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Thèse

présentée en vue de l'obtention du

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Spécialité : Mathématiques Pures

par

Laurent Mazet

Construction de surfaces minimales par résolution du problème de Dirichlet

Soutenue le Jeudi 2 Décembre 2004, devant le jury composé de :

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suite aux rapports de F. Pacard et A. Ros (Professeur, université de Grenade, Espagne).

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à Jules et Sophie.

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Introduction

Ce mémoire de doctorat est consacré à l'étude de la théorie des surfaces minimales. Notre objectif est de construire une grande classe d'exemples de surfaces minimales complètes à courbure totale finie. Ce travail passe par une étude précise de la convergence des suites de solutions de l'équation des surfaces minimales.

Les surfaces minimales sont les surfaces à courbure moyenne nulle, cette condition signifie que celles-ci sont des points critiques de la fonctionnelle d'aire pour les variations à support compact. Les surfaces minimales que nous souhaitons construire sont complètes et à courbure totale finie (*i.e.* l'intégrale de la valeur absolue de la courbure de Gauss le long de la surface est finie). On sait grâce à un résultat de R. Ossermann qu'une telle surface est conformément équivalente à une surface de Riemann compacte privée d'un nombre fini de points et est proprement immergée ; ainsi ces surfaces ont un comportement asymptotique donné par un nombre fini d'anneaux appelés bouts de la surface. Si un bout est plongé, R. Schoen nous apprend que son comportement asymptotique est caractérisé par un vecteur appelé flux du bout qui est colinéaire à la normale limite du bout ; si ce flux est nul le bout est asymptotique à un plan, si le flux est non-nul le bout est asymptotique à un bout de caténoïde ayant le même flux. Nous nous intéressons donc aux surfaces ayant leurs bouts plongés, nous appellerons, dans la suite, r-noïde, une surface minimale complète de courbure totale finie à bouts plongés (r désigne le nombre de bouts). Une condition nécessaire d'existence de r-noïde est alors que la somme des flux des bouts soit nulle, ce que l'on peut interpréter comme une condition d'équilibre sur les bouts.

Dans un article de 2001, C. Cosín et A. Ros s'intéressent à l'espace des r-noïdes Alexandrov-plongés et symétriques par rapport au plan horizontal d'équation $z = 0$. La seconde hypothèse impose que le flux de chaque bout soit un vecteur horizontal, la première dit que celui-ci ne peut pas être nul. Dans cet article, les deux auteurs s'intéressent plus particulièrement aux r-noïdes de genre 0, où le genre d'un r-noide désigne le genre de la surface de Riemann compacte sous-jacente. C. Cosín et A. Ros associent à chaque r-noïde Alexandrov-plongé symétrique de genre 0 un polygone appelé polygone de flux : il s'agit des flux des bouts rangés dans un ordre naturel donné par l'immersion.

Leur travail consiste alors à caractériser les r-noïdes Alexandrov-plongés symétriques de genre 0 en terme de leur polygone de flux. On dira qu'un polygone $(v_1, \ldots, v_r)$ de $\mathbb{R}^2$ borde un disque immergé si il existe une immersion du disque unité fermé dans $\mathbb{R}^2$ envoyant le cercle unité sur le polygone. Enonçons maintenant le résultat d'existence contenu dans le travail de C. Cosín et A. Ros.

Théorème. *Soient $v_1, \ldots, v_r$, r vecteurs non-nuls de $\mathbb{R}^2$, tels que $(v_1, \ldots, v_r)$ soit un polygone de $\mathbb{R}^2$ bordant un disque immergé. Il existe alors un r-noïde Alexandrov-plongé symétrique de genre 0 dont le polygone de flux est le polygone $(v_1, \ldots, v_r)$.*

En fait, les deux auteurs montrent qu'à l'exception du cas $r = 2$ qui correspond aux caténoïdes, la condition de border un disque immergé pour le polygone de flux est une condition qui, en plus d'être suffisante, est nécessaire pour l'existence d'un r-noïde Alexandrov-plongé symétrique de genre 0. Dans une première partie, notre travail consistera à donner une nouvelle démonstration de leur résultat d'existence, notre approche pouvant être qualifiée de plus constructive.

Dans un deuxième temps, nous poursuiverons notre travail en étudiant le cas des r-noïdes Alexandrov-plongés symétriques de genre 1. Nous montrerons que, dans ce cas, on peut aussi définir un polygone de flux. En développant plus profondément les idées mise en œuvre dans notre démonstration du résultat ci-dessus, nous obtiendrons un second résultat d'existence.

Théorème. *Soient $v_1, \ldots, v_r$, r vecteurs non-nuls de $\mathbb{R}^2$, tels que $(v_1, \ldots, v_r)$ soit un polygone de $\mathbb{R}^2$ bordant un disque immergé. Il existe alors un r-noïde Alexandrov-plongé symétrique de genre 1 dont le polygone de flux est le polygone $(v_1, \ldots, v_r)$.*

Les deux théorèmes que nous venons d'énoncer sont les principaux résultats d'existence contenus dans notre travail. Nous démonterons toutefois que des r-noïdes Alexandrov-plongés symétriques de genre 1 existent aussi pour des polygones plus généraux que ceux considérés dans le théorème ci-dessus.

L'idée principale que nous utiliserons pour construire ces r-noïdes est qu'en faisant usage de la symétrie plane, ceux-ci peuvent se voir comme la surface minimale conjuguée d'un graphe. Le graphe d'une fonction est une surface minimale si la fonction satisfait une équation aux dérivées partielles du second ordre appelée équation des surfaces minimales. Le problème de la construction des r-noïdes se résume donc à la résolution du problème de Dirichlet associé à cette équation.

Dans le cas du genre 0, le domaine et les données au bord nous seront donnés par le polygone. Par contre, pour le genre 1, le domaine sera connu, à l'exception d'une singularité. Le problème qui consiste à placer cette singularité de telle sorte que la surface obtenue après conjugaison soit un r-noïde est appelé problème de période.

La construction des solutions du problème de Dirichlet est donnée par un passage à la limite de solutions de l'équation des surfaces minimales que l'on sait construire

sur des domaines élémentaires grâce à un résultat de H. Jenkins et J. Serrin. Nous consacrerons donc une partie importante de ce mémoire à l'étude de la convergence des suites de solutions de l'équation des surfaces minimales. Cette étude consiste à comprendre les possibles divergences de telles suites ; à cet effet, nous avons le résultat fondamental suivant.

Théorème. *Soient Ω un domaine de $\mathbb{R}^2$ et (u_n) une suite de solutions de l'équation des surfaces minimales sur Ω. On désigne par $N_n(Q)$ la normale au graphe de u_n au point $(Q, u_n(Q))$. Soient $P \in \Omega$ et N un vecteur unitaire horizontal, on note L la géodésique de Ω passant par P et normale à N. Si la suite des normales $(N_n(P))$ converge vers N, alors $(N_n(Q))$ converge vers N en tout point Q de L.*

Ce théorème est le principal outil que nous utiliserons dans notre étude de la convergence des suites car il permet, en connaissant les suites le long du bord du domaine, de contrôler leur convergence ou leur divergence. Outre la résolution du problème de Dirichlet, nous utiliserons aussi ce résultat dans l'étude du problème de période associé à la construction de r-noïdes de genre 1.

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Chapitre 1

Les surfaces minimales

Dans ce chapitre, nous présentons différents aspects de la théorie des surfaces minimales. Les trois premières parties sont consacrées à des résultats classiques dont on pourra trouver les preuves dans la littérature. Pour la section 1.1, on se reférera aux livres de R. Osserman [Os2] et de U. Dierkes, S. Hildebrandt, A. Küster et O. Wohlrab [DHKW]. Une bonne référence pour la deuxième partie est l'article de D. Hoffman et H. Karcher [HK]. Quant aux graphes minimaux, on lira le livre de R. Osserman et celui de J.C.C. Nitsche [Ni2]. La dernière partie présente des résultats nouveaux concernant l'étude des graphes minimaux, les preuves se trouvent soit directement dans cette partie soit dans les chapitres suivants.

1.1 Un peu de théorie générale

Dans ce travail, nous considérons des surfaces Σ qui sont immergées dans $\mathbb{R}^3$. Nous adopterons donc le point de vue suivant : $X : \Sigma \to \mathbb{R}^3$ où Σ est une variété orientée de dimension 2 et X est une immersion ; $X(\Sigma)$ est alors une surface immergée dans $\mathbb{R}^3$. Toutefois, par abus de langage, nous dirons parfois que Σ est une surface de $\mathbb{R}^3$.

Nous avons alors la définition d'une surface minimale.

Définition 1.1.1. Considérons une surface immergée $X : \Sigma \to \mathbb{R}^3$. X est une immersion *minimale* si sa courbure moyenne H est nulle en tout point. On dira alors que $X(\Sigma)$ est une *surface minimale* de $\mathbb{R}^3$.

Rappelons que la courbure moyenne est la somme des courbures principales ; autrement dit, il s'agit de la trace de la seconde forme fondamentale.

1.1.1 Fonctions harmoniques et points de branchement

Soit $X : \Sigma \to \mathbb{R}^3$ une surface immergée, on pose $X = (x_1, x_2, x_3)$. Notons $\mathrm{d}s^2$ la métrique induite par X sur Σ, alors

$$H = \Delta_{\mathrm{d}s^2}(x_1, x_2, x_3) \tag{1.1}$$

Ainsi les fonctions coordonnées x_i d'une surface minimale sont des fonctions harmoniques pour la métrique induite.

Par ailleurs, la métrique $\mathrm{d}s^2$ induit une structure de surface de Riemann sur Σ. L'harmonicité étant un invariant conforme, les fonctions coordonnées x_i sont harmoniques en tant que fonctions sur cette surface de Riemann. La métrique induite par X définissant la structure de surface de Riemann de Σ, l'application X est conforme.

Maintenant, si Σ est une surface de Riemann, $X : \Sigma \to \mathbb{R}^3$ est une surface minimale si et seulement si X est une immersion harmonique et conforme. De plus, avec ce point de vue, on peut généraliser la définition des surfaces minimales. Si $X : \Sigma \to \mathbb{R}^3$ est juste supposée harmonique, non-constante et conforme, X est alors une immersion sauf en des points isolés.

Définition 1.1.2. Soit Σ une surface de Riemann. $X : \Sigma \to \mathbb{R}^3$ est *une surface minimale généralisée* si X est harmonique, non-constante et conforme. Les points où X n'est pas une immersion sont appelés *points de branchements* de la surface.

Dans la suite, on dira seulement que Σ est une surface minimale en précisant juste si elle possède ou non des points de branchement.

Les fonctions coordonnées étant harmoniques, elles ne peuvent présenter de maximum en un point intérieur à moins d'être constantes ainsi il n'existe pas d'immersion minimale d'une surface de Riemann Σ compacte sans bord. A titre d'exemple, il n'existe pas, dans $\mathbb{R}^3$, de surface minimale ayant la topologie de la sphère.

1.1.2 Représentation de Weierstrass et application de Gauss

On va maintenant voir comment on peut construire des surfaces minimales.

Soit Σ une surface de Riemann et $X = (x_1, x_2, x_3) : \Sigma \to \mathbb{R}^3$ une immersion minimale. X est alors conforme et les x_i sont harmoniques. Soit alors z une coordonnée locale sur Σ, on note $\Phi_i = \dfrac{\partial x_i}{\partial z}$. On a alors les propriétés suivantes :

1. les fonctions Φ_i sont holomorphes ce qui traduit l'harmonicité des x_i,
2. $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$ ce qui est équivalent à X est conforme,
3. $|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 \neq 0$ ce qui traduit le caractère immergé.

Réciproquement, si on a trois fonctions holomorphes Φ_i vérifiant les deux dernières propriétés et définies sur un ouvert simplement connexe de $\mathbb{C}$, l'expression suivante définit une surface minimale :

$$\Re \int (\Phi_1, \Phi_2, \Phi_3)\, \mathrm{d}z \tag{1.2}$$

Si la propriété 3 n'est pas vérifiée, il s'agit d'une surface minimale avec des points de branchement.

Construire des surfaces minimales revient donc à résoudre l'équation $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$ pour des fonctions holomorphes sous la contrainte $|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 \neq 0$. On montre alors que les solutions sont de la forme :

$$\Phi_1 = \frac{1}{2} f(1 - g^2), \quad \Phi_2 = \frac{i}{2} f(1 + g^2), \quad \Phi_3 = fg$$

avec f une fonction holomorphe et g une fonction méromorphe telles que g ait un pôle d'ordre m en z si et seulement si z est un zéro d'ordre $2m$ pour f. Si f a un zéro d'ordre supérieur en un point, la propriété 3 n'est pas satisfaite et il s'agit d'un point de branchement.

On a donc un résultat important :

Théorème 1.1.3 (Représentation de Weierstrass). *Toute surface minimale simplement connexe peut être paramétrée de la façon suivante :*

$$X = \Re \left(\int \frac{1}{2} f(1 - g^2) \mathrm{d}z, \int \frac{i}{2} f(1 + g^2) \mathrm{d}z, \int fg \mathrm{d}z \right) \tag{1.3}$$

où f est une fonction holomorphe et g une fonction méromorphe définies sur un ouvert simplement connexe de $\mathbb{C}$ telles que g ait un pôle d'ordre m en z si et seulement si z est un zéro d'ordre $2m$ pour f.

Quitte à passer au revêtement universel $\widetilde{\Sigma}$ de Σ, ce théorème permet de représenter toutes les surfaces minimales (on rappelle que le revêtement universel ne peut être compact et est donc un ouvert de $\mathbb{C}$). Cette formule de représentation permet de construire de nombreux exemples de surfaces minimales.

Par ailleurs, on peut retrouver toutes les données géométriques à partir des fonctions f et g. Par exemple, la métrique induite est donnée par :

$$\mathrm{d}s^2 = \left(\frac{|f|(1 + |g|^2)}{2} \right)^2 \mathrm{d}z \mathrm{d}\overline{z} \tag{1.4}$$

On constate que cette métrique est bien conforme à la métrique euclidienne sur $\mathbb{C}$. Par ailleurs, on montre que la normale à la surface est donnée par :

$$N = \left(\frac{2\Re(g)}{|g|^2 + 1}, \frac{2\Im(g)}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right) \tag{1.5}$$

Ceci nous dit que la fonction méromorphe g est le projeté stéréographique de l'application $N : \Sigma \to \mathbb{S}^2$. Ainsi on peut identifier la fonction méromorphe g et l'application N, ce qui nous donne :

Proposition 1.1.4. *Sur une surface minimale Σ, l'application qui, à un point, associe la normale à la surface en ce point est une application conforme de Σ dans $\mathbb{S}^2$. Cette application est appelée application de Gauss.*

1.1.3 Surface minimale conjuguée

Dans cette partie, nous allons voir comment à toute surface minimale on peut associer une autre surface minimale qui lui est localement isométrique. Cette seconde surface est un outil très important dans l'étude des surfaces minimales.

Soit $X : \Sigma \to \mathbb{R}^3$ une surface minimale, nous supposons tout d'abord Σ simplement connexe pour plus de simplicité. Pour $i = 1, 2, 3$, la fonction coordonnée x_i s'écrit $x_i = \Re \int \Phi_i \mathrm{d}z$ et est donc la partie réelle d'une fonction holomorphe, notons alors x_i^* la partie imaginaire de cette fonction holomorphe, x_i^* est appelée fonction harmonique conjuguée de x_i (remarquons que x_i^* n'est définie qu'à une constante additive près). La fonction x_i^* est harmonique sur Σ. On note alors $X^* : \Sigma \to \mathbb{R}^3$ l'application définie par les x_i^*.

$X^* : \Sigma \to \mathbb{R}^3$ est alors une surface minimale car

$$\frac{\partial x_k^*}{\partial z} = -i\frac{\partial x_k}{\partial z} = -i\Phi_k \tag{1.6}$$

la fonction $\dfrac{\partial x_k^*}{\partial z}$ satisfait donc les propriétés 1, 2 et 3 de la section précédente. De plus, ce calcul prouve que les deux surfaces sont localement isométriques.

Pour le cas général, on doit considérer le revêtement universel de Σ pour définir X^*, ceci nous donne :

Définition 1.1.5. Soit $X : \Sigma \to \mathbb{R}^3$ une surface minimale, alors $X^* : \widetilde{\Sigma} \to \mathbb{R}^3$ est appelée *surface minimale conjugée* à $X(\Sigma)$ ($\widetilde{\Sigma}$ désigne le revêtement universel de Σ). La surface minimale conjuguée n'est définie qu'à une translation près.

Si f et g sont les données de Weierstrass pour la surface $X(\Sigma)$, les données pour $X^*(\Sigma)$ sont $-if$ et g. On constate ainsi que la surface originale et sa conjuguée ont la même application de Gauss.

Si $X : \Sigma \to \mathbb{R}^3$, alors X s'écrit :

$$\int (\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3) \tag{1.7}$$

Aux différentielles $\mathrm{d}x_k$, on associe des différentielles $\mathrm{d}x_k^*$ mais pour définir la primitive X^* de celles-ci, nous devons passer, *a priori*, au revêtement universel $\widetilde{\Sigma}$. Maintenant, on peut se demander si l'application X^* n'est pas définie sur un revêtement intermédiaire, entre Σ et $\widetilde{\Sigma}$, voir même sur Σ. Pour cela on doit étudier, pour les lacets γ non triviaux de Σ, la valeur de

$$\int_\gamma (\mathrm{d}x_1^*, \mathrm{d}x_2^*, \mathrm{d}x_3^*) \tag{1.8}$$

Si cette intégrale est nulle, la définition de X^* descend à un revêtement intermédiaire. Si l'intégrale est non-nulle, la surface minimale conjuguée présente alors une invariance par translation. Tout ce problème s'appelle le problème de période et les vecteurs donnés par l'expression (1.8) sont appelés les périodes de $X^*(\widetilde{\Sigma})$.

1.1.4 Principes de réflexion de Schwarz

Intéressons nous maintenant aux symétries apparaissant dans les surfaces minimales. Nous allons noter D le disque unité de $\mathbb{C}$, $D^+ = \{z \in \mathbb{C} |\ |z| < 1,\ \Im z > 0\}$, $D^- = \{z \in \mathbb{C} |\ |z| < 1,\ \Im z < 0\}$ et $I = \{z \in \mathbb{C} |\ |z| < 1,\ z \in \mathbb{R}\}$. On a alors les deux résultats suivants, appelés principes de réflexion.

Proposition 1.1.6. *Soit $X : D^+ \cup I \to \mathbb{R}^3$ continue telle que la restriction de X à D^+ soit une surface minimale (i.e. X est harmonique, non constante et conforme). On suppose que l'image de I par X est incluse dans une droite L_0 de $\mathbb{R}^3$. Alors X peut être étendue à D par réflexion par rapport à L_0 ; $X(D)$ est alors une surface minimale. Précisément sur D^-, X est définie par :*

$$X(z) := R(X(\overline{z})) \tag{1.9}$$

où R est la symétrie par rapport à la droite L_0.

Proposition 1.1.7. *Soit $X : D^+ \cup I \to \mathbb{R}^3$ C^1 telle que la restriction de X à D^+ soit une surface minimale (i.e. X est harmonique, non constante et conforme). On suppose que l'image de I par X est incluse dans un plan P_0 de $\mathbb{R}^3$ tel que X soit normale à P_0 le long de I (la normale à X en un point de I est incluse dans P_0). Alors X peut être étendue à D par réflexion par rapport à P_0 ; $X(D)$ est alors une surface minimale. Précisément sur D^-, X est définie par :*

$$X(z) := R(X(\overline{z})) \tag{1.10}$$

où R est la symétrie par rapport au plan P_0.

Ces deux résultats nous disent sous quelles conditions on peut étendre une surface minimale au dela de son bord.

La Proposition 1.1.6 permet, si on a une partie de droite dans le bord d'une surface minimale, d'étendre cette surface par réflexion par rapport à cette droite. De même, la Proposition 1.1.7 permet, si le bord d'une surface minimale est inclus dans un plan et que la surface arrive perpendiculairement au plan, d'étendre la surface par réflexion par rapport au plan.

Maintenant, si une surface minimale possède une telle symétrie, que se passe-t'il pour la surface minimale conjuguée ? En fait, on montre que si une droite ou une partie de droite est incluse dans une surface minimale alors, dans la surface conjuguée, cette partie de droite correspond à une courbe incluse dans un plan P_0 le long de laquelle la surface est normale à P_0. De même, si une surface minimale Σ contient une courbe incluse dans un plan le long de laquelle Σ est normale au plan alors à cette courbe correspond, dans la surface minimale conjuguée, à une partie de droite. Essentiellement, une surface minimale est invariante par symétrie par rapport à une droite si et seulement si la surface minimale conjuguée est invariante par une symétrie plane. De plus, l'application de Gauss étant conservée par passage à la conjuguée, si une surface minimale contient une droite L_0 alors la courbe correspondante dans la surface conjuguée est incluse dans un plan qui est normal à L_0.

1.1.5 Flux

On va maintenant introduire une notion qui est utile dans la théorie des surfaces minimales.

Soit $X : \Sigma \to \mathbb{R}^3$ une surface minimale et γ un lacet orienté de Σ. On note ν le choix d'une normale à γ dans $X(\Sigma)$ (en général on prend ν la conormale, *i.e.* ν est telle que (N, γ', ν) est un trièdre direct), on note alors :

$$Flux(\gamma) = \int_\gamma \nu(s)\mathrm{d}s \tag{1.11}$$

où s désigne l'abscisse curviligne le long de γ. $X(\Sigma)$ étant minimale, $Flux(\gamma)$ ne dépend que de la classe d'homologie du cycle γ. Par ailleurs, si $\mathrm{d}x_i^*$ désigne la différentielle de la conjuguée de la fonction coordonnée x_i, on a :

$$Flux(\gamma) = -\int_\gamma (\mathrm{d}x_1^*, \mathrm{d}x_2^*, \mathrm{d}x_3^*) \tag{1.12}$$

On constate donc que le flux est lié au problème de période. On reviendra sur la notion de flux dans la suite.

1.1.6 Exemples

En utilisant la représentation de Weierstrass, on va présenter quelques exemples classiques de surfaces minimales.

Le premier exemple consiste à prendre pour données de Weierstrass $g(z) = 1$ et $f(z) = 1$ sur $\mathbb{C}$, la surface minimale que l'on obtient est le **plan**. Il s'agit de l'exemple le plus simple que l'on puisse imaginer.

Le deuxième exemple que l'on considère est celui correspondant aux données de Weierstrass $g(z) = z$ et $f(z) = \frac{\lambda}{z}$ sur $\mathbb{C}^*$ avec $\lambda \in \mathbb{R}_+^*$. *A priori*, $\mathbb{C}^*$ n'étant pas simplement connexe, il y a un problème de période, on montre toutefois que la surface se referme bien. La surface que l'on obtient s'appelle un **caténoïde** (Figure 1.1). En passant au revêtement universel de $\mathbb{C}^*$, on voit que ce caténoïde a pour donnée de Weierstrass sur $\mathbb{C}$, $g(\omega) = \exp(\omega)$ et $f(\omega) = \lambda$. Ainsi on a le paramétrage suivant du caténoide dans des coordonnées conformes :

$$X(u,v) = \begin{cases} \lambda \cosh(u)\cos(v) \\ \lambda \cosh(u)\sin(v) \\ \lambda u \end{cases} \tag{1.13}$$

Les caténoïdes sont donc les surfaces de révolution d'équation $r = \lambda \cosh(\frac{z}{\lambda})$ (où r, θ, z sont les coordonnées cylindriques). Il s'agit en fait des seules surfaces minimales de révolution.

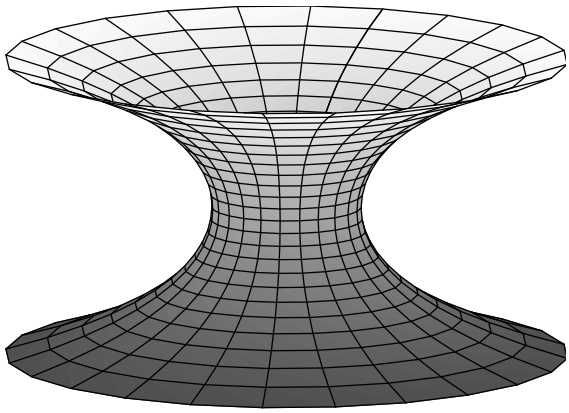

FIG. 1.1 – Le caténoïde

Le dernier exemple que l'on va regarder est celui des **hélicoïdes** (Figure 1.2), il s'agit des surfaces conjuguées aux caténoïdes. Un paramétrage de ces surfaces dans

des coordonnées conformes est donné par :

$$X(u,v) = \begin{cases} \lambda \sinh(u)\cos(v) \\ \lambda \sinh(u)\sin(v) \\ \lambda v \end{cases} \tag{1.14}$$

Géométriquement, on peut décrire ces surfaces par une droite horizontale passant par l'axe vertical d'équation $x = 0 = y$ que l'on fait monter en la tournant à vitesse constante autour de cet axe.

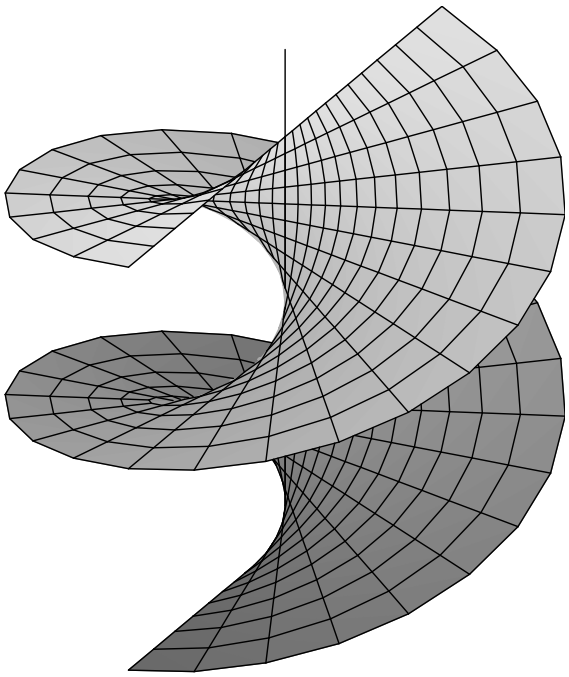

Fig. 1.2 – L'hélicoïde

Avec cette description, on constate qu'un hélicoïde est une surface réglée. Cette surface possède donc de nombreuses symétries d'après les principes de réflexion de Schwarz. Sur les caténoïdes, ces symétries se traduisent par l'invariance par rotation.

1.2 Surfaces minimales à courbure totale finie et problème de Plateau à l'infini

Dans cette section, nous allons nous intéresser aux surfaces minimales $X : \Sigma \to \mathbb{R}^3$ où X est une immersion **propre** et $X(\Sigma)$ est **complète**.

1.2.1 Structure conforme

Définition 1.2.1. Soit $X : \Sigma \to \mathbb{R}^3$ une surface minimale. On dira que cette surface est à *courbure totale finie* si :

$$-\int_\Sigma K \mathrm{d}A < +\infty \tag{1.15}$$

où K désigne la courbure sectionnelle de $X(\Sigma)$.

On rappelle que pour une surface minimale, la courbure sectionnelle K est négative en tout point.

On a alors le résultat suivant qui précise la structure conforme de Σ sous l'hypothèse de courbure totale finie.

Théorème 1.2.2 (R. Osserman [Os2]). *Soit $X : \Sigma \to \mathbb{R}^3$ une surface minimale propre et complète avec une courbure totale finie. Alors :*

1. *il existe $\overline{\Sigma}$ une surface de Riemann compacte et $p_1, \ldots, p_r \in \overline{\Sigma}$ r points tels que Σ soit conforme à $\overline{\Sigma} \backslash \{p_1, \ldots, p_r\}$ et*
2. *l'application de Gauss $N : \Sigma \to \mathbb{S}^2$ s'étend en une application méromorphe à $\overline{\Sigma}$.*

A titre d'exemple, le caténoïde qui est de courbure totale -4π satisfait les hypothèses de ce théorème. On sait que sa structure conforme est celle de $\mathbb{C}^*$, c'est-à-dire la sphère de Riemann privée de deux points.

1.2.2 Comportement asymptotique

Soit $X : \Sigma \to \mathbb{R}^3$ une surface minimale satisfaisant les hypothèses du Théorème 1.2.2. L'immersion X étant propre, le comportement asymptotique de la surface est donné par les voisinages des points p_i. On appelle alors bouts de la surface les images par X des voisinages des points p_i. Le comportement asymptotique de $X(\Sigma)$ se décompose alors en r parties, chaque partie est l'un des bouts. On va maintenant faire une hypothèse supplémentaire sur la surface pour assurer que le comportement des bouts est simple : on suppose que les bouts sont plongés.

Par ailleurs, l'application de Gauss étant méromorphe sur $\overline{\Sigma}$, la normale admet une limite en chaque point p_i. On peut maintenant préciser le comportement asymptotique de $X(\Sigma)$, on note D^* le disque unité pointé.

Théorème 1.2.3 (R. Schoen [Sc]). *Soit $X : D^* \to \mathbb{R}^3$ un bout plongé d'une surface minimale propre complète à courbure totale finie. On suppose que la valeur de N en l'origine est $(0, 0, 1)$. Alors, hors d'un compact, $X(D^*)$ est un graphe (sur un domaine extérieur du plan (x_1, x_2)) avec le comportement suivant :*

$$x_3(x_1, x_2) = \alpha \ln(\rho) + \beta + \rho^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(\rho^{-2}) \tag{1.16}$$

avec $\rho = \sqrt{x_1^2 + x_2^2}$.

Ce que dit ce théorème, c'est que chaque bout de $X(\Sigma)$ est asymptotique à un bout de caténoïde si $\alpha \neq 0$ ou à un plan si $\alpha = 0$. On appelle α la croissance du bout.

Sous les hypothèses du Théorème 1.2.2 et l'hypothèse de bouts plongés, on peut donner une expression de la courbure totale (le calcul est dû à L.P. Jorge et W.H. Meeks [JM]), on a :

$$\int_\Sigma K \mathrm{d}A = -4\pi(g + r - 1) \tag{1.17}$$

où g désigne le genre de la surface de Riemann compacte $\overline{\Sigma}$.

Replaçons nous dans le cadre du Théorème 1.2.3 . Si γ est un générateur du $\pi_1(D^*)$ et ν désigne la normale dans $X(D^*)$ pointant vers la singularité, le développement asymptotique (1.16) permet de calculer le flux de γ et on obtient $(0, 0, 2\pi\alpha)$. Ainsi pour un point p_i et γ un générateur du groupe fondamental du bout, le flux de γ que l'on appelle le flux du bout associé à p_i contient presque toute l'information sur le comportement asymptotique du bout lorsque ce flux est non nul ; en effet, on connaît la normale limite et la croissance du bout.

Par ailleurs, si V_i désigne le flux du bout associé à p_i, on a la formule suivante qui traduit un certain équilibre sur l'ensemble du comportement asymptotique de la surface :

$$V_1 + \cdots + V_r = 0 \tag{1.18}$$

1.2.3 Le problème de Plateau à l'infini

Nous allons maintenant poser un problème appelé problème de Plateau à l'infini qui est étudié dans les chapitres 2 et 3.

Le problème de Plateau classique consiste à déterminer une surface d'aire minimale pour un bord fixé. La question que nous allons nous poser est de trouver une surface minimale ayant un comportement asymptotique prescrit.

Soit r et g deux entiers positifs, on considère $v_1, \ldots, v_r$, r vecteurs unitaires de $\mathbb{R}^3$, et $\alpha_1, \ldots, \alpha_r$, r réels, tels que $\alpha_1 v_1 + \cdots + \alpha_r v_r = 0$. Le problème de Plateau à l'infini pour ces données est alors de trouver une surface minimale $X : \Sigma_g \backslash \{p_1, ..., p_r\} \to \mathbb{R}^3$ (Σ_g est une surface de Riemann compacte de genre g) qui soit propre, complète, à courbure totale finie et à bouts plongés et telle que v_i soit la normale limite en p_i et $\alpha_i v_i$ soit le flux du bout associé à p_i. Une solution de ce problème est appelée un r-noïde.

On peut trouver une étude de ce problème pour le genre 0 dans l'article de S. Kato, M. Umehara et K. Yamada [KUY].

Notre étude va porter sur le cas où le r-noïde $X(\Sigma)$ est Alexandrov-plongé.

Définition 1.2.4. Soit $X : \Sigma \to \mathbb{R}^3$ une immersion propre, $X(\Sigma)$ est *Alexandrov-plongée* si il existe W une variété de dimension 3 telle que $\Sigma = \partial W$ et X s'étende en une immersion propre de W dans $\mathbb{R}^3$.

Sous cette hypothèse, les croissances α_i des bouts sont nécessairement non-nulles. De plus, nous nous intéresserons à un cas très symétrique puisque nous supposerons que les normales limites v_i sont horizontales.

Dans leur article [CR], C. Cosín et A. Ros étudient le problème de Plateau à l'infini sous ces hypothèses dans le cas du genre 0. Leurs principaux résultats sont rappelés dans les sections 2.5 et 3.2. Notre étude vise tout d'abord à donner une nouvelle preuve du résultat de C. Cosín et A. Ros que l'on peut qualifier de constructive. Par la suite, nous utiliserons nos idées pour étudier le cas du genre 1.

Le point de départ est que, sous les hypothèses précisées ci-dessus, une solution du problème de Plateau à l'infini est symétrique par rapport à un plan horizontal. Ainsi, il suffit de construire une moitié de la surface. Le deuxième point important est que la surface conjuguée à cette moitié peut se voir alors comme le graphe d'une fonction au dessus d'un domaine relativement simple ; on connaît de plus la valeur sur le bord de cette fonction. La construction d'une solution du problème de Plateau à l'infini passe donc par la résolution d'un problème de Dirichlet (voir section 1.3.3) ; une partie de notre travail consiste donc en la construction de ces graphes. Ces idées seront précisées dans les chapitres suivants.

1.3 Les graphes minimaux

Comme nous venons de le dire les surfaces minimales qui sont des graphes vont être, dans la suite, les surfaces qui vont nous intéresser. En fait, toute surface immergée dans $\mathbb{R}^3$ est localement le graphe d'une fonction au dessus d'un domaine du plan tangent, ce point de vue est donc très général. Nous allons donc donner certains des principaux résultats sur les graphes de fonctions qui sont des surfaces minimales.

1.3.1 L'équation des surfaces minimales

Considérons $\Omega \subset \mathbb{R}^2$ un domaine du plan et u une fonction que l'on suppose deux fois continûment dérivable. Le graphe de u est une surface immergée de $\mathbb{R}^3$ et cette surface est une surface minimale si la fonction u satisfait l'équation des surfaces minimales :

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{ESM}$$

Avec les notations classiques $u_x = \dfrac{\partial u}{\partial x}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ et autres, cette équation se réécrit :

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$$

Précisons dès à présent que, dans la suite, nous noterons $W = \sqrt{1+|\nabla u|^2}$.

L'équation des surfaces minimales (ESM) est une équation aux dérivées partielles du second ordre ; elle est non-linéaire et elliptique car s'écrivant sous forme divergence (voir [GT]). Ainsi les solutions de cette équation héritent de toutes les propriétés issues de la théorie des équations aux dérivées partielles elliptiques.

Par ailleurs, on sait que les fonctions coordonnées sont harmoniques donc analytiques, on en déduit qu'une solution de (ESM) est une fonction analytique en (x, y).

1.3.2 Estimés et convergence des suites

Comme pour de nombreuses équations aux dérivées partielles elliptiques, on peut estimer les dérivées des solutions de (ESM). Par exemple, on a :

Proposition 1.3.1. *Il existe une constante positive C telle que, si u est une solution positive de* (ESM) *définie sur le disque $x^2+y^2<R^2$, on ait :*

$$W(0) \leq C \exp\left(\frac{\pi u(0)}{2R}\right) \tag{1.19}$$

Proposition 1.3.2. *Il existe une constante positive C telle que, si u est une solution de* (ESM) *définie sur le disque $x^2+y^2<R^2$, on ait :*

$$u_{xx}^2(0) + 2u_{xy}^2(0) + u_{yy}^2(0) \leq \frac{C}{R^2} W^4(0) \tag{1.20}$$

Il existe de nombreux autres estimés pour les solutions de (ESM), on pourra se référer à l'article de J.C.C. Nitsche [Ni1].

Un des intérêts de ces résultats sont les différents théorèmes de compacité sur les solutions de l'équation des surfaces minimales. Le principal résultat sur la convergence des suites de solutions de (ESM) est le suivant :

Théorème 1.3.3. *Soit (u_n) une suite de solutions de l'équation des surfaces minimales définies sur un domaine $\Omega \subset \mathbb{R}^2$. On suppose que la suite (u_n) est uniformément bornée sur tout compact K inclus dans Ω. Alors, il existe une sous-suite de (u_n) qui converge vers u une solution de* (ESM) *pour la convergence C^k uniforme sur tout compact et pour tout $k \in \mathbb{N}$.*

Ce théorème est un résultat important car il est lié à la résolution du problème de Dirichlet que nous allons évoquer dans la partie suivante. Nous reviendrons par la suite sur la convergence des suites de solutions de l'équation des surfaces minimales.

1.3.3 Le problème de Dirichlet

Le problème de Dirichlet est le problème naturel associé à une équation aux dérivées partielles, il s'agit de déterminer une solution u de l'équation des surfaces minimales sur un domaine Ω en se fixant la valeur de u le long du bord de Ω. Ce problème pose deux questions importantes : celle de l'existence de la solution et celle de l'unicité de celle-ci.

Pour des domaines bornés, la réponse la plus générale fut apportée par H. Jenkins et J. Serrin [JS]. Elle porte sur les domaines convexes et des données au bord très diverses puisque les valeurs $+\infty$ et $-\infty$ sont autorisées.

Tout d'abord nous allons donner un lemme qui précise le type de données considérées dans le résultat de H. Jenkins et J. Serrin.

Lemme 1.3.4 (Lemme de la ligne droite). *On considère un domaine borné Ω inclus dans un demi-plan délimité par une droite L. On note $T = L \cap \partial\Omega$ et C la partie complémentaire du bord. Alors pour tout compact $K \subset \Omega \cup C$ il existe une constante N positive telle que, pour toute solution u de* (ESM) *sur Ω telle que $m \leq u \leq M$ sur C, on ait :*

$$m - N \leq u \leq M + N \quad sur\ K \tag{1.21}$$

Ce résultat, avec le Théorème 10.3 de [Os2], prouve qu'une solution u de l'équation des surfaces minimales peut tendre vers $+\infty$ ou $-\infty$ le long d'un arc inclus dans le bord d'un domaine Ω uniquement si cet arc est rectiligne.

Fixons maintenant les notations pour donner le résultat de H. Jenkins et J. Serrin. Soit Ω un domaine convexe borné de $\mathbb{R}^2$, on note alors $A_1, \ldots, A_k$ une famille de segments inclus dans le bord de Ω, $B_1 \ldots, B_l$ une deuxième famille de segments inclus dans le bord de Ω. Les A_j (resp. B_j) correspondent aux segments le long desquels la donnée au bord sera $+\infty$ (resp. $-\infty$). On note $C_1, \ldots, C_n$ les arcs composant le reste du bord de Ω. Par ailleurs, si P est un polygone inclus dans Ω dont les sommets sont parmi les extrémités des segments A_j et B_j, on note α la longueur totale des segments A_j inclus dans P, β la longueur totale des segments B_j inclus dans P et γ le périmètre de P. On peut maintenant donner le résultat de H. Jenkins et J. Serrin [JS] :

Théorème 1.3.5. *Soit Ω un domaine convexe borné de $\mathbb{R}^2$ muni des familles $\{A_j\}$, $\{B_j\}$ et $\{C_j\}$.*

Si la famille $\{C_j\}$ est non vide, il existe une solution de (ESM) *sur Ω qui prend les valeurs $+\infty$ le long des A_j, $-\infty$ le long des B_j et toutes données continues le long des C_j, si et seulement si :*

$$2\alpha < \gamma \quad et \quad 2\beta < \gamma \tag{$*$}$$

pour tout polygone P inclus dans Ω dont les sommets sont parmi les extrémités des A_j et B_j. De plus, si la solution existe, elle est unique.

Si la famille $\{C_j\}$ est vide, une solution existe, si et seulement si :

$$\alpha = \beta \tag{$**$}$$

pour le polygone $P = \partial\Omega$ et $(*)$ est satisfaite pour tout autre polygone P inclus dans Ω dont les sommets sont parmi les extrémités des A_j et B_j. De plus, si la solution existe, elle est unique à l'ajout d'une constante près.

Une première remarque que l'on peut faire est qu'aucune valeur n'est imposée aux extrémités des arcs. Toutefois, ce résultat apporte une réponse complète au problème de Dirichlet puisqu'il traite de l'existence et de l'unicité de la solution. L'article de H. Jenkins et J. Serrin contient une généralisation de ce résultat au cas non-convexe, en fait ils supposent juste que les arcs C_j sont convexes. Dans le chapitre 2, nous donnerons une nouvelle généralisation de ce résultat. Tous ces résultats traitent du cas des domaines bornés, le cas des domaines non-bornés est un problème qui reste totalement ouvert ; on a tout de même des résultats partiels et on sait entre autre que l'on perd, en général, l'unicité de la solution (voir [Co]).

Le problème de l'unicité est relié à l'existence d'un principe de maximum. Pour les surfaces minimales il existe différents principes du maximum ; pour les graphes, celui-ci s'écrit de la façon suivante :

Proposition 1.3.6 (principe du maximum). *Soient Ω un domaine borné de $\mathbb{R}^2$, u et v deux solutions sur Ω de l'équation des surfaces minimales. On suppose que $\lim(u - v) \leq 0$ pour toute suite convergeant vers un point du bord de Ω. Alors $u - v \leq 0$ sur tout Ω.*

On trouvera une démonstration dans [Ni1].

Il existe d'autres versions du principe du maximum, on peut, entre autre, adapter les hypothèses au cadre décrit dans le Théorème 1.3.5.

1.3.4 Principe de réflexion de Schwarz

Nous avons vu dans la section 1.1.4, des résultats permettant d'étendre des surfaces minimales au travers de leur bord. Ces résultats se traduisent pour les graphes minimaux.

Proposition 1.3.7. *Soit Ω un domaine de $\mathbb{R}^3$ que l'on suppose inclus dans $\{y \geq 0\}$. On suppose de plus qu'il existe T, un intervalle ouvert de $\{y = 0\}$, qui soit une partie du bord de Ω. Soit u une solution de l'équation des surfaces minimales sur Ω prenant une valeur $c \in \mathbb{R}$ le long de T, u s'étend alors au domaine $\Omega' = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \in \Omega,\ (x, y) \in T$ ou $(x, -y) \in \Omega\}$ en une solution v de* (ESM) *par $v(x, y) = u(x, y)$ si $y \geq 0$ et $v(x, y) = 2c - u(x, -y)$ si $y \leq 0$.*

Cette proposition peut aussi s'interpréter comme un résultat de régularité au bord du domaine de la solution u.

1.3.5 La fonction conjuguée

Nous allons maintenant introduire un outil essentiel dans l'étude des solutions de (ESM). Considérons Ω un domaine de $\mathbb{R}^2$ et une solution u de l'équation des surfaces minimales. Notons alors $\mathrm{d}\Psi$ la différentielle suivante :

$$\mathrm{d}\Psi = \frac{u_x}{W}\mathrm{d}y - \frac{u_y}{W}\mathrm{d}x \tag{1.22}$$

La fonction u étant une solution de (ESM), la différentielle $\mathrm{d}\Psi$ est fermée et définit, au moins localement, une fonction Ψ.

Définition 1.3.8. La fonction Ψ ainsi définie est appelée *fonction conjuguée à u.*

Tout d'abord, expliquons cette terminologie ; en fait, la différentielle $\mathrm{d}\Psi$ est la différentielle de la conjuguée de la troisième fonction coordonnée sur le graphe de u exprimée dans les variables x et y, donc Ψ correspond à la conjuguée de x_3 exprimée dans les variables x et y. Précisons juste que l'orientation sur le graphe qui est nécessaire pour définir la conjuguée est fixée en imposant que la normale à la surface pointe vers le bas. Ce choix d'orientation sera d'ailleurs celui que l'on fera toujours pour un graphe.

Regardons maintenant les propriétés de la fonction conjuguée à u. Tout d'abord, la fonction Ψ n'est définie que localement et à une constante près, on parlera tout de même de la fonction Ψ lorsqu'il n'y aura pas trop d'ambiguïté. On a :

$$|\nabla\Psi| = \frac{\sqrt{u_x^2 + u_y^2}}{W} < 1 \tag{1.23}$$

Ainsi Ψ est une fonction 1-Lipschitzienne, donc elle s'étend de façon continue au bord du domaine. Calculer Ψ le long du bord est justement une question intéressante et on a les deux résultats suivants :

Lemme 1.3.9. *Soit u une solution de* (ESM) *sur un domaine Ω. On suppose que u est continue le long d'un arc convexe C appartenant au bord de Ω. Alors :*

$$\left|\int_C \mathrm{d}\Psi\right| < |C| \tag{1.24}$$

où $|C|$ désigne la longueur de l'arc C.

Lemme 1.3.10. *Soit Ω un domaine de $\mathbb{R}^2$ bordé en partie par un segment T orienté par la normale sortante. Soit u une solution de* (ESM) *sur Ω tendant vers $+\infty$ en tout point intérieur au segment T. Alors :*

$$\int_T \mathrm{d}\Psi = |T| \tag{1.25}$$

On trouve la preuve de ces deux résultats dans [JS].

Nous reviendrons sur l'utilisation de la fonction conjuguée dans la partie consacrée à la convergence des suites de solutions de l'équation des surfaces minimales.

1.3.6 Exemples

Nous allons présenter quelques exemples de solutions de l'équation des surfaces minimales.

Tout d'abord illustrons le Théorème 1.3.5 grâce à la **surface de Scherk** (Figure 1.3). On se place sur le carré $]-\frac{\pi}{2},\frac{\pi}{2}[^2$ et on considère la fonction définie par :

$$u(x,y) = -\ln(\cos x) + \ln(\cos y) \tag{1.26}$$

On montre que u satisfait l'équation des surfaces minimales, en fait, cette solution apparaît lorsque l'on cherche une solution à variables séparées (*i.e.* qui s'écrit $u_1(x)+u_2(y)$). Pour ses valeurs au bord, on constate que u prend la valeur $+\infty$ le long de $\{-\frac{\pi}{2}\}\times]-\frac{\pi}{2},\frac{\pi}{2}[$ et $\{\frac{\pi}{2}\}\times]-\frac{\pi}{2},\frac{\pi}{2}[$ et la valeur $-\infty$ le long de $]-\frac{\pi}{2},\frac{\pi}{2}[\times\{-\frac{\pi}{2}\}$ et $]-\frac{\pi}{2},\frac{\pi}{2}[\times\{\frac{\pi}{2}\}$. Sur le carré, les conditions du Théorème 1.3.5 sont donc vérifiées.

Par ailleurs, on constate que le graphe de la fonction u, qui est appelé surface de Scherk, est bordé par quatre droites verticales passant par les sommets du carré. Ainsi on peut étendre la surface de Scherk par des réflexions le long de ces droites. On obtient ainsi une surface minimale doublement périodique. Le fait qu'à l'extrémité commune d'un segment de type A_j et d'un segment de type B_j le graphe soit bordé par une droite verticale est un fait général, on démontrera ce résultat dans le chapitre 2.

Nous allons maintenant donner un exemple de solution de (ESM) sur un domaine non borné. Considérons $\Omega = \mathbb{R}_+^* \times]-\frac{\pi}{2},\frac{\pi}{2}[$ et u définie sur Ω par :

$$u(x,y) = x\tan y \tag{1.27}$$

Alors u est une solution de l'équation des surfaces minimales ; en fait, le graphe de u est un morceau d'**hélicoïde** (Figure 1.4) d'axe la droite d'équation $x = 0 = z$. La fonction u peut se voir comme la solution du problème de Dirichlet pour les données au bord : 0 sur $\{0\}\times]-\frac{\pi}{2},\frac{\pi}{2}[$, $+\infty$ sur $\mathbb{R}_+^* \times \{\frac{\pi}{2}\}$ et $-\infty$ sur $\mathbb{R}_+^* \times \{-\frac{\pi}{2}\}$.

Dans la suite, les domaines que nous étudieront seront de la forme suivante : une partie compacte à laquelle on colle des demi-bandes. Les données au bord qui nous

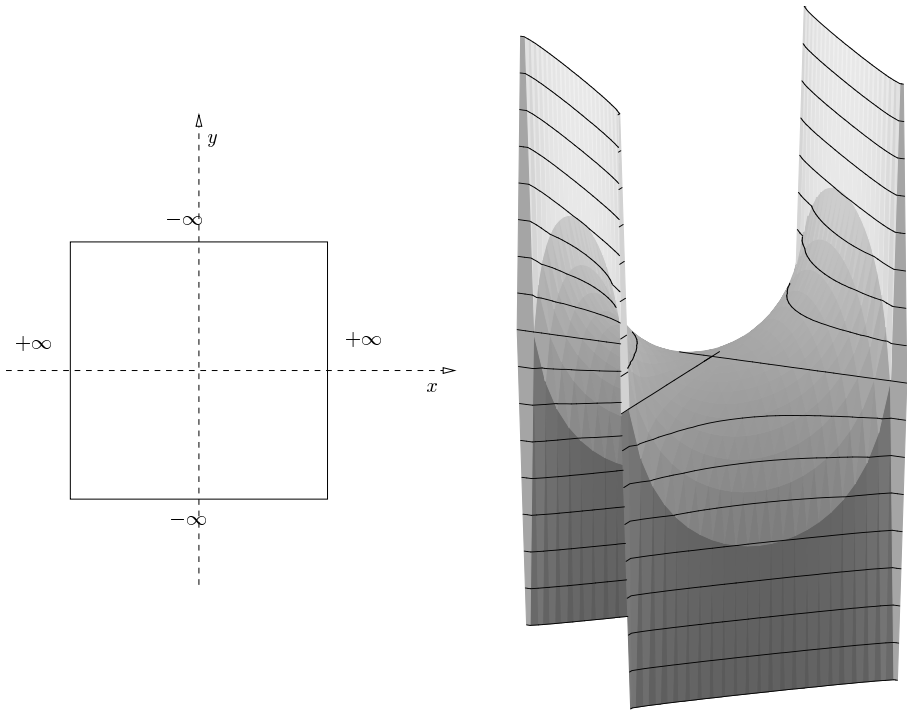

FIG. 1.3 – La surface de Scherk

intéresseront seront $+\infty$ le long d'un bord des demi-bandes et $-\infty$ le long de l'autre. Ainsi le dessin correspondant à ce morceau d'hélicoïde représente la situation que l'on rencontrera par la suite.

1.4 Convergences des suites de solutions de (ESM)

Nous allons consacrer cette dernière partie à l'étude de la convergence des suites (u_n) de solutions de l'équation des surfaces minimales sur un domaine Ω. Nous regroupons en fait des résultats que nous démontrons dans la section 2.4 et l'appendice 3.8.

