# Categorical abstract games

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# Abstract

Goal: to define notions of games (and strategies) encompassing ***concrete*** games (as introduced by Hyland, Ong, Nickau, Abramsky, Jagadeesan, Malacaria) and ready for further implementations.

Basic idea: instead of playing in an arena, play in a category.

Starting from a category we first build a double-category.

Then, from a double-category, we build a category of strategies.

We sketch how to recover HO games in the new framework.

# Summary

- Semantics : some history
- HO-games : overview
- Playing in a ***category***
- Categories of *strategies*
- A new game for $HO$

# 1. History

# Semantics

Semantics should make it possible to study the behavior of programs in a given langage through a mathematical model. The model should be simple enough in order to facilitate proofs (certification, validation).

Wanted: models satisfying properties like coherence and adequacy.

# Properties of semantics

| | denotational | abstract interpretation |
|---|---|---|
| model | $[\![P_1]\!] = [\![P_2]\!]$ | $[\![P_1]\!] \neq [\![P_2]\!]$ |
| | coherence $\downarrow$ | $\downarrow$ soundness |
| programs | $P_1 \overset{?}{=} P_2$ | $P_1 \overset{?}{\neq} P_2$ |

**games**

# Game semantics

- Games = a homogeneous universe sharing coherence with denotational semantics and soundness with abstract interpretation.
- still close to syntax, not too easy for manipulations
- provided good models for functional languages, for imperative languages with fixpoints, for probabilistic languages
- algorithmic insight : programs with functional type of order at most two can be interpreted through regular languages

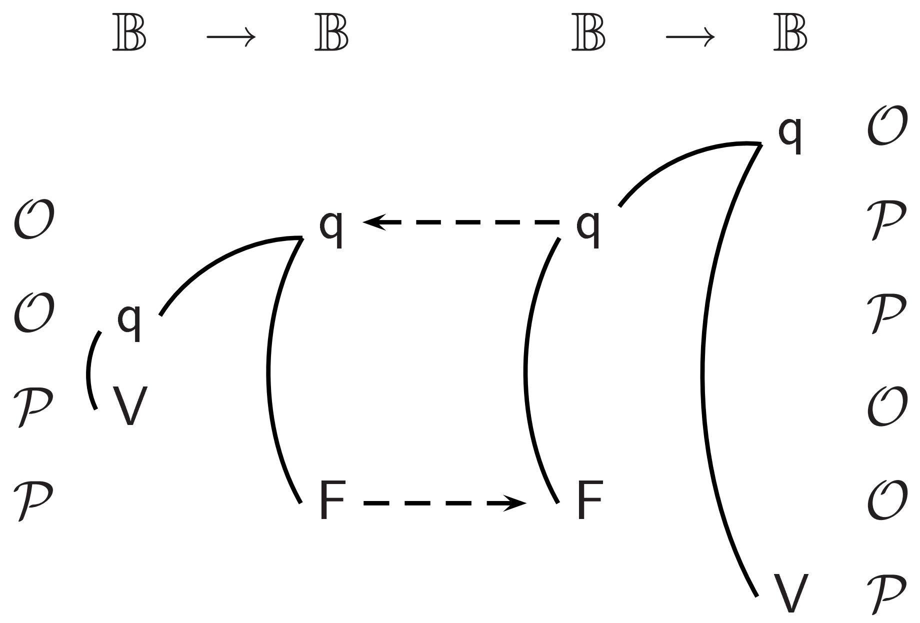
# 2. Some intuition on concrete games

# Boolean games

The `negation` strategy:

$$
\begin{array}{cccc}
\mathbb{B} & \longrightarrow & \mathbb{B} & \\
 & & \mathsf{q} & \mathcal{O} \\
\mathsf{q} & & & \mathcal{P} \\
\mathsf{F} & & & \mathcal{O} \\
 & & \mathsf{V} & \mathcal{P}
\end{array}
\qquad\qquad
\begin{array}{cccc}
\mathbb{B} & \longrightarrow & \mathbb{B} & \\
 & & \mathsf{q} & \mathcal{O} \\
\mathsf{q} & & & \mathcal{P} \\
\mathsf{V} & & & \mathcal{O} \\
 & & \mathsf{F} & \mathcal{P}
\end{array}
$$

# Composing negation with itself




leads to identity in $\mathbb{B} \to \mathbb{B}$.

