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Augusto C. Ponce. Some elliptic problems with singularities. Mathematics [math]. Université Pierre et Marie Curie - Paris VI, 2004. English. NNT: . tel-00009043

**HAL Id: tel-00009043**

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# Quelques problèmes elliptiques avec singularités

## THÈSE

présentée et soutenue publiquement le lundi 16 février 2004

pour l'obtention du

**Doctorat de l'Université Paris 6 — Pierre et Marie Curie**  
(spécialité Mathématiques Appliquées)

par

Augusto PONCE

### Composition du jury

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Mis en page avec la classe thloria.

« ... *reminiscitur Argos* »  
(*Virgile*)



*À mamie*



## Remerciements

C'est avec une immense joie que j'exprime ma reconnaissance envers mon directeur de thèse, Haïm Brezis. Sa clairvoyance et son enthousiasme ont été pour moi une source inépuisable d'inspiration. Et quel bonheur de pouvoir travailler avec lui !

Je suis extrêmement sensible à l'honneur que m'ont fait Giuseppe Buttazzo et Laurent Véron en acceptant d'être les rapporteurs de cette thèse. Je remercie tout particulièrement M. Véron pour sa lecture approfondie de mon travail, ainsi que pour ses nombreux encouragements.

C'est également avec plaisir que j'exprime mes vifs remerciements à Fabrice Bethuel, Etienne Sandier et Itai Shafrir pour l'intérêt qu'ils ont porté à mon travail. Je remercie aussi Petru Mironescu, avec qui j'ai beaucoup apprécié travailler, et Lucio Boccardo, qui a accepté sans hésitation de faire le voyage en France pour participer au jury.

Je remercie la CAPES pour le soutien financier, et aussi le Département de Mathématiques de l'Université de Rutgers, où une partie de cette thèse a été rédigée.

Je tiens à remercier tout le Laboratoire Jacques-Louis Lions. J'adresse ma gratitude à Mmes Boulic, Loriller et Ruprecht, ainsi qu'à M. David. Un merci tout special à mes collègues thésards et jeunes docteurs : Alexandra, Basarab, David, Delphine, Filipa, Hamdi, Jérôme, Karim, Luis-Manuel, Marcela, Martin, Mourad, Pascal, Paul, Radu, Saïma, Stéphane et Vincent.

Mes remerciements vont aussi à mes *frères* aînés, Juan et Louis, pour tout ce qu'ils m'ont appris, et aussi à Julio, le meilleur ami que j'ai au monde.

Je dois une grande partie de cette thèse aux fantômes de mon passé, à cette galerie de personnes et de choses qui ont fait partie de ma vie. Pendant mes quelques moments de détresse, ils ont su me redonner de l'espoir : « *et dulcis moriens reminiscitur Argos.* »

Cette thèse est l'aboutissement de tout ce que j'ai vu et vécu.





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# Introduction générale

## 1 Avant-propos

Cette thèse est une compilation des articles publiés par l'auteur au cours de son doctorat (cf. [32,33,40,46,70–72]), mais nous expliquons aussi des travaux encore en préparation (cf. [30,34]). Quelques modifications ont été introduites à la rédaction originale de façon à établir un rapport entre les différentes parties. Notre but est d'assurer une présentation unifiée de l'ensemble des résultats. Nous avons, en particulier, décidé de regrouper les diverses références bibliographiques, plutôt que de les mettre à la fin de chaque chapitre.

## 2 Description de la thèse

### 2.1 L'inégalité de Kato lorsque $\Delta u$ est une mesure (en collaboration avec H. Brezis)

L'inégalité de Kato est un outil simple mais très puissant dans l'étude des EDP elliptiques du second ordre (cf. [61]). Parmi ses innombrables formulations, l'une des plus classiques est la suivante :

Étant donnée une fonction  $u \in L^1_{\text{loc}}(B_1)$  telle que  $\Delta u \in L^1_{\text{loc}}(B_1)$ , alors  $\Delta u^+ \in \mathcal{M}_{\text{loc}}(B_1)$  et

$$\Delta u^+ \geq \chi_{[u \geq 0]} \Delta u \quad \text{dans } \mathcal{D}'(B_1), \quad (1)$$

c'est-à-dire,

$$\int_{B_1} u^+ \Delta \varphi \geq \int_{[u \geq 0]} \Delta u \varphi, \quad \forall \varphi \in C_c^\infty(B_1), \text{ avec } \varphi \geq 0 \text{ sur } B_1.$$

Nous rappelons que  $\mu \in \mathcal{M}_{\text{loc}}(B_1)$  si et seulement si, pour tout  $\omega \subset\subset B_1$ , il existe  $C_\omega > 0$  tel que

$$\left| \int_{B_1} \varphi d\mu \right| \leq C_\omega \|\varphi\|_{L^\infty}, \quad \forall \varphi \in C_c^\infty(\omega).$$

La condition  $\Delta u \in L^1_{\text{loc}}(B_1)$  impose des restrictions qui ne sont pas naturelles dans plusieurs problèmes que nous avons étudiés, puisque en général  $\Delta u$  est seulement une mesure de Radon. Il nous faut donc étendre (1) lorsque  $\Delta u \in \mathcal{M}_{\text{loc}}(B_1)$ .

D'abord, nous observons que si  $\Delta u \in \mathcal{M}_{\text{loc}}(B_1)$ , alors  $\Delta u^+ \in \mathcal{M}_{\text{loc}}(B_1)$ . En effet, il suffit de montrer que

$$\Delta u^+ \geq -(\Delta u)^- \quad \text{dans } \mathcal{D}'(B_1).$$



Or, cette inégalité s'établit très facilement à partir de l'inégalité de Kato classique, en utilisant un argument de densité.

Ensuite, toute mesure de Radon  $\mu$  s'écrit sous la forme  $\mu = \mu_d + \mu_c$ , où  $\mu_d$  ne charge pas les ensembles de capacité (newtonienne) nulle et  $\mu_c$  est une mesure concentrée sur un ensemble de capacité zéro.

Soit  $u \in L^1_{\text{loc}}(B_1)$  tel que  $\Delta u \in \mathcal{M}_{\text{loc}}(B_1)$ . Dans ce cas, il est possible de montrer que l'inégalité (1) doit être remplacée par :

$$(\Delta u^+)_d \geq \chi_{[u \geq 0]}(\Delta u)_d \quad \text{sur } B_1, \quad (2)$$

$$(-\Delta u^+)_c = (-\Delta u)_c^+ \quad \text{sur } B_1. \quad (3)$$

(En vue du Lemme 2.1 dans le Chapitre 2, la fonction  $u$  est bien définie sauf sur les ensembles de capacité nulle; en particulier, le produit  $\chi_{[u \geq 0]}(\Delta u)_d$  a bien un sens.)

Dans les Chapitres 1–3, nous établissons quelques cas particuliers de (2)–(3). La preuve de (2)–(3), par contre, est assez délicate, et nous ne la présentons pas ici. Le lecteur en trouvera la démonstration dans un article de Brezis et Ponce [34]. Les deux ingrédients principaux sont le principe du maximum « inverse » de Dupaigne et Ponce [46] (voir le paragraphe 2.3 dans cette introduction) et la caractérisation des mesures diffuses par Boccardo, Gallouët et Orsina [15, Théorème 2.1].

Notre travail prend sa source dans un article très intéressant d'Ancona [4] et dans une question qui nous a été posée par Y. Li.

## 2.2 Singularités éliminables (en collaboration avec J. Dávila et L. Dupaigne)

Comme première application de l'inégalité de Kato que nous venons de présenter, nous étudions le problème des singularités éliminables des EDP elliptiques du second ordre. Pour une introduction à ce sujet passionnant, voir le livre de Véron [86].

Un résultat classique dans cette direction est le suivant (cf. [59]) :  
Soient  $K \subset B_1$  un ensemble compact et  $u \in L^1_{\text{loc}}(B_1 \setminus K)$ , avec  $u \geq 0$  p.p. Supposons que

$$-\Delta u \geq 0 \quad \text{dans } \mathcal{D}'(B_1 \setminus K).$$

Si la capacité newtonienne de  $K$  est nulle, alors  $u \in L^1_{\text{loc}}(B_1)$  et

$$-\Delta u \geq 0 \quad \text{dans } \mathcal{D}'(B_1).$$

Nous observons d'abord que nous ne faisons aucune hypothèse sur le comportement de  $u$  au voisinage de  $K$ . Malgré tout, le théorème ci-dessus nous dit que  $u$  a une certaine régularité, à condition que l'ensemble  $K$  soit suffisamment petit.

Dans le premier chapitre, nous étudions ce résultat en détail lorsque  $K$  est une variété de codimension supérieure ou égale à 2. Il est possible alors d'obtenir des formules très intéressantes pour la restriction du laplacien de  $u$  sur  $K$ . Cette méthode a quelques inconvénients. D'abord, elle est limitée à des ensembles  $K$  qui sont très réguliers. Puis, lorsque  $K$  n'est pas une variété régulière, mais juste un compact, il n'est pas clair en quel sens  $K$  doit être petit.

Nous reprenons les singularités éliminables dans le Chapitre 3, où nous étudions le problème dans le cadre général, à savoir sans l'hypothèse de régularité sur l'ensemble  $K$ , et même pour des opérateurs quasi-linéaires au lieu du laplacien. En utilisant les inégalités de Kato et de Harnack, nous étendons un lemme bien connu de Brezis et Lions [29].

### 2.3 Le principe du maximum « inverse » (en collaboration avec L. Dupaigne)

Selon le principe du maximum classique, si  $u : \bar{B}_1 \rightarrow \mathbb{R}$  est une fonction régulière telle que  $-\Delta u \geq 0$  sur  $B_1$  et si  $u \geq 0$  sur  $\partial B_1$ , alors  $u \geq 0$  partout. Dans le Chapitre 3, nous étudions en quelque sorte la réciproque de ce résultat, qui a été conjecturée par H. Brezis et M. Marcus. Voici d'abord un exemple :

Soit  $(a_i)$  une suite de points distincts dans  $\mathbb{R}^3$ . Étant donnée une suite  $(\alpha_i) \subset \mathbb{R}$  telle que  $\sum_i |\alpha_i| < \infty$ , nous définissons

$$u(x) = \sum_i \frac{\alpha_i}{|x - a_i|} \quad \text{pour presque tout } x \in \mathbb{R}^3.$$

Il n'est pas difficile de montrer que si  $u \geq 0$  p.p. sur  $\mathbb{R}^3$ , alors  $\alpha_i \geq 0$ ,  $\forall i \geq 1$ .

Une façon équivalente d'énoncer le même problème est la suivante :

Soit  $u \in L^1_{\text{loc}}(\mathbb{R}^3)$  tel que

$$-\Delta u = \sum_i \alpha_i \delta_{a_i} \quad \text{dans } \mathcal{D}'(\mathbb{R}^3).$$

Si  $u \geq 0$  p.p. sur  $\mathbb{R}^3$ , alors

$$-\Delta u \geq 0 \quad \text{dans } \mathcal{D}'(\mathbb{R}^3).$$

Notre résultat suivant met en évidence le principe général sous-jacent dans l'exemple ci-dessus :

**Principe du maximum « inverse ».** Soit  $u \in L^1_{\text{loc}}(\mathbb{R}^N)$  tel que  $\Delta u \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$ . Si  $u \geq 0$  p.p. sur  $\mathbb{R}^N$ , alors

$$(-\Delta u)_c \geq 0 \quad \text{sur } \mathbb{R}^N. \quad (4)$$

Nous rappelons que  $(-\Delta u)_c$  est la partie de  $(-\Delta u)$  concentrée sur un ensemble de capacité nulle.

En utilisant le principe du maximum « inverse » et l'inégalité de Kato (2)–(3), Brezis, Marcus et Ponce [30] ont étudié l'existence des solutions du problème elliptique non linéaire :

$$\begin{cases} -\Delta u + g(u) = \mu & \text{sur } B_1, \\ u = 0 & \text{sur } \partial B_1, \end{cases}$$

où  $\mu \in \mathcal{M}(B_1)$  et  $g : \mathbb{R} \rightarrow \mathbb{R}$  est une fonction croissante et continue, avec  $g(0) = 0$ . Voir à ce sujet le Théorème 3.3 dans le Chapitre 3.

## 2.4 Le principe du maximum fort (en collaboration avec H. Brezis)

Nous présentons dans le deuxième chapitre une autre application de l'inégalité de Kato : le principe du maximum fort pour l'opérateur  $-\Delta + a(x)$ , avec un potentiel  $a \in L^1(B_1)$ .

Plus précisément, soit  $u \in L^1_{\text{loc}}(B_1)$ ,  $u \geq 0$  p.p. sur  $B_1$ , tel que  $\Delta u \in L^1_{\text{loc}}(B_1)$  et

$$-\Delta u + au \geq 0 \quad \text{p.p. sur } B_1. \quad (5)$$

Nous démontrons que si  $u = 0$  sur un ensemble de mesure positive, alors  $u = 0$  p.p. sur  $B_1$ .

Un cas particulier de ce théorème (lorsque  $u$  est à support compact) a été utilisé par Bénilan et Brezis [6] dans l'étude du problème de Thomas-Fermi.

Rappelons que le principe du maximum classique affirme que si le potentiel  $a$  appartient à  $L^p$ , avec  $p > \frac{N}{2}$ , et si  $u = 0$  en un point, alors  $u \equiv 0$  sur  $B_1$ . Cette version tombe en défaut dans notre situation : la fonction  $u$  peut avoir des zéros à l'intérieur du domaine sans être identiquement nulle. Par exemple, la condition (5) est vérifiée avec  $u(x) = |x|^2$  et  $a(x) = \frac{2N}{|x|^2}$ .

Ce résultat a été établi par Ancona [4] en utilisant des outils de la théorie du potentiel. Nous présentons une démonstration « style EDP ».

## 2.5 Une nouvelle caractérisation des espaces de Sobolev

Dans [20], Bourgain, Brezis et Mironescu ont démontré que

$$\lim_{n \rightarrow \infty} \int_{B_1} \int_{B_1} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy = K_{p,N} \int_{B_1} |\nabla f|^p, \quad \forall f \in W^{1,p}(B_1), \quad (6)$$

où  $(\rho_n) \subset L^1(\mathbb{R}^N)$  est une suite de fonctions radiales positives qui convergent vers la masse de Dirac  $\delta_0$ , et  $K_{p,N}$  est une constante géométrique qui dépend seulement de  $p$  et de la dimension  $N$  de l'espace.

Motivés par ce résultat, nous établissons l'inégalité (voir le Chapitre 4) :

$$\int_{B_1} |f - f_{B_1}|^p \leq C \int_{B_1} \int_{B_1} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy, \quad \forall f \in L^p(B_1), \quad \forall n \geq n_0, \quad (7)$$

valable en toute dimension supérieure ou égale à 2. Cette estimation peut être déduite d'un théorème de compacité dans [20], sous l'hypothèse que les fonctions  $\rho_n$  soient radiales **décroissantes**. Nous montrons que l'inégalité (7) reste vrai sans cette dernière condition. Ce résultat est assez inattendu, car dans [20] les auteurs ont construit un contre-exemple en dimension 1 pour des fonctions  $\rho_n$  qui n'étaient pas décroissantes.

En prenant la limite dans (7) lorsque  $n$  tend vers  $\infty$ , nous retrouvons l'inégalité de Poincaré classique :

$$\int_{B_1} |f - f_{B_1}|^p \leq A_0 \int_{B_1} |\nabla f|^p, \quad \forall f \in W^{1,p}(B_1).$$

Dans le Chapitre 5, nous étendons (6) dans plusieurs directions. Nous étudions cette limite, par exemple, lorsque les fonctions  $\rho_n$  ne sont plus radiales. En utilisant des résultats bien connus de la théorie de la relaxation (voir à ce sujet le livre de Buttazzo [35]), il est possible de démontrer que (6) reste vrai au sens de la  $\Gamma$ -convergence.

## 2.6 Singularités topologiques des applications dans $W^{1,1}(S^2; S^1)$ (en collaboration avec H. Brezis et P. Mironescu)

Notre point de départ est l'article de Bourgain, Brezis et Mironescu [22] sur le problème de minimisation dans  $\mathbb{R}^3$  :

$$\text{Min}_{u \in H_g^1(B^3; S^1)} \int_{B^3} |\nabla u|^2, \quad (8)$$

avec  $g : S^2 \rightarrow S^1$ .

L'analogie de ce problème en 2-d a été étudié par Bethuel, Brezis et Hélein [11], et consiste à minimiser la fonctionnelle d'énergie

$$\int_{B^2} |\nabla u|^2$$

sur toute les fonctions  $u \in H^1(B^2; S^1)$  telles que  $u = g$  sur  $S^1$ , avec  $g : S^1 \rightarrow S^1$  régulier. Or, lorsque  $\deg g \neq 0$ , ce dernier problème de minimisation est impossible, car  $H_g^1(B^2; S^1) = \emptyset$ . En effet, la condition  $\deg g \neq 0$  impose des singularités du type  $\frac{x}{|x|}$  à l'intérieur du domaine, qui ont une énergie  $H^1$  infinie (voir, e.g., les articles de Sandier [75] et de Han et Shafrir [58]). Pour éviter cette obstruction topologique, Bethuel, Brezis et Hélein ont élargi la classe de fonctions admissibles en prenant  $H_g^1(B^2; \mathbb{C})$ . Par contre, ils ont introduit un terme de pénalisation pour compenser le fait que ces fonctions ne prennent plus leurs valeurs dans  $S^1$ . Le problème devient alors de minimiser la fonctionnelle de Ginzburg-Landau :

$$\text{Min}_{u \in H_g^1(B^2; \mathbb{C})} \left\{ \int_{B^2} |\nabla u|^2 + \frac{1}{2\varepsilon^2} \int_{B^2} (1 - |u|^2)^2 \right\},$$

qui a toujours une solution  $u_\varepsilon$ . Dans [11], les auteurs ont étudié le comportement des suites  $(u_{\varepsilon_j})$  lorsque  $\varepsilon_j \downarrow 0$ .

En 3-d, la situation change complètement. En effet, pour toute fonction  $g : S^2 \rightarrow S^1$  **régulière**, l'espace  $H_g^1(B^3; S^1)$  est non vide. Le problème (8) a donc toujours une solution.

De façon à retrouver la même obstruction topologique qu'en dimension 2, F. Bethuel a suggéré qu'il fallait prendre des données au bord  $g$  ayant des singularités de la forme  $\frac{z-a}{|z-a|}$  (modulo rotations) au voisinage de  $a \in S^2$ . Or, en utilisant la théorie du degré pour des fonctions  $H^{1/2}$  (cf. Brezis et Nirenberg [23]), il est possible de démontrer que pour une telle fonction  $g$  nous avons  $H_g^1(B^3; S^1) = \emptyset$ . Le problème (8) devient alors impossible, exactement comme avant.

Dans le cas où  $g$  est régulière sauf en un nombre fini de points, le comportement des minimiseurs de la fonctionnelle de Ginzburg-Landau en 3-d, lorsque  $\varepsilon \downarrow 0$ , a été étudié

par Rivière [74] (voir aussi Lin et Rivière [64]). Bourgain, Brezis et Mironescu [22] ont considéré le cas général lorsque  $g \in H^{1/2}(S^2; S^1)$ . Ce problème est très difficile et passe par une analyse des singularités topologiques de  $g$ .

Rappelons que  $W^{1,1}(S^2; S^1)$  n'est pas inclus dans  $H^{1/2}(S^2; S^1)$  (voir, e.g., [18]). Malgré tout, ces deux espaces ont plusieurs propriétés en commun. Nous observons que toute application  $g \in W^{1,1}(S^2; S^1)$  a un déterminant jacobien  $\text{Det}(\nabla g)$ , bien défini au sens des distributions. En utilisant un résultat de densité de Bethuel et Zheng [12], il est possible de montrer que  $\text{Det}(\nabla g)$  s'écrit toujours sous la forme

$$\text{Det}(\nabla g) = \pi \sum_{j=1}^{\infty} (\delta_{p_j} - \delta_{n_j}) \quad \text{dans } \mathcal{D}'(S^2),$$

pour des points  $p_j, n_j \in S^2$  tels que  $\sum_j d(p_j, n_j) < \infty$ . La longueur de la connexion minimale entre les singularités positives et négatives de  $g$  est définie par

$$L(g) = \frac{1}{\pi} \sup_{|\zeta|_{\text{Lip}} \leq 1} \langle \text{Det}(\nabla g), \zeta \rangle.$$

(Cette définition prend sa source dans un article de Brezis, Coron et Lieb [28].)

