Multifractional Lévy Motions

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Introduction

Les applications des champs autosimilaires se sont considérablement développées au cours de ces dernières années. En effet, de nombreux phénomènes autosimilaires sont observés en hydrologie, en écologie, en mécanique des fluides, en imagerie ou encore en finance. Par suite, de nombreux modèles probabilistes autosimilaires ont été introduits et étudiés pour modéliser ces phénomènes. L'autosimilarité, propriété d'invariance en loi par changement d'échelle, est gouvernée par un indice H. De plus, les trajectoires d'un champ autosimilaire d'ordre H ne sont pas lisses dès que $H < 1$. Dès lors, l'autosimilarité fournit des modèles adaptés à l'étude de phénomènes irréguliers. L'exemple le plus simple de processus autosimilaire est sans doute le mouvement brownien. Ceci étant, ces accroissements étant indépendants, il ne permet pas la modélisation de phénomènes à longue dépendance. Dès lors, B. Mandelbrot et J. Van Ness ont introduit dans [MVN68] un champ généralisant le mouvement brownien et adapté à la modélisation de tels phénomènes : il s'agit du mouvement brownien fractionnaire ou FBM pour l'anglais Fractional Brownian Motion. Ils ont notamment défini le FBM pour pouvoir modéliser les minima annuels du niveau du Nil et expliquer les observations de H. Hurst en 1951 dans [Hur51] sur la série des données de ces minima de l'an 622 à l'an 1284. Ainsi, la vitesse de convergence de la statistique R/S, appelée rescaled adjusted rand, peut s'expliquer par la présence de longue dépendance. Par ailleurs, le FBM avait été implicitement utilisé dans [Kol40] pour construire des «spirales» gaussiennes sur les espaces d'Hilbert. Précisons que le FBM $(B_H(x))_{x\in\mathbb{R}^d}$ est un champ gaussien dont la loi est caractérisée par l'indice fractionnaire H encore appelé indice de Hurst. Depuis son introduction, le FBM a été utilisé comme modèle dans des domaines variés. Cependant, la régularité des trajectoires du FBM B_H ne dépend pas du point où on l'observe. Par suite, un FBM ne peut modéliser que des phénomènes spatialement homogènes. De ce point de vue, le FBM n'est pas adapté à la modélisation d'un relief de montagnes qui est plus ou moins régulier selon la nature du sol. En fait, ce relief doit être modélisé par un champ dont la régularité locale des trajectoires peut varier. D'autres exemples dans lesquels on constate la nécessité d'avoir un exposant fractionnaire variable en fonction de la position sont donnés dans [Fri95] dans le domaine de la turbulence ou encore en finance dans [Man97]. Dès lors, de nombreuses généralisations du FBM ont été introduites. Toutes ces extensions du FBM satisfont une propriété d'autosimilarité et la régularité de leurs trajectoires a été étudiée. Ces études ont notamment été menées dans le cadre des champs introduits dans [BJR97], [PLV96], [ALV00], [BBCI00] et [BCI02]. De plus, la propriété d'autosi-

milarité devient une propriété locale et asymptotique. Cette notion d'autosimilarité locale a été introduite dans [BJR97].

Par ailleurs, le mouvement brownien multifractionnaire (MBM en abrégé), introduit indépendamment dans [PLV96] et [BJR97], est sans doute le plus connu des champs localement autosimilaires généralisant le FBM. Ainsi, il permet la modélisation de phénomènes non homogènes spatialement car son exposant ponctuel de Hölder peut varier le long des trajectoires. Cependant, la régularité des trajectoires d'un MBM ne varie pas de façon très brutale d'un point à l'autre. Dès lors, de nombreux mathématiciens s'intéressent à généraliser le MBM de sorte que la régularité puisse varier le plus erratiquement possible le long des trajectoires. Par exemple, des champs dont la régularité peut varier de façon très brutale ont été introduits dans [ALV00] et [BBCI00]. Ces champs peuvent notamment trouver des applications en traitement d'image où l'on observe des phénomènes de segmentations.

De plus, le FBM est un champ isotrope et de nombreux phénomènes notamment en médecine ne le sont pas. Ainsi, [LP99], [ALP02] ou encore [BE03] se sont intéressés à des champs anisotropes notamment en vue de détecter l'ostéoporose.

Par ailleurs, les phénomènes observés ne sont pas toujours gaussiens en pratique et les champs cités jusqu'ici le sont. Ainsi, en traitement d'image, en turbulence, dans l'étude des réseaux de télécommuncations ou encore en finance, on peut observer des phénomènes non gaussiens. Des exemples sont notamment donnés dans [Mal89], [Sim99] et [Vid99] en traitement d'image. Très peu de champs non gaussiens autosimilaires ou localement autosimilaires ont été introduits. Citons les champs autosimilaires stables étudiés dans [ST94]. Des champs localement autosimilaires ont aussi été définis dans [BCI02] et [BCI04]. Contrairement aux champs stables, ces champs sont du second ordre et ont la même structure de covariance que le FBM. En particulier les champs de Lévy fractionnaires étudiés par [BCI02] généralisent le FBM et ont des propriétés semblables aux siennes. Ils sont définis à partir d'une représentation harmonisable et appelés RHFLMs (abréviation de l'anglais Real Harmonizable Fractional Lévy Motions). Ces champs permettent la modélisation de phénomènes non gaussiens, mais comme dans le cas du brownien fractionnaire, spatialement homogènes.

Dans cette thèse, nous introduisons dans un premier temps des champs non gaussiens localement autosimilaires dont l'exposant ponctuel de Hölder peut varier le long des trajectoires. Plus précisément, ces champs réels, appelés en anglais Real Harmonizable Multifractional Lévy Motions (RHMLMs) ou encore en français champs réels de Lévy multifractionnaires, généralisent les RHFLMs étudiés dans [BCI02] et sont définis au moyen d'une représentation harmonisable. La plus grande partie de cette thèse est consacrée à l'étude de ces champs ainsi qu'à leur simulation.

Enfin, dans un deuxième temps nous nous sommes intéressés plus particulièrement à la propriété d'autosimilarité locale. Cette propriété introduit une notion de champ tangent. Dans la plupart des exemples de champs fractionnaires ou multifractionnaires étudiés dans la littérature, le champ tangent en x est un FBM. Par

exemple, les RHMLMs, même s'ils ne sont pas gaussiens, ont en tout point x un FBM comme champ tangent. Nous pouvons aussi citer les champs gaussiens étudiés dans [ALV00, BCIJ98] : localement tous ces champs ressemblent à un FBM. Il existe toutefois des champs fractionnaires X_H autosimilaires non gaussiens, tels les champs stables fractionnaires, mais alors en tout point le champ tangent est le même et c'est X_H lui-même. Nous introduisons alors un champ $X_{H,\beta}$ localement autosimilaire mais avec un comportement différent en un point x_0 : le champ tangent en x_0 n'est pas du même type que le champ tangent en tout autre point de la trajectoire. En ce point x_0, le comportement de $X_{H,\beta}$ est alors dit atypique.

Le premier chapitre s'intéresse aux champs qui ont conduit à l'introduction des RHMLMs. Plus précisément, les propriétés du FBM ainsi que du MBM sont rappelées. Par ailleurs, la construction des RHFLMs, décrite dans [BCI02], est donnée. La plupart des résultats sont déjà connus.

Le deuxième chapitre, qui a donné lieu à l'article [Lac04a] à paraître aux Annales de l'Institut Henri Poincaré, introduit la classe des RHMLMs. Cette classe contient non seulement les RHFLMs mais aussi le MBM. De plus, les RHMLMs sont des champs du second ordre non gaussiens en général et dont la structure de covariance est celle d'un MBM. Leurs propriétés semblables à celles du MBM étendent le champ d'applications du MBM à des phénomènes non gaussiens. Ainsi, un RHMLM X_h est localement autosimilaire avec les mêmes champs tangents et la même fonction multifractionnaire que le MBM B_h. Dès lors, au vu de sa structure locale, un RHMLM peut localement être vu comme un FBM. Dans le cadre du MBM, cette propriété d'autosimilarité locale a déjà été établie dans [BJR97]. Par ailleurs, un RHMLM X_h a le même exposant ponctuel en x que le MBM B_h. En particulier, l'exposant ponctuel de Hölder d'un RHMLM peut varier le long des trajectoires alors que tel n'est pas le cas pour un RHFLM. Enfin, le chapitre se conclut par l'identification de la fonction multifractionnaire. L'estimateur basé sur les variations quadratiques généralisées proposé par [IL97] dans un cadre gaussien permet l'identification de l'indice fractionnaire d'un RHFLM comme établi dans [BCI02]. Dans le cadre général des RHMLMs, en procédant comme dans [BCI98] qui s'intéresse au MBM, c'est-à-dire en localisant les variations quadratiques généralisées, nous obtenons un estimateur de la fonction multifractionnaire h.

Suite à l'étude des RHMLMs, le chapitre 3, qui constitue l'article [Lac04b], propose une méthode de simulation des RHMLMs. Précisons tout d'abord qu'on ne s'intéresse qu'à la simulation de la partie non gaussienne X_h d'un RHMLM. La méthode proposée est basée sur la représentation sous forme de séries de bruits généralisés des lois indéfiniment divisibles (voir [Ros90]). Dans le cas où la mesure de contrôle du RHMLM est de masse finie, une représentation sous forme de série de bruits de X_h est donnée. Dans le cas contraire, on propose alors de procéder comme le fait [AR01] pour la simulation des processus de Lévy. Plus précisément, on décompose X_h en deux RHMLMs indépendants $X_{\varepsilon,1}$ et $X_{\varepsilon,2}$ de fonction multi-

fractionnaire h. Le champ $X_{\varepsilon,2}$ est alors représenté sous forme d'une série de bruits généralisés et un théorème de convergence en loi donne une approximation de $X_{\varepsilon,1}$ par un MBM. Les erreurs commises par ses diverses approximations sont étudiées. Enfin, des exemples de simulations sont donnés et l'on constate que plus $h(x)$, qui est l'exposant ponctuel en x de X_h, est grand plus la trajectoire est régulière en x.

Enfin, le dernier chapitre introduit un champ $X_{H,\beta}$ localement autosimilaire mais avec un champ tangent en 0 différent du champ tangent en $x \neq 0$. Dans cet exemple, le champ tangent en $x = 0$ dépend de la valeur de β. Mais le cas le plus frappant est le cas où $\beta > d/\alpha$. En effet, alors que le champ tangent en $x \neq 0$ est un FBM d'indice H, le champ tangent à $X_{H,\beta}$ en $x = 0$ est un processus stable fractionnaire défini par une représentation harmonisable. Par suite, même si le champ $X_{H,\beta}$ est un champ du second ordre, son champ tangent en 0 n'en est pas un. De plus, en $x = 0$ le comportement local de $X_{H,\beta}$ est très éloigné de son comportement en $x \neq 0$. D'où un comportement dit atypique du champ $X_{H,\beta}$ en 0. Dans un premier temps, les propriétés d'autosimilarité locale sont établies et ce quel que soit β. De plus, le champ $X_{H,\beta}$ satisfait une propriété d'autosimilarité à grandes échelles. On retrouve alors un phénomène multi-échelles déjà rencontré pour les RHMLMs ou encore dans [BD99]. L'étude de $X_{H,\beta}$ est ensuite complétée par celle de la régularité des trajectoires et de la dimension de Hausdorff de ses graphes. Enfin, des champs avec un comportement atypique en plusieurs points sont introduits.

Chapter 1

Locally asymptotically self-similar fields

Self-similar fields provide powerful models in applied mathematics. Actually, they are often used to model natural phenomena. Then, many examples of applications of self-similarity are given in [Taq86]. Furthermore, the Fractional Brownian Motion is certainly the most famous self-similar field.

1.1 Fractional Brownian Motion

1.1.1 Definitions and properties

Mandelbrot and Van Ness have defined the Fractional Brownian Motion, in [MVN68], as a stochastic integral. However, there exist many ways to characterize the Fractional Brownian Motion. Since it is a Gaussian field, we introduce it thanks to its covariance function.

Definition 1.1.1. Let $0 < H < 1$. A Fractional Brownian Motion $(B_H(x))_{x\in\mathbb{R}^d}$, in short FBM, with index H is a real-valued centered Gaussian field with covariance function

$$R_H(x,y) = \frac{C}{2}\Big(\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\Big), \tag{1.1}$$

where $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $C > 0$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

In addition, B_H is called standard FBM if $C = 1$.

Remark 1.1.2. In the case where $d = 1$ and $H = 1/2$, B_H is a Brownian Motion.

Brownian Motions are the only FBMs whose increments are independent. However, like Brownian Motion, FBMs have stationary increments.

Proposition 1.1.3. *A FBM B_H has stationary increments, i.e*

$$\forall x_0 \in \mathbb{R}^d,\ (B_H(x+x_0) - B_H(x))_{x\in\mathbb{R}^d} \overset{(d)}{=} (B_H(x))_{x\in\mathbb{R}^d},$$

where $\overset{(d)}{=}$ stands for equality in law of finite dimensional margins.

Many properties of a FBM B_H are given by its fractional index H. As an example, H governs the self-similarity property. First, let us precise the definition of a self-similar field.

Definition 1.1.4. A real-valued field $(Y(x))_{x\in\mathbb{R}^d}$ is self-similar with index H if

$$\forall \lambda > 0,\ (Y(\lambda x))_{x\in\mathbb{R}^d} \overset{(d)}{=} \lambda^H (Y(x))_{x\in\mathbb{R}^d}.$$

Proposition 1.1.5. *A FBM B_H is self-similar with index H, i.e.*

$$\forall \lambda > 0,\ (B_H(\lambda x))_{x\in\mathbb{R}^d} \overset{(d)}{=} \lambda^H (B_H(x))_{x\in\mathbb{R}^d}.$$

From this property of self-similarity, an upper bound for the pointwise Hölder exponent can be stated. Let us first recall the definition of the pointwise Hölder exponent of a function.

Definition 1.1.6. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Then,

$$H_f(x) = \sup\left\{\alpha > 0,\ \lim_{y\to 0} \frac{f(x+y) - f(x)}{\|y\|^\alpha} = 0\right\}$$

is called the pointwise Hölder exponent of the function f at point x.

Then, since B_H is H-self-similar,

$$H_{B_H}(x) \leq H \text{ almost surely.} \tag{1.2}$$

In dimension $d = 1$, the increments stationarity and the self-similarity characterize the FBM.

Proposition 1.1.7. *Assume that $d = 1$. Then, FBMs with index H are the only real-valued centered Gaussian fields which are self-similar with index H and have stationary increments.*

When $d \geq 2$, the preceding proposition does not remain true. In fact, there exists anisotropic fields, see for instance [BE03], which are self-similar and have stationary increments.

Using the Kolmogorov criterion, it can be shown that the trajectories of a FBM are continuous.

Proposition 1.1.8. *For every $H' < H$, there exists a modification of the FBM B_H whose sample paths are almost surely locally H'-Hölder continuous on $\mathbb{R}^d$.*

Then, this proposition gives us a lower bound for the pointwise Hölder exponent of a FBM. Hence, using (1.2), one easily proves the following corollary.

Corollary 1.1.9. *Let $x \in \mathbb{R}^d$. Then, almost surely*

$$H_{B_H}(x) = H. \tag{1.3}$$

Remark 1.1.10. Actually, (1.3) holds in a strong sense. More precisely,

$$\mathbb{P}\big(\forall x \in \mathbb{R}^d,\, H_{B_H}(x) = H\big) = 1. \tag{1.4}$$

In fact, [AJT04] have introduced Generalized Multifractional Processes with Random Exponents which generalize the FBM and can have a random pointwise Hölder exponent. Then, in the particular case of FBM, [AJT04] have obtained (1.4).

1.1.2 Harmonizable Representation

There exist many representations of the FBM. Here we only give its harmonizable representation. However, other representations can be found in [ST94].

Proposition 1.1.11. *Let $0 < H < 1$. Then, the standard FBM $(B_H(x))_{x\in\mathbb{R}^d}$ has the following integral representation:*

$$B_H(x) = \frac{1}{C(H)} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}}\, W(d\xi), \tag{1.5}$$

where $W(d\xi)$ is a Wiener measure and

$$C(H) = \left(\int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2H}}\, d\xi \right)^{1/2}, \tag{1.6}$$

with $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$.

Remark 1.1.12. The Wiener measure $W(d\xi)$ is a complex measure which ensures that the field defined by (1.5) is a real-valued field. In fact, $W(d\xi)$ is a Fourier transform of a real Brownian measure, see [ST94] or [Coh99] for more details.

Remark 1.1.13. The constant $C(H)$ can be rewritten, see [ST94] for $d = 1$ and chapter 3 for $d \geq 2$, as follows:

$$C(H) = \left(\frac{\pi^{(d+1)/2}\Gamma(H + 1/2)}{H\Gamma(2H)\sin(\pi H)\Gamma(H + d/2)} \right)^{1/2},$$

where Γ is the Gamma-function.

As noticed in corollary 1.1.9, the pointwise Hölder exponent of a FBM is almost surely equal to a constant, which restricts the field of applications. Also, some generalizations of the FBM which allows the pointwise Hölder exponent to vary along the trajectories have been introduced. The most famous one is certainly the Multifractional Brownian Motion.

1.2 Multifractional Brownian Motion

The Multifractional Brownian Motion, in short MBM, has been introduced independently in [PLV96] and [BJR97]. It can be defined by substituting in the harmonizable representation of the FBM, see (1.5), to the constant parameter H a locally β-Hölder function h. Then, this function governs the properties of the MBM.

Definition 1.2.1. Let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function. Then, a Multifractional Brownian Motion, in short MBM, $(B_h(x))_{x\in\mathbb{R}^d}$ is a real-valued field which admits the representation

$$B_h(x) = \frac{1}{C(h(x))} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} \, W(d\xi), \tag{1.7}$$

where $W(d\xi)$ is a Wiener measure and where $C(\cdot)$ is defined by (1.6).

Since $W(d\xi)$ is a Wiener measure, a MBM B_h is a real-valued centered Gaussian field. Then, since the map $f \longrightarrow \int f \, dW$ is an isometry, i.e.

$$\mathbb{E}\left(\left| \int_{\mathbb{R}^d} f(\xi) \, W(d\xi) \right|^2 \right) = \int_{\mathbb{R}^d} |f(\xi)|^2 \, d\xi.$$

the covariance function of B_h can be evaluated.

Proposition 1.2.2. *Let $(x,y) \in \mathbb{R}^d$. Then,*

$$\mathbb{E}(B_h(x)B_h(y)) = D(h(x),h(y)) \Big(\|x\|^{h(x)+h(y)} + \|y\|^{h(x)+h(y)} - \|y-x\|^{h(x)+h(y)} \Big),$$

where for $(H_1,H_2) \in (0,1)^2$

$$D(H_1,H_2) = \frac{C^2((H_1+H_2)/2)}{C(H_1)C(H_2)}$$

with $C(\cdot)$ defined by (1.6).

Proof. see [ACLV00]. □

Then, the Kolmogorov criterion gives the continuity of the trajectories of a MBM. They are even locally Hölderian.

Proposition 1.2.3. *Let $K \subset \mathbb{R}^d$ be a compact set.*
Then, for every $H' < \min(\min_K h, \beta)$, there exists a modification of the MBM B_h which has H'-Hölder sample paths on K.

Proof. see [BJR97]. □

Consequently, since h is a continuous function, for every $x \in \mathbb{R}^d$,

$$H_{B_h}(x) \geq \min\left(h(x), \beta\right) \tag{1.8}$$

almost surely.

In the case of a FBM, the self-similarity property gives the equality in (1.8). However, a MBM is not self-similar in most cases. Nevertheless, a MBM satisfies a local property of self-similarity introduced in [BJR97].

Definition 1.2.4. A field $(Y(x))_{x\in\mathbb{R}^d}$ is locally asymptotically self-similar, in short lass, at point x with index $h(x)$ if

$$\lim_{\varepsilon\to 0_+}\left(\frac{Y(x+\varepsilon u)-Y(x)}{\varepsilon^{h(x)}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (T_x(u))_{u\in\mathbb{R}^d}, \tag{1.9}$$

where the non-degenerate field $(T_x(u))_{u\in\mathbb{R}^d}$ is called the tangent field at point x of Y and the limit is in distribution for all finite dimensional margins of the fields.

Furthermore, the field Y is lass with multifractional function h if for every $x \in \mathbb{R}^d$, it is lass at point x with index $h(x)$.

Remark 1.2.5. In general, the convergence (1.9) is in distribution for all finite dimensional margins. Nevertheless, in some cases, it can be strengthened. As an example, in the case where Y and T_x are continuous, the convergence (1.9) may be a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets. Also, we will precise each time the sense of the convergence (1.9).

Remark 1.2.6. A lass field looks like locally a self-similar field. More precisely, if a field Y is lass at point x with index $h(x)$ and tangent field T_x, then T_x is self-similar with index $h(x)$. Also, the lass property describes the local behaviour of a field. Furthermore, the local structure, reflected by the tangent fields, is studied in [Fal02] in the framework of continuous fields and in [Fal03] in the case of processes with jumps.

Then, in general, a MBM looks like locally a FBM but with exponent which varies along the trajectory.

Proposition 1.2.7. *Let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function such that*

$$\forall x \in \mathbb{R}^d,\ h(x) < \beta.$$

Then, the MBM B_h is locally asymptotically self-similar with multifractional function h and at each point its tangent field is a FBM. More precisely, for every fixed $x \in \mathbb{R}^d$,

$$\lim_{\epsilon\to 0^+}\left(\frac{B_h(x+\epsilon u)-B_h(x)}{\epsilon^{h(x)}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(B_{h(x)}(u)\right)_{u\in\mathbb{R}^d}, \tag{1.10}$$

where the convergence is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets and $B_{h(x)}$ is a standard FBM with index $h(x)$.

Proof. see [BJR97]. □

Let us remark that in (1.10), the point x is fixed and the limit is a limit of fields indexed by u. Then one can wonder if the limit of the fields indexed by (x,u) exists, i.e. if

$$\left(\widetilde{B}(x,u)\right)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d} \overset{(d)}{=} \lim_{\epsilon\to 0^+}\left(\frac{B_h(x+\epsilon u)-B_h(x)}{\epsilon^{h(x)}}\right)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}$$

exists, where the sense of the limit has to be specified. Owing to proposition 1.2.7, necessarily, for every fixed $x\in\mathbb{R}^d$,

$$\left(\widetilde{B}(x,u)\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(B_{h(x)}(u)\right)_{u\in\mathbb{R}^d}.$$

Then, it remains to know the correlation between the fields $B_{h(x)}$, $x\in\mathbb{R}^d$. Following next proposition, these fields are independent.

Proposition 1.2.8. *Let $h:\mathbb{R}^d\longrightarrow(0,1)$ be a locally β-Hölder function such that*

$$\forall x\in\mathbb{R}^d,\ h(x)<\beta.$$

Then, let $\left(B_{h(x)}\right)_{x\in\mathbb{R}^d}$ be a family of independent standard FBM. In particular, for every fixed $x\in\mathbb{R}^d$, $\left(B_{h(x)}(u)\right)_{u\in\mathbb{R}^d}$ is a standard FBM with index $h(x)$. Let $\widetilde{B}$ be the centered Gaussian field defined by

$$\widetilde{B}(x,u)=B_{h(x)}(u).$$

Then,

$$\lim_{\varepsilon\to 0^+}\left(\frac{B_h(x+\varepsilon u)-B_h(x)}{\varepsilon^{h(x)}}\right)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d} \overset{(d)}{=} \left(\widetilde{B}(x,u)\right)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}, \tag{1.11}$$

where the limit is in distribution for all finite dimensional margins.

Before we prove proposition 1.2.8, let us introduce the field $(B(x,y))_{(x,y)\in\mathbb{R}^d\times(0,1)}$ defined by

$$B(x,y)=\frac{1}{C(y)}\int_{\mathbb{R}^d}\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{y+d/2}}\,W(d\xi),$$

where $C(\cdot)$ is defined by (1.6) and $W(d\xi)$ is a Wiener measure. Then,

$$(B_h(x))_{x\in\mathbb{R}^d} \overset{(d)}{=} (B(x,h(x)))_{x\in\mathbb{R}^d}.$$

Proof. Let

$$Y_\varepsilon(x,u)=\frac{B_h(x+\varepsilon u)-B_h(x)}{\varepsilon^{h(x)}}.$$

Let us split Y_ε into two fields $Y_\varepsilon \overset{(d)}{=} Y_{\varepsilon,1}+Y_{\varepsilon,2}$ where

$$Y_{\varepsilon,1}(x,u)=\frac{B(x+\varepsilon u,h(x))-B(x,h(x))}{\varepsilon^{h(x)}}$$

and

$$Y_{\varepsilon,2}(x,u)=\frac{B(x+\varepsilon u,h(x+\varepsilon u))-B(x+\varepsilon u,h(x))}{\varepsilon^{h(x)}}.$$

Then, the behaviour of $Y_{\varepsilon,1}$, as $\varepsilon\to 0_+$, is first studied.

Step 1 Remark that for every fixed x, $(Y_{\varepsilon,1}(x,u))_{u\in\mathbb{R}^d}$ is a FBM with index $h(x)$. Also, if $x=y$,

$$\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v))=\mathbb{E}\big(B_{h(x)}(u)B_{h(x)}(v)\big),$$

and by definition of $\widetilde{B}$,

$$\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(x,v))=\mathbb{E}\Big(\widetilde{B}(x,u)\widetilde{B}(x,v)\Big).$$

Then, assume $x\neq y$. By isometry,

$$\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v))=\frac{\varepsilon^{-h(x)-h(y)}}{C(h(x))C(h(y))}\int_{\mathbb{R}^d}\frac{\mathrm{e}^{i(y-x)\cdot\xi}\big(\mathrm{e}^{-i\varepsilon u\cdot\xi}-1\big)\big(\mathrm{e}^{i\varepsilon v\cdot\xi}-1\big)}{\|\xi\|^{d+h(x)+h(y)}}\,d\xi.$$

Hence,

$$\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v))=\frac{1}{C(h(x))C(h(y))}\int_{\mathbb{R}^d}\frac{\mathrm{e}^{ia_\varepsilon\cdot\xi}\big(\mathrm{e}^{-iu\cdot\xi}-1\big)\big(\mathrm{e}^{iv\cdot\xi}-1\big)}{\|\xi\|^{d+h(x)+h(y)}}\,d\xi,$$

where $a_\varepsilon=(y-x)/\varepsilon$. Moreover, since $x\neq y$,

$$\lim_{\varepsilon\to 0_+}\|a_\varepsilon\|=+\infty.$$

Then, using the Riemann Lebesgue Lemma, one easily concludes that

$$\lim_{\varepsilon\to 0_+}\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v))=0.$$

Furthermore, since $x\neq y$, $\widetilde{B}(x,u)$ and $\widetilde{B}(x,v)$ are two independent centered Gaussian variables. Hence,

$$\lim_{\varepsilon\to 0_+}\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v))=\mathbb{E}\Big(\widetilde{B}(x,u)\widetilde{B}(y,v)\Big).$$

As a consequence,

$$\lim_{\varepsilon\to 0_+}(Y_{\varepsilon,1}(x,u))_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}\overset{(d)}{=}\Big(\widetilde{B}(x,u)\Big)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d},\tag{1.12}$$

where the limit is in distribution for every finite dimensional margins.

Step 2 Using the isometry property and a Taylor expansion, one can prove

$$\lim_{\varepsilon\to 0_+}\mathbb{E}\big(Y_{\varepsilon,2}^2(x,u)\big)=0,$$

see proof of proposition 2.2.11 in chapter 2 for more details. Hence, in view of (1.12), the proof is done. □

Furthermore, an upper bound for the pointwise Hölder exponent of B_h can be deduced from the lass property. Then, using (1.8), the following corollary is obtained.

Corollary 1.2.9. *Let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function such that*

$$\forall x \in \mathbb{R}^d,\ h(x) < \beta.$$

Then, for every $x \in \mathbb{R}^d$, the pointwise Hölder exponent $H_{B_h}(x)$ of the MBM B_h at point x is almost surely equal to $h(x)$.

Hence, the pointwise Hölder exponent of the MBM B_h is allowed to vary along the trajectories and is governed by the function h. The greater $h(x)$ is, the smoother the trajectories of B_h are in a neighbourhood of x. As an example, the MBM is used as a toy-model for modeling mountains because it allows to take into account the erosion phenomena. However, the trajectories of the MBM are in some sense regular since they are locally Hölderian. Also, [ALV00] and [BBCI00] introduce some generalizations of the MBM in order to have a pointwise Hölder exponent which varies abruptly. Nevertheless, their fields remain lass Gaussian fields and can not model non-Gaussian phenomena. Hence, non-Gaussian lass fields, called Real Harmonizable Fractional Lévy Motions, have been introduced in [BCI02].

1.3 Real Harmonizable Fractional Lévy Motions

Real Harmonizable Fractional Lévy Motions, in short RHFLMs, share many properties with the FBM. In addition, the FBM is a RHFLM. Benassi, Cohen and Istas have defined the class of RHFLMs by substituting in the harmonizable representation of the FBM to the Wiener measure $W(d\xi)$ a Lévy random measure $L(d\xi)$.

1.3.1 Lévy random measure

Heuristically, a Lévy random measure is linked with the increments of a Lévy process. Furthermore, the non-Brownian part of a Lévy process can be represented by a Poisson process. Also, the non-Brownian part of a Lévy random measure is introduced owing to a Poisson random measure.

Let $N(d\xi, dz)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with mean measure $n(d\xi, dz)$, i.e. a random measure such that

- for every Borel set A such that $n(A) < +\infty$, $N(A)$ is a Poisson random variable of intensity $n(A)$.
- if the set I is finite and if the Borel sets A_i, $i \in I$ are pairwise disjoint, then the random variables $N(A_i)$, $i \in I$, are independent.

Notation Throughout this section, $L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$ is the space of complex-valued functions which are square integrable for the measure $n(d\xi, dz)$.

Let us note $\widetilde{N} = N - n$ the compensated Poisson random measure. It is then classical to define the stochastic integral

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\, \widetilde{N}(d\xi, dz)$$

for every function $\varphi : \mathbb{R}^d \times \mathbb{C} \longrightarrow \mathbb{C}$ such that $\varphi \in L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$. Then, by construction, for every $\varphi \in L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$,

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\, \widetilde{N}(d\xi, dz)\right|^2\right] = \int_{\mathbb{R}^d \times \mathbb{C}} |\varphi(\xi, z)|^2\, n(d\xi, dz),$$

i.e. the map $\varphi \longmapsto \int \varphi\, d\widetilde{N}$ is an isometry from $L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$ onto a subset of $L^2(\Omega)$.

Notice that if φ is real so is $\int \varphi\, d\widetilde{N}$. Let us denote by $\Re(z)$ the real part of a complex z and by $\Im(z)$ its imaginary part.

The law of the random variable $\int \varphi\, d\widetilde{N}$ is given by its characteristic function

$$\mathbb{E}\left[\exp\left(i\left(u \int \Re(\varphi)\, d\widetilde{N} + v \int \Im(\varphi)\, d\widetilde{N}\right)\right)\right]$$

which is equal to

$$\exp\left[\int_{\mathbb{R}^d \times \mathbb{C}} \left[\exp\left(i(u\Re(\varphi) + v\Im(\varphi))\right) - 1 - i(u\Re(\varphi) + v\Im(\varphi))\right] d\xi\, \nu(dz)\right],$$

where $(u, v) \in \mathbb{R}^2$. Let us remark that $\int \varphi\, d\widetilde{N}$ is a centered random variable.

Here the mean measure $n(d\xi, dz) = \mathbb{E}N(d\xi, dz) = d\xi\, \nu(dz)$ satisfies:

$$\forall\, p \geqslant 2, \quad \int_{\mathbb{C}} |z|^p\, \nu(dz) < +\infty. \tag{1.13}$$

Moreover ν is a non vanishing measure such that $\nu(\{0\}) = 0$.

Then, in [BCI02], a measure $M(d\xi)$, associated with the Poisson random measure $N(d\xi, dz)$, is defined by

$$\int_{\mathbb{R}^d} f(\xi)\, M(d\xi) \stackrel{def}{=} \int_{\mathbb{R}^d \times \mathbb{C}} [f(\xi)z + f(-\xi)\bar{z}]\, \widetilde{N}(d\xi, dz), \tag{1.14}$$

where $f \in L^2(\mathbb{R}^d)$.

Hence, if

$$\forall \xi \in \mathbb{R}^d, \quad f(-\xi) = \overline{f(\xi)} \tag{1.15}$$

$\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)$ is a real-valued centered random variable. Furthermore, its characteristic function is

$$\mathbb{E}\Big(\mathrm{e}^{iu\int f\, dM}\Big) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}} \left[\exp\left(2iu\Re(f(\xi)z)\right) - 1 - 2iu\Re(f(\xi)z)\right] d\xi\, \nu(dz)\right), \quad (1.16)$$

where $u \in \mathbb{R}$.

In addition, the control measure $\nu(dz)$ is assumed to be rotationally invariant. Let P be the map $P\big(\rho \mathrm{e}^{i\theta}\big) = (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_*^+$. The measure $\nu(dz)$ satisfies the following property:

$$P(\nu(dz)) = d\theta\, \nu_\rho(d\rho), \quad (1.17)$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$. Hence, if f satisfies (1.15), the random variable $\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)$ is symmetric and for every $a \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} f(\xi) \exp\left(ia\cdot\xi\right) M(d\xi) \overset{(d)}{=} \int_{\mathbb{R}^d} f(\xi)\, M(d\xi). \quad (1.18)$$

Moreover, when f satisfies (1.15),

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)\right|^2\right] = 4\pi \|f\|^2_{L^2(\mathbb{R}^d)} \int_0^{+\infty} \rho^2\, \nu_\rho(d\rho). \quad (1.19)$$

Definition 1.3.1. A Lévy random measure associated with a Poisson random measure $N(d\xi, dz)$ is

$$L(d\xi) = aM(d\xi) + bW(d\xi),$$

where $(a, b) \in \mathbb{R}^2$ and $W(d\xi)$ is a Wiener measure independent of $M(d\xi)$. In addition, $aM(d\xi)$ is called a Lévy random measure without Brownian component.

Remark 1.3.2. Let $d = 1$. Then, $(L([0,t]))_{t\geq 0}$ is a Lévy process and $(aM([0,t]))_{t\geq 0}$ is its non-Brownian part.

Hence, in order to study the trajectories regularity of a RHFLM, [BCI02] establishes next proposition which gives an evaluation of the moment

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)\right|^p\right].$$

Proposition 1.3.3. *Let $p \in \mathbb{N}^*$, $f \in L^2\big(\mathbb{R}^d\big) \cap L^{2p}\big(\mathbb{R}^d\big)$ and suppose that f satisfies* (1.15) *then $\int_{\mathbb{R}^d} f(\xi) M(d\xi)$ is in $L^{2p}(\Omega)$ and*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)\right)^{2p}\right] = \sum\nolimits_{m=1}^{p} (2\pi)^m \sum_{L_m} \prod_{q=1}^{m} \frac{(2l_q)! \|f\|^{2l_q}_{2l_q} \int_0^{+\infty} \rho^{2l_q}\, \nu_\rho(d\rho)}{(l_q!)^2},$$

where $\sum_{L_m}$ stands for the sum over the set of partitions L_m of $\{1, \ldots, 2p\}$ in m subsets K_q such that the cardinality of K_q is $2l_q$ with $l_q \geqslant 1$ and where $\|f\|_{2l_q}$ is the $L^{2l_q}(\mathbb{R}^d)$ norm of f.

Under the assumption that $f \in L^2(\mathbb{R}^d) \cap L^{2p}(\mathbb{R}^d)$, [BCI02] obtains the existence and the expression of the moment of order $2p$ of the stochastic integral $\int f\, dM$. Actually, this assumption is also necessary.

Proposition 1.3.4. *Let $f \in L^2(\mathbb{R}^d)$ and $p \in [2, +\infty)$. Assume that f satisfies* (1.15)*. Then,*

$$\mathbb{E}\left(\left|\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)\right|^p\right) < +\infty \iff f \in L^p(\mathbb{R}^d).$$

Proof. By definition, for every $u \in \mathbb{R}^d$,

$$\mathbb{E}\left[\exp\left(iu \int_{\mathbb{R}^d} f(\xi)\, M(d\xi)\right)\right]$$

is equal to

$$\exp\left[\int_{\mathbb{R}^d \times \mathbb{C}} \left[\exp\left(2iu\Re(f(\xi)z)\right) - 1 - 2iu\Re(f(\xi)z)\right] d\xi\, \nu(dz)\right].$$

Then, the law of $\int f\, dM$ is an infinitely divisible distribution. Furthermore, its Lévy measure Λ_f is the push-forward of the measure $n(d\xi, dz) = d\xi\, \nu(dz)$ by the map $(\xi, z) \longmapsto \Re(f(\xi)z)$. Hence, by theorem 25.3 in [Sat99],

$$\mathbb{E}\left(\left|\int_{\mathbb{R}^d} f(\xi)\, M(d\xi)\right|^p\right) < +\infty \iff \int_{\mathbb{R}} |x|^p\, \Lambda_f(dx) < +\infty.$$

Furthermore, by definition of Λ_f,

$$\int_{\mathbb{R}} |x|^p\, \Lambda_f(dx) = 2^p \int_{\mathbb{R}^d \times \mathbb{C}} |\Re(f(\xi)z)|^p\, d\xi\, \nu(dz).$$

Then, since $\nu(dz)$ is a rotationally invariant measure,

$$\int_{\mathbb{R}} |x|^p\, \Lambda_f(dx) = 2^p \int_{\mathbb{R}^d} |f(\xi)|^p\, d\xi \int_{\mathbb{C}} |\Re(z)|^p \nu(dz).$$

Hence, in view of (1.13) and since $\nu(dz)$ is a non vanishing measure,

$$\int_{\mathbb{R}} |x|^p\, \Lambda_f(dx) < +\infty \iff \int_{\mathbb{R}^d} |f(\xi)|^p\, d\xi < +\infty,$$

which concludes the proof. □

1.3.2 Definition and properties

Definition 1.3.5. Assume that the Lévy random measure $M(d\xi)$ satisfies the finite moment assumption (1.13) and the rotational invariance (1.17). Let $(a,b) \in \mathbb{R}^2$ and $H \in (0,1)$. A Real Harmonizable Fractional Lévy Motion, in short RHFLM, with index H is a real-valued field which admits a harmonizable representation

$$X_H(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} L(d\xi),$$

where $L(d\xi) = aM(d\xi) + bW(d\xi)$ is a Lévy random measure in the sense of definition 1.3.1.

Hence, a FBM is a RHFLM obtained for $L(d\xi) = bW(d\xi)$. Therefore, by definition, a RHFLM X_H is the sum of two independent fields: a FBM and the RHFLM Z_H defined by

$$Z_H(x) = a\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} M(d\xi).$$

In general, a RHFLM is not a Gaussian field. Furthermore, by construction, a RHFLM is a second order field and its covariance function is given by the isometry property (1.19).

Proposition 1.3.6. *Let X_H be the RHFLM associated with the Lévy random measure $L(d\xi) = aM(d\xi)+bW(d\xi)$. Then, X_H is an infinitely divisible field. In addition, its increments are stationary and for every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$,*

$$\mathbb{E}\big[(X_H(x) - X_H(y))^2\big] = C_1^2(H)\|x-y\|^{2H}, \tag{1.20}$$

where

$$C_1(H) = \left(a^2 + 4b^2\pi \int_0^{+\infty} \rho^2 \, \nu_\rho(d\rho)\right)^{1/2} C(H)$$

with $C(\cdot)$ defined by (1.6).

Therefore, a RHFLM X_H has the same variogram and then the same structure of covariance as a FBM. However, whereas a FBM has moments of every order, a RHFLM may not.

Proposition 1.3.7. *Assume that X_H is a RHFLM with index H associated with the Lévy random measure $L(d\xi) = aM(d\xi) + bW(d\xi)$ where $a \neq 0$. Let $p \in [2, +\infty)$ and $x \in \mathbb{R}^d \backslash \{0\}$. Then,*

$$\mathbb{E}(|X_H(x)|^p) < +\infty \iff H < 1 - \tfrac{d}{2} + \tfrac{d}{p}.$$

Proof. Proposition 1.3.7 is a simple consequence of proposition 1.3.4. □

Remark 1.3.8. A RHFLM X_H with index H, which is not a FBM, has finite moments of every order if and only if $H \leq 1 - d/2$. In particular, when $d \geq 2$, it does not have finite moment of every order.

Then, like FBM, RHFLMs have locally Hölder sample paths.

Proposition 1.3.9. *For all $H' < H$, there exists a modification of the RHFLM X_H whose sample paths are almost surely locally H'-Hölder continuous on $\mathbb{R}^d$.*

Hence, a lower bound for the pointwise Hölder exponent of a RHFLM X_H is obtained:

$$H_{X_H}(x) \geq H \quad \text{almost surely.} \tag{1.21}$$

Furthermore, a RHFLM X_H is not in general self-similar but is lass with tangent field a FBM at each point. Even a non-Gaussian RHFLM looks like locally a FBM. In addition, at each point x, the index of the FBM is the same.

Proposition 1.3.10. *A RHFLM X_H is locally asymptotically self-similar with parameter H in the sense that for every $x \in \mathbb{R}^d$,*

$$\lim_{\epsilon \to 0^+} \left(\frac{X_H(x+\epsilon u) - X_H(x)}{\epsilon^H} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} (B_H(u))_{u \in \mathbb{R}^d}, \tag{1.22}$$

where the convergence is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets and B_H is a FBM of index H.

Like in the case of the MBM, the limit

$$\lim_{\epsilon \to 0^+} \left(\frac{X_H(x+\epsilon u) - X_H(x)}{\epsilon^H} \right)_{(x,u) \in \left(\mathbb{R}^d\right)^2}$$

exists. However, the proof is quite long and will be given in chapter 2 in a more general framework, see proposition 2.2.11.

In addition, some RHFLMs satisfy an asymptotic self-similarity property when the increments are taken at large scales.