# 3. Where to play ?

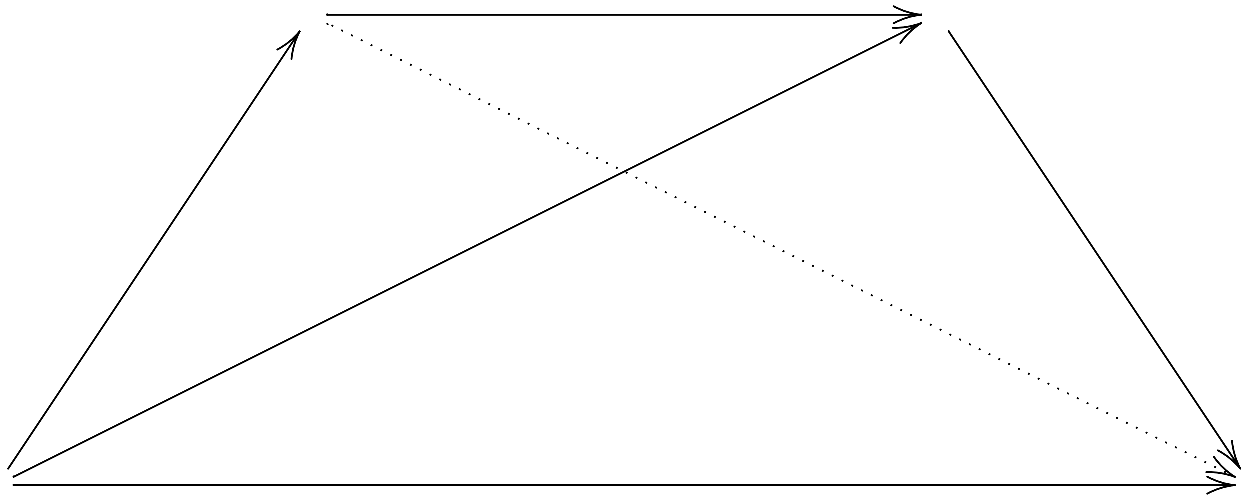
# For convenience

- Here, for simplicity, we do not mention signs. In the general case, we encode some features like *alternation* by putting signs on the objects of our category.
- Here, for simplicity, we start from a category. In the general case, we start from a weak category, and consider cases where composition is multivalued.

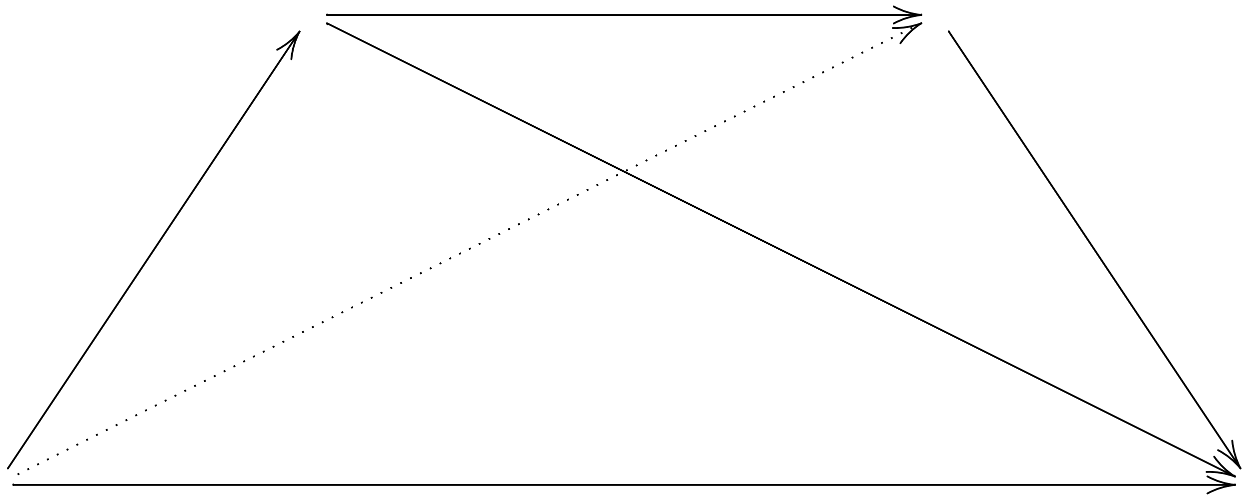
# Playing in a category $\mathfrak{C}$

- positions = arrows of $\mathfrak{C}$
- moves = $f, b \subseteq \mathfrak{C}_1 \longrightarrow$ yield a $G$
- we play at a position $p$
  - $\mathcal{P}$ wants to prove $p \in$ subcategory generated by $f$ and $b$
  - $\mathcal{P}$ tries to decompose $p$
  - $\mathcal{O}$ wants to prove $p \notin$ subcategory generated by $f$ and $b$
  - $\mathcal{O}$ tries to compose $p$

# Associativity diagram



# Associativity diagram



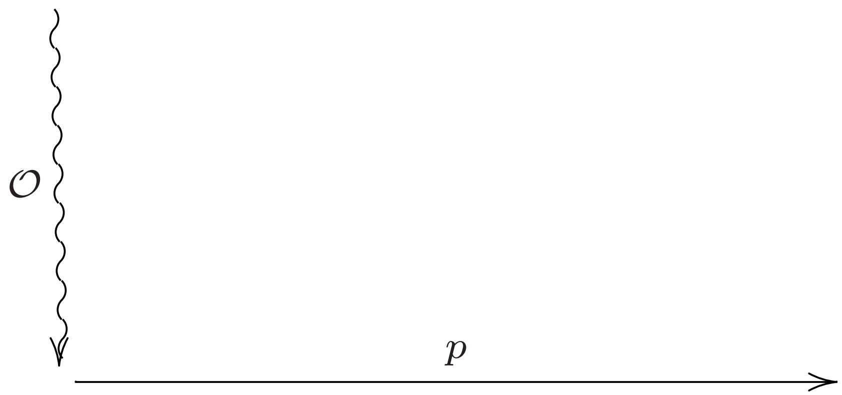
# Dictionary

| concrete games | abstract games |
|---|---|
| $\mathcal{O}/\mathcal{P}$ | `compo / decompo` |
| `translation`<br>`functions` | `associativity`<br>`axiom` |
| `interactions` | `stacks of triplexes` |