Dans le Chapitre 6, nous étudions plusieurs propriétés de  $g$ . Nous démontrons, par exemple, que

$$\min_{\substack{\varphi \in BV(S^2; \mathbb{R}) \\ g = e^{i\varphi} \text{ sur } S^2}} \int_{S^2} |D\varphi| = \int_{S^2} |\nabla g| + 2\pi L(g). \quad (9)$$

L'inégalité  $\leq$  était déjà connue par Demengel et Hadiji [44] (voir aussi [54]) pour des fonctions dans  $W^{1,1}(B^2; S^1)$ . Observons que le membre de gauche dans (9) coïncide avec l'énergie relaxée de  $g$  :

$$E_{\text{rel}}(g) = \text{Inf} \left\{ \liminf_{n \rightarrow \infty} \int_{S^2} |\nabla g_n| ; g_n \in C^\infty(S^2; S^1) \text{ et } g_n \rightarrow g \text{ p.p.} \right\}.$$

Cette notion a été introduite par Bethuel, Brezis et Coron [10] dans leur étude des applications  $H^1(B^3; S^2)$ . L'énergie relaxée joue aussi un rôle très important dans le cadre des courants cartésiens (voir les livres de Giaquinta, Modica et Souček [53, 54]).

Comme corollaire de (9), nous avons l'inégalité

$$\int_{S^2} |D\varphi| \leq 2 \int_{S^2} |\nabla g|, \quad (10)$$

valable pour toute fonction  $\varphi \in BV(S^2; \mathbb{R})$  telle que  $g = e^{i\varphi}$  sur  $S^2$ . Dávila et Ignat [39] ont récemment démontré que l'estimation (10) reste vraie même pour des applications  $g$  dans  $BV$  à valeurs dans  $S^1$ .

Motivés par les travaux de Fonseca, Fusco et Marcellini [50], et aussi de Giaquinta, Modica et Souček [52], nous considérons ensuite le jacobien relaxé de  $g \in W^{1,1}(S^2; S^1)$  :

$$TV(g) = \text{Inf} \left\{ \liminf_{n \rightarrow \infty} \int_{S^2} |\det(\nabla g_n)| ; g_n \in C^\infty(S^2; \mathbb{R}^2) \text{ et } g_n \rightarrow g \text{ dans } W^{1,1} \right\}.$$

Alors, nous montrons que  $TV(g) < \infty$  si et seulement si  $g$  a un nombre fini de singularités. Dans ce cas,

$$\frac{1}{\pi}TV(g) = \text{nombre de singularités topologiques de } g \text{ (multiplicité comprise)}.$$

## 2.7 Les distributions de la forme $\sum_j (\delta_{p_j} - \delta_{n_j})$

Les résultats du paragraphe précédent mettent en lumière l'importance d'étudier les distributions de la forme

$$T = \sum_{j=1}^{\infty} (\delta_{p_j} - \delta_{n_j}) \quad \text{dans } [\text{Lip}(X)]^*, \quad (11)$$

où  $X$  est un espace métrique complet et  $\sum_j d(p_j, n_j) < \infty$ . Plus précisément,  $T$  est donné par

$$\langle T, \zeta \rangle = \sum_{j=1}^{\infty} [\zeta(p_j) - \zeta(n_j)], \quad \forall \zeta \in \text{Lip}(X).$$

La longueur de la connexion minimale de  $T$  est définie comme avant :

$$L = \sup_{|\zeta|_{\text{Lip}} \leq 1} \langle T, \zeta \rangle. \quad (12)$$

Bourgain, Brezis et Mironescu [22] ont montré que  $L$  admet une caractérisation duale :

$$L = \inf_{\substack{(\tilde{p}_j) \\ (\tilde{n}_j)}} \left\{ \sum_j d(\tilde{p}_j, \tilde{n}_j) ; T = \sum_j (\delta_{\tilde{p}_j} - \delta_{\tilde{n}_j}) \text{ dans } [\text{Lip}(X)]^* \right\}. \quad (13)$$

Alors que le supremum dans (12) est toujours atteint, cela n'est pas le cas avec l'infimum dans (13). Par contre, nous démontrons que si  $\mathcal{H}^1(\text{supp } T) = 0$ , alors il existe deux suites  $(\tilde{p}_j)$ ,  $(\tilde{n}_j)$  telles que

$$T = \sum_j (\delta_{\tilde{p}_j} - \delta_{\tilde{n}_j}) \quad \text{et} \quad L = \sum_j d(\tilde{p}_j, \tilde{n}_j).$$

Nous montrons aussi que  $T$  est une mesure, c'est-à-dire qu'il existe  $C > 0$  tel que

$$|\langle T, \zeta \rangle| \leq C \|\zeta\|_{L^\infty}, \quad \forall \zeta \in \text{Lip}(X),$$

si et seulement si  $T$  s'écrit comme une somme finie de dipôles. Ce théorème a été établi par Smets [80], sous l'hypothèse supplémentaire que  $X$  soit localement compact.

Les résultats ci-dessus sont une conséquence de l'existence d'une représentation irréductible de  $T$ , une notion que nous introduisons dans le Chapitre 7.



# Chapitre 1

## Variants of Kato's inequality and removable singularities

### Sommaire

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### 1.1 Introduction<sup>1</sup>

The original motivation for this chapter is the following remark, which is related to Kato's inequality (see [61]). First, let us recall one of its many versions. Consider  $\Omega \subset \mathbb{R}^N$  an open set, and  $v \in L^1(\Omega)$  such that  $\Delta v \in L^1(\Omega)$ ; then,

$$\Delta|v| \geq \text{sign}(v) \Delta v \quad \text{in } \mathcal{D}'(\Omega), \quad (1.1)$$

where  $\text{sign}(s) = 1$  if  $s > 0$ ,  $-1$  if  $s < 0$ , and zero at  $s = 0$ . If we assume in addition that  $v$  is continuous in  $\Omega$ , then it is easy to verify that

$$\Delta|v| = \text{sign}(v) \Delta v \quad \text{in } \mathcal{D}'([v \neq 0]). \quad (1.2)$$

Comparison between (1.1) and (1.2) suggests that the inequality in (1.1) should be a consequence of the fact that  $|v|$  achieves its minimum on the set  $[v = 0]$ , where one has  $\Delta|v| \geq 0$  in a suitable sense.

Motivated by this fact, we proved the following theorem, which gives a positive answer to a question raised by Y. Li. Here and throughout this chapter, we denote by  $\Omega$  an open bounded subset of  $\mathbb{R}^N$ , with  $N \geq 1$ .

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<sup>1</sup>Ce chapitre a été écrit en collaboration avec J. Dávila; le texte original est déjà sorti dans J. Anal. Math. **91** (2003), 143–178.



**Theorem 1.1** *Suppose  $u \in C(\bar{\Omega})$  is such that  $u \geq 0$  in  $\Omega$ , and  $u = 0$  on a compact set  $K \subset \Omega$ . Assume, in addition, that*

$$\Delta u \geq g \quad \text{in } \mathcal{D}'(\Omega \setminus K), \quad (1.3)$$

for some  $g \in L^1(\Omega)$ . Then,

$$\Delta u \geq g \chi_{\Omega \setminus K} \quad \text{in } \mathcal{D}'(\Omega); \quad (1.4)$$

in other words,

$$\int_{\Omega} u \Delta \varphi \geq \int_{\Omega \setminus K} g \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega. \quad (1.5)$$

We would like to emphasize that we do not make any assumptions on the size of the zero set  $K$ ; we do not assume either that  $\Delta u \in L^1_{\text{loc}}(\Omega \setminus K)$ .

A consequence of this theorem is that  $\Delta u + |g|$  is a nonnegative distribution, from which we conclude that  $\Delta u$  is a Radon measure on  $\Omega$  (see [78]). Since  $u$  is uniformly bounded on  $\Omega$ , we have  $u \in H^1_{\text{loc}}(\Omega)$  (see [7]; see also Chapter 3). Theorem 1.1 can be viewed in this way as a removable singularity result. Another possible interpretation, in the spirit of Kato's inequality, is the following :

Assume that equality holds in (1.3); that is, suppose that

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus K). \quad (1.6)$$

We can now define the distribution

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega), \quad (1.7)$$

so that  $\text{supp } \mu \subset K$ . Then, according to Theorem 1.1,  $\mu$  is a nonnegative Radon measure concentrated on  $K \subset [u = 0]$ .

As an example, let  $u(x) := \frac{1}{2}|x_N|$  for  $x \in \mathbb{R}^N$ . Then,

$$\Delta u = \mathcal{H}^{N-1} \llcorner_{[x_N=0]} \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

where  $\mathcal{H}^{N-1}$  denotes the  $(N-1)$ -dimensional Hausdorff measure.

We also notice that if (1.6) holds and  $K$  is sufficiently small, then we can actually conclude that  $\mu = 0$  in  $\Omega$ . In order to make this precise, we first recall that the compact set  $K \subset \Omega$  has zero  $H^1$ -capacity, which we denote by  $\text{cap}_2(K) = 0$ , if there exists a sequence  $(\varphi_n) \subset C_0^\infty(\Omega)$  such that  $\varphi_n \geq 1$  in some neighborhood of  $K$  and

$$\int_{\Omega} |\nabla \varphi_n|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In particular, it follows from Poincaré's inequality that  $|K| = 0$ . See Section 3.2 for other properties of  $K$ .

We then have the following corollary of Theorem 1.1 :

**Corollary 1.1** *Under the hypotheses of Theorem 1.1, suppose in addition that (1.6) holds. If  $\text{cap}_2(K) = 0$ , then*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega). \quad (1.8)$$

A slightly stronger version of Theorem 1.1, without the continuity assumption on  $u$ , has been studied by Dávila and Ponce [40]. In their case<sup>2</sup>, the function  $u$  belongs to  $L^1(\Omega)$ , and the condition  $u = 0$  on  $K$  is to be understood in the sense that

$$\lim_{r \downarrow 0} \frac{1}{r^N} \int_{N_r(K)} u = 0, \quad (1.9)$$

where  $N_r(K)$  denotes the neighborhood of  $K$  of radius  $r$ . This hypothesis is probably too strong, but it is still unclear how to weaken it in this general setting.

In the sequel, we shall restrict our attention to singular sets  $K$  which are smooth manifolds of  $\mathbb{R}^N$ , without boundary, having codimension  $k \geq 1$ . For this reason, we shall use the more natural notation  $M$ , instead of  $K$ . It turns out that, in this special case, we can get explicit formulas for  $\mu$  in terms of the mean value of  $u$  on tubular neighborhoods of  $M$ . As we shall see, the cases  $k = 1$  and  $k \geq 2$  have completely different features, the main reason for that being that manifolds of codimension  $k \geq 2$  always have zero  $H^1$ -capacity (see Corollary 3.4).

In the case where  $K = M \subset \Omega$  is a smooth manifold of codimension 1, we have been able to relax (1.9) by assuming that  $u \in L^1(\Omega)$  satisfies

$$\lim_{r \downarrow 0} \int_{\Xi_r} u = 0,$$

where  $\Xi_r = \Xi_r(M)$  is the tubular neighborhood of  $M$  of radius  $r$ . In other words, for such singular sets, one can replace the factor  $\frac{1}{r^N}$  in (1.9) by  $\frac{1}{r}$ , and still get the same conclusion of Theorem 1.1. More precisely,

**Theorem 1.2** *Suppose  $M \subset \Omega$  has codimension 1. Let  $u \in L^1_{\text{loc}}(\Omega)$ , and assume that there exists  $g \in L^1_{\text{loc}}(\Omega)$  such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

If

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\Xi_r} |u| = 0, \quad (1.10)$$

then, for each  $\varphi \in C_0^\infty(\Omega)$ ,  $\frac{1}{r^2} \int_{\Xi_r} u\varphi$  converges as  $r \downarrow 0$ , and

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u\varphi = \frac{1}{2} \int_{\Omega} (u\Delta\varphi - g\varphi). \quad (1.11)$$

In particular, if we suppose in addition that  $u \geq 0$  a.e. in  $\Omega$ , then

$$\Delta u \geq g \quad \text{in } \mathcal{D}'(\Omega). \quad (1.12)$$

---

<sup>2</sup>Même si ce chapitre est basé sur l'article [40], nous avons décidé d'introduire quelques modifications au texte original. Une version assez différente du Théorème 1.1 y est énoncé. Celui que nous présentons ici met en évidence le rapport entre les Chapitres 1 et 3. Tous les autres résultats restent inchangés.

**Remark 1.1** We conclude *a posteriori* from (1.12) that  $u \in W_{\text{loc}}^{1,p}(\Omega)$  for  $1 \leq p < \frac{N}{N-1}$  (see [7]; see also Corollary 3.2). Thus, condition (1.10) is actually equivalent to  $u = 0$  in  $M$  in the sense of traces.

We next study the case where the singular set  $M$  is a compact manifold of codimension  $k \geq 2$ . It turns out that, in this case, the condition  $u \geq 0$  a.e. in  $\Omega$  already suffices to conclude that  $-\Delta u$  is a (nonnegative) measure on  $M$ . More precisely, we have

**Theorem 1.3** *Suppose  $M \subset \Omega$  has codimension  $k \geq 2$ . Let  $u \in L_{\text{loc}}^1(\Omega)$ ,  $u \geq 0$  a.e. in  $\Omega$ , and assume there exists  $g \in L_{\text{loc}}^1(\Omega)$  such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega), \tag{1.13}$$

which is a distribution supported on  $M$ .

Then,

$$\mu \text{ is a nonpositive measure on } M \tag{1.14}$$

and, for any  $\varphi \in C_0^\infty(\Omega)$ , we have

$$\langle \mu, \varphi \rangle = \begin{cases} -2(k-2) \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u \varphi & \text{if } k \geq 3, \\ -2 \lim_{r \downarrow 0} \frac{1}{r^2 |\log r|} \int_{\Xi_r} u \varphi & \text{if } k = 2. \end{cases} \tag{1.15}$$

A classical result in Potential Theory states that (1.14) still holds if  $M$  is replaced by a compact set of zero capacity  $K$  (see, e.g., [59, Theorem 7.7]). We shall return to this problem later, in Chapter 3, where we extend these results in greater generality. We present in Section 1.5 a completely independent proof of (1.14) in our special case, in order to deduce (1.15).

Even if we do not assume any conditions on the sign of  $u$ , we can still characterize the case where  $\mu$  is a measure in terms of the growth of  $|u|$  near  $M$ . More precisely, we have the following

**Theorem 1.4** *Suppose  $M \subset \Omega$  has codimension  $k \geq 3$ . Let  $u \in L_{\text{loc}}^1(\Omega)$  (here we do **not** assume that  $u \geq 0$  a.e. in  $\Omega$ ), and suppose there exists  $g \in L_{\text{loc}}^1(\Omega)$  such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega).$$

Then,  $\mu$  is a Radon measure if, and only if,

$$\frac{1}{r^2} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0. \tag{1.16}$$

In this case, for all  $\varphi \in C_0^\infty(\Omega)$  we have

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u \varphi \quad \text{exists and equals} \quad - \frac{1}{2(k-2)} \langle \mu, \varphi \rangle. \quad (1.17)$$

Moreover,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u| \quad \text{exists and equals} \quad \frac{1}{2(k-2)} \|\mu\|_{\mathcal{M}}, \quad (1.18)$$

where

$$|\mu|_{\mathcal{M}} := \sup \left\{ \int_M w d\mu : w \in C(M), \|w\|_\infty \leq 1 \right\}$$

denotes the usual norm of Radon measures on  $M$ .

**Remark 1.2** Using a formula deduced in Section 1.3, it is possible to show that (1.17) still holds if one replaces (1.16) by

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\Xi_r} |u| = 0. \quad (1.19)$$

On the other hand, if one takes for instance the function  $u(x) = \frac{x_1}{|x|^3}$  in  $\mathbb{R}^3 \setminus \{0\}$ , then  $\Delta u = c \partial_1 \delta_0$  for some constant  $c \neq 0$ . In the notation of Theorem 1.4, let  $M := \{0\}$ ,  $g := 0$ , and  $\mu := c \partial_1 \delta_0$ , so that  $\mu$  is a distribution of order 1 and

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{B_r} |u| > 0.$$

We do not know if the condition (1.19) implies that  $\mu$  is a measure.

There is also a result analogous to Theorem 1.4 in the case of codimension  $k = 2$  :

**Theorem 1.5** *Suppose  $M \subset \Omega$  has codimension  $k = 2$ . Let  $u \in L_{\text{loc}}^1(\Omega)$  (here we do **not** assume that  $u \geq 0$  a.e. in  $\Omega$ ), and suppose there exists  $g \in L_{\text{loc}}^1(\Omega)$  such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega).$$

Then,  $\mu$  is a Radon measure if, and only if,

$$\frac{1}{r^2 |\log r|} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0.$$

In this case, for all  $\varphi \in C_0^\infty(\Omega)$  we have

$$\lim_{r \downarrow 0} \frac{1}{r^2 |\log r|} \int_{\Xi_r} u \varphi \quad \text{exists and equals} \quad - \frac{1}{2} \langle \mu, \varphi \rangle.$$

Moreover,

$$\lim_{r \downarrow 0} \frac{1}{r^2 |\log r|} \int_{\Xi_r} |u| \quad \text{exists and equals} \quad \frac{1}{2} |\mu|_{\mathcal{M}}.$$

As a consequence of Theorems 1.4 and 1.5, we have :

**Corollary 1.2** *Under the assumptions of Theorem 1.4 (resp. Theorem 1.5 for  $k = 2$ ), we have  $\Delta u \in L^1_{\text{loc}}(\Omega)$  if, and only if,*

$$\begin{aligned} \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u| &= 0, \quad \text{for } k \geq 3, \\ \lim_{r \downarrow 0} \frac{1}{r^2 |\log r|} \int_{\Xi_r} |u| &= 0, \quad \text{for } k = 2. \end{aligned}$$

Applying Corollary 1.2, we obtain the following well-known result about removable singularities :

**Theorem 1.6** *Assume  $M \subset \Omega$  has codimension  $k \geq 3$ . Let  $c \in \mathbb{R}$ , and  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous function satisfying*

$$\liminf_{t \rightarrow \infty} \frac{g(t)}{t^{\frac{k}{k-2}}} > 0. \quad (1.20)$$

*Let  $u \in L^1_{\text{loc}}(\Omega \setminus M)$ ,  $u \geq 0$  a.e. in  $\Omega$ , be such that  $g(u) \in L^1_{\text{loc}}(\Omega \setminus M)$  and*

$$-\Delta u + cu = g(u) \quad \text{in } \mathcal{D}'(\Omega \setminus M). \quad (1.21)$$

*Then,  $u, g(u) \in L^1_{\text{loc}}(\Omega)$  and*

$$-\Delta u + cu = g(u) \quad \text{in } \mathcal{D}'(\Omega). \quad (1.22)$$

Theorem 1.6 was originally proved by Véron [85], and it was further extended by Baras and Pierre [5]. It turns out that condition (1.20) is related to the fact that  $M^{N-k}$  has zero  $W^{2,k/2}$ -capacity; see the book of Véron [86] for details and other results on removable singularities.

**Remark 1.3** The above result may seem misleading at first. Assume, for instance, that  $k \geq 3$ ,  $c = 0$ , and  $g(t) = t^{\frac{k}{k-2}}$ . Although Theorem 1.6 implies

$$-\Delta u = u^{\frac{k}{k-2}} \quad \text{in } \mathcal{D}'(\Omega), \quad (1.23)$$

one cannot conclude solely from this equation that  $u$  is smooth. Theorem 1.6 only asserts that the singularities of  $u$  are not detectable in the distributional level. In fact, let  $M \subset \Omega$  be a compact manifold of codimension  $k$ . In a very interesting paper of Mazzeo and Pacard [69], they have been able to construct nonnegative functions  $u$  satisfying (1.23), which are singular precisely on  $M$ .

**Remark 1.4** Theorem 1.6 also holds for  $k = 2$ , as proved by Vázquez and Véron [84]. In this case, assumption (1.20) should be replaced by

$$\liminf_{t \rightarrow \infty} \frac{g(t)}{e^{at}} > 0, \quad \forall a > 0. \quad (1.24)$$

For the convenience of the reader, we shall give a proof of Theorem 1.6 and Remark 1.4, based on Corollary 1.2 above.

## 1.2 Proof of Theorem 1.1

We first prove the following variant of Kato's inequality :

**Lemma 1.1** *Let  $\omega \subset \mathbb{R}^N$  be a bounded open set and  $h \in L^1(\omega)$ . If  $v \in L^1(\omega)$  satisfies*

$$\Delta v \geq h \quad \text{in } \mathcal{D}'(\omega),$$

then

$$\Delta v^+ \geq h\chi_{[v \geq 0]} \quad \text{in } \mathcal{D}'(\omega).$$

In order to avoid any ambiguity regarding the definition of the set  $[v \geq 0]$ , we can assume that  $v$  itself is the precise representative of the equivalence class of  $v$  in  $L^1(\omega)$  (see, e.g., [47]). This way,  $[v \geq 0]$  is well-defined except for a null set. Also notice that we cannot directly apply Kato's inequality here since we do not assume that  $\Delta v$  is an  $L^1$ -function.

