



On large clusters in percolation

Olivier Couronné

► To cite this version:

Olivier Couronné. On large clusters in percolation. Mathematics [math]. Université Paris Sud - Paris XI, 2004. English. NNT: . tel-00007948

HAL Id: tel-00007948

<https://theses.hal.science/tel-00007948>

Submitted on 7 Jan 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

N^o D'ORDRE : 7743

UNIVERSITÉ DE PARIS-SUD
U.F.R. SCIENTIFIQUE D'ORSAY

THÈSE

présentée pour obtenir le grade de

DOCTEUR EN MATHÉMATIQUES
DE L'UNIVERSITÉ PARIS XI ORSAY

par

Olivier COURONNÉ

Sujet :

**SUR LES GRANDS CLUSTERS
EN PERCOLATION**

Rapporteurs : M. Francis COMETS
M. Geoffrey GRIMMETT

Soutenue le 9 décembre 2004 devant la Commission d'examen composée de :

M. Kenneth	ALEXANDER	
M. Raphaël	CERF	Directeur de thèse
M. Francis	COMETS	Rapporteur
M. Vladas	SIDORAVICIUS	
M. Wendelin	WERNER	

Abstract

This thesis is dedicated to the study of large clusters in percolation and is divided into four articles. Models under consideration are Bernoulli percolation, FK percolation and oriented percolation. Key ideas are renormalization, large deviations, FKG and BK inequalities and mixing properties.

We prove a large deviation principle for clusters in the subcritical phase of Bernoulli percolation. We use FKG inequality for the lower bound. As for the upper bound, we use BK inequality together with a skeleton coarse graining.

We establish large deviations estimates of surface order for the density of the maximal cluster in a box in dimension two for supercritical FK percolation. We use renormalization and we compare a block process with a site-percolation process whose parameter of retention is close to one.

We prove that large finite clusters are distributed accordingly to a Poisson process in supercritical FK percolation and in all dimensions. The proof is based on the Chen–Stein method and it makes use of mixing properties such as the ratio weak mixing property.

We establish a large deviation principle of surface order for the supercritical oriented percolation. The framework is that of the non-oriented case, but difficulties arise despite of the Markovian nature of the oriented process. We give new block estimates, which describe the behaviour of the oriented process. We also obtain the exponential decay of connectivities outside the cone of percolation, which is the typical shape of an infinite cluster.

Keywords: percolation, large deviations, renormalization, FK percolation, oriented percolation

Classification MSC 1991 : 60F10, 60K35, 82B20, 82B43

Remerciements / Acknowledgments

Je tiens à exprimer toute ma reconnaissance à Raphaël Cerf qui m'a fait découvrir la recherche et m'a aidé tout au long de cette thèse. J'ai beaucoup apprécié les sujets de recherche qu'il m'a donnés à étudier. J'ai particulièrement aimé ses conseils multiples, son aide précieuse sur les questions difficiles.

Je remercie Francis Comets d'avoir accepté d'être rapporteur. Ses cours en licence m'ont apporté une vision claire des probabilités.

I wish to thank Geoffrey Grimmett for accepting to be one of the referees. His book on percolation has been an invaluable help to this thesis.

Je remercie Kenneth Alexander, Vladas Sidoravicius et Wendelin Werner pour avoir accepté d'être dans mon jury.

Je remercie Reda–Jürg Messikh, qui m'a apporté un grand soutien durant cette thèse. Ses connaissances dans notre sujet de recherche commun ont souvent été salvatrices.

Mes remerciements vont aux thésards d'Orsay que j'ai cotoyés. Je tiens à remercier en particulier Cédric Boutillier, Béatrice Detilière, Yong Fang et Céline Lévy–Leduc. Ce fut un plaisir de passer ces années avec eux à Orsay.

Je remercie Gaël Benabou, Nicolas Champagnat, Olivier Garet, Mylène Maïda et Régine Marchand. C'est toujours un grand plaisir de les rencontrer lors d'un séminaire ou au hasard d'un colloque.

Je remercie les chercheurs que j'ai rencontrés à Prague, à Eindhoven et à Aussois. Leur compétence et leur gentillesse ont été très appréciables.

Cette thèse a été réalisée avec le soutien affectif de mon entourage. Mes plus vifs remerciements vont à Delphine Gauchet. Ses encouragements et son aide sont pour beaucoup dans le travail contenu dans cette thèse.

Un grand merci à toi, Lecteur, pour l'attention que tu portes à cette thèse.

Table des matières

Remerciements	1
Chapitre 1 : Introduction	
1 Introduction à la percolation	8
2 Des estimés exponentiels en FK percolation	10
3 Un principe de grandes déviations dans le régime sous-critique	14
4 Les grands clusters sont distribués comme un processus de Poisson	17
5 Une étude sur la percolation orientée en dimensions supérieures à 3	20
6 La percolation à orientation aléatoire	23
7 Organisation de la thèse	24
Chapitre 2 : Surface order large deviations for $2D$ FK-percolation and Potts models	
1 Introduction	30
2 Statement of the results	31
3 Preliminaries	32
4 Connectivity in boxes	37
5 Renormalization	42
6 Proof of the surface order large deviations	46
Chapitre 3 : Large deviations for subcritical Bernoulli percolation	
1 Introduction	56
2 The model	57
3 The $\mathcal{H}_\xi^1$ measure and the space of the large deviation principle	59
4 Curves and continua	60
5 The skeletons	61
6 The lower bound	63
7 Coarse graining	65
8 The upper bound	66
Chapitre 4 : Poisson approximation for large finite clusters in the supercritical FK model	
1 Introduction	72
2 Statement of the result	72
3 FK model	74
4 Mixing properties	75
5 The Chen Stein method	76
6 Second moment inequality	78

7	A control on p_x	81
8	Proof of Theorem 2.1	82
9	Proof of Theorem 2.3	83
10	A perturbative mixing result	85

Chapitre 5 : Surface large deviations for supercritical oriented percolation

1	Introduction	90
2	The model	94
3	Block events	97
4	The rescaled lattice	103
5	Surface tension	105
6	The Wulff crystal and the positivity of the surface tension	109
7	Separating sets	114
8	Interface estimate	114
9	An alternative separating estimate	117
10	Geometric tools	121
11	Surface energy	124
12	Approximation of sets	126
13	Local upper bound	126
14	Coarse grained image	128
15	The boundary of the block cluster	129
16	Exponential contiguity	132
17	The $\mathcal{I}$ -tightness	134
18	Lower bound	138
19	The geometry of the Wulff shape and more exponential results	143
20	Exponential decrease of the connectivity function	146
21	A note on the Wulff variational problem	149

Chapitre 1

Introduction

Cette thèse porte sur la percolation, et plus particulièrement sur l'étude des grands clusters. Dans ce chapitre introductif, nous expliquons le processus de percolation dans la section 1 et donnons les différents résultats que nous avons obtenus dans les sections 2, 3, 4 et 5. La section 2 porte sur la FK percolation sur-critique dans une boîte en dimension deux, et contient des estimés d'ordre surfacique sur le comportement du cluster maximal et des clusters intermédiaires. Dans la section 3, nous nous intéressons aux grands clusters en régime sous-critique et nous donnons un principe de grandes déviations. Nous considérons dans la section 4 les grands clusters finis dans le régime surcritique. D'après un résultat que nous établissons, ces clusters sont distribués comme un processus spatial de Poisson. La section 5 porte sur la percolation orientée en régime surcritique. Nous y donnons un principe de grandes déviations pour le cluster de l'origine. La section 6 est une petite note sur la percolation à orientation aléatoire. La section 7 donne le contenu des chapitres suivants.

1 Introduction à la percolation

1.1 Explication physique. La situation initiale est la suivante : une pierre spongieuse est immergée dans de l'eau, comme représenté sur la figure 1, et nous voulons savoir si le centre de la pierre est mouillé. Broadbent et Hammersley ont défini un modèle mathématiques qui permet de répondre à ce genre de question.

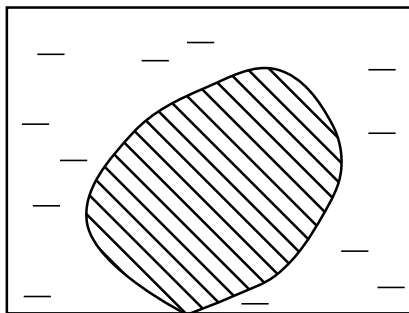


figure 1: La pierre spongieuse immergée.

1.2 Le modèle mathématiques [9]. Considérons $\mathbb{Z}^d$ l'ensemble des vecteurs d'entiers à d coordonnées. Nous le munissons d'une structure de graphe en mettant une arête pour chaque couple de points (x, y) voisins. Nous notons $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ le graphe obtenu. Ce graphe est infini et invariant par les translations entières.

L'espace des configurations pour la percolation sur $\mathbb{Z}^d$ est $\Omega = \{0, 1\}^{\mathbb{E}^d}$. Soit ω un élément de Ω . Une arête e de $\mathbb{E}^d$ est dite *ouverte* dans ω si $\omega(e) = 1$, et fermée si $\omega(e) = 0$.

Nous modélisons donc la pierre spongieuse en assimilant les petits canaux à l'intérieur de la pierre aux arêtes du graphe $\mathbb{L}^d$, un canal laissant passer l'eau uniquement si l'arête est ouverte. La question de savoir si le centre de la pierre est mouillé revient à savoir si il y a un chemin infini partant de l'origine 0 du graphe et ne passant que par les arêtes ouvertes. La figure 2 représente une réalisation du processus de percolation sur $\mathbb{Z}^2$.

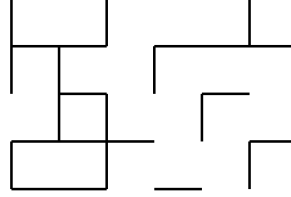


figure 2: exemple de réalisation du processus de percolation

Pour pouvoir répondre à cette question, il nous faut une mesure de probabilité. L'ensemble Ω est muni de la tribu produit $\mathcal{F}$. Soit p un paramètre compris entre 0 et 1. La mesure de percolation P_p est la mesure sur $(\Omega, \mathcal{F})$ telle que les arêtes soient ouvertes avec probabilité p , fermées avec probabilité $1 - p$, et ceci indépendamment les unes des autres. C'est donc le produit tensoriel des mesures de Bernoulli $p\delta_0 + (1 - p)\delta_1$ associées à chaque arête.

Plus le paramètre p est grand, plus la probabilité qu'il y ait un chemin infini d'arêtes ouvertes est grande. Pour la pierre spongieuse, cela signifie que plus il y a de petits canaux, plus le centre de la pierre a de chance d'être atteint par l'eau.

Un cluster est une composante connexe du graphe aléatoire, dont l'ensemble d'arêtes est constitué d'arêtes ouvertes. Nous disons qu'il y a percolation s'il existe un cluster infini, et nous notons $\{0 \rightarrow \infty\}$ l'événement où l'origine est dans un cluster infini. La *probabilité de percolation* est

$$\theta(p) = P_p(0 \rightarrow \infty).$$

1.3 Événements croissants et domination stochastique. Nous définissons un ordre partiel sur Ω en disant que $\omega_1 \leq \omega_2$ si et seulement si $\omega_1(e) \leq \omega_2(e)$ pour toute arête e de $\mathbb{E}^d$. Un événement A est dit croissant si

$$\omega_1 \in A \text{ et } \omega_2 \geq \omega_1 \Rightarrow \omega_2 \in A.$$

Si A^c le complémentaire de A est croissant, alors A est dit décroissant. Une inégalité fondamentale est l'inégalité FKG, qui établit que les événements croissants sont corrélés

positivement : si A et B sont deux événements croissants, alors

$$P(A \cap B) \geq P(A) \times P(B).$$

Une fonction f de Ω dans $\mathbb{R}$ est dite croissante si $\omega_1 \leq \omega_2$ implique $f(\omega_1) \leq f(\omega_2)$. Dire qu'un événement A est croissant est alors équivalent à dire que sa fonction indicatrice 1_A est croissante. Soit μ et ν deux mesures sur Ω . Nous disons que μ est *dominée stochastiquement* par ν si pour toute fonction f croissante de Ω dans $\mathbb{R}$, $\mu(f) \leq \nu(f)$. Nous avons par exemple :

$$\text{pour tous } p, p' \in [0, 1] \quad p \leq p' \Rightarrow P_p \leq P_{p'}.$$

2 Des estimées exponentielles sur le comportement des clusters dans une boîte en FK percolation

2.1 Le modèle FK. Le modèle FK [11] est une extension du modèle de percolation Bernoulli dans lequel les arêtes ne sont plus indépendantes. Pour pouvoir définir ce processus sur $\mathbb{Z}^d$, nous commençons par le définir dans une boîte.

Soit donc Λ une boîte de $\mathbb{Z}^d$. Nous notons $E(\Lambda)$ l'ensemble des arêtes qui sont à l'intérieur de Λ , et nous posons $\Omega_\Lambda = \{0, 1\}^{E(\Lambda)}$ l'ensemble des configurations dans la boîte. Notons $\partial\Lambda$ l'ensemble des sites appartenant à la frontière de Λ :

$$\partial\Lambda = \{x \in \Lambda : \exists y \notin \Lambda, (x, y) \text{ est une arête}\}.$$

Soit π une partition de $\partial\Lambda$. Nous appelons π -cluster une composante connexe de Λ pour laquelle nous considérons que deux points dans la même classe de π sont reliés. Le nombre correspondant de π -clusters dans la configuration ω est noté $\text{cl}^\pi(\omega)$. Pour $p \in [0, 1]$ et $q \geq 1$, nous posons alors

$$\forall \omega \in \Omega_\Lambda \quad \Phi_\Lambda^{\pi, p, q}[\{\omega\}] = \frac{1}{Z_\Lambda^{\pi, p, q}} \left(\prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right) q^{\text{cl}^\pi(\omega)},$$

le terme $Z_\Lambda^{\pi, p, q}$ servant à renormaliser l'expression. Lorsque $q = 1$, les arêtes sont indépendantes et nous retrouvons la mesure de Bernoulli. Ces mesures vérifient l'inégalité FKG (c'est la raison pour laquelle nous imposons $q \geq 1$).

Il y a deux conditions aux bords extrêmes : celle où tous les points de $\partial\Lambda$ sont dans une seule classe est notée w pour *wired*, et celle où chaque classe est constituée d'un seul point est notée f pour *free*. Pour toute partition π de $\partial\Lambda$ et pour toute configuration ω , nous avons

$$\text{cl}^w(\omega) \leq \text{cl}^\pi(\omega) \leq \text{cl}^f(\omega),$$

ainsi que les dominations stochastiques suivantes :

$$\Phi_{\Lambda}^{f,p,q} \preceq \Phi_{\Lambda}^{\pi,p,q} \preceq \Phi_{\Lambda}^{w,p,q}.$$

L'ensemble des mesures FK correspondant aux différentes conditions aux bords est noté $\mathcal{R}(p, q, \Lambda)$.

Par un argument de monotonie, les deux mesures $\Phi_{\Lambda}^{f,p,q}$ et $\Phi_{\Lambda}^{w,p,q}$ convergent faiblement lorsque $\Lambda \rightarrow \mathbb{Z}^d$, vers des mesures sur Ω notées $\Phi_{\infty}^{f,p,q}$ et $\Phi_{\infty}^{w,p,q}$. Ces deux mesures sont égales sauf peut-être pour un ensemble dénombrable de valeurs de p , cet ensemble dépendant du paramètre q . Elles ont donc un point critique commun défini par

$$p_c = \sup \{p : \Phi_{\infty}^{f,p,q}(0 \rightarrow \infty) = 0\} = \sup \{p : \Phi_{\infty}^{w,p,q}(0 \rightarrow \infty) = 0\}.$$

Nous avons besoin de certaines estimées exponentielles. Pour ce faire, nous introduisons le point critique suivant :

$$p_g = \sup \{p : \exists c > 0, \forall x \forall y \in \mathbb{Z}^2, \Phi_{\infty}^{p,q}[x \leftrightarrow y] \leq \exp(-c|x - y|)\}.$$

Le point dual de p_g est le point défini par

$$\hat{p}_g = \frac{q(1 - p_g)}{p_g + q(1 - p_g)} \geq p_c.$$

2.2 Résultats. Nous considérons le modèle FK sur $\mathbb{Z}^2$ dans le régime surcritique. Soit $\Lambda(n)$ le carré $[-n, n]^2$. Nous disons qu'un cluster de $\Lambda(n)$ *traverse* $\Lambda(n)$ s'il intersecte tous les côtés de $\Lambda(n)$. Soit l un entier. Un cluster est *l -intermédiaire* si son cardinal n'est pas maximal parmi les clusters de $\Lambda(n)$, et si son diamètre dépasse l . Nous notons $\mathbb{J}_l$ l'ensemble des clusters l -intermédiaire de $\Lambda(n)$ et nous posons $\theta = \theta(p)$ pour alléger les notations. Soit l'événement

$$\begin{aligned} K(n, \varepsilon, l) = \{ & \exists! \text{ cluster } C_m \text{ dans } \Lambda(n) \text{ qui est maximal pour le volume,} \\ & \text{le cluster } C_m \text{ traverse } \Lambda(n), n^{-2}|C_m| \in]\theta - \varepsilon, \theta + \varepsilon[\\ & \text{et } n^{-2} \sum_{C \in \mathbb{J}_l} |C| < \varepsilon \}. \end{aligned}$$

Nous démontrons le résultat suivant:

THÉOREME 1. : *Soit $q \geq 1$, $1 > p > \hat{p}_g$ et $\varepsilon \in]0, \theta/2[$ fixés. Il existe une constante L telle que*

$$\begin{aligned} -\infty & < \liminf_{n \rightarrow \infty} \frac{1}{n} \log \inf_{\Phi \in \mathcal{R}(p,q,\Lambda(n))} \Phi[K(n, \varepsilon, L)^c] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p,q,\Lambda(n))} \Phi[K(n, \varepsilon, L)^c] < 0. \end{aligned}$$

Ainsi, à des déviations d'ordre surfacique près, la configuration typique dans une grande boîte est un unique cluster qui touche toutes les faces du carré et qui a la même densité que le cluster infini, et un ensemble de clusters de tailles intermédiaires dont le volume total est aussi petit que nécessaire.

Le théorème 1 est l'adaptation en dimension deux d'un résultat de A. Pisztora [14].

2.3 Renormalisation. Soit N un entier. La renormalisation consiste à diviser la boîte $\Lambda(n)$ en boîtes de taille N . Nous posons

$$\Lambda^{(N)} = \{\mathbf{k} \in \mathbb{Z}^2 : N\mathbf{k} +] - N/2, N/2] \subset \Lambda\},$$

comme représentée à la figure 3 (pour simplifier nous supposons que nous obtenons une partition de $\Lambda(n)$).

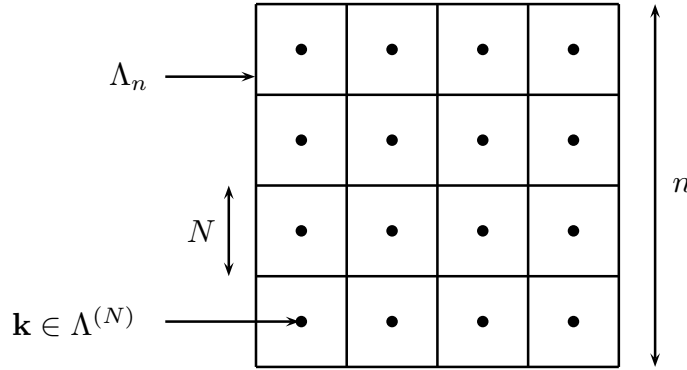


figure 3: le découpage d'une boîte

Pour $\mathbf{i}$ appartenant à $\Lambda^{(N)}$, nous posons $B_{\mathbf{i}} = N\mathbf{i} +] - N/2, N/2]$. Nous allons prendre N fixé mais assez grand pour que avec grande probabilité la configuration dans une boîte $B_{\mathbf{i}}$ soit proche de la configuration typique.

Considérons dans un premier temps la probabilité qu'il existe un cluster dans $\Lambda(n)$ qui soit de cardinal supérieur à $(\theta + \varepsilon)n^2$. Le cardinal d'un cluster dans $\Lambda(n)$ est majoré par le cardinal des clusters de chaque boîte $B_{\mathbf{i}}$, $\mathbf{i} \in \Lambda(N)$, intersectant le bord de $B_{\mathbf{i}}$. Nous notons $Y_{\mathbf{i}}$ ce cardinal. Par un procédé d'isolation des boîtes $B_{\mathbf{i}}$, $\mathbf{i} \in \Lambda(N)$, nous rendons les variables $Y_{\mathbf{i}}$ indépendantes. Nous prenons N assez grand pour que l'espérance de $Y_{\mathbf{i}}/N^2$ soit inférieure à $\theta + \varepsilon/2$. En appliquant le théorème de Cramer, la probabilité qu'un cluster soit de cardinal supérieur à $(\theta + \varepsilon)n^2$ est inférieure à $\exp(-cn^2)$ pour une constante $c > 0$.

Pour les déviations de la densité par en-dessous, nous nous intéressons à un processus de percolation par site sur $\Lambda^{(N)}$, qui va ensuite nous donner des informations sur le processus

de percolation sur $\Lambda(n)$. Pour $\mathbf{i} \in \Lambda^{(N)}$, nous notons $R_{\mathbf{i}}$ l'événement : il existe un unique cluster $C_{\mathbf{i}}^*$ traversant $B_{\mathbf{i}}$ et tout chemin ouvert dans $B_{\mathbf{i}}$ de diamètre supérieur à $\sqrt{N}/10$ est inclus dans $C_{\mathbf{i}}^*$.

Soit Λ une boîte. Nous disons qu'il y a une 1-traversée dans Λ s'il existe un cluster dans Λ qui relie le côté gauche au côté droit. Nous définissons de la même manière les 2-traversées. Pour $\mathbf{i}, \mathbf{j}$ appartenant à $\Lambda^{(N)}$ tels que $|\mathbf{i} - \mathbf{j}|_2 = |\mathbf{i}_r - \mathbf{j}_r| = 1$ avec $r = 1$ ou 2 , nous définissons la boîte

$$D_{\mathbf{i}, \mathbf{j}} = [-N/4, N/4]^2 + (\mathbf{i} + \mathbf{j})N/2,$$

et l'événement

$$K_{\mathbf{i}, \mathbf{j}} = \{\exists r\text{-traversée dans } D_{\mathbf{i}, \mathbf{j}}\}.$$

Pour $\mathbf{i} \in \Lambda^{(N)}$, nous définissons

$$X_{\mathbf{i}} = \begin{cases} 1 & \text{sur } R_{\mathbf{i}} \cap \bigcap_{i \sim j} K_{\mathbf{i}, \mathbf{j}} \\ 0 & \text{sinon.} \end{cases}$$

Prenons $\mathbf{i}$ et $\mathbf{j}$ dans $\Lambda^{(N)}$, voisins et tels que $X_{\mathbf{i}} = X_{\mathbf{j}} = 1$. Comme nous pouvons le voir sur la figure 4, les deux clusters $C_{\mathbf{i}}^*$ et $C_{\mathbf{j}}^*$ sont reliés par l'intermédiaire de $D_{\mathbf{i}, \mathbf{j}}$.

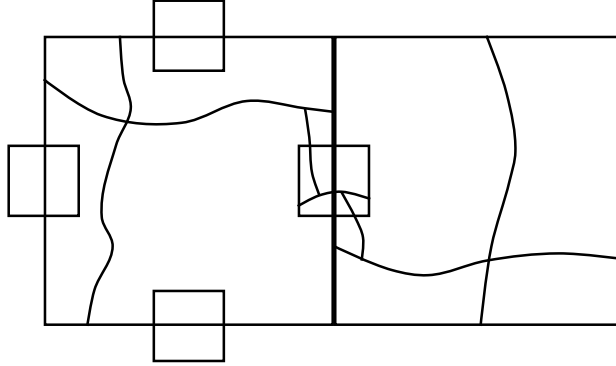


figure 4: les clusters de boîtes voisines sont inter-connectés

Dans [5], il a été démontré que pour p assez proche de 1, il existe une constante $c > 0$ telle que

$$P_p(\exists \mathbf{C} \text{ cluster de site dans } \Lambda^{(N)} \text{ tel que } \frac{N^2}{n^2} |\mathbf{C}| \geq 1 - \varepsilon) \geq 1 - \exp(-cn).$$

Ce cluster macroscopique $\mathbf{C}$ de petites boîtes implique l'existence d'un cluster microscopique C contenant les clusters C_i^* pour i appartenant à $\mathbf{C}$. Pour N assez grand, l'espérance du cardinal de C_i^* est supérieur à $\theta - \varepsilon/2$. Comme précédemment, le résultat est obtenu en rendant ces variables indépendantes et en appliquant le théorème de Cramer.

3 Un principe de grandes déviations dans le régime sous-critique

3.1 La mesure de Hausdorff. Cette mesure a été définie pour répondre à des questions du genre : quelle est la longueur des côtes bretonnes, quelle est la surface d'un flocon de neige, quelle est la dimension d'un mouvement brownien plan ? La mesure de Hausdorff est un outil primordial pour l'études des fractales [7], dont nous rappelons le concept figure 5.



figure 5: représentation d'une fractale

La longueur de la fractale représentée figure 5 est infinie, mais nous ne pouvons pas dire pour autant qu'elle ait une aire. Nous voulons disposer d'une quantité qui caractérise cet ensemble et qui étende les notions classiques de longueur et d'aire.

Soit E un sous-ensemble de $\mathbb{R}^d$. Son diamètre est

$$\text{diam } E = \sup\{|x - y|_2 : x, y \in E\},$$

où $|\cdot|_2$ est la norme euclidienne. Prenons r un réel appartenant à $[0, d]$. Pour $A \subset \mathbb{R}^d$, sa mesure de Hausdorff r -dimensionnelle est

$$\mathcal{H}^r(A) = \sup_{\delta > 0} \inf \left\{ \sum_{i \in I} (\text{diam } E_i)^r : A \subset \bigcup_{i \in I} E_i, \sup_{i \in I} \text{diam } E_i \leq \delta \right\}.$$

La dimension de Hausdorff de l'ensemble A est alors égale à la quantité

$$\dim_H A = \sup\{r : \mathcal{H}^r(A) = \infty\}.$$

Même si r est la dimension de A , $\mathcal{H}^r(A)$ peut prendre les valeurs 0 et $+\infty$. Pour toucher au plus près la structure d'un ensemble, il faut parfois généraliser la définition de la mesure

de Hausdorff, en autorisant d'autres fonctions que les fonctions puissances. Si f est une fonction continue de $\mathbb{R}_+$ dans $\mathbb{R}_+$ avec $f(0) = 0$, nous définissons

$$\mathcal{H}^f(A) = \sup_{\delta > 0} \inf \left\{ \sum_{i \in I} f(\text{diam } E_i) : A \subset \bigcup_{i \in I} E_i, \sup_{i \in I} \text{diam } E_i \leq \delta \right\}.$$

Nous pouvons par exemple prendre $f(x) = x^2/(\ln x)$. La mesure $\mathcal{H}^r$ correspond à la mesure $\mathcal{H}^f$ avec $f(x) = x^r$.

La mesure $\mathcal{H}^1$ correspond à la notion de longueur dans le cadre euclidien. Si nous nous plaçons dans un milieu non isotrope, tel que la distance entre deux points x et y soit définie par $\xi(x - y)$ avec ξ une norme quelconque, nous devons modifier comme suit la définition de $\mathcal{H}^1$ pour garder la correspondance avec la longueur:

$$\mathcal{H}_\xi^1(A) = \sup_{\delta > 0} \inf \left\{ \sum_{i \in I} \xi(E_i) : A \subset \bigcup_{i \in I} E_i, \sup_{i \in I} \xi(E_i) \leq \delta \right\},$$

où $\xi(E_i) = \sup\{\xi(x - y) : x, y \in E_i\}$.

3.2 Nos résultats en percolations sous-critique. En régime sous-critique, la queue de la loi du diamètre des clusters est exponentiellement décroissante : $\exists c > 0$ tel que $\forall n \in \mathbb{N}$,

$$P(\text{diam } C(0) \geq n) \leq \exp(-cn).$$

Nous nous intéressons au problème plus spécifique d'estimer la probabilité que le cluster de l'origine, mis à l'échelle $\frac{1}{n}$, soit proche d'une certaine forme. Nous y répondons en établissant que le cluster de l'origine vérifie un principe de grandes déviations pour la distance de Hausdorff.

Pour x dans $\mathbb{R}^d$, nous notons $\lfloor x \rfloor$ le point de $\mathbb{Z}^d$ situé juste "en dessous et à gauche" de x . Soit ξ la norme sur $\mathbb{R}^2$ définie par

$$\xi(x) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln P(O \rightarrow \lfloor nx \rfloor).$$

Pour K un compact de $\mathbb{R}^d$, nous posons

$$\mathcal{I} = \begin{cases} \mathcal{H}_\xi^1(K) & \text{si le compact } K \text{ est connexe et contient } 0 \\ +\infty & \text{sinon.} \end{cases}$$

Nous appelons énergie de K la quantité $\mathcal{I}(K)$. La distance de Hausdorff entre deux compacts K_1 et K_2 est définie par

$$D_H(K_1, K_2) = \max \left\{ \max_{x_1 \in K_1} d(x_1, K_2), \max_{x_2 \in K_2} d(x_2, K_1) \right\}.$$

Nous notons $\mathcal{K}$ pour l'ensemble des compacts de $\mathbb{R}^d$. La distance de Hausdorff induit une topologie sur l'ensemble $\mathcal{K}$.

THÉORÈME 2. Soit $p < p_c$. Pour tout borélien $\mathcal{U}$ de $\mathcal{K}$,

$$\begin{aligned} -\inf \{ \mathcal{I}(K) : K \in \overset{\circ}{\mathcal{U}} \} &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \ln P(C(0)/n \in \mathcal{U}) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \ln P(C(0)/n \in \mathcal{U}) \\ &\leq -\inf \{ \mathcal{I}(K) : K \in \overline{\mathcal{U}} \}. \end{aligned}$$

La prochaine étape sera de démontrer ce résultat pour la percolation FK.

3.3 Les squelettes. Pour prouver le principe de grandes déviations, nous approximations les clusters par des ensembles de segments appelés squelettes, voir figure 6.

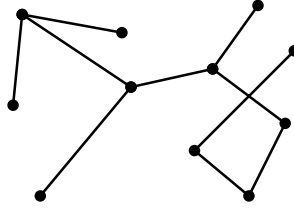


figure 6: un squelette

Pour la borne inférieure, nous prenons un squelette S proche pour la distance de Hausdorff de Γ et tel que $\mathcal{I}(S) \leq \mathcal{I}(\Gamma)$. Ensuite, pour tout segment $[x, y]$ de S , nous imposons que nx soit connecté à ny par un chemin ouvert qui reste proche du segment $[nx, ny]$. Grâce à l'inégalité FKG, la probabilité de cet événement est supérieure à $\exp(-n\mathcal{I}(S))$. Nous montrons ensuite que le cluster contenant ces chemins ouverts reste proche de l'ensemble Γ .

Pour la borne supérieure, nous utilisons l'inégalité BK. Si le cluster de 0 n'est pas dans un ensemble de niveau de la fonction de taux, alors tous les squelettes proches de ce cluster ont une certaine énergie. Pour pouvoir conclure, il faut disposer d'un contrôle sur ce nombre de squelettes. Ceci est réalisé en imposant une longueur minimale pour les segments du squelette.

3.4 La forme typique d'un grand cluster en régime sous-critique. Peu de choses sont connues à son sujet. Contrairement au régime sur-critique, notre principe de grandes déviations ne nous fournit aucun contrôle sur le cardinal du cluster de l'origine. Il n'est de plus pas certain qu'un cluster de cardinal n ait en général un diamètre de l'ordre de n .

J'ai réalisé la simulation suivante sur un ordinateur: prenons un carré de taille 400×400 , et fixons la configuration de départ de telle sorte que toutes les arêtes soient ouvertes. A chaque cycle, prenons aléatoirement une arête. Si elle est fermée, elle devient ouverte avec probabilité $\frac{1}{4}$ (nous prenons arbitrairement ce paramètre qui est inférieur à $\frac{1}{2}$ le point critique de $\mathbb{Z}^2$). Si elle est ouverte, nous vérifions que sa fermeture ne va pas faire descendre le cardinal de $C(0)$ en-dessous de 300. Si le cluster de l'origine reste suffisamment gros malgré la fermeture, nous fermons cette arête avec probabilité $\frac{3}{4}$, sinon nous la laissons ouverte. De cette manière, le cluster $C(0)$ a toujours un cardinal supérieur à 300. Les figures obtenues ont un aspect très irrégulier, de type "fractale".

Il faudrait réussir à donner une notion à la dimension fractale de $C(0)$, si tant est qu'elle existe. Un premier pas serait d'estimer la variable $\text{diam } C(0)$ conditionnellement au fait que le cardinal de $C(0)$ est plus grand que n . Par exemple, trouver le plus grand c tel que

$$P(\text{diam } C(0) \geq n^c \mid |C(0)| \geq n) \rightarrow 1, \quad \text{lorsque } n \rightarrow \infty.$$

4 Les grands clusters sont distribués comme un processus de Poisson

4.1 Le processus de Poisson spatial. Des points sont lancés au hasard dans l'espace euclidien $\mathbb{R}^d$. Pour un des lancers ω , notons $N(\omega, A)$ le nombre de points compris dans l'ensemble $A \subset \mathbb{R}^d$. La variable $N(A)$ est donc une variable aléatoire discrète prenant les valeurs $0, 1, \dots, \infty$. La famille des variables aléatoires $\{N(A) : A \in \mathcal{B}^d\}$ où $\mathcal{B}^d$ est l'ensemble des boréliens de $\mathbb{R}^d$, est un processus ponctuel de $\mathbb{R}^d$.

On appelle processus de Poisson homogène sur $\mathbb{R}^d$ d'intensité λ un processus ponctuel sur $\mathbb{R}^d$ tel que, pour toute famille $\{A_i : 1 \leq i \leq k\}$ de sous-ensembles mesurables de $\mathbb{R}^d$:

- (i) $N(A_i)$ est une variable de Poisson de paramètre $\lambda \mathcal{L}^d(A_i)$
 - (ii) la famille $\{N(A_i) : 1 \leq i \leq k\}$ est une famille de variables aléatoires indépendantes.
- Cette présentation du processus spatial de Poisson est extraite de [2].

Considérons un processus de Bernoulli indexé par $\mathbb{Z}^d$ d'intensité p' . En mettant le réseau $\mathbb{Z}^d$ à l'échelle $\frac{1}{n}$, le processus de Bernoulli induit un processus ponctuel sur $\mathbb{R}^d$: pour $A \subset \mathbb{R}^d$, nous notons $N(A)$ le nombre de points de $\mathbb{Z}^d$ compris dans nA . En faisant tendre p' vers 0 et n vers l'infini de telle sorte que $np' \rightarrow \lambda$, la suite de processus ponctuels sur $\mathbb{R}^d$ converge en loi vers un processus de Poisson sur $\mathbb{R}^d$ d'intensité λ . Le processus de Poisson est ainsi caractéristique de la distribution des événements rares dans l'espace.

4.2 Le processus des grands clusters finis. Dans le régime surcritique de la percolation Bernoulli, les grands clusters finis sont des objets rares. Il existe ainsi une constante $c > 0$ telle que

$$\lim_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(n^d \leq |C(0)| < \infty) = -c. \quad (3)$$

Cela signifie que pour voir dans une boîte un cluster de taille plus grande que n et ne touchant pas les bords, il faut prendre une boîte de taille $\exp(cn^{(d-1)/d})$. Cette taille étant très largement supérieure à la taille des clusters considérés, ces clusters ressemblent à des points lorsque nous ramenons cette boîte à une boîte de taille 1. La discussion précédente nous laisse à penser que ces points sont distribués comme un processus de Poisson.

Nous étudions le processus pontuel défini comme suit. Soit C un cluster fini. Son centre de gravité est

$$M_C = \left\lfloor \frac{1}{|C|} \sum_{x \in C} x \right\rfloor,$$

où $\lfloor y \rfloor$ représente le point de $\mathbb{Z}^d$ en dessous et à gauche de y . Soit Λ une boîte et n un entier. Nous définissons un processus X sur Λ par

$$X(x) = \begin{cases} 1 & \text{si } x \text{ est le centre de gravité d'un cluster fini de cardinal } \geq n \\ 0 & \text{sinon.} \end{cases} \quad (4)$$

Pour Y processus sur Λ à valeurs dans $\mathbb{N}$, la *distance de variation totale* entre X et Y est

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV} = \sup \{ |P(X \in A) - P(Y \in A)|, A \subset \{0, 1\}^\Lambda \}.$$

Soit λ l'espérance du nombre de points x de Λ tels que $X(x) = 1$. Nous prouvons le résultat suivant:

THÉORÈME 5. *Soit $p > p_c$. Il existe une constante $c > 0$ telle que : pour toute boîte Λ , si X est le processus défini par l'équation (4), et si Y est un processus de Bernoulli sur Λ ayant les mêmes marginales que X , i.e. $P(Y(x) = 1) = P(X(x) = 1)$ pour tout x de Λ , alors pour n assez grand*

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV} \leq \lambda \exp(-cn^{(d-1)/d}).$$

Comme corollaire, la loi du nombre de clusters finis de taille plus grande que n intersectant Λ est proche d'une loi de Poisson de paramètre λ si λ n'est pas trop grand.

Nous démontrons en fait le Théorème 5 pour la percolation FK, mais en imposant des conditions supplémentaires sur p .

4.3 La méthode Chen-Stein. La méthode Chen-Stein permet de contrôler la distance de variation totale entre deux processus X, Y sur Λ par des moments de second ordre. Ici Y est un processus de Bernoulli ayant les mêmes marginales que X . Pour $x \in \Lambda$, nous notons

$$p_x := P(X(x) = 1) = P(Y(x) = 1),$$

et pour y appartenant à Λ

$$p_{xy} := P(X(x) = 1, X(y) = 1).$$

Nous définissons trois coefficients b_1 , b_2 et b_3 :

$$\begin{aligned} b_1 &= \sum_{x \in \Lambda} \sum_{y \in B_x} p_x p_y, \\ b_2 &= \sum_{x \in \Lambda} \sum_{y \in B_x \setminus x} p_{xy}, \\ b_3 &= \sum_{x \in \Lambda} E \left| E \left(X(x) - p_x | \sigma(X(y), y \notin B_x) \right) \right|. \end{aligned}$$

Le théorème 2 de [1] établit que

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV} \leq 2(2b_1 + 2b_2 + 2b_3) + \sum_{x \in \Lambda} p_x^2.$$

4.4 Schéma de la preuve. Le travail principal est de contrôler le terme p_{xy} , i.e. les interactions entre les différents clusters. Nous effectuons ceci de deux manières différentes, suivant que $|x - y|_1$ soit de l'ordre de $\ln n$ ou plus grand. Dans le second cas, nous supposons la *ratio weak mixing property*, qui permet de contrôler les interactions à distance et dont voici la définition :

DEFINITION 6. *La mesure Φ a la ratio weak mixing property si il existe $c_1, \mu_1 > 0$, tels que pour tous les ensembles $\Lambda, \Delta \subset \mathbb{Z}^d$,*

$$\begin{aligned} &\sup \left\{ \left| \frac{\Phi(E \cap F)}{\Phi(E)\Phi(F)} - 1 \right| : E \in \mathcal{F}_\Lambda, F \in \mathcal{F}_\Delta, \Phi(E)\Phi(F) > 0 \right\} \\ &\leq c_1 \sum_{x \in \Lambda, y \in \Delta} e^{-\mu_1 |x-y|_1}, \end{aligned}$$

Dans le cas où $|x - y|_1$ est inférieur à $K \ln n$ pour un K donné, nous modifions la configuration pour relier les deux clusters dont les centres de gravité sont x et y (il faut d'ailleurs contrôler la probabilité que deux clusters aient le même centre de gravité). Cette modification est réalisée de telle sorte que le nombre d'antécédants par cette application soit borné par une puissance de n . Nous la représentons figure 7.

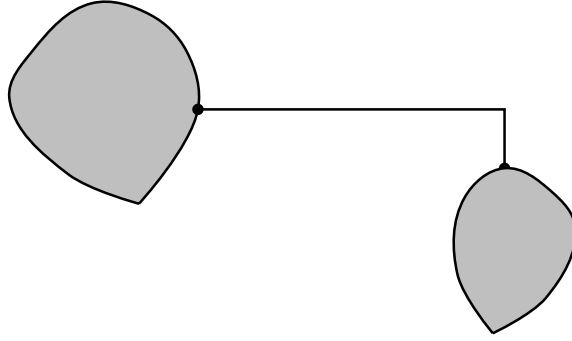


figure 7: les deux clusters sont reliés

5 Une étude sur la percolation orientée en dimensions supérieures à trois

5.1 La percolation orientée. Nous étudions à présent une autre structure de graphe sur $\mathbb{Z}^d$, dans laquelle les arêtes de $\mathbb{Z}^d$ sont toutes orientées dans le sens positif. Nous représentons figure 8 le graphe orienté $\mathbb{Z}^2$.

Les arêtes sont ouvertes avec probabilité p , indépendamment les unes des autres. Il y a percolation dans le graphe orienté s'il existe un chemin infini orienté d'arêtes ouvertes. Pour un point x de $\mathbb{Z}^d$, le cluster de x , noté $C(x, \omega)$ ou $C(x)$, est l'ensemble des points de $\mathbb{Z}^d$ que l'on peut atteindre à partir de x . La densité de percolation est

$$\vec{\theta}(p) = P_p(0 \rightarrow \infty),$$

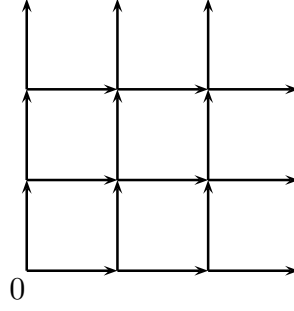
et le point critique de ce modèle est

$$\vec{p}_c = \sup\{p : \vec{\theta}(p) = 0\}.$$

Le point critique $\vec{p}_c$ est compris strictement entre 0 et 1, et de plus $\vec{p}_c > p_c$.

Un cluster infini ne remplit pas tout l'espace comme dans le cas non-orienté, mais ressemble plutôt à un cône [6], appelé *cône de percolation*.

5.2 Principe de grandes déviations en percolation orientée. Dans le cadre non-orienté, un principe de grandes déviations a été prouvé, qui a permis d'estimer la probabilité qu'un cluster soit fini et de cardinal supérieur à n (voir [3]), et de connaître la forme typique d'un tel cluster. Nous démontrons le principe de grandes déviations dans le cas de la percolation orientée.

figure 8: le graphe orienté de $\mathbb{Z}^2$

Nous définissons une tension de surface τ , à laquelle nous adjoignons le cristal de Wulff $\mathcal{W}_\tau$ correspondant, dont nous rappellerons la définition. Soit A un borélien de $\mathbb{R}^d$. Son énergie de surface $\mathcal{I}(A)$ est définie par

$$\mathcal{I}(A) = \sup \left\{ \int_A \operatorname{div} f(x) dx : f \in C_c^1(\mathbb{R}^d, \mathcal{W}_\tau) \right\},$$

où $C_c^1(\mathbb{R}^d, \mathcal{W}_\tau)$ est l'ensemble des fonctions C^1 définies sur $\mathbb{R}^d$ à valeurs dans $\mathcal{W}_\tau$ ayant un support compact et div est l'opérateur usuel de divergence. Cette expression de l'énergie de surface est équivalente par la formule de Stokes à l'écriture plus usuelle suivante :

$$\mathcal{I}(A) = \int_{\partial^* A} \tau(\nu_A(x)) d\mathcal{H}^{d-1}(x),$$

avec $\partial^* A$ représentant la frontière “régulière” de A et pour x appartenant à $\partial^* A$, $\nu_A(x)$ est le vecteur normal extérieur à A en x .

Nous notons $\mathcal{M}(\mathbb{R}_+^d)$ pour l'ensemble des mesures boréliennes σ -finies sur $\mathbb{R}_+^d$. Nous le munissons de la topologie faible : c'est la topologie la plus grossière pour laquelle les fonctions linéaires

$$\nu \in \mathcal{M}(\mathbb{R}_+^d) \rightarrow \int f d\nu, \quad f \in C_c(\mathbb{R}^d, \mathbb{R})$$

sont continues, où $C_c(\mathbb{R}^d, \mathbb{R})$ est l'ensemble des applications continues de $\mathbb{R}^d$ vers $\mathbb{R}$ ayant un support compact. Nous définissons une énergie de surface $\mathcal{I}$ sur $\mathcal{M}(\mathbb{R}_+^d)$ en posant $\mathcal{I}(\nu) = \mathcal{I}(A)$ si $\nu \in \mathcal{M}(\mathbb{R}_+^d)$ est la mesure $\vec{\theta}(p)1_A$ avec A un borélien, et sinon $\mathcal{I}(\nu) = +\infty$.

THÉOREME 7. *Soit $d \geq 3$ et $p > \vec{p}_c$. La suite des mesures aléatoires définies par*

$$C_n = \frac{1}{n^d} \sum_{x \in C(0)} \delta_{\frac{x}{n}}$$

vérifie un principe de grandes déviations sur $\mathcal{M}(\mathbb{R}_+^d)$, de vitesse n^{d-1} et de fonction de taux $\mathcal{I}$, I.E., pour tout borélien $\mathbb{M}$ de $\mathcal{M}(\mathbb{R}_+^d)$,

$$\begin{aligned} -\inf\{\mathcal{I}(\nu) : \nu \in \overset{\circ}{\mathbb{M}}\} &\leq \liminf_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathbb{M}) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathbb{M}) \leq -\inf\{\mathcal{I}(\nu) : \nu \in \overline{\mathbb{M}}\}. \end{aligned}$$

L'un des principaux problèmes vient du fait que la tension de surface τ que nous définissons pour ce modèle n'est pas strictement positive sur toute la sphère $\mathbb{S}^{d-1}$. De plus, les clusters ne correspondent plus à des composantes connexes du graphe, et cela entraîne quelques complications lorsque nous manipulons des unions de clusters dont les cardinaux ne s'additionnent plus.

La borne supérieure est également valide en dimension deux, au contraire de la borne inférieure. La construction pour la borne inférieure utilise des chemins de longueur n , dont la probabilité de l'ordre de $\exp(-cn)$ n'intervient pas dans les estimés à la condition que la dimension d soit supérieure ou égale à trois.

5.3 Autres résultats en percolation orientée. Le résultat suivant est un corollaire du principe de grandes déviations du théorème 7.

THÉORÈME 8. *Soit $d \geq 3$ et $p > \vec{p}_c$. Il existe une constante $c > 0$ telle que*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(n^d \leq |C(0)| < \infty) = -c.$$

A côté du principe de grandes déviations, nous prouvons que la fonction de connectivité décroît exponentiellement vite en dehors du cône de percolation :

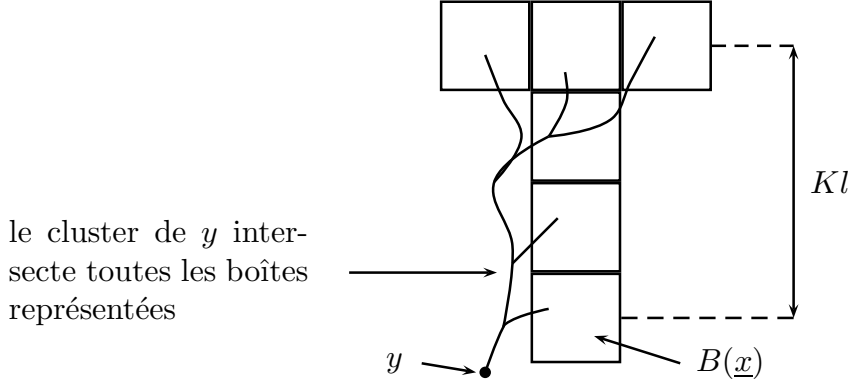
THÉORÈME 9. *Soit $d \geq 3$ et $p > \vec{p}_c$. Soit x n'appartenant pas au cône de percolation. Il existe alors $c > 0$ tel que*

$$P(0 \rightarrow nx) \leq \exp -cn.$$

5.4 Les événements blocs. Nous orientons notre réseau de telle sorte que les arêtes soient dirigées vers le haut. Cela revient en dimension deux à faire une rotation d'angle $\pi/4$. Soit K un entier. Pour $\underline{x}$ appartenant à $\mathbb{Z}^d$, nous notons $B(\underline{x})$ la boîte $] -K/2, K/2]^d + K\underline{x}$. Nous définissons un événement qui décrit l'expansion horizontale des clusters.