1.4.1 Le domaine de convergence

Nous considérons (u_n) une suite de solutions de (ESM) définies sur un domaine $\Omega \subset \mathbb{R}^2$. La question que l'on se pose est celle d'une éventuelle convergence d'une sous-suite de (u_n).

Tout d'abord, remarquons que, si u est une solution de (ESM) alors pour (c_n) une suite de réels, la suite définie par $u_n = u + c_n$ est une suite de solutions de

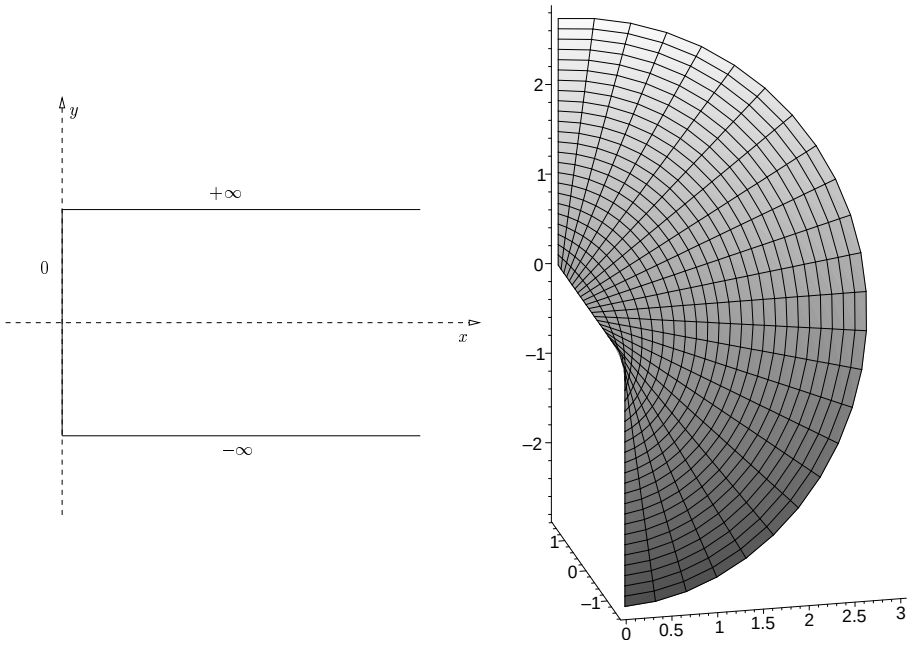

FIG. 1.4 – L'hélicoïde

(ESM) qui ne converge que si (c_n) converge. Nous allons donc nous intéresser aux suites (u_n) à translation verticale près.

Maintenant, si une suite (u_n) converge, nous savons que la suite (W_n) qui contrôle la norme du gradient doit rester bornée. On a alors un premier résultat :

Lemme 1.4.1. *Soient Ω un domaine et (u_n) une suite de solutions de* (ESM) *sur Ω. Soit $P \in \Omega$; on suppose que la suite $(W_n(P))$ est bornée par une constante M. Alors il existe $R > 0$ qui dépend seulement de M et de la distance de P à $\partial\Omega$ tel que (W_n) soit bornée par $2M$ sur le disque de centre P et de rayon R.*

Ce résultat est en fait le Lemme 2.4.1, on en trouve donc la preuve dans la section 2.4. La preuve est essentiellement basée sur l'estimé donné par la Proposition 1.3.2. Nous posons alors la définition suivante :

Définition 1.4.2. Soit (u_n) une suite de solutions de (ESM) définie sur Ω. On pose alors :

$$\mathcal{B}(u_n) = \{Q \in \Omega |\ (W_n(Q)) \text{ est bornée}\} \tag{1.28}$$

$\mathcal{B}(u_n)$ est appelé *domaine de convergence de la suite* (u_n).

Le Lemme 1.4.1 a deux conséquences importantes sur le domaine de convergence de (u_n). Tout d'abord, il implique que $\mathcal{B}(u_n)$ est un ouvert de Ω. Par ailleurs, il

implique que la suite (W_n) est uniformément bornée sur tout compact du domaine de convergence. Ainsi si P est un point de $\mathcal{B}(u_n)$ et C est la composante connexe de $\mathcal{B}(u_n)$ contenant P, il existe une sous-suite $(u_{n'} - u_{n'}(P))$ qui converge vers une solution de l'équation des surfaces minimales sur C (la convergence est celle du Théorème 1.3.3). Ceci justifie le nom donné à $\mathcal{B}(u_n)$. Tout le problème de la convergence se porte donc sur la compréhension de $\mathcal{B}(u_n)$ ou, plus précisément, de son complémentaire.

1.4.2 Les lignes de divergence

Considérons P un point dans le complémentaire du domaine de convergence. Il existe alors une sous-suite $(u_{n'})$ telle que $W_{n'}(P) \to +\infty$. Par ailleurs, la normale au graphe de u_n au point $(P, u_n(P))$ nous est donnée par :

$$N_n(P) = \left(\frac{u_{nx}(P)}{W_n(P)}, \frac{u_{ny}(P)}{W_n(P)}, -\frac{1}{W_n(P)}\right) \tag{1.29}$$

u_{nx} est la dérivée de u_n par rapport à x, on rappelle aussi le choix de la normale pointant vers le bas. Nous pouvons donc supposer que la normale $N_{n'}(P)$ converge vers un vecteur unitaire horizontal N. On a alors le résultat suivant :

Théorème 1.4.3. *Soient Ω un domaine de $\mathbb{R}^2$ et (u_n) une suite de solutions de* (ESM) *sur Ω. Soit $P \in \Omega$ et N un vecteur unitaire horizontal, on note L la géodésique de Ω passant par P et normale à N. Si la suite des normales $(N_n(P))$ converge vers N, alors $(N_n(Q))$ converge vers N pour tout point Q de L.*

Précisons tout d'abord que la géodésique L est juste la composante connexe contenant P de l'intersection de Ω avec la droite de $\mathbb{R}^2$ passant par P et normale à N. Ce résultat est en fait le Théorème 2.4.3, la preuve est basée sur le Théorème 2.4.2 qui est un résultat local.

Définition 1.4.4. Avec les notations ci-dessus, si L est une géodésique de Ω telle que $(N_{n'}(Q))$ converge vers N pour tout point Q de L avec N un vecteur unitaire horizontal et normal à L, L est appelée *une ligne de divergence* de la suite (u_n). On dira, de plus, que la sous-suite d'indice (n') fait apparaître la ligne de divergence L.

On comprend maintenant le complémentaire du domaine de convergence, en effet, si une sous-suite $N_{n'}(Q)$ converge vers un vecteur horizontal, alors $(W_n(Q))$ est non-bornée. Ainsi le complémentaire de $\mathcal{B}(u_n)$ est l'union des lignes de divergence de (u_n).

La convergence de la normale le long d'une ligne de divergence se traduit sur la suite (Ψ_n) des conjuguées des fonctions u_n. Ainsi, considérons L une ligne de divergence avec, par exemple, $N_{n'}(Q) \to N$ pour tout point Q de L et T un intervalle

borné de L orienté par v un vecteur directeur de la droite choisi de telle façon que (N, v) soit une base directe. On a alors :

$$\lim_{n' \to +\infty} \int_T \mathrm{d}\Psi_{n'} = |T| \tag{1.30}$$

D'une manière générale, c'est cette caractérisation des lignes de divergence que l'on utilisera par la suite.

1.4.3 Quelles lignes de divergence existent ?

On vient de voir que l'étude de la convergence d'une suite (u_n) de solutions de l'équation des surfaces minimales passe par la compréhension des lignes de divergence qui lui sont associées. La question qui se pose est, en fait, de comprendre quelles hypothèses sur la suite (u_n) permettent d'interdire à telle ou telle ligne de divergence d'apparaître. Les lignes de divergence étant des géodésiques, elles rencontrent le bord du domaine ou partent éventuellement à l'infini si le domaine est non-borné. Ce sont donc des conditions sur les valeurs de u_n au bord que l'on va imposer ; ce type de condition est naturel puisque l'étude de la convergence d'une suite est un outil pour la résolution du problème de Dirichlet.

Le premier résultat que l'on peut donner concerne le cas où les fonctions u_n prennent toutes une valeur infinie le long d'un segment du bord du domaine.

Proposition 1.4.5. *Soit (u_n) une suite de solutions de* (ESM) *sur $]0,1[^2$ telle que, pour tout n, la fonction u_n tende vers $+\infty$ le long de $\{1\}\times]0,1[$. Alors, aucune ligne de divergence de la suite (u_n) n'a le point $(1, \frac{1}{2})$ pour extrémité.*

Cette proposition nous dit qu'une ligne de divergence ne peut pas avoir pour extrémité un point où la suite prend une valeur infinie. Ce résultat est prouvé dans l'appendice 3.8, il s'agit du Lemme 3.8.3. Intéressons nous maintenant au cas où la valeur au bord est une constante.

Lemme 1.4.6. *Considérons Ω un domaine dont une partie du bord C est rectiligne et (u_n) une suite de solutions de* (ESM) *telle que u_n prenne la valeur $c_n \in \mathbb{R}$ le long de C. Alors aucune ligne de divergence de la suite (u_n) n'a un point intérieur à C pour extrémité.*

Démonstration. Supposons que L soit une ligne de divergence de la suite u_n ayant le point $P \in C$ pour extrémité et, de plus, que $N_n(Q) \to N$ pour tout point $Q \in L$ (N est normal à L). Quitte à considérer un voisinage de P et à faire une homothétie, on peut supposer que le domaine Ω est $D^+ = \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 < 1,\ y > 0\}$, $C =]-1,1[\times\{0\}$ et $P = (0,0)$.

La fonction u_n converge vers c_n le long de C, donc, par réflexion de Schwarz, on peut étendre u_n en une solution de (ESM) définie sur $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Maintenant pour un point Q de L, on a toujours $N_n(Q) \to N$ donc la ligne de divergence L s'étend à tout le disque et $N_n(0,0) \to N$. Par ailleurs, u_n étant constante le long de l'axe des réels, la normale en l'origine a une première coordonnée nulle. Ainsi le vecteur limite N est $(0, \pm 1, 0)$ ce qui est impossible puisque L est incluse dans le demi-disque supérieur. □

On va généraliser ce résultat à d'autres données finies sur le bord ; mais avant cela nous devons préciser le comportement de la suite des graphes au dessus d'une ligne de divergence.

Proposition 1.4.7. *Considérons Ω un domaine, P un point de Ω et (u_n) une suite de solutions de* (ESM). *Supposons que $N_n(P) \to N$ avec N un vecteur unitaire horizontal, on note $2R$ la distance de P au bord du domaine $\partial\Omega$. Alors, si $D_n(R)$ désigne le disque géodésique du graphe de u_n de centre $(P, u_n(P))$ et de rayon R, la suite $(D_n(R))$ converge vers un disque plat de rayon R et normal à N (la convergence se fait à translation verticale près).*

Démonstration. Tout d'abord, quitte à restreindre le domaine et à faire une homothétie, on peut supposer que Ω est le disque unitaire, que P en est le centre et que $N = (1,0,0)$. On se trouve donc dans la situation du Théorème 2.4.2, on va utiliser les mêmes outils que dans la preuve de celui-ci.

Fixons certaines notations, on pose $p_n = \dfrac{\partial u_n}{\partial x}$ et $q_n = \dfrac{\partial u_n}{\partial y}$; on peut alors définir $\Phi_n : (x,y) \mapsto (\xi, \eta)$ avec $\Phi_n(0,0) = (0,0)$ et :

$$\mathrm{d}\xi = \left(1 + \frac{1+p_n^2}{W_n}\right)\mathrm{d}x + \frac{p_n q_n}{W_n}\mathrm{d}y \tag{1.31}$$

$$\mathrm{d}\eta = \frac{p_n q_n}{W_n}\mathrm{d}x + \left(1 + \frac{1+q_n^2}{W_n}\right)\mathrm{d}y \tag{1.32}$$

On sait que Φ_n est un difféomorphisme sur son image car Φ_n augmente les distances et que (ξ, η) sont des paramètres conformes pour le graphe de u_n. Finalement, on note $g_n(\xi + i\eta)$ l'application de Gauss. Comme Φ_n augmente les distances, l'image de Φ_n contient le disque unitaire du plan des paramètres (ξ, η). Notre étude va maintenant se restreindre à ce disque D. De plus, on sait, par hypothèse sur la normale, que la suite (g_n) converge vers 1 sur D (voir la démonstration du Théorème 2.4.2), la convergence est uniforme sur tout compact du disque. Ceci prouve que pour tout $\varepsilon > 0$ et tout $\alpha < 1$, si n est assez grand on a :

$$\frac{p_n}{W_n} > 1 - \varepsilon \quad , \quad \frac{|q_n|}{W_n} < \varepsilon \quad \text{et} \quad \frac{1}{W_n} < \varepsilon \tag{1.33}$$

sur $\Phi_n^{-1}(D(0,\alpha))$.

On note $(x_n, y_n) = \Phi_n^{-1}$ sur le disque unité alors, si $z_n = u_n(x_n, y_n)$, (x_n, y_n, z_n) sont les fonctions coordonnées pour le graphe de u_n dans les paramètres conformes (ξ, η). On a alors :

$$\nabla x_n = \left(\frac{W_n + 1 + q_n^2}{J_n W_n}, -\frac{p_n q_n}{J_n W_n}\right) \circ (x_n, y_n) \tag{1.34}$$

$$\nabla y_n = \left(-\frac{p_n q_n}{J_n W_n}, \frac{W_n + 1 + p_n^2}{J_n W_n}\right) \circ (x_n, y_n) \tag{1.35}$$

$$\nabla z_n = \left(\frac{W_n p_n + p_n}{J_n W_n}, \frac{W_n q_n + q_n}{J_n W_n}\right) \circ (x_n, y_n) \tag{1.36}$$

avec $J_n = \det(\mathrm{d}\Phi_n) = W_n + 2 + \frac{1}{W_n}$. On en déduit que le facteur conforme de la métrique vaut $\frac{J_n}{W_n}$. D'après les équations (1.33), le facteur conforme tend vers 1 uniformément sur tout compact du disque, donc pour n assez grand le disque géodésique $D_n(1/2)$ est inclus dans la partie paramétrée par le disque de centre l'origine et de rayon 3/4 du plan $\xi\eta$. Les majorations (1.33) et les équations (1.34), (1.35) et (1.36) montrent que $\nabla x_n \to (0,0)$, $\nabla y_n \to (0,1)$ et $\nabla z_n \to (1,0)$, les convergences sont uniformes sur le disque de rayon 3/4. Ceci prouve que la suite des disques géodésiques $D_n(1/2)$ converge bien vers un disque plat normal à $N = (1,0,0)$. □

En utilisant ce résultat, nous allons pouvoir démontrer le résultat suivant qui est une généralisation du Lemme 1.4.6 :

Proposition 1.4.8. *Soient Ω un domaine de $\mathbb{R}^2$, C un arc convexe appartenant au bord de Ω et (f_n) une suite de fonctions continues définies sur C, on suppose que (f_n) converge uniformément vers f, une fonction continue sur C. Considérons (u_n) une suite de solutions de l'équation des surfaces minimales sur Ω telle $u_n = f_n$ le long de C ; alors aucune ligne de divergence de la suite (u_n) n'a un point intérieur à C pour extrémité.*

Démonstration. Tout d'abord, d'après le Lemme de la ligne droite 1.3.4 et l'estimé (1.19), le résultat est vrai si C est strictement convexe (*i.e.* ne contient pas de segment).

On supposera donc maintenant que C est un segment. Supposons que L soit une ligne de divergence de la suite (u_n) ayant un point P intérieur à C pour extrémité. Pour simplifier, on supposera que $N_n(Q) \to N$ pour tout point de L. De même que dans le Lemme 1.4.6, quitte à considérer un voisinage de P et à faire une homothétie, on peut supposer que $\Omega = D^+$, $C =]-1,1[\times\{0\}$ et $P = (0,0)$.

Considérons $\varepsilon > 0$, il existe alors $0 < \eta < 1$ tel que sur $]-\eta,\eta[\times\{0\}$, $|f - f(P)| < \varepsilon$. On pose alors, pour tout $n \in \mathbb{N}$, g_n la fonction définie sur C par : sur $]-\eta,\eta[\times\{0\}$,

$g_n = f(P)$ et sur $C\backslash]-\eta,\eta[\times\{0\}$, $g_n = f_n$. Alors en utilisant le Théorème 1.3.5, on définit sur D^+ la solution v_n de l'équation des surfaces minimales satisfaisant $v_n = g_n$ le long de C et $v_n = u_n$ le long du reste du bord du domaine. La preuve consiste à montrer que la suite v_n admet L comme ligne de divergence ce qui est impossible d'après le Lemme 1.4.6.

Tout d'abord, constatons que, pour n assez grand, on a $f_n - 2\varepsilon < g_n < f_n + 2\varepsilon$ ainsi, d'après le principe du maximum, pour n assez grand :

$$u_n - 2\varepsilon < v_n < u_n + 2\varepsilon \tag{1.37}$$

Pour simplifier, dans ce qui suit, on supposera que ces inégalités sont satisfaites pour tout n. Le graphe de v_n est ainsi compris entre le translaté de 2ε vers le bas du graphe de u_n et son translaté de 2ε vers le haut.

Fixons un point Q appartenant à la ligne de divergence L et notons $2R$ la distance de Q au bord de D^+. Alors, grâce à la Proposition 1.4.7, on connait le comportement du graphe de u_n au voisinage du point $(Q, u_n(Q))$. On fixe maintenant ε de sorte que $R > 4\varepsilon$. Nous nous intéressons au comportement des dérivées de v_n au point Q, tout se passe donc à une translation verticale près ; nous supposons donc que $u_n(Q) = 0$.

Supposons tout d'abord que la suite $(W_{v_n}(Q))$ soit bornée, on a $-2\varepsilon < v_n(Q) < 2\varepsilon$, donc il existe une sous-suite $v_{n'}$ qui converge vers v sur un disque centré en Q et de rayon r. On a $-2\varepsilon \leq v(Q) \leq 2\varepsilon$. Par ailleurs, le disque géodésique du graphe de $u_n - 2\varepsilon$ de rayon R et de centre $(Q, -2\varepsilon)$ converge vers le disque plat de centre $(Q, -2\varepsilon)$ et de rayon R normal à N. Comme $R > 4\varepsilon$, ce disque et le graphe de v s'intersectent transversalement et donc pour n' assez grand, les graphes de $v_{n'}$ et de $u_{n'} - 2\varepsilon$ s'intersectent ce qui est impossible d'après (1.37).

Si $(W_{v_n}(Q))$ n'est pas bornée, on peut supposer que la suite des normales aux graphes de v_n, (N_{v_n}), vérifie $N_{v_{n'}} \to N'$ avec N' un vecteur unitaire horizontal. Si $N' = \pm N$ alors L est une ligne de divergence pour la suite v_n, la preuve est alors finie. Montrons que $N' \neq \pm N$ est impossible. D'après la Proposition 1.4.7, les disques géodésiques des graphes de $v_{n'}$ de centre $(Q, v_{n'}(Q))$ et de rayon R convergent vers un disque plat de rayon R, de centre (Q, t) (avec $-2\varepsilon \leq t \leq 2\varepsilon$) et normal à N'. Ce disque plat intersecte transversalement le disque limite des disques géodésiques des graphes de $u_n - 2\varepsilon$ de centre $(Q, -2\varepsilon)$ et rayon R puisque $N' \neq \pm N$ et $R > 4\varepsilon$. Donc pour n' assez grand, les graphes de $v_{n'}$ et $u_{n'} - 2\varepsilon$ s'intersectent, ce qui est impossible d'après (1.37). □

Les propositions 1.4.5 et 1.4.8 permettent d'interdire directement l'existence de certaines lignes de divergence. L'étude du domaine de convergence nécessite aussi parfois de comprendre le comportement d'une éventuelle limite de la suite (u_n). Dans cette direction, on a le résultat suivant :

Proposition 1.4.9. *Soit* (u_n) *une suite de solutions de* (ESM) *sur* $]0,1[^2$ *telle que* (u_n) *converge vers une solution* u *et telle que l'on soit dans l'un des deux cas suivants :*

- *pour tout* $n \in \mathbb{N}$*, la fonction* u_n *tend vers* $+\infty$ *sur* $\{1\}\times]0,1[$*, ou,*
- *pour tout* $n \in \mathbb{N}$*, la fonction* u_n *est la restriction à* $]0,1[^2$ *d'une solution* v_n *de* (ESM) *définie sur* $]0,2[\times]0,1[$ *et, pour tout* $y \in]0,1[$*, on a* $N_n(1,y) \to (1,0,0)$.

Alors, la fonction limite u *tend vers* $+\infty$ *sur* $\{1\}\times]0,1[$.

Il s'agit du Lemme 3.8.4.

1.4.4 Remarques

Nous avons énoncé les résultats ci-dessus dans un certain cadre pour avoir une certaine unité dans les énoncés, nous allons maintenant donner quelques explications sur certains cadres plus généraux où on peut les appliquer.

Tout d'abord, il se peut que le domaine Ω de définition de la suite ne soit pas fixe. Donc considérons la situation d'une suite de domaine (Ω_n) et des solutions (u_n) de (ESM) définies sur Ω_n, on souhaite parler de la convergence de la suite (u_n). Tout d'abord, si la suite de domaine est croissante, on peut appliquer les résultats des sections précédentes sur le domaine limite qu'est la réunion des Ω_n. Lorsque la suite n'est pas croissante, il faut pouvoir donner un sens au domaine limite. Le domaine que l'on considère dans ce cas est l'intérieur de la limite inférieure des Ω_n, c'est-à-dire qu'un point est dans cet ensemble si il existe un voisinage de ce point dans $\mathbb{R}^2$ qui est inclus dans tous les Ω_n pour n assez grand. Pour ce domaine limite on peut alors appliquer les résultats sur les lignes de divergence. On peut, entre autre, adapter les propositions 1.4.5, 1.4.8 et 1.4.9 pour les appliquer dans ce cadre.

La deuxieme remarque que l'on peut faire concerne la généralisation de la notion de domaine. On verra dans les chapitres suivants que, pour traiter un nombre plus grand de surfaces minimales comme des graphes, on introduit la notion de multi-domaine : il s'agit essentiellement d'un espace qui est localement un domaine de $\mathbb{R}^2$. On étudiera, sur ces multi-domaines, les suites de solutions de l'équation des surfaces minimales. Pour faire cette étude, on pourra alors utiliser les lignes de divergences (le Théorème 1.4.3 est d'ailleurs prouvé dans le cadre des multi-domaines).

La dernière remarque que l'on va faire explique que les lignes de divergence étudient plus la suite des dérivées de u_n que la suite (u_n) elle-même. Considérons sur un domaine Ω deux suites de fonctions (p_n) et (q_n) satisfaisant :

1. $\dfrac{\partial p_n}{\partial y} = \dfrac{\partial q_n}{\partial x}$ et
2. $\operatorname{div} \dfrac{(p_n, q_n)}{\sqrt{1+p_n^2+q_n^2}}$.

La première équation nous dit que, localement, il existe une fonction u_n dont p_n et q_n sont les dérivées et la seconde équation nous dit que u_n satisfait (ESM). On peut alors mener l'étude de la convergence de la suite (p_n, q_n) comme celle des solutions de (ESM) : on a un domaine de convergence qui correspond au point où (p_n, q_n) reste bornée et, sur le domaine de convergence, des sous-suites $(p_{n'}, q_{n'})$ convergeant vers (p, q) un couple de fonctions satisfaisant les deux équations ci-dessus. Le complémentaire du domaine de convergence correspond à des lignes de divergence. Ce point de vue est intéressant si le domaine Ω n'est pas simplement connexe, on peut ainsi étudier la suite des dérivées même si celles-ci définissent une fonction multi-valuée sur le domaine.

Chapitre 2

De nouveaux outils pour le problème de Dirichlet et le problème de Plateau à l'infini en genre 0

Ce chapitre reprend un article dont le titre original est «The Dirichlet problem for the minimal surfaces equation and Plateau problem at infinity» ; il apparaît dans la bibliographie à la référence [Ma1].

Le but de ce papier est de donner une nouvelle preuve d'un résultat donné par C. Cosín et A. Ros [CR] qui affirme que si un polygone borde un disque immergé alors ce polygone est le polygone de flux d'un r-noïde de genre 0 Alexandrov plongé (Théorème 2.5.3). Un exemple d'un tel r-noïde est donné par la figure 2.1, il s'agit d'un exemple construit par L.P. Jorge et W.H. Meeks [JM] (l'image est fournie par la biliothèque de surfaces minimales de GRAPE [MSL]). Pour résoudre ce problème, nous introduisons la notion de multi-domaine qui permet de traiter un plus grand nombre de surfaces minimales en tant que graphes. Les deux résultats importants de ce papier sont le Théorème 2.3.2 qui précise le comportement du graphe d'une solution de (ESM) lorsque l'on passe de la donnée au bord $+\infty$ à la donnée $-\infty$ et le Théorème 2.4.3 qui permet d'introduire la notion de ligne de divergence pour une suite de solutions de l'équation des surfaces minimales.

2.1 Introduction

One classical way to construct minimal surfaces in $\mathbb{R}^3$ is to see them as the graph of a function u over a domain $\Omega \subset \mathbb{R}^2$ (see for example the paper of H. Karcher [Ka]). The graph of a function u is a minimal surface if u satisfies the elliptic partial

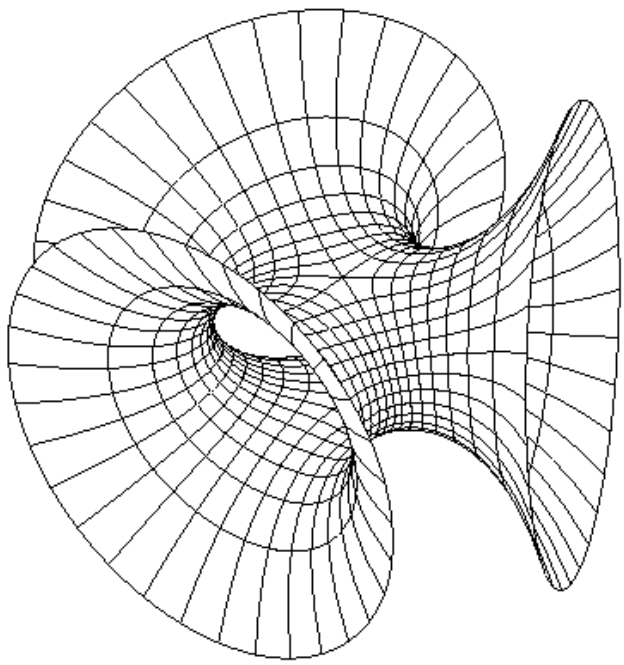

FIG. 2.1 – Un exemple de 3-noïde

differential equation called the minimal surfaces equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0 \tag{MSE}$$

The problem which is associated to this point of view is the Dirichlet problem for the equation (MSE): for a domain Ω and a function f on $\partial\Omega$, this problem consits in finding a continuous function u on $\overline{\Omega}$ which is a solution of the minimal surfaces equation in Ω and such that $u=f$ on the boundary of Ω. One of the most general answers to the Dirichlet problem for bounded domain has been given by H. Jenkins and J. Serrin in [JS]. They give a nice condition on the domain to solve for any function f; moreover, their result allows us to give infinite value for the boundary data f. For unbounded domain, the Dirichlet problem is still an open problem. We know that, in the general case, we lose the uniqueness of solution. In this paper, using a new approach, we develop some tools for the study of this problem.

An other interesting and still open problem concerning minimal surfaces is the Plateau problem at infinity which is the following: finding a minimal surface for a given asymptotic behaviour. More precisely, we know that, if a complete minimal

surface has finite total curvature and embedded ends, each end of this minimal surface is asymptotic to a plane or to a half-catenoid; besides, we can associate to each end a vector in $\mathbb{R}^3$, this vector is called the flux vector of the end. These vectors satisfy the following condition: the sum of the flux vectors over all ends is zero. So the problem is: given a finite number of vectors such that their sum is zero, can we find a minimal surface which has these vectors as flux vectors? Our answer comes from the following idea: seeing a solution of the Plateau problem at infinity as the conjugate surface of a solution of the Dirichlet problem on an unbounded domain.

In [CR], C. Cosín and A. Ros give a description of the space of solutions of the Plateau problem at infinity with an asymptotic behaviour which is symmetric with respect to an horizontal plane (*i.e.* all the flux vectors are horizontal). They also restrict themselves to the case of Alexandrov embedded minimal surfaces; this condition implies that no flux vector is zero and that there is a natural order on the ends of the surface. Since the flux vectors are horizontal and their sum is zero, these vectors draw a polygon in $\mathbb{R}^2$. C. Cosín and A. Ros give a necessary and sufficient condition on this polygon to have a solution. See section 2.5, for more explanations about their work.

In this paper, we give a more constructive proof of the result of C. Cosín and A. Ros. Our method is based on the Dirichlet problem on an unbounded "domain" Ω. When the polygon given by the flux vectors is convex, Ω can be defined as the polygonal domain bounded by the flux polygon to which we glue a half-strip on each edge. We note L_i^+ and L_{i+1}^- the two sides of each half-strip S_i, alternating the sign $+$ and $-$ such that each vertex of the polygon is common to some L_i^- and L_i^+. When the flux polygon is non-convex and satisfies the condition of C. Cosín and A. Ros, we need to use the concept of mutli-domain for defining Ω (see Definition 2.2.1 for this concept).

Our main result for the Dirichlet problem for this kind of domain Ω is then (see Theorem 2.6.1):

There exists a solution u of the minimal surfaces equation on Ω such that u tends to $+\infty$ on L_i^+ and $-\infty$ on L_i^-. Besides, the solution is unique up to an additive constant

This statement is a new result for the Dirichlet problem on unbounded domains that are similar to the ones studied by H. Rosenberg and R. Sa-Earp in [RSE].

The function u in this result is build as the limit of solutions of the Dirichlet problem on bounded domain. We describe the possible divergences that can occur for a sequence of solutions of (MSE). In fact, we prove that if the sequence diverges at a point, it must diverge along a line passing by this point. This result is a generalization of the results that H. Jenkins and J. Serrin use in [JS]. Our result allows us to do the same discussion that H. Jenkins and J. Serrin made in the

particular case of monotone sequences of solutions of (MSE); this is our main tools to prove the existence part of Theorem 2.6.1.

The solution to the Plateau problem at infinity is then the conjugate surface to the graph of u. In order to know the geometry of the conjugate surface along its boundary, we need to understand the behaviour of the graph in the neighborhood of the vertices of Ω which are the vertices of the polygon. Some results are known for such problem concerning the Dirichlet problem in the convex case. For example, consider f a data on the boundary of a domain Ω, we suppose that f has a finite discontinuity at a point P where the boundary is convex (*i.e.* we suppose that $f(Q)$ has a limit if we tend to P by the right hand side or by the left hand side and that the difference of these two limits is finite), then we know that the graph of a solution u over Ω of the Dirichlet problem with f as boundary value, has a vertical segment over P in its boundary, it was proved in [Ni2]. In our case, we can prove that the boundary of the graph is the vertical straight line passing by the vertex ; although the domain is locally an angular sector that not need to be convex and the boundary data takes the values $+\infty$ on one side of the sector and $-\infty$ on the other side.

The paper is organized as follows; in the first section, we define multi-domains and extend the result of H. Jenkins and J. Serrin to bounded multi-domains. The multi-domains are necessary to express the condition of C. Cosín and A. Ros. This result will be our first tool in the proof of our main theorem.

The second section is devoted to the proof of our result concerning the boundary behaviour of solutions of the Dirichlet problem.

In section 2.4, we study the sequences of solutions of (MSE) and define the lines of divergence.

In section 2.5, we explain the result of C. Cosín and A. Ros, and recall some elements of their proof. In the last section, we give the proof of our main result. We then use it to give a new proof of the result of C. Cosín and A. Ros.

Let us fix some notations. In the following, when u is a function on a domain of $\mathbb{R}^2$ we shall note $W = \sqrt{1+|\nabla u|^2}$. We shall also use the classical following notations for partial derivatives: $p = \dfrac{\partial u}{\partial x}$, $q = \dfrac{\partial u}{\partial y}$, $r = \dfrac{\partial^2 u}{\partial x^2}$, $s = \dfrac{\partial^2 u}{\partial x \partial y}$ and $t = \dfrac{\partial^2 u}{\partial y^2}$. Besides, for the graph of u, we shall always chose the downward pointing normal to give an orientation to the graph.

In this paper, we shall call an extraction a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$. If a_n is a sequence and θ is an extraction, the sequence $a_{\theta(n)}$ is a subsequence of a_n and every subsequence of a_n can be written in this way. If θ_1 is an extraction, we shall say that θ_2 is a sub-extraction of θ_1 if there exists an extraction α such that $\theta_2 = \theta_1 \circ \alpha$.

2.2 The Dirichlet problem on multi-domains

In this section, we shall give a generalization of the results of H. Jenkins and J. Serrin [JS] for the Dirichlet problem on bounded domain. First we have to generalize the notion of domain of $\mathbb{R}^2$. Let us consider a pair (Ω, φ) where Ω is a simply-connected 2-dimensional complete flat manifold with piecewise smooth boundary and $\varphi : \Omega \longrightarrow \mathbb{R}^2$ is a local isometry. The map φ is called the *developing map* and the points where the boundary $\partial\Omega$ are not smooth are called *vertices*.

Definition 2.2.1. A pair (Ω, φ), where Ω is a simply-connected 2-dimensional complete flat manifold with piecewise smooth boundary and $\varphi : \Omega \longrightarrow \mathbb{R}^2$ is a local isometry, is a *multi-domain* if each connected component of the smooth part of $\partial\Omega$ is a convex arc.

If (Ω, φ) is as above and a part of $\partial\Omega$ is linear then we add two vertices at the end points of this linear part and we call this new part an *edge*.

Let (Ω, φ) be a multi-domain, if u is a smooth function on Ω we shall call *graph* of u the surface in $\mathbb{R}^3$ given by $\{(\varphi(x), u(x))\}_{x\in\Omega}$. If u is a solution of the minimal surfaces equation (MSE), the graph of u is a minimal surface of $\mathbb{R}^3$. The Dirichlet problem on multi-domain consists in the determination of a function u satisfying the equation (MSE) on Ω and taking on assignated values on the boundary of Ω.

As in the case of a domain in $\mathbb{R}^2$, if u is a solution of (MSE) on Ω, we can define a differential form $\mathrm{d}\Psi_u$ on Ω which corresponds to the differential of the third coordinate of the conjugate surface of the graph of u. In using the charts given by the developing map φ, we have $\mathrm{d}\Psi_u = \frac{p}{W}\mathrm{d}y - \frac{q}{W}\mathrm{d}x$. $\mathrm{d}\Psi_u$ is a closed form by (MSE) and, since Ω is simply connected, we can define a function Ψ_u on Ω which is 1-Lipschitz continuous, we call this function the *conjugate function* to u. One important result concerning $\mathrm{d}\Psi_u$ is the following lemma.

Lemma 2.2.2. *Let Ω be a domain bounded in part by a straight segment T, oriented such that the right hand normal to T is the outer normal to Ω. Let u be a solution of* (MSE) *in Ω which assumes the boundary value $+\infty$ on T. Then*

$$\int_T \mathrm{d}\Psi_u = |T|$$

This is Lemma 4 in [JS]. For other properties of Ψ_u and $\mathrm{d}\Psi_u$, we refer to this paper.

When Ω is compact there is a finite number of connected components of the smooth part of $\partial\Omega$; let us call them $C_1, \ldots, C_n$. When the data on the boundary is bounded, we have this result:

Theorem 2.2.3. *Let (Ω, φ) be a compact multi-domain with boundary arcs $C_1, \ldots, C_n$ and let $u_1, \ldots, u_n$ be bounded continuous functions respectively on $C_1, \ldots, C_n$. Then there exists a unique solution u of the minimal surfaces equation on Ω such that $u_{|C_i} = u_i$.*

Proof. The proof of the uniqueness is a particular case of the proof of Theorem 2.2.5, so we make it later.

The existence of the solution on multi-domain is due to a Perron process, let us recall some elements of this method. If v is a continuous function on Ω and D is a disk in Ω, we note by $u_{v,D}$ the solution of (MSE) in D which takes the value v on ∂D. We also note $M_D[v]$ the continuous function which coincides with v on $\Omega \backslash D$ and $u_{v,D}$ on D. Let $u_1, \ldots, u_n$ be the data on the boundary of Ω; we say that v is a sub-solution of the Dirichlet problem if $v \leq u_i$ on C_i and $v \leq M_D[v]$ for all disks D in Ω. Since the u_i are bounded by a constant M, the class $\mathcal{F}$ of all sub-solutions is non-empty: the constant function $-M$ is in; besides, each sub-solution v verifies $v \leq M$. So we can define a function u by:

$$\forall P \in \Omega \qquad u(P) = \sup_{v \in \mathcal{F}} v(P) \tag{2.1}$$

By standard argument, we can show that u is a solution of (MSE). Since in our definition of multi-domain we suppose that the boundary is locally convex, there exist barrier functions on the boundary (they are constructed in using the Scherk surface). So we can insure that u takes the value u_i on C_i. For more details on Perron process, we can refer to the book of D. Gilbarg and N.S. Trudinger [GT] or the one of R. Courant and D. Hilbert [CH] which illustrate this method for the classical Laplacian Dirichlet problem, there is also the book of J. C. C. Nitsche [Ni2] which studies the case of the minimal surfaces equation. □

The work of H. Jenkins and J. Serrin is to allow infinite data on the boundary. By the Straight Line Lemma [JS], we know that infinite data can only be allowed on linear parts of the boundary.

Definition 2.2.4. Let (Ω, φ) be a multi-domain, a *polygonal domain* $\mathcal{P}$ of Ω is a connected compact subset of Ω such that $(\mathcal{P}, \varphi)$ is a multi-domain, the boundary of $\mathcal{P}$ is only composed of edges and the vertices of $\mathcal{P}$ are drawn from the vertices of Ω.

We want to solve the Dirichlet problem with infinite data so let us call $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ the edges of Ω such that we assign the value $+\infty$ on A_j and $-\infty$ on B_j. We call $C_1, \ldots, C_n$ the remaining arcs on which we assign continuous data.

Let $\mathcal{P}$ be a polygonal domain of Ω. We note, respectively, α and β the total length of the edges A_j and the one of the edges B_j which belong to the boundary of $\mathcal{P}$ and we note γ the perimeter of $\mathcal{P}$. We then have the following generalization of the result of H. Jenkins and J. Serrin.

Theorem 2.2.5. *Let (Ω, φ) be a compact multi-domain with the families $\{A_j\}$, $\{B_j\}$ and $\{C_j\}$ as above.*

If the familly $\{C_j\}$ is non-empty, then there exists a solution of the minimal surface equation in Ω which assumes the value $+\infty$ on each A_j, the value $-\infty$ on each B_j and arbitrarily assignated continuous data on each C_j, if and only if

$$2\alpha < \gamma \quad \textit{and} \quad 2\beta < \gamma \tag{$*$}$$

for each polygonal domain $\mathcal{P}$ of Ω. If a solution exists, it is unique.

If the familly $\{C_j\}$ is empty, then a solution exists, if and only if

$$\alpha = \beta$$

when $\mathcal{P}$ coincides with Ω and $(*)$ holds for all other polygonal domains of Ω. In this case, if a solution exists, it is unique up to an additive constant.

Proof. To prove the existence of a solution, we can use the same arguments than H. Jenkins and J. Serrin, so we refer to [JS].

The proof of the uniqueness in [JS] works also but we give another proof which we can apply in other situations. Let u_1 and u_2 be different solutions of (MSE) with the same data on the boundary. In the case where the familly (C_j) is empty, we suppose that $u_1 - u_2$ is not constant; besides, in considering $u_i - u_i(P)$ (where $P \in \Omega$), we can assume that $\{u_1 < u_2\}$ and $\{u_1 > u_2\}$ are non-empty. In choosing sufficiently small $\varepsilon > 0$, we have $\Omega_\varepsilon = \{u_1 - u_2 > \varepsilon\} \neq \emptyset$, besides the choise of ε is such that $\partial\Omega_\varepsilon$ is regular. We note $\mathrm{d}\tilde{\Psi} = \mathrm{d}\Psi_{u_1} - \mathrm{d}\Psi_{u_2}$, since $\mathrm{d}\tilde{\Psi}$ is closed, we have $\displaystyle\int_{\partial\Omega_\varepsilon} \mathrm{d}\tilde{\Psi} = 0$. Because u_1 and u_2 have the same data on the boundary, $\partial\Omega_\varepsilon$ does not intersect $\cup_j C_j$ so $\partial\Omega_\varepsilon$ is composed of three parts: one is included in $\cup_j A_j \bigcup \cup_j B_j$ on which $\mathrm{d}\tilde{\Psi} = 0$ (this is a consequence of Lemma 2.2.2), one is included in Ω and a last part which is composed of some vertices of Ω but its contribution to the integral is zero. On the second part, let us call it $\widetilde{\partial\Omega_\varepsilon}$, $\nabla u_1 - \nabla u_2$ points in Ω_ε, this part is then oriented by the non-direct normal to $\nabla u_1 - \nabla u_2$ so, by Lemma 2 of P. Collin and R. Krust in [CK], $\displaystyle\int_{\widetilde{\partial\Omega_\varepsilon}} \mathrm{d}\tilde{\Psi} < 0$; this gives us a contradiction. $\square$

2.3 A result of regularity at the vertices

The aim of this section is to understand what geometrically happens at a vertex of a multi-domain where two edges A_j and B_j converge.

For $\beta_1 < \beta_2$ and $R > 0$, we consider:

$$\Omega_{\beta_1}^{\beta_2}(R) = \{(r, \theta) |\ 0 \leq r \leq R,\ \beta_1 \leq \theta \leq \beta_2\}$$

with the metric $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ (we identify all the points $(0,\theta)$ and this point will be called the vertex of $\Omega_{\beta_1}^{\beta_2}(R)$). We define also on $\Omega_{\beta_1}^{\beta_2}(R)$ the map $\varphi : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$. Then $(\Omega_{\beta_1}^{\beta_2}(R), \varphi)$ is a multi-domain, it is a description of a neighborhood of a vertex where two edges converge. We call $L(\beta)$ the set of points in $\Omega_{\beta_1}^{\beta_2}(R)$ such that $\theta = \beta$. We are interested in the geometrical "configuration" of the graph of a solution u of (MSE) such that u tends to $-\infty$ on $L(\beta_2)$ and $+\infty$ on $L(\beta_1)$; such a solution u will be called a solution of the problem $\mathcal{P}$.

The first thing we have to do to understand a solution u of the problem $\mathcal{P}$ is being able to bound the function u on each radius $L(\beta)$. Our arguments are based on the comparison with the Scherk surface.

Let us consider ABC an isosceles triangle ($|AB| = |AC| = R$), we consider the solution w of the Dirichlet problem on ABC such that $w = 0$ on $[A,B]$ and $[A,C]$ and tends to $+\infty$ on $[B,C]$; this function exists by Theorem 2.2.5. When ABC is right-angled w is the Scherk surface, after dilatation, w is given by:

$$w(x,y) = h(x,y) = -\ln\cos x + \ln\cos\ y \tag{2.2}$$

In the general case, the solution w will be called a *pseudo Scherk surface.*

We shall use the Scherk surface to control solutions of the problem $\mathcal{P}$. We first consider the case where ABC is right-angled. In fact, a neighborhood of B in ABC can be isometrically parametrized by $\Omega^0_{-\frac{\pi}{4}}(R)$ and h is a solution of (MSE) on $\Omega^0_{-\frac{\pi}{4}}(R)$ such that $h = 0$ on $L(0)$, $+\infty$ on $L(-\frac{\pi}{4})$ and some positive function on the third part of the boundary. Since we have an expression for h we can see that h is uniformly bounded on $\Omega^0_\alpha(R) \subset \Omega^0_{-\frac{\pi}{4}}(R)$ for every $-\frac{\pi}{4} < \alpha < 0$.

We do not suppose now that ABC is right-angled; but we suppose that the angle at the vertex A is greater than $\frac{\pi}{2}$. In this case we can choose a point A' such that $A'BC$ is isosceles and right-angled and $A'BC$ contains ABC. We consider in ABC the pseudo Scherk surface w and h the Scherk surface on $A'BC$; since h is positive in $A'BC$, we have $h > w$. As above, a neighborhood of B in ABC can be isometrically parametrized by $\Omega^0_\beta(R)$ with $\beta < 0$ and w can be seen as the solution of (MSE) on $\Omega^0_\beta(R)$ such that $w = 0$ on $L(0)$, $+\infty$ on $L(\beta)$ and some positive function on the third part of the boundary. Since $w < h$, w is uniformly bounded on $\Omega^0_\alpha(R)$ for every $\beta < \alpha < 0$.

By our expression for h, there exists $m \in \mathbb{R}$ such that $h \le m$ on $[A,B]$ and $[A,C]$. This proves that $h - m \le w$ in ABC. Then in our parametrization of a neighborhood of B, for every $M \in \mathbb{R}$ there exist α such that $w \ge M$ in $\Omega^\alpha_\beta(R)$.

Lemma 2.3.1. *Let $\beta_1 < \beta_2$ and $R > 0$. We consider a solution u of the problem $\mathcal{P}$ on $\Omega_{\beta_1}^{\beta_2}(R)$. Then for every $\beta_1 < \alpha < \beta_2$, there exist M and M' in $\mathbb{R}$ such that $u \le M$ in $\Omega_\alpha^{\beta_2}(\frac{R}{4})$ and $u \ge M'$ on $\Omega_{\beta_1}^\alpha(\frac{R}{4})$. For every $M \in \mathbb{R}$, there exist α and α' in $]\beta_1, \beta_2[$ such that $u \ge M$ in $\Omega_{\beta_1}^\alpha(\frac{R}{4})$ and $u \le M$ in $\Omega_{\alpha'}^{\beta_2}(\frac{R}{4})$.*

Proof. Let us consider $\alpha_1 < \alpha_2$ and $R' > 0$. We consider v the solution of the problem $\mathcal{P}$ on $\Omega_{\alpha_1}^{\alpha_2}(R')$ such that $v = 0$ on the third part of the boundary; v exists because the hypotheses of Theorem 2.2.5 are fulfilled. The isometry of $\Omega_{\alpha_1}^{\alpha_2}(R') \times \mathbb{R}$ defined by $(r, \theta, z) \mapsto (r, \alpha_1 + \alpha_2 - \theta, -z)$ does not change the boundary data so v is invariant by this isometry because of the uniqueness of such a solution. This proves that $v = 0$ on $L(\frac{\alpha_1+\alpha_2}{2})$. Then, by maximum principle, we have $v > 0$ between $L(\alpha_1)$ and $L(\frac{\alpha_1+\alpha_2}{2})$ and $v < 0$ between $L(\frac{\alpha_1+\alpha_2}{2})$ and $L(\alpha_2)$.

Let us consider $\alpha_1 < \alpha < \alpha_2$. Let us prove that there exists a constant M such that $v \leq M$ in $\Omega_{\alpha}^{\alpha_2}(\frac{R'}{2})$. If $\alpha \geq \frac{\alpha_1+\alpha_2}{2}$, $M = 0$ works. We note $\widetilde{\alpha} = \frac{\alpha_1+\alpha_2}{2}$. We suppose $\alpha \leq \widetilde{\alpha}$ then we take a sufficiently big n such that $\frac{\widetilde{\alpha}-\alpha}{n} \leq \frac{\pi}{4}$ and $\frac{\widetilde{\alpha}-\alpha}{n} \leq \alpha - \alpha_1$. We note B the vertex of $\Omega_{\alpha_1}^{\alpha_2}(R')$. For $k \leq 2n+1$ we note $\alpha(k) = \widetilde{\alpha} - k\dfrac{\widetilde{\alpha} - \alpha}{2n}$ and for $k \leq 2n-1$ we note A_k the points of coordinates $(\frac{R'}{2}, \alpha(k))$ and C_k the point of second coordinate $\alpha(k+2)$ such that A_kBC_k is an isosceles triangle at A_k (see Figure 2.2). We have $v = 0$ on $[B, A_0]$ and v is bounded on $[A_0, C_0]$, then, by adding a constant, we can put a pseudo Scherk surface above v over A_0BC_0. This proves that v is upper-bounded in $\Omega_{\alpha(1)}^{\alpha_2}(\frac{R'}{2})$. Since v is upper-bounded on $[B, A_1]$ and $[A_1, C_1]$, we can put a pseudo Scherk surface above v over A_1BC_1 then v is bounded on $\Omega_{\alpha(2)}^{\alpha_2}(\frac{R'}{2})$. We can do this for every k then we obtain that v is uniformly upper-bounded on $\Omega_{\alpha}^{\alpha_2}(\frac{R'}{2})$.

With the same method, we can prove that there exists M' such that $v \geq M'$ on $\Omega_{\alpha_1}^{\alpha}(\frac{R'}{2})$.

Let us now consider our original problem. We have u and α and we want to prove the existence of M. We consider $\beta_1 < \alpha' < \alpha$, since u tends to $-\infty$ along $L(\beta_2)$, there exists m such that $u \leq m$ at all the points of coordinates $(\frac{R}{2}, \theta)$ with $\alpha' \leq \theta < \beta_2$. We consider on $\Omega_{\alpha'}^{\beta_2}(\frac{R}{2})$ the solution v that we have studied above, by maximum principle, we have $u \leq v + m$ on $\Omega_{\alpha'}^{\beta_2}(\frac{R}{2})$. We then have the existence of M because of the result on v. We construct M' in the same way.

Let us now consider u a solution of $\mathcal{P}$ and $M \in \mathbb{R}$. We consider $\beta_1 < \beta < \beta_2$ such that $\beta - \beta_1 \leq \frac{\pi}{4}$, we consider the point A of coordinates $(\frac{R}{2}, \beta)$ and C the point on $L(\beta_1)$ such that ABC is a isosceles triangle (where B is the vertex of $\Omega_{\beta_1}^{\beta_2}$). By what we have just proved, u is lower-bounded on $[B, A]$ and $[A, C]$ then we can put a pseudo Scherk surface under u. The existence of α is due to the last remark that we made about pseudo Scherk surfaces $\square$

Using this result, we can prove the following geometrical result.