**Proof of Lemma 1.1.** Take  $\rho \in C_0^\infty(B_1)$  such that  $\rho \geq 0$  in  $\mathbb{R}^N$  and  $\int_{\mathbb{R}^N} \rho = 1$ . For any  $\varepsilon > 0$ , define  $\rho_\varepsilon(x) := \varepsilon^{-N} \rho(x/\varepsilon)$  on  $\mathbb{R}^N$  and  $v_\varepsilon := \rho_\varepsilon * v$ . Let  $V \subset\subset \omega$  and  $\delta > 0$ . For  $\varepsilon > 0$  small enough, we have

$$\Delta v_\varepsilon \geq h_\varepsilon \quad \text{in } \mathcal{D}'(V).$$

By the standard Kato's inequality, we get

$$\Delta(v_\varepsilon + \delta)^+ \geq h_\varepsilon \chi_{[v_\varepsilon + \delta \geq 0]} \quad \text{in } \mathcal{D}'(V).$$

In other words,

$$\int_V (v_\varepsilon + \delta)^+ \Delta \varphi \geq \int_V h_\varepsilon \chi_{[v_\varepsilon \geq -\delta]} \varphi, \quad \forall \varphi \in C_0^\infty(V), \varphi \geq 0 \text{ in } V. \quad (1.25)$$

We now fix a sequence  $\varepsilon_n \downarrow 0$ . Clearly,

$$\liminf_{n \rightarrow \infty} \left\{ h_{\varepsilon_n}(x) \chi_{[v_{\varepsilon_n} \geq -\delta]}(x) \right\} \geq h(x) \chi_{[v > -\delta]}(x) - h^-(x) \chi_{[v = -\delta]}(x)$$

for a.e.  $x \in V$ .

Applying Fatou's Lemma to (1.25), with  $\varepsilon = \varepsilon_n$ , we get

$$\int_V (v + \delta)^+ \Delta \varphi \geq \int_V \left\{ h \chi_{[v > -\delta]} - h^- \chi_{[v = -\delta]} \right\} \varphi. \quad (1.26)$$

Since  $h \in L^1(\omega)$ , we must have  $h = 0$  a.e. on the set  $[v = -\delta]$ , for a.e.  $\delta > 0$ . In particular,

$$h^- \chi_{[v = -\delta]} = 0 \quad \text{for a.e. } \delta > 0.$$

Take a sequence  $\delta_j \downarrow 0$  such that this holds; thus, (1.26) simply becomes

$$\int_V (v + \delta_j)^+ \Delta \varphi \geq \int_V h \chi_{[v > -\delta_j]} \varphi.$$

Passing to the limit as  $j \rightarrow \infty$ , we conclude that

$$\Delta v^+ \geq h\chi_{[v \geq 0]} \quad \text{in } \mathcal{D}'(V).$$

Since  $V \subset\subset \omega$  was arbitrary, the result follows.

**Proof of Theorem 1.1.** (We would like to thank H. Brezis for simplifying our original proof.)

We shall split the proof into two steps :

**Step 1.** Under the assumptions of the theorem, we have

$$\Delta u \geq g\chi_{[u > 0]} \quad \text{in } \mathcal{D}'(\Omega). \quad (1.27)$$

Given  $\delta > 0$ , it follows from the previous lemma, applied with  $\omega = [u > 2\delta]$  and  $v = u - 3\delta$ , that

$$\Delta(u - 3\delta)^+ \geq g\chi_{[u \geq 3\delta]} \quad \text{in } \mathcal{D}'([u > 2\delta]).$$

Let  $\psi \in C_0^\infty(\Omega)$  be such that  $0 \leq \psi \leq 1$  in  $\Omega$ ,  $\psi = 1$  in  $[u < \delta]$ , and  $\text{supp } \psi \subset [u < 2\delta]$ . For any  $\varphi \in C_0^\infty(\Omega)$ ,  $\varphi \geq 0$  in  $\Omega$ , we then have

$$\begin{aligned} \int_{\Omega} (u - 3\delta)^+ \Delta \varphi &= \int_{\Omega} (u - 3\delta)^+ \Delta [\varphi \psi + \varphi(1 - \psi)] \\ &= \int_{\Omega} (u - 3\delta)^+ \Delta [\varphi(1 - \psi)] \\ &= \int_{[u > 2\delta]} (u - 3\delta)^+ \Delta [\varphi(1 - \psi)] \\ &\geq \int_{[u > 2\delta]} g\chi_{[u \geq 3\delta]} \varphi(1 - \psi) = \int_{\Omega} g\chi_{[u \geq 3\delta]} \varphi. \end{aligned}$$

In other words,

$$\Delta(u - 3\delta)^+ \geq g\chi_{[u \geq 3\delta]} \quad \text{in } \mathcal{D}'(\Omega).$$

Passing to the limit as  $\delta \downarrow 0$ , we obtain (1.27).

**Step 2.** Proof of Theorem 1.1 completed.

Let  $h \in C^\infty(\mathbb{R}^N)$  be such that  $h = 0$  on  $K$  and  $h > 0$  in  $\mathbb{R}^N \setminus K$ . Applying Step 1 to the function  $u + \lambda h$ , where  $\lambda > 0$ , we get

$$\Delta u + \lambda \Delta h \geq (g + \lambda h)\chi_{[u + \lambda h > 0]} = (g + \lambda h)\chi_{\Omega \setminus K} \quad \text{in } \mathcal{D}'(\Omega).$$

Passing to the limit as  $\lambda \downarrow 0$ , the result follows.

**Proof of Corollary 1.1.** It follows from Theorem 1.1 that

$$\Delta u \geq g\chi_{\Omega \setminus K} = g \quad \text{in } \mathcal{D}'(\Omega).$$

Here, we have used that  $|K| = 0$ . On the other hand, since (1.6) holds and  $\text{cap}_2(K) = 0$ , it is well-known that (see [59]; see also Theorem 3.1)

$$-\Delta u \geq -g \quad \text{in } \mathcal{D}'(\Omega).$$

A comparison between these two inequalities implies that (1.8) holds.

## 1.3 Some useful formulas

Let us first recall some standard results.

Given a compact smooth manifold  $M^{N-k}$  (with or without boundary) embedded in  $\mathbb{R}^N$ , with codimension  $k \geq 1$ , we define its distance function  $d : \mathbb{R}^N \rightarrow \mathbb{R}_+$  by  $d(x) := \text{dist}(x, M)$ . The case  $k = N$  is included, i.e.,  $M$  may be a finite collection of points. It is a well-known fact that for  $\delta > 0$  small enough, the set  $\overline{N_\delta(M)}$  is a smooth manifold with boundary, also called the  $\delta$ -tubular neighborhood of  $M$ , which from now on we shall denote by  $\Xi_\delta(M)$ , and when no confusion arises, simply write  $\Xi_\delta$ . The distance function  $d$  is Lipschitz in  $\mathbb{R}^N$ , it is smooth in  $\Xi_\delta \setminus M$ , and satisfies (for the second property, see [65]) :

$$|\nabla d| = 1 \quad \text{a.e. in } \mathbb{R}^N, \quad (1.28)$$

$$\Delta d = \frac{k-1}{d} + a_0 \quad \text{in } \Xi_\delta \setminus M, \quad (1.29)$$

where  $a_0$  is a bounded function in  $\Xi_\delta \setminus M$ .

For each  $x \in \Xi_\delta$ , there exists a unique element  $\pi(x) \in M$  for which the distance function is realized, i.e., such that  $|x - \pi(x)| = d(x)$ . The projection  $\pi : \Xi_\delta \rightarrow M$  thus defined is also smooth.

For simplicity, from now on we shall assume that  $\Xi_2$  is a smooth tubular neighborhood of  $M$ .

Finally, let us recall that, for every  $v \in L^1(\mathbb{R}^N)$ , we have the coarea formula (see, e.g., [47]) :

$$\int_{\Xi_\delta} v = \int_0^\delta \int_{\partial \Xi_r} v \, d\sigma \, dr.$$

**Lemma 1.2** *Suppose  $M \subset \Omega$  is a compact smooth manifold, without boundary, of codimension  $k \geq 1$ . Let  $u \in L^1_{\text{loc}}(\Omega)$ , and assume there exists  $g \in L^1_{\text{loc}}(\Omega)$  such that*

$$\Delta u = g \quad \text{in } \mathcal{D}'(\Omega \setminus M).$$

Set

$$\mu := \Delta u - g \quad \text{in } \mathcal{D}'(\Omega).$$

For  $k \geq 1$  and  $t, r > 0$  define

$$G_k(r, t) = \begin{cases} \frac{1}{k} & \text{if } 0 < t < r, \\ -\frac{k-1}{k} \frac{r^k}{t^k} & \text{if } 0 < r < t. \end{cases} \quad (1.30)$$

Then, for any  $R \in (0, 1)$  fixed and  $\varphi \in C_0^\infty(\Omega)$ , all the limits below exist and :

a) if  $k \geq 3$ , then

$$\begin{aligned} \frac{1}{2(k-2)} \langle \mu, \varphi \rangle &= \lim_{r \downarrow 0} \left\{ -\frac{1}{r^2} \int_{\Xi_r} u \varphi + \right. \\ &\quad \left. + \frac{1}{r^2} \int_0^R G_k(r, t) \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt \right\}; \end{aligned} \quad (1.31)$$



b) if  $k = 2$ , then

$$\begin{aligned} \frac{1}{2}\langle \mu, \varphi \rangle &= \lim_{r \downarrow 0} \left\{ -\frac{1}{r^2 |\log r|} \int_{\Xi_r} u \varphi + \right. \\ &\quad \left. + \frac{1}{r^2 |\log r|} \int_0^R G_2(r, t) \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt \right\}; \end{aligned} \quad (1.32)$$

c) if  $k = 1$ , then

$$\begin{aligned} -\frac{1}{2}\langle \mu, \varphi \rangle &= \lim_{r \downarrow 0} \left\{ -\frac{1}{r^2} \int_{\Xi_r} u \varphi + \frac{1}{r} \left( \lim_{t \downarrow 0} \int_{\partial \Xi_t} u \varphi \right) \right. \\ &\quad \left. + \frac{1}{r^2} \int_0^r \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt \right\}. \end{aligned} \quad (1.33)$$

**Proof.** We first establish the following

**Claim.** For any  $\varphi \in C_0^\infty(\Omega)$ , the function  $s \mapsto \int_{\partial \Xi_s} u \varphi$  is  $C^1$  on  $(0, 1)$ , and

$$\langle \Delta u, \varphi \rangle = \int_{\Xi_s} u \Delta \varphi - 2 \int_{\partial \Xi_s} u \nabla \varphi \cdot \nabla d + s^{k-1} \frac{d}{ds} \left( \frac{1}{s^{k-1}} \int_{\partial \Xi_s} u \varphi \right) - \int_{\partial \Xi_s} u \varphi a_0. \quad (1.34)$$

**Proof of (1.34).** We first assume that  $u$  is smooth.

Fix a smooth, non-increasing function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Phi(t) = 0$  for  $t \geq 1$  and  $\Phi(t) = 1$  for  $t \leq 0$ . For  $\varepsilon > 0$ , set

$$\Phi_\varepsilon(t) := \Phi\left(\frac{t-1}{\varepsilon}\right).$$

Now let  $\varphi \in C_0^\infty(\Omega)$  and, for  $\varepsilon, s > 0$ , define

$$\varphi_{s,\varepsilon}(x) := \begin{cases} \varphi(x) & \text{if } x \in \Xi_s, \\ \varphi(x) \Phi_\varepsilon(d(x)/s) & \text{if } x \in \Xi_{s(1+\varepsilon)} \setminus \Xi_s, \\ 0 & \text{if } x \in \Omega \setminus \Xi_{s(1+\varepsilon)}. \end{cases}$$

Observe that  $\varphi_{s,\varepsilon} = \varphi$  in  $\Xi_s$  and  $\varphi_{s,\varepsilon} = 0$  in  $\Omega \setminus \Xi_{s(1+\varepsilon)}$ . Using (1.29), we now compute  $\Delta \varphi_{s,\varepsilon}$  in  $\Xi_{s(1+\varepsilon)} \setminus \Xi_s$ :

$$\Delta \varphi_{s,\varepsilon} = \Delta \varphi \Phi_\varepsilon(d/s) + \frac{2}{s} \nabla \varphi \cdot \nabla d \Phi'_\varepsilon(d/s) + \frac{1}{s^2} \varphi \left\{ \Phi''_\varepsilon(d/s) + \frac{s}{d} \Phi'_\varepsilon(d/s) (k-1 + a_0 d) \right\}.$$

Since  $\varphi_{s,\varepsilon}$  is an admissible test function, we obtain

$$\langle \Delta u, \varphi_{s,\varepsilon} \rangle = \int_{\Omega} u \Delta \varphi \Phi_\varepsilon(d/s) + I_1 + I_2 + I_3 + I_4, \quad (1.35)$$

where

$$I_1 = \frac{2}{s} \int_{\Omega} u \nabla \varphi \cdot \nabla d \Phi'_\varepsilon(d/s),$$

$$I_2 = \frac{1}{s^2} \int_{\Omega} u \varphi \Phi''_\varepsilon(d/s),$$

$$I_3 = \frac{k-1}{s} \int_{\Omega} u \frac{\varphi}{d} \Phi'_\varepsilon(d/s),$$

$$I_4 = \frac{1}{s} \int_{\Omega} u \varphi a_0 \Phi'_\varepsilon(d/s).$$

Next, we find the limit as  $\varepsilon \downarrow 0$  of the four previous integrals. For this purpose, we compute

$$I_1 = \frac{2}{\varepsilon s} \int_{\Xi_{s(1+\varepsilon)} \setminus \Xi_s} \Phi' \left( \frac{d/s - 1}{\varepsilon} \right) u \nabla \varphi \cdot \nabla d$$

so, by the coarea formula,

$$\begin{aligned} &= \frac{2}{\varepsilon s} \int_s^{s(1+\varepsilon)} \left\{ \Phi' \left( \frac{r/s - 1}{s} \right) \int_{\partial \Xi_r} u \nabla \varphi \cdot \nabla d \right\} dr \\ &= 2 \int_0^1 \left\{ \Phi'(t) \int_{\partial \Xi_{(1+\varepsilon t)s}} u \nabla \varphi \cdot \nabla d \right\} dt. \end{aligned}$$

We now let  $\varepsilon \downarrow 0$  :

$$\lim_{\varepsilon \downarrow 0} I_1 = 2 \int_0^1 \Phi'(t) \left\{ \int_{\partial \Xi_s} u \nabla \varphi \cdot \nabla d \right\} dt = -2 \int_{\partial \Xi_s} u \nabla \varphi \cdot \nabla d. \quad (1.36)$$

We now proceed with  $I_2$  :

$$\begin{aligned} I_2 &= \frac{1}{s^2} \int_{\Omega} u \varphi \Phi''(d/s) = \frac{1}{\varepsilon^2 s^2} \int_{\Xi_{s(1+\varepsilon)} \setminus \Xi_s} u \varphi \Phi'' \left( \frac{d/s - 1}{\varepsilon} \right) \\ &= \frac{1}{\varepsilon s} \int_0^1 \left\{ \Phi''(t) \int_{\partial \Xi_{s(1+\varepsilon t)}} u \varphi \right\} dt \end{aligned}$$

and, integrating by parts,

$$I_2 = \left[ \frac{1}{\varepsilon s} \Phi'(t) \int_{\partial \Xi_{s(1+\varepsilon t)}} u \varphi \right]_{t=0}^{t=1} - \int_0^1 \left( \Phi'(t) \left[ \frac{d}{d\lambda} \int_{\partial \Xi_\lambda} u \varphi \right]_{\lambda=s(1+\varepsilon t)} \right) dt.$$

Letting  $\varepsilon \downarrow 0$ , we arrive at

$$\lim_{\varepsilon \downarrow 0} I_2 = \frac{d}{ds} \int_{\partial \Xi_s} u \varphi. \quad (1.37)$$

The computations for  $I_3, I_4$  are similar, and they yield

$$\lim_{\varepsilon \downarrow 0} I_3 = -\frac{k-1}{s} \int_{\partial \Xi_s} u \varphi, \quad (1.38)$$

$$\lim_{\varepsilon \downarrow 0} I_4 = - \int_{\partial \Xi_s} u \varphi a_0. \quad (1.39)$$

Thus, passing to the limit as  $\varepsilon \downarrow 0$  in (1.35), and using (1.36)–(1.39), we get

$$\langle \Delta u, \varphi \rangle = \int_{\Xi_s} u \Delta \varphi - 2 \int_{\partial \Xi_s} u \nabla \varphi \cdot \nabla d + \frac{d}{ds} \int_{\partial \Xi_s} u \varphi - \frac{k-1}{s} \int_{\partial \Xi_s} u \varphi - \int_{\partial \Xi_s} u \varphi a_0. \quad (1.40)$$

But

$$\frac{d}{ds} \int_{\partial \Xi_s} u\varphi - \frac{k-1}{s} \int_{\partial \Xi_s} u\varphi = s^{k-1} \frac{d}{ds} \left( \frac{1}{s^{k-1}} \int_{\partial \Xi_s} u\varphi \right), \quad (1.41)$$

and therefore, combining (1.40) with (1.41), we find (1.34).

We now consider  $u$  as in the statement of the lemma, i.e.,  $u \in L^1_{\text{loc}}(\Omega)$  so that  $\mu := \Delta u - g$  is a distribution with support in  $M$ , where  $g \in L^1_{\text{loc}}(\Omega)$ . Using a density argument, and the fact that  $u \in W^{1,1}_{\text{loc}}(\Omega \setminus M)$ , we deduce that the function  $s \mapsto \int_{\partial \Xi_s} u\varphi$  is  $C^1$  on  $(0, 1)$  and that

$$\langle \mu, \varphi \rangle = \int_{\Xi_s} (u\Delta\varphi - g\varphi) - 2 \int_{\partial \Xi_s} u\nabla\varphi \cdot \nabla d + s^{k-1} \frac{d}{ds} \left( \frac{1}{s^{k-1}} \int_{\partial \Xi_s} u\varphi \right) - \int_{\partial \Xi_s} u\varphi a_0. \quad (1.42)$$

At this point, we distinguish the three cases : a)  $k \geq 3$ , b)  $k = 2$ , and c)  $k = 1$ .

**a) Case  $k \geq 3$ .** Fix  $R \in (0, 1)$  and let  $0 < t < R$ . Dividing (1.42) by  $s^{k-1}$  and integrating over  $s \in (t, R)$ , we get

$$\begin{aligned} \frac{1}{(k-2)t^{k-2}} \langle \mu, \varphi \rangle &= o(1) \frac{1}{t^{k-2}} - \frac{1}{t^{k-1}} \int_{\partial \Xi_t} u\varphi + \\ &\quad - \int_t^R \left\{ \frac{1}{s^{k-1}} \int_{\partial \Xi_s} u\varphi a_0 + 2u\nabla\varphi \cdot \nabla d \right\} ds, \end{aligned} \quad (1.43)$$

where  $o(1)$  denotes a quantity that goes to zero as  $t \rightarrow 0$ . Multiplying (1.43) by  $t^{k-1}$  and integrating over  $t \in (0, r)$  with  $0 < r < R$ , we obtain

$$\frac{1}{2(k-2)} \langle \mu, \varphi \rangle = o(1) - \frac{1}{r^2} \int_{\Xi_r} u\varphi - \frac{1}{r^2} \int_0^r t^{k-1} \int_t^R \frac{1}{s^{k-1}} \int_{\partial \Xi_s} v ds dt, \quad (1.44)$$

where we use the notation

$$v := 2u\nabla\varphi \cdot \nabla d + u\varphi a_0. \quad (1.45)$$

We now integrate by parts in the last term on the right-hand side of (1.44) :

$$\begin{aligned} \int_t^R \frac{1}{s^{k-1}} \left( \frac{d}{ds} \int_{\Xi_s} v \right) ds &= \left[ \frac{1}{s^{k-1}} \int_{\Xi_s} v \right]_{s=t}^{s=R} - \int_t^R (1-k) \frac{1}{s^k} \int_{\Xi_s} v ds \\ &= \frac{1}{R^{k-1}} \int_{\Xi_R} v - \frac{1}{t^{k-1}} \int_{\Xi_t} v + (k-1) \int_t^R \frac{1}{s^k} \int_{\Xi_s} v ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{r^2} \int_0^r t^{k-1} \int_t^R \frac{1}{s^{k-1}} \int_{\partial \Xi_s} v ds dt &= \frac{r^{k-2}}{kR^{k-1}} \int_{\Xi_R} v - \frac{1}{r^2} \int_0^r \int_{\Xi_t} v dt + \\ &\quad + \frac{k-1}{r^2} \int_0^r t^{k-1} \int_t^R \frac{1}{s^k} \int_{\Xi_s} v ds dt, \end{aligned} \quad (1.46)$$

and, changing the order of integration in the last term of (1.46), this gives

$$\int_0^r t^{k-1} \int_t^R \frac{1}{s^k} \int_{\Xi_s} v ds dt = \frac{1}{k} \int_0^r \int_{\Xi_s} v ds + \frac{r^k}{k} \int_r^R \frac{1}{s^k} \int_{\Xi_s} v ds. \quad (1.47)$$

Then, (1.46) in combination with (1.47) yields

$$\begin{aligned} \frac{1}{r^2} \int_0^r t^{k-1} \int_t^R \frac{1}{s^{k-1}} \int_{\partial\Xi_s} v \, ds \, dt &= \frac{r^{k-2}}{kR^{k-1}} \int_{\Xi_R} v - \frac{1}{kr^2} \int_0^r \int_{\Xi_t} v \, dt + \\ &+ \frac{k-1}{k} r^{k-2} \int_r^R \frac{1}{s^k} \int_{\Xi_s} v \, ds. \end{aligned} \quad (1.48)$$

Hence, using (1.48) in (1.44), we conclude that

$$\begin{aligned} \frac{1}{2(k-2)} \langle \mu, \varphi \rangle &= o(1) - \frac{1}{r^2} \int_{\Xi_r} u\varphi + \frac{1}{kr^2} \int_0^r \int_{\Xi_t} v \, dt - \frac{k-1}{k} r^{k-2} \int_r^R \frac{1}{s^k} \int_{\Xi_s} v \, ds \\ &= o(1) - \frac{1}{r^2} \int_{\Xi_r} u\varphi + \frac{1}{r^2} \int_0^R \left( G_k(r, t) \int_{\Xi_t} v \right) dt, \end{aligned}$$

where  $G_k$  is given by (1.30).