Soit l un entier > 0 . Soit $\underline{D}_0$ l'ensemble

$$\underline{D}_0(\underline{x}, l) = \left(\bigcup_{0 \leq i \leq l} \{\underline{x} + ie_d\} \right) \cup \left(\bigcup_{1 \leq d-1} \{\underline{x} + le_d \pm e_i\} \right).$$

figure 9: L'événement R

Nous posons alors

$$R(B(\underline{x}), l) = \left\{ \forall y \text{ tel que } C(y) \cap B(\underline{x}) \neq \emptyset \text{ et } |C(y)| \geq K/2, \right. \\ \left. \text{nous avons } \forall \underline{z} \in \underline{D}_0(\underline{x}, l), C(y) \cap B(\underline{z}) \neq \emptyset \right\},$$

comme représenté sur la figure 9.

Nous prouvons que pour l assez grand,

$$P(R(B(\underline{x}), l)) \rightarrow 1 \quad \text{lorsque } K \rightarrow \infty.$$

Pour comprendre l'intérêt de cet événement, définissons une nouvelle structure de graphe $\widehat{\mathbb{L}}^d$ sur $\mathbb{Z}^d$. Nous mettons une arête orientée de $\underline{x}$ vers $\underline{y}$ pour tout couple $(\underline{x}, \underline{y})$ tel que $\underline{y} \in \underline{D}_0(\underline{x}, l)$. Grâce aux arêtes du type $(\underline{x}, \underline{x} + le_d \pm e_i)$ pour $1 \leq i \leq d-1$, la structure de $\widehat{\mathbb{L}}^d$ est suffisamment riche pour que le point critique de la percolation par site sur ce graphe soit strictement inférieur à 1. Nous disons maintenant qu'un site $\underline{x}$ de $\widehat{\mathbb{L}}^d$ est occupé si nous avons l'événement $R(B(\underline{x}), l)$. Si $(\underline{x}_0, \dots, \underline{x}_n)$ est un chemin orienté de sites occupés dans $\widehat{\mathbb{L}}^d$, et si $y \in \mathbb{Z}^d$ est tel que son cluster intersecte $B(\underline{x}_0)$ et $|C(y)| \geq K/2$, alors le cluster de y intersecte toutes les boîtes $B(\underline{x}_j)$ pour $0 \leq j \leq n$.

5.5 Le cristal de Wulff. Soit τ une fonction continue de $\mathbb{S}^{d-1}$ dans $\mathbb{R}_+$. Le cristal de Wulff associé est défini par

$$\mathcal{W}_\tau = \{x \in \mathbb{R}^d : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^{d-1}\}.$$

C'est un ensemble fermé, borné et convexe.

Dans les modèles de percolation, la fonction τ représente le coût d'une surface d'arêtes fermées s'appuyant sur les bords d'un hyper-rectangle. Elle ne dépend que du vecteur

normal à cet hyper-rectangle. En percolation classique, le cristal de Wulff contient 0 en son intérieur, et sa forme varie de la sphère lorsque p est proche de p_c , à l'hypercube lorsque p tend vers 1. Dans le modèle de la percolation orientée, le cristal de Wulff est inclus dans un cône et présente une singularité en 0.

Le cristal de Wulff correspond à la forme typique des grands clusters finis en percolation non-orientée. Pour obtenir ce résultat, il faut disposer d'un principe de grandes déviations et savoir que le cristal de Wulff est l'unique solution d'un principe variationnel. Le théorème 7 fournit la première partie. Malheureusement, le problème variationnel de Wulff n'est résolu que pour des fonctions τ strictement positives. Il faudra donc reprendre la résolution de ce problème dans notre cas pour pouvoir obtenir le cristal de Wulff comme forme d'un grand cluster fini.

6 La percolation à orientation aléatoire

Durant cette thèse je me suis intéressé au modèle à orientation aléatoire décrit ci-après. Cette recherche n'a pas abouti à montrer qu'il y a percolation dans ce modèle dès que la symétrie est brisée.

Dans le graphe $\mathbb{Z}^2$, nous orientons les arêtes positivement avec probabilité p , et négativement avec probabilité $1 - p$. Nous en donnons une réalisation figure 10.

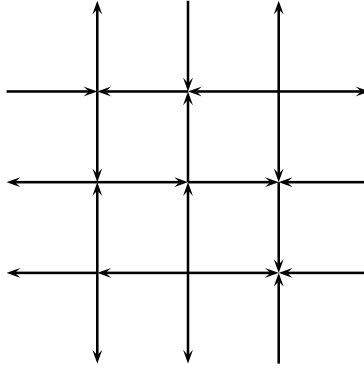


figure 10: des arêtes orientées aléatoirement

Lorsque $p = 1/2$, en comparant avec le modèle classique, nous nous apercevons qu'il n'y a pas percolation. Que pouvons-nous dire lorsque $p > 1/2$? Par comparaison avec le modèle orienté, il y a percolation lorsque $p > \vec{p}_c$. Il est en fait conjecturé qu'il y a des chemins orientés infinis dès que $p > 1/2$. Des simulations numériques semblent le confirmer. En introduisant le dual du processus à orientation aléatoire, nous pouvons montrer que le processus n'est pas sous-critique [10].

L'une des difficultés de ce modèle est que nous ne disposons plus de l'inégalité FKG. Cela peut être résolu comme dans [10] en remplaçant chaque arête de $\mathbb{Z}^2$ par deux arêtes orientées en sens contraire. L'arête qui est dans le sens positif est ouverte avec probabilité

p , celle qui est dans le sens négatif est ouverte avec probabilité $1 - p$. En ce qui concerne l'existence de chemins infinis, les deux modèles sont équivalents. Cependant des questions demeurent spécifiques au modèle à orientation aléatoire. Par exemple, l'inégalité “anti-FKG” suivante devrait être valide : pour tout x, y, z de $\mathbb{Z}^2$,

$$P(x \rightarrow y, y \rightarrow z) \leq P(x \rightarrow y)P(y \rightarrow z).$$

7 Organisation de la thèse

Chacun des chapitres suivant est un article rédigé en anglais. Le chapitre 2 contient l'article “Surface order large deviation for $2D$ FK percolation and Potts models”, qui est un travail réalisé en collaboration avec Réda-Jürg Messikh et correspond à la section 2 de ce chapitre introductif. Le chapitre 3 contient l'article “A large deviation result for Bernoulli percolation” et correspond à la section 3. Le chapitre 4 est constitué de l'article “Poisson approximation for large finite clusters in the supercritical FK model” et correspond à la section 4. Le chapitre 5 contient l'article “Surface large deviations for supercritical oriented percolation” et est consacré à l'étude de la percolation orientée en dimensions supérieures à trois.

Cette thèse a été rédigée en utilisant les logiciels emacs et ams- $\text{\TeX}$. Les deux livres que j'ai utilisés pour l'utilisation de $\text{\TeX}$ sont celui de R. Séroul [15] et le $\text{\TeX}$ book de D. E. Knuth [12], ainsi que sa traduction française réalisée par J.-C. Charpentier.

Bibliography

1. R. Arratia, L. Goldstein and L. Gordon, *Two moments suffice for Poisson approximations: The Chen-Stein method*, Ann. Prob. **17** (1989), 9–25.
2. P. Brémaud, *Introduction aux probabilités*, Springer.
3. R. Cerf, *The Wulff crystal in Ising and Percolation models*, Saint-Flour lecture notes, first version (2004).
4. A. Dembo, O. Zeitouni, *Large deviations techniques and applications*, Second edition, Springer, New York, 1998.
5. J.-D. Deuschel, Á. Pisztora, *Surface order large deviations for high-density percolation*, Probab. Theory Relat. Fields **104** (1996), 467–482.
6. R. Durrett, *Oriented percolation in two dimensions*, Ann. Probab. **12** (1984), 999–1040.
7. K. J. Falconer, *The Geometry of Fractals Sets*, Cambridge.
8. C. Fortuin, P. Kasteleyn and J. Ginibre, *Correlation inequalities on some partially ordered sets*, Commun. Math. Phys. **22** (1971), 89–103.
9. G. R. Grimmett, *Percolation*, Second Edition, vol. 321, Springer, 1999.
10. G. R. Grimmett, *Infinite paths in randomly oriented lattices*, Random Structures Algorithms **18** (2001), 257–266.
11. G. R. Grimmett, *The random cluster model* **110** (2003), Springer, Probability on Discrete Structures. Ed. H. Kesten, Encyclopedia of Mathematical Sciences, 73–123.
12. D. E. Knuth, *The T_EXbook*, Addison Wesley Publishing Company.
13. Y. Kovchegov, S. Sheffield, *Linear speed large deviations for percolation clusters*, Preprint (2003).
14. Á. Pisztora, *Surface order large deviations for Ising, Potts and percolation models*, Probab. Theory Relat. Fields **104** (1996), 427–466.
15. R. Sérout, *Le petit livre de T_EX*, Deuxième édition, Masson.

Chapitre 2

Surface order large deviations for 2D FK-percolation and Potts model

Joint work with Reda-Jürg Messikh

Abstract: By adapting the renormalization techniques of Pisztor, [32], we establish surface order large deviations estimates for FK-percolation on $\mathbb{Z}^2$ with parameter $q \geq 1$ and for the corresponding Potts models. Our results are valid up to the exponential decay threshold of dual connectivities which is widely believed to agree with the critical point.

Keywords: Large deviations, FK-percolation, Potts models.

1991 Mathematics Subject Classification: 60F10, 60K35, 82B20, 82B43.

1 Introduction

In this paper we derive surface order large deviations for Bernoulli percolation, FK-percolation with parameter $q > 1$ and for the corresponding Potts models on the planar lattice $\mathbb{Z}^2$.

In dimension two, surface order large deviations behaviour and the Wulff construction has been established for the Ising model [15, 16, 23, 24, 25, 26, 30, 31, 33, 34, 35, 36], for independent percolation [3, 5] and for the random cluster model [4]. These works include also more precise results than large deviations for the Wulff shape. They are obtained by using the skeleton coarse graining technique to study dual contours which represent the interface. In higher dimensions other methods had to be used to achieve the Wulff construction, [8, 10, 11, 12], where one of the main tools that have been used was the blocks coarse graining of Pisztor [32]. This renormalization technique led to surface order large deviations estimates for FK-percolation and for the corresponding Potts models simultaneously. The results of [32], and thus the Wulff construction in higher dimensions, are valid up to the limit of the slab percolation thresholds. In the case of independent percolation, this threshold has been proved to agree with the critical point [21] and recently it has also been proved in the case $q = 2$ [9]. Otherwise, it is believed to be so for all the FK-percolation models with parameter $q \geq 1$ in dimension greater than two.

Our aim is to import Pisztor's blocks techniques [32] to the two-dimensional lattice as an alternative to the use of contours. It is also worth noting that Pisztor's renormalization technique forms a building block that has been used to answer various other questions related to percolation [6, 7, 28, 29]. The main point in our task is to get rid of the percolation in slabs which is specific to the higher dimensional case. For this we produce estimates analogue to those of theorem 3.1 in [32] relying on the hypothesis that the dual connectivities decay exponentially. This hypothesis is very natural in $\mathbb{Z}^2$, because it is possible to translate events from the supercritical regime to the subcritical regime by planar duality. For Bernoulli percolation, the exponential decay of connectivities is known to hold in all the subcritical regime, see [17] and the references therein. For the random cluster model on $\mathbb{Z}^2$ with $q = 2$ the exponential decay follows from the exponential decay of the correlation function in the Ising model [13], and a proof has also been given when q is greater than 25.72, see [19] and the references therein. Even if not proved, the exponential

decay of the connectivities is widely believed to hold up to the critical point of all the FK-percolation models with $q \geq 1$. In addition to that, we use a property which is specific to the two dimensional case, namely the weak mixing property. This property has been proved to hold for all the random cluster models with $q \geq 1$ in the regime where the connectivities decay exponentially [1]. We need this property in order to use the exponential decay in finite boxes [2].

2 Statement of results

Our results concern asymptotics of FK-measures on finite boxes

$$B(n) = (-n/2, n/2]^2 \cap \mathbb{Z}^2,$$

where n is a positive integer. We will denote by $\mathcal{R}(p, q, B(n))$ the set of these FK-measures defined on $B(n)$ with parameters (p, q) and where we have identified some vertices of the boundary. For $q \geq 1$ and $0 < p \neq p_c(q) < 1$, it is known [20] that there is a unique infinite volume Gibbs measure that we will note $\Phi_\infty^{p,q}$. It is also known that $\Phi_\infty^{p,q}$ is translation invariant and ergodic. In the uniqueness region, we will denote by $\theta = \theta(p, q)$ the density of the infinite cluster. As the exponential-decay plays a crucial rule in our analysis, we will introduce the following threshold¹

$$p_g = \sup\{p : \exists c > 0, \forall x \forall y \in \mathbb{Z}^2, \Phi_\infty^{p,q}[x \leftrightarrow y] \leq \exp(-c|x - y|)\}, \quad (2.1)$$

where $|x - y|$ is the $\mathcal{L}^1$ norm and $\{x \leftrightarrow y\}$ is the event that there exists an open path joining the vertex x to the vertex y .

By the results of [22], it is known that exponential decay holds as soon as the connectivities decay at a sufficient polynomial rate. We thus could replace (2.1) by

$$p_g = \sup\{p : \exists c > 0, \forall x \forall y \in \mathbb{Z}^2, \Phi_\infty^{p,q}[x \leftrightarrow y] \leq c/|x - y|\}.$$

We introduce the point dual to p_g :

$$\widehat{p}_g = \frac{q(1 - p_g)}{p_g + q(1 - p_g)} \geq p_c(q),$$

which is conjectured to agree with the critical point $p_c(q)$.

Our result states that up to large deviations of surface order, there exists a unique biggest cluster in the box $B(n)$ with the same density than the infinite cluster, and that the set of clusters of intermediate size has a negligible volume. To be more precise, we say

¹The notation p_g comes from [19].

that a cluster in $B(n)$ is *crossing* if it intersects all the faces of $B(n)$. For $l \in \mathbb{N}$, we say that a cluster is *l -intermediate* if it is not of maximal volume and its diameter does exceed l . We denote by $\mathbb{J}_l$ the set of l -intermediate clusters. Let us set the event

$$K(n, \varepsilon, l) = \left\{ \begin{array}{l} \exists! \text{ open cluster } C_m \text{ in } B(n) \text{ of maximal volume,} \\ C_m \text{ is crossing, } n^{-2}|C_m| \in (\theta - \varepsilon, \theta + \varepsilon), \\ n^{-2} \sum_{C \in \mathbb{J}_l} |C| < \varepsilon \end{array} \right\}$$

THEOREM 2.2. *Let $q \geq 1, 1 > p > \widehat{p}_g$ and $\varepsilon \in (0, \theta/2)$ be fixed. Then there exists a constant L such that*

$$\begin{aligned} -\infty &< \liminf_{n \rightarrow \infty} \frac{1}{n} \log \inf_{\Phi \in \mathcal{R}(p, q, B(n))} \Phi[K(n, \varepsilon, L)^c] \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p, q, B(n))} \Phi[K(n, \varepsilon, L)^c] < 0. \end{aligned}$$

This result, via the FK-representation, can be used as in [32] to deduce large deviations estimates for the magnetization of the Potts model. We omit this as it would be an exact repetition of theorem 1.1 and theorem 5.4 in [32].

Organization of the paper: In the following section we introduce notation and give a summary of the FK model and of the duality in the plane. In section 1, we study connectivity properties of FK percolation in a large box $B(n)$ and establish estimates that will be crucial for the renormalization à la Pisztora. In section 2, we introduce the renormalization and proof estimates on the N-block process. In section 3, we finally give the proof of theorem 2.2.

3 Preliminaries

In this section we introduce the notation used and the basic definitions.

Norm and the lattice: We use the $\mathcal{L}^1$ -norm on $\mathbb{Z}^2$, that is, $|x - y| = \sum_{i=1,2} |x_i - y_i|$ for any x, y in $\mathbb{Z}^2$. For every subset A of $\mathbb{Z}^2$ and $i = 1, 2$ we define $\text{diam}_i(A) = \sup\{|x_i - y_i| : x, y \in A\}$ and the diameter of A is $\text{diam}(A) = \max(\text{diam}_1(A), \text{diam}_2(A))$. We turn $\mathbb{Z}^2$ into a graph $(\mathbb{Z}^2, \mathbb{E}^2)$ with vertex set $\mathbb{Z}^2$ and edge set $\mathbb{E}^2 = \{\{x, y\}; |x - y| = 1\}$. If x and y are nearest neighbors, we denote this relation by $x \sim y$.

Geometric objects: A box Λ is a finite subset of $\mathbb{Z}^2$ of the form $\mathbb{Z}^2 \cap [a, b] \times [c, d]$. For $\underline{r} \in (0, \infty)^2$, we define the box $B(\underline{r}) = \mathbb{Z}^2 \cap \Pi_{i=1,2}(-r_i/2, r_i/2]$. We say that the box is symmetric if $r_1 = r_2 = r$, and we denote it by $B(r)$. For $t \in \mathbb{R}^+$, we note the set

$\mathcal{H}_2(t) = \{\underline{r} \in \mathbb{R}^2 : r_i \in [t, 2t], i = 1, 2\}$. The set of all boxes in $\mathbb{Z}^2$, which are congruent to a box $B(\underline{r})$ with $\underline{r} \in \mathcal{H}_2(t)$, is denoted by $\mathcal{B}_2(t)$.

Discrete topology: Let A be a subset of $\mathbb{Z}^2$. We define two different boundaries:

- the inner vertex boundary: $\partial A = \{x \in A \mid \exists y \in A^c \text{ such that } y \sim x\}$;
- the edge boundary: $\partial^{edge} A = \{\{x, y\} \in \mathbb{E}^2 \mid x \in A, y \in A^c\}$.

For a box Λ and for each $i = \pm 1, \pm 2$, we define the i th face $\partial_i \Lambda$ of Λ by $\partial_i \Lambda = \{x \in \Lambda \mid x_i \text{ is maximal}\}$ for i positive and $\partial_i \Lambda = \{x \in \Lambda \mid x_{|i|} \text{ is minimal}\}$ for i negative. A *path* γ is a finite or infinite sequence $x_1, x_2, \dots$ of distinct nearest neighbors.

FK percolation.

Edge configurations: The basic probability space for the edge processes is given by $\Omega = \{0, 1\}^{\mathbb{E}^2}$; its elements are called *edge configurations in $\mathbb{Z}^2$* . The natural projections are given by $\text{pr}_e : \omega \in \Omega \mapsto \omega(e) \in \{0, 1\}$, where $e \in \mathbb{E}^2$. An edge e is called open in the configuration ω if $\text{pr}_e(\omega) = 1$, and closed otherwise.

For $E \subseteq \mathbb{E}^2$ with $E \neq \emptyset$, we write $\Omega(E)$ for the set $\{0, 1\}^E$; its elements are called *configurations in E* . Note that there is a one-to-one correspondence between cylinder sets and configurations on finite sets $E \subset \mathbb{E}^2$, which is given by $\eta \in \Omega(E) \mapsto \{\eta\} := \{\omega \in \Omega \mid \omega(e) = \eta(e) \text{ for every } e \in E\}$. We will use the following convention: the set Ω is regarded as a cylinder (set) corresponding to the “empty configuration” (with the choice $E = \emptyset$.) We will sometimes identify cylinders with the corresponding configuration. For $A \subset \mathbb{Z}^2$, we set $\mathbb{E}(A) = \{(x, y) : x, y \in A, x \sim y\}$. Let Ω_A stand for the set of the configurations in $A : \{0, 1\}^{\mathbb{E}(A)}$ and Ω^A for the set of the configurations *outside* $A : \{0, 1\}^{\mathbb{E}^2 \setminus \mathbb{E}(A)}$. In general, for $A \subseteq B \subseteq \mathbb{Z}^2$, we set $\Omega_B^A = \{0, 1\}^{\mathbb{E}(B) \setminus \mathbb{E}(A)}$. Given $\omega \in \Omega$ and $E \in \mathbb{E}^2$, we denote by $\omega(E)$ the restriction of ω to $\Omega(E)$. Analogously, ω_B^A stands for the restriction of ω to the set $\mathbb{E}(B) \setminus \mathbb{E}(A)$.

Given $\eta \in \Omega$, we denote by $\mathcal{O}(\eta)$ the set of the edges of $\mathbb{E}^2$ which are open in the configuration η . The connected components of the graph $(\mathbb{Z}^2, \mathcal{O}(\eta))$ are called η -clusters. The path $\gamma = (x_1, x_2, \dots)$ is said to be η -open if all the edges $\{x_i, x_{i+1}\}$ belong to $\mathcal{O}(\eta)$. We write $\{A \leftrightarrow B\}$ for the event that there exists an open path joining some site in A with some site in B .

If $V \subseteq \mathbb{Z}^2$ and E consists of all the edges between vertices in V , the graph $G = (V, E) \subseteq (\mathbb{Z}^2, \mathbb{E}^2)$ is called the maximal subgraph of $(\mathbb{Z}^2, \mathbb{E}^2)$ on the vertices V . Let ω be an edge configuration in $\mathbb{Z}^2$ (or in a subgraph of $(\mathbb{Z}^2, \mathbb{E}^2)$). We can look at the *open clusters in V* or alternatively the open V -clusters. These clusters are simply the connected components of the random graph $(V, \mathcal{O}(\omega(E)))$, where $\omega(E)$ is the restriction of ω to E .

For $A \subseteq B \subseteq \mathbb{Z}^2$, we use the notation $\mathcal{F}_B^A$ for the σ -field generated by the finite-dimensional cylinders associated with configurations in Ω_B^A . If $A = \emptyset$ or $B = \mathbb{Z}^2$, then we omit them from the notation. *Stochastic domination* There is a partial order $\preceq$ in Ω given by $\omega \preceq \omega'$ iff $\omega(e) \leq \omega'(e)$ for every $e \in \mathbb{E}^2$. A function $f : \Omega \rightarrow \mathbb{R}$ is called *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega \preceq \omega'$. An event is called increasing if its characteristic function

is increasing. Let $\mathcal{F}$ be a σ -field of subsets of Ω . For a pair of probability measures μ and ν on $(\Omega, \mathcal{F})$, we say that μ (*stochastically*) *dominates* ν if for any $\mathcal{F}$ -measurable increasing function f the expectations satisfy $\mu(f) \geq \nu(f)$. *FK measures* Let $V \subseteq \mathbb{Z}^2$ be finite and $E = \mathbb{E}(V)$. We first introduce (*partially wired*) *boundary conditions* as follows. Consider a partition π of the set ∂V , say $\{B_1, \dots, B_n\}$. (The sets B_i are disjoint nonempty subsets of ∂V with $\bigcup_{i=1, \dots, n} B_i = \partial V$.) We say that $x, y \in \partial V$ are π -wired, if $x, y \in B_i$ for an $i \in \{1, \dots, n\}$. Fix a configuration $\eta \in \Omega_V$. We want to count the η -clusters in V in such a way that π -wired sites are considered to be connected. This can be done in the following formal way. We introduce an equivalence relation on V : x and y are said to be $\pi \cdot \eta$ -wired if they are η -connected or if they are both joined by η -open paths to (or identical with) sites $x', y' \in \partial V$ which are themselves π -wired. The new equivalence classes are called $\pi \cdot \eta$ -clusters, or η -clusters in V with respect to the boundary condition π . The number of η -clusters in V with respect to the boundary condition π (i.e., the number of $\pi \cdot \eta$ -clusters) is denoted by $\text{cl}^\pi(\eta)$. (Note that cl^π is simply a random variable). For fixed $p \in [0, 1]$ and $q \geq 1$, the *FK measure on the finite set $V \subset \mathbb{Z}^2$ with parameters (p, q) and boundary conditions π* is a probability measure on the σ -field $\mathcal{F}_V$, defined by the formula

$$\forall \eta \in \Omega_V \quad \Phi_V^{\pi, p, q}[\{\eta\}] = \frac{1}{Z_V^{\pi, p, q}} \left(\prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)} \right) q^{\text{cl}^\pi(\eta)}, \quad (3.1)$$

where $Z_V^{\pi, p, q}$ is the appropriate normalization factor. Since $\mathcal{F}_V$ is an atomic σ -field with atoms $\{\eta\}, \eta \in \Omega_V$, formula (3.1) determines a unique measure on $\mathcal{F}_V$. Note that every cylinder has nonzero probability. There are two extremal b.c.s: the free boundary condition corresponds to the partition f defined to have exactly $|\partial V|$ classes, and the wired b.c corresponds to the partition w with only one class. The set of all such measures called FK (or random cluster) measures corresponding to different b.c.s will be denoted by $\mathcal{R}(p, q, V)$. The stochastic process $(\text{pr}_e)_{e \in \mathbb{E}(V)} : \Omega \rightarrow \Omega_V$ given on the probability space $(\Omega, \mathcal{F}, \Phi_V^{\pi, p, q})$ is called *FK percolation with boundary conditions π* . We list some useful properties of FK measures with different b.c.s. There is a partial order on the set of partitions of ∂V . We say that π *dominates* π' , $\pi \geq \pi'$, if x, y π' -wired implies that they are π -wired. We then have $\Phi_V^{\pi', p, q} \preceq \Phi_V^{\pi, p, q}$. This implies immediately that for each $\Phi \in \mathcal{R}(p, q, V)$,

$$\Phi_V^{f, p, q} \preceq \Phi \preceq \Phi_V^{w, p, q}.$$

Next we discuss properties of conditional FK measures. For given $U \subseteq V$ and $\omega \in \Omega$, we define a partition $W_V^U(\omega)$ of ∂U by declaring $x, y \in \partial U$ to be $W_V^U(\omega)$ -wired if they are joined by an ω_V^U -open path. Fix a partition π of ∂V . We define a new partition of ∂U to be $\pi \cdot W_V^U(\omega)$ -wired if they are $W_V^U(\omega)$ -wired, or if they are both joined by ω_V^U -open paths to (or identical with) sites x', y' , which are themselves π -wired. Then, for every $\mathcal{F}_U$ -measurable function f ,

$$\Phi_V^{\pi, p, q}[f | \mathcal{F}_V^U](\omega) = \Phi_V^{\pi \cdot W_V^U(\omega), p, q}[f], \quad \Phi_V^{\pi, p, q} \text{ a.s.} \quad (3.2)$$

Note that formula (3.2) can be interpreted as a kind of Markov property. A direct consequence is the *finite-energy property*. Fix an edge e of $\mathbb{E}(V)$ and denote by $\mathcal{F}_V^e$ the σ -algebra generated by the random variables $\{pr_b; b \in \mathbb{E}(V) \setminus \{e\}\}$. Then

$$\Phi_V^{\pi,p,q}[e \text{ is open} | \mathcal{F}_V^e](\omega) = \begin{cases} p & \text{if the endpoints of } e \text{ are } \pi \cdot W_V^e\text{-wired,} \\ p/[p + q(1 - p)] & \text{otherwise.} \end{cases} \quad (3.3)$$

The equality (3.2) leads to volume monotonicity for FK-measures. Let $U \subset V$, for every increasing function $g \in \mathcal{F}_U$ and $\Phi_V \in \mathcal{R}(p, q, V)$, we have

$$\begin{aligned} \Phi_U^{f,p,q}[g] &\leq \Phi_V[g | \mathcal{F}_V^U] \leq \Phi_U^{w,p,q}[g] \quad \Phi_V \text{ a.s.}, \\ \Phi_U^{f,p,q}[g] &\leq \Phi_V^{f,p,q}[g] \leq \Phi_V^{w,p,q}[g] \leq \Phi_U^{w,p,q}[g]. \end{aligned}$$

Planar duality for FK-measures: Because of its importance in our note, we recall the duality property for planar FK-measures, see for example [18]. To this end, we first begin with the following simple but useful observation.

LEMMA 3.4. *For all $0 < p < 1, q > 0$ and for any finite box $B \subset \mathbb{Z}^2$ we have that*

$$\forall \omega \in \Omega_B : \Phi_B^{w,p,q}[\omega] = \Phi_{\mathbb{E}(B) \setminus \mathbb{E}(\partial B)}^{w,p,q}[\omega^{\partial B}] \prod_{e \in \mathbb{E}(\partial(B))} p^{\omega_{\partial B}(e)} (1 - p)^{1 - \omega_{\partial B}(e)}$$

PROOF. Each $\omega \in \Omega_B$ is the concatenation of $\omega_{\partial B}$ and $\omega^{\partial B}$ and the result follows from (3.2) by observing that $\text{cl}^w(\omega)$ does not depend on $\omega_{\partial B}$ and is equal to $\text{cl}^w(\omega^{\partial B})$. $\square$

This observation states that:

- The σ -algebras $\mathcal{F}^{\partial B}$ and $\mathcal{F}_{\partial B}$ are independent under $\Phi_B^{w,p,q}$.
- The law of $\omega_{\partial B}$ under $\Phi_B^{w,p,q}$ is the independent percolation of parameter p on $\mathbb{E}(\partial B)$.
- The law of $\omega^{\partial B}$ under $\Phi_B^{w,p,q}$ is the wired FK-measure on $\mathbb{E}(B) \setminus \mathbb{E}(\partial B)$. To construct the dual model we associate to a box B the set $\widehat{B} \subset \mathbb{Z}^2 + (1/2, 1/2)$, which is defined as the smallest box of $\mathbb{Z}^2 + (1/2, 1/2)$ containing B , see figure 1 below.

To each edge $e \in \mathbb{E}(B)$ we associate the edge $\widehat{e} \in \mathbb{E}(\widehat{B})$ that crosses the edge e . Note that $\{e' \in \mathbb{E}(\widehat{B}) : \exists e \in \mathbb{E}(B), \widehat{e} = e'\} = \mathbb{E}(\widehat{B}) \setminus \mathbb{E}(\partial \widehat{B})$.

This allows us to build a bijective application from Ω_B to $\Omega_{\widehat{B}}^{\partial \widehat{B}}$ that maps each original configuration $\omega \in \Omega_B$ into its dual configuration $\widehat{\omega} \in \Omega_{\widehat{B}}^{\partial \widehat{B}}$ such that

$$\forall e \in \mathbb{E}(B) : \widehat{\omega}(\widehat{e}) = 1 - \omega(e).$$

And the duality property is:

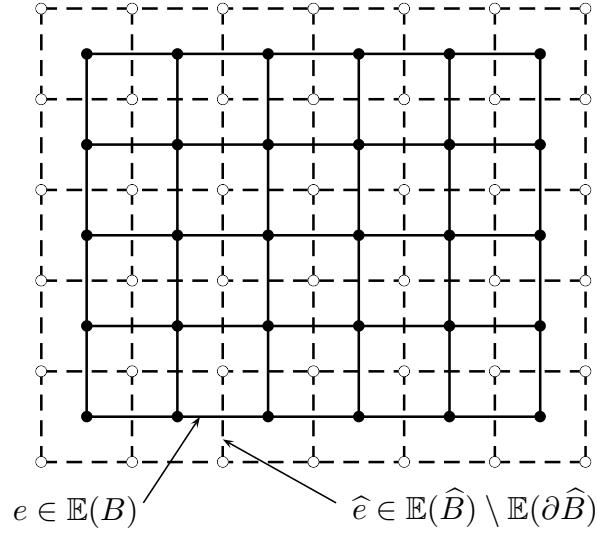


figure 1: A box and its dual

PROPOSITION 3.5. *For all $0 < p < 1, q > 0$ and for all $\omega_d \in \Omega_{\widehat{B}}^{\partial \widehat{B}}$ we have that*

$$\Phi_B^{f,p,q}[\{\omega \in \Omega_B : \widehat{\omega} = \omega_d\}] = \Phi_{\mathbb{E}(\widehat{B}) \setminus \mathbb{E}(\partial \widehat{B})}^{w,\widehat{p},q}[\omega_d],$$

where $\widehat{p}$ is the dual point of p : $\widehat{p} = q(1-p)/(p+q(1-p))$.

PROOF. First we observe that the number of connected components $c(\widehat{\omega})$ of the graph $\widehat{G}(\widehat{\omega}) = (\widehat{B}, \{\widehat{e} \in \mathbb{E}(\widehat{B}) \setminus \mathbb{E}(\partial \widehat{B}) : \widehat{\omega}(\widehat{e}) = 1\} \cup \mathbb{E}(\partial \widehat{B}))$ is equal to $\text{cl}^w(\widehat{\omega})$. Similarly the number of connected components $c(\omega)$ of the graph $G(\omega) = (B, \{e \in \mathbb{E}(B) : \omega(e) = 1\})$ is equal to $\text{cl}^f(\omega)$.

Also one may observe that the number of faces $f(\widehat{\omega})$ of $\widehat{G}(\widehat{\omega})$ is equal to $\text{cl}^f(\omega)$. So that by Euler's formula we get

$$\text{cl}^f(\omega) = \text{cl}^w(\widehat{\omega}) - |\widehat{B}| + |\mathbb{E}(\partial \widehat{B})| + \sum_{\widehat{e} \in \mathbb{E}(\widehat{B}) \setminus \mathbb{E}(\partial \widehat{B})} \widehat{\omega}(\widehat{e}).$$

Thus, for all $\omega \in \Omega_B$ we have

$$q^{\text{cl}^f(\omega)} \prod_{e \in \mathbb{E}(B)} p^{\omega(e)} (1-p)^{1-\omega(e)} = q^{|\mathbb{E}(\partial \hat{B})| - |\hat{B}|} q^{\text{cl}^w(\hat{\omega})} \times \prod_{\hat{e} \in \mathbb{E}(\hat{B}) \setminus \mathbb{E}(\partial \hat{B})} p(q(1-p)/p)^{\hat{\omega}(\hat{e})}.$$

Finally, the parameter $\hat{p}$ such that $q(1-p)/p = \hat{p}/(1-\hat{p})$ is the one given in the proposition and this concludes the proof. $\square$

COROLLARY 3.6. *For any $0 < p < 1, q > 0$, any $\mathcal{F}_B$ -measurable event A we have*

$$\Phi_B^{f,p,q}[A] = \Phi_{\hat{B}}^{w,\hat{p},q}[\hat{A}],$$

where $\hat{A} = \{\eta \in \Omega_{\hat{B}} : \exists \omega \in A, \hat{\omega} = \eta^{\partial \hat{B}}\} \subset \Omega_{\hat{B}}^{\partial \hat{B}}$ is the dual event of A and $\hat{p}$ is given in proposition 3.5.

PROOF. This is a direct consequence of proposition 3.5 and lemma 3.4. $\square$

Remark When we translate an $\mathcal{F}_B$ -measurable event A into its dual $\hat{A}$, we obtain an event which is in $\mathcal{F}_{\hat{B}}^{\partial \hat{B}}$. Thus by lemma 3.4, $\Phi_{\hat{B}}^{w,\hat{p},q}[\hat{A}]$ is independent of the states of the edges in $\mathbb{E}(\partial \hat{B})$.

4 Connectivity in boxes

In this section we establish preliminary estimates on crossing events in boxes. We rely on the exponential decay of the connectivities in the dual subcritical model. The usual definition of the exponential decay is based on the infinite volume FK-measure $\Phi_{\infty}^{p,q}$. But we are concerned by asymptotics of finite volume measures and we would like to use the exponential decay in finite boxes. In order to translate the exponential decay to the finite volume measures we need a control on the effects of boundary conditions. As shown in [1], the infinite FK-measure on $\mathbb{Z}^2$ satisfies the weak mixing property as soon as the connectivities decay exponentially. That is to say for all events A, B which are respectively $\mathcal{F}_{\Lambda}$ measurable and $\mathcal{F}_{\Gamma}$ measurable with $\Lambda, \Gamma \subseteq \mathbb{Z}^2$ then $|\Phi_{\infty}^{p,q}[A|B] - \Phi_{\infty}^{p,q}[A]|$ decreases exponentially in the distance between Λ and Γ . This weak mixing property implies, as proved in [2], that we have exponential decay in finite boxes as soon as the exponential decay for the infinite volume measure holds ($p < p_g$):

PROPOSITION 4.1. *([Theorem 1.2 of [2]]) Let $q \geq 1$ and $p < p_g$. There exists two positive constants c and λ such that for all boxes $\Lambda \subset \mathbb{Z}^2$ and for all x, y in Λ , we have that*

$$\Phi_{\Lambda}^{w,p,q}[x \leftrightarrow y \text{ in } \Lambda] \leq \lambda \exp(-c|x - y|).$$

In fact, theorem 1.2 of [2] is more general and applies to sets Λ which are not boxes and to general boundary conditions. From this result, we get that

LEMMA 4.2. *Let $q \geq 1$ and $p < p_g$. There exists a positive constant c such that for all positive integers n and for l large enough, we have*

$$\sup_{\underline{n} \in \mathcal{H}_2(n)} \Phi_{B(\underline{n})}^{w,p,q}[\exists \text{ an open path in } B(\underline{n}) \text{ of diameter } \geq l] \leq n^2 \exp(-cl).$$

PROOF. Let us fix n and l , then we have

$$\begin{aligned} & \sup_{\underline{n} \in \mathcal{H}_2(n)} \Phi_{B(\underline{n})}^{w,p,q}[\exists \text{ an open path in } B(\underline{n}) \text{ of diameter } \geq l] \\ & \leq 4n^2 \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{x \in B(\underline{n})} \Phi_{B(\underline{n})}^{w,p,q}[x \leftrightarrow \partial B(x, 2l) \text{ in } B(\underline{n})] \\ & \leq 32n^2 l \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{x \in B(\underline{n})} \sup_{y \in \partial B(x, 2l)} \Phi_{B(\underline{n})}^{w,p,q}[x \leftrightarrow y \text{ in } B(\underline{n})] \\ & \leq 32\lambda n^2 l \exp(-cl), \end{aligned}$$

where we used proposition 4.1 in the last line. The result follows by taking l large enough. $\square$

As a first consequence of the exponential decay in finite boxes, we obtain:

LEMMA 4.3. *For $p > \hat{p}_g$ we have,*

$$\lim_{n \rightarrow \infty} \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] = \theta(p, q).$$

PROOF. Let $N < n$, then

$$\begin{aligned} & \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N)] - \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N), 0 \leftrightarrow \partial B(n)] \\ & = \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] \leq \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N)]. \end{aligned} \tag{4.4}$$

Now we estimate $\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N), 0 \leftrightarrow \partial B(n)]$: by symmetry,

$$\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N), 0 \leftrightarrow \partial B(n)] \leq 4\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial_1 B(N), 0 \leftrightarrow \partial B(n)].$$

Then for N large enough we have that

$$\begin{aligned}
\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial_1 B(N), 0 \leftrightarrow \partial B(n)] &\leq \Phi_{\widehat{B}(n)}^{w,\widehat{p},q} \left[\begin{array}{l} \exists k > 0 \exists j \in \mathbb{Z} : \exists \text{ an open} \\ \text{path from } (-k + \frac{1}{2}, \frac{1}{2}) \\ \text{to } (N + \frac{1}{2}, j + \frac{1}{2}) \end{array} \right] \\
&\leq \sum_{k>0, j \in \mathbb{Z}} \exp(-c(N + k + |j|)) \\
&\leq \exp(-cN),
\end{aligned} \tag{4.5}$$

for a certain positive constant c . The second inequality follows from lemma 4.2.

By taking the limit $n \rightarrow \infty$ in (4.5) we get

$$\begin{aligned}
\Phi_{\infty}^{p,q}[0 \leftrightarrow \partial B(N)] - 4e^{-dN} &\leq \liminf_{n \rightarrow \infty} \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] \\
&\leq \limsup_{n \rightarrow \infty} \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] \leq \Phi_{\infty}^{p,q}[0 \leftrightarrow \partial B(N)],
\end{aligned}$$

finally by taking the limit $N \rightarrow \infty$, we get the desired result. $\square$

Next, we define events that will be crucial in the renormalization procedure. For this, we introduce the notion of *crossing*. Let $B \subset \mathbb{Z}^2$ be a finite box. For $i = 1, 2$ we say that a i -*crossing* occurs in B , if $\partial_{-i} B$ and $\partial_i B$ are joined by an open path in B . In addition to that, we say that a cluster C of B is crossing in B , if C contains a 1-crossing path and a 2-crossing path.

For $\underline{n} \in \mathcal{H}_2(n)$, we set

$$U(\underline{n}) = \{\exists! \text{ open cluster } C^* \text{ crossing } B(\underline{n})\}.$$

For a monotone, increasing function $g : \mathbb{N} \rightarrow [0, \infty)$ with $g(n) \leq n$, let us define

$$R^g(\underline{n}) = U(\underline{n}) \cap \left\{ \begin{array}{l} \text{every open path } \gamma \subset B(\underline{n}) \text{ with} \\ \text{diam}(\gamma) \geq g(n) \text{ is contained in } C^* \end{array} \right\}.$$

And finally we set

$$O^g(\underline{n}) = R^g(\underline{n}) \cap \left\{ \begin{array}{l} C^* \text{ crosses every sub-box} \\ \mathcal{Q} \in \mathcal{B}_2(g(n)) \text{ contained in } B(\underline{n}) \end{array} \right\}.$$

The next theorem gives the desired estimates on the above mentioned events.

THEOREM 4.6. Assume $p > \widehat{p}_g$. We have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{\Phi \in \mathcal{R}(p, q, B(\underline{n}))} \Phi[U(\underline{n})^c] < 0. \quad (4.7)$$

Also, there exists a constant $\kappa = \kappa(p, q) > 0$ such that $\liminf_{n \rightarrow \infty} g(n)/\log n > \kappa$ implies

$$\limsup_{n \rightarrow \infty} \frac{1}{g(n)} \log \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{\Phi \in \mathcal{R}(p, q, B(\underline{n}))} \Phi[R^g(\underline{n})^c] < 0. \quad (4.8)$$

There exists a constant $\kappa' = \kappa'(p, q) > 0$ such that $\liminf_{n \rightarrow \infty} g(n)/\log n > \kappa'$ implies

$$\limsup_{n \rightarrow \infty} \frac{1}{g(n)} \log \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{\Phi \in \mathcal{R}(p, q, B(\underline{n}))} \Phi[O^g(\underline{n})^c] < 0. \quad (4.9)$$

Note that in dimension two, if there is a crossing cluster then it is unique.

PROOF.. As $U(\underline{n})^c$ is decreasing we have for every $\Phi \in \mathcal{R}(p, q, B(\underline{n}))$ that

$$\begin{aligned} \Phi[U(\underline{n})^c] &\leq \Phi_{B(\underline{n})}^{f, p, q}[U(\underline{n})^c] \\ &\leq \Phi_{B(\underline{n})}^{f, p, q}[\nexists \text{ 1-crossing for } B(\underline{n})] + \Phi_{B(\underline{n})}^{f, p, q}[\nexists \text{ 2-crossing for } B(\underline{n})] \\ &\leq \sum_{i=1,2} \Phi_{\widehat{B}(\underline{n})}^{w, \widehat{p}, q}[\partial_{-i} \widehat{B}(\underline{n}) \leftrightarrow \partial_i \widehat{B}(\underline{n}) \text{ in } \widehat{B}(\underline{n}) \setminus \partial \widehat{B}(\underline{n})], \end{aligned}$$

the last inequality follows from planar duality: if there is no 1-crossing in the original lattice then $\partial_{-2} \widehat{B}(\underline{n}) \leftrightarrow \partial_2 \widehat{B}(\underline{n})$ in $\widehat{B}(\underline{n}) \setminus \partial \widehat{B}(\underline{n})$ for the corresponding dual configuration. The same argument works for the 2-crossing. Thus, we have that

$$\Phi[U(\underline{n})^c] \leq 2\Phi_{\widehat{B}(\underline{n})}^{w, \widehat{p}, q}[\exists \text{ an open path in } \widehat{B}(\underline{n}) \text{ of diameter } \geq n],$$

and (4.7) follows from lemma 4.2.

For the second inequality, let us note that

$$R^g(\underline{n})^c \subset U(\underline{n})^c \cup \left(U(\underline{n}) \cap \left\{ \exists \text{ an open path } \gamma \text{ of } B(\underline{n}) \text{ with } \begin{array}{l} \text{diam}(\gamma) \geq g(n) \\ \text{not contained in } C^* \end{array} \right\} \right).$$

By (4.7), we only have to deal with the second term.

We consider the dual event of

$$U(\underline{n}) \cap \left\{ \begin{array}{l} \exists \text{ an open path } \gamma \text{ of } B(\underline{n}) \text{ with} \\ \text{diam}(\gamma) \geq g(n) \text{ not contained in } C^* \end{array} \right\}$$

which is $\mathcal{F}_{\widehat{B}(\underline{n})}^{\partial \widehat{B}(\underline{n})}$ -measurable. By the remark after corollary 3.6 we can consider all the edges of $\mathbb{E}(\partial \widehat{B}(\underline{n}))$ as open. Then by proposition 11.2 of [17] there is a unique innermost open circuit in $\widehat{B}(\underline{n})$ containing γ in its interior. From this circuit, we extract an open path living in the graph $(\widehat{B}(\underline{n}), \mathbb{E}(\widehat{B}(\underline{n})) \setminus \mathbb{E}(\partial \widehat{B}(\underline{n})))$ of diameter greater than $g(n)$: without loss of generality, we can suppose that $\text{diam}(\gamma) = \text{diam}_1(\gamma)$ and that $\gamma \leftrightarrow \partial_2 B(\underline{n})$. Among the vertices of the dual circuit surrounding γ , let $\widehat{x}$ be the highest vertex among the most on the left, and let $\widehat{y}$ be the highest vertex among the most on the right. Then there is an arc joining $\widehat{x}$ and $\widehat{y}$ in $(\widehat{B}(\underline{n}), \mathbb{E}(\widehat{B}(\underline{n})) \setminus \mathbb{E}(\partial \widehat{B}(\underline{n})))$. This arc is of diameter larger than $g(n)$. Thus by lemma 4.2 there is a positive constant c such that for n large enough we have that

$$\Phi \left[U(\underline{n}) \cap \left\{ \begin{array}{l} \exists \text{ an open path } \gamma \text{ of } B(\underline{n}) \text{ with} \\ \text{diam}(\gamma) \geq g(n) \text{ not contained in } C^* \end{array} \right\} \right] \leq n^2 \exp[-cg(n)].$$

Take $\alpha > 0$ such that $\alpha c > 1$. Then for g such that $g(n) > 2\alpha \log n / (\alpha c - 1)$ we have

$$\limsup_{n \rightarrow \infty} \frac{1}{g(n)} \log(n^2 \exp[-cg(n)]) < -\frac{1}{\alpha},$$

which concludes the proof of (4.8).