Theorem 2.3.2. *Let (Ω, φ) be a multi-domain and P a vertex of Ω such that two edges L_1 and L_2 have P as end point (L_1 and L_2 are enumerated with respect to the orientation). Let u be a solution of* (MSE) *on Ω such that u tends to $-\infty$ on L_1 and $+\infty$ on L_2. We consider Ψ_u the conjugate function to u normalized such that*

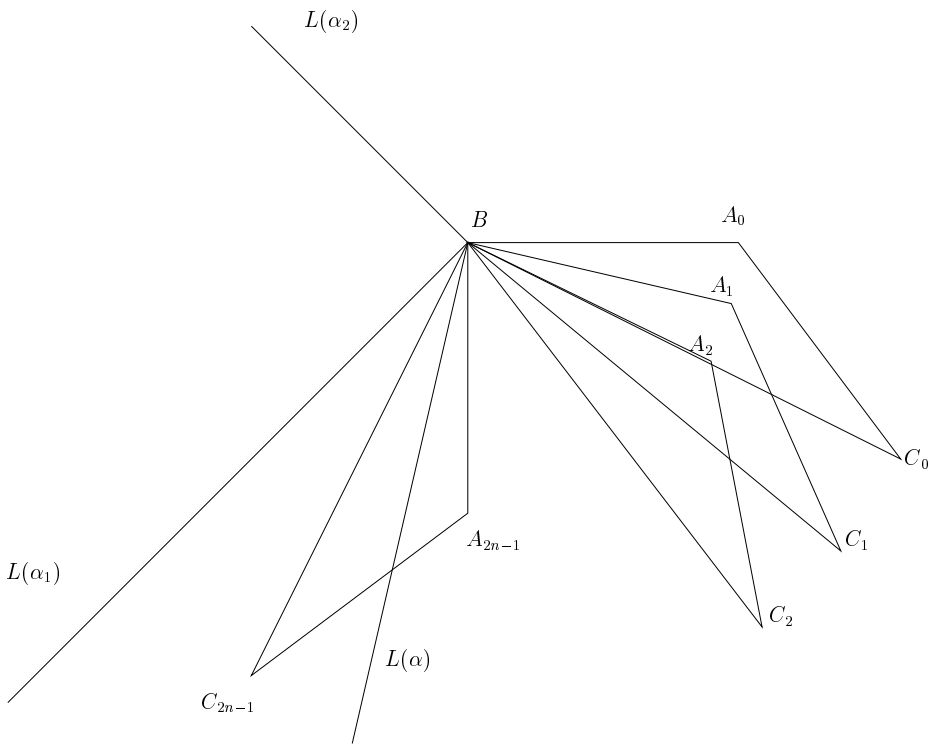

Figure 2.2:

$\Psi_u(P) = 0$. *Then, if* Ψ_u *is non-negative in a neighborhood of* P*, the vertical straight line passing through* $\varphi(P)$ *is the boundary of the graph of* u *above a neighborhood of* P*.*

First, we remark that, if Q is a point on L_1 or L_2, then $\Psi_u(Q) = |PQ| \geq 0$ by Lemma 2.2.2. This proves that, if the angle at P is strictly less than π, the hypothesis on Ψ_u is always verified; so we have the result for a convex corner.

Proof. By a translation and a rotation, we can isometrically parametrized a neighborhood of P by $\Omega^0_\beta(R)$ for $\beta < 0$ and R small enough. Then u can be seen as a solution of the problem $\mathcal{P}$. We suppose that $\Psi_u \geq 0$ in $\Omega^0_\beta(R)$.

First part

First, we prove that there exists $M_1 \in \mathbb{R}$ such that $\varphi(P)\times]-\infty, M_1[$ is a part of the boundary of the graph. We take $-\frac{\pi}{2} < \alpha < 0$, we suppose that $\alpha > \frac{\beta}{2}$. Then $\Omega^0_\alpha(R) \subset \Omega^0_\beta(R)$ can be parametrized by euclidean parameters (x, y), in fact $\Omega^0_\alpha(R)$ is embedded in $\mathbb{R}^2$. The idea is to see the part of the graph which is over $\Omega^0_\alpha(R)$ as a graph over the vertical plane given by the equation $y = 0$. Let $R' < \frac{R}{2}$, then for

all $Q \in \Omega^0_\alpha(R')$ the nearest point from Q on $\partial\Omega$ is on L_1. If we take R' small enough and α such that $\tan\alpha > -\frac{1}{8}$, then every point of $\Omega^0_\alpha(R')$ verifies the hypothesis of Lemma 1 in [JS]. This lemma implies that, at every point of $\Omega^0_\alpha(R')$, $q = \dfrac{\partial u}{\partial y} < 0$. Using our euclidean parameters, we note, for $(x,y) \in \Omega^0_\alpha(R')$, $\Theta(x,y) = (x, u(x,y))$. We have:

$$\mathrm{d}\Theta = \begin{pmatrix} 1 & 0 \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \end{pmatrix}$$

Since $q < 0$, this proves that Θ is a local diffeomorphism. Since u strictly decreases when y increases, Θ is injective. By Lemma 2.3.1, we know that there exists $K \in \mathbb{R}$ such that $u \geq K$ on $L(\alpha)$, we put $x_1 = R'\cos\alpha$. We then have $]0, x_1[\times]-\infty, K[\subset \Theta\left(\Omega^0_\alpha(R')\right)$. We note $\chi = \Theta^{-1}$ on $]0, x_1[\times]-\infty, K[$; we then have $y = \chi_{(2)}(x,z)$ on the graph of u (we note $\chi_{(2)}$ the second coordinate function of χ) then $\chi_{(2)}$ verifies (MSE). When $x \longrightarrow 0$, we have $y = \chi_{(2)}(x,z) \longrightarrow 0$, it is due to the shape of $\Omega^0_\alpha(R')$. From Lemma 2.3.1, there exist $\alpha < \alpha' < 0$ and r such that $\Omega^0_{\alpha'}(r) \subset \operatorname{im}\chi$. By results of boundary regularity, $\chi_{(2)}$ is regular at the boundary, actually we can extend $\chi_{(2)}$ by making a reflection with respect to the axis $\{x = 0, y = 0\}$. We now show that a part of this axis is a part of the boundary of the whole graph. By Lemma 2.3.1, there exists M' such that $u \geq M'$ in $\Omega^{\alpha'}_\beta(r)$. We note $M_1 = M' - 1 < K$; then if a sequence of points of the graph of u over $\Omega^0_\beta(r)$ tends to a point of $\varphi(P)\times]-\infty, M_1[$, we have (x,y) in $\operatorname{im}\chi$ after a certain rank. Then the graph of u over $\operatorname{im}\chi$ is a neighborhood of $\varphi(P)\times]-\infty, M_1[$; as $\chi_{(2)}$ is regular through the boundary, $\varphi(P)\times]-\infty, M_1[$ is a part of the boundary.

With the same arguments, we can show that there exists M_2 such that $\varphi(P)\times]M_2, +\infty[$ is a part of the boundary.

<u>Second part</u>

The first part proves that outside a compact the graph of u has a good behaviour above the point $\varphi(P)$. Now, we prove that we can extend, by reflection, this compact part through the verical straight line passing by $\varphi(P)$.

From what we have just done, there exist $\beta < \alpha_2 < \alpha_1 < 0$ such that the graph above $\Omega^{\alpha_2}_\beta(R)$ and $\Omega^0_{\alpha_1}(R)$ is regular above P. We choose M_1 and M_2 as in the first part such that $(\varphi(P), M_1) \in \partial\operatorname{Graph}(u|_{\Omega^0_{\alpha_1}(R)})$ and $(\varphi(P), M_2) \in \partial\operatorname{Graph}(u|_{\Omega^{\alpha_2}_\beta(R)})$. We shall construct a curve Γ as follow: we start from the point $A_1 = (0,0,M_1)$ ($(0,0) = \varphi(P)$), we go down vertically to the point $A_2 = (0,0,M_1 - 1)$, then we go to some point $A_3 = (\varepsilon\cos\theta, \varepsilon\sin\theta, M_1 - 1)$ in following the level curve $\{u = M_1 - 1\}$ (we suppose ε small and $\theta > \alpha_1$), we then follow the curve

$$t \mapsto (\varepsilon\cos t, \varepsilon\sin t, u(\varepsilon\cos t, \varepsilon\sin t)),$$

we let t decreases to some $\theta' < \alpha_2$ such that $u(\varepsilon\cos\theta', \varepsilon\sin\theta') = M_2 + 1$ (we note A_4 the end point), following the level curve $\{u = M_2 + 1\}$ we go to the point $A_5 = (0, 0, M_2 + 1)$ and finally we go down to the point $A_6 = (0, 0, M_2)$. We can smooth Γ at the points A_2, A_3, A_4 and A_5 such that A_2 and A_5 are always in the smooth Γ and the new Γ is embedded in the graph of u. The vertical projection of Γ on $\Omega^0_\beta(R)$ bounds a domain $\widetilde{\Omega}$. We note Σ the graph of u above $\widetilde{\Omega}$. Because of our choice of Γ, Σ extends in a minimal surface Σ' through Γ (The only problem is through $[A_1, A_2]$ and $[A_5, A_6]$, but the first part says us that we can extend Σ through these two segments by symmetry). Because Σ is a graph, Σ is simply connected and its boundary is not empty; the same is true for Σ'. This remark says us that we have conformal parametrization $h_1 : D \longrightarrow \Sigma'$ and $h_2 : D \longrightarrow \Sigma$ (D is the unit disk). We put $\widetilde{D} = h_1^{-1}(\Sigma)$ and $\widetilde{h} : \widetilde{D} \longrightarrow D$ defined by $\widetilde{h} = h_2^{-1} \circ h_1$; $\widetilde{h}$ is a biholomorphic map. As $h_1^{-1}(\Gamma)$ is embedded in D, the property of Schönflies is verified at every point ; by the Carathéodory's Theorem, $\widetilde{h}$ extends to an homeomorphism of $\widetilde{D} \cup h_1^{-1}(\Gamma)$ into $D \cup V$ where V is part of the boundary of D (for all this argument we refer to appendix 2.7). This proves that we can extend h_2 in an homeomorphism of $D \cup V$ into $\Sigma \cup \Gamma$. Let us consider $f : D \longrightarrow D^-$ ($D^- = \{(x, y) \in D |\ y < 0\}$) a biholomorphic map, then f extends to the boundary. Let us consider the following points on Γ: $A_{1.5} = (0, 0, M_1 - 0.5)$ and $A_{5.5} = (0, 0, M_2 + 0.5)$. We note $X = h_2 \circ H \circ f^{-1}$ where H is a Moebius transformation of the unit disk. We note $B_i = X^{-1}(A_i)$ for every i. Then, for a suitable choice of H, we can have the situation described by Figure 2.3.

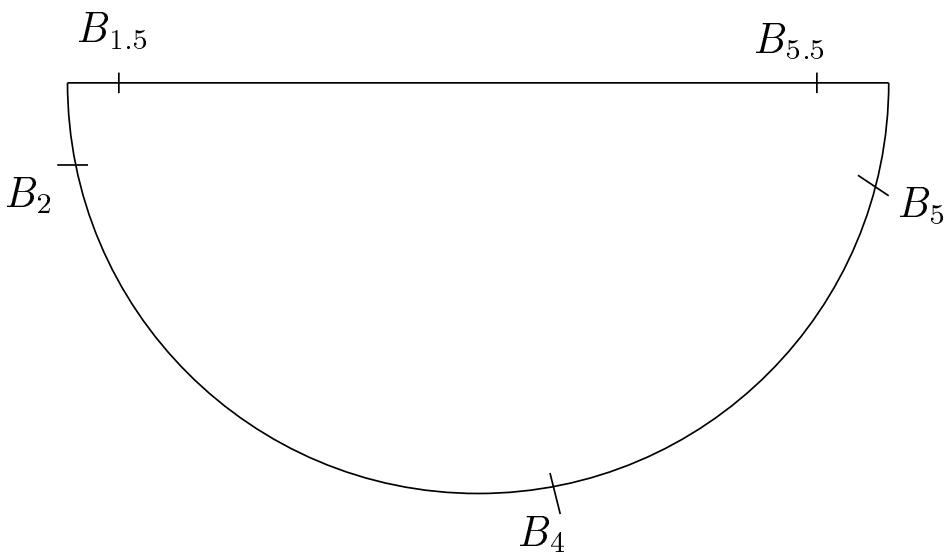

Figure 2.3:

Let us show that X extends to the whole disk. We shall note x_1, x_2 and x_3 the

three coordinates of X, this three functions are harmonic since Σ is minimal. First we observe that x_1 and x_2 tend to 0 when $z \in D^-$ tends to $D^0 = \{z \in D|\ z \in \mathbb{R}\}$, this is due to the shape of $\widetilde{\Omega}$. Then, by Schwarz reflection principle, x_1 and x_2 extend to D in harmonic functions. Let us consider x_3^* the harmonic conjugate to x_3 on D^-, we normalized x_3^* by $x_3^*(B_2) = 0$. By our choice of normalization, for every $z \in D^-$, we have $\Psi_u(X(z)) = x_3^*(z)$; this proves that x_3^* tends to 0 when z tends to D^0, we can extend x_3^* by reflection to D. By taking the conjugate function, we have proved that x_3 extends to D. We then have constructed a minimal immersion X on D, then Σ extends through $[A_1, A_6]$. This extention is given by the reflection with respect to $[A_1, A_6]$.

Third part

The last thing we have to show is that the minimal immersion X has no branch point. If it has a branch point then it must be on D^0, since, on the other part, the surface is a graph and then there is no branch point. Let z_0 be a branch point, then $\nabla x_3^*(z_0) = 0$. Since x_3^* is harmonic, its local behaviour is quite similar to the one of $\Re(z-z_0)^p$ with $p \geq 2$ (in fact, in some holomorphic chart we have $x_3^*(z) = \Re(z-z_0)^p$). This implies that there exists z in D^- such that $x_3^*(z) < 0$, but this contradicts our hypothesis $\Psi_u \geq 0$.

We then have proved that there is no branch point so the vertical straight line passing by $\varphi(P)$ is the boundary of the graph. □

Remark 2.3.3. We can remark that in the first two parts we do not use the hypothesis on Ψ_u. So in such a situation we can always extend the graph by making a reflection with respect to the vertical axis. But what we obtain is a minimal surface with, may be, a finite number of branch points on the vertical axis.

Remark 2.3.4. We can make an other remark. If we consider a vertex P, two edges L_1 and L_2 having P as end point, and u such that u assumes the data $+\infty$ (or $-\infty$) on L_1 and L_2, the hypothesis on Ψ_u did not have any more sense and the angle at the vertex P must be greater than π. But we can always apply the two first parts of the proof. The only problem is that we need a result similar to Lemma 2.3.1; this is given by Theorem 10.3 in [Os2]. So we can affirm that on the boundary of the graph of u we have a half straight line with a finite number of branch points. Obviously, we must have a branch point at the end point of the half straight line.

2.4 Convergence and divergence of sequence of solutions of (MSE)

In this section we shall explain what we can say when we have a sequence (u_n) of solutions of (MSE) about its convergence: can we make converge a subsequence by

some compactness result? What are the different ways of divergence? In [JS], it is shown that for a monotone sequence, it appears lines which separate domains of convergence and domains of divergence (this works only for subsequence). We shall show that such lines always appear (Theorem 2.4.3).

First, we have to determine the domain on which we can make converge a sequence, since each surface is a graph, if we want the limit to be a graph, the normal to the surface needs to stay close by the vertical unit vector and then W_n have to be bounded. We have then the following lemma.

Lemma 2.4.1 (Lemma 1.4.1). *Let Ω be a domain and (u_n) a sequence of solutions of* (MSE) *on Ω. Let $P \in \Omega$; we suppose that $W_n(P)$ is bounded by a constant M; then there exists $R > 0$ which depends only of M and the distance of P to $\partial\Omega$ such that W_n is bounded by $2M$ on the disk of center P and radius R.*

Proof. We fix an index n. We know (see [Ni1]) that there exists a constant c such that if u is a solution of (MSE) on the disk $\{(x,y)|\ x^2+y^2<R^2\}$ we have:

$$r^2(0)+2s^2(0)+t^2(0) \leq \frac{c}{R^2}W^4(0) \tag{2.3}$$

Let R be such that $2R = d(P,\partial\Omega)$ then, for all Q in $D(P,R)$, the above equation gives $r^2(Q)+2s^2(Q)+t^2(Q) \leq \frac{c}{R^2}W^4(Q)$. We have $\nabla W = (\frac{rp+sq}{W}, \frac{sp+tq}{W})$, so, in $D(P,R)$, we have $||\nabla W|| \leq \widetilde{C}W^2$ with $\widetilde{C}$ which depends only of R. Let z be the function such that $z(0)=M$ and $z' = \widetilde{C}z^2$, z is defined on $[0, \frac{1}{M\widetilde{C}}[$ by:

$$\frac{1}{M} - \frac{1}{z} = \widetilde{C}r \tag{2.4}$$

Because of our estimate on $||\nabla W||$, we have, in polar coordinates, $W(r,\theta) \leq z(r)$. Then W is bounded by $2M$ on $D(P, \min(R, \frac{1}{2M\widetilde{C}}))$. □

Let (u_n) be a sequence of solutions of (MSE) on a domain Ω. We then define $\mathcal{B}(u_n)$ as the set of the point $Q \in \Omega$ such that $(W_n(Q))$ is bounded. Lemma 2.4.1 says us that $\mathcal{B}(u_n)$ is an open set and that W_n is uniformly bounded on each compact inclued in $\mathcal{B}(u_n)$. Then if D is a connected component of $\mathcal{B}(u_n)$ and $P \in D$ there exists an extraction θ such that $u_{\theta(n)} - u_{\theta(n)}(P)$ converges uniformly on each compact of D to a solution u of (MSE); here, we use some classical compactness results (see [Ni1]). This proves that the divergence of the sequence is due to the behaviour of the sequence over $\Omega\backslash\mathcal{B}(u_n)$.

If $P \in \Omega\backslash\mathcal{B}(u_n)$, there exists a subsequence $u_{n'}$ such that $W_{n'}(P) \longrightarrow +\infty$. As the normal N_n to the graph at $(P, u_n(P))$ is given by:

$$N_n(P) = \left(\frac{p_n}{W_n}(P), \frac{q_n}{W_n}(P), -\frac{1}{W_n}(P)\right) \tag{2.5}$$

we can suppose that $N_{n'}(P)$ converges to an horizontal unit vector. The following theorem describes what locally happens.

Theorem 2.4.2. *Let r be positive. Let (u_n) be a sequence of solutions of* (MSE) *on the disk $D(0,r)$. We suppose that $N_n(0)$ converges to $(1,0,0)$. Let $\alpha \in]0,1[$, then there exists an extraction θ such that $N_{\theta(n)}$ converges to $(1,0,0)$ at almost every point of $\{0\} \times [-\alpha r, \alpha r]$.*

Proof. Let $n \in \mathbb{N}$, we know (see [Os2] and [JS]) that there exists $\Phi_n : (x,y) \mapsto (\xi,\eta)$ with $\Phi_n(0,0) = (0,0)$ and:

$$\mathrm{d}\xi = \left(1 + \frac{1+p_n^2}{W_n}\right)\mathrm{d}x + \frac{p_n q_n}{W_n}\mathrm{d}y \tag{2.6}$$

$$\mathrm{d}\eta = \frac{p_n q_n}{W_n}\mathrm{d}x + \left(1 + \frac{1+q_n^2}{W_m}\right)\mathrm{d}y \tag{2.7}$$

We know that Φ_n increases distance so it is bijective on its image. This image contains the disk of center $(0,0)$ and radius r. Besides, we know that (ξ,η) are conformal parameters for the graph of u_n. On the $\xi\eta$ disk $D(0,r)$ we then have the Gauss map $g_n(\xi + i\eta)$ which corresponds to the stereographic projection of N_n; g_n is holomorphic. We note $z_n = g_n(0)$, by hypothesis we have $z_n \longrightarrow 1$. We note $z = \xi + i\eta$; by our choice of normal $g_n : D(0,r) \longrightarrow D(0,1)$, then there exists $h_n : D(0,r) \longrightarrow D(0,1)$ holomorphic with $h_n(0) = 0$ such that:

$$g_n(z) = \frac{h_n(z) + z_n}{1 + \overline{z_n} h_n(z)} \tag{2.8}$$

Since $z_n \longrightarrow 1$, the sequence of holomorphic functions $z \longmapsto \dfrac{z + z_n}{1 + \overline{z_n} z}$ converges simply to 1 on $D(0,1)$ and uniformly on the disk $D(0,\alpha)$ for all $\alpha < 1$. But by Schwarz Lemma, we have, for all $n \in \mathbb{N}$, $h_n\big(D(0,\alpha r)\big) \subset D(0,\alpha)$, we then have uniform convergence of g_n to 1 on $D(0,\alpha r)$. In using (2.5), this proves that for every ε, if n is big enough, we can say that: $\dfrac{p_n}{W_n} \geq 1 - \varepsilon$ and $\dfrac{|q_n|}{W_n} \leq \varepsilon$ in $\Phi_n^{-1}\left(D(0,\alpha r)\right)$. So to conclude, we need to understand the shape of $\Phi_n^{-1}\left(D(0,\alpha r)\right)$; we shall see that these sets are concentrating along the segment $\{0\} \times [-\alpha r, \alpha r]$.

For all $n \in \mathbb{N}$, we consider, in the $\xi\eta$ disk, the path $\gamma_n :]-r, r[\longrightarrow D(0,r)$ defined by $\gamma_n(0) = 0$ and $\gamma_n' = \frac{\nabla y_n}{||\nabla y_n||}$ where y_n is the second coordinate of Φ_n^{-1}. We have (see [Os2]):

$$\nabla y_n = \left(-\frac{p_n q_n}{J_n W_n}, \frac{W_n + 1 + p_n^2}{J_n W_n}\right) \tag{2.9}$$

where $J_n = \det(\mathrm{d}\Phi_n) = W_n + 2 + \dfrac{1}{W_n}$.Because (2.5), ∇y_n converges uniformly to $(0,1)$ on the disk $D(0,\alpha r)$ for all $\alpha < 1$.

Let $\alpha \in]0,1[$, we note A_α and B_α the points in the xy-disk $D(0,r)$ of respective coordinates $(0,-\alpha r)$ and $(0,\alpha r)$. In the following, we prove that $\displaystyle\int_{[A_\alpha,B_\alpha]} \mathrm{d}\Psi_{u_n} \longrightarrow 2\alpha r$.

Let $\widetilde{\alpha} > \alpha$, then for n big enough, we have $||\nabla y_n|| > \dfrac{\alpha}{\widetilde{\alpha}}$ in $D(0,\widetilde{\alpha} r)$. Because, for $t \in [-\widetilde{\alpha} r, \widetilde{\alpha} r]$, $\gamma_n(t) \in D(0,\widetilde{\alpha} r)$, there exists $\widetilde{\alpha} r \leq t_0^n < t_1^n \leq \widetilde{\alpha} r$ such that $y_n(\gamma_n(t_0^n)) = -\alpha r$ and $y_n(\gamma_n(t_1^n)) = \alpha r$. Along $[t_0^n, t_1^n]$, $y_n \circ \gamma_n$ increases strictly from $-\alpha r$ to αr, then the path $\Gamma_n = \Phi_n^{-1} \circ \gamma_n$ on $[t_0^n, t_1^n]$ can be parametrized by $y \in [-\alpha r, \alpha r]$: we have a function f_n on $[-\alpha r, \alpha r]$ such that for $t \in [t_0^n, t_1^n]$ $x_n(\gamma_n(t)) = f_n\big(y_n(\gamma_n(t))\big)$. We have:

$$\begin{aligned} |x_n \circ \gamma_n(t)| &\leq \left| \int_0^t |(x_n \circ \gamma_n)'(u)| \mathrm{d}u \right| \\ &\leq \left| \int_0^t ||\nabla x_n||(\gamma_n(u)) \mathrm{d}u \right| = \left| \int_0^t \left(\frac{1+q_n^2}{(1+W_n)^2} \right)^{\frac{1}{2}} (\Gamma_n(u)) \mathrm{d}u \right| \end{aligned} \tag{2.10}$$

(for the last equality see [Os2]), then for n big enough $|x_n \circ \gamma_n|$ on $[t_0^n, t_1^n]$ can be bounded by a constant uniformly small; this is due to the fact that $\dfrac{|q_n|}{W_n} \leq \varepsilon$ for big n. We then have proved that the path Γ_n is close by the segment $[A_\alpha, B_\alpha]$ for big n.

Let us now calculate $\displaystyle\int_{\Gamma_n} \mathrm{d}\Psi_{u_n}$. We have:

$$\int_{\Gamma_n} \frac{p_n}{W_n} \mathrm{d}y - \frac{q_n}{W_n} \mathrm{d}x = \int_{-\alpha r}^{\alpha r} \left(\frac{p_n}{W_n}(f_n(y), y) - \frac{q_n}{W_n}(f_n(y), y) f_n'(y) \right) \mathrm{d}y \tag{2.11}$$

We have $|f_n'\big(y_n(\gamma_n(t))\big)| \leq \dfrac{||\nabla x_n||}{||\nabla y_n||}(\gamma_n(t)) \longrightarrow 0$, the convergence is uniform so f_n' tends uniformly to 0 on $[-\alpha r, \alpha r]$. This proves that $\displaystyle\int_{\Gamma_n} \mathrm{d}\Psi_{u_n} \longrightarrow 2\alpha r$

Let us consider the path $\widetilde{\Gamma_n}$ which consists on the segment $[A_\alpha, B_\alpha]$, then the segment $[B_\alpha, \Gamma(t_1^n)]$, then $-\Gamma_n$, then, finally, the segment $[\Gamma_n(t_0^n), A_\alpha]$. Let $\varepsilon > 0$, for n big enough, we can suppose that $\displaystyle\int_{\Gamma_n} \mathrm{d}\Psi_{u_n} \geq 2\alpha r - \varepsilon$ and $\left(\dfrac{1+q_n^2}{(1+W_n)^2} \right)^{\frac{1}{2}} \leq \varepsilon$

on $\Phi_n^{-1}\big(D(0,\widetilde{\alpha}r)\big)$. As $\mathrm{d}\Psi_{u_n}$ is closed $\displaystyle\int_{\widetilde{\Gamma_n}} \mathrm{d}\Psi_{u_n} = 0$; we then have:

$$\begin{aligned}
2\alpha r \geq \int_{[A_\alpha,B_\alpha]} \mathrm{d}\Psi_{u_n} &= -\int_{[B_\alpha,\Gamma(t_1^n)]} \mathrm{d}\Psi_{u_n} - \int_{-\Gamma_n} \mathrm{d}\Psi_{u_n} - \int_{[\Gamma_n(t_0^n),A_\alpha]} \mathrm{d}\Psi_{u_n} \\
&\geq \int_{\Gamma_n} \mathrm{d}\Psi_{u_n} - |x_n\big(\gamma(t_1^n)\big)| - |x_n\big(\gamma(t_0^n)\big)| \qquad (2.12)\\
&\geq 2\alpha r - \varepsilon - \varepsilon|t_1^n| - \varepsilon|t_0^n| \\
&\geq 2\alpha r - \varepsilon(1+2\widetilde{\alpha}r)
\end{aligned}$$

This proves that $\displaystyle\int_{[A_\alpha,B_\alpha]} \mathrm{d}\Psi_{u_n} \longrightarrow 2\alpha r$. We have $\displaystyle\int_{[A_\alpha,B_\alpha]} \mathrm{d}\Psi_{u_n} = \int_{[A_\alpha,B_\alpha]} \frac{p_n}{W_n}\mathrm{d}y$. Because $\dfrac{p_n}{W_n} \leq 1$, the preceding equalities prove that $\dfrac{p_n}{W_n}$ converges to 1 in $\mathbb{L}^1([A_\alpha,B_\alpha])$. Then there exists an extraction θ such that $\dfrac{p_{\theta(n)}}{W_{\theta(n)}}$ converges simply to 1 at almost every point in $[A_\alpha,B_\alpha]$; thus the proposition is proved. □

This theorem gives us a local result and we have the following global result

Theorem 2.4.3 (Theorem 1.4.3). *Let (Ω,φ) be a multi-domain. Let (u_n) be a sequence of solutions of* (MSE) *on Ω. Let $P \in \Omega$ and N a unit horizontal vector and D the geodesic of Ω passing at P and normal to N. If the sequence $(N_n(P))$ converges to N, then $N_n(Q)$ converges to N at every point of D.*

As Ω is locally isometric to $\mathbb{R}^2$, we have allowed us to call N the parallel transport of N along D.

Proof. We first get a parametrization of D by arc-length with P as origin-point; then D is parametrized by $]a,b[$, $]-\infty,b[$, $]a,+\infty[$ or $]-\infty,+\infty[$, we shall suppose that we are in the case $]a,+\infty[$ (the other cases are similar). We then consider the set $\mathcal{F}$ of $t \in \mathbb{R}_+^*$ such, if θ_1 is an extraction, there exists a sub-extraction θ_2 such that $N_{\theta_2(n)}(Q)$ converge to N at almost every Q of the part of D parametrized by $]a+\dfrac{|a|}{t+1},t[$ $(a<0)$. Let us prove that $\mathcal{F}=\mathbb{R}_+^*$. First, we observe that, if $t_1 \in \mathcal{F}$ and $t_2 < t_1$, $t_2 \in \mathcal{F}$. From Theorem 2.4.2, there exists $t>0$ such that $t \in \mathcal{F}$. Let $t_0 = \sup \mathcal{F}$ and suppose that $t_0 < +\infty$. We consider P_1 and P_2 the points on D parametrized by $a + \dfrac{|a|}{1+t_0}$ and t_0. We choose $R>0$ such that $D(P_i,R) \subset \Omega$ for $i=1,2$. Let θ_1 be an extraction. Since $t_0 = \sup\mathcal{F}$, there exist $Q_1 \in D(P_1,\frac{R}{3}) \cap D$, $Q_2 \in D(P_2,\frac{R}{3}) \cap D$ and a sub-extraction θ_2 such that $N_{\theta_2(n)}$ converges to N at Q_1 and Q_2. We have $D(Q_i,\frac{2R}{3}) \subset \Omega$, we then apply Theorem 2.4.2 to points Q_1 and Q_2 with $\alpha = \frac{3}{4}$. We then have a sub-extraction θ_3 such that N_{θ_3} converges to N at

almost every point of $D(Q_i, \frac{R}{2}) \cap D$ for $i = 1, 2$; this proves that t_0 is not $\sup \mathcal{F}$, because N_{θ_3} converges to N at almost every point of the part of D parametrized by an open interval that contains the segment $[a + \dfrac{|a|}{t_0 + 1}, t_0]$.

By a standard diagonal process, we can then construct an extraction θ such that $N_{\theta(n)}$ converges to N at almost every point of D. Let Q be in D and we consider N' a cluster point of the sequence $N_{\theta(n)}(Q)$, if the third coordinate of N' is negative then there exists a sub-extraction θ' such that $W_{\theta'(n)}(Q)$ is bounded but this is impossible since, by Lemma 2.4.1, $W_{\theta'(n)}$ would be bounded in a neighborhood of Q and $W_{\theta(n)}$ diverges at almost every point of D. Thus the third coordinate of N' is 0; if $N' \neq N$, applying what we have already proved, it appears a second geodesic D' passing by Q normal to N' and an extraction θ' such that $N_{\theta'(n)}$ converges to N' at almost every point of D'. We parametrized D and D' by arc-length in using the orientation given by the direct normal to N and N', we choose Q as origin point. Let $\varepsilon > 0$; we note A the point on D of coordinate $-\varepsilon$ and B the point on D' of coordinate ε. For ε small enough, the triangle AQB is in Ω and then $\displaystyle\int_{AQB} \mathrm{d}\Psi_{u_{\theta'(n)}} = 0$. We let n tends to $+\infty$ and then obtain $|AC| + |BC| \leq |AB|$ which contradicts the triangle inequality. We then have proved that $N_{\theta(n)}$ converges to N at every point of D. We then have proved that for every extraction θ we can construct a sub-extraction θ' such that $N_{\theta'(n)}$ converges to N at every point of D

To finish the proof, we take a point Q in D and suppose that $N_{\alpha(n)}(Q)$ converge to N' with α an extraction. Since $N_{\alpha(n)}(P) \longrightarrow N$, there exists a sub-extraction α' such that $N_{\alpha'(n)}$ converges to N at every point of D, in particular at Q, then $N = N'$. □

Remark 2.4.4. We then understand the structure of the complementary of $\mathcal{B}(u_n)$, it is a set of geodesics of Ω; one of these geodesics will be called a *line of divergence*. Then when we have a sequence of solutions of (MSE), the problem of the convergence of the sequence is linked to the understanding of: which lines of divergence are possible? The answer is, in general, given by the behaviour at the boundary. The existence of such lines will permit us to use arguments that are similar to the ones used by H. Jenkins and J. Serrin in [JS].

The behaviour of the normal along a line of divergence says us that the limit of $\displaystyle\int_T \mathrm{d}\Psi_{u_n}$, where T is a segment of a line of divergence with the orientation given by the limit normal, is $|T|$. In the following, we shall draw this limit normal on the figures to explain our arguments.

2.5 The Plateau problem at infinity

In this section, we explain the problem studied by C. Cosín and A. Ros in [CR] and give the main results of their paper with some elements of proofs.

Let M be a complete minimal surface with finite total curvature in $\mathbb{R}^3$; we know that M is isometric to a compact Riemann surface minus a finite number of points (we can refer to [Os2]). M then has a finite number of annular ends; when these ends are embedded they are asymptotic either to a half-catenoid or to a plane. A properly immersed minimal surface with r embedded ends will be called a *r-noid*. We can associate to each end a vector which caracterizes the direction and the growth of the asymptotic half-catenoid (when the end is asymptotic to a plane this vector is zero); this vector is called the *flux* of the end (for a precise definition of the flux see [HK]). If $v_1, \ldots, v_r$ are the fluxes at each end, we have the following balancing condition:

$$v_1 + \cdots + v_r = 0 \tag{2.13}$$

This condition tells us that the total flux of the system vanishes. If $v_1, \ldots, v_r$ are vectors in $\mathbb{R}^3$ such that (2.13) is verified and g is a non-negative integer, the Plateau problem at infinity for these data is to find an r-noid of genus g which has $v_1, \ldots, v_r$ as fluxes at its ends.

Let $\psi : M \longrightarrow \mathbb{R}^3$ be an r-noid. M is conformally equivalent to a compact surface $\overline{M}$ minus a finite number of points $p_1, \ldots, p_r$. We will say that M is *Alexandrov-embedded* if $\overline{M}$ bounds a compact 3-manifold $\overline{\Omega}$ and the immersion ψ extends to a proper local diffeomorphism $f : \overline{\Omega} \backslash \{p_1, \ldots, p_r\} \longrightarrow \mathbb{R}^3$. An Alexandrov-embedded surface has a canonical orientation; we choose the Gauss map to be the outward pointing normal. An Alexandrov-embedded r-noid can not have a planar end (see [CR]). We call $\mathcal{M}_r$ the space of Alexandrov-embedded r-noids of genus 0 and r horizontal catenoidal ends. We identify two elements in $\mathcal{M}_r$ which differ by a translation. In [CR], C. Cosín and A. Ros give a nice description of the space $\mathcal{M}_r$.

Let $\psi : M \longrightarrow \mathbb{R}^3$ be a nonflat immersion of a connected orientable surface M and Π be a plane in $\mathbb{R}^3$, normalized to be $\{x_3 = 0\}$. We note by S the Euclidiean symmetry with respect to Π and consider the subsets:

$$\begin{aligned} M^+ &= \{p \in M | x_3(p) > 0\} \\ M^- &= \{p \in M | x_3(p) < 0\} \\ M^0 &= \{p \in M | x_3(p) = 0\} \end{aligned}$$

With these notation we have:

Definition 2.5.1. We shall say that M is strongly symmetric with respect to Π if

- There exists an isometric involution $s : M \longrightarrow M$ such that $\psi \circ s = S \circ \psi$.

- $\{p \in M | s(p) = p\} = M^0$.
- The third coordinate N_3 of the Gauss map of M takes positive (resp. negative) values on M^+ (resp. M^-).

In [CR], C. Cosín and A. Ros prove

Proposition 2.5.2. *Let M be an r-noid with horizontal ends. Then M is strongly symmetric with respect to an horizontal plane if and only if M is Alexandrov-embedded.*

We then use the notion of strong symmetry to study $\mathcal{M}_r$; in the following, we always suppose that the plane of strong symmetry is the plane $\{x_3 = 0\}$. If $M \in \mathcal{M}_r$, s extends to $\overline{M} = \overline{\mathbb{C}}$, the involution s is $z \mapsto \dfrac{1}{\overline{z}}$ and the points $p_1, \ldots, p_r$ are in $\{z \in \mathbb{C} | \ |z| = 1\}$. We then have an order on $\{p_1, \ldots, p_r\}$, let us suppose that $p_1, \ldots, p_r$ are put in this order. Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^3$ such that $2v_i$ is the flux at the end p_i. We have $v_1 + \cdots + v_r = 0$, so if we draw the vectors consecutively in the plane, we get a piece-wise linear closed curve: a polygon. We note $F(M)$ this polygon.

We say that a polygon V bounds an immersed polygonal disk if there exists a compact multi-domain $(\mathcal{P}, \varphi)$ such that $\partial\mathcal{P}$ is only composed of edges and $\varphi(\partial\mathcal{P}) = V$.

Then the most important result in [CR] is

Theorem 2.5.3. *Let $v_1, \ldots, v_r$ be horizontal vectors such that $v_1 + \cdots + v_r = 0$ and V the associated polygon, then there exists $M \in \mathcal{M}_r$ such that $F(M) = V$ if, and only if, V bounds an immersed polygonal disk*

Besides, we have as much $M \in \mathcal{M}_r$ such that $F(M) = V$ as immersed polygonal disks bounded by V. Let V be a polygon and $(\mathcal{P}, \varphi)$ a compact multi-domain such that $\varphi(\mathcal{P})$ is an immersed polygonal disk bounded by V. Let $P_1, \ldots, P_r$ be the vertices of $\mathcal{P}$ which are identified with the ones of V; we put $P_1 = P_{r+1}$. Let $i \in \{1, \ldots, r\}$, we can glue to $\mathcal{P}$ along $[P_i, P_{i+1}]$ a half-strip S_i isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$. We get a multi-domain which we call $\Omega(\mathcal{P})$; the boundary of $\Omega(\mathcal{P})$ is composed of $2r$ half straight-lines, we call L_i^- (resp. L_i^+) the half line in the boundary which has P_i as end point and is in S_{i-1} (resp. S_i).

Let M be in $\mathcal{M}_r$, we consider $(M^+)^*$ the conjugate surface to M^+ for the outward pointing normal. In [CR], the authors prove that it exists $(\mathcal{P}, \varphi)$ a multi-domain bounded by $F(M)$ such that $(M^+)^*$ is a graph over the multi-domain $\Omega(\mathcal{P})$; the normal to the graph is the upward pointing normal by Definition 2.5.1. If u is the function on $\Omega(\mathcal{P})$ that gives $(M^+)^*$, they prove that u tends to $+\infty$ (resp. $-\infty$) on L_i^+ (resp. L_i^-). C. Cosín and A. Ros use these arguments to prove that

if the Plateau problem at infinity has a solution the flux polygon $F(M)$ bounds an immersed polygonal disk. For the other implication, they prove that the map $F : M \mapsto F(M)$ is a covering map to conclude, they use a compactness argument and prove that the space $\mathcal{M}_r$ has a smooth structure.

In the next section, we shall solve on $\Omega(\mathcal{P})$ the Dirichlet problem for the boundary data $+\infty$ on L_i^+ and $-\infty$ on L_i^-. We shall then take the conjugate of the graph of the solution for the downward pointing normal and so build the solution to the Plateau problem at infinity. The change of orientation makes that we get the surface we want.

2.6 The construction of a solution to the Plateau problem at infinity

The first part of this section will be devoted to the proof of our main result.

Theorem 2.6.1. *Let V be a polygon which bounds an immersed polygonal disk $(\mathcal{P}, \varphi)$, we define $\Omega(\mathcal{P})$ as in the preceding section. Then there exists a solution u of* (MSE) *on $\Omega(\mathcal{P})$ such that u tends to $+\infty$ on L_i^+ and $-\infty$ on L_i^-. Besides, the solution is unique up to an additive constant.*

Let us first consider u a solution of (MSE) on the half-strip $[0, a] \times \mathbb{R}_+$ such that u tends to $-\infty$ on $\{a\} \times \mathbb{R}_+^*$ and $+\infty$ on $\{0\} \times \mathbb{R}_+^*$. This situation describes the behaviour in the r half-strips S_i. Then by Lemma 1 in [JS] we have:

$$\frac{|q|}{W}(x, y) \geq 1 - \frac{a^2}{x^2} \tag{2.14}$$

$$\frac{|p|}{W}(x, y) \leq \sqrt{2}\frac{a}{x} \tag{2.15}$$

when $x \geq 4a$. We consider now the general problem.

We begin in proving the uniqueness part of Theorem 2.6.1. Let u_1 and u_2 be two different solutions of the problem (*i.e.* $u_1 - u_2$ is non-constant). As in the proof of Theorem 2.2.5, we can suppose that $\{u_1 > u_2\}$ and $\{u_1 < u_2\}$ are non-empty. Let us call Ω_l the subset of $\Omega(\mathcal{P})$ which is the union of $\mathcal{P}$ and the set of points in each S_i that are at a distance less than l from $[P_i, P_{i+1}]$; we define $\Omega_l^+ = \Omega_l \cap \{u_1 > u_2\}$. Let us consider:

$$I = \int_{\partial\Omega_l^+} \mathrm{d}\tilde{\Psi} \tag{2.16}$$

where $\mathrm{d}\tilde{\Psi} = \mathrm{d}\Psi_{u_1} - \mathrm{d}\Psi_{u_2}$. Since $\mathrm{d}\tilde{\Psi}$ is closed, we have $I = 0$. $\partial\Omega_l^+$ is composed of a part which is included on $\left(\cup_i L_i^+\right) \bigcup \left(\cup_i L_i^-\right)$ where $\mathrm{d}\tilde{\Psi} = 0$, a part included in the

interior of $\Omega(\mathcal{P})$, noted Γ_l, and a part in $I_{i,l}$ which is the part in S_i parametrized by $[P_i, P_{i+1}] \times \{l\}$. On the part included in $I_{i,l}$ if l is big enough the integral of $\mathrm{d}\tilde{\Psi}$ is less than $2\sqrt{2}\frac{|P_iP_{i+1}|^2}{l}$ by (2.15). We then have:

$$0 = I \leq \int_{\Gamma_l} \mathrm{d}\tilde{\Psi} + \sum_{i=1}^{r} 2\sqrt{2}\frac{|P_iP_{i+1}|^2}{l} \tag{2.17}$$

By Lemma 2 in [CK], $\displaystyle\int_{\Gamma_l} \mathrm{d}\tilde{\Psi}$ is negative and decreases as l increases. Because $\displaystyle\sum_{i=1}^{r} \sqrt{2}\frac{|P_iP_{i+1}|^2}{l} \xrightarrow[l\to+\infty]{} 0$, we get a contradiction. This proves that, if u_1 and u_2 are two solutions of our Dirichlet problem, there exists $c \in \mathbb{R}$ such that $u_1 = u_2 + c$.

We now prove the existence of the solution. We fix a point P_0 in $\mathcal{P}$. Let us consider in S_i the point Q_i^k which is the middle point of $I_{i,k}$, we then define Ω_k to be the compact subdomain of $\Omega(\mathcal{P})$ bounded by the segments $[P_i, Q_i^k]$ and $[Q_i^k, P_{i+1}]$. Let $\mathcal{G}_i^k$ be the set of the points Q in Ω_k such that $d(Q, Q_i^k) < d(P_i, Q_i^k)$; if k is big enough the sets $\mathcal{G}_i^k$ are disjoint, this proves that the conditions of Theorem 2.2.5 are fulfilled for big k. Then by Theorem 2.2.5, we can build a function u_k on Ω_k such that u_k tends to $-\infty$ (resp. $+\infty$) on $[Q_i^k, P_{i+1}]$ (resp. on $[P_i, Q_i^k]$) and $u_k(P_0) = 0$. Following Remark 2.4.4 in section 2.4, we shall prove that this sequence (u_k) of solutions of (MSE) has no line of divergence, then the limit u of (u_k) will be our solution. We shall make discussions that are similar to the ones made by H. Jenkins and J. Serrin. We note $\mathrm{d}\Psi_{u_k} = \mathrm{d}\Psi_k$. We recall that, if T is a segment included in a line of divergence, $\left|\displaystyle\int_T \mathrm{d}\Psi_k\right|$ converge to the length of T for a subsequence.

Suppose there exists a line of divergence L. We first prove that L can not have an end point in the interior of a L_i^+ or a L_i^-. Suppose that L has an end point D in L_i^- (the same argument works for L_i^+). Let A be a point in $L \cap S_{i-1}$, we orient L by $\overrightarrow{AD}$, we suppose that the limit normal along L points on the right-hand side of L. We chose a point B of L_i^- on the right-hand side of D. Because of the triangle inequality, there exists a point C on $[A, D]$ such that $|AC| + |DB| > |CD| + |BA|$. for k big enough we have A and C in Ω_k we then put $D_k = [A, D] \cap [P_i, Q_{i-1}^k]$ and $B_k = [A, B] \cap [P_i, Q_{i-1}^k]$ (see Figure 2.4).

Let T_k be the triangle AD_kB_k with this orientation. We then have:

$$\begin{aligned} 0 = \int_{T_k} \mathrm{d}\Psi_k &\geq \int_{[A,C]} \mathrm{d}\Psi_k - |CD_k| + |D_kB_k| - |B_kA| \\ &\geq \int_{[A,C]} \mathrm{d}\Psi_k - |CD| + |D_kB_k| - |BA| \end{aligned} \tag{2.18}$$

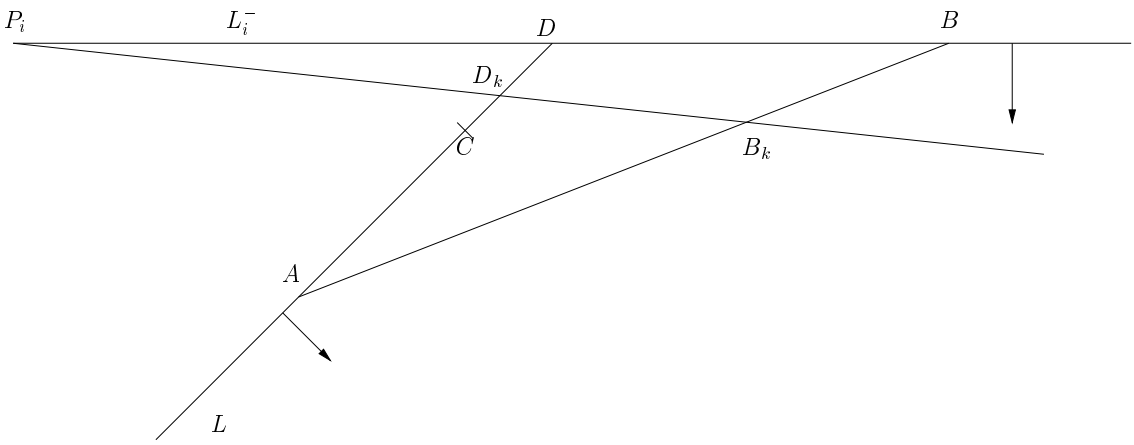

Figure 2.4:

But $|D_kB_k| \longrightarrow |DB|$ and $\displaystyle\int_{[A,C]} \mathrm{d}\Psi_k \longrightarrow |AC|$ for the subsequence that makes L appear; this gives us a contradiction.

We have now only a finite number of possibilities for a line of divergence. If it has an end point, it must be one P_i. By construction, we have $\displaystyle\int_{[P_i,P_{i+1}]} \mathrm{d}\Psi_k = 0$ (because the integral of $\mathrm{d}\Psi_k$ along the triangle $P_iQ_i^kP_{i+1}$ is zero and we know $\mathrm{d}\Psi_k$ along $[P_iQ_i^k]$ and $[Q_i^kP_{i+1}]$ by Lemma 2.2.2) so if Γ is a curve joining P_i to P_j we have $\displaystyle\int_\Gamma \mathrm{d}\Psi_k = 0$. Then, by passing to the limit, if Γ is a line of divergence, we obtain $|P_iP_j| = 0$ which is not possible. This proves that a line of divergence has at most one end point. Suppose that a line of divergence L has no end point, we are in the situation of Figure 2.5. Let A and B be point on L as in Figure 2.5 such that $|AB| > |P_iP_{i+1}|$. We note D (resp. C) the projection of A (resp. B) on L_{i+1}^-. For k big enough we note $C_k = [B,C] \cap [P_{i+1}, Q_i^k]$ and $D_k = [A,D] \cap [P_{i+1}, Q_i^k]$. We then have:

$$\begin{aligned} 0 = \int_{ABC_kD_k} \mathrm{d}\Psi_k &\geq \int_{[A,B]} \mathrm{d}\Psi_k - |BC_k| + \int_{[C_k,D_k]} \mathrm{d}\Psi_k - |D_kA| \\ &\geq \int_{[A,B]} \mathrm{d}\Psi_k - 2|P_i, P_{i+1}| + |C_kD_k| \end{aligned} \tag{2.19}$$

We have $|C_kD_k| \longrightarrow |CD| = |AB|$ and $\displaystyle\int_{[A,B]} \mathrm{d}\Psi_k \longrightarrow |AB|$ for a subsequence, so we get a contradiction and a line of divergence must have one end point.

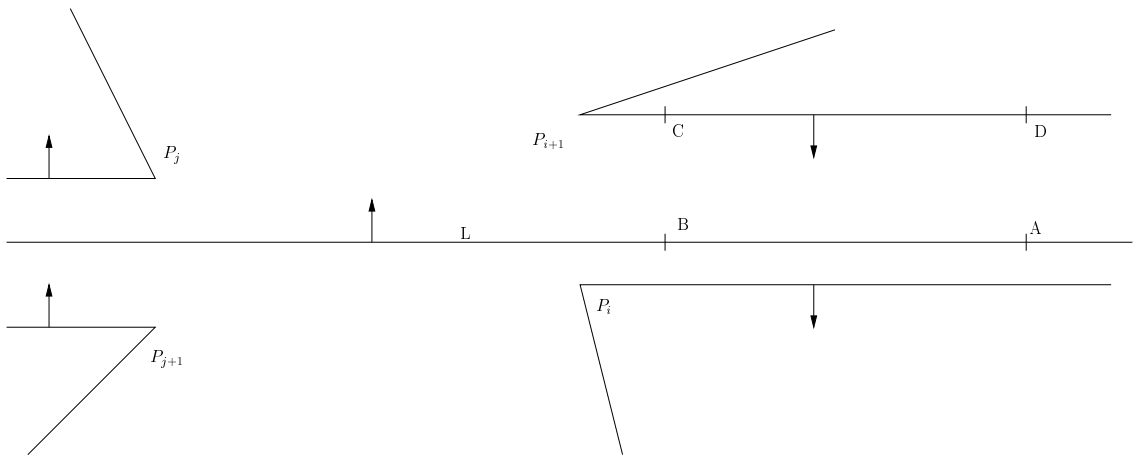

Figure 2.5:

Let L be a line of divergence, we know that we are in the case where L has P_i as end point and goes to infinity in one S_j. By what we have done just above, we have only one possiblity for the limit normal: we are in the same situation as in the semi-strip S_j in Figure 2.5. Then, by changing L if necessary, we can suppose that the part of L in S_j is parametrized by $\{A\} \times \mathbb{R}_+$ with $A \in]P_j, P_{j+1}[$ and the domain $\widetilde{\Omega}$ parametrized by $[A, P_{j+1}] \times \mathbb{R}_+$ is in $\mathcal{B}(u_k)$. Let θ be an extraction that makes L appear, since $\widetilde{\Omega} \subset \mathcal{B}(u_k)$, there exists an extraction θ' such that $u_{\theta'(k)}$ is a subsequence of $u_{\theta(k)}$ and $u_{\theta'(k)} - u_{\theta'(k)}(K)$ (where $K \in \widetilde{\Omega}$) converges to v a solution of (MSE) on $\widetilde{\Omega}$.