This establishes (1.31).

**b) Case  $k = 2$ .** Note that (1.42) is still valid and, since  $k = 2$ , it takes the form

$$\langle \mu, \varphi \rangle = o(1) - \int_{\partial\Xi_s} v + s \frac{d}{ds} \left( \frac{1}{s} \int_{\partial\Xi_s} u\varphi \right),$$

where  $v$  is given by (1.45). Dividing the last equation by  $s$  and integrating over  $s \in (t, R)$ , we get

$$(\log R - \log t) \langle \mu, \varphi \rangle = o(1) |\log t| - \int_t^R \frac{1}{s} \int_{\partial\Xi_s} v \, ds - \frac{1}{t} \int_{\partial\Xi_t} u\varphi.$$

Multiplying by  $t$  and integrating over  $t \in (0, r)$ , we obtain

$$\frac{1}{2} \langle \mu, \varphi \rangle = o(1) - \frac{1}{r^2 |\log r|} \int_0^r t \int_t^R \frac{1}{s} \int_{\partial\Xi_s} v \, ds \, dt - \frac{1}{r^2 |\log r|} \int_{\Xi_r} u\varphi. \quad (1.49)$$

But, integrating by parts,

$$\begin{aligned} \int_t^R \frac{1}{s} \int_{\partial\Xi_s} v \, ds &= \left[ \frac{1}{s} \int_{\Xi_s} v \right]_{s=t}^{s=R} + \int_t^R \frac{1}{s^2} \int_{\Xi_s} v \, ds \\ &= \frac{1}{R} \int_{\Xi_R} v - \frac{1}{t} \int_{\Xi_t} v + \int_t^R \frac{1}{s^2} \int_{\Xi_s} v \, ds. \end{aligned}$$

Hence, using Fubini, we get

$$\begin{aligned} \int_0^r t \int_t^R \frac{1}{s} \int_{\partial\Xi_s} v \, ds &= \frac{r^2}{2R} \int_{\Xi_R} v - \int_0^r \int_{\Xi_t} v \, dt + \int_0^r t \int_t^R \frac{1}{s^2} \int_{\Xi_s} v \, ds \, dt \\ &= \frac{r^2}{2R} \int_{\Xi_R} v - \frac{1}{2} \int_0^r \int_{\Xi_t} v \, dt + \frac{r^2}{2} \int_r^R \frac{1}{s^2} \int_{\Xi_s} v \, ds. \end{aligned} \quad (1.50)$$

So, from (1.49) and (1.50), we infer that

$$\begin{aligned} \frac{1}{2}\langle \mu, \varphi \rangle &= o(1) - \frac{1}{r^2|\log r|} \int_{\Xi_r} u\varphi + \frac{1}{2r^2|\log r|} \int_0^r \int_{\Xi_t} v dt - \frac{1}{2|\log r|} \int_r^R \frac{1}{t^2} \int_{\Xi_t} v dt \\ &= o(1) - \frac{1}{r^2|\log r|} \int_{\Xi_r} u\varphi + \frac{1}{r^2|\log r|} \int_0^R \left( G_2(r, t) \int_{\Xi_t} v \right) dt. \end{aligned}$$

This proves (1.32).

c) **Case  $k = 1$ .** This time (1.42) becomes

$$\langle \mu, \varphi \rangle = \int_{\Omega} (u\Delta\varphi - g\varphi) - \int_{\partial\Xi_s} v + \frac{d}{ds} \int_{\partial\Xi_s} u\varphi.$$

Integrate the previous relation over  $s \in (t, \lambda)$  :

$$(\lambda - t)\langle \mu, \varphi \rangle = o(1) - \int_{\Xi_\lambda \setminus \Xi_t} v + \int_{\partial\Xi_\lambda} u\varphi - \int_{\partial\Xi_t} u\varphi. \quad (1.51)$$

where  $o(1) \rightarrow 0$  as  $\lambda \rightarrow 0$ . Since  $v = 2u\nabla\varphi \cdot \nabla d + u\varphi a_0 \in L^1_{\text{loc}}(\Omega)$ , by letting  $t \downarrow 0$  in (1.51) we see that  $\lim_{t \downarrow 0} \int_{\partial\Xi_t} u\varphi$  exists and

$$\lambda\langle \mu, \varphi \rangle = o(1) - \int_{\Xi_\lambda} v + \int_{\partial\Xi_\lambda} u\varphi - \left( \lim_{t \downarrow 0} \int_{\partial\Xi_t} u\varphi \right). \quad (1.52)$$

We now integrate (1.52) over  $\lambda \in (0, r)$  and divide by  $r^2$  to find

$$\frac{1}{2}\langle \mu, \varphi \rangle = o(1) + \frac{1}{r^2} \int_{\Xi_r} u\varphi - \frac{1}{r} \left( \lim_{t \downarrow 0} \int_{\partial\Xi_t} u\varphi \right) - \frac{1}{r^2} \int_0^r \int_{\Xi_t} v dt,$$

which concludes the proof of the lemma.

## 1.4 Proof of Theorem 1.2

Set  $\mu := \Delta u - g$ . Suppose that (1.10) holds. Then, since  $\lim_{r \downarrow 0} \int_{\partial\Xi_r} u\varphi$  exists by Lemma 1.2, we conclude that

$$\lim_{r \downarrow 0} \int_{\partial\Xi_r} u\varphi = 0, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (1.53)$$

On the other hand, given  $\varepsilon > 0$ , (1.10) implies that there exists  $\delta > 0$  such that

$$\int_{\Xi_r} |u| \leq \varepsilon r, \quad \forall r \in (0, \delta).$$

Therefore, we have

$$\begin{aligned} \left| \int_0^r \left( \int_{\Xi_t} 2u\nabla\varphi \cdot \nabla d + u\varphi a_0 \right) dt \right| &\leq C \int_0^r \left( \int_{\Xi_t} |u| \right) dt \\ &\leq C \int_0^r \varepsilon t dt = \varepsilon \frac{r^2}{2}, \quad \forall r \in (0, \delta). \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, we deduce that

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_0^r \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt = 0. \quad (1.54)$$

Inserting (1.53) and (1.54) into (1.33), we get

$$\frac{1}{2} \langle \mu, \varphi \rangle = \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u \varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

The limit in (1.11) now follows since, by definition,  $\mu = \Delta u - g$ . This completes the proof of the theorem.

## 1.5 Proof of Theorem 1.3

We shall give a proof of Theorem 1.3 only for the case of codimension  $k \geq 3$ , the case  $k = 2$  being entirely analogous.

Using the fact that  $u \geq 0$  a.e. in  $\Omega$ , we have

$$\begin{aligned} \left| \int_0^R G_k(r, t) \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt \right| &\leq \\ &\leq C \int_0^r \left( \int_{\Xi_t} u \right) dt + C \int_r^R \frac{r^k}{t^k} \left( \int_{\Xi_t} u \right) dt \\ &\leq Cr \int_{\Xi_r} u + C \int_r^R \frac{r^k}{t^k} \left( \int_{\Xi_t} u \right) dt, \quad \forall r \in (0, R). \end{aligned} \quad (1.55)$$

Choose  $R_1 \in (0, R)$  small, so that  $CR_1 < \frac{1}{2}$ . We now apply (1.31) with  $R := R_1$  and  $\varphi \in C_0^\infty(\Omega)$ ,  $\varphi = 1$  on  $\Xi_{R_1}$ . Then, by (1.55) and our choice of  $R_1$ , we get

$$\frac{1}{2} \int_{\Xi_r} u - C \int_r^{R_1} \frac{r^k}{t^k} \left( \int_{\Xi_t} u \right) dt \leq Cr^2, \quad \forall r \in (0, R_1). \quad (1.56)$$

We shall use (1.56) and a bootstrap argument to conclude that

$$\int_{\Xi_r} u \leq Cr^2, \quad \forall r \in (0, R_1). \quad (1.57)$$

In fact, since  $\int_{\Xi_t} u$  is uniformly bounded for  $t \in (0, R_1)$ , we have

$$\int_r^{R_1} \frac{r^k}{t^k} \left( \int_{\Xi_t} u \right) dt \leq Cr, \quad \forall r \in (0, R_1). \quad (1.58)$$

In particular, (1.56) and (1.58) imply that

$$\frac{1}{2} \int_{\Xi_r} u \leq Cr, \quad \forall r \in (0, R_1),$$

so that

$$\int_r^{R_1} \frac{r^k}{t^k} \left( \int_{\Xi_t} u \right) dt \leq Cr^2, \quad \forall r \in (0, R_1). \quad (1.59)$$

Therefore, by (1.56) and (1.59), we conclude that estimate (1.57) holds.

It then follows from (1.57) and (1.55), with  $R$  replaced by  $R_1$ , that the right-hand side in (1.55) is bounded by  $Cr^3$ ,  $\forall r \in (0, R_1)$ . In particular,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \left\{ \int_0^{R_1} G_k(r, t) \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt \right\} = 0. \quad (1.60)$$

By (1.31) and (1.60), we have

$$-\frac{1}{2(k-2)} \langle \mu, \varphi \rangle = \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u \varphi, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (1.61)$$

If we now apply (1.61) with estimate (1.57), then we conclude that  $\mu$  is a measure. Since  $u \geq 0$  a.e. in  $\Omega$ , (1.61) implies that  $\mu$  is nonpositive.

## 1.6 Proof of Theorems 1.4 and 1.5

**Proof of Theorem 1.4.** We shall split the proof of the theorem into 3 steps :

**Step 1.** If

$$\frac{1}{r^2} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0, \quad (1.62)$$

then  $\mu$  is a measure and

$$\langle \mu, \varphi \rangle = -2(k-2) \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u \varphi, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (1.63)$$

It is easy to see that condition (1.62) implies that

$$\lim_{r \downarrow 0} \frac{1}{r^2} \left\{ \int_0^{R_1} G_k(r, t) \left( \int_{\Xi_t} 2u \nabla \varphi \cdot \nabla d + u \varphi a_0 \right) dt \right\} = 0.$$

From the limit above and (1.31), we deduce that (1.63) holds. In particular, it follows from (1.62) and (1.63) that  $\mu$  is a measure and

$$|\mu|_{\mathcal{M}} \leq 2(k-2) \liminf_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u|. \quad (1.64)$$

**Step 2.** If  $\mu$  is a measure, then

$$\frac{1}{r^2} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0, \quad (1.65)$$

and

$$|\mu|_{\mathcal{M}} \geq 2(k-2) \limsup_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u|. \quad (1.66)$$

In this step, we shall use an estimate given in the proof of Theorem 1.3, and also the representation of the solutions of  $\Delta v = \nu$  in  $\mathbb{R}^N$  in terms of the fundamental solution. More precisely, let  $E(x) = \frac{c_N}{|x|^{N-2}}$  be the fundamental solution of  $-\Delta$  in  $\mathbb{R}^N$ ,  $N \geq 3$ , where the constant  $c_N$  is chosen so that  $-\Delta E = \delta_0$ . If  $\nu$  is a Radon measure in  $\mathbb{R}^N$ , then

$$v := E * \nu \in L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{and} \quad -\Delta v = \nu \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Now let  $\nu := g + \mu$  in  $\Omega$ . Next, we decompose  $\nu = \nu^+ - \nu^-$  in its positive and negative parts, where  $\nu^\pm = g^\pm + \mu^\pm$ . Let  $v^\pm := E * \nu^\pm$ . As we observed above, we have

$$-\Delta v^\pm = \nu^\pm = g^\pm + \mu^\pm \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Moreover, note that  $v^\pm \geq 0$  a.e. in  $\mathbb{R}^N$ . In particular, the functions  $v^\pm$  satisfy the assumptions of Theorem 1.3, so that (1.14) holds with  $u$  and  $\mu$  replaced by  $v^\pm$  and  $-\mu^\pm$ , respectively. In other words, we have

$$\int_{\Xi_r} v^\pm \leq Cr^2, \quad \forall r \in (0, 1), \quad (1.67)$$

and

$$\frac{1}{2(k-2)} \langle \mu^\pm, \varphi \rangle = \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} v^\pm \varphi, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (1.68)$$

On the other hand, it is easy to see that  $u = v^- - v^+ + w$  a.e. in  $\Omega$ , for some harmonic function  $w$ . Since  $w$  is bounded in a neighborhood of  $M$ , we have

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |w| = 0. \quad (1.69)$$

In particular, (1.65) follows from (1.67) and (1.69). Moreover, if we apply (1.68) with a test function  $\varphi$  such that  $\varphi = 1$  in some neighborhood of  $M$ , then we have

$$\begin{aligned} \frac{1}{2(k-2)} |\mu|_{\mathcal{M}} &= \frac{1}{2(k-2)} (\langle \mu^+, 1 \rangle + \langle \mu^-, 1 \rangle + 0) \\ &= \lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} (v^+ + v^- + |w|) \geq \limsup_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u|. \end{aligned}$$

This concludes the proof of Step 2.

**Step 3.** Proof of Theorem 1.4 completed.

By Steps 1 and 2 we know that  $\mu$  is a measure if and only if

$$\frac{1}{r^2} \int_{\Xi_r} |u| \quad \text{remains bounded as } r \downarrow 0,$$

in which case formula (1.17) holds. Moreover, applying (1.64) and (1.66) we get

$$|\mu|_{\mathcal{M}} \leq 2(k-2) \liminf_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u| \leq 2(k-2) \limsup_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} |u| \leq |\mu|_{\mathcal{M}},$$



so that all the inequalities are reduced to equalities in the estimate above and (1.18) holds.

**Proof of Theorem 1.5.** The proof of Theorem 1.5 follows along the same lines and shall be omitted.

**Remark 1.5** Let us mention that formula (1.17) in Theorem 1.4 holds under weaker conditions; namely, that if

$$\frac{1}{r} \int_{\Xi_r} |u| \rightarrow 0 \quad \text{as } r \downarrow 0, \quad (1.70)$$

then

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u \varphi = -\frac{1}{2(k-2)} \langle \mu, \varphi \rangle, \quad \forall \varphi \in C_0^\infty(\Omega).$$

This formula can be easily deduced from Lemma 1.2.

## 1.7 Proof of Theorem 1.6

It follows from (1.20) and (1.21) that

$$-\Delta u = g(u) - cu \geq -C \quad \text{in } \mathcal{D}'(\Omega \setminus M),$$

for some constant  $C > 0$  sufficiently large. Since  $\text{cap}_2(M) = 0$ , we then conclude that  $u \in L_{\text{loc}}^1(\Omega)$  and

$$-\Delta u \geq -C \quad \text{in } \mathcal{D}'(\Omega).$$

In particular,  $\Delta u$  is a Radon measure in  $\Omega$ . Using this property and (1.21), it is easy to see that  $g(u) \in L_{\text{loc}}^1(\Omega)$  (see, e.g., Section 3.3); in particular,

$$u \in L_{\text{loc}}^{\frac{k}{k-2}}(\Omega).$$

Recall that  $M$  has codimension  $k$ ; thus,  $|\Xi_r| \sim r^k$  as  $r \downarrow 0$ . It then follows from Hölder's inequality that

$$\int_{\Xi_r} u \leq |\Xi_r|^{2/k} \left( \int_{\Xi_r} u^{\frac{k}{k-2}} \right)^{\frac{k-2}{k}} \leq Cr^2 \left( \int_{\Xi_r} u^{\frac{k}{k-2}} \right)^{\frac{k-2}{k}}.$$

Therefore,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \int_{\Xi_r} u = 0.$$

By Corollary 1.2, we must have  $\mu = \Delta u - cu + g(u) = 0$  in  $M$ . Hence, (1.22) holds.

**Proof of Remark 1.4.** Let us assume that

$$\liminf_{t \rightarrow \infty} \frac{g(t)}{e^{at}} > 0 \quad \text{for every } a > 0 \text{ sufficiently large.}$$

As before, we have  $u, g(u) \in L_{\text{loc}}^1(\Omega)$  and (1.21) is satisfied. In particular,

$$e^{au} \in L_{\text{loc}}^1(\Omega), \quad \forall a > 0. \quad (1.71)$$

By Jensen's inequality and (1.71), we know that

$$e^{\frac{1}{|\Xi_r|} \int_{\Xi_r} au} \leq \frac{1}{|\Xi_r|} \int_{\Xi_r} e^{au} \leq \frac{C_a}{|\Xi_r|}, \quad \forall r > 0 \text{ small},$$

where  $C_a > 0$  is a constant depending on  $a$ . We conclude that

$$\frac{a}{|\Xi_r|} \int_{\Xi_r} u \leq \log \frac{C_a}{|\Xi_r|}. \quad (1.72)$$

Let  $0 < \alpha_1 \leq \alpha_2$  be such that  $\alpha_1 r^2 \leq |\Xi_r| \leq \alpha_2 r^2$  for all  $r > 0$  small. From (1.72), we get

$$\frac{a}{\alpha_2 r^2 \log 1/r} \int_{\Xi_r} u \leq \frac{\log(C_a/\alpha_1 r^2)}{\log 1/r} = 2 + \frac{\log(C_a/\alpha_1)}{\log 1/r}.$$

By letting  $r \downarrow 0$ , we deduce that

$$\limsup_{r \downarrow 0} \frac{1}{r^2 |\log r|} \int_{\Xi_r} u \leq \frac{2\alpha_2}{a}.$$

If we take  $a \rightarrow \infty$ , then we have

$$\lim_{r \downarrow 0} \frac{1}{r^2 |\log r|} \int_{\Xi_r} u = 0.$$

We now invoke Corollary 1.2 to conclude that Theorem 1.6 also holds for  $k = 2$ .



# Chapitre 2

## Remarks on the strong maximum principle

### Sommaire

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### 2.1 Introduction<sup>3</sup>

The strong maximum principle asserts that if  $u$  is smooth,  $u \geq 0$  and  $-\Delta u \geq 0$  in a connected domain  $\Omega \subset \mathbb{R}^N$ , then either  $u \equiv 0$  or  $u > 0$  in  $\Omega$ . The same conclusion holds when  $-\Delta$  is replaced by  $-\Delta + a(x)$  with  $a \in L^p(\Omega)$ ,  $p > \frac{N}{2}$  (this is a consequence of Harnack's inequality; see, e.g., [81], and also [83, Corollary 5.3]). Another formulation of the same fact says that if  $u(x_0) = 0$  for some point  $x_0 \in \Omega$ , then  $u \equiv 0$  in  $\Omega$ . A similar conclusion fails, however, when  $a \notin L^p(\Omega)$ , for any  $p > \frac{N}{2}$ . For instance,  $u(x) = |x|^2$  satisfies  $-\Delta u + a(x)u = 0$  in  $B_1$  with  $a = \frac{2N}{|x|^2} \notin L^{N/2}(B_1)$ .

If  $u$  vanishes on a larger set, one may still hope to conclude, under some weaker condition on  $a$ , that  $u \equiv 0$  in  $\Omega$ . Such a result was obtained by B enilan and Brezis [6, Appendix D] (with a contribution by R. Jensen) in the case where  $a \in L^1(\Omega)$  and  $\text{supp } u$  is a compact subset of  $\Omega$ . Their maximum principle has been further extended by Ancona [4], who proved Theorem 2.1 below.

We will assume throughout this chapter that  $\Omega$  is a bounded connected domain in  $\mathbb{R}^N$ , with  $N \geq 2$ .

We recall that a function  $v : \Omega \rightarrow \mathbb{R}$  is quasicontinuous if there exists a sequence of open subsets  $(\omega_n)$  of  $\Omega$  such that  $v|_{\Omega \setminus \omega_n}$  is continuous for every  $n \geq 1$ , and  $\text{cap}_2 \omega_n \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\text{cap}_2 \omega_n$  denotes the  $H^1$ -capacity of  $\omega_n$  (see Definition 3.2).

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<sup>3</sup>Ce chapitre a  et e  ecrit en collaboration avec H. Brezis; le texte original est d ej a sorti dans *Differential Integral Equations* **16** (2003), 1–12.