# When $\mathcal{O}$ plays on the left

$$\xrightarrow{\quad\quad\quad p \quad\quad\quad}$$

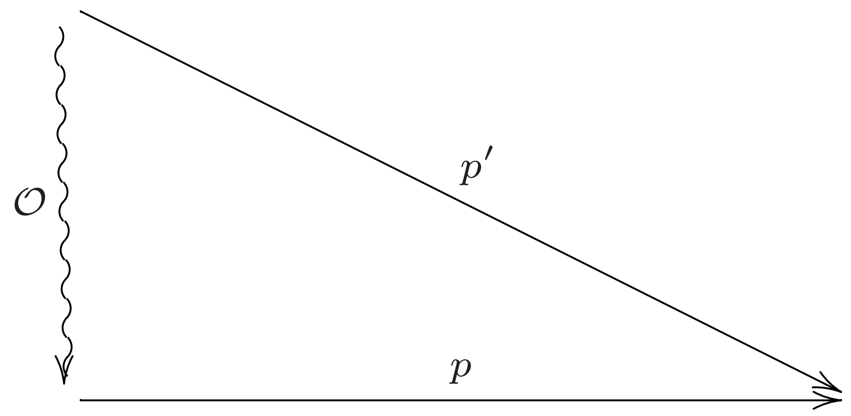
# When $\mathcal{O}$ plays on the left



# When $\mathcal{O}$ plays on the left

**Remarque 1 – Coherence**

$$p \in V \to p' \in V$$



# When $\mathcal{O}$ plays on the right

$$\xrightarrow{\qquad\qquad\qquad p \qquad\qquad\qquad}$$

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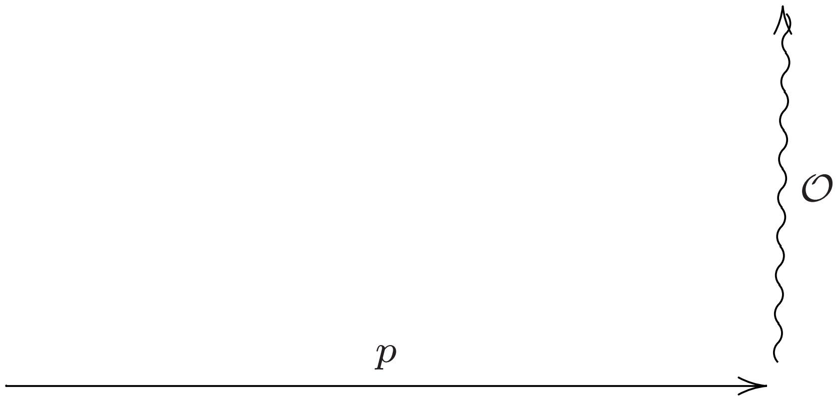
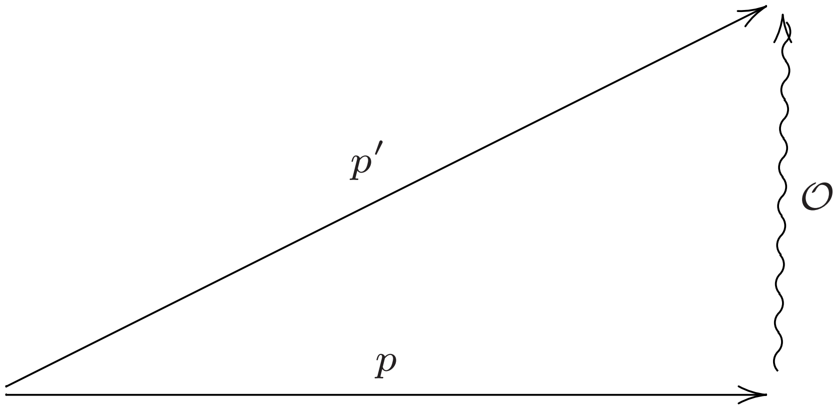


