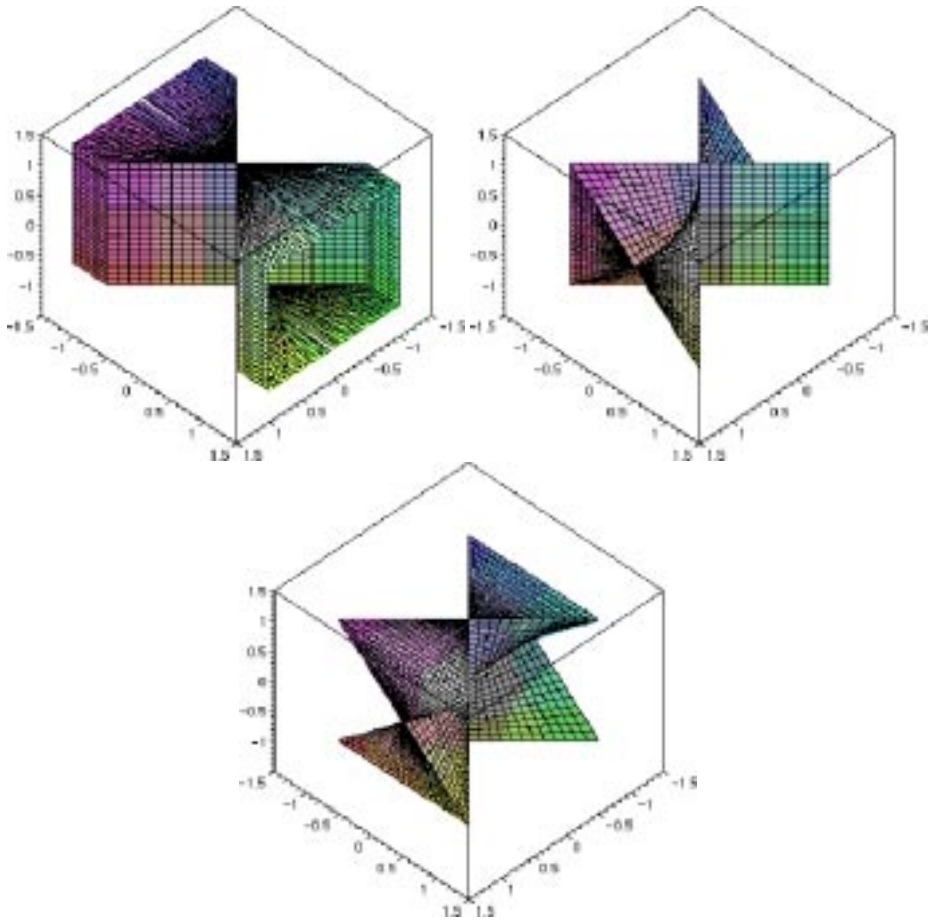


Figure 9.2: The figure shows the three sectors computed by applying blowing-ups to the system in example 19.

yield solution curves for the sectors

$$U_2^{(0)} = \{X : |X|^{(-1,1,0)} \leq \epsilon,\ |X|^{(1,-1,1)} \leq \epsilon,\ |x_1| \leq \epsilon\}$$
$$U_3^{(0)} = \{X : |X|^{(-1,1,-1)} \leq \epsilon,\ |x_3| \leq \epsilon,\ |x_1| \leq \epsilon\}\ .$$

As no further singularities except the one in $\tilde{X} = 0$ appear on the exceptional divisors of the power transformations we can choose $\epsilon = 1$. The sets $U_1^{(0)}$, $U_1^{(0)}$ and $U_1^{(0)}$ cover an entire neighbourhood of $X = 0$.

The sectors computed for this example are sketched in figure 9.2

- $\exists k \in \mathbb{N} : \lambda_1 = k\lambda_2$)
  $M = \{\alpha \begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} :\ \alpha, \beta \in \mathbb{N}\}.$

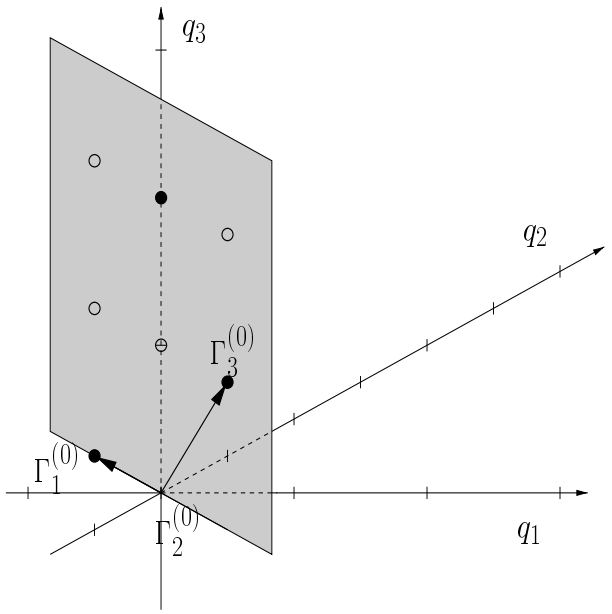


Figure 9.3: The support and some points of the set M for the system treated in example 19.

The Newton diagram has the two vertices

$$\Gamma_1^{(0)} = (-1,\ k,\ \beta_1),\ \beta_1 \in \mathbb{N},$$
$$\Gamma_2^{(0)} = (0,\ 0,\ 0)\ .$$

A procedure similar to the one used in the previous case yields systems that verify all conditions for a further local integration. As the used power transformations are blowing-ups, the computed solution curves are only valid in sectors.

- $\nexists k \in \mathbb{N} : \lambda_1 = k\lambda_2$)
  $M = \{\alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} :\ \alpha \in \mathbb{N}\}$
  All points in M are on a line. That means that the algorithms for $m = 1$ proposed in section 9.1 can be applied.

5. $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0$
   The notations

$$k_1 := min\{k \in \mathbb{Q} : \frac{\lambda_1}{k} \in \mathbb{Z} \wedge \frac{\lambda_2}{k} \in \mathbb{Z}\}$$
$$k_2 := min\{k \in \mathbb{Q} : \frac{\lambda_2}{k} \in \mathbb{Z} \wedge \frac{\lambda_3}{k} \in \mathbb{Z}\}$$

   with $Q_1 := \begin{pmatrix} \frac{-\lambda_3}{k_1} \\ 0 \\ \frac{\lambda_1}{k_1} \end{pmatrix}$ and $Q_2 := \begin{pmatrix} 0 \\ \frac{-\lambda_3}{k_2} \\ \frac{\lambda_2}{k_2} \end{pmatrix}$ are used to simplify the representation of the set M in this case.

- $\exists k \in \mathbb{N} : \lambda_1 = k\lambda_2$ and

$$\exists a, b > 0 : a\lambda_1 - \lambda_2 + b\lambda_3 = 0 \tag{9.9}$$

$$M = \{\begin{pmatrix} a \\ -1 \\ b \end{pmatrix} + \alpha Q_1 + \beta Q_2,\ \begin{pmatrix} -1 \\ k_1 \\ 0 \end{pmatrix} + \alpha Q_1 + \beta Q_2 :\ \alpha, \beta \in \mathbb{N}\}$$

We will show that in this case $k_1 = k_2$ and that the considered normal form can be simplified using a power transformation $X = \tilde{X}^{A^T}$ defined by the matrix A with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\lambda_1}{k_1} & -\frac{\lambda_2}{k_1} & -\frac{\lambda_3}{k_1} \end{pmatrix},\ A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\lambda_1}{\lambda_3} & -\frac{\lambda_2}{\lambda_3} & -\frac{k_1}{\lambda_3} \end{pmatrix}.$$

A and A^{-1} verify all conditions required and $det(A) = -\frac{\lambda_3}{k_1}$.

Proof 5 $(k_1 = k_2)$ *This needs to be shown only for* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{N}$ *otherwise* $k_1 = k_2 = 1$ *or equation (9.9) is not verified.*
suppose $k_1 \neq k_2$. $\lambda_1 = k\lambda_2$ *yields* $k_1 = gcd(k\lambda_2, -\lambda_3) = n\ gcd(\lambda_2, -\lambda_3) = n\ k_2,\ n \in N,\ n > 1$. *(9.9) can be written as*

$$n\ k_2(a\frac{k\lambda_2}{n\ k_2} + b\frac{\lambda_3}{n\ k_2}) = k_2\frac{\lambda_2}{k_2}.$$

Dividing by nk_2 *yields* $a\frac{k\lambda_2}{n\ k_2} + b\frac{\lambda_3}{n\ k_2} = \frac{\lambda_2}{n\ k_2}$ *where the right hand side is in* $\mathbb{N}$ *but the left hand side is not (if* $n > 1$*). This is a contradiction so* $k_1 = k_2$.

Proof 6 *The points with negative entries in* $supp(F)$ *can be written in the form*

$$Q = \begin{pmatrix} -1 \\ k_1 \\ 0 \end{pmatrix} + \beta Q_2 = \frac{k_1}{\lambda_3}Q_1 - \frac{\lambda_2 k_1}{\lambda_1\lambda_3}Q_2 + \beta Q_2$$

or

$$Q = \begin{pmatrix} a \\ -1 \\ b \end{pmatrix} + \alpha Q_1 = \frac{k_1}{\lambda_3}Q_2 - \frac{ak_1}{\lambda_3}Q_1 + \alpha Q_1$$

after having applied the transformation $X = \tilde{X}^{A^T}$ *the points in the support of the new system are given by*

$$\tilde{Q} = AQ = \frac{k_1}{\lambda_3}AQ_1 + (\beta - \frac{\lambda_2 k_1}{\lambda_1\lambda_3})AQ_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + (\beta\frac{\lambda_3}{k_1} - k)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\tilde{Q} = AQ = \frac{k_1}{\lambda_3}AQ_2 + (\alpha - \frac{a\ k_1}{\lambda_3})AQ_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + (\alpha\frac{\lambda_3}{k_1} - a)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

That means that all new points are in N. *The matrix* A *is choosen such that the points in* $N_1 - \mathbb{N}^3$ *only appear in the first equation and all points in* $N_2 - \mathbb{N}^3$ *appear only in the second equation of the new system.*

- $\exists k \in \mathbb{N} : \lambda_1 = k\lambda_2$ and $\nexists a, b > 0 : a\lambda_1 - \lambda_2 + b\lambda_3 = 0$
  $M = \{\begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix} + \alpha Q_1 + \beta Q_2 : \ \alpha, \beta \in \mathbb{N}\}$
  The matrix
  $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\lambda_1}{k_2} & -\frac{\lambda_2}{k_2} & -\frac{\lambda_3}{k_2} \end{pmatrix}$$
  and the power transformation $X = \tilde{X}^{A^T}$ can be used to simplify the considered system. This can be proved like above.

  Example 20 *Consider the system*
  $$\dot{X} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & -1 \end{pmatrix} X + \begin{pmatrix} x_1^2 x_3 \\ -x_1^2 + x_1^3 x_3 - x_2^2 x_3^2 \\ 0 \end{pmatrix}$$
  that is in normal form. The power transformation $X = \tilde{X}^{A^T}$ defined by the matrix
  $$A^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$
  with the vectors $Q_1 = (1,\ 0,\ 1)$ and $Q_2 = (0,\ 1,\ 2)$ yields a system
  $$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tilde{X} + \begin{pmatrix} \tilde{x}_1^2 \\ -\tilde{x}_1^2 + \tilde{x}_1^3 - \tilde{x}_2^2 \\ 0 \end{pmatrix}$$
  with positive exponents that can be integrated. The resonant plane of the initial system and of the resulting system are sketched in figure 9.4.

- $\nexists k \in \mathbb{N} : \lambda_1 = k\lambda_2$ and $\exists a, b > 0 : -\lambda_1 + a\lambda_2 + b\lambda_3 = 0$
  $M = \{\begin{pmatrix} -1 \\ a \\ b \end{pmatrix} + \alpha Q_1 + \beta Q_2 : \ \alpha, \beta \in \mathbb{N}\}$
  The power transformation $X = \tilde{X}^{A^T}$ defined by the matrix
  $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\lambda_1}{\gamma} & -\frac{\lambda_2}{\gamma} & -\frac{\lambda_3}{\gamma} \end{pmatrix}, \ \gamma = gcd(k_1, k_2)$$
  simplifies the system. This can be proved like above.

- $\nexists k \in \mathbb{N} : \lambda_1 = k\lambda_2$ and $\nexists a, b > 0 : -\lambda_1 + a\lambda_2 + b\lambda_3 = 0$
  $M = \{\alpha Q_1 + \beta Q_2 : \ \alpha, \beta \in \mathbb{N}\}$
  In this case the matrix A for the power transformation can be found like in example 18 as the vectors Q_1 and Q_2 span a cone V that contains $supp(F)$.

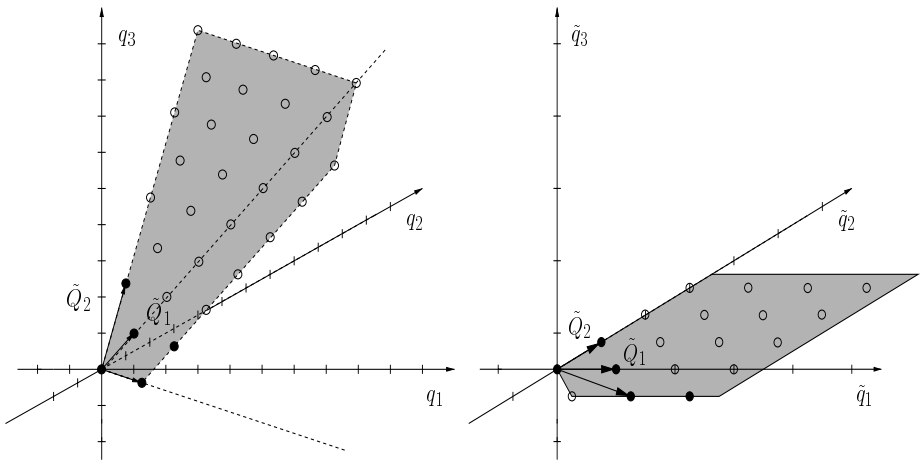


Figure 9.4: This figure shows the resonant planes of the initial system and the reduced system from example 20.

6. $\lambda_1 > 0,\ \lambda_2 < 0,\ \lambda_3 < 0$
   In this case the reduction to a two dimensional system is similar to the previous point.

All cases where non-rational eigenvalues appear can be derived from the cases treated above. If there exists an eigenvalue $\lambda_i \notin \mathbb{Q}$ resonances appear if there exist vectors $Q \in \mathcal{N}$ such that

$$\langle \Lambda, Q \rangle = \sum \lambda_i q_i = 0 \ .$$

In this case $m = 1$ or $m = 2$. Further there exists a vector $\tilde{\Lambda} = (\tilde{\lambda}_1,\ \tilde{\lambda}_2,\ \tilde{\lambda}_3)$ such that $M \subset \{Q \in \mathcal{N} :< Q, \tilde{\Lambda} >= 0\}$. Finding $\tilde{\Lambda}$ is equivalent to finding a vector in $\mathbb{Q}^3$ that is orthogonal to one or two vectors in $\mathcal{N}$. As for the resonance equation it makes no difference if Λ or $\tilde{\Lambda}$ is used. All cases with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ have a corresponding case with $\lambda_1, \lambda_3, \lambda_3 \in \mathbb{Q}$.

9.3.3 Real three-dimensional systems

This section deals with systems that have real coefficients but a pair of complex conjugated eigenvalues. In certain cases the reduction to a two-dimensional system can be performed such that it yields a system that has real coefficients.

Without loss of generality it can be supposed that $\lambda_1 = \overline{\lambda_2}$. Three cases have to be considered.

- $\mathrm{Re}(\lambda_1) = 0$ and $\lambda_3 \neq 0$.
  In this case the resonant plane M contains only one linearly independent vector. This case has alredy been treated for n-dimensional systems in section 9.2.

- $\mathrm{Re}(\lambda_1) = 0$ and $\lambda_3 = 0$.
  The reductions in this case are similar to those applied in section 9.2. However here

the resonant plane contains two linearly independent vectors. As a consequence the reduced system is not integrable directly but we can integrate a corresponding two-dimensional system.