**Theorem 2.1 (Ancona [4])** *Let  $u \in L^1(\Omega)$ ,  $u \geq 0$  a.e. in  $\Omega$ , be such that  $\Delta u$  is a Radon measure on  $\Omega$ . Then, there exists  $\tilde{u} : \Omega \rightarrow \mathbb{R}$  quasicontinuous such that  $u = \tilde{u}$  a.e. in  $\Omega$ .*

*Let  $a \in L^1(\Omega)$ ,  $a \geq 0$  a.e. in  $\Omega$ . If*

$$-\Delta u + au \geq 0 \quad \text{in } \Omega,$$

*in the following sense*

$$\int_E \Delta u \leq \int_E au \quad \text{for every Borel set } E \subset \Omega, \quad (2.1)$$

*and if  $\tilde{u} = 0$  on a set of positive  $H^1$ -capacity in  $\Omega$ , then  $u = 0$  a.e. in  $\Omega$ .*

The proof given by Ancona is purely based on Potential Theory. Our aim in this chapter will be to present one in the spirit of PDEs. We also carefully discuss the meaning of the condition  $-\Delta u + au \geq 0$  in  $\Omega$ .

The next two corollaries follow immediately from the theorem above :

**Corollary 2.1** *Let  $u$  and  $a$  be as in Theorem 2.1, and suppose (2.1) is satisfied.*

*If  $u = 0$  on a subset of  $\Omega$  with positive measure, then  $u = 0$  a.e. in  $\Omega$ .*

*If  $u$  is continuous in  $\Omega$  and  $u = 0$  on a subset of  $\Omega$  with positive  $H^1$ -capacity, then  $u \equiv 0$  in  $\Omega$ .*

**Corollary 2.2** *Let  $u$  and  $a$  be as in Theorem 2.1. Suppose that  $\Delta u \in L^1(\Omega)$ .*

*If*

$$-\Delta u + au \geq 0 \quad \text{a.e. in } \Omega$$

*and  $u = 0$  on a subset of  $\Omega$  with positive measure, then  $u = 0$  a.e. in  $\Omega$ .*

The next corollary follows from Theorem 2.1 and Remark 2.3 :

**Corollary 2.3** *Let  $u$  and  $a$  be as in Theorem 2.1. Suppose that  $au \in L^1_{\text{loc}}(\Omega)$ .*

*If*

$$-\Delta u + au \geq 0 \quad \text{in } \mathcal{D}'(\Omega),$$

*i.e.*

$$\int_{\Omega} u \Delta \varphi \leq \int_{\Omega} au \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega,$$

*and  $u = 0$  on a subset of  $\Omega$  with positive measure, then  $u = 0$  a.e. in  $\Omega$ .*

**Remark 2.1** In view of Corollary 2.3 above, it would seem natural to replace condition (2.1) in Theorem 2.1 by

$$\int_{\Omega} u \Delta \varphi \leq \int_{\Omega} au \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega,$$

which makes sense even if  $au \notin L^1_{\text{loc}}(\Omega)$  (note that  $au \varphi \geq 0$  a.e., so that the right-hand side is always well-defined, possibly taking the value  $+\infty$ ). However, the strong maximum principle is no longer true in general; see Remark 2.4.

There are several interesting questions related to Theorem 2.1 :

**Open Problem 1** In the statement of Theorem 2.1, suppose in addition that  $\text{supp } u \subset \Omega$  is a compact set. Can one replace the assumption  $a \in L^1_{\text{loc}}$  by a weaker condition, for example  $a^{1/2} \in L^1_{\text{loc}}$  (or  $a^{1/2} \in L^p_{\text{loc}}$  for some  $p > 1$ ), and still conclude that  $u = 0$  a.e. in  $\Omega$ ?

Note that one cannot hope to go below  $L^{1/2}$ . For instance, the  $C^2$ -function  $u$  given by

$$u(x) = \begin{cases} (1 - |x|^2)^4 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

satisfies  $-\Delta u + au \geq 0$  for some function  $a(x)$  such that  $a(x) \sim \frac{1}{(1 - |x|)^2}$  for  $|x| \lesssim 1$ .

Here,  $a^\alpha \in L^1$  for every  $\alpha < 1/2$ , but  $a^{1/2} \notin L^1$ .

Here is another one :

**Open Problem 2** Assume  $u \in C^0$ ,  $u \geq 0$ , and  $a \in L^q_{\text{loc}}$  for some  $1 < q \leq \frac{N}{2}$ ,  $a \geq 0$  a.e., satisfy (2.1). Suppose that  $u = 0$  on a set  $E$  with  $\text{cap}_{2q}(E) > 0$ , where  $\text{cap}_{2q}$  refers to the capacity associated with the Sobolev space  $W^{1,2q}$  (see Definition 3.1). Can one conclude that  $u \equiv 0$  in  $\Omega$ ? Same question if  $E$  has positive  $W^{2,q}$ -capacity.

Theorem 2.1 above shows that the answer is positive when  $q = 1$ . It is also true when  $q > \frac{N}{2}$ , by the strong maximum principle mentioned above. Note that if  $q > \frac{N}{2}$  and  $x_0$  is any point, then  $\text{cap}_{2q}(\{x_0\}) > 0$ ; see Lemma 3.1.

## 2.2 Some comments about condition (2.1)

Since in the statement of Theorem 2.1 it may happen that  $au \notin L^1_{\text{loc}}(\Omega)$ , and so  $au$  is not necessarily a distribution, one should be very careful in order to give a precise meaning to the inequality

$$\Delta u \leq au \quad \text{in } \Omega.$$

More generally, let  $\mu$  be a Radon measure on  $\Omega$  and  $f$  a measurable function,  $f \geq 0$  a.e. in  $\Omega$ . Here are two possible definitions for the inequality  $\mu \leq f$  in  $\Omega$  :

**Definition 2.1** We shall write

$$\mu \leq_1 f \quad \text{in } \Omega,$$

if

$$\int_E d\mu \leq \int_E f \quad \text{for every Borel set } E \subset \Omega.$$

**Definition 2.2** We shall write

$$\mu \leq_2 f \quad \text{in } \Omega,$$

if

$$\int_\Omega \varphi d\mu \leq \int_\Omega f\varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega.$$

In the first definition, we view  $f$  as the nonnegative measure  $f dx$ , while in the second one  $f$  is treated as if it were a distribution (which need not be the case in general, unless  $f \in L^1_{\text{loc}}(\Omega)$ ).

**Remark 2.2** If  $\mu \leq_1 f$  in  $\Omega$ , then  $\mu \leq_2 f$  in  $\Omega$ . However, the converse is not true in general; see Remark 2.4 below.

**Remark 2.3** If we assume in addition that  $f \in L^1_{\text{loc}}(\Omega)$ , then  $\mu \leq_1 f$  in  $\Omega$  if, and only if,  $\mu \leq_2 f$  in  $\Omega$ .

**Remark 2.4** Theorem 2.1 above is no longer true in general (even for the case where  $\Delta u \in L^1(\Omega)$ ) if we replace (2.1) by

$$-\Delta u + au \geq_2 0 \quad \text{in } \Omega.$$

In fact, let  $N \geq 2$ . Take  $v \in L^1(\mathbb{R}^N)$ ,  $v \geq 0$  a.e. in  $\mathbb{R}^N$ , such that  $\text{supp } v \subset B_1$ ,  $\Delta v \in L^1(\mathbb{R}^N)$ , but  $v$  is unbounded (this is possible since  $N \geq 2$ ). In particular, there exists  $b \in L^1(\mathbb{R}^N)$ ,  $b \geq 0$  a.e. in  $\mathbb{R}^N$ , such that  $bv \notin L^1(\mathbb{R}^N)$ .

Let  $(x_j) \subset B_1$  be a dense sequence in  $B_1$  and, for each  $j \geq 1$ , let

$$\gamma_j := \min \left\{ \frac{1}{j}, \frac{1 - |x_j|}{2} \right\}.$$

We define

$$u(x) := \sum_{j=1}^{\infty} \frac{1}{2^j \gamma_j^{N-2}} v \left( \frac{x - x_j}{\gamma_j} \right),$$

$$a(x) := \sum_{j=1}^{\infty} \frac{1}{2^j \gamma_j^N} b \left( \frac{x - x_j}{\gamma_j} \right).$$

Then,

$$u \in L^1(\mathbb{R}^N), \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N,$$

$$\Delta u \in L^1(\mathbb{R}^N),$$

$$a \in L^1(\mathbb{R}^N), \quad a \geq 0 \text{ a.e. in } \mathbb{R}^N.$$

Note that, by construction,  $\int_{\omega} au = +\infty$  for every open set  $\omega \subset B_1$ . Thus, if  $\varphi \in C_0^\infty(\Omega)$ ,  $\varphi \geq 0$  in  $\mathbb{R}^N$ , then we have

$$\int_{\mathbb{R}^N} au\varphi = \begin{cases} +\infty & \text{if } \text{supp } \varphi \cap B_1 \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that  $u$  satisfies

$$\int_{\mathbb{R}^N} u\Delta\varphi \leq \int_{\mathbb{R}^N} au\varphi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geq 0 \text{ in } \mathbb{R}^N,$$

but  $\text{supp } u \subset \overline{B_1}$  and  $u \not\equiv 0$  in  $\mathbb{R}^N$ . In view of Theorem 2.1, it follows that the inequality  $\Delta u \leq_1 au$  is **not** satisfied in  $\mathbb{R}^N$ .

From now on, we shall always consider the inequality  $\Delta u \leq au$  in the sense of Definition 2.1. In particular, we shall omit the subscript 1 in the symbol  $\leq_1$ .

## 2.3 Proof of the quasicontinuity of $u$ in Theorem 2.1

Before proving the first part of Theorem 2.1 (see Lemma 2.1 below), we make the following remark :

**Remark 2.5** If  $v \in H_{\text{loc}}^1(\Omega)$ , then there exists  $\tilde{v} : \Omega \rightarrow \mathbb{R}$  quasicontinuous such that  $v = \tilde{v}$  a.e. in  $\Omega$  (see, e.g., [63]). In addition,  $\tilde{v}$  is well-defined modulo polar subsets of  $\Omega$ , i.e. if  $\tilde{v}_1$  and  $\tilde{v}_2$  are two quasicontinuous functions such that  $\tilde{v}_1 = v = \tilde{v}_2$  a.e. in  $\Omega$ , then there exists a polar set  $P \subset \Omega$  such that  $\tilde{v}_1(x) = \tilde{v}_2(x)$  for every  $x \in \Omega \setminus P$  (see [45]). (Recall that a set  $P$  is polar if it has zero  $H^1$ -capacity.)

**Notation.** Given  $k > 0$ , we denote by  $T_k : \mathbb{R} \rightarrow \mathbb{R}$  the truncation function

$$T_k(s) := \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } -k < s < k, \\ -k & \text{if } s \leq -k. \end{cases}$$

The existence of a quasicontinuous function  $\tilde{u} : \Omega \rightarrow \mathbb{R}$  such that  $u = \tilde{u}$  a.e. in  $\Omega$ , as in the statement of Theorem 2.1, is a consequence of Lemma 2.1 below :

**Lemma 2.1** *Assume  $u \in L^1(\Omega)$  is such that  $\Delta u$  is a Radon measure on  $\Omega$ . Then,*

$$T_k(u) \in H_{\text{loc}}^1(\Omega), \quad \forall k > 0, \quad (2.2)$$

and, for each open subset  $A \subset \subset \Omega$ , there exists  $C_A > 0$  so that

$$\int_A |\nabla T_k(u)|^2 \leq k \left( \int_{\Omega} |\Delta u| + C_A \int_{\Omega} |u| \right), \quad \forall k > 0. \quad (2.3)$$

Moreover, there exists  $\tilde{u} : \Omega \rightarrow \mathbb{R}$  quasicontinuous such that  $u = \tilde{u}$  a.e. in  $\Omega$ .

**Proof.** We shall split the proof of Lemma 2.1 into two steps :

**Step 1.** Proof of (2.2) and (2.3).

Let  $\rho \in C_0^\infty(B_1)$  be a radial, nonnegative, mollifier. Set

$$u_\varepsilon(x) := \rho_\varepsilon * u(x) = \int_{\Omega} \rho_\varepsilon(x-y)u(y) dy, \quad \forall x \in \Omega.$$

For  $k > 0$  fixed, we have  $T_k(u_\varepsilon) \in H^1(\Omega)$  and

$$\nabla T_k(u_\varepsilon) = \nabla u_\varepsilon \chi_{[|u_\varepsilon| < k]}, \quad (2.4)$$

where  $\chi_{[|u_\varepsilon| < k]}$  denotes the characteristic function of the set  $[|u_\varepsilon| < k]$ .



Given an open set  $A \subset\subset \Omega$ , let  $\varphi \in C_0^\infty(\Omega)$  be such that  $0 \leq \varphi \leq 1$  in  $\Omega$  and  $\varphi = 1$  on  $A$ . Using (2.4) and integrating by parts, we have

$$\begin{aligned} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^2 \varphi &= \int_{\Omega} \nabla T_k(u_\varepsilon) \cdot (\nabla u_\varepsilon) \varphi \\ &= - \int_{\Omega} T_k(u_\varepsilon) (\Delta u_\varepsilon) \varphi - \int_{\Omega} T_k(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi. \end{aligned} \quad (2.5)$$

On the other hand,

$$\begin{aligned} \int_{\Omega} T_k(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi &= - \int_{\Omega} u_\varepsilon \nabla T_k(u_\varepsilon) \cdot \nabla \varphi - \int_{\Omega} u_\varepsilon T_k(u_\varepsilon) \Delta \varphi \\ &= - \int_{\Omega} T_k(u_\varepsilon) \nabla T_k(u_\varepsilon) \cdot \nabla \varphi - \int_{\Omega} u_\varepsilon T_k(u_\varepsilon) \Delta \varphi \\ &= - \frac{1}{2} \int_{\Omega} \nabla [T_k(u_\varepsilon)]^2 \cdot \nabla \varphi - \int_{\Omega} u_\varepsilon T_k(u_\varepsilon) \Delta \varphi \\ &= \frac{1}{2} \int_{\Omega} [T_k(u_\varepsilon)]^2 \Delta \varphi - \int_{\Omega} u_\varepsilon T_k(u_\varepsilon) \Delta \varphi \\ &= - \int_{\Omega} T_k(u_\varepsilon) \left( u_\varepsilon - \frac{1}{2} T_k(u_\varepsilon) \right) \Delta \varphi \\ &\geq -k \int_{\Omega} |u_\varepsilon| |\Delta \varphi|. \end{aligned} \quad (2.6)$$

It follows from (2.5) and (2.6) that

$$\int_A |\nabla T_k(u_\varepsilon)|^2 \leq \int_{\Omega} |\nabla T_k(u_\varepsilon)|^2 \varphi \leq k \left( \int_{\text{supp } \varphi} |\Delta u_\varepsilon| + \|\Delta \varphi\|_{L^\infty} \int_{\text{supp } \varphi} |u_\varepsilon| \right).$$

In particular, for every  $0 < \varepsilon < \text{dist}(\text{supp } \varphi, \partial\Omega)$ ,

$$\int_A |\nabla T_k(u_\varepsilon)|^2 \leq k \left( \int_{\Omega} |\Delta u| + \|\Delta \varphi\|_{L^\infty} \int_{\Omega} |u| \right).$$

Letting  $\varepsilon \downarrow 0$ , we conclude that  $T_k(u) \in H^1(A)$  and (2.3) holds with  $C_A = \|\Delta \varphi\|_{L^\infty}$ .

**Step 2.** Under the assumptions of the lemma, there exists a function  $\tilde{u} : \Omega \rightarrow \mathbb{R}$  quasi-continuous such that  $u = \tilde{u}$  a.e. in  $\Omega$ .

By (2.2) and Remark 2.5, for each  $k > 0$  there exists  $\widetilde{T_k(u)} : \Omega \rightarrow \mathbb{R}$  quasicontinuous such that  $T_k(u) = \widetilde{T_k(u)}$  a.e. in  $\Omega$ .

Let  $v_k := \frac{1}{k} T_k(u)$ , so that

$$v_k \rightarrow 0 \quad \text{in } L^q(\Omega), \quad \forall q \in [1, \infty)$$

and, by (2.3),

$$\int_A |\nabla v_k|^2 \rightarrow 0, \quad \forall A \subset\subset \Omega.$$

In particular,  $v_k \rightarrow 0$  in  $H_{\text{loc}}^1(\Omega)$ , which implies there exists a polar set  $P \subset \Omega$  such that

$$\tilde{v}_k(x) = \frac{1}{k} \widetilde{T_k(u)}(x) \rightarrow 0, \quad \forall x \in \Omega \setminus P. \quad (2.7)$$

Set

$$w(x) := \begin{cases} \sup_{k \in \mathbb{N}} \left\{ \widetilde{T_k(u)}(x) \right\} & \text{if } \sup_{k \in \mathbb{N}} \left| \widetilde{T_k(u)}(x) \right| < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

so that  $w = u$  a.e. in  $\Omega$ . By (2.7) and the quasicontinuity of the functions  $\widetilde{T_k(u)}$ , it is easy to see that  $w$  is quasicontinuous in  $\Omega$ . This concludes the proof of the lemma.

## 2.4 A variant of Kato's inequality when $\Delta u$ is a Radon measure

We start with the following (see [4])

**Lemma 2.2** *Assume  $u \in L^1(\Omega)$ ,  $u \geq 0$  a.e. in  $\Omega$ , is such that  $\Delta u$  is a Radon measure on  $\Omega$ . Then,*

$$\Delta T_k(u) \quad \text{is a Radon measure,} \quad \forall k > 0.$$

Moreover, for any  $a \in L^\infty(\Omega)$ ,  $a \geq 0$  a.e. in  $\Omega$ , we have

$$\Delta T_k(u) - aT_k(u) \leq (\Delta u - au)^+ \quad \text{in } \mathcal{D}'(\Omega). \quad (2.8)$$

**Proof.** We shall use the same notation as in the proof of Lemma 2.1. By the standard  $L^1$ -version of Kato's inequality (see [61]) we have (note that  $T_k|_{\mathbb{R}_+}$  is concave)

$$\Delta T_k(u_\varepsilon) \leq t_k(u_\varepsilon) \Delta u_\varepsilon \quad \text{in } \Omega, \quad \forall \varepsilon > 0, \quad (2.9)$$

where the function  $t_k : \mathbb{R}_+ \rightarrow \mathbb{R}$  is given by

$$t_k(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq k, \\ 0 & \text{if } s > k. \end{cases}$$

Since  $T_k(s) \geq t_k(s)s$  for every  $s \geq 0$ , and  $a \geq 0$  a.e. in  $\Omega$ , it follows from (2.9) that

$$\Delta T_k(u_\varepsilon) - aT_k(u_\varepsilon) \leq t_k(u_\varepsilon)(\Delta u_\varepsilon - au_\varepsilon) \leq (\Delta u_\varepsilon - au_\varepsilon)^+ \quad \text{in } \mathcal{D}'(\Omega).$$

In other words, we have

$$\int_{\Omega} \left[ T_k(u_\varepsilon) \Delta \varphi - aT_k(u_\varepsilon) \varphi \right] \leq \int_{\Omega} (\Delta u_\varepsilon - au_\varepsilon)^+ \varphi, \quad (2.10)$$

for every  $\varphi \in C_0^\infty(\Omega)$ , such that  $\varphi \geq 0$  in  $\Omega$ .

For  $\lambda > 0$ , let  $\Omega_\lambda := \{x \in \Omega : d(x, \partial\Omega) > \lambda\}$ . Thus, if  $0 < \varepsilon < \lambda$ , then we get

$$\begin{aligned} \Delta u_\varepsilon - au_\varepsilon &= (\Delta u - au)_\varepsilon + (au)_\varepsilon - au_\varepsilon \\ &\leq \rho_\varepsilon * (\Delta u - au)^+ + |(au)_\varepsilon - au| + |au - au_\varepsilon| \quad \text{in } \Omega_\lambda. \end{aligned}$$

Therefore, for any  $\varphi \in C_0^\infty(\Omega)$ ,  $\varphi \geq 0$  in  $\Omega$ , and  $0 < \varepsilon < \text{dist}(\text{supp } \varphi, \partial\Omega)$ , we may write

$$\begin{aligned} \int_\Omega (\Delta u_\varepsilon - au_\varepsilon)^+ \varphi &\leq \int_\Omega \rho_\varepsilon * (\Delta u - au)^+ \varphi + \\ &\quad + \|\varphi\|_{L^\infty} \left\{ \|(au)_\varepsilon - au\|_{L^1} + \|a\|_{L^\infty} \|u - u_\varepsilon\|_{L^1} \right\} \quad (2.11) \\ &= \int_\Omega (\rho_\varepsilon * \varphi) (\Delta u - au)^+ + o(1). \end{aligned}$$

Since  $\rho_\varepsilon * \varphi \rightarrow \varphi$  uniformly in  $\Omega$  and  $(\Delta u - au)^+$  is a Radon measure in  $\Omega$ , by letting  $\varepsilon \downarrow 0$  in (2.10) and (2.11), we conclude that

$$\int_\Omega \left[ T_k(u) \Delta \varphi - a T_k(u) \varphi \right] \leq \int_\Omega (\Delta u - au)^+ \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \text{ in } \Omega.$$

Thus,  $\Delta T_k(u)$  is a Radon measure (take for instance  $a = 0$ ), and (2.8) holds.