# When $\mathcal{O}$ plays on the right

**Remarque 2 – Coherence**

$$p \in V \to p' \in V$$



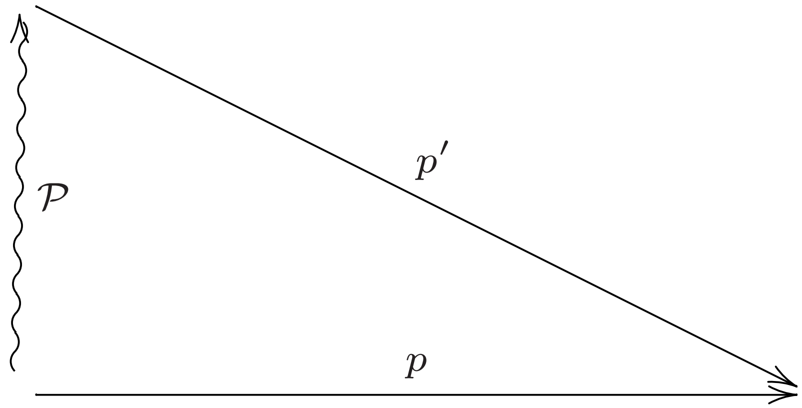
# When $\mathcal{P}$ plays on the left

$$\xrightarrow{\qquad\qquad p \qquad\qquad}$$

# When $\mathcal{P}$ plays on the left

**Remarque 3 – Coherence**

$$p' \in V \to p \in V$$



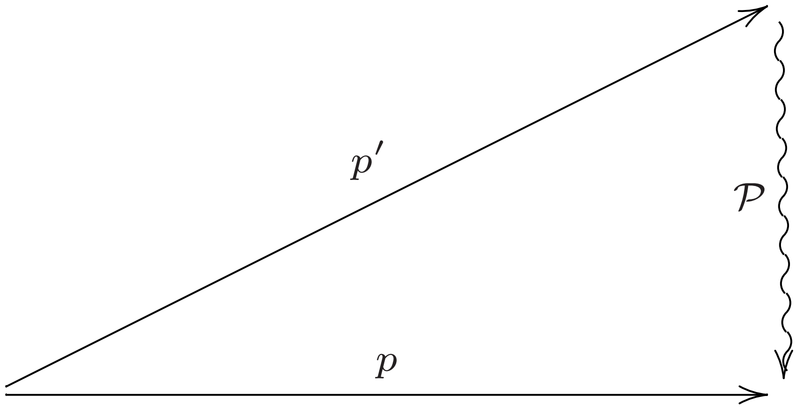
# When $\mathcal{P}$ plays on the right

$$\xrightarrow{\qquad\qquad p \qquad\qquad}$$

# When $\mathcal{P}$ plays on the right

**Remarque 4 – Coherence**

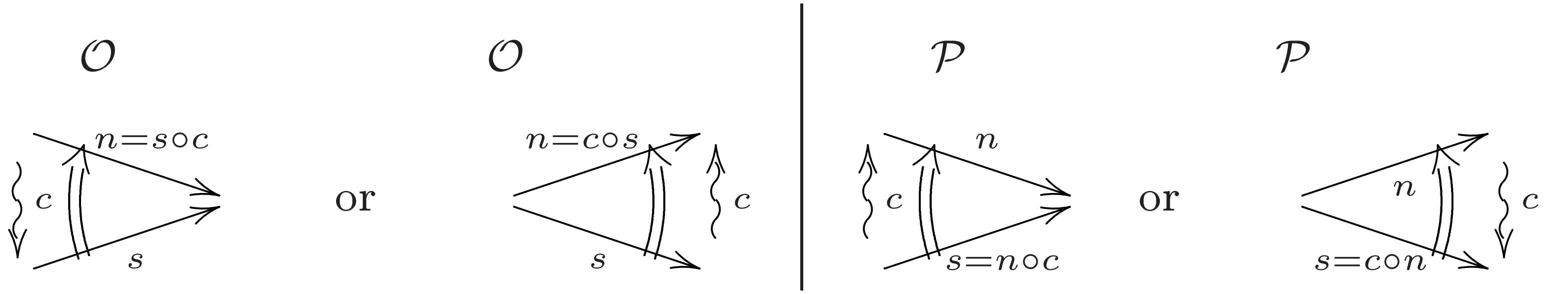
$$p' \in V \rightarrow p \in V$$



# Plays (1)

A ***move*** is also :

- a commutative triangle in $\mathfrak{C}$
- one small side of which is a move in $G$



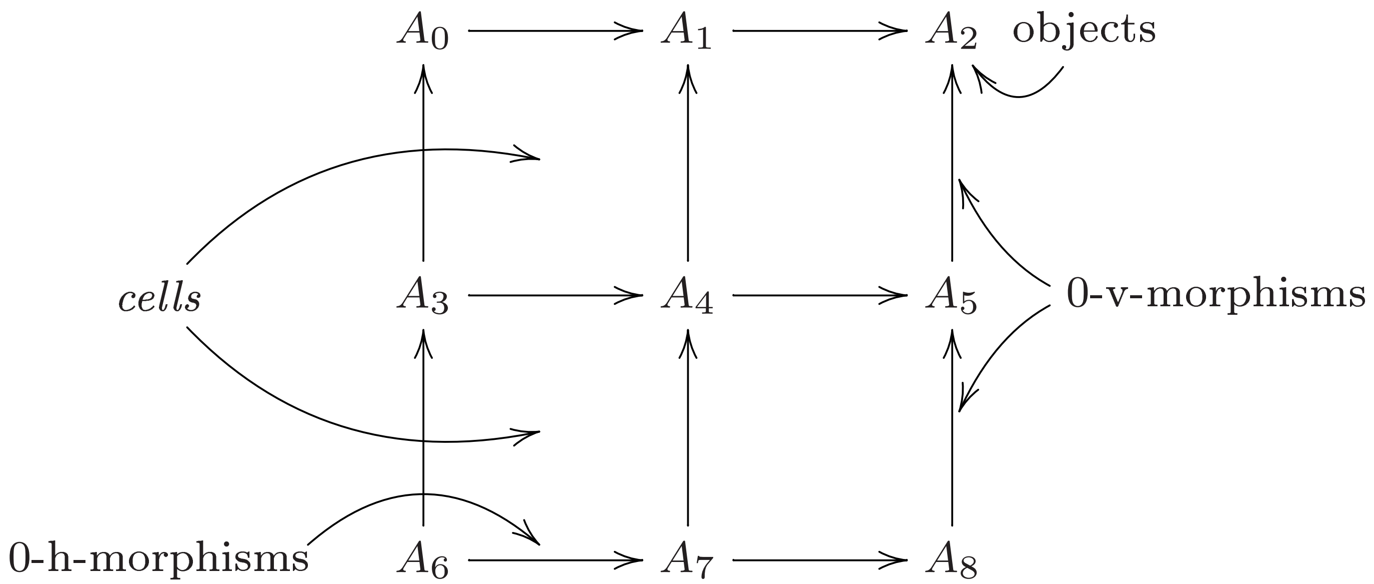
# The *switching* condition

For convenience, we also impose the ***switching condition*** :

- the opponent is not allowed to switch side.

# 4. The double catégory of plays

# Double catégory

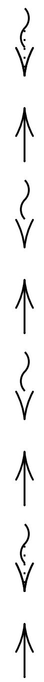



- in 0-h-morphisms 0 refers to the dimension of contact

for instance cells are 1-v as well as 1-h morphisms.