Consider without loss of generality that the linear part of the considered system is in real Jordan form. The transformation to Jordan form by the transformation $X = P\tilde{X}$ and the transformation of the resulting system to Poincaré-Dulac normal form yields the system

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} i\alpha & & \\ & -i\alpha & \\ & & 0 \end{pmatrix} \hat{X} + \begin{pmatrix} \hat{x}_1 \sum_k \sum_l a_{1kl} (\hat{x}_1\, \hat{x}_2)^k \hat{x}_3^l \\ \hat{x}_2 \sum_k \sum_l a_{2kl} (\hat{x}_1\, \hat{x}_2)^k \hat{x}_3^l \\ \sum_k \sum_l a_{3kl} (x_1\, x_2)^k x_3^l \end{pmatrix}$$

with the particularity that $\forall k, l: \ a_{1kl} = \overline{a_{2kl}}$ and $\forall k, l: \ \mathrm{Im}(a_{3kl}) = 0$. For this reason the transformation $\hat{X} = P^{-1}\tilde{\tilde{X}}$ yields the new real system

$$\frac{\partial \tilde{\tilde{X}}}{\partial t} = \begin{pmatrix} 0 & \alpha & \\ -\alpha & 0 & \\ & & 0 \end{pmatrix} \tilde{\tilde{X}} + \begin{pmatrix} \sum_k \sum_l (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k \tilde{\tilde{x}}_3^l (\mathrm{Re}(a_{1kl}) \tilde{\tilde{x}}_1 - \mathrm{Im}(a_{1kl}) \tilde{\tilde{x}}_2) \\ \sum_k \sum_l (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k \tilde{\tilde{x}}_3^l (\mathrm{Im}(a_{1kl}) \tilde{\tilde{x}}_1 + \mathrm{Re}(a_{1kl}) \tilde{\tilde{x}}_2) \\ \sum_k \sum_l (\tilde{\tilde{x}}_1^2 + \tilde{\tilde{x}}_2^2)^k \tilde{\tilde{x}}_3^l a_{3kl} \end{pmatrix} .$$

Introducing polar coordinates

$$\begin{aligned} \tilde{\tilde{x}}_1 &= r \sin \phi \\ \tilde{\tilde{x}}_2 &= r \cos \phi \end{aligned}$$

yields the real system

$$\begin{cases} \frac{\partial \phi}{\partial t} = f_1(r, \tilde{\tilde{x}}_3) \\ \frac{\partial r}{\partial t} = f_2(r, \tilde{\tilde{x}}_3) \\ \frac{\partial \tilde{\tilde{x}}_3}{\partial t} = f_3(r, \tilde{\tilde{x}}_3) \ . \end{cases} \tag{9.10}$$

The system (9.10) is either directly integrable or a two-dimensional system in the variables r and $\tilde{\tilde{x}}_3$ can be split from it.

Example 21 *Consider the system*

$$\dot{X} = \begin{pmatrix} 0 & 4 & \\ -4 & 0 & \\ & & 0 \end{pmatrix} X + \begin{pmatrix} 3\, {x_1}^2\, x_2 - {x_3}^3 \\ 0 \\ {x_1}^2\, x_2\, x_3 - x_1\, x_2 \end{pmatrix}$$

that is transformed to Jordan form by the linear transformation $X = P\tilde{X}$ and to Poincaré-Dulac normal form

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} 4I & & \\ & -4I & \\ & & 0 \end{pmatrix} \tilde{X} + \begin{pmatrix} \frac{3}{8} I\, \tilde{x}_1^2\, \tilde{x}_2 - \frac{27}{1024} I\, \tilde{x}_1^3\, \tilde{x}_2^2 + \dots \\ -\frac{3}{8} I\, \tilde{x}_1\, \tilde{x}_2^2 + \frac{27}{1024} I\, \tilde{x}_1^2\, \tilde{x}_2^3 + \dots \\ \frac{1}{8} \tilde{x}_3^4\, \tilde{x}_1\, \tilde{x}_2 + \dots \end{pmatrix} .$$

The transformation $\tilde{X} = P^{-1}\hat{X}$ yields

$$\frac{\partial \hat{X}}{\partial t} = \begin{pmatrix} 0 & 4 & \\ -4 & 0 & \\ & & 0 \end{pmatrix} \hat{X} + \begin{pmatrix} \frac{3}{8} \hat{x}_2^3 + \frac{3}{8} \hat{x}_1^2\, \hat{x}_2 + \dots \\ -\frac{3}{8} \hat{x}_1\, \hat{x}_2^2 - \frac{3}{8} \hat{x}_1^3 + \dots \\ \frac{1}{8} \hat{x}_2^2\, \hat{x}_3^4 + \frac{1}{8} \hat{x}_3^4\, \hat{x}_1^2 + \dots \end{pmatrix} .$$

Introducing polar coordinates yields the system

$$\begin{cases} \frac{\partial \phi}{\partial t} &= 4 - \frac{3}{32}\,\hat{x}_3^5 + \frac{3}{8}\,r^2 - \frac{27}{1024}\,r^4 + \dots \\ \frac{\partial r}{\partial t} &= -\frac{3}{32}\,r^3\,\hat{x}_3^3 + \dots \\ \frac{\partial \hat{x}_3}{\partial t} &= \frac{1}{8}\,r^2\,\hat{x}_3^4 + \dots\ . \end{cases}$$

- $\mathrm{Re}(\lambda_1) \neq 0$.
  In this case a construction like in the previous cases is not possible. This is due to the fact that the transformation matrix P is more complex. However if no resonances occur ($m = 0$) the normal form is linear. Therefore the computation of a real normal form is possible. (Compare section 7.2). If $M \neq 0$ the computations have to be performed in $\mathbb{C}^3$.

9.4 Reduction of n-dimensional normal forms for $m > 2$

The classification in the previous section has shown that any three-dimensional Poincaré-Dulac normal form can be reduced to a system of the dimension of its resonant plane. In this section these results are generalized to n-dimensional normal forms. This generalization yields some additional problems as the cone containing $supp(F)$ can be spanned by any number of vectors for $m > 2$. The algorithms are illustrated by examples for four-dimensional normal forms with $m = 3$. The basic procedure is similar to the procedure used for three-dimensional systems in section 9.3.1.

Suppose that the vectors $Q_1, \dots, Q_m \in M$ form a basis for $\overline{M}$. Then the normal form (9.1) can be reduced to a system

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} \tilde{x}_1 g_1(\tilde{x}_1, \dots, \tilde{x}_m) \\ \dots \\ \tilde{x}_n g_n(\tilde{x}_1, \dots, \tilde{x}_m) \end{pmatrix} . \tag{9.11}$$

by a power transformation defined by the matrix

$$A^* = (Q_1 | \dots | Q_m | \dots)$$

where the vectors $Q_i,\ i = 1, \dots, m$ define the row vectors of A^*. The first m lines of the system in equation (9.11) only depend on the variables $\tilde{x}_1, \dots, \tilde{x}_m$. That means that it is formally integrable if the m-dimensional system

$$\begin{cases} \frac{\partial \tilde{x}_1}{\partial t} = \tilde{x}_1 g(\tilde{x}_1, \dots, \tilde{x}_m) \\ \dots \\ \frac{\partial \tilde{x}_m}{\partial t} = g_m(\tilde{x}_1, \dots, \tilde{x}_m) \end{cases} \tag{9.12}$$

can be integrated formally. The solution curves $\tilde{X}(t)$ for equation (9.11) can be obtained from the solution curves $(\tilde{x}_1(t), \dots, \tilde{x}_m(t))$ for equation (9.12) by an integration of the $n - m$ scalar differential equations

$$\frac{\partial log\, \tilde{x}_i}{\partial t} = g_i(\tilde{x}_1(t), \dots, \tilde{x}_m(t)), \quad i = m+1, \dots, n\ .$$

That means that the problem of finding solution curves for the initial normalized system (9.1) has been reduced to the problem of finding an appropriate power transformation or an appropriate matrix A. Further, the system (9.12) has to be integrated. In general the system (9.12) will have a nonelementary singular point in $\tilde{X} = 0$. Therefore, in practice we will only be able to integrate systems (9.12) of dimension $m = 2$. Nevertheless, the used methods will be developped using approaches that can be generalized to $m > 2$. In those cases the problems that the integration of system (9.12) might cause are not considered.

However, some essential properties are required for the system (9.11) in order to integrate it. Those properties affect the choice of the vectors $Q_1, \ldots, Q_m$ and therefore the computaion of the matrix A.

9.4.1 Conditions for the choice of the matrix A

The power transformation $X = \tilde{X}^{A^T}$ that transforms the initial normal form to a system of the form (9.11) is defined by the matrix A. This matrix must be chosen such that the power transformation and the new system verify the following properties:

- the vectors $Q_1, \ldots, Q_m$ have to form a basis of M. Otherwise the reduced system does not have the form (9.11).
- the coefficients of the system (9.11) have to be integer and positive. Otherwise the resulting system can not be integrated by the previously introduced algorithms.
- the power transformation $X = \tilde{X}^{A^T}$ has to be a diffeomorphism. This is verified if the matrix A is unimodular. If A is a non-unimodular invertible matrix the constructions introduced in section 2.3 can be used to define a corresponding bijective diffeomorphism.
- if the power transformation $X = \tilde{X}^{A^T}$ is not a blowing-up, the solutions of the resulting systems are valid in the whole neighbourhood of $X = 0$.

  If the power transformation $X = \tilde{X}^{A^T}$ is a blowing-up, the reduction and integration yield solutions that are only valid in a sector. As a single transformation is not sufficient to cover a neighbourhood of $X = 0$ with sectors, a serie of blowing-ups controlled by a Newton diagram are needed.

To verify whether an appropriate matrix A exists or not it is useful to consider 3 possible cases for the position of the resonant plane M in the space of exponents:

- $M \subset \mathbb{N}^n$
  In this case all vectors in M lie within the first quadrant. An appropriate matrix A exits if the support of F is included in a cone V spanned by m vectors $Q_1, \ldots, Q_m \in M$. Otherwise, the considered system can be reduced by defining a so called virtual Newton diagram and by using blowing-ups.
- $M \not\subset \mathbb{N}^n$ and the set $M \cap \mathbb{N}^n$ contains m linearly independent vectors.
  To compute a power transformation that is not a blowing-up yields the same problem as the previous case. Further the matrix must be chosen such that the vectors in $M - \mathbb{N}^n$ are transformed to points that lie within $N - \mathbb{N}^n$. This matrix does not always exist.

The use of blowing-ups controlled by the Newton diagram of the considered system is more appropriate here.

- $M \not\subset \mathbb{N}^n$ and the set $M \cap \mathbb{N}^n$ contains less than m linearly independent vectors. Any basis of M contains at least one vector with negative entries. Therefore the concerned systems can only be reduced by blowing-ups.

Those three possibilities for the position of the resonant plane M will be considered more closely in the following sections.

9.4.2 The resonant plane lies entirely within $\mathbb{N}^n$

Consider all cones V that contain the support of F and that are spanned by vectors $Q_i \in M$. The problem arises from fact that all cones V might be spanned by more than m vectors.

Let V be a cone that contains $supp(F)$ and that is defined by the minimum number of vectors $Q_1, \dots Q_k \in M$ with $k \geq m$. We have to distinguish two cases :

- If $k = m$ the first rows of the matrix

$$A^* = (Q_1 | \dots | Q_m | \dots)$$

are defined by the m vectors $Q_1, \dots, Q_m$. In this case all conditions for the choice of A are verified.

Proof 7 *Any vector $Q \in supp(F)$ can be written as*

$$Q = \sum \alpha_i Q_i, \;\; \alpha_i \in \mathbb{R}, \;\; \alpha_i \geq 0 \; .$$

According to theorem 3 in section 2.2 Q is transformed to the point

$$\tilde{Q} = \sum \alpha_i \det A e_i \; .$$

Therefore its coordinates are positive and the first m lines in the new system only depend on the variables $\tilde{x}_1, \dots, \tilde{x}_m$. As A has only integer coefficients the coordinates of $\tilde{Q}$ are integer.

As all Q_i have only positive entries A^* can be chosen such that $X = \tilde{X}^{A^T}$ is not a blowing-up.

- If $k > m$ the vectors $Q_1, \dots, Q_k$ are linearly dependent. As the matrix A^* has to be invertible only m vectors $R_1, \dots, R_m \in \mathbb{Z}^n \cap \overline{M}$ can be used for the construction of the matrix

$$A^* = (R_1 | \dots | R_m | \dots) \; .$$

The cone W spanned by those vectors has to contain all $Q \in supp(F)$ and therefore the cone V. Otherwise some points $Q \in supp(F)$ are transformed to points $\tilde{Q} = AQ$ with negative coordinates and the corresponding system has negative exponents. As W has to include all $Q \in supp(F)$ at least one vectors $R_i, \; i = 1, \dots, m$ has negative coordinates. As a consequence the power transformation $X = \tilde{X}^{A^T}$ is a blowing-up.

For this reason the concerned systems can only be reduced by blowing-ups. If blowing-ups are used they have to be controlled by a Newton diagram. However the Newton diagram for the concerned systems consists of a single vertex $\Gamma_1^{(0)} = 0$. For this reason an additional construction, called the virtual Newton diagram is used.

The virtual Newton diagram

Consider a normalized system that has its support on the resonant plane M with m linearly independent vectors. Any cone W that contains $supp(F)$ and that is defined by the vectors $R_1, \ldots, R_m$ has at least one vector with negative coordinates. Therefore the matrices

$$A^* = (R_1| \ldots |R_m| \ldots)$$

define blowing-ups. For this reason the solution curves computed for the reduced systems are only valid in a sector of the initial neighbourhood U. Several sectors and several blowing-ups are needed to cover the entire neighbourhood U. For this reason several cones W are required too. Those cones are computed by the virtual Newton diagram. The name "virtual Newton diagram" is used as the employed techniques yield a structure that is similar to the Newton diagram.

The idea of the virtual Newton diagram is simple. The only vertex $\Gamma_1^{(0)} = 0$ of the Newton diagram of the considered system (9.1) is considered as a set of identical vertices $\tilde{\Gamma}_i^{(0)}$, edges $\tilde{\Gamma}_i^{(1)}$ of length 0 and higher dimensional faces $\tilde{\Gamma}_i^{(n)}$ without spacial expansion. By computing the virtual Newton diagram those faces are visualized.

Any virtual vertex $\tilde{\Gamma}_i^{(0)}$ is joined by m faces. Those faces are not necessarily faces of the virtual Newton diagram but faces of the convex hull of the considered set of points. The intersection of those m faces define m vectors $R_{i1}, \ldots, R_{im}$ that define the cone W_i and that are used for the construction of the matrix

$$A^* = (R_{i1}| \ldots |R_{im}| \ldots) \ .$$

For any virtual vertex $\tilde{\Gamma}_i^{(0)}$ the virtual Newton diagram yields a cone W_i that contains $supp(F)$ and that defines the matrices A^* and A for a blowing-up. This blowing-up is applied to the initial normal form (9.1) without using a time transformation as the only real vertex of the considered system is $\Gamma_1^{(0)} = 0$.

The virtual Newton diagram is not unique. The defined blowing-ups yield no further solutions. For this reason we can let $\epsilon = 1$ in the definition of the corresponding sectors.

The construction of the virtual Newton diagram is simple. It is based on the computation of the complex hull of a set of points. Possible algorithms for the computation of the convex hull are for example the gift wraping method [48]. Computing the convex hull of all points $Q \in supp(F)$ yield k faces of dimension $m-1$ that are joining in $\Gamma_1^{(0)} = 0$ and that limit the cone V.

The cone V is defined by the k vectors $Q_1, \ldots, Q_k$. Now take m of the faces joining in $\Gamma_1^{(0)}$ and choose $m-1$ of them at a time. The intersections of those $m-1$ faces are lines that are either characterized by a vector $Q_i,\ i \in \{1, \ldots, k\}$ or a vector $P_i \notin \{Q_1, \ldots, Q_k\}$. For all choice of $m-1$ faces this yields a set of vectors $P_i \in M,\ i = 1, \ldots, l$. Now we

compute the new set of points

$$\{Q : Q \in supp(F)\} \cup \{Q : \exists j, Q_0 \in supp(F) : Q = P_j + Q_0\} \ , \tag{9.13}$$

and its convex hull. The Newton diagram of the set of points (9.13) contains the vertices

$$\hat{\Gamma}_i^{(0)} = P_i, \ i = 1, \ldots, l \ and \ \hat{\Gamma}_{l+1}^{(0)} = 0 \ .$$

All of these vertices $\hat{\Gamma}_i^{(0)}$, that are the virtual vertices of the initial system, are considered separately. If the convex hull of (9.13) defines m faces joining the vertex $\hat{\Gamma}_i^{(0)}$ they define m vectors $R_1, \ldots, R_m$ that can be used for the construction of the matrix A^*. If more than m faces are joining the vertex $\hat{\Gamma}_i^{(0)}$ they define $\tilde{k} > m$ vectors those faces can be used for a recursive call of the entire algorithm.

The virtual Newton diagram allows the computation of blowing-ups for systems with $k > m$. This is illustrated by the following example.

Example 22 *Consider the 4 dimensional system*

$$\dot{X} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} X + \begin{pmatrix} x_1^2 x_2 \\ x_2^2 x_4 \\ x_3^2 x_4 + x_3^2 x_1 \\ 0 \end{pmatrix}$$

that is in normal form. The set

$$supp(F) = \{v_1, v_2, v_3, v_4\} \subset M$$

with

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

is rectangular to the vector $(-1, 1, 1, -1)$. None of the points in $supp(F)$ can be expressed by a positive linear combination of the other three points. Therefore the 4 vectors in $supp(F)$ form a convex cone V that is not included in any cone spanned by only three of the vectors $v_1, \ldots, v_4$.