**Lemma 2.3** *Assume  $u \in L^1(\Omega)$ ,  $u \geq 0$  a.e. in  $\Omega$ , is such that  $\Delta u$  is a Radon measure on  $\Omega$ . Let  $a \in L^1(\Omega)$ ,  $a \geq 0$  a.e. in  $\Omega$ . If*

$$-\Delta u + au \geq 0 \quad \text{in } \Omega,$$

*in the following sense*

$$\int_E \Delta u \leq \int_E au \quad \text{for every Borel set } E \subset \Omega, \quad (2.12)$$

*then*

$$-\Delta T_k(u) + a T_k(u) \geq 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \forall k > 0. \quad (2.13)$$

**Proof.** By the preceding lemma applied with  $a_i := T_i(a)$ , where  $i$  is a positive integer, we know that

$$\Delta T_k(u) - a_i T_k(u) \leq (\Delta u - a_i u)^+ \quad \text{in } \mathcal{D}'(\Omega). \quad (2.14)$$

On the other hand, from (2.12) we get

$$\int_E (\Delta u - a_i u) \leq \int_E (a - a_i) u \quad \text{for every Borel set } E \subset \Omega. \quad (2.15)$$

Since  $(a - a_i)u \geq 0$  a.e. in  $\Omega$ , (2.15) implies that

$$0 \leq \int_E (\Delta u - a_i u)^+ \leq \int_E (a - a_i) u \quad \text{for every Borel set } E \subset \Omega. \quad (2.16)$$

Hence,  $(\Delta u - a_i u)^+$  is a nonnegative measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, we have

$$(\Delta u - a_i u)^+ \in L^1(\Omega), \quad \forall i = 1, 2, \dots \quad (2.17)$$

We now return to (2.16) to conclude that

$$0 \leq (\Delta u - a_i u)^+ \leq (a - a_i)u \quad \text{a.e. in } \Omega.$$

In particular,

$$(\Delta u - a_i u)^+ \downarrow 0 \quad \text{a.e. in } \Omega \text{ as } i \rightarrow \infty. \quad (2.18)$$

It follows from (2.17) and (2.18) that

$$(\Delta u - a_i u)^+ \rightarrow 0 \quad \text{in } L^1(\Omega) \text{ as } i \rightarrow \infty. \quad (2.19)$$

Finally, for any  $\varphi \in C_0^\infty(\Omega)$ ,  $\varphi \geq 0$  in  $\Omega$ , by (2.14) and (2.19) we have

$$\int_{\Omega} \left[ T_k(u) \Delta \varphi - a T_k(u) \varphi \right] \leq \int_{\Omega} T_k(u) \Delta \varphi - a_i T_k(u) \varphi \leq \int_{\Omega} (\Delta u - a_i u)^+ \varphi \rightarrow 0$$

as  $i \rightarrow \infty$ , so that (2.13) holds.

## 2.5 Proof of Theorem 2.1 completed

It follows from Lemma 2.1 in Section 2.3 above that there exists  $\tilde{u} : \Omega \rightarrow \mathbb{R}$  quasicontinuous such that  $u = \tilde{u}$  a.e. in  $\Omega$ . Let us assume that  $\tilde{u} = 0$  on a set of positive capacity  $E \subset \Omega$ . We shall prove that  $u = 0$  a.e. in  $\Omega$ .

We split the proof into two steps :

**Step 1.** Assume in addition that  $u \in L^\infty(\Omega)$ , then  $u = 0$  a.e. in  $\Omega$ .

Since  $u \in L^\infty(\Omega)$ , we have  $au \in L^1(\Omega)$ . It follows from (2.1) and Remark 2.3 that

$$-\Delta u + au \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Let  $0 < \varepsilon < \lambda$ . In  $\Omega_\lambda$ , we have

$$\begin{aligned} \Delta u_\varepsilon &\leq (au)_\varepsilon = au_\varepsilon + [(au)_\varepsilon - au_\varepsilon] \\ &\leq au_\varepsilon + [(au)_\varepsilon - au_\varepsilon]^+ =: au_\varepsilon + f_\varepsilon. \end{aligned} \quad (2.20)$$

Since  $(au)_\varepsilon \rightarrow au$  in  $L^1(\Omega)$ ,  $u_\varepsilon \rightarrow u$  a.e. in  $\Omega$ , and  $u$  is bounded,

$$f_\varepsilon \rightarrow 0 \quad \text{in } L^1(\Omega). \quad (2.21)$$

Let  $\delta > 0$  be a fixed number. Multiplying (2.20) by  $\frac{1}{u_\varepsilon + \delta}$ , we get

$$\frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \leq a + \frac{f_\varepsilon}{\delta} \quad \text{in } \Omega_\lambda, \quad \forall \varepsilon \in (0, \lambda). \quad (2.22)$$

We also remark that

$$\frac{\nabla u_\varepsilon}{(u_\varepsilon + \delta)^2} = -\nabla \left( \frac{1}{u_\varepsilon + \delta} \right) \quad \text{in } \Omega. \quad (2.23)$$

Let  $\varphi \in C_0^\infty(\Omega)$  and  $0 < \varepsilon < \text{dist}(\text{supp } \varphi, \partial\Omega)$ . We now multiply both sides of (2.23) by  $\varphi^2$  and integrate by parts on  $\Omega$  to get

$$\begin{aligned} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 &= - \int_\Omega \nabla u_\varepsilon \cdot \nabla \left( \frac{1}{u_\varepsilon + \delta} \right) \varphi^2 \\ &= \int_\Omega \frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \varphi^2 + \int_\Omega \frac{2\varphi \nabla \varphi \cdot \nabla u_\varepsilon}{u_\varepsilon + \delta} \\ &\leq \int_\Omega \left( a + \frac{f_\varepsilon}{\delta} \right) \varphi^2 + \frac{1}{2} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 + 2 \int_\Omega |\nabla \varphi|^2, \end{aligned}$$

where in the last inequality we applied (2.22) and Cauchy-Schwarz. Therefore,

$$\frac{1}{2} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2} \varphi^2 \leq \int_\Omega \left( a + \frac{f_\varepsilon}{\delta} \right) \varphi^2 + 2 \int_\Omega |\nabla \varphi|^2.$$

Since

$$\nabla \log \left( \frac{u_\varepsilon}{\delta} + 1 \right) = \frac{\nabla u_\varepsilon}{u_\varepsilon + \delta},$$

the estimate above may be rewritten as

$$\frac{1}{2} \int_\Omega \left| \nabla \log \left( \frac{u_\varepsilon}{\delta} + 1 \right) \right|^2 \varphi^2 \leq \int_\Omega \left( a + \frac{f_\varepsilon}{\delta} \right) \varphi^2 + 2 \int_\Omega |\nabla \varphi|^2. \quad (2.24)$$

We now let  $\varepsilon \downarrow 0$  in (2.24). It follows from (2.21) that

$$\log \left( \frac{u}{\delta} + 1 \right) \in H_{\text{loc}}^1(\Omega), \quad \forall \delta > 0,$$

and

$$\frac{1}{2} \int_\Omega \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \varphi^2 \leq \int_\Omega (a\varphi^2 + 2|\nabla \varphi|^2), \quad \forall \varphi \in C_0^\infty(\Omega). \quad (2.25)$$

Let  $E \subset \Omega$  be a set of positive capacity such that  $\tilde{u} = 0$  on  $E$ . Without any loss of generality, we may assume that  $E \subset \Omega_\lambda$  for some  $\lambda > 0$  sufficiently small.

Assume  $\omega \subset\subset \Omega$  is an open connected set containing  $E$ . Let  $\varphi_0 \in C_0^\infty(\Omega)$  be a fixed test function such that  $\varphi = 1$  on  $\omega$ .

By (2.25), we have

$$\int_\omega \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2 \leq 2 \int_\Omega (a\varphi_0^2 + 2|\nabla \varphi_0|^2). \quad (2.26)$$

Since the quasicontinuous representative of  $\log \left( \frac{u}{\delta} + 1 \right)$ ; namely,

$$\widetilde{\log \left( \frac{u}{\delta} + 1 \right)} = \log \left( \frac{\tilde{u}}{\delta} + 1 \right),$$

equals 0 on  $E \subset \Omega$  with  $\text{cap}_2(E) > 0$ , it follows from a variant of Poincaré's inequality (easily proved by contradiction) that there exists  $C > 0$  (depending only on  $E$  and  $\Omega$ ) such that

$$\int_{\omega} \log^2 \left( \frac{u}{\delta} + 1 \right) \leq C \int_{\omega} \left| \nabla \log \left( \frac{u}{\delta} + 1 \right) \right|^2, \quad \forall \delta > 0. \quad (2.27)$$

Equations (2.26) and (2.27) yield

$$\int_{\omega} \log^2 \left( \frac{u}{\delta} + 1 \right) \leq 2C \int_{\Omega} (a\varphi_0^2 + 2|\nabla\varphi_0|^2), \quad \forall \delta > 0. \quad (2.28)$$

In particular, the integral in the left-hand side remains bounded as  $\delta \downarrow 0$ . On the other hand,

$$\log^2 \left( \frac{u}{\delta} + 1 \right) \rightarrow +\infty \quad \text{a.e. in } \omega \setminus [u = 0] \text{ as } \delta \downarrow 0. \quad (2.29)$$

By (2.28) and (2.29), we conclude that  $u = 0$  a.e. in  $\omega$ . Since  $\omega$  is an arbitrary connected neighborhood of  $E$  in  $\Omega_\lambda$  for all  $\lambda > 0$  small, we conclude that  $u = 0$  a.e. in  $\Omega$ .

**Step 2.** Proof of Theorem 2.1 completed.

From Lemma 2.3, we know that  $\Delta T_1(u)$  is a Radon measure and

$$-\Delta T_1(u) + aT_1(u) \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

In addition,  $\widetilde{T_1(u)} = T_1(\tilde{u}) = 0$  on  $E \subset \Omega$  with  $\text{cap}_2(E) > 0$ .

By Step 1, we have  $T_1(u) = 0$  a.e. in  $\Omega$ , and so  $u = 0$  a.e. in  $\Omega$ .



# Chapitre 3

## Singularities of positive supersolutions in elliptic PDEs

### Sommaire

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### 3.1 Introduction<sup>4</sup>

When can the set of singularities of a solution to a linear (or quasi-linear) elliptic equation be removed? To shed some light on this question, let us first recall a classical result in Potential Theory.

Throughout this chapter, we assume that  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , is a bounded domain and  $\Sigma \subset \Omega$  is a compact subset.

Let us assume that  $\text{cap}_2(\Sigma) = 0$ , where  $\text{cap}_2$  denotes the standard  $H^1$ -capacity (see Section 3.2 below). Let  $u \in H_{\text{loc}}^1(\Omega \setminus \Sigma)$  be a nonnegative function such that

$$-\Delta u \geq 0 \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma).$$

Note that no information is given about  $u$  on the set  $\Sigma$ . Nevertheless, it is well-known that the function  $u$  actually belongs to  $L_{\text{loc}}^1(\Omega)$  and satisfies

$$-\Delta u \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.1}$$

See, e.g., [59, Theorem 7.7]. Note that if  $\text{cap}_2(\Sigma) > 0$ , then (3.1) will no longer hold in general.

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<sup>4</sup>Ce chapitre a été écrit en collaboration avec L. Dupaigne; le texte original est déjà sorti dans *Selecta Math. (N.S.)* **10** (2004), 341–358.



Our first theorem extends this classical result to the operator  $-\Delta + c$ , with  $c \in \mathbb{R}$ . It also generalizes a previous work of Brezis and Lions [29] (see also [40]), who considered the case where  $\Sigma$  is a point :

**Theorem 3.1** *Assume that  $\text{cap}_2(\Sigma) = 0$ . Let  $c \in \mathbb{R}$  and  $f \in L^1_{\text{loc}}(\Omega)$ . If  $u \in L^1_{\text{loc}}(\Omega \setminus \Sigma)$ ,  $u \geq 0$  a.e. in  $\Omega$ , satisfies*

$$-\Delta u + cu \geq f \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma), \quad (3.2)$$

then  $u \in L^1_{\text{loc}}(\Omega)$  and

$$-\Delta u + cu \geq f \quad \text{in } \mathcal{D}'(\Omega). \quad (3.3)$$

We would like to emphasize that above we do **not** assume that  $\Delta u \in L^1_{\text{loc}}(\Omega \setminus \Sigma)$ .

**Remark 3.1** It follows from Theorem 3.1 that  $\Delta u \in \mathcal{M}_{\text{loc}}(\Omega)$ ; thus,  $u \in W^{1,p}_{\text{loc}}(\Omega)$  for all  $1 \leq p < \frac{N}{N-1}$  (see Corollary 3.2 below). This regularity result is very standard and just follows from (3.3).

An interesting consequence of Theorem 3.1 is the following :

**Corollary 3.1** *Assume that  $\text{cap}_2(\Sigma) = 0$ . Let  $c \in \mathbb{R}$  and  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous nonnegative function.*

*Let  $u \in L^1_{\text{loc}}(\Omega \setminus \Sigma)$ ,  $u \geq 0$  a.e. in  $\Omega$ , be such that  $g(u) \in L^1_{\text{loc}}(\Omega \setminus \Sigma)$  and*

$$-\Delta u + cu \geq g(u) \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma). \quad (3.4)$$

Then,  $u, g(u) \in L^1_{\text{loc}}(\Omega)$  and

$$-\Delta u + cu \geq g(u) \quad \text{in } \mathcal{D}'(\Omega). \quad (3.5)$$

This corollary can be interpreted as a linear version of a very general result of Baras and Pierre [5] about removable singularities. Note that we do not impose any asymptotic behavior on  $g(t)$  as  $t \rightarrow \infty$ .

We recall that any Radon measure  $\mu$  in  $\mathbb{R}^N$  can be decomposed as a sum  $\mu = \mu_a + \mu_s$ , where  $\mu_a$  and  $\mu_s$  are the absolutely continuous and the singular parts of  $\mu$  with respect to the Lebesgue measure. There are several other possible decompositions of  $\mu$ , however. A less standard one is given by (see [14] and also [51])

$$\mu = \mu_d + \mu_c,$$

where

$$\begin{aligned} \mu_d(A) &= 0 && \text{for any Borel measurable set } A \subset \Omega \text{ such that } \text{cap}_2(A) = 0, \\ |\mu_c|(\Omega \setminus F) &= 0 && \text{for some Borel measurable set } F \subset \Omega \text{ such that } \text{cap}_2(F) = 0. \end{aligned}$$

In particular, the Radon measures  $\mu_d$  and  $\mu_c$  are singular with respect to each other.

Using the above notation, we have

**Theorem 3.2 (“Inverse” maximum principle)** *Let  $u \in L^1_{\text{loc}}(\Omega)$  with  $u \geq 0$  a.e. in  $\Omega$ . Assume that  $\Delta u$  is a Radon measure in  $\Omega$ . Then,*

$$(-\Delta u)_c \geq 0 \quad \text{in } \Omega. \quad (3.6)$$

This “inverse” maximum principle plays a very important role in the study of the nonlinear problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \quad (3.7)$$

where  $\mu$  is a nonnegative Radon measure and  $g$  is a continuous increasing function with  $g(0) = 0$ . In a joint work with Brezis and Marcus [30], we established that (3.7) need not have a solution for every measure  $\mu \geq 0$ . However, the following holds :

**Theorem 3.3 (Brezis-Marcus-Ponce [30])** *For every Radon measure  $\mu \geq 0$ , there exists a largest positive measure  $\mu^* \leq \mu$  for which (3.7) has a (unique) solution. Moreover, we always have  $\mu^*_d = \mu_d$ .*

We refer the reader to [30] for a proof of Theorem 3.3. We point out that  $\mu^*$  strongly depends on the nonlinearity  $g$ . In [30], we also study several properties of the mapping  $\mu \mapsto \mu^*$ ; for instance, monotonicity, contraction, etc.

We now return to Theorem 3.1. This result can be extended to other second order linear elliptic operators. In what follows, we use Einstein’s summation convention.

**Theorem 3.4** *Assume that  $\text{cap}_2(\Sigma) = 0$ . For  $i, j \in \{1, \dots, N\}$ , let  $a^{ij}, b^i, c \in L^\infty(\Omega)$ ,  $f \in L^1(\Omega)$ , and  $g^i \in L^2(\Omega)$ , where the coefficients  $a^{ij}$  are locally Lipschitz continuous in  $\Omega \setminus \Sigma$  and satisfy the uniform ellipticity condition*

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^N,$$

for some  $\lambda > 0$ .

If  $u \in W^{1,1}_{\text{loc}}(\Omega \setminus \Sigma)$ ,  $u \geq 0$  a.e. in  $\Omega$ , is such that

$$-\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu \geq f + \partial_i g^i \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma),$$

then  $u \in W^{1,1}_{\text{loc}}(\Omega)$  and

$$-\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu \geq f + \partial_i g^i \quad \text{in } \mathcal{D}'(\Omega).$$

Theorem 3.4 can be further generalized to the setting of quasi-linear elliptic equations as follows :

Let  $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  and  $B : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  be two Carathéodory functions. A weakly differentiable function  $v$  in  $\omega \subset \Omega$  is a supersolution of

$$-\text{div } A(x, v, \nabla v) \geq B(x, v, \nabla v) \quad \text{in } \mathcal{D}'(\omega)$$

if  $A^i(x, v, \nabla v), B(x, v, \nabla v)$  are locally integrable in  $\omega$  and

$$\int_{\omega} A^i(x, v, \nabla v)\partial_i \varphi \geq \int_{\omega} B(x, v, \nabla v)\varphi, \quad \forall \varphi \in C_0^\infty(\omega), \varphi \geq 0 \text{ in } \omega.$$

We shall assume in the sequel that  $1 < p \leq N$ , and that for a.e.  $x \in \Omega$  and for every  $r \geq 0$ ,  $q \in \mathbb{R}^N$ , we have

$$|A(x, r, q)| \leq a_0|q|^{p-1} + a_1(x)r^{p-1} + g(x), \quad (3.8)$$

$$-B(x, r, q) \leq b_0(x)|q|^{p-1} + b_1(x)r^{p-1} + f(x), \quad (3.9)$$

$$A(x, r, q) \cdot q \geq |q|^p - c_1(x)r^p - c_2(x), \quad (3.10)$$

where  $a_0 \geq 0$  is constant,  $a_i, b_i, c_i \in L^\infty(\Omega)$ ,  $f \in L^1(\Omega)$ , and  $g \in L^{p/(p-1)}(\Omega)$  are nonnegative functions.

Under these assumptions, we have the following

**Theorem 3.5** *Suppose that  $\text{cap}_p(\Sigma) = 0$ . If  $u \in W_{\text{loc}}^{1,p}(\Omega \setminus \Sigma)$ ,  $u \geq 0$  a.e. in  $\Omega$ , satisfies*

$$-\text{div } A(x, u, \nabla u) \geq B(x, u, \nabla u) \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma), \quad (3.11)$$

then

$$u^{p-1}, |\nabla u|^{p-1} \in L_{\text{loc}}^1(\Omega). \quad (3.12)$$

Furthermore,  $A^i(x, u, \nabla u), B(x, u, \nabla u) \in L_{\text{loc}}^1(\Omega)$  and  $u$  satisfies

$$-\text{div } A(x, u, \nabla u) \geq B(x, u, \nabla u) \quad \text{in } \mathcal{D}'(\Omega). \quad (3.13)$$

Here,  $\text{cap}_p(\Sigma)$  denotes the  $W^{1,p}$ -capacity of  $\Sigma$  (see Definition 3.1 below).

**Remark 3.2** The meaning of  $\nabla u$  in  $\Omega$  requires some clarification. In fact, since  $u$  belongs to  $W_{\text{loc}}^{1,p}(\Omega \setminus \Sigma)$  and  $|\Sigma| = 0$ , then  $\nabla u$  is well-defined a.e. in  $\Omega$ . We take this as the **definition** of  $\nabla u$  in  $\Omega$ , even if  $u$  is not (locally) weakly differentiable in the whole domain  $\Omega$ . By Corollary 3.2 below, if  $p > 2 - \frac{1}{N}$ , then  $|\nabla u| \in L_{\text{loc}}^1(\Omega)$ . In this case, we can conclude that  $u \in W_{\text{loc}}^{1,1}(\Omega)$  and  $\nabla u$  is the weak derivative of  $u$  in  $\Omega$  (see Lemma 3.3).

**Remark 3.3** The fact that  $A^i(x, u, \nabla u) \in L_{\text{loc}}^1(\Omega)$  is a direct consequence of (3.8) and (3.12). The corresponding property for  $B(x, u, \nabla u)$  requires some additional argument.