# The 0-vertical category

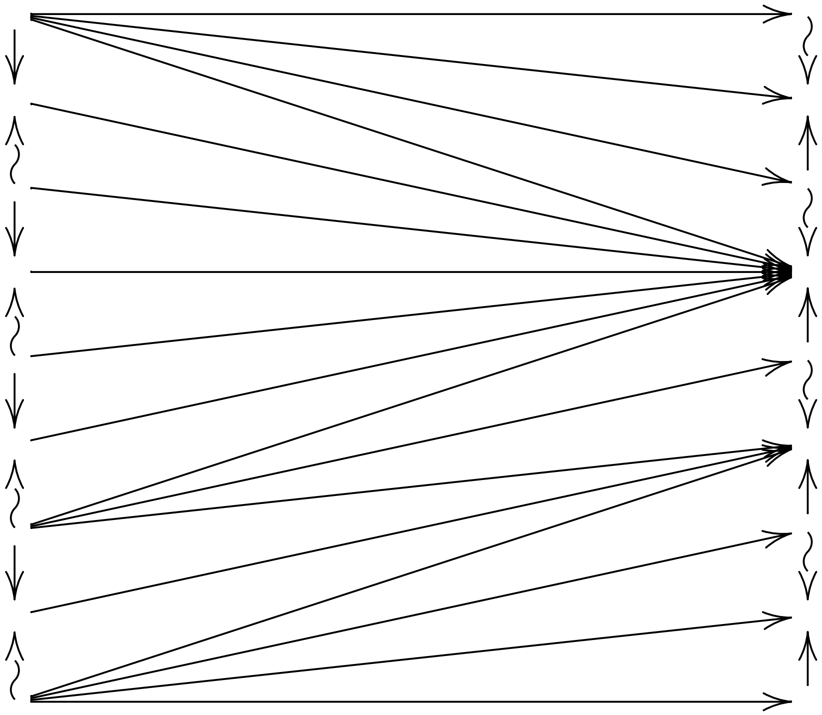
As 0-vertical category, we take a kind of free category generated by the graph $G$ of moves :



- ▶ ▶ objects = objects of $\mathfrak{C}$
- ▶ ▶ morphisms = stacks of moves in $G$
- ▶ ▶ composition = concatenation

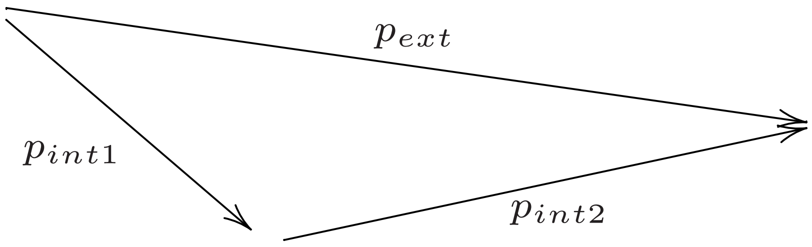
# Plays

- $\{\mathcal{O}$ `and` $\mathcal{P}$ `play at a position`$\}$



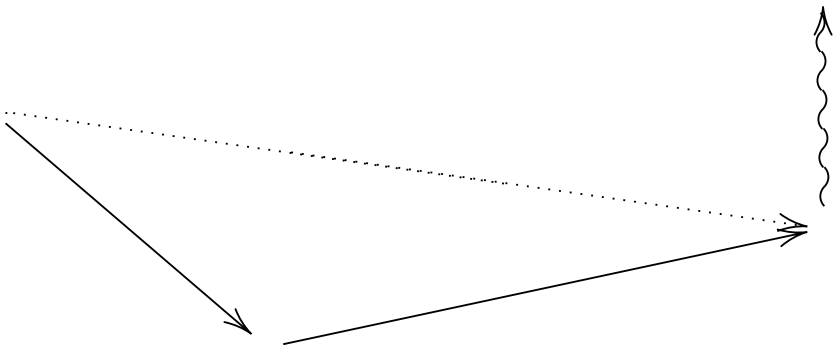
# Interactions (1)

- $\mathcal{P}_1$ et $\mathcal{P}_2$ cooperate against $\mathcal{O}$ on a triangle



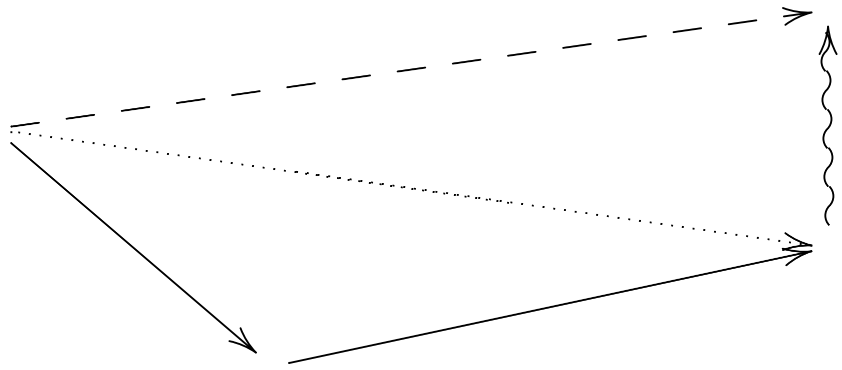
# Interactions (2)

- $\mathcal{O}$ `plays on the external projection`



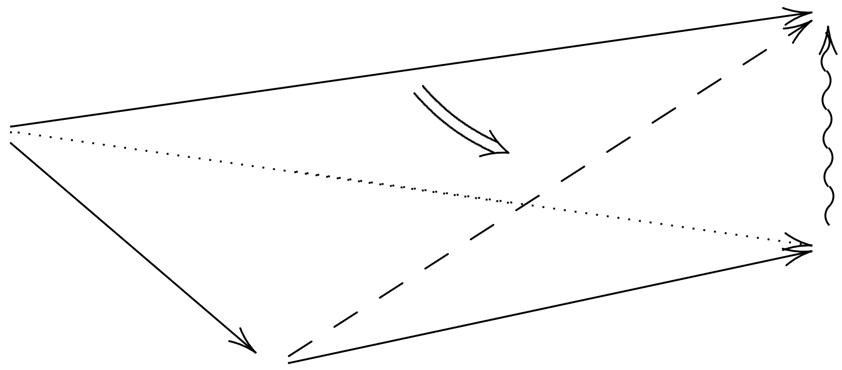
# Interactions (2)