To construct a power transformation a cone $\tilde{W}$ described by only three vectors that contains all points in $supp(F)$ is needed.

We can try to construct W by using the planes $p_i, \ i = 1 \ldots 4$ spanned by two vectors in $supp(F)$. This cone is limited by three planes and spanned by the 3 vectors on the intersection of the planes. (Two arbitrarily chosen planes always intersect in the origin and therefore in a line passing through the origin.) It is obvious that W contains all points in $supp(F)$.

The planes p_i are given by

$$\begin{aligned} p_1 &= \alpha v_1 + \beta v_3 \\ p_2 &= \alpha v_3 + \beta v_2 \\ p_3 &= \alpha v_2 + \beta v_4 \\ p_4 &= \alpha v_4 + \beta v_1 \end{aligned}$$

with $\alpha, \beta \in \mathbb{R}$. We obtain 4 possible cones W

$$\begin{aligned} W_1 &= \alpha v_2 + \beta v_3 + \gamma v_5 \\ W_2 &= \alpha v_1 + \beta v_4 - \gamma v_5 \\ W_3 &= \alpha v_2 + \beta v_4 + \gamma v_6 \\ W_4 &= \alpha v_1 + \beta v_3 - \gamma v_6 \end{aligned}$$

with $\alpha, \beta.\gamma \geq 0$ where $v_5 = (0, 1, -1, 0)$ and $v_6 = (1, 0, 0, -1)$. v_5 and v_6 are the vectors lying on the intersection of the planes p_1 and p_3 and the intersection of the planes p_2 and p_4 respectively.

That means all cones containing supp(F) contain at least one vector with negative entries. Therefore any power transformation $X = \tilde{X}^{A^T}$ defined by

$$A^* = (v_{i1}|v_{i2}|v_{i3}|\ldots) \tag{9.14}$$

is a blowing up. In equation (9.14) the vectors $v_{i1}, \ldots, v_{i3}$ are the three vectors spanning the cone W_i, $i = 1, \ldots, 3$.

To control those blowing-ups a virtual Newton diagram is constructed. It is computed using the three planes p_1, p_2 and p_3. Their intersections yield the lines

$$\begin{aligned} p_1 \cap p_2 &= \alpha v_3 \\ p_1 \cap p_3 &= \alpha v_5 \\ p_2 \cap p_3 &= \alpha v_2 \end{aligned}$$

wit $\alpha \in \mathbb{R}$. The Newton diagram of the set $\{Q_1 = (0, 0, 0, 0), Q_2 = (0, 1, -1, 0)\}$ yields the virtual vertices $\tilde{\Gamma}_1^{(0)} = Q_1$ and $\tilde{\Gamma}_2^{(0)} = Q_2$.For those vertices the cones including the supp(F) are the cones W_1 and W_2. Those cones define the matrices

$$A_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } A_2^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The blowing-ups yield the new systems

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} \tilde{x}_1^2 + \tilde{x}_2\,\tilde{x}_1 \\ \tilde{x}_2\,\tilde{x}_1 + \tilde{x}_2^2 + \tilde{x}_3\,\tilde{x}_2^2 \\ \tilde{x}_3^2\,\tilde{x}_1 - \tilde{x}_1\,\tilde{x}_3 - \tilde{x}_2\,\tilde{x}_3 \\ -\tilde{x}_4 \end{pmatrix}$$

and

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} \tilde{x}_1^2 + \tilde{x}_2\,\tilde{x}_1 \\ \tilde{x}_2^2 \\ \tilde{x}_3^2\,\tilde{x}_1 + \tilde{x}_2\,\tilde{x}_3^2 - \tilde{x}_2\,\tilde{x}_3 \\ -\tilde{x}_4 \end{pmatrix}.$$

Solution curves computed for those systems are valid in the sectors

$$\begin{aligned} U_1^{(0)} &= \{X : |X|^{(0,1,-1,0)} \leq \epsilon, \ldots\} \\ U_2^{(0)} &= \{X : |X|^{(0,-1,1,0)} \leq \epsilon, \ldots\} \end{aligned}$$

with $\epsilon = 1$. The virtual Newton diagram is sketched in 3 dimensions in figure 9.6 ane figure 9.5 shows the cones V, W_1 and W_2.

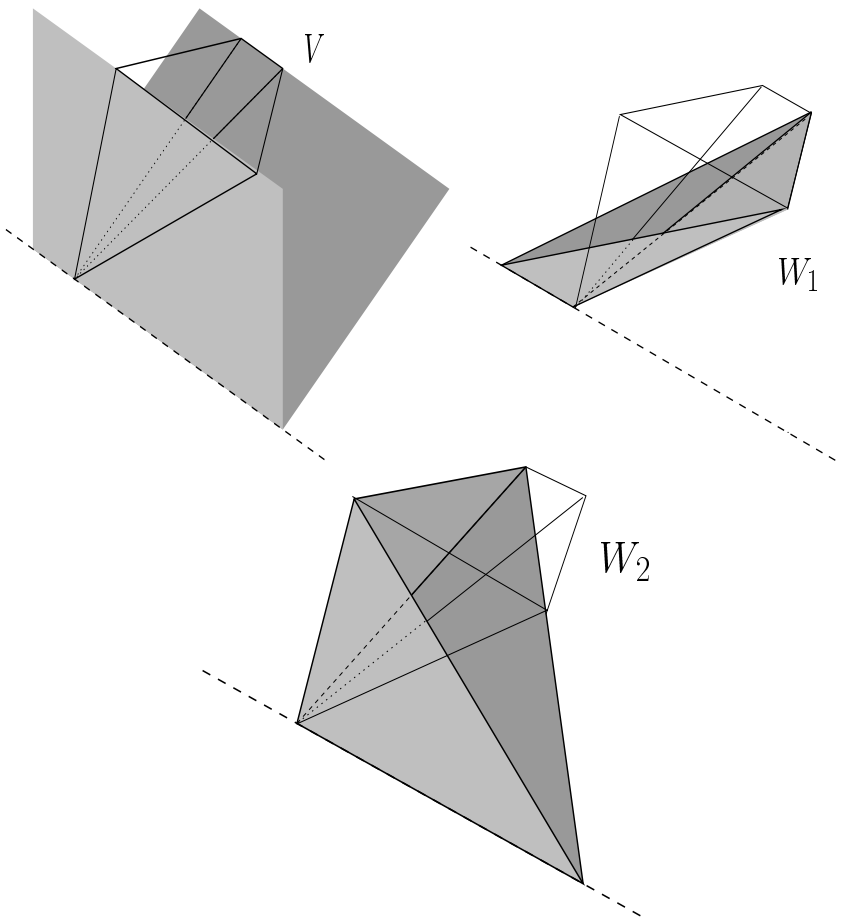


Figure 9.5: The suport of the normal forms treated in example 22 lies within a cone V that is spanned by 4 vectors. As only 3 of these vectors can be used to define the power transformation $X = \tilde{X}^{A^T}$, the cones W_1 and W_2 are used instead.

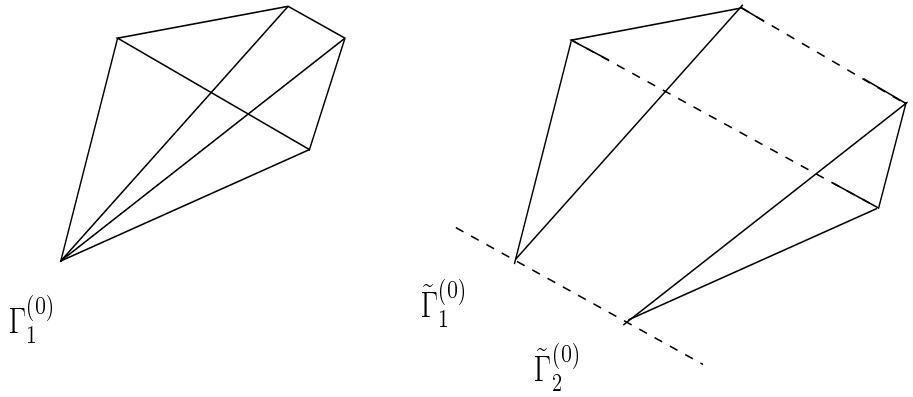


Figure 9.6: To obtain an appropriate control structure for the definition of the blowing-ups used in example 22 the virtual Newton diagram that consists of two vertices and one edge is computed..

9.4.3 The set $M \cap \mathbb{N}^n$ contains m linearly independent vectors

In this case there exits a basis $Q_1, \ldots, Q_n \in M \cap \mathbb{N}^n$ for $\overline{M}$. For this reason for some particular normal forms the reduction by a power transformation that is not a blowing-up is possible. However the matrix A that defines this transformation has to verify some very strict conditions. In general it does not exist.

Therefore the use of blowing-ups is the more appropriate method for this case. The Newton diagram of the concerned systems consists of more than one vertex. For this reason it can be used to control the blowing-ups. However the Newton diagram of the system (9.1) might contain higher-dimensional faces than edges and vertices. Blowing-ups for those faces have not been defined.

The time transformation

$$d\tilde{t} = X^{\Gamma_i^{(0)}} dt$$

that tranforms the vertex $\Gamma_i^{(0)}$ to the point $Q = 0$ yields a new system

$$\frac{\partial X}{\partial \tilde{t}} = \tilde{F} \ .$$

The k faces of the convex hull of $supp(F)$ that are joining in $Q = 0$ define k vectors $Q_1, \ldots, Q_k$. Those vectors define the cone V that contains the support of $\tilde{F}$. As the number of vectors spanning V might exceed m the construction of a virtual Newton diagram in for the concerned vertex might be necessary.

The following examples ilustrate this case. In the first example a power transformation that is not a blowing-up can be used for the reduction of the concerned system. This is not possible for the system treated in the second example were blowing-ups are used.

Example 23 *Consider the normalized system*

$$\dot{X} = F(X) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} X + \begin{pmatrix} x_3\, {x_4}^2 \\ {x_2}^2\, x_4 + {x_3}^2\, x_1 \\ {x_3}^2\, x_4 + {x_3}^2\, x_1 \\ 0 \end{pmatrix}$$

that also has the points $(-1, 0, 1, 2)$ *and* $(1, -1, 2, 0)$ *in its support. The set* $M \cap \mathbb{N}^4$ *lies within a cone* V *spanned by the vectors* $(1, 0, 1, 0)$, $(0, 1, 0, 1)$ *and* $(0, 0, 1, 1)$. *Therefore the power transformation defined by the matrices*

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

is used to simplify the considered system. This yields the new system

$$\frac{\partial \tilde{X}}{\partial t} = \begin{pmatrix} \tilde{x}_1^2 + \tilde{x}_3\, \tilde{x}_1 + \tilde{x}_3^2 \\ \tilde{x}_3\, \tilde{x}_1 + \tilde{x}_2^2 \\ \tilde{x}_3\, \tilde{x}_1 + \tilde{x}_3^2 \\ -\tilde{x}_4 \end{pmatrix} \ .$$

Example 24 *Consider the normalized system*

$$\dot{X} = F(X) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} X + \begin{pmatrix} x_3\, {x_4}^2 \\ {x_2}^2\, x_4 + {x_3}^2\, x_1 \\ {x_3}^2\, x_4 + {x_3}^2\, x_1 + {x_2}^2\, x_4 \\ 0 \end{pmatrix}$$

that also has the points $(-1, 0, 1, 2)$, $(0, 2, -1, 1)$ *and* $(1, -1, 2, 0)$ *in its support. Reducing this system by a power transformation defined by the matrix* A *used in the previous example is not possible.*

9.4.4 The set $M \cap \mathbb{N}^n$ contains less than m linearly independent vectors

In this case any basis $Q_1, \ldots, Q_m$ of M has at least one vectror with negative coordinates. As a consequence matrix A defined via its adjoint matrix

$$A^* = (Q_1 | \ldots | Q_m | \ldots)$$

that does not define a blowing-up does not exist. The only possibilty to reduce the concerned system to a m-dimensional system is to use blowing-ups controlled by the Newton diagram of the system (9.1). Eventually the virtual Newton diagram can be used if the use of the Newton diagram is not sufficient.

9.5 Conclusion

In this chapter it has been shown that three-and higher-dimensional normal forms can be reduced to m-dimensional systems. m denotes the maximum number of linearly independent vectors in the resonant plane M of the concerned normalized system. The reduction is performed by a power transformation. If the power transformation is a blowing-up it has to be controlled either by the Newton diagram or the virtual Newton diagram.

However in certain cases blowing-ups associated to higher-dimensional faces of the Newton diagram are needed. Those transformations have not yet been defined. They are studied in some examples in the following chapter.

The reduction of n-dimensional normal forms yields m-dimensional systems with a nonelementary singular point. The integration of those systems is not always possible for $m > 2$ (see [30]). For $m = 2$ however any normal form can be integrated by the proposed algorithms.

Chapter 10

Three-dimensional nonelementary singular points

In chapter 9 blowing-ups have already been used for the reduction of three-and higher-dimensional non-nilpotent systems. In particular the virtual Newton diagram has shown to be an efficient tool to define blowing-ups. In this chapter blowing-ups will be used to reduce three-dimensional systems in the neighbourhood of a nonelementary singular point. However the use of blowing-ups for three-dimensional systems yields problems. This has been shown by J. Jouanolou [38], X. Gómez-Mont and I. Luengo [30] and mentioned in section 4.3. Therefore it will be presumed that the considered differential equation

$$\dot{X} = F(X) \tag{10.1}$$

is non-dicritical. That means that at each step of the desingularization the noncritical case occurs. It has been shown by F. Cano and D. Cerveau [10] that those systems can be reduced to a finite number of systems with regular or elementary singular points by a finite number of successive blowing-ups. The reduction of three-dimensional systems (10.1) by blowing-up is based on the construction of quasi-homogeneous blowing-ups in section 4.2. Further the Newton diagram and the virtual Newton diagram will be used. As in the case of a two-dimensional nonelementary singular point the elementary operations as translations and time and power transformations. The used power transformatins are diffeomorphisms (see section 2.3).

In the study of a three-dimensional nonelementary singular point we will focus on the algorithmic point of view and on the construction of the sectors. The definition of the sectors is affected by the choice of the blowing-ups and it is not unique. It will be shown that the sectors can be constructed such that they cover the entire considered neighbourhood

$$U = \{X : |X|_\infty \leq \epsilon\}$$

of the singularity in $X = 0$. The sectors and the algorithms are constructed according to the faces of the Newton diagram.

10.1 The vertices

Consider the vertex $\Gamma_i^{(0)}$ of the Newton diagram of the initial system (10.1). As in the two-dimensional case a time transformation

$$d\tilde{t} = X^{\Gamma_i^{(0)}} dt$$

translates $\Gamma_i^{(0)}$ to the point $Q = 0$. The resulting system

$$\dot{X} = \frac{F(X)}{X^{\Gamma_i^{(0)}}} = \tilde{F}(X)$$

is a class $\mathcal{V}$ system. It has its support in a cone that is defined by m vectors $v_1, \ldots, v_m$. Three cases have to be distinguished.

- $m = 3$
  The cone V containing $supp(\tilde{F})$ if defined by the 3 vectors v_1, v_2 and v_3. They define the columns of the matrix
  $$A^* = (v_1|v_2|v_3)$$
  that defines the power transformation $X = \bar{X}^{A^T}$. The power transformation transforms all points in $supp(\tilde{F})$ to points with positive integer coordinates. For this reason the resulting system has positive integer exponents. As it has a non-nipotent linear part it can be integrated using the methods introduced in chapter 9.

  All further reductions and integrations are valid locally in a neighbourhood $\bar{U} = \{\bar{X} : |\bar{X}|_\infty \leq \bar{\epsilon}\}$. For this reason in the initial coordinates the solution curves are valid in the sector
  $$U_i^{(0)} = \{X : |X|^{v_1} \leq \bar{\epsilon}^{\frac{1}{det\, A}},\ |X|^{v_2} \leq \bar{\epsilon}^{\frac{1}{det\, A}},\ |X|^{v_3} \leq \bar{\epsilon}^{\frac{1}{det\, A}}\}\ .$$

- $m < 3$
  This case has already been treated partly in chapter 9 as the support of $\tilde{F}$ lies on a plane or on a line. The matrix A^* is defined by the column vectors $v_1, \ldots, v_m$ and completed to an invertible matrix by the vectors $v_{m+1}, \ldots, v_3 \in \mathbb{N}^3$. The condition $v_k \in \mathbb{N}^3$ guarantees that the corresponding surface $|X|^{v_k} = \epsilon$ that limits the sector does not pass through the origin.