To study $O^g(\underline{n})$, we remark that the number of boxes $\mathcal{Q}$ of $\mathcal{B}_2(g(n))$ contained in $B(\underline{n})$ is bounded by $16n^4$. This implies that for every $\Phi \in \mathcal{R}(p, q, B(\underline{n}))$ one gets

$$\begin{aligned} \Phi[O^g(\underline{n})^c] &\leq \Phi[R^g(\underline{n})^c] + 16n^4 \sup_{\mathcal{Q} \in \mathcal{B}_2(g(n))} \Phi[\nexists \text{ crossing in } \mathcal{Q}] \\ &\leq \Phi[R^g(\underline{n})^c] + 16n^4 \sup_{\mathcal{Q} \in \mathcal{B}_2(g(n))} \Phi_{B(\underline{n})}^{f,p,q}[\nexists \text{ crossing in } \mathcal{Q}] \\ &\leq \Phi[R^g(\underline{n})^c] + 16n^4 \sup_{\mathcal{Q} \in \mathcal{B}_2(g(n))} \Phi_{\mathcal{Q}}^{f,p,q}[\nexists \text{ crossing in } \mathcal{Q}]. \end{aligned}$$

To deduce the last inequality, we notice that $\{\nexists \text{ crossing in } \mathcal{Q}\}$ is a decreasing event and that all the $\mathcal{Q} \in \mathcal{B}_2(g(n))$ are smaller than $B(n)$, thus for all $\mathcal{Q} \in \mathcal{B}_2(g(n))$ that are included in $B(n)$ we have that

$$\Phi_{B(\underline{n})}^{f,p,q}[\nexists \text{ crossing in } \mathcal{Q}] \leq \Phi_{\mathcal{Q}}^{f,p,q}[\nexists \text{ crossing in } \mathcal{Q}].$$

The first term in the r.h.s. has been treated previously. By (4.7) the second term is bounded by $n^4 \exp[-cg(n)]$ for a certain positive constant c and we conclude the proof as before. $\square$

5 Renormalization

In this section we adapt the renormalization procedure introduced in [32] to the two dimensional case. To do this, let $N \geq 24$ be an integer. We say that a subset Λ of $\mathbb{Z}^2$ is a *N-large box* if Λ is a finite box containing a symmetric box of scale-length $3N$, i.e., if $\Lambda = \mathbb{Z}^2 \cap \prod_{i=1,2}(a_i, b_i]$ where $b_i - a_i \geq 3N$ for $i = 1, 2$. When Λ is a *N-large box*, one can partition it with blocks of $\mathcal{B}(N)$. We first define the *N-rescaled box* of Λ : $\Lambda^{(N)} = \{\mathbf{k} \in \mathbb{Z}^2 \mid T_{N\mathbf{k}}(-N/2, N/2]^2 \subseteq \Lambda\}$; where T_a is the translation in $\mathbb{Z}^2$ by a vector $a \in \mathbb{Z}^2$. We turn $\Lambda^{(N)}$ into a graph by endowing it with the set of edges $\mathbb{E}(\Lambda^{(N)})$. Then we define the partitioning blocks:

- If $\mathbf{k} \in \Lambda^{(N)} \setminus \partial\Lambda^{(N)}$ then $B_{\mathbf{k}} = T_{N\mathbf{k}}(-N/2, N/2]^2$.
- If $\mathbf{k} \in \partial\Lambda^{(N)}$ then some care is needed in order to get a partition. In this case we define the set

$$\mathcal{M}(\mathbf{k}) = \{\mathbf{l} \in \mathbb{Z}^2 \mid \mathbf{l} \sim \mathbf{k}, T_{N\mathbf{l}}(-N/2, N/2]^2 \cap \Lambda \neq \emptyset, \\ T_{N\mathbf{l}}(-N/2, N/2]^2 \cap \Lambda^c \neq \emptyset\},$$

and the corresponding blocks become

$$B_{\mathbf{k}} = T_{N\mathbf{k}}(-N/2, N/2]^2 \cup \bigcup_{\mathbf{l} \in \mathcal{M}(\mathbf{k})} (T_{N\mathbf{l}}(-N/2, N/2]^2 \cap \Lambda).$$

The collection of sets $\{B_{\mathbf{k}}, \mathbf{k} \in \Lambda^{(N)}\}$ is a partition of Λ into blocks included in $\mathcal{B}(N)$, see figure 2

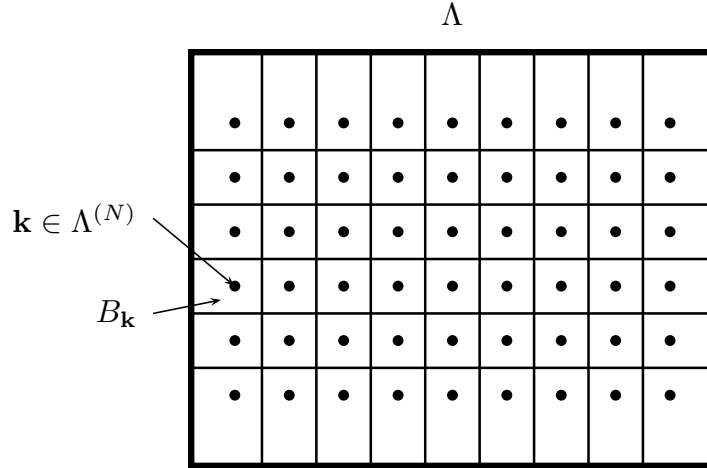


figure 2: The partition of Λ

In addition to the boxes $\{B_{\mathbf{k}}, \mathbf{k} \in \Lambda^{(N)}\}$ we associate to each edge $(\mathbf{k}, \mathbf{l})$ of $\mathbb{E}(\Lambda^{(N)})$ the box $D_{\mathbf{k}, \mathbf{l}}$. More precisely, for $(\mathbf{k}, \mathbf{l}) \in \mathbb{E}(\Lambda^{(N)})$ such that

$$\sum_{j=1,2} |\mathbf{k}_j - \mathbf{l}_j| = \mathbf{k}_i - \mathbf{l}_i = 1,$$

we define

$$m(\mathbf{l}, \mathbf{k}) = T_{N\mathbf{l}}(\lfloor N/2 \rfloor e^{(i)}),$$

where $(e^{(1)}, e^{(2)})$ is the canonical orthonormal base of $\mathbb{Z}^2$ and $\lfloor r \rfloor$ denotes the integer part of r . The point $m(\mathbf{l}, \mathbf{k})$ represents the middle of the i -th face of $B_{\mathbf{l}}$. Then we define the box

$$D_{(\mathbf{l}, \mathbf{k})} = D_{(\mathbf{k}, \mathbf{l})} = T_{m(\mathbf{l}, \mathbf{k})}(B(\lfloor N/4 \rfloor)).$$

Now we have all the needed geometric objects to construct our renormalized (dependent) site percolation process on $(\Lambda^{(N)}, \mathbb{E}(\Lambda^{(N)}))$. This process will depend on the original FK-percolation process only through a number of events defined in the boxes $(B_{\mathbf{k}})_{\mathbf{k} \in \Lambda^{(N)}}$ and $(D_e)_{e \in \mathbb{E}(\Lambda^{(N)})}$. These events are:

- For all $(\mathbf{k}, \mathbf{l}) \in \mathbb{E}(\Lambda^{(N)})$ such that $\sum_{j=1,2} |\mathbf{k}_j - \mathbf{l}_j| = \mathbf{k}_i - \mathbf{l}_i = 1$, we define

$$K_{\mathbf{k}, \mathbf{l}} = \{\exists \text{ } i\text{-crossing in } D_{\mathbf{k}, \mathbf{l}}\}, \quad K_{\mathbf{k}} = \bigcap_{\mathbf{j} \in \Lambda^{(N)} : \mathbf{j} \sim \mathbf{k}} K_{\mathbf{k}, \mathbf{j}}.$$

- For all $\mathbf{i} \in \Lambda^{(N)}$, we define

$$R_{\mathbf{i}} = \{\exists! \text{ a crossing cluster } C_{\mathbf{i}}^* \text{ in } B_{\mathbf{i}}\} \cap \left\{ \text{every open path } \gamma \subset B_{\mathbf{i}} \text{ with } \text{diam}(\gamma) \geq \frac{\sqrt{N}}{10} \text{ is included in } C_{\mathbf{i}}^* \right\}.$$

Finally our renormalized process is the indicator of the occurrence of the above mentioned events:

$$\forall \mathbf{k} \in \Lambda^{(N)} \quad X_{\mathbf{k}} = \begin{cases} 1 & \text{on } R_{\mathbf{k}} \cap K_{\mathbf{k}} \\ 0 & \text{otherwise} \end{cases}$$

We also call the process $\{X_{\mathbf{k}}, \mathbf{k} \in \Lambda^{(N)}\}$ the N -block process and whenever $X_{\mathbf{k}} = 1$, we say that the block $B_{\mathbf{k}}$ is occupied. As explained in [32], the N -block process has the following important geometrical property: if $C^{(N)}$ is a cluster of occupied blocks then there is a unique cluster C of the underlying microscopic FK-percolation process that crosses all the blocks $\{B_{\mathbf{k}}, \mathbf{k} \in C^{(N)}\}$. Moreover, the events involved in the definition of the N -block process become more probable as the size of the blocks increases. This leads us to the following stochastic domination result:

PROPOSITION 5.1. *Let $q \geq 1$ and $p > \hat{p}_g$. Then for N large enough, every N -large box Λ and every measure $\Phi^\pi \in \mathcal{R}(p, q, \Lambda)$, the law of the N -block process $(X_{\mathbf{i}})_{\mathbf{i} \in \Lambda^{(N)}}$ under Φ^π , stochastically dominates independent site percolation on $\Lambda^{(N)}$ with parameter $p(N) = 1 - \exp(-C\sqrt{N})$, where C is a positive constant.*

PROOF. According to [27], it is sufficient to establish that for N large enough and for all $\mathbf{i} \in \Lambda^{(N)}$ the following inequality holds:

$$\Phi^\pi[X_{\mathbf{i}} = 0 \mid \sigma(X_{\mathbf{j}} : |\mathbf{j} - \mathbf{i}| > 1)] \leq \exp(-C\sqrt{N}). \quad (5.2)$$

In what follows, we use the same notation for positive constants that may differ from one line to another. In order to prove (5.2), we consider the set

$$E_{\mathbf{i}} = B_{\mathbf{i}} \cup \bigcup_{\mathbf{j} \sim \mathbf{i}} D_{\mathbf{i}, \mathbf{j}},$$

as drawn in figure 3.

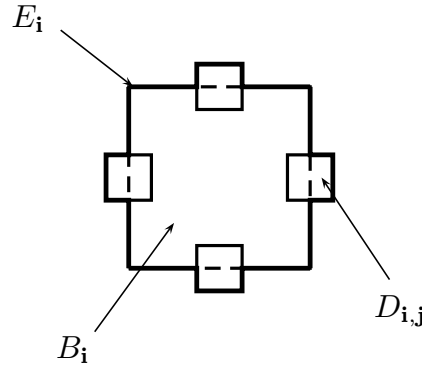


figure 3: The region $E_{\mathbf{i}}$

The σ -algebra $\mathcal{F}_{\Lambda}^{E_{\mathbf{i}}}$ is finer than $\sigma(X_{\mathbf{j}} : |\mathbf{j} - \mathbf{i}| > 1)$, thus it suffices to prove (5.2) for $\Phi^\pi[X_{\mathbf{i}} = 0 \mid \mathcal{F}_{\Lambda}^{E_{\mathbf{i}}}]$. Clearly $\mathcal{F}_{\Lambda}^{E_{\mathbf{i}}}$ is atomic and its atoms are of the form $\{\eta\}$, where $\eta \in \Omega_{\Lambda}^{E_{\mathbf{i}}}$. So let us consider such a $\eta \in \Omega_{\Lambda}^{E_{\mathbf{i}}}$, then we have that

$$\Phi^\pi[X_{\mathbf{i}} = 0 \mid \eta] \leq \sum_{\mathbf{j} \sim \mathbf{i}} \Phi^\pi[K_{\mathbf{i}, \mathbf{j}}^c \mid \eta] + \Phi^\pi[R_{\mathbf{i}}^c \mid \eta]. \quad (5.3)$$

For each $\mathbf{i}, \mathbf{j} \in \Lambda^{(N)}$ such that $\mathbf{i} \sim \mathbf{j}$, let us fix $\eta' \in \Omega_{E_{\mathbf{i}}}^{B_{\mathbf{i}}}$, $\eta'' \in \Omega_{E_{\mathbf{i}}}^{D_{\mathbf{i}, \mathbf{j}}}$ in order to construct $\eta\eta' \in \Omega_{\Lambda}^{B_{\mathbf{i}}}$ and $\eta\eta'' \in \Omega_{\Lambda}^{D_{\mathbf{i}, \mathbf{j}}}$, which are the concatenation of η with η' , respectively with η'' :

$$\eta\eta'(e) = \eta'(e) \text{ for } e \in \mathbb{E}(E_{\mathbf{i}}) \setminus \mathbb{E}(B_{\mathbf{i}}), \quad \eta\eta''(e) = \eta(e) \text{ for } e \in \mathbb{E}(\Lambda) \setminus \mathbb{E}(E_{\mathbf{i}});$$

and

$$\eta\eta''(e) = \eta''(e) \text{ for } e \in \mathbb{E}(E_{\mathbf{i}}) \setminus \mathbb{E}(D_{\mathbf{i},\mathbf{j}}), \quad \eta\eta''(e) = \eta(e) \text{ for } e \in \mathbb{E}(\Lambda) \setminus \mathbb{E}(E_{\mathbf{i}}).$$

Then, by theorem 4.9, there exist an integer $N_0 > 0$ and a real number $C > 0$ such that for all $N > N_0$

$$\begin{aligned} \Phi^\pi[R_{\mathbf{i}}^c \mid \eta\eta'] &= \Phi^{\pi \cdot W_\Lambda^{B_{\mathbf{i}}}(\eta\eta')}[R_{\mathbf{i}}^c] \leq \exp(-C\sqrt{N}), \\ \Phi^\pi[K_{\mathbf{i},\mathbf{j}}^c \mid \eta\eta''] &= \Phi^{\pi \cdot W_\Lambda^{D_{\mathbf{i},\mathbf{j}}}(\eta\eta'')}[K_{\mathbf{i},\mathbf{j}}^c] \leq \exp(-CN). \end{aligned}$$

Finally, by averaging over all the η' and η'' we get from these estimates that

$$\begin{aligned} \Phi^\pi[X_{\mathbf{i}} = 0 \mid \eta] &\leq 4 \exp(-CN) + \exp(-C\sqrt{N}) \\ &\leq \exp(-CN^{1/2}), \end{aligned}$$

for N large enough. $\square$

We end this section by proving a useful estimate on the renormalized process. Let $B(n)$ be a N -large box, consider its N -partition and the corresponding N -block process. The rescaled box $B(n)^{(N)}$ will be denoted by $\mathbf{B}$. For $\delta > 0$ we consider the event

$$Z(n, \delta, N) = \left\{ \begin{array}{l} \exists! \text{ crossing cluster of blocks } \tilde{\mathbf{C}} \\ \text{in } \mathbf{B} \text{ with } |\tilde{\mathbf{C}}| \geq (1 - \delta)|\mathbf{B}| \end{array} \right\}. \quad (5.4)$$

Remark: The event $Z(n, \delta, N)$ has the following interesting property: the presence of the crossing cluster of blocks $\tilde{\mathbf{C}}$ induces a set of clusters $\{\tilde{C}_{\mathbf{i}} \text{ crossing for } B_{\mathbf{i}} : \mathbf{i} \in \tilde{\mathbf{C}}\}$ in the original FK-percolation process. These clusters are connected and form a crossing cluster $\tilde{C}$ for $B(n)$.

PROPOSITION 5.5. *Let $p > \hat{p}_g$ and $q \geq 1$. Then for each $\delta > 0$ and $N > 0$ large enough*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p, q, B(n))} \Phi[Z(n, \delta, N)^c] < 0.$$

PROOF. By theorem 1.1 of [14], there exists $p_0 \in (0, 1)$ such that for all $p > p_0$,

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log \sup_{\underline{m} \in \mathcal{H}_2(m)} P_{B(\underline{m}), \text{site}}^{p, \text{indpt}} \left[\begin{array}{l} \nexists \text{ crossing cluster } \tilde{C} \text{ with} \\ |\tilde{C}| \geq (1 - \delta)|B(\underline{m})| \end{array} \right] < 0. \quad (5.6)$$

Now choose N such as in proposition 5.1 and such that $p(N) > p_0$. Then by proposition 5.1 and by (5.6) we have that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p, q, B(n))} \Phi \left[\begin{array}{l} \text{\textit{A} crossing cluster of blocks } \tilde{\mathbf{C}} \\ \text{in } \mathbf{B} \text{ with } |\tilde{\mathbf{C}}| \geq (1 - \delta)|\mathbf{B}| \end{array} \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{\mathbf{B}, \text{site}}^{p(N), \text{indpt}} \left[\begin{array}{l} \text{\textit{A} crossing cluster } \tilde{\mathbf{C}} \text{ with} \\ |\tilde{\mathbf{C}}| \geq (1 - \delta)|\mathbf{B}| \end{array} \right] < 0. \quad \square \end{aligned}$$

6 Proof of the surface order large deviations

In this section we finally establish theorem 2.2. We begin by stating two lemmas. The first one deals with large deviations from above. Let $\mathbb{B}(n)$ denote the set of clusters in $B(n)$ intersecting $\partial B(n)$. Note that if the crossing cluster exists then it is in $\mathbb{B}(n)$.

LEMMA. *Let $q \geq 1$ and $p \in [0, 1]$. For $\delta > 0$, we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log \sup_{\Phi \in \mathcal{R}(p, q, B(n))} \Phi \left[\sum_{C \in \mathbb{B}(n)} |C| > (\theta + \delta)n^2 \right] < 0.$$

We omit the proof as it would be an exact repetition of Lemma 5.1 in [32].

The second lemma is about large deviations from below and is of surface order, in contrast to lemma 6.0. In section 3, we introduced the event

$$U(n) = \{\exists! \text{ open cluster } C^* \text{ crossing } B(n)\}.$$

For $\delta > 0$, let us define the event

$$V(n, \delta) = U(n) \cap \{|C^*| > (\theta - \delta)n^2\}.$$

LEMMA 6.1. *Let $q \geq 1$ and $p > \hat{p}_g$. Then for each $\delta > 0$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p, q, B(n))} \Phi[V(n, \delta)^c] < 0.$$

PROOF.. From lemma 4.3, we have the inequality:

$$\liminf_{n \rightarrow \infty} \Phi_{B(N)}^f \left[N^{-2} \sum_{C; \text{diam}(C) \geq \sqrt{N}} |C| \right] \geq \theta.$$

Take N such that $\Phi_{B(N)}^f [\sum_{C; \text{diam}(C) \geq \sqrt{N}} |C|] \geq (\theta - \delta/4)N^2$, let $B(n)$ be a N -large box and consider its N -partition and the corresponding N -block process. The rescaled box $B(n)^{(N)}$ will be denoted by $\mathbf{B}$. By proposition 5.5, it suffices to give an upper bound on the probability of the event

$$W(n) = Z(n, \delta/8, N) \cap \{|\tilde{C}| \leq (\theta - \delta)n^2\},$$

where N is large enough and $Z(n, \delta/8, N)$ is defined in (5.4). By remark 5.4, on the event $Z(n, \delta/8, N)$ the crossing cluster $\tilde{C}$ contains all the $B_{\mathbf{i}}$ -crossing clusters $\tilde{C}_{\mathbf{i}}$, where $\mathbf{i} \in \tilde{\mathbf{C}}$ and $\{B_{\mathbf{i}}, \mathbf{i} \in \mathbf{B}\}$ are the partitioning N -blocks. For each $\mathbf{i} \in \mathbf{B}$, set $Y_{\mathbf{i}} = \sum_{C; \text{diam } C \geq N^{1/2}} |C|$, where C is a cluster of $B_{\mathbf{i}}$. Since for $\mathbf{i} \in \tilde{\mathbf{C}}$, $Y_{\mathbf{i}} = |\tilde{C}_{\mathbf{i}}|$, we obtain the following lower bound

$$|\tilde{C}| \geq \sum_{\mathbf{i} \in \tilde{\mathbf{C}}} Y_{\mathbf{i}} \geq \sum_{\mathbf{i} \in \mathbf{B}} Y_{\mathbf{i}} - \sum_{\mathbf{i} \in \mathbf{B} \setminus \tilde{\mathbf{C}}} |B_{\mathbf{i}}| \geq \sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} - (\delta/2)n^2,$$

where $\overset{\circ}{\mathbf{B}} = \mathbf{B} \setminus \partial \mathbf{B}$. Hence on $W(n)$ we have that $\sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} \leq (\theta - \delta/2)n^2$. Denote by $E(n)$ the event that for each $\mathbf{i} \in \overset{\circ}{\mathbf{B}}$ every edge in $\partial^{\text{edge}} B_{\mathbf{i}}$ is closed. Observing that $\sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}}$ is an increasing function, we have for each $\Phi \in \mathcal{R}(p, q, B(n))$,

$$\Phi[W(n)] \leq \Phi_{B(n)}^f \left[\sum_{\mathbf{i} \in \overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} < (\theta - \delta/2)n^2 \middle| E(n) \right] \leq \exp(-C(\delta, \theta, N)n^2),$$

where $C(\delta, \theta, N)$ is a positive constant. The last inequality is an application of Cramér's large deviations theorem, as the variables $(Y_{\mathbf{i}}, \mathbf{i} \in \overset{\circ}{\mathbf{B}})$ are i.i.d. with respect to the conditional measure, with an expected value larger than $(\theta - \delta/4)N^2$. This completes the proof. $\square$

Proof of Theorem 2.2 First we prove the upper bound. By lemma 6.0, we can replace the condition $n^{-2}|C_m| \in (\theta - \varepsilon, \theta + \varepsilon)$ in the definition of $K(n, \varepsilon, l)$ by $n^{-2}|C_m| > (\theta - \varepsilon)$ and denote the new but otherwise unchanged event by $K'(n, \varepsilon, l)$. Set

$$T(n, \varepsilon, N) = Z(n, \varepsilon/4, N) \cap \{|\tilde{C}| > (\theta - \varepsilon)n^2\},$$

where $Z(n, \varepsilon/4, N)$ is defined by (5.4). Fix $\varepsilon < \theta/2$ and N such as in proposition 5.5 and such that $\sqrt{N} \geq 32/\varepsilon$.

Then by proposition 5.5 and by lemma 6.1, we have

$$\limsup_{n \rightarrow \infty} \sup_{\Phi \in \mathcal{R}(p, q, B(n))} \frac{1}{n} \log \Phi[T(n, \varepsilon, N)^c] < 0. \quad (6.2)$$

Set $n \geq 64N/\varepsilon$ and $L = 2N$, we claim that $T(n, \varepsilon, N) \subset K'(n, \varepsilon, L)$. This fact, together with (6.2), implies the upper bound. Therefore, to complete the upper bound we will proof that the cluster $\tilde{C}$ of $T(n, \varepsilon, N)$, is the unique cluster with maximal volume and that the L -intermediate clusters have a negligible volume. So suppose that $T(n, \varepsilon, N)$ occurs. As $\varepsilon < \theta/2$ we have that $L^2 \leq (\theta - \varepsilon)n^2$, thus the clusters of diameter less than L , have a smaller volume than $\tilde{C}$. To control the size of the clusters different from $\tilde{C}$ and of diameter greater than L , we define the following regions:

$$\begin{aligned} \forall \mathbf{i} \in \mathbf{B} : \quad G_{\mathbf{i}} &= \{x \in B_{\mathbf{i}} \mid \text{dist}(x, \partial B_{\mathbf{i}}) \leq \sqrt{N}\} \quad \text{and} \quad Q_{\mathbf{i}} = B_{\mathbf{i}} \setminus G_{\mathbf{i}}, \\ G &= \bigcup_{\mathbf{i} \in \mathbf{B}} G_{\mathbf{i}}, \end{aligned}$$

as shown in figure 4.

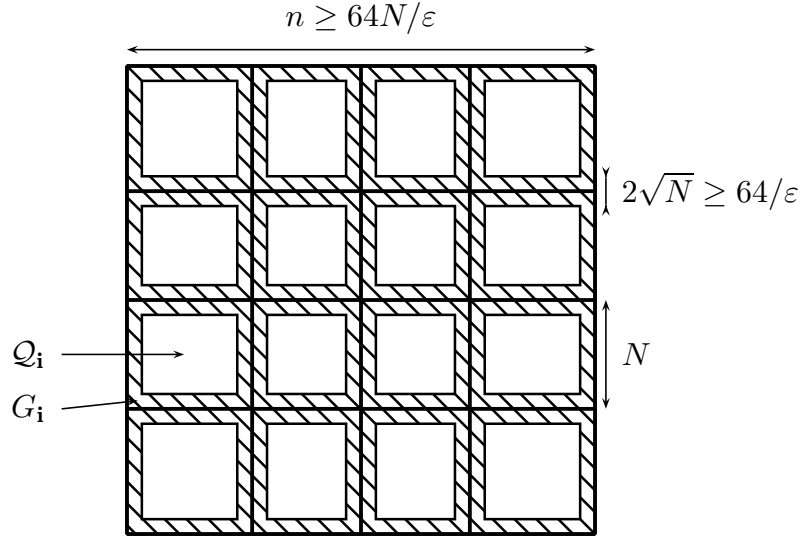


figure 4: The regions $G_{\mathbf{i}}$ and $Q_{\mathbf{i}}$

Then, as $n \geq 64N/\varepsilon$, we have

$$\sum_{\mathbf{i} \in \partial \mathbf{B}} |B_{\mathbf{i}}| \leq 16nN \leq \frac{\varepsilon}{4} n^2,$$

and, as $\sqrt{N} \geq 32/\varepsilon$

$$|G| \leq 8 \frac{n^2}{\sqrt{N}} \leq \frac{\varepsilon}{4} n^2.$$

Take a cluster C of diameter greater than L and different from $\tilde{C}$. Then C touches at least two blocks. However, it may not touch the set $\cup \mathcal{Q}_{\mathbf{i}}$ where $\mathbf{i}$ runs over $\tilde{\mathbf{C}}$; otherwise we would have that $\text{diam}(C \cap B_{\mathbf{i}}) \geq \sqrt{N}$ for an occupied block $B_{\mathbf{i}}$, and therefore we would have that $C = \tilde{C}$. Hence all the clusters of diameter greater than L must lie in the set $G \cup (\cup_{\mathbf{i} \in \tilde{\mathbf{C}}^c} B_{\mathbf{i}})$. Let us estimate the volume of this set:

$$|\cup_{\mathbf{i} \in \tilde{\mathbf{C}}^c} B_{\mathbf{i}}| \leq \sum_{\mathbf{i} \in \partial \mathbf{B}} |B_{\mathbf{i}}| + N^2 |\tilde{\mathbf{C}}^c| < \frac{\varepsilon}{2} n^2.$$

Thus

$$|G \cup (\cup_{\mathbf{i} \in \tilde{\mathbf{C}}^c} B_{\mathbf{i}})| \leq \frac{3\varepsilon}{4} n^2.$$

Since $(3\varepsilon/4)n^2 < (\theta - \varepsilon)n^2$, $\tilde{C}$ is the unique cluster of maximal volume and the L -intermediate class $\mathbb{J}_L$ has a total volume smaller than $(3\varepsilon/4)n^2$. This proves that

$$T(n, \varepsilon, L) \subset K'(n, \varepsilon, L)$$

and completes the proof of the upper bound.

For the lower bound, it suffices to close all the horizontal edges in $B(n)$ intersecting the vertical line $x = 1/2$. This implies that there is no crossing cluster in $B(n)$. By (3.3) and FKG inequality, the probability of this event is bounded from below by $(1 - p)^n$. $\square$

We would like to thank R. Cerf for suggesting the problem and for many helpful discussions.

Bibliography

1. K. S. Alexander, *On weak mixing in lattice models*, Probab. Theory Relat. Fields **110** (1998), 441–471.
2. K. S. Alexander, *Mixing properties and exponential decay for lattice systems in finite volumes*, Annals of Probability **32** (2004), 441–487.
3. K. S. Alexander, *Stability of the Wulff minimum and fluctuations in shape for large finite clusters in two-dimensional percolation*, Probab. Theory Related Fields **91** (1992), 507–532.
4. K. S. Alexander, *Cube-root boundary fluctuations for droplets in random cluster models* Comm. Math. Phys. **224** (2001) 733–781.
5. K. S. Alexander, J. T. Chayes, L. Chayes, *The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation* Comm. Math. Phys. **131** (1990) 1–50.
6. P. Antal, Á. Pisztor, *On the chemical distance for supercritical Bernoulli percolation*, Ann. Probab. **24** (1996), 1036–1048.
7. M. T. Barlow, *Random walks on supercritical percolation clusters* To appear in Ann. Probab.
8. T. Bodineau, *The Wulff construction in three and more dimensions* Comm. Math. Phys. **207** (1999) 197–229.
9. T. Bodineau, *Slab percolation for the Ising model*, available at www.arxiv.org/abs/math.PR/0309300.
10. R. Cerf, *Large deviations for three-dimensional supercritical percolation*, Astérisque **267** (2000).
11. R. Cerf, Á. Pisztor, *On the Wulff crystal in the Ising model*, Ann. Probab. **28** (2000), 947–1017.
12. R. Cerf, Á. Pisztor, *Phase coexistence in Ising, Potts and percolation models*, Ann. I. H. P. **PR 37** (2001), 643–724.
13. J. T. Chayes, L. Chayes, R. H. Schonmann, *Exponential decay of connectivities in the two-dimensional Ising model* J. Stat. Phys. **49** (433–445).
14. J.-D. Deuschel, Á. Pisztor, *Surface order large deviations for high-density percolation* Probab. Theory Relat. Fields **104** (1996) 467–482.
15. R. K. Dobrushin, O. Hryniv, *Fluctuations of the phase boundary in the 2D Ising ferromagnet* Comm. Math. Phys. **189** (1997) 395–445.

16. R. L. Dobrushin, R. Kotecký, S. B. Shlosman, *Wulff construction: a global shape from local interaction* *Amer. Math. Soc. Transl. Ser.* (1992).
17. G. R. Grimmett, *Percolation* Springer, *Grundlehren der mathematischen Wissenschaften* **321** (1999).
18. G. R. Grimmett, *Percolation and disordered systems in Lectures on Probability Theory and Statistics. Lectures from the 26th Summer school on Probability Theory held in Saint Flour, August 19-September 4, 1996* (P. Bertrand, ed.) *Lecture Notes in Mathematics* **1665** (1997).
19. G. R. Grimmett, *The random cluster model* Springer, *Probability on Discrete Structures*, ed. H. Kesten, *Encyclopedia of Mathematical Sciences* **110** (2003) 73–123.
20. G. R. Grimmett, *The stochastic random-cluster process and the uniqueness of random-cluster measures* *Ann. Probab.* **23** (1995) 1461–1510.
21. G. R. Grimmett, J. M. Marstrand, *The supercritical phase of percolation is well behaved* *Proc. R. Soc. Lond. Ser. A* **430** (1990) 439–457.
22. G. R. Grimmett, M. S. T. Piza, *Decay of correlations in subcritical Potts and random-cluster models* *Comm. Math. Phys.* **189** (1997) 465–480.
23. O. Hryniv, *On local behaviour of the phase separation line in the 2D Ising model* *Probab. Theory Related Fields* **102** (1998) 411–432.
24. D. Ioffe, *Large deviation for the 2D Ising model: a lower bound without cluster expansions* *J. Stat. Phys.* **74** (1993) 411–432.
25. D. Ioffe, *Exact large deviation bounds up to T_c for the Ising model in two dimensions* *Probab. Theory Related Fields* **102** (1995) 313–330.
26. D. Ioffe, R. Schonmann, *Dobrushin-Kotecký-Shlosman Theorem up to the critical temperature* *Comm. Math. Phys.* **199** (1998) 117–167.
27. T. M. Liggett, R. H. Schonmann, A. M. Stacey, *Domination by product measures* *Ann. Probab.* **25** (1997) 71–95.
28. P. Mathieu, E. Remy, *Isoperimetry and heat kernel decay on percolation clusters* *Preprint* (2003).
29. M. D. Penrose, Á. Pisztor, *Large deviations for discrete and continuous percolation* *Adv. in Appl. Probab.* **28** (1996) 29–52.
30. C. E. Pfister, *Large deviations and phase separation in the two-dimensional Ising model* *Helv. Phys. Acta* **64** (1991) 953–1054.
31. C. E. Pfister, Y. Velenik, *Large deviations and continuum limit in the 2D Ising model* *Probab. Theory Related Fields* **109** (1997) 435–506.
32. Á. Pisztor, *Surface order large deviations for Ising, Potts and percolation models* *Probab. Theory Relat. Fields* **104** (1996) 427–466.
33. R. H. Schonmann, *Second order large deviation estimates for ferromagnetic systems in the phase coexistence region* *Comm. Math. Phys.* **112** (1987) 409–422.
34. R. H. Schonmann, S. B. Shlosman, *Constrained variational problem with applications to the Ising model* *J. Stat. Phys.* **83** (1996) 867–905.

- 35. R. H. Schonmann, S. B. Shlosman, *Complete analyticity for the 2D Ising model completed* *Comm. Math. Phys.* **179** (1996) 453–482.
- 36. R. H. Schonmann, S. B. Shlosman, *Wulff droplets and the metastable relaxation of kinetic Ising models* *Comm. Math. Phys.* **194** (1998) 389–462.

Chapitre 3

Large deviations for subcritical Bernoulli percolation

Abstract: We consider subcritical Bernoulli percolation in dimensions two and more. If C is the open cluster containing the origin, we prove that the law of C/N satisfies a large deviation principle with respect to the Hausdorff metric.

1991 Mathematics Subject Classification: 60K35

Keywords: subcritical percolation, large deviations

1 Introduction

Consider the cluster C of the origin in the subcritical phase of Bernoulli percolation in $\mathbb{Z}^d$. This is a random object of the space $\mathcal{K}_c$ of connected compact sets in $\mathbb{R}^d$. We let D_H be the Hausdorff distance on $\mathcal{K}_c$. Let

$$\xi = \lim_{N \rightarrow \infty} \frac{1}{N} \ln P(0 \text{ is connected to } Nx).$$

be the inverse correlation length. Assume that $\mathcal{H}_\xi^1$ is the one-dimensional Hausdorff measure on $\mathbb{R}^d$ constructed from ξ .

In the supercritical regime, large deviation principles have been proved for the law of C/N [3,4]. In two dimensions, it relies on estimates of the law of dual clusters, which are subcritical. More precisely, let Γ be a contour in $\mathbb{R}^2$ enclosing an area. The probability that a dual cluster is close for the Hausdorff distance to $N\Gamma$ behaves like $\exp(-N\mathcal{H}_\xi^1(\Gamma))$. But what happens if we consider more general connected sets than contours ?

In this note we establish a large deviation principle for the law of C/N in the subcritical regime in dimensions two and more. Let $\overline{\mathcal{K}}_c$ denote the set of connected compact sets of $\mathbb{R}^d$ quotiented by the translation equivalence. The usual distance between compact sets is the Hausdorff distance. We denote it by $\overline{D}_H$ when considered as a distance on $\overline{\mathcal{K}}_c$. Let C be still the open cluster containing the origin. Write $\overline{C}$ for the equivalent class of C in $\overline{\mathcal{K}}_c$. Let P be the measure and p_c be the critical point of the Bernoulli percolation process. The formulation of our large deviation principle is the following:

THEOREM 1.1. *Let $p < p_c$. Under P , the family of the laws of $(\overline{C}/N)_{N \geq 1}$ on the space $\overline{\mathcal{K}}_c$ equipped with the Hausdorff metric $\overline{D}_H$ satisfies a large deviation principle with good rate function $\mathcal{H}_\xi^1$ and speed N : for every borel subset $\overline{\mathcal{U}}$ of $\overline{\mathcal{K}}_c$,*

$$\begin{aligned} -\inf\{\mathcal{H}_\xi^1(\overline{U}) : \overline{U} \in \text{interior}(\overline{\mathcal{U}})\} &\leq \liminf_{N \rightarrow \infty} \frac{1}{N} \ln P(\overline{C}/N \in \overline{\mathcal{U}}) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \ln P(\overline{C}/N \in \overline{\mathcal{U}}) \\ &\leq -\inf\{\mathcal{H}_\xi^1(\overline{U}) : \overline{U} \in \text{closure}(\overline{\mathcal{U}})\}, \end{aligned}$$

where the interior and the closure are taken with respect to the Hausdorff metric on $\overline{\mathcal{K}_c}$.

The proof of the lower bound relies on the FKG inequality; we use it to construct a cluster close to a given large connected set with a sufficient high probability. Concerning the upper bound, the proof is based on the skeleton coarse graining technique and on the BK inequality; it follows the lines of the proof in [3] with slight adaptations.

We underline that in supercritical percolation the large deviation principles lead to estimates of the shape of large finite clusters. In fact, there exists a shape called the Wulff crystal, which minimizes the rate function under a volume constraint. Unfortunately, the large deviation principle does not allow us to describe the typical shape of a large cluster in the subcritical phase. In this regime, computing simulations of large clusters show very irregular objects.

We note furthermore that our main result has been obtained independently by Kovchegov, Sheffield [11]. Their approach is quite different and makes use of Steiner trees to approximate connected compact sets.

In the next section we recall the definition and basic results of the percolation model. Then we define the measure $\mathcal{H}_\xi^1$ and the space $\overline{\mathcal{K}_c}$. Geometric results required about connected compact sets are given in Section 4. In Section 5 we introduce skeletons, and use them to approximate connected compact sets. The proof of the lower bound follows in Section 6. The coarse graining technique is given in Section 7, and the proof of the upper bound follows in Section 8.

2 The model

We consider the site lattice $\mathbb{Z}^d$ where d is a fixed integer larger than or equal to two. We use the euclidian norm $|\cdot|_2$ on $\mathbb{Z}^d$. We turn $\mathbb{Z}^d$ into a graph $\mathbb{L}^d$ by adding edges between all pairs x, y of points of $\mathbb{Z}^d$ such that $|x - y|_2 = 1$. The set of all edges is denoted by $\mathbb{E}^d$. A path in $(\mathbb{Z}^d, \mathbb{E}^d)$ is an alternating sequence $x_0, e_0, \dots, e_{n-1}, x_n$ of distinct vertices x_i and edges e_i where e_i is the edge between x_i and x_{i+1} .

Let p be a parameter in $(0, 1)$. The edges of $\mathbb{E}^d$ are open with probability p , and closed otherwise, independently from each others. We denote by P the product probability measure on the configuration space $\Omega = \{0, 1\}^{\mathbb{E}^d}$. The measure P is the classic Bernoulli bond percolation measure. Two sites x and y are said *connected* if there is a path of open edges linking x to y . We note this event $\{x \leftrightarrow y\}$. A *cluster* is a connected component of the random graph.

The model exhibits a phase transition at a point p_c , called the *critical point*: for $p < p_c$ the clusters are finite and for $p > p_c$ there exists a unique infinite cluster. We work with a fixed value $p < p_c$.

The following properties describe the behaviour of the tail distribution of the law of a cluster (for a proof see [9]).

LEMMA 2.1. *Let $p < p_c$ and let C be the cluster of the origin. There exists $a_0 > 0$ and $a_1 > 0$ such that for all n*

$$P(|C| \geq n) \leq \exp(-a_0 n), \quad (2.2)$$

$$P(\text{diam } C \geq n) \leq \exp(-a_1 n). \quad (2.3)$$

We briefly recall two fundamental correlation inequalities. To a configuration ω , we associate the set $K(\omega) = \{e \in \mathbb{E}^2 : \omega(e) = 1\}$. Let A and B be two events. The disjoint occurrence $A \circ B$ of A and B is the event

$$\left\{ \begin{array}{l} \omega \text{ such that there exists a subset } H \text{ of } K(\omega) \text{ such that if} \\ \omega', \omega'' \text{ are the configurations determined by } K(\omega') = H \\ \text{and } K(\omega'') = K(\omega) \setminus H, \text{ then } \omega' \in A \text{ and } \omega'' \in B. \end{array} \right\}$$

There is a natural order on Ω defined by the relation: $\omega_1 \leq \omega_2$ if and only if all open edges in ω_1 are open in ω_2 . An event is said to be increasing (respectively decreasing) if its characteristic function is non decreasing (respectively non increasing) with respect to this partial order.

Suppose A and B are both increasing (or both decreasing). The Harris–FKG inequality [7,10] says that $P(A \cap B) \geq P(A)P(B)$. The van den Berg–Kesten inequality [1] says that $P(A \circ B) \leq P(A)P(B)$.

For x, y two sites we consider $\{x \leftrightarrow y\}$ the event that x and y are connected. In the subcritical regime the probability of this event decreases exponentially: for any x in $\mathbb{R}^d$, we denote by $\lfloor x \rfloor$ the site of $\mathbb{Z}^d$ whose coordinates are the integer part of those of x . Then

PROPOSITION 2.4. *The limit*

$$\xi(x) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln P(0 \leftrightarrow \lfloor Nx \rfloor)$$

exists and is > 0 , see [9, section 6.2]. The function ξ thus obtained is a norm on $\mathbb{R}^d$. In addition for every site x in $\mathbb{Z}^d$, we have

$$P(0 \leftrightarrow x) \leq \exp(-\xi(x)). \quad (2.5)$$

Since ξ is a norm there exists a positive constant $a_2 > 0$ such that for all x in $\mathbb{R}^d$,

$$a_2 |x|_2 \leq \xi(x). \quad (2.6)$$

3 The $\mathcal{H}_\xi^1$ measure and the space of the large deviation principle

With the norm ξ , we construct the one-dimensional Hausdorff measure $\mathcal{H}_\xi^1$. If U is a non-empty subset of $\mathbb{R}^d$ we define the ξ -*diameter* of U as

$$\xi(U) = \sup\{\xi(x - y) : x, y \in U\}.$$

If $E \subset \cup_{i \in I} U_i$ and $\xi(U_i) < \delta$ for each i , we say that $\{U_i\}_{i \in I}$ is a δ -cover of E . For every subset E of $\mathbb{R}^d$, and every real $\delta > 0$ we write

$$\mathcal{H}_{\xi, \delta}^1(E) = \inf \sum_{i=1}^{\infty} \xi(U_i),$$

where the infimum is taken over all countable δ -covers of E . Then we define the *one-dimensional Hausdorff measure* of E as

$$\mathcal{H}_\xi^1(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_{\xi, \delta}^1(E).$$

For a study of the Hausdorff measure, see e.g. [6].

We denote by $\mathcal{K}$ the collection of all compact sets of $\mathbb{R}^d$. The euclidian distance between a point and a set E is

$$d(x, E) = \inf\{|x - y|_2 : y \in E\}.$$

We endow $\mathcal{K}$ with the Hausdorff metric $\overline{D}_H$:

$$\forall K_1, K_2 \in \mathcal{K}, \overline{D}_H(K_1, K_2) = \max \left\{ \max_{x_1 \in K_1} d(x_1, K_2), \max_{x_2 \in K_2} d(x_2, K_1) \right\}$$

Let $\mathcal{K}_c$ be the subset of $\mathcal{K}$ consisting of connected sets. An element of $\mathcal{K}_c$ is called a continuum. We define an equivalence on $\mathcal{K}_c$ by: K_1 is equivalent to K_2 if and only if K_1 is a translate of K_2 . We denote by $\overline{\mathcal{K}}_c$ the quotient set of classes of $\mathcal{K}_c$ associated to this relation, and by $\overline{D}_H$ the resulting quotient metric:

$$\overline{D}_H(\overline{K}_1, \overline{K}_2) = \inf_{x_1, x_2 \in \mathbb{R}^d} D_H(K_1 + x_1, K_2 + x_2) = D_H(K_1, \overline{K}_2).$$

We finally define the Hausdorff measure on $\overline{\mathcal{K}}_c$ by

$$\forall \overline{K} \in \overline{\mathcal{K}}_c \quad \mathcal{H}_\xi^1(\overline{K}) = \mathcal{H}_\xi^1(K),$$

which makes sense since $\mathcal{H}_\xi^1$ is invariant by translation on $\mathcal{K}_c$.

Now we state an essential property required by the large deviation principle.

PROPOSITION 3.1. *The measure $\mathcal{H}_\xi^1$ is a good rate function on the space $\overline{\mathcal{K}}_c$.*

PROOF. The lower semicontinuity is due to Golab and the proof can be found in [6, p 39]. We follow now the proof of the proposition 5 in [3]. Let $t > 0$ and let $(\overline{K}_n, n \in \mathbb{N})$ be a sequence in $\overline{\mathcal{K}}_c$ such that $\mathcal{H}_\xi^1(\overline{K}_n) \leq t$ for all n in $\mathbb{N}$. For each n we can assume that the origin belongs to K_n . Since the diameter of an element of $\mathcal{K}_c$ is bounded by a constant time its $\mathcal{H}_\xi^1$ -measure, there exists a bounded set B such that

$$K \in \mathcal{K}_c, 0 \in K, \mathcal{H}_\xi^1(K) \leq t \Rightarrow K \subset B.$$

Thus, the sets K_n are subsets of B . For every compact set K_0 the subset $\{K \in \mathcal{K} : K \subset K_0\}$ is itself compact with respect to the metric D_H [2]. Hence $(K_n)_{n \in \mathbb{N}}$ admits a subsequence converging for the metric D_H ; the same subsequence of $(\overline{K}_n)_{n \in \mathbb{N}}$ converges for the metric $\overline{D}_H$. $\square$

4 Curves and continua

A *curve* is a continuous injection $\Gamma : [a, b] \rightarrow \mathbb{R}^d$, where $[a, b] \subset \mathbb{R}$ is a closed interval. We write also Γ for the image $\Gamma([a, b])$. We call $\Gamma(a)$ the *first point* of the curve and $\Gamma(b)$ its *last point*. Any curve is a continuum. We say that a curve is *rectifiable* if its $\mathcal{H}_\xi^1$ -measure is finite.

We state a simple lemma:

LEMMA 4.1. *For each curve $\Gamma : [a, b] \rightarrow \mathbb{R}^d$,*

$$\mathcal{H}_\xi^1(\Gamma) \geq \mathcal{H}_\xi^1([\psi(a), \psi(b)]) = \xi(\psi(a) - \psi(b)).$$

Next, we associate to a continuum a finite family of curves in two different manners. With the first one, we shall prove the lower bound, and with the second one, we shall prove the upper bound.

DEFINITION 4.2. *A family of curves $\{\gamma_i\}_{i \in I}$ is said *hardly disjoint* if for all $i \neq j$, the curve γ_j can intersect γ_i only on one of the endpoints of γ_i .*

PROPOSITION 4.3. *Let Γ be a continuum with $\mathcal{H}_\xi^1(\Gamma) < \infty$. Then for all parameter $\delta > 0$, there exists a finite family $\{\Gamma_i\}_{i \in I}$ of rectifiable curves included in Γ such that $D_H(\Gamma, \cup_{i \in I} \Gamma_i) < \delta$, $\cup_{i \in I} \Gamma_i$ is connected and the family $\{\Gamma_i\}_{i \in I}$ is hardly disjoint.*

Furthermore, there exists a deterministic way to choose the Γ_i 's such that if Γ' is a translate of Γ , the resultant Γ'_i 's are the translates of the Γ_i 's by the same vector.

PROPOSITION 4.4. *Let Γ be a continuum with $\mathcal{H}_\xi^1(\Gamma) < \infty$. Then for all parameter $\delta > 0$, there exists a finite family $\{\Gamma_i\}_{i \in I}$ of rectifiable curves included in Γ such that $D_H(\Gamma, \cup_{i \in I} \Gamma_i) < \delta$, with the following properties: the euclidian diameter of Γ_i is larger than δ for all i in I , $\cup_{i=1}^l \Gamma_i$ is connected for all $l \geq 1$, and the first point of Γ_l is in $\cup_{k < l} \Gamma_k$.*

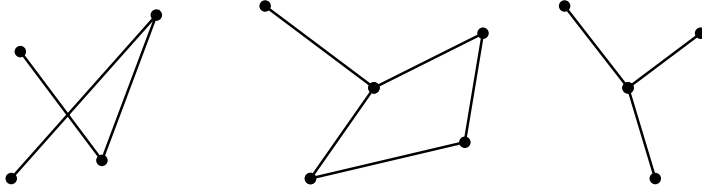
Propositions 4.3 and 4.4 are corollaries of lemma 3.13 of [6] in which we have stated the additional facts coming from the proof. $\square$

We often think of $\mathbb{L}^d$ as embedded in $\mathbb{R}^d$, the edges $\{x, y\}$ being straight line segments $[x, y]$. An *animal* is a finite connected subgraph of $\mathbb{L}^d$ containing the origin. The Hausdorff distance between an animal and its corresponding cluster is $\frac{1}{2}$. So, to prove the large deviation principle we shall consider the animal of the origin instead of the cluster. The point is that an animal is a continuum. Hence we shall be able to apply Propositions 4.3 and 4.4 to an animal.

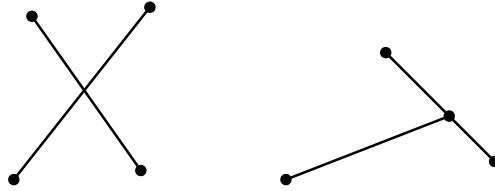
5 The skeletons

DEFINITION 5.1. *A skeleton S is a finite family of segments that are linked by their endpoints. We denote by $E(S)$ the set of the vertices of the segments of S and by $\text{card } S$ the cardinal of $E(S)$. We define $\mathcal{HS}_\xi^1(S)$ as the sum of the ξ -length of the segments of S . A point is also considered as a skeleton.*

Examples:



Counter-examples: the following families of two segments are not skeletons



Sometimes a skeleton S is simply understood as the union of its segments, and so is a compact connected subset of $\mathbb{R}^d$. This is the case when we write $\mathcal{H}_\xi^1(S)$. We always have

$$\mathcal{H}_\xi^1(S) \leq \mathcal{HS}_\xi^1(S). \quad (5.2)$$

If S_1 and S_2 are two skeletons which have a vertex in common, then $S = S_1 \cup S_2$ is also a skeleton, and

$$\mathcal{HS}_\xi^1(S) = \mathcal{HS}_\xi^1(S_1) + \mathcal{HS}_\xi^1(S_2). \quad (5.3)$$

LEMMA 5.4. *For every Γ continuum with $\mathcal{H}_\xi^1(\Gamma) < \infty$, for all $\delta > 0$, there exists a skeleton S such that*

$$D_H(S, \Gamma) < \delta, \quad \mathcal{HS}_\xi^1(S) \leq \mathcal{H}_\xi^1(\Gamma).$$

The skeleton S is said to δ -approximate Γ .