The proof of Theorem 3.5 relies on a standard Moser iteration technique in the spirit of [82]. The same idea has been used by Serrin [79] to show that if  $u$  is a **solution** of

$$-\text{div } A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma)$$

and  $\text{cap}_p(\Sigma) = 0$ , then  $u$  is Hölder continuous in  $\Omega$  and satisfies

$$-\text{div } A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \mathcal{D}'(\Omega).$$

Back to the case of supersolutions of (3.11), once (3.13) is established, then it is well-known that the regularity result (3.12) can be improved. As we shall see in Section 3.5, we have

**Corollary 3.2** *Under the assumptions of Theorem 3.5, if  $u \in W_{\text{loc}}^{1,p}(\Omega \setminus \Sigma)$ ,  $u \geq 0$  a.e. in  $\Omega$ , satisfies (3.11), then*

$$u^{p-1} \in L_{\text{loc}}^q(\Omega), \quad \forall 1 \leq q < \frac{N}{N-p}, \quad (3.14)$$

$$|\nabla u|^{p-1} \in L_{\text{loc}}^r(\Omega), \quad \forall 1 \leq r < \frac{N}{N-1}. \quad (3.15)$$

Theorem 3.5 extends results of Bidaut-Véron [13] and also of Kilpeläinen [62] on removable singularities for the  $p$ -Laplace operator :

**Corollary 3.3** *Assume that  $\text{cap}_p(\Sigma) = 0$ . Let  $c \in \mathbb{R}$  and  $f \in L_{\text{loc}}^1(\Omega)$ . If  $u \in W_{\text{loc}}^{1,p}(\Omega \setminus \Sigma)$ ,  $u \geq 0$  a.e. in  $\Omega$ , satisfies*

$$-\Delta_p u + cu^{p-1} \geq f \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma),$$

then

$$u^{p-1}, |\nabla u|^{p-1} \in L_{\text{loc}}^1(\Omega)$$

and we have

$$-\Delta_p u + cu^{p-1} \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

## 3.2 Some remarks about the $p$ -capacity

Given  $1 \leq p < +\infty$ , we first recall the definition of the  $p$ -capacity :

**Definition 3.1** *The  $p$ -capacity of a compact set  $\Sigma \subset \Omega$  is defined as*

$$\text{cap}_p(\Sigma) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ in some neighborhood of } \Sigma \right\}.$$

It follows from Definition 3.1 that if  $\text{cap}_p(\Sigma) = 0$ , then  $\text{cap}_q(\Sigma) = 0$  for every  $1 \leq q < p$ . We next point out that in this definition we could have restricted ourselves to a smaller class of functions  $\varphi$ . Namely, we have

$$\text{cap}_p(\Sigma) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p : \varphi \in C_0^\infty(\Omega), 0 \leq \varphi \leq 1 \text{ in } \Omega, \varphi = 1 \text{ in some ngbd of } \Sigma \right\}.$$

Let indeed  $(\varphi_n) \subset C_0^\infty(\Omega)$ ,  $\varphi_n \geq 1$  near  $\Sigma$ , be a minimizing sequence for  $\text{cap}_p(\Sigma)$ . Define  $v_n = \min\{\varphi_n^+, 1\}$  and observe that  $v_n = 1$  in a neighborhood of  $\Sigma$ . Denoting by  $(\rho_\varepsilon)$  a sequence of standard mollifiers, it follows that for  $\varepsilon = \varepsilon_n$  small enough,  $w_n := v_n * \rho_{\varepsilon_n}$  also satisfies  $w_n = 1$  in a neighborhood of  $\Sigma$ . Also  $w_n \in C_0^\infty(\Omega)$ ,  $w_n \geq 0$  in  $\Omega$ , and

$$\int |\nabla w_n|^p \leq \int |\nabla \varphi_n|^p \rightarrow \text{cap}_p(\Sigma).$$

We also observe that if  $\text{cap}_p(\Sigma) = 0$ , then  $|\Sigma| = 0$ . Indeed, it follows from Poincaré's inequality that for any nonnegative  $\varphi \in C_0^\infty(\Omega)$  such that  $\varphi = 1$  on  $\Sigma$ , we have

$$|\Sigma| \leq \int_{\Omega} \varphi^p \leq C \int_{\Omega} |\nabla \varphi|^p.$$

Taking the infimum with respect to  $\varphi$ , we conclude that  $|\Sigma| = 0$ .

This result can be refined in more geometric terms (see [49] and also [47]) :

- Lemma 3.1** (i)  $\mathcal{H}^{N-1}(\Sigma) = 0$  if, and only if,  $\text{cap}_1(\Sigma) = 0$  ;  
(ii) if  $1 < p \leq N$  and  $\mathcal{H}^{N-p}(\Sigma) < \infty$ , then  $\text{cap}_p(\Sigma) = 0$  ;  
(iii) if  $1 < p \leq N$  and  $\text{cap}_p(\Sigma) = 0$ , then  $\mathcal{H}^s(\Sigma) = 0$  for every  $s > N - p$  ;  
(iv) if  $p > N$  and  $\text{cap}_p(\Sigma) = 0$ , then  $\Sigma = \emptyset$ .

Note that (iv) is just a consequence of Morrey's inequality. In fact, if  $p > N$  and  $(\varphi_n) \subset C_0^\infty(\Omega)$  is such that  $\int |\nabla \varphi_n|^p \rightarrow 0$ , then  $(\varphi_n)$  converges uniformly to 0 as  $n \rightarrow \infty$ . Since  $\varphi_n \geq 1$  on  $\Sigma$ , we must have  $\Sigma = \emptyset$ . This shows in particular why, as mentioned earlier, we restrict ourselves to the case  $p \leq N$ .

As a corollary of Lemma 3.1 (ii), we have the following :

**Corollary 3.4** Let  $1 < p \leq N$ . If  $\Sigma$  is contained in some manifold of codimension  $k \geq p$ , then  $\text{cap}_p(\Sigma) = 0$ .

We shall make use of the following two simple lemmas :

**Lemma 3.2** Suppose  $\text{cap}_p(\Sigma) = 0$ . Given  $\psi \in C_0^\infty(\Omega)$ , there exists  $(\psi_n) \subset C_0^\infty(\Omega \setminus \Sigma)$  such that

$$|\psi_n| \leq |\psi| \quad \text{in } \Omega \quad \text{and} \quad \psi_n \rightarrow \psi \quad \text{in } W^{1,p}(\Omega).$$

If  $\psi \geq 0$  in  $\Omega$ , then  $(\psi_n)$  can be chosen so that each  $\psi_n$  is nonnegative in  $\Omega$ .

**Lemma 3.3** Suppose  $\text{cap}_1(\Sigma) = 0$ . If  $u \in W^{1,p}(\Omega \setminus \Sigma)$ , then  $u \in W^{1,p}(\Omega)$ .

**Proof of Lemma 3.2.** It suffices to take  $\psi_n := (1 - \varphi_n)\psi$ , where  $(\varphi_n) \subset C_0^\infty(\Omega)$  is such that  $0 \leq \varphi_n \leq 1$  in  $\Omega$ ,  $\varphi_n = 1$  in some neighborhood of  $\Sigma$ , and  $\int |\nabla \varphi_n|^p \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proof of Lemma 3.3.** We split the proof into two steps :

**Step 1.** Assume in addition that  $u$  is bounded.

We first show that  $u$  is weakly differentiable in  $\Omega$ . In fact, since  $u$  is weakly differentiable in  $\Omega \setminus \Sigma$ , for each  $i = 1, \dots, N$  we have

$$\int_{\Omega} u \partial_i \varphi = - \int_{\Omega} \partial_i u \varphi, \quad \forall \varphi \in C_0^\infty(\Omega \setminus \Sigma).$$

Given  $\psi \in C_0^\infty(\Omega)$ , it follows from the previous lemma that we can find a uniformly bounded sequence  $(\psi_n)$  in  $C_0^\infty(\Omega \setminus \Sigma)$  converging to  $\psi$  in  $W^{1,1}(\Omega)$ . We now replace  $\varphi$  by  $\psi_n$  in the above identity. Passing to the limit as  $n$  goes to  $\infty$ , we find

$$\int_{\Omega} u \partial_i \psi = - \int_{\Omega} \partial_i u \psi, \quad \forall \psi \in C_0^\infty(\Omega).$$

In particular,  $\partial_i u$  gives the weak derivative of  $u$  in  $\Omega$ . Since

$$\int_{\Omega} |\nabla u|^p = \int_{\Omega \setminus \Sigma} |\nabla u|^p < \infty,$$

we conclude that  $u \in W^{1,p}(\Omega)$ .

**Step 2.** Proof of the lemma completed.

By working with the positive and negative parts of  $u$ , we may always assume that  $u \geq 0$ . For every  $k > 0$ , let now  $u_k = \min\{u, k\}$ , so that  $u_k \in W^{1,p}(\Omega \setminus \Sigma)$ . It then follows from the previous step that  $u_k \in W^{1,p}(\Omega)$  and

$$\int_{\Omega} u_k \partial_i \psi = - \int_{\Omega} \partial_i u_k \psi = - \int_{[u \leq k]} \partial_i u \psi, \quad \forall \psi \in C_0^\infty(\Omega).$$

Note that  $\partial_i u \in L^p(\Omega \setminus \Sigma) = L^p(\Omega)$  for every  $i = 1, \dots, N$ . As  $k \rightarrow \infty$ , we conclude that  $u$  is weakly differentiable in  $\Omega$  and  $u \in W^{1,p}(\Omega)$ .

We now extend the definition of the  $p$ -capacity for any measurable subset of  $\Omega$ . For simplicity, we only consider the case  $p = 2$ .

**Definition 3.2** *Given an open set  $\omega \subset \Omega$ , we define*

$$\text{cap}_2(\omega) := \sup \left\{ \text{cap}_2(K) : K \text{ is compact and } K \subset \omega \right\}.$$

*For any Borel measurable set  $F \subset \Omega$ , we let*

$$\text{cap}_2(F) := \inf \left\{ \text{cap}_2(\omega) : \omega \text{ is open and } F \subset \omega \subset \Omega \right\}.$$

One can easily see that Definition 3.2 agrees with Definition 3.1 when  $F \subset \Omega$  is compact. We also observe that if  $F_1 \subset F_2 \subset \Omega$ , then  $\text{cap}_2(F_1) \leq \text{cap}_2(F_2)$ .

### 3.3 Proof of Theorems 3.1 and 3.4

The proof of Theorem 3.1 (and also of Theorem 3.4) is essentially contained in Section 3.5. However, it is enlightening to go through this special case before proving the more general result.

Below, we shall denote by  $u_k$  the function  $\min\{u, k\}$ . Let us first state and prove the following lemma :

**Lemma 3.4** *Let  $u \in L^1(\omega)$ ,  $u \geq 0$  a.e. in  $\omega$ , be such that*

$$-\Delta u \geq h \quad \text{in } \mathcal{D}'(\omega), \quad (3.16)$$

*where  $h \in L^1(\omega)$ . Then,  $u_k \in H_{\text{loc}}^1(\omega)$  and*

$$-\Delta u_k \geq h \chi_{[u < k]} \quad \text{in } \mathcal{D}'(\omega). \quad (3.17)$$

*Moreover, for every  $k > 0$ , we have*

$$\int_{\omega} |\nabla u_k|^2 \varphi^2 \leq 2(k+1)^2 \int_{\omega} (|h| \varphi^2 + 2|\nabla \varphi|^2), \quad \forall \varphi \in C_0^\infty(\omega), \varphi \geq 0 \text{ in } \omega. \quad (3.18)$$

**Proof.** We already know from Lemma 1.1 that

$$-\Delta u_k \geq h\chi_{[u < k]} \quad \text{in } \mathcal{D}'(\omega). \quad (3.19)$$

Moreover, since  $\Delta u \in \mathcal{M}_{\text{loc}}(\omega)$ , we have  $u_k \in H_{\text{loc}}^1(\omega)$  by Lemma 2.1. In order to conclude the proof of the lemma, we only need to show that (3.18) holds. Note that this improves our previous estimate (2.3), since (3.18) does not make use of the full norm  $|\Delta u|_{\mathcal{M}}$ .

We first multiply both sides of (3.19) by  $\frac{\varphi^2}{u_k + 1}$ , where  $\varphi \in C_0^\infty(\omega)$ . Integrating by parts the resulting expression, we get

$$\int_{\omega} \nabla u_k \cdot \nabla \left( \frac{\varphi^2}{u_k + 1} \right) \geq - \int_{\omega} |h|\varphi^2. \quad (3.20)$$

The left-hand side of (3.20) can be estimated by

$$\begin{aligned} \int_{\omega} \nabla u_k \cdot \nabla \left( \frac{\varphi^2}{u_k + 1} \right) &= - \int_{\omega} \frac{|\nabla u_k|^2}{(u_k + 1)^2} \varphi^2 + 2 \int_{\omega} \frac{\nabla u_k \cdot \nabla \varphi}{u_k + 1} \varphi \\ &\leq - \frac{1}{2} \int_{\omega} \frac{|\nabla u_k|^2}{(u_k + 1)^2} \varphi^2 + 2 \int_{\omega} |\nabla \varphi|^2. \end{aligned} \quad (3.21)$$

Since  $u_k + 1 \leq k + 1$ , we conclude from (3.20) and (3.21) that

$$\frac{1}{2} \int_{\omega} |\nabla u_k|^2 \varphi^2 \leq (k + 1)^2 \int_{\omega} (|h|\varphi^2 + 2|\nabla \varphi|^2).$$

This is precisely (3.18).

**Proof of Theorem 3.1.** Applying the previous lemma with  $\omega = \Omega \setminus \Sigma$ , we see that  $u_k \in H_{\text{loc}}^1(\Omega \setminus \Sigma)$  for every  $k > 0$  and

$$-\Delta u_k + cu\chi_{[u < k]} \geq f\chi_{[u < k]} \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma). \quad (3.22)$$

In addition,

$$\frac{1}{2} \int_{\Omega} |\nabla u_k|^2 \varphi^2 \leq (k + 1)^2 \int_{\Omega} (|\tilde{h}|\varphi^2 + 2|\nabla \varphi|^2), \quad \forall \varphi \in C_0^\infty(\Omega \setminus \Sigma), \quad (3.23)$$

where  $\tilde{h} = (f - cu)\chi_{[u < k]}$ .

Let  $\psi \in C_0^\infty(\Omega)$ ,  $\psi \geq 0$  in  $\Omega$ . Since  $\text{cap}_2(\Sigma) = 0$ , it follows from Lemma 3.2 that there exists a sequence  $(\varphi_n) \subset C_0^\infty(\Omega \setminus \Sigma)$  such that  $0 \leq \varphi_n \leq \psi$  and  $\varphi_n \rightarrow \psi$  in  $H^1(\Omega)$ . We now replace  $\varphi$  by  $\varphi_n$  in (3.23). Passing to the limit as  $n$  goes to  $\infty$ , we conclude that

$$\frac{1}{2} \int_{\Omega} |\nabla u_k|^2 \psi^2 \leq (k + 1)^2 \int_{\Omega} (|\tilde{h}|\psi^2 + 2|\nabla \psi|^2).$$

Take for instance  $\psi = 1$  in some neighborhood of  $\Sigma$ ; Lemma 3.3 then implies that  $u_k \in H_{\text{loc}}^1(\Omega)$ .

We now use  $\varphi_n$  as a test function in (3.22) :

$$\int_{\Omega} \nabla u_k \cdot \nabla \varphi_n + c \int_{\Omega} u \chi_{[u < k]} \varphi_n \geq \int_{\Omega} f \chi_{[u < k]} \varphi_n.$$

Since  $u_k \in H_{\text{loc}}^1(\Omega)$ , as  $n \rightarrow \infty$  we find that

$$\int_{\Omega} \nabla u_k \cdot \nabla \psi + c \int_{\Omega} u \chi_{[u < k]} \psi \geq \int_{\Omega} f \chi_{[u < k]} \psi, \quad \forall \psi \in C_0^\infty(\Omega), \psi \geq 0 \text{ in } \Omega.$$

In other words,

$$-\Delta u_k + cu \chi_{[u < k]} \geq f \chi_{[u < k]} \quad \text{in } \mathcal{D}'(\Omega). \quad (3.24)$$

Assume for the moment that  $u \in L_{\text{loc}}^1(\Omega)$ . In this case, we are allowed to take  $k \rightarrow \infty$  in (3.24), from which (3.3) follows.

Thus, in order to conclude the proof of Theorem 3.1, we only need to prove that  $u \in L_{\text{loc}}^1(\Omega)$ , which requires a Harnack-type estimate. For this, we multiply both sides of (3.24) by  $\frac{\varphi^2}{(u_k+1)^{2/N}}$ , where  $\varphi \in C_0^\infty(\Omega)$ . Proceeding exactly as in the previous lemma, we obtain

$$\frac{1}{N} \int_{\Omega} \frac{|\nabla u_k|^2}{(u_k+1)^{\frac{N+2}{N}}} \varphi^2 \leq |c| \int_{\Omega} u_k^{\frac{N-2}{N}} \varphi^2 + \int_{\Omega} |f| \varphi^2 + N \int_{\Omega} (u_k+1)^{\frac{N-2}{N}} |\nabla \varphi|^2.$$

We claim that, by choosing  $\varphi$  appropriately, this inequality implies that  $u \in L_{\text{loc}}^1(\Omega)$ . The argument in this case is essentially the same as in the more general setting. For this reason, we shall postpone the details until the next section (see Steps 2 and 3). This concludes the proof of Theorem 3.1.

The proof of Theorem 3.4 follows along the same lines (although a little more technical) and we shall omit it.

## 3.4 Proof of Corollary 3.1 and Theorem 3.2

**Proof of Corollary 3.1.** Since  $g(u) \geq 0$  a.e. in  $\Omega$ , the function  $u$  satisfies

$$-\Delta u + cu \geq 0 \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma).$$

Applying Theorem 3.1 with  $f = 0$ , we conclude that  $u \in L_{\text{loc}}^1(\Omega)$  and

$$-\Delta u + cu \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

In particular,  $\Delta u$  is a Radon measure in  $\Omega$ . By taking a smaller open set if necessary, we may assume that  $\int_{\Omega} |\Delta u| < \infty$ .

Let  $(\varphi_n)$  be an increasing sequence of nonnegative test functions such that  $0 \leq \varphi_n \leq 1$  in  $\Omega$  and  $\varphi_n(x) \uparrow 1$  for every  $x \in \Omega \setminus \Sigma$ . It follows from (3.4) that

$$\int_{\Omega} g(u) \varphi_n \leq - \int_{\Omega} \varphi_n \Delta u + c \int_{\Omega} \varphi_n u \leq \int_{\Omega} |\Delta u| + |c| \int_{\Omega} |u|.$$



As  $n \rightarrow \infty$ , we conclude that

$$\int_{\Omega} g(u) \leq \int_{\Omega} |\Delta u| + |c| \int_{\Omega} |u| < \infty$$

(recall that  $|\Sigma| = 0$ ). Thus,  $g(u) \in L^1_{\text{loc}}(\Omega)$  and, clearly, (3.5) holds.

Before establishing Theorem 3.2, we state the following variant of Lemma 3.4 :

**Lemma 3.5** *Let  $u \in L^1(\omega)$ ,  $u \geq 0$  a.e. in  $\omega$ , be such that  $\Delta u$  is a Radon measure on  $\Omega$ . Then,  $u_k \in H^1_{\text{loc}}(\omega)$ ,  $\Delta u_k$  is a Radon measure on  $\Omega$ , and*

$$\Delta u_k \leq (\Delta u)^+ \quad \text{in } \mathcal{D}'(\omega). \quad (3.25)$$

**Proof of Theorem 3.2.** It follows from the previous lemma applied with  $\omega = \Omega$  that  $u_k \in H^1_{\text{loc}}(\Omega)$ ,  $\forall k > 0$ .