We shall now prove that v tends to $-\infty$ on L^-_{j+1} and $+\infty$ on L. Let $B \in [A, P_{j+1}]$ and C and D be the points which are respectively parametrized by (B, c) and (B, d) $(c < d)$. We note:

$$\begin{gathered} E \text{ the projection of } D \text{ on } L^-_{j+1} \\ F \text{ the projection of } C \text{ on } L^-_{j+1} \\ G \text{ the projection of } D \text{ on } L \\ H \text{ the projection of } C \text{ on } L \end{gathered}$$

We note also, when k is big enough, $E_k = [D, E] \cap [P_{j+1}, Q^k_j]$ and $F_k = [C, F] \cap$

$[P_{j+1}, Q_j^k]$. Because $\mathrm{d}\Psi_k$ is closed we have:

$$\left| \int_{[C,D]} \mathrm{d}\Psi_{\theta'(k)} - |F_{\theta'(k)} E_{\theta'(k)}| \right| \leq 2|BP_{j+1}| \tag{2.20}$$

$$\left| \int_{[C,D]} \mathrm{d}\Psi_{\theta'(k)} - \int_{[H,G]} \mathrm{d}\Psi_{\theta'(k)} \right| \leq 2|BA| \tag{2.21}$$

Thus, in letting k tends to infinity, we obtain:

$$\left| \int_{[C,D]} \mathrm{d}\Psi_v - |CD| \right| \leq 2|BP_{j+1}| \tag{2.22}$$

$$\left| \int_{[C,D]} \mathrm{d}\Psi_v - |CD| \right| \leq 2|BA| \tag{2.23}$$

So we can calculate $\mathrm{d}\Psi_v$ on L and L_{j+1}^-, we remark that $\mathrm{d}\Psi_v$ has the same behaviour as if v assumes the boundary values $+\infty$ on L and $-\infty$ on L_{j+1}^-. We prove that this is, in fact, the case. We consider now two points A_1 and A_2 on $L \cap \widetilde{\Omega}$ and two points A_3 and A_4 on L_{j+1}^-. There exists a solution v' of (MSE) on the domain bounded by the polygon $A_1A_2A_3A_4$ such that $v' = v$ on $[A_1, A_4]$ and $[A_2, A_3]$, v' tends to $+\infty$ on $[A_1, A_2]$ and tends to $-\infty$ on $[A_3, A_4]$. Since we know the value of $\mathrm{d}\Psi_v$ on $[A_1, A_2]$ and $[A_3, A_4]$, the uniqueness part of the proof of Theorem 2.2.5 proves that $v = v'$. We then have proved that v tends to $+\infty$ on L and $-\infty$ on L_{j+1}^-.

We shall now get a contradiction to the existence of the line of divergence L. We have $\displaystyle\int_{[A,P_{j+1}]} \mathrm{d}\Psi_v = \int_{\widetilde{\Omega} \cap I_{j,l}} \mathrm{d}\Psi_v$, then, by (2.15) and letting l tends to infinity, we get $\displaystyle\int_{[A,P_{j+1}]} \mathrm{d}\Psi_v = 0$. If we follow L between P_i and A and the segment $[A, P_{j+1}]$, we get a path joining P_i to P_{j+1}. Then we have:

$$0 = \int_{[P_i,A]} \mathrm{d}\Psi_{\theta'(k)} + \int_{[A,P_{j+1}]} \mathrm{d}\Psi_{\theta'(k)} \tag{2.24}$$

Let k tend to infinity, we get $0 = |P_iA| + \displaystyle\int_{[A,P_{j+1}]} \mathrm{d}\Psi_v = |P_iA|$; but $P_i \notin [P_j, P_{j+1}]$, this is our contradiction.

We then have prove that $\mathcal{B}(u_k) = \Omega(\mathcal{P})$, as $u_k(P_0) = 0$ for all k there exists a subsequence $u_{k'}$ which converges to a solution u of (MSE). The same arguments that we used just above for v prove that u tends to $+\infty$ (resp. $-\infty$) on L_i^- (resp. L_i^+); we have then established Theorem 2.6.1. □

We are then able to build the solution to the Plateau problem at infinity. Let V be a polygon and $(\mathcal{P}, \varphi)$ a polygonal disk bounded by V. We consider the solution u of the Dirichlet problen given by Theorem 2.6.1. We note $P_1, \ldots, P_r$ the vertices of V, we consider Ψ_u normalized by $\Psi_u(P_1) = 0$, from the proof above, we have $\Psi_u(P_i) = 0$ for all i. Then on L_i^+ and L_i^- we have $\Psi_u(Q) = |QP_i|$; since Ψ_u is 1-Lipschitz continuous, we have $\Psi_u(Q) \geq 0$ for all $Q \in S_i$. Suppose that $\{Q \in \Omega(\mathcal{P}) |\ \Psi_u(Q) \leq 0\}$ is not reduced to $\{P_1, \ldots, P_r\}$ then there exists a point in the interior of $\mathcal{P}$ such that Ψ_u is minimal at this point. But Ψ_u corresponds to x_3^*, the third coordinate on M^* the conjugate surface to the graph of u; since x_3^* is harmonic on M^*, it can not have a minimum in the interior of M^*. We then have proved that $\Psi_u > 0$ in the interior of $\Omega(\mathcal{P})$.

By Theorem 2.3.2, the boundary of the graph of u is composed of the r vertical lines over the points $\varphi(P_i)$. Let M be the graph of u with these r vertical lines. We consider M^* the conjugate surface to M. The boundary of M^* is composed of r horizontal planar geodesic curves, since $\Psi_u(P_i) = 0$ for all i the r curves are all in the plane $\{x_3 = 0\}$. Finally, we consider Σ the union of M^* and of its symmetry by $\{x_3 = 0\}$. The surface Σ is a regular minimal surface, it is complete and its flux polygon is V by construction. By construction, we know also that Σ is strongly symmetric with respect to $\{x_3 = 0\}$.

The last thing we have to prove about Σ for being sure that it is the solution of the Plateau problem at infinity is that it has finite total curvature.

We know (see [Os2]) that there exists a constant c such that if u is a solution of (MSE) on a domain D and $A \in D$, M is the graph of u and d is the distance along M of the point in M over A to the boundary of S then the curvature K of M at the point over A is bounded by $\dfrac{c}{d^2 W^2(A)}$.

Let us consider a half-strip $S = [0, a] \times \mathbb{R}_+$ and u a solution of (MSE) on S such that u takes the value $+\infty$ (resp. $-\infty$) on $\{0\} \times \mathbb{R}_+^*$ (resp. $\{a\} \times \mathbb{R}_+^*$). The boundary of the graph of u is over $[0, a] \times \{0\}$. We then have $K(x, y) \leq \dfrac{c}{x^2 W^2(x, y)}$. We consider the part $S' \subset S$ such that $x \geq x_0 > 0$. For a domain D we note $K(D)$ the total curvature of the graph over D. We then have:

$$\begin{aligned} K(S') &= \int_{S'} K(x, y) W(x, y) \mathrm{d}x \mathrm{d}y \\ &\leq \int_{S'} \frac{c}{x^2 W(x, y)} \mathrm{d}x \mathrm{d}y \\ &\leq \int_{x_0}^{+\infty} \frac{ca}{x^2} \mathrm{d}x = \frac{ca}{x_0} < +\infty \end{aligned}$$

We now use arguments that are similar to the first part of the proof of Theorem

2.3.2. We consider, for $\alpha \in]0, \frac{\pi}{2}[$,

$S(\alpha) = \{(x,y) \in S |\ a - y \leq x \tan\alpha\}$. Let us take α such that $\tan\alpha < \frac{1}{8}$, then Lemma 1 of [JS] proves that for every $(x,y) \in S(\alpha)$ $q(x,y) < 0$. We note $L(\alpha)$ the segment in $S(\alpha)$ such that $a - y = x\tan\alpha$. By Lemma 2.3.1, u is lower-bounded by m_1 on $L(\alpha)$ and upper bounded by m_2 on the part of L such that $x < x_0 < a$. We then define $\Theta : (x,y) \mapsto (x, u(x,y))$. Θ is a diffeomorphism of $S(\alpha)$ into its image $\operatorname{im}\Theta$. We define $\chi = \Theta^{-1}$ then $\chi_{(2)}$ is a solution of (MSE) on $\operatorname{im}\Theta$. We observe that χ extends smoothly to $\Theta(L)$. We have $\chi_{(2)}(x,z)$ tends to a as x tends to 0 so we can extend $\chi_{(2)}$ by symmetry to $\operatorname{im}\Theta \cup \{(x,z) \in \mathbb{R}^2 | (-x,z) \in \operatorname{im}\Theta\}$. To compute the total curvature of the graph of u over $D(\alpha) \cap \{x < x_0\}$, we use its parametrization as a graph over $\operatorname{im}\Theta \cap \{0 \leq x \leq x_0\}$:

$$\begin{aligned}
K(D(\alpha) \cap \{x < x_0\}) &= K(\operatorname{im}\Theta \cap \{0 \leq x \leq x_0\}) \\
&= \int_{\operatorname{im}\Theta \cap \{0 \leq x \leq x_0\} \cap \{z \leq m_1 - 1\}} K(x,z)W(x,z)\mathrm{d}x\mathrm{d}z \\
&\quad + \int_{\operatorname{im}\Theta \cap \{0 \leq x \leq x_0\} \cap \{z \geq m_1 - 1\}} K(x,z)W(x,z)\mathrm{d}x\mathrm{d}z \\
&\leq \int_{\operatorname{im}\Theta \cap \{0 \leq x \leq x_0\} \cap \{z \leq m_1 - 1\}} \frac{c}{(z - m_1)^2 W(x,z)}\mathrm{d}x\mathrm{d}z \\
&\quad + \int_{\operatorname{im}\Theta \cap \{0 \leq x \leq x_0\} \cap \{z \geq m_1 - 1\}} K(x,z)W(x,z)\mathrm{d}x\mathrm{d}z \\
&\leq \int_1^{+\infty} \frac{cx_0}{z^2}\mathrm{d}z + C < +\infty
\end{aligned}$$

The last line is true because $\overline{\operatorname{im}\Theta} \cap \{0 \leq x \leq x_0\} \cap \{z \geq m_1 - 1\}$ is compact.

We can do the same work for $\{(x,y) \in S |\ y \leq x\tan\alpha\}$.

We then control the curvature on each semi-strip S_i. There is a last part in $\Omega(\mathcal{P})$. This part is compact and by Lemma 2.3.1, u is bounded on this part; besides the graph is regular at the boundary. So the graph above this last part is a compact part of the whole graph then it has finite total curvature. We then have proved that the graph M has finite total curvature. Since M^* is isometric to M, it has finite total curvature and then Σ has finite total curvature because it is twice as many as the one of M^*.

2.7 Appendix : The Carathéodory's Theorem

In this section, we give some explanations on an argument of the proof of Theorem 2.3.2. The problem is: when we have a biholomorphic map between two open sets of $\mathbb{C}$, can we extend it to the boundary?

We consider U an open set included in $\mathbb{C}$ and P a point of ∂U. We say that P has the property of Schönflies if, for all radii R, there exists a radius $r = r(R)$ such that for all two points in $U \cap D(P, r)$ there exists a path in $U \cap D(P, R)$ joining these two points.

We then have the following theorem that we use in our proof.

Theorem 2.7.1 (Carathéodory). *Let U be a simply connected open set in $\mathbb{C}$ and V an open set of the boundary of U. We consider $f : U \longrightarrow D = \{z \in \mathbb{C} | \ |z| < 1\}$ a biholomorphic map. We suppose that every point of V has the property of Schönflies , then f extends to an homeomorphism from $U \cup V$ into $D \cup C$ where $C \subset \partial D$.*

A proof of this theorem can be found in [He]. In our proof, we have to verify the property of Schönflies at the points of a part of the boundary. We know that this part of the boundary is embedded in $\mathbb{C}$ so we can build neighborhoods of every point of the boundary in using ε-tubular neighborhood of the boundary. These neighborhoods prove that we have the property of Schönflies.

Chapitre 3

Le problème de Plateau à l'infini en genre 1

Ce chapitre reprend un article dont le titre original est «The Plateau problem at infinity for horizontal ends and genus 1» ; il apparaît dans la bibliographie à la référence [Ma2].

Dans le chapitre précédent, nous avons étudié le problème de Plateau à l'infini en genre 0, le but de celui-ci est de faire une étude similaire pour le genre 1. Le résultat principal est le Théorème 3.2.11 qui affirme que sous la même hypothèse qu'en genre 0 (border un disque immergé) un polygone est le polygone de flux d'un r-noïde de genre 1. Le Théorème 3.7.1 donne lui une autre hypothèse sur les polygones sous laquelle un polygone est le polygone de flux d'un r-noide de genre 1. La figure 3.1 montre un exemple de 3-noïde de genre 1, le polygone de flux correspondant est un triangle équilatéral (l'image est fournie par la bibliothèque de surfaces minimales de GRAPE [MSL]).

Comme au chapitre précédent, la première partie du travail consiste en la résolution d'un problème de Dirichlet pour l'équation (ESM) puis en l'étude de la régularité du graphe de la solution. La nouvelle difficulté est que, pour construire des r-noides de genre 1, on doit résoudre un problème de période, c'est le travail qui est fait dans la section 3.6.

3.1 Introduction

The classical Plateau problem consists in finding a surface of least area bounded by a given closed curve in $\mathbb{R}^3$, such a surface satisfies that its mean curvature vanishes. A surface in $\mathbb{R}^3$ with zero mean curvature will be called a minimal surface. The Plateau problem has been solved by T. Radó in 1930. A generalization of this problem is: finding a minimal surface for a given asymptotic behaviour. We first

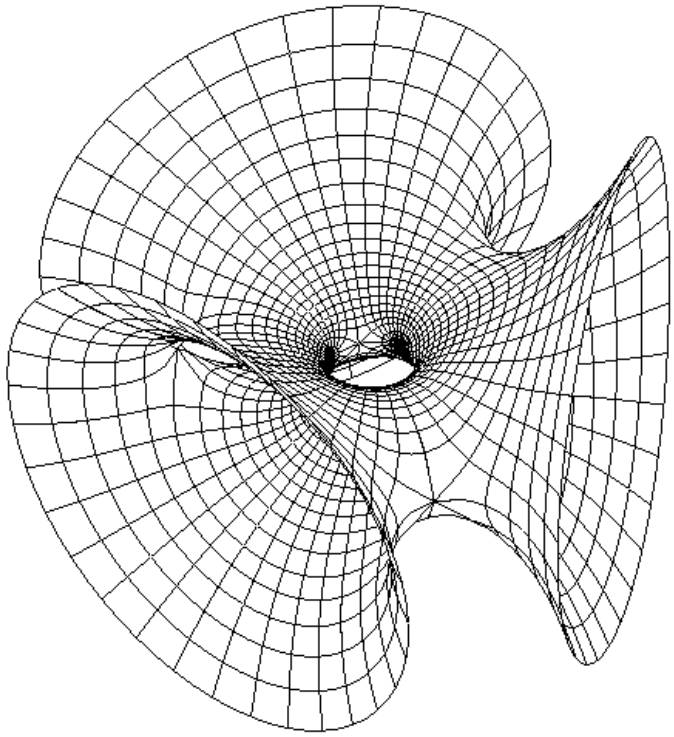

FIG. 3.1 – Un exemple de 3-noïde de genre 1

give sense to this question.

We know that, if a complete minimal surface M has finite total curvature and r embedded ends (such a surface is called an r-noid), each end of this minimal surface is asymptotic to a plane or to a half-catenoid; besides, we can associate to each end a vector in $\mathbb{R}^3$, this vector is called the flux vector of the end. These vectors satisfy the following condition: the sum of the flux vectors over all ends is zero. So the generalization of the Plateau problem is: given a finite number of vectors such that their sum is zero, can we find an r-noid which has these vectors as flux vectors? This problem is called the Plateau problem at infinity. Besides, we know that M is conformally equivalent to a compact Riemann surface $\overline{M}$ minus r points, the punctures of M, and what we call the genus of M is in fact the genus of $\overline{M}$.

In this paper, we shall study a particular case of this problem. In [CR], C. Cosín and A. Ros give a description of the space of solutions of the Plateau problem at infinity with an asymptotic behaviour which is symmetric with respect to an

horizontal plane (*i.e.* all the flux vectors are horizontal) for genus 0 (the Riemann surface $\overline{M}$ is the Riemann sphere $\mathbb{S}^2$). In the genus 0 case, there is a natural order on the ends. Then, since the flux vectors are horizontal and their sum is zero, the flux vectors draw a polygon in $\mathbb{R}^2$; this polygon is called the flux polygon of M. C. Cosín and A. Ros give a necessary and sufficient condition on this polygon for having a solution to the Plateau problem at infinity. In our work, we study the case where $\overline{M}$ is of genus 1 (*i.e.* $\overline{M}$ is a torus).

For the case of genus 1, we need to distinguish two types of r-noid with horizontal ends; this classification depends on the place of the punctures on the torus: when M is of the first type, there is a natural order on the punctures and for the second type, there is not. If M is an r-noid of genus 1 and horizontal ends of the first type, since there is a natural order on the punctures, we can define as in the genus 0 case the flux polygon associated to M. Then our main result can be state as follow (see Theorem 3.2.11).

Let M be an r-noid with genus 0 *and horizontal ends, then there exists Σ an r-noid with genus* 1 *and horizontal ends of the first type which have the same flux polygon as M.*

In Corollary 3.7.2, we give examples of polygons which are the flux polygons of r-noid with genus 1 but not the flux polygons of r-noid with genus 0.

The r-noids M, we consider, are symmetric with respect to a horizontal plane that we can normalized to be $\{z = 0\}$. Then to build them, it is sufficient to build the part M^+ of M included in $\mathbb{R}^2 \times \mathbb{R}_+$. The proof of our result is then based on the fact that M^+ is the conjugate of a minimal surface that can be seen as the graph of a function u over a planar "domain" which depends on the flux polygon of M. This "domain" is in fact a multi-domain (see definitions in Section 3.2).

If Ω is a domain in $\mathbb{R}^2$ and u is a function on Ω, the graph of u is a minimal surface in $\mathbb{R}^3$ iff u satisfies the elliptic partial equation called the minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{MSE}$$

Like every partial differential equation, we can associate to (MSE) the Dirichlet problem that consists in finding a function u on Ω which is a solution of the minimal surface equation and takes on assignated values on the boundary of Ω.

The first step to build a r-noid with genus 1 having a polygon V as flux polygon is then to consider a multi-domain Ω associated to this polygon and solve a Dirichlet problem on Ω. The boundary data is such that the graph of the solution can be the conjugate of M^+. As an example if V is a triangle ABC, we can glue along the three

edges of V three half-strips ($[A,B] \times \mathbb{R}_+$ along $[A,B]$, $[B,C] \times \mathbb{R}_+$ along $[B,C]$...), we get an unbounded domain D. Then the multi-domain Ω is the universal cover of $D \backslash \{Q\}$ where Q is a point in the triangle ABC. The boundary value we take on Ω is $\pm\infty$ alternating the sign such that for every half-strip in Ω one side has $+\infty$ and the other has $-\infty$.

If Ω is a multi-domain associated to the polygon V, the conjugate surface to the graph of the solution of the Dirichlet problem is a minimal surface which is invariant under a translation by a horizontal vector. If this vector is non zero, this minimal surface is not the piece M^+ of an r-noid M, then we must choose a multi-domain associated to V such that the corresponding vector is zero. In our example of the triangle, the only choice we have is the position of the point Q in ABC, then we need to prove that there exists $Q \in ABC$ such that the corresponding vector vanishes. This problem is called the period problem.

In the case of the triangle, the idea to solve the period problem is the following. To each point Q in the interior of the triangle ABC, we can associate an horizontal vector, in fact, this defines a continuous map from the triangle to $\mathbb{R}^2$ and we want to show that this map vanishes; this map is the period map. Then we whall compute the degree of the period map along the boundary of the triangle and find a non-zero number, then the period map must vanishes at one point. The proof of our main result is based on a generalization of this arguement.

The paper is organized as follows; in the first section, we define all the notions of multi-domain we use in the following and the objects associated to a function on such domains. We also precise what kind of r-noid we study in this paper and give the first results concerning them.

Section 3.3 is devoted to the resolution of the Dirichlet problem on a multi-domain Ω associated to a polygon V. In this section, we make use of tools developped in section 1.4 and recalled in Appendix 3.8.

In the third section, we study the regularity of the graph of the solution of the Dirichlet problem of the preceding section near the singularity point of the multi-domain Ω.

In Section 3.5, we explain what is the period problem for the construction of a r-noid. We also generalize the notion of the period map and give the proof of our main result (Theorem 3.2.11) in using Theorem 3.6.11 which is proved in Section 3.6.

Section 3.6 is devoted to an extension of the period map and the computation of its degree.

In section 3.7, we give examples of r-noids with genus 1 that are not given by Theorem 3.2.11. In particular, we consider the case where the polygon V is a regular polygon.

Let us fix some notations. In the following, when u is a function on a domain

of $\mathbb{R}^2$ we shall note $W = \sqrt{1+|\nabla u|^2}$. We shall also use the classical following notations for partial derivatives: $p = \dfrac{\partial u}{\partial x}$ and $q = \dfrac{\partial u}{\partial y}$. Besides, for the graph of u, we shall always chose the downward pointing normal to give an orientation to the graph.

3.2 Preliminaries

3.2.1 Graph on multi-domains

In this section,we give several generalizations of the notion of domain in $\mathbb{R}^2$, since our aim is to describe some minimal surfaces as the graph of a function which is a solution of (MSE) (see the example of the half helicoid). We use notions introduced in [CR] and chapter 2.

Let us consider a pair (D, ψ) where D is 2-dimensionnal flat manifold with piecewise smooth boundary and $\psi : D \to \mathbb{R}^2$ is a local isometry. The map ψ is called the *developing map* and the points where the boundary ∂D is not smooth are called vertices. If a part of the boundary of D is linear then this part will be called an *edge* of the boundary.

Definition 3.2.1. A pair (D, ψ), where D is a simply-connected 2-dimensionnal complete flat manifold with piecewise smooth boundary and $\psi : D \to \mathbb{R}^2$ is a local isometry, is a *multi-domain* if each connected component of the smooth part of ∂D is a convex arc.

Let D be a complete metric space and Q a point of D we say that D admits a *cone singularity* at Q of angle α if $D\backslash\{Q\}$ is a 2-dimensional flat manifold and if there exist $\rho_0 > 0$ such that $\{M \in D|\ d(M,Q) < \rho_0\}$ is isometric to $\{(\rho,\theta)|\, 0 \leq \rho < \rho_0,\ 0 \leq \theta \leq \alpha\}$ with the polar metric $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2$ where all the points $(0,\theta)$ are identified and where for every ρ we identified $(\rho, 0)$ with (ρ, α) (the isometry sends Q to $(0,0)$).

Definition 3.2.2. A triplet (D, Q, ψ) is a *multi-domain with a cone singularity at* Q if

1. D is a simply-connected complete metric space,
2. there exists $q \in \mathbb{N}$ such that D admits a cone singularity at Q of angle $2q\pi$,
3. D has piecewise smooth convex boundary and
4. $\psi : D \to \mathbb{R}^2$ is a local isometry outside Q.

We can remark that a multi-domain (D, ψ) can be seen as a multi-domain with a cone singularity at Q if Q is some point of D. The angle at the singularity is 2π.

Let (D, ψ) be a compact multi-domain such that its boundary is only composed of edges. The developing map allows us to see ∂D included in $\mathbb{R}^2$ since there are only edges $\psi(\partial D)$ is a polygon in $\mathbb{R}^2$. The same thing can be done for (D, Q, ψ) a multi-domain with cone singularity. We then say that a polygon V bounds a multi-domain (with perhaps a cone singularity) if there exists (D, ψ) or (D, Q, ψ) such that $V = \psi(\partial D)$. When V bounds a multi-domain (D, ψ) we shall also say that v bounds an immersed polygonal disk as in [CR].

The last generalization we need is to give a sense to a cone singularity with infinite angle.

Let us consider $\mathcal{D} = \{(\rho, \theta) |\ \rho \in \mathbb{R}_+,\ \theta \in \mathbb{R}\}$ with the polar metric and where all the points $(0, \theta)$ are identified, this point will be called the vertex of $\mathcal{D}$ and noted $\mathcal{O}$. The space $\mathcal{D}$ is a simply-connected complete metric space and $\mathcal{D} \backslash \mathcal{O}$ is a 2-dimensional flat manifold.

Definition 3.2.3. A triplet $(\Omega, \mathcal{Q}, \varphi)$ is a *multi-domain with a logarithmic singularity at* $\mathcal{Q}$ if

1. Ω is a simply-connected complete metric space,
2. $\mathcal{Q} \in \Omega$,
3. $\Omega \backslash \mathcal{Q}$ is a 2-dimensional flat manifold with piecewise smooth convex boundary,
4. $\varphi : \Omega \to \mathcal{D}$ is a local isometry such that $\varphi(\mathcal{Q}) = \mathcal{O}$ and
5. there exist a neighborhood $\mathcal{N}$ of $\mathcal{Q}$ in Ω and $\rho > 0$ such that $\varphi|_{\mathcal{N}}$ is an isometry into $\{M \in \mathcal{D} \mid d(\mathcal{O}, M) < \rho\}$.

Let us define $R_\alpha : \mathcal{D} \to \mathcal{D}$ by $R_\alpha(r, \theta) = (r, \theta + \alpha)$, R_α is an isometry of $\mathcal{D}$.

Definition 3.2.4. A multi-domain with a logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ is *periodic* if there exists $f : \Omega \to \Omega$ an isometry and $n \in \mathbb{N}^*$ such that

$$\varphi \circ f = R_{2n\pi} \circ \varphi \tag{$*$}$$

The period of Ω is then $2\pi q$ where q is the smallest n such that there exists f making $(*)$ true.

The first example of a multi-domain with a logarithmic singularity, we can give, is $(\mathcal{D}, \mathcal{O}, \mathrm{id})$. This multi-domain is periodic of period 2π.

Construction 1. Let us consider (D, ψ) a multidomain and Q a point in D. We then note $\Omega \xrightarrow{\pi} D \backslash Q$ a universal cover of $D \backslash Q$. We can pull back to Ω the flat

metric of D. The metric completion of Ω is just $\Omega \cup \{\mathcal{Q}\} = \overline{\Omega}$ where $\mathcal{Q}$ is a point "above" Q (*i.e.* if $\mathcal{A}_n \to \mathcal{Q}$, we have $\pi(\mathcal{A}_n) \to Q$). If (ρ, θ) are the polar coordinates on $\mathbb{R}^2$ of center $\psi(Q)$ then, on Ω the 1-forms $(\pi \circ \psi)^* \mathrm{d}\rho$ and $(\pi \circ \psi)^* \mathrm{d}\theta$ are exact and by integration we can define a map $\varphi : \Omega \cup \{\mathcal{Q}\} \to \mathcal{D}$ such that $(\overline{\Omega}, \mathcal{Q}, \varphi)$ is a multi-domain with a logarithmic singularity. The multi-domain, we have just build, is a periodic one of period 2π.

In fact, we can do the same work for (D, Q, ψ) a multi-domain with a cone singularity at Q of angle $2q\pi$. We get $(\overline{\Omega}, \mathcal{Q}, \varphi)$ a periodic multi-domain with a logarithmic singularity (the period is less than $2q\pi$) and a covering map $\pi : \overline{\Omega} \to D$ with $\pi(\mathcal{Q}) = Q$.

Construction 2. The inverse construction is also possible. Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with a logarithmic singularity of period $2q\pi$ and isometry f. Then by taking the quotient of Ω by the group $\{f^n\}_{n \in \mathbb{Z}}$, we build a multi-domain with a cone singularity at Q, the image of $\mathcal{Q}$ in the quotient, and angle $2q\pi$.

Remark 3.2.5. We make a remark about these two constructions. Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with a logarithmic singularity of period $2q\pi$ and isometry f. If we considere the quotient of Ω by the group $\{f^{an}\}_{n \in \mathbb{Z}}$ for $a \in \mathbb{N}^*$, we get a multi-domain with a cone singularity (D, Q, ψ). The cone singularity at Q of D has $2qa\pi$ as angle. But if we apply Construction 1 to D we get Ω which have a period less than $2qa\pi$ if $a > 1$.

Let $V = (v_1, \ldots, v_r)$ be a polygon which, for example, bounds an immersed polygonal disk (D, ψ). If $Q \in D$, we make Construction 1 and we get a multi-domain with logarithmic singularity $(\overline{\Omega}, \mathcal{Q}, \varphi)$. The quotient (D', Q', φ') of $\overline{\Omega}$ by $\{f^{2n}\}_{n \in \mathbb{Z}}$ is a multi-domain with cone singularity of angle 4π; besides this multi-domain bounds the polygon $(v_1, \ldots, v_r, v_1, \cdots, v_r)$. But since Construction 1 gives $\overline{\Omega}$ for D and D', the two polygons V and $(v_1, \ldots, v_r, v_1, \cdots, v_r)$ will not be distinguished in the following.

Let $(\Omega, \mathcal{Q}, \varphi)$ be a multi-domain with logarithmic singularity and A be a point in $\mathbb{R}^2$. We then can define the map $\varphi_A : \Omega \to \mathbb{R}^2$ by $\varphi_A = G \circ \varphi$ with $G(\rho, \theta) = A + (\rho \cos\theta, \rho \sin\theta)$. If Ω is given by Construction 1 we always choose $A = \psi(Q)$.

Let Ω such that (Ω, φ) or (Ω, Q, φ) corresponds to one of the three definitions of multi-domain given above. Let u be a function defined on Ω or Ω minus its singularity. The *graph* of u is then the surface in $\mathbb{R}^3$ defined by $\{\varphi(x), u(x)\}_{x \in \Omega}$ or $\{\varphi_A(x), u(x)\}_{x \in \Omega \setminus \{\mathcal{Q}\}}$ with $A \in \mathbb{R}^2$. In the following the function u will often be a solution of the minimal surface equation, in this case the graph of u becomes a minimal surface of $\mathbb{R}^3$. The fact that u is a solution of (MSE) allows us to define a closed 1-form $\mathrm{d}\Psi_u$ on Ω, $\mathrm{d}\Psi_u$ is the inner product $\frac{\nabla u}{W} \lrcorner \mathrm{d}V$ where $\mathrm{d}V$ is the volume form on Ω. Since $\mathrm{d}\Psi_u$ is closed we can define locally a function Ψ_u (obviously Ψ_u is well defined only if we fix its value at one point). Ψ_u is locally defined in the interior of Ω minus the possible singularity, but Ψ_u is 1-Lipschitz countinuous then

it can be countinuously extended to the singularity and the boundary, then since Ω is simply connected Ψ_u is then globally defined on Ω. In fact Ψ_u correponds to the third coordinates of the conjugate surface to the graph of u; Ψ_u is called the conjugate function to u. For other properties on Ψ_u we refer to [JS] and Appendix 3.8.

In [JS] and chapter 2, we can find the most general answer to the Dirichlet problem on compact multi-domain (D, ψ): the Dirichlet problem consists in finding a solution u on D of (MSE) knowing its value on the boundary.

On multi-domain with cone or logarithmic singularity there is no general answer. To give an exemple of solution of (MSE) on a multi-domain with logarithmic singularity, let us consider the function u defined on $(\mathcal{D}, \mathcal{O}, \mathrm{id})$ by $u(\rho, \theta) = \theta$, it is obvious that the graph of u is the half of an helicoid; more precisely it is the surface given in isothermal coordinate by $(a, b) \mapsto (\sinh a \cos b, \sinh a \sin b, b)$ for $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$. The function u is then a solution of the minimal surface equation.

3.2.2 The r-noids of genus 1

In this section we give precise definitions of r-noids, flux at one end and other objects linked to the Plateau problem at infinity. We also give some results concerning this problem and explain how we can build solutions in the genus 1 case.

Let M be a complete minimal surface with finite total curvature in $\mathbb{R}^3$; we know that M is isometric to a compact Riemann surface $\overline{M}$ minus a finite number of points (we can refer to [Os2]). Then M has a finite number of annular ends; when these ends are embedded they are asymptotic either to a half-catenoid or to a plane. A properly immersed minimal surface with r embedded ends will be called a *r-noid*. We can associate to each end a vector which caracterizes the direction and the growth of the asymptotic half-catenoid (when the end is asymptotic to a plane this vector is zero); this vector is called the *flux* of the end (for a precise definition of the flux see [HK]). If $v_1, \ldots, v_r$ are the fluxes at each end, we have the following balancing condition:

$$v_1 + \cdots + v_r = 0 \tag{3.1}$$

This condition tells us that the total flux of the system vanishes. If $v_1, \ldots, v_r$ are vectors in $\mathbb{R}^3$ such that (3.1) is verified and g is a non-negative integer, the Plateau problem at infinity for these data is to find an r-noid of genus g which has $v_1, \ldots, v_r$ as fluxes at its ends (the genus g is the genus of $\overline{M}$).

Let $X : M \longrightarrow \mathbb{R}^3$ be an r-noid. M is conformally equivalent to a compact surface $\overline{M}$ minus r points $p_1, \ldots, p_r$. We will say that M is *Alexandrov-embedded* if $\overline{M}$ bounds a compact 3-manifold $\overline{\Omega}$ and the immersion X extends to a proper local diffeomorphism $f : \overline{\Omega} \backslash \{p_1, \ldots, p_r\} \longrightarrow \mathbb{R}^3$. An Alexandrov-embedded surface has a canonical orientation; we choose the Gauss map to be the outward pointing normal.

An Alexandrov-embedded r-noid can not have a planar end (see [CR]).

We are interested in the case where $X : M \longrightarrow \mathbb{R}^3$ is an Alexandrov-embedded r-noid of genus g and r horizontal ends (*i.e.* the flux at each end is an horizontal vector).

Let $X : M \longrightarrow \mathbb{R}^3$ be a nonflat immersion of a connected orientable surface M and Π be a plane in $\mathbb{R}^3$, normalized to be $\{x_3 = 0\}$. We note by S the Euclidiean symmetry with respect to Π and consider the subsets:

$$\begin{aligned} M^+ &= \{p \in M | x_3(p) > 0\} \\ M^- &= \{p \in M | x_3(p) < 0\} \\ M^0 &= \{p \in M | x_3(p) = 0\} \end{aligned}$$

With these notation we have:

Definition 3.2.6. We shall say that M is strongly symmetric with respect to Π if

- There exists an isometric involution $s : M \longrightarrow M$ such that $\psi \circ s = S \circ \psi$.
- $\{p \in M | s(p) = p\} = M^0$.
- The third coordinate N_3 of the Gauss map of M takes positive (resp. negative) values on M^+ (resp. M^-).

In [CR], C. Cosín and A. Ros prove

Proposition 3.2.7. *Let M be an r-noid with horizontal ends. Then M is strongly symmetric with respect to an horizontal plane if and only if M is Alexandrov-embedded.*

The notion of strong symmetry is then important for the study of the Alexandrov-embedded r-noid.

The case of genus 0 was studied by C. Cosín and A. Ros in [CR]; they show that in this case there is a natural order on the ends. Let M be an Alexandrov-embedded r-noid of genus 0 if $2v_1, \ldots, 2v_r$ are the fluxes of M ordered as the ends, the polygon $(v_1, \cdots, v_r)$ is called the *flux polygon* of M and is noted $F(M)$. We then have

Theorem 3.2.8. *Let $v_1, \ldots, v_r$ be horizontal vectors such that $v_1 + \cdots + v_r = 0$ and V the associated polygon, then there exists M an Alexandrov-embedded r-noid of genus 0 such that $F(M) = V$ if, and only if, V bounds an immersed polygonal disk. Besides there is a bijection from the set of M such that $F(M) = V$ and the set of the immersed polygonal bounded by V.*

In the following, when $(\mathcal{P}, \psi)$ is an immersed polygonal disk we shall call $\Sigma(\mathcal{P})$ the r-noid of genus 0 associated to $\mathcal{P}$ by this bijection; we refer to [CR] and chapter 2 for more explanations on this theorem.

In this paper we are interested in the case of genus 1. Let M be an Alexandrov-embedded r-noif of genus 1 with horizontal ends. M is conformally a torus $\overline{M}$ minus r points $p_1, \ldots, p_r$. By Proposition 3.2.7, M is strongly symmetric with respect to the plane $\{z = 0\}$ (in fact M is strongly symmetric with respect to an horizontal plane that we can normalize to be $\{z = 0\}$). As in [CR], the punctures p_i are fixed by the antiholomorphic involution s given by Definition 3.2.6; we have the following lemma.

Lemma 3.2.9. *Let $\overline{M}$ be a conformal torus and s an antiholomorphic involution on $\overline{M}$. We suppose that s has a fixed point then the set of the fixed points of s is two separated circles.*

We can then give the following definition.

Definition 3.2.10. Let M be an Alexandrov-embedded r-noid with genus one and horizontal ends and p_i the punctures of M. We note s the antiholomorphic involution associated to M, we then have C_1 and C_2 the two circles of fixed points of s. With this notations, we shall say that:

- M is of *type I* if all the p_i are in one of the two circles C_1 and C_2,
- M is of *type II* if not.

We suppose now that M is of type I, the circles of fixed points are the boundary of $M^+ \cup M^0$, this minimal surface is oriented by the outward pointing normal and then its boundary has a natural orientation. Then the circle that contains the points p_i has a natural orientation and we suppose that the points p_i are numbered with respect to this orientation. We note $2v_i$ the flux vector associated to p_i. We know that we have $\sum_{i=1}^{r} v_i = 0$, so, as in the case $g = 0$, the list $(v_1, \ldots, v_r)$ defines a polygon which we call the *flux polygon* of M and we note it $F(M)$. We can remark that if M is of type II we do not have such an easy definition.

In the following, we use some arguements developped by C. Cosín and A. Ros in [CR] to the study of the genus 0 case. Let $X : \overline{M} \backslash \{p_1, \ldots, p_r\} \to M$ be an Alexandrov-embedded r-noid with genus one and horizontal ends of type I. The surface $M^+ \cup M^0$ is topologically an annulus, so by passing to the universal cover, we can define its conjugate surface $M^*_{0,+}$. The surface $M^*_{0,+}$ is a periodic minimal surface and its period vector is $\int_\gamma \mathrm{d}X^*$ where γ is a path in $M^+ \cup M^0$ that generates $\pi_1(M^+ \cup M^0)$. Since M is of type I all the p_i are in one circle of fixed points, then

the other circle of fixed points generates $\pi_1(M^+ \cup M^0)$ and since the normal along this circle is horizontal the first two coordinates of the period vector are zero. This proves that the period vector is a vertical vector.

The symmetry curves M^0 consists of r complete strictly convex curves in the plane $\{z = 0\}$ (they are the images by X of the arcs $p_i p_{i+1}$) and the image by X of the circle of fixed points that contains no p_i. In the conjugate surface, these curves are transformed in vertical lines; the image of the arc $p_i p_{i+1}$ in the conjugate surface are vertical straight-lines over the vertices of the polygon $F(M)$

Since on M^+ the third coordinate of the normal is positive, the projection map Π from M_+^*, the conjugate of M^+, into the plane $\{z = 0\}$ is a local diffeomorphism. So we can pull back to M_+^* the flat metric of this plane. Besides M_+^* is stable by the translation t of vector $\displaystyle\int_\gamma \mathrm{d}X^*$. Since this vector is vertical, $\Pi \circ t = \Pi$; t is then an isometry of M_+^* with the flat metric $\Pi^*(\mathrm{d}s^2_{\mathbb{R}^2})$.

Using arguements of C. Cosín and A. Ros, we can prove that there exists $(\Omega, \mathcal{Q}, \varphi)$ a multi-domain with logarithmic singularity such that M_+^* with the flat metric can be seen as the interior of Ω minus the singularity point $\mathcal{Q}$. Besides Ω is periodic because of the existence of the isometry t. The boundary of Ω is composed of half-lines. M_+^* is then a graph over Ω such that the line which is the conjugate of the circle of fixed points that contains no p_i is the part of the boundary of the graph which is above $\mathcal{Q}$ and the conjugates of the arcs $p_i p_{i+1}$ are lines over the vertices of Ω.

The quotient of Ω by the group $(t^n)_{n\in\mathbb{Z}}$ is then a multi-domain with cone singularity (D', Q, ψ). D' has r vertices $P_1, \ldots, P_r$ and is bounded by $2r$ half-lines; more precisely, D' is a multi-domain with cone singularity D which is bounded by the flux polygon $F(M) = (v_1, \ldots, v_r)$ (we have $v_i = \overrightarrow{\psi(P_i)\psi(P_{i+1})}$) to which we have glued r half-strips (along the edge $[P_i, P_{i+1}]$, we glue a half-strip isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$). This proves that the flux polygon $F(M)$ bounds a multi-domain with cone singularity.

In Section 3.3 and Section 3.4, we shall prove many results that explains, when a polygon V bounds a multi-domain with cone singularity, how we can build a candidate for a surface M_+^* such that $F(M) = V$.

The main result we prove concerning the Plateau problem at infinity for genus one is the following.

Theorem 3.2.11. *Let $v_1, \ldots, v_r$ be r non zero vectors of $\mathbb{R}^2$ such that $(v_1, \ldots, v_r)$ is a polygon that bounds an immersed polygonal disk. Then there exists M an Alexandrov-embedded r-noid with genus one and horizontal ends of type I such that $F(M) = (v_1, \ldots, v_r)$.*

This implies that every polygon that can be realized as the flux polygon of an

r-noid of genus 0 is also the flux polygon of an r-noid of genus 1.

3.3 A Dirichlet problem

Let $v_1, \ldots, v_r$ be r non zero vectors of $\mathbb{R}^2$ such that $V = (v_1, \ldots, v_r)$ bounds a multi-domain with cone singularity (D, Q, ψ). Let us note $P_1, \ldots, P_r$ the vertices of the polygon, we put $P_{r+1} = P_1$; in the following, we suppose that the orientation on the polygon V is the one given as boundary of D (We remark that, in the following, the vertices of V will be identified with the vertices of D). Following Construction 1, we get a multi-domain with logarithmic singularity $(\mathcal{W}, \mathcal{Q}, \varphi)$ with a projection map $\pi : \mathcal{W} \to D$. Because of Remark 3.2.5, we suppose that the period $2q\pi$ of $\mathcal{W}$ is equal to the angle of the cone singularity of D, in fact we suppose that D is the quotient of $\mathcal{W}$ by the isometry f; In the following we shall say that D satisfies the hypothesis H.

Construction 3. Let $i \in \{1, \ldots, r\}$ and consider E an edge of $\mathcal{W}$ which is send by π to the edge $[P_i, P_{i+1}]$ of D; we can glue to $\mathcal{W}$ along E a half strip S_i isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+^*$ (if $A \in E$ the point A is identified with $(\pi(A), 0)$). Making this for every i and every edges E, we get a new multi-domain with a logarithmic singularity at $\mathcal{Q}$ that we note $(\Omega, \mathcal{Q}, \varphi)$ (we keep the same notation for the developping map since it is an extension of the original developping map). Since $\mathcal{W}$ is periodic, Ω is periodic and have the same period, we note f the corresponding isometry.

Let us note, for $i \in \{1, \ldots, r\}$, $\mathcal{L}_i^+$ (resp. $\mathcal{L}_i^-$) the union of the straight-lines corresponding to $\{P_i\} \times \mathbb{R}_+^*$ in the half-strips glued along the edges E such that $\pi(E) = [P_i, P_{i+1}]$ (resp. $\pi(E) = [P_{i-1}, P_i]$). $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ are a countable union of half straight-lines. We note $\mathcal{V}$ the set of the vertices of Ω this set is $\pi^{-1}\{P_1, \ldots, P_r\}$.

We then have the following existence and uniqueness result.

Theorem 3.3.1. *Let (D, Q, ψ) be a multi-domain with cone singularity that bounds a polygon V, Constructions 1 and 3 give us a periodic multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$; we suppose that the period of Ω is the cone angle of D at Q. Then there exists a solution u of the minimal surface equation on Ω such that*

1. *u tends to $+\infty$ along $\mathcal{L}_i^+$ and $-\infty$ along $\mathcal{L}_i^-$ and*
2. *$\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{V})$,*

where $\mathcal{L}_i^+$, $\mathcal{L}_i^-$ and $\mathcal{V}$ are the notations given in Construction 3. Besides, the solution is unique up to an additive constant.

The conditions imposed to a function u solution of this Plateau problem make that the graph of u is a good candidate for giving a solution to the Plateau problem at infinity with V as flux polygon.

Corollary 3.3.2. *If u is a solution on Ω of the Dirichlet problem asked in Theorem 3.3.1, there exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$.*

Proof. Let u be a solution of the Dirichlet problem asked in Theorem 3.3.1, then $u \circ f$ is also a solution of this Dirichlet problem. This proves that $u \circ f - u$ is constant. □

The following of this section is devoted to the proof of Theorem 3.3.1

3.3.1 Notations

First, the period of Ω is $2q\pi$. We can suppose that P_1 is such that $d(Q, P_1) = \min_i d(Q, P_i)$. Let us consider a vertex in $\mathcal{V}$ which is a lift of P_1 and denote it $\mathcal{P}_1(0)$. Since $d(M, P_1)$ is minimal, the geodesic joining $\mathcal{Q}$ to $\mathcal{P}_1(0)$ is embedded, this implies that the first coordinate of $\varphi(\mathcal{P}_1(0))$ is $d(\mathcal{Q}, \mathcal{P}_1(0)) = d(Q, P_1)$. By considering $R_\alpha \circ \varphi$ instead of φ, we can suppose that $\varphi(\mathcal{P}_1(0)) = (d(\mathcal{Q}, \mathcal{P}_1(0), 0)$. Let us note $\mathcal{P}_1(k) = f^k(\mathcal{P}_1(0))$ for $k \in \mathbb{Z}$; we then have $\varphi(\mathcal{P}_1(k)) = (d(M, P_0), 2kq\pi)$. Then $\{\mathcal{P}_1(i), i \in \mathbb{Z}\}$ is the set of the vertex of Ω corresponding to P_1 (*i.e.* $\{\mathcal{P}_1(i), i \in \mathbb{Z}\} = \pi^{-1}(P_1)$).

If we remove the geodesics $[\mathcal{Q}, \mathcal{P}_1(k)]$ and $[\mathcal{Q}, \mathcal{P}_1(l)]$ $(k < l)$ from Ω, we get a space which have three connected components; one of them is such that its intersection with $\mathcal{N}$, the neighborhood of $\mathcal{Q}$ introduced in Definition 3.2.3, is isometric with $]0, r[\times]2kq\pi, 2lq\pi[$ by φ; we shall note this part Ω_k^l. For $n \in \mathbb{Z}$ we have $f^n(\Omega_k^l) = \Omega_{k+n}^{l+n}$. In Ω_0^1, there is exactly one lift of every P_i $(2 \leq i \leq r)$, we note $\mathcal{P}_i(0)$ the lift of P_i that is in Ω_0^1. We then note for $k \in \mathbb{Z}$ $\mathcal{P}_i(k) = f^k(\mathcal{P}_i(0))$; we then have that $\mathcal{P}_i(k)$ is the only lift of P_i in Ω_k^{k+1}. A part Ω_k^{k+1} of Ω is called a *period* of Ω.

For $i \in \{1, \ldots, r\}$ and $k \in \mathbb{Z}$, let us note $\mathcal{L}_i^+(k)$ (resp. $\mathcal{L}_i^-(k)$) the half straight-line included in $\mathcal{L}_i^+$ (resp. $\mathcal{L}_i^-$) and having $\mathcal{P}_i(k)$ as end-point.

Let us consider k and l in $\mathbb{Z}$ such that $k < l$, then Ω_k^l is a multi-domain. Its vertices are $\mathcal{Q}$, the $\mathcal{P}_1(m)$ for $k \leq m \leq l$ and the $\mathcal{P}_i(m)$ for $2 \leq i \leq r$ and $k \leq m < l$. Its boundary is composed of two segments $[\mathcal{Q}, \mathcal{P}_1(k)]$ and $[\mathcal{Q}, \mathcal{P}_1(l)]$ and $2r(l-k)$ half straight-lines which are the $\mathcal{L}_1^+(m)$ for $k \leq m < l$, the $\mathcal{L}_1^-(m)$ for $k < m \leq l$ and the $\mathcal{L}_i^\pm(m)$ for $2 \leq i \leq r$ and $k \leq m < l$. From Ω_k^l, we now define a new multi-domain: we can glue to Ω_k^l two half-strips, one is isometric to $[\mathcal{Q}, \mathcal{P}_1(k)] \times \mathbb{R}_+$ and is glued along $[\mathcal{Q}, \mathcal{P}_1(k)]$, the second is isometric to $[\mathcal{P}_1(l), \mathcal{Q}] \times \mathbb{R}_+$ and is glued along $[\mathcal{P}_1(l), \mathcal{Q}]$. We note $\widetilde{\Omega}_k^l$ this new multi-domain. We note $\widetilde{\mathcal{L}}_k^-$ (resp. $\widetilde{\mathcal{L}}_l^+$) the new half straight-line in the boundary of $\widetilde{\Omega}_k^l$ with $\mathcal{P}_1(k)$ (resp. $\mathcal{P}_1(l)$) as end-point. We also note $\widetilde{\mathcal{L}}^+$ and $\widetilde{\mathcal{L}}^-$ the two half straight-lines in the boundary with $\mathcal{Q}$ as end-point such that $\widetilde{\mathcal{L}}^+$ is in the same half-strip than $\widetilde{\mathcal{L}}_k^-$.

3.3.2 Proof of the existence

We shall now prove the existence part of Theorem 3.3.1. First, for n in $\mathbb{N}^*$, we prove that there exists a function u_n on $\widetilde{\Omega}^n_{-n}$ which:

1. is a solution of (MSE),
2. tends to $+\infty$ on $\widetilde{\mathcal{L}}^+$, $\widetilde{\mathcal{L}}^+_n$ and all the $\mathcal{L}^+_i(k)$ that are in the boundary of $\widetilde{\Omega}^n_{-n}$ and
3. tends to $-\infty$ on $\widetilde{\mathcal{L}}^-$, $\widetilde{\mathcal{L}}^-_{-n}$ and all the $\mathcal{L}^-_i(k)$ that are in the boundary of $\widetilde{\Omega}^n_{-n}$.

Let us consider the following polygon:

$$(\overrightarrow{QP_1}, \underbrace{v_1, \ldots, v_r, \ldots\ldots, v_1, \ldots, v_r}_{2n \text{ times}}, \overrightarrow{P_1Q})$$

If we remove to $\widetilde{\Omega}^n_{-n}$ all the half-strips, we get a multi-domain which is bounded by the above polygon. Then, the existence of the solution u_n is ensured by Theorem 2.6.1.

Let us now only consider the restriction of u_n to Ω^n_{-n}. We then have a sequence of solutions of (MSE) defined in an increasing sequence of domains Ω^n_{-n} and $\bigcup_{n\in\mathbb{N}^*} \Omega^n_{-n} = \Omega$. We can then consider that (u_n) is a sequence of functions on Ω. We want (u_n) to converge, so we shall prove that there is no line of divergence (see Appendix 3.8 and Section 1.4).

First we give some preliminary results on u_n.