Proposition 1.3.11. *Let $0 < \alpha < 2$ and assume that the control measure $\nu(dz)$ is associated with*

$$\nu_\rho(d\rho) = \frac{\mathbf{1}_{0<\rho<1}}{\rho^{1+\alpha}} d\rho$$

by (1.17) and $L(d\xi) = M(d\xi)$. Moreover, assume that $\widetilde{H} = H + d/2 - d/\alpha$ is such that $\widetilde{H} > 0$. Then, the RHFLM X_H with index H and control measure $\nu(dz)$ is asymptotically self-similar with parameter $\widetilde{H}$. More precisely,

$$\lim_{R \to +\infty} \left(\frac{X_H(Ru)}{R^{\widetilde{H}}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} \left(S_{\widetilde{H}}(u)\right)_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $S_{\widetilde{H}}$ is a Real Harmonizable Fractional Stable Motion that has representation

$$S_{\widetilde{H}}(u) = 2\Re\left(\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{\widetilde{H}+d/\alpha}} M_\alpha(d\xi) \right), \tag{1.23}$$

with M_α a complex isotropic stable α-symmetric random measure.

Remark 1.3.12. In the previous proposition, the RHFSM $S_{\widetilde{H}}$ is well-defined by (1.23) since $0 < \widetilde{H} < H < 1$.

Then, at large scales, the behaviour of a RHFLM can be very different from the Gaussian model and this even in the case where the RHFLM has moments of every order.

However, since a RHFLM has stationary increments, the distribution of its pointwise Hölder exponent at point x does not depend on x. Furthermore, the lass property and the lower bound (1.21) gives its value.

Proposition 1.3.13. *At every point x, the pointwise Hölder exponent $H_{X_H}(x)$ of the RHFLM X_H is almost surely equal to H.*

In addition, since a RHFLM is an isotropic field, the distribution of its directional pointwise Hölder at point x does not depend on the direction. Let us first recall the definition of the directional pointwise Hölder exponent.

Definition 1.3.14. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function, $x \in \mathbb{R}^d$ and $u \in S^{d-1}$, where S^{d-1} is the unit sphere of $\mathbb{R}^d$.

Then, the pointwise Hölder exponent of the function f at point x in direction u is

$$H_f(x,u) = \sup\left\{\alpha > 0,\ \lim_{\varepsilon \to 0_+} \frac{f(x+\varepsilon u) - f(x)}{\varepsilon^\alpha} = 0\right\}.$$

Also, thanks to the isotropy property of RHFLMs, the following corollary is a simple consequence of proposition 1.3.13.

Corollary 1.3.15. *Let $(x,u) \in \mathbb{R}^d \times S^{d-1}$. Then, the pointwise Hölder exponent $H_{X_H}(x,u)$ of a RHFLM X_H at point x in direction u is almost surely equal to H.*

From this corollary, one can deduce that the restriction of a RHFLM along a straight line is not a RHFLM as soon as the non-Brownian part does not vanish. It is well known that if B_H is a FBM, then its restriction $(B_H(\lambda u))_{\lambda \in \mathbb{R}}$ is a FBM with exponent H. It does not remain true for RHFLMs.

Proposition 1.3.16. *Let $d \geq 2$ and $X_H = (X_H(x))_{x \in \mathbb{R}^d}$ be a RHFLM with index H associated with the Lévy random measure $L(d\xi) = aM(d\xi) + bW(d\xi)$. Assume $a \neq 0$. Let $(x,u) \in \mathbb{R}^d \times S^{d-1}$. Then,*

$$(X_H(x+\lambda u) - X_H(x))_{\lambda \in \mathbb{R}}$$

is not a RHFLM.

Proof. For every $\lambda \in \mathbb{R}$, let

$$Y(\lambda) = X_H(x+\lambda u) - X_H(x).$$

Since X_H has stationary increments, we can suppose that $x = 0$. In this case,

$$Y(\lambda) = X_H(\lambda u).$$

Let us fixed $\lambda \neq 0$. By applying proposition 1.3.7 to the RHFLM X_H, if $q \geq 2$,

$$\mathbb{E}(|Y(\lambda)|^q) < +\infty \iff H < 1 + \frac{d}{q} - \frac{d}{2}.$$

Remark that since $d \geq 2$, $Y(\lambda)$ does not have moment of every order.

Furthermore, if Y were a RHFLM, by corollary 1.3.15, its pointwise Hölder exponent at 0 would be equal to H and then its index would be equal to H too. Therefore, by applying proposition 1.3.7 to Y, we would have:

$$\mathbb{E}(|Y(\lambda)|^q) < +\infty \iff H < \frac{1}{q} + \frac{1}{2}.$$

However, since $d \geq 2$, for every $q > 2$

$$\frac{1}{q} + \frac{1}{2} > 1 + \frac{d}{q} - \frac{d}{2}.$$

Then, since $0 < H < 1$, one can choose q such that

$$1 + \frac{d}{q} - \frac{d}{2} < H < \frac{1}{q} + \frac{1}{2}$$

and Y is not a RHFLM. □

As a conclusion, RHFLMs are non-Gaussian fields which share many properties with the FBM. However, their pointwise Hölder exponent does not depend on the position, which restricts the field of applications. Hence, next chapter introduces a generalization of RHFLMs which allows the pointwise Hölder exponent to vary along the trajectories.

Chapter 2

Real Harmonizable Multifractional Lévy Motions: definition and properties

Real Harmonizable Fractional Lévy Motions (in short RHFLMs), introduced by Benassi, Cohen and Istas in [BCI02], make up a class of lass fields which includes non-Gaussian fields and the FBM. However, their increments are stationary and their Hölder exponent is almost surely equal to a constant. Then, this chapter introduces some non-Gaussian lass fields whose Hölder exponent varies along the trajectories. These fields are called Real Harmonizable Multifractional Lévy Motions, in short RHMLMs.

The MBM has been defined in [PLV96] and [BJR97] by substituting in a representation of the FBM to the constant parameter H a function h. Here, RHMLMs are introduced in the same way from the definition of RHFLMs. Let us recall that a RHFLM X_H of index H $(0 < H < 1)$ is defined as the stochastic integral:

$$X_H(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} \, L(d\xi), \tag{2.1}$$

where $\|\xi\|$ is the Euclidean norm of ξ and $L(d\xi)$ is a Lévy random measure. Then RHMLMs are defined by substituting in (2.1) to the constant parameter H a locally Hölder function h. When $L(d\xi)$ is a Wiener measure $W(d\xi)$, one obtains the harmonizable representation of the MBM. Hence, the class of RHMLMs generalizes the MBM and the RHFLMs. One of its main interest is that it contains some non-Gaussian fields which share many properties with the MBM which is a powerful model in applied mathematics. Actually, RHMLMs have locally Hölder sample paths and are locally asymptotically self-similar with FBM as tangent field at each point. Hence, RHMLMs have the same local structure as the MBM: locally they look like FBMs.

In addition, [BCI02] studies the asymptotic at large scale of RHFLMs. Also, RHFLMs can have different asymptotic at low and at large scales. More precisely, some RHFLMs satisfy an asymptotic self-similarity at infinity with tangent field a

Real Harmonizable Fractional Stable Motion, in short RHFSM. Whereas RHFLMs locally look like a FBM, at large scales they can look like RHFSM, which does not have even second order moments. Actually, the same behaviour at large scale holds for a larger class of RHMLMs.

Like in the case of MBM, the properties of a RHMLM X_h are governed by the multifractional function h. In particular, it controls the local regularity of its sample paths. Also, in view of applications, a central problem is to identify this function h. In the case of a RHFLM X_H, i.e. in the case where h is equal to a constant, the fractional index H is identified in [BCI02] thanks to an estimator based on generalized quadratic variations. One of the main interest of this estimator is that it is constructed owing to discrete observations of one sample path of the RHFLM X_H. It has first been introduced in a general Gaussian framework in [IL97] and studied in the case of Filtered White Noises in [BCIJ98]. Nevertheless, whereas it cannot be applied to RHFSMs (see [ADF00, APPT00, Dur01] as for estimation of RHFSMs), it allows to identify the fractional index of RHFLMs which are second order fields. However, as for the MBM B_h, this estimator only gives the minimum of the function h. Also, in the framework of RHMLMs, in order to identify $h(x)$, the generalized quadratic variations are localized in a neighbourhood of x as in [BCI98]. Hence, the estimation of the multifractional function h can be performed using only one discrete sample paths of the RHMLM X_h.

The next section is devoted to the construction of RHMLMs. In section 2.2, the regularity of the sample paths of RHMLMs and the asymptotic self-similarity properties are studied. The last section deals with the identification of the multifractional function, using localized generalized quadratic variations.

2.1 Construction of non-Gaussian Multifractional Fields

In this part, $M(d\xi)$ is a Lévy random measure without Brownian component in the sense of definition 1.3.1. More precisely, $M(d\xi)$ is represented by a Poisson random measure $N(d\xi, dz)$ on $\mathbb{R}^d \times \mathbb{C}$ whose mean measure $n(d\xi, dz) = d\xi\, \nu(dz)$ satisfies the the finite moment assumption (1.13) , i.e. is such that

$$\forall\, p \geqslant 2, \quad \int_{\mathbb{C}} |z|^p \, \nu(dz) < +\infty.$$

Here, $\nu(dz)$ is a non vanishing measure such that $\nu(\{0\}) = 0$. Moreover, as in [BCI02], the control measure $\nu(dz)$ is assumed to be rotationally invariant. Then, $\nu(dz)$ is associated to a measure $\nu_\rho(d\rho)$ on $(0, +\infty)$ by (1.17).

Let us now introduce Real Harmonizable Multifractional Lévy Motions.

Definition 2.1.1. Let $\beta > 0$ and let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function on $\mathbb{R}^d$. Let $(a,b) \in \mathbb{R}^2$. A Real Harmonizable Multifractional Lévy Motion, in short RHMLM, is a real-valued field which admits a harmonizable representation

$$X_h(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} L(d\xi),$$

where $L(d\xi) = aM(d\xi) + bW(d\xi)$ is a Lévy random measure in the sense of definition 1.3.1.

Consequently, X_h is the sum of two independent fields, one of which is a Multifractional Brownian Motion. In particular, the MBM is the RHMLM obtained for $L(d\xi) = W(d\xi)$. Furthermore, when h is equal to a constant H, X_h is a RHFLM. Then, the class of RHMLMs contains the MBM and RHFLMs. In addition, let us notice that by construction X_h is an infinitely divisible field.

As noticed in [BCI02], RHFLMs have the same structure of covariance as the FBM. Also, by the isometry property (1.19), a RHMLM has the same structure of covariance as a MBM. However, whereas the FBM and the MBM have moments of every order, RHFLMs and RHMLMs do not in general.

Proposition 2.1.2. *Let $a \neq 0$ and X_h be a RHMLM associated with the Lévy measure $L(d\xi) = aM(\xi) + bW(d\xi)$. Let $x \in \mathbb{R}^d\backslash\{0\}$ and $p \in [2, +\infty)$. Then,*

$$\mathbb{E}(|X_h(x)|^p) < +\infty \iff h(x) < 1 - \frac{d}{2} + \frac{d}{p}.$$

Proof. It is a simple consequence of proposition 1.3.4. □

In particular, $X_h(x)$ has moments of every order if and only if $h(x) \leq 1 - d/2$. Then, as soon as $d \geq 2$ or $h(x) > 1/2$, $X_h(x)$ does not have moment of every order.

Assuming (1.17), RHFLMs have stationary increments. It does not remain true for RHMLMs. As an example, the MBM is a RHMLM whose increments are not stationary in most cases.

Throughout this chapter, $h : \mathbb{R}^d \longrightarrow (0,1)$ is a locally β-Hölder function on $\mathbb{R}^d$ and X_h is the RHMLM associated with h and the Lévy random measure $L(d\xi)$.

2.2 Sample paths regularity and Asymptotic self-similarity

This section deals with two properties that RHMLMs share with the MBM. On the one hand, the RHMLM X_h has locally H-Hölder sample paths on a neighbourhood of x for every $H < \min(h(x), \beta)$. On the other hand, if for every x, $0 < h(x) < \beta$,

then X_h is lass with tangent FBM at point x. From these two properties, one deduces that the Hölder exponent of X_h at point x is almost surely equal to $h(x)$. As these properties have already be shown in [BJR97] in the case of the MBM, i.e. in the case where $L(d\xi) = W(d\xi)$, we suppose for the sake of simplicity in the proofs that $L(d\xi) = M(d\xi)$.

Furthermore, the asymptotic at large scales is studied. Actually, some RHMLMs satisfy an asymptotic property when the increments are taken at large scales and the limit field is a fractional stable model.

2.2.1 Preliminary Lemmas

Usually, to get the regularity of the trajectories and the lass property, one estimates

$$\mathbb{E}[|X_h(x) - X_h(y)|^q],$$

where $q \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Nevertheless when $h(x) > 1 - d/2$ and $q \geq 3$, $X_h(x)$ may have an infinite moment of order q. Thus the field X_h is split into two fields $X_h = X_h^+ + X_h^-$ where X_h^+ has moments of every order and X_h^- has almost surely locally β-Hölder sample paths. Then one can estimate

$$\mathbb{E}\big[\big|X_h^+(x) - X_h^+(y)\big|^q\big],\ q \in \mathbb{N}^*.$$

Let $n \in \mathbb{N}$ and

$$P_n(t) = \sum_{k=1}^{n} \frac{t^k}{k!},\ \text{ with convention } P_0(t) = 0.$$

Then

$$g_n^+(x,\xi) = \frac{\mathrm{e}^{-ix\cdot\xi} - 1 - P_n(-ix\cdot\xi)\,\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{h(x)+d/2}} \tag{2.2}$$

and

$$g_n^-(x,\xi) = \frac{P_n(-ix\cdot\xi)\,\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{h(x)+d/2}} \tag{2.3}$$

are in $L^2\big(\mathbb{R}^d\big)$ for every $x \in \mathbb{R}^d$.
Therefore $X_h = X_{h,n}^+ + X_{h,n}^-$ with

$$X_{h,n}^+(x) = \int_{\mathbb{R}^d} g_n^+(x,\xi)\, L(d\xi) \tag{2.4}$$

$$\text{and}\quad X_{h,n}^-(x) = \int_{\mathbb{R}^d} g_n^-(x,\xi)\, L(d\xi). \tag{2.5}$$

Notice that

$$X_h = X_{h,0}^+.$$

Moreover $g_n^+(x,\cdot) \in L^q(\mathbb{R}^d)$ for every $q \geq 2$ such that

$$(n + 1 - d/2 - h(x))q > -d.$$

Consequently, when $n \geq d/2$, $g_n^+(x,\cdot) \in L^q(\mathbb{R}^d)$ for every $q \geq 2$ and every $x \in \mathbb{R}^d$. In this case, by proposition 1.3.4, $X_{h,n}^+$ has moments of every order. Furthermore, thanks to proposition 1.3.3, proved in [BCI02], one can compute

$$\mathbb{E}\Big[\big(X_{h,n}^+(x) - X_{h,n}^+(y)\big)^{2p}\Big]$$

with the help of some L^{2q}-norms of the deterministic map $\xi \longmapsto g_n(x,y,\xi)$, where

$$g_n(x,y,\xi) = g_n^+(x,\xi) - g_n^+(y,\xi).$$

To estimate these norms, g_n is split into $g_n = g_{n,1} + g_{n,2}$ with

$$g_{n,1}(x,y,\xi) = \frac{\mathrm{e}^{-ix\cdot\xi} - \mathrm{e}^{-iy\cdot\xi} + [P_n(-iy\cdot\xi) - P_n(-ix\cdot\xi)]\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{h(y)+d/2}} \tag{2.6}$$

and

$$g_{n,2}(x,y,\xi) = \frac{\mathrm{e}^{-ix\cdot\xi} - 1 - P_n(-ix\cdot\xi)\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{d/2}}\left[\frac{1}{\|\xi\|^{h(x)}} - \frac{1}{\|\xi\|^{h(y)}}\right]. \tag{2.7}$$

If X_h is a RHFLM of index H, i.e. if h is equal to a constant H, notice that $g_{n,2} = 0$. One of the main difference between the studies of RHFLMs and RHMLMs lies in the study of the properties of $g_{n,2}$.

Lemma 2.2.1. *Let $K \subset \mathbb{R}^d$ be a compact set. Suppose that $q \geq 2$ is such that $q(n - d/2) \geq -d$. Then there exists a non negative constant $C = C(K,q)$ such that*

$$\forall (x,y) \in K^2,\ \|g_{n,1}(x,y,\cdot)\|_q^q \leq C\|x-y\|^{qh(y)}.$$

Proof. Let $(x,y) \in K^2$ and let us note $I_1 = \|g_{n,1}(x,y,\cdot)\|_q^q$.

$$I_1(x,y) = I_{11}(x,y) + I_{12}(x,y)$$

where

$$I_{11}(x,y) = \int_{\|\xi\|\leq 1} \frac{\left|\mathrm{e}^{-ix\cdot\xi} - \mathrm{e}^{-iy\cdot\xi} + P_n(-iy\cdot\xi) - P_n(-ix\cdot\xi)\right|^q}{\|\xi\|^{qh(y)+qd/2}}\,d\xi,$$

$$\text{and}\quad I_{12}(x,y) = \int_{\|\xi\|\geq 1} \frac{\left|\mathrm{e}^{-ix\cdot\xi} - \mathrm{e}^{-iy\cdot\xi}\right|^q}{\|\xi\|^{qh(y)+qd/2}}\,d\xi.$$

By Taylor expansion,

$$\left|\mathrm{e}^{-ix\cdot\xi} - \mathrm{e}^{-iy\cdot\xi} + P_n(-iy\cdot\xi) - P_n(-ix\cdot\xi)\right| \leq C\|x-y\|\|\xi\|^{n+1}.$$

Let us define $M_K = \max_K h$. If $0 < \|\xi\| \leq 1$, then

$$\frac{1}{\|\xi\|^{h(y)}} \leq \frac{1}{\|\xi\|^{M_K}}.$$

Consequently,

$$I_{11}(x,y) \leq C \left\|x-y\right\|^q \int_{\|\xi\|\leq 1} \frac{1}{\left\|\xi\right\|^{q(M_K-n-1)+qd/2}}\, d\xi.$$

This last integral is defined since $q(n-d/2) \geq -d$ and so

$$I_{11}(x,y) \leq C\|x-y\|^q.$$

It remains to study I_{12}. Unfortunately Taylor expansion gives an infinite bound. Let us suppose that $x \neq y$. One splits $I_{12}(x,y)$ into the integrals

$$J_1(x,y) = \int_{\substack{\|x-y\|\|\xi\|\geq 1 \\ \|\xi\|\geq 1}} \frac{\left|\mathrm{e}^{-i(x-y)\cdot\xi}-1\right|^q}{\left\|\xi\right\|^{qh(y)+qd/2}}\, d\xi$$

$$\text{and} \quad J_2(x,y) = \int_{1\leq\|\xi\|\leq\frac{1}{\|x-y\|}} \frac{\left|\mathrm{e}^{-i(x-y)\cdot\xi}-1\right|^q}{\left\|\xi\right\|^{qh(y)+qd/2}}\, d\xi.$$

Then one can easily see that

$$J_1(x,y) \leq C\|x-y\|^{d(q/2-1)+qh(y)}.$$

By Taylor expansion,

$$J_2(x,y) \leq C\|x-y\|^q \int_1^{\frac{1}{\|x-y\|}} \rho^{d-1-qh(y)+q-qd/2}\, d\rho.$$

Then

$$J_2(x,y) \leq C\|x-y\|^{q(1+h(y)-M_K)} \int_1^{\frac{1}{\|x-y\|}} \rho^{d-1-qM_K+q-qd/2}\, d\rho.$$

Consequently, by evaluating the last integral, one obtains that

$$J_2(x,y) \leq C\|x-y\|^{qh(y)}.$$

□

Let us now study $g_{n,2}$.

Lemma 2.2.2. *Let $K \subset \mathbb{R}^d$ be a compact set. Let us note $m_K = \min_K h$ and $M_K = \max_K h$.*

Suppose that $q \geq 2$ is such that $q(n-d/2) \geq -d$. Then

$$\forall (x,y) \in K^2,\ \|g_{n,2}(x,y,\cdot)\|_q^q \leq C\|x-y\|^{q\beta},$$

where

$$C = \|h\|_{\beta,K}^q \sup_{u\in K} J(u) \in \mathbb{R}^+$$

with

$$J(u)=\int_{\mathbb{R}^d}\frac{\left|\mathrm{e}^{-iu\cdot\xi}-1-P_n(-iu\cdot\xi)\mathbf{1}_{\|\xi\|\leq 1}\right|^q|\ln\|\xi\||^q}{\|\xi\|^{dq/2}}\left[\frac{1}{\|\xi\|^{m_K}}+\frac{1}{\|\xi\|^{M_K}}\right]^q d\xi$$

and $\|h\|_{\beta,U}=\sup\limits_{\substack{(u,v)\in K^2\\ u\neq v}}\dfrac{|h(u)-h(v)|}{\|u-v\|^\beta}$.

Proof. Let $(x,y)\in K^2$ and let us note $I_2=\|g_{n,2}(x,y,\cdot)\|_q^q$. By the Mean Value Theorem,

$$\frac{1}{\|\xi\|^{h(x)}}-\frac{1}{\|\xi\|^{h(y)}}=\frac{-(h(x)-h(y))\ln\|\xi\|}{\|\xi\|^{c_{\xi,x,y}}},$$

where $c_{\xi,x,y}$ is between $h(x)$ and $h(y)$. Therefore $m_K\leq c_{\xi,x,y}\leq M_K$.
Furthermore

$$\frac{1}{\|\xi\|^{c_{\xi,x,y}}}\leq\frac{1}{\|\xi\|^{m_K}}\mathbf{1}_{\|\xi\|\geq 1}+\frac{1}{\|\xi\|^{M_K}}\mathbf{1}_{\|\xi\|<1}.$$

Then

$$\frac{1}{\|\xi\|^{c_{\xi,x,y}}}\leq\frac{1}{\|\xi\|^{m_K}}+\frac{1}{\|\xi\|^{M_K}}.$$

Consequently,

$$\left|\frac{1}{\|\xi\|^{h(x)}}-\frac{1}{\|\xi\|^{h(y)}}\right|\leq|h(x)-h(y)||\ln\|\xi\||\left[\frac{1}{\|\xi\|^{m_K}}+\frac{1}{\|\xi\|^{M_K}}\right].$$

Therefore $I_2(x,y)\leq|h(x)-h(y)|^qJ(x)$, where $J(x)$ is equal to

$$\int_{\mathbb{R}^d}\frac{\left|\mathrm{e}^{-ix\cdot\xi}-1-P_n(-ix\cdot\xi)\mathbf{1}_{\|\xi\|\leq 1}\right|^q|\ln\|\xi\||^q}{\|\xi\|^{dq/2}}\left[\frac{1}{\|\xi\|^{m_K}}+\frac{1}{\|\xi\|^{M_K}}\right]^q d\xi.$$

It is straightforward to prove that $\sup_{u\in K}J(u)<+\infty$. Moreover since h is a locally β-Hölder function on $\mathbb{R}^d$,

$$|h(x)-h(y)|\leq\|h\|_{\beta,K}\|x-y\|^\beta$$

and then

$$I_2(x,y)\leq\|h\|_{\beta,K}^q\sup_{u\in K}J(u)\,\|x-y\|^{\beta q}.$$

□

Since $g_n=g_{n,1}+g_{n,2}$, by applying the Minkowski inequality, proposition 1.3.3 and lemmas 2.2.1 and 2.2.2, one can proves:

Lemma 2.2.3. *Let $K\subset\mathbb{R}^d$ be a compact set and $n\geq d/2$, then for every $p\in\mathbb{N}^*$, there exists a non negative constant $C=C(K,p)$ such that*

$$\forall(x,y)\in K^2,\ \mathbb{E}\Big[\big(X_{h,n}^+(x)-X_{h,n}^+(y)\big)^{2p}\Big]\leqslant C\|y-x\|^{2pm},$$

where $m=\min(h(y),\beta)$.

Let us notice that by symmetry, m can be replaced by

$$m' = \min\left(\max\left(h(x), h(y)\right), \beta\right).$$

It remains now to study $X^-_{h,n}$. Please note that as usual the regularity of the trajectories is given for a modification $\widetilde{X}^-$ of $X^-_{h,n}$, i.e. for a field $\widetilde{X}^-$ such that

$$\forall x \in \mathbb{R}^d,\ \mathbb{P}\Big(\widetilde{X}^-(x) = X^-_{h,n}(x)\Big) = 1.$$

Lemma 2.2.4. *There exists a modification of the field $X^-_{h,n}$ that has, with probability one, locally β-Hölder sample paths.*

Remark 2.2.5. When h is $\mathcal{C}^1$, there exists a modification of the field $X^-_{h,n}$ such that with probability one, $X^-_{h,n} \in \mathcal{C}^1$.

Proof. Notice that for every $x \in \mathbb{R}^d$

$$X^-_{h,n}(x) = Z_n(x, h(x)),$$

where the field $(Z_n(x,y))_{(x,y)\in\mathbb{R}^d\times(0,1)}$ is defined as follows:

$$Z_n(x,y) = \int_{\|\xi\|\le 1} \frac{P_n(-ix\cdot\xi)}{\|\xi\|^{y+d/2}}\, L(d\xi). \tag{2.8}$$

Then since h is a locally β-Hölder map with values in $(0,1)$, it is sufficient to prove that there exists a modification of the field Z_n such that $\mathbb{P}(Z_n \in \mathcal{C}^1) = 1$.

Let us define

$$Y_\alpha(y) = \int_{\|\xi\|\le 1} \frac{i^{|\alpha|}\prod_{j=1}^d \xi_j^{\alpha_j}}{\|\xi\|^{y+d/2}}\, L(d\xi),$$

where $\alpha \in \mathbb{N}^d$ is such that $1 \le |\alpha| = \sum_{j=1}^n \alpha_j \le n$ and $y \in (0,1)$. One shows that for every α, the field Y_α admits a modification which has almost surely $\mathcal{C}^1$-sample paths on $(0,1)$. Then since P_n is a polynomial, the same holds for Z_n.

Let $\alpha \in \mathbb{N}^d$ such that $1 \le \sum_{j=1}^n \alpha_j \le n$ and $\eta \in (0,1)$. One can prove with Taylor expansion the existence of a constant $C > 0$ such that

1. $\mathbb{E}\big[|Y_\alpha(y+\delta) - Y_\alpha(y)|^2\big] \le C|\delta|^2$, for every $y \in [\eta, 1-\eta]$ and every δ such that $y + \delta \in [\eta, 1-\eta]$,

2. $\mathbb{E}\big[|Y_\alpha(y+\delta) + Y_\alpha(y-\delta) - 2Y_\alpha(y)|^2\big] \le C|\delta|^4$, for every $y \in [\eta, 1-\eta]$ and every δ such that $(y+\delta, y-\delta) \in [\eta, 1-\eta]^2$.

According to [CL67], see page 69, these statements imply the existence of a modification of Y_α which has almost surely $\mathcal{C}^1$-sample paths. And so the same holds for Z_n. □

2.2.2 Trajectories Regularity

There exist modifications of the MBM whose sample paths are locally Hölder. Here an analogous result in the case of RHMLMs is shown.

Proposition 2.2.6. *Let $K \subset \mathbb{R}^d$ be a compact set.*
Then for every $H < \min(\min_K h, \beta)$, there exists a modification of the RHMLM X_h which has H-Hölder sample paths on K.

Proof. Let $H < \min(\min_K h, \beta)$ and $n \geq d/2$.
As a consequence of lemma 2.2.3 and of the Kolmogorov Theorem, there exists a modification of the field $X^+_{h,n}$ whose sample paths are H-Hölder on K. Then since $X^-_{h,n}$ has almost surely locally β-Hölder sample paths (see lemma 2.2.4) and $X_h = X^+_{h,n} + X^-_{h,n}$, the proof is done. □

Then proposition 2.2.6 gives us a lower bound for the pointwise Hölder exponent $H_{X_h}(x)$ of X_h at point x:

$$H_{X_h}(x) \geq \min(h(x), \beta). \tag{2.9}$$

2.2.3 Asymptotic Self-Similarity

Like RHFLMs and MBM, a RHMLM is not self-similar but locally asymptotically self-similar. However, a RHMLM looks like locally a FBM whose index varies along the trajectory. In fact, its tangent field at each point is a FBM.

Proposition 2.2.7. *Suppose that for every $x \in \mathbb{R}^d$, $h(x) < \beta$. Then the Real Harmonizable Multifractional Lévy Motion X_h is locally self-similar with multifractional function h in the sense that for every fixed $x \in \mathbb{R}^d$:*

$$\lim_{\varepsilon \to 0^+} \left(\frac{X_h(x + \varepsilon u) - X_h(x)}{\varepsilon^{h(x)}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} C_1(h(x)) \big(B_{h(x)}(u) \big)_{u \in \mathbb{R}^d}, \tag{2.10}$$

where the convergence is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets, $B_{h(x)}$ is a standard FBM of index $h(x)$ and

$$C_1(H) = \left[4a^2\pi \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) + b^2 \right]^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \frac{\left| e^{ie_1 \cdot \xi} - 1 \right|^2}{\|\xi\|^{d+2H}} \, d\xi \right)^{\frac{1}{2}}, \tag{2.11}$$

with $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$.

Before we prove proposition 2.2.7, the field

$$Y(x, y) = \int_{\mathbb{R}^d} \frac{e^{-ix \cdot \xi} - 1}{\|\xi\|^{y + d/2}} \, L(d\xi), \quad x \in \mathbb{R}^d, \, y \in (0, 1) \tag{2.12}$$

is introduced. Then, notice that

$$\forall x \in \mathbb{R}^d,\ X_h(x) = Y(x, h(x)). \tag{2.13}$$

The field Y will be used to split the increments of X_h into increments of Y which only involves one variation of its two variables.

Proof. Since this proposition has already been proved in [BJR97] in the case where $L(d\xi) = W(d\xi)$, i.e. in the case of the MBM, we can assume for the sake of simplicity $L(d\xi) = M(d\xi)$.

We first prove the convergence of the finite dimensional margins. Next a tightness property is shown. It is a direct consequence of lemmas 2.2.3 and 2.2.4.

Let us fixed $x \in \mathbb{R}^d$. Then for every $\varepsilon > 0$ and $u \in \mathbb{R}^d$,

$$Y_\varepsilon(u) = \frac{X_h(x + \varepsilon u) - X_h(x)}{\varepsilon^{h(x)}}.$$

Convergence of the finite dimensional margins:

The field Y_ε is split into two fields $Y_{\varepsilon,1}$ and $Y_{\varepsilon,2}$ such as $Y_{\varepsilon,1}$ only involves a variation of Y in its first variable. Hence, $Y_{\varepsilon,2}$ is an increment of Y in its second variable. More precisely, $Y_\varepsilon = Y_{\varepsilon,1} + Y_{\varepsilon,2}$ where

$$Y_{\varepsilon,1}(u) = \frac{Y(x + \varepsilon u, h(x)) - X_h(x)}{\varepsilon^{h(x)}}$$

and

$$Y_{\varepsilon,2}(u) = \frac{X_h(x + \varepsilon u) - Y(x + \varepsilon u, h(x))}{\varepsilon^{h(x)}}.$$

Then, by (2.13),

$$Y_{\varepsilon,1}(u) = \frac{Y(x + \varepsilon u, h(x)) - Y(x, h(x))}{\varepsilon^{h(x)}}.$$

Moreover, $(Y(v, h(x)))_{v \in \mathbb{R}^d}$ is a RHFLM with index $h(x)$ and it is associated with the Lévy random measure $L(d\xi)$. Also, thanks to proposition 3.1 in [BCI02],

$$\lim_{\varepsilon \to 0_+} (Y_{\varepsilon,1}(u)) \overset{(d)}{=} C_1(h(x)) \big(B_{h(x)}(u)\big)_{u \in \mathbb{R}^d}, \tag{2.14}$$

where the limit is in distribution for all finite dimensional margins of the fields, $B_{h(x)}$ is a standard FBM of index $h(x)$ and $C_1(\cdot)$ is defined by (2.11). Actually, [BCI02] establishes that the convergence (2.14) is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets. However, we will not directly use this result.

Furthermore,

$$Y_{\varepsilon,2}(u) = \frac{1}{\varepsilon^{h(x)}} \int_{\mathbb{R}^d} g_{0,2}(x + \varepsilon u, x, \xi)\, L(d\xi),$$

where $g_{0,2}$ is defined by (2.7). Therefore, using the isometry property (1.19),

$$\mathbb{E}\big(Y_{\varepsilon,2}^2(u)\big) = \frac{4\pi}{\varepsilon^{h(x)}} \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) \|g_{0,2}(x+\varepsilon u, x, \cdot)\|_2^2.$$

Hence, by lemma 2.2.2, for every $\varepsilon \leq 1$

$$\mathbb{E}\big(Y_{\varepsilon,2}^2(u)\big) \leq C\varepsilon^{2(\beta-h(x))}.$$

Then as $\beta > h(x)$, $\lim_{\varepsilon\to 0_+} \mathbb{E}\big(Y_{\varepsilon,2}^2(u)\big) = 0$. Hence,

$$\lim_{\varepsilon\to 0_+} \left(Y_{\varepsilon,2}(u)\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} 0, \tag{2.15}$$

where the limit is in distribution for all finite dimensional margins of the fields.

Then, since $Y_\varepsilon = Y_{\varepsilon,1} + Y_{\varepsilon,2}$, (2.14) and (2.15) gives the convergence of the finite dimensional margins of Y_ε.

Tightness: Let $n \geq d/2$,

$$Y_\varepsilon^+(u) = \frac{X_{h,n}^+(x+\varepsilon u) - X_{h,n}^+(x)}{\varepsilon^{h(x)}} \quad \text{and} \quad Y_\varepsilon^-(u) = \frac{X_{h,n}^-(x+\varepsilon u) - X_{h,n}^-(x)}{\varepsilon^{h(x)}}.$$

Then $Y_\varepsilon = Y_\varepsilon^+ + Y_\varepsilon^-$.

Since $h(x) < \beta$ and since $X_{h,n}^-$ has locally β-Hölder sample paths, it is clear that

$$\lim_{\varepsilon\to 0_+} \left(\frac{X_{h,n}^-(x+\varepsilon u) - X_{h,n}^-(x)}{\varepsilon^{h(x)}} \right)_{u\in\mathbb{R}^d} \overset{(d)}{=} 0,$$

where the convergence is a convergence in distribution in the space of continuous functions endowed with the topology of the uniform convergence on compact sets. Consequently, $(Y_\varepsilon^-)_{\varepsilon>0}$ is a tight family. Let us now prove the tightness of $(Y_\varepsilon^+)_\varepsilon$. Notice that $Y_\varepsilon^+(0) = 0$ and so that $(Y_\varepsilon^+(0))_\varepsilon$ is tight.

Let $K \subset \mathbb{R}^d$ be a compact set and $r_0 > 0$ such that when $\varepsilon \leq 1$, $x + \varepsilon K \subset K_0$ where $K_0 = \overline{B(x, r_0)}$. The lemma 2.2.3 is applied to the compact set K_0. Then for every $p \in \mathbb{N}^*$, there exists a constant $C = C(p, K_0, x) > 0$ such that for every $\varepsilon \in (0, 1]$ and every $(u, v) \in K^2$,

$$\mathbb{E}\Big[\big(Y_\varepsilon^+(u) - Y_\varepsilon^+(v)\big)^{2p}\Big] \leq C\varepsilon^{2p(h(x+\varepsilon v)-h(x))} \|u-v\|^{2ph(x+\varepsilon v)}.$$

Moreover as h is a locally β-Hölder map,

$$\forall \varepsilon \in (0,1],\ \forall v \in K,\ \varepsilon^{2p(h(x+\varepsilon v)-h(x))} \leq C.$$

Therefore for every $p \in \mathbb{N}^*$, there exists a constant $C = C(p, K, x) > 0$ such that for every $\varepsilon \in (0, 1]$ and every $(u, v) \in K^2$,

$$\mathbb{E}\Big[\big(Y_\varepsilon^+(u) - Y_\varepsilon^+(v)\big)^{2p}\Big] \leq C\|u-v\|^{2p \min_{K_0} h}.$$

One can choose $p \in \mathbb{N}^*$ such that $2p \min_{K_0} h > d$, which concludes the proof. □

On the one hand, the behaviour of the increments of Y in its second variable is governed by the regularity of the function h since they involve increments of this function. On the other hand, the behaviour of the increments of Y in its first variable are given by the value of $h(x)$. Hence, the tangent field at X_h at point x depends whether $h(x) < \alpha_h(x)$ or not. Actually, the convergence (2.10) holds as soon as $h(x) < \alpha_h(x)$. Then, one can wonder what happens when $h(x) > \alpha_h(x)$. In this case, the preponderant term is due to the increments in the second variable of Y.

Proposition 2.2.8. *Let $x \in \mathbb{R}^d$. Assume that $h(x) > \alpha_h(x)$ and that*

$$l(x) = \lim_{y \to 0} \frac{h(x+y) - h(x)}{\|y\|^{\alpha_h(x)}} \tag{2.16}$$

exists. Then

$$\lim_{\varepsilon \to 0^+} \left(\frac{X_h(x+\varepsilon u) - X_h(x)}{\varepsilon^{\alpha_h(x)}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} l(x) X^*(x) \left(\|u\|^{\alpha_h(x)} \right)_{u \in \mathbb{R}^d}, \tag{2.17}$$

where the limit is in distribution for all finite margins of the fields and where

$$X^*(x) = \int_{\mathbb{R}^d} \frac{\left(\mathrm{e}^{-ix\cdot\xi} - 1\right) \ln \|\xi\|}{\|\xi\|^{h(x)+d/2}} \, L(d\xi).$$

Remark 2.2.9. When $x \neq 0$ and $l(x) \neq 0$, the limit field in (2.17) is not degenerate and then (2.17) means that X_h is lass at x with exponent $\alpha_h(x)$. Let us remark that in (2.17), the field $\left(\|u\|^{\alpha_h(x)} \right)_{u \in \mathbb{R}^d}$ is deterministic. Then the randomness of the tangent field is only due to the real-valued random variable $X^*(x)$ which does not depend on u.

Proof. For the sake of simplicity, we assume that $L(d\xi) = M(d\xi)$. Then, one proceeds as in the proof of (2.10) replacing $\varepsilon^{h(x)}$ by $\varepsilon^{\alpha_h(x)}$. Let

$$Y_\varepsilon(u) = \frac{X_h(x+\varepsilon u) - X_h(x)}{\varepsilon^{\alpha_h(x)}}.$$

and let us split Y_ε into $Y_\varepsilon = Y_{\varepsilon,1} + Y_{\varepsilon,2}$ where

$$Y_{\varepsilon,1}(u) = \frac{Y(x+\varepsilon u, h(x)) - X_h(x)}{\varepsilon^{\alpha_h(x)}}$$

and

$$Y_{\varepsilon,2}(u) = \frac{X_h(x+\varepsilon u) - Y(x+\varepsilon u, h(x))}{\varepsilon^{\alpha_h(x)}}.$$

First, by proposition 3.1 in [BCI02], the RHFLM $(Y(v, h(x)))_{v \in \mathbb{R}^d}$ is lass with index $h(x)$. Hence, since $X(x) = Y(x, h(x))$ and since $h(x) > \alpha_h(x)$,

$$\lim_{\varepsilon \to 0_+} \left(Y_{\varepsilon,1}(u) \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} 0.$$

Therefore, it remains to study the field $Y_{\varepsilon,2}$ which is here the preponderant term. Let us recall that

$$Y_{\varepsilon,2}(u) = \frac{1}{\varepsilon^{\alpha_h(x)}} \int_{\mathbb{R}^d} g_{0,2}(x+\varepsilon u, x, \xi)\, L(d\xi)$$

where

$$g_{0,2}(x+\varepsilon u, x, \xi) = \frac{\mathrm{e}^{-i(x+\varepsilon u)\cdot\xi} - 1}{\|\xi\|^{d/2}} \left(\frac{1}{\|\xi\|^{h(x+\varepsilon u)}} - \frac{1}{\|\xi\|^{h(x)}} \right).$$

Furthermore, (2.16) leads to

$$\widetilde{g}(x,u,\xi) = \lim_{\varepsilon \to 0_+} \frac{g_{0,2}(x+\varepsilon u, x, \xi)}{\varepsilon^{\alpha_h(x)}} = -\frac{\left(\mathrm{e}^{-ix\cdot\xi} - 1\right) \ln \|\xi\|}{\|\xi\|^{h(x)+d/2}}\, l(x) \|u\|^{\alpha_h(x)}.$$

Hence, by a dominated convergence argument,

$$\lim_{\varepsilon \to 0_+} \left\| \frac{g_{0,2}(x+\varepsilon u, x, \cdot)}{\varepsilon^{\alpha_h(x)}} - \widetilde{g}(x,u,\cdot) \right\|_2 = 0.$$

Then, the isometry property (1.19) yields that

$$\mathbb{E}\left(\left| Y_{\varepsilon,2}(u) + l(x)\|u\|^{\alpha_h(x)} X^*(x) \right|^2 \right) = 0,$$

where $X^*(x) = \displaystyle\int_{\mathbb{R}^d} \frac{\left(\mathrm{e}^{-ix\cdot\xi} - 1\right) \ln \|\xi\|}{\|\xi\|^{h(x)+d/2}}$.

As a consequence,

$$\lim_{\varepsilon \to 0^+} \left(Y_{\varepsilon,2}(u)\right)_{u\in\mathbb{R}^d} \stackrel{(d)}{=} -l(x) X^*(x) \left(\|u\|^{\alpha_h(x)} \right)_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite margins of the fields. Furthermore, by (1.18), $X^*(x)$ is a symmetric random variable, which concludes the proof. □

In addition, since $X^*(0) = 0$, the limit field in (2.17) is degenerate if $x = 0$. Hence, the convergence (2.17) does not establish the lass property at $x = 0$. Actually, the variation of Y in its second variable

$$Y(\varepsilon u, h(x+\varepsilon u)) - Y(\varepsilon u, h(x))$$

does not give the tangent field; its convergence is accelerated by the convergence to zero of its first variable εu as ε tends to zero. However, the RHMLM X_h remains lass at $x = 0$ and its tangent field is governed by the behaviour of Y in its first component. Actually, the limit field is given by proposition 2.2.7 without assumption on the value of $h(0)$ as stated in next proposition.