- $\mathcal{O}$ `plays on the external projection`



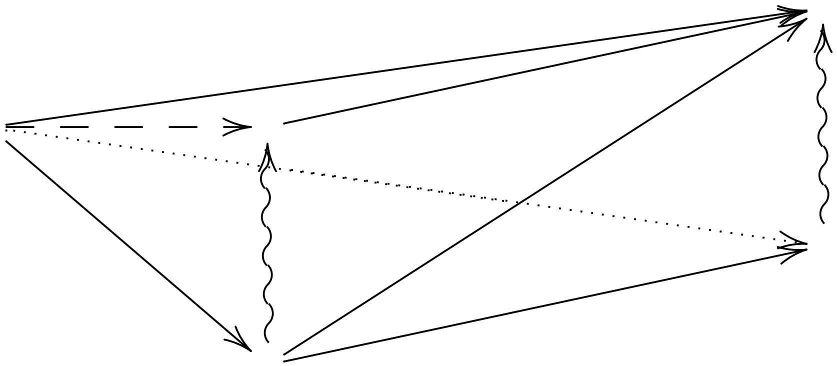
# Interactions (2)

- …and associativity translates the move into $p_{int2}$



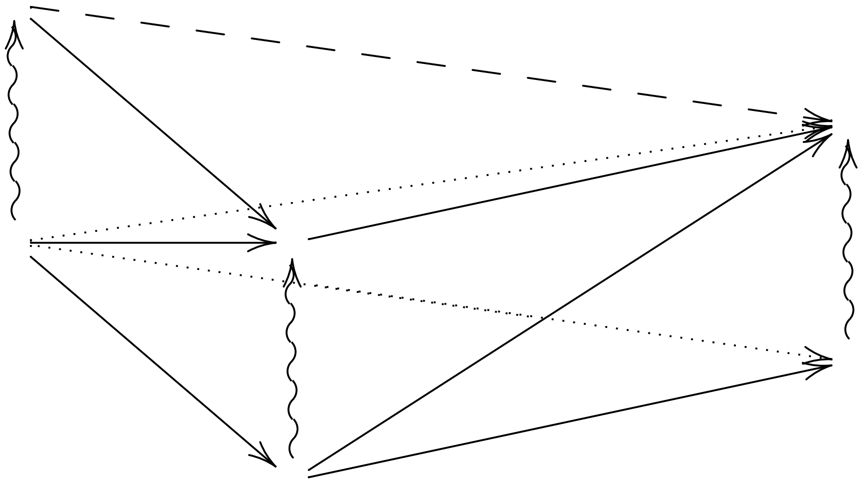
# Interactions (3)

- $\mathcal{P}_2$ `plays on` $p_{int2}$ and associativity translates the move into $p_{ext}$



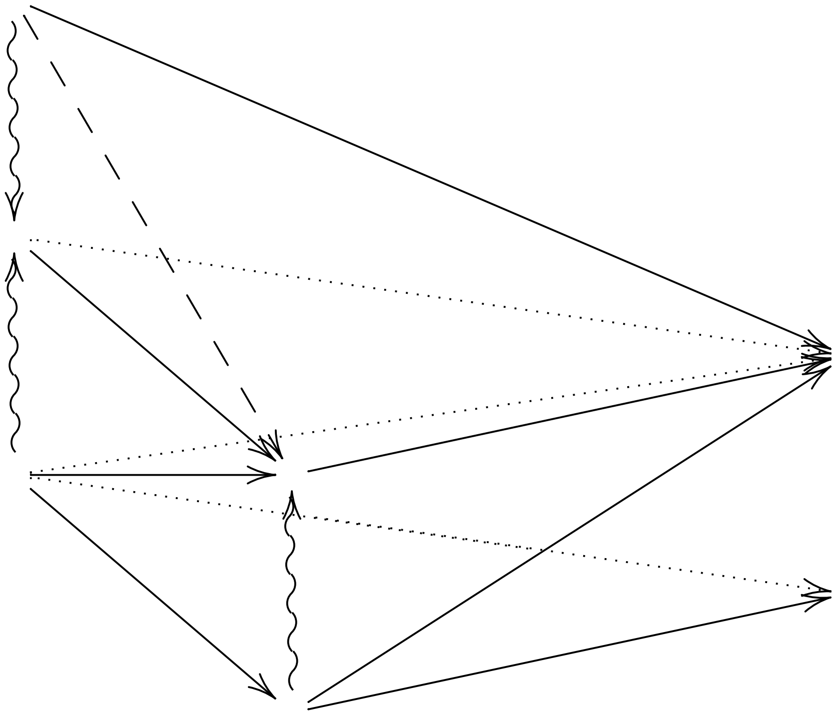
# Interactions (4)

- $\mathcal{P}_1$ answers $\mathcal{P}_2$ on $p_{int1}$ and associativity translates the move into $p_{ext}$