Lemma 3.3.3. *Let u be a solution of* (MSE) *on the half-strip $[0,a]\times\mathbb{R}_+$ such that u tends to $-\infty$ on $\{0\}\times\mathbb{R}_+$ and $+\infty$ on $\{a\}\times\mathbb{R}_+$. Then, for $y \geq 4a$, we have:*

$$\frac{|p|}{W}(x,y) \geq 1 - \frac{a^2}{y^2}$$
$$\frac{|q|}{W}(x,y) \leq \sqrt{2}\frac{a}{y}$$

Proof. We note A the point of coordinates (x,y) and B a point in the boundary of the half-strip which realizes the distance from A to this boundary. Since $y \geq 4a$ the coordinates of B are $(0,y)$ or (a,y); besides the distance $|AB|$ is less than $a/2$. Because of the value of u on the boundary, the distance along the graph from the point above A to the boundary of the graph is bigger than $4a$. The ratio of this two distances is less than $1/8$, then we can apply Lemma 1 in [JS]; this gives the lemma. □

Corollary 3.3.4. *Let u be a solution of* (MSE) *on the half-strip* $[0,a]\times\mathbb{R}_+$ *such that u tends to $-\infty$ on $\{0\}\times\mathbb{R}_+$ and $+\infty$ on $\{a\}\times\mathbb{R}_+$. We have $\Psi_u(0,y)=\Psi_u(a,y)$ and if $\Psi_u(0,0)=0$ then $\Psi_u\geq 0$ in the half-strip.*

Proof. Because of the value of u on the boundary we have $\Psi_u(0,y)=\Psi_u(0,0)+y$ and $\Psi_u(a,y)=\Psi_u(a,0)+y$. This implies that for $y\geq 4a$ we have

$$\begin{aligned}|\Psi_u(0,0)-\Psi_u(a,0)| &= |\Psi_u(0,y)-\Psi_u(a,y)|\\ &= |\int_0^a -\frac{q}{w}(x,y)\mathrm{d}x|\\ &\leq \int_0^a \sqrt{2}\frac{a}{y}\mathrm{d}x \leq \sqrt{2}\frac{a^2}{y}\end{aligned}$$

Then by letting y goes to $+\infty$ we have $\Psi_u(0,0)=\Psi_u(a,0)$ and then $\Psi_u(0,y)=\Psi_u(a,y)$.

If $\Psi_u(0,0)=0$ then $\Psi_u(0,y)=y$. If A is in the interior of the half-strip, there exists y such that A is at a distance less than y from the point $(0,y)$; then, since Ψ_u is 1-Lipschitz continuous $\Psi_u(A)\geq 0$. □

This corollary implies that, if we choose Ψ_{u_n} such that $\Psi_{u_n}(\mathcal{Q})=0$, we have $\Psi_{u_n}(\mathcal{P}_i(k))=0$ for all $\mathcal{P}_i(k)\in\Omega^n_{-n}$ and $\Psi_{u_n}\geq 0$ in the half-trips contained in $\widetilde{\Omega}^n_{-n}$. Since Ψ_{u_n} satisfies a maximum principle, we have $\Psi_{u_n}\geq 0$ in Ω^n_{-n}.

Assume that there exists a line of divergence L. By Proposition 1.4.5, L can not have an end point in the interior of one of the half straight-lines that compose the boundary of Ω; this prove that if it has end-points it must be $\mathcal{Q}$ or one $\mathcal{P}_i(k)$.

If L has no end point, let $\mathcal{A}$ be a point of L and let us note d the distance between $\mathcal{A}$ and $\mathcal{Q}$, since $\Psi_{u_n}(\mathcal{Q})=0$ and Ψ_{u_n} is 1-Lipschitz, we have $0\leq\Psi_{u_n}(\mathcal{A})\leq d$. Let us fix an orientation to L such that the limit normal along this line of divergence is the right-hand unit normal. We then note s the arc-length along L with $\mathcal{A}$ as origin. Let us consider $\mathcal{A}'$ the point on L of arc-length $s=-2d$. Then we know that, for the subsequence that makes L appear, we have $\Psi_{u_{n'}}(\mathcal{A})-\Psi_{u_{n'}}(\mathcal{A}')\to 2d$ but, since $\Psi_{u_n}(\mathcal{Q})\leq d$ this implies that $\Psi_{u_{n'}}(\mathcal{A}')<0$ for big n'; this is a contradiction.

If L is a segment (L has two end-points noted $\mathcal{A}_1$ and $\mathcal{A}_2$), we know that for each n we have $\Psi_{u_n}(\mathcal{A}_1)=0=\Psi_{u_n}(\mathcal{A}_2)$, but for the subsequence that make L appear we have $|\Psi_{u_{n'}}(\mathcal{A}_1)-\Psi_{u_{n'}}(\mathcal{A}_2)|\to d(\mathcal{A}_1,\mathcal{A}_2)>0$; this gives us a contradiction.

We then can suppose that L has only one end-point that we note $\mathcal{F}$ and goes to infinity in one half-strip that we can parametrized isometricaly by $[0,a]\times\mathbb{R}_+$. We remark that for one half-strip in Ω the number of such lines L is finite. We have, for each n, $\Psi_{u_n}(\mathcal{F})=0$. There exists $b\in]0,a[$ such that the part of L in the half-strip is $\{b\}\times\mathbb{R}_+$. By changing L if necessary we can suppose that the part $]b,a[\times\mathbb{R}_+$ is included in $\mathcal{B}(u_n)$. Let $\mathcal{A}=(b,0)$, $\mathcal{B}=(b,2(a-b))$, $\mathcal{C}=(a,2(a-b))$ and $\mathcal{D}=(a,0)$ be four points in the half-strip. Since $\mathrm{d}\Psi_{u_n}$ is closed we have:

$$\begin{aligned}\int_{[\mathcal{A},\mathcal{B}]} \mathrm{d}\Psi_{u_n} &= -\int_{[\mathcal{B},\mathcal{C}]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{C},\mathcal{D}]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{D},\mathcal{A}]} \mathrm{d}\Psi_{u_n} \\ &= 2(a-b) - \int_{[\mathcal{B},\mathcal{C}]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{D},\mathcal{A}]} \mathrm{d}\Psi_{u_n} \\ &\geq 2(a-b) - (a-b) - (a-b) = 0\end{aligned}$$

This implies that we have only one possibility for the limit normal . Since $]b,a[\times\mathbb{R}_+ \subset \mathcal{B}(u_n)$ we can suppose that a subsequence $(u_{n'})$ converges to a function v on $]b,a[\times\mathbb{R}_+$ (n' is chosen such that the line of divergence L appears). v is a solution of (MSE) and by Proposition 1.4.9, we know that v tends to $+\infty$ along $\{b\}\times\mathbb{R}_+^*$ and $-\infty$ along $\{a\}\times\mathbb{R}_+^*$. Then by Lemma 3.3.4, $\Psi_v(\mathcal{A}) = \Psi_v(\mathcal{D}) = \lim \Psi_{u_n}(\mathcal{D}) = 0$. But, for the subsequence, $\Psi_{u_{n'}}(\mathcal{A}) - \Psi_{u_{n'}}(\mathcal{F}) \to d(\mathcal{F},\mathcal{A}) > 0$, this contradicts the fact that $\Psi_{u_n}(\mathcal{F}) = 0$ and $\lim \Psi_{u_{n'}}(\mathcal{A}) = 0$.

We then have prove that $\mathcal{B}(u_n) = \Omega$, and, by taking a subsequence, we can suppose that u_n converges to a solution u of (MSE) on Ω. By Proposition 1.4.9, u is such that u tends to $+\infty$ along the $\mathcal{L}_i^+(k)$ and $-\infty$ along the $\mathcal{L}_i^-(k)$. By construction, we have also that $\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{P}_i(k))$ and $\Psi_u \geq 0$.

3.3.3 Property of the solution 1

Before proving the uniqueness, we need to give some properties of a solution of the Dirichlet problem asked in Theorem 3.3.1. We recall that there exist ρ_0 and a neighborhood $\mathcal{N}$ of $\mathcal{Q}$ such that φ is an isometry from $\mathcal{N}$ into $\{M \in \mathcal{D} | d(\mathcal{O},M) \leq \rho_0\}$. We then can use the polar coordinates for v a function defined on $\mathcal{N}$.

Proposition 3.3.5. *Let ε be a positive number. There exists $d > 0$ such that for every $\alpha \in \mathbb{R}$ and every v solution of* (MSE) *on $\mathcal{N} \cap \{\alpha - \pi < \theta < \alpha + \pi\}$:*

$$\begin{aligned}\sup_{0<\rho\leq\rho_0} v(\rho,\alpha) &< \sup_{[d,\rho_0]\times[\alpha-3\pi/4,\alpha+3\pi/4]} v(\rho,\theta) + \varepsilon \\ \inf_{0<\rho\leq\rho_0} v(\rho,\alpha) &> \inf_{[d,\rho_0]\times[\alpha-3\pi/4,\alpha+3\pi/4]} v(\rho,\theta) - \varepsilon\end{aligned}$$

The proof is based on the idea used by R. Osserman to prove Theorem 10.3 in [Os2].

Proof. First let us assume that $\alpha = \pi/2$. Then $\mathcal{N} \cap \{\alpha - \pi < \theta < \alpha + \pi\}$ is isometric to the disk of center the origin and radius ρ_0 minus the segment joining the origin to $(0, -\rho_0)$. Then we only prove the first inequality since the second is a consequence of the first by substituting v for $-v$.

Let $\varepsilon > 0$ and d be a positive number, we note A the point of coordinates $(0, -d)$ and D the domain in $D(0, \rho_0)$ delimited by the segment joining $(-\sqrt{\rho_0^2 - d^2}, -d)$ to $(-d, -d)$, the half circle of center A and radius d: $\theta \mapsto (d\cos\theta, d(\sin\theta - 1))$ for $\theta \in [0, \pi]$ and the segment joining $(d, -d)$ to $(\sqrt{\rho_0^2 - d^2}, -d)$ and containing $(0, \rho_0/2)$. On the domain D we consider the function $c : M \mapsto d\left(-\operatorname{argch}(\frac{|AM|}{d}) + \operatorname{argch}(\frac{\rho_0+d}{d})\right)$. The graph of c is a piece of a catenoid and c is a positive function upper bounded by $d\operatorname{argch}(\frac{\rho_0+d}{d})$. Let d be small enough such that this upper-bound is less than ε.

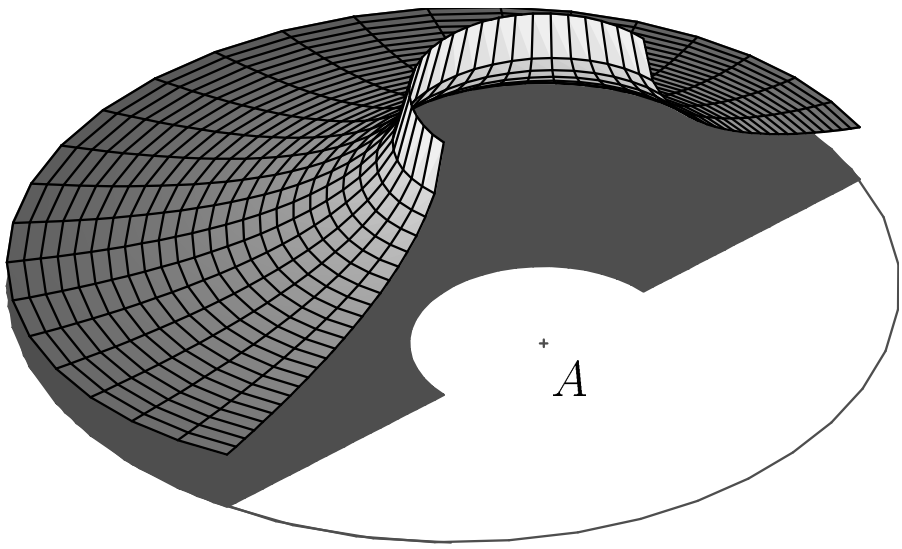

Figure 3.2: the graph of the function c

On the part of the boundary of D which is a half circle of center A, the derivatives $\nabla c \cdot n$, where n is the outward pointing normal, takes the value $+\infty$. This implies, by Lemma 10.2 in [Os2], that $c + K \geq v$ where K is the supremum of v on the boundary of D minus the half circle of center A. Since this part of the boundary is included in the set of points of polar coordinates included in $[d, \rho_0] \times [-\pi/4, 5\pi/4]$ and D contains the segment joining the origin to $(0, 1)$ the proposition is established. $\square$

This proposition is an important fact since it implies that if u is a solution of the dirichlet problem asked in Theorem 3.3.1, u is bounded on $\mathcal{N} \cap \Omega_k^l$ for every k and l.

3.3.4 Proof of the uniqueness

Let us now prove the uniqueness part of Theorem 3.3.1. We consider u and v two solutions of the Dirichlet problem asked in Theorem 3.3.1 and we assume that these

two solutions are different: $u - v$ is not constant. By changing v by $v + c$ where c is a real constant we can suppose:

1. $\{u - v > 0\} \cap \Omega_0^{+\infty}$ and $\{u - v < 0\} \cap \Omega_0^{+\infty}$ are non-empty ($\Omega_0^{+\infty}$ is $\bigcup_{n>0} \Omega_0^n$) and
2. the segment $[\mathcal{Q}, \mathcal{P}_1(0)]$ is included in $\{u - v > 0\}$.

The first assertion is due to the fact that $u - v$ is non-constant and that we can exchange u and v, the second assertion is a consequence of the fact that u and v are bounded on the segment $[\mathcal{Q}, \mathcal{P}_1(0)]$ by Proposition 3.3.5 and Lemma 2.3.1.

We note $\Delta = \{u - v < 0\} \cap \Omega_0^{+\infty}$ We then note Δ^n the intersection of Δ and Ω_0^n. The boundary $\partial\Delta^n$ is composed of a part included in the boundary of Ω, a part included in the interior of Ω_0^n and a part included in the segment $[\mathcal{Q}, \mathcal{P}_1(n)]$. Along the first part $\mathrm{d}\widetilde{\Psi} = \mathrm{d}\Psi_u - \mathrm{d}\Psi_v = 0$ and along the second part $\mathrm{d}\widetilde{\Psi}$ is positive by Lemma 2 in [CK]; the union of this first two parts will be noted $\partial^n\Delta$.

We note Δ_n^l the intersection of Δ_n with the part of Ω_0^n such that a point A is in this part if either A is not in one half-strip included in Ω_0^n or $A = (x, y)$ is in one of these half-strips which is isometricaly parametrized by $[0, a] \times \mathbb{R}_+$ and $y \leq l$. The boundary of Δ_n^l is composed of three parts: the first is $\partial\Delta_n \cap [\mathcal{Q}, \mathcal{P}_1(n)]$, the second is $\partial^n\Delta \cap \overline{\Delta_n^l}$ and the third is included in the union of the segments parametrized by $[0, a] \times \{l\}$ in each half-strip; we note Γ^l this third part. Since $\mathrm{d}\widetilde{\Psi}$ is closed we have:

$$\int_{\partial\Delta_n \cap [\mathcal{Q}, \mathcal{P}_1(n)]} \mathrm{d}\widetilde{\Psi} + \int_{\partial^n\Delta \cap \overline{\Delta_n^l}} \mathrm{d}\widetilde{\Psi} + \int_{\Gamma^l} \mathrm{d}\widetilde{\Psi} = 0 \tag{3.2}$$

We then have

$$\int_{\partial^n\Delta \cap \overline{\Delta_n^l}} \mathrm{d}\widetilde{\Psi} \leq |\mathcal{Q}\mathcal{P}_1(n)| + \left| \int_{\Gamma^l} \mathrm{d}\widetilde{\Psi} \right|$$

Since $\mathrm{d}\widetilde{\Psi} \geq 0$ along $\partial^n\Delta$, $\int_{\partial^n\Delta \cap \overline{\Delta_n^l}} \mathrm{d}\widetilde{\Psi}$ increases as l increases. By Lemma 3.3.3, $\int_{\Gamma^l} \mathrm{d}\widetilde{\Psi} \to 0$ as l goes to $+\infty$. Then the above equation implies that $\mathrm{d}\widetilde{\Psi}$ is integrable on $\partial^n\Delta$ and by passing to the limit in (3.2), we obtain

$$\int_{\partial\Delta_n \cap [\mathcal{Q}, \mathcal{P}_1(n)]} \mathrm{d}\widetilde{\Psi} + \int_{\partial^n\Delta} \mathrm{d}\widetilde{\Psi} = 0 \tag{3.3}$$

This equation implies that

$$\int_{\partial^n\Delta} \mathrm{d}\widetilde{\Psi} \leq |\mathcal{Q}\mathcal{P}_1(n)| = |\mathcal{Q}\mathcal{P}_1(0)|$$

Then $\mathrm{d}\widetilde{\Psi}$ is integrable on $\partial\Delta$ and $\int_{\partial\Delta}\mathrm{d}\widetilde{\Psi} > 0$ since the part of $\partial\Delta$ included in the interior of Ω is non empty. We have

$$\int_{\partial\Delta}\mathrm{d}\widetilde{\Psi} = \sum_{n=0}^{+\infty}\int_{\partial\Delta\cap\Omega_n^{n+1}}\mathrm{d}\widetilde{\Psi}$$

Then $\displaystyle\int_{\partial\Delta\cap\Omega_n^{n+1}}\mathrm{d}\widetilde{\Psi} \longrightarrow 0$. We then want to understand what happens on Ω_n^{n+1} for big n.

Let us consider, for $n \in \mathbb{N}$, $u_n = u \circ f^{-n}$ and $v_n = v \circ f^{-n}$; the restriction of u_n to Ω_0^1 is equal to the restriction of u to Ω_n^{n+1} and the same is true for v. With the same arguements as in the proof of existence, we can prove that there exist two real sequences (a_n) and (b_n) such that $u_{n'} - a_{n'} \to \tilde{u}$ and $v_{n'} - b_{n'} \to \tilde{v}$ on Ω (in fact $a_n = u_n(P)$ and $b_n = v_n(P)$ for a point P in Ω). The functions $\tilde{u}$ and $\tilde{v}$ are two solutions of the Dirichlet problem asked in Theorem 3.3.1 and the convergence is uniform for every derivative on compact parts. By changing our subsequence if necessary, we can assume that $b_{n'} - a_{n'} \to \pm\infty$ or $b_{n'} - a_{n'} \to c \in \mathbb{R}$. We are interested in $\{u_n - v_n < 0\} \cap \Omega_0^1$ and, in a certain sense, these sets "converge" to $\{\tilde{u} - \tilde{v} < \lim b_{n'} - a_{n'}\} \cap \Omega_0^1$.

Let $\gamma : [0, |\mathcal{Q}\mathcal{P}_1(1)|] \to [\mathcal{Q}, \mathcal{P}_1(1)]$ the parametrization by arc-length of the segment $[\mathcal{Q}, \mathcal{P}_1(1)]$, with $\gamma(0) = \mathcal{Q}$.

First we assume that $b_{n'} - a_{n'} \to \pm\infty$. Let $\varepsilon > 0$ such that ε is less than $\displaystyle\frac{1}{3}\int_{\partial^n\Delta}\mathrm{d}\widetilde{\Psi}$ and $\frac{1}{2}|\mathcal{Q}, \mathcal{P}_1(1)|$. Since $\tilde{u}$ and $\tilde{v}$ are bounded on $[\mathcal{Q}, \mathcal{P}_1(1)]$ and that we have uniform convergence on $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$, we can ensure that, for n' big enough, $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ is included in $\{u_{n'} - v_{n'} < 0\}$ or does not intersect $\{u_{n'} - v_{n'} < 0\}$ following the sign of the limit of $b_{n'} - a_{n'}$. For every n, $\displaystyle\int_{[\mathcal{Q},\mathcal{P}_1(n)]}\mathrm{d}\widetilde{\Psi} = 0$ because of our hypotheses on Ψ_u and Ψ_v, we then have for big n'

$$\left|\int_{\partial\Delta_{n'+1}\cap[\mathcal{Q},\mathcal{P}_1(n'+1)]}\mathrm{d}\widetilde{\Psi}\right| \leq 2\varepsilon \tag{3.4}$$

This equality implies that $\displaystyle\int_{\partial^{n'+1}\Delta}\mathrm{d}\widetilde{\Psi} \leq 2\varepsilon$ for big n'. Then by passing to the limit we obtain a contradiction because of our hypothesis on ε.

We assume now that $b_{n'} - a_{n'} \to c$. Let ε be as above. If the segment $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ is included in $\{\tilde{u} - \tilde{v} < c\}$ or does not intersect $\{\tilde{u} - \tilde{v} < c\}$ then the same property is true for $\{u_{n'} - v_{n'} < 0\}$ for big n. Then same arguements as above give us a contradiction. We then can ensure that there is a point in $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ where $\tilde{u} - \tilde{v} = c$. If $\tilde{u} = \tilde{v} + c$ on Ω, we have $\mathrm{d}\Psi_{u_{n'}} - \mathrm{d}\Psi_{v_{n'}} \to 0$ uniformly on

$\gamma([\varepsilon, |\mathcal{QP}_1(1)| - \varepsilon])$. Then for n' big enough, the inequality (3.4) is true, this gives us a countradiction. Then $\tilde{u} \neq \tilde{v} + c$ on Ω and there is a compact part of the boundary of $\{\tilde{u} - \tilde{v} < c\}$ that crosses the segment $\gamma([\varepsilon, |\mathcal{QP}_1(1)| - \varepsilon])$, let us note Γ this part of the boundary (Γ is oriented as the boundary of this set). We have $\int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}} > 0$ by Lemma 2 in [CK]. The curve Γ is included in a compact part of Ω_0^2 and since $(u_{n'} - v_{n'})$ converges to $\tilde{u} - \tilde{v} - c$ (the convergence is uniform for each derivative on every compact part) we can ensure that for n' big enough

$$\int_{\partial\{u_{n'} - v_{n'} < 0\} \cap \Omega_0^2} \mathrm{d}\Psi_{u_{n'}} - \mathrm{d}\Psi_{v_{n'}} \geq \frac{1}{2} \int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}}$$

This implies that

$$\int_{\partial\Delta \cap \Omega_{n'}^{n'+2}} \mathrm{d}\widetilde{\Psi} \geq \frac{1}{2} \int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}} > 0$$

We have then a contradiction with $\int_{\partial\Delta \cap \Omega_n^{n+1}} \mathrm{d}\widetilde{\Psi} \longrightarrow 0$. We have then proved that our two solutions u and v differ only by a real additive constant.

3.3.5 Property of the solution 2

From Corollary 3.3.2, if u is a solution of the Dirichlet problem asked in Theorem 3.3.1, there exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$. We have the following result for such a situation.

Proposition 3.3.6. *Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with logarithmic singularity, we note f the isometry associated to the periodicity. Let v be a solution of the minimal surface equation on Ω such that:*

1. *there exists a constant $c \in \mathbb{R}$ such that $v \circ f = v + c$ and*
2. *if Ψ_v is the conjugate function to v with $\Psi_v(\mathcal{Q}) = 0$, Ψ_v is non-negative.*

Under these hypotheses, the constant c is non zero.

Proof. Let us suppose that the constant c is zero and consider v only on the neighborhood $\mathcal{N}$ of the singularity point $\mathcal{Q}$. Since $v \circ f = v$, by taking the quotient with respect to f, the function v can be seen as a function $\widetilde{v}$ defined on $\mathcal{C}$ a flat disk of radius ρ_0 with a cone singularity at its center of angle $2q\pi$ minus the singularity point (*i.e.* $\mathcal{C}$ is $\{(\rho, \theta), \rho \in (0, \rho_0), \theta \in [0, 2q\pi]\}$ where $(\rho, 0)$ and $(\rho, 2q\pi)$ are identified, the metric on $\mathcal{C}$ is the polar metric). The graph of $\widetilde{v}$ is a minimal surface of $\mathcal{C} \times \mathbb{R}$. This surface is topologicaly an annulus then it is conformally

parametrized by a Riemann surface R which is an annulus: there are two harmonic maps $X : R \to \mathcal{C}$ and $x_3 : R \to \mathbb{R}$ such that $x_3 = \widetilde{v} \circ X$. We can suppose that R is either $\{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$ for $1 < a \leq +\infty$ or $\{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$ for $0 < a \leq +\infty$. On $\mathcal{C}$, the function $\Psi_{\widetilde{v}}$ is well defined and satisfies $\Psi_{\widetilde{v}} \geq 0$ and $\Psi_{\widetilde{v}} = 0$ at the cone singularity by hypothesis. Since x_3 is harmonic, on R we can define its harmonic conjugate x_3^*; *a priori* x_3^* is multi-valuated but, for a good choice of x_3^*, we have $x_3^* = \Psi_{\widetilde{v}} \circ X$ then x_3^* is well defined on R. As $\Psi_{\widetilde{v}} = 0$ at the cone singularity, $x_3^*(\zeta)$ converges to zero as either $|\zeta|$ tends to 1 when $R = \{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$ or $|\zeta|$ tends to zero when $R = \{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$.

In the second case ($R = \{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$), x_3^* is a harmonic function on a pointed disk, which has a continuous extension to the whole disk. But, an harmonic function can not have an isolated singularity, then the continuous extension is harmonic and have an extremum at the origin, since $x_3^* \geq 0$ on R; this gives a contradiction.

In the other case ($R = \{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$), since $x_3^* = 0$ on the unit circle, the function x_3^* extends to an harmonic function on $\{\zeta \in \mathbb{C} | \frac{1}{a} < |\zeta| < a\}$ by Schwartz reflection principle (the extension is defined by $x_3^*(\zeta) = -x_3^*(\frac{1}{\overline{\zeta}})$; see [ABR]). By taking the harmonic conjugate, x_3 also extends to $\{\zeta \in \mathbb{C} | \frac{1}{a} < |\zeta| < a\}$. Along the unit circle $\mathbb{S}^1$, $\nabla x_3^* \cdot n = 0$, with n the unit tangent vector to $\mathbb{S}^1$, since $x_3^* = 0$ along this circle. Besides by Rolle's Theorem there is a point on $\mathbb{S}^1$ where $\nabla x_3 \cdot n = 0$. At this point, $\nabla x_3^* = 0$ and the local structure of a critical point of a non constant harmonic function implies that x_3^* must be negative on R in the neighborhood of this point, this contradicts $x_3^* \geq 0$ on R. □

Remark 3.3.7. This proposition proves that, for a solution u of the Dirichlet problem asked in Theorem 3.3.1, the constant c such that $u \circ f = u + c$ is non-zero, then u can not pass to the quotient by f. But since $u \circ f = u + c$ the derivatives of u are invariant by f then the derivatives of u are well defined on the quotient. Besides, this implies that the function Ψ_u is also invariant by f.

An other property of such a solution u is that u can be build by the way used in the proof of the existence. As a consequence, the conjugate function Ψ_u is non-negative on Ω.

3.4 The regularity over the singularity point

As in the preceding section we consider V a polygon that bounds a multi-domain with cone singularity (D, Q, ψ); using Construction 1 and 3, we get a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$; we suppose that D satisfies the hypothesis H. Theorem 3.3.1 allows us to construct a minimal surface in $\mathbb{R}^3$ which is a graph over Ω; we want to use this surface to build a r-noid with genus 1 and horizontal

ends of type I with V as flux polygon therefore the graph needs to be regular up to its boundary. In this section, we understand the behaviour of this minimal surface over the singularity point $\mathcal{Q}$: we shall prove the following result.

Theorem 3.4.1. *Let u be a solution of the Dirichlet problem asked in Theorem 3.3.1. Then the surface $\{\varphi_{\psi(Q)}(x), u(x)\}$ for $x \in \mathcal{N}$ is a minimal surface with boundary and its boundary is $\{\varphi_{\psi(Q)}(x), u(x)\}$ for $x \in \partial\mathcal{N}$ and the vertical straight line passing through $\psi(Q)$.*

In fact, this surface can also be seen as a minimal surface in $\mathcal{D} \times \mathbb{R}$; with this point of view, Theorem 3.4.1 can be stated as follows.

Theorem 3.4.2. *Let u be a solution of the Dirichlet problem asked in Theorem 3.3.1. Then the surface $\{\varphi(x), u(x)\}_{x\in\mathcal{N}} \subset \mathcal{D} \times \mathbb{R}$ is a minimal surface with boundary and its boundary is $\{\varphi(x), u(x)\}$ for $x \in \partial\mathcal{N}$ and the vertical straight line $\{\mathcal{O}\} \times \mathbb{R}$.*

It is easy to see that Theorem 3.4.2 implies Theorem 3.4.1, then we shall prove Theorem 3.4.2. The problem in this theorem is the behaviour near the singularity point not along $\partial\mathcal{N}$.

3.4.1 A compactness result

In the proof of Theorem 3.4.2, we shall need to make converge a sequence of minimal surfaces. The following theorem will be an important tool.

Theorem 3.4.3. *Let ζ_1, ζ_2 and ζ_3 be three different points in $\mathbb{S}^1$. Let $\gamma_n : \mathbb{S}^1 \to \mathcal{D} \times \mathbb{R}^{k_1}$ be a sequence of Jordan curves in $\mathcal{D} \times \mathbb{R}^{k_1}$ and $f_n : \mathbb{S}^1 \to \mathbb{R}^{k_2}$ be a sequence of continuous maps; we note $g_n = (\gamma_n, f_n)$. We note $\Gamma_n = \gamma_n(\mathbb{S}^1)$ and put, for $1 \le i \le 3$, $p_i^n = \gamma_n(\zeta_i)$. We suppose that:*

1. *$I = \inf_{n\in\mathbb{N}}\{d(p_1^n, p_2^n), d(p_1^n, p_3^n), d(p_2^n, p_3^n)\} > 0$,*

2. *for every $m > 0$ there exists $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$, if $\zeta, \zeta' \in \mathbb{S}^1$ and $d(g_n(\zeta), g_n(\zeta')) < \varepsilon$, one component of $\Gamma_n \backslash \{\gamma_n(\zeta), \gamma_n(\zeta')\}$ is of diameter less than m and*

3. *there exist $M > 0$ and a sequence $X_n : \Delta \to \mathcal{D} \times \mathbb{R}^{k_1+k_2}$ (Δ is the unit disk) such that $g_n = X_n|_{\mathbb{S}^1}$ and*

$$\int_\Delta (|X_{nx}|^2 + |X_{ny}|^2)\mathrm{d}x\mathrm{d}y < M$$

The family $\{\gamma_n\}_{n\in\mathbb{N}}$ is then equicontinuous.

By Arzela's theorem, this implies that if, for example, the sequence (p_1^n) converges there exists a subsequence $\gamma_{n'}$ that converges for uniform convergence. Theorem 3.4.3 is in fact very similar to a classical result used in the resolution of the classical Plateau problem (see for example, Lemma 3.2 in [Cou] or [Hi]).

Proof. The proof is based on the following lemma.

Lemma 3.4.4 (Courant-Lebesgue). *Let X be of class $C^0(\Delta, \mathcal{D}\times\mathbb{R}^k)\cap C^1(\mathring{\Delta}, \mathcal{D}\times\mathbb{R}^k)$ ($\mathring{\Delta}$ is the interior of Δ) and satisfy*

$$\int_\Delta (|X_x|^2 + |X_y|^2)\mathrm{d}x\mathrm{d}y < M$$

for some $M \in \mathbb{R}_+$. Then, for every $\zeta_0 \in \mathbb{S}^1$ and for each $\delta \in (0,1)$, there exists a number $\rho \in (\delta, \sqrt{\delta})$ such that the distance between the images $X(\zeta)$, $X(\zeta')$ of the two intersection points ζ and ζ' of $\mathbb{S}^1$ with the circle $\partial D_\rho(\zeta_0)$ can be estimated by

$$d(X(\zeta), X(\zeta')) \leq \left(\frac{4M\pi}{\ln(1/\delta)}\right)^{1/2}$$

The proof of this lemma can be found in [Hi].

Let e be a positive number, we suppose that $e < I$. By the second hypothesis, there exists ε such that, for every $n \in \mathbb{N}$, if $\zeta, \zeta' \in \mathbb{S}^1$ and $d(g_n(\zeta), g_n(\zeta')) < \varepsilon$, one component of $\Gamma_n \backslash \{\gamma_n(\zeta), \gamma_n(\zeta')\}$ is of diameter less than e. Let $\delta > 0$ be such that

$$\left(\frac{4M\pi}{\ln(1/\delta)}\right)^{1/2} \leq \varepsilon$$

and for every $\zeta \in \mathbb{S}^1$ we have $|\zeta - \zeta_i| > \delta$ for at least two of the points $\zeta_1, \zeta_2, \zeta_3$.

Let $n \in \mathbb{N}$ and $\zeta_0 \in \mathbb{S}^1$, by the third hypothesis and the Courant-Lebesgue Lemma, there exists $\delta < \rho < \sqrt{\delta}$ such that, if ζ and ζ' are the two intersections points of $\mathbb{S}^1$ with the circle $\partial D_\rho(\zeta_0)$, $d(X_n(\zeta), X_n(\zeta')) \leq \varepsilon$. $\mathbb{S}^1 \backslash \{\zeta, \zeta'\}$ is composed of two arcs: one, A', contains ζ_0 and all the points that are at a distance less than δ from ζ_0, the second arc, A'', contains two of the three points $\zeta_1, \zeta_2, \zeta_3$. Since $d(X_n(\zeta), X_n(\zeta')) \leq \varepsilon$, one of the two arcs $\gamma_n(A')$, $\gamma_n(A'')$ is of diameter less than e; but $\gamma_n(A'')$ contains two of the points p_1^n, p_2^n, p_3^n and $e < I$, then it is $\gamma_n(A')$ that is of diameter less than e. We then have proved that if $|\zeta - \zeta_0| < \delta$, $d(\gamma_n(\zeta), \gamma_n(\zeta_0)) \leq e$; this proves that the family $\{\gamma_n\}$ is equicontinuous. □

3.4.2 Proof of Theorem 3.4.2

Preliminaries

Let u be a solution of the Dirichlet problem asked in Theorem 3.3.1. We use the notations introduced in Section 3.3. The function u is contructed as the limit of a sequence (u_n) where u_n is a solution of a Dirichlet problem on $\widetilde{\Omega}^n_{-n}$. Since we are only interested in the behaviour on $\mathcal{N}$ we shall use the polar coordinates given by φ; u is then defined on $[0,\rho_0]\times\mathbb{R}$ and u_n is defined on $[0,\rho_0]\times[-2nq\pi-\frac{\pi}{2}, 2nq\pi+\frac{\pi}{2}]$. Because of the periodicity of u, to prove Theorem 3.4.2, it is enough to make the proof on a period $[0,\rho_0]\times[0,2q\pi]$.

By construction, u_n takes the value $+\infty$ on $(0,\rho_0)\times\{-2nq\pi-\frac{\pi}{2}\}$ and the value $-\infty$ on $(0,\rho_0)\times\{2nq\pi+\frac{\pi}{2}\}$, besides, by taking $\Psi_{u_n}(\mathcal{Q})=0$, we have $\Psi_{u_n}\geq 0$ on $\widetilde{\Omega}^n_{-n}$, this proves, by Theorem 2.3.2, that the minimal surface $\{\varphi(x),u_n(x)\}_{x\in\mathcal{N}\cap\widetilde{\Omega}^n_{-n}}\subset\mathcal{D}\times\mathbb{R}$ has the vertical straight-line passing by $\mathcal{O}$ as boundary. The idea of the proof of Theorem 3.4.2 is to follow the behaviour of the graph near this vertical straight-line when n goes to $+\infty$.

We now need a result on the behaviour of graph bounded by vertical line.

Lemma 3.4.5. *Let v be a solution of* (MSE) *on a sector of $\mathcal{D}$ $\{(r,\theta)\in\mathcal{D}|\ r\leq r_0,\ \alpha_1\leq\theta\leq\alpha_2\}$ $(\alpha_1<0<\alpha_2)$. Suppose that the graph of v in $\mathbb{R}^3$ is a complete minimal surface with boundary and the part of the boundary over the origin is an interval of the vertical straight-line passing by the origin. Then if v is bounded on $\{\theta=0\}$, $\lim_{r\to 0}v(r,0)$ exists. Besides the normal to the graph v at the points $(0,0,\lim_{r\to 0}v(r,0))$ is $\pm(0,1,0)$.*

Proof. We note Σ the graph of v. All the cluster points of $(r,0,v(r,0))$ as r goes to 0 are in the boundary of the graph and more precisely in the part of the boundary consisting in the interval of the vertical straight-line passing by the origin. Besides, the curve $r\mapsto(r,0,v(r,0))$ is in the intersection of the vertical plane $\{y=0\}$ and the graph of v, this curve is then in the intersection of two minimal surfaces. Let $(0,0,a)$ be a cluster point of $(r,0,v(r,0))$ as r goes to 0, since the boundary of Σ is a vertical straight-line near $(0,0,a)$ the normal to Σ at this point is horizontal and is $(\cos\alpha,\sin\alpha,0)$ for some $-\pi\leq\alpha<\pi$. Near $(0,0,a)$, Σ is then a graph over the vertical plane $\{x\cos\alpha+y\sin\alpha=0\}$ and is tangent to this plane at $(0,0,a)$. Then, if $|\alpha|\neq\pi/2$, the intersection of Σ and $\{y=0\}$ is only the vertical straight-line near $(0,0,a)$ then no $(r_n,0,v(r_n,0))$ can converge to $(0,0,a)$. This implies that the normal at $(0,0,a)$ is $\pm(0,1,0)$ and then since Σ is a graph over $\{y=0\}$ the intersection of Σ with this plane near $(0,0,a)$ is the vertical straight-line and some smooth curves passing by $(0,0,a)$ such that $x\neq 0$ along them. One of this curve is then $(r,0,v(r,0))$, by continuity; this prove that $\lim_{r\to 0}v(r,0)=a$. $\square$

We want to understand the convergence of the graph of u_n near the singularity point $\mathcal{O}$. let $\alpha \in \mathbb{R}$, to simplify our notation in the following we suppose that $\alpha \in [0, 2q\pi]$, let $k \in \mathbb{N}$ and note $\alpha_1 = \alpha - 2kq\pi$ and $\alpha_2 = \alpha + 2kq\pi$. We know that $u_n(r, \alpha_1)$ and $u_n(r, \alpha_2)$ are bounded by Lemma 2.3.1. Then since the graph of u_n has a vertical straight-line as boundary, $u_n(r, \alpha_1) \to a_n$, $u_n(r, \alpha_2) \to b_n$ and $u_n(r, \alpha) \to c_n$ as r goes to 0, for some $c_n \in (b_n, a_n)$. The graph $\Sigma_n \subset \mathcal{D} \times \mathbb{R}$ of u_n over $U = [0, \rho_0] \times [\alpha_1, \alpha_2]$ is a minimal surface bounded by $\rho \mapsto (\rho, \alpha_1, u_n(\rho, \alpha_1))$ for $0 \leq \rho \leq \rho_0$, $\theta \mapsto (\rho_0, \theta, u_n(\rho_0, \theta))$ for $\alpha_1 \leq \theta \leq \alpha_2$, $\rho \mapsto (\rho, \alpha_2, u_n(\rho, \alpha_2))$ for $\rho_0 \geq \rho \geq 0$ and the segment $\{\mathcal{O}\} \times [b_n, a_n]$. Σ_n is a minimal surface of $\mathcal{D} \times \mathbb{R}$ which is of the type of the disk then we can parametrize conformally it by the upper half-disk $\Delta^+ = \{\zeta \in \Delta | \Im(\zeta) \geq 0\}$: there exist an harmonic map $X_n : \Delta^+ \to \mathcal{D}$ and an harmonic function $z_n : \Delta^+ \to \mathbb{R}$ such that the surface Σ_n is $\{(X_n(\zeta), z_n(\zeta)), \zeta \in \Delta^+\}$. We choose X_n and z_n such that $(X_n(-1), z_n(-1)) = (0, 0, a_n)$, $(X_n(0), z_n(0)) = (0, 0, c_n)$ and $(X_n(1), z_n(1)) = (0, 0, b_n)$. We want to apply Theorem 3.4.3 to the sequence $(X_n, z_n)|_{\partial\Delta^+}$, but we can not do this directly.

Proposition 3.4.6. *Let (X_n, z_n) be as above then , if k is big enough, a subsequence $(X_{n'})$ converges uniformly to an harmonic map $X : \Delta^+ \to \mathcal{D}$ and $(z_{n'})$ converges uniformly on each compact subset to an harmonic function $z : \overset{\circ}{\Delta}{}^+ \to \mathbb{R}$ (where $\overset{\circ}{\Delta}{}^+$ is the interior of Δ^+). On $\overset{\circ}{\Delta}{}^+$, X and z satisfy $z = u \circ X$.*

Proof. In fact the function z_n is not the good function to consider. On $U = [0, \rho_0] \times [\alpha_1, \alpha_2]$, the function Ψ_{u_n} is defined with $\Psi_{u_n}(0, 0) = 0$. Ψ_{u_n} corresponds to the harmonic conjugate of z_n. More precisely, if $\zeta_0 \in \overset{\circ}{\Delta}{}^+$ and z_n^* is the conjugate function to z_n such that $z_n^*(\zeta_0) = \Psi_{u_n}(X_n(\zeta_0))$, we have $z_n^* = \Psi_{u_n} \circ X_n$. *A priori*, the preceding equality is true only in the interior of Δ^+, but, since Ψ_{u_n} is Lipschitz continuous, z_n^* can be extended to the boundary such that the equality is true everywhere.

By a result of J.C.C. Nitsche [Ni1], the area of Σ_n is bounded by $Area(U) + \int_{\partial U} |u_n|$. Since u_n converges to u uniformly on each compact subset of $\Omega \backslash \{\mathcal{Q}\}$, the u_n are uniformly bounded functions on U by Proposition 3.3.5. This proves that the areas of Σ_n are uniformly bounded by a constant M. Since (X_n, z_n) are conformal:

$$\int_{\Delta^+} |X_{nx}|^2 + |X_{ny}|^2 + |\nabla z_n|^2 \mathrm{d}x\mathrm{d}y = Area(\Sigma_n)^2 < M^2$$

Let $F_n : \Delta^+ \to \mathcal{D} \times \mathbb{R}^3$ be the map defined by $F_n(\zeta) = (X_n(\zeta), z_n^*(\zeta), x, y)$ with $\zeta = x + iy$. We shall apply Theorem 3.4.3 with γ_n the restriction of F_n to the boundary of Δ^+, $f_n = z_n$ and with $\zeta_1 = -1$, $\zeta_2 = 0$ and $\zeta_3 = 1$. We note $G_n = (F_n, z_n)$. Since z_n^* is conjugate to z_n, $|\nabla z_n^*| = |\nabla z_n|$; then

$$\int_{\Delta^+} \left(|G_{nx}|^2 + |G_{ny}|^2\right) \mathrm{d}x\mathrm{d}y \leq 2M^2 + 2Area(\Delta^+)$$

We have the third hypothesis of Theorem 3.4.3.

Because of the function x in F_n, $d(F_n(\zeta_i), F_n(\zeta_j)) \geq 1$ for $i \neq j$, this is the first hypothesis of Theorem 3.4.3.

To prove that (γ_n, f_n) satisfies the second hypothesis of Theorem 3.4.3, we need a lemma.

Lemma 3.4.7. *If k is big enough, for big n we have:*

$$1 \leq \inf_{(0,\rho_0]\times[-2kq\pi,-2(k-1)q\pi]} u_n - \sup_{(0,\rho_0]\times[0,2q\pi]} u_n \tag{3.5}$$

$$1 \leq \inf_{(0,\rho_0]\times[0,2q\pi]} u_n - \sup_{(0,\rho_0]\times[2kq\pi,2(k+1)q\pi]} u_n \tag{3.6}$$

Proof. There exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$, by Proposition 3.3.6, $c \neq 0$ and, by construction, $c < 0$, this is due to the value $+$ and $-\infty$ on $\widetilde{\mathcal{L}}^+$ and $\widetilde{\mathcal{L}}^-$ for the function u_n. This implies that

$$\sup_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u - \sup_{(0,\rho_0]\times[2mq\pi,2(m+1)q\pi]} u = c(l-m)$$

$$\inf_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u - \inf_{(0,\rho_0]\times[2mq\pi,2(m+1)q\pi]} u = c(l-m)$$

This implies that for a k big enough:

$$2 \leq \inf_{(0,\rho_0]\times[-2(k+1)q\pi,-2(k-2)q\pi]} u - \sup_{(0,\rho_0]\times[-2q\pi,4q\pi]} u$$

$$2 \leq \inf_{(0,\rho_0]\times[-2q\pi,4q\pi]} u - \sup_{(0,\rho_0]\times[2(k-1)q\pi,2(k+2)q\pi]} u$$

Let us apply Proposition 3.3.5 with $\varepsilon = 1/4$, there exists then d such that for every $l \in \mathbb{Z}$ and every $n > l + 2$:

$$\sup_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u_n \leq \sup_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n + \frac{1}{4}$$

$$\inf_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u_n \geq \inf_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n - \frac{1}{4}$$

But since $u_n \to u$ uniformly on every compact subset of $\Omega\backslash\{\mathcal{Q}\}$ we have for big n:

$$\sup_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n \leq \sup_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u + \frac{1}{4}$$

$$\inf_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n \geq \inf_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u - \frac{1}{4}$$

Then in using these inequalities with $l = 0$ and $l = -k$, we have for n big enough:

$$\begin{aligned}
\inf_{(0,\rho_0]\times[-2kq\pi,-2(k-1)q\pi]} u_n &\geq \inf_{[d,\rho_0]\times[-2(k+1)q\pi,-2(k-2)q\pi]} u_n - \frac{1}{4} \\
&\geq \inf_{[d,\rho_0]\times[-2(k+1)q\pi,-2(k-2)q\pi]} u - \frac{1}{2} \\
&\geq \sup_{[d,\rho_0]\times[-2q\pi,4q\pi]} u + \frac{3}{2} \\
&\geq \sup_{[d,\rho_0]\times[-2q\pi,4q\pi]} u_n + \frac{5}{4} \\
&\geq \sup_{(0,\rho_0]\times[0,2q\pi]} u_n + 1
\end{aligned}$$

This shows (3.5). With $l = 0$ and $l = k$, we get (3.6):

$$\inf_{(0,\rho_0]\times[0,2q\pi]} u_n \geq \sup_{(0,\rho_0]\times[2kq\pi,2(k+1)q\pi]} u_n + 1$$

□

Let $k \in \mathbb{N}$ given by Lemma 3.4.7. In $\mathcal{D}$, the curves $\zeta \mapsto X_n(\zeta)$ for $\zeta \in \partial\Delta^+$ have the same image Γ for every n. Γ is a Jordan curve in $\mathcal{D}$ so for every $m > 0$ there exists $\varepsilon > 0$ such that, if $p', p'' \in \Gamma$ and $d(p', p'') < \varepsilon$, one of the components of $\Gamma\backslash\{p', p''\}$ is of diameter less than m. Let δ be $\min\{\rho_0/2, 1/2\}$ if $\mathcal{A}$ is the point of coordinates (ρ_0, α_1) then 2δ is less than the distance between $\mathcal{O}$ and $\mathcal{A}$ and if $m < \delta$, in the above property, there is only one component of $\Gamma\backslash\{p', p''\}$ with diameter less than m.

Let $0 < m < \delta$, there exists $\varepsilon > 0$ that satisfies the above property, there also exists η such that if $\zeta', \zeta'' \in \partial\Delta^+$ and $|\zeta' - \zeta''| < \eta$, one of the components of $\partial\Delta^+\backslash\{\zeta', \zeta''\}$ is of diameter less than m. Since $m < \delta \leq 1/2$ this component is unique. Let $n \in \mathbb{N}$ be big enough such that (3.5) and (3.6) are satisfied. Let ζ' and ζ'' in $\partial\Delta^+$ such that the distance between $G_n(\zeta')$ and $G_n(\zeta'')$ is less than $\min\{\varepsilon, \eta, 1/2\}$. Let us note $F_n(\zeta') = (p', \Psi_{u_n}(p'), x', y')$ and $F_n(\zeta'') = (p'', \Psi_{u_n}(p''), x'', y'')$. We have $d(p', p'') \leq \varepsilon$, then there exists one component of $\Gamma\backslash\{p', p''\}$ with diameter less than m. Let I be the part of $\partial\Delta^+$ that parametrize this component; the end points of I are $\zeta' = x' + iy'$ and $\zeta'' = x'' + iy''$. Since $|\zeta' - \zeta''| < \eta$, I or its complemantary in $\partial\Delta^+$ is of diameter less than m; let us prove that it is I. If $\mathcal{O} \notin X_n(I)$ then $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\} \cap I = \emptyset$, but $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ is of diameter 2 then I is of diameter less than m. If $\mathcal{O} \in X_n(I)$ then $\mathcal{A} \notin X_n(I)$ and the points p' and p'' are at a distance less than m from $\mathcal{O}$ then the point p' and p'' can not be on the part of $\Gamma : \theta \mapsto (\rho_0, \theta)$. There are different possible cases. First, I can be included in $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ then since $|\zeta' - \zeta''| < \eta$, I is of diameter less than m. The

second case is when $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ is included in I then we can suppose that $p' \in \{(\rho, \alpha_1)\}_{0\leq\rho\leq\rho_0}$ and $p'' \in \{(\rho, \alpha_2)\}_{0\leq\rho\leq\rho_0}$ this implies, by Lemma 3.4.7, that $|z_n(\zeta') - z_n(\zeta'')| \geq 2$ which is impossible since the distance between $G_n(\zeta')$ and $G_n(\zeta'')$ is less than $\min\{\varepsilon, \eta, 1/2\} < 2$. For the third case, we can suppose that $\zeta' \in \{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ and $p'' \in \{(\rho, \alpha_2)\}_{0\leq\rho\leq\rho_0}$ (all the other cases are given by permutating ζ' and ζ'' and α_1 and α_2). Since $\mathcal{A} \notin X_n(I)$, we have $1 \in I$. Besides z_n is decreasing along $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$, then, since $c_n - z_n(\zeta'') \geq 1$, by Lemma 3.4.7, and $|z_n(\zeta') - z_n(\zeta'')| \leq 1/2$, $\zeta' \in [0, 1]$. Since $|\zeta' - \zeta''| \leq 1/2$, ζ'' is in the part of $\partial\Delta^+$ with $\Re(\zeta) \geq 0$ then the shortest component of $\partial\Delta^+ \backslash \{\zeta', \zeta''\}$ is the one that contains 1 then it is I.

So it was proved that I and $X_n(I)$ are of diameter less than m. Since Ψ_{u_n} is 1-Lipschitz continuous and $z_n^* = \Psi_{u_n} \circ X_n$ the set $z_n^*(I)$ is of diameter less than m. Then, if $\zeta', \zeta'' \in \partial\Delta^+$ are such that the distance between $G_n(\zeta')$ and $G_n(\zeta'')$ is less than $\min\{\varepsilon, \eta, 1/2\}$, one component of $F_n(\partial\Delta^+) \backslash \{F_n(\zeta'), F_n(\zeta'')\}$ is of diameter less than $\sqrt{3}m$; this is the second hypothesis.