PROOF. Let Γ be a continuum with $\mathcal{H}_\xi^1(\Gamma) < \infty$. Let $\{\Gamma_k\}_{k \in I}$ be the sequence of rectifiable curves coming from proposition 4.3 with parameter $\delta/2$. Consider Γ_1 . We take $t_0 = 0$, $x_0 = \Gamma_1(0)$ and for $n \geq 0$

$$t_{n+1} = \inf \{t > t_n : |\Gamma_1(t) - \Gamma_1(t_n)| \geq \delta/2\}.$$

If t_{n+1} is finite then $x_{n+1} = \Gamma_1(t_{n+1})$. Otherwise, we take for x_{n+1} the last point of Γ_1 if it is different from x_n , and we stop the sequence of the x_i 's. Since Γ_1 is rectifiable and because of lemma 4.1, this sequence is finite. We call S_1 the family of the segments $[x_i, x_{i+1}]$ for $i = 0$ to $n - 1$. By construction S_1 is a skeleton, the endpoints of Γ_1 are vertices of S_1 and S_1 $\delta/2$ -approximates Γ_1 . We construct in the same way the other S_i 's for i in I . By assumption, the Γ_i 's are connected by their endpoints. Since these endpoints are vertices of S_i 's, the union of the S_i 's denoted by S is also a skeleton. We control the $\mathcal{HS}_\xi^1$ measure of S by

$$\mathcal{HS}_\xi^1(S) = \sum_{i \in I} \mathcal{HS}_\xi^1(S_i) \leq \sum_{i \in I} \mathcal{H}_\xi^1(\Gamma_i) \leq \mathcal{H}_\xi^1(\Gamma),$$

where we use (5.3) and lemma 4.1. The Hausdorff distance between S and Γ is controlled by

$$D_H(S, \Gamma) < D_H(S, \cup_{i \in I} \Gamma_i) + \delta/2 < \sup_{i \in I} D_H(S_i, \Gamma_i) + \delta/2 < \delta. \quad \square$$

Remark: if Γ' is the image of Γ by a translation of vector $\vec{u}$, then the skeleton S' constructed as above from Γ' is the image by the same translation of the skeleton S constructed from Γ .

6 The lower bound

We prove in this section the lower bound stated in Theorem 1.1. By a standard argument [5], it is equivalent to prove that for all $\delta > 0$, all Γ in $\mathcal{K}_c$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln P(\overline{D}_H(\overline{C}/N, \overline{\Gamma}) < \delta) \geq -\mathcal{H}_\xi^1(\Gamma).$$

We introduce two notations. The r -neighbourhood of a set E is the set

$$\mathcal{V}(E, r) = \{x \in \mathbb{R}^d : d(x, E) < r\}.$$

Let E_1, E_2 be two subsets of $\mathbb{R}^d$. We define

$$e(E_1, E_2) = \inf \{r > 0 : E_2 \subset \mathcal{V}(E_1, r)\}.$$

We now take Γ in $\overline{\Gamma}$ such that the origin is a vertex of the skeleton S constructed from Γ , as described in the proof of lemma 5.4. This can be done because of the previous remark. First observe that

$$\begin{aligned} P(\overline{D}_H(\overline{C}/N, \overline{\Gamma}) < \delta) &\geq P(D_H(C/N, \Gamma) < \delta) \\ &\geq P(\{e(C/N, \Gamma) < \delta/2\} \cap \{e(\Gamma, C/N) < \delta\}). \end{aligned}$$

We let

$$\begin{aligned} G(N, \delta/2, \Gamma) &= \{\exists \text{ a connected set } C' \text{ of the percolation process,} \\ &\quad \text{containing 0, such that } D_H(C'/N, \Gamma) < \delta/2\}. \end{aligned}$$

We have $G(N, \delta/2, \Gamma) \subset \{e(C/N, \Gamma) < \delta/2\}$. So

$$\begin{aligned} P(D_H(\overline{C}/N, \overline{\Gamma}) < \delta) &\geq P(G(N, \delta/2, \Gamma) \cap \{e(\Gamma, C/N) < \delta\}) \\ &\geq P(G(N, \delta/2, \Gamma)) \times \\ &\quad P(e(\Gamma, C/N) < \delta \mid G(N, \delta/2, \Gamma)). \end{aligned} \tag{6.1}$$

We study the first term of the product. Let r be positive and let x and y be two sites. The event that there exists an open path from x to y whose Hausdorff distance to the segment $[x, y]$ is less than r is denoted by $x \xrightarrow{r} y$. We restate lemma 8 in Section 5 of [3]:

LEMMA 6.2. *Let $\phi(n)$ be a function such that $\lim_{n \rightarrow \infty} \phi(n) = \infty$. For every point x , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} P(0 \xleftrightarrow{\phi(n)} \lfloor nx \rfloor) = -\xi(x).$$

Take the skeleton S which $\delta/4$ -approximates Γ , as in lemma 5.4. We have carefully chosen Γ such that the origin is a vertex of S . We label $x_1, \dots, x_n$ the vertices of S . We note $i \sim j$ if $[x_i, x_j]$ is a segment of S . Then

$$\begin{aligned} P(G(N, \delta/2, \Gamma)) &\geq P(G(N, \delta/4, S)) \\ &\geq P(\lfloor Nx_i \rfloor \xleftrightarrow{N\delta/4} \lfloor Nx_j \rfloor, \forall i < j \text{ such that } i \sim j). \end{aligned}$$

The fact that the origin is a vertex of S is used in the last inequality. Since the events last considered are increasing, the FKG inequality leads to

$$P(G(N, \delta/2, \Gamma)) \geq \prod_{i < j, i \sim j} P(\lfloor Nx_i \rfloor \xleftrightarrow{N\delta/4} \lfloor Nx_j \rfloor).$$

But by lemma 6.2

$$\begin{aligned} \lim \frac{1}{N} \ln \prod_{i < j, i \sim j} P(\lfloor Nx_i \rfloor \xleftrightarrow{N\delta/4} \lfloor Nx_j \rfloor) &= - \sum_{i < j, i \sim j} \mathcal{H}_\xi^1([x_i, x_j]) \\ &= -\mathcal{HS}_\xi^1(S). \end{aligned}$$

Hence

$$\liminf \frac{1}{N} \ln P(G(N, \delta/2, \Gamma)) \geq -\mathcal{HS}_\xi^1(S) \geq -\mathcal{H}_\xi^1(\Gamma). \quad (6.3)$$

Now we analyze the second term $P(e(\Gamma, C/N) < \delta \mid G(N, \delta/2, \Gamma))$ of the product in (6.1). First observe that the event

$$\{e(\Gamma, C/N) \geq \delta\} \cap G(N, \delta/2, \Gamma)$$

is included in

$$\begin{aligned} \{ \exists \text{ an open path of length } \geq N\delta/2 \text{ lying in} \\ \mathcal{V}(N\Gamma, N\delta) \setminus \mathcal{V}(N\Gamma, N\delta/2) \} \cap G(N, \delta/2, \Gamma). \end{aligned}$$

The two events appearing in the last intersection are independent, since they depend on disjoint sets of bonds. So

$$\begin{aligned} P(e(\Gamma, C/N) \geq \delta \mid G(N, \delta/2, \Gamma)) \\ \leq P(\exists \text{ an open path of length } \geq N\delta/2 \text{ lying in } \mathcal{V}(N\Gamma, N\delta)) \\ \leq c_1 (\mathcal{H}_\xi^1(\Gamma) + \delta) \delta^{d-1} N^d \exp(-a_0 N\delta/2), \end{aligned}$$

for a certain constant $c_1 > 0$. In the last inequality, we use (2.2) and a bound of the cardinality of $\mathcal{V}(N\Gamma, N\delta) \cap \mathbb{Z}^d$. The member on the RHS tends to 0 as N tends to infinity. Hence

$$\lim_{N \rightarrow \infty} P(e(\Gamma, C/N) < \delta | G(N, \delta/2, \Gamma)) = 1. \quad (6.4)$$

By limits (6.3) and (6.4), the inequality (6.1) yields to the lower bound. $\square$

7 Coarse graining

Now we associate a skeleton to an animal. By a counting argument it will yield to the desired upper bound.

DEFINITION 7.1. *Let $S = \{T_i\}_{i \in I}$ be a skeleton, and let C be an animal. We say that S fits C if $E(S)$ is included in the set of vertices of C , if for all i in I there exists a curve γ_i such that γ_i is included in C and has the same endpoints than T_i , and if the family $\{\gamma_i\}_{i \in I}$ is hardly disjoint.*

LEMMA 7.2. *Let $s > 4$. For all animal C with $\text{diam}(C) > s$, there exists a skeleton S such that $\mathcal{HS}_\xi^1(S) \geq a_2(s/8)\text{card } S$, $D_H(C, S) < s$, and the skeleton S fits the animal C .*

Such a skeleton is said to be *s-compatible* with the animal C .

PROOF. We recall that an animal is also a continuum. Let $\{\Gamma_k\}_{k \in I}$ be a sequence of rectifiable curves as in proposition 4.4 with parameter $s/2$. Consider for example Γ_1 . We take $x_0 = \Gamma_1(0)$ and $t_0 = 0$. For $n \geq 0$, let

$$t_{n+1} = \inf\{t > t_n : \Gamma_1(t) \in \mathbb{Z}^d, |\Gamma_1(t) - \Gamma_1(t_n)|_2 \geq s/4\}.$$

If t_{n+1} is finite, then $x_{n+1} = \Gamma_1(t_{n+1})$. Otherwise, we erase x_n , we put x_n the last point of Γ_1 and we stop the sequence. Note that t_1 cannot be infinite.

We call S'_1 the family of the segments $[x_j, x_{j+1}]$. The set S'_1 is a skeleton, and is called the *s-skeleton* of Γ_1 . For the other i 's in I we construct S'_i the *s-skeleton* of Γ_i in the same way. For each i in I we have

$$\mathcal{HS}_\xi^1(S'_i) \geq (\text{card } S'_i - 1)a_2(s/4).$$

Since the euclidian diameter of Γ_i is larger than s for each i in I , we have $\text{card } S'_i \geq 2$. Since $s > 4$, it follows that $\mathcal{HS}_\xi^1(S'_i) \geq a_2(s/8)\text{card } S'_i$, for each i in I .

We now refine the skeleton S'_i into another skeleton S_i . For each $j > i$ such that the first point of Γ_j , say z , is in Γ_i but is not a vertex of S'_i , we take the segment of S'_i whose endpoints x and y surround z on Γ_i . We replace in S'_i the segment $[x, y]$ by the two segments $[x, z]$ and $[z, y]$. When we have done this for all j we rename S'_i by S_i .

The set S_i is always a skeleton which satisfies $D_H(S_i, \Gamma_i) < s/2$. By triangular inequality, $\mathcal{H}_\xi^1(S_i) \geq \mathcal{H}_\xi^1(S'_i)$. We denote by S the concatenation of the S_i 's. By induction, S is a skeleton. Furthermore, each vertex of S is a vertex of S'_i for a certain i .

Now we check that S fulfills the good properties. We have

$$\mathcal{H}_\xi^1(S) = \sum_{i \in I} \mathcal{H}_\xi^1(S_i) \geq \sum_{i \in I} a_2(s/8) \text{card } S'_i \geq a_2(s/8) \text{card } S,$$

and

$$D_H(S, \Gamma) < \sup_{i \in I} D_H(S_i, \Gamma_i) + s/2 < s. \quad \square$$

The next statement gives the interest of such a construction. For a given skeleton S we let $\mathcal{A}(S)$ be the event that S is s -compatible with an animal.

LEMMA 7.3. *For all scales $s > 4$,*

$$P(\mathcal{A}(S)) \leq \exp\{-\mathcal{H}_\xi^1(S)\}.$$

PROOF. If S is compatible with an animal, we have the disjoint occurrences of the events $\{x_i \leftrightarrow x_j\}$ for all $i < j$ such that $[x_i, x_j]$ is a segment of S . The BK inequality implies

$$P(\mathcal{A}(S)) \leq \prod_{\substack{i < j \\ [x_i, x_j] \text{ is a segment of } S}} P(x_i \leftrightarrow x_j).$$

The last sentence of proposition 2.5 yields to the desired result. $\square$

8 The upper bound

We prove here the upper bound stated in Theorem 1.1. Consider the animal C containing the origin. Let $\Phi_H(u) = \{\bar{K} \in \bar{\mathcal{K}}_c : \mathcal{H}_\xi^1(\bar{K}) \leq u\}$. We prove that $\forall u \geq 0, \forall \delta > 0, \forall \alpha > 0, \exists N_0$ such that $\forall N \geq N_0$,

$$P(D_H(C/N, \Phi_H(u)) \geq \delta) \leq \exp -Nu(1 - \alpha).$$

This is the Freidlin-Wentzell presentation of the upper bound of our large deviation principle, see [8].

Let c be a positive constant to be chosen later, and take $s = 8c \ln N$. For N large enough, $D_H(C/N, \Phi_H(u)) \geq \delta$ implies $\text{diam } C > s$. By lemma 7.2, we can take S a skeleton that s -approximates C . We have $D_H(C/N, S) \leq 8c \ln N/N$, so for N large enough,

$$P(D_H(C/N, \Phi_H(u)) \geq \delta) \leq P(D_H(S/N, \Phi_H(u)) \geq \delta/2).$$

Since S is an element of $\mathcal{K}_c$, the inequality $D_H(S/N, \Phi_H(u)) \geq \delta/2$ implies that $\mathcal{H}_\xi^1(S) \geq uN$ and so $\mathcal{HS}_\xi^1(S) \geq uN$ by (5.2).

Let a be such that $a > u/a_1$. We have

$$\begin{aligned} P(\mathcal{HS}_\xi^1(S) \geq uN) \\ \leq P(\mathcal{HS}_\xi^1(S) \geq uN, \text{diam } C \leq aN) + P(\text{diam } C > aN). \end{aligned}$$

But $P(\text{diam } C > aN) < \exp -a_1 aN$ by inequality (2.3). Since $a > u/a_1$, we have $P(\text{diam } C > aN) < \exp -uN$.

We estimate now the term $P(\mathcal{HS}_\xi^1(S) \geq uN, \text{diam } C \leq aN)$. Let $\mathcal{A}(n, u, a, N)$ be the set of skeletons T such that $\mathcal{HS}_\xi^1(T) \geq uN$, $E(T)$ is included in $\mathbb{Z}^d$, $\text{card } T = n$, and there exists a connected set of sites containing the origin of diameter less than aN that is s -compatible with the skeleton T . We have

$$P(\mathcal{HS}_\xi^1(S) \geq uN, \text{diam } C \leq aN) \leq \sum_n \sum_{T \in \mathcal{A}(n, u, a, N)} P(S = T).$$

The number of skeletons we can construct from n points is bounded by $(n^n)^2$. Take a skeleton in $\mathcal{A}(n, u, a, N)$. All its vertices are in a box centered at 0, of side length $2(aN + c \ln N)$. So the cardinal of $\mathcal{A}(n, u, a, N)$ is less than $2^{dn}(aN + c \ln N)^{dn}(n^n)^2$, and moreover $n \leq 2^d(aN + c \ln N)^d$. Hence there exists $a_3 > 0$ such that

$$|\mathcal{A}(n, u, a, N)| \leq \exp a_3 n \ln N$$

Take $b > 0$ a constant such that $a_3 - a_2 b < 0$. We assume now that $c > b$. We have

$$\begin{aligned} \mathcal{HS}_\xi^1(T) &= \mathcal{HS}_\xi^1(T)(1 - b/c) + b/c \mathcal{HS}_\xi^1(T) \\ &\geq uN(1 - b/c) + a_2 b n \ln N \end{aligned}$$

because $\mathcal{HS}_\xi^1(T) \geq a_2(s/8)\text{card } T$. Then by lemma 7.3, for N large enough

$$\begin{aligned} P(\mathcal{HS}_\xi^1(S) \geq uN, \text{diam } C \leq aN) \\ \leq \sum_n \sum_{T \in \mathcal{A}(n, u, a, N)} \exp -\mathcal{HS}_\xi^1(T) \\ \leq \sum_n \sum_{T \in \mathcal{A}(n, u, a, N)} \exp(-uN(1 - b/c) - a_2 b n \ln N) \\ \leq \exp(-uN(1 - b/c)) \sum_n \exp((a_3 - a_2 b)n \ln N) \\ \leq \exp -uN(1 - a_4/c) \end{aligned}$$

for any $a_4 > b$ and N large enough. We take c such that $a_4/c < \alpha$ and this concludes the proof. $\square$

Bibliography

1. J. van den Berg, H. Kesten, *Inequalities with applications to percolation and reliability theory*, J. Appl. Prob. **22** (1985), 556–569.
2. Castaing, C. and Valadier, M., *Convex analysis and measurable multifunctions*, Lectures Notes in Math. **580** (1977), Springer.
3. R. Cerf, *Large Deviations of the Finite Cluster Shape for Two-Dimensional Percolation in the Hausdorff and L^1 Metric*, Journ. of Theo. Prob. **13** (2000).
4. R. Cerf, *Large deviations for three-dimensional supercritical percolation*, Astérisque **267** (2000).
5. A. Dembo, O. Zeitouni, *Large deviations techniques and applications*, Second edition, Springer, New York, 1998.
6. K. J. Falconer, *The Geometry of Fractals Sets*, Cambridge.
7. C. Fortuin, P. Kasteleyn and J. Ginibre, *Correlation inequalities on some partially ordered sets*, Commun. Math. Phys. **22** (1971), 89–103.
8. M.I. Freidlin, A.D. Wentzell, *Random perturbations of dynamical systems*, Springer–Verlag, New York, 1984.
9. G. Grimmett, *Percolation*, Second Edition, vol. 321, Springer, 1999.
10. T.E. Harris, *A lower bound for the critical probability in a certain percolation process*, Proc. Camb. Phil. Soc. **56** (1960), 13–20.
11. Y. Kovchegov, S. Sheffield, *Linear speed large deviations for percolation clusters*, Electron. Comm. Probab. **8** (2003), 179–183.

Chapitre 4

Poisson approximation for large finite clusters in the supercritical FK model

Abstract: Using the Chen-Stein method, we show that the spatial distribution of large finite clusters in the supercritical FK model approximates a Poisson process when the ratio weak mixing property holds.

Keywords: FK model, ratio weak mixing

1991 Mathematics Subject Classification: 60K35, 82B20.

1 Introduction

We consider here the behaviour of large finite clusters in the supercritical FK model. In dimension two and more, their typical structure is described by the Wulff shape [4, 5, 6, 8, 9, 10, 11]. An interesting issue is the spatial distribution of these large finite clusters. Because of their rarity, a Poisson process naturally comes to mind. Indeed, we prove that the point process of the mass centers of large finite clusters sharply approximates a Poisson process. Furthermore, considering large finite clusters in a large box such that their mean number is not too large, we observe Wulff droplets distributed according to this Poisson process.

Redig and Hostad have recently studied the law of large finite clusters in a given box [20]. Their aim was different, in that they obtained accurate estimates on the law of the maximal cluster in the box, but intermediate steps are similar. In the supercritical regime they considered only Bernoulli percolation and not FK percolation.

As in [1, 13, 15, 20], our main result is based on a second moment inequality. We have to control the interaction between two clusters. To do this, we suppose that ratio weak mixing holds [2]. The ratio weak mixing holds for p large enough in dimension two [2], but such a result is not available in higher dimensions. Hence, we will prove some intermediate inequalities with the weaker assumption that weak mixing holds, or with the assumption that p is close enough to 1 in dimensions three and more. Once we obtain these inequalities, we apply the Chen-Stein method to get the approximation by a Poisson process.

The following section is devoted to the statement of our results. In section 3, we define the FK model. We recall the weak and the ratio weak mixing properties and we state a perturbative mixing result in section 4. Section 5 contains the definition of our point process and the description of the Chen-Stein method. The core of the article is section 6, where we study a second moment inequality. In section 7, we deal with the probability of having a large finite cluster with its center at the origin. In section 8, we treat the case of distant clusters and we finish the proof of Theorem 1. The proof of Theorem 3 is done in section 9, and the proof of the perturbative mixing result is done in section 10.

2 Statement of the results

We consider the FK measure Φ on the d -dimensional lattice $\mathbb{Z}^d$ and in the supercritical regime. The point $\hat{p}_c$ stands for $\hat{p}_g$ in dimension two, and for p_c^{slab} in dimensions three and

more. For $q \geq 1$ we let $\mathcal{U}(q)$ be the set such that there exists a unique FK measure on $\mathbb{Z}^d$ of parameters p and q if p is not in $\mathcal{U}(q)$. By [17] this set is at most countable.

Let Λ be a large box in $\mathbb{Z}^d$. We fix n an integer and we consider the finite clusters of cardinality larger than n . We call them *n-large clusters*. Let C be a finite cluster. The *mass center* of C is

$$M_C = \left\lfloor \frac{1}{|C|} \sum_{x \in C} x \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the site of $\mathbb{Z}^d$ whose coordinates are the integer part of those of x . We define a process X on Λ by

$$X(x) = \begin{cases} 1 & \text{if } x \text{ is the mass center of a } n\text{-large cluster } C \\ 0 & \text{otherwise.} \end{cases}$$

Let λ be the expected number of sites x in Λ such that $X(x) = 1$. We denote by $\mathcal{L}(X)$ the law of a process X . For Y a process on Λ , we let $\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV}$ be the total variation distance between the laws of the processes X and Y [7].

THEOREM 2.1. *Let $q \geq 1$ and $p > \hat{p}_c$ with $p \notin \mathcal{U}(q)$. Let Φ be the FK measure on $\mathbb{Z}^d$ of parameters p and q . We suppose that Φ is ratio weak mixing. There exists a constant $c > 0$ such that: for any box Λ , letting X be defined as above, and letting Y be a Bernoulli process on Λ with the same marginals than X , we have for n large enough*

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV} \leq \lambda \exp(-cn^{(d-1)/d}).$$

As a corollary, the number of large clusters in Λ is approximated by a Poisson variable.

COROLLARY 2.2. *Let Φ be as in Theorem 2.1. Let N be the number of large finite clusters whose mass centers are in the box Λ . Let Z be a Poisson variable of mean λ , and let $c > 0$ be the same constant as in Theorem 2.1. Then for any $A \subset \mathbb{Z}^+$ and for n large enough,*

$$|P(N \in A) - P(Z \in A)| \leq \lambda \exp(-cn^{(d-1)/d}).$$

We provide next a control of the shape of the large finite clusters. Let $\mathcal{W}$ be the Wulff crystal, let θ be the density of the infinite cluster, and let $\mathcal{L}^d(\cdot)$ be the Lebesgue measure on $\mathbb{R}^d$. Let

$$W = \frac{1}{(\theta \mathcal{L}^d(\mathcal{W}))^{1/d}} \mathcal{W}$$

be the renormalized Wulff crystal. For $l > 0$, let $V_\infty(C, l)$ be the neighbourhood of C of width l for the metric $|\cdot|_\infty$. For two sets A and B , the notation $A \triangle B$ stands for the symmetric difference between A and B .

THEOREM 2.3. *Let Φ be as in Theorem 2.1. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be such that $f(n)/n \rightarrow 0$ and $f(n)/\ln n \rightarrow \infty$ as n goes to infinity. Let $(\Lambda_n)_n$ be a sequence of boxes in $\mathbb{Z}^d$, and let λ_n be the expected number of mass centers of n -large clusters in Λ_n . For all $\delta > 0$, there exists $c > 0$ such that if $\limsup 1/n^{(d-1)/d} \ln \lambda_n \leq c$.*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{(d-1)/d}} \ln \Phi \left[\mathcal{L}^d \left(\left(\bigcup_{\substack{x \in \Lambda_n \\ X(x)=1}} (x+W) \right) \Delta \left(n^{-1} \bigcup_{\substack{C \text{ } n\text{-large} \\ C \cap \Lambda_n \neq \emptyset}} V_\infty(C, f(n)) \right) \right) \geq \delta |\{x : X(x) = 1\}| \right] < 0.$$

For clarity, we omit the subscript n on X .

3 FK model

We consider the lattice $\mathbb{Z}^d$ with $d \geq 2$. We turn it into a graph by adding bonds between all pairs x, y of nearest neighbours. We write $\mathbb{E}$ for the set of bonds and we let Ω be the set $\{0, 1\}^{\mathbb{E}}$. A *bond configuration* ω is an element of Ω . A bond e is *open* in ω if $\omega(e) = 1$, and *closed* otherwise.

A *path* is a sequence $(x_0, \dots, x_n)$ of distinct sites such that $\langle x_i, x_{i+1} \rangle$ is a bond for each i , $0 \leq i \leq n-1$. A subset Δ of $\mathbb{Z}^d$ is *connected* if for every x, y in Δ , there exists a path included in Δ connecting x and y . If all bonds of a path are open in ω , we say that the path is *open* in ω . A *cluster* is a connected component in $\mathbb{Z}^d$ when we keep only open bonds. It is usually denoted by C . Let x be a site. We write $C(x)$ for the cluster containing x .

To define the FK measure, we first consider finite volume FK measures. Let Λ be a box included in $\mathbb{Z}^d$. We write $\mathbb{E}(\Lambda)$ for the set of bonds $\langle x, y \rangle$ with $x, y \in \Lambda$. Let $\Omega_\Lambda = \{0, 1\}^{\mathbb{E}(\Lambda)}$ be the space of bonds configuration in Λ . Let $\mathcal{F}_\Lambda$ be its σ -field, that is the set of subsets of Ω_Λ . For ω in Ω_Λ , we define $\text{cl}(\omega)$ as the number of clusters of the configuration ω .

For $p \in [0, 1]$ and $q \geq 1$, the FK measure in Λ with parameters p, q and free boundary condition is the probability measure on Ω_Λ defined by

$$\forall \omega \in \Omega_\Lambda \quad \Phi_\Lambda^{f,p,q}(\omega) = \frac{1}{Z_\Lambda^{f,p,q}} \left(\prod_{e \in \mathbb{E}(\Lambda)} p^{\omega(e)} (1-p)^{1-\omega(e)} \right) q^{\text{cl}(\omega)},$$

where $Z_\Lambda^{f,p,q}$ is the appropriate normalization factor.

We also define FK measures for arbitrary boundary conditions. For this, let $\partial\Lambda$ be the *boundary* of Λ ,

$$\partial\Lambda = \{x \in \Lambda \text{ such that } \exists y \notin \Lambda, \langle x, y \rangle \text{ is a bond}\}.$$

For a partition π of $\partial\Lambda$, a π -cluster is a cluster of Λ when we add open bonds between the pairs of sites that are in the same class of π . Let $\text{cl}_\pi(\omega)$ be the number of π -clusters in ω . To define $\Phi_\Lambda^{\pi,p,q}$ we replace $\text{cl}(\omega)$ by $\text{cl}_\pi(\omega)$ and $Z_\Lambda^{f,p,q}$ by $Z_\Lambda^{\pi,p,q}$ in the above formula.

There exists a countable subset $\mathcal{U}(q)$ in $[0, 1]$ such that the following holds. As Λ grows and invades the whole lattice $\mathbb{Z}^d$, the finite volume measures converge weakly toward the same infinite measure $\Phi_\infty^{p,q}$ for all $p \notin \mathcal{U}(q)$ [17]. We will always suppose that this occurs, that is $p \notin \mathcal{U}(q)$. We shall drop the superscript and the subscript on $\Phi_\infty^{p,q}$, and simply write Φ . It is known that the FK measure Φ is translation-invariant.

The measure Φ verify the *finite energy property*: for each p in $(0, 1)$, there exists $\delta > 0$ such that for every finite-dimensional cylinders ω_1 and ω_2 that differ by only one bond,

$$\Phi(\omega_1)/\Phi(\omega_2) \geq \delta. \quad (3.1)$$

The random cluster model has a phase transition. There exists $p_c \in (0, 1)$ such that there is no infinite cluster Φ -almost surely if $p < p_c$, and an infinite cluster Φ -almost surely if $p > p_c$. Other critical points have been introduced in order to work with 'fine' properties. In dimension two, we define $\hat{p}_g$ as the critical point for the exponential decay of dual connectivities, see [14, 17]. In three and more dimensions, let p_c^{slab} be the limit of the critical points for the percolation in slabs [22]. For brevity, $\hat{p}_c$ will stand for $\hat{p}_g$ in dimension two, and for p_c^{slab} in dimensions three and more. It is believed that $\hat{p}_c = p_c$ in all dimensions and for all $q \geq 1$, but in most cases we only know that $\hat{p}_c \geq p_c$.

We now state Theorem 17 of [12], applied to FK measures.

If $q \geq 1$, $p > \hat{p}_c$ and $p \notin \mathcal{U}(q)$,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < \infty) = -w_1, \quad (3.2)$$

where $C(0)$ is the cluster of the origin, and $w_1 > 0$.

4 Mixing properties

Let x and y be two points in $\mathbb{Z}^d$ and let $(x_i)_{i=1}^d$ and $(y_i)_{i=1}^d$ be their coordinates. Write $|x - y|_1 = \sum_{i=1}^d |x_i - y_i|$.

DEFINITION 4.1. *Following [3], we say that Φ has the weak mixing property if for some $c, \mu > 0$, for all sets $\Lambda, \Delta \subset \mathbb{Z}^d$,*

$$\begin{aligned} & \sup \left\{ \left| \Phi(E \mid F) - \Phi(E) \right| : E \in \mathcal{F}_\Lambda, F \in \mathcal{F}_\Delta, \Phi(F) > 0 \right\} \\ & \leq c \sum_{x \in \Lambda, y \in \Delta} e^{-\mu|x-y|_1}. \end{aligned} \quad (4.2)$$

DEFINITION 4.3. *Following [3], we say that Φ has the ratio weak mixing property if for some $c_1, \mu_1 > 0$, for all sets $\Lambda, \Delta \subset \mathbb{Z}^d$,*

$$\begin{aligned} & \sup \left\{ \left| \frac{\Phi(E \cap F)}{\Phi(E)\Phi(F)} - 1 \right| : E \in \mathcal{F}_\Lambda, F \in \mathcal{F}_\Delta, \Phi(E)\Phi(F) > 0 \right\} \\ & \leq c_1 \sum_{x \in \Lambda, y \in \Delta} e^{-\mu_1 |x-y|_1}, \end{aligned} \tag{4.4}$$

Roughly speaking, the influence of what happens in Δ on the state of the bonds in Λ decreases exponentially with the distance between Λ and Δ .

In dimension two, the measure Φ is ratio weak mixing as soon as $p > \hat{p}_g$ [3], but such a result is not available in dimension larger than three. We provide a perturbative mixing result, which is valid for all dimensions larger than three, and which is similar to the weak mixing property.

LEMMA 4.5. *Let $d \geq 3$ and $q \geq 1$. There exists $p_1 < 1$ and $c > 0$ such that: for all $p > p_1$, all connected sets Γ, Δ with $\Gamma \subset \Delta$, every boundary conditions η, ξ on Δ , every event E supported on Γ ,*

$$|\Phi_\Delta^{\eta,p,q}(E) - \Phi_\Delta^{\xi,p,q}(E)| \leq 2|\partial\Delta| \exp(-c \inf \{|x-y|_1, x \in \Gamma, y \in \Delta\}).$$

We are not aware of a particular reference of this result, and we give a sketch of the proof in Section 10.

5 The Chen-Stein method

From the percolation process, we want to extract a point process describing the occurrence of large finite clusters. For a point x in $\mathbb{R}^d$, let $\lfloor x \rfloor$ denotes the site of $\mathbb{Z}^d$ whose coordinates are the integer parts of those of x . Assume that C is a finite subset of $\mathbb{Z}^d$. Then the *mass center* of C is

$$M_C = \left\lfloor \frac{1}{|C|} \sum_{x \in C} x \right\rfloor.$$

Let $n \in \mathbb{N}$. A *n-large cluster* is a finite cluster of cardinality larger than n . Let Λ be a box in $\mathbb{Z}^d$. We define a process X on Λ by

$$X(x) = \begin{cases} 1 & \text{if } x \text{ is the mass center of a } n\text{-large cluster } C \\ 0 & \text{otherwise.} \end{cases}$$

In order to apply the Chen-Stein method, we define for x, y in $\mathbb{Z}^d$,

$$\begin{aligned} p_x &= \Phi(X(x) = 1), \\ p_{xy} &= \Phi(\exists C, C' \text{ two clusters such that: } C \cap C' = \emptyset, \\ &\quad n \leq |C|, |C'| < \infty, M_C = x \text{ and } M_{C'} = y), \end{aligned}$$

and we let $B_x = B(x, n^2)$ be the box centered at x of side length n^2 . Let λ be the expected number of sites x in Λ such that $X(x) = 1$. We have $\lambda = \sum_{x \in \Lambda} p_x$ and, because of the translation-invariance of Φ , for each site x in Λ

$$\lambda = |\Lambda| \cdot p_x. \quad (5.1)$$

We introduce three coefficients b_1, b_2, b_3 by:

$$\begin{aligned} b_1 &= \sum_{x \in \Lambda} \sum_{y \in B_x} p_x p_y, \\ b_2 &= \sum_{x \in \Lambda} \sum_{y \in B_x \setminus x} p_{xy}, \\ b_3 &= \sum_{x \in \Lambda} E \left| E \left(X(x) - p_x \mid \sigma(X(y), y \notin B_x) \right) \right|. \end{aligned}$$

Let Z_1 and Z_2 be two Bernoulli processes on Λ . The *total variation distance* between the laws of the processes Z_1 and Z_2 [7] is

$$\|\mathcal{L}(Z_1) - \mathcal{L}(Z_2)\|_{TV} = \sup \{ |P(Z_1 \in A) - P(Z_2 \in A)|, A \text{ subset of } \{0, 1\}^\Lambda \}.$$

Let Y be a Bernoulli process on Λ such that the $Y(x)$'s are iid and

$$P(Y(x) = 1) = p_x.$$

The Chen-Stein method provides a control of the total variation distance between X and Y in terms of the b_i 's. Indeed Theorem 2 of [7] asserts that

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV} \leq 2(2b_1 + 2b_2 + 2b_3) + \sum_{x \in \Lambda} p_x^2. \quad (5.2)$$

To prove Theorem 2.1, we shall provide an upper bound on each term b_i . The ratio weak mixing property is essential to our proof of the bound of b_2 . Nevertheless, we believe that one can prove the following inequality, without any mixing assumption:

$$\Phi[n \leq C(x) < \infty, n \leq C(y) < \infty, C(x) \cap C(y) = \emptyset] \leq \Phi(2n \leq C(0) < \infty). \quad (5.3)$$

Let us give now an upper bound on p_x . By [16], there exists a constant $c > 0$ such that:

$$\Phi(n \leq |C(0)| < \infty) \leq \exp(-cn^{(d-1)/d}).$$

But

$$\begin{aligned} p_x &\leq \sum_{k \geq n} \Phi(\exists C, |C| = k, M_C = x) \\ &\leq \sum_{k \geq n} \sum_{y \in B(x, 2k)} \Phi(|C(y)| = k) \\ &\leq \sum_{k \geq n} (2k)^d \exp(-cn^{(d-1)/d}). \end{aligned}$$

Hence there exists a constant $c > 0$ such that for n large enough

$$p_x \leq \exp(-cn^{(d-1)/d}). \quad (5.4)$$

6 Second moment inequality

In this section we bound the term p_{xy} with the help of the ratio weak mixing property. First we introduce a local version of p_{xy} . We define $\tilde{p}_{xy}$ by

$$\begin{aligned} \tilde{p}_{xy} &= \Phi(\exists C, C' \text{ two clusters such that} \\ &\quad n \leq |C| < n^2, n \leq |C'| < n^2, M_C = x, \text{ and } M_{C'} = y). \end{aligned}$$

The distance between two sets Γ and $\Delta \subset \mathbb{Z}^d$ is

$$d(\Gamma, \Delta) = \inf\{|x - y|_1, x \text{ in } \Gamma, y \text{ in } \Delta\},$$

and it is the length of the shortest path in $\mathbb{Z}^d$ connecting Γ to Δ .

We divide the term $\tilde{p}_{xy}$ into two parts. Let μ_1 be the constant appearing in the definition of the ratio weak mixing property and let $K > 5/\mu_1$. We define $\tilde{p}_{xy}^c$ by

$$\begin{aligned} \tilde{p}_{xy}^c &= \Phi(\exists C, C' \text{ two clusters such that } d(C, C') \leq K \ln n, \\ &\quad n \leq |C| < n^2, n \leq |C'| < n^2, M_C = x, \text{ and } M_{C'} = y). \end{aligned}$$

We define also $\tilde{p}_{xy}^d$ by

$$\tilde{p}_{xy} = \Phi(\exists C, C' \text{ two clusters such that } d(C, C') > K \ln n, \\ n \leq |C| < n^2, n \leq |C'| < n^2, M_C = x, \text{ and } M_{C'} = y).$$

The superscripts c and d stand for *close* and *distant*. So $\tilde{p}_{xy} = \tilde{p}_{xy}^c + \tilde{p}_{xy}^d$ and we study separately these two terms.

First we focus on $\tilde{p}_{xy}^d$. We have

$$\tilde{p}_{xy}^d \leq \sum_{C, C' \text{ distant}} \Phi(C \text{ and } C' \text{ are clusters}),$$

where the sum is over the couples (C, C') of connected subsets of $\mathbb{Z}^d$ such that

$$n \leq |C| < n^2, n \leq |C'| < n^2, \\ M_C = x, M_{C'} = y, \text{ and } d(C, C') > K \ln n.$$

Let c_1, μ_1 be the constants appearing in the definition of the ratio weak mixing property. Let (C, C') be a couple appearing in the sum above. We have

$$\sum_{u \in C, v \in C'} e^{-\mu_1 |u-v|} \leq n^4 \exp(-\mu_1 K \ln n),$$

so for n large enough

$$c_1 \sum_{u \in C, v \in C'} e^{-\mu_1 |u-v|} \leq 1.$$

So for n large enough

$$\Phi(C \text{ and } C' \text{ are clusters}) \leq 2\Phi(C \text{ is a cluster}) \cdot \Phi(C' \text{ is a cluster}),$$

by the ratio weak mixing property (4.4). Hence there exists $c > 0$ such that for n large enough

$$\begin{aligned} \tilde{p}_{xy}^d &\leq \sum_{u \in B(x, 2n^2), v \in B(y, 2n^2)} 2\Phi(n \leq |C(u)| < \infty) \cdot \Phi(n \leq |C(v)| < \infty) \\ &\leq \exp(-cn). \end{aligned} \tag{6.1}$$

Now we consider $\tilde{p}_{xy}^c$. We have

$$\tilde{p}_{xy}^c \leq \sum_{C, C' \text{ close}} \Phi(C \text{ and } C' \text{ are clusters}),$$

where the sum is over the couples (C, C') of subsets of $\mathbb{Z}^d$ such that

$$\begin{aligned} n \leq |C| < n^2, n \leq |C'| < n^2, \\ M_C = x, M_{C'} = y, \text{ and } d(C, C') \leq K \ln n. \end{aligned}$$

For n large enough, the event $\{C \text{ and } C' \text{ are clusters}\}$ is $\mathcal{F}_{B(x, 3n^2)}$ -measurable. So we only consider bonds configurations in $B(x, 3n^2)$.

We give a deterministic total order on the pairs (u, v) of $\mathbb{Z}^d$ in such a way that if $|u_1 - v_1|_1 < |u_2 - v_2|_1$, then $(u_1, v_1) < (u_2, v_2)$. Let (C, C') be a pair of sets appearing in the above sum. Take a configuration ω in $B(x, 3n^2)$ such that C and C' are clusters in ω . We change the configuration ω as follows.

To start with, we take the pair (u, v) such that $u \in C$, $v \in C'$ and (u, v) is the first such pair for the order above. For $0 \leq i \leq d$, we define t_i the point whose $d - i$ first coordinates are equal to those of u , and the others are equal to those of v . Hence $t_0 = u$, $t_d = v$, and t_i and t_{i+1} differ by only one coordinate. We consider the shortest path $(u_0, \dots, u_k)$ connecting u to v through the t_i 's. It is composed of the segments $[t_i, t_{i+1}]$ for $0 \leq i \leq d - 1$.

We open all the bonds $\langle u_i, u_{i+1} \rangle$ for $i = 0 \dots k - 1$. In the same time, we close all the bonds incident to u_i for $i = 1 \dots k - 1$ distinct from the previous bonds $\langle u_j, u_{j+1} \rangle$. Let $\tilde{\omega}$ be the new configuration in $B(x, 3n^2)$. We denote by $\tilde{C}$ the set $C \cup C' \cup \{u_i\}_{i=1}^{k-1}$. By construction, $\tilde{C}$ is a cluster in $\tilde{\omega}$. We have

$$2n \leq \tilde{C} < 4n + K \ln n.$$

The number of bonds we have changed is bounded by $2dK \ln n$. By the finite energy property (3.1):

$$\Phi(\tilde{\omega}) \geq n^{2dK \ln \delta} \Phi(\omega),$$

for a certain constant δ in $(0, 1)$.

Now we control the number of antecedents by our transformation. Take a configuration $\tilde{\omega}$ of $B(x, 3n^2)$. To get an antecedent of $\tilde{\omega}$, we have to

- (a) choose two sites u, v in $B(x, 3n^2)$, with $|u - v|_1 \leq K \ln n$
- (b) take the path connecting u to v along the coordinate axis
- (c) choose the state of the bonds that have an endpoint on this path.

In step (a) we have less than $(3n^2)^d (2K \ln n)^d$ choices. In step (b) we have just one choice. In step (c) the number of choices is bounded by $2^{2dK \ln n}$. Hence for n large enough the number of antecedents of $\tilde{\omega}$ is bounded by n^{4dK} .

Finally,

$$\sum_{C, C' \text{ close}} \Phi(C \text{ and } C' \text{ are clusters}) \leq n^{4dK} \cdot n^{2dK \ln \delta} \sum_{\tilde{C}} \Phi(\tilde{C} \text{ is a cluster}),$$

where the sum is over connected subsets $\tilde{C}$ of $\mathbb{Z}^d$ such that $2n \leq |\tilde{C}| < 5n$ and $\tilde{C}$ is contained in $B(x, 3n^2)$. This sum is bounded by

$$|B(x, 3n^2)| \cdot \Phi(2n \leq |C(0)| < 5n).$$

Thus by (3.2), there exists $c_2 > w_1$ such that for n large enough,

$$\tilde{p}_{xy}^c \leq \exp(-c_2 n^{(d-1)/d}). \quad (6.2)$$

To conclude, remark that

$$p_{xy} - \tilde{p}_{xy} \leq \Phi(\exists C \text{ a cluster such that } n^2 \leq |C| < \infty, M_C = x).$$

By (5.4), there exists c such that for n large enough the difference between p_{xy} and $\tilde{p}_{xy}$ is bounded by $\exp(-cn^{2(d-1)/d})$. So by (6.1) there exists $c > 0$ such that $p_{xy} \leq \tilde{p}_{xy}^c + \exp(-cn)$. Since in (6.2) the constant c_2 is strictly larger than w_1 , there exists $c_3 > w_1$ such that for n large enough

$$p_{xy} \leq \exp(-c_3 n^{(d-1)/d}). \quad (6.3)$$

7 A control on p_x

We compare p_x and $\Phi(n \leq |C(0)| < \infty)$.

LEMMA 7.1. *If $q \geq 1$, $p > \hat{p}_c$, and $p \notin \mathcal{U}(q)$, then*

$$\lim_{n^{(d-1)/d}} \frac{1}{n^{(d-1)/d}} \ln p_x = -w_1.$$

We note that in [20], the authors take the left endpoints of clusters instead of mass centers and they get the same limit.

PROOF OF LEMMA 7.1. We begin with a lower bound for p_x . We recall that for all x in $\mathbb{Z}^d$, $p_x = \Phi(X(0) = 1)$. Let $\alpha > 1$. Because of (3.2), we have

$$\lim_{n^{(d-1)/d}} \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < \infty) = \lim_{n^{(d-1)/d}} \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < n^\alpha).$$

Then

$$\begin{aligned}\Phi(n \leq |C(0)| < n^\alpha) &\leq \sum_{x \in B(0, n^\alpha)} \Phi(n \leq |C(0)| < n^\alpha, M_C = x) \\ &\leq |B(0, n^\alpha)| \Phi(X(0) = 1).\end{aligned}$$

We give next an upper bound:

$$\begin{aligned}\Phi(X(0) = 1) &= \Phi(\exists C \text{ a cluster}, M_C = 0, n \leq |C| < n^\alpha) \\ &\quad + \Phi(\exists C \text{ a cluster}, M_C = 0, n^\alpha \leq |C| < \infty) \\ &\leq \sum_{x \in B(0, n^\alpha)} \Phi(n \leq |C(x)| < \infty) \\ &\quad + \sum_{k \geq n^\alpha} \Phi(\exists C \text{ a cluster}, |C| = k, C \cap B(0, 2k) \neq \emptyset) \\ &\leq |B(0, n^\alpha)| \Phi(n \leq |C(x)| < \infty) + \sum_{k \geq n^\alpha} |B(0, 2k)| \Phi(|C(0)| = k).\end{aligned}$$

Finally, we use the limit (3.2) to get

$$\lim_{n \rightarrow \infty} \frac{1}{n^{(d-1)/d}} \ln p_x = \lim_{n \rightarrow \infty} \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < \infty) = -w_1. \quad \square$$

8 Proof of Theorem 2.1

We recall that Λ is a box and λ is the expected number of the mass centers in Λ of n -large clusters. We write $\mathcal{F}_\Lambda^{B_x}$ for the σ -field $\mathcal{F}_{\Lambda \setminus B_x}$. First, we bound the term

$$E|E(X(x) - p_x | \mathcal{F}_\Lambda^{B_x})|.$$

Let $\tilde{X}(x)$ be equal to 1 if x is the mass center of a cluster C , with C such that $n \leq |C| < n^2/4$, and equal to 0 otherwise. Let $\tilde{p}_x = \Phi(\tilde{X}(x))$. We have

$$\begin{aligned}E|E(X(x) - p_x | \mathcal{F}_\Lambda^{B_x})| &\leq E|E(X(x) - \tilde{X}(x) | \mathcal{F}_\Lambda^{B_x})| \\ &\quad + E|E(\tilde{X}(x) - \tilde{p}_x | \mathcal{F}_\Lambda^{B_x})| + E|E(\tilde{p}_x - p_x | \mathcal{F}_\Lambda^{B_x})|. \end{aligned} \tag{8.1}$$

Since the quantity $X(x) - \tilde{X}(x)$ is always positive,

$$\begin{aligned}E|E(X(x) - \tilde{X}(x) | \mathcal{F}_\Lambda^{B_x})| &= E[E(X(x) - \tilde{X}(x) | \mathcal{F}_\Lambda^{B_x})] \\ &= p_x - \tilde{p}_x.\end{aligned}$$

We have also

$$E|E(\tilde{p}_x - p_x | \mathcal{F}_\Lambda^{B_x})| = p_x - \tilde{p}_x.$$

But

$$p_x - \tilde{p}_x = \Phi(\exists C \text{ a cluster, } n^2/4 \leq |C| < \infty, M_C = x),$$

so by (5.4) there exists $c > 0$ such that $p_x - \tilde{p}_x \leq \exp(-cn^2)$.

The variable $\tilde{X}(x)$ is $\mathcal{F}_{B(x, n^2/4)}$ -measurable. The distance between $B(x, n^2/4)$ and the complementary region of B_x is of order n^2 . If Φ is weak mixing, or by lemma 4.5 if p is close enough to 1, there exists a constant $c > 0$ such that for n large enough

$$E|E(\tilde{X}(x) - \tilde{p}_x | \mathcal{F}_\Lambda^{B_x})| \leq \exp(-cn^2).$$

Putting together the estimates of the three terms on the right-hand side of (8.1), we conclude that there exists $c > 0$ such that for n large enough

$$E|E(X(x) - p_x | \mathcal{F}_\Lambda^{B_x})| \leq \exp(-cn^2). \quad (8.2)$$

Now observe that $|\Lambda| = \lambda p_x^{-1}$. Using inequality (6.3) and the limit of Lemma 7.1, there exists $c > 0$ such that

$$b_2 \leq \lambda p_x^{-1} \exp(-c_3 n^{(d-1)/d}) \leq \lambda \exp(-cn^{(d-1)/d}).$$

Because of (8.2), there exists $c > 0$, $c' > 0$ such that

$$b_3 \leq \lambda p_x^{-1} \exp(-cn^2) \leq \lambda \exp(-c'n^2).$$

The term b_1 is controlled by Lemma 7.1. We apply finally the Chen-Stein inequality (5.2) to obtain Theorem 2.1. $\square$

9 Proof of Theorem 2.3

The Wulff crystal is the typical shape of a large finite cluster in the supercritical regime. The crystal is built on a surface tension τ . The surface tension is a function from $\mathbb{S}^{d-1}$, the $(d-1)$ -dimensional unit sphere of $\mathbb{R}^d$, to $\mathbb{R}^+$. It controls the exponential decay of the probability for having a large separating surface in a certain direction, with all bonds closed. We refer the reader to [9, 12] for an extended survey of this function.