Proposition 2.2.10. *The Real Harmonizable Multifractional Lévy Motion X_h is locally asymptotically self-similar at $x = 0$ with index $h(0)$ in the sense that*

$$\lim_{\varepsilon \to 0^+} \left(\frac{X_h(\varepsilon u)}{\varepsilon^{h(0)}} \right)_{u\in\mathbb{R}^d} \stackrel{(d)}{=} C_1(h(0)) \left(B_{h(0)}(u) \right)_{u\in\mathbb{R}^d}, \tag{2.18}$$

where the limit is in distribution for all finite margins of the fields, $B_{h(0)}$ is a standard FBM of index $h(0)$ and $C_1(\cdot)$ is defined by (2.11).

Proof. For the sake of simplicity, we suppose that $L(d\xi) = M(d\xi)$. However, the proposition remains true for a general Lévy random measure $L(d\xi)$. We directly prove the result without using the decomposition of the increments of X_h.

Let $p \in \mathbb{N}^*$, $u = (u_1, \ldots, u_p) \in \left(\mathbb{R}^d\right)^p$, $v = (v_1, \ldots, v_p) \in \mathbb{R}^p$ and

$$g_0^\varepsilon(\xi, z) = \frac{2}{\varepsilon^{h(0)}} \Re\left(z \sum_{k=1}^{p} v_k \, \frac{\mathrm{e}^{-i\varepsilon u_k \cdot \xi} - 1}{\|\xi\|^{h(\varepsilon u_k)}} \right).$$

Then since $\displaystyle\sum_{k=1}^{p} v_k \, Y_\varepsilon(u_k) = \int_{\mathbb{R}^d} g_0^\varepsilon(\xi, z) \, \widetilde{N}(d\xi, dz)$,

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^{p} v_k \, Y_\varepsilon(u_k) \right) \right] = \exp\left(\psi_\varepsilon(u, v)\right)$$

with $\displaystyle\psi_\varepsilon(u, v) = \int_{\mathbb{R}^d \times \mathbb{C}} \left[\exp\left(i g_0^\varepsilon(\xi, z)\right) - 1 - i g_0^\varepsilon(\xi, z)\right] d\xi \, \nu(dz)$.

Therefore, the change of variable $\lambda = \varepsilon\xi$ leads to:

$$\psi_\varepsilon(u, v) = \int_{\mathbb{R}^d \times \mathbb{C}} l^\varepsilon(\lambda, z) \, d\lambda \, \nu(dz),$$

where $l_\varepsilon(\lambda, z) = \varepsilon^{-d}[\exp\left(i \widetilde{g_0}^\varepsilon(\xi, z)\right) - 1 - i \widetilde{g_0}^\varepsilon(\lambda, z)]$ with

$$\widetilde{g_0}^\varepsilon(\xi, z) = 2\varepsilon^{d/2} i \Re\left(z \sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k \cdot \lambda} - 1}{\|\lambda\|^{h(\varepsilon u_k) + d/2}} \varepsilon^{h(\varepsilon u_k) - h(0)} \right).$$

Hence, since h is locally Hölder, $\lim\limits_{\varepsilon \to 0_+} \varepsilon^{h(\varepsilon u_k) - h(0)} = 1$ and then

$$\lim_{\varepsilon \to 0_+} l_\varepsilon(\lambda, z) = -4\Re^2\left(z \sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k \lambda} - 1}{\|\lambda\|^{h(0) + d/2}} \right).$$

Then, a dominated convergence argument gives the convergence of the characteristic function and the conclusion. □

Furthermore, in the convergence (2.10), the point x is fixed. Actually, the convergence of

$$\left(\frac{X_h(x + \varepsilon u) - X_h(x)}{\varepsilon^{h(x)}} \right)_{(x,u) \in \mathbb{R}^d \times \mathbb{R}^d}$$

is established in next proposition. This result generalizes proposition 1.2.8 to the class of RHMLMs. In particular, it holds for any RHFLMs.

Proposition 2.2.11. *Let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function such that*

$$\forall x \in \mathbb{R}^d,\ h(x) < \beta.$$

Then, let $\big(B_{h(x)}\big)_{x\in\mathbb{R}^d}$ be a family of independent standard FBM. In particular, for every $x \in \mathbb{R}^d$, $\big(B_{h(x)}(u)\big)_{u\in\mathbb{R}^d}$ is a standard FBM with index $h(x)$. Let $\widetilde{B}$ be the centered real-valued Gaussian field defined by

$$\widetilde{B}(x,u) = B_{h(x)}(u).$$

Then,

$$\lim_{\varepsilon\to 0^+}\left(\frac{X_h(x+\varepsilon u)-X_h(x)}{\varepsilon^{h(x)}}\right)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d} \overset{(d)}{=} \Big(C_1(h(x))\widetilde{B}(x,u)\Big)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}, \quad (2.19)$$

where the limit is in distribution for all finite dimensional margins and $C_1(\cdot)$ is defined by (2.11).

The proof of this proposition is quite long and is based on the following lemma.

Lemma 2.2.12. *Let $p \in \mathbb{N}$ such that $p \geq 2$. Let $u_0 = 0$ and $(u_1,\ldots,u_p) \in \mathbb{R}^p$. Then,*

$$\mathrm{e}^{i\sum_{k=1}^p u_k} - \sum_{k=1}^p \mathrm{e}^{iu_k} + p - 1 = \sum_{k=0}^{p-2} \mathrm{e}^{i\sum_{j=0}^k u_j}\big(\mathrm{e}^{iu_{k+1}}-1\big)\sum_{l=k+2}^p \big(\mathrm{e}^{iu_l}-1\big).$$

Proof. By expanding the right hand side term, one easily obtains the result. □

Proof of proposition 2.2.11. First, the convergence of the Brownian part is given by proposition 1.2.8. Then, assume that $L(d\xi) = M(d\xi)$. Let $p \in \mathbb{N}^*$ and $(x_1,\ldots,x_p)$ be a family of pairwise different elements of $\mathbb{R}^d$. Then for each integer k such that $1 \leq k \leq p$, let $p_k \in \mathbb{N}^*$, $(u_{k1},\ldots,u_{kp_k}) \in \big(\mathbb{R}^d\big)^{p_k}$ and $(v_{k1},\ldots,v_{kp_k}) \in \mathbb{R}^{p_k}$. In addition, let

$$Y_\varepsilon(x,u,v) = \sum_{k=1}^p\sum_{j=1}^{p_k} v_{kj}\frac{X_h(x_k+\varepsilon u_{kj})-X_h(x_k)}{\varepsilon^{h(x_k)}}.$$

The proposition will be proved as soon as we establish that

$$\lim_{\varepsilon\to 0_+} Y_\varepsilon(x,u,v) \overset{(d)}{=} \sum_{k=1}^p\sum_{j=1}^{p_k} v_{kj}\widetilde{B}(x_k,u_{kj}). \quad (2.20)$$

When $k=1$, proposition 2.2.7 implies (2.20). Then, assume that $k \geq 2$.

Let us split $Y_\varepsilon = Y_{\varepsilon,1} + Y_{\varepsilon,2}$ into two fields defined by

$$Y_{\varepsilon,1}(x,u,v) = \sum_{k=1}^p\sum_{j=1}^{p_k} v_{kj}\frac{Y(x_k+\varepsilon u_{kj},h(x_k))-Y(x_k,h(x_k))}{\varepsilon^{h(x_k)}}$$

and

$$Y_{\varepsilon,2}(x,u,v)=\sum_{k=1}^{p}\sum_{j=1}^{p_k}v_{kj}\frac{Y(x_k+\varepsilon u_{kj},h(x_k+\varepsilon u_{kj}))-Y(x_k+\varepsilon u_{kj},h(x_k))}{\varepsilon^{h(x_k)}}.$$

Step 1 Study of $\boldsymbol{Y_{\varepsilon,1}}$: Let

$$G_\varepsilon(\xi)=\sum_{k=1}^{p}\mathrm{e}^{-ix_k\cdot\xi}F_k^\varepsilon(\xi),$$

where

$$F_k^\varepsilon(\xi)=\sum_{j=1}^{p_k}v_{kj}\frac{\mathrm{e}^{-i\varepsilon u_{kj}\cdot\xi}-1}{\varepsilon^{h(x_k)}\|\xi\|^{h(x_k)+d/2}}$$

Then, since $Y_{\varepsilon,1}(x,u,v)=\displaystyle\int_{\mathbb{R}^d}G_\varepsilon(\xi)\,M(d\xi)$,

$$\mathbb{E}[\exp\left(iY_{\varepsilon,1}(x,u,v)\right)]=\exp\left(\psi(\varepsilon)\right)$$

with

$$\psi(\varepsilon)=\int_{\mathbb{R}^d\times\mathbb{C}}[\exp\left(2i\Re(G_\varepsilon(\xi)z)\right)-1-2i\Re(G_\varepsilon(\xi)z)]\,d\xi\,\nu(dz).$$

Thus,

$$\psi(\varepsilon)=\sum_{k=1}^{p}I_k(\varepsilon)+J_p(\varepsilon)$$

where

$$I_k(\varepsilon)=\int_{\mathbb{R}^d\times\mathbb{C}}\left[\exp\left(2i\Re\left(\mathrm{e}^{-ix_k\cdot\xi}F_k^\varepsilon(\xi)z\right)\right)-1-2i\Re\left(\mathrm{e}^{-ix_k\cdot\xi}F_k^\varepsilon(\xi)z\right)\right]d\xi\,\nu(dz)$$

and

$$J_p(\varepsilon)=\int_{\mathbb{R}^d\times\mathbb{C}}\left[\exp\left(2i\Re(G_\varepsilon(\xi)z)\right)+p-1-\sum_{k=1}^{p}\exp\left(2i\Re\left(\mathrm{e}^{-ix_k\cdot\xi}F_k^\varepsilon(\xi)z\right)\right)\right]d\xi\,\nu(dz).$$

Hence, by rotational invariance of the measure $\nu(dz)$,

$$I_k(\varepsilon)=\int_{\mathbb{R}^d\times\mathbb{C}}[\exp\left(2i\Re(F_k^\varepsilon(\xi)z)\right)-1-2i\Re(F_k^\varepsilon(\xi)z)]\,d\xi\,\nu(dz).$$

Therefore, by definition of F_k^ε,

$$\exp\left(I_k(\varepsilon)\right)=\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p_k}v_{kj}\widetilde{X}_{h(x_k)}(\varepsilon u_{kj})\right)\right],$$

where $\widetilde{X}_{h(x_k)}$ is a RHFLM with index $h(x_k)$ and associated with the Lévy measure $L(d\xi) = M(d\xi)$. Consequently, by proposition 2.2.7,

$$\lim_{\varepsilon \to 0_+} \sum_{k=1}^{p} I_k(\varepsilon) = -\frac{1}{2} \sum_{k=1}^{p} \mathrm{Var}\left(\sum_{j=1}^{p_k} v_{kj} C_1(h(x_k)) \widetilde{B}(x_k, u_{kj}) \right). \tag{2.21}$$

Let us now prove that

$$\lim_{\varepsilon \to 0_+} J_p(\varepsilon) = 0. \tag{2.22}$$

First, by applying the change of variable $\lambda = \varepsilon \xi$, $J_p(\varepsilon)$ is equal to

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varepsilon^{-d} \left[\exp\Big(2i\varepsilon^{d/2} \Re\Big(\widetilde{G}_\varepsilon(\lambda) z \Big) \Big) + p - 1 - \sum_{k=1}^{p} \exp\big(2i\varepsilon^{d/2} \Re\big(\mathrm{e}^{-ix_k \cdot \lambda/\varepsilon} F_k(\lambda) z \big) \big) \right] d\lambda \, \nu(dz),$$

where

$$F_k(\lambda) = F_k^1(\lambda) = \sum_{j=1}^{p_k} v_{kj} \frac{\mathrm{e}^{-iu_{kj} \cdot \lambda} - 1}{\|\lambda\|^{h(x_k) + d/2}}$$

and

$$\widetilde{G}_\varepsilon(\lambda) = \sum_{k=1}^{p} \mathrm{e}^{-ix_k \cdot \lambda/\varepsilon} F_k(\lambda).$$

Then, lemma 2.2.12 leads to

$$J_p(\varepsilon) = \sum_{k=0}^{p} \sum_{k'=k+2}^{p} A_{k,k'}$$

where

$$A_{k,k'}(\varepsilon) = \int_{\mathbb{R}^d \times \mathbb{C}} K_\varepsilon(\xi, z) \, d\lambda \, \nu(dz)$$

with

$$K_\varepsilon(\lambda, z) = \varepsilon^{-d} \exp\left[2i\varepsilon^{d/2} \Re\left(\sum_{j=1}^{k} \mathrm{e}^{-ix_j \cdot \lambda/\varepsilon} F_j(\lambda) z \right) \right] K_{\varepsilon,1}(\lambda, z)$$

and

$$K_{\varepsilon,1}(\lambda, z) = \big[\exp\big(2i\varepsilon^{d/2} \Re\big(\mathrm{e}^{-ix_{k+1} \cdot \lambda/\varepsilon} F_{k+1}(\lambda) z \big) \big) - 1 \big] \big[\exp\big(2i\varepsilon^{d/2} \Re\big(\mathrm{e}^{-ix_{k'} \cdot \lambda/\varepsilon} F_{k'}(\lambda) z \big) \big) - 1 \big].$$

Hence, in order to obtain (2.22), it is sufficient to prove that

$$\lim_{\varepsilon \to 0_+} A_{k,k'}(\varepsilon) = 0. \tag{2.23}$$

Then, let

$$\widetilde{K}_\varepsilon(\lambda, z) = 4 \Re\big(\mathrm{e}^{-ix_{k+1} \cdot \lambda/\varepsilon} F_{k+1}(\lambda) z \big) \Re\big(\mathrm{e}^{-ix_{k'} \cdot \lambda/\varepsilon} F_{k'}(\lambda) z \big).$$

Hence,

$$\lim_{\varepsilon \to 0_+} \Big(K_\varepsilon(\lambda, z) + \widetilde{K}_\varepsilon(\lambda, z) \Big) = 0$$

and a dominated convergence argument yields that

$$\lim_{\varepsilon\to 0_+}\int_{\mathbb{R}^d\times\mathbb{C}}\Big(K_\varepsilon(\lambda,z)+\widetilde{K}_\varepsilon(\lambda,z)\Big)\,d\lambda\,\nu(dz)=0.$$

Therefore, in order to prove (2.23), it remains to show that

$$\lim_{\varepsilon\to 0_+}\int_{\mathbb{R}^d\times\mathbb{C}}\widetilde{K}_\varepsilon(\lambda,z)\,d\lambda\,\nu(dz)=0. \tag{2.24}$$

In addition, since the map $\varphi\longmapsto\int\varphi d\widetilde{N}$ is an isometry,

$$\int_{\mathbb{R}^d\times\mathbb{C}}\widetilde{K}_\varepsilon(\lambda,z)\,d\lambda\,\nu(dz)=\mathbb{E}(L_{k+1}L_{k'}),$$

where

$$L_l=2\int_{\mathbb{R}^d\times\mathbb{C}}\Re\big(\mathrm{e}^{-ix_l\cdot\lambda/\varepsilon}F_l(\lambda)z\big)\,\widetilde{N}(d\xi,dz).$$

Hence, since $F_l(-\lambda)=\overline{F_l(\lambda)}$,

$$L_l=\int_{\mathbb{R}^d}\mathrm{e}^{-ix_l\cdot\lambda/\varepsilon}F_l(\lambda)\,M(d\xi).$$

Also, using (1.19), one easily shows that

$$\int_{\mathbb{R}^d\times\mathbb{C}}\widetilde{K}_\varepsilon(\lambda,z)\,d\lambda\,\nu(dz)=-4\pi\int_0^{+\infty}\rho^2\,\nu_\rho(d\rho)\int_{\mathbb{R}^d}\mathrm{e}^{i(x_{k'}-x_{k+1})\cdot\lambda/\varepsilon}F_{k+1}(\lambda)\overline{F_{k'}(\lambda)}\,d\lambda.$$

Then, since $x_{k'}\neq x_{k+1}$ $(k'>k+1)$, by applying the Riemann Lebesgue lemma, one obtains (2.24). Consequently,

$$\lim_{\varepsilon\to 0_+}\psi(\varepsilon)=-\frac{1}{2}\sum_{k=1}^{p}\mathrm{Var}\left(\sum_{j=1}^{p_k}v_{kj}C_1(h(x_k))\widetilde{B}(x_k,u_{kj})\right).$$

Therefore, by definition of $\widetilde{B}$,

$$\lim_{\varepsilon\to 0_+}Y_{\varepsilon,1}(x,u,v)\overset{(d)}{=}\sum_{k=1}^{p}\sum_{j=1}^{p_k}v_{kj}\widetilde{B}(x_k,u_{kj}). \tag{2.25}$$

Step 2 Study of $\boldsymbol{Y_{\varepsilon,2}}$ Let us remark that

$$\mathbb{E}\big[(Y(x_k+\varepsilon u_{kj},h(x_k+\varepsilon u_{kj}))-Y(x_k+\varepsilon u_{kj},h(x_k)))^2\big]=C\|g_{0,2}(x_k+\varepsilon u_{kj},x_k,\cdot)\|_2^2,$$

with $C=4\pi\int_0^{+\infty}\rho^2\,\nu(d\rho)$. Hence, since $h(y)<\beta$ for every $y\in\mathbb{R}^d$, by the Minkowski inequality and lemma 2.2.2,

$$\mathbb{E}\big(|Y_{\varepsilon,2}(x,u,v)|^2\big)=0. \tag{2.26}$$

Conclusion The conclusion is then given by (2.25) and (2.26). □

In the case of RHFLMs associated with a stable control measure, there exists an asymptotic self-similarity at infinity. A similar property holds for more general RHMLMs. Then, when the increments are taken at large scales, the limit field is a Real Harmonizable Fractional Stable Motion. Actually, it holds when RHMLMs are associated with a truncated α-stable control measure. Heuristically, this truncation disappears at large scales.

Proposition 2.2.13. *Let $L(d\xi) = M(d\xi)$, $0 < \alpha < 2$ and assume that the control measure $\nu(dz)$ is associated with*

$$\nu_\rho(d\rho) = \frac{\mathbf{1}_{0<\rho<1}}{\rho^{1+\alpha}} d\rho$$

by (1.17). *Moreover, assume that*

$$\textbf{H1} \quad \exists h_\infty \in \mathbb{R}, \lim_{\|x\|\to+\infty} (h(x) - h_\infty) \ln \|x\| = 0,$$

and

$$\textbf{H2} \quad \widetilde{h}_\infty = h_\infty + d/2 - d/\alpha > 0.$$

Then, the RHMLM X_h is asymptotically self-similar with parameter $\widetilde{h}_\infty$ at infinity. More precisely,

$$\lim_{R\to+\infty} \left(\frac{X_h(Ru)}{R^{\widetilde{h}_\infty}} \right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(S_{\widetilde{h}_\infty}(u) \right)_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $S_{\widetilde{h}_\infty}$ is a Real Harmonizable Fractional Stable Motion, in short RHFSM, that has representation

$$S_{\widetilde{h}_\infty}(u) = D(\alpha) \Re \left(\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{\widetilde{h}_\infty + d/\alpha}} M_\alpha(d\xi) \right), \tag{2.27}$$

with M_α a complex isotropic stable α-symmetric random measure with control measure the Lebesgue measure (in the sense of [ST94]) and

$$D(\alpha) = \left[2^{\alpha+1} \pi \int_0^{+\infty} (1 - \cos(r)) \frac{dr}{r^{1+\alpha}} \right]^{1/\alpha}.$$

Remark 2.2.14. Since $0 < \widetilde{h_\infty} < h_\infty \leq 1$, the RHFSM $S_{\widetilde{h}_\infty}$ is well-defined.

Remark 2.2.15. In the case where h is equal to a constant H, the assumption **H1** is fulfilled and the preceding proposition has been proved in [BCI02]. However, there exists non-constant function h which satisfies **H1** and **H2** such as

$$h(x) = C \exp\left(-\frac{1}{\|x\|} \right),$$

where α et C are chosen such that $d/\alpha - d/2 < C \leq 1$. Then, the preceding proposition holds with $\widetilde{h}_\infty = C + d/2 - d/\alpha$. Hence, there exists RHMLMs associated with a non-constant function h and with a RHFSM as tangent field at infinity.

Proof. Let $p \in \mathbb{N}^*$, $u = (u_1, \ldots, u_p) \in \left(\mathbb{R}^d\right)^p$, $v = (v_1, \ldots, v_p) \in \mathbb{R}^p$ and

$$g_0^R(\xi, z) = \frac{2}{R^{\widetilde{h}_\infty}} \Re\left(z \sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iRu_k\cdot\xi} - 1}{\|\xi\|^{h(Ru_k)+d/2}} \right).$$

Then since $\displaystyle\sum_{k=1}^{p} v_k\, X_h(Ru_k) = \int_{\mathbb{R}^d} g_0^R(\xi, z)\, \widetilde{N}(d\xi, dz)$,

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^{p} v_k\, X_h(Ru_k) \right) \right] = \exp\left(\psi(R)\right)$$

with

$$\psi(R) = \int_{\mathbb{R}^d\times\mathbb{C}} \left[\exp\left(i g_0^R(\xi, z) \right) - 1 - i g_0^R(\xi, z) \right] d\xi\, \nu(dz).$$

First, since $\nu(dz)$ is a rotationally invariant measure and since $g_0^R(\xi, -z) = -g_0^R(\xi, z)$,

$$\psi(R) = \int_{\mathbb{R}^d\times\mathbb{C}} \left[\cos\left(g_0^R(\xi, z) \right) - 1 \right] d\xi\, \nu(dz).$$

Hence, by definition of $\nu(dz)$,

$$\psi(R) = \int_{\mathbb{R}^d} \int_0^{+\infty} \int_0^{2\pi} \left[\cos\left(g_0^R\left(\xi, \rho \mathrm{e}^{i\theta}\right) \right) - 1 \right] \mathbf{1}_{0<\rho<1}\, d\theta\, \frac{d\rho}{\rho^{1+\alpha}}\, d\xi.$$

Therefore, the changes of variable $\lambda = R\xi$ and $r = R^{d/\alpha}\rho$ leads to:

$$\psi(R) = \int_{\mathbb{R}^d} \int_0^{2\pi} \int_0^{+\infty} l_R(\lambda, r, \theta)\, dr\, d\theta\, d\xi,$$

where

$$l_R(\lambda, r, \theta) = \frac{1}{r^{1+\alpha}} \left[\cos\left(\widetilde{g}_0^R(\lambda, r, \theta) \right) - 1 \right] \mathbf{1}_{0<r<R^{d/\alpha}}.$$

with

$$\widetilde{g}_0^R(\lambda, r, \theta) = 2r\Re\left(\mathrm{e}^{i\theta} \sum_{i=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{h(Ru_k)+d/2}} R^{h(Ru_k)-h_\infty} \right).$$

Furthermore, the assumption **H1** implies that for every k such that $u_k \neq 0$,

$$\lim_{R\to+\infty} R^{h(Ru_k)-h_\infty} = 1.$$

Therefore,

$$\widetilde{g}(\lambda, r, \theta) = \lim_{R\to+\infty} \widetilde{g}_0^R(\lambda, r, \theta) = 2r\Re\left(\mathrm{e}^{i\theta} \sum_{i=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{h_\infty+d/2}} \right).$$

Then, let m and M such that $0 < m < h_\infty < M$. Hence, there exists R_0 such that for every $R \geq R_0$ and $1 \leq k \leq p$,

$$m \leq h(Ru_k) \leq M \text{ and } R^{h(Ru_k)-h_\infty} \leq 2.$$

Therefore, for every $R \geq R_0$,

$$\left|\widetilde{g}_0^R(\lambda, r, \theta)\right| \leq G(\lambda) r, \tag{2.28}$$

where

$$G(\lambda) = 2\sum_{k=1}^{p} |v_k| \left[\frac{\|u_k\| \mathbf{1}_{\|\lambda\| \leq 1}}{\|\lambda\|^{M-1+d/2}} + \frac{\mathbf{1}_{\|\lambda\|>1}}{\|\lambda\|^{m+d/2}} \right].$$

Moreover, by (2.28), for every $R \geq R_0$,

$$|l_R(\lambda, r, \theta)| \leq l(\lambda, r, \theta),$$

with

$$l(\lambda, r, \theta) = \frac{1}{2} G^2(\lambda) r^{1-\alpha} \mathbf{1}_{G(\lambda)r \leq 1} + \frac{2}{r^{1+\alpha}} \mathbf{1}_{G(\lambda)r>1}.$$

Let us recall that $0 < \widetilde{h}_\infty = h_\infty + d/2 - d/\alpha < h_\infty \leq 1$. Then m and M can be chosen such that

$$0 < m + d/2 - d/\alpha < M + d/2 - d/\alpha < 1.$$

Hence, for this choice, one easily proves that $l \in L^1\big(\mathbb{R}^d \times (0, +\infty) \times [0, 2\pi]\big)$. Consequently, a dominated convergence argument yields that

$$\lim_{R \to +\infty} \psi(R) = \int_{\mathbb{R}^d} \int_0^{2\pi} \int_0^{+\infty} \left[\cos\left(\widetilde{g}(\lambda, r, \theta)\right) - 1\right] \frac{dr}{r^{1+\alpha}} \, d\theta \, d\lambda.$$

Moreover, for every $x \in \mathbb{R}$,

$$\int_0^{+\infty} \left[\cos(xr) - 1\right] \frac{dr}{r^{1+\alpha}} = -C(\alpha)|x|^\alpha$$

where $C(\alpha) = \displaystyle\int_0^{+\infty} (1 - \cos(r)) \frac{dr}{r^{1+\alpha}}$. Therefore,

$$\lim_{R \to +\infty} \psi(R) = -C(\alpha) \int_0^{2\pi} |2\cos(\theta)|^\alpha \, d\theta \int_{\mathbb{R}^d} \left| \sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k \cdot \lambda} - 1}{\|\lambda\|^{h_\infty + d/2}} \right|^\alpha d\lambda.$$

As a conclusion, since $h_\infty + d/2 = \widetilde{h}_\infty + d/\alpha$,

$$\lim_{R \to +\infty} \mathbb{E}\left[\exp\left(i \sum_{k=1}^{p} v_k \frac{X_h(Ru_k)}{R^{\widetilde{h}_\infty}}\right)\right] = \mathbb{E}\left[\exp\left(i \sum_{k=1}^{p} v_k S_{\widetilde{h}_\infty}(u_k)\right)\right],$$

where $S_{\widetilde{h}_\infty}$ is defined by (2.27), which concludes the proof. □

2.3 Pointwise Hölder exponent

The pointwise Hölder exponent of a RHFLM, and then in particular of a FBM, does not vary along a sample path. Then, the MBM is a model which generalizes the FBM and allows the pointwise Hölder exponent to vary along the trajectories, which extends the field of applications. More generally, in most cases, the pointwise Hölder exponent of RHMLMs is not equal to a constant.

Proposition 2.3.1. *Suppose that for every $x \in \mathbb{R}^d$, $h(x) < \beta$. Then for every $x \in \mathbb{R}^d$, the pointwise Hölder exponent of the RHMLM X_h at point x is almost surely equal to $h(x)$.*

Remark 2.3.2. This proposition generalizes corollary 1.2.9.

Proof. It is classical, see proposition 3.3 page 109 in [BCI02], to deduce from the lass property that the pointwise Hölder exponent $H_{X_h}(x)$ of the RHMLM X_h at point x satisfies:

$$H_{X_h}(x) \leq h(x).$$

Then by (2.9), $H_{X_h}(x) = h(x)$. □

The corollary 1.3.15 establishes that the directional pointwise Hölder exponent of a RHFLM does not depend on the direction. It is due to the increments stationarity and the isotropy property of RHFLMs. In general, a RHMLM does not have stationary increments and is not an isotropic field. Nevertheless, in most cases, its directional pointwise Hölder exponent does not depend on the direction.

Proposition 2.3.3. *Let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function such that*

$$\forall x \in \mathbb{R}^d,\ h(x) < \beta.$$

Let us fixed $(x,u) \in \mathbb{R}^d \times S^{d-1}$. Then, the pointwise Hölder exponent $H_{X_h}(x,u)$ of a RHFLM X_h at point x in direction u is almost surely equal to $h(x)$.

Remark 2.3.4. The proof of proposition 2.3.3 uses the same arguments as in the proof of proposition 2.3.1. Then, an upper bound for $H_{X_h}(x,u)$ is given by the lass property.

Proof. By definition, $H_{X_h}(x,u) \geq H_{X_h}(x)$. Hence, by proposition 2.3.1,

$$H_{X_h}(x,u) \geq h(x) \tag{2.29}$$

almost surely.

Let $H' > h(x)$. Then, the lass property, stated in proposition 2.2.7, implies:

$$\lim_{\varepsilon \to 0_+} \frac{X_h(x+\varepsilon u) - X_h(x)}{\varepsilon^{h(x)}} \overset{(d)}{=} C_1(h(x)) B_{h(x)}(u).$$

Moreover, since $u \neq 0$, $\mathbb{P}\big(B_{h(x)}(u) = 0\big) = 0$. Hence, since $C_1(h(x)) \neq 0$,

$$\lim_{\varepsilon \to 0_+} \frac{\varepsilon^{H'}}{|X_h(x+\varepsilon u) - X_h(x)|} \stackrel{(d)}{=} 0.$$

This last convergence is also a convergence in probability. Therefore, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$ and

$$\lim_{n\to+\infty} \frac{X_h(x+\varepsilon_n u) - X_h(x)}{\varepsilon_n} = +\infty \text{ almost surely.}$$

Therefore, by definition, $H_{X_h}(x,u) \leq H'$ almost surely, which concludes the proof. □

Then, the multifractional function h gives the local regularity of the RHMLM X_h. Furthermore, it also governs its local structure in terms of tangent fields. From a statistical point of view, a central problem is then to identify this function which characterizes the properties of the RHMLM.

2.4 Identification

In this section, $L(d\xi) = M(d\xi) + \sigma\, W(d\xi)$ is a Lévy random measure which satisfies the assumptions of the definition 2.1.1. In particular, W and M are two independent measures.

Let $h : \mathbb{R}^d \longrightarrow (0,1)$ be a locally β-Hölder function and X_h be the RHMLM associated with $L(d\xi)$ and h. Then

$$X_h(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}}\, L(d\xi).$$

Our aim is to identify the multifractional function h from discrete observations of the field X_h on $[0,1]^d$. The variance σ^2 and the control measure $\nu(dz)$ are unknown. The field is observed at sampling points $\big(\frac{k_1}{N}, \ldots, \frac{k_d}{N}\big)$, $0 \leq k_j \leq N$, $1 \leq j \leq d$. The multifractional function is then identified owing to one sample path.

Then, one uses localized generalized quadratic variations, method introduced in [BCI98]. For any $x \in (0,1)^d$, we define a (ε, N)-neighbourhood of x by

$$\mathcal{V}_{\varepsilon,N}(x) = \left\{ \boldsymbol{p} \in \mathbb{Z}^d,\ \max_{j=1\ldots d} \left| \frac{p_j}{N} - x_j \right| < \varepsilon \right\}.$$

Let $(a_\ell)_{\ell=0\ldots K}$ be a real valued sequence such that:

$$\sum_{\ell=0}^{K} a_\ell = \sum_{\ell=0}^{K} \ell a_\ell = 0. \tag{2.30}$$

As an example, one can take $K = 2,\ a_0 = 1,\ a_1 = -2,\ a_2 = 1$.

For every $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, $a_{\boldsymbol{k}} = \prod_{j=1}^{d} a_{k_j}$. Let us note

$$\boldsymbol{\mathcal{K}} = \big\{\boldsymbol{k} \in \mathbb{N}^d,\ 0 \leq k_j \leq K\ j = 0 \ldots K\big\}.$$

$$\Delta X_{\boldsymbol{p}} = \sum_{\boldsymbol{k} \in \boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}}\, X_h\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}\right) = \sum_{k_1,\ldots,k_d=0}^{K} \prod_{j=1}^{d} a_{k_j}\, X_h\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}\right)$$

where $\boldsymbol{p} \in \mathcal{V}_{\varepsilon,N}(x)$, are the increments of X_h associated with the sequence a.

Then the localized generalized quadratic variation at point x is equal to

$$V_{\varepsilon,N}(x) = \sum_{\boldsymbol{p} \in \mathcal{V}_{\varepsilon,N}(x)} (\Delta X_{\boldsymbol{p}})^2.$$

Our aim is to show that

$$\widehat{h}_N(x) = \frac{1}{2}\left[\mathrm{Log}_2\left(\frac{V_{\varepsilon,N/2}(x)}{V_{\varepsilon,N}(x)}\right) + d\right]$$

is a consistent estimator of $h(x)$.

Notations Let $(v_m)_m$ be a deterministic real valued sequence, $(Z_m)_m$ and $(R_m)_m$ be two sequences of real random variables.

- $v_m = O(1)$ means that the sequence $(v_m)_m$ is bounded.
- $Z_m = O_{\mathbb{P}}(1)$ if and only if

$$\forall \varepsilon > 0,\ \exists M > 0,\ \sup_m \mathbb{P}(|Z_m| > M) < \varepsilon.$$

- $Z_m = O_{\mathbb{P}}(R_m)$ means $Z_m = R_m U_m$ with $U_m = O_{\mathbb{P}}(1)$.

Theorem 2.4.1. *Let h be a β-Hölder function on $[0,1]^d$ and suppose that*

$$\max_{y \in [0,1]^d} h(y) < \beta.$$

Let $\varepsilon = N^{-\alpha}$ with $0 < \alpha < 1$. Then, as $N \to +\infty$,

$$\widehat{h}_N(x) \overset{(\mathbb{P})}{\to} h(x)\ ,$$

where $\overset{(\mathbb{P})}{\to}$ means a convergence in probability.

If $d \geq 2$ and $0 < \alpha < 1 - 1/d$, then as $N \to +\infty$

$$\widehat{h}_N(x) \to h(x) \quad \textit{almost surely.}$$

Moreover for $\alpha = d/(d+2\beta)$ and for every $\gamma < \alpha\beta$,

$$\widehat{h}_N(x) = h(x) + O_{\mathbb{P}}\big(N^{-\min(\beta - h(x), \gamma)} \ln N\big). \tag{2.31}$$

Remark 2.4.2. Since the rate of convergence in (2.31) depends on β and $h(x)$, it is only given here as a curiosity! Nevertheless we can remark that when β is known, the last claim of theorem 2.4.1 explains how to choose α. Moreover when $\beta = 1$, which is the case when h is $\mathcal{C}^1$ for instance, it is the same choice as in [BCI98].

In order to prove this theorem, one studies the asymptotic of $V_{\varepsilon,N}(x)$. Let us recall that

$$X_h(x) = Y(x, h(x))$$

where

$$Y(x,y) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{y+d/2}} \, L(d\xi), \quad x \in \mathbb{R}^d, \ y \in]0,1[.$$

Hence,

$$V_{\varepsilon,N}(x) = \sum_{\boldsymbol{p}\in\mathcal{V}_{\varepsilon,N}(x)} \left[\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, Y\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N} \right) \right) \right]^2. \tag{2.32}$$

Like in [ALV02] and [ABCLV02], one replaces $Y\big(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\big(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}\big)\big)$ by $Y\big(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\big(\frac{\boldsymbol{p}}{N}\big)\big)$ in (2.32):

$$W_{\varepsilon,N}(x) = \sum_{\boldsymbol{p}\in\mathcal{V}_{\varepsilon,N}(x)} \left[\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, Y\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\Big(\frac{\boldsymbol{p}}{N} \Big) \right) \right]^2. \tag{2.33}$$

The study of the asymptotic of $W_{\varepsilon,N}(x)$ is simpler than for $V_{\varepsilon,N}(x)$. Then it remains to compare $W_{\varepsilon,N}(x)$ to $V_{\varepsilon,N}(x)$. In order to obtain the asymptotic of $W_{\varepsilon,N}(x)$, one evaluates $\mathbb{E}(W_{\varepsilon,N}(x))$ and $\operatorname{Var} W_{\varepsilon,N}(x)$. At first, let us give useful expressions of them. Please notice that because of (2.30),

$$\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, Y\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\Big(\frac{\boldsymbol{p}}{N} \Big) \right) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-i\frac{\boldsymbol{p}}{N}\cdot\xi}}{\|\xi\|^{\frac{d}{2}+h\left(\frac{\boldsymbol{p}}{N}\right)}} \sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, \mathrm{e}^{-i\frac{\boldsymbol{k}}{N}\cdot\xi} \, L(d\xi).$$

Then because of (1.19),

$$\mathbb{E}(W_{\varepsilon,N}(x)) = A \sum_{\boldsymbol{p}\in\mathcal{V}_{\varepsilon,N}(x)} I_{\boldsymbol{p},\boldsymbol{p}} \tag{2.34}$$

where $A = \sigma^2 + 4\pi \int_0^{+\infty} \rho^2 \, \nu_\rho(d\rho)$ and

$$I_{\boldsymbol{p},\boldsymbol{p}'} = \int_{\mathbb{R}^d} \mathrm{e}^{i\frac{(\boldsymbol{p}'-\boldsymbol{p})}{N}\cdot\xi} \frac{\left| \sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, \mathrm{e}^{-i\frac{\boldsymbol{k}}{N}\cdot\xi} \right|^2}{\|\xi\|^{d+h\left(\frac{\boldsymbol{p}}{N}\right)+h\left(\frac{\boldsymbol{p}'}{N}\right)}} \, d\xi. \tag{2.35}$$

Moreover one can prove as it is done in the last section of [BCI02] that

$$\operatorname{Var} W_{\varepsilon,N}(x) = \sum_{\boldsymbol{p},\boldsymbol{p}'\in\mathcal{V}_{\varepsilon,N}(x)} \left[2A^2 (I_{\boldsymbol{p},\boldsymbol{p}'})^2 + 2B J_{\boldsymbol{p},\boldsymbol{p}'} + B L_{\boldsymbol{p},\boldsymbol{p}'} \right], \tag{2.36}$$

where $B = 4\pi \int_0^{+\infty} \rho^4 \, \nu_\rho(d\rho)$,

$$J_{\boldsymbol{p},\boldsymbol{p'}} = \int_{\mathbb{R}^d} \frac{\left|\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, \mathrm{e}^{-i\frac{\boldsymbol{k}}{N}\cdot\xi}\right|^4}{\|\xi\|^{2d+2h\left(\frac{\boldsymbol{p}}{N}\right)+2h\left(\frac{\boldsymbol{p'}}{N}\right)}} \, d\xi \tag{2.37}$$

and

$$L_{\boldsymbol{p},\boldsymbol{p'}} = \int_{\mathbb{R}^d} \mathrm{e}^{2i\frac{(\boldsymbol{p'}-\boldsymbol{p})}{N}\cdot\xi} \frac{\left|\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, \mathrm{e}^{-i\frac{\boldsymbol{k}}{N}\cdot\xi}\right|^4}{\|\xi\|^{2d+2h\left(\frac{\boldsymbol{p}}{N}\right)+2h\left(\frac{\boldsymbol{p'}}{N}\right)}} \, d\xi. \tag{2.38}$$

Before proving theorem 2.4.1, let us study $I_{\boldsymbol{p},\boldsymbol{p'}}$, $J_{\boldsymbol{p},\boldsymbol{p'}}$ and $L_{\boldsymbol{p},\boldsymbol{p'}}$.

Lemma 2.4.3. *For every $\delta > 0$ such that $h(x) > \delta$, there exist $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}^*$ such that for $\varepsilon \in (0, \varepsilon_0]$, $N \geq N_0$ and $(\boldsymbol{p}, \boldsymbol{p'}) \in \mathcal{V}_{\varepsilon,N}(x)^2$,*

$$\left| I_{\boldsymbol{p},\boldsymbol{p'}} - \frac{1}{N^{2h(x)}} F_{2h(x)}(\boldsymbol{p'} - \boldsymbol{p}) \right| \leq \frac{C\varepsilon^\beta \ln N}{N^{2\delta} \prod_{j=1}^{d} \left(1 + |p'_j - p_j|\right)},$$

where C is a non negative constant which only depends on δ and x, and where

$$F_\gamma(u) = \int_{\mathbb{R}^d} \mathrm{e}^{iu\cdot\xi} \frac{\left|\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, \mathrm{e}^{i\boldsymbol{k}\cdot\xi}\right|^2}{\|\xi\|^{d+\gamma}} \, d\xi, \quad \gamma \in (0,2), \quad u \in \mathbb{R}^d.$$

Proof. Let $r > 0$ such that $K_0 = \overline{B(x,r)} \subset (0,1)^d$ and $m = \min_{K_0} h > \delta$. Let $M = \max_{K_0} h$. Then let $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}^*$ such that for every $\boldsymbol{p} \in \mathcal{V}_{\varepsilon,N}(x)$ and $\boldsymbol{k}$,

$$\frac{\boldsymbol{k} + \boldsymbol{p}}{N} \in K_0.$$

Let us remark that

$$I_{\boldsymbol{p},\boldsymbol{p'}} = \frac{1}{N^{h\left(\frac{\boldsymbol{p}}{N}\right)+h\left(\frac{\boldsymbol{p'}}{N}\right)}} F_{h\left(\frac{\boldsymbol{p}}{N}\right)+h\left(\frac{\boldsymbol{p'}}{N}\right)}(\boldsymbol{p'} - \boldsymbol{p}).$$

Notice that for every $u \in \mathbb{R}^d$, the map $\gamma \longmapsto \frac{1}{N^\gamma} F_\gamma(u)$ is $\mathcal{C}^1$ on $(0,2)$. Then a Taylor expansion gives:

$$\left| I_{\boldsymbol{p},\boldsymbol{p'}} - \frac{1}{N^{2h(x)}} F_{2h(x)}(\boldsymbol{p'} - \boldsymbol{p}) \right| \leq C\varepsilon^\beta \max_{2m\leq\gamma\leq 2M} |A(\boldsymbol{p}, \boldsymbol{p'}, \gamma)|,$$

where $A(\boldsymbol{p}, \boldsymbol{p'}, \gamma) = \frac{1}{N^\gamma} A_1(\boldsymbol{p}, \boldsymbol{p'}, \gamma) + \frac{\ln N}{N^\gamma} F_\gamma(\boldsymbol{p'} - \boldsymbol{p})$,

$$\text{with} \quad A_1(\boldsymbol{p}, \boldsymbol{p'}, \gamma) = \int_{\mathbb{R}^d} \frac{\ln \|\xi\| \, \mathrm{e}^{i(\boldsymbol{p'}-\boldsymbol{p})\cdot\xi} \left|\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \, \mathrm{e}^{i\boldsymbol{k}\cdot\xi}\right|^2}{\|\xi\|^{d+\gamma}} \, d\xi.$$

Notice that $A(\boldsymbol{p},\boldsymbol{p},\gamma)=A(\gamma)$. Then suppose that $\boldsymbol{p}\neq\boldsymbol{p'}$ and proceed like [BCI02] or [BCIJ98] by integrating by parts, which leads to

$$|A_1(\boldsymbol{p},\boldsymbol{p'},\gamma)|\leq C\prod_{\{j\,/\,p_j\neq p_{j'}\}}\frac{1}{\left|p'_j-p_j\right|}\leq C\prod_{j=1}^{d}\frac{1}{1+\left|p'_j-p_j\right|}.$$

Moreover one can also prove, using integrations by parts that

$$\max_{2m\leq\gamma\leq 2M}|F_\gamma(\boldsymbol{p'}-\boldsymbol{p})|\leq C\prod_{j=1}^{d}\frac{1}{1+\left|p'_j-p_j\right|}.$$

Then since $m>\delta$,

$$\left|I_{\boldsymbol{p},\boldsymbol{p'}}-\frac{1}{N^{2h(x)}}F_{2h(x)}(\boldsymbol{p'}-\boldsymbol{p})\right|\leq C\,\frac{\varepsilon^\beta\ln N}{N^{2\delta}}\prod_{j=1}^{d}\frac{1}{1+\left|p'_j-p_j\right|}.$$

□

Lemma 2.4.4. *For every $\delta>0$ such that $h(x)>\delta$, there exist $\varepsilon_0>0$, $N_0\in\mathbb{N}^*$ such that for $\varepsilon\in(0,\varepsilon_0]$, $N\geq N_0$ and $(\boldsymbol{p},\boldsymbol{p'})\in\mathcal{V}_{\varepsilon,N}(x)^2$,*

$$|J_{\boldsymbol{p},\boldsymbol{p'}}|\leq C\left[\frac{1}{N^{d+4h(x)}}+\frac{\varepsilon^\beta\ln N}{N^{d+4\delta}}\right],$$

where C is a non negative constant which only depends on δ and x.