Let us apply the result of Theorem 3.4.3, the family $\{F_n|_{\partial\Delta^+}\}$ is equicontinuous then there exists a subsequence $(F_{n'}|_{\partial\Delta^+})$ that converges uniformly. Since the F_n are harmonic maps, the sequence $(F_{n'})$ converges uniformly on Δ^+. Then $X_{n'} \to X$ uniformly on Δ^+ and $z_{n'}^* \to z^*$ with X and z^* harmonic map. The fact that z_n and z_n^* are harmonic conjugates implies that $(z_{n'})$ converges to the harmonic conjugate z of z^* and the convergence is uniform on each compact subset of $\overset{\circ}{\Delta}{}^+$. The equality $z = u \circ X$ is just the limit of the equality $z_n = u_n \circ X_n$. This ends the proof of Proposition 3.4.6. □

Construction 4. Let us summarize what we have done above. We take $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^*$. For every $n \in \mathbb{N}$, we consider (X_n, z_n) a conformal parametrization of the graph of u_n over $[0, \rho_0] \times [\alpha - 2kq\pi, \alpha + 2kq\pi]$ by the upper-half disk Δ^+ such that the part parametrized by $\Delta^+ \cap \mathbb{R}$ is the vertical segment in the boundary which is above $\mathcal{O}$. Then Proposition 3.4.6 says us that if we take k big enough we have a subsequence $(X_{n'}, z_{n'})$ that converges to (X, z) as described in Proposition 3.4.6. We also have the sequence (z_n^*) and $z_{n'}^* \to z^*$ uniformly on Δ^+. X satisties the following property.

Corollary 3.4.8. *Let X be constructed as in Construction 4. Then*

$$X(\Delta^+) = [0, r_0] \times [\alpha_1, \alpha_2]$$

Proof. Let $\mathcal{A} \in [0, r_0] \times [\alpha_1, \alpha_2]$ then for every n there exists ζ_n such that $X_n(\zeta_n) = \mathcal{A}$. We have $X_{n'} \to X$ uniformly on Δ^+ and, since Δ^+ is compact, a subsequence n'' of n' is such that $\zeta_{n''} \to \zeta \in \Delta^+$. Then $\mathcal{A} = X_{n''}(\zeta_{n''}) \to X(\zeta)$ and $\mathcal{A} = X(\zeta)$. □

The proof

Before proving Theorem 3.4.2, we need a remark

Proposition 3.4.9. *Let $\alpha \in \mathbb{R}$ then $u(\rho, \alpha)$ converges as ρ goes to* 0.

Proof. Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ be big enough such that Construction 4 can be done. By construction, we have X_n, X, z_n^* and z^* defined on Δ^+. Let U be a compact neighborhood of 0 in Δ^+ such that $X(U) \subset [0, \rho_0] \times (\alpha - \pi/2, \alpha + \pi/2)$. Using the projection $H : (\rho, \theta) \mapsto (\rho \cos\theta, \rho \sin\theta)$ we can consider X as a harmonic map from Δ^+ to $\mathbb{R}^2$. For n big enough, $X_n(U) \subset [0, r_0] \times (\alpha - \pi/2, \alpha + \pi/2)$, the maps X_n, X, z_n^* and z^* can be extended to U' where U' is the union of U and $\{\zeta \in \mathbb{C} |\ \bar{\zeta} \in U\}$. X_n and z_n^* are extended by Schwarz reflection principle to harmonic map. Since $X_{n'} \to X$ and $z_{n'}^* \to z^*$ on U we have the same convergence for their extensions to U'. In the same way, the functions z_n and z can be extended to U' and the convergence $z_{n'} \to z$ is uniform on each compact of U'. The map (X, z) on U' gives a minimal surface of $\mathbb{R}^3$. Since the point $(X_n(0), z_n(0))$ is for every $n \in \mathbb{N}$ the limit point of $(\rho, \alpha, u_n(\rho, \alpha))$ as ρ goes to 0. The normal to the minimal surface (X_n, z_n) at the origin is $(\sin\alpha, -\cos\alpha, 0)$. Then, the normal to the minimal surface (X, z) is also $(\sin\alpha, -\cos\alpha, 0)$ at the origin. Then the intersection of the minimal surface $\{X(\zeta), z(\zeta)\}_{\zeta \in U}$ with the vertical plane of equation $x \sin\alpha - y \cos\alpha = 0$ in the neighborhood of the point $(X(0), z(0))$ is composed of a piece of the vertical straight line passing by this point and a curve which is above $\{(\rho, \alpha)\}_{\rho > 0}$ with $(X(0), z(0))$ as end point. Then this curve is, in fact, $\{\rho, \alpha, u(\rho, \alpha)\}_{\rho > 0}$, this implies that $\lim\limits_{\rho \to 0} u(\rho, \alpha) = z(0)$. □

We can then make the proof of Theorem 3.4.2.

Proof of Theorem 3.4.2. . Let $k \in \mathbb{N}$ be big enough such that we can make Construction 4 with $\alpha = 0$. We then get a map $X : \Delta^+ \to \mathcal{N}$ and two functions $z : \mathring{\Delta}^+ \to \mathbb{R}$ and $z^* : \Delta^+ \to \mathbb{R}$. Let I be the part of $\partial\Delta^+$ such that $\zeta \in I \Leftrightarrow X(\zeta) = \mathcal{O}$. Since X is monotone, I is connected and by construction $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\} \subset I$. Let ζ_1 and ζ_2 denote the two end-points of I and consider the biholomorphic map $h : \Delta^+ \to \Delta^+$ such that $h(-1) = \zeta_1$, $h(0) = (0)$ and $h(1) = \zeta_2$. If $\widetilde{X} = X \circ h$, $\widetilde{z} = z \circ h$ and $\widetilde{z}^* = z^* \circ h$ (it is obvious that the conjugate harmonic function to $\widetilde{z}$ is $\widetilde{z}^*$), we make only a reparametrization of the minimal surface (X, z). For $\zeta \in \partial\Delta^+$, $\widetilde{X}(\zeta) = \mathcal{O} \Leftrightarrow \zeta \in \mathbb{R}$. Then let consider $\zeta_0 \in \partial\Delta^+ \backslash \mathbb{R}$. We have $\widetilde{X}(\zeta_0) \neq \mathcal{O}$ then since $\widetilde{z}(\zeta) = u \circ \widetilde{X}(\zeta)$ we can define $\widetilde{z}(\zeta_0)$ by making ζ converging to ζ_0. $\widetilde{z}$ can be also defined at -1 and 1: as $\zeta \in \partial\Delta^+ \backslash \mathbb{R}$ goes to -1 or 1, $X(\zeta)$ goes to $\mathcal{O}$ along $\{(\rho, \alpha_1)\}$ or $\{(\rho, \alpha_2)\}$ and then $\widetilde{z}(\zeta)$ goes to $\lim\limits_{\rho \to 0} u(\rho, \alpha_1) = z(-1)$ or $\lim\limits_{\rho \to 0} u(\rho, \alpha_2) = z(1)$. Since $\widetilde{X} = \mathcal{O}$ on $\mathbb{R} \cap \Delta^+$, $\widetilde{z}^* = 0$ on the same set. then we can extend $\widetilde{z}^*$ to the whole

disk by Schwartz reflection principle, this prove that $\widetilde{z}$ can be also extended to the interior of the whole disk by reflection. Since we have define $\widetilde{z}$ on the circular part of the boundary of Δ^+, $\widetilde{z}$ is then defined also on the boundary of the disk and is continuous on the boundary then $\widetilde{z}$ is the harmonic extention to the disk of this countinuous function on the circle. This proves that $\widetilde{z}$ is a continuous function on Δ^+. Then we have $(\widetilde{X}, \widetilde{z})_{\Delta^+}$ which is a parametrization of the graph of u above $[0, \rho_0] \times [\alpha_1, \alpha_2]$ (because $\widetilde{z} = u \circ \widetilde{X}$) and $(\widetilde{X}, \widetilde{z})_{\Delta^+}$ has a boundary such that the part of this boundary which is above $\mathcal{O}$ is a vertical segment. This proves Theorem 3.4.2 because $[0, \rho_0] \times [\alpha_1, \alpha_2]$ contains several period of the domain $\mathcal{N}$. □

3.4.3 Property of the solution 3

Let u be a solution of the Dirichlet problem asked in Theorem 3.3.1. We then can understand the boundary behaviour of the graph of u in $\mathbb{R}^3$.

From Theorem 3.4.1, we know that, over the neighborhood $\mathcal{N}$ of $\mathcal{Q}$, the graph is bounded by the vertical straight-line passing by the point $\psi(Q)$.

The other points where there are boundary components for the graph of u are the vertices of Ω. This points statisfy the hypotheses of Theorem 2.3.2. Then the graph of u is bounded by vertical straight-lines near the vertices.

The last remark we can make is the following. We have $\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{V})$; this implies that the conjugate surface of the graph of u, which is bounded by the conjugate of the boundary of the graph of u, has its boundary included in the plane $\{z = 0\}$.

This remark implies that, if Σ is the graph of u over Ω_0^1, a period of Ω, its conjugate surface can be extend by symmetry with respect to the plane $\{z = 0\}$, we note Σ^* this symmetric surface. We then have the following result.

Lemma 3.4.10. *Σ^* is of finite total curvature and its total curvature is $4\pi r$.*

Proof. Using arguements given in Section 2.6, we can prove that Σ is of finite total curvature, this implies that the same is true for Σ^*. Then, as in Proposition 2.2 in [HK], each catenoidal end gives a contribution of 2π to the total curvature (see [JM], for the original arguements) and, using Gauss-Bonnet Theorem, we compute the value of the total curvature and get $4\pi r$. □

3.5 The period problem

3.5.1 The general case

We now try to build a r-noid with genus 1 and horizontal ends of type I for a given polygon of flux.

Let $V = (v_1, \ldots, v_r)$ be a polygon that bounds a multi-domain with cone singularity (D, Q, ψ). Using Construction 1 and Construction 3 as in Section 3.3, we get a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ and a solution u on Ω of the Dirichlet problem asked in Theorem 3.3.1; as in Section 3.3, we assume that the period $2q\pi$ of Ω is the angle at the cone singularity of D. Let us consider Ω_0^1 one period of the multi-domain Ω and Σ the graph of u over Ω_0^1. We know, because of the result of the preceding sections, that Σ is a minimal surface bounded by $r-1$ vertical lines passing by the vertices P_i $(i \neq 1)$ of the polygon V, two vertical half-lines over P_1, a vertical segment over $\psi(Q)$ and two curves over the segment $[\psi(Q), \psi(P_1)]$. The conjugate surface of Σ is included in $\{z \geq 0\}$ and the conjugates of the $r-1$ vertical lines, the two vertical falf-lines and the vertical segment are exactly the intersection of the conjugate surface with the plane $\{z = 0\}$. We then can extend the conjugate surface by symmetry with respect to this plane, we get a new surface that we note Σ^*. Σ^* is a solution for the Plateau problem at infinity for the data V if the two components of boundary of Σ^*, coming, by conjugation, from the two curves which are over $[\psi(Q), \psi(P_1)]$, glue together such a way that Σ^* has no boundary.

In fact this two components of boundary differ from a translation, how can we compute the vector of the translation? From the function u we can derive three closed 1-forms $\mathrm{d}X_1^*$, $\mathrm{d}X_2^*$ and $\mathrm{d}X_3^*$ on Ω which are the differential of the three coordinate functions of the conjugate surface to the graph of u (These 1-forms depend only on the first derivatives of u). For example, we have $\mathrm{d}X_3^* = \mathrm{d}\Psi_u$. In $\mathcal{N}$, the neighborhood of $\mathcal{Q}$ in Ω, we consider the path $\Gamma : \theta \mapsto (\rho, \theta)$ for some $\rho < \rho_0$ and $\theta \in [0, 2q\pi]$; Γ is a lift of a generator of $\pi_1(D \backslash \{Q\})$. Then the two components of boundary of Σ^* differ from the following vector, called the *period vector*:

$$\left(\int_\Gamma \mathrm{d}X_1^*, \int_\Gamma \mathrm{d}X_2^*, \int_\Gamma \mathrm{d}X_3^* \right)$$

Since $\mathrm{d}X_3^* = \mathrm{d}\Psi_u$ and as Ψ_u is invariant by f, $\displaystyle\int_\Gamma \mathrm{d}X_3^* = 0$. Obviously, the value of the integrals is the same for every Γ which is the lift of a generator of $\pi_1(D \backslash \{Q\})$; in fact, from Remark 3.3.7, since $\mathrm{d}X_i^*$ depends only on the derivatives of u, $\mathrm{d}X_i^*$ is well defined on $D \backslash Q$

Then the question of the existence of a solution to the Plateau problem at infinity for the data V becomes: knowing if there exists (D, Q, ψ) bounded by V such that the associated period vector is zero; this is the period problem.

Remark 3.5.1. We now give some explanations on Remark 3.2.5 and the hypothesis on D saying that its angle at the cone singularity is the period of Ω. Let $V = (v_1, ..., v_r)$, (D, Q, ψ), $(\Omega, \mathcal{Q}, \varphi)$ and u be as above (D sastisfies the hypothesis H). We also suppose the the period vector associated to D is zero. Then Σ^* which is the conjugate surface to Σ, the graph of u over Ω_0^1, extended by symmetry is a r-noid with genus 1 and horizontal ends of type I having V as flux polygon. Let f be the

isometry of Ω associated to its periodicity. Let $a \in \mathbb{N}^*$, then the quotient of $\mathcal{W}$ by the group $\{f^{an}\}_{n\in\mathbb{Z}}$ ($\mathcal{W}$ is given by Construction 1 applied to D) is a multi-domain with cone singularity that bounds the polygon

$$V_a = (\underbrace{v_1, \dots, v_r, \dots\dots, v_1, \dots, v_r}_{a \text{ times}})$$

Besides, if Σ_a is the graph of u on Ω_0^a, the conjugate surface Σ_a^* of Σ_a extended by symmetry is a ar-noid with genus 1 and horizontal ends of type I having V_a as flux polygon. In fact Σ_a^* is just Σ^* that we cover a times. Then we also can find solution to the Plateau problem at infinity for D that does not satisfy the hypothesis H.

3.5.2 The period map and the proof of Theorem 3.2.11

In this subsection we explain how we shall prove Theorem 3.2.11.

Let $V = (v_1, \dots, v_r)$ be a polygon bounded by an immersed polygonal disk $(\mathcal{P}, \psi)$. For each A in the interior of $\mathcal{P}$, Construction 1 gives us $(\mathcal{W}_A, \mathcal{A}, \varphi_A)$ a multidomain with a logarithmic singularity . Then by applying Construction 3 and Theorem 3.3.1, we get a periodic multi-domain with logarithmic singularity $(\Omega_A, \mathcal{A}, \varphi_A)$ and a function u_A defined on Ω_A. We then have the three closed 1-forms $\mathrm{d}X_i^*(A)$. To prove Theorem 3.2.11, we need to find a point $A \in \overset{\circ}{\mathcal{P}}$ such that the period vector associated to the above constuction for A is 0.

In fact, we have a map from the interior $\mathcal{P}$ to $\mathbb{R}^2$ which associates to every point $A \in \overset{\circ}{\mathcal{P}}$ the vector $\left(\int_\Gamma \mathrm{d}X_1^*(A),\ \int_\Gamma \mathrm{d}X_2^*(A)\right)$, this map is the period map and will be noted Per. Then to prove Theorem 3.2.11, the problem is to prove that this map vanishes at one point of $\overset{\circ}{\mathcal{P}}$. The period map satisfies the following proposition.

Proposition 3.5.2. *The period map Per is continuous on the interior of $\mathcal{P}$.*

Proof. Let us consider a sequence (A_n) of points in $\overset{\circ}{\mathcal{P}}$ that converges to $A \in \overset{\circ}{\mathcal{P}}$. Let Γ be a closed path in $\mathcal{P}\setminus\{A, A_0, A_1, \dots, A_n, \dots\}$ such that for every n, Γ is a generator of $\pi_1(\mathcal{P}\setminus\{A_n\})$ and Γ is a generator of $\pi_1(\mathcal{P}\setminus\{A\})$. Since for A (or every A_n) we have $u_A \circ f = u_A + c$ the derivatives of u_A are well defined on $\mathcal{P}\setminus\{A\}$. Then the two closed 1-forms $\mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A)$ are well defined on $\mathcal{P}\setminus\{A\}$. The same is true for A_n. We then have:

$$Per(A_n) = \left(\int_\Gamma \mathrm{d}X_1^*(A_n),\ \int_\Gamma \mathrm{d}X_2^*(A_n)\right)$$

On $\mathcal{P}$, we have the sequence of the derivatives of u_{A_n} and these derivatives converge to the derivatives of u_A if there is no line of divergence (a line of divergence

is a phenomenon linked to the behaviour of the first derivatives so we can use this arguement in this case). Since the arguements used in the proof of the existence part of Theorem 3.3.1 are always true, there is no line of divergence and $\mathrm{d}X_1^*(A_n) \to \mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A_n) \to \mathrm{d}X_2^*(A)$, the convergence is uniform along Γ. This proves that $Per(A_n) \to Per(A)$ by integration. □

The idea to prove that the period map vanishes at one point in the interior of $\mathcal{P}$ is then to extend continuously Per to the boundary of $\mathcal{P}$ and show that the degree of the period map along the boundary of $\mathcal{P}$ is non zero. In fact, we shall use a modified boundary of $\mathcal{P}$. Then, using Proposition 3.20 in [Fu], this proves that there exists a point $A \in \overset{\circ}{\mathcal{P}}$ where $Per(A) = 0$. The following section is devoted to the extension of Per to the boundary and to the proof of Theorem 3.6.11 that establishes that the degree of the period map is non-zero along the boundary.

To extend the period map on the boundary we make a renormalization of the map Per: let A be a point of $\overset{\circ}{\mathcal{P}}$, if $||Per(A)|| \leq 1$ then we do not change the value of $Per(A)$ but if $||Per(A)|| \geq 1$ the new value of $Per(A)$ is $\dfrac{Per(A)}{||Per(A)||}$. The new period map is always continuous and for every point the norm of the period at this point is less than one.

3.6 The period map on the boundary of $\mathcal{P}$

We use the notations introduced in Subsection 3.5.2.

3.6.1 The behaviour on the edges

Proposition 3.6.1. *Let (A_n) be a sequence in $\overset{\circ}{\mathcal{P}}$ such that $A_n \to A$ where A is a point in the interior of one edge of the boundary of $\mathcal{P}$. Then $Per(A_n)$ converges to $\mathrm{d}\psi|_A(N)$ where N is the outer unit normal to the edge at A, we recall that ψ is the developping map of $\mathcal{P}$.*

Proof. As in chapter 2, we note $\Omega(\mathcal{P})$ the multi-domain obtained when we glue to every edge $[P_i, P_{i+1}]$ a half strip isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$. Then, for every A_n the covering map $\pi : \mathcal{W}_{A_n} \to \mathcal{P}$ extends to a covering map $\pi : \Omega_{A_n} \to \Omega(\mathcal{P})$ and the derivatives of u_{A_n} are then well defined on $\Omega(\mathcal{P})\backslash\{A_n\}$, by Remark 3.3.7. Suppose that the point A is in the interior of the edge $[P_1, P_2]$ and that $|P_1P_2| = 2$. By choosing a good chart, we can suppose that $[-1, 1] \times \mathbb{R}_+$ is the half-strip glued to this edge. We note D_r the domain in $\mathbb{R}^2$ which is the intersection of the domain $y \leq 0$ and the disk of center $(0, r)$ and radius $\sqrt{r^2 + 1}$. By choosing a $r > 0$ big enough, D_r is a neighborhood of the edge $[P_1, P_2]$ in $\mathcal{P}$.

We suppose that A is the point $(a,0)$ $(-1<a<1)$ and the points A_n lie in D_r and have coordinates (a_n,b_n) $(b_n<0)$. We have $a_n\to a$ and $b_n\to 0$. For every n, we note L_n the half straight-line $\{(a_n,b_n+t)\}_{t\geq 0}$ and L the half straight-line $\{(a,t)\}_{t\geq 0}$. By using the covering map π and u_{A_n}, we can define on $\Omega(\mathcal{P})\backslash L_n$ a function u_n which has the same derivatives as u_{A_n}; u_n is solution of the minimal surface equation and has the value $+\infty$ (resp. $-\infty$) on $\pi(\mathcal{L}_i^+)$ (resp. $\pi(\mathcal{L}_i^-)$). We then want to understand the convergence of u_n. By the same arguement as in the proof of Theorem 3.3.1, there is no line of divergence in $\Omega(\mathcal{P})\backslash L$; this proves that, for a subsequence, (u_n) converges to a solution u of (MSE) on $\Omega(\mathcal{P})\backslash L$. By Proposition 1.4.9, u takes the value $+\infty$ (resp. $-\infty$) on $\pi(\mathcal{L}_i^+)$ (resp. $\pi(\mathcal{L}_i^-)$). We then need to understand its behaviour near L to know the function u.

Since by convention $\Psi_{u_n}(A_n)=0$, we fix $\Psi_u(A)=0$

Lemma 3.6.2. *With this convention, $\Psi_u(a,t)=t$ for $t\geq 0$.*

Proof. Since Ψ_u is 1-Lipschitz continuous, $\Psi_u(a,t)\leq t$; let us suppose that for some t_0 we have $\Psi_u(a,t_0)=t_0-\varepsilon$ with $\varepsilon>0$, then for $t>t_0$ $\Psi_u(a,t)\leq t-\varepsilon$. We have $\Psi_u(1,t)=t$ for every $t\geq 0$ because u takes the value $+\infty$ along $\pi(\mathcal{L}_0^+)$. We have:

$$\Psi_u(1,t)-\Psi_u(a,t)=\lim_{n\to+\infty}\Psi_{u_n}(1,t)-\Psi_{u_n}(a,t)=\lim_{n\to+\infty}\int_{[(a,t),(1,t)]}\mathrm{d}\Psi_{u_n}$$

By Lemma 3.3.3, the integral is always less than $2\sqrt{2}\frac{1-a}{t}$ for big t. So for t big enough, this upper-bound is less than ε: this give us a contradiction. □

The result of Lemma 3.6.2 says us, by Proposition 1.4.9, that u takes the value $+\infty$ on one side of L and $-\infty$ on the other side; more pricisely, $u(a+\eta,t)$ tends to $+\infty$ (resp. $-\infty$) if η tends to 0 by negative value (resp. positive value). There is only one solution for the Dirichlet problem for such boundary condition (we apply Theorem 2.6.1 with the polygon $(\overrightarrow{P_1A},\overrightarrow{AP_2},v_2,\cdots,v_r)$), this proves that the limit for the subsequences of (u_n) is unique then the sequence (u_n) must converge to the function u.

In $[-1,1]\times\mathbb{R}_+\cup D_r$, the 1-form $\mathrm{d}X_1^*(A_n)$ and $\mathrm{d}X_2^*(A_n)$ are given by:

$$\mathrm{d}X_1^*(A_n)=\frac{q_np_n}{W_n}\mathrm{d}x+\frac{1+q_n^2}{W_n}\mathrm{d}y$$
$$\mathrm{d}X_2^*(A_n)=-\frac{1+p_n^2}{W_n}\mathrm{d}x-\frac{p_nq_n}{W_n}\mathrm{d}y$$

with p_n and q_n the first derivatives of u_n (see [Os2]). Using this expressions, we can also define the 1-forms $\mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A)$.

Let η_1 be a small positive number, $\eta_2<\eta_1$ and l positive numbers. Let Γ be the closed path which consists in the segment $[(a+\eta_1,l),(a-\eta_1,l)]$, the segment

$[(a-\eta_1,l),(a-\eta_1,0)]$, the half circle in D_r of center A and radius η_1 and the segment $[(a+\eta_1,0),(a+\eta_1,l)]$. Besides we call Γ_1 the part of Γ consisting in the two vertical segments and the half circle, Γ_2 the union of the two segments $[(a+\eta_1,l),(a+\eta_2,l)]$ and $[(a-\eta_2,l),(a-\eta_1,l)]$ and Γ_3 the segment $[(a+\eta_2,l),(a-\eta_2,l)]$ (see Figure 3.3). There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and every η_1, η_2 and l, the period for the point A_n is computed by

$$\left(\int_\Gamma \mathrm{d}X_1^*(A_n), \int_\Gamma \mathrm{d}X_2^*(A_n)\right)$$

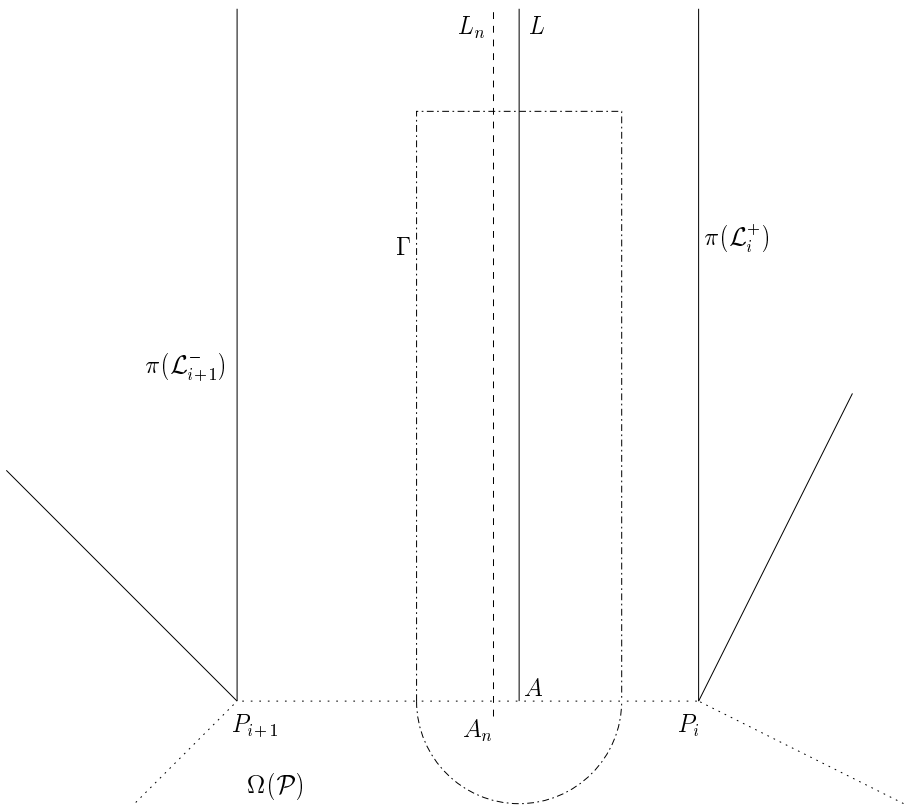

Figure 3.3:

Let $0 < \alpha < 1$. By Lemma 3.3.3, we have $\dfrac{|q_n|}{|p_n|} \leq \sqrt{2}\dfrac{2}{l}\dfrac{1}{1-\frac{4}{l^2}}$ on $\Gamma_2 \cup \Gamma_3$. Then in choosing l big enough, we can ensure that:

$$\left|\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_1^*(A_n)\right| < \frac{\alpha}{2}\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)$$

We have:

$$\lim_{n\to+\infty}\int_{\Gamma_1} \mathrm{d}X_1^*(A_n) = \int_{\Gamma_1} \mathrm{d}X_1^*(A)$$
$$\lim_{n\to+\infty}\int_{\Gamma_1} \mathrm{d}X_2^*(A_n) = \int_{\Gamma_1} \mathrm{d}X_2^*(A)$$

the same is true on Γ_2 and as η_2 tends to 0, we have $\displaystyle\int_{\Gamma_2} \mathrm{d}X_2^*(A) \to +\infty$. The last assertion is due to Lemma 1 in [JS] which implies that, as η goes to zero $\dfrac{p}{W}(a+\eta,l) \longrightarrow 1$ and $p(a+\eta,l) \geq \dfrac{C}{\eta}$ for some constant C. We then can choose η_2 such that for big n, we have

$$\frac{\left|\int_{\Gamma_1} \mathrm{d}X_1^*(A_n)\right|}{\int_{\Gamma_2} \mathrm{d}X_2^*(A_n)} < \frac{\alpha}{2} \quad \text{and} \quad \frac{\left|\int_{\Gamma_1} \mathrm{d}X_2^*(A_n)\right|}{\int_{\Gamma_2} \mathrm{d}X_2^*(A_n)} < \frac{\alpha}{8}$$

This implies first that $\displaystyle\lim_{n\to+\infty}\int_{\Gamma} \mathrm{d}X_2^*(A_n) = +\infty$, since $\displaystyle\int_{\Gamma_3} \mathrm{d}X_2^*(A_n) \geq 0$; then the period at A_n, for big n, is renormalized and must have a non negative second coordinate. Secondly, for big n, we have:

$$\begin{aligned}
\frac{\left|\int_{\Gamma} \mathrm{d}X_1^*(A_n)\right|}{\left|\int_{\Gamma} \mathrm{d}X_2^*(A_n)\right|} &\leq \frac{\left|\int_{\Gamma_1} \mathrm{d}X_1^*(A_n)\right| + \left|\int_{\Gamma_2} \mathrm{d}X_1^*(A_n)\right| + \left|\int_{\Gamma_3} \mathrm{d}X_1^*(A_n)\right|}{-\left|\int_{\Gamma_1} \mathrm{d}X_2^*(A_n)\right| + \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \int_{\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\leq \frac{\frac{\alpha}{2}\int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \frac{\alpha}{2}\int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \alpha\int_{\Gamma_3} \mathrm{d}X_2^*(A_n)}{-\frac{\alpha}{8}\int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + (1-\frac{\alpha}{8})\int_{\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\leq \frac{\alpha\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)}{(1-\frac{\alpha}{8})\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\leq \frac{\alpha}{1-\frac{\alpha}{8}}
\end{aligned}$$

This proves that the renormalized period converges to the vector $(0,1)$; this is what we want to prove. □

3.6.2 The behaviour at the vertices

Because of Proposition 3.6.1, it is clear that we can not extend the period map to the vertices and obtain a continuous map on the boundary. In fact the idea to solve this problem is to make a blowing-up of $\mathcal{P}$ at its vertices.

Let P_i be a vertex of $\mathcal{P}$, there exists $\alpha > 0$ such that a neighborhood of P_i is isometric to $\{(\rho,\theta),\ 0 \leq \rho < \mu, 0 \leq \theta \leq \alpha\}$ with the polar metric $\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2$. A blowing-up at P_i consists in remplacing the point P_i with the segment of all the points $(P_i,\theta)_{0\leq\theta\leq\alpha}$ and if (A_n) is a sequence of points of $\overset{\circ}{\mathcal{P}}$ converging to P_i in the original topology we shall say that (A_n) converges to (P_i,θ) if $\theta_n \to \theta$ where (ρ_n,θ_n) are the coordinates of A_n near P_i. If we make this blowing-up at all the vertices, we get a new topological space that we note $\widetilde{\mathcal{P}}$; $\widetilde{\mathcal{P}}$ is always a topological space homeomorphic to the closed unit disk and its interior is equal to the interior of $\mathcal{P}$. Then the question is to understand what is the limit of $Per(A_n)$ when A_n tend to some (P_i,θ).

Let us consider the case where $i = 1$; a neighborhood of P_1 in $\mathcal{P}$ is $\{(\rho,\theta),\ 0 \leq \rho < \mu, 0 \leq \theta \leq \alpha\}$. We know that there is a bijection between the Alexandrov-embedded r-noid with genus 0 and horizontal ends and the polygonal immersed disk (see [CR]). In this bijection, the corresponding r-noid to $\mathcal{P}$ will be noted $\Sigma(\mathcal{P})$ and $\Sigma(\mathcal{P})^+$ is the conjugate of a graph over the multi-domain $\Omega(\mathcal{P})$ which contains $\mathcal{P}$ and has as vertices the vertices of $\mathcal{P}$. Besides this graph is bounded by r vertical straight-lines passing by the vertices of $\mathcal{P}$ (see [CR] and chapter 2). Let us consider $\mathcal{C}$ the conjugate of the straight-line passing by P_1. $\mathcal{C}$ is a strictly convex curve and there exists $\gamma : (-\pi/2, \alpha + \pi/2) \to \{z = 0\}$ a parametrization of $\mathcal{C}$ by its normal. We then have the following result.

Proposition 3.6.3. *Let (A_n) be a sequence of points in the interior of $\mathcal{P}$ converging to the point (P_1,θ) of $\partial\widetilde{\mathcal{P}}$. Then $Per(A_n)$ converges to:*

- $(0,-1)$ *if* $\theta = 0$,
- $(-\sin\alpha, \cos\alpha)$ *if* $\theta = \alpha$,
- $\overrightarrow{\gamma(\theta-\pi/2)\gamma(\theta+\pi/2)}$ *or* $\dfrac{\overrightarrow{\gamma(\theta-\pi/2)\gamma(\theta+\pi/2)}}{||\overrightarrow{\gamma(\theta-\pi/2)\gamma(\theta+\pi/2)}||}$, *following the sign of*

$$||\overrightarrow{\gamma(\theta-\pi/2)\gamma(\theta+\pi/2)}|| - 1, \ \textit{if } \theta \in (0,\alpha).$$

Preliminaries

Let (A_n) be a sequence of points in $\overset{\circ}{\mathcal{P}}$ converging to (P_1,θ) and we suppose that a neighborhood of P_1 in $\mathcal{P}$ is $\{(\rho,\theta),\ 0 \leq \rho < \mu, 0 \leq \theta \leq \alpha\}$. In $\Omega(\mathcal{P})$, a neighborhood of P_1 is then $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \mu) = \{(\rho,\theta),\ 0 \leq \rho < \mu, -\frac{\pi}{2} \leq \theta \leq \alpha + \frac{\pi}{2}\}$. Let u_n be the restriction of the solution u_{A_n} to the period $\Omega^1_{A_0}$; we can remark that the period $\Omega^1_{A_0}$ can be identified with $\Omega(\mathcal{P})\backslash[A_n, P_1]$ in using the covering map π.

We note Σ_n the graph in $\mathbb{R}^3$ of u_n and Σ_n^* the minimal surface consisting in the union of the conjugate surface of Σ_n with its symetric with respect to $\{z = 0\}$

(the conjugate surface to Σ_n is normalized such that the conjugates of the vertical lines satisfy $z = 0$). We also note $\widetilde{\Sigma}_n^*$ the periodic minimal surface consisting in the union of the conjugate surface of the graph of u_{A_n} with its symetric with respect to $\{z = 0\}$. In a certain way, Σ_n^* is a period of $\widetilde{\Sigma}_n^*$, and the vector that lets $\widetilde{\Sigma}_n^*$ invariant is the non renormalized $Per(A_n)$.

From Lemma 3.4.10, we remark that the total curvature of Σ_n^* does not depend on n and is $4\pi r$.

The convergence of the graphs

To understand the behaviour of the surface Σ_n^* when n goes to $+\infty$ we need to know the behaviour of the sequence (u_n). We consider u_n as a function on $\Omega(\mathcal{P})\backslash[A_n, P_1]$; then the study of the convergence is on the limit multi-domain $\Omega(\mathcal{P})$. Using the arguements of the proof of Theorem 3.3.1, we see that there is no line of divergence so a subsequence $(u_{n'})$ converges to a function u solution of (MSE) on $\Omega(\mathcal{P})$ and taking the value $+\infty$ on $\pi(\mathcal{L}_i^+)$ and $-\infty$ on $\pi(\mathcal{L}_i^-)$ (more precisely, if B is a point in $\mathcal{P}$ we have $u_{n'} - u_{n'}(B) \to u$). By Theorem 2.6.1, such a solution u is unique; then, in fact, the sequence (u_n) converges to u. The graph of this function is the conjugate surface to $\Sigma(\mathcal{P})^+$.

We also need to study the convergence of (u_n) near the point P_1, and to do this we shall renormalize a neighborhood of P_1. u_n is defined on the neighborhood $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \mu)\backslash[A_n, P_1]$ of P_1; more precisely, the derivatives of u_n are well defined on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \mu)\backslash\{A_n\}$. We have $A_n = (\rho_n, \theta_n)$ with $\theta_n \to \theta$ and $\rho_n \to 0$. We then renormalized by $\frac{1}{\rho_n}$: we get a function v_n defined on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \frac{\mu}{\rho_n})\backslash\{(\rho, \theta_n), \rho \in [0, 1]\}$ by $v_n(\rho, \beta) = \frac{1}{\rho_n} u_n(\rho_n \rho, \beta)$. v_n is a solution of the minimal surfaces equation. We want to understand the asymptotic behaviour of v_n. Since $\frac{\mu}{\rho_n} \to +\infty$ the limit multi-domain is $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)\backslash\{(1, \theta)\}$ for the derivatives of v_n. In the following we note $B(\beta)$ the point of polar coordinates $(1, \beta)$ and $L(\beta)$ the half straight-line $\{(\rho, \beta)\}_{\rho > 0}$.

First we must study the lines of divergence. We know that v_n takes the value $+\infty$ on $L(-\frac{\pi}{2})$ and the value $-\infty$ on $L(\alpha + \frac{\pi}{2})$; we have $\Psi_{v_n}(P_1) = 0$, $\Psi_{v_n}(B(\theta_n)) = 0$ and $\Psi_{v_n} \geq 0$. Let L be a line of divergence, L must have an end-point, otherwise we can apply the arguement of the proof of Theorem 3.3.1 with the point P_1. This end point can not be on $L(\alpha + \frac{\pi}{2})$ or $L(-\frac{\pi}{2})$ because of Proposition 1.4.5. Then the end point must be P_1 or $B(\theta)$. If the line of divergence has two end-points, it is the segment $[P_0, B(\theta)]$ then we have

$$0 = |\Psi_{v_n}(B(\theta_n)) - \Psi_{v_n}(P_1)| = \left| \int_{[P_1, B(\theta_n)]} \mathrm{d}\Psi_{v_n} \right| \longrightarrow \left| \int_{[P_1, B(\theta)]} \lim \mathrm{d}\Psi_{v_n} \right|$$
$$\longrightarrow 1$$

This is a contradiction.

We then can ensure that L is a half straight-line with P_1 or $B(\theta)$ as end-point. Suppose that the end-point is P_1 then L is some $L(\beta)$.

Lemma 3.6.4. *Let L be a line of divergence with P_1 as end-point, L is some $L(\beta)$. Then $\beta \notin (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$*

Proof. Let us suppose that $\beta \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$. We note C_ρ the point of $L(\beta)$ with coordinates (ρ, β). Since $\mathrm{d}\Psi_{v_n}$ is closed,

$$\int_{[P_1, B(\theta_n)]} \mathrm{d}\Psi_{v_n} + \int_{[B(\theta_n), C(\rho)]} \mathrm{d}\Psi_{v_n} + \int_{[C(\rho), P_1]} \mathrm{d}\Psi_{v_n} = 0$$

The first integral is always zero, then

$$\left| \int_{[C(\rho), P_1]} \mathrm{d}\Psi_{v_n} \right| \leq |B(\theta_n)C(\rho)|$$

Then by passing to the limit for a subsequence making L appears we get, $\rho \leq |B(\theta)C(\rho)|$, but $|B(\theta)C(\rho)| = \sqrt{\rho^2 + 1 - 2\rho\cos(\beta - \theta)}$ then for big ρ the inequality is not true. □

We suppose now that the end-point of L is $B(\theta)$. We note (ρ', γ') the polar coordinates on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)$ with $B(\theta)$ as origin point; γ' is chosen such that the coordinates of P_0 in this new coordinates are $(1, \pi)$. In this polar coordinates, the line of divergence L is some $L'(\beta) = \{\gamma' = \beta,\ \rho' > 0\}$.

Lemma 3.6.5. *Let L be a line of divergence with $B(\theta)$ as end point, L is some $L'(\beta)$. Then $\beta \notin (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.*

Proof. The proof is the same as the one of Lemma 3.6.4 in exchanging P_1 and $B(\theta)$. □

In the following, we shall prove that, in fact, all the lines of divergence that we have not excluded by Lemma 3.6.4 and 3.6.5 yet appear. We first observe that the allowed lines of divergence do not intersect themselves. Since, for every n, $\Psi_{v_n}(P_1) = 0 = \Psi_{v_n}(B(\theta_n))$ and $\Psi_{v_n} \geq 0$, there is only one possibility for the limit normal on each line of divergence. Besides, we know that $\mathcal{B}(v_n) = \{P \in T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty) |\ |\nabla v_n(P)|$ is bounded$\}$ contains a strip which is delimited by the two half straight-lines with P_1 as end-point $L(\theta - \frac{\pi}{2})$ and $L(\theta + \frac{\pi}{2})$ and the two half straight-lines with $B(\theta)$ as end-point $L'(-\frac{\pi}{2})$ and $L'(\frac{\pi}{2})$ for the polar coordinates centred on $B(\theta)$. If all the lines of divergence appear, $\mathcal{B}(v_n)$ is exactly this strip (see Figure 3.4).

Let us suppose that one allowed line of divergence L do not appear, *i.e.* $L \subset \mathcal{B}(v_n)$. Then the connected component $\mathcal{B}$ of $\mathcal{B}(v_n)$ that contains L is then a multi-domain which is such that there exists a subset K such that $\mathcal{B}\backslash K$ is isometric to an angular sector (let us observe that the angle at the vertex can be greater than 2π) minus the set of the points at a distance less than d from the vertex of the angular sector (d is a positive number). On $\mathcal{B}$ we have a subsequence $v_{n'}$ that converges to some function v. Since $\mathcal{B}$ is bounded by lines of divergence or by the boundary of $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)$ the value of v is $+\infty$ on one side of the angular sector and $-\infty$ on the other side, by Proposition 1.4.9. Besides $\Psi_v \geq 0$ since $\Psi_{v_n} \geq 0$ for every n. Then, the function v satisfies many conditions that contradict Theorem 4.4.2; this proves there is no sub-sequence such that one of the allowed lines of divergence does not appear. We then know the limit of the normal to the graph of v_n for all the points outside the strip.

On the strip, there is a sub-sequence $(v_{n'})$ that converges to some function v (in fact $v_{n'}$ is not well defined on the strip since it is defined only outside $[P_1, B(\theta_n)]$, but the derivatives are well defined and converge to the derivatives of some function v). The function v takes the value $-\infty$ on $L(\theta + \frac{\pi}{2})$ and $L'(-\frac{\pi}{2})$ and the value $+\infty$ on $L(\theta - \frac{\pi}{2})$ and $L'(\frac{\pi}{2})$ by Proposition 1.4.9; such a solution v is unique and is a peace of helicoid. More precisely, if the strip is isometricaly parametrized by $\mathbb{R} \times [-1/2, 1/2]$ with $P_1 = (0, 1/2)$ and $B(\theta) = (0, -1/2)$ then we have $v(x, y) = x \tan(\pi y)$.

Remark 3.6.6. Let P be a point in the strip, the curvature of the helicoid at this point is non zero, then the curvature of the graph of v_n over the point P goes to the curvature of the helicoid at this point. This implies that there exists a sequence of point $p_n \in \Sigma_n$ such that the curvature of Σ_n at p_n goes to $+\infty$.

Remark 3.6.7. Besides, we know that there exists, for each n, a constant c_n such that $u_{A_n} \circ f_n = u_{A_n} + c_n$, then the result above proves that $\dfrac{c_n}{\rho_n} \to +\infty$.

We also want to know what occurs when we homothetically expand the sequence u_n in a general way. Let (M_n) be a sequence of point in $\Omega(\mathcal{P})$ such that, for every n, $M_n \neq A_n$ and (λ_n) a sequence of positive number such that λ_n goes to $+\infty$. Let h_n be the homothety of center M_n and ratio λ_n, by applying the dilatation h_n we get a function v_n on $h_n(\Omega(\mathcal{P}))\backslash\{A_n\}$ defined by $v_n(M) = \lambda_n u_n(h_n^{-1}(M))$. The question we ask is: what asymptotic behaviours are possible? By taking a subsequence, if it is necessary, we can suppose that we are in one of the following cases.

Case 1. For every i, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to +\infty$. In this case the limit multi-domain for the sequence v_n is $\mathbb{R}^2$ then if there is no line of divergence a sub-sequence $(v_{n'})$ must converge to a linear function by Bernstein Theorem. If there is a line of divergence L, each connected component of the domain of convergence of a subsequence $(v_{n'})$is a strip or a half-plane and $(v_{n'})$ converges to a function v with the value $+\infty$ on one side and $-\infty$ on the other side; but such solution of (MSE) does not exist by Proposition 4.2.1 so the domain of convergence is empty

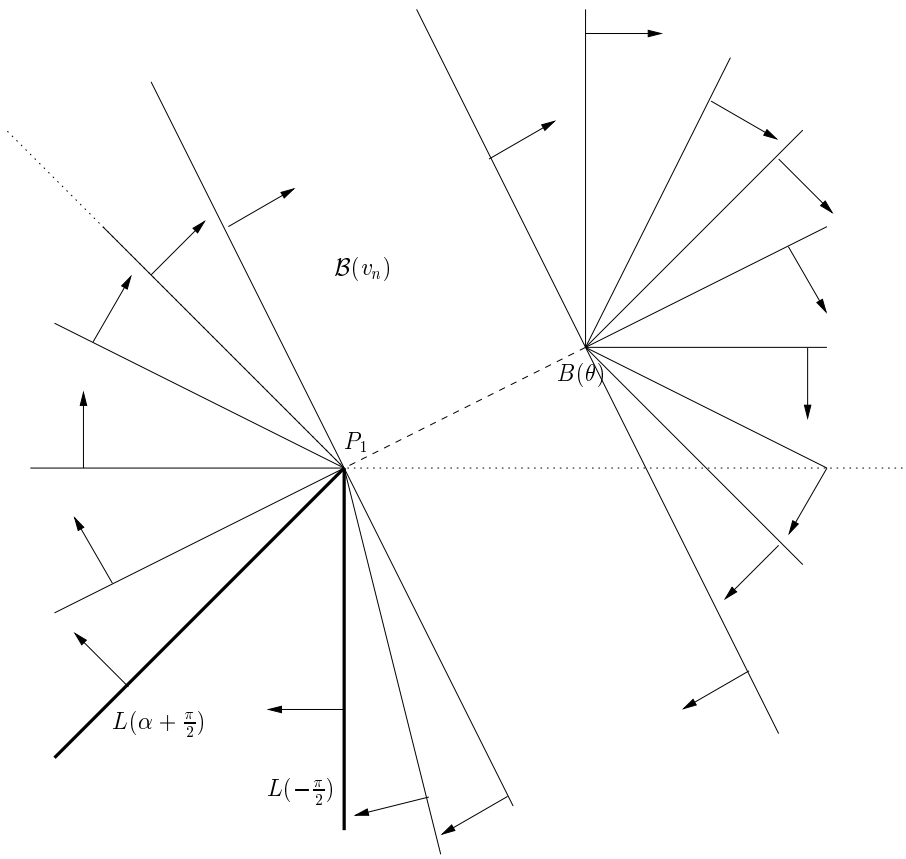

Figure 3.4: the asymptotic behaviour of v_n

and we have only lines of divergence which are all parallel to L and the limit normal is constant on $\mathbb{R}^2$.

Case 2. For every i, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to d \geq 0$. In this case the limit multi-domain is $\mathbb{R}^2$ minus the limit point A' of $h_n(A_n)$. Since $d(M_n, h_n(P_1)) \to +\infty$, we have $d(h_n(A_n), h_n(P_1)) \to +\infty$ then $\lambda_n \rho_n \to +\infty$; by Remark 3.6.7, this implies that $\lambda_n c_n$, which is the vertical jump over $[h_n(A_n), h_n(P_1)]$ for the function v_n, goes to $+\infty$. Then the derivatives of v_n can not converge on $\mathbb{R}^2 \backslash \{A'\}$ and there are lines of divergence. Since $\Psi_{v_n}(h_n(A_n)) = 0$ and $\Psi_{v_n} \geq 0$, using arguements that we have already seen, we can ensure that a line of divergence must be a half straight-line with A' as end-point and that there is only one possibility for the limit normal along the line of divergence. If the domain of convergence $\mathcal{B}(v_n)$ is non empty, each connected component of it is an angular sector of $\mathbb{R}^2$; then on one component, a subsequence $(v_{n'})$ converges to a solution v of (MSE) with the value $+\infty$ on one side and $-\infty$ on the other side. By Proposition 4.2.3, such a solution

does not exist and we have only the lines of divergence as asymptotic behaviour.
Case 3. There exists $j \neq 1$ such that $d(M_n, h_n(P_j)) \to c \geq 0$. This implies that, for $i \neq j$, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to +\infty$; then the limit multi-domain is an angular sector isometric to some $T(0, \beta, +\infty)$ with $P'_j = \lim h_n(P_j)$ as vertex. As above, the lines of divergence must be half straight-lines with P'_j as end-point and the limit normal on a line of divergence is given by the condition $\Psi_{v_n} \geq 0$. As in Case 2, $\mathcal{B}(v_n)$ is empty and the asymptotic behaviour is given by the lines of divergence.
Case 4. $d(M_n, h_n(P_1)) \to c \geq 0$ and $d(h_n(P_1), h_n(A_n)) \to 0$ or $+\infty$. In this case, the limit multi-domain is an angular sector with $P'_1 = \lim h_n(P_1)$ as vertex and the asymptotic behaviour is the same as in Case 3.
Case 5. $d(M_n, h_n(P_1)) \to c \geq 0$ and $d(h_n(P_1), h_n(A_n)) \to c > 0$. The limit-multi-domain is then an angular sector, with P'_1 as vertex, minus the point $A' = \lim h_n(A_n)$. We are in the situation studied above and we know that the asymptotic behaviour is a domain of convergence which is a strip where (v_n) converges to a piece of helicoid and, outside the strip, lines of divergence with A' or P'_1 as end-point (see Figure 3.4).

The convergence of Σ_n^*

We first fix the notations. Let Q be a point in the interior of $\mathcal{P}$ and q_n the corresponding point in Σ_n. For every n, the value $\Psi_{u_n}(Q)$ is well defined and we normalize Σ_n^* such that q_n^*, the corresponding point to q_n, has coordinates $(0, 0, \Psi_{u_n}(Q))$; Σ_n^* is then still symetric with respect to the plane $\{z = 0\}$. We want to determine the limit of the sequence of minimal surfaces (Σ_n^*).

Let M be a surface in $\mathbb{R}^3$. In the following, when we shall talk about a geodesical disk $D(m, \mu)$ of center $m \in M$ and radius μ, we shall consider the disk of radius μ in the tangent plane to M at m with the exponential map $\exp_m : T_m M \to \mathbb{R}^3$. Besides, we shall sometimes identify a point in the tangent plane with its image by $\exp_m$.