In the regime $p > \hat{p}_c$ and $p \notin \mathcal{U}(q)$, the surface tension is positive, continuous, and satisfies the weak simplex inequality. We denote by $\mathcal{W}$ the Wulff shape associated to τ ,

$$\mathcal{W} = \{x \in \mathbb{R}^d, x \cdot u \leq \tau(u) \text{ for all } u \text{ in } \mathbb{S}^{d-1}\}.$$

The Wulff shape is a main ingredient in the proof of (3.2).

Let $\theta = \Phi(0 \leftrightarrow \infty)$ be the density of the infinite cluster. Let $f : \mathbb{N} \rightarrow \mathbb{N}$, such that $f(n)/n \rightarrow 0$ and $f(n)/\ln n \rightarrow \infty$ as n goes to infinity. Let x and y be two points of $\mathbb{R}^d$, and let $(x_i)_{i=1}^d$ and $(y_i)_{i=1}^d$ be their coordinates. We write $|x - y|_\infty = \max_{1 \leq i \leq d} |x_i - y_i|$. We define a neighbourhood of a cluster C by

$$V_\infty(C, f(n)) = \{x \in \mathbb{R}^d, \exists y \in C, |x - y|_\infty \leq f(n)\}.$$

Let $(\Lambda_n)_{n \geq 0}$ be a sequence of boxes in $\mathbb{Z}^d$, and let λ_n be the expected number of mass centers of n -large clusters in Λ_n . In Theorem 3, we consider the event

$$\left\{ \mathcal{L}^d \left(\bigcup_{\substack{x \in \Lambda_n \\ X(x)=1}} (x + \theta \mathcal{L}^d(\mathcal{W})^{-1/d} \mathcal{W}) \triangle \right. \right. \\ \left. \left. n^{-1} \bigcup_{\substack{C \text{ } n\text{-large} \\ C \cap \Lambda_n \neq \emptyset}} V_\infty(C, f(n)) \right) \geq \delta |\{x : X(x) = 1\}| \right\}. \quad (9.1)$$

It is included in the event

$$\left\{ \begin{array}{l} \text{there exists } C \text{ a } n\text{-large cluster such that } M_C \in \Lambda_n, \\ \mathcal{L}^d \left((M_C + \theta \mathcal{L}^d(\mathcal{W})^{-1/d} \mathcal{W}) \triangle (n^{-1} V_\infty(C, f(n))) \right) \geq \delta \end{array} \right\}.$$

Taking the logarithm of its probability and dividing by $n^{d-1/d}$, we may show that for n large it is equivalent to the logarithm divided by $n^{d-1/d}$ of the following quantity:

$$\lambda_n \Phi \left[\mathcal{L}^d \left((M_{C(0)} + \theta \mathcal{L}^d(\mathcal{W})^{-1/d} \mathcal{W}) \triangle (n^{-1} V_\infty(C(0), f(n))) \right) \geq \delta \mid n \leq |C(0)| < \infty \right].$$

By [9, 12], there exists $c > 0$ such that if

$$\limsup 1/n^{(d-1)/d} \ln \lambda_n \leq c,$$

then the inequality in Theorem 2.3 holds. $\square$

10 A perturbative mixing result

We prove lemma 4.5, following the proof of the uniqueness of the FK measure for p close enough to 1 in [18]. The difference is that we consider not just one but two independent FK measures. The idea of using two independent copies of a measure comes from [19].

Let Δ be a connected subset of $\mathbb{Z}^d$. There is a partial order $\preceq$ in Ω_Δ given by $\omega \preceq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for every bond e . A function $f : \Omega_\Delta \rightarrow \mathbb{R}$ is called *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega \preceq \omega'$. An event is an element of Ω_Δ . An event is called increasing if its characteristic function is increasing. For a pair of probability measures μ and ν on $(\Omega_\Delta, \mathcal{F}_\Delta)$, we say that μ (*stochastically dominates*) ν if for any $\mathcal{F}_\Delta$ -measurable increasing function f the expectations satisfy $\mu(f) \geq \nu(f)$ and we denote it by $\mu \succeq \nu$. Let P_p be the Bernoulli bond-percolation measure on $\mathbb{Z}^d$ of parameter p . The FK measures on Δ dominate stochastically a certain Bernoulli measure restricted on $\mathbb{E}(\Delta)$:

$$\Phi_\Delta^{\eta, p, q} \succeq P_{p/[p+q(1-p)]}|_{\mathbb{E}(\Delta)}. \quad (10.1)$$

For $(\omega_1, \omega_2) \in \Omega^2$, we call a site x *white* if $\omega_1(e)\omega_2(e) = 1$ for all bond e incident with x , and *black* otherwise. We define a new graph structure on $\mathbb{Z}^d$. Take two sites x and y and label x_i, y_i their coordinates. If $\max_{i=1\dots d} |x_i - y_i| = 1$, then $\langle x, y \rangle$ is a $\star$ -bond and y is a $\star$ -neighbour of x . A $\star$ -path is a sequence $(x_0, \dots, x_n)$ of distinct sites such that $\langle x_i, x_{i+1} \rangle$ is a $\star$ -bond for $0 \leq i \leq n-1$.

For any set V of sites, the *black cluster* $B(V)$ is the union of V together with the set of all x_0 for which there exists a $\star$ -path $x_0, \dots, x_n$ such that $x_n \in V$ and $x_0, \dots, x_{n-1}$ are all black. Let Γ, Δ be two connected sets with $\Gamma \subset \Delta$. The 'interior boundary' $D(B(\partial\Delta))$ of $B(\partial\Delta)$ is the set of sites x satisfying:

- (a) $x \notin B(\partial\Delta)$
- (b) there is a $\star$ -neighbour of x in $B(\partial\Delta)$
- (c) there exists a path from x to Γ that does not use a site in $B(\partial\Delta)$.

Let I be the set of sites x_0 for which there exists a path $x_0, \dots, x_n$ with $x_n \in \Gamma$, $x_i \notin B(\partial\Delta)$ for all i , see figure 1.

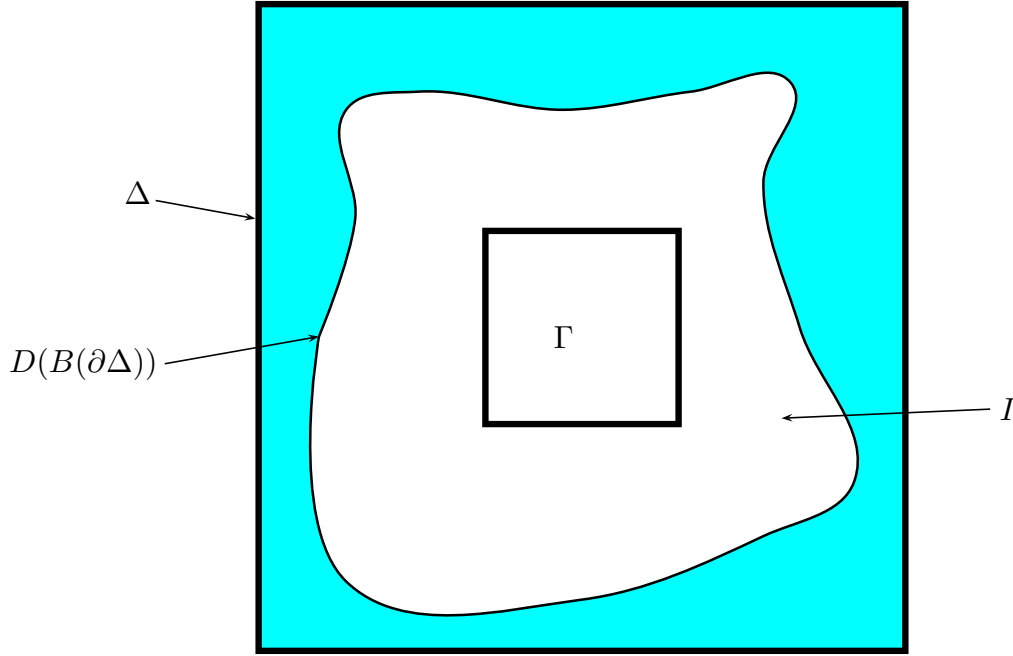
Let

$$K_{\Gamma, \Delta} = \{(B(\partial\Delta) \cup D(B(\partial\Delta))) \cap \Gamma = \emptyset\}.$$

If $K_{\Gamma, \Delta}$ occurs, we have the following facts:

- (a) $D(B(\partial\Delta))$ is connected
- (b) every site in $D(B(\partial\Delta))$ is white
- (c) $D(B(\partial\Delta))$ is measurable with respect to the colours of sites in $\mathbb{Z}^d \setminus \Gamma$
- (d) each site in ∂I is adjacent to some site of $D(B(\partial\Delta))$.

These claims have been established in the proof of Theorem 5.3 in [18].

figure 1: The set I inside Δ

Pick η, ξ two boundary conditions of Δ . For brevity let $\mathcal{P} = \Phi_{\Delta}^{\eta,p,q} \times \Phi_{\Delta}^{\xi,p,q}$. We shall write X, Y for the two projections from $\Omega_{\Delta} \times \Omega_{\Delta}$ to Ω_{Δ} . Then for any $E \in \mathcal{F}_{\Gamma}$, we have by the claims above

$$\mathcal{P}(X \in E, K_{\Gamma, \Delta}) = \mathcal{P}(Y \in E, K_{\Gamma, \Delta}) = \mathcal{P}(\Phi_I^{w,p,q}(E) 1_{K_{\Gamma, \Delta}}).$$

Hence

$$|\Phi_{\Delta}^{\eta,p,q}(E) - \Phi_{\Delta}^{\xi,p,q}(E)| \leq 2(1 - \mathcal{P}(K_{\Gamma, \Delta})).$$

Because of inequality (10.1) and by the stochastic domination result in [21], the process of black sites is stochastically dominated by a Bernoulli site-percolation process whose parameter is independent of $\Gamma, \Delta, \eta, \xi$ and decreases to 0 as p goes to 1. There exists $p_1 < 1$ such that this Bernoulli process is subcritical for the $\star$ -graph structure of $\mathbb{Z}^d$ and for $p \geq p_1$. Hence there exists $c > 0$ such that for $p > p_1$, for all $\Gamma, \Delta, \eta, \xi$,

$$\mathcal{P}(K_{\Gamma, \Delta}) \geq 1 - |\partial\Delta| \exp(-c d(\Gamma, \partial\Delta)). \quad \square$$

Bibliography

1. M. Abadi, J.-R. Chazottes, F. Redig, E. Verbitskiy, *Exponential distribution for the occurrence of rare patterns in Gibbsian random fields*, Comm. Math. Phys. **246** (2004), 296–294.
2. K. S. Alexander, *On weak mixing in lattice models*, Probab. Theory Relat. Fields **110** (1998), 441–471.
3. K. S. Alexander, *Mixing properties and exponential decay for lattice systems in finite volumes*, Ann. Probab. **32** (2004), 441–487.
4. K. S. Alexander, *Stability of the Wulff minimum and fluctuations in shape for large finite clusters in two-dimensional percolation*, Probab. Theory Related Fields **91** (1992), 507–532.
5. K. S. Alexander, *Cube-root boundary fluctuations for droplets in random cluster models*, Comm. Math. Phys. **224** (2001), 733–781.
6. K. S. Alexander, J. T. Chayes, L. Chayes, *The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation*, Comm. Math. Phys. **131** (1990), 1–50.
7. R. Arratia, L. Goldstein and L. Gordon, *Two moments suffice for Poisson approximations: The Chen-Stein method*, Ann. Prob. **17** (1989), 9–25.
8. T. Bodineau, *The Wulff construction in three and more dimensions*, Comm. Math. Phys. **207** (1999), 197–229.
9. R. Cerf, *Large deviations for three-dimensional supercritical percolation*, Astérisque **267** (2000).
10. R. Cerf, Á. Pisztora, *On the Wulff crystal in the Ising model*, Ann. Probab. **28** (2000), 947–1017.
11. R. Cerf, Á. Pisztora, *Phase coexistence in Ising, Potts and percolation models*, Ann. I. H. P. **PR 37** (2001), 643–724.
12. R. Cerf, *The Wulff crystal in Ising and Percolation models*, Saint-Flour lecture notes, first version (2004).
13. J.-R. Chazottes, F. Redig, *Occurrence, repetition and matching of patterns in the low-temperature Ising model*, Preprint (2003).
14. O. Couronné, R.-J. Messikh, *Surface order large deviations for 2D FK-percolation and Potts models*, Stoch. Proc. Appl. **113** (2004), 81–99.
15. P. A. Ferrari, P. Picco, *Poisson approximation for large-contours in low-temperature Ising models*, Physica A: Statistical Mechanics and its Applications **279** (2000), Issues 1–4, 303–311.

16. G. R. Grimmett, *Percolation. Second Edition*, Springer, Grundlehren der mathematischen Wissenschaften **321** (1999).
17. G. R. Grimmett, *The random cluster model*, Springer, Probability on Discrete Structures, ed. H. Kesten, Encyclopedia of Mathematical Sciences **110** (2003), 73–123.
18. G. R. Grimmett, *The stochastic random-cluster process and the uniqueness of random-cluster measures*, Ann. Probab. **23** (1995), 1461–1510.
19. H.-O. Georgii, O. Haggstrom, C. Maes, *The random geometry of equilibrium phases*, Phase Transit. Crit. Phenom. **18** (2001), 1–142.
20. R. van der Hofstad, F. Redig, *Maximal clusters in non-critical percolation and related models*, Preprint (2004).
21. T. M. Liggett, R. H. Schonmann, A. M. Stacey, *Domination by product measures*, Ann. Probab. **25** (1997), 71–95.
22. Á. Pisztora, *Surface order large deviations for Ising, Potts and percolation models*, Probab. Theory Relat. Fields **104** (1996), 427–466.

Chapitre 5

Surface large deviations for supercritical oriented percolation

Abstract: We prove a large deviation principle of surface order for supercritical oriented percolation on $\mathbb{Z}^d$, $d \geq 3$, which leads to asymptotics of the finite cluster distribution.

1991 Mathematics Subject Classification: 60K35, 82B20

Keywords: oriented percolation, large deviations, Wulff crystal

1 Introduction

In this article we adapt the arguments of [4], in order to derive a large deviation principle for supercritical oriented percolation. We consider oriented percolation on $\mathbb{Z}^d$ with $d \geq 3$. We let p_c be the corresponding critical point, and we let $C(0)$ be the cluster of the origin.

THEOREM 1.1. *Let $d \geq 3$. For every $p > p_c$, there exists a constant $c > 0$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(n^d \leq |C(0)| < \infty) = -c.$$

This limit gives the answer to a question raised in [10] for oriented percolation in dimension two.

Theorem 1.1 is a consequence of a large deviation principle. We shall define a tension surface τ for the oriented percolation process, and we denote by $\mathcal{W}_\tau$ the corresponding Wulff crystal. With the help of the Wulff crystal, we define the surface energy $\mathcal{I}(A)$ of a Borel set A as

$$\mathcal{I}(A) = \sup \left\{ \int_A \operatorname{div} f(x) dx : f \in C_c^1(\mathbb{R}^d, \mathcal{W}_\tau) \right\},$$

where $C_c^1(\mathbb{R}^d, \mathcal{W}_\tau)$ is the set of C^1 vector functions defined on $\mathbb{R}^d$ with values in $\mathcal{W}_\tau$ having compact support and div is the usual divergence operator.

Consider $\mathcal{M}(\mathbb{R}_+^d)$ the set of finite Borel measures on $\mathbb{R}_+^d$. We equip $\mathcal{M}(\mathbb{R}_+^d)$ with the weak topology, that is the coarsest topology for which the linear functionals

$$\nu \in \mathcal{M}(\mathbb{R}_+^d) \rightarrow \int f d\nu, \quad f \in C_c(\mathbb{R}^d, \mathbb{R})$$

are continuous, where $C_c(\mathbb{R}^d, \mathbb{R})$ is the set of the continuous maps from $\mathbb{R}^d$ to $\mathbb{R}$ having compact support.

For $\nu \in \mathcal{M}(\mathbb{R}_+^d)$, we define $\mathcal{I}(\nu) = \mathcal{I}(A)$ if ν is the measure with density $\theta 1_A$ with respect to the Lebesgue measure, where A is a Borel subset of $\mathbb{R}^d$, and $\mathcal{I}(\nu) = \infty$ otherwise.

THEOREM 1.2. *Let $d \geq 3$ and let $p > p_c$. The sequence of random measures*

$$\mathcal{C}_n = \frac{1}{n^d} \sum_{x \in C(0)} \delta_{\frac{x}{n}}$$

satisfies a large deviation principle in $\mathcal{M}(\mathbb{R}_+^d)$ with speed n^{d-1} and rate function $\mathcal{I}$, i.e., for every Borel subset $\mathbb{M}$ of $\mathcal{M}(\mathbb{R}_+^d)$,

$$\begin{aligned} -\inf\{\mathcal{I}(\nu) : \nu \in \overset{\circ}{\mathbb{M}}\} &\leq \liminf_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathbb{M}) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathbb{M}) \leq -\inf\{\mathcal{I}(\nu) : \nu \in \overline{\mathbb{M}}\}. \end{aligned}$$

Under the conditional probability $\hat{P}(\cdot) = P(\cdot \mid |C(0)| < \infty)$ we have the enhanced large deviation upper bound: for any Borel subset $\mathbb{M}$ of $\mathcal{M}(\mathbb{R}^d)$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln \hat{P}(\mathcal{C}_n \in \mathbb{M}) &\leq \\ &\quad - \sup_{f, \delta} \inf \{ \mathcal{I}(\rho) : \rho(\mathbb{R}^d) < \infty, \exists \nu \in \mathbb{M} \quad |\rho(f) - \nu(f)| < \delta \} \end{aligned}$$

where the supremum is taken over $\delta > 0$ and the functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ that are bounded and continuous.

G. R. Grimmett submitted the Wulff shape problem for oriented percolation to R. Cerf back in 1995. One could believe that the oriented case should be easier to tackle than the unoriented one [3]. However, we were surprised to deal with delicate proofs, despite the Markov property of the oriented process.

In [3], the large deviation principle is stated with the conditional measure $\hat{P}$, which is enough to prove the result of Theorem 1.1. The statement of the large deviation principle with the percolation measure P in [4] requires no more effort. Let us keep in mind that in the usual percolation process, the surface tension is bounded away from 0, so that there is a linear relation between the perimeter and the surface energy.

This relation still remains for bounded Borel subsets of $\mathbb{R}^d$ in the oriented case. On the other hand, when we focus on a bounded region, we find that there is no more equivalence between the perimeter and the surface energy restricted to that region. This leads to extra work in order to prove the $\mathcal{I}$ -tightness under P of the random measure $\mathcal{C}_n$. Theorem 1.2 is stronger than what we need for Theorem 1.1. We establish the large deviation principle with the measure P in order to keep the result of [4], and to highlight a difference between the oriented case and the non-oriented one.

This article is devoted to the proof of the (weak) large deviation principle stated in Theorem 1.2, and follows the schemes of [4]. We do not give the proofs of the enhanced upper bound and of Theorem 1.1, as it would be a repetition of [4]. Also, we often recall lemmas from [4].

Beside the large deviation principle, we get other results on the percolation process by using block arguments. We state these results in the following three theorems. In the supercritical oriented percolation model, an infinite cluster does not fill the whole space but looks like a deterministic cone. This cone is called the cone of percolation, and we shall show that the percolation process inside this cone is supercritical in section 19:

THEOREM 1.3. *Let $d \geq 3$ and $p > p_c$. Let O be an open subset of $\mathbb{R}^{d-1}$ such that the cone $\{(tO, t) : t \geq 0\}$ is included in the cone of percolation. Then with probability one there is an infinite path in $\{(tO, t) : t \geq 0\}$.*

The next result deals with the positivity of the surface tension. The relevant cone for the surface tension is a cone orthogonal to the cone of percolation, and which we call the cone of positivity, see figure 1.

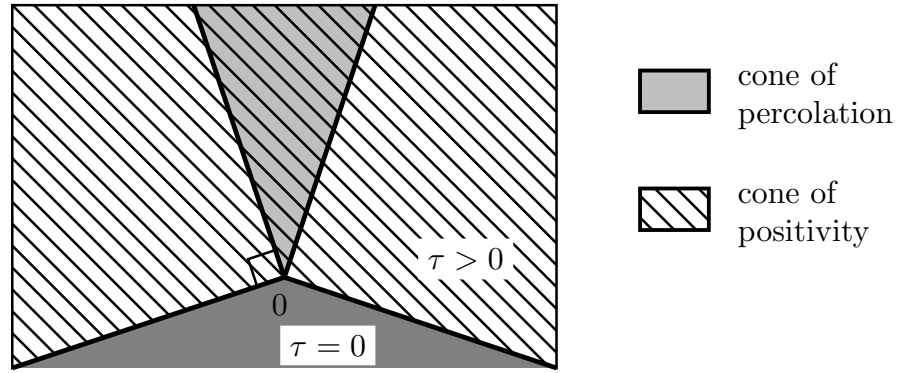


figure 1: The cone of positivity

THEOREM 1.4. *Let $d \geq 3$ and $p > p_c$. The surface tension τ is strictly positive in the cone of positivity and null outside.*

We also prove that the connectivity function $P(0 \rightarrow x)$ decreases exponentially outside the cone of percolation in section 20:

THEOREM 1.5. *Let x be not in the cone of percolation. There exists $c > 0$ such that*

$$P(0 \rightarrow nx) \leq \exp(-cn).$$

The cone of percolation is defined in section 2, and the cone of positivity is defined in section 5.

As we have noted before, the surface tension is null in a whole angular sector. Hence, the corresponding Wulff crystal does not contain 0 in its interior. Indeed, the Wulff crystal

is contained in the cone of percolation, and it has a singularity at 0. Nevertheless, we prove that $\mathcal{W}_\tau$ has a non-empty interior. Unfortunately, the proofs that the Wulff crystal is the unique solution which minimizes the surface energy under a volume constraint, always rely on the strict positivity of the surface tension. Thus, to obtain the Wulff shape for large finite clusters as in [4], one has to resolve the Wulff variational problem for a convex function whose Wulff crystal has a positive Lebesgue measure. This problem has not been solved yet.

Most of our results are based on a block argument, and we now describe the basic idea which leads to the definition of our block events. The graph is oriented so that the process goes upward. The oriented percolation process has a Markovian structure, and we sometimes think of this process as a process indexed by the last coordinate. In the supercritical regime, clusters tend to spread horizontally with linear speed, and most of the block events that we consider assert that the “block process” increases in typical configurations. In that way, we can estimate the price to pay to restrain the block process in a given region.

We give a short review of the main points of this article. Two block events are defined in section 3. They control the increase of the (Markovian) oriented percolation process from below. Another block estimate, given in section 19, provides a control from above of the increase of the oriented percolation process.

The proof of the upper bound is divided into three parts: a local upper bound, the definition of a set of blocks which is exponentially contiguous to the cluster of the origin, and the $\mathcal{I}$ -tightness of this set of blocks.

The local upper bound relies on a local estimate, provided in section 8 and in section 9. The arguments in section 8 are similar to those in [4]. However, the result of section 9 in which we consider the density has still a counterpart in [4], but the proof is much longer and it relies on a static renormalization much like [20]. The point is that when we consider a family of clusters, the clusters can intersect so that the cardinality of their union is not the summation of their cardinals.

Because of the lack of equivalence between the perimeter and the surface energy in a bounded domain, our proof of the $\mathcal{I}$ -tightness is more involved. In order to control the proportion of bad blocks in the boundary of the block process, our definition of block events will depend on the domain under consideration, as well as the size of the blocks.

The proof of the lower bound is also more delicate, because the percolation process does not naturally fill a given shape. We put some seeds at the “bottom” of the shape to solve this problem.

The following is a sequential description of our article. We first describe the oriented percolation process and then give background results in section 2. Section 3 is devoted to the study of two block events, and we define block processes in section 4. We define a surface tension in section 5. In section 6 we introduce the Wulff crystal and we study the

positivity of the surface tension. In section 7, we estimate the probability of the existence of a separating set near a hypersurface. Section 8 is devoted to the proof of the interface estimate, which provides the link between the surface tension and the large deviation upper bound. Section 9 contains an alternative separate estimate, which is more relevant for the local large deviation upper bound. In section 10, we introduce the Caccioppoli sets, which are the natural objects for our large deviation principle. The definition of their surface energy follows in section 11, and we give two ways for approximating Caccioppoli sets in section 12. A local upper bound follows in section 13. In section 14 we build a block cluster and a block measure from the cluster $C(0)$. Section 15 is devoted to the study of the boundary of the block cluster. The exponential contiguity between the block measure and the measure $\mathcal{C}_n$ is proved in section 16, and the $\mathcal{I}$ -tightness of $\mathcal{C}_n$ is proved in section 17. In section 18 we build with sufficiently high probability the cluster $C(0)$ near a given shape, in order to obtain the lower bound. We discuss the geometry of the Wulff shape and finish the study of the positivity of the surface tension in section 19. We prove that the connectivity function decreases exponentially outside the cone of percolation in section 20. To finish, section 21 contains a little note on the Wulff variational problem.

2 The model

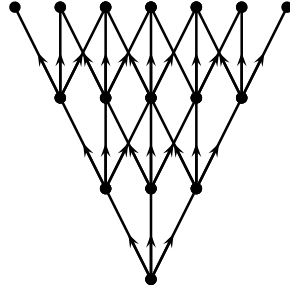
Let $\mathbb{Z}^d$ be the set of all d -vectors $x = (x_1, \dots, x_d)$ of integers. For $x, y \in \mathbb{Z}^d$, we define

$$|x - y| = \sum_{i=1}^d |x_i - y_i|.$$

We let e_i be the i^{th} coordinate vector, for $1 \leq i \leq d$. We refer to vectors in $\mathbb{Z}^d$ as *vertices*, and we turn $\mathbb{Z}^d$ into a graph by adding an undirected edge between every pair x, y of vertices such that $|x - y| = 1$. The resulting graph is denoted $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$. The *origin* of this graph is the vertex $0 = (0, \dots, 0)$.

We will consider the following oriented graph. Each vertex $x = (x_1, \dots, x_d)$ may be expressed as $x = (\mathbf{x}, t)$ where $\mathbf{x} = (x_1, \dots, x_{d-1})$ and $t = x_d$. Consider the directed graph with vertex set $\mathbb{Z}^d$ and with a directed edge joining two vertices $x = (\mathbf{x}, t)$ and $y = (\mathbf{y}, u)$ whenever $\sum_{i=1}^{d-1} |y_i - x_i| \leq 1$ and $u = t + 1$. As in [17], we write $\vec{\mathbb{L}}_{\text{alt}}^d = (\mathbb{Z}^d, \vec{\mathbb{E}}_{\text{alt}}^d)$ for the ensuing directed graph, represented in figure 2. We shall concentrate on this model for notational convenience, but our results apply also to the conventional oriented model [8].

Let $G = (V, E)$ be a graph. The configuration space for percolation on G is the set $\Omega = \{0, 1\}^E$. For $\omega \in \Omega$, we call an edge $e \in E$ *open* if $\omega(e) = 1$ and *closed* otherwise. With Ω we associate the σ -field $\mathcal{F}$ of subsets generated by the finite-dimensional cylinders. For $0 \leq p \leq 1$, we let P_p or simply P be the product measure on $(\Omega, \mathcal{F})$ with density p . When the graph G has translations, the measure P is invariant under translation and is even ergodic.


 figure 2: The graph $\vec{\mathbb{L}}_{\text{alt}}^2$

There is a natural order on Ω defined by the relation $\omega_1 \leq \omega_2$ if and only if all open edges in ω_1 are open in ω_2 . An event is said to be increasing (respectively decreasing) if its characteristic function is non-decreasing (respectively non-increasing) with respect to this partial order. Suppose the events A, B are both increasing or both decreasing. The Harris–FKG inequality [16] says that

$$P(A \cap B) \geq P(A)P(B). \quad (2.1)$$

We shall compare a block process with a Bernoulli–site process with the help of stochastic domination. Let μ, ν be two measures on Ω . We say that μ is *stochastically dominated* by ν , which we denote by $\mu \preceq \nu$, if $\mu(f) \leq \nu(f)$ for every bounded increasing measurable function $f : \Omega \rightarrow \mathbb{R}$. For $p \in [0, 1]$, we let Z^p be the Bernoulli site process on G with density p .

Let $\omega \in \Omega$. An *open path* is an alternating sequence $x_0, e_0, x_1, e_1, x_2, \dots$ of distinct vertices x_i and open edges e_i such that $e_i = [x_i, x_{i+1}]$ for all i . If the path is finite, it has two endvertices x_0, x_n , and it is said to connect x_0 to x_n . If the path is infinite, it is said to connect x_0 to infinity. A vertex x is said to be connected to a vertex y , written $x \rightarrow y$, if there exists an open path connecting x to y . For $A, B \subset \mathbb{Z}^d$, we say that A is connected to B , or B is connected from A , if there exists $a \in A$ and $b \in B$ such that $a \rightarrow b$; in this case, we write $A \rightarrow B$.

For $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, we write

$$C(x) = C(x, \omega) = \{y \in \mathbb{Z}^d : x \rightarrow y\}.$$

The percolation probability is defined as the function

$$\theta(p) = P(0 \rightarrow \infty).$$

We introduce the critical point

$$p_c = \sup\{p : \theta(p) = 0\}.$$

By [2,17], we know that $\theta(p_c) = 0$.

For $A \subset \mathbb{Z}^{d-1}$ and $n \in \mathbb{N}$, we define

$$\xi_n^A = \{\mathbf{x} \in \mathbb{Z}^{d-1} : A \times \{0\} \rightarrow (\mathbf{x}, n)\}.$$

Let $\mathbf{x} \in \mathbb{Z}^{d-1}$. We define $\xi_n^A(\mathbf{x}) = 1$ if $\mathbf{x} \in \xi_n^A$, and 0 otherwise. We let

$$H_n = \cup_{m \leq n} \xi_m^0 \quad \text{and} \quad K_n = \{\mathbf{x} : \xi_n^0(\mathbf{x}) = \xi_{\mathbb{Z}^{d-1}}^0(\mathbf{x})\}.$$

We define

$$\bar{H}_n = \bigcup_{\mathbf{x} \in H_n} x + [-\frac{1}{2}, \frac{1}{2}]^{d-1}, \quad \bar{K}_n = \bigcup_{\mathbf{x} \in K_n} x + [-\frac{1}{2}, \frac{1}{2}]^{d-1}.$$

We let

$$\Omega_\infty = \{\xi_n^0 \neq \emptyset \text{ for all } n\},$$

and

$$\tau = \inf\{n : \xi_n^0 = \emptyset\}.$$

We state a shape theorem for oriented percolation from [2,6,7]

PROPOSITION 2.2. *Let $p > p_c$. There exists a convex subset $\mathbf{U}$ of $\mathbb{R}^{d-1}$ such that, for any $\varepsilon > 0$, for almost all $\omega \in \Omega_\infty$,*

$$(1 - \varepsilon)n\mathbf{U} \subset (\bar{H}_n \cap \bar{K}_n) \subset (1 + \varepsilon)n\mathbf{U},$$

for n large enough.

We shall need some exponential estimates on the supercritical oriented percolation (see [9, 11, 18]). For $A \subset \mathbb{Z}^{d-1}$, we let

$$\tau_A = \inf\{n : \xi_n^A = \emptyset\}.$$

PROPOSITION 2.3. *Let $p > p_c$. There exists a strictly positive constant γ such that, for n large enough*

$$P(n < \tau < \infty) \leq \exp(-\gamma n),$$

and, for $A \subset \mathbb{Z}^{d-1}$,

$$P(\tau_A < \infty) \leq \exp(-\gamma|A|).$$

PROPOSITION 2.4. *Let $p > p_c$. There exist strictly positive constants γ and $\hat{\delta}$ such that, for n large enough, for all $\mathbf{x} \in \mathbb{Z}^{d-1}$ such that $|\mathbf{x}| < \hat{\delta}n$,*

$$P(\mathbf{x} \notin H_n, \tau = \infty) \leq \exp(-\gamma n),$$

$$P(\mathbf{x} \notin K_n, \tau = \infty) \leq \exp(-\gamma n).$$

DEFINITION 2.5. *Let $p > p_c$ and let $\mathbf{U}$ be the convex set introduced in proposition 2.2. The cone of percolation is the set*

$$\mathbf{F} = \cup_{t \geq 0} \{(t\mathbf{U}, t)\}.$$

For $\alpha > 0$, we define also

$$\mathbf{F}(\alpha) = \cup_{t \geq 0} \{(\alpha t\mathbf{U}, t)\}.$$

We shall need the following generalizations of the process ξ

DEFINITION 2.6. *Let $y = (\mathbf{y}, t)$ in $\mathbb{Z}^d$. We define*

$$\xi_n^y = \{\mathbf{x} \in \mathbb{Z}^{d-1} : y \rightarrow (\mathbf{y} + \mathbf{x}, t + n)\},$$

and

$$\xi_n^{\mathbb{Z}^{d-1}, y} = \{\mathbf{u} \in \mathbb{Z}^{d-1} : \exists \mathbf{x} \in \mathbb{Z}^{d-1} (\mathbf{x}, t) \rightarrow (\mathbf{y} + \mathbf{u}, t + n)\}.$$

The process $\xi_n^{\mathbb{Z}^{d-1}, y}$ is the process $\xi_n^{\mathbb{Z}^{d-1}}$ translated by y .

3 Block events

In this section we introduce two events which describe the typical behaviour of the oriented percolation process. The first one handles the density of a cluster in a box, the second one shows that a large cluster typically looks like the cone of percolation $\mathbf{F}$.

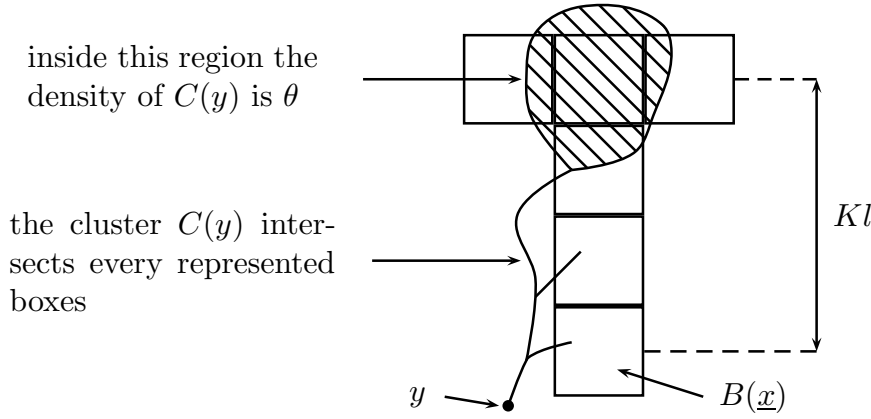
We let K be a positive integer. For $\underline{x}$ in $\mathbb{Z}^d$, we define $B(\underline{x}) =] - K/2, K/2]^d + K\underline{x}$. The graph structure of the set of boxes $\{B(\underline{x}), \underline{x} \in \mathbb{Z}^d\}$ will be studied in the next section.

Let $\varepsilon > 0$, and let l be a positive integer. We introduce a region of blocks:

$$\underline{D}_0(\underline{x}, l) = \left(\bigcup_{0 \leq i \leq l} \{\underline{x} + ie_d\} \right) \cup \left(\bigcup_{1 \leq i \leq d-1} \{\underline{x} + le_d \pm e_i\} \right).$$

We define

$$\begin{aligned} R(B(\underline{x}), l, \varepsilon) = \{ & \forall y \text{ such that } C(y) \cap B(\underline{x}) \neq \emptyset \text{ and } |C(y)| \geq K/2 : \\ & (\theta - \varepsilon)K^d \leq |C(y) \cap B(\underline{x} + le_d)| \leq (\theta + \varepsilon)K^d, \\ & \text{and } \forall \underline{z} \in \underline{D}_0(\underline{x}, l), C(y) \cap B(\underline{z}) \neq \emptyset \}, \end{aligned}$$

figure 3: The event R

see figure 3.

PROPOSITION 3.1. *There exists $l > 0$ such that for all $\varepsilon > 0$,*

$$P(R(B(\underline{x}), l, \varepsilon)) \rightarrow 1 \quad \text{as } K \rightarrow \infty.$$

PROOF. For A a subset of $\mathbb{R}^d$ and $r > 0$, the notation $\mathcal{V}_\infty(A, r)$ stands for the r -neighbourhood of A for the norm $|\cdot|_\infty$ as described in section 10. Let $\mathcal{D}$ be the region

$$\mathcal{D} = \mathcal{V}_\infty(B(\underline{x}), K/2).$$

For z in $\mathbb{R}^d$, we let $\mathbf{F}(\widehat{\delta}, z)$ stand for $z + \mathbf{F}(\widehat{\delta})$. We take l large enough so that the box $B(\underline{x} + le_d)$ is included in $\mathbf{F}(\widehat{\delta}, z)$ for every z in $\mathcal{D}$. Let η , $0 < \eta < 1/2$, and define

$$\mathcal{D}'(\eta) = \mathcal{V}_\infty(B(\underline{x}), \eta K).$$

Let η be small enough such that

$$\forall z \in \mathcal{D}'(\eta), \quad \mathbf{F}(\widehat{\delta}/2, z) \cap B(\underline{x} + e_d) \neq \emptyset. \quad (3.2)$$

Let y be such that $C(y) \cap B(\underline{x}) \neq \emptyset$ and $|C(y)| \geq K/2$. There exists z in $\mathcal{D}'(\eta) \cap C(y)$ such that $|C(z)| \geq \eta K/2$. This is evident in the case $y \in \mathcal{D}'(\eta)$, and if $y \notin \mathcal{D}'(\eta)$, then pick Υ a path from y to $B(\underline{x})$ and take for z the first point in $\Upsilon \cap \mathcal{D}'(\eta)$. By propositions 2.3 and 2.4, there exists $\gamma > 0$ such that for all K

$$P(|C(z)| < \infty \mid |C(z)| \geq \eta K/2) \leq \exp(-\gamma \eta K), \quad (3.3)$$

and for all $n \in \mathbb{N}$, for all $\mathbf{x} \in \mathbb{Z}^{d-1}$ such that $|\mathbf{x}| \leq \widehat{\delta}n$,

$$P(\mathbf{x} \notin H_n \cap K_n, |C(0)| = \infty) \leq \exp(-\gamma n). \quad (3.4)$$

Let $E_0(\underline{x})$ be the event

$$\begin{aligned} E_0(\underline{x}) = \{ & \forall z \in \mathcal{D}'(\eta) \text{ such that } |C(z)| \geq \eta K, \\ & \forall n \geq K/2, \forall \mathbf{u} \in \mathbb{Z}^{d-1} \text{ such that } |\mathbf{u}| \leq \widehat{\delta}n, \text{ we have} \\ & \xi_n^z(\mathbf{u}) = \xi_n^{\mathbb{Z}^{d-1}, z}(\mathbf{u}) \}. \end{aligned}$$

By (3.3) and (3.4),

$$P(E_0(\underline{x})) \rightarrow 1 \quad \text{as } K \rightarrow \infty. \quad (3.5)$$

Observe that for every y with $y \cdot e_d \geq K(\underline{x} \cdot e_d + 1)$ and for every z in $\mathcal{D}'(\eta)$, we have the following implications:

$$(\mathbb{Z}^{d-1} \times \{0\} + K(\underline{x} - e_d)) \rightarrow y \quad \Rightarrow \quad (\mathbb{Z}^{d-1} \times \{0\} + z) \rightarrow y, \quad (3.6)$$

and

$$(\mathbb{Z}^{d-1} \times \{0\} + z) \rightarrow y \quad \Rightarrow \quad (\mathbb{Z}^{d-1} \times \{0\} + K(\underline{x} + e_d)) \rightarrow y, \quad (3.7)$$

see figure 4.

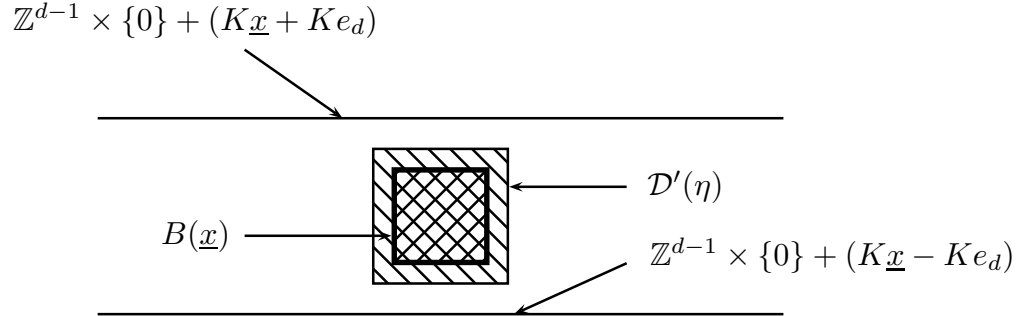


figure 4: the set $\mathcal{D}'(\eta)$

Let $\varepsilon' > 0$. We partition the top of $B(\underline{x} + le_d)$ with hypersquares of side length $\varepsilon'K$. We denote by $\mathcal{S}$ the collection of these hypersquares. By (3.2), we can take $\varepsilon' > 0$ small enough such that for each z in $\mathcal{D}'(\eta)$, there is a hypersquare in $\mathcal{S}$ included in $\mathbf{F}(\widehat{\delta}, z)$. We

can adapt proposition 2.3 by inverting the orientation of the graph, to obtain that for every hypersquare s in $\mathcal{S}$,

$$P(s \not\leftarrow \mathbb{Z}^{d-1} \times \{0\} + (K\underline{x} - Ke_d)) \leq \exp(-cK^{d-1}),$$

where $c > 0$ is a constant independent of K . Hence there exists $c > 0$ such that

$$P(\exists s \in \mathcal{S} \text{ such that } \mathbb{Z}^{d-1} \times \{0\} + (K\underline{x} - Ke_d) \not\leftarrow s) \leq \exp(-cK^{d-1}).$$

By (3.6), if $E_0(\underline{x})$ occurs, then for all y such that $C(y) \cap B(\underline{x}) \neq \emptyset$ and such that $|C(y)| \geq K/2$, the cluster $C(y)$ intersects $B(\underline{x} + e_d)$. We repeat the same procedure for the other boxes.

We turn now to the study of the density inside the box $B(\underline{x} + le_d)$. By the Birkhoff ergodic theorem, we have P almost surely

$$\frac{1}{K^d} \left| \{y \in B(le_d) : \mathbb{Z}^{d-1} \times \{0\} \rightarrow y\} \right| \rightarrow \theta \quad \text{as } K \rightarrow \infty.$$

Thus, for all $\varepsilon_1 > 0$, for K large enough

$$P\left(\frac{1}{K^d} \left| \{y \in B(\underline{x} + le_d) : (\mathbb{Z}^{d-1} \times \{0\} + (K\underline{x} - Ke_d)) \rightarrow y\} \right| \geq \theta - \varepsilon\right) \geq 1 - \varepsilon_1, \quad (3.8)$$

and

$$P\left(\frac{1}{K^d} \left| \{y \in B(\underline{x} + le_d) : (\mathbb{Z}^{d-1} \times \{0\} + (K\underline{x} + Ke_d)) \rightarrow y\} \right| \leq \theta + \varepsilon\right) \geq 1 - \varepsilon_1. \quad (3.9)$$

By the definition of $E_0(\underline{x})$, the density of the clusters considered in the event R is controlled from below by inequality (3.6) and estimate (3.8), and is controlled from above by inequality (3.7) and estimate (3.9). The limit (3.5) yields to the desired result. $\square$

Let $\varepsilon > 0$, $\alpha > 0$, and let l, r be positive integers. We introduce two regions of blocks:

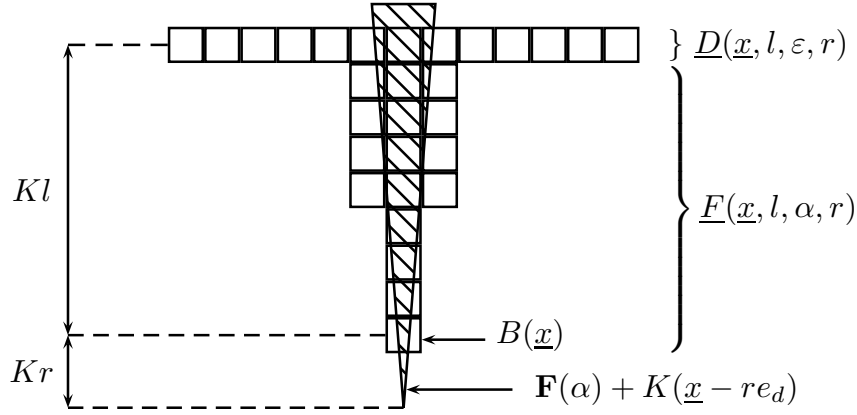
$$\underline{D}(\underline{x}, l, \varepsilon, r) = \{\underline{y} : (\underline{y} - \underline{x}) \cdot e_d = l, B(\underline{y}) \cap (\mathbf{F}(1 - \varepsilon) + K(\underline{x} - re_d)) \neq \emptyset\}, \quad (3.10)$$

and

$$\underline{F}(\underline{x}, l, \alpha, r) = \{\underline{y} : 0 \leq (\underline{y} - \underline{x}) \cdot e_d < l, B(\underline{y}) \cap (\mathbf{F}(\alpha) + K(\underline{x} - re_d)) \neq \emptyset\}. \quad (3.11)$$

These two regions are represented on figure 5. Let $V(B(\underline{x}), l, \varepsilon, \alpha, r)$ be the event

$$V(B(\underline{x}), l, \varepsilon, \alpha, r) = \left\{ \text{for all } y \text{ such that } C(y) \cap B(\underline{x}) \neq \emptyset \text{ and } |C(y)| \geq K/2, \right. \\ \left. \text{we have } \forall \underline{z} \in \underline{F}(\underline{x}, l, \alpha, r) \cup \underline{D}(\underline{x}, l, \varepsilon, r), B(\underline{z}) \cap C(y) \neq \emptyset \right\}.$$


 figure 5: The sets $\underline{D}$ and $\underline{F}$

PROPOSITION 3.12. $\forall r > 0 \exists \alpha > 0 \forall \varepsilon > 0 \exists l > 0$ such that

$$\lim_{K \rightarrow \infty} P(V(B(\underline{x}), l, \varepsilon, \alpha, r)) = 1.$$

PROOF. For simplicity we do the proof for $r = 0$. The integer r will be used in the proof of the $\mathcal{I}$ -tightness, where we shall place a cone similar to the cone of percolation $\mathbf{F}$ such that the cone contains the box $B(\underline{x})$.