The same inequality holds for $L_{\boldsymbol{p},\boldsymbol{p'}}$.

Proof. Notice that

$$|L_{\boldsymbol{p},\boldsymbol{p'}}|\leq J_{\boldsymbol{p},\boldsymbol{p'}}.$$

Therefore it is sufficient to prove the lemma for $J_{\boldsymbol{p},\boldsymbol{p'}}$. Let $K_0=\overline{B(x,r)}$, ε_0 and N_0 be such as in the proof of the previous lemma.

Please notice that

$$J_{\boldsymbol{p},\boldsymbol{p'}}=\frac{1}{N^{d+2h\left(\frac{\boldsymbol{p}}{N}\right)+2h\left(\frac{\boldsymbol{p'}}{N}\right)}}\,G_{2h\left(\frac{\boldsymbol{p}}{N}\right)+2h\left(\frac{\boldsymbol{p'}}{N}\right)}(0),$$

where

$$G_\gamma(u)=\int_{\mathbb{R}^d}\mathrm{e}^{-2iu\cdot\xi}\,\frac{\left|\sum_{\boldsymbol{k}\in\mathcal{K}}a_{\boldsymbol{k}}\,\mathrm{e}^{i\boldsymbol{k}\cdot\xi}\right|^4}{\|\xi\|^{2d+\gamma}}\,d\xi.$$

Since the map $\gamma\longmapsto\frac{1}{N^\gamma}G_\gamma(0)$ is $\mathcal{C}^1$ on $]0,4[$, a Taylor expansion leads to

$$\left|J_{\boldsymbol{p},\boldsymbol{p'}}-\frac{1}{N^{d+4h(x)}}\,G_{4h(x)}(0)\right|\leq\frac{C\varepsilon^\beta\ln N}{N^{d+4\delta}}.$$

□

Let us now prove theorem 2.4.1.

Proof of theorem 2.4.1. Let us take $\varepsilon = N^{-\alpha}$ with $\alpha \in (0,1)$. Let $\delta > 0$ such that $\delta < h(x)$.

Step 1: Asymptotic of $W_{\varepsilon,N}(x)$

One deduces from the equality (2.34) and from lemma 2.4.3 that

$$\mathbb{E}(W_{\varepsilon,N}(x)) = \varepsilon^d N^{d-2h(x)} D(x) + O\big(N^{-2h(x)}\big) + O\big(\varepsilon^{d+\beta} N^{d-2\delta} \ln N\big), \tag{2.39}$$

where $D(x) = 2^d A\, F_{2h(x)}(0)$.

It remains to study $\operatorname{Var} W_{\varepsilon,N}(x)$. In the sequel, lemmas 2.4.3 and 2.4.4 are applied in order to estimate this variance. Let us first recall that

$$\big|F_{2h(x)}(\boldsymbol{p'} - \boldsymbol{p})\big| \leq C \prod_{j=1}^{d} \frac{1}{1 + \big|p'_j - p_j\big|},$$

and remark that

$$\sum_{\boldsymbol{p},\boldsymbol{p'} \in \mathcal{V}_{\varepsilon,N}(x)} \prod_{j=1}^{d} \frac{1}{\big(1 + \big|p'_j - p_j\big|\big)^2} \leq C \varepsilon^d N^d.$$

Then by applying lemmas 2.4.3 and 2.4.4 to (2.36) with $\delta > h(x) - \alpha\beta/2$,

$$\operatorname{Var} W_{\varepsilon,N}(x) = O\big(\varepsilon^d N^{d-4h(x)}\big). \tag{2.40}$$

Consequently, if $d(1-\alpha) > 1$, then by the Borel-Cantelli lemma, as $N \to +\infty$,

$$\varepsilon^{-d} N^{-d+2h(x)}\, W_{\varepsilon,N}(x) \to D(x) \quad \text{almost surely.} \tag{2.41}$$

Furthermore, because of (2.39) and (2.40),

$$W_{\varepsilon,N}(x) = \varepsilon^d N^{d-2h(x)} D(x) + O_{\mathbb{P}}\big(\varepsilon^{d/2} N^{d/2-2h(x)}\big) + O\big(\varepsilon^{d+\beta} N^{d-2\delta} \ln N\big).$$

At first, suppose that $\alpha \leq d/(d+2\beta)$, then since $\delta < h(x)$,

$$(d+\beta)\alpha - d + 2\delta < d(\alpha - 1)/2 + 2h(x).$$

Consequently, since $\varepsilon = N^{-\alpha}$,

$$W_{\varepsilon,N}(x) = \varepsilon^d N^{d-2h(x)} D(x) + O_{\mathbb{P}}\big(\varepsilon^{d+\beta} N^{d-2\delta} \ln N\big). \tag{2.42}$$

Else suppose that $\alpha > d/(d+2\beta)$, then one can choose

$$\delta > h(x) - (d+2\beta)\alpha/4 + d/4.$$

For this choice of δ,

$$W_{\varepsilon,N}(x) = \varepsilon^d N^{d-2h(x)} D(x) + O_{\mathbb{P}}\big(\varepsilon^{d/2} N^{d/2-2h(x)}\big). \tag{2.43}$$

Step 2: Comparison of $W_{\varepsilon,N}(x)$ and $V_{\varepsilon,N}(x)$

Since Y may have infinite moments, it is split into $Y = Y_n^+ + Z_n$ where $n \geq d/2$ and Z_n is defined by (2.8), see page 36. Notice that

$$X_{h,n}^+(y) = Y_n^+(y, h(y)) \text{ and that } X_{h,n}^-(y) = Z_n(y, h(y)).$$

Moreover

$$\left|V_{\varepsilon,N}^{1/2}(x) - W_{\varepsilon,N}^{1/2}(x)\right|^2 \leq 2A(\varepsilon, N) + 2B(\varepsilon, N),$$

$$\text{where } A(\varepsilon,N) = \sum_{\boldsymbol{p}\in\mathcal{V}_{\varepsilon,N}(x)} \left[\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \left(X_{h,n}^+\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}\right) - Y_n^+\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\Big(\frac{\boldsymbol{p}}{N}\Big)\right)\right)\right]^2$$

$$\text{and } B(\varepsilon,N) = \sum_{\boldsymbol{p}\in\mathcal{V}_{\varepsilon,N}(x)} \left[\sum_{\boldsymbol{k}\in\boldsymbol{\mathcal{K}}} a_{\boldsymbol{k}} \left(X_{h,n}^-\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}\right) - Z_n\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\Big(\frac{\boldsymbol{p}}{N}\Big)\right)\right)\right]^2.$$

Since Z_n has $\mathcal{C}^1$-sample paths,

$$|B(\varepsilon, N)| \leq C\varepsilon^d N^{d-2\beta}, \tag{2.44}$$

where C is a random variable.

Let us now study the moments of $A(\varepsilon, N)$. Notice that

$$X_{h,n}^+\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}\right) - Y_n^+\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, h\Big(\frac{\boldsymbol{p}}{N}\Big)\right) = \int_{\mathbb{R}^d} g_{n,2}\left(\frac{\boldsymbol{k}+\boldsymbol{p}}{N}, \frac{\boldsymbol{p}}{N}, \xi\right) L(d\xi),$$

where $g_{n,2}$ is defined by (2.7), see page 33. Then by applying proposition 1.3.3 and lemma 2.2.2, one obtains that

$$\mathbb{E}(A(\varepsilon, N)^q) \leq C\varepsilon^{dq} N^{q(d-2\beta)}.$$

Then since $\beta > h(x)$, one can choose $q \in \mathbb{N}^*$ such that $2q(\beta - h(x)) > 1$. The Borel-Cantelli lemma leads to:

$$\varepsilon^{-d} N^{-d+2h(x)} A(\varepsilon, N) \to 0 \text{ almost surely, as } N \to +\infty.$$

Moreover $A(\varepsilon, N) = O_{\mathbb{P}}\big(\varepsilon^d N^{d-2\beta}\big)$. Consequently,

$$V_{\varepsilon,N}^{1/2}(x) - W_{\varepsilon,N}^{1/2}(x) = O_{\mathbb{P}}\big(\varepsilon^{d/2} N^{d/2-\beta}\big) \tag{2.45}$$

and as $N \to +\infty$

$$\varepsilon^{-d/2} N^{-d/2+h(x)} \Big(V_{\varepsilon,N}^{1/2}(x) - W_{\varepsilon,N}^{1/2}(x)\Big) \to 0 \text{ almost surely.} \tag{2.46}$$

Step 3: Conclusion

Notice that

$$V_{\varepsilon,N}(x) = W_{\varepsilon,N}(x) + 2\, W_{\varepsilon,N}^{1/2}(x)\, D_{\varepsilon,N}(x) + D_{\varepsilon,N}^2(x),$$

where $D_{\varepsilon,N}(x) = \Big(V_{\varepsilon,N}^{1/2}(x) - W_{\varepsilon,N}^{1/2}(x)\Big)$. Then when $d(1-\alpha) > 1$, because of (2.41) and (2.46), and since $AF_{2h(x)}(0) \neq 0$,

$$\widehat{h}_N(x) \to h(x) \text{ almost surely as } N \to +\infty.$$

Even if we do not need the asymptotic of $V_{\varepsilon,N}(x)$ for every α to prove the theorem, let us state it. Using the equalities (2.42), (2.43) and (2.45), one can prove:

- if $\alpha > d/(d+2\beta)$, then
  - if $h(x) \leq \beta - d(1-\alpha)/2$,
    $$V_{\varepsilon,N}(x) = 2^d A\varepsilon^d N^{d-2h(x)} F_{2h(x)}(0) + O_{\mathbb{P}}\big(\varepsilon^{d/2} N^{d/2-2h(x)}\big),$$
  - else $\quad V_{\varepsilon,N}(x) = 2^d A\varepsilon^d N^{d-2h(x)} F_{2h(x)}(0) + O_{\mathbb{P}}\big(\varepsilon^d N^{d-h(x)-\beta}\big)$.
- if $\alpha \leq d/(d+2\beta)$, then
  - if $h(x) \geq \beta - \alpha\beta$,
    $$V_{\varepsilon,N}(x) = 2^d A\varepsilon^d N^{d-2h(x)} F_{2h(x)}(0) + O_{\mathbb{P}}\big(\varepsilon^d N^{d-h(x)-\beta}\big),$$
  - else $\quad V_{\varepsilon,N}(x) = 2^d A\varepsilon^d N^{d-2h(x)} F_{2h(x)}(0) + O_{\mathbb{P}}\big(\varepsilon^{d+\beta} N^{d-2\delta} \ln N\big)$

  where $\delta < h(x)$.

Let us now take $\alpha = d/(d+2\beta)$ and explain in few words this choice. Let us first recall that

$$W_{\varepsilon,N}(x) = \varepsilon^d N^{d-2h(x)} D(x) + O_{\mathbb{P}}\big(\varepsilon^{d/2} N^{d/2-2h(x)}\big) + O\big(\varepsilon^{d+\beta} N^{d-2\delta} \ln N\big).$$

For a given δ, since $\varepsilon = N^{-\alpha}$, the best choice of α is obtained by solving the following equation in α:

$$\frac{-\alpha d + d}{2} - 2h(x) = -(d+\beta)\alpha + d - 2\delta,$$

which gives

$$\alpha = \frac{d - 4\delta + 4h(x)}{d+2\beta}.$$

Then the two errors terms are of the same order up to a logarithmic factor. But since the error term is a non-increasing function of δ and since δ can be arbitrarily chosen closed to $h(x)$, one choose

$$\alpha = \frac{d}{d+2\beta}.$$

In this case,

$$V_{\varepsilon,N}(x) = 2^d A\varepsilon^d N^{d-2h(x)} F_{2h(x)}(0) + O_{\mathbb{P}}\big(N^{-\eta(x)} \ln N\big), \tag{2.47}$$

where

$$\eta(x) = \min\left((d+\beta)\alpha - d + 2\delta, d\alpha - d + h(x) + \beta\right).$$

Then using lemma 2.12 of [vdV98], one can prove that

$$\mathrm{Log}_2 V_{\varepsilon,N}(x) = \mathrm{Log}_2\big(2^d A \varepsilon^d N^{d-2h(x)} F_{2h(x)}(0)\big) + O_{\mathbb{P}}\big(N^{-\min(\beta\alpha+2\delta-2h(x),\beta-h(x))} \ln N\big),$$

and conclude that

$$\widehat{h}_N(x) = h(x) + O_{\mathbb{P}}\big(N^{-\min(\beta\alpha+2\delta-2h(x),\beta-h(x))} \ln N\big).$$

Notice that with the same argument one can give a rate of convergence for every choice of α. □

As a conclusion, RHMLMs are in general non-Gaussian lass fields which pointwise Hölder exponent is allowed to vary along the trajectories. Furthermore, the multifractional function h governs the properties of a RHMLM X_h. In particular, it gives the local regularity of the RHMLM X_h. In addition, even in the case of non-Gaussian RHMLMs, the multifractional function h can be identified owing to generalized quadratics variations whereas it is not possible in the case of RHFSMs. Also, RHMLMs share many properties with the MBM and then extends the fields of applications of the MBM to non-Gaussian phenomena. Thus, in view of applications, next chapter proposes a method for generating RHMLMs owing to generalized shot noise series.

Chapter 3

Series Representation and Simulation of Multifractional Lévy Motions

In this chapter, we describe a method for generating the sample paths of RHMLMs and thereby to give an account of the sample paths theoretical roughness.

3.1 Introduction

Let us recall that a RHMLM Z_h with multifractional function h is defined as the stochastic integral:

$$Z_h(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} \, L(d\xi),$$

where $\|\xi\|$ is the Euclidean norm of ξ and $L(d\xi)$ is a Lévy random measure. In fact,

$$Z_h = aB_h + bX_h,$$

where $(a, b) \in \mathbb{R}^2$, B_h is a Multifractional Brownian Motion and

$$X_h(x) = \int_{\mathbb{R}^d} f(x, \xi) \, M(d\xi),$$

with $M(d\xi)$ a Lévy random measure without Brownian component and

$$f(x, \xi) = \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}}.$$

Let us notice that B_h and X_h are two independent fields.

In this chapter, only the generation of the non-Brownian part X_h is discussed. As for the FBM, many authors have already studied the simulation of its sample paths, see for example [WC94] or [AS96]. Implementations of severals methods of simulation of the FBM can be found in [Coe00]. On the other hand, methods of generating sample paths of the MBM are given in [PLV96] and in [CW98].

Many authors have already been interested in the simulation of non-Gaussian fields defined as stochastic integrals. As an example, simulation of solutions of stochastic differential equations driven by Lévy processes are studied in [Rub03]. Furthermore, simulation of stochastic integrals with respect to Lévy processes are discussed in [Wik02]. However in the case of RHMLMs, the stochastic integral is on whole $\mathbb{R}^d$ and not just on a compact interval of $\mathbb{R}$. In [Dur01], a method for generating symmetric α-stable processes based on a multiresolution analysis is proposed. Nevertheless, the stochastic integral is truncated. On the other hand, X_h is an infinitely divisible field and infinitely divisible laws can be represented as generalized shot noise series. An overview of these representations is given in [Ros01a]. Moreover the simulation implementation is discussed in [Bon82]. Furthermore, series representations are also studied in [Ros01b] in the case of Lévy processes and in [Ros89]. Then, generalized shot noise series allow us to represent X_h as series without truncating the stochastic integral. Nevertheless, it only works if the control measure $\nu(dz)$ of the Lévy random measure $M(d\xi)$ has finite mass. In this case, the generalized shot noise series

$$Y_h(x) = 2\sum_{n=1}^{+\infty} \Re\left\{ f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_n \right\},$$

for a suitable choice of random variables (T_n, Z_n, U_n), converges almost surely. Moreover for this choice,

$$\{X_h(x) : x \in \mathbb{R}^d\} \overset{(d)}{=} \{Y_h(x) : x \in \mathbb{R}^d\},$$

where $\overset{(d)}{=}$ denotes equality in distribution. In practice, one then simulates the sample paths of Y_h which is equal in law to X_h. When ν is not a finite measure, the approximation of X_h is closely related to those of Lévy processes with infinite Lévy measure given in [AR01]. In this case, X_h is split into two independent RHMLMs

$$X_h = X_{\varepsilon,1} + X_{\varepsilon,2}$$

where $X_{\varepsilon,2}$ is associated with a finite control measure $\nu_{\varepsilon,2}(dz)$. Thus $X_{\varepsilon,2}$ can be approximated by a generalized shot noise series. It remains to generate $X_{\varepsilon,1}$. As it is done in [AR01] in the case of Lévy processes, a functional Central Limit Theorem leads to a normal approximation of $X_{\varepsilon,1}$.

Then next section is devoted to the case of a finite control measure. The rate of convergence of the shot noise series is studied. Sufficient conditions to establish a Central Limit Theorem are discussed in section 3.3. Finally some simulation examples are given.

3.2 Generalized shot noise series

In this section, ν is supposed to be a finite measure. The random field X_h can be then represented as a generalized shot noise series.

Let us first introduce some notations that will be used throughout the chapter.

Notation Let $(Z_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ be independent sequences of random variables.

- $(Z_n)_{n\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables with common law
$$\mathcal{L}(Z_n) = \frac{\nu(dz)}{\nu(\mathbb{C})}.$$
- $(U_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that U_1 is uniformly distributed on the unit sphere S^{d-1} of the Euclidean space $\mathbb{R}^d$. Let σ_{d-1} be the uniform measure on S^{d-1}. Then
$$\mathcal{L}(U_n) = \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$
Let us recall that
$$\sigma_{d-1}\big(S^{d-1}\big) = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$
where Γ is the Gamma-function. Moreover let us introduce
$$c_d = \frac{\sigma_{d-1}\big(S^{d-1}\big)}{d}$$
the volume of the unit ball of $\mathbb{R}^d$.
- T_n is the nth arrival time of a Poisson process with intensity 1.

Remark 3.2.1. When $d = 1$, U_n is a symmetric Bernoulli random variable.

Proposition 3.2.2. *Let $K \subset \mathbb{R}^d$ be a compact set. Then almost surely, the series*
$$Y_h(\cdot) = 2\sum_{n=1}^{+\infty} \Re\left\{ f\left(\cdot\,, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n \right) Z_n \right\} \tag{3.1}$$
converges uniformly on K and
$$\{X_h(x) : x \in K\} \overset{(d)}{=} \{Y_h(x) : x \in K\}.$$

Remark 3.2.3. Actually, $Y_{h,N}$ converges almost surely to Y_h in the space of continuous functions endowed with the topology of the uniform convergence on compact sets.

Therefore, since Y_h is equal in law to X_h, in practice one simulates the sample paths of Y_h. These sample paths can be approximated on K owing to (3.1). For the sake of simplicity, in dimension $d = 1$, they are simulated on a compact interval K.

Proof. The series (3.1) can be rewritten as the generalized shot noise series

$$\sum_{n=1}^{+\infty} H(T_n, V_n)(x),$$

where $V_n = (U_n, Z_n)$ and

$$H(r,v)(x) = 2\Re\left\{ f\left(x, \left(\frac{r}{c_d\,\nu(\mathbb{C})}\right)^{1/d} u\right) z\right\}, \quad r>0 \quad v=(u,z).$$

$(V_n)_{n\geq 1}$ is a sequence of i.i.d. random variables which is independent of $(T_n)_{n\geq 1}$. In order to obtain the convergence of (3.1), one shall verify the conditions of Theorem 2.4 in [Ros90]. This theorem implies the convergence in E_K the space of real-valued continuous functions on K endowed with the uniform norm $\|\cdot\|_K$: for every $g \in E_K$, $\|g\|_K = \sup_{x\in K} |g(x)|$. Endowed with this norm, E_K is a separable Banach space. Moreover

$$\begin{aligned} H \;:\; \mathbb{R}_*^+ \times \mathcal{D} &\longrightarrow E_K \\ (r,v) &\longmapsto H(r,v), \end{aligned}$$

where $\mathcal{D} = S^{d-1} \times \mathbb{C}$, is a Borel measurable map. Define a measure F_K on the Borel σ-ring $\mathcal{B}_{E_K}$ of E_K by

$$F_K(A) = \int_0^{+\infty}\!\!\int_{\mathcal{D}} \mathbf{1}_{A\setminus\{0\}}(H(r,v))\,\lambda(dv)\,dr,$$

where λ is the law of V_1. Since ν is a symmetric measure, so is F_K. First one proves that F_K is a Lévy measure. Let E_K' be the dual of E_K and for $y' \in E_K'$, $y \in E_K$ denote $< y', y >= y'(y)$. Let us recall, see [Ros90], that F_K is a Lévy measure if for every $y' \in E_K'$,

$$\int_{E_K} \left(< y', y >^2 \wedge 1\right) F_K(dy) < +\infty \tag{3.2}$$

and if

$$\begin{aligned} \Phi_K \;:\; E_K' &\longrightarrow \mathbb{C} \\ y' &\longmapsto \exp\left\{ \int_{E_K} \left(\mathrm{e}^{i<y',y>} - 1 - i < y', y > \mathbf{1}_{\|y\|_K \leq 1} \right) F_K(dy) \right\} \end{aligned}$$

is the characteristic function of a probability on E_K.

Notice that

$$\int_{E_K} \|y\|_K^2\, F_K(dy) = \int_{]0,+\infty[\times\mathcal{D}} \|H(r,v)\|_K^2\, dr\, \lambda(dv).$$

Therefore by applying the change of variable $\rho = (r/(c_d\,\nu(\mathbb{C})))^{1/d}$,

$$\int_{E_K} \|y\|_K^2\, F_K(dy) = 4 \int_{\mathbb{C}} \int_{S^{d-1}} \int_0^{+\infty} g^2(\rho u, z)\, \rho^{d-1} d\rho\, \sigma_{d-1}(du)\, \nu(dz)$$

where $g(\xi, z) = \sup_{x\in K} |\Re(f(x,\xi)z)|$. Hence using polar coordinates,

$$\int_{E_K} \|y\|_K^2\, F_K(dy) = 4 \int_{\mathbb{R}^d\times\mathbb{C}} g^2(\xi, z)\, d\xi\, \nu(dz).$$

As a result,

$$\int_{E_K} \|y\|_K^2\, F_K(dy) \le 4 \int_{\mathbb{C}} |z|^2\, \nu(dz) \int_{\mathbb{R}^d} \tilde{g}^2(\xi)\, d\xi$$

with $\tilde{g}(\xi) = \sup_{x\in K} |f(x,\xi)|$. Furthermore,

$$|\tilde{g}(\xi)| \le \frac{C_K}{\|\xi\|^{M_K - 1 + d/2}} \mathbf{1}_{\|\xi\|\le 1} + \frac{2}{\|\xi\|^{m_K + d/2}} \mathbf{1}_{\|\xi\|>1},$$

where $C_K = \max_{u\in K} \|u\|$, $m_K = \min_K h$ and $M_K = \max_K h$. Therefore, since $0 < m_K < 1$ and $0 < M_K < 1$, $\tilde{g} \in L^2\big(\mathbb{R}^d\big)$. Then by (1.13),

$$\int_{E_K} \|y\|_K^2\, F_K(dy) < +\infty, \tag{3.3}$$

which implies (3.2). Moreover since the integral in (3.3) is finite,

$$\int_{\|y\|_K\ge 1} < y', y > \; F_K(dy)$$

is well defined and is equal to 0 by symmetry of F_K. Hence

$$\Phi_K(y') = \exp\left\{ \int_{E_K} \Big(\mathrm{e}^{i<y',y>} - 1 - i < y', y > \Big)\, F_K(dy) \right\}.$$

Let us recall, see proposition 2.2.6, that the sample paths of X_h are almost surely continuous. Thus, we can consider $< y', X_{h|_K} >$. Let us first extend y' to the space of continuous functions on K with values in $\mathbb{C}$. For every continuous function $g : K \longrightarrow \mathbb{C}$, let $< y', g > = < y', \Re(g) > + i < y', \Im(g) >$. In particular, by definition, $\Re(< y', g >) = < y', \Re(g) >$ and $\Im(< y', g >) = < y', \Im(g) >$. Then, since

$$< y', X_{h|_K} > = \int_{\mathbb{R}^d} < y', f(\cdot,\xi)_{|_K} > \; M(d\xi), \tag{3.4}$$

by (1.16), see page 22, $\mathbb{E}\Big(\mathrm{e}^{i<y', X_{h|_K}>} \Big)$ is equal to

$$\exp\left\{ \int_{\mathbb{R}^d\times\mathbb{C}} \Big(\mathrm{e}^{2i\Re\left(<y' f(\cdot,\xi)_{|_K}> z\right)} - 1 - 2i\Re\Big(< y', f(\cdot,\xi)_{|_K} > z \Big) \Big)\, d\xi\, \nu(dz) \right\}.$$

Since $2\Re\Big(< y', f(\cdot,\xi)_{|_K} > z\Big) =< y', 2\Re\Big(f(\cdot,\xi)_{|_K} z\Big) >$, then by definition of F_K,

$$\Phi_K(y') = \mathbb{E}\Big(\mathrm{e}^{i<y', X_{h|_K}>}\Big). \tag{3.5}$$

As a consequence, Φ_K is the characteristic function of $X_{h|_K}$. Then F_K is a Lévy measure on E_K. According to Theorem 2.4 in [Ros90],

$$\sum_{j=1}^{n} H(T_j, V_j) - A(T_n),$$

where for $s \geq 0$

$$A(s) = \int_0^s \int_{\mathcal{D}} H(r,v)\mathbf{1}_{\|H(r,v)\|_K \leq 1}\, \lambda(dv)\, dr,$$

is convergent in E_K. Since ν is a symmetric measure, for every $s \geq 0$, $A(s) = 0$, which gives the convergence of (3.1) in E_K. Then let Y_h be its limit. In view of Theorem 2.4 in [Ros90], the characteristic function of $(Y_h(x))_{x\in K}$ is Φ_K. Therefore

$$\{X_h(x),\, x \in K\} \overset{(d)}{=} \{Y_h(x),\, x \in K\}$$

follows from (3.5), which concludes the proof. □

Remark 3.2.4. In view of (3.3) and corollary 2.5 in [Ros90] applied with $p = 2$, the series (3.1) converges also in $\mathrm{L}^2(E_K)$, i.e.

$$\lim_{N\to+\infty} \mathbb{E}\left\{\left(\sup_{x\in K} |Y_{h,N}(x) - Y_h(x)|\right)^2\right\} = 0,$$

where

$$Y_{h,N}(x) = 2\sum_{n=1}^{N} \Re\left\{f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_n\right\}. \tag{3.6}$$

In simulation, Y_h is approximated by $Y_{h,N}$. Then one is interested in the rate of convergence of (3.1). We first consider one-dimensional distribution. The next proposition studies the error in L^q. From this, an almost sure error is deduced and stated in corollary 3.2.8. Then a functional result is established.

Proposition 3.2.5. *Let $q \geq 2$ and $x \in \mathbb{R}^d$. Then for every $N \in \mathbb{N}^*$ such that $N > q/2 + qh(x)/d - 1$, $Y_h(x) - Y_{h,N}(x) \in L^q$ and*

$$\mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q) \leq C_{q,x}\, \frac{D_{N,q}(h(x))}{N^{qh(x)/d}}, \tag{3.7}$$

where $C_{q,x}$ does not depend on N and for $0 < s < 1$ and $n > q/2 + qs/d$

$$D_{n,q}(s) = \frac{\Gamma(n+1-q/2-qs/d)\,(n+1)^{q/2+qs/d}}{\Gamma(n+1)}. \tag{3.8}$$

Remark 3.2.6. By the Stirling formula, $\lim_{N\to+\infty} D_{N,q}(s) = 1$. Therefore thanks to (3.7), the rate of convergence of $Y_{h,N}(x)$ in L^q is at least $N^{h(x)/d}$. Actually, we will prove that the rate of convergence in (3.7) is optimal, see proposition 3.2.7.

For simulation reasons, it can be useful to explicitly provide a constant $C_{q,x}$ for which (3.7) holds. In fact one can take

$$C_{q,x} = 2^{1+3q/2}\, B_q^q \left(\frac{d}{h(x)}\right)^{q/2} \mathbb{E}(|\Re(Z_1)|^q)\, (c_d \nu(\mathbb{C}))^{q/2+qh(x)/d}, \tag{3.9}$$

where

$$B_q = \begin{cases} 1 & \text{when } q = 2, \\ \sqrt{2}\left(\dfrac{\Gamma((q+1)/2)}{\sqrt{\pi}}\right)^{1/q} & \text{when } q > 2. \end{cases} \tag{3.10}$$

Proof. Let

$$\xi_n(x) = 2\Re\left\{ f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right) Z_n \right\},\ n \in \mathbb{N}^*,$$

and

$$R_{N,P}(x) = Y_{h,P}(x) - Y_{h,N}(x) = \sum_{n=N+1}^{P} \xi_n(x),\ 1 \le N < P.$$

Then $\lim_{P\to+\infty} R_{N,P}(x) = Y_h(x) - Y_{h,N}(x)$ almost surely. In fact as $P \to +\infty$, $R_{N,P}$ converges in E_K for any compact subset K of $\mathbb{R}^d$.

As $(\xi_n(x))_{n\ge1}$ is a symmetric sequence of random variables, by proposition 2.3 pages 47-48 in [LT91],

$$\mathbb{E}\left(\max_{N+1\le M\le P} |R_{N,M}(x)|^q\right) \le 2\mathbb{E}(|R_{N,P}(x)|^q).$$

Moreover since T_n, U_n and Z_n are independent random variables and since the law of Z_n is rotationally invariant,

$$\xi_n(x) \overset{(d)}{=} \widetilde{\xi}_n(x)$$

where

$$\widetilde{\xi}_n(x) = 2\left| f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right)\right| \Re(Z_n).$$

Furthermore for given values of $(U_n)_{n\ge1}$ and $(T_n)_{n\ge1}$, $(\xi_n(x))_{n\ge1}$ and $(\widetilde{\xi}_n(x))_{n\ge1}$ are sequences of independent random variables. Therefore

$$(\xi_n(x))_{n\ge1} \overset{(d)}{=} \left(\widetilde{\xi}_n(x)\right)_{n\ge1}.$$

As a consequence,

$$\mathbb{E}(|R_{N,P}(x)|^q) = \mathbb{E}\left\{\left(\sum_{n=N+1}^{P} \widetilde{\xi}_n(x)\right)^q\right\}.$$

Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of i.i.d. symmetric Bernoulli random variables which is independent of $(Z_n, U_n, T_n)_{n\geq 1}$. Hence by symmetry of the law of Z_n, Z_n can be replaced in $R_{N,P}$ by $\varepsilon_n Z_n$. Then by applying the Khintchine inequality with constant B_q and conditionally with respect to $(Z_n, U_n, T_n)_{n\geq 1}$,

$$\mathbb{E}(|R_{N,P}(x)|^q) \leq B_q^q\, \mathbb{E}\left\{\left(\sum_{n=N+1}^{P} \widetilde{\xi}_n^{\,2}(x)\right)^{q/2}\right\}. \tag{3.11}$$

Notice that the best possible constant B_q in the Khintchine inequality is known. According to [Haa81], the best constant is

$$B_q = \begin{cases} 1 & \text{when } q = 2, \\ \\ \sqrt{2}\left(\dfrac{\Gamma((q+1)/2)}{\sqrt{\pi}}\right)^{1/q} & \text{when } q > 2. \end{cases}$$

Then by the Minkowski inequality,

$$\mathbb{E}(|R_{N,P}(x)|^q) \leq B_q^q \left\{\sum_{n=N+1}^{P} \mathbb{E}\Big(\Big|\widetilde{\xi}_n(x)\Big|^q\Big)^{2/q}\right\}^{q/2}.$$

Let us now evaluate $\mathbb{E}\Big(\Big|\widetilde{\xi}_n(x)\Big|^q\Big)$. By independence and since $Z_1 \overset{(d)}{=} Z_n$,

$$\mathbb{E}\Big(\Big|\widetilde{\xi}_n(t)\Big|^q\Big) = 2^q\, \mathbb{E}(|\Re(Z_1)|^q)\mathbb{E}\left\{\left|f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right)\right|^q\right\}$$

Moreover

$$\mathbb{E}\left\{\left|f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})}\right)^{1/d} U_n\right)\right|^q\right\} \leq 2^q\, (c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d} \int_0^{+\infty} \frac{r^{n-1-q/2-qh(x)/d}}{(n-1)!}\, \mathrm{e}^{-r}\, dr.$$

As a result, when $n > qh(x)/d + q/2$, $\widetilde{\xi}_n(x) \in L^q$ and

$$\mathbb{E}\Big(\Big|\widetilde{\xi}_n(x)\Big|^q\Big) \leq 2^{2q}\, (c_d\, \nu(\mathbb{C}))^{q/2+qh(x)/d}\, \mathbb{E}(|\Re(Z_1)|^q)\, \frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}.$$

Then let $N > qh(x)/d + q/2 - 1$. Therefore

$$\mathbb{E}\Big(\max_{N+1\leq M\leq P} |R_{N,M}(x)|^q\Big) \leq A_{q,x}\left\{\sum_{n=N+1}^{P} \left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)}\right)^{2/q}\right\}^{q/2}, \tag{3.12}$$

where $A_{q,x} = 2^{2q+1} B_q^q \, (c_d \, \nu(\mathbb{C}))^{q/2+qh(x)/d} \, \mathbb{E}(|\Re(Z_1)|^q)$.
Then by the Stirling Formula, as $P \to +\infty$,

$$\sum_{n=N+1}^{P} \left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)} \right)^{2/q}$$

converges and

$$\sum_{n=N+1}^{+\infty} \left(\frac{\Gamma(n - q/2 - qh(x)/d)}{\Gamma(n)} \right)^{2/q} \sim \sum_{n=N+1}^{+\infty} \frac{1}{n^{1+2h(x)/d}} \sim \frac{d}{2h(x) N^{2h(x)/d}}.$$

Owing to (3.12) and to a monotone convergence argument,

$$\mathbb{E}\left(\sup_{M \geq N+1} |R_{N,M}(x)|^q \right) \leq A_{q,x} \sup_{n \geq N+1} D_{n-1,q}(h(x)) \left(\sum_{n=N+1}^{+\infty} \frac{1}{n^{1+2h(x)/d}} \right)^{q/2},$$

where for $0 < s < 1$ and $n > q/2 + qs/d - 1$,

$$D_{n,q}(s) = \frac{\Gamma(n + 1 - q/2 - qs/d) \, (n+1)^{q/2+qs/d}}{\Gamma(n+1)}.$$

As a consequence,

$$\mathbb{E}\left(\sup_{M \geq N+1} |R_{N,M}(x)|^q \right) \leq \frac{C_{q,x}}{N^{qh(x)/d}} \sup_{n \geq N} D_{n,q}(h(x)),$$

where

$$C_{q,x} = \left(\frac{d}{2h(x)} \right)^{q/2} A_{q,x}.$$

Therefore by an argument of dominated convergence,

$$\mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q) \leq \frac{C_{q,x}}{N^{qh(x)/d}} \sup_{n \geq N} D_{n,q}(h(x)).$$

Moreover

$$\frac{D_{n+1,q}(h(x))}{D_{n,q}(h(x))} = \exp(g(n+1))$$

where for $y > q/2 + qh(x)/d$,

$$g(y) = \frac{q(d + 2h(x))}{2d} \ln\left(1 + \frac{1}{y}\right) + \ln\left(1 - \frac{q(d+2h(x))}{2dy}\right).$$

A simple study of g shows that $g(x) < 0$. Therefore $(D_{n,q}(h(x)))_{n \geq [q/2+qh(x)/d]+1}$ is a non-increasing sequence. As a result,

$$\sup_{n \geq N} D_{n,q}(h(x)) = D_{N,q}(h(x)),$$

which concludes the proof. □

The next proposition studies the asymptotic of the mean square error. From this proposition, we deduce that the rate of convergence in (3.7) is optimal. For the sake of clearness, the proof of this proposition is given in appendix, see section 3.5.

Proposition 3.2.7. *Let $x \in \mathbb{R}^d \backslash \{0\}$. Then,*

$$\lim_{N \to +\infty} N^{2h(x)/d}\, \mathbb{E}\big(|Y_h(x) - Y_{h,N}(x)|^2\big) = \frac{C_{2,x}}{4},$$

where $C_{2,x}$ is defined by (3.9).

The following corollary gives a rate of almost sure convergence of $Y_{h,N}$. It is deduced from the proposition 3.2.5 and the Borel-Cantelli lemma.

Corollary 3.2.8. *Let $x \in \mathbb{R}^d$ and $H' < h(x)$. Then there exists a finite positive random variable C such that, almost surely,*

$$\forall N \geq 1,\ |Y_h(x) - Y_{h,N}(x)| \leq \frac{C}{N^{H'/d}}.$$

The two previous results study the approximation errors for a one-dimensional distribution. However the sample paths are approximated on the compact set K. Therefore errors in term of norms on E_K are studied.

Let $K \subset \mathbb{R}^d$ be a compact set and let us endow E_K with a L^p-norm:

$$\|g\|_{p,K} = \left\{ \int_K |g(\xi)|^p \, d\xi \right\}^{\frac{1}{p}}, \tag{3.13}$$

where $p \geq 1$. By applying the following proposition with $q = 2$ and $p = 2$, the mean integrated square error is evaluated.

Proposition 3.2.9. *Let $q \geq \max(p, 2)$, $m_K = \min_K h$ and $M_K = \max_K h$. Then for every integer $N \in \mathbb{N}^*$ such that $N \geq q/2 + qM_K/d + 1$,*

$$\mathbb{E}\Big(\|Y_h - Y_{h,N}\|_{p,K}^q \Big) \leq C_q \frac{D_{N,q}(m_K)}{N^{qm_K/d}},$$

where C_q does not depend on N and $D_{N,q}$ is defined by (3.8).

Remark 3.2.10. In fact proposition 3.2.9 holds with

$$C_q = 2^{1+3q/2}\, B_q^q \left(\frac{d}{m_K} \right)^{q/2} \mathbb{E}(|\Re(Z_1)|^q)\, d_K^{q/p} \max_{x \in K} \Big\{ (c_d\, \nu(\mathbb{C}))^{q/2 + qh(x)/d} \Big\},$$

where B_q is defined by (3.10) and d_K is the volume of the compact set K.

Notice that

$$\max_{x \in K} (c_d\, \nu(\mathbb{C}))^{qh(x)/d} = \begin{cases} (c_d\, \nu(\mathbb{C}))^{qM_K/d} & \text{if } c_d\, \nu(\mathbb{C}) \geq 1 \\ (c_d\, \nu(\mathbb{C}))^{qm_K/d} & \text{if } c_d\, \nu(\mathbb{C}) \leq 1. \end{cases}$$

Proof. Let $N \in \mathbb{N}^*$ such that $N > q/2 + qM_K - 1$. Since $q \geq p$, by the Hölder inequality,

$$\|Y_h - Y_{h,N}\|_{p,K} \leq {d_K}^{1/p-1/q} \|Y_h - Y_{h,N}\|_{q,K},$$

where d_K is the volume of K. Therefore

$$\mathbb{E}\Big(\|Y_h - Y_{h,N}\|_{p,K}^q\Big) \leq {d_K}^{q/p-1} \int_K \mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q)\, dx.$$

Then proposition 3.2.5 is not directly applied. However, in what follows the notation is the same as in the proof of proposition 3.2.5. Let us recall, see inequality (3.12), that

$$\mathbb{E}\Big(\max_{N+1\leq M\leq P} |R_{N,P}(x)|^q\Big) \leq A_{q,x}\left\{\sum_{n=N+1}^{P}\left(\frac{\Gamma(n-q/2-qh(x)/d)}{\Gamma(n)}\right)^{2/q}\right\}^{q/2},$$

where $A_{q,x} = 2^{2q+1} B_q^q \,(c_d\,\nu(\mathbb{C}))^{q/2+qh(x)/d}\, \mathbb{E}(|\Re(Z_1)|^q)$. Therefore since

$$n - q/2 - qm_K/d \geq N + 1 - q/2 - qh(x)/d \geq N + 1 - q/2 - qM_K/d \geq 2,$$

and since Γ is an increasing function on $[2, +\infty[$

$$\mathbb{E}\Big(\max_{N+1\leq M\leq P} |R_{N,P}(x)|^q\Big) \leq A_{q,x}\left\{\sum_{n=N+1}^{P}\left(\frac{\Gamma(n-q/2-qm_K/d)}{\Gamma(n)}\right)^{2/q}\right\}^{q/2}.$$

Furthermore the arguments used in the proof of the proposition 3.2.5 leads to

$$\mathbb{E}(|Y_h(x) - Y_{h,N}(x)|^q) \leq \left(\frac{d}{2m_K}\right)^{q/2} \sup_{x\in K} A_{q,x}\, \frac{D_{N,q}(m_K)}{N^{qm_K/d}}.$$

As a result,

$$\mathbb{E}\Big(\|Y_h - Y_{h,N}\|_{p,K}^q\Big) \leq C_q \frac{D_{N,q}(m_K)}{N^{qm_K/d}},$$

where $C_q = d_K^{q/p}\Big(\frac{d}{2m_K}\Big)^{q/2} \sup_{x\in K} A_{q,x}$.