- $m > 3$
  The cone V containig $supp(F)$ is spanned by more than three vectors. However for the construction of A via A^* a maximum of 3 linearly independent vectors is needed. Therefore a vector W thet contains V is constructed such that W is defined by the three vectors $w_1, \ldots, w_3$.

  The power transformation $X = \bar{X}^{A^T}$ with
  $$A^* = (w_1|w_2|w_3)$$
  yields a new system with positive integer exponents. The resulting system can be integrated with previously introduced methods. Its solution curves are valid in the sector
  $$U_i^{(0)}(W) = \{X : |X|^{w_1} \leq \bar{\epsilon}^{\frac{1}{det\, A}},\ |X|^{w_2} \leq \bar{\epsilon}^{\frac{1}{det\, A}},\ |X|^{w_3} \leq \bar{\epsilon}^{\frac{1}{det\, A}}\}$$

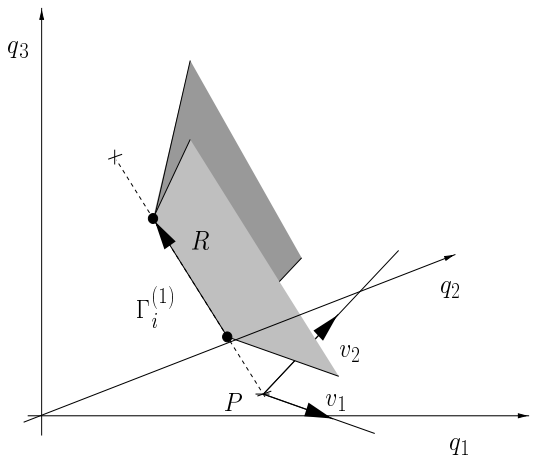


Figure 10.1: This figure shows the cone V constructed to define the blowing-up applied in the case of an edge $\Gamma_i^{(1)}$.

that depends on the choice of W. This choice and therefore the construction of A^* is not unique.

The proposed algorithm is similar to the methods used for the reduction of two-dimensional systems. However in section 10.4 it will be shown that for the case $m > 3$ some additional constructions are needed to guarantee an entire covering of the initial neighbourhood U by sectors.

10.2 The edges

Consider the edge $\Gamma_i^{(1)}$ of the Newton diagram and the vector $R = (r_1, r_2, r_3)$ with $gcd(r_1, r_2, r_3) = 1$ that lies on that edge. According to the results from section 4.2 we will construct a quasihomogeneous blowing-up that straightens that edge. Further the blowing-up yields a system with positive integer exponents after having applied a time transformation that translates the edge $\Gamma_i^{(1)}$ to the set

$$\{Q : q_1 = 0,\ q_2 = 0\}\ . \tag{10.2}$$

Any blowing-up defined in the previous section for the case of a vertex $\Gamma_i^{(0)} \in \Gamma_i^{(0)}$ with $w_3 = R$ or $v_3 = R$ can be used to perform the blowing-up for an edge.

However there are different possbilities to construct the matrix A^* that defines the blowing-up.

If $r_3 \neq 0$ we can presume without loss of generality that the vector R is chosen such that $r_3 > 0$. Otherwise $-R$ can be chosen instead of R. The line $Q_0 + \alpha R$ with $Q_0 \in \Gamma_i^{(1)}$, $\alpha \in \mathbb{R}$ cuts tha q_1q_2-plane in the point P. The vector R leads away from that point. Consider that the $\Gamma_i^{(2)}$ and $\Gamma_j^{(2)}$ are the faces of the Newton diagram that join in the edge $\Gamma_i^{(1)}$. They intersect the q_1q_2 plane in the lines $P + \alpha v_1$ and $P + \alpha v_2$ with $\alpha \in \mathbb{R}$

respectively. Without loss of generality it can be presumed that the vectors v_1 and v_2 verify

$$v_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ 0 \end{pmatrix} \; with \; v_{11} > 0 \; and \; v_{12} \leq 0, \; v_2 = \begin{pmatrix} v_{11} \\ v_{12} \\ 0 \end{pmatrix} \; with \; v_{11} > 0 \; and \; v_{12} \leq 0 \; .$$

The vectors v_1, v_2 and R form a cone V such that $Q_0 + V$ contains $supp(F)$. Therefore the blowing-up $X = \tilde{X}^{A^T}$ defined by the matrix

$$A^* = (v_1|v_2|R)$$

straightens the edge $\Gamma_i^{(1)}$. The straightened edge is parallel to the set (10.2). A time transformation

$$d\tilde{t} = \tilde{x}_1^{\tilde{q}_{01}} \tilde{x}_2^{\tilde{q}_{02}} dt$$

with $\tilde{Q}_0 = (\tilde{q}_{01}, \tilde{q}_{02}, \tilde{q}_{03}) = A\, Q_0$. $Q_0 \in \Gamma_i^{(1)}$ translates the straightened edge to the set (10.2). The exponents of the resulting system

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \tilde{F}(\tilde{X}) \tag{10.3}$$

are positive and integer.

An example for such a cone V is shown in figure 10.1.

If the edge $\Gamma_i^{(1)}$ is an extremal edge the faces $\Gamma_i^{(2)}$ or $\Gamma_j^{(2)}$ might not exist. Then the vectors $v_1 \in \Gamma_i^{(2)}$ and $v_2 \in \Gamma_j^{(2)}$ can be replaced by the vectors e_1 and e_2 respectively.

If $r_3 = 0$ the vector R can be chosen such that $r_2 > 0$. The line $Q_0 + \alpha R$ cuts the $q_1 q_3$ plane in the point P. The vectors $v_1 \in \Gamma_i^{(2)}$ and $v_2 \in \Gamma_j^{(2)}$ are defined such that

$$v_1 = \begin{pmatrix} v_{11} \\ 0 \\ v_{13} \end{pmatrix} \; with \; v_{11} > 0 \; and \; v_{13} \leq 0, \; v_2 = \begin{pmatrix} v_{11} \\ 0 \\ v_{13} \end{pmatrix} \; with \; v_{11} > 0 \; and \; v_{13} \leq 0 \; .$$

The matrix A^* and the time transformation can be defined like above. They yield equivalent results. In fact any cone V defined by the vectors v_1, v_2 and R such that

$$\forall Q \in supp(F) : \; Q \in P + V$$

with $P = Q_0 + \alpha R$, $Q_0 \in \Gamma_i^{(1)}$ can be used to define a blowing-up that straightenes the edge $\Gamma_i^{(1)}$.

For any blowing-up for the edge $\Gamma_i^{(1)}$ the resulting system (10.3) is studied on the set

$$S = \{\tilde{X} : \tilde{x} = 0, \tilde{y} = 0\} \tag{10.4}$$

that is a part of the exceptional divisor. The singularities of the system (10.3) on the set (10.4) are identic to the singularities of the quasihomogeneous part

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \tilde{F}_i^{(1)} = \begin{pmatrix} \tilde{x}_1 \tilde{f}_1(\tilde{X}) \\ \tilde{x}_2 \tilde{f}_2(\tilde{X}) \\ \tilde{x}_3 \tilde{f}_3(\tilde{X}) \end{pmatrix} \tag{10.5}$$

on the set (10.4). The quasihomogeneous par of a system (10.3) has been defined in section 4.2. It contains only the monomials associated to the points on the straightened edge. It is used to distinguish two cases.

- $\tilde{f}_3 \not\equiv 0$
  This is the noncritical case. The points $\tilde{X}_0 \in S$ with $\tilde{f}_3(\tilde{X}_0) = 0$ are the singularities of system (10.5) and system (10.3). All other points are regular points. In the neighbourhood of these regular points the solution curves are parallel to the exceptional divisor. Further the exceptional divisor itself is a solution curve for the blown-up system (10.3).

- $\tilde{f}_3 \equiv 0$
  This is the dicritical case. For the resulting system there might not exist a serie of blowing-ups that entirely reduce the nilpotent system to a finite number of non-nilpotent or regular systems (see J. Jouanolou [38], X. Gómez-Mont and I. Luengo [30]). However the dicritical case also yields problems in it's algorithmic aspects.

  The entire set S is a non-isolated singularity of the system (10.5). However a time transformation

  $$d\bar{t} = \tilde{x}_1 d\tilde{t}$$

  or

  $$d\bar{t} = \tilde{x}_2 d\tilde{t}$$

  might transform the system (10.5) into a system with isolated singularities. The resulting system has the form

  $$\frac{\partial \tilde{X}}{\partial \bar{t}} = \begin{pmatrix} \tilde{f}_1(\tilde{X}) \\ \frac{\tilde{x}_2}{\tilde{x}_1}\tilde{f}_2(\tilde{X}) \\ \frac{\tilde{x}_3}{\tilde{x}_1}\tilde{f}_3(\tilde{X}) \end{pmatrix} \tag{10.6}$$

  or

  $$\frac{\partial \tilde{X}}{\partial \bar{t}} = \begin{pmatrix} \frac{\tilde{x}_1}{\tilde{x}_2}\tilde{f}_1(\tilde{X}) \\ \tilde{f}_2(\tilde{X}) \\ \frac{\tilde{x}_3}{\tilde{x}_2}\tilde{f}_3(\tilde{X}) \end{pmatrix} . \tag{10.7}$$

  These systems might have negative exponents. Therefore a further reduction and integration using the methods proposed here is not always possible.

  The singularities of the system (10.6) or (10.7) are identic to the singularities of the system

  $$\frac{\partial \tilde{X}}{\partial \bar{t}} = \frac{1}{\tilde{x}_1}\tilde{F}(\tilde{X})$$

  or

  $$\frac{\partial \tilde{X}}{\partial \bar{t}} = \frac{1}{\tilde{x}_2}\tilde{F}(\tilde{X}) .$$

  They are given by the points $\tilde{X}_0 \in S$ that yield a vanishing right hand side in the equations (10.6) or (10.7). All othe points are regular points. They are either tangencies or there exists a solution curve passing through $\tilde{X}_0$. Therefore the dicritical case yields an infinity of solution curves passing through $X = 0$ fr the initial system (10.1).

The system (10.3) is studied in a neighbourhood of a finite part of the set S. This neighbourhood is denoted by

$$\tilde{U} = \{\tilde{X} : |\tilde{x}_1| \leq \tilde{\epsilon},\ |\tilde{x}_2| \leq \tilde{\epsilon},\ \tilde{\epsilon} \leq |\tilde{x}_3| \leq \tilde{\epsilon}^{-1}\} .$$

This study yields results that are valid in a sector

$$U_i^{(1)} = \{X : |X|^{v_1} \leq \tilde{\epsilon}^{\frac{1}{det\,A}},\ |X|^{v_2} \leq \tilde{\epsilon}^{\frac{1}{det\,A}},\ \tilde{\epsilon}^{\frac{1}{det\,A}} \leq |X|^R \leq \tilde{\epsilon}^{\frac{-1}{det\,A}}\}$$

of the initial neighbourhood U.

10.3 The faces

Consider the face $\Gamma_i^{(2)}$. It can be straightened by a blowing-up $X = \tilde{X}^{A^T}$ defined by the matrix

$$A^* = (v_1|v_2|v_3)$$

with the column vectors $v_2, v_3 \in \Gamma_i^{(2)}$. The vectors v_1 and v_2 are linearly independent. The matrix A^* is completed by a vector $v_3 \in \mathbb{Z}^3$. v_3 is chosen such that the support of F lies within the set $R + V$ where R is a point that lies on the plane passing through $\Gamma_i^{(2)}$. V is the cone spanned by the three vectors v_1, v_2 and v_3.

The face $\Gamma_i^{(2)}$ is transformed to a face that is parallel to the set

$$\{Q : q_1 = 0\} . \tag{10.8}$$

The time transformation

$$d\tilde{t} = \tilde{x}^{\tilde{q}_{01}} dt$$

with $\tilde{Q}_0 = (\tilde{q}_{01}, \tilde{q}_{02}, \tilde{q}_{03}) = AQ_0,\ Q_0 \in \Gamma_i^{(2)}$ yields the new system

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \tilde{F}(\tilde{X}) \tag{10.9}$$

that has positive integer exponents. The straightened face has been translated to the set (10.8).

The new system (10.9) is studdied in a neighbourhood of a finite part of the set

$$S = \{\tilde{X} : \tilde{x}_1 = 0\} . \tag{10.10}$$

The singularities of the new system (10.9) on S are identic to the singularities of the quasihomogeneous system

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \tilde{F}_i^{(2)}(\tilde{X}) = \begin{pmatrix} \tilde{x}_1 \tilde{f}_1(\tilde{X}) \\ \tilde{x}_2 \tilde{f}_2(\tilde{X}) \\ \tilde{x}_3 \tilde{f}_3(\tilde{X}) \end{pmatrix} . \tag{10.11}$$

The system (10.11) is used to distinguish the noncritical and the dicritical case:

- $\tilde{f}_2 \not\equiv 0$ or $\tilde{f}_3 \not\equiv 0$
  The points $\tilde{X}_0 \in S$ are singularities of the system (10.9) if $\tilde{f}_2(\tilde{X}_0) = 0$ and $\tilde{f}_3(\tilde{X}_0) = 0$.

  All other points are regular points. The solution curves in the neighbourhood of all regular points are parallel to S.

- $\tilde{f}_2 \equiv 0$ and $\tilde{f}_3 \equiv 0$
  In this case the nonelementary singularity of the initial system (10.1) might not be reduceable by finite successive blowing-up.

  As S represents a non-isolated singularity for the system (10.11) a time transformation
  $$d\bar{t} = x d\tilde{t}$$
  is used to transform the system (10.11) to the new system
  $$\frac{\partial \tilde{X}}{\partial \bar{t}} = \begin{pmatrix} \tilde{f}_1(\tilde{X}) \\ \frac{\tilde{x}_2}{\tilde{x}_1}\tilde{f}_2(\tilde{X}) \\ \frac{\tilde{x}_3}{\tilde{x}_1}\tilde{f}_3(\tilde{X}) \end{pmatrix}$$
  that has no negative exponents. Its singularities are given by the points $\tilde{X}_0$ with $\tilde{f}_1(\tilde{X}_0) = \tilde{f}_2(\tilde{X}_0) = \tilde{f}_3(\tilde{X}_0) = 0$.

  All other points $\tilde{X}_0$ are regular points. If $\tilde{f}_1(\tilde{X}_0) = 0$ they are tangencies. Otherwise there exists a solution curves passing through X_0.

The system (10.9) is studdied in a neighbourhood of a finite part of the set (10.10). This neighbourhood is given by
$$\tilde{U} = \{\tilde{X} : |x_1| \le \tilde{\epsilon},\ \tilde{\epsilon} \le |x_2| \le \tilde{\epsilon}^{-1},\ \tilde{\epsilon} \le |x_3| \le \tilde{\epsilon}^{-1}\}\ .$$
Therefore the sectors associated to the faces are given by
$$U_i^{(2)} = \{X : \tilde{\epsilon}^{\frac{1}{det\,A}} \le |X|^{v_1} \le \tilde{\epsilon}^{\frac{-1}{det\,A}},\ \tilde{\epsilon}^{\frac{1}{det\,A}} \le |X|^{v_2} \le \tilde{\epsilon}^{\frac{-1}{det\,A}},\ |X|^{v_3} \le \tilde{\epsilon}^{\frac{1}{det\,A}}\}$$
where v_3 represents the column vector used to complete the matrix A^*.

10.4 The sectors

As it has been shown previously the definition of blowing-ups for vertices, edges and faces of the Newton diagram also affects the shape of the sectors. In this section it will be shown that there exist blowing-up constructions such that a concerned neighbourhood
$$U = \{X : |X|_\infty \le \delta\}$$
with δ sufficiently small can be covered. First we will presume that in any vertex $\Gamma_i^{(0)}$ of the Newton diagram the cone V is defined by 3 vectors. For the vertex $\Gamma_i^{(0)}$, the edge $\Gamma_i^{(1)}$ and the face $\Gamma_i^{(2)}$ the corresponding sectors are defined as
$$\begin{aligned} U_i^{(0)} &= \{X : |X|^{v_1^{(0)}} \le \epsilon,\ |X|^{v_2^{(0)}} \le \epsilon, |X|^{v_2^{(0)}} \le \epsilon\} \\ U_i^{(1)} &= \{X : |X|^{v_1^{(1)}} \le \epsilon,\ |X|^{v_2^{(1)}} \le \epsilon, \epsilon \le |X|^R \le \epsilon^{-1}\} \\ U_i^{(2)} &= \{X : \epsilon \le |X|^{v_2^{(2)}} \le \epsilon^{-1},\ \epsilon \le |X|^{v_3^{(2)}} \le \epsilon^{-1}, |X|^{v_1^{(2)}} \le \epsilon\} \end{aligned} \tag{10.12}$$

The constants $\tilde{\epsilon}$ that define the size of the sets $\tilde{U}$ in section 10.1, 10.2 and 10.3 are chosen such that $\tilde{\epsilon} = \epsilon^{\det A}$.