We concentrate on the region $\underline{D}$, the region $\underline{F}$ being handled as in proposition 3.1. Let $\varepsilon > 0$, and let $\underline{x}$ be in $\mathbb{Z}^d$. Let $\varepsilon' > 0$, and let l_1 be the constant given by proposition 3.1. We define $E(\underline{x})$ as

$$E(\underline{x}) = \left\{ \forall y \text{ such that } |C(y)| \geq K/2 \text{ and } C(y) \cap B(\underline{x}) \neq \emptyset, \right. \\ \left. \text{we have } \left| \{z \in C(y) \cap B(\underline{x} + l_1 e_d) : |C(z)| \geq K/4\} \right| \geq 4\varepsilon' K^d \right\}.$$

We claim that for ε' small enough,

$$P(E(\underline{x})) \rightarrow 1 \quad \text{as } K \rightarrow \infty. \quad (3.13)$$

PROOF OF (3.13). The events $\{\mathbb{Z}^{d-1} \times \{0\} \rightarrow z\}$ and $|C(z)| \geq K/4$ are independent (we could also use the FKG inequality), and of probability larger than θ . We adapt (3.8) in the following way. For all $\varepsilon > 0$,

$$P\left(\frac{1}{K^d} \left| \{z \in B(\underline{x} + l_1 e_d) : (\mathbb{Z}^{d-1} \times \{0\} + (K\underline{x} - K e_d)) \rightarrow z \right. \right. \\ \left. \left. \text{and } |C(z)| \geq K/4\} \right| \geq \theta^2 - \varepsilon \right) \rightarrow 1, \quad (3.14)$$

as K goes to infinity. From the estimates (3.6) and (3.5) we get the limit (3.13). $\square$

Let $\varepsilon_1 > 0$. Pick $\varepsilon' > 0$ and K large enough such that

$$P(E(\underline{x})) \geq 1 - \varepsilon_1. \quad (3.15)$$

We now introduce the event that a cluster is near the cone of percolation. Let $y = (\mathbf{y}, t)$ in $\mathbb{Z}^d$, and let n be a positive integer. We recall that

$$\xi_n^y = \{\mathbf{x} \in \mathbb{Z}^{d-1} : y \rightarrow (\mathbf{y} + \mathbf{x}, t + n)\}.$$

We define $\overline{H}_n^y$ and $\overline{K}_n^y$ in the same way as $\overline{H}_n$ and $\overline{K}_n$ before proposition 2.2. Let n_0 in $\mathbb{N}$, and let y in $\mathbb{Z}^d$. We define

$$A(y, \varepsilon, n_0) = \{\forall n \geq n_0, (\overline{H}_n^y \cap \overline{K}_n^y) \supset (1 - \varepsilon)n\mathbf{U}\}.$$

By proposition 2.2, for all $\varepsilon > 0$, there exists n_0 such that

$$P(A(0, \varepsilon, n_0) \mid |C(0)| = \infty) \geq 1 - \varepsilon'.$$

Let $\varepsilon > 0$, and take n_0 such that the above inequality holds. With the help of the exponential estimates of proposition 2.3 on the law of $|C(0)|$, we obtain that there exists K_0 in $\mathbb{N}$ such that, for all $K \geq K_0$,

$$P(A(0, \varepsilon, n_0) \mid |C(0)| \geq K/4) \geq 1 - 2\varepsilon'.$$

Hence, by the ergodic theorem [22], for all $\varepsilon_1 > 0$, for K large enough,

$$P\left(\left|\{y \in B(\underline{x} + l_1 e_d) : |C(y)| \geq K/4 \text{ and } A^c(y, \varepsilon, n_0)\}\right| \geq 3\varepsilon' K^d\right) \leq \varepsilon_1. \quad (3.16)$$

Take $\varepsilon' > 0$ such that $\varepsilon' < \theta/8$. Putting together inequalities (3.15) and (3.16), we obtain

$$\begin{aligned} P(\forall y \text{ such that } |C(y)| \geq K/4 \text{ and } C(y) \cap B(\underline{x}) \neq \emptyset, \\ \exists z \in B(\underline{x} + l_1 e_d) \cap C(y) \text{ such that } A(z, \varepsilon, n_0) \text{ occurs}) \geq 1 - 2\varepsilon_1. \end{aligned} \quad (3.17)$$

We take l such that $lK \geq 2n_0$, and such that for every z in $B(\underline{x} + l_1 e_d)$,

$$\underline{D}(\underline{x}, l, 2\varepsilon, 0) \subset \underline{\mathbf{F}}(z, 1 - \varepsilon). \quad (3.18)$$

By the ergodic theorem, the definition of K_n , and by the inclusion (3.18), for K large enough,

$$\begin{aligned} P(\forall z \in B(\underline{x} + l_1 e_d) \text{ such that } A(z, \varepsilon, n_0) \text{ occurs,} \\ C(z) \text{ intersects every box in } \underline{D}(\underline{x}, l, 2\varepsilon, 0)) \geq 1 - \varepsilon_1. \end{aligned} \quad (3.19)$$

The estimates (3.17) and (3.19) yield that, for K large enough,

$$P(V(B(\underline{x}), l, 2\varepsilon, \widehat{\delta}, 0)) \geq 1 - 3\varepsilon_1. \quad \square$$

4 The rescaled lattice

Let K be an integer. We divide $\mathbb{Z}^d$ into small boxes called blocks of size K in the following way. For $\underline{x} \in \mathbb{Z}^d$, we define the *block* indexed by $\underline{x}$ as

$$B(\underline{x}) =] - K/2, K/2]^d + K\underline{x}.$$

Note that the blocks partition $\mathbb{R}^d$. Let A be a region in $\mathbb{R}^d$. We define the rescaled region $\underline{A}$ as

$$\underline{A} = \{\underline{x} \in \mathbb{Z}^d : B(\underline{x}) \cap A \neq \emptyset\}.$$

In general, we use underline in the notation to emphasize that we are dealing with rescaled objects.

We define the sets $\mathbb{E}^d, \mathbb{E}^{d,\infty}$ by

$$\mathbb{E}^d = \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\},$$

$$\mathbb{E}^{d,\infty} = \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_\infty = 1\}.$$

The rescaled lattice is isomorphic to $\mathbb{Z}^d$ and we equip it with the graph structures corresponding to $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, or $\mathbb{L}^{d,\infty} = (\mathbb{Z}^d, \mathbb{E}^{d,\infty})$.

Let A be a subset of $\mathbb{Z}^d$. We define the *inner boundary* $\partial^{in} A$ of A as

$$\partial^{in} A = \{x \in A : \exists y \notin A, |x - y| = 1\}.$$

The *residual components* of A are the connected components of the graph $(A^c, \mathbb{E}^d(A^c))$. Let R be a residual component of A . The exterior boundary of R (in A) is

$$\{x \in \partial^{in} A : \exists y \in R, |x - y| = 1\}.$$

The importance of the graph $\mathbb{L}^{d,\infty}$ lies in the fact that the exterior boundary of R is $\mathbb{L}^{d,\infty}$ -connected.

Let $X(\underline{x})$ be a site process on $\mathbb{Z}^d$. We say that a box is *good* if $X(\underline{x}) = 1$, and *bad* otherwise. For $\underline{A}$ a subset of $\mathbb{Z}^d$, we denote by $N_2(\underline{A})$ the number of bad boxes in $\underline{A}$ (we will use N_1 as the number of good boxes later). Let $\varepsilon > 0$. We say that $\underline{A}$ is ε -bad, if the proportion of bad blocks in $\underline{A}$ is larger than ε , that is if

$$N_2(\underline{A})/|\underline{A}| > \varepsilon.$$

LEMMA 4.1. *There exists a dimension dependent constant $b(d) > 0$ such that, for every bounded open set O , every integers $s, t > 0$, every $\delta, \varepsilon > 0$, if $X \succ Z^{1-\delta}$, then*

$$P\left(\exists (\underline{A}_i)_{i \in I} \text{ a family of disjoint } \mathbb{L}^{d,\infty}\text{-connected components,}\right. \\ \left.\sum_{i \in I} |\underline{A}_i| \geq s, \text{ for all } i \in I, \underline{A}_i \cap \underline{Q} \neq \emptyset, |\underline{A}_i| \geq t, \text{ and } \cup_{i \in I} \underline{A}_i \text{ is } \varepsilon\text{-bad}\right) \\ \leq 2 \sum_{j \geq s} \exp j \left(\frac{1}{t} \ln \mathcal{L}^d(\mathcal{V}(O, d)) + \ln b + \Lambda^*(\varepsilon, \delta) \right)$$

where

$$\Lambda^*(\varepsilon, \delta) = \varepsilon \ln \frac{\varepsilon}{\delta} + (1 - \varepsilon) \ln \frac{1 - \varepsilon}{1 - \delta}$$

is the Fenchel–Legendre transform of the logarithmic moment generating function of a Bernoulli variable with parameter δ .

PROOF. The inequality follows as in [4] from a counting Peierls argument and from the theorem of Cramer [5]. $\square$

We return to the block events R and V that we introduced in the previous section. The events $R(B(\underline{x}), l, \varepsilon)$ and $V(B(\underline{x}), l, \varepsilon, \alpha, r)$ depend only on edges in the set $\cup_{|\underline{y}-\underline{x}| < 2l} B(\underline{y})$. Hence we can apply the domination result of [19] to our block processes:

LEMMA 4.2. *Let $X(\underline{x})$ be the indicator variable of either the event $R(B(\underline{x}), l, \varepsilon)$ or $V(B(\underline{x}), l, \varepsilon, \alpha)$ with ε, α , and l as in propositions 3.1 or 3.12. For every $\delta > 0$, there exists K_0 such that for all integer $K \geq K_0$, the process X dominates stochastically the Bernoulli site-process $Z^{1-\delta}$ of intensity $1 - \delta$.*

With the help of lemma 4.2 we shall use the estimate in lemma 4.1 for the events R and V . In [4], the author does not use this domination estimate. Indeed, he considers the event that all blocks in a certain region $\underline{A}$ are bad. He can partition the lattice $\mathbb{Z}^d$ into a fixed number N of distinct classes such that in each class, the variables are mutually independent, hence there exists a class whose intersection with the set $\underline{A}$ has a cardinality larger than $N^{-1}|\underline{A}|$, and all the blocks in this intersection are bad. In our case, we can not control the proportion of bad blocks in an intersection, thus we make appeal to the domination result of [19].

5 Surface tension

Let $x = (x_1, \dots, x_d)$ be a point of $\mathbb{R}^d$ and let w be a vector in the unit sphere S^{d-1} . The hyperplane containing x with normal vector w is

$$\text{hyp}(x, w) = \{y \in \mathbb{R}^d : (y - x) \cdot w = 0\}.$$

Let A be a subset of $\mathbb{R}^d$ of linear dimension $d - 1$, that is A spans a hyperplane of $\mathbb{R}^d$, which we denote $\text{hyp } A$. We call such a set a hyperset. By $\text{nor } A$ we denote one of the two unit vectors orthogonal to $\text{hyp } A$. The cylinder of basis A is the set

$$\text{cyl } A = \{x + t \text{ nor } A : t \in \mathbb{R}, x \in A\}.$$

Let w be a unit vector and $r > 0$. We define

$$\text{cyl}^-(A, w, r) = \{x - tw : t > r, x \in A\},$$

$$\text{cyl}^+(A, w, r) = \{x + tw : t > r, x \in A\}.$$

For $r > 0$, the r -neighbourhood $\mathcal{V}(A, r)$ of a subset A of $\mathbb{R}^d$ is

$$\mathcal{V}(A, r) = \{x \in \mathbb{R}^d : \inf_{y \in A} |x - y| < r\}.$$

We fix a real number $\zeta > 2d$. We define two regions:

$$R^-(A, w, \zeta) = \text{cyl}^-(A, w, \zeta) \cap \mathcal{V}(\mathbb{R}^d \setminus \text{cyl } A, \zeta),$$

$$R^+(A, w, \zeta) = \text{cyl}^+(A, w, \zeta) \cap \mathcal{V}(\mathbb{R}^d \setminus \text{cyl } A, \zeta),$$

as represented on figure 6.

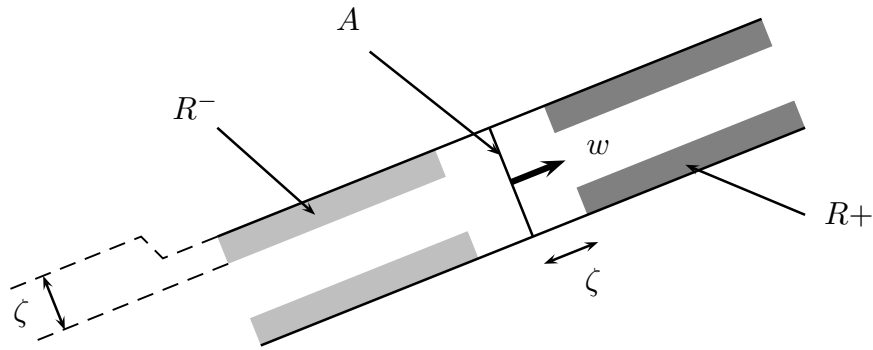


figure 6: the regions R^- and R^+ .

DEFINITION 5.1. *Let A be a closed hyperrectangle, let w be a unit vector and let s be positive or infinite. We denote by $W(\partial A, w, s, \zeta)$ the event that there exists a finite set of closed edges E inside $\mathcal{V}(\text{hyp } A, s)$ such that in the graph $(\mathbb{Z}^d \cap \text{cyl } A, \vec{\mathbb{E}}_{\text{alt}}^d)$, there is no oriented open path from $R^-(A, w, \zeta)$ to $R^+(A, w, \zeta)$.*

Loosely speaking, the “boundary” of the interface E is “pinned down” at ∂A within a distance ζ .

PROPOSITION 5.2. *Let $p \in]0, 1[$. Let A be a hyperrectangle and let w be $\text{nor } A$ or $-\text{nor } A$. Let $\Phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that $\lim_{n \rightarrow \infty} \Phi(n) = \infty$. The limit*

$$\lim_{n \rightarrow \infty} -\frac{1}{\mathcal{H}^{d-1}(nA)} \ln P(W(\partial nA, w, \Phi(n), \zeta))$$

exists in $[0, \infty]$ and depends only on w . We denote it by $\tau(w)$ and call it the surface tension in the direction w .

PROOF. The proof relies on the same subadditivity argument of [4]. From now on, we drop ζ in the notations. $\square$

Here is a heuristical comment of the reason we alter the definition of the surface tension given in [4]. If we use our definition of the surface tension for non-oriented percolation, then we obtain the same function as in [4]. On the other hand, we can not use the definition of [4] in our case, because it is too easy to find a set of edges which cuts the cylinder $\text{cyl } A$ in two parts in the oriented case. For example, if w in S^{d-1} is such that $w \cdot e_d < \sqrt{2}/2$, then there is no oriented path from $-\infty$ to $+\infty$ in $\text{cyl } A$. Let W' be the event considered in [4]. The point is that, in [4], the event W' implies that for all $\varepsilon > 0$, with probability tending to 1 as n goes to ∞ , the number of vertices in $\text{cyl}^+ A$ joined by $\text{cyl}^- A$ is less than εn^d . This property is crucial to obtain the upper bound. Now consider the oriented case and a hyperrectangle A which is normal to e_1 . As previously noted, we have $P(W'(\partial A, e_1, 2n, \zeta)) = 1$. But as we may see in figure 7, there exists $\alpha > 0$ such that with probability tending to 1 as n goes to ∞ , the number of vertices in $\text{cyl}^+ A$ attained by $\text{cyl}^- A$ is larger than αn^d . Thus, with the definition of [4], we would not have the large deviation upper bound.

We derive now some basic properties of the surface tension. The surface tension τ inherits automatically some symmetry properties from the model. For instance, if f is a linear isometry of $\mathbb{R}^r$ such that $f(0) = 0$, $f(\mathbb{Z}^d) = \mathbb{Z}^d$, and $f(e_d) = e_d$, then $\tau \circ f = \tau$. Note that there is less symmetry than in the unoriented model. Since the function τ is not symmetric, we have to take care on the orientation of the vectors when we state the following weak triangle inequality:

PROPOSITION 5.3 (WEAK TRIANGLE INEQUALITY). *Let (ABC) be a non degenerate triangle in $\mathbb{R}^d$. In the plane spanned by A, B, C , let ν_A be the exterior normal unit vector*

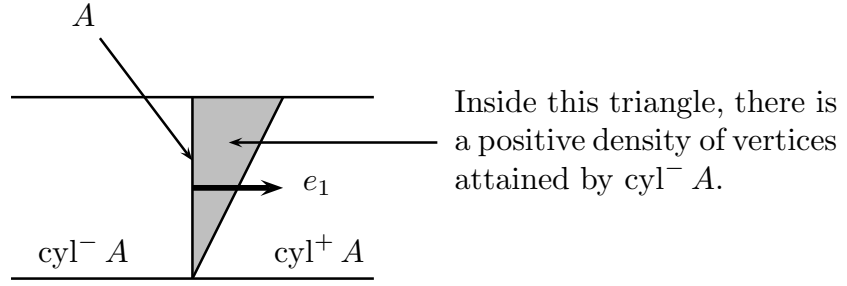


figure 7: why we should prevent connections from $\text{cyl}^- A$ to $\text{cyl}^+ A$.

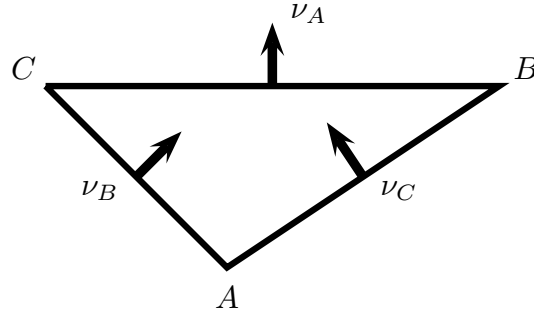


figure 8: the three normal vectors of a triangle.

to $[BC]$, and let ν_B, ν_C be the interior normal unit vectors to the sides $[AC], [AB]$, see figure 8. Then

$$\mathcal{H}^1([BC])\tau(\nu_A) \leq \mathcal{H}^1([AC])\tau(\nu_B) + \mathcal{H}^1([AB])\tau(\nu_C). \quad (5.4)$$

PROOF. The proof is the same as in [4], except that we have to take care about the orientation of the vectors. $\square$

PROPOSITION 5.5. *The homogeneous extension τ_0 of τ to $\mathbb{R}^d$ defined by $\tau_0(0) = 0$ and*

$$\forall w \in \mathbb{R}^d \setminus \{0\} \quad \tau_0(w) = |w|_2 \tau(w/|w|_2)$$

is finite everywhere and is a convex continuous function.

PROOF. The convexity of τ_0 is a consequence of the weak triangle inequality (5.4): let (A, B, C) be a non-degenerate triangle, and let (A', B', C') be the image of the triangle (A, B, C) by the rotation of angle $\pi/2$ in the plane spanned by A, B, C (we choose the orientation of the plane such that the triangle is oriented counter-clockwise). Let ν_A be the exterior normal vector to $[BC]$, and let ν_B and ν_C be the interior normal vectors to the sides $[AC]$, $[AB]$. Then $\tau_0(\overrightarrow{A'B'}) = [AB]\tau(\nu_C)$, $\tau_0(\overrightarrow{C'A'}) = [AC]\tau(\nu_B)$, and $\tau_0(\overrightarrow{C'B'}) = [BC]\tau(\nu_A)$. It follows that

$$\tau_0(\overrightarrow{A'B'}) \leq \tau_0(\overrightarrow{A'C'}) + \tau_0(\overrightarrow{C'B'}),$$

and this holds for every A', B', C' . Then for every $\lambda \in [0, 1]$, for all $\vec{u}, \vec{v}$,

$$\tau_0(\lambda\vec{u} + (1-\lambda)\vec{v}) \leq \tau_0(\lambda\vec{u}) + \tau_0((1-\lambda)\vec{v}) \leq \lambda\tau_0(\vec{u}) + (1-\lambda)\tau_0(\vec{v}).$$

The finiteness is checked as in [4], and the continuity is then a consequence [21]. $\square$

Let $\mathbf{G} \subset S^{d-1}$ be the set

$$\mathbf{G} = \{w \in S^{d-1} : \text{hyp}(0, w) \cap \mathbf{F} \neq \{0\}\},$$

and denote by $\widehat{\mathbf{G}}$ its corresponding cone:

$$\widehat{\mathbf{G}} = \{tw; t \geq 0, w \in \mathbf{G}\}.$$

The two cones $\mathbf{F}$ and $\widehat{\mathbf{G}}$ are represented on figure 9. We call $\widehat{\mathbf{G}}$ the *cone of positivity*, partly because of the next proposition.

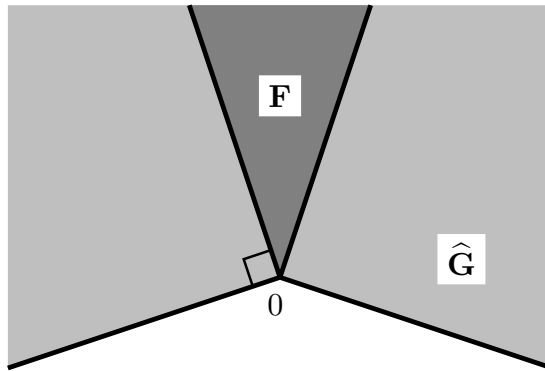


figure 9: The two cones $\mathbf{F}$ and $\widehat{\mathbf{G}}$

PROPOSITION 5.6. *Let $p > p_c$. The surface tension is equal to 0 outside $\mathbf{G}$.*

PROOF. By proposition 2.2, for all $\varepsilon > 0$, $\varepsilon' > 0$, there exists n_0 such that, for all $n \geq n_0$,

$$P\left(0 \rightarrow ((1 + \varepsilon)n(\mathbb{R}^{d-1} \setminus \mathbf{U}), n)\right) \leq \varepsilon',$$

and the nullity outside $\mathbf{G}$ follows. $\square$

6 The Wulff crystal and the positivity of the surface tension

We begin with the definition of the Wulff set.

DEFINITION 6.1. *The Wulff crystal of τ is the set*

$$\mathcal{W}_\tau = \{x \in \mathbb{R}^d : x \cdot w \leq \tau(w) \text{ for all } w \text{ in } S^{d-1}\}.$$

The Wulff crystal is a closed and convex set containing 0. Since τ is bounded, the Wulff crystal is also bounded. The nullity of τ outside the region $\mathbf{G}$ implies that $\mathcal{W}_\tau$ is included in the cone of percolation $\mathbf{F}$. From $\mathcal{W}_\tau$ we can recover the function τ :

PROPOSITION 6.2. *The surface tension τ is the support function of its Wulff crystal, that is,*

$$\forall \nu \in S^{d-1} \quad \tau(\nu) = \sup\{x \cdot \nu : x \in \mathcal{W}_\tau\}.$$

The crystal $\mathcal{W}_\tau$ admits a unit outwards normal vector $\nu_{\mathcal{W}_\tau}(x)$ at $\mathcal{H}^{d-1}$ almost all points $x \in \partial\mathcal{W}_\tau$ and

$$\tau(\nu_{\mathcal{W}_\tau}(x)) = x \cdot \nu_{\mathcal{W}_\tau}(x) \text{ for } \mathcal{H}^{d-1} \text{ almost all } x \in \partial\mathcal{W}_\tau.$$

PROOF. The proof in [14] relies on the strict positivity of the function τ and do not make any assumption of convexity. Besides, the proof in [4] only relies on the convexity of τ_0 . $\square$

We want to show that the Wulff crystal has a non-empty interior. This will follow from the positivity of the surface tension inside a sufficiently large angular sector:

PROPOSITION 6.3. *There exist $\varepsilon > 0$ and $\eta > 0$ such that, for each w in S^{d-1} , if $w \cdot e_d > -\eta$, then $\tau(w) \geq \varepsilon$.*

PROOF. The first step is to prove that

$$\tau(e_d) > 0. \tag{6.4}$$

PROOF OF (6.4). Let A be the hyperrectangle $[-n, n]^{d-1} \times \{0\}$. Let $\varepsilon > 0$, and let A' be the hyperrectangle $[-n/2, n/2]^{d-1} \times \{-n/4\}$. Consider the event

$$W(\partial A, e_d, n/8).$$

Because of the graph structure of $\vec{\mathbb{L}}_{\text{alt}}^d$, each oriented path joining A' to $A + (n/8)e_d$ lies inside $\text{cyl}(A)$. Hence, the event $W(\partial A, e_d, n/8)$ implies that the set A' is not connected to the infinity. By proposition 2.3, we conclude that

$$P(W(\partial A, e_d, n/8)) \leq \exp(-\gamma n^{d-1}),$$

with $\gamma > 0$ independent of n . $\square$

Let us return to the proof of proposition 6.3. Suppose that there exists w in S^{d-1} such that $w \cdot e_d > 0$ and $\tau(w) = 0$. Let $\hat{w}$ be the image of w by the symmetry of axis e_d . Because of the symmetry properties of τ , we have $\tau(\hat{w}) = 0$. By the convexity of τ_0 , it follows that $\tau(e_d) = 0$, which contradicts (6.4).

Now suppose that there exists w in S^{d-1} such that $w \cdot e_d = 0$ and $\tau(w) = 0$. In that case the symmetries of the graph and the convexity of τ_0 imply that for all w' in S^{d-1} such that $w \cdot e = 0$, we have $\tau(w') = 0$. We now prove that

$$\tau(e_1) > 0. \tag{6.5}$$

PROOF OF INEQUALITY (6.5). Let A be the hyperrectangle $\{n\} \times [0, n]^{d-1}$. Let $\varepsilon > 0$, and let A' be the hyperrectangle $[\varepsilon n, (1 - \varepsilon)n]^{d-1} \times \{0\}$. We define the regions $\mathcal{K}_i^\pm$, $1 \leq i \leq d-1$, by

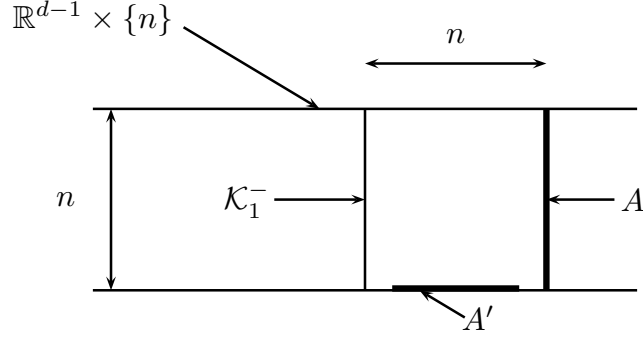
$$\begin{aligned} \mathcal{K}_i^+ &= [0, n]^{i-1} \times \{n\} \times [0, n]^{d-i} \\ \mathcal{K}_i^- &= [0, n]^{i-1} \times \{0\} \times [0, n]^{d-i}, \end{aligned}$$

and we let

$$\mathcal{K} = \bigcup_{1 \leq i \leq d-1} \mathcal{K}_i^\pm,$$

see figure 10. Note that $\mathcal{K}_1^+ = A$.

Let K be an integer. We work with the lattice rescaled by K . We denote by $\underline{C}(A')$ the set of blocks intersecting $C(A')$ the cluster of A' . Consider the event $R'(B(\underline{x}, l))$ defined as the event $R(B(\underline{x}, l, \varepsilon))$ except that we do not require any density property. We pick $l > 0$ such that the limit in proposition 3.1 holds. We call the blocks good or bad accordingly to the event R' .


 figure 10: the set $\mathcal{K}$ surrounding A'

We introduce notations in order to count the good and bad blocks of the boundary. A block $B(\underline{x})$ is *at height i* if $\underline{x} \cdot e_d = i$.

- a_i = number of blocks at height i that are in $\underline{C}(A')$,
- b_i = number of good blocks at height i that are in $\partial^{in}\underline{C}(A')$,
- b'_i = number of good blocks at height i that are in $\partial^{in}\underline{C}(A')$,
and that have a neighbour at height i that is not in $\underline{C}(A')$,
- c_i = number of bad blocks at height i that are in $\partial^{in}\underline{C}(A')$.

For $i \geq 0$, let Y_i be the family of blocks in $\underline{C}(A')$ at height i . The process $(Y_i)_{i \geq 0}$ can be view as a contact process. Boxes in Y_i that are not in the boundary of $\underline{C}(A')$ or that are good are still in Y_{i+1} . Hence

$$a_{i+1} \geq a_i - c_i.$$

Moreover, a good box in $\partial^{in}\underline{C}(A')$ and counted in b'_i gives “birth” to at least one box in Y_{i+l} because of the definition of the event $R'(B(\underline{x}), l)$. We have to care about the fact that several boxes counted in b'_i can give birth to the same box in Y_{i+l} . Actually, the maximal number of boxes giving birth to the same box is bounded by $2(d-1)$. Therefore, for all i in $[0, n/K]$,

$$a_{i+l} \geq a_i + \frac{b'_i}{2(d-1)} - c_i - c_{i+1} - \dots - c_{i+l-1},$$

see figure 11.

Furthermore, $a_0 \geq ((1-2\varepsilon)n/K)^{d-1}$, and $a_i \leq (n/K)^{d-1}$ for all i in $[0, n/K]$. We let $B'_k = \sum_{i=0}^{n/(Kl)} b'_{k+il}$, $B' = \sum_{i=0}^{n/K} b'_i$, and we let $C = \sum_{i=0}^{n/K} c_i$. Summing the previous

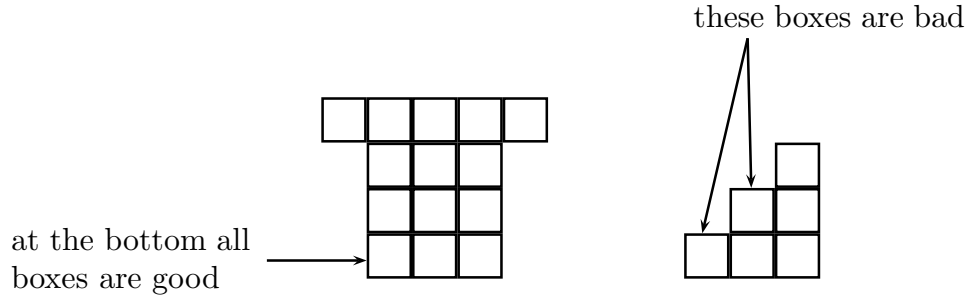


figure 11: examples of block configurations.

inequality over i with step l , we obtain

$$2C \geq \frac{1}{2(d-1)} B'_k - (2\varepsilon n/K)^{d-1},$$

for all k . But there exist $k \in \{0, \dots, l-1\}$, such that $B'_k \geq \frac{1}{l} B'$. Hence

$$2C \geq \frac{1}{2(d-1)l} B' - (2\varepsilon n/K)^{d-1}.$$

Now let $b''_i = b_i - b'_i$. For each box counted in b''_{i+1} , there is a box counted in b'_i , and a box counted in b'_i can give no more than $2(d-1)$ boxes counted in b''_{i+1} , thus

$$b''_{i+1} \leq 2(d-1)b'_i, \quad (6.6)$$

see figure 12.

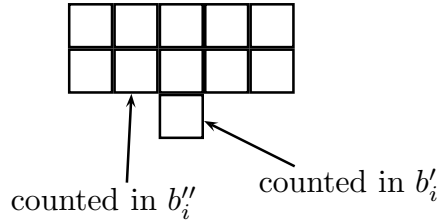


figure 12: different boundary boxes

Denote by B the number

$$B = \sum_{i=0}^{n/K} b_i.$$

From (6.6), it follows that $B' \geq \frac{1}{4(d-1)}B$, and we get that

$$2C \geq \frac{1}{8(d-1)^2 l} B - 2\varepsilon n/K)^{d-1}.$$

Hence, if A' is not joined to $\mathcal{K}$, there exists a $\mathbb{L}^{d,\infty}$ connected component of cardinality larger than $(n/(2K))^{d-1}$ intersecting $[0, n]^{d-1} \times \{0\}$, which has a proportion of bad boxes larger than $1/(20(d-1)^2 l)$ for ε small enough. By a counting Peierls argument, there exists $C > 0$ such that for K large enough,

$$P(A' \not\rightarrow \mathcal{K}) \leq \exp(-cn^{d-1}).$$

On the other hand, because of the symmetry of the graph, $P(A' \rightarrow \mathcal{K}_i^\pm)$ does not depend on i nor on the sign. By the FKG inequality (2.1),

$$\begin{aligned} P(A' \not\rightarrow \mathcal{K}) &= P\left(\bigcap_{1 \leq i \leq d-1} \{A' \not\rightarrow \mathcal{K}_i^+\} \cap \bigcap_{1 \leq i \leq d-1} \{A' \not\rightarrow \mathcal{K}_i^-\}\right) \\ &\geq P(A' \not\rightarrow A)^{2(d-1)}. \end{aligned}$$

Thus

$$P(A \rightarrow A') \leq \exp(-cn^{d-1}/(2(d-1))). \quad \square$$

With the help of the continuity of τ_0 , we get the desired positivity result of proposition 6.3. $\square$

COROLLARY 6.7. *The Wulff crystal $\mathcal{W}_\tau$ has a non empty interior and a strictly positive Lebesgue measure.*

PROOF. This is a straightforward consequence of the continuity of τ_0 , of the positivity property stated in lemma 6.3, and of the definition of the Wulff crystal. $\square$

7 Separating sets

We need more flexibility on the localization of the set E which separates the cylinder of A in two parts in definition 5.1. Let A be a hyperset in $\mathbb{R}^d$ and let r be positive. We denote by $S(A, w, r)$ the event that there exists a finite set of closed edges in $\text{cyl } A \cap \mathcal{V}(\text{hyp } A, r)$ such that there is no oriented open path in the graph $(\mathbb{Z}^d \cap \text{cyl } A, \vec{\mathbb{E}}_{\text{alt}}^d)$ from $\text{cyl}^-(A, w, r)$ to $\text{cyl}^+(A, w, r)$. From now on, we work with a fixed value of ζ larger than $2d$. We now recall some result on separating sets from [4]

LEMMA 7.1. *Let O be an open hyperset in $\mathbb{R}^d$, let w be one of the two unit vectors orthogonal to $\text{hyp } O$, and let $\Phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that $\lim_{n \rightarrow \infty} \Phi(n) = \infty$. We have*

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(S(nO, w, \Phi(n))) \geq -\mathcal{H}^{d-1}(O)\tau(w).$$

For r an integer, we let α_r be the volume of the r -dimensional unit ball.

LEMMA 7.2. *There exists a positive constant $c = c(d, \zeta)$ such that, for each x in $\mathbb{R}^d$, all positive ρ, η with $\eta < \rho$, every w in S^{d-1} ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(S(n \text{disc}(x, \rho, w), w, n\eta)) \leq -\alpha_{d-1}\rho^{d-1}\tau(w) + c\eta\rho^{d-2}.$$

LEMMA 7.3. *Let F be a $d-1$ dimensional set such that $\mathcal{H}^{d-2}(\partial F) < \infty$, and let w be nor F or $-\text{nor } F$. We define $\text{wall}(F, w, n)$ as the event*

$$\begin{aligned} \text{wall}(F, w, n) = & S(nF, w, \ln n) \cap \\ & \{ \text{all the edges in } \mathcal{V}(\text{cyl } \partial nF, 2d) \cap \mathcal{V}(\text{hyp } nF, \ln n) \text{ are closed} \}. \end{aligned}$$

Then

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\text{wall}(F, w, n)) \geq -\mathcal{H}^{d-1}(F)\tau(w).$$

8 Interface estimate

Let x be a point of $\mathbb{R}^d$. The closed ball of center x and Euclidian radius $r > 0$ is denoted by $B(x, r)$. We denote by α_d the volume of the d -dimensional unit ball. For w in the unit sphere S^{d-1} , we define the half balls

$$B_-(x, r, w) = B(x, r) \cap \{y \in \mathbb{R}^d : (y - x) \cdot w \leq 0\},$$

$$B_+(x, r, w) = B(x, r) \cap \{y \in \mathbb{R}^d : (y - x) \cdot w \geq 0\}.$$

The open $B(nx, nr)$ -clusters are the open clusters in the configuration restricted to the ball $B(nx, nr)$. Let $\text{Sep}(n, x, r, w, \delta)$ be the following event: there exists a collection $\mathcal{C}$ of open $B(nx, nr)$ -clusters such that

$$\left| \bigcup_{C \in \mathcal{C}} C \cap B_-(nx, nr, w) \right| \geq (1 - \delta) \mathcal{L}^d(B_-(nx, nr, w)),$$

$$\left| \bigcup_{C \in \mathcal{C}} C \cap B_+(nx, nr, w) \right| \leq \delta \mathcal{L}^d(B_+(nx, nr, w)).$$

LEMMA 8.1. *Let $p \in]0, 1[$ and let $\alpha > 0$ be a parameter. There exists $c = c(p, d, \zeta, \alpha)$ such that for every $x \in \mathbb{R}^d$, every $r \in]0, 1[$, every unit vector $w \in S^{d-1}$ with $\tau(w) \geq \alpha$, and every $\delta \in]0, 1[$:*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\text{Sep}(n, x, r, w, \delta)) \leq -\alpha_{d-1} r^{d-1} \tau(w) (1 - c\delta^{1/2}).$$

PROOF. We adapt the proof of D. Barbato [1] to oriented percolation. Suppose that the event $\text{Sep}(n, x, r, w, \delta)$ occurs, and let $\mathcal{C}$ be a collection of open $B(nx, nr)$ -clusters realizing it. We let E_- be the set of the open edges in $B_-(nx, nr, w)$ which do not belong to a cluster $C \in \mathcal{C}$. Symmetrically, let E_+ be the set of the open edges in $B_+(nx, nr, w)$ which belong to a cluster $C \in \mathcal{C}$. For $h \in \mathbb{R}$, let $\pi(h)$ be the hyperplane

$$\pi(h) = \{y \in \mathbb{R}^d : (y - x) \cdot w = h\}.$$

Let $\rho = r\sqrt{1 - \delta}$ and $\eta = \sqrt{\delta}r/3$. The projection on the line $x + \mathbb{R}w$ of the segment joining the endpoints of an edge has length at most 1, hence

$$\int_0^{\eta n} |e \in E_+ : e \cap \pi(h) \neq \emptyset| dh \leq |E_+|$$

and therefore there exists $h \in [0, \eta n]$ such that $\pi(h) \cap \mathbb{Z}^d = \emptyset$ and

$$|\{e \in E_+ : e \cap \pi(h) \neq \emptyset\}| \leq \frac{2d\delta}{\eta} n^{d-1} r^d \alpha_d.$$

Let h^* be the infimum in $[0, \eta n]$ of the real numbers h satisfying this inequality. We can take $\varepsilon > 0$ small enough so that $\pi(h) \cap \mathbb{Z}^d \cap B(n) = \emptyset$ for $h \in]h^*, h^* + \varepsilon[$. The set $\{e \in E_+ : e \cap \pi(h) \neq \emptyset\}$ is then constant in the interval $h \in]h^*, h^* + \varepsilon[$. We fix a value h_+ in this interval. Then the above inequality holds for h_+ , and every edge of E_+ which

intersects $\pi(h_+)$ has an endpoint in each of the two half spaces delimited by $\pi(h_+)$. Let V_+ be the set

$$V_+ = \{y \in \mathbb{Z}^d : (y - x) \cdot w > h_+, \\ y \text{ is the endpoint of an edge of } E_+ \text{ intersecting } \pi(h_+)\}.$$

Let F_+ be the set

$$F_+ = \{e \in \vec{\mathbb{E}}_{\text{alt}}^d : \text{one of the endpoint of } e \text{ is in } V_+, e \text{ does not intersect } \pi(h_+)\}.$$

We define in the same way the sets V_- and F_- . For $y \in \mathbb{R}^d$, w in the unit sphere S^{d-1} , and r_1, r_2 in $\mathbb{R} \cup \{-\infty, +\infty\}$, we define

$$\text{slab}(y, w, r_1, r_2) = \{z \in \mathbb{R}^d : r_1 \leq (z - y) \cdot w \leq r_2\}.$$

We define the following subsets of $B(nx, nr)$:

$$\begin{aligned} Z &= \text{cyl}(n \text{ disc}(x, \rho, w)), \\ D &= Z \cap \text{slab}(nx, w, -n\eta - \zeta, n\eta + \zeta), \\ D^+ &= Z \cap \text{slab}(nx, w, 1, n\eta + \zeta), \\ D^- &= Z \cap \text{slab}(nx, w, -n\eta - \zeta, 0), \\ \partial^+ D &= Z \cap \text{slab}(nx + n\eta w, w, -\zeta, \zeta), \\ \partial^- D &= Z \cap \text{slab}(nx - n\eta w, w, -\zeta, \zeta), \\ \partial^- D^+ &= Z \cap \text{slab}(nx, w, 1, 1 + \zeta), \\ \partial^+ D^- &= Z \cap \text{slab}(nx, w, -\zeta, 0). \end{aligned}$$

Let γ be an oriented open path in D joining $\partial^- D$ to $\partial^+ D$. Consider the last edge e of γ intersecting $\pi(h^+)$. There are two possibilities: either e is an edge of a cluster $C \in \mathcal{C}$ or not.

- In the first case the edge e is in E^+ . After the edge e , the path γ has to go through an edge of F_+ .
- In the second case, the fact that there is no cluster $C \in \mathcal{C}$ containing e implies that all the edges of γ before e are not in a cluster $C \in \mathcal{C}$. Let f be the first edge of γ intersecting $\pi(h^-)$. We know that $f \in E_-$. Before f , the path γ has to go through an edge of F_- .

In conclusion, all open path in D joining $\partial^- D$ to $\partial^+ D$ has to go through an edge of $F_- \cup F_+$. We perform the same surgery as in [4], and we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\text{Sep}(n, x, r, w, \delta)) \leq \\ 8d^2 \frac{\delta}{\eta} r^d \alpha_d \ln \frac{2}{1-p} - \alpha_{d-1} \rho^{d-1} \tau(w) + c\eta \rho^{d-2}.$$

Since we impose $\tau(w) > \alpha$ with $\alpha > 0$, there exists a constant $c'' = c''(p, d, \zeta, \alpha)$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\text{Sep}(n, x, r, w, \delta)) \leq -\alpha_{d-1} r^{d-1} \tau(w) (1 - c'' \delta^{1/2}),$$

for every $x \in \mathbb{R}^d$, $r \in]0, 1[$, $\delta \in]0, 1[$ and w in S^{d-1} such that $\tau(w) > \alpha$. $\square$

9 An alternative separating estimate

In the proof of the local upper bound, we shall not deal directly with the event “Sep”. We denote by $\partial^{in} B(nx, nr)$ the set

$$\partial^{in} B(nx, nr) = \{z \in B(nx, nr) \cap \mathbb{Z}^d : \exists y \notin B(nx, nr) \mid |z - y| = 1\}.$$

Let $\text{Sep}_\theta(n, x, r, w, \delta)$ be the following event: there exists a collection $\mathcal{C}$ of open $B(nx, nr)$ -clusters coming from $\partial^{in} B(nx, nr)$, and such that

$$\left| \bigcup_{C \in \mathcal{C}} C \cap B_-(nx, nr, w) \right| \geq (\theta - \delta) \mathcal{L}^d(B_-(nx, nr, w)),$$

$$\left| \bigcup_{C \in \mathcal{C}} C \cap B_+(nx, nr, w) \right| \leq \delta \mathcal{L}^d(B_+(nx, nr, w)).$$

LEMMA 9.1. *Let $p \in]0, 1[$. For every $\varepsilon > 0$, there exists $\delta_0 \in]0, 1[$ such that the following holds. For every $x \in \mathbb{R}^d$, every $r \in]0, 1[$, every unit vector $w \in S^{d-1}$, and every $\delta \in]0, \delta_0[$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\text{Sep}_\theta(n, x, r, w, \delta) \setminus \text{Sep}(n, x, r, w, \delta + \varepsilon)) = -\infty.$$

PROOF. The aim is to find a set of y 's in $B_-(nx, nr, w)$, such that $\mathcal{D} \cup \mathcal{C}$ satisfies the event $\text{Sep}(n, x, r, w, \delta)$, where $\mathcal{D}$ is the collection of the clusters of the y 's. To do this, we partition $B_-(nx, nr, w)$ with boxes of size K . For each box we consider box a little smaller and included in it. A box B is good if all vertices in the smaller box joined by the boundary of ∂B are in $\mathcal{C}$. Then the set of the y 's is the union of the smaller boxes that are included in a good box. In that way, the intersection of $\mathcal{D} \cup \mathcal{C}$ with $B_+(nx, nr, w)$ does not change. In the following we give the details of this argument.

The proof is quite long, and has much to do with the exponential estimates of volume order of [20]. We partition the half-ball $B_-(nx, nr, w)$ with large boxes of fixed size. The number of these boxes is of order n^d , and we show that if a typical event arises in most of the boxes partitioning $B_-(nx, nr, w)$, then the new event $\text{Sep}_\theta(n, x, r, w, \delta)$ is included in $\text{Sep}(n, x, r, w, \delta + \varepsilon)$ for a certain $\varepsilon > 0$ and for δ small enough.

Let $\varepsilon > 0$, and let $M > 0$ be such that $\frac{M-1}{M+3} > 1 - \varepsilon$. Take $\delta > 0$. Let $\alpha > 0$ and $\beta > 0$. Pick $\widehat{\delta} > 0$ the constant appearing in lemma 2.2, and fix an integer $K > 0$. For a box B , we define

$$\partial_-^{in} B = \{y = (\mathbf{y}, t) \in \partial^{in} B : (\mathbf{y}, t-1) \notin B\}.$$

We denote by π_B the hyperplane spanned by $\partial_-^{in} B$. We say that a box B of side length K is good if the following five conditions hold:

- (i) $|y \in B : \partial^{in} B \rightarrow y| \leq (\theta + \delta)K^d$.
- (ii) there is no open path γ in B such that $|\gamma| > \beta K$ and $\pi_B \nrightarrow \gamma$.
- (iii) for all x in $\partial_-^{in} B$ such that $|C(x)| \geq \beta K$, $\xi^x(y) = \xi^{\pi_B}(y)$ for all $y = (\mathbf{y}, t)$ such that $\beta K \leq t < K$ and $|\mathbf{y}| \leq \widehat{\delta}t$.
- (iv) for every $z \in \partial_-^{in} B$ such that $|C(z)| \geq K$, there exists $\widehat{z} \in C(z)$ such that

$$\pi_B \cdot e_d + \beta \leq \widehat{z} \cdot e_d \leq \pi_B \cdot e_d + 2\beta,$$

and

$$|C(\widehat{z}) \cap \mathbf{F}(\widehat{\delta}, \beta K, \widehat{z}) \cap B| \geq \alpha \beta^d K^d.$$

- (v) for all hypersquare A of side length $\geq \beta K$ and included in $\partial_-^{in} B$, $\tau_A \geq K$.

By propositions 2.3 and 2.4 and by the Birkhoff's ergodic theorem [22], there exists $\alpha > 0$, such that for β small enough

$$P(B \text{ is good}) \rightarrow 1 \quad \text{as } K \rightarrow \infty. \quad (9.2)$$

Let $\eta \in]0, 1/4[$. The interior of a box B of side length K is defined by

$$B^{int}(\eta) = B \setminus \mathcal{V}(\partial B, \eta K),$$

as represented in figure 13.

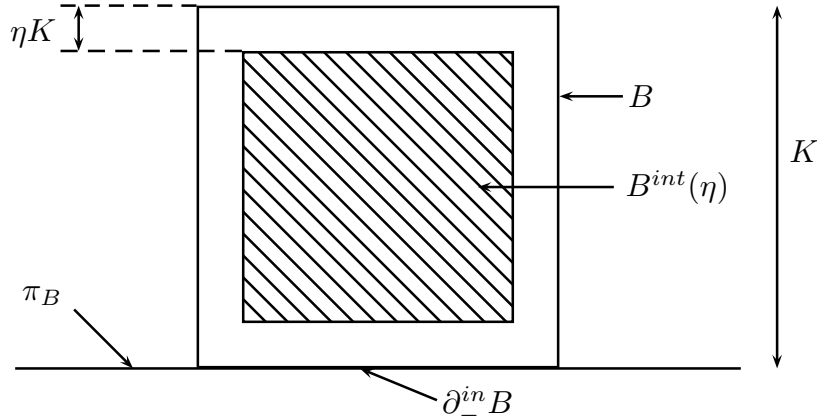


figure 13: details of a box B

LEMMA 9.3. *For all $\eta > 0$, there exists $\beta > 0$ and $\delta_0 > 0$, such that the following holds. Let $\delta \in]0, \delta_0[$. If $\mathcal{C}$ is a set of clusters in B coming from π_B such that*

$$|\mathcal{C}| \in [\theta - (M + 2)\delta, \theta + \delta]K^d,$$

if B is a good box, and if $y \in B^{int}(\eta)$ is such that $\pi_B \rightarrow y$, then $y \in \mathcal{C}$.

PROOF. Figure 14 shows what happens in a good box. Let $\eta > 0$. Take $\alpha > 0$ and $\beta > 0$ small enough such that the limit (9.2) holds. Furthermore, let β be small enough, such that

$$[-\beta/2, \beta/2]^{d-1} \subset \eta\hat{\delta}/2\mathbf{U}. \quad (9.4)$$

Moreover, assume that β is small enough, so that for all $\mathbf{y} \in \mathbb{Z}^{d-1}$ with $|\mathbf{y}| < 4\beta K$, we have

$$(\mathbf{F}(\hat{\delta}/2) \cap (\mathbb{R}^{d-1} \times \{t : t \geq \eta K\})) \subset (\mathbf{F}(\hat{\delta}, (\mathbf{y}, 0)) \cap (\mathbb{R}^{d-1} \times \{t : t \geq \eta K\})). \quad (9.5)$$

Now let $y \in B^{int}(\eta)$ such that $\pi_B \rightarrow y$. By condition (v) and (9.4), there exists z in $\partial_-^{in} B$ such that $y \in \mathbf{F}(\hat{\delta}/2, z)$ and $|C(z)| \geq K$. Because of condition (iii), since $\pi_B \rightarrow y$, we have $y \in C(z)$. We pick $\hat{z} \in C(z)$, accordingly to condition (iv).