□

Consequently, the Borel-Cantelli lemma yields the following corollary.

Corollary 3.2.11. *Let $H' < \min_K h$. Then there exists a finite positive random variable C such that, almost surely,*

$$\forall N \geq 1,\ \|Y_h - Y_{h,N}\|_{p,K} \leq \frac{C}{N^{H'/d}}.$$

In practice, when ν is a finite measure, one simulates the sample paths of Y_h which is equal in law to X_h owing to a generalized shot noise series representation. However when ν is not a finite measure, it becomes more complicated. The field X_h is split into two fields $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$. First $X_{\varepsilon,2}$ can be simulated as a generalized shot noise series. Then, the next section gives conditions to approximate $X_{\varepsilon,1}$ by a Multifractional Brownian Motion.

3.3 Normal Approximation

In this part ν is not supposed to be finite. Then let $\varepsilon > 0$ and let us split

$$X_h = X_{\varepsilon,1} + X_{\varepsilon,2}$$

into two random fields where

$$X_{\varepsilon,1}(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re(f(x,\xi)z) \mathbf{1}_{|z|<\varepsilon} \widetilde{N}(d\xi, dz), \tag{3.14}$$

and

$$X_{\varepsilon,2}(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re(f(x,\xi)z) \mathbf{1}_{|z|\geq\varepsilon} \widetilde{N}(d\xi, dz). \tag{3.15}$$

Let us consider the two independent Poisson random measures

$$N_{\varepsilon,1}(d\xi, dz) = \mathbf{1}_{|z|<\varepsilon} N(d\xi, dz)$$

and

$$N_{\varepsilon,2}(d\xi, dz) = \mathbf{1}_{|z|\geq\varepsilon} N(d\xi, dz).$$

Let $M_{\varepsilon,i}$ be the Lévy random measure associated with $N_{\varepsilon,i}$ by (1.14). Remark that $X_{\varepsilon,i}$ is a RHMLM associated with $M_{\varepsilon,i}$. Moreover $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$ are independent. Notice that the control measure $\nu_{\varepsilon,2}(dz) = \mathbf{1}_{|z|\geq\varepsilon}\, \nu(dz)$ of $M_{\varepsilon,2}$ is finite. Therefore $X_{\varepsilon,2}$ can be simulated as a generalized shot noise series, see section 3.2. It now remains to approximate $X_{\varepsilon,1}$. In [AR01], it is proposed to simulate the small jump part of a Lévy process by a Brownian Motion. Here proposition 3.3.1 gives sufficient conditions to approximate $X_{\varepsilon,1}$ by a Multifractional Brownian Motion. These conditions are closely related to those given in [AR01] for Lévy processes.

Let us introduce $\nu_{\varepsilon,1}(dz) = \mathbf{1}_{|z|<\varepsilon}\, \nu(dz)$ the control measure of $M_{\varepsilon,1}$ and

$$\sigma(\varepsilon) = \left(\int_{0<\rho<\varepsilon} \rho^2 \, \nu_\rho(d\rho) \right)^{1/2}. \tag{3.16}$$

The following proposition can be established under simpler conditions, see corollary 3.3.4. However in some simple cases, the assumptions of this proposition are satisfied whereas those of corollary 3.3.4 are not. An example that comes from [AR01] will be given below.

Proposition 3.3.1. *Suppose that for each $\kappa > 0$,*

$$\mathbf{H1} \quad \sigma(\kappa\sigma(\varepsilon) \wedge \varepsilon) \sim \sigma(\varepsilon) \qquad \text{as } \varepsilon \to 0_+$$

then

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{\varepsilon,1}(x)}{\sigma(\varepsilon)} \right)_{x \in \mathbb{R}^d} \overset{(d)}{=} (C_h(x) B_h(x))_{x \in \mathbb{R}^d}, \tag{3.17}$$

where the limit is in distribution for all finite-dimensional margins, B_h is a standard Multifractional Brownian Motion with multifractional function h and

$$C_h(x) = \left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(h(x)+1/2)}{h(x)\,\Gamma(2h(x))\sin(\pi h(x))\,\Gamma(h(x)+d/2)} \right)^{1/2}. \tag{3.18}$$

Let us recall that h is a locally β-Hölder function. Let $K \subset \mathbb{R}^d$ be a compact set, $m_K = \min_K h$ and $p_K = 1 + \left[\frac{d}{2\min(m_K,\beta)} \right]$. Then if **H1** *and*

$$\textbf{H2} \quad \exists \varepsilon_0 > 0,\ \exists C \in (0,+\infty),\ \forall \varepsilon \leq \varepsilon_0,\ \int_{0<\rho<\varepsilon} \rho^{2p_K}\,\nu_\rho(d\rho) \leq C\sigma^{2p_K}(\varepsilon)$$

are satisfied, the convergence (3.17) *is a convergence in distribution on the space E_K of continuous functions on K endowed with the topology of uniform convergence.*

For the sake of clearness, the proof of this proposition is given in appendix, see section 3.6.

Remark 3.3.2. When $d = 1$ and $m_K > 1/2$, $p_K = 1$ and **H2** is fulfilled.

Remark 3.3.3. Suppose that **H2** is satisfied for all closed balls of $\mathbb{R}^d$. Then under **H1**, the convergence (3.17) is a convergence in distribution on the space of continuous functions endowed with the topology of uniform convergence on compact sets.

As it is done in [AR01], a simpler convergence condition can be given. It compares the second moment of $\nu_{\varepsilon,1}(dz)$ to the level of truncation ε. Actually, as the level of truncation tends to zero, the dispersion $\sigma(\varepsilon)$ also converges to zero but slower.

Corollary 3.3.4. *If*

$$\textbf{H3} \quad \lim_{\varepsilon \to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty,$$

then the assumptions **H1** *and* **H2** *are satisfied.*

Proof. The comparison between **H3** and **H1** has already been done in [AR01]. Moreover for $k \geq 1$,

$$\int_{0<\rho<\varepsilon} \rho^{2k}\,\nu_\rho(d\rho) \leq \varepsilon^{2k-2}\sigma^2(\varepsilon).$$

Thanks to this inequality, **H3** implies **H2**.

□

It can be shown by the same way that **H2** is satisfied as soon as

$$\liminf_{\varepsilon \to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} > 0.$$

Let us give an example which satisfies the assumptions **H1** and **H2** of proposition 3.3.1 whereas it does not satisfy **H3**. In fact it is the Example 2.1 in [AR01].

Example 3.3.5. Let $(a_n)_{n\geq 1}$ be a decreasing sequence such that

$$\lim_{n\to+\infty} a_n = 0 \text{ and } \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 0.$$

Assume that $a_1 = 1$. Let ν_ρ be a Lévy measure on $(0, +\infty)$ such that

$$\sigma(\varepsilon) = \begin{cases} a_n & \text{for } \varepsilon \in]a_{n+1}, a_n] \\ 1 & \text{for } \varepsilon \geq 1. \end{cases}$$

First, the Lévy measure ν associated to ν_ρ by (1.17) satisfies (1.13). Moreover, since $\liminf_{\varepsilon\to 0_+} \frac{\sigma(\varepsilon)}{\varepsilon} = 1$, **H3** is not fulfilled whereas **H2** is. On the other hand, according to [AR01], **H1** is satisfied. As a result, proposition 3.3.1 can be applied.

Let us now discuss the rate of convergence in terms of Berry-Esseen bounds. For the sake of simplicity, $X_{\varepsilon,1}(x)$ is supposed to have a moment of order three, which allows us to apply the classical Berry-Esseen inequality. However a generalization of the classical Berry-Esseen inequality, see Theorem 5.7 in [Pet95], under a weaker moment assumption, allows us to extend the following results. Then an estimation in term of Berry-Esseen bounds is deduced from the next lemma which is a consequence of the classical Berry-Esseen inequality.

Lemma 3.3.6. *Let μ be an infinitely divisible law on $\mathbb{R}$ such that $\int_{\mathbb{R}} x\,\mu(dx) = 0$ and $\int_{\mathbb{R}} |x|^3\,\mu(dx) < +\infty$. Then*

$$\sup_{x\in\mathbb{R}} \left| \mu((-\infty, x)) - \int_{-\infty}^{x/\sigma} \mathrm{e}^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq 0.7975\,\sigma^{-3} \int_{\mathbb{R}} |x|^3\,\Lambda(dx),$$

where $\sigma = \left(\int_{\mathbb{R}} |x|^2\,\mu(dx)\right)^{1/2}$ and Λ is the Lévy measure of μ.

Proof. Let $(Z(t))_{t\geq 0}$ be a Lévy process such that μ is the law of $Z(1)$. Then proceed as in the proof of Theorem 3.1 page 487 in [AR01], i.e. write

$$Z(1) = \sum_{k=1}^{n} \left(Z\left(\frac{k}{n}\right) - Z\left(\frac{k-1}{n}\right) \right).$$

Hence $Z(1)$ is a sum of i.i.d real-valued random variables with mean zero and variance σ^2/n. Moreover, according to [AR01],

$$\lim_{n\to+\infty} n\mathbb{E}\left(\left| Z\left(\frac{1}{n}\right) \right|^3 \right) = \int_{\mathbb{R}} |x|^3\,\Lambda(dx).$$

Therefore the conclusion is given by the classical Berry-Esseen inequality. □

Then an estimation in term of Berry-Esseen bounds of the rate of convergence stated in proposition 3.3.1 can be given.

Proposition 3.3.7. *Let $K \subset \mathbb{R}^d$ be a compact set, $y' \in E'_K$ and assume that $\max_K h < 1 - d/6$. Then $\sup_{x\in K} |f(x,\xi)| \in L^3\big(\mathbb{R}^d\big)$ and*

$$\sup_{u\in\mathbb{R}} \Big|\mathbb{P}\Big(< y', X_{\varepsilon,1|_K} >\leq u\Big) - \mathbb{P}\Big(< y', \sigma(\varepsilon)(C_h B_h)_{|_K} >\leq u\Big)\Big| \leq C(y')\,\frac{m_3^3(\varepsilon)}{\sigma^3(\varepsilon)}$$

where $m_3^3(\varepsilon) = \int_{0<\rho<\varepsilon} \rho^3\,\nu_\rho(d\rho)$ and

$$C(y') = \frac{8\,A\displaystyle\int_{\mathbb{R}^d} \Big|< y', f(\cdot,\xi)_{|_K} >\Big|^3 d\xi}{3\left(\pi\displaystyle\int_{\mathbb{R}^d} \Big|< y', f(\cdot,\xi)_{|_K} >\Big|^2 d\xi\right)^{3/2}},$$

with $A = 0.7975$.

Proof. $< y', X_{\varepsilon,1|_K} >$ is a real-valued infinitely divisible random variable. Its Lévy measure Λ is the push forward of $\nu_{\varepsilon,1}$ by

$$(\xi, z) \longmapsto 2\Re\Big(< y', f(\cdot,\xi)_{|_K} > z\Big).$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left(\Big|< y', X_{\varepsilon,1|_K} >\Big|^2\right) &= 4\int_{\mathbb{R}^d\times\mathbb{C}} \Big|2\Re\Big(< y', f(\cdot,\xi)_{|_K} > z\Big)\Big|^2\,\mathbf{1}_{|z|<\varepsilon}\,d\xi\,\nu(dz)\\
&= 4\pi\,\sigma^2(\varepsilon)\int_{\mathbb{R}^d} \Big|< y', f(\cdot,\xi)_{|_K} >\Big|^2 d\xi\\
&= \mathrm{Var}\Big(< y', \sigma(\varepsilon)(C_h B_h)_{|_K} >\Big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}} |x|^3\,\Lambda(dx) &= 4\,m_3^3(\varepsilon)\int_0^{2\pi} |\cos\theta|^3\,d\theta\int_{\mathbb{R}^d} \Big|< y', f(\cdot,\xi)_{|_K} >\Big|^3 d\xi\\
&= \tfrac{64}{3}\,m_3^3(\varepsilon)\int_{\mathbb{R}^d} \Big|< y', f(\cdot,\xi)_{|_K} >\Big|^3 d\xi.
\end{aligned}$$

One concludes by applying the lemma 3.3.6. □

Remark 3.3.8. In proposition 3.3.7, the assumption $\max_K h < 1-d/6$ means that for every $x \in K$, $X_h(x)$ has moment of order three. However, the preceding proposition can be generalized to any RHMLM. In fact there always exists $\delta \in (0,1)$ such that $\max_K h < 1-d/2+d/(2+\delta)$. Then, for every $x \in K$, $\mathbb{E}\Big(|X_{\varepsilon,1}(x)|^{2+\delta}\Big) < +\infty$. As a consequence, a generalization of proposition 3.3.7 is obtained owing to Theorem 5.7 in [Pet95].

Hence, in the case where ν is not a finite measure, X_h can be approximated in law as soon as ν satisfies assumptions of proposition 3.3.1. In this case, according to propositions 3.3.1 and 3.2.2, an approximation is given by

$$Y_{\varepsilon,N}(x) = \sigma(\varepsilon)C(h(x))B_h(x) + 2\sum_{n=1}^{N} \Re\left(f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_{\varepsilon,n} \right),$$

where B_h, T_n, U_n and $Z_{\varepsilon,n}$ are independent. T_n and U_n are defined in section 3.2. We have that $(Z_{\varepsilon,n})_n$ are i.i.d. random variables with common law $\nu_{\varepsilon,2}(dz)/\nu_{\varepsilon,2}(\mathbb{C})$. It is supposed that $\nu_{\varepsilon,2}(\mathbb{C}) \neq 0$, which is the case for ε sufficiently small.

Then the approximations given in section 3.2 and section 3.3 are used in the next part to generate sample paths of X_h. First, examples of RHMLMs with finite control measure are given since it is the simplest case of simulation.

3.4 Simulation

3.4.1 Case of finite measure

Suppose here that $0 < \nu(\mathbb{C}) < +\infty$. Let us recall that

$$Y_{h,N}(x) = 2\sum_{n=1}^{N} \Re\left(f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_n \right),$$

with

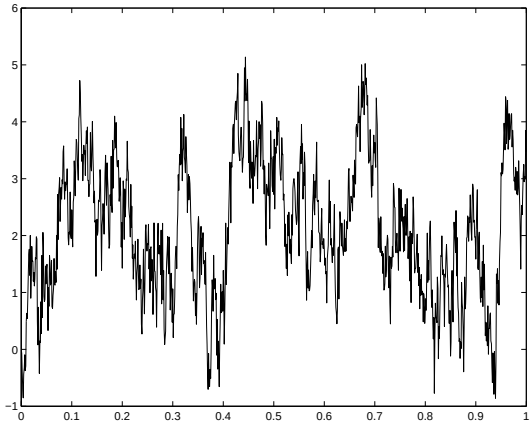
$$f(x,\xi) = \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}}.$$

Then, $Y_{h,N}$ converges almost surely to Y_h which is equal in law to X_h. Therefore the sample paths of Y_h are simulated and are approximated by $Y_{h,N}$.

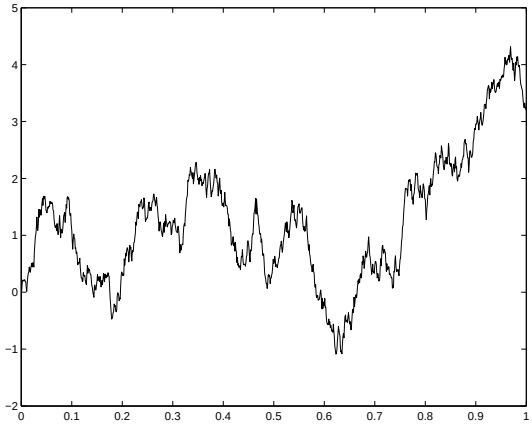
Let us present some examples. Suppose that ν is the uniform law on the unit circle of $\mathbb{C}$. Then $\nu(\mathbb{C}) = 1$.

First assume that h is constant to H, which means that X_h is a RHFLM. According to [BCI02], the sample paths of $X_H = X_h$ are locally Hölderian. Furthermore, the pointwise Hölder exponent does not vary along the trajectory and is almost surely equal to H. Figure 3.1 yields illustrations of these facts.

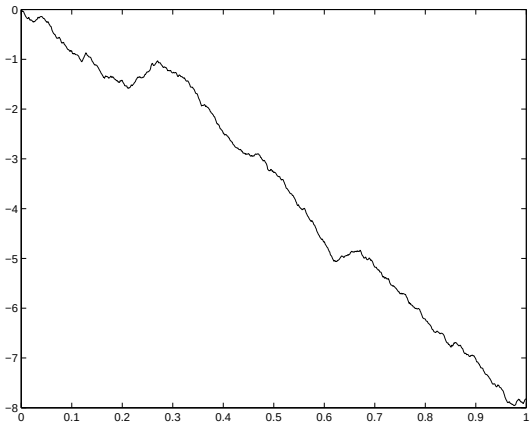
Let now $h : \mathbb{R}^d \longrightarrow]0,1[$ be a locally β-Hölder function. The pointwise Hölder exponent of a RHFLM is constant but for a general RHMLM it is not. More precisely, if $h(x) < \beta$, the pointwise Hölder exponent at point x of X_h is almost surely equal to $h(x)$. In figure 3.2, examples of RHMLMs in dimension $d = 1$ are given. Furthermore, figure 3.3 present an example of RHMLM in dimension $d = 2$ such that the function $h : \mathbb{R}^2 \longrightarrow]0,1[$ is periodic in its first variable x_1 and equal to a constant in its second variable x_2. One can observe that the regularity varies along the trajectory as h does. The greater $h(x)$ is, the smoother the trajectories are on a neighbourhood of x. When ν is a finite measure, simulations exhibit the same smoothness properties as the theoretical model.



$H = 0.3$

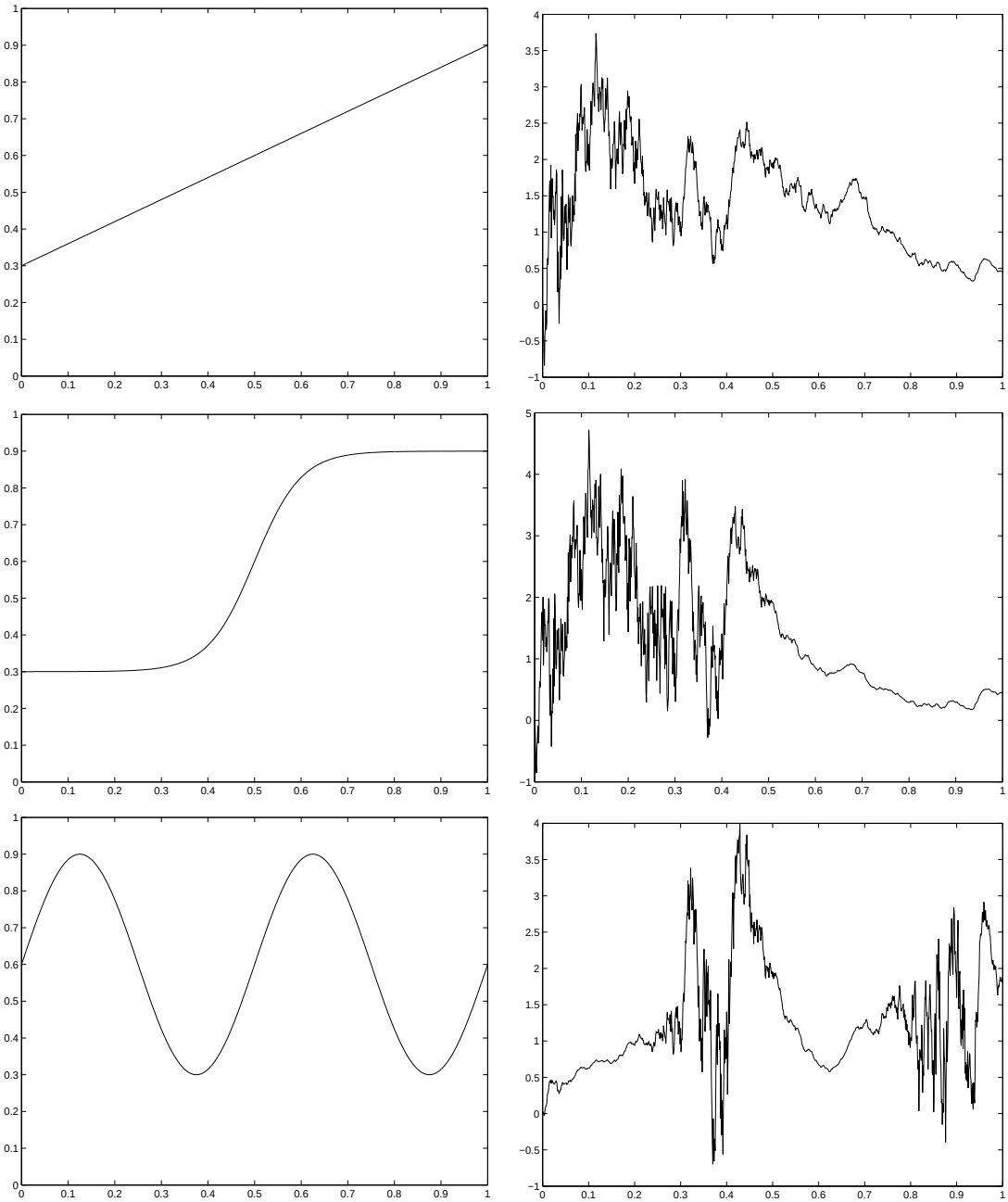


$H = 0.5$



$H = 0.8$

Figure 3.1: Examples of RHFLMs



Multifractional function h | Trajectory of a RHMLM X_h

Figure 3.2: Examples of RHMLMs

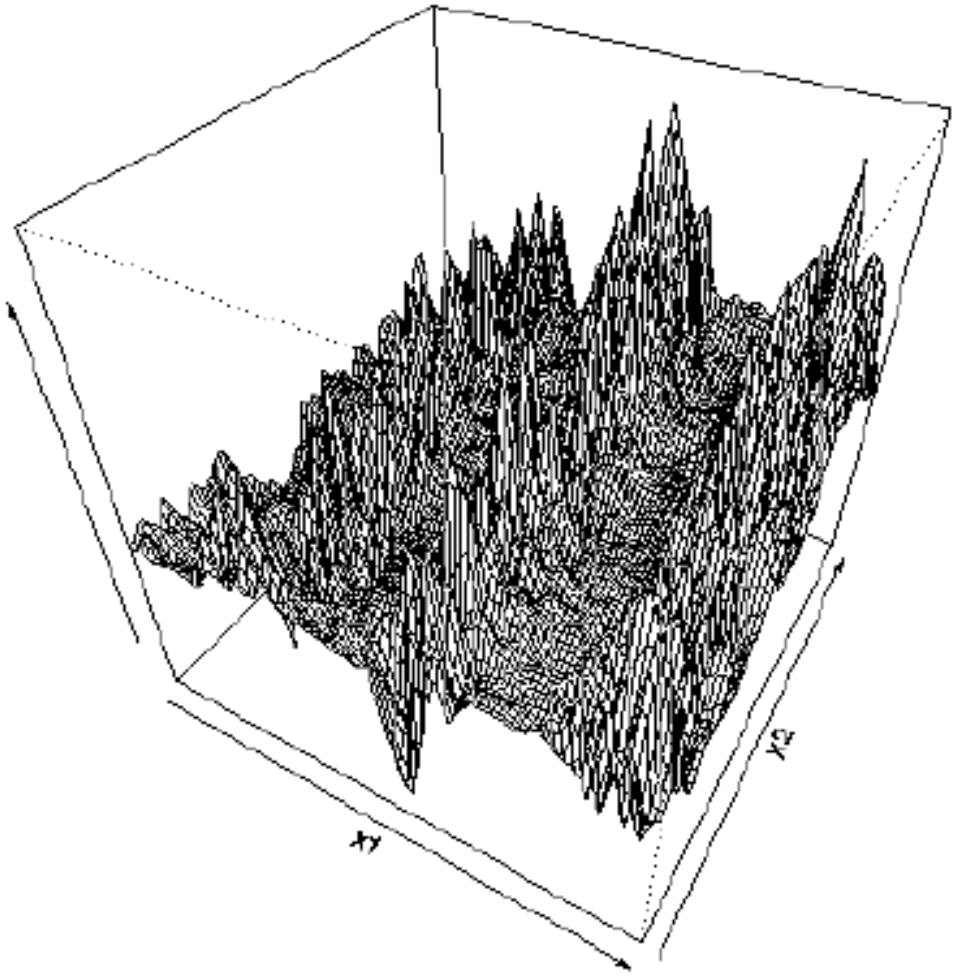


Figure 3.3: Example of RHMLM

3.4.2 Case of infinite measure

Suppose that ν is a measure which satisfies the assumptions of proposition 3.3.1. Hence X_h is split into two independent RHMLMs $X_{\varepsilon,1}$ and $X_{\varepsilon,2}$, defined by (3.14) and (3.15). Then for ε sufficiently small, $0 < \nu_{\varepsilon,2}(\mathbb{C}) < +\infty$.

Let $(Z_{\varepsilon,n})_n$ be a sequence i.i.d. random variables with common law $\nu_{\varepsilon,2}(dz)/\nu_{\varepsilon,2}(\mathbb{C})$ and B_h a standard Multifractional Brownian Motion. Let us recall that $(U_n)_n$ is a sequence of i.i.d uniform random variables on S^{d-1} and T_n is the nth arrival time of a Poisson process with intensity 1. Assume that B_h, $(Z_{\varepsilon,n})_n$, $(U_n)_n$ and $(T_n)_n$ are independent. As a consequence,

$$Y_{\varepsilon,N}(x) = \sigma(\varepsilon)C_h(x)B_h(x) + 2\sum_{n=1}^{N} \Re\left(f\left(x, \left(\frac{T_n}{c_d\, \nu(\mathbb{C})} \right)^{1/d} U_n \right) Z_{\varepsilon,n} \right),$$

approximates a field Y_h which is equal in law to X_h.

In order to simulate B_h, we use J.-F. Coeurjolly's programs which are available on

http://www-lmc.imag.fr/SMS/software.html.

More precisely, the Fractional Brownian Motion (case h constant) is generated by the method of circulant embedding, introduced in [WC94].

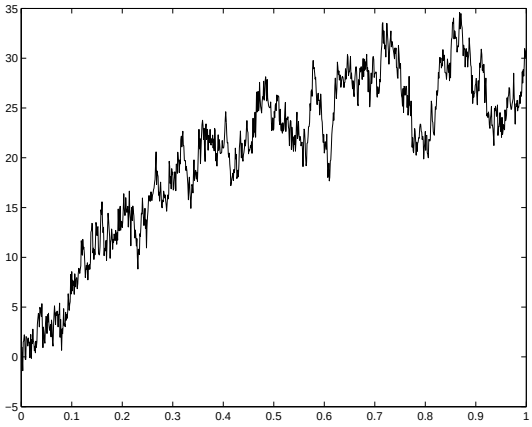
Let us now give some examples. Assume that $\nu_\rho(d\rho)$ is a truncated α-stable measure. More precisely, suppose that

$$\nu_\rho(d\rho) = \mathbf{1}_{0<\rho<1} \frac{d\rho}{\rho^{1+\alpha}},$$

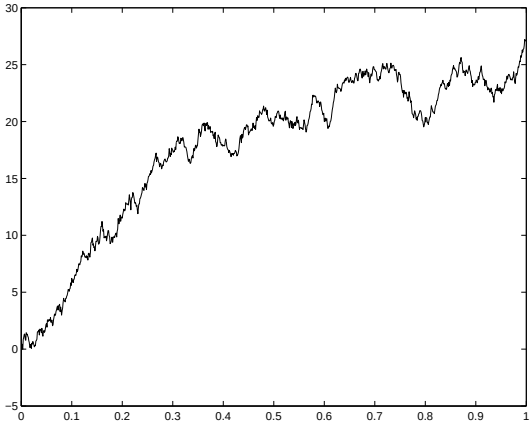
with $0 < \alpha < 2$. Let $0 < \varepsilon < 1$. Then

$$\nu_{\varepsilon,2}(\mathbb{C}) = \frac{2\pi(\varepsilon^{-\alpha} - 1)}{\alpha} \text{ and } \sigma^2(\varepsilon) = \frac{\varepsilon^{2-\alpha}}{2-\alpha}.$$

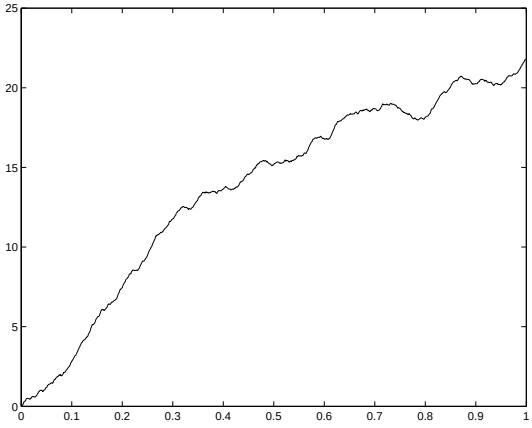
Figures 3.4 and 3.5 present some examples of RHMLMs with control measure ν. Like in the case of finite measures, the theoretical smoothness of the trajectory is observed. Actually, the local regularity of the sample paths follows the variation of the multifractional function h.



$H = 0.3$



$H = 0.5$



$H = 0.8$

Figure 3.4: Examples of RHFLMs

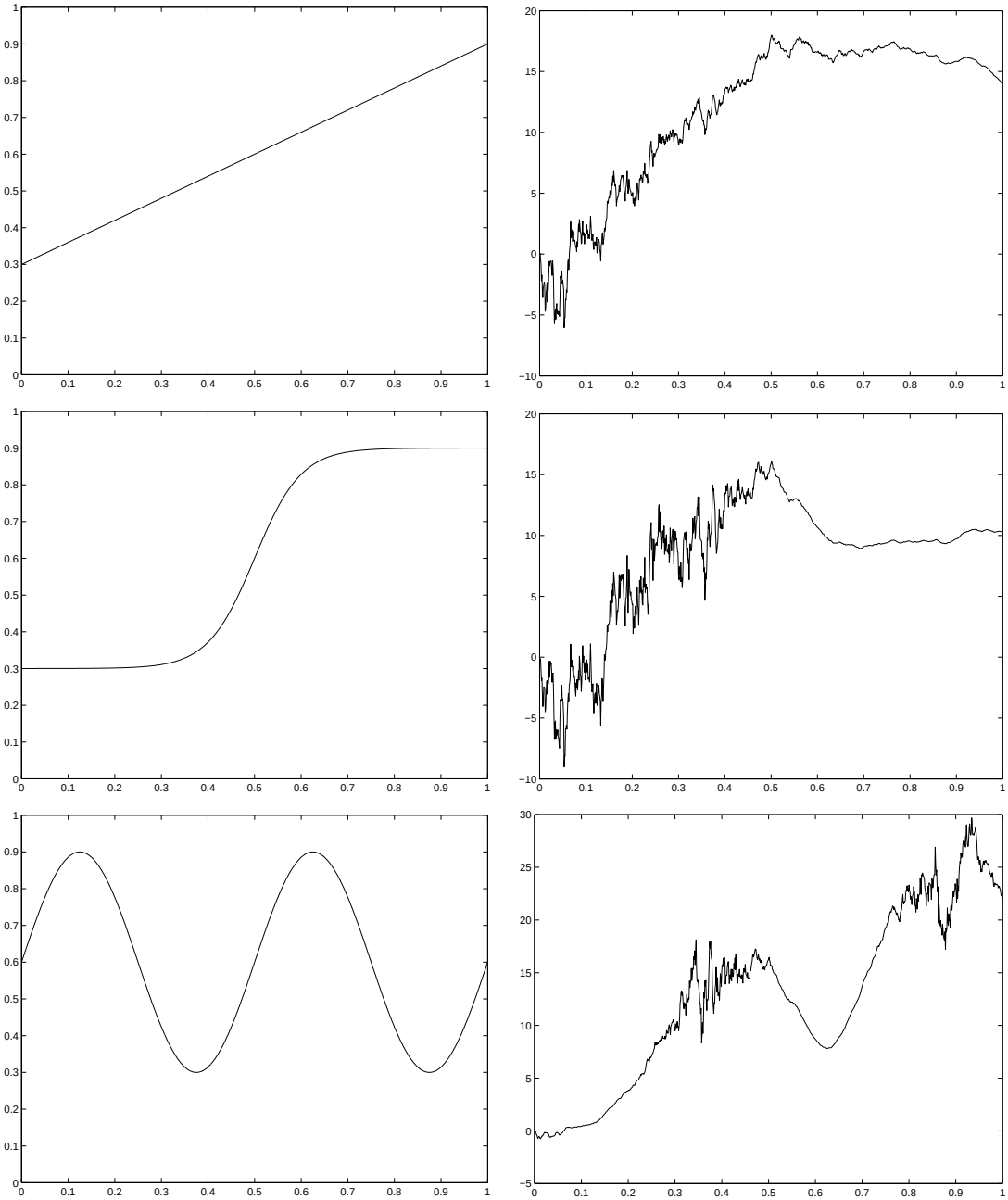


Figure 3.5: Examples of RHMLMs

3.5 Proof of proposition 3.2.7

In the case where $d \geq 2$, the proof of the proposition 3.2.7 is based on the study of the asymptotic of

$$K_n(y,b) = \int_0^{+\infty} \mathrm{e}^{ir^{1/d}y} r^{n-b} \mathrm{e}^{-r}\, dr, \tag{3.19}$$

where $y \in \mathbb{R}$ and $b > 0$. Else, if $d = 1$, the proof of the proposition 3.2.7 is simpler. The next lemma gives the asymptotic of K_n.

Lemma 3.5.1. *Let $d \geq 2$, $y \in \mathbb{R}$ and $b > 0$. Then*

$$\lim_{n\to+\infty} \frac{K_n(y,b)}{n^{n-b+1/2}\,\mathrm{e}^{-n}\mathrm{e}^{in^{1/d}y}} = \begin{cases} \sqrt{2\pi}\,\mathrm{e}^{-y^2/8} & \text{if } d = 2, \\ \sqrt{2\pi} & \text{if } d \geq 3. \end{cases}$$

Proof. Let

$$\widetilde{K_n}(y,b) = \frac{K_n(y,b)}{n^{n-b+1/2}\,\mathrm{e}^{-n}\mathrm{e}^{in^{1/d}y}}.$$

By applying the change of variables $s = \frac{r-n}{\sqrt{n}}$,

$$\widetilde{K_n}(y,b) = \mathrm{e}^{-in^{1/d}y} \int_{-\sqrt{n}}^{+\infty} \mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} \left(1+\frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}}\, ds.$$

A Taylor expansion leads to

$$\lim_{n\to+\infty} \left(1+\frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}} = \mathrm{e}^{-s^2/2}.$$

Moreover, using a Taylor expansion,

$$\lim_{n\to+\infty} \mathrm{e}^{-in^{1/d}y} \mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} = \begin{cases} \mathrm{e}^{isy/2} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$$

Consequently,

$$\lim_{n\to+\infty} \mathrm{e}^{-in^{1/d}y} \mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y} \left(1+\frac{s}{\sqrt{n}}\right)^{n-b} \mathrm{e}^{-s\sqrt{n}} = \begin{cases} \mathrm{e}^{isy/2}\mathrm{e}^{-s^2/2} & \text{if } d = 2, \\ \mathrm{e}^{-s^2/2} & \text{if } d \geq 3. \end{cases}$$

We do not directly apply a dominated convergence argument. Let us first write

$$\widetilde{K_n}(y,b) = \widetilde{K_{n,1}}(y,b) + \widetilde{K_{n,2}}(y,b),$$

where

$$\widetilde{K_{n,1}}(y,b) = \int_{\sqrt{n}}^{+\infty} g_n(y,b,s)\, ds$$

and

$$\widetilde{K_{n,2}}(y,b) = \int_{-\sqrt{n}}^{\sqrt{n}} g_n(y,b,s)\,ds$$

with $g_n(y,b,s) = \mathrm{e}^{-in^{1/d}y}\,\mathrm{e}^{in^{1/d}\left(1+s/\sqrt{n}\right)^{1/d}y}\left(1+\dfrac{s}{\sqrt{n}}\right)^{n-b}\mathrm{e}^{-s\sqrt{n}}$.

Study of $\widetilde{K_{n,1}}(y,b)$

Remark that

$$\begin{aligned}\left|\widetilde{K_{n,1}}(y,b)\right| &\leq \int_{\sqrt{n}}^{+\infty}\left(1+\frac{s}{\sqrt{n}}\right)^{n-b}\mathrm{e}^{-s\sqrt{n}}\,ds \\ &\leq \int_{\sqrt{n}}^{+\infty}\left(1+\frac{s}{\sqrt{n}}\right)^{n}\mathrm{e}^{-s\sqrt{n}}\,ds,\end{aligned}$$

since $b>0$ and $1+s/\sqrt{n}\geq 1$. Furthermore, integrating by parts leads to

$$\int_{\sqrt{n}}^{+\infty}\left(1+\frac{s}{\sqrt{n}}\right)^{n}\mathrm{e}^{-s\sqrt{n}}\,ds \leq \left(2^{n+1}-1\right)\frac{\mathrm{e}^{-n}}{\sqrt{n}}.$$

As a consequence,

$$\lim_{n\to+\infty}\widetilde{K_{n,1}}(y,b) = 0. \tag{3.20}$$

Study of $\widetilde{K_{n,2}}(y,b)$

Let us recall that

$$\lim_{n\to+\infty} g_n(y,b,s) = \begin{cases} \mathrm{e}^{isy/2}\mathrm{e}^{-s^2/2} & \text{if } d=2, \\ \mathrm{e}^{-s^2/2} & \text{if } d\geq 3. \end{cases}$$

Let $n>b$. Let us notice that for every $|x|<1$, $\ln(1+x)\leq x-x^2/6$. Then,

$$\begin{aligned}|g_n(y,b,s)|\mathbf{1}_{|s|<\sqrt{n}} &\leq \mathrm{e}^{-\frac{bs}{\sqrt{n}}}\mathrm{e}^{-\frac{n-b}{6n}s^2} \\ &\leq \mathrm{e}^{b}\mathrm{e}^{-\frac{n-b}{6n}s^2}\end{aligned}$$

since $\left|\frac{bs}{\sqrt{n}}\right|\leq b$. Then for n sufficiently large,

$$|g_n(y,b,s)|\mathbf{1}_{|s|<\sqrt{n}} \leq \mathrm{e}^{b}\mathrm{e}^{-\frac{s^2}{12}},$$

for every $s\in\mathbb{R}$. Therefore, using a dominated convergence argument, one concludes that

$$\lim_{n\to+\infty}\widetilde{K_{n,2}}(y,b) = \begin{cases} \displaystyle\int_{\mathbb{R}}\mathrm{e}^{isy/2}\mathrm{e}^{-s^2/2}\,ds & \text{if } d=2, \\ \displaystyle\int_{\mathbb{R}}\mathrm{e}^{-s^2/2}\,ds & \text{if } d\geq 3. \end{cases}$$

As a consequence,

$$\lim_{n\to+\infty} \widetilde{K_{n,2}}(y,b) = \begin{cases} \sqrt{2\pi}\,\mathrm{e}^{-y^2/8} & \text{if } d=2, \\ \\ \sqrt{2\pi} & \text{if } d\geq 3. \end{cases} \tag{3.21}$$

The conclusion is then given by (3.20) and (3.21). □

Let us now prove the proposition 3.2.7.

Proof. Let $x \in \mathbb{R}^d \backslash \{0\}$. In the following, the notation is the same as in the proof of proposition 3.2.5. Let us first recall that

$$\mathbb{E}\big(|R_{N,P}(x)|^2\big) = \mathbb{E}\left\{ \left(\sum_{n=N+1}^{P} \widetilde{\xi}_n(x) \right)^2 \right\},$$

where $\widetilde{\xi}_n(x) = 2\left| f\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})} \right)^{1/d} U_n \right) \right| \Re(Z_n)$.
Therefore, since $\left(\Re(Z_n)\right)_n$ is a sequence of i.i.d symmetric random variables and since this sequence is independent of $(U_n, T_n)_n$,

$$\mathbb{E}\big(|R_{N,P}(x)|^2\big) = \sum_{n=N+1}^{P} \mathbb{E}\Big(\widetilde{\xi}_n^{\,2}(x)\Big).$$

Moreover, $R_{N,P}(x)$ converges to $Y_h(x) - Y_{h,N}(x)$ in quadratic mean as $P \to +\infty$. Therefore,

$$\begin{aligned} \mathbb{E}\big(|Y_h(x) - Y_{h,N}(x)|^2\big) &= \sum_{n=N+1}^{+\infty} \mathbb{E}\Big(\widetilde{\xi}_n^{\,2}(x)\Big) \\ &= 4\,\mathbb{E}(\Re^2(Z_1)) \sum_{n=N+1}^{+\infty} \mathbb{E}\left\{ \left| f\left(x, \left(\frac{T_n}{c_d\,\nu(\mathbb{C})} \right)^{1/d} U_n \right) \right|^2 \right\}. \end{aligned}$$

Then, by definition of the sequence $(T_n, U_n)_{n\geq 1}$,

$$\mathbb{E}\big(|Y_h(x) - Y_{h,N}(x)|^2\big) = 4(c_d\nu(\mathbb{C}))^{1+2h(x)/d}\,\mathbb{E}\big(\Re^2(Z_1)\big) \sum_{n=N+1}^{+\infty} I_n\left(\frac{x}{(c_d\nu(\mathbb{C}))^{1/d}} \right), \tag{3.22}$$

where for every $v \in \mathbb{R}^d$,

$$I_n(v) = \int_{S^{d-1}} \int_0^{+\infty} \left| \mathrm{e}^{-ir^{1/d} v\cdot u} - 1 \right|^2 r^{n-2-2h(x)/d}\,\mathrm{e}^{-r}\,\frac{dr}{(n-1)!}\,\frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

Let $n > 1 + 2h(x)$. Let us remark that

$$I_n(v) = \frac{2\Gamma(n-1-2h(x)/d)}{\Gamma(n)} - \frac{2}{\Gamma(n)}\Re(J_n(v))$$

with

$$J_n(v) = \int_{S^{d-1}}\int_0^{+\infty} r^{n-2-2h(x)/d}\, \mathrm{e}^{ir^{1/d}v\cdot u - r}\, dr\, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

Let us assume in the following that $v \neq 0$.