Let m_n^* be a sequence of point of $\mathbb{R}^3$ such that m_n^* lies in the part of Σ_n^* which is the conjugate of Σ_n (the third coordinate of m_n^* is non-negative). We make the following assumption: $K_{\widetilde{\Sigma}_n^*}(m_n^*) \to +\infty$. Because of Remark 3.6.6, we know that such a sequence exists. Let μ be a positive number, and, for each n, we consider $D(m_n^*, \mu)$ the closed geodesical disk of center m_n^* and radius μ in $\widetilde{\Sigma}_n^*$. On $D(m_n^*, \mu)$, the function $a \mapsto |K_{\widetilde{\Sigma}_n^*}(a)|(\mu - d(a, m_n^*))^2$ admits a maximum in the interior of the disk (the distance d is the distance in the tangent space) and let p_n^* be a point where the maximum is reached. Let us note $\lambda_n = \sqrt{|K_{\widetilde{\Sigma}_n^*}(p_n^*)|}$ and $\mu_n = (\mu - d(p_n^*, m_n^*))$. The geodesical disk $D(p_n^*, \mu_n)$ is included in $D(m_n^*, \mu)$ (in fact the image by the exponential map in $\widetilde{\Sigma}_n^*$ of the geodesical disk $D(p_n^*, \mu_n)$ is

included in the image of $D(m_n^*, \mu)$); besides since p_n^* realizes the maximum we have $\lambda_n^2\mu_n^2 \geq K_{\widetilde{\Sigma}_n^*}(m_n^*)\mu^2$, then $\lambda_n\mu_n \to +\infty$ and $\lambda_n \to +\infty$. By translating p_n^* to the origin and homothetically expanding the disk $D(p_n^*, \mu_n)$ by the factor λ_n, we obtain a new geodesical disk $D'_n = D(0, \lambda_n\mu_n)$, (D'_n) is a sequence of minimal surfaces. We have $|K_{D'_n}(0)| = 1$. Let R be a positive number and $\tilde{a}$ a point in the geodesical disk $D(0, R)$ of D'_n; we then note a the point in $D(p_n^*, \mu_n)$ corresponding to $\tilde{a}$. $\exp_{p_n^*}(a)$ is included in the image of the disk $D(m_n^*, \mu)$. Then there exists $a' \in D(m_n^*, \mu)$ such that $\exp_{m_n^*}(a') = \exp_{p_n^*}(a)$ and $d(m_n^*, a') \leq d(m_n^*, p_n^*) + d(p_n^*, a)$; we have $K_{\widetilde{\Sigma}_n^*}(a) = K_{\widetilde{\Sigma}_n^*}(a')$. With this notation we then have:

$$\begin{aligned}
|K_{D'_n}(\tilde{a})|(\lambda_n\mu_n - R)^2 \leq |K_{D'_n}(\tilde{a})|(\lambda_n\mu_n - d(\tilde{a}, 0))^2 &= |K_{\widetilde{\Sigma}_n^*}(a)|(\mu_n - d(a, p_n^*))^2 \\
&\leq |K_{\widetilde{\Sigma}_n^*}(a')|(\mu - d(a', m_n^*))^2 \\
&\leq \lambda_n^2\mu_n^2
\end{aligned}$$

the equality is due to the fact that the function $|K_{\widetilde{\Sigma}_n^*}(\cdot)|(\mu_n - d(\cdot, p_n^*))^2$ is invariant under rescaling. Thus the curvature on D'_n is uniformly bounded on the sequence of geodesical disks $D(0, R)$ of D'_n. Then there exists a subsequence $(D'_{n'})$ that converges to a complete minimal surface that we denote D'_∞; this surface is complete since $\lambda_n\mu_n \to +\infty$. Besides D'_∞ is non flat since at the origin its curvature is -1. Since D'_∞ is non flat there is a point $\tilde{a}$ where the normal has a negative third coordinate; there exists a neighborhood U of $\tilde{a}$ in the tangent plane to D'_∞ at $\tilde{a}$ such that D'_∞ and $D'_{n'}$, for big n', are graphs over U and $D'_{n'} \to D'_\infty$ as graphs. Let $\tilde{a}_{n'}$ be the sequence of points in $D'_{n'}$ over $\tilde{a}$ as graphs. The normal at $\tilde{a}_{n'}$ to $D'_{n'}$ has a negative third coordinate then before the rescaling $\tilde{a}_{n'}$ correponds to a point $a_{n'}$ which lies in the conjugate of $\Sigma_{n'}$. Let $b_{n'}$ be the point in $\Sigma_{n'}$ corresponding to $a_{n'}$ and $B_{n'}$ the projection on $\Omega(\mathcal{P})$ of $b_{n'}$. The convergence of $D_{n'}$ to D'_∞ near $\tilde{a}$ says us that $\mathcal{B}(v_{n'})$ is non empty where $v_{n'}$ is the rescaled function of $u_{n'}$ with the factor $\lambda_{n'}$ and $B_{n'}$ as origin points. We are then in Case 1 or Case 5 of the preceding subsection, but if it is Case 1 the limit graph would be a plane which is impossible since D'_∞ is non flat. Then it is Case 5 and the limit graph is a piece of helicoid then this implies that D'_∞ is a catenoid. This also implies that $\lambda_{n'} \sim \frac{c}{\rho_{n'}}$ with c some real constant (we recall that $\rho_n = |P_1A_n|$).

Remark 3.6.8. If for example, we take m_n^* such that $K_{\Sigma_n}(m_n^*) = \max K_{\Sigma_n}(\cdot)$ the above arguements show that $\max K_{\Sigma_n}(\cdot) = O(\left(\frac{1}{\rho_n}\right)^2)$.

The boundary of Σ_n^* is composed of two closed paths Γ_n^1 and Γ_n^2, we want to understand the behaviour of this boundary when n goes to $+\infty$. The boundary of the graph Σ_n is composed of $r-1$ straight-lines, they are over the points P_i for $i \neq 1$, and a curve which consists in a half straight-line over P_1 that goes down from the infinity to some point called t_n^1, a curve which is a graph over the segment $[P_1, A_n]$

joining t_n^1 to some point t_n^2, a vertical segment $[t_n^2, t_n^3]$ over A_n, a curve which is a graph over the segment $[P_1, A_n]$ joining t_n^3 to some point t_n^4 (by a vertical translation, this curve is the same as the curve joining t_n^1 to t_n^2) and a half straight-line over P_1 with t_n^4 as end-point and going down to the infinity. Then the path Γ_n^1 (resp. Γ_n^2) consists in the conjugate of the curve joining t_n^1 to t_n^2 (resp. t_n^3 to t_n^4) with its symetric with respect to the plane $\{z = 0\}$.

As in Subsection 3.6.2, we consider v_n the rescaled function with factor $\frac{1}{\rho_n}$ on $T(-\frac{\pi}{2}, \alpha+\frac{\pi}{2}, \frac{r}{\rho_n})\backslash[P_1, B(\theta_n)]$. By Remark 3.6.8, there exists a constant which bounds the curvature on the graph of v_n for every n. Let p_n be a point in the graph of v_n which is above the middle of the segment $[P_1, B(\theta_n)]$ and $\mathcal{Y}_n$ the conjugate of the graph of v_n that we extend by symmetry and periodicity such that the conjugate point of p_n is the origin of $\mathbb{R}^3$. Let R be a positive number and consider the sequence of geodesical disks $D(0, R)$ in $\mathcal{Y}_n$. Since the curvature is uniformly bounded on the disk there exists a subsequence such that the sequence of disks converges, this implies that, using a Cantor diagonal process, there exists a subsequence $(\mathcal{Y}_{n'})$ which converges to some minimal surface $\mathcal{Y}$. Since the graphs of v_n converge near p_n to a piece of an helicoid, $\mathcal{Y}$ is a catenoid whose flux is the vector $2\overrightarrow{P_1B(\theta)}$. There is only one possible limit for subsequences of $(\mathcal{Y}_n)$ so the sequence $(\mathcal{Y}_n)$ converges to $\mathcal{Y}$. Let us consider the catenoid given in cylindrical coordinates by $(u, v) \mapsto (\frac{1}{\pi}u, v, \frac{1}{\pi}\operatorname{argch}(u))$. Thus $\mathcal{Y}$ is the translated by $(0, 0, -\frac{1}{\pi})$ of the image by a rotation of axis $\{z = 0, x\cos\theta + y\sin\theta = 0\}$ and angle $\frac{\pi}{2}$ of this catenoid. Suppose that p_n is the point in the graph of v_n which is the limit of the points $(1/2, \beta, v_n(1/2, \beta))$ when $\beta \to \theta_n$ with $\beta < \theta_n$. We note $\mathcal{Y}_n'$ the conjugate of the graph of v_n that we extend by symmetry but not by periodicity such that the conjugate of the point p_n is the origin. What we have proved just above implies that $(\mathcal{Y}_n')$ converges to the part of the catenoid $\mathcal{Y}$ included in $\{x\cos\theta + y\sin\theta \leq 0\}$; this part is noted $\mathcal{Y}_-$. If p_n is build with $\beta > \theta_n$ we get the half catenoid included in $\{x\cos\theta + y\sin\theta \geq 0\}$.

This proves that the rescaled paths $\frac{1}{\rho_n}\Gamma_n^1$ and $\frac{1}{\rho_n}\Gamma_n^2$ converge to two circles of the same radius. Let us call s_n^i the point in Σ_n^* which is the conjugate of t_n^i for $1 \leq i \leq 4$. The above proof shows that for every $\varepsilon > 0$ and for big n the ball of center s_n^1 (resp. s_n^4) and radius ε contains Γ_n^1 (resp. Γ_n^2) and even a bigger and bigger part of a surface near a half catenoid. We consider the case where p_n is build with $\beta < \theta_n$. If $B(R)$ is the ball of center the origin and radius R, we have $B(R) \cap \mathcal{Y}_n'$ is near from $B(R) \cap \mathcal{Y}_-$ for big n. Besides for big n, $\rho_n R < \varepsilon$ then the ball $B(s_n^1, \varepsilon)$ contains the homothetic by ρ_n of a surface near $B(R) \cap \mathcal{Y}_-$. The same is true for p_n build with $\beta > \theta_n$. This implies that the sequence of total curvatures of the part of Σ_n^* included in this ball has an lower limit bigger than 2π, since the total curvature of an half catenoid is 2π. In fact, we have to remember that near the points s_n^1 and s_n^4 the surface Σ_n^* behaves like small half-catenoid for big n.

We know that (Σ_n^*) is a sequence of symetric minimal surface with finite total

curvature $4\pi r$. Each surface Σ_n^* has two connected components of boundary Γ_n^1 and Γ_n^2. We know that the diameter of each component goes to zero and if B_n is a sequence of balls of diameter ε centred at s_n^2 then the sequence of total curvatures of $\Sigma_n^* \cap B_n$ has an lower limit bigger than 2π. By taking a subsequence we can suppose that (s_n^2) and (s_n^3) diverge to the infinity or converge to s_∞^2 and s_∞^3. Then by results explained in [CR], there exist a finite number of distinct properly and simply immersed branched minimal surfaces $M_1, \ldots, M_k \subset \mathbb{R}^3$ with finite total curvature, a finite subset $X \in \mathbb{R}^3$ contained in $M = M_1 \cup \cdots \cup M_k$ and a subsequence of (Σ_n^*), that we still call (Σ_n^*), such that:

1. (Σ_n^*) converges to M (with finite multiplicity) on compact subsets of $\mathbb{R}^3 \backslash (X \cup \{s_\infty^2, s_\infty^3\})$ in the C^m-topology for any positive integer m;

2. on each M_i the multiplicity m_i is well defined and is such that

$$m_1 C(M_1) + \cdots + m_k C(M_k) \leq C(\Sigma_n^*);$$

3. X is the singular set of the limit $\Sigma_n^* \to M$. Given a point $p \in X$, the amount of total curvature of the sequence (Σ_n^*) which disappears through the point p is a positive multiple of 4π.

Since at s_∞^2 and s_∞^3, there is 2π of total curvature that disappear, we have $m_1 C(M_1) + \cdots + m_k C(M_k) \leq C(\Sigma_n^*) - 4\pi = 4\pi(r-1)$ even if s_n^2 or s_n^3 diverges. We know that the sequence of functions u_n on $\Omega(\mathcal{P}) \backslash [P_1, A_n]$ converge on $\Omega(\mathcal{P})$ to the solution u of the Dirichlet problem asked in Theorem 2.6.1. Then q_n converges to q_∞ the point in the graph of u which is above Q. The conjugate surface to the graph of u, after being extended by symetry with respect to the plane $\{z = 0\}$, is the solution $\Sigma(\mathcal{P})$ to the plateau problem at infinity with genus 0 for the data $\mathcal{P}$. Let q_∞^* be the point in $\Sigma(\mathcal{P})$ that correspond to q_∞. We then have $q_n^* \to q_\infty^*$ and, in a neighborhood of q_∞^*, Σ_n^* converges to $\Sigma(\mathcal{P})$, then $\Sigma(\mathcal{P})$ is one M_i. We then can suppose that $M_1 = \Sigma(\mathcal{P})$, since $C(\Sigma(\mathcal{P})) = 4\pi(r-1)$, $m_1 = 1$ and for, $i \neq 1$, M_i is a plane and besides X is empty. In fact, in [CR], C. Cosín and A. Ros proves that, in such a convergence, no plane can appear. Then finally, $\Sigma_n^* \to \Sigma(\mathcal{P})$.

Since the problems of convergence appear only near the points s_∞^2 and s_∞^3, the curves in $\Sigma(\mathcal{P})$ which are the conjugates of the $r-1$ straight-lines that are over the points P_i, for $i \neq 1$, are the respective limits of the curves in Σ_n^* which are the conjugates of the $r-1$ straight-lines in Σ_n that are over the points P_i, for $i \neq 1$.

The convergence of $Per(A_n)$

$Per(A_n)$ corresponds to the vector that defines the translation under which $\widetilde{\Sigma}_n^*$ is invariant. Then $Per(A_n)$ is $\overrightarrow{s_n^2 s_n^3}$ or $\frac{\overrightarrow{s_n^2 s_n^3}}{||\overrightarrow{s_n^2 s_n^3}||}$, following the value of $||\overrightarrow{s_n^2 s_n^3}||$.

As above, we consider $\mathcal{C}$ the curve in $\Sigma(\mathcal{P})$ which is the conjugate of the vertical straight-line which is over P_1 in the graph of u. This curve is strictly convex then we can parametrized $\mathcal{C}$ by its normal. Then there exists a parametrization $\gamma : (-\frac{\pi}{2}, \alpha + \frac{\pi}{2}) \to \{z = 0\}$ of $\mathcal{C}$ such that the normal to $\gamma(\beta)$ is $(\sin\beta, -\cos\beta, 0)$; we cover $\mathcal{C}$ as we cover the straight-line over P_1 when we go down. We know that $\mathcal{C}$, outside s^2_∞ and s^3_∞, is the limit, in the Hausdorff topology, of a part of the conjugate of the boundary of the graph Σ_n of u_n which is included in $\{z = 0\}$. For each n this set is composed of three strictly convex arc, we note $\mathcal{C}^1_n$, $\mathcal{C}^2_n$ and $\mathcal{C}^3_n$: $\mathcal{C}^1_n$ is the conjugate of the vertical half straight-line that have t^1_n as end-point ($\mathcal{C}^1_n$ has s^1_n as end-point), $\mathcal{C}^2_n$ is the conjugate of the vertical segment $[t^2_n, t^3_n]$ ($\mathcal{C}^2_n$ is joining s^2_n to s^3_n) and $\mathcal{C}^3_n$ is the conjugate of the vertical half straight-line that have t^4_n as end-point ($\mathcal{C}^3_n$ has s^4_n as end-point). As $\mathcal{C}$, this three strictly convex arcs can be parametrized by their normal, then there exist $\gamma^1_n : (-\frac{\pi}{2}, \theta_n] \to \{z = 0\}$, $\gamma^2_n : [\theta_n - \pi, \theta_n + \pi] \to \{z = 0\}$ and $\gamma^3_n : [\theta_n, \alpha + \frac{\pi}{2}) \to \{z = 0\}$ such that γ^i_n parametrized $\mathcal{C}^i_n$ and the normal at the point $\gamma^i_n(\beta)$ is $(\sin\beta, -\cos\beta, 0)$. We have $\gamma^1_n(\theta_n) = s^1_n$, $\gamma^2_n(\theta_n - \pi) = s^2_n$, $\gamma^2_n(\theta_n + \pi) = s^3_n$ and $\gamma^3_n(\theta_n) = s^4_n$. We note I^i_n the definition set of γ^i_n. We then have $I^i_n \to I^i$ where $I^1 = (\frac{\pi}{2}, \theta]$, $I^2 = [\theta - \pi, \theta + \pi]$ and $I^3 = [\theta, \alpha + \frac{\pi}{2})$. The question is: is γ^i_n converging on I^i?

Since there is a half catenoid that appears near the points s^i_n for big n, if $\beta \in (\theta - \frac{\pi}{2}, \theta] \subset I^1$ the curvature at $\gamma^1_n(\beta)$ becomes infinite, the same is true for the sequence of point $(\gamma^2_n(\beta))$ for $\beta \in [\theta - \pi, \theta - \frac{\pi}{2}) \cup (\theta + \frac{\pi}{2}, \theta + \pi] \subset I^2$ and the sequence of points $(\gamma^3_n(\beta))$ for $\beta \in [\theta, \theta + \frac{\pi}{2}) \subset I^3$.

The set $\mathcal{C} \backslash \{s^2_\infty, s^3_\infty\}$ is composed of a finite number of convex arcs. Let a be a point in $\mathcal{C} \backslash \{s^2_\infty, s^3_\infty\}$, at this point the convergence of Σ^*_n to $\Sigma(\mathcal{P})$ well behaves so there exists a neighborhood U of a in the tangent plane to $\Sigma(\mathcal{P})$ at a (this plane is vertical) such that, over U, $\Sigma(\mathcal{P})$ and Σ^*_n, for big n, are graphs and, as graphs, Σ^*_n converge to $\Sigma(\mathcal{P})$. Over U, there is a neighborhood of a in $\mathcal{C} \backslash \{s^2_\infty, s^3_\infty\}$ and this neighborhood is the limit of the part of Σ^*_n which is included in $\{z = 0\}$. Since each unit vector is reached a finite number of times on $\mathcal{C}^1_n \cup \mathcal{C}^2_n \cup \mathcal{C}^3_n$, by taking a subsequence n', there exist β, ε and $i \in \{1, 2, 3\}$ such that, for every n', the part of $\Sigma^*_{n'} \cap \{z = 0\}$ which is over U contains $\gamma^i_{n'}(\beta - \varepsilon, \beta + \varepsilon)$. Then the convergence as graphs of Σ^*_n implies that on $(\beta - \varepsilon, \beta + \varepsilon)$ the sequence $(\gamma^i_{n'})$ converges to some map $\widetilde{\gamma}$ that parametrized a neighborhood of a in $\mathcal{C}$ such that the normal to the point $\widetilde{\gamma}(\omega)$ is $(\sin\omega, -\cos\omega, 0)$ (the convergence is in the C^m topology for every m). Since $\gamma^i_{n'} \to \widetilde{\gamma}$, for every $\omega \in (\beta - \varepsilon, \beta + \varepsilon)$ the curvature at the point $\gamma^i_{n'}(\omega)$ remains bounded. Then, if $i = 1$, $(\beta - \varepsilon, \beta + \varepsilon) \cap [\theta - \frac{\pi}{2}, \theta] = \emptyset$, if $i = 2$, $(\beta - \varepsilon, \beta + \varepsilon) \cap \big([\theta - \pi, \theta - \frac{\pi}{2}] \cup [\theta + \frac{\pi}{2}, \theta + \pi]\big) = \emptyset$ and, if $i = 3$, $(\beta - \varepsilon, \beta + \varepsilon) \cap [\theta, \theta + \frac{\pi}{2}] = \emptyset$.

Then by applying the above arguement to a countable number of points and constructing a subsequence by diagonal Cantor process, there exist

1. k_3 open intervals in $I = (-\frac{\pi}{2}, \alpha+\frac{\pi}{2})$: $J_1, \dots, J_{k_1} \subset I^1 \backslash [\theta-\frac{\pi}{2}, \theta]$, $J_{k_1+1}, \dots, J_{k_2} \subset I^2 \backslash \left([\theta-\pi, \theta-\frac{\pi}{2}] \cup [\theta+\frac{\pi}{2}, \theta+\pi]\right)$ and $J_{k_2+1}, \dots, J_{k_3} \subset I^3 \backslash [\theta, \theta+\frac{\pi}{2}]$, these intervals satisfy $J_j \cap J_l = \emptyset$ if $j \neq l$, and

2. a map $\widetilde{\gamma}$ with value in $\{z = 0\}$, defined on the union of the k_3 intervals, $\widetilde{\gamma}$ parametrizes $\mathcal{C} \backslash \{s^2_\infty, s^3_\infty\}$ by its normal,

3. a subsequence n',

such that, on J_j, the sequence $(\gamma^i_{n'})$ (for the corresponding i) converges to $\widetilde{\gamma}$; besides, the J_j are maximal in the sense that $\widetilde{\gamma}$ restricted to J_j parametrized one of the convex arcs that composed $\mathcal{C} \backslash \{s^2_\infty, s^3_\infty\}$. The curve $\mathcal{C}$ has total curvature $\alpha + \pi$ and since $\widetilde{\gamma}$ parametrizes by the normal $\mathcal{C} \backslash \{s^2_\infty, s^3_\infty\}$, which has the same total curvature, we have

$$\alpha + \pi = \sum_{j=1}^{k_3} l(J_j) \leq l(I) = \alpha + \pi \tag{3.7}$$

where $l(J_j)$ is the length of the interval J_j.

• We now assume that $\theta \notin \{0, \alpha\}$. Under this hypothesis,the computation (3.7) implies that there exist points

$$-\frac{\pi}{2} = \beta_0 < \beta_1 < \cdots < \beta_{k_1} = \theta - \frac{\pi}{2} < \beta_{k_1+1} < \cdots$$
$$\cdots < \beta_{k_2} = \theta + \frac{\pi}{2} < \beta_{k_2+1} < \cdots < \beta_{k_3} = \alpha + \frac{\pi}{2}$$

such that, for every j, $J_j = (\beta_{j-1}, \beta_j)$.

Lemma 3.6.9. *for $j \notin \{0, k_1, k_2, k_3\}$, the sequence $(\gamma^i_{n'})$ (for the corresponding i) converges in a neighborhood of β_j in the C^1 topology, then we can extend the definition of $\widetilde{\gamma}$ at β_j.*

Proof. We apply Proposition 3.9.4, if there is no C^1 convergence near β_j, since $\Sigma^*_n \to \Sigma(\mathcal{P})$ in the Hausdorff topology, $\Sigma(\mathcal{P})$ contains a segment. But this is not true then we can extend the definition of $\widetilde{\gamma}$ at β_j. □

We then have $\gamma^1_{n'} \to \widetilde{\gamma}$ on $(-\frac{\pi}{2}, \theta - \frac{\pi}{2})$, $\gamma^2_{n'} \to \widetilde{\gamma}$ on $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ and $\gamma^3_{n'} \to \widetilde{\gamma}$ on $(\theta + \frac{\pi}{2}, \alpha + \frac{\pi}{2})$.

We shall then study the behaviour near $\theta - \frac{\pi}{2} = \theta'$.

Let $\beta < \frac{\pi}{4}$ be a small positive angle and $\varepsilon > 0$. We apply Proposition 3.9.2 to $\gamma^1_{n'}$ on $(\theta' - \beta, \theta' + \beta)$, then the set $\gamma^1_{n'}(\theta' - \beta, \theta' + \beta)$ is included in an angular sector, we note S, of vertex $\gamma^1_{n'}(\theta' - \beta)$ and angle 2β. We have $\gamma^1_{n'}(\theta' + \beta) \in S$. Then, for big n', because of the behaviour of the sequence $(\Sigma^*_{n'})$ near the point $s^2_{n'} = \gamma^2_{n'}(\theta)$, we have the distance between $\gamma^1_{n'}(\theta' + \beta)$ and $s^2_{n'}$ less than ε and the distance between

$\gamma^1_{n'}(\theta'+\beta)$ and $\gamma^2_{n'}(\theta'-\beta)$ less than ε. Besides by applying Proposition 3.9.2 to $\gamma^2_{n'}$ on $(\theta'-\beta,\theta'+\beta)$, we have that $\gamma^2_{n'}(\theta'-\beta,\theta'+\beta)$ is included in an angular sector of vertex $\gamma^2_{n'}(\theta'-\beta)$ and angle 2β. Since at $\gamma^1_{n'}(\theta'-\beta)$ and $\gamma^2_{n'}(\theta'-\beta)$, the two strictly convex curves $\mathcal{C}^1_{n'}$ and $\mathcal{C}^2_{n'}$ have the same normal and their curvature have the same sign, the angular sector of vertex $\gamma^2_{n'}(\theta'-\beta)$ is just the translation of S by the vector $\overrightarrow{\gamma^1_{n'}(\theta'-\beta)\gamma^2_{n'}(\theta'-\beta)}$. This second angular sector is then included in the set S_ε of points of $\mathbb{R}^2$ which are at a distance less than ε from S. Then by passing to the limit $n'\to+\infty$, we get that $\widetilde{\gamma}\left((\theta'-\beta,\theta'+\beta)\backslash\{\theta'\}\right)$ is included in the set of points in $\mathbb{R}^2$ which are at a distance less than ε from an angular sector which have $\widetilde{\gamma}(\theta'-\beta)$ as vertex and 2β as angle. The points $s^2_{n'}$ are also in this set for big n' (see Figure 3.5).

We can do the same work in starting from the point $\gamma^2_{n'}(\theta'+\beta)$ and in covering the curves $\mathcal{C}^1_{n'}$ and $\mathcal{C}^2_{n'}$ in the opposite sense. We then get that $\widetilde{\gamma}\left((\theta'-\beta,\theta'+\beta)\backslash\{\theta'\}\right)$ and $s^2_{n'}$, for big n', are included in the set of points in $\mathbb{R}^2$ which are at distance less than ε from an angular sector which have $\widetilde{\gamma}(\theta'+\beta)$ as vertex and 2β as angle. Since β is less than $\frac{\pi}{4}$, the intersection of the two sets we have just build is a compact subset of $\mathbb{R}^2$ which contains $s^2_{n'}$, then s^2_∞ exists. Then $\widetilde{\gamma}(t)$ admits two limits: one when $t\to\theta'$, $t<\theta'$ and one when $t\to\theta'$, $t>\theta'$. This two limit points are in $\mathcal{C}$ and $\widetilde{\gamma}$ parametrized $\mathcal{C}$ except for two points where the normal is different. Since the normals at this two limit points are the same, the two limit points are, in fact, a unique point and then the definition of $\widetilde{\gamma}$ extend at θ'. Now, letting β and ε goes to zero, we see that the limit compact is just $\widetilde{\gamma}(\theta')$ then $\widetilde{\gamma}(\theta')=s^2_\infty$.

In the same way we see that s^3_∞ exists and $\widetilde{\gamma}(\theta+\frac{\pi}{2})=s^3_\infty$. In fact this prove that $\widetilde{\gamma}$ parametrizes $\mathcal{C}$ by its normal on I and then $\widetilde{\gamma}=\gamma$. Then the sequence $(Per(A_n))$ has only one possible cluster point then the sequence converges to the limit given in Proposition 3.6.3.

• We now study the case $\theta=0$ (the case $\theta=\alpha$ can be done in the same way). In this case $k_1=0$ and using the same arguements as above we can prove that: s^3_∞ exists, $\gamma=\widetilde{\gamma}$ and $\gamma(\frac{\pi}{2})=s^3_\infty$. Because of the asymptotic behaviour of the curve $\mathcal{C}$, we know that there exists $0<\beta<\frac{\pi}{2}$ such that for every $t\in(-\frac{\pi}{2},-\frac{\pi}{2}+\beta)$, $|\gamma(t)s^3_\infty|>2$ and $\frac{\overrightarrow{\gamma(t)s^3_\infty}}{|\gamma(t)s^3_\infty|}$ is at a distance less than ε from the vector $(0,-1)$. Let $0<\beta'<\beta$, we consider the angular sector S of vertex $\gamma(-\frac{\pi}{2}+\beta')$ and angle 2β which contains $\gamma(-\frac{\pi}{2},-\frac{\pi}{2}+\beta')$ and has, as a part of its boundary, the half straight-line tangent to $\mathcal{C}$ at $\gamma(-\frac{\pi}{2}+\beta')$ and with $\gamma(-\frac{\pi}{2}+\beta')$ as end-points; S exists because of Proposition 3.9.2. Then, because of the asymptotic behaviour of $\mathcal{C}$, there exist $0<\beta'<\beta$ and ε' such that, for every point s at a distance less than ε' from S, we have $|ss^3_\infty|>2$ and $\frac{\overrightarrow{ss^3_\infty}}{|ss^3_\infty|}$ is at a distance less than ε from $(0,-1)$.

We apply now Proposition 3.9.2 to $\gamma^2_{n'}$ on $(-\frac{\pi}{2}-\beta,-\frac{\pi}{2}+\beta)$ that we cover in the opposite sense. We then obtain that $\gamma^2_{n'}(-\frac{\pi}{2}-\beta,-\frac{\pi}{2}+\beta)$ is included in an angular

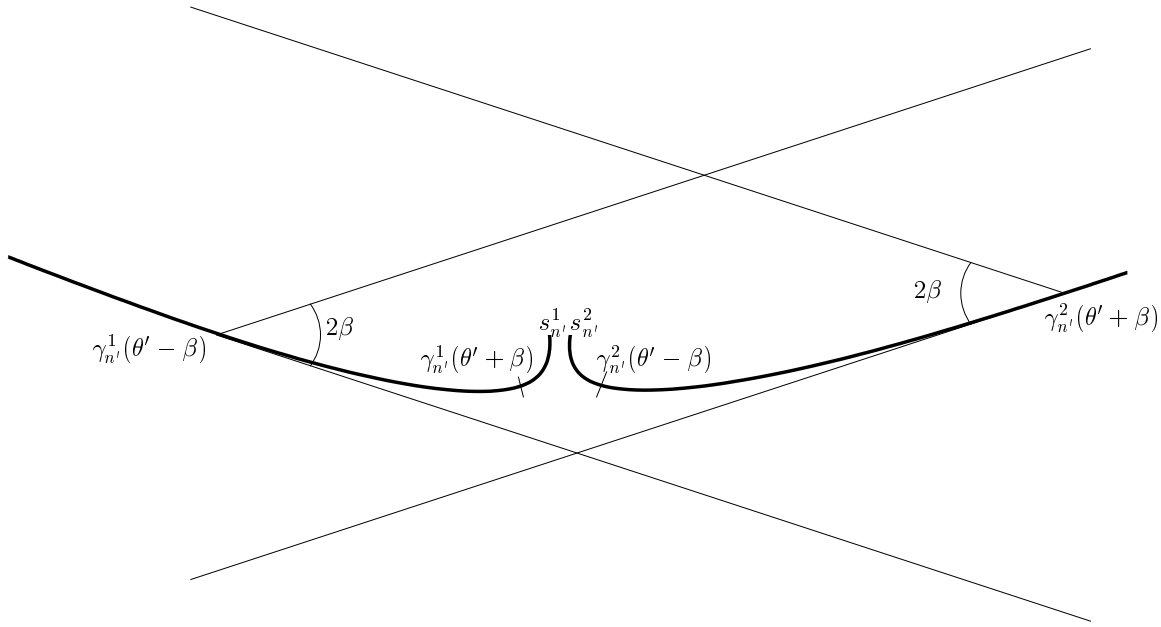

Figure 3.5: the local behaviour near $s^2_{n'}$

sector with vertex $\gamma^2_{n'}(-\frac{\pi}{2}+\beta')$ and angle 2β. Since $\gamma^2_{n'} \to \gamma$ on $(-\frac{\pi}{2}, -\frac{\pi}{2}+\beta)$, these angular sector converges to S, in fact, the angular sectors we have build are just the translations of S by the vector $\overrightarrow{\gamma(-\frac{\pi}{2}+\beta')\gamma^2_{n'}(-\frac{\pi}{2}+\beta')}$ which goes to zero. Besides we know that the distance between $\gamma^2_{n'}(-\frac{\pi}{2}-\beta')$ and $s^2_{n'}$ goes to zero. Then for big n', $s^2_{n'}$ is at a distance less than ε' from S. Then, using that $s^3_{n'}$ goes to s^3_∞, this proves that for big n', $|s^2_{n'}s^3_{n'}| > 2$ and $\frac{\overrightarrow{s^2_{n'}s^3_{n'}}}{|s^2_{n'}s^3_{n'}|} \to (0,-1)$.

Since in each case, there is only one possible limit for $(Per(A_{n'}))$ this proves that $(Per(A_n))$ converges to this limit. Proposition 3.6.3 is then established.

3.6.3 Conclusion

We use the preceding subsections to extend Per to $\widetilde{\mathcal{P}}$ then we get a continuous map on $\widetilde{\mathcal{P}}$. First we have the following remark.

Proposition 3.6.10. *The period map does not vanish on the boundary of* $\widetilde{\mathcal{P}}$

Proof. The only points where Per can vanish are points in the vertices and if $Per(A) = 0$ we have $\overrightarrow{\gamma(\theta-\frac{\pi}{2})\gamma(\theta+\frac{\pi}{2})} = 0$ for some strictly convex curve γ. But γ on $(\theta-\frac{\pi}{2}, \theta+\frac{\pi}{2})$ is a graph over a straight-line then the above vector can not be 0. $\square$

This proves that we can compute the degree of the period map along the boundary of $\widetilde{\mathcal{P}}$.

Theorem 3.6.11. *The degree of the period map along the boundary of* $\widetilde{\mathcal{P}}$ *is* $-(r-1)$

Proof. The edges of $\mathcal{P}$ does not contribute toward the degree. So only the behaviour at the vertices is interesting for the degree. Let us compute the contribution of the vertex P_1; we use the notation of the preceding section. Since the curve γ is strictly convex, the map $\theta \mapsto \frac{\overrightarrow{\gamma(\theta-\frac{\pi}{2})\gamma(\theta+\frac{\pi}{2})}}{|\gamma(\theta-\frac{\pi}{2})\gamma(\theta+\frac{\pi}{2})|}$ is a monotone map. Let θ be in $[0,\alpha]$, the unit vector tangent to γ at $\gamma(\theta)$ is $(\cos\theta, \sin\theta)$ which turns in the clockwise sense, when θ increases. Besides, for $0 < \theta < \alpha$, $\gamma([\theta-\pi/2, \theta+\pi/2])$ is a graph over a straight-line generated by $(\cos\theta, \sin\theta)$. This implies that $Per(P_1,\theta)\cdot(\cos\theta,\sin\theta)$ never vanishes ((P_1,θ) is a point of the boundary of $\widetilde{\mathcal{P}}$); by looking at the behaviour for small θ we have $Per(P_1,\theta)\cdot(\cos\theta,\sin\theta) \geq 0$. Besides, for $\theta = 0$, the basis composed of $(1,0)$ and $Per(P_1,0)$ is an indirect one and, for $\theta = \alpha$, the basis $((\cos\alpha,\sin\alpha), Per(P_1,\alpha)$ is a direct one. Then, when θ increases from 0 to α, $Per(P_1,\theta)$ describes an angle α (since the unit tangent describes this angle) plus π (since the basis composed of the unit tangent at $\gamma(\theta)$ and $Per(P_1,\theta)$ is an orthonormal indirect one for $\theta = 0$ and an orthonormal direct one for $\theta = \alpha$). Since, when we describe $\partial\widetilde{\mathcal{P}}$ in the clockwise sense, θ decreases, the contribution of the vertex P_1 towards the degree is $-\frac{1}{2\pi}(\alpha+\pi)$.

The degree is then $-\dfrac{1}{2\pi}(r\pi + \displaystyle\sum_{i=0}^{r-1}\alpha_i)$ where α_i is the inner angle at the vertex P_i. Then by applying Gauss-Bonnet Theorem to $\mathcal{P}$, $\displaystyle\sum_{i=0}^{r-1}\alpha_i = r\pi - 2\pi$ and then the degree is $-(r-1)$. □

Theorem 3.6.11 then proves that the degree of the period map is non zero and then there exists $A \in \overset{\circ}{\mathcal{P}}$ such that $Per(A) = 0$ and then Theorem 3.2.11 is proved.

3.7 An other example of solution to the Plateau problem at infinity

Theorem 3.2.11 gives a wide class of solutions of the Plateau problem at infinity with genus 1. But it gives no example of polygon which is the the flux polygon of an r-noid of genus 1 but not the flux polygon of an r-noid of genus 0. Corollary 3.7.2 gives such polygons. In fact we study the case where the multi-domain with cone singularity bounded by the polygon is invariant under a "rotation".

Theorem 3.7.1. *Let V be a polygon that bounds a multi-domain with cone singularity (D, Q, ψ). We suppose that D satisfies the hypothesis H and that there exists an isometry h of D such that $\psi \circ h = R \circ \psi$ where R is the rotation in $\mathbb{R}^2$ with center $\psi(Q)$ and angle $\alpha \in (0, 2\pi)$. Then the period vector associated to D vanishes and there exists an Alexandrov-embedded r-noid with genus 1 and horizontal ends having V as flux polygon.*

Proof. From Construction 1 and 3, we have a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ and Theorem 3.3.1 gives us a function u on Ω which is unique up to an additive constant. By construction, the map h can be lifted to Ω to a map $\tilde{h}$ which is an isometry of Ω such that $\varphi_{\psi(Q)} \circ \tilde{h} = R \circ \varphi_{\psi(Q)}$. Since $\tilde{h}$ is an isometry of Ω, $u \circ \tilde{h}$ is a solution of the same Dirichlet problem as u then $u \circ \tilde{h} = u + c$ with $c \in \mathbb{R}$. Then the two equations $\varphi_{\psi(Q)} \circ \tilde{h} = R \circ \varphi_{\psi(Q)}$ and $u \circ \tilde{h} = u + c$ imply that:

$$\tilde{h}^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) = R(\mathrm{d}X_1^*, \mathrm{d}X_2^*)$$

with $\mathrm{d}X_1^*$ and $\mathrm{d}X_2^*$ the 1-forms associated to u as in Subsection 3.5.1. Then, if Γ is a lift of a generator of $\pi_1(D \backslash \{Q\})$, we have:

$$\begin{aligned}
\int_\Gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*) &= \int_{\tilde{h}(\Gamma)} (\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\
&= \int_\Gamma \tilde{h}^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\
&= \int_\Gamma R(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\
&= R\left(\int_\Gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*)\right)
\end{aligned}$$

The first equality is due to the fact that Γ and $\tilde{h}(\Gamma)$ are lifts of two closed pathes that give the same generator of $\pi_1(D \backslash \{Q\})$. Besides R has a unique fixed point, since $\alpha \in (0, 2\pi)$, which is 0. Then the period vector vanishes.

□

We then can give some examples of polygons V that satify this condition; we consider the case where V is a regular convex polygon or a regular star polygon (see [Cox]).

Corollary 3.7.2. *Let q and r be in $\mathbb{N}^*$ such that $\gcd(q, r) = 1$ and $2q < r$. We note, for $i = 1, \ldots, r+1$, $P_i = e^{2(i-1)\sqrt{-1}\frac{q}{r}\pi} \in \mathbb{C} = \mathbb{R}^2$ ($P_1 = P_{r+1}$) (see Figure 3.6). Then there exists an Alexandrov-embedded r-noid of genus 1 and horizontal ends with $(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \ldots, \overrightarrow{P_rP_{r+1}})$ as flux polygon.*

Proof. The idea of the proof is to build a multi-domain with cone singularity bounded by the above polygon and satisfying the hypothesis of Theorem 3.7.1. Let $\mathbb{R}^2$ be identified with $\mathbb{C}$. We note T the set of $(\rho,\theta) \in \mathbb{R}_+ \times [0,\pi]$ such that $\rho e^{i\theta}$ is in the triangle P_1P_2O, where O is the origin. We note D the set of (ρ,θ) in $\mathbb{R}^+ \times [0,2q\pi]$ such that (ρ,θ) is in D if (ρ,θ') is in T where $\theta = n\left(2\frac{q}{r}\pi\right) + \theta'$ is the only writing with $n \in \mathbb{Z}$ and $\theta' \in [0, 2\frac{q}{r}\pi)$ (like an euclidean division of θ by $2\frac{q}{r}\pi$). We remark $\{0\} \times [0,2q\pi] \subset D$ and $(\rho,0) \in D$ iff $(\rho,2q\pi) \in D$ iff $\rho \in [0,1]$. Because of these remarks, we can consider the polar metric on D (we identify all the points $(0,\theta)$ and call the new point Q) and indentify the point $(\rho,0)$ with $(\rho,2q\pi)$ for $0 \leq \rho \leq 1$. Then if we consider on D the map $\psi : (\rho,\theta) \mapsto (\rho\cos\theta, \rho\sin\theta)$, the triplet (D,Q,ψ) is a multi-domain with cone singularity which is bounded by the polygon $(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \ldots, \overrightarrow{P_rP_{r+1}})$. The angle of D is $2q\pi$. Besides on D the map

$$h : (\rho,\theta) \longmapsto \begin{cases} (\rho, \theta + 2\frac{q}{r}\pi) & \text{if } \theta \leq 2q\pi - 2\frac{q}{r}\pi \\ (\rho, \theta - 2\frac{r-1}{r}q\pi) & \text{if } \theta \geq 2q\pi - 2\frac{q}{r}\pi \end{cases}$$

is well defined and is an isometry of D. Besides $\psi \circ h = R \circ \psi$ with R the rotation of center O and angle $2\frac{q}{r}\pi < \pi$. Then we can apply Theorem 3.7.1. □

When $q = 1$, the polygon is a convex regular polygon and the multi-domain D that we build is in fact an immerssed polygonal disk, then Theorem 3.2.11 gives also the result but what we now know is that in the proof of Theorem 3.2.11 we can choose A as the isobarycenter of the polygon.

When $q > 1$, the polygon $(\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}, \ldots, \overrightarrow{P_rP_{r+1}})$ does not bound an immersed polygonal disk. Then we get new examples of polygons which are flux polygons of an Alexandrov-embedded r-noid with genus 1.

Remark 3.7.3. In [JM], L. P. Jorge and W. H. Meeks give Weierstrass data for r-noid with genus 0 and horizontal ends having as flux polygon the polygon studied in Corollary 3.7.2 with $q = 1$. Then Corollary 3.7.2 gives r-noids similar to the Jorge-Meeks examples for the genus 1 case. In fact this examples are known by H. Karcher which gave in [Ka] Weierstrass data for some r-noids which correspond to the ones we have just built. These Weierstrass data are expressed in terms of the Weierstrass $\mathfrak{p}$-function.

3.8 Appendix : Convergence of sequences of solution of (MSE) and line of divergence

The aim of this appendix is to explain some results on the convergence of sequences of solutions of the minimal surface equation.

Let (u_n) be a sequence of solution of (MSE) on a multi-domain D (D has possibly a cone or logarithmic singularity), the first result is then :

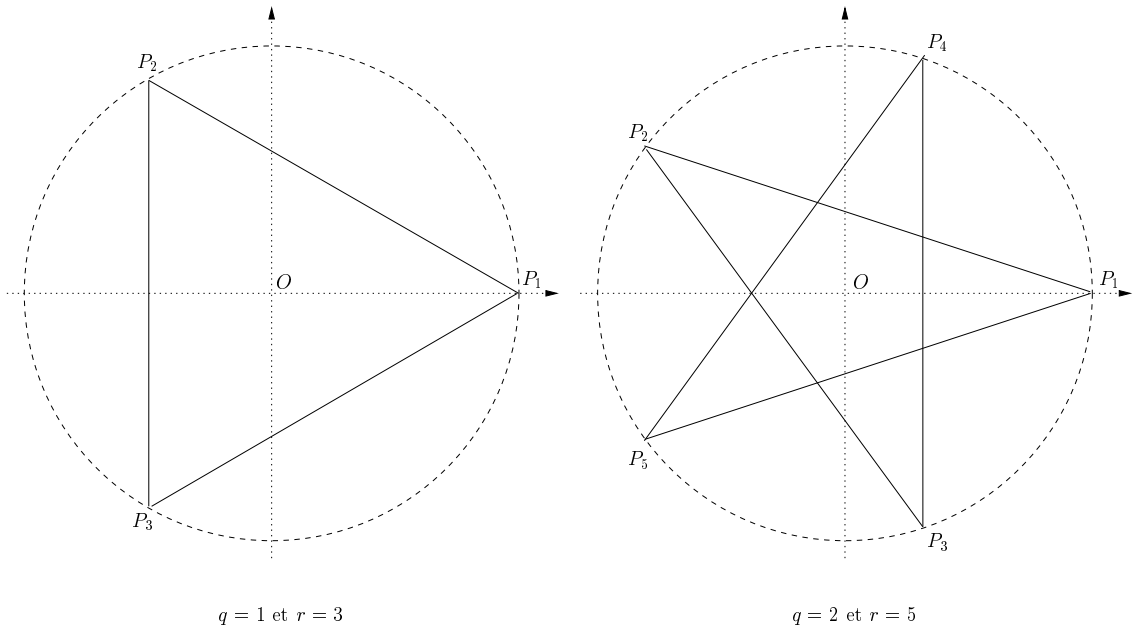

Figure 3.6: Examples of polygons studied in Corollary 3.7.2

Proposition 3.8.1. *If (u_n) is an uniformly bounded sequence on D, there exists a subsequence which converges to a solution of* (MSE) *on D. The convergence is uniform on every compact subset of D.*

This result proves that if the sequence $(||\nabla u_n||)$ is uniformly bounded on every compact subset of D there exists a subsequence $(u_{n'} - u_{n'}(P))$, where P is a point in D, which converges to a solution of (MSE) on D (D is connected). We then can define the domain of convergence of the sequence (v_n) as the set of the points $P \in D$ where the sequence $(||\nabla u_n(P)||)$ is bounded, we note this set $\mathcal{B}(u_n)$. We know that $\mathcal{B}(v_n)$ is an open subset of D such that $(||\nabla u_n||)$ is uniformly bounded on every compact subset of $\mathcal{B}(u_n)$ (see Section 1.4). Then, on every connected component of the domain of convergence, we can make converge a subsequence $(u_{n'} - u_{n'}(P))$. In fact, in our paper we often write that a subsequence $(u_{n'})$ converges instead of $(u_{n'} - u_{n'}(P))$ but, since the value on the boundary are often infinite, this does not matter; when it is necessary we use $(u_{n'} - u_{n'}(P))$. The question is to understand the domain of convergence, in fact we shall understand $D \backslash \mathcal{B}(u_n)$.

Let P be in $D \backslash \mathcal{B}(u_n)$, then for a subsequence $W_{n'} \to +\infty$ and since the normal to the graph at the point over P is

$$N_{n'}(P) = \left(\frac{p_{n'}}{W_{n'}}(P), \frac{q_{n'}}{W_{n'}}(P), -\frac{1}{W_{n'}}(P) \right)$$

(we use euclidean coordinates in a neighborhood of P) we can suppose that $N_{n'}(P) \to N$ where N is a unit horizontal vector. We then have the following result.

Theorem 3.8.2 (Theorem 1.4.3). *Let (D, ψ) be a multi-domain. Let (u_n) be a sequence of solutions of* (MSE) *on D. Let $P \in D$ and N be a unit horizontal vector and L the geodesic of D passing by P and normal to N. If the sequence $(N_n(P))$ converges to N, then $N_n(A)$ converges to N at every point A of L*

Since D is locally isometric to $\mathbb{R}^2$ we can see N as a vector in $\mathbb{R}^2$ and then the vector N is well defined at all the points of D in fact this definition coincides with the parallel transport of N. The proof of this theorem is in chapter 2.

Then in our situation $N_{n'}(P)$ is converging to a unit horizontal vector then $N_{n'}$ converges to this vector along a straight-line L. Such a line L is called a *line of divergence* of the sequence since L must be included in $D \backslash \mathcal{B}(u_n)$. Then the set $D \backslash \mathcal{B}(u_n)$ is an union of geodesics of D and the question is what are the possible lines of divergence: the following lemmas give some tools to answer to this discussion.

First, we observe that the existence of a line of divergence has a consequence on the sequence of 1-forms $\mathrm{d}\Psi_{u_n}$. If $N_n \to N$ along a line of divergence and T is a segment included in this line of divergence then $\displaystyle\int_T \mathrm{d}\Psi_{u_n}$ converges to $|T|$ the length of T if the orientation of T is such that N is the right-hand normal. Since (Ψ_{u_n}) is a sequence of 1-Lipschitz continuous function, we can always take a subsequence and suppose that it has a limit Ψ, then the above remark on $\mathrm{d}\Psi_{u_n}$ allows us to make some calculations on Ψ.

As in Figure 3.4, when we make a picture to explain the convergence of a sequence of solutions of (MSE), we draw the limit normal along the lines of divergence to explain the asymptotical behaviour.

We then have the two following results concerning the lines of divergence and the convergence of sequences of solutions of (MSE)

Lemma 3.8.3 (Proposition 1.4.5). *Let (u_n) be a sequence of solutions of* (MSE) *on $[0,1]^2$ such that, for all n, the function u_n tends to $+\infty$ on $\{1\} \times]0,1[$. Then, no line of divergence of the sequence (u_n) has $(1, \frac{1}{2})$ as end-point.*

Proof. Let us suppose that there exists such a line of divergence L. We note $A = (1,0)$, $B = (1, \frac{1}{2})$ and $C = (1,1)$. Let us consider a point $M \in L$. We suppose that the limit normal along L is such that the basis composed of $\overrightarrow{MB}$ and the limit normal is direct. Since $\mathrm{d}\Psi_{u_n}$ is closed, we then have:

$$\int_{[A,B]} \mathrm{d}\Psi_{u_n} + \int_{[B,M]} \mathrm{d}\Psi_{u_n} + \int_{[M,A]} \mathrm{d}\Psi_{u_n} = 0$$

Since u_n takes the value $+\infty$ on $[A,B]$, this equality proves that:

$$|AB| + \int_{[B,M]} \mathrm{d}\Psi_{u_n} \leq |MA|$$

But, by our choice of limit normal, we have $\int_{[B,M]} \mathrm{d}\Psi_{u_n} \to |BM|$ for a subsequence, then $|AB| + |BM| \leq |MA|$ which contradicts the triangle inequality. If the limit normal is the opposite of the one we consider, we can do the same arguement with the triangle BCM. □

Lemma 3.8.4 (Proposition 1.4.9). *Let (u_n) be a sequence of solutions of* (MSE) *on $[0,1]^2$ such that (u_n) converges in the interior of the square to a solution u and such that we are in one of the following two cases*

- *for all n, the function u_n tends to $+\infty$ on $\{1\}\times]0,1[$ or,*
- *for all n, u_n is the restriction to $[0,1]^2$ of a solution v_n of* (MSE) *defined on $[0,2]\times[0,1]$ and for $y \in]0,1[$ we have $N_n(1,y) \to (1,0,0)$.*

Then, the limit function u tends to $+\infty$ on $\{1\}\times]0,1[$.

Proof. let $0 < \varepsilon < \frac{1}{2}$, for $x \in [0,1]$ we note $A_x = (x,\varepsilon)$ and $B_x = (x, 1-\varepsilon)$. Since $\mathrm{d}\Psi_{u_n}$ is closed, for $x < 1$ we have:

$$\int_{[A_x,B_x]} \mathrm{d}\Psi_{u_n} - \int_{[A_1,B_1]} \mathrm{d}\Psi_{u_n} = \int_{[B_x,B_1]} \mathrm{d}\Psi_{u_n} + \int_{[A_1,A_x]} \mathrm{d}\Psi_{u_n}$$

Then, by passing to the limit and using that $\int_{[A_1,B_1]} \mathrm{d}\Psi_{u_n} \to 1-2\varepsilon$ in the two cases, we obtain that:

$$\left| \int_{[A_x,B_x]} \mathrm{d}\Psi_u - (1-2\varepsilon) \right| \leq 2(1-x)$$

This proves that $\mathrm{d}\Psi_u = \mathrm{d}y$ on $[A_1,B_1]$. Let $x \in]0,1[$ and v be the solution of (MSE) on $A_xB_xB_1A_1$ such that v tends to $+\infty$ on $[A_1,B_1]$ and $v = u$ on the rest of the boundary we shall prove that $u = v$.