To show that the entire neighbourhood U is covered by the sectors (10.12) the singularity is approached for $t \to \infty$ on class $\mathcal{W}$ curves

$$\mathcal{F} : \begin{cases} x_1(t) = t^{\alpha_1}(c_1 + O(1/t)) \\ \dots \\ x_3(t) = t^{\alpha_3}(c_3 + O(1/t)) \end{cases} .$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ and $\alpha_1, \alpha_2, \alpha_3 \leq 0$. The neighbourhood U is entirely covered if for any α and any $c = (c_1, c_2, c_3)$ and $t \to \infty$ the curve $\mathcal{F}$ lies within a sector (10.12).

Consider any vector v. The point $X = 0$ is approached on a curve $X = \mathcal{F}(t)$. The condition

$$|X|^v \leq \epsilon$$

holds if $\langle \alpha, v \rangle < 0$ or if $\langle \alpha, v \rangle = 0$ and $|c|^v \leq \epsilon$. The condition

$$\epsilon \leq |X|^v \leq \epsilon^{-1}$$

holds if $\langle \alpha, v \rangle = 0$ and $\epsilon \leq |c|^v \leq \epsilon^{-1}$. That means that $\mathcal{F}$ lies within the sector $U_i^{(0)}$ associated to the vertex $\Gamma_i^{(0)}$ if α verifies the conditions

$$\begin{aligned} \langle \alpha, v_1^{(0)} \rangle &\leq 0 \\ \langle \alpha, v_2^{(0)} \rangle &\leq 0 \\ \langle \alpha, v_3^{(0)} \rangle &\leq 0 \ . \end{aligned} \tag{10.13}$$

Further the conditions

$$|c|^{v_1^{(0)}} \leq \epsilon, \ |c|^{v_2^{(0)}} \leq \epsilon, \ |c|^{v_3^{(0)}} \leq \epsilon \tag{10.14}$$

have to be verified if the corresponding equalities in equation (10.13) are verified. The condition (10.13) is equivalent to the condition that α has to lie in the dual cone associated to the cone defined by the vectors $v_1^{(0)}$, $v_2^{(0)}$ and $v_3^{(0)}$. This dual cone is associated to the vertex $\Gamma_i^{(0)}$.

The curve $\mathcal{F}$ lies within the sector $U_i^{(1)}$ if the conditions

$$\begin{aligned} \langle \alpha, v_1^{(1)} \rangle &\leq 0 \\ \langle \alpha, v_2^{(1)} \rangle &\leq 0 \\ \langle \alpha, R \rangle &= 0 \end{aligned} \tag{10.15}$$

and

$$\epsilon \leq |c|^R \leq \epsilon^{-1} \tag{10.16}$$

hold. Further the conditions

$$|c|^{v_1^{(1)}} \leq \epsilon, \ |c|^{v_2^{(1)}} \leq \epsilon \tag{10.17}$$

have to be verified if the corresponding equalities in equation (10.15) are verified. The condition (10.15) is equivalent to the condition that α has to lie in the dual cone of the degenerate cone

$$V = \alpha v_1^{(1)} + \beta v_2^{(2)} + \gamma R, \ \alpha, \beta, \gamma \in \mathbb{R}, \ \alpha, \beta \geq 0$$

that is defined by the vectors $v_1^{(1)}$, $v_2^{(1)}$ and R. This dual cone is associated to the edge $\Gamma_i^{(1)}$.

The curve $\mathcal{F}$ lies within the sector $U_i^{(2)}$ if the conditions

$$\begin{aligned} \langle \alpha, v_2^{(2)} \rangle &= 0 \\ \langle \alpha, v_3^{(2)} \rangle &= 0 \end{aligned} \tag{10.18}$$

and

$$\begin{aligned} \epsilon \leq |c|^{v_1^{(2)}} &\leq \epsilon^{-1} \\ \epsilon \leq |c|^{v_2^{(2)}} &\leq \epsilon^{-1} \end{aligned} \tag{10.19}$$

hold. The inequality

$$\langle \alpha, v_3^{(2)} \rangle < 0$$

is always verified for t sufficient large if (10.18) holdes. The condition (10.18) is equivalent to the condition that α has to lie in the dual cone for the degenerate cone

$$V = \alpha v_2^{(2)} + \beta v_3^{(2)} + \gamma v_1^{(2)},\ \alpha, \beta, \gamma \in \mathbb{R},\ \gamma \geq 0$$

defined by the vectors $v_1^{(2)}$, $v_2^{(2)}$ and $v_3^{(2)}$. This dual cone is associated to the face $\Gamma_i^{(2)}$.

However the dual cones associated to the cones defined for the faces of the Newton diagram entirely cover $\{Q \in \mathbb{R}^3 : q_1, q_2, q_3 \leq 0\}$. Therefore every class $\mathcal{W}$ curve $\mathcal{F}$ lies within a sector if the following conditions are verified :

- All vectors $c \in \mathbb{R}^3$ or $\mathbb{C}^3$ lie within the set denoted by (10.14) or the set denoted by (10.16) if α lies in the dual cone associated to an edge and not in a dual cone associated to a neighbouring face.

  This can easily be shown as there always exist 2 vertices $\Gamma_i^{(0)}$ and $\Gamma_j^{(0)} \in \Gamma_i^{(1)}$. The cones associated to those vertices contain the vector R or $-R$ that defines the edge. Therefor either the condition (10.14) for the edge $\Gamma_i^{(1)}$ or the condition (10.16) for one of the edges $\Gamma_i^{(0)}$ ok $\Gamma_j^{(0)}$ holds.

- All vectors $c \in \mathbb{R}^3$ or $\mathbb{C}^3$ lie within the set denoted by (10.14) or (10.17) and (10.16) or (10.19) if α lies within the dual cone associated to the face $\Gamma_i^{(2)}$.

  This is not always true as the computations for simple examples show. To solve this problem a number of methods can be taken into consideration. The parameters ϵ and $\tilde{\epsilon}$ can be varied for certain faces. Another possibility is to extend the solutions computed in the neighbourhood of the sets that are not covered by the sectors. This is not possible if those sets contain singularities. Further the sets $\tilde{U}$ in the sections 10.1, 10.2 and 10.3 can be choosen in a different way. This includes a smaller $\tilde{\epsilon}$ or a different shape of the sectors.

It has been shown that the sectors defined by the blowing-ups mainly cover U if $m = 3$ for all concerned vertices. If $m > 3$ it is obvious that this is no longer true. Therfore we will introduce an additional construction to the Newton diagram that adds virtual vertices and edges such that $m = 3$ is verified for any real and virtual vertex.

10.5 The virtual Newton diagram

The previous section has shown that with the proposed methods an entire study of a three-dimensional system is possible if the Newton diagram has a regular structure. The structure of a Newton diagramm will be called regular if in any vertex $\Gamma_i^{(0)}$ is regular. That means that the cone V that verifies $\Gamma_i^{(0)} + V \supset supp(F)$ is defined by 3 vectors. Those vectors lie on adjoining edges if the vertex is not an extremal vertex. Otherwise the three vectors are defined by the edges adjoining $\Gamma_i^{(0)}$ and by vectors from the set $\{e_1, e_2, e_3\}$. Therefore for any regular vertex the corresponding blowing-up is uniquely defined up to a permutation of the column vectors in the matrix A^*.

In general the Newton diagram of a given system does not yield such a regular structure. The blowing-ups associated to any non-regular vertex are not uniquely defined. The dual cones of the cones computed to define the blowing-ups corresponding to non-regular vertices do not allow to entirely cover the set $\{Q \in \mathbb{R}^3 : q_1, q_2, q_3 \leq 0\}$. Therefore the sectors resulting form the blowing-up construction do not cover the entire neighbourhood U. To solve this problem we will introduce an additional construction that allows to compute a Newton diagram that extends the conventinal diagram such that it has a regular structure. This additional construction is called the virtual Newton diagram.

In a first step the Newton diagram is computed. Its non-regular vertices are considered as a finite number of identic vertices connected by edges of zero length. Any of those virtual vertices owns a cone V that is defined by 3 vectors. The blowing-ups associated to all real and virtual faces yield sectors that cover the neighbourhood U as mentioned in section 10.4. It is obvious that the blowing-ups associated to virtual edges yield no singularities on the set S.

10.5.1 The construction of the virtual Newton diagram

The construction of the virtual Newton diagram is based on the computation of the convex hull and the Newton diagram of a set of points. An algorithm for the computation of the convex hull is for example the gift wraping method (see for example F. Preparata and M. Shamos [48]).

Consider the vertex $\Gamma_i^{(0)}$ and presume that it lies on the intersection of $k > 3$ faces of the convex hull of F. For non-extremal vertices those faces also belong to the Newton diagram of F. The intersections of two of those faces define the edges joining in $\Gamma_i^{(0)}$ and the vectors defining the associated cone.

Now choose 3 of those faces and compute the intersection of each two of them. This yields 3 lines that intersect in $\Gamma_i^{(0)}$. They can be used to define three vectors v_1, v_2, v_3 such that these vectors define a convex cone V. The cone verifies $supp(F) \subset \Gamma_i^{(0)} + V$. Two possible cases have to be considered for each of the vectors $v_k,\ k = 1, \ldots, 3$:

- v_k lies on the convex hull of F. In this case v_k does not define a virtual edge.
- v_i does not lie on the convex hull of F. That means that the vector v_k defines a virtual edge.

If none of the computed vectors $v_k, k = 1, \ldots, 3$ defines a virtual edge the introduction of virtual vertices is not necessary. Otherwise the algorithm defines the virtual vertices

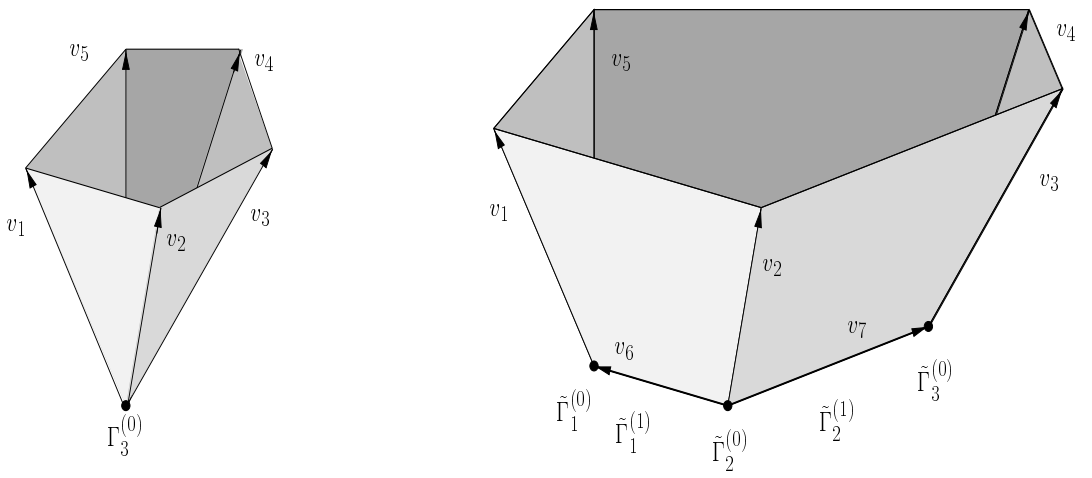


Figure 10.2: The virtual Newton diagram allows to replace the cone W for $\Gamma_3^{(0)}$ by the three cones V_1, V_2 and V_3 for the virtual vertices $\tilde{\Gamma}_1^{(0)}$, $\tilde{\Gamma}_2^{(0)}$ and $\tilde{\Gamma}_3^{(0)}$.

$\Gamma_i^{(0)} + v_k$ where the v_k are the vectors defining virtual edges. Now the Newton diagram of the set of points

$$supp(F) \cup \{Q + v_k : Q \in supp(F),\ v_k \ defines\ a\ virtual\ edge\}$$

is computed. The virtual edges and vertices for the initial differential equation are real edges and vertices of the computed Newton diagram. If the resulting cones associated to the vertices $\tilde{\Gamma}_i^{(0)}$ are still defined by more that three vectors the whole algorithm is repeated for the concerned vertices.

The virtual Newton diagram yields vertices, edges and faces and their associated cones. These cones can be blown-up with the constructions introduced previously. The resulting sectors cover the initial neighbourhood U with the restrictions mentioned in section 10.5.

10.6 Examples for the reduction of three-dimensional nilpotent systems by blowing-ups

Consider the three-dimensional system of differential equations

$$\dot{X} = \begin{pmatrix} {x_1}^6 + {x_1}^3\,{x_2}^2 + {x_1}^2\,x_2\,x_3 \\ {x_2}^6 + {x_2}^3\,{x_3}^2 \\ {x_3}^6 \end{pmatrix} . \tag{10.20}$$

Its Newton diagram contains 5 faces, 10 edges and 6 vertices. All constructions introduced previously can be illustrated by this example.

Example 25 (The vertex $(1,1,1)$.) *As shown in figure 10.2 5 edges are joinig in the vertex $\Gamma_3^{(0)} = (1,1,1)$. Therefore $\Gamma_3^{(0)}$ is replaced by the 3 virtual vertices $\tilde{\Gamma}_1^{(0)} = \tilde{\Gamma}_2^{(0)} = \tilde{\Gamma}_3^{(0)}$*

that coincide with $\Gamma_3^{(0)}$ *and the virtual edges* $\tilde{\Gamma}_1^{(1)}$ *and* $\tilde{\Gamma}_2^{(1)}$ *defined by the vectors* v_6 *and* v_7 *respectively. The virtual vertices and edges have been sketched in figure 10.2. The cones* $V_k,\ k = 1,\dots 3$ *for the virtual vertices are defined by the vectors* $v_1, v_5, -v_6$ *for* $\tilde{\Gamma}_1^{(0)}$*, by* v_2, v_6, v_7 *for* $\tilde{\Gamma}_2^{(0)}$ *and by* $v_3, v_4, -v_7$ *for* $\tilde{\Gamma}_3^{(0)}$*. The vectors* $v_1, \dots, v_7$ *are given by*

$$\begin{aligned} v_1 &= (-1,\ 4,\ -1) \\ v_2 &= (1,\ 1,\ -1) \\ v_3 &= (4,\ -1,\ -1) \\ v_4 &= (-1,\ -1,\ 4) \\ v_5 &= (-1,\ 1,\ 1) \\ v_6 &= (-11,\ 19,\ -1) \\ v_7 &= (3,\ -7,\ 3) \end{aligned}$$

They characterize the matrices for the blowing-ups. For the vertex $\tilde{\Gamma}_1^{(0)}$ *that yields the blowing-up* $X = \tilde{X}^{A^T}$ *by the matrices*

$$A^* = \begin{pmatrix} -1 & -1 & 11 \\ 1 & 4 & -19 \\ 1 & -1 & 1 \end{pmatrix},\ A = \begin{pmatrix} 15 & 10 & 25 \\ 20 & 12 & 8 \\ 5 & 2 & 3 \end{pmatrix}\ .$$

The blowing-up and an apropriate time transformation yields the system

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \begin{pmatrix} -\frac{1}{20}\, x_1 + \frac{1}{20}\, x_1\, {x_2}^{20} + \frac{1}{20}\, {x_1}^{21} + \dots \\ -\frac{1}{20}\, x_2 + \frac{1}{5}\, {x_2}^{21} + \frac{1}{5}\, {x_1}^{20}\, x_2 + \dots \\ \frac{11}{20}\, x_3 - \frac{19}{20}\, {x_2}^{20}\, x_3 - \frac{19}{20}\, {x_1}^{20}\, x_3 + \dots \end{pmatrix}\ .$$

Example 26 (The edge $\Gamma_1^{(1)}$**.)** *Consider the edge* $\Gamma_1^{(1)}$ *defined by the vector* $R = (-4, 1, 1)$ *and the vertices* $\Gamma_1^{(0)} = (5,\ 0,\ 0)$*,* $\Gamma_3^{(0)} = (1,\ 1,\ 1) \in \Gamma_1^{(1)}$*. The simplest way to compute a blowing-up for the edge* $\Gamma_1^{(1)}$ *is to use the blowing-up associated to the virtual vertex* $\tilde{\Gamma}_3^{(0)}$ *It is given by* $X = \tilde{X}^{A^T}$ *with*

$$A^* = \begin{pmatrix} -3 & -1 & 4 \\ 7 & -1 & -1 \\ -3 & 4 & -1 \end{pmatrix},\ A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 3 & 5 \\ 5 & 3 & 2 \end{pmatrix}$$

To straighten the edge $\Gamma_1^{(1)}$ *the vector* $-R = v_4$ *has to appear in the third column of the matrix* A^**.*

After an appropriate time transformation this yields the system

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \begin{pmatrix} -\frac{1}{5}\, \tilde{x}_1 - \frac{1}{5}\, \tilde{x}_1^4\, \tilde{x}_3^6 + \frac{7}{15}\, \tilde{x}_1^4\, \tilde{x}_2^6 + \dots \\ -\frac{1}{15}\, \tilde{x}_2 - \frac{1}{15}\, \tilde{x}_1^3\, \tilde{x}_2\, \tilde{x}_3^6 - \frac{1}{15}\, \tilde{x}_1^3\, \tilde{x}_2^7, + \dots \\ \frac{4}{15}\, \tilde{x}_3 + \frac{4}{15}\, \tilde{x}_1^3\, \tilde{x}_3^7 - \frac{1}{15}\, \tilde{x}_1^3\, \tilde{x}_2^6\, \tilde{x}_3 + \dots \end{pmatrix}\ .$$

The vertices $\Gamma_1^{(0)}$ *and* $\Gamma_3^{(0)}$ *have been transformed to the points* $(5, 10, 15)$ *and* $(5, 10, 10)$ *respectively. As the first and second coordinates of those points are identic the edge* $\Gamma_1^{(1)}$ *has been straightened up.*

Example 27 (Blowing-up of the face $\Gamma_1^{(2)}$) *Consider the face $\Gamma_1^{(2)}$ defined by the normal vector $(-2, -3, -5)$ and the vertices $\Gamma_1^{(0)}$, $\Gamma_3^{(0)}$, $\Gamma_2^{(0)} = (2, 2, 0) \in \Gamma_1^{(2)}$. The cone V defined by the three vectors v_2, v_3 and $v_8 = (-13, 2, 7)$ verifies $supp(F) \subset \Gamma_3^{(0)} + V$. Therefore it can be used to define the matrices*

$$A^* = \begin{pmatrix} -13 & 1 & 4 \\ 2 & 1 & -1 \\ 7 & -1 & -1 \end{pmatrix}, A = \begin{pmatrix} 2 & 3 & 5 \\ 5 & 15 & 5 \\ 9 & 6 & 15 \end{pmatrix}$$

for the blowing-up. The vectors v_2 and v_3 represent the second and third column vectors in A^*.