Step 1 Case $d = 1$ Using the characteristic function of a Gamma-distribution, we obtain that

$$J_n(v) = \left(1+v^2\right)^{h(x)/d+(1-n)/2} \cos\left((n-1-2h(x)/d)\arctan v\right)\Gamma(n-1-2h(x)/d).$$

Hence, since $v \neq 0$,

$$I_n(v) \sim \frac{2\Gamma(n-1-2h(x)/d)}{\Gamma(n)}. \tag{3.23}$$

as $n \to +\infty$.

Step 2 Case $d \geq 2$ We prove that the equivalence (3.23) remains true for $d \geq 2$. It is then sufficient to prove that

$$\lim_{n\to+\infty} \frac{J_n(v)}{\Gamma(n-1-2h(x)/d)} = 0. \tag{3.24}$$

Remark that

$$J_n(v) = \int_{S^{d-1}} K_n(v\cdot u, 2+2h(x)/d)\, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})},$$

where K_n is defined by (3.19).
Since

$$\Gamma(n-1-2h(x)/d) \sim \sqrt{2\pi} n^{n-3/2-2h(x)/d}\mathrm{e}^{-n},$$

by applying the lemma 3.5.1,

$$\lim_{n\to+\infty} \frac{K_n(v\cdot u, 2+2h(x)/d)}{\Gamma(n-1-2h(x)/d)}\mathrm{e}^{-in^{1/d}v\cdot u} = l_d(v\cdot u),$$

where $l_d(y) = \begin{cases} \mathrm{e}^{-y^2/8} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$

Furthermore,

$$\frac{J_n(v)}{\Gamma(n-1-2h(x)/d)} = J_{n,1}(v) + J_{n,2}(v),$$

where

$$J_{n,1}(v) = \int_{S^{d-1}} \mathrm{e}^{in^{1/d}v\cdot u}\, l_d(v\cdot u)\, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}$$

and

$$J_{n,2}(v) = \int_{S^{d-1}} \left(\frac{K_n(v \cdot u, 2 + 2h(x)/d)}{\Gamma(n - 1 - 2h(x)/d)} \mathrm{e}^{-in^{1/d} v \cdot u} - l_d(v \cdot u) \right) \mathrm{e}^{in^{1/d} v \cdot u} \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}.$$

Let us first remark that by definition,

$$0 \leq \frac{|K_n(v \cdot u, 2 + 2h(x)/d)|}{\Gamma(n - 1 - 2h(x)/d)} \leq 1.$$

Then, a dominated convergence argument leads to

$$\lim_{n \to +\infty} J_{n,2}(y) = 0.$$

Furthermore, by rotational invariance of the measure $\sigma_{d-1}(du)$,

$$J_{n,1}(v) = \int_{S^{d-1}} \mathrm{e}^{in^{1/d} \|v\| e_1 \cdot u} \, l_d(\|v\| e_1 \cdot u) \, \frac{\sigma_{d-1}(du)}{\sigma_{d-1}(S^{d-1})}$$

with $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. As a consequence,

$$J_{n,1}(v) = \frac{\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \int_{-1}^{1} \mathrm{e}^{in^{1/d} \|v\| s} \, l_d(\|v\| s) \big(1 - s^2\big)^{(d-3)/2} \, ds.$$

Let us recall that $v \neq 0$. Therefore, by the Riemann-Lebesgue lemma,

$$\lim_{n \to +\infty} J_{n,1}(y) = 0.$$

As a result,

$$\lim_{n \to +\infty} \frac{J_n(v)}{\Gamma(n - 1 - 2h(x)/d)} = 0,$$

which implies (3.23).

Step 3 Conclusion By applying (3.23) and the Stirling formula,

$$I_n(v) \sim \frac{2}{n^{1+2h(x)/d}}.$$

Then, because of (3.22), as $N \to +\infty$,

$$\mathbb{E}\big(|Y_h(x) - Y_{h,N}(x)|^2\big) \sim 4(c_d \nu(\mathbb{C}))^{1+2h(x)/d} \, \mathbb{E}\big(\Re^2(Z_1)\big) \sum_{n=N+1}^{+\infty} \frac{2}{n^{1+2h(x)/d}},$$

which gives the conclusion. □

3.6 Proof of proposition 3.3.1

This part is devoted to the proof of the proposition 3.3.1.

Proof of proposition 3.3.1. Let $Y_\varepsilon(x) = \frac{X_{\varepsilon,1}(x)}{\sigma(\varepsilon)}$.

Convergence of the finite dimensional margins

Let $r \in \mathbb{N}^*$, $u = (u_1, \cdots, u_r) \in \left(\mathbb{R}^d\right)^r$ and $v = (v_1, \cdots, v_r) \in \mathbb{R}^r$. Then

$$\sum_{k=1}^{r} v_k Y_\varepsilon(u_k) = \int_{\mathbb{R}^d} \frac{g(\xi, u, v)}{\sigma(\varepsilon)} M_{\varepsilon,1}(d\xi),$$

where

$$g(\xi, u, v) = \sum_{k=1}^{r} v_k f(u_k, \xi)$$

Thus by (1.16),

$$\mathbb{E}\left(\sum_{k=1}^{r} v_k Y_\varepsilon(u_k)\right) = \exp\left(\Psi_\varepsilon(u, v)\right)$$

with

$$\Psi_\varepsilon(u, v) = \int_{\mathbb{R}^d \times \mathbb{C}} \left(\exp\left(\frac{2i\Re(g(\xi, u, v)z)}{\sigma(\varepsilon)} \right) - 1 - \frac{2i\Re(g(\xi, u, v)z)}{\sigma(\varepsilon)} \right) d\xi \, \nu_{\varepsilon,1}(dz).$$

Then by rotational invariance of ν,

$$\Psi_\varepsilon(u, v) = \int_{\mathbb{R}^d \times [0, 2\pi]} I_\varepsilon(2|g(\xi, u, v)| \cos(\theta)) \, d\xi \, d\theta,$$

where for every $y \in \mathbb{R}$,

$$I_\varepsilon(y) = \int_0^{+\infty} \left(\mathrm{e}^{i\frac{y\rho}{\sigma(\varepsilon)}} - 1 - i\frac{y\rho}{\sigma(\varepsilon)} \right) \mathbf{1}_{0<\rho<\varepsilon} \, \nu_\rho(d\rho).$$

Under the assumption **H1**, see [AR01],

$$\lim_{\varepsilon \to 0_+} I_\varepsilon(y) = \frac{-y^2}{2}.$$

Moreover, since

$$\forall y \in \mathbb{R}, \, |I_\varepsilon(y)| \leq \frac{y^2}{2},$$

a dominated convergence argument yields

$$\begin{aligned} \lim_{\varepsilon \to 0_+} \Psi_\varepsilon(u, v) &= -2\pi \int_{\mathbb{R}^d} \left| \sum_{k=1}^{r} \frac{v_k \left(\mathrm{e}^{-iu_k \cdot \xi} - 1 \right)}{\|\xi\|^{h(u_k) + d/2}} \right|^2 d\xi \\ &= -\tfrac{1}{2} \mathrm{Var}\left(\sum_{k=1}^{r} v_k C_h(u_k) B_h(u_k) \right), \end{aligned}$$

where B_h is a standard Multifractional Brownian Motion and where for every $x \in \mathbb{R}^d$,

$$C_h(x) = \left(4\pi \int_{\mathbb{R}^d} \frac{\left| \mathrm{e}^{-ie_1 \cdot \xi} - 1 \right|^2}{\|\xi\|^{d+2h(x)}} \, d\xi \right)^{1/2},$$

with $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$.

Let us now prove the formula (3.18) i.e.

$$C_h(x) = \left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(h(x)+1/2)}{h(x)\,\Gamma(2h(x))\sin(\pi h(x))\,\Gamma(h(x)+d/2)} \right)^{1/2},$$

which will conclude the proof of the convergence of the finite dimensional margins. In fact, when $d = 1$, this formula is already known, see pages 328-329 in [ST94]. Suppose that $d \geq 2$. Then, thanks to polar coordinates,

$$C_h^2(x) = \sigma_{d-2}\big(S^{d-2}\big) \int_{\mathbb{R}} \frac{\left| \mathrm{e}^{-ir} - 1 \right|^2}{r^{1+2h(x)}} \, dr \int_0^{\pi/2} (\cos\theta)^{2h(x)} (\sin\theta)^{d-2} \, d\theta.$$

By applying the formula (3.18) for $d = 1$,

$$C_h^2(x) = \frac{\pi^{(d+3)/2} B(H+1/2, (d-1)/2)}{h(x)\Gamma(2h(x))\sin(\pi h(x))\Gamma((d-1)/2)},$$

where B is the Beta-function. Then, the formula (3.18) is deduced from the relationship between the Beta-function and the Gamma-function.

Tightness Now assume that **H1** and **H2** are satisfied. Let us prove that the family $\big\{(Y_\varepsilon(x))_{x\in K},\, \varepsilon > 0\big\}$ is tight in E_K. The field Y_ε is split into two fields $Y_\varepsilon = Y_\varepsilon^+ + Y_\varepsilon^-$ where

$$Y_\varepsilon^+(x) = \frac{1}{\sigma(\varepsilon)} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1 - P_n(-ix\cdot\xi)\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{h(x)+d/2}} \, M_{\varepsilon,1}(d\xi)$$

and

$$Y_\varepsilon^-(x) = \frac{1}{\sigma(\varepsilon)} \int_{\|\xi\|\leq 1} \frac{P_n(-ix\cdot\xi)}{\|\xi\|^{h(x)+d/2}} \, M_{\varepsilon,1}(d\xi)$$

with $n \in \mathbb{N}^*$ such that $n > d/2$ and

$$P_n(t) = \sum_{k=1}^{n} \frac{t^k}{k!}.$$

According to chapter 2, the sample paths of Y_ε^+ and Y_ε^- are continuous. Therefore to prove the tightness of $(Y_\varepsilon)_\varepsilon$ in E_K, it is sufficient to prove the tightness of $(Y_\varepsilon^+)_\varepsilon$ and $(Y_\varepsilon^-)_\varepsilon$. Since Y_ε^+ has moments of every order, the tightness of $(Y_\varepsilon^+)_{\varepsilon>0}$ is shown owing to the Kolmogorov criterion.

Step 1: Tightness of $(Y_\varepsilon^+)_\varepsilon$
Let $x_0 \in K$. Notice that

$$\mathbb{E}\Big(\big|Y_\varepsilon^+(x_0)\big|^2\Big) = \mathbb{E}\Big(\big|Y_1^+(x_0)\big|^2\Big) < +\infty.$$

As a result, $(Y_\varepsilon^+(x_0))_{\varepsilon>0}$ is a tight family of random variables.
Moreover by proposition 2.2 in [BCI02],

$$\big\|Y_\varepsilon^+(u) - Y_\varepsilon^+(v)\big\|_{2p_K}^{2p_K} = \sum_{m=1}^{p_K} (2\pi)^m \sum_{L_m} \prod_{q=1}^{m} \frac{(2l_q)!\,\|g_n(u,v,\cdot)\|_{2l_q}^{2l_q}\, m_{2l_q}^{2l_q}(\varepsilon)}{\sigma^{2l_q}(\varepsilon) l_q!}, \tag{3.25}$$

where $m_{2l_q}^{2l_q}(\varepsilon) = \int_{0<\rho<\varepsilon} \rho^{2l_q}\, \nu_\rho(d\rho)$, $g_n(u,v,\xi) = g_n^+(u,\xi) - g_n^+(v,\xi)$ with

$$g_n^+(y,\xi) = \frac{\mathrm{e}^{-iy\cdot\xi} - 1 - P_n(-iy\cdot\xi)\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{h(y)+d/2}},$$

and where $\sum_{L_m}$ stands for the sum over the set of partitions L_m of $\{1,\ldots,2p_K\}$ in m subsets K_q such that the cardinality of K_q is $2l_q$ with $l_q \geqslant 1$.
Moreover, by lemmas 2.2.1 and 2.2.2 in chapter 2, there exists $C>0$ such that

$$\forall (u,v) \in K^2,\ \|g_n(u,v,\cdot)\|_{2l_q}^{2l_q} \leq C\|u-v\|^{2l_q \min(m_K,\beta)}. \tag{3.26}$$

Let us now study $\frac{m_{2l_q}^{2l_q}(\varepsilon)}{\sigma^{2l_q}(\varepsilon)}$ for $1 \leq l_q \leq p_K$. Since

$$\frac{1}{2p_K} \leq \frac{1}{2l_q} \leq \frac{1}{2},$$

there exists $\theta \in [0,1]$ such that

$$\frac{1}{2l_q} = \frac{\theta}{2} + \frac{1-\theta}{2p_K}.$$

Therefore by the Hölder inequality,

$$\left(\int_{0<\rho<\varepsilon} \rho^{2l_q}\, \nu_\rho(d\rho)\right)^{\frac{1}{2l_q}} \leq \left(\int_{0<\rho<\varepsilon} \rho^2\, \nu_\rho(d\rho)\right)^{\frac{\theta}{2}} \left(\int_{0<\rho<\varepsilon} \rho^{2p_K}\, \nu_\rho(d\rho)\right)^{\frac{1-\theta}{2p_K}}.$$

Consequently by **H2**,

$$m_{2l_q}(\varepsilon) \leq C^{1-\theta}\sigma(\varepsilon). \tag{3.27}$$

Then owing to (3.25), (3.26) and (3.27), there exists $C>0$ such that

$$\forall \varepsilon \leq \varepsilon_0,\ \forall (u,v) \in K^2,\ \big\|Y_\varepsilon^+(u) - Y_\varepsilon^+(v)\big\|_{2p_K}^{2p_K} \leq C\|u-v\|^{2p_K \min(m_K,\beta)}.$$

As $2p_K \min(m_K,\beta) > d$, $(Y_\varepsilon^+)_{\varepsilon\leq\varepsilon_0}$ is a tight family in E_K.

Step 2: Tightness of $(Y_\varepsilon^-)_\varepsilon$

Let $x_0 \in K$. $\mathbb{E}\Big(|Y_\varepsilon^-(x_0)|^2\Big)$ does not depend on ε and is finite, which gives the tightness of $(Y_\varepsilon^-(x_0))_{\varepsilon>0}$. For the sake of clearness, for $\alpha = (\alpha_1 \cdots, \alpha_d) \in \mathbb{N}^d$ and $z = (z_1 \cdots, z_d) \in \mathbb{C}^d$, let

$$|\alpha| = \sum_{j=1}^{d} \alpha_j \text{ and } z^\alpha = \prod_{j=1}^{d} {z_j}^{\alpha_j}.$$

For $u \in \mathbb{R}^d$ and $0 < y < 1$,

$$Z_{\varepsilon,n}(u,y) = \frac{1}{\sigma(\varepsilon)} \int_{\|\xi\|\le 1} \frac{P_n(-iu\cdot\xi)}{\|\xi\|^{y+d/2}} M_{\varepsilon,1}(d\xi).$$

Therefore $Y_\varepsilon^-(u) = Z_{\varepsilon,n}(u, h(u))$. On the other hand,

$$Z_{\varepsilon,n}(u,y) = \sum_{1\le|\alpha|\le n} C_\alpha u^\alpha\, Y_\varepsilon^\alpha(y).$$

where for $\alpha = (\alpha_1 \cdots, \alpha_d) \in \mathbb{N}^d$,

$$Y_\varepsilon^\alpha(y) = \frac{1}{\sigma(\varepsilon)} \int_{\|\xi\|\le 1} \frac{(-i\xi)^\alpha}{\|\xi\|^{y+d/2}} M_{\varepsilon,1}(d\xi).$$

Then

$$\begin{aligned} |Y_\varepsilon^-(u) - Y_\varepsilon^-(v)| &\le |Z_{\varepsilon,n}(u,h(u)) - Z_{\varepsilon,n}(v,h(u))| + |Z_{\varepsilon,n}(v,h(u)) - Z_{\varepsilon,n}(v,h(v))| \\ &\le \sum_{1\le|\alpha|\le n} |C_\alpha Y_\varepsilon^\alpha(h(u))||u^\alpha - v^\alpha| + \sum_{1\le|\alpha|\le n} |C_\alpha v^\alpha||Y_\varepsilon^\alpha(h(u)) - Y_\varepsilon^\alpha(h(v))|. \end{aligned}$$

Since there exists $C > 0$ such that for every $(u,v) \in K^2$, $|u^\alpha - v^\alpha| \le C\|u - v\|$ and $|v^\alpha| \le C$, then

$$\big|Y_\varepsilon^-(u) - Y_\varepsilon^-(v)\big| \le C\left(\|u-v\| \sum_{1\le|\alpha|\le n} |Y_\varepsilon^\alpha(h(u))| + \sum_{1\le|\alpha|\le n} |Y_\varepsilon^\alpha(h(u)) - Y_\varepsilon^\alpha(h(v))|\right)$$

Therefore by continuity of h, to obtain the tightness of $(Y_\varepsilon^-)_\varepsilon$, it is sufficient to show that for each $\alpha \in \mathbb{N}^d$ such that $1 \le |\alpha| \le n$,

1. $(Y_\varepsilon^\alpha)_\varepsilon$ is tight in the space of the continuous functions on $[m_K, M_K]$, where $M_K = \max_K h$.
2. the family $\big(\sup_{[m_K,M_K]} |Y_\varepsilon^\alpha(y)|\big)_\varepsilon$ is tight.

According to [Bil68], 2 is a consequence of 1. Moreover,

$$\mathbb{E}\big(|Y_\varepsilon^\alpha(m_K)|^2\big) = \mathbb{E}\big(|Y_1^\alpha(m_K)|^2\big) < +\infty$$

and by a Taylor expansion,

$$\forall \varepsilon > 0,\ \forall (y,y') \in [m_K, M_K]^2,\ \mathbb{E}\Big(|Y_\varepsilon^\alpha(y) - Y_\varepsilon^\alpha(y')|^2\Big) \le C|y-y'|^2,$$

which by the Kolmogorov criterion gives 1 and concludes the proof. □

Chapter 4

Fields with exceptional tangent fields

In this chapter, we propose to define a lass field which has a special local structure at $x = 0$. More precisely, the behaviours of this field at $x = 0$ and at $x \neq 0$ are very different from each other.

4.1 Introduction

The lass property is a local property which gives the behaviour of a field at each point. As already noticed, tangent fields, in the sense of the lass property, are self-similar. Furthermore, general properties of tangent fields are given in [Fal02] in the framework of continuous fields and in [Fal03] in the case of processes with jumps. In particular, in these frameworks, for almost all point x, a tangent field at point x must have stationary increments. In addition, many lass fields has been studied and their local structure identified. The most famous lass Gaussian field which generalized the FBM is certainly the Real Harmonizable Multifractional Brownian Motion, in short MBM, introduced independently in [BJR97] and in [PLV96]. However, many other examples of lass Gaussian fields have been studied, see for instance [ALV00, BBCI00, BCI98, BCIJ98]. All these fields share many properties with the FBM. Furthermore, at each point x, the tangent field is a FBM. The lass Gaussian fields introduced in [BE03] do not have FBM as tangent field; however, these Gaussian fields have stationary increments and then at each point, the tangent field is the same Gaussian field. In addition, Real Harmonizable Fractional Lévy Motions are in general non-Gaussian lass fields with FBM as tangent field, see [BCI02]. In the same way, Real Harmonizable Multifractional Lévy Motions remains lass fields with FBM as tangent field at each point. Furthermore, the lass fields studied in [BCI04] have stationary increments and then the same tangent field at each point. However, this tangent field is a Fractional Stable Motion and and is not Gaussian. Furthermore, for most of the previous fields, the trajectories regularity is known. In general, these fields are locally Hölder continuous. Moreover, [BCI03] gives sufficient conditions to evaluate the Hausdorff dimension of the graphs of a field. In addition, these conditions are linked to the trajectories regularity and the lass property. Furthermore, in [BCI03], the Hausdorff dimension of the graph is given for some of the above-mentioned fields.

However, in all encountered examples, the local behaviour is the same at each point. Also, in this chapter, we propose to define a field $X_{H;\beta}$ with a special behaviour at $x = 0$. The field $X_{H,\beta}$ is a non-Gaussian lass field which has finite second order moment. More precisely, its tangent field at $x \neq 0$ is a FBM. Furthermore, its tangent field at $x = 0$ depends on the value of β. Especially, when $\beta > d/\alpha$, it is a Fractional Stable Motion, in short de RHFSM. Then, in this case, the behaviour of $X_{H,\beta}$ at $x = 0$ is very different from the one at $x \neq 0$. In fact, even if $X_{H,\beta}$ has finite second order moments, its tangent field at $x = 0$ does not. Hence, the tangent field is of the same kind at each point x except at $x = 0$. Considering this, 0 is an exceptional point for the field $X_{H,\beta}$. In addition, when $\beta \leq d/\alpha$, the field $X_{H,\beta}$ remains lass at $x = 0$ but in most cases the tangent field is a Gaussian field. However, in general, the tangent field at $x = 0$ does not have stationary increments and then is not a FBM. Also, in general, when $\beta \leq d/\alpha$, for one point the tangent field does not have stationary increments.

Like RHFLMs, $X_{H,\beta}$ is defined as a stochastic integral. The definition of $X_{H,\beta}$ and of RHFLMs are very similar. Then let us first recall that a RHFLM Z_H with index H is defined as follows:

$$Z_H(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} z \right) \widetilde{N}(d\xi, dz), \tag{4.1}$$

where $\|\xi\|$ is the Euclidean norm of ξ and $\widetilde{N}(d\xi, dz)$ is the compensated Poisson random measure associated with the Poisson random measure $N(d\xi, dz)$ whose intensity measure is $n(d\xi, dz) = d\xi\, \nu(dz)$. Moreover, the previous stochastic integral is defined as soon as

$$\int_{\mathbb{C}} |z|^2 \, \nu(dz) < +\infty.$$

However, in this chapter, $\nu(dz)$ does not have a second order moment and then Z_H may not be defined. Actually, $\nu(dz)$ is a rotationally invariant measure associated with a symmetric α-stable measure. More precisely, let P be the map

$$P\big(\rho \mathrm{e}^{i\theta}\big) = (\theta, \rho) \in [0, 2\pi) \times (0, +\infty).$$

Then, throughout this chapter,

$$P(\nu(dz)) = d\theta \, \frac{d\rho}{\rho^{1+\alpha}}, \tag{4.2}$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$, $d\rho$ is the Lebesgue measure on $(0, +\infty)$ and $0 < \alpha < 2$. Then, $\nu(dz)$ does not have any finite moment. Thus, we introduce a function ψ such that

$$\int_{\mathbb{C}} |z\, \psi(|z|)|^2 \, \nu(dz) < +\infty.$$

Hence, the field $X_{H,\beta}$ is defined by

$$X_{H,\beta}(x) = 2 \int_{\mathbb{R}^d \times \mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} z \right) \psi\Big(\|x\|^\beta |z| \Big) \, \widetilde{N}(d\xi, dz).$$

In fact, the function ψ has a compact support and then allows us to truncate the support of the measure $\nu(dz)$. Moreover, the truncation of the support in the definition of $X_{H,\beta}(x)$ depends on x. In addition, if $X_{H,\beta}$ were a Gaussian field, i.e. if $N(d\xi, dz)$ were a Gaussian measure with intensity $n(d\xi, dz)$, it would be self-similar with index $H - \beta(1 - \alpha/2)$ by homogeneity of the integrand and of the mean measure $n(d\xi, dz)$. Furthermore, following [Ros89], the invariance of the mean measure $n(d\xi, dz)$ by a map L implies an invariance in law of the Poisson random measure $N(d\xi, dz)$ i.e.

$$N(d\xi, dz) \overset{(d)}{=} N \circ L^{-1}(d\xi, dz).$$

In addition, for every $\varepsilon > 0$, the mean measure $n(d\xi, dz)$ is invariant by the map

$$L(\xi, z) = \big(\varepsilon\xi, \varepsilon^{d/\alpha} z\big).$$

Hence, by applying the changes of variable $\lambda = \varepsilon\xi$ and $z' = \varepsilon^{d/\alpha} z$,

$$X_{H,\beta}(\varepsilon x) \overset{(d)}{=} 2\,\varepsilon^{H+d/2-d/\alpha} \int_{\mathbb{R}^d} \Re\left(\frac{\mathrm{e}^{-ix\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} z'\right) \psi\Big(\varepsilon^{\beta-d/\alpha} \|x\|^{\beta} |z'|\Big)\, \widetilde{N}(d\lambda, dz'). \quad (4.3)$$

Also, the behaviour of $X_{H,\beta}$ at $x = 0$ is not the same whether $\beta = d/\alpha$, $\beta > d/\alpha$ or $\beta < d/\alpha$. In particular, if $\beta = d/\alpha$, the field $X_{H,\beta}$ is self-similar with index $H + d/2 - d/\alpha$. Furthermore, when $\beta > d/\alpha$, the truncation due to the function ψ heuristically disappears in (4.3) as ε tends to 0. Then, when $\beta > d/\alpha$, the tangent field at $x = 0$ is a RHFSM with index $H + d/2 - d/\alpha$. Remark that the index does not depend on the value of β. However, when $\beta < d/\alpha$, $X_{H,\beta}$ remains a lass field at $x = 0$ with index $H - \beta(1 - \alpha/2)$ as it would be expected in a Gaussian framework. Moreover, in this case, the tangent field is Gaussian but its increments are not stationary as soon as $\beta \neq 0$.

The next section is devoted to the construction of $X_{H,\beta}$. Then, in section 4.3, the lass property is established. Moreover, an asymptotic self-similarity property at infinity is stated. In section 4.4, the trajectories regularity and the Hausdorff dimension of the graph are studied. Thus, section 4.5 is devoted to the study of the Gaussian field $W_{H,\beta}$ which has the same covariance as $X_{H,\beta}$. In addition, some generalizations of the field $X_{H,\beta}$ are introduced in section 4.6. Finally, some remarks on the construction are given.

4.2 Definition

Let $N(d\xi, dz)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with mean measure

$$n(d\xi, dz) = \mathbb{E}(N(d\xi, dz)) = d\xi\, \nu(dz),$$

where $\nu(dz)$ is defined by (4.2). Then, let $\widetilde{N} = N - n$ be its Poisson compensated random measure and $\varphi \in L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$ for the measure $n(d\xi, dz)$. Let us recall that

the map $\varphi \mapsto \int \varphi \, d\widetilde{N}$ is an isometry, which means that

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d\times\mathbb{C}} \varphi(\xi,z)\,\widetilde{N}(d\xi,dz)\right|^2\right] = \int_{\mathbb{R}^d\times\mathbb{C}} |\varphi(\xi,z)|^2\, n(d\xi,dz), \tag{4.4}$$

Furthermore, the characteristic function of $\int \varphi \, d\widetilde{N}$ is known. In particular, suppose that φ is a real-valued function, then $\int \varphi \, d\widetilde{N}$ is a real-valued random variable and for every $u \in \mathbb{R}$,

$$\mathbb{E}\left[\mathrm{e}^{iu\int \varphi\, d\widetilde{N}}\right] = \exp\left[\int_{\mathbb{R}^d\times\mathbb{C}} \left[\exp\left(iu\varphi(\xi,z)\right) - 1 - iu\varphi(\xi,z)\right] d\xi\, \nu(dz)\right]. \tag{4.5}$$

Notation Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$-function such that

$$\psi(u) = \begin{cases} 1 & \text{if } |u| \leq \frac{1}{2} \\ 0 & \text{if } |u| \geq 1. \end{cases} \tag{4.6}$$

Then, in particular, ψ and its derivatives are with compact support and then bounded.

Let us now define the field $X_{H,\beta}$ which is a real-valued centered field.

Definition 4.2.1. Let $\beta \in \mathbb{R}$. Then, for every $x \in \mathbb{R}^d$,

$$X_{H,\beta}(x) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}} z\right) \psi\Big(\|x\|^\beta |z|\Big)\, \widetilde{N}(d\xi,dz)$$

with convention $0^0 = 1$ and if $\beta < 0$ and $x = 0$, $\psi\Big(\|x\|^\beta|z|\Big) = 0$.

Remark 4.2.2. When $\beta = 0$, $X_{H,\beta}$ is a RHFLM with control measure the push-forward of $\nu(dz)$ by the map $z \mapsto z\psi(|z|)$.

Since the map $\varphi \mapsto \int \varphi \, d\widetilde{N}$ is an isometry from $L^2\big(\mathbb{R}^d \times \mathbb{C}\big)$, for the measure $n(d\xi, dz)$, onto $L^2(\Omega)$, $X_{H,\beta}$ is a field with finite second order moments and one can evaluate its covariance function.

Proposition 4.2.3. *Let γ be the covariance function of $X_{H,\beta}$, i.e.*

$$\gamma(x,y) = \mathbb{E}(X_{H,\beta}(x)X_{H,\beta}(y)),$$

for every $(x,y) \in \mathbb{R}^d \times \in \mathbb{R}^d$. Then, if $x \neq 0$ and $y \neq 0$, $\gamma(x,y)$ is equal to

$$\frac{C^2(H)}{2}\Big(\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\Big)\int_0^{+\infty} \rho^{1-\alpha}\psi\Big(\|x\|^\beta\rho\Big)\psi\Big(\|y\|^\beta\rho\Big)\, d\rho,$$

where

$$C(H) = \left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(H+1/2)}{H\,\Gamma(2H)\sin(\pi H)\,\Gamma(H+d/2)}\right)^{1/2}. \tag{4.7}$$

Otherwise, i.e. if $x = 0$ or $y = 0$, $\gamma(x,y) = 0$.

Especially, for every $x \in \mathbb{R}^d\backslash\{0\}$,

$$\mathbb{E}\big(X_{H,\beta}^2(x)\big) = C^2(H)\int_0^{+\infty} \rho^{1-\alpha}\psi^2(\rho)\, d\rho\, \|x\|^{2H-\beta(2-\alpha)}.$$

Consequently, if $\beta > 2H/(2-\alpha)$, $X_{H,\beta}$ is not continuous in quadratic mean at 0.

Proof. Since $X_{H,\beta}(0) = 0$, if $x = 0$ or $y = 0$, $\gamma(x,y) = 0$. Then, let $x \in \mathbb{R}^d\backslash\{0\}$, and $y \in \mathbb{R}^d\backslash\{0\}$. Since the map $\varphi \mapsto \int \varphi \, d\widetilde{N}$ is an isometry,

$$\gamma(x,y) = 4 \int_{\mathbb{R}^d \times \mathbb{C}} \Re(f(x,\xi)z)\Re(f(y,\xi)z)\, \psi\Big(\|x\|^\beta |z|\Big)\psi\Big(\|y\|^\beta |z|\Big)\, d\xi\, \nu(dz),$$

where

$$f(u,\xi) = \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{H+d/2}}.$$

Then, using that $2\Re(z_1)\Re(z_2) = \Re(z_1 z_2) + \Re(z_1\overline{z_2})$,

$$\gamma(x,y) = 2\Re(I_1(x,y)) + 2\Re(I_2(x,y)),$$

where

$$I_1(x,y) = \int_{\mathbb{C}} z^2 \psi\Big(\|x\|^\beta |z|\Big)\psi\Big(\|y\|^\beta |z|\Big)\, \nu(dz) \int_{\mathbb{R}^d} f(x,\xi) f(y,\xi)\, d\xi$$

and

$$I_2(x,y) = \int_{\mathbb{C}} |z|^2 \psi\Big(\|x\|^\beta |z|\Big)\psi\Big(\|y\|^\beta |z|\Big)\, \nu(dz) \int_{\mathbb{R}^d} f(x,\xi)\overline{f(y,\xi)}\, d\xi.$$

Moreover, since $\nu(dz)$ is rotationally invariant,

$$\int_{\mathbb{C}} z^2 \psi\Big(\|x\|^\beta |z|\Big)\psi\Big(\|y\|^\beta |z|\Big)\, \nu(dz) = 0$$

and then $I_1(x,y) = 0$.
Furthermore, by definition of $\nu(dz)$ and of the function f,

$$I_2(x,y) = \frac{C^2(H)}{2}\, \mathbb{E}(B_H(x)B_H(y)) \int_0^{+\infty} \rho^{1-\alpha} \psi\Big(\|x\|^\beta \rho\Big)\psi\Big(\|y\|^\beta \rho\Big)\, d\rho,$$

where B_H is a standard FBM and

$$C(H) = \left(4\pi \int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2H}}\, d\xi\right)^{1/2}$$

with $e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$. Therefore, $\gamma(x,y)$ is equal to

$$\frac{C^2(H)}{2}\Big(\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\Big) \int_0^{+\infty} \rho^{1-\alpha} \psi\Big(\|x\|^\beta \rho\Big)\psi\Big(\|y\|^\beta \rho\Big)\, d\rho.$$

Furthermore, according to chapter 3, see proof of proposition 3.3.1,

$$C(H) = \left(\frac{4\,\pi^{(d+3)/2}\,\Gamma(H+1/2)}{H\,\Gamma(2H)\sin(\pi H)\,\Gamma(H+d/2)}\right)^{1/2}.$$

Moreover, in the case where $x = y$, the change of variable $r = \|x\|^\beta \rho$ leads to:

$$\mathbb{E}\big(X_{H,\beta}^2(x)\big) = C^2(H) \int_0^{+\infty} r^{1-\alpha} \psi^2(r)\, dr\, \|x\|^{2H-\beta(2-\alpha)},$$

which concludes the proof. □

Let $\varepsilon > 0$. Then, by a simple change of variable,

$$\gamma(\varepsilon x, \varepsilon y) = \varepsilon^{2H-\beta(2-\alpha)} \gamma(x, y).$$

Also, if the centered field $X_{H,\beta}$ were Gaussian, it would be self-similar with index $H - \beta(1 - \alpha/2)$.

Throughout this chapter, $0 < H < 1$, $\beta \in \mathbb{R}$ and $X_{H,\beta}$ is the field associated with (H, β) by the definition 4.2.1.

4.3 Asymptotic Self-Similarity

In this section, the asymptotic self-similarity properties of the field $X_{H,\beta}$ are studied. Like RHFLMs, $X_{H,\beta}$ is locally asymptotically self-similar at each point $x \in \mathbb{R}^d$. However, RHFLMs have stationary increments and then the same tangent field at each point. It does not remain true in the case of $X_{H,\beta}$ whose increments are not stationary in general. First, in the case where $\beta = d/\alpha$, the field $X_{H,\beta}$ is self-similar. Then, the nature of the tangent field at $x = 0$ depends on β whereas it does not at $x \neq 0$. Hence, the lass property at $x \neq 0$ is first stated. Then, the lass property at $x = 0$ is given according to the value of β. In particular, when $\beta > d/\alpha$, the tangent field at $x = 0$ is a RHFSM whereas it is a FBM at $x \neq 0$. In this case the behaviour at $x = 0$ is very different from the behaviour at $x \neq 0$. In addition, when $\beta < d/\alpha$, the tangent field at $x = 0$ is Gaussian but it is not in general a FBM. Also, in this case, the tangent field at $x = 0$ and $x \neq 0$ are two Gaussian fields. However, whereas, in general tangent fields have stationary increments, the tangent field at $x = 0$ for $\beta < d/\alpha$ does not, as soon as $\beta \neq 0$. Furthermore, this section is concluded by the study of the asymptotic of the field when the increments are taken at large scales.

4.3.1 Case $\beta = d/\alpha$

First, as already noticed, if $X_{H,\beta}$ were a centered Gaussian field, it would be self-similar with exponent $H - \beta(1 - \alpha/2)$ because of its structure of covariance. However, in general $X_{H,\beta}$ is not self-similar but locally asymptotically self-similar. Nevertheless, in the case where $\beta = d/\alpha$, by homogeneity of the mean measure of $N(d\xi, dz)$, $X_{H,\beta}$ is self-similar with index $H - \beta(1 - d/\alpha)$ as it would be expected if it were Gaussian. Then, the field $X_{H,\beta}$ satisfies a global property of self-similarity only for $\beta = d/\alpha$.

Proposition 4.3.1. *Assume that $\beta = d/\alpha$. Then, $X_{H,\beta}$ is self-similar with index $\widetilde{H} = H + d/2 - d/\alpha$, i.e.*

$$\forall \varepsilon > 0,\ \left(\frac{X_{H,\beta}(\varepsilon u)}{\varepsilon^{H+d/2-d/\alpha}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(X_{H,\beta}(u)\right)_{u\in\mathbb{R}^d}.$$

Proof. Let $p \in \mathbb{N}\backslash\{0\}$, $(v_1,\dots,v_p) \in \mathbb{R}^p$, $(u_1,\dots,u_p) \in \left(\mathbb{R}^d\right)^p$ and

$$K_\varepsilon(\xi,z) = 2\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-i\varepsilon u_k\cdot\xi}-1}{\varepsilon^{\widetilde{H}}\|\xi\|^{H+d/2}} z\psi\Big(\varepsilon^\beta \|u_k\|^\beta |z|\Big)\right).$$

Then, since $\displaystyle\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{\widetilde{H}}} = \int_{\mathbb{R}^d\times\mathbb{C}} K_\varepsilon(\xi,z)\,\widetilde{N}(d\xi,dz)$,

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{\widetilde{H}}}\right)\right] = \exp\left(\varphi(\varepsilon)\right)$$

where

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} \left[\exp\left(iK_\varepsilon\big(\xi,\rho\mathrm{e}^{i\theta}\big)\right) - 1 - iK_\varepsilon\big(\xi,\rho\mathrm{e}^{i\theta}\big)\right]\frac{d\rho}{\rho^{1+\alpha}}\,d\theta\,d\xi.$$

Hence, since $\widetilde{H} = H + d/2 - d/\alpha$, by applying the changes of variables $\lambda = \varepsilon\xi$ and $r = \varepsilon^{d/\alpha}\rho$,

$$\varphi(\varepsilon) = \varphi(1),$$

which concludes the proof. □

Hence, when $\beta = d/\alpha$, $X_{H,\beta}$ is lass at $x = 0$ with exponent $H + d/2 - d/\alpha$ and tangent field itself. However, $X_{H,\beta}$ remains lass at $x \neq 0$ but the tangent field is not the same as $x = 0$.

4.3.2 Asymptotic self-similarity at $x \neq 0$

Before we study the field $X_{H,\beta}$, let us first define

$$Y_H(x,y) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}} z\right)\psi(y|z|)\,\widetilde{N}(d\xi,dz), \tag{4.8}$$

where $x \in \mathbb{R}^d$ and $y > 0$. Furthermore, let $Y_H(0,0) = 0$.
Then, for every $x \in \mathbb{R}^d$,

$$X_{H,\beta}(x) = Y_H\Big(x, \|x\|^\beta\Big). \tag{4.9}$$

The theorem 4.3.3 states the lass property at $x = 0$. It will be proved using (4.9) and the lass property of RHFLMs, established in [BCI02]. In fact, for every $y > 0$, the field $\left(Y_H(x,y)\right)_{x\in\mathbb{R}^d}$ is a RHFLM with index H. Therefore, following [BCI02], $\left(Y_H(x,y)\right)_{x\in\mathbb{R}^d}$ is a lass field and its tangent field at point x is known. Hence, let us first explain why $\left(Y_H(x,y)\right)_{x\in\mathbb{R}^d}$ is a RHFLM and give the Poisson random measure it is associated with.

Lemma 4.3.2. *Let $y > 0$. Then, the field $(Y_H(x,y))_{x\in\mathbb{R}^d}$ is a RHFLM with index H associated with a Poisson random measure $N_y(d\xi, dz)$ whose mean measure is $n_y(d\xi, dz) = d\xi\,\nu(y, dz)$, where $\nu(y, dz)$ is the push-forward of $\nu(dz)$ by the map $z \longmapsto z\,\psi(y|z|)$.*

Proof. Let $y > 0$. Thus, for every $g \in L^2\big(\mathbb{R}^d\big)$, let

$$M_y(g) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re(g(\xi)z)\,\psi(y|z|)\,\widetilde{N}(d\xi, dz).$$

Then, for every $u \in \mathbb{R}$,

$$\mathbb{E}\big(\mathrm{e}^{iuM_y(g)}\big) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}} \big[\mathrm{e}^{2iu\Re(g(\xi)z)} - 1 - 2iu\Re(g(\xi)z)\big]\,d\xi\,\nu(y, dz)\right), \tag{4.10}$$

where $\nu(y, dz)$ is the push-forward of $\nu(dz)$ by the map $z \longmapsto z\,\psi(y|z|)$. As a consequence,

$$M_y(g) \overset{(d)}{=} 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re(g(\xi)z)\,\widetilde{N_y}(d\xi, dz),$$

where $\widetilde{N_y}(d\xi, dz) = N_y(d\xi, dz) - n_y(d\xi, dz)$, with $N_y(d\xi, dz)$ a Poisson random measure whose mean measure is $n_y(d\xi, dz) = d\xi\,\nu(y, dz)$. Moreover, since $y \neq 0$, then, for every $p \geq 2$,

$$\int_{\mathbb{C}} |z|^p\,\nu(y, dz) = \int_{\mathbb{C}} |z\,\psi(y|z|)|^p\,\nu(dz) < +\infty.$$

Furthermore, since $\nu(dz)$ is a rotationally invariant measure, the measure $\nu(y, dz)$ is rotationally invariant too. Hence, the assumptions done in [BCI02] are fulfilled and we can consider the RHFLM $Z_{H,y}$ defined by

$$Z_{H,y}(x) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}}z\right)\widetilde{N_y}(d\xi, dz).$$

Therefore, in view of (4.10),

$$\big\{Y_H(x,y),\, x \in \mathbb{R}^d\big\} \overset{(d)}{=} \big\{Z_{H,y}(x),\, x \in \mathbb{R}^d\big\},$$

which concludes the proof. □

Let us now state the lass property at point $x \neq 0$.