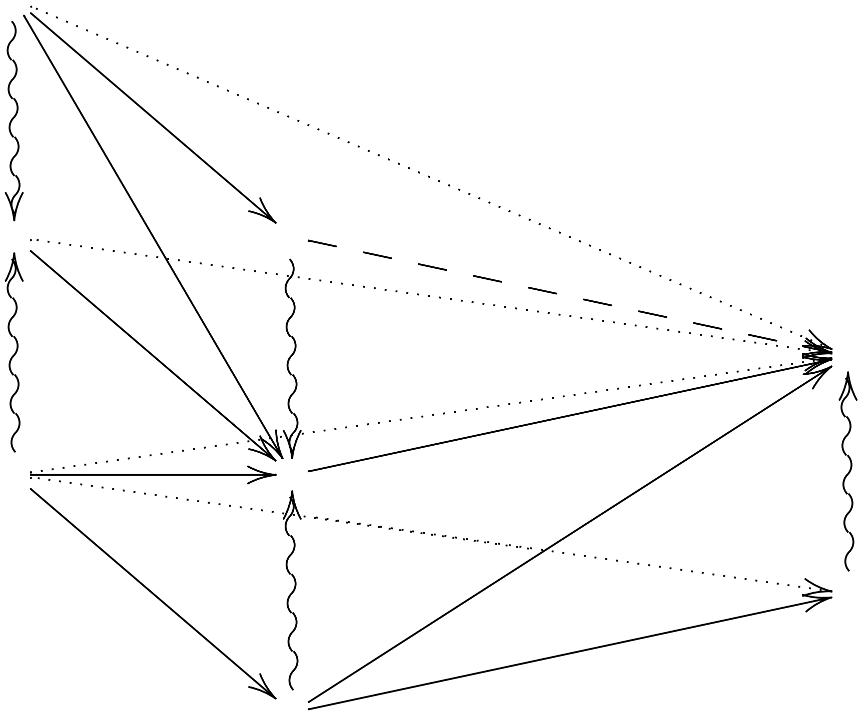
# Interactions (5)

- $\mathcal{O}$ `answers on` $p_{ext}$ and associativity translates the move into $p_{int1}$



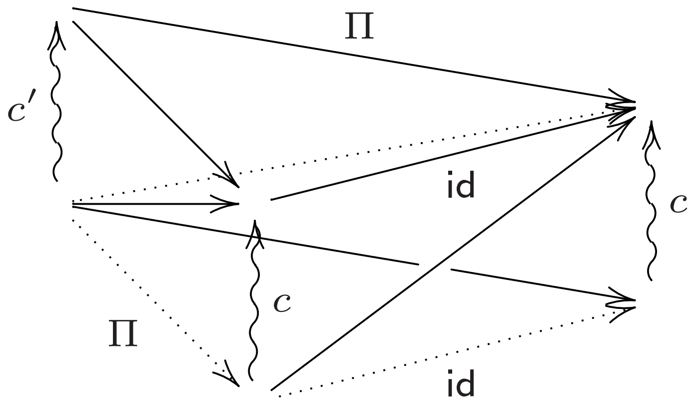
# Interactions (6)

- $\mathcal{P}_1$ `answers on` $p_{int1}$ and associativity translates the move into $p_{int2}$