After having applied an appropriate time change the system resulting from the power transformation $x = \tilde{X}^{A^T}$ has the form

$$\frac{\partial \tilde{X}}{\partial \tilde{t}} = \begin{pmatrix} -\frac{13}{15}\tilde{x}_1 - \frac{13}{15}\tilde{x}_1\,\tilde{x}_3^{15} - \frac{13}{15}\tilde{x}_1\,\tilde{x}_2^{15} + \dots \\ \frac{1}{15}\tilde{x}_2^{16} + \frac{1}{15}\tilde{x}_2 + \frac{1}{15}\tilde{x}_2\,{x_3}^{15} + \dots \\ \frac{4}{15}\tilde{x}_2^{15}\,x_3 + \frac{4}{15}\tilde{x}_3^{16} + \frac{4}{15}\tilde{x}_3 + \dots \end{pmatrix}$$

The vertices $\Gamma_1^{(0)}, \Gamma_2^{(0)}$ and $\Gamma_3^{(0)}$ are transformed to the points $(10, 15, 45), (10, 40, 30)$ and $(10, 25, 30)$ respectively.

The previous sections have shown how blowing-up can be reduced for the reduction of three-dimensional nilpotent vector fields. Further it has been shown how the study of the blown-up system in a part of the exceptional divisor yields sectors for the initial coordinates. Therefore solution curves within the sectors can be computed by calculating solution curves in a neighbourhood of the concerned parts of the exceptional divisors. Those neighbourhoods are denoted by $\tilde{U}$ in the sections 10.1, 10.2 and 10.3. To compute solution curves in $\tilde{U}$ the sets $\tilde{U}$ are divided into subsectors associated to regular and simple points of the reduced system on the exceptional divisor. The solution curves for those subsectors are computed by a recursive call of the entire algorithm. However it can not be guaranteed for all three-dimensional systems that they can be entirely reduced by a finite number of blowing-ups. Further algorithmic problems may occur if the dicritical case is verified for an edge of the Newton diagram.

10.7 Higher-dimensional nonelementary singular points

In the previous sections algorithms for the reduction of three-dimensional dynamical systems have been introduced. Especially the virtual Newton diagram has shown to be a powerful tool. Those results can be extended to higher-dimensional problems. The main difficulties for higher dimensional blowing-ups are the same as for three-dimensional systems. The virtual Newton diagram can be used to solve large number of those problems. It can be used to enlarge the conventional Newton diagram by introducing virtual faces of dimension $n-2$ and lower. This new structure allows well directed manipulation on the cones and a sector definition that covers the studied neighbourhood U.

Chapter 11

The FRIDAY package

In the previous chapters several algorithms for the reduction and the integration of two-, three- and higher-dimensional systems of autonomous differential equations

$$\dot{X} = F(X) \tag{11.1}$$

were introduced. Those algorithms are implemented in the FRIDAY [1] MAPLE package. In particular the FRIDAY package contains procedures for the computation of n-dimensional normal forms for systems of the form (11.1) in the neighbourhood of regular and elementary singular points. Further, it allows to integrate any real and complex two-dimensional system and real three-dimensional systems in the neighbourhood of elementary singular points.

The intention in implementing the FRIDAY package was to design a program that is easy to use and that can handle a large number of systems (11.1). Therefore the data structure is conceived object-like because object-oriented programming can't be realized in MAPLE. The organisation in modules simplifies the addition of new procedures and functions.

Primitives act on the object-like data structure and perform elementary operations. These primitives are used by the control structure to execute the different steps of the algorithm. The control structure of the program is split into 4 main parts. According to the classification of dynamical systems three modules deal with the case of regular points, elementary singular point and nonelementary singular points. One module performs the classification of the considered system (11.1).

Only a few procedures are visible and can be manipulated by the user. The main part of the functions are capsuled. The procedure `FRIDAY` reduces a given system (11.1) as far as possible and performs eventual integrations. Besides this main procedure in particular the procedures for normal form computations are accessible. They allow computations of Poincaré-Dulac normal forms, normal forms for systems with real coefficients and complex eigenvalues and normal forms for systems in the neighbourhood of a regular point. Further some procedures that handle the obtained solutions, the transformations and the sectors are available. A large number of examples in section 11.3 show how these procedures can be used.

[1] FRIDAY stands for **F**ormal **R**eduction and **I**ntegration of **D**ynamical **A**utonomous S**y**stems.

There is a restriced number of possible tests for the computed solutions. These methods were already mentioned in section 1.4. They were performed for a large number of arbitrarilly chosen systems.

11.1 Organisation

The FRIDAY package is split into 4 main modules. According to the classification of dynamical systems the modules `SP_n`, `ESP_n` and `NESP_n` deal with the cases of regular points, elementary singular points and nonelementary singular points respectively. The module `separate_cases` performs the classification of the considered system.

The procedure `separate_cases` distinguishes the different possible cases as it has been introduced in chapter 5. Further it performs eventual translations and defines neighbourhoods. Therefore it uses the primitives `newtrans_n` and `initial_sector_n`.

The module `SP_n` reduces and integrates n-dimensional vector fields in the neighbourhood of regular points. The application of the flow-box theorem reduces the given system to a normal form that can easily be integrated. The program is based on the algorithms introduced in chapter 6.

The module `ESP_n` is based on the algorithms defined in the chapters 7 and 9. It integrates two- and three-dimensional systems in the neighbourhood of elementary singular points. Therefore it computes normal forms using the primitive functions `jordan_sys`, `PDNF_n`, `GNF_n` and `PI_NF_n`. All two- and a large number three-dimensional normal forms are integrable. The integrations are performed by a large number of elementary integration procedures. The reduction of the remaining three-dimensional normal forms is controlled by the procedure `two_d_solutions`. If the reduction is performed without blowing-ups an appropriate matrix A for the power transformation $X = \tilde{X}^{A^T}$ is computed by the function `find_matrix_A`. The power transformation is applied to the initial system in the procedure `power_trans_n` and the recursion is defined within the module `two_d_solutions`. If blowing-ups have to be used their application is controlled by the module `two_d_solutions` that uses the Newton diagram computed using the function `ND`. Once solution curves for two-dimensional systems have been computed, those solution curves, the sectors and the transformations are extended to the three dimensions by the function `solution_23`. Therefore the projection of the sectors on the $x_1 - x_2$-plane has to be reversed. If problems of bijectivity occur the definition of the sectors is adapted. Further the equation remaining from the splitting of the two-dimensional system is integrated to complete the solution curves.

The module `NESP_n` handels the integration of two-dimensional systems in the neighbourhood of nonelementary singular points. It is based on the algorithms introduced in chapter 8. It uses the primitive `ND` for the compuatation of the Newton diagram. Any of the computed faces are used to define blowing-ups that reduce the system and enables the computation of solution curves. The main transformation used in this context are the elementary operations as time- and power transformations. They are defined in the functions `ntt_n` and `power_trans_n`. The used methods lead to a recursive call if the entire algorithm. The definition of the recursion is based on the computations of the new singularities on the exceptional divisor and on the computation of subsectors. Those operations are performed for real and complex problems by the elementary procedure `subsectors`.

The primitive functions `transform_sector_n`, `vi_sector_2` and `vi_sector_3` are not used by the main procedure but they represent helpful tools for handling sectors and for displaying them. Their use will be illustrated by examples in section 11.3.

Only a part of the implemented modules are visible to the user. All integrations and reductions for two- and three-dimensional systems can be performed using the main procedure `FRIDAY`. The use of the visible procedures is illustrated by examples in section 11.3.

11.2 Using the package

The FRIDAY package is written in MAPLE V, release 5. It is available as a MAPLE package or as source code files. If the package form is used, the package has to be installed using the command

```
> with(FRIDAY);
```

This makes all visible procedures and functions available. The `with` command can only be executed if the variable

$$libname := libname,\ `user/packages/FRIDAY`:$$

in the file `.mapleinit` has been set to the directory that contains the package. The package version contains help topics that explain the use of the most important functions that are visible to the user. Those help pages can be consulted using the command

```
> ?function
```

If the source code of the FRIDAY package is available the program can be installed using the command

```
> read(FRIDAY);
```

or an equivalent command if the concerned files are not in the working directory. Informations on the current state of the computations can be displayed during runtime. Therefore the constant `infolevel[FRIDAY]` has to be set to a value between 1 and 3. For `infolevel[FRIDAY]` = 1 only basic informations are displayed. For higher values those informations are more and more precise. is set

The FRIDAY MAPLE package and its source code are available on the internet. More informations are available on the following adress

$$http://www-lmc.imag.fr/CF/logiciel.html\ .$$

The package version adresses to users that are only interested in computation results. The source code allows a user to perform only elementary operations and to modify or extend the FRIDAY package.

11.3 Introducing examples

The way the FRIDAY package works is best illustrated by some examples. In this section examples for the use of procedures contained in the FRIDAY package are given for a number of representative problems. The procedures are capable of treating more complex problems and are not restricted to the given simple examples.

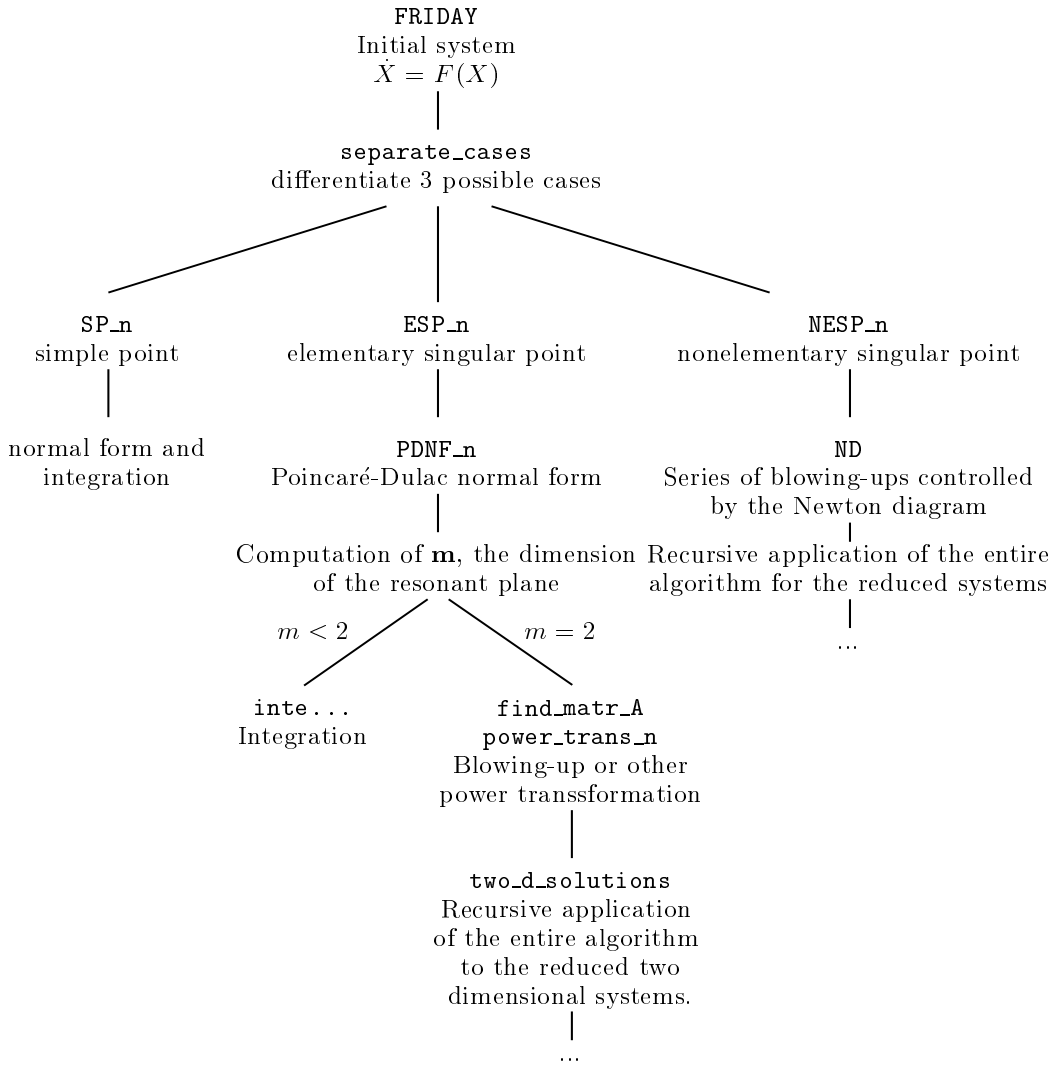


Figure 11.1: The sheme of the reduction tree. It shows the evolution of the computations. The concerned algorithms are specified in the chapters 5 to 9.

11.3.1 Integration of n-dimensional systems in the neighbourhood of a regular point

The algorithm that is used to compute solution curves for n-dimensional systems in the neighbourhood of a regular point has been introduced in chapter 6. The solution curves can be calculated using the main procedure `FRIDAY`.