Theorem 4.3.3. *Let $x \in \mathbb{R}^d$, $x \neq 0$. Then, the field $X_{H,\beta}$ is locally asymptotically self-similar with index H and tangent FBM at point x. More precisely,*

$$\lim_{\varepsilon\to 0_+} \left(\frac{X_{H,\beta}(x+\varepsilon u) - X_{H,\beta}(x)}{\varepsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} C(x,H)(B_H(u))_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields, B_H is a standard FBM with index H and

$$C(x,H) = \frac{C(H)}{\|x\|^{\beta(1-\alpha/2)}}\left(\int_0^{+\infty} \rho^{1-\alpha}\psi^2(\rho)\,d\rho\right)^{1/2}$$

where $C(\cdot)$ is defined by (4.7).

Proof. Let

$$V_\varepsilon(u) = \frac{X_{H,\beta}(x+\varepsilon u) - X_{H,\beta}(x)}{\varepsilon^H}.$$

Since $x \neq 0$, $V_\varepsilon(u)$ can be split into

$$V_{\varepsilon,1}(u) = \frac{Y_H\Big(x+\varepsilon u, \|x\|^\beta\Big) - X_{H,\beta}(x)}{\varepsilon^H}$$

and

$$V_{\varepsilon,2}(u) = \frac{X_{H,\beta}(x+\varepsilon u) - Y_H\Big(x+\varepsilon u, \|x\|^\beta\Big)}{\varepsilon^H}.$$

Then, the asymptotic of $V_{\varepsilon,1}$ is first studied.

Step 1 For every $u \in \mathbb{R}^d$,

$$V_{\varepsilon,1}(u) = \frac{Y_H\Big(x+\varepsilon u, \|x\|^\beta\Big) - Y_H\Big(x, \|x\|^\beta\Big)}{\varepsilon^H}.$$

Furthermore, since $x \neq 0$, according to lemma 4.3.2, the field $(Y_H(v, \|x\|^\beta))_{v\in\mathbb{R}^d}$ is a RHFLM associated with a Poisson random measure $N_{\|x\|^\beta}(d\xi, dz)$. In addition, the mean measure of $N_{\|x\|^\beta}(d\xi, dz)$ is $n_{\|x\|^\beta}(d\xi, dz) = d\xi\,\nu(\|x\|^\beta, dz)$, where $\nu(\|x\|^\beta, dz)$ is the push-forward of the measure $\nu(dz)$ by the map $z \longmapsto z\,\psi(\|x\|^\beta|z|)$. Then, by applying proposition 3.1 in [BCI02],

$$\lim_{\varepsilon\to 0_+} (V_{\varepsilon,1}(u))_{u\in\mathbb{R}^d} \overset{(d)}{=} C(x,H)(B_H(u))_{u\in\mathbb{R}^d}, \tag{4.11}$$

where B_H is a standard FBM with index H and

$$C(x,H) = \left(4\int_0^{+\infty} \Re^2(z)\,\nu\Big(\|x\|^\beta, dz\Big)\int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2H}}\,d\xi\right)^{1/2}$$

with $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. Therefore, by definition of $\nu\Big(\|x\|^\beta, dz\Big)$,

$$C(x,H) = \left(\frac{4\pi}{\|x\|^{\beta(2-\alpha)}}\int_0^{+\infty} \rho^{1-\alpha}\psi^2(\rho)\,d\rho\int_{\mathbb{R}^d} \frac{\left|\mathrm{e}^{-ie_1\cdot\xi} - 1\right|^2}{\|\xi\|^{d+2H}}\,d\xi\right)^{1/2}.$$

Step 2 Now let us prove that

$$\lim_{\varepsilon\to 0_+} (V_{\varepsilon,2}(u))_{u\in\mathbb{R}^d} \overset{(d)}{=} 0. \tag{4.12}$$

Let us fix $u \in \mathbb{R}^d$. By definition,

$$V_{\varepsilon,2}(u) = \frac{1}{\varepsilon^H}\int_{\mathbb{R}^d\times\mathbb{C}} g_2(x+\varepsilon u, x, \xi, z)\,\widetilde{N}(d\xi, dz),$$

where

$$g_2(a,b,\xi,z) = 2\Re\left(\frac{\mathrm{e}^{-ia\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\right)\Big(\psi\Big(\|a\|^\beta|z|\Big)-\psi\Big(\|b\|^\beta|z|\Big)\Big).$$

Therefore by (4.4),

$$\mathbb{E}\big(V_{\varepsilon,2}^2(u)\big) = \frac{1}{\varepsilon^{2H}}\int_{\mathbb{R}^d\times\mathbb{C}}|g_2(x+\varepsilon u,x,\xi,z)|^2\,d\xi\,\nu(dz).$$

Then, by definition of $\nu(dz)$, which is rotationally invariant,

$$\mathbb{E}\big(V_{\varepsilon,2}^2(u)\big) \;=\; \frac{4\pi\, I_\varepsilon(u)}{\varepsilon^{2H}}\int_{\mathbb{R}^d}\frac{\left|\mathrm{e}^{-i(x+\varepsilon u)\cdot\xi}-1\right|^2}{\|\xi\|^{d+2H}}\,d\xi,$$

where $I_\varepsilon(u) = \displaystyle\int_0^{+\infty}\rho^{1-\alpha}\Big(\psi\Big(\|x+\varepsilon u\|^\beta\rho\Big)-\psi\Big(\|x\|^\beta\rho\Big)\Big)^2\,d\rho.$

Hence,

$$\mathbb{E}\big(V_{\varepsilon,2}^2(u)\big) = \frac{C^2(H)\,\|x+\varepsilon u\|^{2H}\,I_\varepsilon(u)}{\varepsilon^{2H}}.$$

Remark that $I_\varepsilon(0)=0$. Then, suppose that $u\neq 0$. Let

$$K_\varepsilon = \big\{y\in\mathbb{R}^d\,/\,\|y-x\|\le\varepsilon\|u\|\big\}$$

and $\varepsilon_0>0$ such that $0\notin K_{\varepsilon_0}$. Hence, for every $\varepsilon\le\varepsilon_0$, $K_\varepsilon\subset K_{\varepsilon_0}$ and $x+\varepsilon u\in K_{\varepsilon_0}$. Let $m_{K_{\varepsilon_0}} = \min_{v\in K_{\varepsilon_0}}\|v\|^\beta$. Since K_{ε_0} is a compact set such that $0\notin K_{\varepsilon_0}$, $m_{K_{\varepsilon_0}}>0$. Then, in view of the support of ψ, for every $\varepsilon\le\varepsilon_0$ and every $\rho\ge m_{K_{\varepsilon_0}}^{-1}$,

$$\psi\Big(\|x+\varepsilon u\|^\beta\rho\Big) = \psi\Big(\|x\|^\beta\rho\Big) = 0$$

and so $\psi\Big(\|x+\varepsilon u\|^\beta\rho\Big)-\psi\Big(\|x\|^\beta\rho\Big)=0$.

Furthermore, by a Taylor expansion, one proves that there exists $D\in\mathbb{R}_+$, which does not depend on ε, such that

$$\forall\varepsilon\le\varepsilon_0,\ \Big|\psi\Big(\|x+\varepsilon u\|^\beta\rho\Big)-\psi\Big(\|x\|^\beta\rho\Big)\Big|\le D\rho\varepsilon\,\mathbf{1}_{[0,m_{K_{\varepsilon_0}}^{-1}]}(\rho).$$

Since $m_{K_{\varepsilon_0}}>0$ and since $0<\alpha<2$,

$$\int_0^{+\infty}\rho^{3-\alpha}\,\mathbf{1}_{[0,m_{K_{\varepsilon_0}}^{-1}]}(\rho)\,d\rho<+\infty.$$

Therefore,

$$\forall\varepsilon\le\varepsilon_0,\ 0\le I_\varepsilon(u)\le D'\varepsilon^2,$$

where D' does not depend on ε. As a consequence, since $H<1$,

$$\forall u\in\mathbb{R}^d,\ \lim_{\varepsilon\to 0_+}\mathbb{E}\big(V_{\varepsilon,2}^2(u)\big)=0,$$

which implies (4.12).

Conclusion (4.11) and (4.12) gives the conclusion. □

Then, $X_{H,\beta}$ is lass at each point $x \in \mathbb{R}^d \backslash \{0\}$ with tangent FBM. Furthermore, in the case where $\beta = d/\alpha$, since $X_{H,\beta}$ is self-similar, it is lass at $x = 0$ with tangent field itself. Then, the next subsection shows that $X_{H,\beta}$ remains lass at $x = 0$ for any $\beta \neq d/\alpha$ but with two different behaviours whether $\beta > d/\alpha$ or $\beta < d/\alpha$.

4.3.3 Exceptional point $x = 0$

Let us first study the case where $\beta > d/\alpha$. In this case, the behaviour of $X_{H,\beta}$ at $x = 0$ is very far from its behaviour at $x \neq 0$. Actually, the tangent field at $x = 0$ is not a Gaussian model whereas it is at $x \neq 0$. The tangent field at $x = 0$ does not have even finite second order moment.

Theorem 4.3.4. *Let $\beta > d/\alpha$ and $\widetilde{H} = H + d/2 - d/\alpha$. Assume that $\widetilde{H} > 0$. Then, $X_{H,\beta}$ is locally asymptotically self-similar at point $x = 0$ with index $\widetilde{H}$ and tangent field RHFSM, in the sense that*

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\beta}(\varepsilon u)}{\varepsilon^{\widetilde{H}}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} \left(S_{\widetilde{H}}(u) \right)_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $S_{\widetilde{H}}$ is a RHFSM that has the representation

$$S_{\widetilde{H}}(u) = D(\alpha)\, \Re\left(\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{\widetilde{H}+d/\alpha}} M_\alpha(d\xi) \right) \tag{4.13}$$

with M_α a complex isotropic symmetric α-stable random measure with control measure the Lebesgue measure and

$$D(\alpha) = \left[2^{\alpha+1}\pi \int_0^{+\infty} (1 - \cos(r)) \frac{dr}{r^{1+\alpha}} \right]^{1/\alpha}.$$

Let us remark that $\widetilde{H} < H < 1$. Then, since $\widetilde{H} > 0$, by assumption, the RHFSM $S_{\widetilde{H}}$ is well defined.

Proof. Let $p \in \mathbb{N}\backslash\{0\}$, $(v_1, \ldots, v_p) \in \mathbb{R}^p$, $(u_1, \ldots, u_p) \in \left(\mathbb{R}^d\right)^p$ and

$$K_\varepsilon(\xi, z) = 2\Re\left(\sum_{k=1}^p v_k \frac{\mathrm{e}^{-i\varepsilon u_k\cdot\xi} - 1}{\varepsilon^{\widetilde{H}} \|\xi\|^{H+d/2}} z\psi\left(\varepsilon^\beta \|u_k\|^\beta |z| \right) \right).$$

Then, since $\displaystyle\sum_{k=1}^p v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{\widetilde{H}}} = \int_{\mathbb{R}^d \times \mathbb{C}} K_\varepsilon(\xi, z)\, \widetilde{N}(d\xi, dz)$,

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^p v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{\widetilde{H}}} \right) \right] = \exp\left(\varphi(\varepsilon) \right)$$

where

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d \times \mathbb{C}} [\exp\left(iK_\varepsilon(\xi, z)\right) - 1 - iK_\varepsilon(\xi, z)] d\xi \, \nu(dz).$$

First, since $\nu(dz)$ is a rotationally invariant measure,

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d \times \mathbb{C}} [\cos\left(K_\varepsilon(\xi, z)\right) - 1] d\xi \, \nu(dz).$$

Then, by definition of $\nu(dz)$,

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d} \int_0^{2\pi} \int_0^{+\infty} \left[\cos\left(K_\varepsilon\big(\xi, \rho \mathrm{e}^{i\theta}\big)\right) - 1\right] \frac{d\rho}{\rho^{1+\alpha}} \, d\theta \, d\xi.$$

Hence, since $\widetilde{H} = H + d/2 - d/\alpha$, by applying the changes of variables $\lambda = \varepsilon\xi$ and $r = \varepsilon^{d/\alpha}\rho$,

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d} \int_0^{2\pi} \int_0^{+\infty} \widetilde{\varphi_\varepsilon}(\lambda, r, \theta) \, dr \, d\theta \, d\lambda,$$

where

$$\widetilde{\varphi_\varepsilon}(\lambda, r, \theta) = \frac{1}{r^{1+\alpha}} \Big[\cos\Big(\widetilde{K_\varepsilon}(\lambda, r, \theta)\Big) - 1\Big].$$

with

$$\widetilde{K_\varepsilon}(\lambda, r, \theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{e}^{i\theta} \psi\Big(\varepsilon^{\beta - d/\alpha} \|u_k\|^\beta r\Big)\right).$$

Remark that since $\beta > d/\alpha$, $\lim\limits_{\varepsilon \to 0_+} \varepsilon^{\beta - d/\alpha} = 0$. Thus, by continuity of ψ and since $\psi(0) = 1$,

$$\widetilde{K}(\lambda, r, \theta) = \lim_{\varepsilon \to 0_+} \widetilde{K_\varepsilon}(\lambda, r, \theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{e}^{i\theta}\right).$$

Then, let us define

$$F(\lambda) = \sum_{k=1}^{p} |v_k| \frac{\left|\mathrm{e}^{-iu_k\cdot\lambda} - 1\right|}{\|\lambda\|^{H+d/2}}.$$

Furthermore,

$$|\widetilde{\varphi_\varepsilon}(\lambda, r, \theta)| \leq \widetilde{\varphi}(\lambda, r, \theta)$$

with

$$\widetilde{\varphi}(\lambda, r, \theta) = F^2(\lambda) r^{1-\alpha} \, \mathbf{1}_{F(\lambda)r \leq 1} + \frac{2}{r^{1+\alpha}} \, \mathbf{1}_{F(\lambda)r > 1}.$$

Using that $0 < \widetilde{H} < 1$, one easily shows that $\widetilde{\varphi} \in L^1\big(\mathbb{R}^d \times (0, +\infty) \times [0, 2\pi]\big)$. Consequently, by a dominated convergence argument, $\lim\limits_{\varepsilon \to 0_+} \varphi(\varepsilon)$ is equal to

$$\int_{\mathbb{R}^d} \int_0^{2\pi} \int_0^{+\infty} [\cos\left(K(\lambda, r, \theta)\right) - 1] \frac{dr}{r^{1+\alpha}} \, d\theta \, d\lambda.$$

Moreover, for every $x \in \mathbb{R}$,

$$\int_0^{+\infty} [\cos(xr) - 1] \frac{dr}{r^{1+\alpha}} = -C(\alpha)|x|^\alpha,$$

where $C(\alpha) = \displaystyle\int_0^{+\infty} (1 - \cos(r)) \frac{dr}{r^{1+\alpha}}$. Therefore,

$$\lim_{\varepsilon \to 0_+} \varphi(\varepsilon) = -C(\alpha) \int_0^{2\pi} |2\cos(\theta)|^\alpha \, d\theta \int_{\mathbb{R}^d} \left| \sum_{k=1}^p v_k \frac{\mathrm{e}^{-iu_k \cdot \lambda} - 1}{\|\lambda\|^{H+d/2}} \right|^\alpha d\lambda.$$

As a consequence, since $H + d/2 = \widetilde{H} + d/\alpha$,

$$\lim_{\varepsilon \to 0_+} \mathbb{E}\left[\exp\left(i \sum_{k=1}^p v_k \frac{X_{H,\beta}(u_k)}{\varepsilon^{\widetilde{H}}} \right) \right] = \mathbb{E}\left[\exp\left(i \sum_{k=1}^p v_k S_{\widetilde{H}}(u_k) \right) \right],$$

where $S_{\widetilde{H}}$ is defined by (4.13), which concludes the proof. □

Therefore, when $\beta > d/\alpha$ and when $0 < \widetilde{H} < 1$, $X_{H,\beta}$ is lass with multifractional function

$$\begin{array}{rccl} h : & \mathbb{R}^d & \longrightarrow & (0,1) \\ & x & \longmapsto & \left\{ \begin{array}{ll} H & \text{if } x \neq 0 \\ \widetilde{H} & \text{if } x = 0. \end{array} \right. \end{array}$$

More precisely, for every $x \in \mathbb{R}^d$, $X_{H,\beta}$ is lass at point x with index $h(x)$. On the one hand, even if $X_{H,\beta}$ has finite second order moments, the tangent field at $x = 0$, which is a RHFSM, does not. On the other hand, the tangent field at $x \neq 0$ is a FBM, which is a Gaussian field. Therefore the behaviours of $X_{H,\beta}$ at $x = 0$ and $x \neq 0$ are very far from each other.

Let us now study the lass property at $x = 0$ when $\beta > d/\alpha$. In this case, the tangent field at $x = 0$ is Gaussian. However, if $\beta \neq 0$, this tangent field does not have stationary increments and then it is not a FBM. Furthermore, $X_{H,\beta}$ is lass at $x = 0$ with index $H - \beta(1 - \alpha/2)$ as it would be expected for the centered Gaussian field that has the same covariance as $X_{H,\beta}$.

Theorem 4.3.5. *Let $\beta < d/\alpha$. Then, the field $X_{H,\beta}$ is locally asymptotically self-similar at $x = 0$ with index $H - \beta(1 - \alpha/2)$, in the sense that*

$$\lim_{\varepsilon \to 0_+} \left(\frac{X_{H,\beta}(\varepsilon u)}{\varepsilon^{H - \beta(1-\alpha/2)}} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} (W_{H,\beta}(u))_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $W_{H,\beta}$ is a centered Gaussian field which has the same covariance as $X_{H,\beta}$.

Proof. Let $p \in \mathbb{N}\backslash\{0\}$, $(v_1, \ldots, v_p) \in \mathbb{R}^p$, $(u_1, \ldots, u_p) \in \left(\mathbb{R}^d\right)^p$ and

$$K_\varepsilon(\xi, z) = 2\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-i\varepsilon u_k\cdot\xi} - 1}{\varepsilon^{H-\beta(1-\alpha/2)}\|\xi\|^{H+d/2}} z\psi\Big(\varepsilon^\beta \|u_k\|^\beta |z|\Big)\right).$$

Then,

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{H-\beta(1-\alpha/2)}}\right)\right] = \exp\left(\varphi(\varepsilon)\right)$$

where

$$\varphi(\varepsilon) = \int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} \left[\exp\left(iK_\varepsilon\big(\xi, \rho\mathrm{e}^{i\theta}\big)\right) - 1 - iK_\varepsilon\big(\xi, \rho\mathrm{e}^{i\theta}\big)\right] \frac{d\rho}{\rho^{1+\alpha}}\, d\theta\, d\xi.$$

Therefore, by applying the changes of variables $\lambda = \varepsilon\xi$ and $r = \varepsilon^\beta\rho$, $\varphi(\varepsilon)$ is equal to

$$\int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} \varepsilon^{-d+\beta\alpha}\Big[\exp\Big(i\varepsilon^{\frac{d-\beta\alpha}{2}} K(\lambda, r, \theta)\Big) - 1 - i\varepsilon^{\frac{d-\beta\alpha}{2}} K(\lambda, r, \theta)\Big] \frac{dr}{r^{1+\alpha}}\, d\theta\, d\lambda,$$

where

$$K(\lambda, r, \theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{e}^{i\theta}\psi\Big(\|u_k\|^\beta r\Big)\right).$$

Moreover, since $\beta < d/\alpha$, $\lim\limits_{\varepsilon\to 0_+} \varepsilon^{\frac{d-\beta\alpha}{2}} = 0$. Consequently,

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-d+\beta\alpha}\Big[\exp\Big(i\varepsilon^{\frac{d-\beta\alpha}{2}} K(\lambda, r, \theta)\Big) - 1 - i\varepsilon^{\frac{d-\beta\alpha}{2}} K(\lambda, r, \theta)\Big] = -\frac{1}{2}K^2(\lambda, r, \theta).$$

Then, using a dominated convergence argument, one easily concludes that

$$\lim_{\varepsilon\to 0_+} \varphi(\varepsilon) = -\frac{1}{2}\int_{\mathbb{R}^d}\int_0^{2\pi}\int_0^{+\infty} |K(\lambda, r, \theta)|^2\, d\lambda\, d\theta\, \frac{dr}{r^{1+\alpha}}.$$

Furthermore, by definition of $W_{H,\beta}$, it is straightforward to prove that

$$\lim_{\varepsilon\to 0_+} \mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k \frac{X_{H,\beta}(\varepsilon u_k)}{\varepsilon^{H-\beta(1-\alpha/2)}}\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{k=1}^{p} v_k W_{H,\beta}(u_k)\right)\right],$$

which concludes the proof. □

Hence, when $\beta < d/\alpha$ $X_{H,\beta}$ is lass with multifractional function

$$\begin{array}{rccl} h: & \mathbb{R}^d & \longrightarrow & (0,1) \\ & x & \longmapsto & \left\{\begin{array}{ll} H & \text{if } x \neq 0 \\ H - \beta(1-\alpha/2) & \text{if } x = 0. \end{array}\right. \end{array}$$

In this case, the tangent fields at $x = 0$ and $x \neq 0$ are two Gaussian models. In particular, when $\beta = 0$, the tangent field is the same at $x = 0$ and at $x \neq 0$. In fact,

when $\beta = 0$, $X_{H,\beta}$ is a RHFLM. As noticed in [BCI02], a RHFLM has stationary increments and then the same tangent field at each point. In addition, when $\beta \neq 0$, the tangent field at $x = 0$ does not have stationary increments.

As a conclusion, $X_{H,\beta}$ is lass at each point. However, its behaviour at $x = 0$ depends on β. The three possible behaviours at $x = 0$ are very different. Furthermore, whereas at $x \neq 0$, the tangent field is a FBM, it is not at $x = 0$ except for $\beta = 0$.

4.3.4 Asymptotic self-similarity property at large scales

Like in the case of RHFLMs, the asymptotic self-similarity is now study at large scales. Heuristically, in view of (4.3), the behaviour at infinity is given by the behaviour at $x = 0$. More precisely, there is a symmetry: the tangent field at infinity when $\beta > d/\alpha$ (respectively when $\beta < d/\alpha$) and the tangent field at $x = 0$ when $\beta < d/\alpha$ (respectively when $\beta > d/\alpha$) are two of a kind. Actually, when the increments are taken at large scales, the limit is a Gaussian model when $\beta > d/\alpha$ and a RHFSM when $\beta < d/\alpha$. Then, by homogeneity of the Poisson random measure $N(d\xi, dz)$, the kind of the tangent fields at infinity and at $x = 0$ are the same but the conditions on β are exchanged.

Proposition 4.3.6. *Let $\beta > d/\alpha$. Then, the field $X_{H,\beta}$ is asymptotically self-similar with index $H - \beta(1 - \alpha/2)$, in the sense that*

$$\lim_{R\to+\infty} \left(\frac{X_{H,\beta}(Ru)}{R^{H-\beta(1-\alpha/2)}} \right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (W_{H,\beta}(u))_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $W_{H,\beta}$ is a centered Gaussian field which has the same covariance as $X_{H,\beta}$.

Remark 4.3.7. The limit field $W_{H,\beta}$ is not a FBM: its increments are not stationary.

Proof. The proof is analogous to the proof of theorem 4.3.4. As noticed in the proof of theorem 4.3.4,

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^{p} v_k \frac{X_{H,\beta}(Ru_k)}{R^{H-\beta(1-\alpha/2)}} \right) \right] = \exp\left(\varphi(R)\right)$$

where

$$\varphi(R) = \int_{\mathbb{R}^d} \int_0^{2\pi} \int_0^{+\infty} R^{-d+\beta\alpha} \Big[\exp\Big(iR^{\frac{d-\beta\alpha}{2}} K(\lambda, r, \theta) \Big) - 1 - iR^{\frac{d-\beta\alpha}{2}} K(\lambda, r, \theta) \Big] \frac{dr}{r^{1+\alpha}}\, d\theta\, d\lambda,$$

with

$$K(\lambda, r, \theta) = 2r\Re\left(\sum_{k=1}^{p} v_k \frac{\mathrm{e}^{-iu_k\cdot\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{e}^{i\theta} \psi\Big(\|u_k\|^{\beta} r \Big) \right).$$

Then, since $\beta > d/\alpha$, $\lim_{R\to+\infty} R^{\frac{d-\beta\alpha}{2}} = 0$ and one concludes as in the proof of theorem 4.3.4. □

In the same way, the field $X_{H,\beta}$ satisfies an asymptotic property at infinity when $\beta < d/\alpha$. However, in this case the limit field is a RHFSM. This phenomenon has already been encountered in the case of RHFLMs associated with the mean measure

$$\mathbf{1}_{|z|\leq 1}\, n(d\xi, dz)$$

in [BCI02]. Actually, when $\beta = 0$, $X_{H,\beta}$ is a RHFLM associated with a quite similar mean measure. The only difference with [BCI02] is that we have truncated the measure $n(d\xi, dz)$ owing to a smooth function.

Proposition 4.3.8. *Let $\beta < d/\alpha$ and $\widetilde{H} = H + d/2 - d/\alpha$. Assume that $\widetilde{H} > 0$. Then, $X_{H,\beta}$ is asymptotically self-similar with index $\widetilde{H}$, in the sense that*

$$\lim_{R\to+\infty} \left(\frac{X_{H,\beta}(Ru)}{R^{\widetilde{H}}} \right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(S_{\widetilde{H}}(u) \right)_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $S_{\widetilde{H}}$ is a RHFSM that has the representation

$$S_{\widetilde{H}}(u) = D(\alpha)\, \Re\left(\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{\widetilde{H}+d/\alpha}}\, M_\alpha(d\xi) \right)$$

with M_α a complex isotropic symmetric α-stable random measure with control measure the Lebesgue measure and

$$D(\alpha) = \left[2^{\alpha+1}\pi \int_0^{+\infty} (1 - \cos(r))\, \frac{dr}{r^{1+\alpha}} \right]^{1/\alpha}.$$

Proof. analogous to the proof of theorem 4.3.5. □

As a conclusion, the field $X_{H,\beta}$ satisfies different asymptotic self-similarities at low and large scales. These properties has already been remarked in the case of RHFLMs in [BCI02]. Furthermore, this phenomenon has already been encountered in [BD99]. However, in [BD99], the limit field at low scales and the limit field at large scales are two FBMs but with different index. In addition, the field $X_{H,\beta}$ has three different behaviours: its local structure at $x = 0$, $x \neq 0$ and at large scales are not the same in most cases.

4.4 Trajectories regularity and Hausdorff dimension of the graphs

In section 4.4.1, the trajectories regularity of $X_{H,\beta}$ is studied. In addition, [BCI03] has linked the Hausdorff dimension to the lass property and the trajectories regularity. Therefore, section 4.4.2 is devoted to the study of the Hausdorff dimension of the graphs of $X_{H,\beta}$.

4.4.1 Trajectories Regularity

In general, in order to study the trajectories regularity of a field $(Y(x))_{x\in\mathbb{R}^d}$, one evaluates the moments of its increments

$$\mathbb{E}(|Y(x)-Y(y)|^q)$$

and applies the Kolmogorov criterion. However, $X_{H,\beta}$ may not have moments of order greater than two. In fact, since the law of $X_{H,\beta}$ is an infinitely divisible law, one easily proves that for every $q\geq 2$ and every $x\in\mathbb{R}^d\backslash\{0\}$,

$$\mathbb{E}(|X_{H,\beta}(x)|^q)<+\infty \quad\Leftrightarrow\quad H<1-\tfrac{d}{2}+\tfrac{d}{q},$$

thanks to Theorem 25.3 in [Sat99]. Then, following [BCI02], $X_{H,\beta}$ is split into two fields $X^+_{H,\beta}$ and $X^-_{H,\beta}$ where $X^+_{H,\beta}$ has moments of every order and $X^-_{H,\beta}$ has almost surely $\mathcal{C}^1$-sample paths on $\mathbb{R}^d\backslash\{0\}$.

The proposition 4.4.3 and its corollary give the trajectories regularity of $X_{H,\beta}$ on $\mathbb{R}^d\backslash\{0\}$. Furthermore, we first study the trajectories regularity of $X^+_{H,\beta}$ and $X^-_{H,\beta}$, see lemmas 4.4.1 and 4.4.2. Then, the proposition 4.4.3 is an immediate consequence of these two lemmas. Hence, let us quickly define $X^+_{H,\beta}$ and $X^-_{H,\beta}$.

Let $n\in\mathbb{N}$ such that $n\geq d/2$ and

$$P_n(t)=\sum_{k=1}^{n}\frac{t^k}{k!}.$$

Let us define

$$g_n^+(x,\xi)=\frac{\mathrm{e}^{-ix\cdot\xi}-1-P_n(-ix\cdot\xi)\,\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{H+d/2}}. \tag{4.14}$$

Then, for every $q\in\mathbb{N}$ such that $q\geq 2$, the map

$$(\xi,z)\longmapsto g_n^+(x,\xi)z\psi\Big(\|x\|^\beta|z|\Big)$$

is in $L^q\big(\mathbb{R}^d\times\mathbb{C}\big)$ for the measure $n(d\xi,dz)=d\xi\,\nu(dz)$. Therefore,

$$X^+_{H,\beta}(x)=2\int_{\mathbb{R}^d\times\mathbb{C}}\Re\big(g_n^+(x,\xi)z\big)\psi\Big(\|x\|^\beta|z|\Big)\,\widetilde{N}(d\xi,dz),$$

is defined and has moments of every order.
Hence, $X_{H,\beta}=X^+_{H,\beta}+X^-_{H,\beta}$ where the field $\big(X^-_{H,\beta}(x)\big)_{x\in\mathbb{R}^d}$ is defined by

$$X^-_{H,\beta}(x)=2\int_{\mathbb{R}^d\times\mathbb{C}}\Re\left(\frac{P_n(-ix\cdot\xi)\,\mathbf{1}_{\|\xi\|\leq 1}}{\|\xi\|^{H+d/2}}z\right)\psi\Big(\|x\|^\beta|z|\Big)\,\widetilde{N}(d\xi,dz).$$

Furthermore, the trajectories regularity of $X^+_{H,\beta}$ is first studied, see lemma 4.4.1. Then, by differentiation of the characteristic function given by (4.5), every moment of order $2p$, $p \in \mathbb{N}\backslash\{0\}$,

$$\mathbb{E}\Big[\big(X^+_{H,\beta}(x) - X^+_{H,\beta}(y)\big)^{2p}\Big]$$

can be computed with the help of some L^{2q}-norm of the deterministic function $(\xi, z) \longmapsto \widetilde{g_n}(x, y, \xi, z)$ where

$$\widetilde{g_n}(x, y, \xi, z) = 2\Re\Big(g_n^+(x, \xi) z \psi\Big(\|x\|^\beta |z|\Big) - g_n^+(y, \xi) z \psi\Big(\|y\|^\beta |z|\Big)\Big). \tag{4.15}$$

As a consequence, an evaluation of these norms will give us an estimation of the moments of the increments of $X^+_{H,\beta}$. Then, the Kolmogorov criterion will be applied.

Lemma 4.4.1. *For every $H' < H$, there exists a modification of $X^+_{H,\beta}$ whose sample paths are almost surely locally H'-Hölder on $\mathbb{R}^d\backslash\{0\}$.*

Proof. Let $K \subset \mathbb{R}^d$ be a compact set such that $0 \notin K$. Then, let $p \in \mathbb{N}\backslash\{0\}$. By differentiation of the characteristic function of $X^+_{H,\beta}(x) - X^+_{H,\beta}(y)$, given by (4.5), one proves that

$$\mathbb{E}\Big[\big(X^+_{H,\beta}(x) - X^+_{H,\beta}(y)\big)^{2p}\Big] = \sum_{q=1}^{p} \sum_{l \in \mathcal{P}_q} \prod_{n=1}^{p} \int_{\mathbb{R}^d \times \mathbb{C}} |\widetilde{g_n}(x, y, \xi, z)|^{2l_n} d\xi \, \nu(dz), \tag{4.16}$$

where $\widetilde{g_n}$ is defined by (4.15) and

$$\mathcal{P}_q = \Big\{ l = (l_1, \ldots, l_q) \in (\mathbb{N}\backslash\{0\})^d,\ l_1 + \cdots + l_q = p \Big\}.$$

Therefore, let us study

$$I_q(x, y) = \int_{\mathbb{R}^d \times \mathbb{C}} |\widetilde{g_n}(x, y, \xi, z)|^{2q} \, d\xi \, \nu(dz)$$

for every $q \in \mathbb{N}\backslash\{0\}$. In order to estimate this norm, $\widetilde{g_n}$ is first split into two functions

$$g^+_{n,1}(x, y, \xi, z) = 2\Re\big(g_n^+(x, \xi) z - g_n^+(y, \xi) z\big) \psi\Big(\|x\|^\beta |z|\Big)$$

and

$$g^+_{n,2}(x, y, \xi, z) = 2\Re\big(g_n^+(y, \xi) z\big) \Big(\psi\Big(\|x\|^\beta |z|\Big) - \psi\Big(\|y\|^\beta |z|\Big)\Big).$$

By the Minkowski inequality,

$$I_q(x, y) \leq (A_q(x, y) + B_q(x, y))^q, \tag{4.17}$$

where

$$A_q(x, y) = \left(\int_{\mathbb{R}^d \times \mathbb{C}} \big| g^+_{n,1}(x, y, \xi, z) \big|^{2q} \, d\xi \, \nu(dz) \right)^{1/q}$$

and

$$B_q(x,y) = \left(\int_{\mathbb{R}^d \times \mathbb{C}} \left| g^+_{n,2}(x,y,\xi,z) \right|^{2q} d\xi \, \nu(dz) \right)^{1/q}.$$

Therefore, let us study $A_q(x,y)$ and $B_q(x,y)$.

Step 1 Since $\nu(dz)$ is rotationally invariant,

$$A^q_q(x,y) = 2^{2q} \widetilde{I}_q(x,y) \int_{\mathbb{C}} \Re^{2q}(z) \psi^{2q}\Big(\|x\|^\beta |z| \Big) \, \nu(dz),$$

where

$$\widetilde{I}_q(x,y) = \int_{\mathbb{R}^d} \frac{\left| \mathrm{e}^{-ix\cdot\xi} - \mathrm{e}^{-iy\cdot\xi} + [P(-iy\cdot\xi) - P(-ix\cdot\xi)] \mathbf{1}_{\|\xi\|\leq 1} \right|^{2q}}{\|\xi\|^{dq+2qH}} \, d\xi.$$

Moreover, according lemma 2.2.1, see chapter 2 page 33, there exists a constant $D \in \mathbb{R}_+$ such that

$$\forall (x,y) \in K^2, \ \widetilde{I}_q(x,y) \leq D \|x-y\|^{2qH}.$$

Furthermore, if $x \neq 0$,

$$\int_{\mathbb{C}} \Re^{2q}(z) \psi^{2q}\Big(\|x\|^\beta |z| \Big) \, \nu(dz) = \frac{1}{\|x\|^{\beta(2q-\alpha)}} \int_0^{2\pi} \cos^{2q}(\theta) \, d\theta \int_0^{+\infty} \rho^{2q-\alpha-1} \psi^{2q}(\rho) \, d\rho.$$

Then, let $m_K = \min_{v\in K} \|v\|^\beta$. Since $0 \notin K$ and since K is a compact set, $m_K > 0$. Therefore, as $(2q-\alpha) \geq 0$, for every $x \in K$,

$$\frac{1}{\|x\|^{\beta(2q-\alpha)}} \leq \frac{1}{m_K^{2q-\alpha}}.$$

As a consequence, there exists a constant $D \in \mathbb{R}_+$ such that for every $(x,y) \in K^2$,

$$A^q_q(x,y) \leq D \|x-y\|^{2qH}. \tag{4.18}$$

Step 2 By definition of $\nu(dz)$,

$$B^q_q(x,y) = J(y) \int_0^{+\infty} \rho^{2q-\alpha-1} \left(\psi\Big(\|x\|^\beta \rho \Big) - \psi\Big(\|y\|^\beta \rho \Big) \right)^{2q} d\rho,$$

where $J(y) = 2^{2q} \displaystyle\int_0^{2\pi} \cos^{2q}(\theta) \, d\theta \int_{\mathbb{R}^d} \frac{\left| \mathrm{e}^{-iy\cdot\xi} - 1 - P(-iy\cdot\xi) \mathbf{1}_{\|\xi\|\leq 1} \right|^{2q}}{\|\xi\|^{dq+2qH}} \, d\xi.$

It is straightforward to prove that $\sup_{v\in K} J(v) < +\infty$. Then, it remains to study

$$\widetilde{B}^q_q(x,y) = \int_0^{+\infty} \rho^{2q-\alpha-1} \left(\psi\Big(\|x\|^\beta \rho \Big) - \psi\Big(\|y\|^\beta \rho \Big) \right)^{2q} d\rho.$$

One proceeds as in step 2 of the proof of theorem 4.3.3. First, when $(x,y) \in K^2$, then for every $\rho \geq m_K^{-1}$,

$$\psi\Big(\|x\|^\beta \rho\Big) - \psi\Big(\|y\|^\beta \rho\Big) = 0,$$

because of the support of ψ. Then by a Taylor expansion, one proves that there exists a constant $D \in \mathbb{R}_+$, such that for every $(x,y) \in K^2$,

$$\Big|\psi\Big(\|x\|^\beta \rho\Big) - \psi\Big(\|y\|^\beta \rho\Big)\Big| \leq D\,\rho\|x-y\|\,\mathbf{1}_{[0,m_K^{-1}]}(\rho).$$

Moreover, since $m_K > 0$ and $0 < \alpha < 2$,

$$\int_0^{+\infty} \rho^{3-\alpha}\,\mathbf{1}_{[0,m_K^{-\beta}]}(\rho)\,d\rho < +\infty.$$

As a consequence, there exists a constant $D \in \mathbb{R}_+$ such that for every $(x,y) \in K^2$,

$$B_q^q(x,y) \leq D\|x-y\|^{2q}. \tag{4.19}$$

Step 3 Owing to (4.17), (4.18) and (4.19), there exists a constant $D \in \mathbb{R}_+$ such that for every $(x,y) \in K^2$,

$$I_q(x,y) \leq D\|x-y\|^{2qH}.$$

Then, by definition of $I_q(x,y)$ and by (4.16), for every $(x,y) \in K^2$,

$$\mathbb{E}\Big[\big(X_{H,\beta}^+(x) - X_{H,\beta}^+(y)\big)^{2p}\Big] \leq D\|x-y\|^{2pH},$$

where $D \in \mathbb{R}_+$ is a constant which only depends on p and K. Since p can be chosen such that $2pH > d$, the Kolmogorov criterion gives the conclusion. □

Hence, it remains to study the field $X_{H,\beta}^-$. Actually, the trajectories of this field are smooth.

Lemma 4.4.2. *There exists a modification of $X_{H,\beta}^-$ which has $\mathcal{C}^1$-sample paths on $\mathbb{R}^d\backslash\{0\}$.*

Proof. This proof is based on the same scheme as the proof of lemma 2.2.4, see chapter 2 page 36. For the sake of clearness, let us recall this scheme.

Notice that for every $x \in \mathbb{R}^d\backslash\{0\}$

$$X_{H,\beta}^-(x) = Y_{H,\beta}^-\Big(x, \|x\|^\beta\Big),$$

where the field $\big(Y_{H,\beta}^-(x,y)\big)_{x\in\mathbb{R}^d,\,y>0}$ is defined as follows:

$$Y_{H,\beta}^-(x,y) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{P_n(-ix\cdot\xi)}{\|\xi\|^{H+d/2}} z\right) \psi(y|z|)\,\mathbf{1}_{\|\xi\|\leq 1}\,\widetilde{N}(d\xi,dz). \tag{4.20}$$

Let us define

$$Z_\gamma(y) = 2\int_{\mathbb{R}^d\times\mathbb{C}} \Re\left(\frac{i^{|\gamma|}\prod_{j=1}^d \xi_j^{\gamma_j}}{\|\xi\|^{H+d/2}} z\right)\psi(y|z|)\,\mathbf{1}_{\|\xi\|\le 1}\,\widetilde{N}(d\xi, dz),$$

where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$ is such that $1 \le |\gamma| = \sum_{j=1}^n \gamma_j \le n$ and $y > 0$.
Since P_n is a polynomial, it is sufficient to prove that Z_γ admits a modification which has $\mathcal{C}^1$-sample paths on $(0, +\infty)$. Then let η and M such that $0 < \eta < M$. One can prove with Taylor expansion the existence of a constant $D > 0$ such that

1. $\mathbb{E}\big[|Z_\gamma(y+\delta) - Z_\gamma(y)|^2\big] \le D|\delta|^2$, for every $y \in [\eta, M]$ and every δ such that $y + \delta \in [\eta, M]$,
2. $\mathbb{E}\big[|Z_\gamma(y+\delta) + Z_\gamma(y-\delta) - 2Z_\gamma(y)|^2\big] \le D|\delta|^4$, for every $y \in [\eta, M]$ and every δ such that $(y+\delta, y-\delta) \in [\eta, M]^2$.

Therefore, according to [CL67], see page 69, there exists a modification of Z_γ which has almost surely $\mathcal{C}^1$-sample paths on $(0, +\infty)$, which concludes the proof. □

The following proposition is an immediate consequence of the lemmas 4.4.1 and 4.4.2.

Proposition 4.4.3. *For every $H' < H$, there exists a modification of $X_{H,\beta}$ whose sample paths are almost surely locally H'-Hölder on $\mathbb{R}^d\backslash\{0\}$.*

From the preceding proposition and the lass property, the pointwise Hölder exponent of $X_{H,\beta}$ at point $x \neq 0$ can be given. Even if the increments of $X_{H,\beta}$ are not stationary, the pointwise Hölder exponent is the same at each point $x \neq 0$.

Corollary 4.4.4. *At every point $x \neq 0$, the pointwise Hölder exponent $H_{X_{H,\beta}}(x)$ of $X_{H,\beta}$ is almost surely equal to H.*

Proof. see proof of proposition 2.3 in [BCI02]. □

Remark 4.4.5. When $\beta > d/\alpha$, because of the lass property at $x = 0$, it can be proved that, almost surely,

$$H_{X_{H,\beta}}(0) \le \widetilde{H}.$$

Hence, $H_{X_{H,\beta}}(0) < H$. In particular, for every H' such that $\widetilde{H} < H' < H$, the trajectories of $X_{H,\beta}$ are not locally H'-Hölder on whole $\mathbb{R}^d$. The same holds has soon as $\beta > 0$.