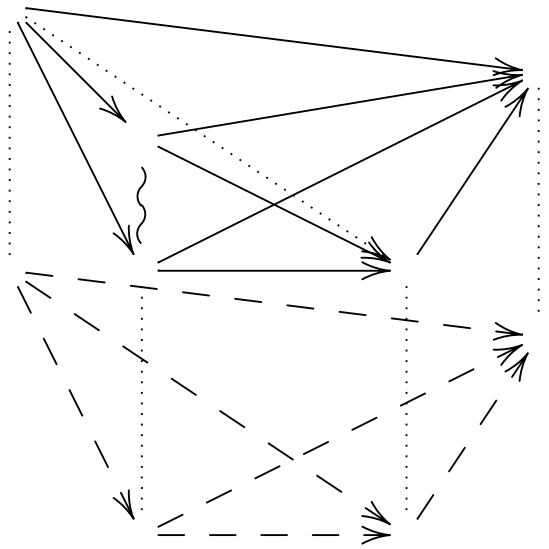
# Identities

$$\mathsf{id}(c) \circ \Pi = \Pi$$



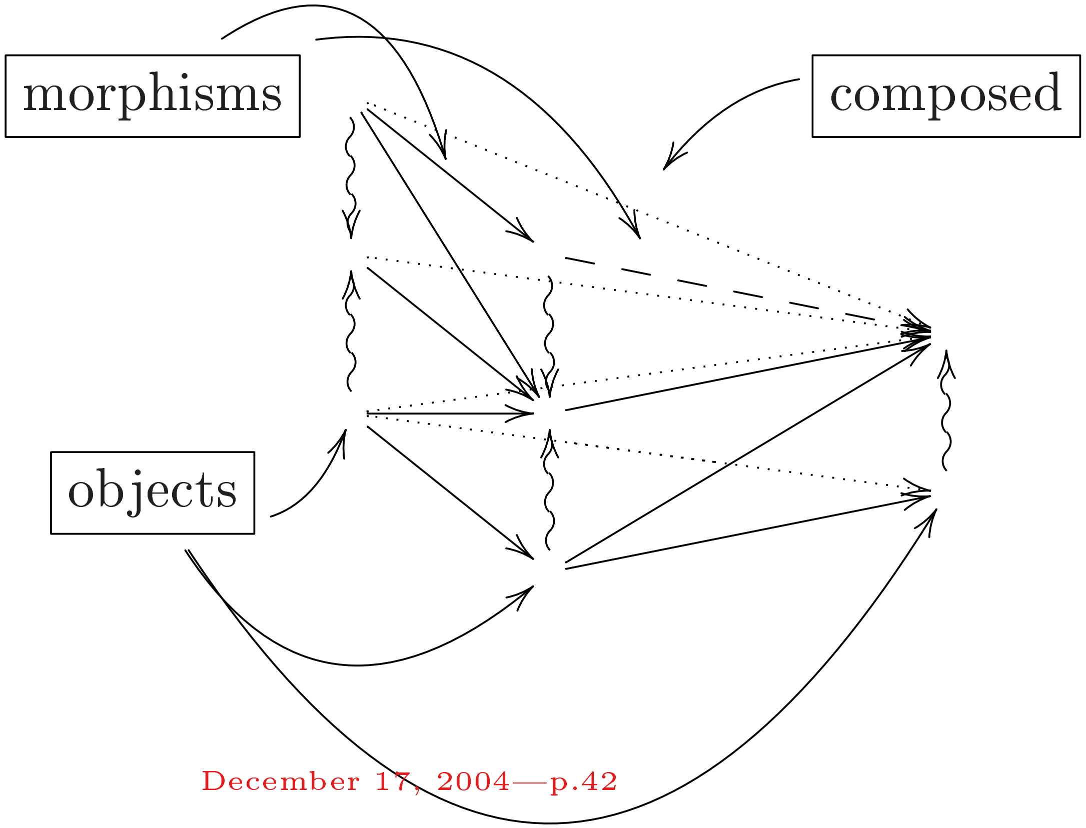
# Interactions for 3

- {tetrahedrons, stacks of quadriplexes, concatenation}



# The horizontal category of plays

***Horizontal*** plays :



# 5. Categories of strategies

# Strategies

▶ If $c = c_1 \circ_1^v c_2$ then $c_1$ is said *prefix* of $c$.

$\mathfrak{S}$ : double-cat; $A, B, C$: objects of $\mathfrak{S}$; $p$ position from $A$ to $B$

- A ***strategie*** for $p_{A \to B}$ is a set of cells :
  - starting at $p$;
  - prefix-closed;
  - containing the vertical identity of $p$.
- ***The identity*** $\rightarrow$ The *copycat(A)* strategy is for $\mathsf{id}(A)$ and consists of cells which are horizontal identities starting at $\mathsf{id}(A)$.
- The ***composition*** of two strategies $A \xrightarrow{\sigma}_p B \xrightarrow{\sigma'}_{p'} C$ consists of cells which are composite of a cell in $\sigma$ and a cell in $\sigma'$.

# Composition of strategies

- For the copycat sets of cells to be strategies, we require that prefixes of horizontql identities are horizontal identities
- For the composition of two strategies to be a strategy again, we require that prefixes of a composite play is a composite of prefixes

# Strategical double-catégories

A (signed)double-catégorie $\mathfrak{S}$ is said ***strategical*** if its subcategory of even plys satisfies :

- $\mathsf{Pref}_{id}$ : vertical prefixes of horizontal identities are horizontal identities again
- $\mathsf{Win}$ : prefixes of a (horizontally) composite cell is a composite of prefixes.

# The category

$\longrightarrow$ strategies on a strategical double-category $\mathfrak{S}$ yield a category denoted by $\mathcal{S}(\mathfrak{S})$.

$\longrightarrow$ the double-category build out from a game is strategical

# 6. The *HO* game

## $\mathcal{T}$ : the baby game of "ho-timings".

- objects := finite signed trees, initial nodes are positive, father=justifier, justification switches sign.
- parity of an object := parity of its total number of nodes
- 1-timing $_{A \to B}$ := map from $init(A)$ to $init(B)$
- composition of 1-timings := composition of maps
- forward move := addition of a positive node
- backward move := substraction of a negative node

## $\mathcal{T}$ : the baby game of "ho-timings" (2)

- 0-v-morphism := addition of an alternating sequence of nodes
- 1-v-morphism := addition of an alternating/switching sequence of nodes (with the justification for initial nodes on the right).
- h-initial (empty) object.
- 0-v-morphism from the empty timing= skeleton of a 0-game= timing=0-timing.
- empty morphism := identity of empty object.
- 1-v-morphism from the empty morphism= skeleton of a 1-game= 1-timing.

# Arenas and parties

Given a classical arena $A$, and a timing $T$, a *play* in $A$ *scheduled* by $T$ is:

a map from $T$ to $A$ compatible with initiality and justifications (=morphism).

To each classical arena $A$, assign the functor (still denoted $A$) on $\mathcal{T}_v$:

$$T \mapsto A(T) := \text{set of parties in } A \text{ scheduled by } T.$$

We call arena any functor from $\mathcal{T}_v^o$ to Set.

Warning: the functor represented by a timing does not come from a classical arena in general (it selects injective parties).

## Yoneda-upgrading

>From the baby game $\mathcal{T}$, build a new one, $\hat{\mathcal{T}}$.

$\mathbf{y} : \mathcal{T}_v \to \mathcal{A} := \hat{\mathcal{T}}$ the Yoneda embedding (think of arenas).

- object of $\hat{\mathcal{T}}$ := (T,A,p) with $T \in \mathcal{T}$, $A \in \mathcal{A}$, $p \in A(T)$
- parity inherited from $T$
- 0-h-morphisms from $(T, A, p)$ to $(T', A', p')$ := morphism from $T$ to $T'$
- forward move := addition of a positive node to $T$ and corresponding "move" to $p$; $A = A'$.
- backward move := substraction of a negative node from $T$ and corresponding "move" from $p$; $A = A'$.

# HO games (summary).

- start with the baby game $\mathcal{T}$,
- upgrade it through Yoneda,
- take the associated double-cat,
- take the associated cat of strats ,
- restrict it to empty objects $(\emptyset, a, \emptyset)$

you get essentially the $HO$-category of arenas & strategies.