First the input variables have to be defined in the vectors `F`, `X` and `S`. `F` defines the considered vector field, `X` the used variables and `S` the coordinates of the point of interest. Further the constant ϵ is needed for the definition of the sectors and neighbourhoods. The constant `index` that denotes the degree of approximation for all normal form computations is set to 4.

```
> F := [3+x[1]+x[2],3*x[1]*x[2]-2*x[1]^2];
> X:=[x[1],x[2]]; S:= [0,0];epsilon:=1/2;index:=4;
```

$$F := [3 + x_1 + x_2,\ 3\,x_1\,x_2 - 2\,{x_1}^2]$$

$$X := [x_1,\ x_2]$$

$$S := [0,\ 0]$$

$$\epsilon := 1/2$$

$$index := 4$$

With these definitions the main procedure can be executed. The varibles `c` and `t` are used for the computed solution curves.

```
> sol:=FRIDAY(F,X,S,c,t,epsilon,index):
```

The number of computed solution curves is 1.

```
> nops(sol);
```

$$1$$

The computed solution is represented by the list `sol[1]` with 5 elements.

```
> nops(sol[1]);
```

$$5$$

The first element of the list `sol[1]` represents the reduced system.

```
> print(sol[1][1]);
```

$$[1,\ 0]$$

This system can be integrated. The integration results are given in the second element of the list.

```
> print(sol[1][2]);
```

$$[t_1,\ c_1]$$

The third element of the list represents the transformation $X = H(\tilde{X})$ that was used to reduce the initial system.

```
> print(sol[1][3]);
```

$$[3\,x_1 + x_1\,x_2 + \frac{3}{2}\,{x_1}^2 + \frac{1}{2}\,{x_1}^2\,x_2 + \frac{1}{2}\,{x_1}^3 + \frac{5}{3}\,{x_1}^3\,x_2 - \frac{11}{8}\,{x_1}^4,$$
$$x_2 + \frac{9}{2}\,{x_1}^2\,x_2 + \frac{3}{2}\,{x_1}^2\,{x_2}^2 - 6\,{x_1}^3 - \frac{5}{2}\,{x_1}^3\,x_2 - \frac{9}{2}\,{x_1}^4]$$

The fourth element `sol[1]` contains the neighbourhood in the new coordinates that denotes the set where the computed solution is valid.

```
> print(sol[1][4]);
```

$$[[\frac{1}{2},\ t_1,\ [\frac{-1}{2},\ \frac{1}{2}]],\ [\frac{-1}{2},\ t_1,\ [\frac{-1}{2},\ \frac{1}{2}]],\ [t_1,\ \frac{1}{2},\ [\frac{-1}{2},\ \frac{1}{2}]],\ [t_1,\ \frac{-1}{2},\ [\frac{-1}{2},\ \frac{1}{2}]]]$$

For a simple point this sector is the entire neighbourhood $\tilde{U} = \{\tilde{X} : |\tilde{x}_1| \leq \epsilon, |\tilde{x}_2| \leq \epsilon\}$. $\tilde{U}$ is given by a list of 4 curves that denote the borders of U. Each curve is given in a parametrized form. The first curve for example is given by $\tilde{X}(t_1) = (1/2, t_1)$ with $t_1 = -1/2 \ldots 1/2$.

```
> print(sol[1][4][1]);
```

$$[\frac{1}{2},\ t_1,\ [\frac{-1}{2},\ \frac{1}{2}]$$

In the case of an elementary singular point the sectors in the initial coordinates are obtained by transforming the neighbourhood $\tilde{U}$. to the initial coordinates. The transformation used for this purpose is given as last element of the list.

```
> print(sol[1][5]);
```

$$[x_1,\ x_2]$$

In the case of a simple point this transformation is identity. The integration results computed for the reduced system can be transformed to the initial coordinates using the procedure `newtrans_n` and the transformation contained in the third element of the list.

```
> sol2:=newtrans_n(sol[1][3],X,sol[1][2]);
```

$$\left[3\,t_1 + t_1\,c_1 + \frac{3}{2}\,{t_1}^2 + \frac{1}{2}\,{t_1}^2\,c_1 + \frac{1}{2}\,{t_1}^3 + \frac{5}{3}\,{t_1}^3\,c_1 - \frac{11}{8}\,{t_1}^4,\right.$$
$$\left. c_1 + \frac{9}{2}\,{t_1}^2\,c_1 + \frac{3}{2}\,{t_1}^2\,{c_1}^2 - 6\,{t_1}^3 - \frac{5}{2}\,{t_1}^3\,c_1 - \frac{9}{2}\,{t_1}^4\right]$$

As for the computation of solution curves for simple points no time transformations are used the computed curves are the aproximated exact solutions of the initial system. The precision of the algorithm can be tested by substituting $X =$`sol2` in the equation

$$\dot{X} - F(X) = \texttt{inaccuarcy} \tag{11.2}$$

The lowest degree in t_1 and c_1 of the terms remaining in `inaccuarcy` denotes the degree of approximation. As the computations were executed with the index 4 the lowest degree of the remainig terms is 4 in both lines of equation (11.2).

```
> inaccuarcy := diff(sol2[1],t[1])-subs(x[1]=sol2[1],x[2]=sol2[2],F[1]):
> ldegree(expand(inaccuarcy),[t[1],c[1]]);
```

4

```
> inaccuarcy := diff(sol2[2],t[1])-subs(x[1]=sol2[1],x[2]=sol2[2],F[2]):
> ldegree(expand(inaccuarcy),[t[1],c[1]]);
```

4

11.3.2 Computation of n-dimensional normal forms for non-nilpotent singular vector fields

Non-nilpotent singular vector fields can be transformed to normal form. In the `FRIDAY` package there exist 3 procedures `PDNF_n`, `PI_NF_n` and `GNF_n` that compute normal forms for different purposes.

Computation of the Poincaré-Dulac normal form

The procedure `PDNF_n` computes the Poincaré-Dulac normal form for a given vector field. It is based on the algorithms introduced in section 3.4. The procedure `PDNF_n` can be used separately from the main procedure and allows the computation of n-dimensional normal forms for vector fields F if the matrix `A` $= D$`F`(0) is in Jordan form. If this is not the case the procedure `jordan_sys` that is based on the Maple function `jordan` can be used to compute a new system `JF` such that its linear part `B` $= D$`JF`(0) is in Jordan form. The linear transformation that transforms `F` to `JF` is assigned to the optional variable `tr1`.

```
> F :=[x[1]+2*x[1]-x[2]+2*x[1]*x[2]-x[3]*x[2],x[2]-3*x[4],
> x[3]+2*x[4],-x[3]+x[4]];
```

$$F := [3\,x_1 - x_2 + 2\,x_2\,x_1 - x_3\,x_2,\ x_2 - 3\,x_4,\ x_3 + 2\,x_4,\ -x_3 + x_4]$$

```
> A:=lin_part_n(F,X):print(A);
```

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

```
> JF:=jordan_sys(F,X,'tr1'):B:=lin_part_n(JF,X):print(B);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1+I\sqrt{2} & 0 \\ 0 & 0 & 0 & 1-I\sqrt{2} \end{bmatrix}$$

For the given example the procedure `PDNF_n` computes the Poincaré-Dulac normal form of the vector field `JF` up to order 3. Further the used transformation is assigned to the optional variable `tr`. The transformation `trans` that transforma the initial vector field `F` into its Poincaré-Dulac normal form `NF` can be computed from `tr1` and `tr` using the procedure `newtrans_n`.

```
> NF:=PDNF_n(JF,X,3,'tr');
```

$$NF := [x_1,\ 3\,x_2 + \frac{34}{3}\,x_1\,x_3\,x_4 + 12\,{x_1}^3,\ (1+I\sqrt{2})\,x_3,\ (1-I\sqrt{2})\,x_4]$$

```
> trans:=newtrans_n(tr1,X,tr):print(trans);
```

$$\left[-\frac{1}{6}\,x_2 + \frac{1}{2}\,x_1 - \frac{1}{3}\,x_2\,{x_1}^2 + \ldots\right]$$

Normal forms for real systems

In example 13 in section 7.2 the Poincaré-Dulac normal form for the pendulum equation in the neighbourhood of the singularity $X = 0$ was computed. The considered sysem has only real coefficients but two complex conjugated, purely imaginary eigenvalues. The particularity of those computations are that parameters are allowed in the linear part of the system as the resonance equation can be solved explicitely.

The computation of the normal form for systems with 2 complex conjugated, purely imaginary eigenvalues can be performed using the procedure `PI_NF_n`. In the considered example the computations are performed up to order 6. The last input variable in the function call indicates weather polar coordinates

$$\begin{aligned} x_1 &= \tilde{x}_1 \sin \tilde{x}_2 \\ x_2 &= \tilde{x}_1 \cos \tilde{x}_2 \\ x_3 &= x_3 \\ &\ldots \end{aligned}$$

should be introduced of not.

```
> F :=vector([x[2],convert(taylor(-g/l*sin(x[1]),x[1],4),polynom)]);
```

$$F := \left[x_2,\ -\frac{g\,x_1}{l} + \frac{1}{6}\,\frac{g\,{x_1}^3}{l}\right]$$

```
> sol:=PI_NF_n(F,X,6,false);
```

$$sol := [-\frac{g\,x_2}{l} + \frac{1}{16}\,\frac{g\,x_2\,{x_1}^2}{l} + \frac{1}{16}\,\frac{g^2\,{x_2}^3}{l^2} + \frac{17}{3072}\,\frac{{x_1}^4\,g\,x_2}{l} + \frac{17}{1536}\,\frac{g^2\,{x_1}^2\,{x_2}^3}{l^2} + \frac{17}{3072}\,\frac{g^3\,{x_2}^5}{l^3},$$
$$x_1 - \frac{1}{16}\,\frac{g\,x_1\,{x_2}^2}{l} - \frac{1}{16}\,{x_1}^3 - \frac{17}{3072}\,\frac{g^2\,x_1\,{x_2}^4}{l^2} - \frac{17}{3072}\,{x_1}^5 - \frac{17}{1536}\,\frac{g\,{x_2}^2\,{x_1}^3}{l}]$$

```
> sol:=PI_NF_n(F,X,6,true);
```

$$sol := \left[0,\ \frac{1}{3072}\,\frac{g\,(-3072\,l^2 + 192\,g\,{x_1}^2\,l + 17\,{x_1}^4\,g^2)}{\sqrt{\frac{g}{l}}\,l^3}\right]$$

The generalized normal form

Another kind of normal form for n-dimensional systems can be computed using the procedure `GNF_n`. The implemented algorithm is based on the matrix representation method and the Maple optimisation procedure `leastsqrs`. It reduces a maximum of nonlinear terms, even if the matrix $DF(0)$ if not in Jordan form. If the matrix $DF(0)$ is in Jordan form the algorithm yields the Poincaré-Dulac normal form. However the used algorithm is less efficient than the algorithms implemented in `PDNF_n` and for the example of the pendulum it does not yield the optimal normal form (compare section 7.2).

```
> g:=1:l:=2:sol:=GNF_n(F,X,6);
```

$$sol := [x_2 - \frac{2}{161}\,x_2\,{x_1}^2 - \frac{3}{161}\,{x_2}^3 - \frac{22156}{129475395}\,x_2\,{x_1}^4 - \frac{11078}{129475395}\,{x_2}^3\,{x_1}^2 - \frac{5539}{25895079}\,{x_2}^5,$$
$$-\frac{1}{2}\,x_1 + \frac{12}{161}\,{x_1}^3 + \frac{2}{161}\,x_1\,{x_2}^2 + \frac{22156}{129475395}\,{x_1}^3\,{x_2}^2 + \frac{44312}{25895079}\,{x_1}^5$$
$$+ \frac{11078}{129475395}\,{x_2}^4\,x_1]$$

In the main procedure `FRIDAY` the procedure `GNF_n` is used if it is known that the normal form of a given vector field has no nonlinear resonant terms (see section 7.2).

11.3.3 Integration of two-dimensional elementary singular points

The integration of two-dimensional vector fields `F` in the neighbourhood of an elementary singular point is handeled by the main procedure `FRIDAY`. The concerned vector fields can have real or complex coefficient and parameters if they do not affect the resonance equation. The implemented algorithms have been introduced in chapter 7.

```
> F := vector([x[2],convert(taylor(-g/l*sin(x[1]),x[1],4),polynom)]);
```

$$F := \left[x_2,\ -\frac{g\,x_1}{l} + \frac{1}{6}\,\frac{g\,{x_1}^3}{l}\right]$$

```
> sol:=FRIDAY(F,X,S,c,t,1/2,4):
```

WARNING : parameters in the system !
WARNING : parameters in linear part of the system !

Like in the example of a regular point in section 11.3.1 the returned solution list `sol` contains the reduced system `sol[1][1]`, the integration result `sol[1][2]` and the used transformation `sol[1][3]`.

```
> sol[1][1];
```

$$\left[0,\ \frac{1}{16}\,\frac{g\,(-16\,l+{x_1}^2\,g)}{\sqrt{\frac{g}{l}}\,l^2}\right]$$

```
> sol[1][2];
```

$$[c_1,\ t_1]$$

```
> sol[1][3];
```

$$\left[\frac{1}{192}\,\frac{x_1\,\sqrt{\frac{g}{l}}\,\sin(x_2)\,(192\,l+5\,{x_1}^2\,g+4\,g\,{x_1}^2\,\cos(x_2)^2)}{l},\right.$$
$$\left.-\frac{1}{64}\,\frac{g\,x_1\,\cos(x_2)\,(64\,l-5\,{x_1}^2\,g+4\,g\,{x_1}^2\,\cos(x_2)^2)}{l^2}\right]$$

11.3.4 Integration of two-dimensional nonelementary singular points

The integration of two-dimensional vector fields `F` in the neighbourhood of a nonlelementary singular point is based on the method introduced in chapter 8. The vector field can have real or complex coefficients. The integration is handeled by the main procedure `FRIDAY`. The variable `infolevel[FRIDAY]` has been set to 1 to give some informations on the computations during runtime. Those informations indicate the current state of the computations.

```
> F:=[-x[1]^4+x[1]^3*x[2],13/9*x[1]^6*x[2]^2-x[1]^2*x[2]^2+x[1]*x[2]^3];
```

$$F := [-{x_1}^4+{x_1}^3\,x_2,\ \frac{13}{9}\,{x_1}^6\,{x_2}^2-{x_1}^2\,{x_2}^2+x_1\,{x_2}^3]$$

```
> sol:=FRIDAY(F,X,S,c,t,1/2,4):
```

```
separate_cases_23:   NONELEMENTARY SINGULAR POINT
separate_cases_23:   NONELEMENTARY SINGULAR POINT
separate_cases_23:   SIMPLE POINT
NF_SP_n:   simple point calculated
separate_cases_23:   SIMPLE POINT
NF_SP_n:   simple point calculated
```

For the considered example, that has already been studied in example 15 in section 8.2, 5 different solutions are computed. The sectors returned in the variable `sol[i][4]` for $i = 1, \ldots, 5$ can be transformed to the initial coordinates by the procedure `transform_sector_n` and the transformation in `sol[i][5]` for $i = 1, \ldots 5$.

```
> nops(sol);
```

$$5$$

```
> sol[1][4];
```

$$[[\frac{-1}{2},\ t_1,\ [\frac{-1}{2},\ \frac{1}{2}]],\ [\frac{1}{2},\ t_1,\ [\frac{-1}{2},\ \frac{1}{2}]],\ [t_1,\ \frac{1}{2},\ [\frac{-1}{2},\ \frac{1}{2}]],\ [t_1,\ \frac{-1}{2},\ [\frac{-1}{2},\ \frac{1}{2}]]]$$

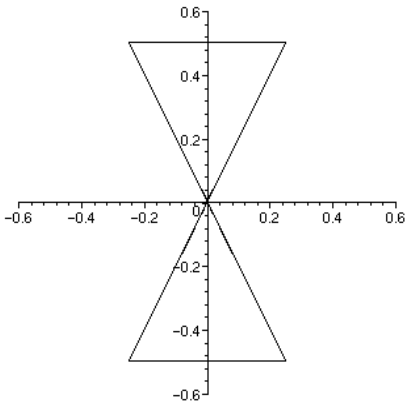
```
> print(sol[1][5]);
```

$$[x_1\ x_2,\ x_2]$$

```
> sect:=transform_sector_n(sol[1][4],X,sol[1][5],1/2):
```

The resulting sectors can be visualized by the procedure `vi_sectors_2`. All 5 sectors are sketched in figure 8.8 in section 8.2.

```
> vi_sectors_2(sect,0.6);
```



If the optional variable `complex` is set in the call of the procedure `FRIDAY`, complex solutions are computed. This yields 8 solutions as the definition of subsectors in $\mathbb{C}^2$ yield more subsectors than in $\mathbb{R}^2$ (compare section 8.2). The sectors are defined by the curves $(x_1(t_1, t_2, t_3), x_2(t_1, t_2, t_3)),\ t_1, t_2, t_3 \in \mathbb{R}$.

```
> read(bruno_2):sol:=FRIDAY(F,X,S,c,t,1/2,4,complex):
> nops(sol);
```

$$8$$

```
> sol[1][4];
```

$$\left[\left[-\tfrac{1}{2}-I\,t_3,\ t_1+I\,t_2\right],\ \left[\tfrac{1}{2}+I\,t_3,\ t_1+I\,t_2\right],\left[t_1+I\,t_2,\ \tfrac{1}{2}+I\,t_3\right]\ \left[t_1+I\,t_2,\ -\tfrac{1}{2}-I\,t_3\right],\right.$$
$$\left.\left[-\tfrac{1}{2}\,I-t_3,\ t_1+I\,t_2\right],\ \left[\tfrac{1}{2}\,I+t_3,\ t_1+I\,t_2\right],\left[t_1+I\,t_2,\ \tfrac{1}{2}\,I+t_3\right],\ \left[t_1+I\,t_2,\ -\tfrac{1}{2}\,I-t_3\right]\right]$$

11.3.5 Integration and Reduction of three-dimensional elementary singular points

The procedures for the integration of three-dimensional vector fields with elementary singular points is based on methods introduced in chapter 9. They can treat systems with real coefficients. The integration of those vector fields is handeled by the main procedure `FRIDAY`. The variable `infolevel[FRIDAY]` is set to 1 to give some informations on the computations during runtime.

The support of the considered vector field `F` lies within a two-dimensional cone and within the first quadrant in the space of exponents. Therfore it can be integrated.