Suppose there exists $\hat{y}$ in $\partial_-^{in} B(\underline{x})$ such that $C(\hat{y}) \cap \mathbf{F}(\hat{\delta}, \beta K, \hat{z}) \neq \emptyset$. Then $|C(\hat{y})| \geq \beta K$, and $|z - \hat{y}| \leq 4\beta K$. By condition (9.5) on the choice of β , $y \in \mathbf{F}(\hat{\delta}, \hat{y})$. Since $\pi_B \rightarrow y$ and by the condition (iii), this implies that $\hat{y} \rightarrow y$. Hence if $y \notin \mathcal{C}$, then $\mathbf{F}(\hat{\delta}, \hat{z}) \cap \mathcal{C} = \emptyset$, and the density of $\{y' \in B(\underline{x}) : \partial_-^{in} B \rightarrow y'\}$ inside B is larger than $\theta - (M + 2)\delta + \alpha\beta^d$. On the other hand, by condition (i), this density is less than $\theta + \delta$. By taking

$$\delta_0 = \frac{1}{2} \frac{\alpha\beta^d}{M + 3},$$

we conclude that $y \in \mathcal{C}$. $\square$

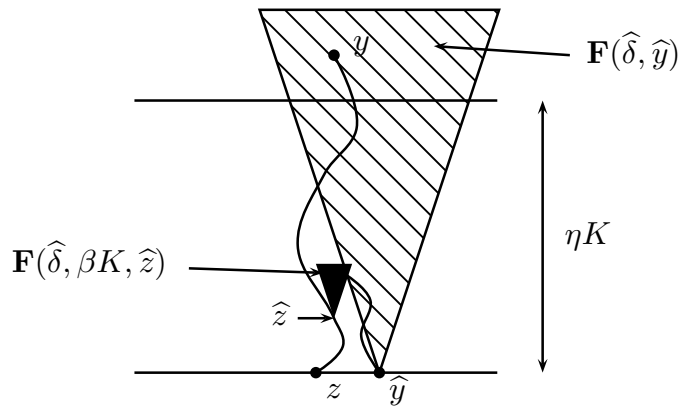


figure 14: in a good box

Let $\eta > 0$. Take $\delta > 0$ and $\beta > 0$ as in lemma 9.3, such that $\delta < \theta/M$ and $\beta < \eta$. Let K be large enough so that the process of good boxes stochastically dominates the Bernoulli-site process $Z^{1-\delta/2}$. For n large enough, there exists a subset $\underline{E}$ of $\mathbb{Z}^d$ such that

$$\mathcal{L}^d\left(B_-(nx, nr, w) \setminus \left(\bigcup_{\underline{x} \in \underline{E}} B_n(\underline{x})\right)\right) \leq \delta \frac{\varepsilon}{2} |\underline{E}|/n^d,$$

and

$$d\left(\bigcup_{\underline{x} \in \underline{E}} B_n(\underline{x}), \partial^{in} B_-(nx, nr, w)\right) \geq 2K/n, \quad (9.6)$$

where $d(\cdot, \cdot)$ is the distance associated to the norm $|\cdot|$.

By the theorem of Cramer [5], there exists a constant $c > 0$ such that for n large enough,

$$P(\text{the proportion of bad boxes in } \underline{E} \text{ is larger than } \delta) \leq \exp(-cn^d).$$

Denote by $\mathcal{E}$ the event that the proportion of bad boxes in $\underline{E}$ is less than δ . Suppose that $\mathcal{E} \cap \text{Sep}_\theta(n, x, r, w, \delta)$ occurs, and let $\underline{E}_1$ be

$$\underline{E}_1 = \{\underline{x} \in \underline{E} : |\mathcal{C} \cap B_n(\underline{x})| \in [\theta - (M+2)\delta, \theta + \delta], B_n(\underline{x}) \text{ is good}\}.$$

The family $\mathcal{C}$ satisfies $|\mathcal{C}| \geq (\theta - \delta)K^d|\underline{E}|$. On the other hand, we have the bound

$$|\mathcal{C}|/K^d \leq |\underline{E}_1|(\theta + \delta) + \delta|\underline{E}| + (|\underline{E}| - |\underline{E}_1|)(\theta - (M+2)\delta) + \delta\varepsilon/2|\underline{E}|.$$

Hence

$$(M-1)|\underline{E}| \leq (M+3)|\underline{E}_1|.$$

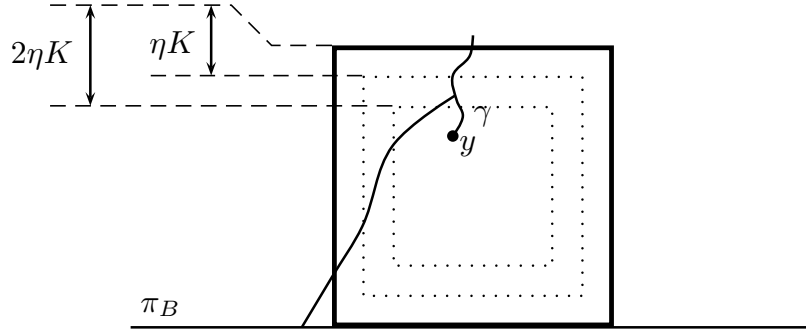
By the choice of M , we have $|\underline{E}_1| \geq (1-\varepsilon)|\underline{E}|$. Let

$$\mathcal{D} = \bigcup_{\underline{x} \in \underline{E}_1} \{C(y) : y \in B^{int}(\underline{x}, 2\eta)\}.$$

Let $\underline{x} \in \underline{E}_1$. Because of the structure of the graph $\vec{\mathbb{L}}_{\text{alt}}^d$ and because of condition (9.6), every path coming from $\partial^{in} B_-(nx, nr, w)$ and intersecting $B_n(\underline{x})$ has to intersect the hyperplane spanned by $\partial^{in} B_n(\underline{x})$. Hence, if γ is a path from $y \in B^{int}(\underline{x}, 2\eta)$ with $\underline{x} \in \underline{E}_1$, and which goes outside $B(\underline{x})$, then the part of γ outside $B(\underline{x})$ is included in a cluster of the family $\mathcal{C}$ because of the definition of a good block and of lemma 9.3, as represented on figure 15.

Thus

$$\left| \bigcup_{C \in \mathcal{C} \cup \mathcal{D}} C \cap B_+(nx, nr, w) \right| \leq \delta \mathcal{L}^d(B_-(nx, nr, w)).$$


 figure 15: the path γ is joined by $\mathcal{C}$

Now let y in $B_-(nx, nr, w) \setminus (\mathcal{C} \cup \mathcal{D})$. This implies that

$$y \in \left(\bigcup_{\underline{x} \in \underline{E}_1} \mathcal{V}(\partial B_n(\underline{x}), 2\eta K) \right) \cup \left(\bigcup_{\substack{\underline{x} \in \underline{E} \\ X(\underline{x})=0}} B_n(\underline{x}) \right) \\ \cup \left(B_-(nx, nr, w) \setminus (\cup_{\underline{x} \in \underline{E}} B_n(\underline{x})) \right).$$

The volume of that set is bounded by

$$(4d\eta + \delta + \varepsilon/2) \mathcal{L}^d(B_-(nx, nr, w)),$$

and so we have

$$\left| \bigcup_{C \in \mathcal{C} \cup \mathcal{D}} C \cap B_-(nx, nr, w) \right| \geq (1 - (\delta + 4d\eta + \varepsilon/2)) \mathcal{L}^d(B_-(nx, nr, w)).$$

Hence $\mathcal{C} \cup \mathcal{D}$ is a set which satisfies the event $\text{Sep}(n, x, r, w, \delta + \varepsilon/2 + 4d\eta)$. $\square$

10 Geometric tools

We introduce here the geometric background we need to deal with the Wulff theorem. For A and B two subsets of $\mathbb{R}^d$, the *distance* between A and B is

$$d(A, B) = \inf\{|x - y| : x \in A, y \in B\}.$$

For E a subset of $\mathbb{R}^d$, we define its diameter as

$$\text{diam } E = \sup\{|x - y|_2 : x, y \in E\},$$

where $|\cdot|_2$ is the usual Euclidian norm. We shall use also the ∞ -diameter defined by

$$\text{diam}_\infty E = \sup\{|x - y|_\infty : x, y \in E\},$$

where $|\cdot|_\infty$ is the usual supremum norm. Let $r > 0$. The ∞ -neighbourhood is defined by

$$\mathcal{V}_\infty(E, r) = \{x \in \mathbb{R}^d : \inf\{|x - y|_\infty : y \in E\} \leq r\}.$$

Let k be an integer. We denote by α_k the volume of the unit ball of $\mathbb{R}^k$. For every $A \subset \mathbb{R}^d$, the k -dimensional Hausdorff measure $\mathcal{H}^k(A)$ of A is defined by [13]

$$\mathcal{H}^k(A) = \sup_{\delta > 0} \inf \left\{ \frac{\alpha_k}{2^k} \sum_{i \in I} (\text{diam } E_i)^k : A \subset \bigcup_{i \in I} E_i, \sup_{i \in I} \text{diam } E_i \leq \delta \right\}.$$

We would like to work with a subset of Borel subsets of $\mathbb{R}^d$ that has good compactness properties. As quoted in [4], it is natural to work with Caccioppoli sets which we introduce now. See for example [12, 24]. For O an open subset of $\mathbb{R}^d$, let $C_c^\infty(O, B(0, 1))$ be the set of C^∞ vector functions from O to $B(0, 1)$ having a compact support included in O . We let div be the usual divergence operator, defined for a C^1 vector function f with scalar components $(f_1, \dots, f_d)$ as

$$\text{div } f = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_d}{\partial x_d}.$$

DEFINITION 10.1. *The perimeter of a Borel set E of $\mathbb{R}^d$ in an open set O is defined as*

$$\mathcal{P}(E, O) = \sup \left\{ \int_E \text{div } f(x) d\mathcal{L}^d(x) : f \in C_c^\infty(O, B(0, 1)) \right\}.$$

The set E is a Caccioppoli set if $\mathcal{P}(E, O)$ is finite for every bounded open set O of $\mathbb{R}^d$.

Let E be a Caccioppoli set, χ_E be its characteristic function, and $\nabla \chi_E$ be the distributional derivative of χ_E . The *reduced boundary* $\partial^* E$ consists of the points x such that

- $\|\nabla \chi_E\|(B(x, r)) > 0$ for every $r > 0$
- if $\nu_r(x) = -\nabla \chi_E(B(x, r)) / \|\nabla \chi_E\|(B(x, r))$ then, as r goes to 0, $\nu_r(x)$ converges toward a limit $\nu_E(x)$ such that $|\nu_E(x)|_2 = 1$. The vector $\nu_E(x)$ is called the *exterior normal vector* of E at x .

For every Borel set A of $\mathbb{R}^d$,

$$\|\nabla \chi_E\|(A) = \mathcal{H}^{d-1}(A \cap \partial^* E),$$

and for every open set O of $\mathbb{R}^d$,

$$\|\nabla \chi_E\|(O) = \mathcal{P}(E, O).$$

DEFINITION 10.2. We denote by $\mathcal{B}(\mathbb{R}^d)$ the set of Borel subsets of $\mathbb{R}^d$, and we denote by $\triangle$ the symmetric difference: for A and B in $\mathcal{B}(\mathbb{R}^d)$,

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

We say that a sequence $(E_n)_{n \in \mathbb{N}}$ converges in L^1 towards $E \in \mathcal{B}(\mathbb{R}^d)$ if $\mathcal{L}^d(E_n \triangle E)$ converges to 0 as n goes to ∞ .

The next geometric lemma will be used to control the perimeter of a set by the surface of its projection along the last coordinate vector.

LEMMA 10.3. Let O be an open ball in $\mathbb{R}^d$, and let A be a Caccioppoli set. Consider the image O' of O by the orthogonal projection on $\mathbb{R}^{d-1} \times \{0\}$. We have

$$\mathcal{H}^{d-1}(O') \geq \left| \int_{\partial^* A \cap O} e_d \cdot \nu_A(x) d\mathcal{H}^{d-1}(x) \right|.$$

PROOF. We apply the Gauss–Green theorem to the set $A \cap O$ and we get

$$\int_{\partial^*(A \cap O)} e_d \cdot \nu_{A \cap O}(x) d\mathcal{H}^{d-1}(x) = 0.$$

The reduced boundary $\partial^*(A \cap O)$ is composed of $\partial^* A \cap O$ plus a set included in ∂O . Consider

$$\left| \int_E e_d \cdot \nu_O(x) d\mathcal{H}^{d-1}(x) \right|$$

for E a borelian subset of ∂O . This integral is maximal in absolute value when E is the lower half part of ∂O , that is to say for

$$\partial_- O = \{x \in \partial O : \nu_O(x) \cdot e_d \leq 0\}.$$

Hence we have

$$\left| \int_{\partial_- O} e_d \cdot \nu_O(x) d\mathcal{H}^{d-1}(x) \right| \geq \left| \int_{\partial^* A \cap O} e_d \cdot \nu_A(x) d\mathcal{H}^{d-1}(x) \right|.$$

Pick $r > 0$ a real number such that $O \cap \mathbb{R}^{d-1} \times -r'$ is empty for $r' \geq r$. We apply now the Gauss–Green theorem to the set of points that are between $\partial_- O$ and $O' \times r$ to obtain that

$$\left| \int_{\partial_- O} e_d \cdot \nu_O(x) d\mathcal{H}^{d-1}(x) \right| = \mathcal{H}^{d-1}(O'). \quad \square$$

11 Surface energy

We recall that τ is the surface tension and is a function from S^{d-1} to $\mathbb{R}^+$, and that $\mathcal{W}_\tau$ is the associated Wulff crystal (see definition 6.1). Now we define the surface energy of a Borel set.

DEFINITION 11.1. *The surface energy $\mathcal{I}(A, O)$ of a Borel set A of $\mathbb{R}^d$ in an open set O is defined as*

$$\mathcal{I}(A, O) = \sup \left\{ \int_A \operatorname{div} f(x) d\mathcal{L}^d(x) : f \in C_c^1(O, \mathcal{W}_\tau) \right\}.$$

For a fixed function f in $C_c^1(O, \mathcal{W}_\tau)$, the map

$$A \in \mathcal{B}(\mathbb{R}^d) \rightarrow \int_A \operatorname{div} f(x) d\mathcal{L}^d(x)$$

is continuous for the L^1 convergence of sets. Thus $\mathcal{I}(\cdot, O)$, being the supremum of all these maps, is lower semicontinuous. Furthermore, let $\tau_{\max}$ be the supremum of τ over S^{d-1} . Since $C_c^1(O, \mathcal{W}_\tau) \subset B(0, \tau_{\max})$, we have

$$\mathcal{I}(A, O) \leq \tau_{\max} \mathcal{P}(A, O).$$

The next proposition asserts that the surface energy is the integral of the surface tension over the reduced boundary.

PROPOSITION 11.2. *The surface energy $\mathcal{I}(A, O)$ of a Borel set A of $\mathbb{R}^d$ of finite perimeter in an open set O is equal to*

$$\mathcal{I}(A, O) = \int_{\partial^* A \cap O} \tau(\nu_A(x)) d\mathcal{H}^{d-1}(x).$$

This formula for the surface energy allows us to define the function $\mathcal{I}(\cdot, E)$ for E a Borel set not necessary open.

In order to deduce the upper bound from the $\mathcal{I}$ -tightness and from the local upper bound, the function $\mathcal{I}$ has to be a good rate function.

PROPOSITION 11.3. *For every open ball O of $\mathbb{R}^d$, the functional $\mathcal{I}(\cdot, O)$ is a good rate function on $\mathcal{B}(O)$ endowed with the topology of L^1 convergence, i.e., for every λ in $\mathbb{R}^+$, the level set*

$$\{E \in \mathcal{B}(O) : \mathcal{I}(E, O) \leq \lambda\}$$

is compact.

PROOF. For every bounded open O and every $\lambda > 0$, the collection of sets $\{E \in \mathcal{B}(O) : \mathcal{P}(E) \leq \lambda\}$ is compact for the topology L^1 . For a proof see for example theorem 1.19 in

[15]. So we just have to prove that there exists a constant $c'(O)$ depending on the open ball O and another constant $c > 0$, such that

$$\mathcal{I}(A, O) \geq -c'(O) + c\mathcal{P}(A, O). \quad (11.4)$$

Suppose that $\mathcal{I}(A, O)$ is finite. By proposition 6.3, we can pick $\eta > 0$ and $\alpha > 0$ such that: if w is a unit vector of S^{d-1} with $\tau(w) \leq \alpha$, then $w \cdot e_d \leq -\eta$. Define

$$\partial_\alpha^* A = \{x \in \partial^* A, \tau(\nu_A(x)) > \alpha\}.$$

Let H be the hyperplane $\{x : x \cdot e_d = 0\}$. Define O' to be the orthogonal projection on H of O . We have

$$\begin{aligned} \mathcal{H}^{d-1}(O') &\geq \left| \int_{\partial^* A \cap O} e_d \cdot \nu_A(x) d\mathcal{H}^{d-1}(x) \right| \\ &\geq \left| \int_{(\partial^* A \setminus \partial_\alpha^* A) \cap O} e_d \cdot \nu_A(x) d\mathcal{H}^{d-1}(x) \right| - \left| \int_{\partial_\alpha^* A \cap O} e_d \cdot \nu_A(x) d\mathcal{H}^{d-1}(x) \right| \\ &\geq \mathcal{H}^{d-1}((\partial^* A \setminus \partial_\alpha^* A) \cap O) \times \eta - \int_{\partial_\alpha^* A \cap O} \tau(\nu_A(x)) d\mathcal{H}^{d-1}(x) \\ &\geq \mathcal{H}^{d-1}((\partial^* A \setminus \partial_\alpha^* A) \cap O) \times \eta - \mathcal{I}(A, O). \end{aligned}$$

The first inequality holds because O is a ball and by lemma 10.3. Furthermore

$$\mathcal{H}^{d-1}(\partial_\alpha^* A \cap O) \leq \frac{1}{\alpha} \mathcal{I}(A, \partial_\alpha^* A \cap O).$$

Thus

$$\mathcal{I}(A, O) + \eta \mathcal{H}^{d-1}(\partial_\alpha^* A \cap O) \geq \mathcal{H}^{d-1}(\partial^* A \cap O) \times \eta - \mathcal{H}^{d-1}(O')$$

which implies

$$\mathcal{I}(A, O) + \frac{\eta}{\alpha} \mathcal{I}(A, \partial_\alpha^* A \cap O) \geq \eta \mathcal{H}^{d-1}(\partial_\alpha^* A \cap O) - \mathcal{H}^{d-1}(O'),$$

and we can conclude

$$\mathcal{I}(A, O) \geq -\frac{\alpha}{\eta + \alpha} \mathcal{H}^{d-1}(O') + \frac{\eta\alpha}{\eta + \alpha} \mathcal{H}^{d-1}(\partial^* A \cap O). \quad \square$$

Here is another consequence of inequality (11.4).

COROLLARY 11.5. *If a set A has a finite energy in an open ball O , then it has a finite perimeter in O . Hence the sets that have a finite energy in every open bounded subset of $\mathbb{R}^d$ are exactly the Caccioppoli sets.*

12 Approximation of sets

In order to prove the large deviation principle, we use two kinds of approximation of Caccioppoli sets. The first one is used in the proof of the local upper bound (for a proof see [4]).

LEMMA 12.1. *Let A be a Caccioppoli set and let O be an open bounded subset of $\mathbb{R}^d$. For every $\varepsilon > 0$, $\delta > 0$, and $\eta \geq 0$, there exists a finite collection of disjoint balls $B(x_i, r_i)$, $i \in I$, such that: for every i in I , x_i belongs to $\partial^* A$, r_i belongs to $]0, 1[$, $B(x_i, r_i)$ is included in O ,*

$$\mathcal{L}^d((A \cap B(x_i, r_i)) \Delta B_-(x_i, r_i, \nu_A(x_i))) \leq \delta \alpha_d r_i^d,$$

$$\left| \mathcal{I}(A, \partial_\eta^* A \cap O) - \sum_{i \in I} \alpha_{d-1} r_i^{d-1} \tau(\nu_A(x_i)) \right| \leq \varepsilon,$$

and

$$\forall i \in I \quad \alpha_{d-1} r_i^{d-1} \tau(\nu_A(x_i)) \leq \varepsilon.$$

The second result says that a Caccioppoli set can be approximated by a polyhedral set [4]. A Borel subset of $\mathbb{R}^d$ is polyhedral if its boundary is included in a finite union of hyperplanes of $\mathbb{R}^d$.

LEMMA 12.2. *Let A be a Caccioppoli set and let O be an open bounded subset of $\mathbb{R}^d$. There exists a sequence (A_n) of polyhedral sets of $\mathbb{R}^d$ converging to A for the topology L^1 over $\mathcal{B}(O)$, such that $\mathcal{I}(A_n, O)$ converges to $\mathcal{I}(A, O)$ as n goes to ∞ .*

13 Local upper bound

LEMMA 13.1. *Let $\nu \in \mathcal{M}(\mathbb{R}^d)$ be such that $\mathcal{I}(\nu) < \infty$. for every $\varepsilon > 0$, there exists a weak neighbourhood $\mathcal{U}$ of ν in $\mathcal{M}(\mathbb{R}^d)$ such that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathcal{U}) \leq -(1 - \varepsilon) \mathcal{I}(\nu).$$

PROOF. By definition of $\mathcal{I}$, since $\mathcal{I}(\nu) < \infty$, there exists a Borel subset A of $\mathbb{R}^d$ such that ν is the measure with density $\theta 1_A$ with respect to the Lebesgue measure and $\mathcal{I}(\nu) = \mathcal{I}(A)$. If $\mathcal{I}(A) = 0$ there is nothing to prove. Suppose that $\mathcal{I}(A) > 0$. For $\varepsilon > 0$, set $\varepsilon' = \varepsilon(1 + 1/\mathcal{I}(A))^{-1}$.

Now we skip out parts of $\partial^* A$ which contribute to the energy only a little. Let η be positive and let $\partial_\eta^* A$ be the set

$$\partial_\eta^* A = \{x \in \partial^* A : \tau(\nu_A(x)) > \eta\}.$$

There exists $\eta > 0$ such that

$$\sup \left\{ \int_{\partial^* A \setminus \partial_\eta^* A} f(x) \cdot \nu_A(x) d\mathcal{H}^{d-1}(x) : f \in C_c^1(\mathbb{R}^d, \mathcal{W}_\tau) \right\} < \varepsilon'/4.$$

Let c be the constant appearing in the interface lemma for the parameter η and let $\varepsilon_1 > 0$ such that $c\sqrt{\varepsilon_1} < \varepsilon'/2$. Let $\delta_0 \in]0, 1[$ be the constant given in lemma 9.1 with parameter ε_1 , and such that $c\sqrt{\delta_0 + \varepsilon_1} < \varepsilon'$.

Let O be an open bounded ball of $\mathbb{R}^d$, such that

$$\mathcal{I}(A, \partial_\eta^* A \cap O) \geq \mathcal{I}(A, \partial_\eta^* A) - \varepsilon'/4.$$

By lemma 12.1, there exists a finite collection $B(x_i, r_i)$, $i \in I$ of disjoint balls such that: for every i in I , x_i belongs to $\partial_\eta^* A$, r_i belongs to $]0, 1[$,

$$\mathcal{L}^d((A \cap B(x_i, r_i)) \Delta B_-(x_i, r_i, \nu_A(x_i))) \leq \delta_0/3\alpha_d r_i^d,$$

$$\left| \mathcal{I}(A, \partial_\eta^* A \cap O) - \sum_{i \in I} \alpha_{d-1} r_i^{d-1} \tau(\nu_A(x_i)) \right| \leq \varepsilon'/4,$$

and

$$\forall i \in I \quad \alpha_{d-1} r_i^{d-1} \tau(\nu_A(x_i)) \leq \varepsilon'/4.$$

Let $\mathcal{U}$ be the weak neighbourhood of ν in $\mathcal{M}(\mathbb{R}^d)$ defined by

$$\mathcal{U} = \left\{ \rho \in \mathcal{M}(\mathbb{R}^d) : \forall i \in I \quad \rho(\overset{\circ}{B}_-(x_i, r_i, \nu_A(x_i))) \geq (\theta - \delta_0)\alpha_d r_i^d/2, \right. \\ \left. \rho(\overline{B}_+(x_i, r_i, \nu_A(x_i))) \leq \delta_0\alpha_d r_i^d/2 \right\},$$

where as usual $\overset{\circ}{B}_-$ and $\overline{B}_+$ denote the interior and the closure of the half balls. Suppose that $\mathcal{C}_n \in \mathcal{U}$. Define

$$I_0 = \{i \in I : 0 \notin B(nx_i, nr_i)\}.$$

The set $I \setminus I_0$ is either $\emptyset$ or a singleton. For $i \in I_0$, the intersection of $C(0)$ with the ball $B(nx_i, nr_i)$ splits into a collection $\mathcal{C}(i)$ of $B(nx_i, nr_i)$ -clusters which all come from the boundary $\partial^{in} B(nx_i, nr_i)$. We conclude that

$$P(\mathcal{C}_n \in \mathcal{U}) \leq P\left(\bigcap_{i \in I_0} \text{Sep}_\theta(n, x_i, r_i, \nu_A(x_i), \delta_0)\right).$$

The events on the right-hand side are independent since the balls are compact and disjoint. We apply the interface lemma 9.1

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathcal{U}) &\leq - \sum_{i \in I_0} \alpha_{d-1} r_i^{d-1} \tau(\nu_A(x_i)) (1 - c\sqrt{\delta_0 + \varepsilon_1}) \\
&\leq -\mathcal{I}(A)(1 - \varepsilon') + \varepsilon'/4 + \varepsilon'/4 + \varepsilon'/4 + \varepsilon'/4 \\
&= \mathcal{I}(\nu)(1 - \varepsilon),
\end{aligned}$$

and we are done. $\square$

14 Coarse grained image

In order to prove the $\mathcal{I}$ -tightness of the random measure $\mathcal{C}_n$, we build an auxiliary random measure $\tilde{\mathcal{C}}_n$ which is exponentially contiguous to $\mathcal{C}_n$, and we prove the $\mathcal{I}$ -tightness for the measure $\tilde{\mathcal{C}}_n$. To this end, we first define for $n \geq 1$,

$$\forall \underline{x} \in \mathbb{Z}^d \quad B_n(\underline{x}) = \frac{1}{n} B(\underline{x}),$$

and we let

$$\underline{\mathcal{C}}_n = \{\underline{x} \in \mathbb{Z}^d : \mathcal{C}_n(B_n(\underline{x})) > 0\}.$$

We now fill the small holes of $\underline{\mathcal{C}}_n$ which do not create any surface energy. We look at the residual component of $\underline{\mathcal{C}}_n$, that is the $\mathbb{L}^{d,\infty}$ -connected component of $\mathbb{Z}^d \setminus \underline{\mathcal{C}}_n$. If $\text{diam}_\infty C(0) \leq K \ln n$ we set $\text{fill } \underline{\mathcal{C}}_n = \emptyset$; if $\text{diam}_\infty C(0) > K \ln n$, we define

$$\text{fill } \underline{\mathcal{C}}_n = \underline{\mathcal{C}}_n \cup \{\underline{R} : \underline{R} \text{ is a finite residual component of } \underline{\mathcal{C}}_n, \text{diam}_\infty \underline{R} < \ln n\}.$$

By construction, we have $\partial^{in} \text{fill } \underline{\mathcal{C}}_n \subseteq \partial^{in} \underline{\mathcal{C}}_n$. If $K \ln n < \text{diam}_\infty C(0) < \infty$, then each $\mathbb{L}^{d,\infty}$ -connected component of $\partial^{in} \text{fill } \underline{\mathcal{C}}_n$ has cardinality strictly larger than $\ln n$.

Let

$$C_n = \bigcup_{\underline{x} \in \text{fill } \underline{\mathcal{C}}_n} B_n(\underline{x}).$$

The measure $\tilde{\mathcal{C}}_n$ is then the measure with density $\theta 1_{C_n}$ with respect to the Lebesgue measure $\mathcal{L}^d$.

15 The boundary of the block cluster

In the article of R. Cerf [4], all blocks in $\partial^{in}\underline{\mathcal{C}}_n$ were bad. In the context of oriented percolation, we provide a control on the proportion of bad blocks in $\partial^{in}\underline{\mathcal{C}}_n$.

LEMMA 15.1. *Let O be an bounded open subset of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial O) < \infty$. Let $\varepsilon > 0$ and let l be a positive integer. Consider the event $R(B(\underline{x}), l, \varepsilon)$, and call the blocks good and bad accordingly. Let N_1 be the number of good boundary blocks of $\underline{\mathcal{C}}_n$ intersecting O , and let N_2 be the number of bad boundary blocks of $\underline{\mathcal{C}}_n$ intersecting O . There exists a constant $c'(O)$ depending on O and a constant $c > 0$ depending only on l , such that*

$$N_1 \leq c'(O)n^{d-1} + cN_2.$$

PROOF. The argument to prove this lemma is the same as the one we used for the positivity of τ in (6.5). There is nevertheless some differences, because we work in a bounded domain whereas the cluster $C(0)$ is not restricted in that domain. For clarity, we redo the full proof.

For i an integer, we say that a box $B(\underline{x})$ is *at height i* , if $\underline{x} \cdot e_d = i$.

Heuristically, we consider the block cluster as a process on $\mathbb{Z}^{d-1}$ indexed by the height. If a block in the boundary of this process is good, then at time l the block gives birth to blocks around itself, and the process “increases”. The process “decreases” when the process goes outside O , or when a block is bad in such a way that the block disappears in time 1. Such a block lies in the boundary of the block cluster.

We introduce notations in order to count the good and bad blocks of the boundary:

$$\begin{aligned} a_i &= \text{number of blocks in } \underline{\mathcal{C}}_n \text{ at height } i \text{ intersecting } O, \\ b_i &= \text{number of good blocks in } \partial^{in}\underline{\mathcal{C}}_n \text{ at height } i \text{ intersecting } O, \\ b'_i &= \text{number of good blocks in } \partial^{in}\underline{\mathcal{C}}_n \text{ at height } i \text{ intersecting } O, \\ &\quad \text{and that have a neighbour at height } i \text{ that is not in } \underline{\mathcal{C}}_n, \\ c_i &= \text{number of bad blocks in } \partial^{in}\underline{\mathcal{C}}_n \text{ at height } i \text{ intersecting } O. \end{aligned}$$

Because of the definition of the event $R(B(\underline{x}), l, \varepsilon)$, we have

$$\begin{aligned} a_{i+l} &\geq a_i + \frac{b'_i}{2(d-1)} - c_i - c_{i+1} - \dots - c_{i+l-1} \\ &\quad - 2(n/K)^d \mathcal{H}^d(\mathcal{V}(\partial O, 3K/n) \cap (\mathbb{R}^{d-1} \times K[i, i+l])). \end{aligned} \tag{15.2}$$

Here we have bounded the number of boxes that “disappear” outside O by two times $(n/K)^d$ times the volume of $\mathcal{V}(\partial O, 3K/n)$. Since we have supposed $\mathcal{H}^{d-1}(\partial O) < \infty$, there exists $c'(O) < \infty$ such that

$$2(n/K)^d \mathcal{H}^d(\mathcal{V}(\partial O, 3K/n)) < c'(O)(n/K)^{d-1}.$$

We let $B'_k = \sum_{i=-\infty}^{\infty} b'_{k+il}$, and $B' = \sum_{i=-\infty}^{\infty} b'_i$. We have $N_2 = \sum_{i=-\infty}^{\infty} c_i$. Summing inequality (15.2) over i with step l , we obtain

$$c'(O)(n/K)^{d-1} + 2N_2 \geq \frac{1}{2(d-1)} B'_k,$$

for all k . But there exist $k \in \{0, \dots, l-1\}$, such that $B'_k \geq \frac{1}{l} B'$. Hence

$$c'(O)(n/K)^{d-1} + 2N_2 \geq \frac{1}{2(d-1)l} B'.$$

Now let $b''_i = b_i - b'_i$. For each box counted in b''_{i+1} and not included in $\mathcal{V}(\partial O, 3K/n)$, there is a box counted in b'_i . We recall that a box counted in b'_i can give no more than $2(d-1)$ boxes counted in b''_{i+1} , and thus

$$b''_{i+1} \leq 2(d-1)b'_i + 2(n/K)^d \mathcal{H}^d(\mathcal{V}(\partial O, 3K/n)). \quad (15.3)$$

Remark that

$$N_1 = \sum_{i=-\infty}^{\infty} b_i.$$

From (15.3), it follows that

$$N_1 \leq 4(d-1)B' + c'(O)(n/K)^{d-1},$$

and we get that

$$2c'(O)(n/K)^{d-1} + 2N_2 \geq \frac{1}{8(d-1)^2 l} N_1. \quad \square$$

Remark: In lemma 15.1, we could replace $\mathcal{C}_n$ by $\text{fill } \mathcal{C}_n$.

We can now control the perimeter of C_n :

LEMMA 15.4. *Let O be an open bounded subset of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial O) < \infty$. There exists $c > 0$ such that for each function $f(n)$ from n to $\mathbb{R}_+$ tending to ∞ as n goes to ∞ , for n large enough*

$$P(\mathcal{P}(C_n, O) > f(n)) \leq \exp -cf(n)n^{d-1}.$$

PROOF. Let $X(\underline{x})$ be the indicator function of the event $R(B(\underline{x}), l, \varepsilon)$. Let N be the number of boundary boxes of C_n in O , and let N_2 be the number of those boundary boxes that are bad, i.e. $X(\underline{x}) = 0$. Pick $\delta \in]0, 1[$, and let K_0 be an integer such that

$X \succeq Z^\delta$. Denote by N the number of boundary blocks in $\text{fill } \underline{\mathcal{C}}_n$ intersecting O . The event $\mathcal{P}(C_n, O) > \varepsilon n$ implies that

$$N \geq f(n)(n/K)^{d-1}.$$

But for a certain constant $c > 0$,

$$N_2(1+c) \geq N - c'(O)(n/K)^{d-1},$$

so

$$\frac{N_2}{N} \geq \frac{1}{1+c} \left(1 - \frac{c'(O)}{f(n)}\right).$$

Thus for n large enough,

$$\frac{N_2}{N} \geq \frac{1}{2(1+c)}.$$

Let b be the constant appearing in lemma 4.1. We take δ small enough so that

$$\ln b + \Lambda^*(1/(2(1+c)), \delta)$$

is negative. We take K large enough such that $X \succeq Z^{1-\delta}$, and we apply lemma 4.1 with $s = c\varepsilon n^d/K^d$. $\square$

We now give a version of lemma 15.1 for the event V , in which the constant c will not depend on l .

LEMMA 15.5. *Let O be a bounded open subset of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial O) < \infty$. Let $\varepsilon > 0$, $\alpha > 0$, and let l, r be positive integers. Consider the event $V(B(\underline{x}), l, \varepsilon, \alpha, r)$, and call the blocks good and bad accordingly. Let N_1 be the number of good boundary blocks of $\underline{\mathcal{C}}_n$ intersecting O , and let N_2 be the number of bad boundary blocks of $\underline{\mathcal{C}}_n$ intersecting O . There exists a constant $c'(O)$ depending on O and a constant $c > 0$ independent of n, l , and r , such that*

$$N_1 \leq c'(O)n^{d-1} + cN_2.$$

PROOF. Let $\widehat{l} > 0$ be the smallest integer such that

$$\forall j, 1 \leq j \leq d-1, \quad B(\underline{x} + \widehat{l} \pm e_j) \subset \underline{F}(\underline{x}, l, \alpha, r),$$

where $\underline{F}(\underline{x}, l, \alpha, r)$ is the region defined before proposition 3.12. The integer $\widehat{l} > 0$ depends only on α . When we consider the event V instead of R , we replace the first inequality in the proof of lemma 15.1 by

$$a_{i+\widehat{l}} \geq a_i + \frac{1}{2(d-1)}(b'_i - c_i) - c_i - c_{i+1} - \dots - c_{i+l-1} - (n/K)^d \mathcal{H}^d(\mathcal{V}(\partial O, 2K/n) \cap (\mathbb{R}^{d-1} \times [i, i+l])). \quad \square$$

Remark: As before, we can replace $\underline{\mathcal{C}}_n$ by $\text{fill } \underline{\mathcal{C}}_n$ in the statement of lemma 15.5.

16 Exponential contiguity

Let us fix $f \in C_c(\mathbb{R}^d, \mathbb{R})$. We shall estimate $|\mathcal{C}_n(f) - \tilde{\mathcal{C}}_n(f)|$, using for the blocks the scale $L = K \ln n$. So we work with the lattice rescaled by a factor L . Let l be the constant given in proposition 3.1 for the event R , and let $\varepsilon > 0$. For $\underline{y} \in \mathbb{Z}^d$, the block variable $Y(\underline{y})$ is the indicator function of the event $R(B(\underline{y}), l, \varepsilon)$. We write $\text{supp}(f)$ for the support of the function f . Since f is continuous and has a compact support, it is uniformly continuous. We suppose that lL/n is less than 1 and small enough so that

$$\forall x, y \in \mathbb{R}^d \quad |x - y| \leq L/n \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

Let O be an open bounded subset of $\mathbb{R}^d$ containing $\mathcal{V}(\text{supp}(f), 2d)$, and let

$$\underline{A} = \{\underline{y} \in \mathbb{Z}^d : B_n(\underline{y}) \cap \text{supp}(f) \neq \emptyset\}.$$

Since $L/n \leq 1$, for each $\underline{y} \in \underline{A}$, we have $B_n(\underline{y}) \subset O$, thus $|\underline{A}|K^d \leq n^d \mathcal{L}^d(O)$. As in [4], we have

$$|\mathcal{C}_n(f) - \tilde{\mathcal{C}}_n(f)| \leq 2\varepsilon \mathcal{L}^d(O) + \|f\|_\infty \sum_{\underline{y} \in \underline{A}} |\mathcal{C}_n(B_n(\underline{y})) - \tilde{\mathcal{C}}_n(B_n(\underline{y}))|. \quad (16.1)$$

We study the last term in the above quantity. If the diameter of $C(0)$ is less than $K \ln n$, then the number of blocks contributing to the sum is less than $(\ln n + 1)^d$ and the sum is bounded by $(\ln n + 1)^d (K/n)^d$. From now on, we suppose that the diameter of $C(0)$ is strictly larger than $K \ln n$. If $\underline{y} \in \underline{A}$ is such that $B_n(\underline{y})$ does not intersect C_n , then $\mathcal{C}_n(B_n(\underline{y})) = \tilde{\mathcal{C}}_n(B_n(\underline{y})) = 0$ and the corresponding term in the sum vanishes. So we need only to consider the blocks $B_n(\underline{y})$ intersecting C_n . Let $\underline{y} \in \underline{A}$ such that $B_n(\underline{y}) \cap C_n \neq \emptyset$. We distinguish several cases. If $Y(\underline{y} - le_d) = 0$, then

$$|\mathcal{C}_n(B_n(\underline{y})) - \tilde{\mathcal{C}}_n(B_n(\underline{y}))| \leq \frac{1}{n^d} |B_n(\underline{y})| 1_{Y(\underline{y} - le_d) = 0}.$$

Suppose next that $Y(\underline{y}) = 1$. Several subcases arise:

- $B_n(\underline{y}) \not\subset C_n$. Then we bound

$$|\mathcal{C}_n(B_n(\underline{y})) - \tilde{\mathcal{C}}_n(B_n(\underline{y}))| \leq \frac{1}{n^d} |B_n(\underline{y})|.$$

By [4], the total volume of such $B_n(\underline{y})$ is bounded by the quantity

$$5^{d+1} \frac{L^{d-1}}{n} \mathcal{P}(C_n, O). \quad (16.2)$$

- $B_n(\underline{y}) \subset C_n$ and $\mathcal{C}_n(B_n(\underline{y})) = 0$. These conditions implies that $B_n(\underline{y})$ is included in one of the small holes of $\underline{C}_n$. Since the diameter of $B_n(\underline{y})$ is strictly larger than the diameters of these small holes, this case can not occur.
- $\mathcal{C}_n(B_n(\underline{y})) > 0$ and $\mathcal{C}_n(B_n(\underline{y} - le_d)) = 0$. Here $B_n(\underline{y})$ is included in $\mathcal{V}(\partial C_n \cap O, l)$. The total volume of such $B_n(\underline{y})$'s is thus bounded by

$$2^d l \frac{L^{d-1}}{n} \mathcal{P}(C_n, O). \quad (16.3)$$

- $(\underline{y} - le_d) \cdot e_d \leq 1$. Only $B(\underline{0})$ is in this case.
- $\mathcal{C}_n(B_n(\underline{y})) > 0$, $\mathcal{C}_n(B_n(\underline{y} - le_d)) > 0$, and $(\underline{y} - le_d) \cdot e_d \geq 1$. The definition of the block event associated to the variable Y implies that

$$|\mathcal{C}_n(B_n(\underline{y})) - \tilde{\mathcal{C}}_n(B_n(\underline{y}))| = |\mathcal{C}_n(B_n(\underline{y})) - \frac{\theta}{n^d} |B_n(\underline{y})|| \leq \frac{\varepsilon}{n^d} |B_n(\underline{y})|. \quad (16.4)$$

Summing the previous inequalities (16.2), (16.3), and (16.4) over $\underline{y} \in \underline{A}$ in (16.1), we get

$$|\mathcal{C}_n(f) - \tilde{\mathcal{C}}_n(f)| \leq \varepsilon \mathcal{L}^d(O) \left(2 + \|f\|_\infty \left(1 + \frac{1}{|\underline{A}|} \sum_{\underline{y} \in \underline{A}} 1_{Y(\underline{y})=0} \right) \right) + \|f\|_\infty 7^{d+1} \frac{L^{d-1}}{n} l \mathcal{P}(C_n, O).$$

The sum in the above quantity is controlled *via* the Cramer's theorem of large deviations [5]. The probability that the perimeter $\mathcal{P}(C_n, O)$ is larger than $\varepsilon' n / L^{d-1}$ for $\varepsilon' > 0$ is bounded with the help of lemma 15.4. Hence we obtain the following result:

LEMMA 16.5. *Let K be large enough. For every continuous function f having a compact support, there exists a positive constant $c(f)$ and an integer $n(f)$ such that,*

$$\forall n \geq n(f) \quad \forall \varepsilon > 0 \quad P(|\mathcal{C}_n(f) - \tilde{\mathcal{C}}_n(f)| > \varepsilon) \leq c(f) \exp \left(- c(f) \varepsilon \frac{n^d}{(K \ln n)^d} \right).$$

This lemma implies the exponential contiguity between the measures $\mathcal{C}_n$ and $\tilde{\mathcal{C}}_n$.

17 The $\mathcal{I}$ -tightness

We show that the sequence of random measures $\tilde{\mathcal{C}}_n$ is $\mathcal{I}$ -tight, that is there exist two constants $c > 0$ and $\lambda_0 \geq 0$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\forall \nu \in \mathcal{I}^{-1}([0, \lambda]) \quad |\tilde{\mathcal{C}}_n(f) - \nu(f)| > \eta) \leq -c\lambda, \quad (17.1)$$

for every $\lambda \geq \lambda_0$, every $\eta > 0$ and each $f \in C_c(\mathbb{R}^d, \mathbb{R})$.

Let us fix $\eta > 0$ and $f \in C_c(\mathbb{R}^d, \mathbb{R})$. Let O be an open bounded subset of $\mathbb{R}^d$ containing the support of f . Near the set $C_n \cap O$ we shall build a set S with a control on the energy of S in O . Let $c'(O)$ be the constant appearing in lemma 15.1, and let $\varepsilon_1 < c'(O)^{-1}$. Because of the continuity of the surface tension, there exists $\varepsilon > 0$, such that for all x in $\partial^*(\mathbf{F}(1 - \varepsilon))$, $\tau(\nu_{\mathbf{F}(1 - \varepsilon)}(x)) < \varepsilon_1$. We choose such an ε in $]0, \frac{1}{2}[$. Let $r > 0$ be such that $[-\frac{1}{2}, \frac{1}{2}]^d$ is included in $\mathbf{F}(1 - \varepsilon) - re_d$, and take $\alpha > 0$ as in proposition 3.12. We pick an integer $l > 0$ such that

$$\mathcal{V}((l + r)(1 - \varepsilon)\mathbf{U}, 2d) \subset l(1 - \varepsilon/2)\mathbf{U},$$

where $\mathbf{U}$ is the convex subset of $\mathbb{R}^{d-1}$ introduced in proposition 2.2. We let

$$\Gamma = (\mathbf{F}(1 - \varepsilon) - rKe_d) \cap (\mathbb{R}^{d-1} \times [-K/2, lK + K/2]), \quad (17.2)$$

as represented in figure 16. Observe that the top of Γ is included in the union of the boxes in the set $\underline{D}(\underline{x}, l, \varepsilon, r)$ defined in (3.10). Let $X(\underline{x})$ be the indicator function of the event $V(B(\underline{x}), l, \varepsilon, \alpha, r)$.

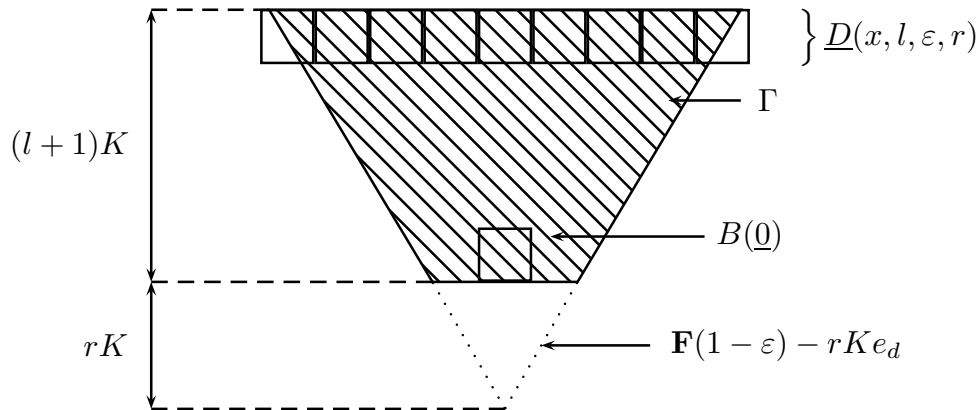


figure 16: the truncated cone Γ

We define the set S by

$$S = C_n \cup \bigcup_{\substack{\underline{x} \in \partial^{in} \text{ fill } \underline{C}_n \\ X(\underline{x})=1}} \frac{1}{n}(\Gamma + K\underline{x}).$$

We may think of S as a try to transform C_n such that C_n locally looks like the cone of percolation $\mathbf{F}$.

Let us make a comment on the sets Γ and S . The boundary of Γ is composed of three parts: the bottom, the side, and the top. The bottom of Γ has no surface energy because $\tau(-e_d) = 0$. For all unit exterior normal vector w to the side of Γ , we have $\tau(w) < \varepsilon_1$. The top of Γ is included in C_n by the definition of a good box. So the surface energy of S comes from the surface energy of the sides of Γ 's that we add, and from bad boxes that are in the boundary. The surface energy of the side of Γ is bounded by $cKl\varepsilon_1$ with a constant $c > 0$. Since we have no control on the term $l\varepsilon_1$, the bound we get on $\mathcal{I}(S)$ is of the form

$$\mathcal{I}(S) \leq l\varepsilon_1 c'(O) + cN_2.$$

This bound depends on the open set O and does not provide a sufficient control on the surface energy of S . We have taken into account the surface energy of the whole sides of all the Γ 's. To obtain a more accurate bound, we divide the set S into slabs of thickness K , and we study the boundaries of these slabs.

We let N_1 be the number of good boxes in $\partial^{in} \text{ fill } \underline{C}_n$, N_2 the number of bad boxes in $\partial^{in} \text{ fill } \underline{C}_n$, and $N = N_1 + N_2$. We consider the set S floor by floor. For $h \in \mathbb{N}$, we define

$$H_{h,n} = \{B_n(\underline{x}) : \underline{x} \cdot e_d = h\}.$$

Let $\mathcal{S}^h$ be the set

$$\mathcal{S}^h = S \cap (\mathbb{R}^{d-1} \times \{Kh/2\}),$$

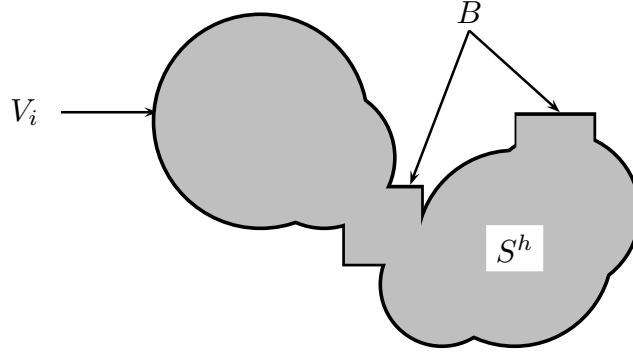
define C_n^h by

$$C_n^h = C_n \cap (\mathbb{R}^{d-1} \times \{Kh/2\}),$$

and let $O^h = O \cap (\mathbb{R}^{d-1} \times \{Kh/2\})$. We let N_2^h be the number of bad blocks in $\partial^{in} \text{ fill } \underline{C}_n \cap C_n^h$, and we let N^h be the number of blocks in $\partial^{in} \text{ fill } \underline{C}_n \cap C_n^h$. We have for a certain constant $c > 0$,

$$\mathcal{I}(S, \overset{\circ}{H}_{h,n} \cap O) \leq cK\varepsilon_1 \mathcal{P}(\mathcal{S}^h, O^h) + cN_2^h/n^{d-1}.$$

We shall control $\mathcal{P}(\mathcal{S}^h, O^h)$ by N^h . Observe that $\mathcal{S}^h$ is composed of a finite union $\cup_{i \in I} U_i$ of dilations of $\mathbf{U}$ together with hypersquares coming from bad boxes. Denote by V_i the set

figure 17: the set S^h

$\partial S^h \cap U_i$, and let $J \subset I$ be the set of indices i such that $V_i \neq \emptyset$. Let B be the part of ∂S^h coming from bad boxes. The boundary ∂S^h is decomposed as

$$\partial S^h = \left(\bigcup_{i \in J} V_i \right) \cup B,$$

see figure 17.