If $u \neq v$ there exist $\eta \neq 0$ such that $\Omega = \{u - v > \eta\}$ is non-empty. The boundary of Ω is composed of one part included in the interior of $A_xB_xB_1A_1$ and one part included in $[A_1,B_1]$. Since $\mathrm{d}\Psi_u$ and $\mathrm{d}\Psi_v$ are closed we have

$$\int_{\partial\Omega} \mathrm{d}\Psi_u - \mathrm{d}\Psi_v = 0$$

But on the part of the boundary included in $[A_1,B_1]$ the integral is 0 since $\mathrm{d}\Psi_u = \mathrm{d}y = \mathrm{d}\Psi_v$ and on the part included in the interior of $A_xB_xB_1A_1$ the integral is negative by Lemma 2 in [CK]. Then we have a contradiction and $u = v$ then u goes to $+\infty$ on $[A_1,B_1]$. □

3.9 Appendix : Convex curves in $\mathbb{R}^2$

A convex curve in $\mathbb{R}^2$ is a curve such that its geodesical curvature has always the same sign. A curve is strictly convex if this geodesical curvature is positive or negative.

If a curve $c : s \mapsto c(s) \in \mathbb{R}^2$ is convex, the map with value in $\mathbb{S}^2$ that associates to the parameter s the normal to c at $c(s)$ is a monotone map and, if c is strictly convex, the above map is strictly monotone. This implies that, if $c : I \to \mathbb{R}^2$ is a strictly convex curve, there exists $h : J \to I$ a diffeomorphism such that for $\beta \in J$ the normal to c at the point $c \circ h(\beta)$ is $(\sin\beta, -\cos\beta)$. We then say that c is parametrized by its normal.

Proposition 3.9.1. *Let $c : I \to \mathbb{R}^2$ be a strictly convex curve parametrized by its normal. If $I \subset (\beta_0 + \frac{\pi}{2}, \beta_0 + \frac{3\pi}{2})$, the curve c is a graph over the straight line $y\cos\beta_0 - x\sin\beta_0 = 0$.*

Proof. We suppose that $\beta_0 = 0$. If $c(\beta) = (x(\beta), y(\beta))$, the map $\beta \mapsto x(\beta)$ is a local diffeomorphism since the second coordinate of the normal is always positive. Besides $\beta \mapsto x(\beta)$ is injective: if $x(\beta') = x(\beta'')$ there would exist, by Rolle's Theorem, $\beta \in (\beta', \beta'')$ such that the normal at $c(\beta)$ is horizontal, it is impossible. This proof that $\beta \mapsto x(\beta)$ is a global diffeomorphism then c is a graph over $y = 0$. □

Proposition 3.9.2. *Let c be a strictly convex curve parametrized by its normal on $(\theta - \varepsilon, \theta + \beta]$ with $\beta \leq \frac{\pi}{2}$. Let v be the unit tangent vector to c at $c(\theta)$ and n' the unit vector which is the normal to c at $c(\theta)$ if the curvature is positive or the opposite of the normal if the curvature is negative. Let w be the unit vector $\cos\beta v + \sin\beta n'$. Then the curve $c([\theta, \theta + \beta])$ is in the angular sector delimited by the two half straight-lines with $c(\theta)$ as end-point and respectively generated by v and w (see Figure 3.7).*

Proof. We can suppose that $\theta = 0$, the curvature is negative and $c(0) = (0, 0)$. In this case $c([0, \beta])$ is the graph of a function f over a segment $[0, a]$ by Proposition 3.9.1. By hypotheses, f is a convex function and $f'(0) = 0$. This implies first that $f \geq 0$. If $\beta = \frac{\pi}{2}$ the proposition is proved. If $\beta < \frac{\pi}{2}$ and $c([0, \beta])$ is not in $\{(x, y) |\ x \geq 0,\ y \geq 0,\ y \leq x\tan\beta\}$ there is a parameter $b \in [0, a]$ such that $f(b) = b\tan\beta$ and then there exists $d < b$ such that $f'(d) = \tan\beta$. Since the normal map is injective on $[0, \beta]$, we have $(d, f(d)) = c(\beta)$ which is impossible since $d < a$. The proposition is then proved □

Lemma 3.9.3. *Let c be a strictly convex curve parametrized by its normal on $[\beta - \varepsilon, \beta)$. We suppose that the curve c is included in a compact of $\mathbb{R}^2$. Then $c(t)$ converges when $t \to \beta$.*

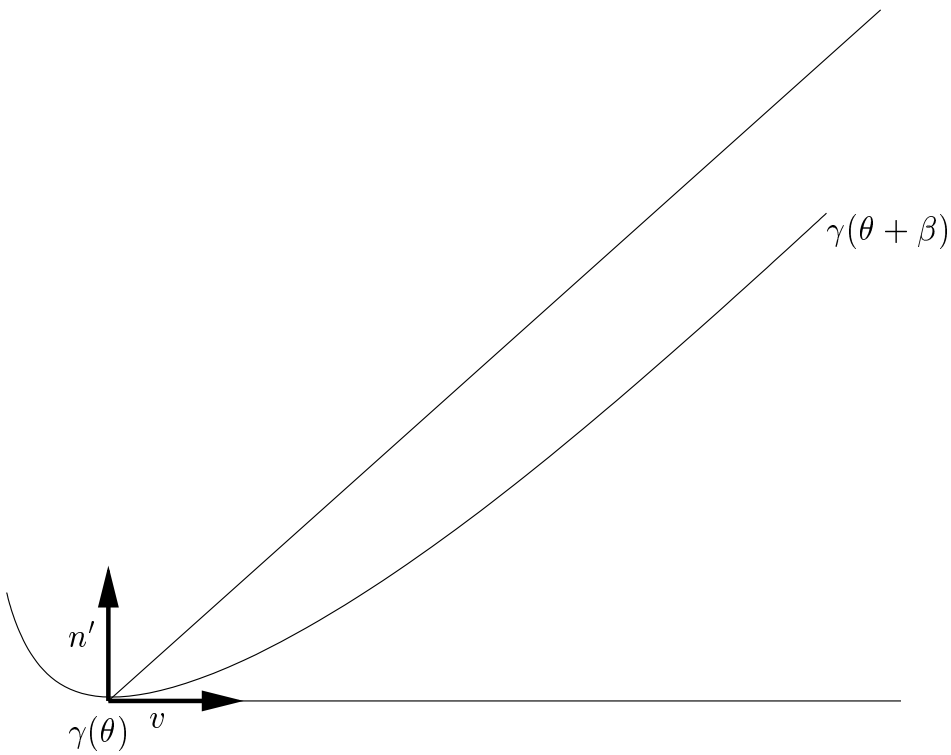

Figure 3.7: the situation of Proposition 3.9.2

Proof. Since we are in a compact part of $\mathbb{R}^2$, it is enough to prove that there is only one cluster point for $c(t)$. Let us suppose that $c(t_n) \to p_1$ and $c(s_n) \to p_2$ and, for every n, $t_n < s_n < t_{n+1}$. We then apply Proposition 3.9.2 on $[t_n, \beta)$ and $[s_n, \beta)$. This proves that p_1 is in an angular sector with vertex $c(s_n)$ and $\beta - s_n$ as angle and p_2 is in an angular sector with vertex $c(t_n)$ and $\beta - t_n$ as angle. Letting n goes to $+\infty$, we obtain that p_1 is in the half straight-line with p_2 as end-point and generated by v (where v is the limit unit tangent vector) and p_2 is in the half straight-line with p_1 as end-point and generated by v. This situation is possible only if $p_1 = p_2$. □

Proposition 3.9.4. *Let (c_n) be a sequence of stricly convex curves parametrized by their normal on $(\theta - \varepsilon, \theta + \varepsilon)$. We suppose that (c_n) converges to $\widetilde{c}$ on $(\theta - \varepsilon, \theta) \cup (\theta, \theta + \varepsilon)$ in the C^1 topology. Then:*

- *we have $\widetilde{c}(t) \to p_1$ when $t \to \theta$, $t < \theta$ and $\widetilde{c}(t) \to p_2$ when $t \to \theta$, $t > \theta$,*
- *as sets, $c_n(\theta - \varepsilon, \theta + \varepsilon)$ converges to $\widetilde{c}(\theta - \varepsilon, \theta) \cup \widetilde{c}(\theta, \theta + \varepsilon) \cup [p_1, p_2]$.*

If $p_1 = p_2$, we have moreover that (c_n) converges to $\widetilde{c}$ (that we extend by $\widetilde{c}(\theta) = p_1$) on $(\theta - \varepsilon, \theta + \varepsilon)$ in the C^1 topology.

Proof. ε is supposed to be small and we choose $\varepsilon' < \varepsilon$. We then apply Proposition 3.9.2 to $c_n(\theta-\varepsilon', \theta+\varepsilon')$ at the points $c_n(\theta-\varepsilon')$ and $c_n(\theta+\varepsilon')$ and we get that, for every n, $c_n(\theta-\varepsilon', \theta+\varepsilon')$ is included in an angular sector of vertex $c_n(\theta-\varepsilon')$ and angle $2\varepsilon'$ and an angular sector of vertex $c_n(\theta+\varepsilon')$ and angle $2\varepsilon'$ (here, we apply Proposition 3.9.2 to the curve c that we cover in the opposite sense). Letting n goes to $+\infty$, we get that $\widetilde{c}((\theta-\varepsilon', \theta) \cup (\theta, \theta+\varepsilon'))$ is included in the intersection of two angular sectors of angle $2\varepsilon'$, one has $\widetilde{c}(\theta-\varepsilon')$ as vertex the other has $\widetilde{c}(\theta+\varepsilon')$ as vertex. The intersection of this two angular sector is a compact; then, by Lemma 3.9.3, p_1 and p_2 exists. We then have also proved that the cluster points of the sequence of curves (c_n) are $\widetilde{c}((\theta-\varepsilon, \theta) \cup (\theta, \theta+\varepsilon))$ and points included in the intersection of the two angular sectors. If we let ε' goes to 0 the intersections of the two sectors converge to the segment $[p_1, p_2]$ which have $(\sin\theta, -\cos\theta)$ as normal. We then must show that all the points of the segment $[p_1, p_2]$ are the limit of a sequence $(c_n(t_n))$.

We suppose now that $\theta = 0$. By Proposition 3.9.1, the curves c_n are graphs over $\{y = 0\}$. Let a and b be the respective first coordinates of $\widetilde{c}(-\varepsilon/2)$ and $\widetilde{c}(\varepsilon/2)$; we suppose $a < b$. Since $c_n \to \widetilde{c}$, we then can ensure that, over the segment $[a, b]$, the curves c_n are graphs for big n. We have $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$; since the segment $[p_1, p_2]$ is horizontal, $y_1 = y_2$. Besides by convergence of (c_n), $a < x_1 \leq x_2 < b$. Let $x \in [x_1, x_2]$; since c_n is a graph over $[a, b]$ for big n, there exist one parameter t_n such that $c_n(t_n)$ has x as first coordinate. Then the only possible cluster point for the sequence $(c_n(t_n))$ is (x, y_1) since every cluster point must have x as first coordinate and be in the segment $[p_1, p_2]$ or in the curve $\widetilde{c}$ but all this set is a graph over $\{y = 0\}$. This then proves that all the points of the segment $[p_1, p_2]$ is in the limit set.

If $p_1 = p_2$, (c_n) converges in the C^1 topology since the normal at the point $\widetilde{c}(\theta) = p_1$ is $(\sin\theta, -\cos\theta)$ by continuity. $\square$

Chapitre 4

Quelques résultats de non-existence pour l'équation des surfaces minimales

Ce chapitre reprend un article ayant le même titre ; il apparaît dans la bibliographie à la référence [Ma3].

Dans cet article, nous démontrons un résultat (Théorème 4.4.2) qui est nécessaire dans la preuve de la Proposition 3.6.3. Il s'agit, en fait, de prouver que certains problèmes de Dirichlet pour (ESM) n'admettent pas de solutions.

4.1 Introduction

Une fonction u définie sur un domaine Ω de $\mathbb{R}^2$ satisfait l'équation des surfaces minimales si :

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{ESM}$$

Cette équation aux dérivées partielles traduit le fait que le graphe de la fonction u est une surface minimale de $\mathbb{R}^3$. A cette équation, on associe le problème de Dirichlet : il s'agit de trouver une solution u de l'équation des surfaces minimales sur un domaine Ω en imposant la valeur de u sur le bord de Ω.

Plusieurs solutions de l'équation (ESM) sont connues ; ainsi sur la demi-bande $\mathbb{R}_+ \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, la fonction $h(x,y) = x \tan y$ est solution de (ESM). Le graphe de h est un morceau d'hélicoïde. On constate que si (x_n, y_n) converge vers un point de $\mathbb{R}_+^* \times \{\frac{\pi}{2}\}$ la suite $(h(x_n, y_n))$ converge vers $+\infty$ et si (x_n, y_n) converge vers un point de $\mathbb{R}_+^* \times \{-\frac{\pi}{2}\}$, on a alors $\lim\limits_{n\to+\infty} h(x_n, y_n) = -\infty$. La fonction h se présente donc

comme une solution du problème de Dirichlet sur la demi-bande avec les valeurs 0 sur $\{0\}\times]-\frac{\pi}{2},\frac{\pi}{2}[$, $+\infty$ sur $\mathbb{R}_+^*\times\{\frac{\pi}{2}\}$ et $-\infty$ sur $\mathbb{R}_+^*\times\{-\frac{\pi}{2}\}$.

Maintenant la question que l'on se pose est de savoir ce qu'il se passe si les deux demi-droites appartenant au bord du domaine ne sont plus parallèles mais forment un angle $2\alpha > 0$. Plus précisément, si Ω est le domaine $\{(x,y)\in[1,+\infty[\times\mathbb{R}|-\tan\alpha<\frac{y}{x}<\tan\alpha\}$, existe-t'il une solution u de (ESM) sur Ω telle que u prenne la valeur 0 sur $\Omega\cap\{x=1\}$, tende vers $+\infty$ sur $\Omega\cap\{x\sin\alpha-y\cos\alpha=0\}$ et tende vers $-\infty$ sur $\Omega\cap\{x\sin\alpha+y\cos\alpha=0\}$?

On connait de nombreux résultats concernant le problème de Dirichlet sur le secteur ainsi, dans [RSE], H. Rosenberg et R. Sa-Earp construisent des solutions pour des données continues sur le bord d'un domaine convexe inclus dans un secteur angulaire. Dans [Ni1], J.C.C. Nitsche se pose la question d'un principe du maximum pour le secteur, un tel principe est démontré dans [RSE]. Ceci implique, d'après le résultat de P. Collin et R. Krust dans [CK], l'unicité des solutions ayant des données bornées sur le bord. On a d'autres résultats d'unicité, ainsi C.-C. Lee, dans [Le], démontre l'unicité en imposant des conditions sur les dérivées au bord.

La réponse à la question posée plus haut est apportée par le Théorème 4.3.1 et cette réponse est négative : une solution de (ESM) prenant la valeur $+\infty$ sur un des cotés d'un secteur et $-\infty$ sur l'autre ne peut exister.

Le Théorème 4.3.1 est le principal résultat de cet article. Il concerne les domaines Ω tels que, hors d'un disque de $\mathbb{R}^2$, $\Omega=\{r\cos\theta,r\sin\theta)\in\mathbb{R}^2\,|\,r>r_0,\ -\alpha<\theta<\alpha\}$ avec $0<\alpha<\pi$. Le Théorème 4.3.1 affirme alors qu'il n'existe pas de solution u de (ESM) sur Ω telle que u tende vers $+\infty$ sur l'un des cotés de ce secteur angulaire et $-\infty$ sur l'autre.

Dans une première partie, nous donnons quelques résultats préliminaires. Parmi ceux-ci, le plus important est la Proposition 4.2.4 qui donne le résultat du Théorème 4.3.1 dans le cas de petits angles α.

La deuxième partie est consacrée à la démonstration du Théorème 4.3.1. L'idée principale de la preuve est que l'existence de fonctions u contredisant le résultat pour de grands angles implique l'existence de fonctions v contredisant le résultat pour de petits angles. On entrerait ainsi en contradiction avec la Proposition 4.2.4.

La troisième partie donne une généralisation de ce résultat dans le cas où l'angle α est supérieur à π.

Dans la suite, si u est une fonction définie sur un domaine Ω on notera $W=\sqrt{1+|\nabla u|^2}$.

4.2 Quelques cas particuliers

4.2.1 Préliminaires

Proposition 4.2.1. *Soit Ω un domaine de $\mathbb{R}^2$ tel qu'une composante connexe de son bord soit une droite L ; on suppose de plus qu'il existe une partie Δ de $\mathbb{R}^2$ telle que $\Omega\backslash\Delta$ soit isométrique à la bande $]-\varepsilon, 0[\times\mathbb{R}$ ($\varepsilon > 0$) où L est la droite $\{0\}\times\mathbb{R}$. Alors il n'existe pas de solution u de l'équation des surfaces minimales qui prend la valeur $+\infty$ sur L (de même pour $-\infty$).*

Démonstration. Supposons, par l'absurde, que u soit une solution de l'équation des surfaces minimales sur $]-\varepsilon, 0[\times\mathbb{R}$ valant $+\infty$ sur L. Alors le graphe de u contredit le principe du demi-espace, plus précisément, la démonstration du Théorème 1 dans [HM] contredit l'existence du graphe de u. □

Par la suite nous allons avoir besoin d'estimées sur les dérivées de solutions de l'équation des surfaces minimales. On a alors le résultat suivant.

Lemme 4.2.2. *Soient Ω un domaine convexe du plan $\mathbb{R}^2$ et u une solution de l'équation des surfaces minimales sur Ω. On considère p un point de Ω et d la distance du point p au bord de Ω. On note $q \in \partial\Omega$ un point qui réalise cette distance, n le vecteur unitaire $\frac{\overrightarrow{pq}}{d}$ et n' un vecteur unitaire normal à n. On note maintenant Σ le graphe de u et P le point de Σ au dessus de p. Soit r la distance le long de la surface Σ du point P au bord de Σ. Alors si le rapport $\frac{d}{r}$ est inférieur à $\frac{1}{8}$, on a au point p :*

$$\frac{|n\cdot\nabla u(p)|}{W} \geq 1 - 4\frac{d^2}{r^2}$$
$$\frac{|n'\cdot\nabla u(p)|}{W} \leq 2\sqrt{2}\frac{d}{r}$$

Il s'agit du Lemme 1 dans [JS].

4.2.2 Le cas du secteur angulaire

Pour $0 < \alpha < \pi$, on considère le domaine de $\mathbb{R}^2$:

$$D(\alpha) = \{(r\cos\theta, r\sin\theta)\}_{r\in\mathbb{R}_+, \theta\in[-\alpha,\alpha]}$$

Pour tout θ, on note $L(\theta) = \{(r\cos\theta, r\sin\theta)\}_{r\in\mathbb{R}_+^*}$; on a alors la proposition suivante :

Proposition 4.2.3. *Soit $\alpha \in]0,\pi[$. Il n'existe pas de solution u de l'équation des surfaces minimales sur $D(\alpha)$ telle que u prenne la valeur $+\infty$ sur $L(\alpha)$ et $-\infty$ sur $L(-\alpha)$.*

Démonstration. Supposons qu'une telle solution u existe. Alors d'après le Théorème 2.3.2 et la Remarque 2.3.3, on peut prolonger le graphe de la fonction u par symétrie par rapport à la droite verticale de $\mathbb{R}^3$: $L = \{x = 0\} \cap \{y = 0\}$. La surface Σ ainsi obtenue est une surface minimale complète simplement connexe ayant un nombre fini de points de branchement le long de la droite verticale de symétrie L. Comme Σ est un graphe au dessus de $D(\alpha)$, les points de Σ ayant une normale horizontale sont les points de L.

D'après le Lemme 3.4.5, la courbe $r \mapsto (r\cos\theta, r\sin\theta, u(r\cos\theta, r\sin\theta))$ appartenant à Σ a pour extrémité, lorsque r tend vers 0, un point de L où la normale vaut $(\sin\theta, -\cos\theta, 0)$ ou $(-\sin\theta, \cos\theta, 0)$. De plus, tout point de L est l'extrémité d'une telle courbe. Ceci implique que pour $\theta \in]\alpha - \pi, \pi - \alpha[$ l'un des vecteurs suivant $(\sin\theta, -\cos\theta, 0)$ ou $(-\sin\theta, \cos\theta, 0)$ n'est jamais la normale à Σ, en effet, ces deux vecteurs ne peuvent être la normale à Σ qu'en l'extrémité de $r \mapsto (r\cos\theta, r\sin\theta, u(r\cos\theta, r\sin\theta))$; ceci ne fait qu'un seul point. On sait ainsi que la normale à Σ omet un nombre infini de points de la sphère, et même un segment dans la sphère par continuité de la normale. Or ceci est impossible pour une surface minimale complète simplement connexe ayant un nombre fini de points de branchement (Théorème 8.2 dans [Os2]). □

4.2.3 Le cas des petits angles

Soit $\alpha \in]0, \frac{\pi}{2}[$ et a un nombre réel positif. On note alors $D_a(\alpha) = D(\alpha) \cap \{x \geq a\}$. Par abus de notation, on continue à noter $L(\theta)$ l'intersection de $L(\theta)$ avec $D_a(\alpha)$.

Proposition 4.2.4. *Il existe $\alpha_0 \in]0, \frac{\pi}{2}[$ tel que, si $0 < \alpha < \alpha_0$ et $a \in \mathbb{R}_+$, il n'existe pas de solution u de l'équation des surfaces minimales sur $D_a(\alpha)$ telle que u prenne la valeur $+\infty$ sur $L(\alpha)$ et $-\infty$ sur $L(-\alpha)$.*

Démonstration. Supposons qu'une telle solution u existe. On considère p un point de $D_a(\alpha)$ de coordonnées (x, y). Soit $x_0 > a$ tel que $x_0 \sin\alpha = x_0 - a$. Si le point p vérifie $x \geq x_0$, le point q du bord de $D_a(\alpha)$ qui réalise la distance de p au bord de $D_a(\alpha)$ n'est pas situé sur $\{x = a\}$ mais sur $L(\alpha)$ ou $L(-\alpha)$ et on a $|pq| \leq x\sin\alpha$. Le long du graphe de u la distance de P, le point de coordonnées $(x, y, u(x, y))$, au bord du graphe est minorée par $x - a$. Le rapport entre ces deux longueurs est donc majoré par

$$\frac{x\sin\alpha}{x-a} = \sin\alpha \frac{1}{1 - \frac{a}{x}}$$

Posons $x_1 = \max\{x_0, 10a\}$, alors, si p vérifie $x \geq x_1$, le rapport entre ces deux longueurs est majoré par $\frac{10}{9}\sin\alpha$. Notons $\alpha_1 \in [0, \frac{\pi}{2}]$ le nombre vérifiant $\frac{10}{9}\sin\alpha_1 = \frac{1}{8}$. Donc si $\alpha < \alpha_1$ on peut appliquer le Lemme 4.2.2 au point p. Le vecteur $\dfrac{\overrightarrow{pq}}{|pq|}$ est $(-\sin\alpha, \cos\alpha)$ ou $(-\sin\alpha, -\cos\alpha)$. On en déduit alors, par le Lemme 4.2.2, que

$$\begin{aligned}\frac{|u_y|}{W} &\geq (1 - 4\left(\frac{10}{9}\right)^2 \sin^2\alpha)\cos\alpha - 2\sqrt{2}\frac{10}{9}\sin^2\alpha \\ &\geq \cos\alpha - \sin^2\alpha\left(4\left(\frac{10}{9}\right)^2\cos\alpha + 2\sqrt{2}\frac{10}{9}\right)\end{aligned}$$

Il existe donc $\alpha_0 < \alpha_1$ tel que si $\alpha \leq \alpha_0$ on a $|u_y(p)| \neq 0$, pour tout point $p = (x, y)$ avec $x \geq x_1$, et donc $u_y > 0$ (On remarque que α_0 ne dépend pas de a).

On se place maintenant avec $\alpha \leq \alpha_0$ et sur $D_a(\alpha) \cap \{x \geq x_1\}$ on pose $\Phi(x, y) = (x, u(x, y))$. On définit ainsi une application à valeurs dans $[x_1, +\infty[\times\mathbb{R}$. La différentielle de Φ est

$$\mathrm{d}\Phi(x, y) = \begin{pmatrix} 1 & u_x \\ 0 & u_y \end{pmatrix}$$

dont le jacobien est non nul puisque sa valeur est $u_y \neq 0$. Φ est donc un difféomorphisme local ; en fait, il s'agit d'un difféomorphisme global de $D_a(\alpha) \cap \{x \geq x_1\}$ sur $[x_1, +\infty[\times\mathbb{R}$ car u croit strictement de $-\infty$ à $+\infty$ le long des courbes $x = \text{cste}$.

On voit ainsi que le graphe de u au dessus de $D_a(\alpha) \cap \{x \geq x_1\}$ peut être vu comme le graphe d'une fonction v solution de (ESM) au dessus du demi-plan vertical $\{y = 0, x \geq x_1\}$. La fonction v est alors bornée sur $\{x = x_1\}$; donc, d'après le résultat de P. Collin et R. Krust (Théorème 3.4 dans [CK]), il existe d_1 et d_2 tels que l'on ait $cx + d_1 \leq v(x, z) \leq cx + d_2$, ce qui est impossible de par la forme de $D_a(\alpha)$. □

4.3 Le résultat général

Nous avons maintenant un résultat similaire à celui de la Proposition 4.2.4 mais sans limite sur l'angle α.

Théorème 4.3.1. *Soient Ω un domaine de $\mathbb{R}^2$ tel qu'il existe un compact K de $\mathbb{R}^2$ et $\alpha \in]0, \pi[$ vérifiant $\Omega\backslash K = D(\alpha)\backslash D(0, r)$ où $D(0, r)$ désigne le disque de centre l'origine et de rayon r. Alors, il n'existe pas de solution u de l'équation des surfaces minimales sur Ω telle que u prenne la valeur $+\infty$ sur $L(\alpha)$ et $-\infty$ sur $L(-\alpha)$.*

Démonstration. Supposons qu'une telle solution existe, nous avons donc une fonction u sur $D(\alpha)\backslash D(0, 1)$ (on suppose $r = 1$) qui prend la valeur $+\infty$ (resp. $-\infty$) sur

$L(\alpha)$ (resp. $L(-\alpha)$). L'idée de la preuve est de se servir de u pour construire une fonction qui contredirait la Proposition 4.2.4.

Soit $\beta > 0$ tel que β soit inférieur à α et à $\frac{\pi}{4}$. On note alors A_r, B_r, C_r et D_r les points de $D(\alpha)$ de coordonnées polaires respectives (r, α), $(r, \alpha - \beta)$, $(r, -\alpha + \beta)$ et $(r, -\alpha)$. On a alors le résultat suivant

Lemme 4.3.2. *Il existe $\beta \leq \min\{\alpha, \frac{\pi}{4}, 2\alpha_0\}$ (α_0 est l'angle donné par la Proposition 4.2.4) et $r_0 > 1$ tels que la fonction u soit croissante*

- *le long des deux arcs de cercle de centre l'origine joignant B_r à A_r et D_r à C_r et*
- *le long des deux segments $[B_r, A_r]$ et $[D_r, C_r]$*

pour $r > r_0$.

Démonstration. Par symétrie du problème, on ne considérera que le segment $[B_r, A_r]$ et l'arc de cercle joignant B_r à A_r. Soit p un point de coordonnées polaires (ρ, θ) avec $\theta \in [\alpha - \beta, \alpha]$ et $\rho \geq \frac{2}{2-\sqrt{2}}$; alors le point q du bord de $D(\alpha) \backslash D(0,1)$ qui réalise la distance de p au bord appartient à $L(\alpha)$. La distance $|pq|$ est donc majorée par $\rho \sin \beta$ et la distance le long du graphe de u de P, le point du graphe situé au dessus de p, au bord est minorée par $\rho - 1$. Le rapport entre ces deux longueurs est donc majoré par $\sin \beta \frac{\rho}{\rho - 1}$. Ainsi, pour $\rho \geq 10$, il est majoré par $\frac{10}{9} \sin \beta$; soit β_1 tel que $\frac{10}{9} \sin \beta_1 = \frac{1}{8}$ alors pour $\beta < \beta_1$ ce rapport est inférieur à $\frac{1}{8}$.

On suppose $\beta < \beta_1$. On définit r_0 tel que $r_0 \cos \frac{\pi}{8} = 10$. Soit $r \geq r_0$, on suppose maintenant que p appartienne au segment $[B_r, A_r]$; on a alors $\rho \geq 10$ par définition de r_0. Le vecteur $\frac{\overrightarrow{pq}}{|pq|}$ est $(-\sin \alpha, \cos \alpha)$ et le vecteur unitaire directeur de $[B_r, A_r]$ est $n = (-\sin(\alpha - \frac{\beta}{2}), \cos(\alpha - \frac{\beta}{2}))$. D'après le Lemme 4.2.2, on a donc

$$\frac{|\nabla u \cdot n|}{W}(p) \geq (1 - 4(\frac{10}{9} \sin \beta)^2)(-\sin \alpha, \cos \alpha) \cdot n - 2\sqrt{2}\frac{10}{9} \sin \beta$$

On constate que losque β tend vers 0 le minorant tend vers 1. Il existe donc $\beta_2 \in]0, \beta_1[$ tel que pour tout $\beta < \beta_2$: $\frac{|\nabla u \cdot n|}{W}(p) > 0$.

Si maintenant p est un point de l'arc de cercle joignant B_r à A_r, le vecteur unitaire tangent à l'arc de cercle en p est $n'(\theta) = (-\sin \theta, \cos \theta)$. D'après le Lemme 4.2.2, on a donc

$$\begin{aligned}\frac{|\nabla u \cdot n'(\theta)|}{W}(p) &\geq (1 - 4(\frac{10}{9} \sin \beta)^2)(-\sin \alpha, \cos \alpha) \cdot n'(\theta) - 2\sqrt{2}\frac{10}{9} \sin \beta \\ &\geq (1 - 4(\frac{10}{9} \sin \beta)^2)(-\sin \alpha, \cos \alpha) \cdot n'(\beta) - 2\sqrt{2}\frac{10}{9} \sin \beta\end{aligned}$$

De même que précédemment, le minorant tend vers 1 lorsque β tend vers 0. Il existe donc $\beta_3 \in]0, \beta_2[$ tel que pour tout $\beta < \beta_3$: $\frac{|\nabla u \cdot n'(\theta)|}{W}(p) > 0$. Alors tout $\beta < \min\{\beta_3, 2\alpha_0\}$ répond au lemme. □

On fixe maintenant β et r_0 tels que le Lemme 4.3.2 soit satisfait. On a alors le résultat suivant :

Lemme 4.3.3. *Il existe $r_1 \geq r_0$ tel que, si $p \in D(\alpha)$ est le point de coordonnées polaires (r, θ) avec $r \geq r_1$, on a $\dfrac{\partial u}{\partial \theta}(p) > 0$.*

Démonstration. Tout d'abord le Lemme 4.3.2 nous dit que la dérivée est positive si $\theta \in]-\alpha, -\alpha+\beta] \cup [\alpha-\beta, \alpha[$. Maintenant si le lemme n'est pas vrai, il existe une suite de points p_n, de coordonnées polaires (r_n, θ_n), telle que r_n tend vers $+\infty$ et $\dfrac{\partial u}{\partial \theta}(p_n) \leq 0$. On a $\theta_n \in [-\alpha+\beta, \alpha-\beta]$ et, quitte à extraire, on peut supposer que θ_n tend vers $\theta_\infty \in [-\alpha+\beta, \alpha-\beta]$. On définit alors Ω_n le domaine de $\mathbb{R}^2$: $\Omega_n = \{(r\cos\theta, r\sin\theta) |\ r \geq {r_n}^{-1},\ \theta \in]-\alpha+\theta_\infty-\theta_n, \alpha+\theta_\infty-\theta_n[\}$. Sur Ω_n on définit, en coordonnées polaires, la fonction v_n : $v_n(r,\theta) = {r_n}^{-1} u(r_n r, \theta+\theta_\infty-\theta_n)$. v_n est une solution de l'équation des surfaces minimales qui vaut $+\infty$ sur $L(\alpha+\theta_\infty-\theta_n)$ et $-\infty$ sur $L(-\alpha+\theta_\infty-\theta_n)$. L'hypothèse sur u au point p_n se traduit par $\dfrac{\partial v_n}{\partial \theta}(1, \theta_\infty) \leq 0$.

On va maintenant étudier la limite de la suite (v_n) ; pour cela on va étudier les lignes de divergence de (v_n) (voir la section 1.4). Le domaine limite est $D(\alpha)$. Comme v_n prend des valeurs infinies sur $L(\alpha+\theta_\infty-\theta_n)$ et $L(-\alpha+\theta_\infty-\theta_n)$, les seules lignes de divergence possibles sont les droites incluses dans $D(\alpha)$ et les demi-droites ayant l'origine pour extrémité ; on utilise ici le Lemme 1.4.5.

Considérons $L(\gamma) \subset D(\alpha)$ et supposons que $L(\gamma)$ soit une ligne de divergence. On considère Ψ_n la fonction conjuguée de v_n normalisée (en coordonnées polaires) par $\Psi_n({r_n}^{-1}, \theta_\infty) = 0$. Alors en choisissant la bonne sous-suite, on a Ψ_n tend vers Ψ avec Ψ qui vérifie : $\Psi = 0$ en l'origine, $\Psi(\rho, \alpha) = -\rho$, $\Psi(\rho, -\alpha) = -\rho$ et $\Psi(\rho, \gamma) = \pm\rho$ suivant la valeur de la normale limite. Mais si $\gamma \geq 0$, comme Ψ est 1-lipschitzienne on a $|\Psi(\rho,\alpha) - \Psi(\rho,\gamma)| < 2\rho$; donc $\Psi(\rho,\gamma) = -\rho$ et la normale limite le long de $L(\gamma)$ est $(-\sin\gamma, \cos\gamma)$; de même si $\gamma \leq 0$. Ainsi pour $L(\gamma)$ il n'y a qu'une possibilité pour la normale limite.

Supposons que la droite $L \subset D(\alpha)$ soit une ligne de divergence ; on a alors $N_{n'}$ qui converge vers une normale à L le long de L. Soit L' la parallèle à L passant par l'origine. On s'intéresse maintenant à la suite $(v_{n'})$. Si une ligne de divergence pour $(v_{n'})$ existe dans la bande comprise entre L et L' ce doit être une droite parallèle à L. Si un point de la bande comprise entre L et L' appartient au domaine de convergence de $(v_{n'})$, la composante connexe de $\mathcal{B}(v_{n'})$ qui contient ce point vérifie alors les hypothèses de la Proposition 4.2.1 et la limite de la suite $v_{n'}$ sur cette composante prend une valeur infinie sur une droite parallèle à L ; ceci est impossible par la Proposition 4.2.1. Ainsi en tout point de la bande comprise entre L et L' passe une ligne de divergence ; de plus en tout point de cette bande $N_{n'}$ converge vers le même vecteur unitaire que le long de L, ceci est entre autre vrai le long de L' sauf pour l'origine. Ainsi il existe γ tel que $L(\gamma)$ et $L(\gamma-\pi)$ soient deux lignes

de divergence avec la même normale limite ce qui est impossible d'après le résultat du paragraphe précédent. Donc L ne peut être une ligne de divergence et les $L(\gamma)$ sont les seules lignes de divergence possibles.

Si le point de coordonnées polaires $(1, \theta_\infty)$ est un point d'une ligne de divergence, la normale limite en ce point est $(-\sin\theta_\infty, \cos\theta_\infty)$, ce qui contredit que $\dfrac{\partial v_n}{\partial \theta}(1, \theta_\infty) \leq 0$. Ce point appartient donc au domaine de convergence, or la composante connexe du domaine de convergence le contenant est un secteur angulaire compris entre $L(\gamma_1)$ et $L(\gamma_2)$ (avec $-\alpha \leq \gamma_1 < \gamma_2 \leq \alpha$) et la limite v de la suite (v_n) sur cette composante vaut $+\infty$ sur $L(\gamma_2)$ et $-\infty$ sur $L(\gamma_1)$ d'après les normales limites de long de ces deux lignes de divergence (Lemme 1.4.9). Une telle solution v n'existe pas par la Proposition 4.2.3. On vient ainsi d'établir une contradiction, le lemme est donc démontré. □

Nous pouvons maintenant finir la preuve du théorème. On considère le domaine $D_{r_1 \sin\frac{\beta}{2}}(\frac{\beta}{2})$ (avec β donné par le Lemme 4.3.2 et r_1 donné par le Lemme 4.3.3) et on note E_r le point de coordonnées polaires $(r, \frac{\beta}{2})$ et F_r le point de coordonnées polaires $(r, -\frac{\beta}{2})$. Soit n un entier positif, les polygones $A_{r_1}B_{r_1}B_{r_1+n}A_{r_1+n}$, $C_{r_1}D_{r_1}D_{r_1+n}C_{r_1+n}$ et $E_{r_1}F_{r_1}F_{r_1+n}E_{r_1+n}$ se déduisent alors les uns des autres par des déplacements et peuvent être ainsi identifiés. En utilisant ces identifications on construit sur $E_{r_1}F_{r_1}F_{r_1+n}E_{r_1+n}$ la solution w_n de l'équation des surfaces minimales valant $u_{|[A_{r_1},B_{r_1}]}$ sur $[E_{r_1}, F_{r_1}]$, $-\infty$ sur $[F_{r_1}, F_{r_1+n}]$, $u_{|[A_{r_1+n},B_{r_1+n}]}$ sur $[E_{r_1+n}, F_{r_1+n}]$ et $+\infty$ sur $[E_{r_1}, E_{r_1+n}]$ (w_n existe d'après le Théorème 1.3.5). D'après le principe du maximum et le Lemme 4.3.3, on a alors :

$$u_{|C_{r_1}D_{r_1}D_{r_1+n}C_{r_1+n}} \leq w_n \leq u_{|A_{r_1}B_{r_1}B_{r_1+n}A_{r_1+n}}$$

Ainsi (w_n) est uniformément bornée sur tout compact de $D_{r_1 \sin\frac{\beta}{2}}(\frac{\beta}{2})$. Donc une sous suite $(w_{n'})$ converge sur $D_{r_1 \sin\frac{\beta}{2}}(\frac{\beta}{2})$ vers une solution w de (ESM) prenant la valeur $+\infty$ sur $L(\frac{\beta}{2})$ et $-\infty$ sur $L(-\frac{\beta}{2})$; ainsi w contredit la Proposition 4.2.4 et le théorème est démontré. □

4.4 Une généralisation au cas $\alpha \geq \pi$

On souhaite pouvoir donner un sens au problème d'existence lorsque α est plus grand que π, et voir si le Théorème 4.3.1 reste vrai dans ce cas.

Pour cela posons $D(\alpha_1, \alpha_2) = \{(r, \theta) |\ r \geq 0, \alpha_1 \leq \theta \leq \alpha_2\}$ muni de la métrique polaire $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ (tous les points de la forme $(0, \theta)$ sont identifiés, on appelle 0 ce nouveau point : c'est le sommet de $D(\alpha_1, \alpha_2)$). Soit φ l'application définie sur $D(\alpha_1, \alpha_2)$ par $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, φ est alors une isométrie locale

de $D(\alpha_1, \alpha_2)$ sur $\mathbb{R}^2$. Le couple $(D(\alpha_1, \alpha_2), \varphi)$ est un multi-domaine (voir chapitre 2). Pour $\alpha > 0$, on note $D(\alpha) = D(-\alpha, \alpha)$; on constate que, pour $\alpha < \pi$, cette définition est équivalente à celle de la section 4.2. Comme précédemment, on note $L(\theta) = \{(r, \theta) | \ r > 0\}$.

Pour pouvoir généraliser le Théorème 4.3.1, nous allons devoir rajouter des hypothèses.

4.4.1 Le cas du secteur angulaire

Dans cette partie, nous généralisons la Proposition 4.2.3, mais pour cela, nous devons rajouter une hypothèse sur Ψ_u.

Proposition 4.4.1. *Soit $\alpha \geq \pi$. Il n'existe pas de solution u de l'équation des surfaces minimales sur $D(\alpha)$ telle que u prenne la valeur $+\infty$ sur $L(\alpha)$, $-\infty$ sur $L(-\alpha)$ et telle que $\Psi_u \leq 0$ (Ψ_u est normalisée par $\Psi_u(0) = 0$).*

Démonstration. Supposons qu'une telle solution u existe. De même que dans la preuve de la Proposition 4.2.3, on peut prolonger le graphe de u par symétrie par rapport à l'axe vertical $L = \{x = 0, y = 0\}$. La surface Σ ainsi obtenue est une surface minimale complète de $\mathbb{R}^3$, elle est simplement connexe et régulière car $\Psi_u \leq 0$. On sait de plus que la normale à Σ est horizontale uniquement le long de L. Par le Lemme 3.4.5, le vecteur horizontal $(\sin\theta, -\cos\theta, 0)$ ne peut être la normale à Σ qu'en l'extrémité de la courbe incluse dans Σ : $r \mapsto (r\cos(\theta+k\pi), r\sin(\theta+k\pi), u(r, \theta+k\pi))$ avec $-\alpha < \theta + k\pi < \alpha$. Ainsi tout vecteur unitaire horizontal n'est normal à Σ qu'en un nombre fini de point. Ces vecteurs s'identifient à l'équateur de la sphère unité, or l'équateur est un ensemble de capacité logarithmique non nulle, Σ est donc de courbure totale finie d'après le Corollaire du Théorème 2.3 dans [Os1]. Comme Σ est de courbure totale finie, Σ est conformément équivalent au plan complexe $\mathbb{C}$ (Théorème 9.1 dans [Os2]). De plus, on peut supposer que la symétrie par rapport à L corresponde sur $\mathbb{C}$ à l'application $\zeta \mapsto \bar{\zeta}$. Sur $\mathbb{C}$, l'application de Gauss g est définie ; g est une application méromorphe. Comme la normale le long de L est horizontale, on a $|g(\zeta)| = 1$ pour $\zeta \in \mathbb{R}$. De plus la symétrie par rapport à L nous dit que $g(\bar{\zeta}) = \dfrac{1}{\overline{g(\zeta)}}$. Comme la courbure totale est finie on sait enfin que g est une fraction rationnelle, en regroupant tous ces arguments on a :

$$g(\zeta) = e^{i\beta} \frac{\prod_{j=1}^{n} (\zeta - \zeta_j)}{\prod_{j=1}^{n} (\zeta - \bar{\zeta}_j)}$$

avec $\beta \in [-3\pi/2, \pi/2[$.

Supposons tout d'abord que $\alpha = \pi$ et $\beta = -3\pi/2$. On considère la suite de points du graphe de u de coordonnées $(0, 0, n)$. On sait que la normale au graphe en ces points converge vers $(0, -1, 0)$ lorsque n tend vers $+\infty$ d'après les valeurs de u sur le bord du domaine. Or si $\zeta_n \in \mathbb{C}$ est le point correspondant à $(0, 0, n)$, ζ_n diverge dans $\mathbb{C}$ et donc $g(\zeta_n)$ tend vers $e^{i\beta} = i$ ce qui nous donne que la valeur limite de la normale est $(0, 1, 0)$; on a donc une contradiction et on suppose, dans la suite, que $\alpha \neq \pi$ ou $\beta \neq -3\pi/2$.

On définit maintenant, sur $D(\alpha)$, la suite (u_n) des dilatés de u par $u_n(r, \theta) = \frac{1}{n}u(nr, \theta)$. On note Ψ_n la fonction conjuguée de u_n normalisée par $\Psi_n(0) = 0$; on a alors $\Psi_n \leq 0$. On s'intéresse à la limite de la normale au graphe de u_n au dessus du point $(1, \beta + \pi/2) \in D(\alpha)$ (l'appartenance à $D(\alpha)$ est due à l'hypothèse sur α et β). Cette normale est égale à la normale au graphe de u en P_n le point situé au dessus de $p_n = (n, \beta + \pi/2)$. P_n diverge dans Σ donc, si ζ_n est le point de $\mathbb{C}$ correspondant, on a $g(\zeta_n)$ tend vers $e^{i\beta}$. Donc la normale au graphe de u_n au dessus de $(1, \beta + \pi/2)$ converge vers $(\cos\beta, \sin\beta, 0)$. On a donc une ligne de divergence qui est $L(\beta + \pi/2)$ et $\Psi_n(1, \beta + \pi/2)$ tend vers 1 ce qui est impossible puisque $\Psi_n \leq 0$. On vient donc de prouver que u n'existe pas. □

4.4.2 Le cas général

On a maintenant la généralisation suivante du Théorème 4.3.1.

Théorème 4.4.2. *Soit Ω un multi-domaine tel qu'il existe une partie $K \subset \Omega$ et $\alpha \geq \pi$ tels que $\Omega \backslash K$ est isométrique à $D(\alpha) \backslash D(0, r)$ où $D(0, r)$ est l'ensemble des points de $D(\alpha)$ à distance inférieure à r de 0. Alors il n'existe pas de solution u de l'équation des surfaces minimales telle que u prenne la valeur $+\infty$ sur la demi-droite correspondant à $L(\alpha)$ et $-\infty$ sur la demi-droite correspondant à $L(-\alpha)$ et $\Psi_u \leq 0$ sur $\Omega \backslash K$ pour un choix de normalisation pour Ψ_u.*

Démonstration. La preuve est identique à celle du Théorème 4.3.1 ; les seules modifications apparaissent dans la preuve de l'équivalent du Lemme 4.3.3. On va donc réécrire la preuve du Lemme 4.3.3 dans ce nouveau cadre.

On suppose à nouveau que $r = 1$. Si le lemme n'est pas vrai on a une suite $p_n = (r_n, \theta_n) \in D(\alpha)$ avec $r_n \to +\infty$, $\theta_n \to \theta_\infty \in]-\alpha, \alpha[$ et $\dfrac{\partial u}{\partial \theta}(p_n) < 0$. On définit alors sur $D(-\alpha + \theta_\infty - \theta_n, \alpha + \theta_\infty - \theta_n) \backslash D(0, {r_n}^{-1})$ la fonction v_n par $v_n(r, \theta) = {r_n}^{-1} u(r_n r, \theta - \theta_n + \theta_\infty)$. v_n est une solution de (ESM) qui vaut $+\infty$ sur $L(\alpha + \theta_\infty - \theta_n)$ et $-\infty$ sur $L(-\alpha + \theta_\infty - \theta_n)$. On a alors $\dfrac{\partial v_n}{\partial \theta}(1, \theta_\infty) < 0$. Si on pose $\Psi_n = \Psi_{v_n}$ normalisée par $\Psi_n({r_n}^{-1}, \theta_\infty) = {r_n}^{-1}\Psi_u(1, \theta_n)$, on a $\Psi_n \leq 0$ et $\Psi_n({r_n}^{-1}, \theta_\infty)$ tend vers 0.

On étudie alors les lignes de divergence de la suite (v_n) sur le multi-domaine limite qui est $D(\alpha)$. Comme plus haut, les seules lignes de divergence sont les $L(\gamma)$

ou les droites incluses dans $D(\alpha)$. Si $L(\gamma)$ est une ligne de divergence on sait que pour une sous suite $\Psi_{n'} \to \Psi$ avec $\Psi(0) = 0$ et $\Psi(\rho, \gamma) = \pm\rho$. Or $\Psi \leq 0$ donc $\Psi(\rho, \gamma) = -\rho$ et la normale limite le long de $L(\gamma)$ est $(-\sin\gamma, \cos\gamma)$. Il y a donc une seule normale limite possible. Comme dans la preuve originale ceci implique que les $L(\gamma)$ sont les seules lignes de divergence possibles.

Comme $\dfrac{\partial v_n}{\partial \theta}(1, \theta_\infty) \leq 0$, le point $(1, \theta_\infty)$ appartient à $\mathcal{B}(v_n)$ le domaine de convergence de v_n. La composante connexe de $\mathcal{B}(v_n)$ contenant $(1, \theta_\infty)$ est alors $D(\gamma_1, \gamma_2)$ pour $\gamma_1 < \gamma_2$. Soit alors v une limite d'une sous-suite de (v_n) sur $D(\gamma_1, \gamma_2)$, on a $\Psi_v = \lim \Psi_{n'} \leq 0$ et v vaut $+\infty$ sur $L(\gamma_2)$ et $-\infty$ sur $L(\gamma_1)$. Ainsi v est une solution de (ESM) qui contredit la Proposition 4.4.1. L'équivalent du Lemme 4.3.3 est donc prouvé. $\square$

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Résumé

Le cadre de cette thèse est la théorie des surfaces minimales. En 2001, C. Cosín et A. Ros démontrent que, si un polygone borde un disque immergé, ce polygone est le polygone de flux d'un r-noïde Alexandrov-plongé symétrique de genre 0. Leur démonstration se fonde sur l'étude de l'espace de ces surfaces minimales. Notre travail présente une démonstration plus constructive de leur résultat.

Notre méthode repose sur la résolution du problème de Dirichlet pour l'équation des surfaces minimales. A cette fin, nous étudions la convergence de suites de solutions de cette équation. Nous définissons la notion de lignes de divergence de la suite qui sont les points où la suite des gradients est non-bornées. L'étude de ces lignes permet de conclure sur la convergence d'une suite. Les r-noïdes sont alors construits comme les surfaces conjuguées aux graphes de solutions du problème de Dirichlet sur des domaines fixés par les polygones.

Dans une seconde partie, nous montrons que, sous l'hypothèse de border un disque immergé, un polygone est aussi le polygone de flux d'un r-noïde Alexandrov-plongé symétrique de genre 1. La démonstration repose sur une amélioration des idées de celle du premier résultat, elle nécessite entre autre la résolution d'un problème de période. Cette résolution passe par l'étude du comportement limite de certaines suites de surfaces minimales.

Abstract

This Ph.D. thesis deals with the theory of minimal surfaces. In 2001, C. Cosín and A. Ros show that, if a polygon bounds an immersed disk, this polygon is the flux polygon of a symmetric Alexandrov-embedded r-noid with genus 0. Their proof is based on the study of the space of these minimal surfaces. Our work gives a more constructive proof of their result.

Our method is based on the solving of the Dirichlet problem for the minimal surface equation. To this end, we study the convergence of sequences of the equation's solutions. We define the lines of divergence which are the points where the gradients' sequence is unbounded. The study of these lines allows us to conclude on the sequences' convergence. The r-noids are then build as the conjugate surfaces to the graphs of Dirichlet problem's solutions on domains fixed by the polygons.

In a second part, we show that, under the "immersed disk bounding" hypothesis, a polygon is also the flux polygon of a genus 1 symmetric Alexandrov-embedded r-noid. The proof is based on the same ideas as the first result, however it requires the solving of a period problem. This solving uses the study of the limit behaviour of some minimal surfaces sequences.