```
> F := [-2*x[1]+x[1]^2*x[3]^2,x[2],x[3]+x[1]*x[2]^2*x[3]];
```

$$F := [-2\,x_1 + {x_1}^2\,{x_3}^2,\ x_2,\ x_3 + x_1\,{x_2}^2\,x_3]$$

```
> sol:=FRIDAY(F,X,[0,0,0],c,t,1/2,4):

separate_cases_23:   ELEMENTARY SINGULAR POINT
two_d_solutions:   2d procedure
separate_cases_23:   NONELEMENTARY SINGULAR POINT
separate_cases_23:   SIMPLE POINT
separate_cases_23:   SIMPLE POINT
two_d_solutions:   2d solution calculated !
two_d_solutions:   2d procedure
separate_cases_23:   NONELEMENTARY SINGULAR POINT
separate_cases_23:   SIMPLE POINT
separate_cases_23:   SIMPLE POINT
two_d_solutions:   2d solution calculated !
```

The vector field F is reduced to a two-dimensional vector field with a nonelementary singular point. This vector field can be integrated but the obtained solution curves are only valid in sectors. Those sectors are extended to three-dimensional sectors. They can be transformed to the initial coordinates using the procedure `transform_sector_n` and visualized in three dimensionas by the procedure `vi_sectors_3`.

```
> sect := transform_sector_n(sol[i][4],X,sol[i][5],1/2);
> vi_sector_3(sect,t,2):
> a:=1/2:  for i from 1 to nops(sol) do
> display(p[i],view=[-a..a,-a..a,-a..a]); od;
```

The 12 resulting sectors are shown in the figures 11.2 and 11.3. It can be observed that in every figure three identic sectors appear. This is due to the fact that in addition to the general solutions computed for a sector two further solutions, the so called particular solutions, have to be considered. (See also case 4 in section 7.1).

The neighbourhood of $X = 0$ is decomposed into 6 sectors for $x_1 > 0$ and 6 sectors for $x_1 < 0$. That shows that the power transformation, that is used to reduce the initial three-dimensional system, is not injective. Therefore additional constructions as they were introduced in section 2.3 are used.

For the considered system `F` the structure of the two-dimensional sectors is well preserved by the extention to three coordinates and the retransformation to the initial coordinates. This is not always true as the inverse of the power transformation, that is used to simplify the vector field `F`, is a blowing-up.

11.4 Tests

A large number of tests have been run to ensure the viability of the FRIDAY package. The possible tests for the computed solutions are the following :

- Introducing the solution curves into the initial differential equation and computing the accuarcy of the results yields an evaluation for the precision of the computations. This test can only be used if the computed results are approsimations of the exact solutions. It has for example be used to proove the viability of the computation of normal forms for systems with simple points. If time transformations are used at any step of the computations, this test fails.

- The computed solution curves can be introduced into the energy function H for Hamiltonian systems or into the scalar differential equation associated the two-dimensional systems. This yields another indicator for the accuarcy of the computed results. This method also works if time transformations were applied to the considered system. However it fails if power transformation were used. Power transformations cause problems as negative exponents may appear in the retransformed solution curves.

Due to the limited possiblities it is in general not possible to test the computed results. However the primitives can be tested individually and the tests proposed above can be used for some particular exampes. As far as possible every module was tested for arbitrarilly chosen vector fields.

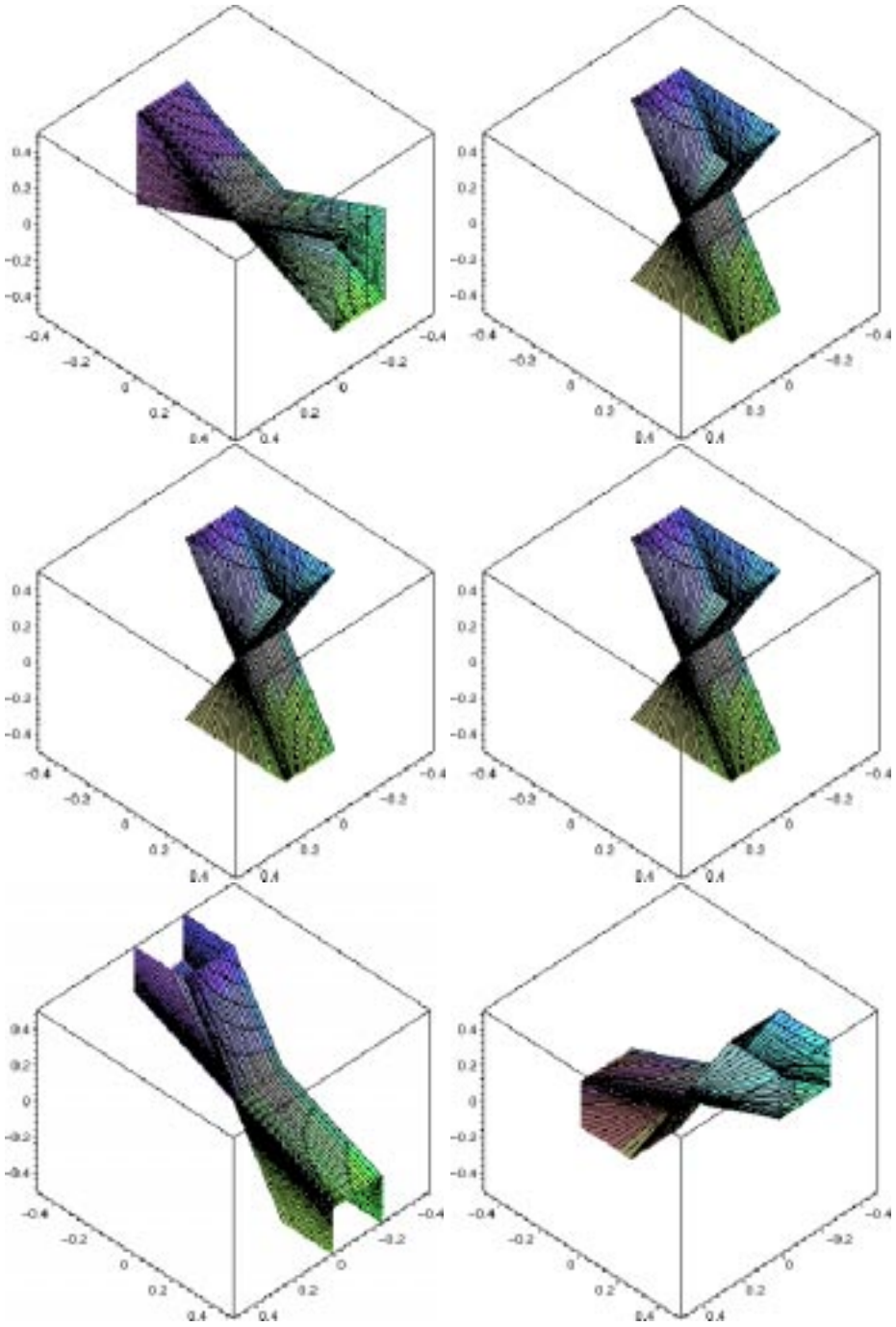


Figure 11.2: This figure shows 6 sectors computed for the example in section 11.3.5. They form a decomposition of a neighbourhood of $X = 0$ for $x_1 < 0$.

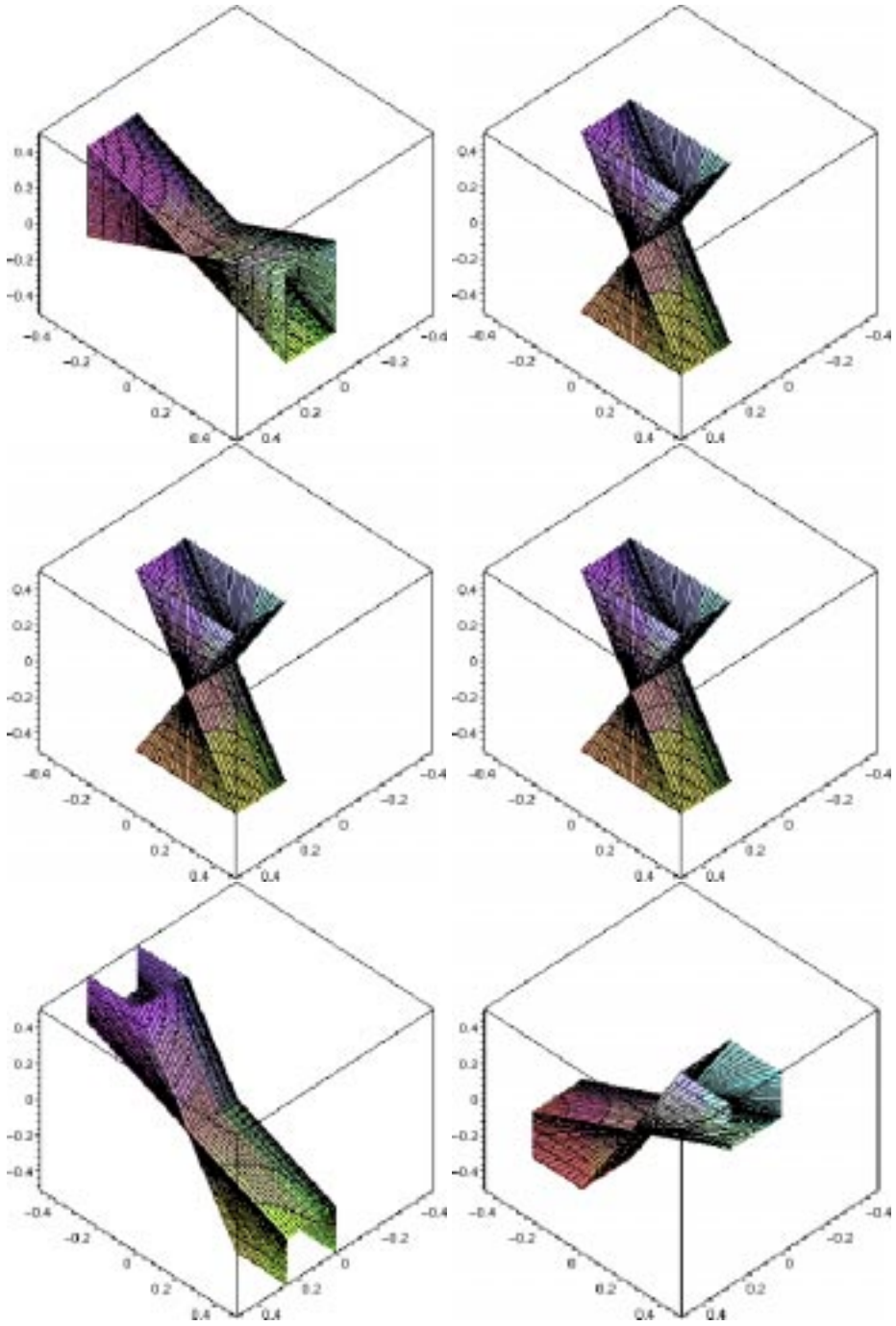


Figure 11.3: This figure shows 6 sectors computed for the example in section 11.3.5. They form a decomposition of a neighbourhood of $X = 0$ for $x_1 > 0$.

Conclusion

The objective of this thesis is the study of the theoretical and practical aspects of the reduction and formal integration of two- and three-dimensional systems of autonomous differential equations. The case of two-dimensional systems has been solved completely. Any considered system can be reduced and integrated by the proposed algorithms.

The three-dimensional case yields much more problems. Reductions are only possible for some particular cases. We have introduced an algorithm that allows the formal integration of any three-dimensional system in the neighbourhood an elementary singular point. These results can be obtained due to a generalization of power transformations and blowing-ups.

The used transformations can be interpreted geometrically. This interpretation allows a very efficient handling of all reductions. All transformations are interpreted geometrically in the space of exponents as manipulations on the support of the system. The use of those geometric methods also allows to cover the concerned neighbourhood entirely by sectors and to compute all solutions.

The generalization of the proposed algorithms to higher-dimensional systems with elementary singular points is possible. For this purpose the virtual Newton diagram has been introduced. The virtual Newton diagram completes the information obtained for the considered system by the Newton diagram. Therefore it allows controlled blowing-ups and an entire covering of the concerned neighbourhood by sectors. However the integration of the reduced systems is only possible if the reduction of higher-dimensional nilpotent systems can be controlled.

The reduction of nilpotent three- and higher-dimensional systems by blowing-ups is only possible for some particular cases. In these cases the virtual Newton diagram can be used to construct blowing-ups that yield sectors that entirely cover the concerned neighbourhood.

The proposed algorithms have been implemented in the FRIDAY Maple package. This program formally integrates any two-dimensional and a large number of three-dimensional systems.

The problem of convergence is always arising in the context of the reduction of dynamical systems by symbolic computations. Although many results on the resummation of formal power series are known, many problems of convergence for normalizing transformations remain unsolved. Further problems are encountered in generalizing the obtained results to higher dimensional problems. These problems offer many possibilities for further work in this domain.

Three and higher dimensional blowing-ups certainly cause the main theoretical problems as it has been shown that finite succesive blowing-up can not entirely reduce any

nonelementary singularity. For an entire study of those cases the use of some further methods might be necessary.

Three dimensional blowing-ups also cause many algorithmic problems as it has been shown for the dicritical case of an edge. This problem will have to be considered more closely to entirely cover the case of a nonelementary singular point.

In this work an algorithm for the computation of the virtual Newton diagram has been proposed. However this algorithm performs several computations of the convex hull of a set of points. As this is not very efficient this algorithm could be replaced by a more appropriate method.

The additional construction introduced for power transformations makes the use of several similar transformations necessary. As the solutions computed for the different transformations are similar too, some simplifications in the algorithms might yield a much more efficient program.

The MAPLE package FRIDAY works very efficiently for two dimensional and some three dimensional problems. However generalizations to higher dimensional problems will need some basic modifications of the sector notation. Handling sectors in several recursions and different dimensions causes implementation problems. Handling the sectors is much more complicated than handling the solutions and the successive transformations.

The possible applications of the proposed methods are the studies of bifurcations as normal forms play a very important role in this domain. As the use of parameters is possible up to a certain degree this represents an important advantage. The use of parameters could also be used to perform algebraic optimization on simple physical models.

Conclusion

L'objectif de cette thèse était d'étudier les aspects théoriques et pratiques de la réduction et de l'intégration des systèmes d'équations différentielles ordinaires en deux et trois dimensions. Le cas des systèmes en deux dimensions a été résolu entièrement. Tout système en deux dimension peut être intégré par les algorithmes que nous avons proposés.

Dans le cas des systèmes en trois dimensions nous rencontrons des problèmes qui rendent l'étude de tels systèmes beaucoup plus complexe. Nous proposons un algorithme permettant l'intégration de tout système au voisinage d'un point singulier élémentaire. Ces résultats sont basés sur la généralisation de l'utilisation des transformations quasi-monomiales et des éclatements.

La généralisation des algorithmes proposés aux systèmes en dimension supérieure est possible. Les problèmes rencontrés peuvent être résolus grâce au diagramme de Newton virtuel que nous avons introduit auparavant. Le diagramme de Newton virtuel complète les informations obtenues par le diagramme de Newton. Il permet de contrôler les éclatements utilisés et définit un ensemble de secteurs couvrant entièrement le voisinage concerné. L'intégration des systèmes que nous obtenons grâce à ces méthodes n'est possible que si les systèmes nilpotents obtenus sont intégrables.

La réduction des systèmes nilpotents en trois dimensions en utilisant des éclatements n'est possible que pour certains cas particuliers. Dans ces cas nous pouvons également utiliser le diagramme de Newton virtuel qui permet de définir des éclatements couvrant entièrement le voisinage concerné par un ensemble de secteurs.

Les algorithmes que nous avons proposés dans cette thèse ont été implantés en Maple dans le package FRIDAY. Ce logiciel permet l'intégration formelle de tout système en deux dimensions et d'une large partie des systèmes en trois dimensions.

Im me semble que les points suivants peuvent compléter ce travail. Du point de vue théorique, de nombreux problèmes restent à résoudre pour des systèmes de dimension trois et supérieure. L'aspect de la resommation et de la convergence devient essentiel si nous voulons utiliser les méthodes proposées pour résoudre des problèmes réels. Mais les travaux connus sur la resommation ne permettent pas encore de formuler des algorithmes ou d'implanter un logiciel.

Du point de vue du code de calcul, la partie des formes normales et le calcul de l'enveloppe convexe nécessite une réécriture dans un langage compilé. L'algorithme introduit pour le calcul du diagramme de Newton virtuel n'est pas très rapide. Il pourrait être remplacé par un algorithme plus sophistiqué.

Le logiciel FRIDAY est très efficace pour résoudre des problèmes en deux et trois dimensions. En généralisant les procedures existantes à des dimensions supérieures, les secteurs utilisés vont devenir de plus en plus compliqués. Un problème fondamental est

donc celui de la gestion des secteurs en plusieurs étapes récursives et en différentes dimensions.

Les applications possibles des méthodes proposées sont surtout les problèmes des bifurcations, car les formes normales jouent un rôle essentiel dans ce domaine. Le fait que des paramètres soient partiellement permis peut représenter un avantage non négligeable. Il permet également d'optimiser les paramètres d'une équation issue d'un problème de modélisation.

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