There exists $c > 0$ such that $\mathcal{H}^{d-2}(B) \leq cN_2^h/n^{d-1}$. Therefore

$$\mathcal{P}(S^h, O^h) \leq \sum_{i \in J} \mathcal{H}^{d-2}(V_i) + cN_2^h/n^{d-1}. \quad (17.3)$$

We suppose that for $i \neq j$ in J , we have $\mathcal{H}^{d-2}(V_i \cap V_j) = 0$. This is the case if $\mathbf{U}$ is strictly convex. If it is not, we order the set J and for every $i \in J$ we replace V_i by $V_i \setminus (\cup_{j \leq i} V_j)$. Let x_i be the center of U_i . We define

$$W_i = [x_i, V_i] := \{x_i + ty : t \in [0, 1], y \in V_i\}.$$

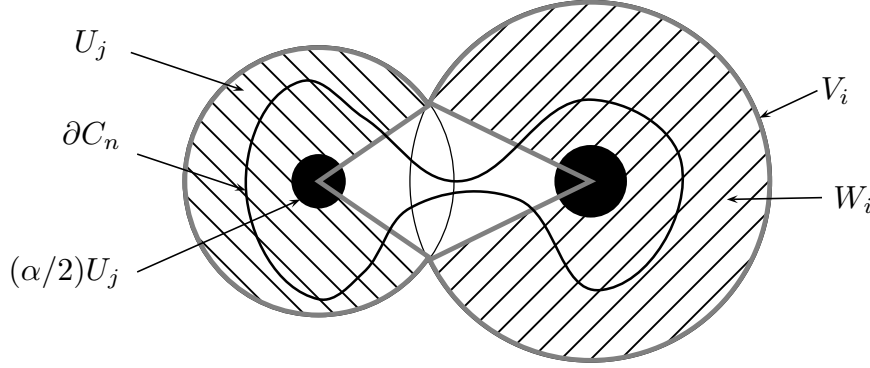
We consider the set S^h as embedded in $\mathbb{R}^{d-1}$. For the topology of $\mathbb{R}^{d-1}$, the set $\mathbf{U}$ is a symmetric convex set with non-empty interior. So, for all $i \neq j$ in J , we have $\mathcal{H}^{d-2}(W_i \cap W_j) = 0$.

Let α be the constant independent of l given in proposition 3.12. By definition of a good block,

$$C_n^h \supset \bigcup_{i \in J} (\alpha/2)U_i.$$

For $i \in J$, consider the set $Z_i = (\partial C_n^h) \cap W_i$. Since V_i is a part of the boundary of S^h , the set Z_i separates topologically in W_i the sets $(\alpha/2)U_i \cap W_i$ and V_i . By the Gauss–Green theorem, there exists a constant $c(\alpha)$ depending only on α such that

$$\mathcal{H}^{d-2}(Z_i) \geq c(\alpha)\mathcal{H}^{d-2}(V_i), \quad (17.4)$$


 figure 18: The boundaries of S and of C_n

see figure 18.

Since the Z_i 's are included in the W_i 's, we have for all $i \neq j$ in J ,

$$\mathcal{H}^{d-2}(Z_i \cap Z_j) = 0.$$

Recalling that the Z_i ' are parts of the boundary of C_n^h , there exists therefore a constant $c > 0$ such that

$$(c/n^{d-2})N^h \geq \sum_{i \in J} \mathcal{H}^{d-2}(Z_i).$$

Hence by (17.3) and (17.4) we have proved that

$$\mathcal{I}(S, \overset{\circ}{H}_{h,n} \cap O) \leq cK/n^{d-1}(\varepsilon_1 N_1^h + N_2^h), \quad (17.5)$$

with c independent of n and l . Summing (17.5) over h in $\mathbb{N}$, this implies that there exists $c_1 > 0$ independent of n and l such that

$$\mathcal{I}(S, O) \leq (\varepsilon_1 c_1/n^{d-1})N_1 + (c_1/n^{d-1})N_2. \quad (17.6)$$

Furthermore, by lemma 15.5, there exists a constant c_2 independent of l and n such that

$$N_1 \leq c'(O)n^{d-1} + c_2 N_2. \quad (17.7)$$

Since we have taken ε_1 such that $\varepsilon_1 c'(O) < 1$, inequalities (17.6) and (17.7) imply

$$\mathcal{I}(S, O) \leq c_1 + (\varepsilon_1 c_2 c_1 + c_1)N_2/n^{d-1}.$$

We conclude that there exists $c_3 > 0$ independent of O such that: for all $u \geq 1$, if $N_2 \leq un^{d-1}$, then

$$\mathcal{I}(S, O) \leq c_3 u.$$

Consider now the symmetric difference between $S \cap O$ and $C_n \cap O$. We add the set Γ only for good boundary boxes, so there exists a constant $c(l)$ depending on l such that

$$\mathcal{L}^d(S \Delta C_n) \leq c(l)N_1/n^d.$$

By lemma 15.1, if we have $N_2 \leq un^{d-1}$ for a certain $u > 0$, then the above quantity tends to 0 as n goes to infinity, and so

$$|\mathcal{C}_n(f) - \theta 1_S(f)| \rightarrow 0 \quad \text{as } n \text{ goes to } \infty.$$

The conclusion is that for all $u \geq 1$, for all $\eta > 0$, for all $f \in C_c(\mathbb{R}^d, \mathbb{R})$, if we have

$$\forall \nu \in \mathcal{I}^{-1}([0, c_3 u]) \quad |\mathcal{C}_n(f) - \nu(f)| > \eta,$$

then for n large enough there is at least un^{d-1} bad boundary boxes in $C_n \cap O$. Hence the proportion of bad boxes in ∂^{in} fill $\underline{C}_n \cap O$ is larger than $u/((c'(O) + c_2 u) \geq 1/(c'(O) + c_2)$.

Let b be the constant appearing in lemma 4.1. We pick $\varepsilon_2 > 0$, such that

$$\ln \mathcal{L}^d(\mathcal{V}(O, d)) + \ln b + \Lambda^*(1/(c'(O) + c_2), \varepsilon_2) < 0. \quad (17.8)$$

By proposition 3.12, we can take K large enough such that the block process X dominates stochastically the Bernoulli-site process $Z^{1-\varepsilon_2}$. Hence, for K large enough, we obtain the $\mathcal{I}$ -tightness property (17.1) with the help of lemma 4.1 and by the choice of ε_2 in (17.8).

18 Lower bound

LEMMA 18.1. *Let $\nu \in \mathcal{M}(\mathbb{R}_+^d)$. For every weak neighbourhood $\mathcal{U}$ of ν in $\mathcal{M}(\mathbb{R}_+^d)$, we have*

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{C}_n \in \mathcal{U}) \geq -\mathcal{I}(\nu).$$

PROOF. Heuristically, we want to show that the cluster of the origin fills a given shape [figure 19] with a certain probability. The cluster of 0 will be restricted into that shape by putting separating surfaces on the boundary as in [4]. Actually, the core of the proof is to make sure that $C(0)$ fills this given shape. The solution is to put a collection of seeds at the bottom of the shape. We denote by S the collection of the seeds and we put a truncated cone starting at each s in S . Furthermore, we partition the shape with boxes

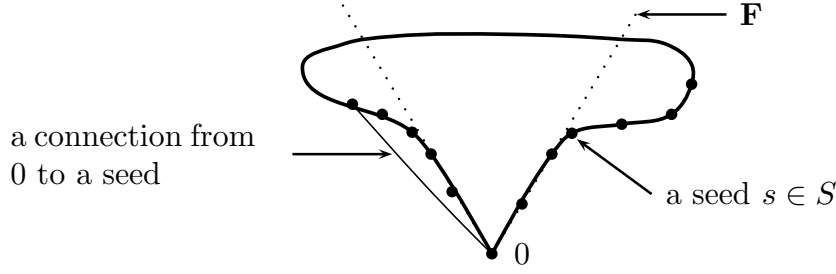


figure 19: the shape we want to obtain

of a linear size, and we take block events such that clusters spread vertically. The cluster $C(0)$ spreads as follows: first the origin is connected to a seed s , then the cluster spreads in the corresponding truncated cone, and then the cluster spreads vertically with the help of good blocks. Now we turn to the detailed proof.

If $\mathcal{I}(\nu) = +\infty$, there is nothing to prove. Let $\nu \in \mathcal{M}(\mathbb{R}^d)$ be such that $\mathcal{I}(\nu) < \infty$. By definition of $\mathcal{I}$, there exists a Borel set A of $\mathbb{R}^d$ such that ν is the measure with density $\theta 1_A$ with respect to the Lebesgue measure and $\mathcal{I}(\nu) = \mathcal{I}(A)$. Let $\mathcal{U}$ be a weak neighbourhood of ν and let $\varepsilon > 0$.

Let $f \in C_c(\mathbb{R}^d, \mathbb{R})$. Let h be an integer such that the supports of f and $\mathcal{U}$ are contained in $\mathbb{R}^{d-1} \times [-h, h]$. Let O be an open bounded subset of $\mathbb{R}^d$ containing

$$\{(\mathbf{x}, t) : 0 \leq t, |\mathbf{x}| \leq t\} \cap (\mathbb{R}^{d-1} \times [-h, h]).$$

By lemma 12.2, there exists a polyhedral set D in $\mathbb{R}_+^d$ such that the measure ψ with density $\theta 1_D$ with respect to the Lebesgue measure belongs to $\mathcal{U}$ and moreover $\mathcal{I}(D, O) \leq \mathcal{I}(A, O) + \varepsilon$.

We are going to estimate the probability that $|\mathcal{C}_n(f) - \psi(f)|$ is small. Let $\varepsilon > 0$. Since f is continuous and has a compact support, it is uniformly continuous.

Let $\widehat{\delta}$ be as in proposition 2.4. For a point s in $\mathbb{R}^d$ and $\varepsilon_1 > 0$, we let $\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s)$ be the set

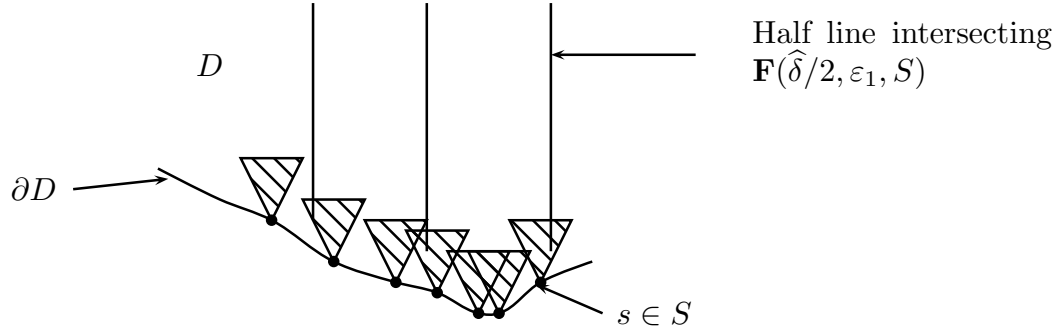
$$\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s) = s + \{t\widehat{\delta}/2\mathbf{U} + te_d, 0 \leq t \leq \varepsilon_1\}.$$

Finally, for a set S of points in $\mathbb{R}^d$, we define

$$\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, S) = \bigcup_{s \in S} \mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s).$$

We call the *downward boundary* of D the set

$$\partial^- D = \{x \in \partial^* D, \nu_D(x) \cdot e_d < 0\}.$$

figure 20: a representation of S

We can take a set S included in $\mathcal{V}(\partial^- D, 2d/n) \cap (\mathbb{Z}^d/n)$ such that for each x in $D \setminus \mathcal{V}(\mathbb{R}^d \setminus D, 2\varepsilon_1)$, the half line $\{x - te_d : t \geq 0\}$ intersects $\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, S)$ before leaving D , see figure 20. Furthermore $|S| \leq c$ where c is a constant independent of n .

We let $\alpha \in]0, 1[$ be small enough so that

$$\mathcal{L}^d(\mathcal{V}(\partial D, 4d\alpha)) \leq \varepsilon,$$

$$\forall x, y \in \mathbb{R}^d \quad |x - y| \leq \alpha \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

We work with the lattice rescaled by a factor $\lfloor \alpha n \rfloor$. For α small, ε_1 small and n large enough, we can pick a set $\underline{E}_1$ such that

$$d\left(\bigcup_{\underline{x} \in \underline{E}_1} B_n(\underline{x}), \mathbb{R}^d \setminus D\right) \geq 4\varepsilon_1, \quad \mathcal{L}^d\left(D \setminus \bigcup_{\underline{x} \in \underline{E}_1} B_n(\underline{x})\right) \leq \varepsilon,$$

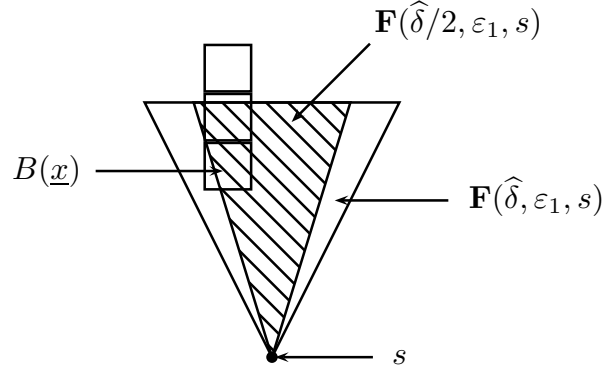
and moreover $|\underline{E}_1| \leq c$ where c is a constant independent of n . Let $\underline{x}$ in $\mathbb{Z}^d$, and let s in S . We suppose that α is small enough such that, if $B_n(\underline{x} + ed) \cap \mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s) \neq \emptyset$, and if $B_n(\underline{x} + 2ed) \cap \mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s) = \emptyset$, then $B_n(\underline{x}) \subset \mathbf{F}(\widehat{\delta}, \varepsilon_1, s)$ (see figure 21).

We build a set $\underline{E}_2$ as follows. First let $\underline{E}_2 = \emptyset$. Then for each $\underline{x}$ in $\underline{E}_1$, we go downward along the last coordinate axis until we get a box $B_n(\underline{y})$ which intersects $\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, S)$. We add to $\underline{E}_2$ all the vertices between $\underline{x}$ and $\underline{y} - e_d$ which are not in $\underline{E}_1$. Note that for all $\underline{z} \in \underline{E}_2$, we have $B_n(\underline{z}) \subset D$.

Let $s \in S$. We define the downward half line of s as

$$N(s) = \{s - te_d : t \geq 0\}.$$

Let A' be a closed and bounded subset of $\mathbb{R}^d$, and let t^* be the larger $t \geq 0$ such that $s - te_d$ is in $A' \cap \mathbb{Z}^d/n$. We call the last point of $N(s)$ in A' the vertex $s - t^*e_d$.


 figure 21: a box included in $\mathbf{F}(\widehat{\delta}, \varepsilon_1, s)$

We define the sets $\underline{E}_3$ and Γ as follows. For each s in S , we go downward along the last coordinate axis. There is three cases

- We intersect a box $B_n(\underline{x})$ with $\underline{x} \in \underline{E}_1$. In this case we go upward and we add to $\underline{E}_3$ all the $\underline{y}$'s until the box $B_n(\underline{y})$ is included in $\mathbf{F}(\widehat{\delta}, \varepsilon_1, s)$. Let $\widehat{s}$ be the last point of $N(s)$ in $B_n(\underline{x} + e_d)$. We take for γ_s the segment $[s, \widehat{s}]$.
- We intersect the set $\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s')$ for $s' \in S$. We let $\widehat{s}$ be the last point of $N(s)$ in $\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, s')$. We define $\gamma(s) = [s, \widehat{s}]$. We add to $\underline{E}_3$ all the boxes intersecting γ_s . We represent this case on figure 22.
- In the case where we do not intersect the boxes of $\underline{E}_1$ nor the set $\mathbf{F}(\widehat{\delta}/2, \varepsilon_1, S)$, we take x the intersection of $N(s)$ with the set

$$\{y = (\mathbf{y}, t) \in \mathbb{R}^d, |\mathbf{y}| = t\}.$$

Note that x is in $\mathbb{Z}^d/n$. We take for γ_s one of the path from 0 to x , union the segment $[x, s]$.

The set S' is the subset of S for which the third case occurs. We let Γ be the following set of edges:

$$\Gamma = \left(\cup_{s \in S'} \gamma_s \right).$$

We define D' as

$$D' = D \cup \bigcup_{s \in S} \mathcal{V}_\infty(\gamma_s, 4\varepsilon_1).$$

For every x in $\underline{E}_1 \cup \underline{E}_2 \cup \underline{E}_3$, the box $B_n(x)$ is included for n large enough in $D \setminus \mathcal{V}(\mathbb{R}^d \setminus D, 3\varepsilon_1)$. The set Γ is also included in that set. Observe that the set D' is polyhedral. By definition of a polyhedral element, $\partial D'$ is the union of a finite number of $d - 1$ dimensional sets $F_1, \dots, F_r$. For $1 \leq j \leq r$, we denote by $\text{nor}(F_j, D')$ the exterior normal vector to D' at F_j . Since the cardinal of S is bounded by a constant independent of n , the set we add

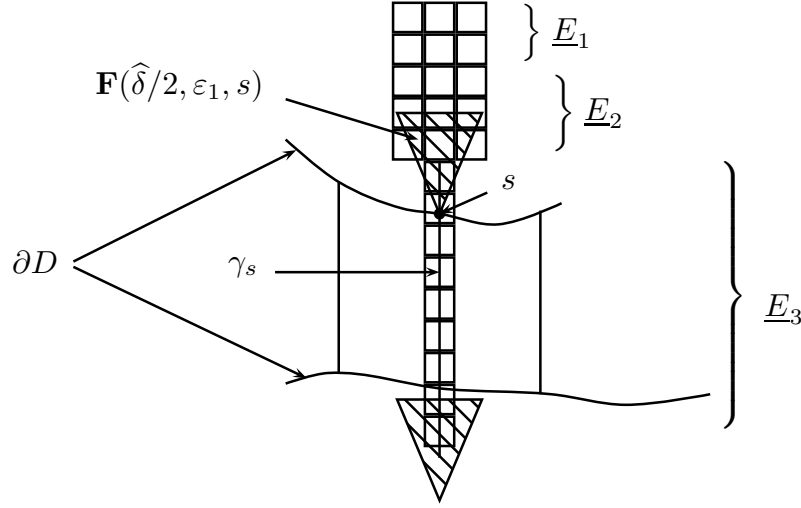


figure 22: a construction for the lower bound

to D do not create too much energy surface for ε_1 small. Thus, for ε_1 small enough,

$$\sum_{1 \leq j \leq r} \mathcal{H}^{d-1}(F_j) \tau(\text{nor}(F_j, D')) \leq \mathcal{I}(A) + 2\varepsilon.$$

Moreover, for each i in $\{1, \dots, r\}$, the relative boundary ∂F_i has a finite $d-2$ dimensional Hausdorff measure.

For $\underline{x}$ in $\mathbb{Z}^d$, we let $Y(\underline{x})$ be the indicator function of the event

$$\left\{ \begin{array}{l} \text{for every } y \text{ such that } |C(y) \cap B_n(\underline{x} - le_d)| \geq \alpha n, \text{ we have} \\ |C(y) \cap B_n(\underline{x} + e_d)| \geq \alpha n \text{ and } |C(y) \cap B_n(\underline{x})| \in (\alpha n)^d [\theta - \varepsilon, \theta + \varepsilon] \end{array} \right\}.$$

We let $Z(\underline{x})$ be the indicator function of the event

$$\left\{ \begin{array}{l} \text{for every } y \text{ such that } |C(y) \cap B_n(\underline{x})| \geq \alpha n, \\ \text{we have } |C(y) \cap B_n(\underline{x} + e_d)| \geq \alpha n \end{array} \right\}.$$

For $s \in S$, we write $T(s)$ for the event

$$\left\{ \text{for every } \underline{x} \text{ such that } B_n(\underline{x}) \subset \mathbf{F}(\hat{\delta}, \varepsilon_1, s), \text{ we have } |C(s) \cap B_n(\underline{x})| \geq \alpha n \right\}.$$

Let $\mathcal{E}$ be the intersection of the events

$$\begin{aligned} & \{ \text{all bonds in } \Gamma \text{ are open} \}, \quad \{ Y(\underline{x}) = 1, \underline{x} \in \underline{E}_1 \}, \quad \bigcap_{s \in S} T(s) \\ & \{ Z(\underline{x}) = 1, \underline{x} \in \underline{E}_2 \text{ or } \underline{x} \in \underline{E}_3 \}, \quad \text{wall}(F_j, n), 1 \leq j \leq r. \end{aligned}$$

The variables $Y(\underline{x})$, $\underline{x} \in \underline{E}_1$, do not depend on what happen in the region Γ and on the events $T(s)$ for s in S . The probabilities that the variables Y and Z are equal to 1 tend to 1 as n goes to infinity. Furthermore, the events represented by the variables $Z(\underline{x})$ are increasing, so we may apply the FKG inequality together with the events $T(s)$ for s in S , and with the event that all bonds in Γ are open. By the choice of D' , the events wall are independent of the other events in $\mathcal{E}$ for n large enough. Hence, as in [4], for all $\varepsilon > 0$, for α small enough,

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{d-1}} \ln P(\mathcal{E}) \geq -\mathcal{I}(D) - \varepsilon.$$

As in [4], the occurrence of $\mathcal{E}$ implies that $|\mathcal{C}_n(f) - \psi(f)|$ is small, and the lower bound is proved. $\square$

19 The geometry of the Wulff shape and more exponential results

In this section, we finish the description of the surface tension we started in proposition 5.6. To do this, we first study the percolation process in a cone “included” in the cone of percolation $\mathbf{F}$, and prove an equivalent statement to theorem 1.3.

PROPOSITION 19.1. *Let $\eta > 0$ and w be a unit vector. We define*

$$K(\eta, w) = \{tx + tw : t \geq 0, x \in \eta \mathbf{U}\}.$$

If w is in $\overset{\circ}{\mathbf{F}}$, then the oriented percolation process on $K(\eta, w)$ is supercritical: there exists x in $K(\eta, w)$ such that

$$P(x \rightarrow \infty \text{ in } K(\eta, w)) > 0.$$

PROOF. Let w in $\overset{\circ}{\mathbf{F}}$ and $\eta > 0$. We use another rescaled lattice. We pick $e'_1, \dots, e'_d$, an orthonormal basis of $\mathbb{R}^d$, such that $e'_d = w$. Let K be an integer. For x in $\mathbb{R}^d$, we let $x'_1, \dots, x'_d$ be its coordinates in the new basis $(e'_1, \dots, e'_d)$. Let $\underline{x}$ in $\mathbb{Z}^d$. We define

$$B'(\underline{x}) = \{y \in \mathbb{R}^d : \forall i, 1 \leq i \leq d, -K/2 < y'_i \leq K/2\}.$$

Now let l be a positive integer and let $\mathcal{D}'$ be the similar set introduced in the proof of proposition 3.12:

$$\mathcal{D}' = B'(\underline{x}) \cup \bigcup_{1 \leq i \leq d} B'(\underline{x} \pm e'_i).$$

We define the event $R_1(\underline{x}, l)$ as

$$V_1(\underline{x}, l) = \left\{ \begin{array}{l} \text{for all } y \text{ in } \mathcal{D} \text{ such that } |C(y)| \geq K \text{ and } C(y) \cap B'(\underline{x}) \neq \emptyset, \\ \text{we have } \forall j, 1 \leq j \leq d, C(y) \cap B'(\underline{x} + lw \pm e_j) \neq \emptyset \end{array} \right\}.$$

By proposition 3.12, there exists an integer l such that

$$P(V_1(\underline{x}, l)) \rightarrow 1 \quad \text{as } K \rightarrow \infty.$$

We also assume that l is large enough so that

$$B'(\underline{x}) \cap K(\eta, w) \neq \emptyset \Rightarrow \forall i, 1 \leq i \leq d-1 \quad B'(\underline{x} + le'd \pm e'_i) \subset K(\eta, w).$$

We call the blocks *good* and *bad*, accordingly to the event V_1 , and we write $X(\underline{x})$ for the indicator function of the event V_1 . Let $\underline{x}$ in $\mathbb{Z}^d$ such that $B'(\underline{x})$ is included in $K(\eta, w)$. We build a graph as follow: We let $\underline{x}$ be the first vertex of the graph. If $\underline{y}$ in a vertex of the graph, we add the two vertices $\underline{y} + le'd \pm e'_1$, and we put oriented edges from $\underline{y}$ to $\underline{y} + le'd \pm e'_1$. This new graph is isomorphic to the two-dimensional oriented graph $\mathbb{Z}_+^2$. We study the percolation process by site $X(\underline{x})$ on the new graph, For every $p' < 1$ and for K large enough, this process dominates stochastically the Bernoulli percolation process by site on the oriented graph $\mathbb{Z}_+^2$. Hence there is an infinite path on the macroscopic graph with strictly positive probability for K large enough. But this infinite path implies the existence of an infinite path in the underlying microscopic graph. Thus the oriented percolation process on $K(\eta, w)$ is supercritical. $\square$

We can now complete proposition 5.6 by proving theorem 1.4 which we restate:

COROLLARY 19.2. *The surface tension τ is strictly positive in the whole angular sector G .*

PROOF. Let w in G and take A a hyperrectangle normal to w . Let $\varepsilon > 0$, and let $w' \in S^{d-1}$ such that

$$\mathcal{H}^{d-1}(\{x \in \partial \text{cyl } A \cap \text{cyl}^-(A, w, \varepsilon) : \{x + tw' : t \geq 0\} \cap \partial \text{cyl } A \cap \text{cyl}^+(A, w, \varepsilon)\}) > 0.$$

Let $\eta > 0$, and let $\underline{A}' \subset \mathbb{Z}^d$ such that $\underline{A}'$ is a translate of $[0, \eta n/K]^{d-1} \times \{0\}$ in the new graph given above with $e'_d = w'$. Let $\alpha > 0$. We define

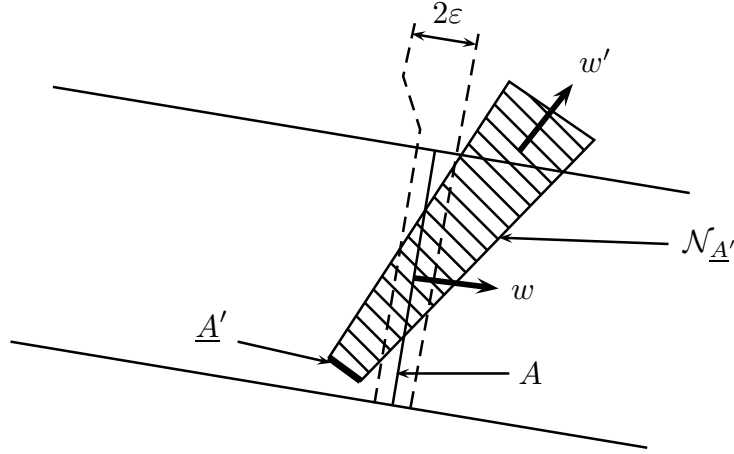
$$\mathcal{N}_{\underline{A}'} = \{\alpha ty + (t-1)w' : t \geq 1, y \in \bigcup_{\underline{x} \in \underline{A}'} B'(\underline{x})\}.$$

We take ε , η , and α small enough such that

$$\mathcal{N}_{\underline{A}'} \cap \mathcal{V}(nA, \varepsilon n) \cap \partial \text{cyl } nA = \emptyset,$$

and

$$\mathcal{N}_{\underline{A}'} \cap \partial \text{cyl } nA \cap \partial \text{cyl}^+ nA \neq \emptyset,$$


 figure 23: the set $\mathcal{N}_{\underline{A}'}$

see figure 23.

We take l large enough so that

$$B'(\underline{x}) \cap \mathcal{N}_{\underline{A}'} \neq \emptyset \Rightarrow \forall 1 \leq i \leq d-1 B'(\underline{x} + le'_d \pm e'_i) \subset \mathcal{N}_{\underline{A}'}.$$

We build a new graph $\tilde{\mathbb{L}} = (\tilde{\mathcal{V}}, \tilde{\mathbb{E}})$. First we set the vertex set at $\underline{A}'$. Then for each $\underline{x} \in \tilde{\mathcal{V}}$, we add the vertices $\underline{x} + le'_d \pm e'_i$ for $1 \leq i \leq d-1$, and we put an oriented edge between $\underline{x}$ and the new vertices.

A vertex $\underline{x}$ in $\tilde{\mathbb{L}}$ is *occupied* if $V_1(\underline{x}, l)$ occurs. If $W(\partial A, w, \varepsilon n)$ occurs, then $\underline{A}' \not\rightarrow \infty$ in the graph $\tilde{\mathbb{L}}$ for this percolation process. Since the probability that a vertex is occupied can be as close to 1 as we want, and since the percolation process in $\tilde{\mathbb{L}}$ is similar to the oriented site percolation process on $\tilde{\mathbb{L}}_{\text{alt}}^d$, by proposition 2.3, for K large enough,

$$P(\underline{A}' \not\rightarrow \infty \text{ in } \tilde{\mathbb{L}}) \leq \exp(-cn^{d-1}),$$

for a constant $c > 0$. $\square$

Therefore, by the continuity of τ , for all w in $\mathring{\mathbf{F}} \cap S^{d-1}$, there exists $t > 0$ such that $tw \in \mathcal{W}_\tau$. Actually, we would like a more precise result:

CONJECTURE 19.3. *We believe that the Wulff crystal $\mathcal{W}_\tau$ is tangent to $\mathbf{F}$ at 0, see figure 24.*

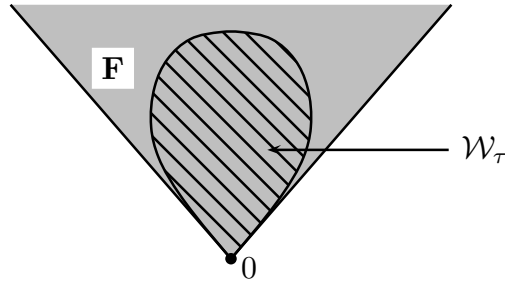


figure 24: a representation of the Wulff crystal

20 Exponential decrease of the connectivity function

The next proposition asserts that the oriented percolation process is subcritical outside the cone of percolation.

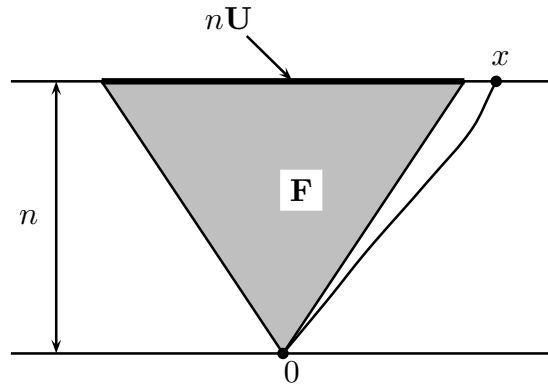
PROPOSITION 20.1. *Let $\varepsilon > 0$. There exists $c > 0$, such that for all $\mathbf{x} \notin (1 + \varepsilon)\mathbf{U}$.*

$$P(0 \rightarrow (\mathbf{x}, n)) \leq \exp(-cn), \quad (20.2)$$

or equivalently

$$P(0 \rightarrow (n(1 + \varepsilon)\mathbf{U}, n)) \leq \exp(-cn). \quad (20.3)$$

This is equivalent to theorem 1.5, and we represent in figure 25 such an improbable connection.

figure 25: a connection outside the cone $\mathbf{F}$

PROOF. It is straightforward that (20.3) implies (20.2). Conversely, the number of vertices in $(n(1 + \varepsilon)\mathbf{U}, n)$ that can be reached by 0 is bounded by a constant times n^{d-1} because of the graph structure of $\vec{\mathbb{L}}_{\text{alt}}^d$.

We turn now to the proof of (20.2). Let K be an integer. We work with the lattice rescaled by K . Let $\underline{x}$ in $\mathbb{Z}^d$, and let $\mathcal{D} = \mathcal{V}_\infty(B(\underline{x}), K)$. We introduce the region of blocks

$$\underline{D}_1(\underline{x}, l, \varepsilon) = \{\underline{y} : (\underline{y} - \underline{x}) \cdot e_d = l, B(\underline{y}) \cap (\mathbf{F}(1 + \varepsilon)^c + K\underline{x}) \neq \emptyset\}.$$

Let us define the event

$$V_1(\underline{x}, l, \varepsilon) = \{\forall y \text{ in } \mathcal{D}, \text{ such that } C(y) \cap B(\underline{x}) \neq \emptyset, \\ \text{we have } \underline{C}(y) \cap \underline{D}_1(\underline{x}, l, \varepsilon) = \emptyset\}.$$

For every $\varepsilon > 0$, for l large enough, we have

$$P(V_1(\underline{x}, l, \varepsilon)) \rightarrow 1 \quad \text{as } K \rightarrow \infty. \quad (20.4)$$

PROOF OF LIMIT (20.4). The proof of (20.4) is similar to the proof of proposition 3.12. Let $\underline{x}$ in $\mathbb{Z}^d$, and let $\varepsilon > 0$. As before, the region $\mathcal{D}$ is the set $\mathcal{V}_\infty(B(\underline{x}), K)$. The *inversed cluster* of a vertex y is the set

$$\overleftarrow{C}(y) = \{z \in \mathbb{Z}^d : z \rightarrow y\}.$$

We introduce $\mathcal{D}_1$ the set of vertices in $\mathbb{Z}^{d-1} \times \{0\} + K(\underline{x} + 2e_d)$ joined by vertices in $B(\underline{x})$:

$$\mathcal{D}_1 = \{z \in \mathbb{Z}^{d-1} \times \{0\} + K(\underline{x} + 2e_d) : \exists y \in \mathcal{D} \text{ such that} \\ C(y) \cap B(\underline{x}) \neq \emptyset \text{ and } z \in C(y)\}.$$

For every $z \in \mathcal{D}_1$, we have $|\overleftarrow{C}(z)| \geq K/2$. Because of the graph structure of $\vec{\mathbb{L}}_{\text{alt}}^d$, there exists a deterministic set $\tilde{\mathcal{D}}_1$ and $\alpha > 0$ such that $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_1$ with $\tilde{\mathcal{D}}_1 \leq \alpha K^{d-1}$. By proposition 3.1, there exists l_1 such that for ε' small enough, for K large enough,

$$P(\forall z \in \mathcal{D}_1 : |\overleftarrow{C}(z) \cap B(\underline{x} - l_1 e_d)| \geq 3\varepsilon' K^d) \geq 1 - \varepsilon_1. \quad (20.5)$$

Now let

$$A_1(y, \varepsilon, n_0) = \{\forall n \geq n_0, (\overline{H_n^y} \cap \overline{K_n^y}) \subset (1 + \varepsilon)\mathbf{U}\}.$$

We let $\varepsilon > 0$. With the help of proposition 2.2, we can pick n_0 such that

$$P(A_1(0, \varepsilon, n_0) \mid |C(0)| = \infty) \geq 1 - \varepsilon'.$$

By the FKG inequality (2.1), this implies that $P(A_1(0, \varepsilon, n_0)) \geq 1 - \varepsilon'$. By the ergodic theorem [22], for K large enough,

$$P(|z \in B(\underline{x} - l_1 e_d) : A_1^c(z, \varepsilon, n_0) \text{ occurs}| \geq 2\varepsilon') \leq \varepsilon_1. \quad (20.6)$$

Thus by (20.5) and (20.6)

$$P(\forall z \in \mathcal{D}_1, \exists s \in \tilde{C}(z) \cap B(\underline{x} - l_1 e_d) \text{ such that } A_1(s, \varepsilon, n_0) \text{ occurs}) \geq 1 - 2\varepsilon_1. \quad (20.7)$$

We represent on figure 26 a cluster starting in $B(\underline{x})$, which is joined in $\mathcal{D}_1$ by a cluster starting in $B(\underline{x} - l_1 e_d)$. We take l large enough, so that for every z in $\tilde{\mathcal{D}}_1$, every s in $B(\underline{x} - l_1 e_d)$, we have

$$(\mathbf{F}(z, 1 + \varepsilon) \cap (\mathbb{R}^{d-1} \times \{K(l-1)\} + K\underline{x})) \supset (\mathbf{F}(y, 1 + \varepsilon) \cap (\mathbb{R}^{d-1} \times \{K(l-1)\} + K\underline{x})).$$

We suppose in addition that $lK \geq 2n_0$, and that for every z in $B(\underline{x} - l_1 e_d)$,

$$\mathbf{F}(z, 1 + \varepsilon) \cap \underline{D}_1(\underline{x}, l, 2\varepsilon) = \emptyset.$$

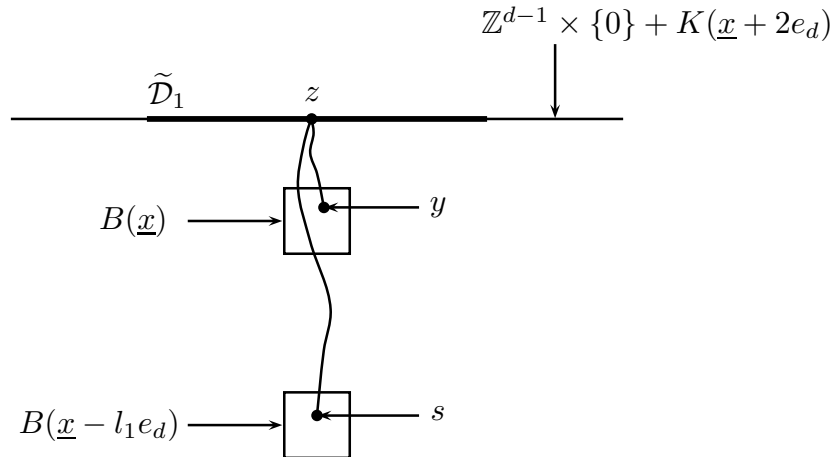


figure 26: the cluster $C(y)$ is joined by $C(s)$ at z

Hence suppose that the event in (20.7) occurs. Let y in $\mathcal{D}$. If $|C(y)| < 2K$, then there is nothing to do. So consider the case $|C(y)| \geq 2K$. There exists z in $\mathcal{D}_1$ such that $z \in C(y)$. But for all z in $\mathcal{D}_1$, there exists s in $B(\underline{x} - l_1 e_d) \cap \bar{C}(z)$ such that $A_1(s, \varepsilon, n_0)$ occurs. Thus for all z in $\mathcal{D}_1$, we have

$$C(z) \cap \underline{D}_1(\underline{x}, l, 2\varepsilon) = \emptyset,$$

and it follows that

$$C(y) \cap \underline{D}_1(\underline{x}, l, 2\varepsilon) = \emptyset.$$

Therefore we have obtained

$$P(V_1(\underline{x}, l, 2\varepsilon)) \geq 1 - 2\varepsilon_1. \quad \square$$

Let $\mathbf{x} \notin (1 + 3\varepsilon)\mathbf{U}$ such that $0 \rightarrow (n\mathbf{x}, n)$, and let γ be an oriented open path from 0 to x . Let l be such that the limit (20.4) holds. We say that a box $B(\underline{y})$ is good if $V_1(\underline{y}, l, \varepsilon)$ occurs. Define $\underline{\gamma}$ as the set of boxes intersecting γ . We introduce a function f from $\mathbb{N}$ to $\mathbb{R}_+$ by

$$f(i) = \max \left\{ \min \{ |\underline{z} - \underline{y}|, (\underline{y}, i) \in \underline{\mathbf{F}}(1 + 2\varepsilon) \}, (\underline{z}, i) \in \underline{\gamma} \right\}.$$

Let i be an integer. If for every $\underline{z} = (\underline{z}, i)$ in $\underline{\gamma}$, $B(\underline{z})$ is good, then $f(i + 1) \leq f(i)$. Moreover, the number of $\underline{y}$ in $\mathbb{Z}^{d-1}$ such that $(\underline{y}, i) \in \underline{\gamma}$ is bounded by 2^d . Hence there exists a positive density of bad boxes in $\underline{\gamma}$, and the proof of proposition 20.3 is finished by using a Peierls argument. $\square$

21 A note on the Wulff variational problem

We study the following variational problem:

$$(W) \quad \text{minimize } \mathcal{I}(E) \text{ under the constraint } \mathcal{L}^d(\mathcal{W}_\tau) \leq \mathcal{L}^d(E) < +\infty.$$

PROPOSITION 21.1. *The Wulff crystal defined in section 6 is a solution of the Wulff variational problem (W).*

This result has already been proved under the assumption that the function τ strictly positive, see [4] for a discussion on this subject. In fact, one may check that in the proof in [4], the strict positivity is not required when the function τ is convex. Here we just redo the proof that for every bounded polyhedral set A in $\mathbb{R}^d$,

$$\mathcal{I}(A) \geq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mathcal{L}^d(A + \varepsilon \mathcal{W}_\tau) - \mathcal{L}^d(A)) \leq \mathcal{I}(A). \quad (21.2)$$

PROOF OF (21.2). By definition, the boundary of A is the union of a finite number of $d - 1$ dimensional bounded polyhedral sets F_i , $i \in I$, so that

$$\mathcal{I}(A) = \sum_{i \in I} \mathcal{H}^{d-1}(F_i) \tau(\nu_A(F_i)),$$

where $\nu_A(F_i)$ is the unit outward normal vector to A along the interior points of the face F_i . Let $S = \partial A \setminus \partial^* A$ be the set of the singular points of ∂A ; it is a $d - 2$ dimensional set. We claim that, for ε small enough,

$$(A + \varepsilon \mathcal{W}_\tau) \setminus \mathcal{V}(S, \varepsilon(2\|\tau\|_\infty + 1)) \cup \bigcup_{i \in I} \text{cyl}(F_i, \nu_A(F_i), \varepsilon \tau(\nu_A(F_i))).$$

Indeed, let $x = a + \varepsilon w$ where $a \in F_i$, $w \in \mathcal{W}_\tau$, and $x \notin A$. There are two cases:

- $w \cdot \nu_A(F_i) \geq 0$. We let y be the orthogonal projection of x on the hyperplane containing F_i . Then

$$|a - y| \leq |\varepsilon w| \leq \varepsilon(\|\tau\|_\infty + 1),$$

$$|(x - y) \cdot \nu_A(F_i)| = |\varepsilon w \cdot \nu_A(F_i)| \leq \varepsilon \tau(\nu_A(F_i)).$$

If x does not belong to $\mathcal{V}(S, \varepsilon(2\|\tau\|_\infty + 1))$, then $a \in F_i \setminus \mathcal{V}(S, \varepsilon(2\|\tau\|_\infty + 1))$ and $y \in F_i$, whence x is in $\text{cyl}(F_i, \nu_A(F_i), \varepsilon \tau(\nu_A(F_i)))$.

- $w \cdot \nu_A(F_i) < 0$. Since $a + \varepsilon w \notin A$, there exists a polyhedral set F_j such that $[a, a + \varepsilon w]$ intersects F_j and $\tau(\nu_A(F_j)) \cdot w \geq 0$. Let $a' = [a, a + \varepsilon w] \cap F_j$, and let $\varepsilon' \leq \varepsilon$ such that $a' + \varepsilon' w = x$. As in the first case, the point x is in $\text{cyl}(F_j, \nu_A(F_j), \varepsilon \tau(\nu_A(F_j)))$, or in $\mathcal{V}(S, \varepsilon(2\|\tau\|_\infty + 1))$.

Thus

$$\mathcal{L}^d(A + \varepsilon \mathcal{W}_\tau) - \mathcal{L}^d(A) \leq \mathcal{L}^d(\mathcal{V}(S, \varepsilon(2\|\tau\|_\infty + 1))) + \sum_{i \in I} \mathcal{H}^{d-1}(F_i) \tau(\nu_A(F_i)).$$

Sending ε to 0, we get equation (21.2). $\square$

Bibliography

1. D. Barbato, *Tesi di Laurea*, Universita di Pisa (2002).
2. C. Bezuidenhout, G. Grimmett, *The critical contact process dies out*, Ann. Probab. **18** (1990), 1462–1482.
3. R. Cerf, *Large deviations for three-dimensional supercritical percolation*, Astérisque **267** (2000).
4. R. Cerf, *The Wulff crystal in Ising and Percolation models*, Saint-Flour lecture notes, first version (2004).
5. A. Dembo, O. Zeitouni, *Large Deviations Techniques and Applications*, Second Edition, Springer-Verlag, Berlin.
6. R. Durrett, D. Griffeath, *Contact process in several dimensions*, Z. Wahrsch. verw. Gebiete **59** (1982), 535–552.
7. R. Durrett, D. Griffeath, *Supercritical contact process on Z* , Ann. Probab. **11** (1983), 1–15.
8. R. Durrett, *Oriented percolation in two dimensions*, Ann. Probab. **12** (1984), 999–1040.
9. R. Durrett, *Stochastic growth models*, Percolation Theory and Ergodic Theory of Infinite Particle Systems (H. Kesten, ed.) (1987), 85–119. Springer.
10. R. Durrett, R. H. Schonmann, *Large Deviations for the Contact Process in Two Dimensional Percolation*, Probab. Theory Relat. Fields **77** (1988), 583–603.
11. R. Durrett, *The contact process 1974–1989*, Mathematics of Random Media, Lectures in Applied Mathematics **27** (1991), 1–18.
12. L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, Boca Raton: CRC Press (1992).
13. K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics 85, Cambridge Univ. Press (1985).
14. I. Fonseca, *The Wulff theorem revisited*, Proc. R. Soc. Lond. Ser. A **432 No. 1884** (1991), 125–145.
15. E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser (1984).
16. G. Grimmett, *Percolation*, Second edition, Springer-Verlag, Berlin **321** (1999).
17. G. Grimmett, P. Hiemer, *Directed percolation and random walk*, Preprint 2001.
18. T. M. Liggett, *Stochastic Interacting systems: contact, voter and exclusion processes*, Springer, 1999.

19. T. M. Liggett, R. H. Schonmann, A. M. Stacey, *Domination by product measures*, Ann. Probab. **25** (1997), 71–95.
20. Á. Pisztora, *Surface order large deviations for Ising, Potts and percolation models*, Probab. Theory Relat. Fields **104** (1996), 427–466.
21. R. T. Rockafellar, *Convex analysis*, Princeton University Press (1970).
22. A. N. Shiryaev, *Probability, Second edition*, Springer **95** (1989).
23. G. Wulff, *Zur Frageder Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen*, Z. Kristallogr. **34** (1902), 449–530.
24. W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, GTM 120 Springer–Verlag (1989).

N⁰ d'impression 2625
4^{ème} trimestre 2004

Sur les grands clusters en percolation

Résumé : Cette thèse est consacrée à l'étude des grands clusters en percolation et se compose de quatre articles distincts. Les différents modèles étudiés sont la percolation Bernoulli, la percolation FK et la percolation orientée. Les idées clés sont la renormalisation, les grandes déviations, les inégalités FKG et BK, les propriétés de mélange.

Nous prouvons un principe de grandes déviations pour les clusters en régime sous-critique de la percolation Bernoulli. Nous utilisons l'inégalité FKG pour démontrer la borne inférieure du PGD. La borne supérieure est obtenue à l'aide de l'inégalité BK combinée avec des squelettes, les squelettes étant des sortes de lignes brisées approximant les clusters.

Concernant la FK percolation en régime sur-critique, nous établissons des estimés d'ordre surfacique pour la densité du cluster maximal dans une boîte en dimension deux. Nous utilisons la renormalisation et comparons un processus sur des blocs avec un processus de percolation par site dont le paramètre de rétention est proche de un.

Pour toutes les dimensions, nous prouvons que les grands clusters finis de la percolation FK sont distribués dans l'espace comme un processus de Poisson. La preuve repose sur la méthode Chen–Stein et fait appel à des propriétés de mélange comme la *ratio weak mixing property*.

Nous établissons un principe de grandes déviations surfaciques dans le régime sur-critique du modèle orienté. Le schéma de la preuve est similaire à celui du cas non-orienté, mais des difficultés surgissent malgré l'aspect Markovien du réseau orienté. De nouveaux estimés blocs sont donnés, qui décrivent le comportement du processus orienté. Nous obtenons également la décroissance exponentielle des connectivités en dehors du cône de percolation, qui représente la forme typique d'un cluster infini.

Mots clés : percolation, grandes déviations, renormalisation, percolation FK, percolation orientée.

Classification MSC 1991 : 60F10, 60K35, 82B20, 82B43