Corollary 4.4.6. *Let $u \in S^{d-1}$ and $x \in \mathbb{R}^d$ such that $x \neq 0$. Then, the pointwise Hölder exponent $H_{X_{H,\beta}}(x, u)$ of $X_{H,\beta}$ at point x in direction u is almost surely equal to H.*

Proof. see proof of proposition 2.3.3 page 50 in chapter 2. □

4.4.2 Hausdorff dimension

Owing to [BCI03], the Hausdorff dimension of the graphs of $X_{H,\beta}$ can be computed. Let us first recall the definition of the graphs of $X_{H,\beta}$. Let $U \subset \mathbb{R}^d$ be a compact set. Then, the graph of $X_{H,\beta}$ on U is

$$\operatorname{graph}\left(X_{H,\beta}{}_{|_U}\right) = \{(x, X_{H,\beta}(x)),\, x \in U\}.$$

The Hausdorff dimension of $\operatorname{graph}\left(X_{H,\beta}{}_{|_U}\right)$ is denoted by $\dim_{\mathcal{H}} X_{H,\beta}{}_{|_U}$.

Proposition 4.4.7. *Let $K = \prod_{i=1}^{d}[a_i, b_i]$, $(a_i, b_i) \in \mathbb{R}^2$ and $a_i < b_i$ Then the Hausdorff dimension of the graph of $X_{H,\beta}$ on K is almost surely equal to $d + 1 - H$.*

Proof. In fact, this proposition is a consequence of Theorem 2.1 in [BCI03]. However, when $0 \in K$, for $\varepsilon > 0$ sufficiently small, the sample paths of $X_{H,\beta}$ are not $(H - \varepsilon)$-Hölder continuous on K. Consequently, when $0 \in K$, the assumptions of Theorem 2.1 in [BCI03] are not fulfilled. Nevertheless,

$$\dim_{\mathcal{H}} X_{H,\beta}{}_{|_K} = \dim_{\mathcal{H}} \{(x, X_{H,\beta}(x)),\, x \in K\backslash\{0\}\} = \dim_{\mathcal{H}} X_{H,\beta}{}_{|_{K^*}},$$

where $K^* = K\backslash\{0\}$. Then, since

$$\dim_{\mathcal{H}} \bigcup_{i\in\mathbb{N}} E_i = \sup_{i\in\mathbb{N}} \dim_{\mathcal{H}} E_i,$$

for every $E_i \subset \mathbb{R}^d$ Borel set, it is sufficient to prove the proposition when $0 \notin K$. In fact, when $0 \in K$, K^* can be written as a countable union of blocks which do not contain 0.

Therefore, let us now assume that $0 \notin K$. Then one must prove that $X_{H,\beta}$ satisfies the assumptions of Theorem 2.1 in [BCI03]. Firstly, by proposition 4.4.3, the sample paths of $X_{H,\beta}$ are $(H - \varepsilon)$-Hölder continuous on K for every $\varepsilon \in (0, H)$. Hence, it remains to prove the existence of a L^1-function Φ and of $\delta_0 > 0$, such that for all $u \in \mathbb{R}$ and every $(x, y) \in K^2$ such that $\|x - y\| \leq \delta_0$,

$$\left|\mathbb{E}\left(\mathrm{e}^{iv\frac{X_{H,\beta}(x) - X_{H,\beta}(y)}{\|x-y\|^H}}\right)\right| \leq \Phi(v). \tag{4.21}$$

Let $(x, y) \in (K)^2$ and $v \in \mathbb{R}$. Then by (4.5),

$$\mathbb{E}\left[\exp\left(iv\frac{X_{H,\beta}(x) - X_{H,\beta}(y)}{\|x - y\|^H}\right)\right] = \exp\left(\varphi(x, y, v)\right),$$

where

$$\varphi(x, y, v) = \int_{\mathbb{R}^d\times\mathbb{C}} \left[\exp\left(ivG(x, y, \xi, z)\right) - 1 - ivG(x, y, \xi, z)\right] d\xi\, \nu(dz)$$

with

$$G(x,y,\xi,z)=\frac{2}{\|x-y\|^H}\,\Re\!\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\,\psi\Big(\|x\|^\beta|z|\Big)-\frac{\mathrm{e}^{-iy\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\,\psi\Big(\|y\|^\beta|z|\Big)\right).$$

As $\nu(dz)$ is a rotationally invariant measure,

$$\varphi(x,y,v)=\int_{\mathbb{R}^d\times\mathbb{C}}[\cos\left(vG(x,y,\xi,z)\right)-1]\,d\xi\,\nu(dz).$$

Put $M_K=\max_K\|u\|^\beta$ and let $T=M_K^{-1}/2$. Then, by (4.6), for every $u\in K$, if $|z|\le T$, $\psi\Big(\|u\|^\beta|z|\Big)=1$ since $\|u\|^\beta|z|\le 1/2$. Furthermore,

$$\varphi(x,y,v)\le\int_{\mathbb{R}^d\times\mathbb{C}}[\cos\left(vG(x,y,\xi,z)\right)-1]\,\mathbf{1}_{[0,T]}(|z|)\,d\xi\,\nu(dz).$$

Therefore, by definition of ψ, the last inequality can be rewritten as follows:

$$\varphi(x,y,v)\le\widetilde{\varphi}(x,y,v),$$

where

$$\widetilde{\varphi}(x,y,v)=\int_{\mathbb{R}^d\times\mathbb{C}}\Big[\cos\Big(v\widetilde{G}(x,y,\xi,z)\Big)-1\Big]\,\mathbf{1}_{[0,T]}(|z|)\,d\xi\,\nu(dz)$$

with

$$\widetilde{G}(x,y,\xi,z)=\frac{2}{\|x-y\|^H}\,\Re\!\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}}z-\frac{\mathrm{e}^{-iy\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\right).$$

As a consequence,

$$0\le\mathbb{E}\left[\exp\left(iv\frac{X_{H,\beta}(x)-X_{H,\beta}(y)}{\|x-y\|^H}\right)\right]\le\exp\left(\widetilde{\varphi}(x,y,v)\right).\tag{4.22}$$

Moreover, notice that

$$\exp\left(\widetilde{\varphi}(x,y,v)\right)=\mathbb{E}\left[\exp\left(iv\frac{\widetilde{X}_H(x)-\widetilde{X}_H(y)}{\|x-y\|^H}\right)\right],$$

where $\widetilde{X}_H$ is a RHFLM with control measure the push forward of $\nu(dz)$ by the map $z\longmapsto z\,\mathbf{1}_{[0,T]}(|z|)$. Furthermore, according to [BCI03], there exists Φ and δ_0 such that $\widetilde{X}_H$ satisfies (4.21). Then, because of (4.22), (4.21) is also fulfilled by $X_{H,\beta}$. As a result, the assumptions of Theorem 2.1 in [BCI03] are fulfilled and then

$$\dim_{\mathcal{H}}X_{H,\beta}{}_{|_K}=d+1-H\quad\text{almost surely.}$$

□

Hence, the Hausdorff dimension of the graphs of $X_{H,\beta}$ does not depend on β. Actually, except at one point, $X_{H,\beta}$ locally looks like FBM with index H. Then, the Hausdorff dimension of its graphs is the same as the Hausdorff dimension of the graphs of a FBM with index H.

RHFLMs have the same structure of covariance and the same local structure as the FBM. In next section, the local structure of $W_{H,\beta}$ is studied. In particular, even if $W_{H,\beta}$ and $X_{H,\beta}$ have the same structure of covariance, their local structure are different when $\beta > d/\alpha$.

4.5 Gaussian Model

This section is devoted to the study of the real-valued Gaussian field $W_{H,\beta}$ which has the same covariance as $X_{H,\beta}$.

Definition 4.5.1. Let $\beta \in \mathbb{R}$. The field $W_{H,\beta}$ is the real-valued centered Gaussian field with covariance function γ given by proposition 4.2.3.

Remark 4.5.2. When $\beta = 0$, $W_{H,\beta}$ is a FBM with index H.

4.5.1 Asymptotic self-similarity

As we have already noticed, thanks to its covariance structure, it is easy to verify that $W_{H,\beta}$ is self-similar.

Proposition 4.5.3. *The field $W_{H,\beta}$ is self-similar with index $H - \beta(1-\alpha/2)$, i.e.*

$$\forall \varepsilon > 0, \left(\frac{W_{H,\beta}(\varepsilon u)}{\varepsilon^{H-\beta(1-\alpha/2)}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} (W_{H,\beta}(u))_{u \in \mathbb{R}^d}.$$

However, in general, $W_{H,\beta}$ does not have stationary increments. Then, even if the self-similarity is a global property, proposition 4.5.3 only implies that $W_{H,\beta}$ is lass at $x = 0$. Nevertheless, the Gaussian field $W_{H,\beta}$ remains lass at each point. Furthermore, it has the same tangent field as $X_{H,\beta}$ at $x \neq 0$.

Theorem 4.5.4. *Let $x \in \mathbb{R}^d$, $x \neq 0$. Then, the field $W_{H,\beta}$ is locally asymptotically self-similar with index H and tangent FBM at point x. More precisely,*

$$\lim_{\varepsilon \to 0_+} \left(\frac{W_{H,\beta}(x + \varepsilon u) - W_{H,\beta}(x)}{\varepsilon^H} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} C(x,H)(B_H(u))_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields, B_H is a standard FBM with index H and

$$C(x,H) = \frac{C(H)}{\|x\|^{\beta(1-\alpha/2)}} \left(\int_0^{+\infty} \rho^{1-\alpha} \psi^2(\rho)\, d\rho \right)^{1/2}$$

with $C(\cdot)$ is defined by (4.7).

Proof. Let

$$\widetilde{V}_\varepsilon(u) = \frac{W_{H,\beta}(x+\varepsilon u) - W_{H,\beta}(x)}{\varepsilon^H}.$$

and

$$V_\varepsilon(u) = \frac{X_{H,\beta}(x+\varepsilon u) - X_{H,\beta}(x)}{\varepsilon^H}.$$

Then, since $W_{H,\beta}$ and $X_{H,\beta}$ are two centered fields with the same structure of covariance,

$$\mathbb{E}\Big(\widetilde{V}_\varepsilon(u)\widetilde{V}_\varepsilon(v)\Big) = \mathbb{E}(V_\varepsilon(u)V_\varepsilon(v)).$$

Then, we use the decomposition $V_\varepsilon = V_{\varepsilon,1} + V_{\varepsilon,2}$ introduced in the proof of theorem 4.3.3. Let us first recall that for every $y \in \mathbb{R}^d$,

$$\forall y \in \mathbb{R}^d,\ \lim_{\varepsilon \to 0_+} \mathbb{E}\big(V_{\varepsilon,2}^2(y)\big) = 0. \tag{4.23}$$

Moreover, $V_{\varepsilon,1}$ is a RHFLM and then have the same structure of covariance as a FBM. Furthermore, the proof of theorem 4.3.3 gives the Poisson random measure which defines $V_{\varepsilon,1}$. Then, using the isometry property induced by this Poisson random measure,

$$\mathbb{E}(V_{\varepsilon,1}(u)V_{\varepsilon,1}(v)) = C^2(x,H)\mathbb{E}(B_H(u)B_H(v)),$$

where B_H is a standard FBM with index H and

$$C(x,H) = \frac{C(H)}{\|x\|^{\beta(1-\alpha/2)}}\left(\int_0^{+\infty} \rho^{1-\alpha}\psi^2(\rho)\,d\rho\right)^{1/2}$$

with $C(\cdot)$ is defined by (4.7). Hence, by (4.23),

$$\lim_{\varepsilon \to 0_+} \mathbb{E}\Big(\widetilde{V}_\varepsilon(u)\widetilde{V}_\varepsilon(v)\Big) = C^2(x,H)\mathbb{E}(B_H(u)B_H(v)),$$

which concludes the proof since $W_{H,\beta}$ and B_H are centered Gaussian fields. □

Hence, $W_{H,\beta}$ and $X_{H,\beta}$ have the same tangent fields at $x \neq 0$. However, in general, their tangent fields at $x = 0$ are not the same.

4.5.2 Trajectories Regularity

Using the Kolmogorov criterion, the trajectories regularity of $W_{H,\beta}$ can be given on $\mathbb{R}^d\backslash\{0\}$. Since $W_{H,\beta}$ is a Gaussian field, it is sufficient to control

$$\mathbb{E}\big[(W_{H,\beta}(x) - W_{H,\beta}(y))^2\big].$$

Proposition 4.5.5. *For every $H' < H$, there exists a modification of $W_{H,\beta}$ whose sample paths are almost surely locally H'-Hölder on $\mathbb{R}^d\backslash\{0\}$.*

Proof. Let $K \subset \mathbb{R}^d$ be a compact set such that $0 \notin K$. Then,

$$\mathbb{E}\big[(W_{H,\beta}(x) - W_{H,\beta}(y))^2\big] = \int_{\mathbb{R}^d \times \mathbb{C}} |\widetilde{g_0}(x,y,\xi,z)|^2 d\xi\, \nu(dz), \tag{4.24}$$

where

$$\widetilde{g_0}(x,y,\xi,z) = 2\Re\Big(g_0(x,\xi)z\psi\Big(\|x\|^\beta|z|\Big) - g_0(y,\xi)z\psi\Big(\|y\|^\beta|z|\Big)\Big).$$

with

$$g_0(u,\xi) = \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{H+d/2}}.$$

Then, one proceeds as in the proof of proposition 4.4.1. Also, there exists a constant D such that for every $(x,y) \in K^2$,

$$\mathbb{E}\big[(W_{H,\beta}(x) - W_{H,\beta}(y))^2\big] \leq D\|x-y\|^{2H}.$$

Hence, since $W_{H,\beta}$ is a centered Gaussian field, for every $p \in \mathbb{N}^*$, there exists a constant D_p such that for every $(x,y) \in K^2$,

$$\mathbb{E}\big[(W_{H,\beta}(x) - W_{H,\beta}(y))^{2p}\big] \leq D_p\|x-y\|^{2pH}.$$

Since p can be chosen such that $2pH > d$, the Kolmogorov criterion gives the conclusion. □

Then, the pointwise Hölder exponent at $x \neq 0$ is deduced from the lass property and proposition 4.5.5.

Corollary 4.5.6. *At every point $x \neq 0$, the pointwise Hölder exponent $H_{W_{H,\beta}}(x)$ of $W_{H,\beta}$ is almost surely equal to H.*

Proof. see proof of proposition 2.3 in [BCI02]. □

Remark 4.5.7. In view of the lass property satisfied by $W_{H,\beta}$ at $x = 0$,

$$H_{W_{H,\beta}}(0) \leq H - \beta(1 - \alpha/2).$$

almost surely. Hence, for every $\widetilde{H} < H' < H$, the trajectories of $W_{H,\beta}$ are not locally H'-Hölder on whole $\mathbb{R}^d$ as soon as $\beta > 0$. We have already obtained the same property for $X_{H,\beta}$.

Corollary 4.5.8. *Let $u \in S^{d-1}$ and $x \in \mathbb{R}^d$ such that $x \neq 0$. Then, the pointwise Hölder exponent $H_{X_{H,\beta}}(x,u)$ of $X_{H,\beta}$ at point x in direction u is almost surely equal to H.*

Proof. see proof of proposition 2.3.3 page 50 in chapter 2. □

4.5.3 Hausdorff dimension

Owing to [BCI03], the Hausdorff dimension of the graphs of $X_{H,\beta}$ have been given in section 4.4. Actually, it has been deduced from Theorem 2.1 in [BCI03]. Let us recall that this theorem links the Hausdorff dimension to the lass property and the trajectories regularity. Then, since $W_{H,\beta}$ and $X_{H,\beta}$ satisfy the same lass property at $x \neq 0$ and have the same pointwise Hölder exponent at $x \neq 0$, we expect that the Hausdorff dimension of their graphs are the same. Then, in view of next proposition and of proposition 4.4.7, the Hausdorff dimension of $W_{H,\beta}$ is equal to the Hausdorff dimension of $X_{H,\beta}$ and does not depend on β.

Proposition 4.5.9. *Let $K = \prod_{i=1}^{d}[a_i, b_i]$, $(a_i, b_i) \in \mathbb{R}^2$ and $a_i < b_i$. Then the Hausdorff dimension of the graph of $W_{H,\beta}$ on K is almost surely equal to $d+1-H$.*

Proof. As explained in the proof of proposition 4.5.9, it is sufficient to prove the proposition when $0 \notin K$. Therefore, let us now assume that $0 \notin K$.

Then, by proposition 4.5.5, the sample paths of $W_{H,\beta}$ are $(H-\varepsilon)$-Hölder continuous on K for every $\varepsilon \in (0, H)$. Hence, let us prove that there exists a L^1-function Φ and $\delta_0 > 0$ such that for all $u \in \mathbb{R}$ and every $(x, y) \in K^2$ such that $\|x - y\| \leq \delta_0$,

$$\left|\mathbb{E}\left(\mathrm{e}^{iv\frac{W_{H,\beta}(x)-W_{H,\beta}(y)}{\|x-y\|^H}}\right)\right| \leq \Phi(v). \tag{4.25}$$

Let $(x, y) \in \left(\mathbb{R}^d\right)^2$ and $v \in \mathbb{R}$. Then, since $W_{H,\beta}$ is a centered Gaussian field,

$$\mathbb{E}\left[\exp\left(iv\frac{W_{H,\beta}(x) - W_{H,\beta}(y)}{\|x-y\|^H}\right)\right] = \exp\left(-v^2\varphi(x,y)\right),$$

where

$$\varphi(x,y) = \mathbb{E}\left[(W_{H,\beta}(x) - W_{H,\beta}(y))^2\right] = \mathbb{E}\left[(X_{H,\beta}(x) - X_{H,\beta}(y))^2\right].$$

Therefore,

$$\varphi(x,y) = \int_{\mathbb{R}^d\times\mathbb{C}} G^2(x,y,\xi,z)\, d\xi\, \nu(dz)$$

with

$$G(x,y,\xi,z) = \frac{2}{\|x-y\|^H}\,\Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}} z\,\psi\left(\|x\|^\beta|z|\right) - \frac{\mathrm{e}^{-iy\cdot\xi}-1}{\|\xi\|^{H+d/2}} z\,\psi\left(\|y\|^\beta|z|\right)\right).$$

Put $M_K = \max_K \|u\|^\beta$ and let $T = M_K^{-1}/2$. Then,

$$\varphi(x,y) \geq \int_{\mathbb{R}^d\times\mathbb{C}} G^2(x,y,\xi,z)\,\mathbf{1}_{[0,T]}(|z|)\, d\xi\, \nu(dz).$$

Therefore, by definition of ψ, the last inequality can be rewritten as follows:

$$\varphi(x,y) \geq \widetilde{\varphi}(x,y),$$

where

$$\widetilde{\varphi}(x,y) = \int_{\mathbb{R}^d\times\mathbb{C}} \widetilde{G}^2(x,y,\xi,z)\,\mathbf{1}_{[0,T]}(|z|)\,d\xi\,\nu(dz)$$

with

$$\widetilde{G}(x,y,\xi,z) = \frac{2}{\|x-y\|^H}\,\Re\left(\frac{\mathrm{e}^{-ix\cdot\xi}-1}{\|\xi\|^{H+d/2}}z - \frac{\mathrm{e}^{-iy\cdot\xi}-1}{\|\xi\|^{H+d/2}}z\right).$$

As a consequence,

$$0 \leq \mathbb{E}\left[\exp\left(iv\frac{W_{H,\beta}(x)-W_{H,\beta}(y)}{\|x-y\|^H}\right)\right] \leq \exp\left(-v^2\widetilde{\varphi}(x,y)\right). \tag{4.26}$$

Moreover, notice that

$$\exp\left(\widetilde{\varphi}(x,y,v)\right) = \mathbb{E}\left[\exp\left(iv\frac{B_H(x)-B_H(y)}{\|x-y\|^H}\right)\right],$$

where B_H is a FBM. Furthermore, according to [BCI03], there exists Φ and δ_0 such that B_H satisfies (4.25). Then, because of (4.26), (4.25) is also fulfilled by $W_{H,\beta}$. As a result, the assumptions of Theorem 2.1 in [BCI03] are fulfilled and then

$$\dim_{\mathcal{H}} W_{H,\beta|_K} = d+1-H \quad \text{almost surely.}$$

□

4.6 Generalization

In this chapter, a lass field $X_{H,\beta}$ with a special behaviour at $x=0$ has been defined. More precisely, its tangent field at $x=0$ is not a FBM and for β enough large, it is not even a Gaussian field. Nevertheless, $X_{H,\beta}$ only admits the point $x=0$ as exceptional point. Also, one can wonder if there exists a field $\widetilde{X}_{H,\beta}$ with several exceptional points. Actually, let $\left\{x_i \in \mathbb{R}^d,\, i\in I\right\}$ be a family; we would like to define a lass field with a tangent field at $x=x_i$ which is not a FBM. Furthermore, the tangent field at $x=x_i$ would be a RHFSM. In this part, we give a solution of this problem when the family $\{x_i,\, i\in I\}$ does not have any accumulation point. There are many ways to proceed. By the following one, the properties of the constructed field $\widetilde{X}_{H,\beta}$ are immediate consequences of the properties of the field $X_{H,\beta}$.

Before we give the definition of the field $\widetilde{X}_{H,\beta}$, let us precise the framework and the notation.

Assumption H Let $\left\{x_i \in \mathbb{R}^d,\, i\in I\right\}$ be a family of pairwise different elements of $\mathbb{R}^d$. Assume that this family does not have any accumulation point.

Hence, the set I is at most countable.

Then, let us consider a family of functions $\{\varphi_i,\, i \in I\}$ which will be used in order to localized the field. Let us first introduced

$$r_i = \inf_{\substack{j \in I \\ j \neq i}} \|x_i - x_j\|.$$

Under the assumption **H**, $r_i > 0$ for every $i \in I$.
Notation For every $i \in I$, let $\varphi_i : \mathbb{R}^d \longrightarrow [0,1]$ be a $\mathcal{C}^\infty$-function with compact support. More precisely,

$$\varphi_i(x) = \begin{cases} 1 & \text{if } \|x - x_i\| \leq \frac{r_i}{8} \\ 0 & \text{if } \|x - x_i\| \geq \frac{r_i}{4}. \end{cases}$$

Then, in particular, for every x, there exists at most one i such that $\varphi_i(x) \neq 0$. Furthermore, let us define $\varphi : \mathbb{R}^d \longrightarrow [0,1]$ by

$$\varphi(x) = 1 - \sum_{i \in I} \varphi_i(x), \in \mathbb{R}^d.$$

The function φ is well-defined since in the last equation the sum is reduced at most at one point. Furthermore, on a neighbourhood of a fixed x,

$$\varphi(y) = 1 \text{ or } 1 - \varphi_i(y)$$

for a certain fixed i. Then, φ is a $\mathcal{C}^\infty$-function.

Thus, let us now introduced the field $\widetilde{X}_{H,\beta}$.

Definition 4.6.1. Let $\beta \in \mathbb{R}$ and let B_H be a standard FBM independent of $X_{H,\beta}$. Then, for every $x \in \mathbb{R}^d$,

$$\widetilde{X}_{H,\beta}(x) = \varphi(x) B_H(x) + \sum_{i \in I} \varphi_i(x) X_{H,\beta}(x - x_i).$$

Remark 4.6.2. For every $i \in I$, $\widetilde{X}_{H,\beta}(x_i) = 0$.

Also, we will easily deduce the properties of $\widetilde{X}_{H,\beta}$ from the properties of $X_{H,\beta}$ using the following lemma.

Lemma 4.6.3. *Let $x \in \mathbb{R}^d$ and $J(x) = \{i \in I,\, \varphi_i(x) \neq 0\}$. Then, $J(x)$ is empty or reduced to one point. Moreover, there exists $r > 0$ such that for every y such that $\|y - x\| < r$,*

$$\widetilde{X}_{H,\beta}(y) = \varphi(y) B_H(y) + \sum_{i \in J(x)} \varphi_i(y) X_{H,\beta}(y - x_i).$$

Proof. In view of the definition of the supports of the continuous functions φ_i, this lemma is straightforward. □

Hence, the lass properties of B_H and $X_{H,\beta}$ at $x \neq 0$ can be established.

Theorem 4.6.4. *Let $x \in \mathbb{R}^d$, $x \notin \{x_i,\, i \in I\}$. Then, the field $\widetilde{X}_{H,\beta}$ is locally asymptotically self-similar with index H and tangent FBM at point x. More precisely,*

$$\lim_{\varepsilon \to 0_+} \left(\frac{\widetilde{X}_{H,\beta}(x+\varepsilon u) - \widetilde{X}_{H,\beta}(x)}{\varepsilon^H} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} \widetilde{C}(x,H) \Big(\widetilde{B}_H(u) \Big)_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields, $\widetilde{B}_H$ is a standard FBM with index H and

$$\widetilde{C}(x,H) = \varphi(x) + \left(\int_0^{+\infty} \rho^{1-\alpha} \psi^2(\rho)\, d\rho \right)^{1/2} C(H) \sum_{i \in I} \frac{\varphi_i(x)}{\|x - x_i\|^{\beta(1-\alpha/2)}}$$

with $C(\cdot)$ is defined by (4.7).

Proof. Let us first study

$$Y_{\varepsilon,1}(u) = \frac{\varphi(x+\varepsilon u) B_H(x+\varepsilon u) - \varphi(x) B_H(x)}{\varepsilon^H}$$

and

$$Y^i_{\varepsilon,2}(u) = \frac{\varphi_i(x+\varepsilon u) X_{H,\beta}(x+\varepsilon u) - \varphi_i(x) X_{H,\beta}(x)}{\varepsilon^H}.$$

Moreover,

$$Y_{\varepsilon,1}(u) = \varphi(x+\varepsilon u) \frac{B_H(x+\varepsilon u) - B_H(x)}{\varepsilon^H} + \frac{\varphi(x+\varepsilon u) - \varphi(x)}{\varepsilon^H} B_H(x).$$

Then, since φ is a $\mathcal{C}^1$-function and since $H < 1$,

$$\lim_{\varepsilon \to 0_+} \frac{\varphi(x+\varepsilon u) - \varphi(x)}{\varepsilon^H} = 0.$$

Hence, since the deterministic function φ is continuous, the lass property satisfied by B_H leads to:

$$\lim_{\varepsilon \to 0_+} \left(Y_{\varepsilon,1}(u) \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} \varphi(x) \Big(\widetilde{B}_H(u) \Big)_{u \in \mathbb{R}^d}, \tag{4.27}$$

where the limit is in distribution for all finite dimensional margins of the fields and $\widetilde{B}_H$ is a standard FBM with index H.

In the same way, thanks to the lass property satisfied by $X_{H,\beta}$ at $y \neq 0$, one establishes that

$$\lim_{\varepsilon \to 0_+} \left(Y^i_{\varepsilon,2}(u) \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} \varphi_i(x) C(x - x_i, H) \Big(\widetilde{B}_H(u) \Big)_{u \in \mathbb{R}^d}, \tag{4.28}$$

where

$$C(y,H) = \frac{C(H)}{\|y\|^{\beta(1-\alpha/2)}} \left(\int_0^{+\infty} \rho^{1-\alpha} \psi^2(\rho)\, d\rho \right)^{1/2}$$

with $C(\cdot)$ defined by (4.7).

Furthermore, by lemma 4.6.3, on a neighbourhood of x, the sum on I in the definition of $\widetilde{X}_{H,\beta}$ is in fact a sum on $J(x)$ which contains at most one point. Also, since B_H and $X_{H,\beta}$ are independent, (4.27) and (4.28) gives the conclusion. □

Furthermore, the field $\widetilde{X}_{H,\beta}$ remains a lass field at each $x = x_i$. Let us state this lass property in the case where $\beta > d/\alpha$.

Theorem 4.6.5. *Let $i \in I$, $\beta > d/\alpha$ and $\widetilde{H} = H + d/2 - d/\alpha$. Assume that $\widetilde{H} > 0$. Then, $\widetilde{X}_{H,\beta}$ is locally asymptotically self-similar at point $x = x_i$ with index $\widetilde{H}$ and tangent field RHFSM, in the sense that*

$$\lim_{\varepsilon \to 0_+} \left(\frac{\widetilde{X}_{H,\beta}(x_i + \varepsilon u)}{\varepsilon^{\widetilde{H}}} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} \left(S_{\widetilde{H}}(u) \right)_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $S_{\widetilde{H}}$ is a RHFSM that has the representation

$$S_{\widetilde{H}}(u) = D(\alpha)\, \Re\left(\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-iu\cdot\xi} - 1}{\|\xi\|^{\widetilde{H}+d/\alpha}}\, M_\alpha(d\xi) \right)$$

with M_α a complex isotropic symmetric α-stable random measure with control the Lebesgue measure and

$$D(\alpha) = \left[2^{\alpha+1}\pi \int_0^{+\infty} (1 - \cos(r))\, \frac{dr}{r^{1+\alpha}} \right]^{1/\alpha}.$$

Proof. In view of lemma 4.6.3, in a neighborhood of x_i, $\widetilde{X}_{H,\beta}(y) = X_{H,\beta}(y - x_i)$. Then, the lass property of $\widetilde{X}_{H,\beta}(y)$ at $x = x_i$ is the same as the lass property satisfied by $X_{H,\beta}$ at $x = 0$. □

In the same way, the field $X_{H,\beta}$ remains lass at $x = x_i$ when $\beta < d/\alpha$. However, in this case the tangent field is a Gaussian field whose increments are not stationary except for $\beta = 0$.

Theorem 4.6.6. *Let $i \in I$ and $\beta < d/\alpha$. Then, $\widetilde{X}_{H,\beta}$ is locally asymptotically self-similar at point $x = x_i$ with index $H - \beta(1 - \alpha/2)$, in the sense that*

$$\lim_{\varepsilon \to 0_+} \left(\frac{\widetilde{X}_{H,\beta}(x_i + \varepsilon u)}{\varepsilon^{H-\beta(1-\alpha/2)}} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} \left(W_{H,\beta}(u) \right)_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields and $W_{H,\beta}$ is a centered Gaussian field which has the same covariance as $X_{H,\beta}$.

Proof. see proof of theorem 4.6.5. □

Then, one can wonder what happens when $\beta = d/\alpha$. In this case, whereas $X_{H,\beta}$ is self-similar, $\widetilde{X}_{H,\beta}$ is not. Nevertheless, it remains lass at $x = x_i$ with $X_{H,\beta}$ as tangent field.

Theorem 4.6.7. *Let $i \in I$ and $\beta = d/\alpha$. Then, $\widetilde{X}_{H,\beta}$ is locally asymptotically self-similar at point $x = x_i$ with index $H - \beta(1-\alpha/2)$, in the sense that*

$$\lim_{\varepsilon \to 0_+} \left(\frac{\widetilde{X}_{H,\beta}(x_i + \varepsilon u)}{\varepsilon^{H-\beta(1-\alpha/2)}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} (X_{H,\beta}(u))_{u \in \mathbb{R}^d},$$

where the limit is in distribution for all finite dimensional margins of the fields.

Proof. see proof of theorem 4.6.5. □

Furthermore, the trajectories regularity of $\widetilde{X}_{H,\beta}$ and the Hausdorff dimension of its graphs can be easily given using lemma 4.6.3.

Proposition 4.6.8. *For every $H' < H$, there exists a modification of $\widetilde{X}_{H,\beta}$ whose sample paths are almost surely locally H'-Hölder on $\mathbb{R}^d \backslash \{0\}$.*

Then, the pointwise Hölder exponent at $x \neq 0$ is deduced from the lass property and proposition 4.6.8.

Corollary 4.6.9. *At every point $x \neq 0$, the pointwise Hölder exponent $H_{\widetilde{X}_{H,\beta}}(x)$ of $\widetilde{X}_{H,\beta}$ is almost surely equal to H.*

Like in the case of $X_{H,\beta}$, the directional pointwise Hölder exponent can be given and does not depend on the direction. Let us now be interested in the Hausdorff dimension of the graphs of $\widetilde{X}_{H,\beta}$.

Proposition 4.6.10. *Let $K = \prod_{i=1}^{d} [a_i, b_i]$, $(a_i, b_i) \in \mathbb{R}^2$ and $a_i < b_i$. Then the Hausdorff dimension of the graph of $W_{H,\beta}$ on K is almost surely equal to $d+1-H$.*

Moreover, there exist other fields which satisfy the same properties as $\widetilde{X}_{H,\beta}$. As an example, the FBM can be replaced by a lass field which admits B_H as tangent field at each point. However, one may suppose that its trajectories are locally Hölderian and take into account the Hausdorff dimension of its graphs. As an example, the FBM B_H can be replaced by a RHFLM X_H with index H.

In addition, one can also study the field $(Y(x))_{x \in \mathbb{R}^d}$ defined by

$$Y(x) = \varphi(x) B_H(x) + \sum_{i \in I} \varphi_i(x) X_{H_i,\beta_i}(x - x_i),$$

where for example the fields B_H and X_{H_i,β_i}, $i \in I$, are independent. Then, playing with the parameters, one obtains several behaviours.

4.7 Notes

In this chapter, a field $X_{H,\beta}$ with two different behaviours have been built. More precisely, when $\beta > d/\alpha$, its tangent field at $x = 0$ is a RHFSM $S_{\widetilde{H}}$ with index

$\widetilde{H} = H + d/2 - d/\alpha$ whereas at $x \neq 0$, it is a FBM with index H. Hence, since $\widetilde{H} < H$, the index of self-similarity at $x = 0$ is strictly lower than the index of self-similarity at $x \neq 0$.

In addition, as soon as we give us $0 < \widetilde{H} < H < 1$, there exists $\alpha \in (0, 2)$ such that $\widetilde{H} = H + d/2 - d/\alpha$ and then a field $X_{H,\beta}$ of second order which is lass with a RHFSM of index $\widetilde{H}$ as tangent field at $x = 0$ and a FBM of index H at $x \neq 0$. Also, we can wonder if there exists such a field when $\widetilde{H} \geq H$. In the sequel, we easily construct such a field. However, the construction depends if $\widetilde{H} = H$ or not. Furthermore, the lass property are taken in the sense of the convergence in distribution for all finite dimensional margins.

Notation Throughout this section, $S_{\widetilde{H}}$ is a RHFSM with index $\widetilde{H}$ and B_H is a FBM with index H. Furthermore, a and b are two real-valued $\mathcal{C}^1$-function on $\mathbb{R}^d$ such that

$$b(0) = 0,\ a(0) \neq 0 \text{ and } b(x) \neq 0 \text{ for every } x \neq 0.$$

Remark 4.7.1. The RHFSM $S_{\widetilde{H}}$ and the FBM B_H are not supposed independent.

Let us first give the construction in the case where $\widetilde{H} < H$.

Proposition 4.7.2. *Assume that $\widetilde{H} < H$ and define the field Y_1 by*

$$Y_1(x) = a(x)S_{\widetilde{H}}(x) + b(x)B_H(x),\ x \in \mathbb{R}^d.$$

Then, the field Y_1 is locally asymptotically self-similar. Furthermore, its tangent field at $x = 0$ is the field $a(0)S_{\widetilde{H}}$ which is a RHFSM and its tangent field at $x \neq 0$ is the FBM $b(x)B_H$.

Remark 4.7.3. The assumptions $a(0) \neq 0$ and $b(x) \neq 0$ are only done to ensure that the limit fields $a(0)S_{\widetilde{H}}$ and $b(x)B_H$ are not degenerate.

Proof. Let us fixed $x \in \mathbb{R}^d$. Then, since $S_{\widetilde{H}}$ is self-similar with index $\widetilde{H}$ and since a is a $\mathcal{C}^1$- function, one easily establishes that

$$\lim_{\varepsilon \to 0_+} \left(\frac{a(x + \varepsilon u)S_{\widetilde{H}}(x + \varepsilon u) - a(x)S_{\widetilde{H}}(x)}{\varepsilon^{\widetilde{H}}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} a(x)\big(S_{\widetilde{H}}(u)\big)_{u \in \mathbb{R}^d}.$$

In the same way, since B_H is self-similar with index H and since b is a $\mathcal{C}^1$- function,

$$\lim_{\varepsilon \to 0_+} \left(\frac{b(x + \varepsilon u)B_H(x + \varepsilon u) - b(x)B_H(x)}{\varepsilon^{H}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} b(x)(B_H(u))_{u \in \mathbb{R}^d}.$$

Also, since $\widetilde{H} < H$,

$$\lim_{\varepsilon \to 0_+} \left(\frac{Y_1(x + \varepsilon u) - Y_1(x)}{\varepsilon^{H}} \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} b(x)(B_H(u))_{u \in \mathbb{R}^d}. \tag{4.29}$$

Actually, if $x \neq 0$, $b(x) \neq 0$ and $b(x)B_H$ is not degenerate. Hence, (4.29) establishes the lass property at $x \neq 0$. However, since $b(0) = 0$, (4.29) does not give the tangent field at $x = 0$. However, in the case where $x = 0$, since b is deterministic,

$$\left(\frac{b(\varepsilon u)B_H(\varepsilon u)}{\varepsilon}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(\frac{b(\varepsilon u)B_H(u)}{\varepsilon^{1-H}}\right)_{u\in\mathbb{R}^d}.$$

Hence,

$$\lim_{\varepsilon\to 0_+}\left(\frac{b(\varepsilon u)B_H(\varepsilon u)}{\varepsilon}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} 0$$

and then

$$\lim_{\varepsilon\to 0_+}\left(\frac{Y_1(\varepsilon u)}{\varepsilon^{\widetilde{H}}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} a(0)\big(S_{\widetilde{H}}(u)\big)_{u\in\mathbb{R}^d}, \tag{4.30}$$

which gives the lass property at $x = 0$ since $a(0) \neq 0$. □

In the case where $\widetilde{H} = H$, i.e. the case where the FBM B_H and the RHFSM $S_{\widetilde{H}}$ have the same index, we can not deduce the local behaviour of Y_1 at $x \neq 0$ from these of B_H and $S_{\widetilde{H}}$ in general.

Proposition 4.7.4. *Assume that $\widetilde{H} = H$.*
Then, let us fixed H' such that $0 < H < H'$. Then, let $\alpha \in (0, 2)$ such that $H = H' + d/2 - d/\alpha$. We consider the field $X_{H',\beta}$, defined in section 4.2. Let

$$Y_2(x) = a(x)X_{H',\beta}(x) + b(x)B_H(x).$$

Then, the field Y_2 is a locally asymptotically self-similar field. Furthermore, its tangent field at $x = 0$ is is a RHFSM with index $\widetilde{H}$ whereas it is a FBM with index H at $x \neq 0$.

Proof. One proceeds as in proof of proposition 4.7.2 using the lass property of $X_{H',\beta}$ instead of the self-similarity of the RHFSM $S_{\widetilde{H}}$. □

As a conclusion, for any $(H, \widetilde{H}) \in (0, 1)^2$, we can build a field which admits 0 as an exceptional point. More precisely, the tangent field at $x = 0$ is a RHFSM whereas it is a FBM at $x \neq 0$.

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Multifractional Lévy Motions

Abstract

In a first part, the class of Real Harmonizable Multifractional Lévy Motions, in short RHMLMs, is introduced. This class is a generalization of the Multifractional Brownian Motion, in short MBM, and of the class of Real Harmonizable Fractional Lévy Motions. This class contains some non-Gaussian second order fields which share many properties with the MBM. Especially, RHMLMs are locally asymptotically self-similar and their pointwise Hölder exponent is allowed to vary along the trajectory. Moreover, their properties are governed by their multifractional function which can be estimated with the localized generalized quadratic variations as in the case of the MBM.

The second part deals with the simulation of the non-Gaussian part of a RHMLM. Actually, the method for generating the sample paths of RHMLMs is based on a generalized shot-noise series expansion. However, in some cases, one part of the RHMLM is approximated by a MBM.

The last part introduces a locally asymptotically self-similar field $X_{H,\beta}$ with a special behaviour at $x = 0$. More precisely, at $x \neq 0$, the tangent field is a Fractional Brownian Motion, in short FBM. However, in most cases, the tangent field at $x = 0$ is not a FBM and can even be non-Gaussian. In addition, the field $X_{H,\beta}$ is asymptotically self-similar at infinity with a Gaussian field, which is not a FBM, as tangent field. Finally, the trajectories regularity and the Hausdorff dimension of the graphs of $X_{H,\beta}$ are studied.

Keywords: Asymptotic self-similarity, identification, generalized shot noise series, infinitely divisible distribution, simulation.
Field: Mathematics, Statistics and Probability Theory.

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Résumé

Dans un premier temps, nous introduisons une classe de champs réels appelés champs de Lévy multifractionnaires au moyen d'une représentation harmonisable. Cette classe contient à la fois celle des champs de Lévy fractionnaires et le mouvement brownien multifractionnaire (MBM en abrégé). Elle fournit notamment des exemples de champs non gaussiens du second ordre ayant des propriétés semblables à celles du MBM. En particulier, les champs de Lévy multifractionnaires sont localement autosimilaires et leur exposant de Hölder ponctuel peut varier le long d'une trajectoire. Par ailleurs, leurs propriétés sont gouvernées par leur fonction multifractionnaire. Par suite, d'un point de vue statistique, un problème naturel est l'identification de cette fonction. Comme dans le cas du MBM, elle peut être identifiée au moyen des variations quadratiques localisées et généralisées.

Dans la deuxième partie, nous nous sommes intéressés à la simulation de la partie non gaussienne d'un champ de Lévy multifractionnaire. La méthode proposée est basée sur une représentation en série de bruits généralisés. Cependant, dans certains cas, on approche aussi une partie du champ de Lévy multifractionnaire par un MBM.

Enfin, la dernière partie introduit un champ $X_{H,\beta}$ localement autosimilaire avec un comportement atypique en 0. En effet, alors qu'en tout point $x \neq 0$, le champ tangent à $X_{H,\beta}$ est un mouvement brownien fractionnaire, en général en $x = 0$ le champ tangent à $X_{H,\beta}$ est de nature bien différente. De plus, le champ $X_{H,\beta}$ satisfait une propriété d'autosimilarité à grandes échelles et son étude est ensuite complétée par celle de la régularité des trajectoires et de la dimension de Hausdorff de ses graphes.

Mots clés : Autosimilarité locale, identification, séries de bruits généralisés, loi indéfiniment divisible, simulation.
Discipline : Mathématiques, Probabilités et Statistique.

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