Let us simply denote  $\Delta u$  by  $\mu$  in  $\Omega$ . We fix a compact set  $K \subset F$ , where  $F$  is a set of zero  $H^1$ -capacity such that  $|\mu_c|(\Omega \setminus F) = 0$ ; in particular,  $\text{cap}_2(K) = 0$ . Applying Lemma 3.5 with  $\omega = \Omega \setminus K$ , we have

$$\Delta u_k \leq \mu^+ \quad \text{in } \mathcal{D}'(\Omega \setminus K). \quad (3.26)$$

Given  $\psi \in C_0^\infty(\Omega)$ ,  $\psi \geq 0$  in  $\Omega$ , let  $(\varphi_n) \subset C_0^\infty(\Omega \setminus K)$  be such that  $0 \leq \varphi_n \leq \psi$  in  $\Omega$  and  $\varphi_n \rightarrow \psi$  in  $H^1(\Omega)$ . Then,

$$\int_{\Omega} \nabla u_k \cdot \nabla \varphi_n \xrightarrow{n \rightarrow \infty} \int_{\Omega} \nabla u_k \cdot \nabla \psi \quad \text{and} \quad \int_{\Omega} \varphi_n d\mu^+ \leq \int_{\Omega \setminus K} \psi d\mu^+, \quad \forall n \geq 1. \quad (3.27)$$

Combining (3.26) and (3.27), we conclude that

$$-\int_{\Omega} \nabla u_k \cdot \nabla \psi \leq \int_{\Omega \setminus K} \psi d\mu^+, \quad \forall \psi \in C_0^\infty(\Omega), \psi \geq 0 \text{ in } \Omega;$$

in other words,

$$\Delta u_k \leq \chi_{\Omega \setminus K} \mu^+ \quad \text{in } \mathcal{D}'(\Omega).$$

As  $k \rightarrow \infty$ , we get

$$\mu = \Delta u \leq \chi_{\Omega \setminus K} \mu^+ \quad \text{in } \mathcal{D}'(\Omega).$$

Thus,

$$\mu_c \llcorner_K = \mu \llcorner_K \leq 0 \quad \text{in } \Omega.$$

Recall that  $K \subset \Omega$  was an arbitrary compact subset of  $F$ . By the inner regularity of Radon measures, we finally conclude that

$$\mu_c \leq 0 \quad \text{in } \Omega.$$

### 3.5 Proof of Theorem 3.5

By assumption, we know that  $A^i(x, u, \nabla u), B(x, u, \nabla u) \in L^1_{\text{loc}}(\Omega \setminus \Sigma)$ , and

$$\int_{\Omega} A^i(x, u, \nabla u) \partial_i \varphi \geq \int_{\Omega} B(x, u, \nabla u) \varphi, \quad \forall \varphi \in C_0^\infty(\Omega \setminus \Sigma), \varphi \geq 0 \text{ in } \Omega.$$

Since  $u \in W^{1,p}_{\text{loc}}(\Omega \setminus \Sigma)$  and  $A$  satisfies (3.8), we actually have

$$A^i(x, u, \nabla u) \in L^{p/(p-1)}_{\text{loc}}(\Omega \setminus \Sigma).$$

It follows from a density argument that

$$\int_{\Omega} A^i(x, u, \nabla u) \partial_i v \geq \int_{\Omega} B(x, u, \nabla u) v \quad (3.28)$$

for every  $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$  such that  $v \geq 0$  a.e. in  $\Omega$  and  $\text{supp } v \subset \Omega \setminus \Sigma$ .

After replacing  $u$  by  $u + 1$ , we can assume that  $u \geq 1$  a.e. in  $\Omega$ . Indeed, the function  $v := u + 1$  satisfies

$$-\text{div } \tilde{A}(x, v, \nabla v) \geq \tilde{B}(x, v, \nabla v) \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma),$$

where  $\tilde{A}(x, r, q) = A(x, r - 1, q)$  and  $\tilde{B}(x, r, q) = B(x, r - 1, q)$ . The functions  $\tilde{A}$  and  $\tilde{B}$  clearly verify assumptions (3.8)–(3.10).

We shall split the proof of Theorem 3.5 into three steps :

**Step 1.** For every  $k \geq 1$ ,  $u_k \in W^{1,p}_{\text{loc}}(\Omega)$ . Moreover, given  $0 \leq \sigma < p - 1$ , we have

$$\int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \psi^p \leq C \left\{ \int_{\Omega} u_k^\sigma (\psi^p + |\nabla \psi|^p) + \int_{\Omega} g^{p/(p-1)} \psi^p + \int_{\Omega} f \psi^p \right\} \quad (3.29)$$

for all  $\psi \in C_0^\infty(\Omega)$ ,  $\psi \geq 0$  in  $\Omega$ , where  $C = C(p, \sigma, \|a_i\|_\infty, \|b_i\|_\infty, \|c_i\|_\infty)$ .

Let  $\varphi \in C_0^\infty(\Omega \setminus \Sigma)$  be such that  $\varphi \geq 0$  in  $\Omega$ . We first apply (3.28) with

$$v = w_k \varphi^p := \left( \frac{1}{u_k^{p-\sigma-1}} - \frac{1}{k^{p-\sigma-1}} \right) \varphi^p \quad \text{in } \Omega.$$

Note in particular that  $v \geq 0$  in  $\Omega$ , and  $v = 0$  a.e. on the set  $[u \geq k]$ ; hence,  $\partial_i v = 0$  a.e. on  $[u \geq k]$ . We have

$$\begin{aligned} \int_{\Omega} B(x, u, \nabla u) v &\leq \int_{\Omega} A^i(x, u, \nabla u) \partial_i v \\ &= \int_{[u < k]} A^i(x, u_k, \nabla u_k) \partial_i (w_k \varphi^p) \\ &= -(p - \sigma - 1) \int_{\Omega} \frac{A^i(x, u_k, \nabla u_k) \partial_i u_k}{u_k^{p-\sigma}} \varphi^p + \\ &\quad + p \int_{\Omega} A^i(x, u_k, \nabla u_k) w_k \partial_i \varphi \varphi^{p-1}. \end{aligned} \quad (3.30)$$

We now apply (3.10) with  $r = u_k$  and  $q = \nabla u_k$ . Multiplying the resulting inequality by  $\frac{\varphi^p}{u_k^{p-\sigma}}$  and integrating over  $\Omega$ , we get

$$\int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p \leq \int_{\Omega} \frac{A^i(x, u_k, \nabla u_k) \partial_i u_k}{u_k^{p-\sigma}} \varphi^p + \int_{\Omega} c_1 u_k^{\sigma} \varphi^p + \int_{\Omega} c_2 \frac{\varphi^p}{u_k^{p-\sigma}}. \quad (3.31)$$

Combining (3.30) and (3.31) yields

$$\int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p \leq I + II + \int_{\Omega} c_1 u_k^{\sigma} \varphi^p + \int_{\Omega} c_2 \varphi^p,$$

where

$$I = -\frac{1}{p-\sigma-1} \int_{\Omega} B(x, u, \nabla u) v, \quad (3.32)$$

$$II = \frac{p}{p-\sigma-1} \int_{\Omega} A^i(x, u_k, \nabla u_k) w_k \partial_i \varphi \varphi^{p-1}. \quad (3.33)$$

We first estimate (3.32). Since  $\sigma < p-1$ , we can apply (3.9) to get

$$I \leq C \int_{\Omega} \left( b_0 |\nabla u|^{p-1} + b_1 u^{p-1} + f \right) v.$$

Recall that  $v = 0$  a.e. on  $[u \geq k]$  and  $v \leq \frac{\varphi^p}{u_k^{p-\sigma-1}}$  a.e. in  $\Omega$ . We then have

$$\begin{aligned} I &\leq C \int_{[u < k]} \left( b_0 |\nabla u|^{p-1} + b_1 u^{p-1} + f \right) \frac{\varphi^p}{u_k^{p-\sigma-1}} \\ &\leq C \left\{ \int_{\Omega} b_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^p + \int_{\Omega} b_1 u_k^{\sigma} \varphi^p + \int_{\Omega} f \frac{\varphi^p}{u_k^{p-\sigma-1}} \right\} \\ &\leq C \left\{ \int_{\Omega} b_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^p + \int_{\Omega} b_1 u_k^{\sigma} \varphi^p + \int_{\Omega} f \varphi^p \right\}. \end{aligned} \quad (3.34)$$

We now estimate the first integral in the right-hand side of (3.34). We first write

$$\int_{\Omega} b_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^p = \int_{\Omega} b_0 \frac{|\nabla u_k|^{p-1}}{u_k^{\frac{(p-\sigma)p-1}{p}}} \varphi^{p-1} \cdot u_k^{\sigma/p} \varphi.$$

For an arbitrary  $\delta > 0$ , it follows from Young's inequality that

$$\int_{\Omega} b_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^p \leq \delta \int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p + C_{\delta} \int_{\Omega} u_k^{\sigma} \varphi^p.$$

Inserting this into (3.34), we obtain

$$I \leq \delta \int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p + C_{\delta} \left\{ \int_{\Omega} u_k^{\sigma} \varphi^p + \int_{\Omega} f \varphi^p \right\}. \quad (3.35)$$

We now consider (3.33). Using (3.8) and arguing as above, we have

$$\begin{aligned}
II &= \frac{p}{p-\sigma-1} \int_{\Omega} A^i(x, u_k, \nabla u_k) w_k \partial_i \varphi \varphi^{p-1} \\
&\leq C \int_{[u < k]} \left( a_0 |\nabla u|^{p-1} + a_1 u^{p-1} + g \right) w_k |\nabla \varphi| \varphi^{p-1} \\
&\leq C \left\{ \int_{\Omega} a_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^{p-1} |\nabla \varphi| + \int_{\Omega} a_1 u_k^{\sigma} \varphi^{p-1} |\nabla \varphi| + \int_{\Omega} g \frac{\varphi^{p-1}}{u_k^{p-\sigma-1}} |\nabla \varphi| \right\} \\
&\leq C \left\{ \int_{\Omega} a_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^{p-1} |\nabla \varphi| + \int_{\Omega} a_1 u_k^{\sigma} \varphi^{p-1} |\nabla \varphi| + \int_{\Omega} g \varphi^{p-1} |\nabla \varphi| \right\}.
\end{aligned} \tag{3.36}$$

On the other hand, given  $\delta > 0$ , it follows from Young's inequality that

$$\int_{\Omega} a_0 \frac{|\nabla u_k|^{p-1}}{u_k^{p-\sigma-1}} \varphi^{p-1} |\nabla \varphi| \leq \delta \int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p + C_{\delta} \int_{\Omega} u_k^{\sigma} |\nabla \varphi|^p. \tag{3.37}$$

In addition,

$$\int_{\Omega} a_1 u_k^{\sigma} \varphi^{p-1} |\nabla \varphi| \leq C \left\{ \int_{\Omega} u_k^{\sigma} \varphi^p + \int_{\Omega} u_k^{\sigma} |\nabla \varphi|^p \right\}, \tag{3.38}$$

$$\int_{\Omega} g \varphi^{p-1} |\nabla \varphi| \leq \int_{\Omega} |\nabla \varphi|^p + \int_{\Omega} g^{p/(p-1)} \varphi^p. \tag{3.39}$$

We now apply (3.36)–(3.39). Since  $u_k \geq 1$  a.e. in  $\Omega$ , we get

$$II \leq \delta \int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p + C_{\delta} \left\{ \int_{\Omega} u_k^{\sigma} (\varphi^p + |\nabla \varphi|^p) + \int_{\Omega} g^{p/(p-1)} \varphi^p \right\}. \tag{3.40}$$

Choosing  $\delta > 0$  sufficiently small, we conclude from (3.31), (3.35) and (3.40) that

$$\int_{\Omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \varphi^p \leq C \left\{ \int_{\Omega} u_k^{\sigma} (\varphi^p + |\nabla \varphi|^p) + \int_{\Omega} g^{p/(p-1)} \varphi^p + \int_{\Omega} f \varphi^p \right\}. \tag{3.41}$$

Let  $\psi \in C_0^{\infty}(\Omega)$ ,  $\psi \geq 0$  in  $\Omega$ . Applying Lemma 3.2, we can find a sequence of nonnegative functions  $(\psi_n)$  in  $C_0^{\infty}(\Omega \setminus \Sigma)$  converging to  $\psi$  in  $W^{1,p}(\Omega)$  and a.e. in  $\Omega$ , such that  $\psi_n \leq \psi$  in  $\Omega$ . Replacing  $\varphi$  by  $\psi_n$  in (3.41) and then letting  $n \rightarrow \infty$  we find (3.29).

Let  $\omega \subset\subset \Omega$  be some fixed open set containing  $\Sigma$ . We now take  $\psi_0 \in C_0^{\infty}(\Omega)$  so that  $\psi_0 = 1$  on  $\omega$  and  $0 \leq \psi_0 \leq 1$  in  $\Omega$ . Applying (3.29) with  $\sigma = 0$ , we obtain

$$\int_{\omega \setminus \Sigma} |\nabla u_k|^p \leq C k^p \left\{ \int_{\Omega} (1 + |\nabla \psi_0|^p) + \int_{\Omega} g^{p/(p-1)} + \int_{\Omega} f \right\}.$$

In particular,  $u_k \in W^{1,p}(\omega \setminus \Sigma)$ . It follows from Lemma 3.3 that  $u_k \in W^{1,p}(\omega)$ . This concludes the first step of the proof.

**Step 2.** Given  $0 < \sigma < p - 1$ , we can find an open  $\omega \subset\subset \Omega$  containing  $\Sigma$  so that

$$\|u_k\|_{L^{\sigma \frac{N}{N-p}}(\omega)}^{\sigma} + \|\nabla u_k\|_{L^{\sigma \frac{N}{N-1}}(\omega)}^{\sigma} \leq C \left\{ \int_{\Omega \setminus \bar{\omega}} u_k^{\sigma} + \int_{\Omega} g^{p/(p-1)} + \int_{\Omega} f \right\}, \quad \forall k \geq 1, \tag{3.42}$$

where  $C = C(p, \sigma, \omega, \Omega, \|a_i\|_\infty, \|b_i\|_\infty, \|c_i\|_\infty)$ .

Let  $\omega \subset\subset \Omega$  be a neighborhood of  $\Sigma$  with measure  $|\omega|$  small enough to be chosen later on (recall that  $|\Sigma| = 0$ , so that such  $\omega$  actually exists). We then take  $\psi_0 \in C_0^\infty(\Omega)$ , so that  $\text{supp } \psi_0 \subset \Omega$  and  $\psi_0 = 1$  on  $\omega$ . Since  $u_k \in W_{\text{loc}}^{1,p}(\Omega)$  and  $u_k \geq 1$  a.e. in  $\Omega$ , we have  $u_k^{\sigma/p} \in W_{\text{loc}}^{1,p}(\Omega)$  and

$$\nabla(u_k^{\sigma/p} \psi_0) = \frac{\sigma}{p} \frac{\nabla u_k}{u_k^{1-\sigma/p}} \psi_0 + u_k^{\sigma/p} \nabla \psi_0 \quad \text{in } \Omega.$$

It follows from (3.29) that

$$\int_{\Omega} \left| \nabla(u_k^{\sigma/p} \psi_0) \right|^p \leq C \left\{ \int_{\Omega} u_k^\sigma (\psi_0^p + |\nabla \psi_0|^p) + \int_{\Omega} g^{p/(p-1)} \psi_0^p + \int_{\Omega} f \psi_0^p \right\}.$$

Since  $\nabla \psi_0 = 0$  on  $\omega$ , we get

$$\begin{aligned} \int_{\Omega} \left| \nabla(u_k^{\sigma/p} \psi_0) \right|^p &\leq C_1 \int_{\omega} u_k^\sigma + C_2 \left\{ (1 + \|\nabla \psi_0\|_{L^\infty}^p) \int_{\Omega \setminus \bar{\omega}} u_k^\sigma + \int_{\Omega} g^{p/(p-1)} + \int_{\Omega} f \right\} \\ &\leq C_1 \int_{\omega} u_k^\sigma + C_2 K, \end{aligned}$$

where  $K$  denotes the term in brackets and  $C_1, C_2$  are positive constants independent of  $k$ ; note also that  $C_1$  does not depend on  $\omega$ .

Applying the Sobolev inequality, we find

$$\left( \int_{\omega} u_k^{\sigma \frac{N}{N-p}} \right)^{\frac{N-p}{N}} \leq \left( \int_{\Omega} u_k^{\sigma \frac{N}{N-p}} \psi_0^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}} \leq \tilde{C}_1 \int_{\omega} u_k^\sigma + \tilde{C}_2 K \quad \text{if } 1 < p < N, \quad (3.43)$$

$$\left( \int_{\omega} u_k^{\sigma q} \right)^{\frac{1}{q}} \leq \tilde{C}_1 \int_{\omega} u_k^\sigma + \tilde{C}_2 K, \quad \forall q \in [1, \infty) \quad \text{if } p = N, \quad (3.44)$$

where  $\tilde{C}_1$  is independent of  $\omega$ .

We shall assume in the sequel that  $1 < p < N$ , since the case  $p = N$  can be dealt with in a similar way.

From Hölder's inequality, we know that

$$\int_{\omega} u_k^\sigma \leq |\omega|^{p/N} \left( \int_{\omega} u_k^{\sigma \frac{N}{N-p}} \right)^{\frac{N-p}{N}}. \quad (3.45)$$

Inserting (3.45) into (3.43), we find

$$(1 - |\omega|^{p/N} \tilde{C}_1) \int_{\omega} u_k^\sigma \leq |\omega|^{p/N} \tilde{C}_2 K.$$

We now choose  $\omega$  so that  $|\omega|^{p/N} \tilde{C}_1 < 1/2$ . Thus, we have

$$\frac{1}{2} \int_{\omega} u_k^\sigma \leq |\omega|^{p/N} \tilde{C}_2 K. \quad (3.46)$$

We finally conclude from (3.43) and the above that

$$\left( \int_{\omega} u_k^{\sigma \frac{N}{N-p}} \right)^{\frac{N-p}{N}} \leq (2|\omega|^{p/N} \tilde{C}_1 + 1) \tilde{C}_2 K \leq 2\tilde{C}_2 K. \quad (3.47)$$

This gives the estimate for the first term in the left-hand side of (3.42). We now estimate the second one.

Applying Hölder's inequality, we have

$$\begin{aligned} \int_{\omega} |\nabla u_k|^{\sigma \frac{N}{N-1}} &= \int_{\omega} \frac{|\nabla u_k|^{\sigma \frac{N}{N-1}}}{u_k^{\frac{(p-\sigma)\sigma}{p} \frac{N}{N-1}}} \cdot u_k^{(p-\sigma)\frac{\sigma}{p} \frac{N}{N-1}} \\ &\leq \left( \int_{\omega} \frac{|\nabla u_k|^p}{u_k^{p-\sigma}} \right)^{\frac{\sigma}{p} \frac{N}{N-1}} \left( \int_{\omega} u_k^{\sigma \frac{N}{N-p/(p-\sigma)}} \right)^{1-\frac{\sigma}{p} \frac{N}{N-1}}. \end{aligned}$$

By (3.29) and (3.46), the first integral is bounded by  $CK$ . Note that  $\frac{N}{N-p/(p-\sigma)} < \frac{N}{N-p}$  for  $\sigma < p-1$ ; thus, by Hölder's inequality, the second integral can be estimated by  $CK^{\frac{N}{N-p/(p-\sigma)}}$ . Therefore,

$$\int_{\omega} |\nabla u_k|^{\sigma \frac{N}{N-1}} \leq (CK)^{\frac{\sigma}{p} \frac{N}{N-1}} (CK^{\frac{N}{N-p/(p-\sigma)}})^{1-\frac{\sigma}{p} \frac{N}{N-1}} = CK^{\frac{N}{N-1}}.$$

This concludes the proof of Step 2.

**Step 3.** Proof of (3.12).

Since  $u \in W_{\text{loc}}^{1,p}(\Omega \setminus \Sigma)$ , it suffices to show that  $u^{p-1}, |\nabla u|^{p-1}$  are integrable in some neighborhood of  $\Sigma$ .

Given  $0 < \sigma < p-1$ , it follows from the previous step that (3.42) holds for some small open set  $\omega$  containing  $\Sigma$ . In particular,

$$\int_{\omega} u_k^{\sigma \frac{N}{N-p}} \leq C \left\{ \int_{\Omega \setminus \bar{\omega}} u^{\sigma} + \int_{\Omega} g^{p/(p-1)} + \int_{\Omega} f \right\}^{\frac{N}{N-p}}, \quad \forall k \geq 1. \quad (3.48)$$

(Shrinking the domain  $\Omega$  if necessary, we can always assume that  $\int_{\Omega \setminus \bar{\omega}} u^{\sigma} < \infty$ .)

By making the special choice  $\sigma = (p-1)\frac{N-p}{N}$  in (3.48), we immediately see that  $u^{p-1} \in L^1(\omega)$ .

Note also that, according to (3.42), we have

$$\int_{\omega} |\nabla u_k|^{\sigma \frac{N}{N-1}} \leq C \left\{ \int_{\Omega \setminus \bar{\omega}} u^{\sigma} + \int_{\Omega} g^{p/(p-1)} + \int_{\Omega} f \right\}^{\frac{N}{N-1}}, \quad \forall k \geq 1. \quad (3.49)$$

Take in particular  $\sigma = (p-1)\frac{N-1}{N}$ . Since  $|\Sigma| = 0$  and  $u \in W_{\text{loc}}^{1,p}(\Omega \setminus \Sigma)$ , we have  $\nabla u_k = \chi_{[u < k]} \nabla u$  a.e. in  $\Omega$ . Applying the Monotone Convergence Theorem to (3.49), we conclude that (3.12) holds.

The argument above actually shows that (3.14) and (3.15) hold; moreover, we have

$$\|u\|_{L^\sigma \frac{N}{N-p}(\omega)}^\sigma + \|\nabla u\|_{L^\sigma \frac{N}{N-1}(\omega)}^\sigma \leq C \left\{ \int_{\Omega \setminus \bar{\omega}} u^\sigma + \int_{\Omega} g^{p/(p-1)} + \int_{\Omega} f \right\},$$

where  $C = C(p, \sigma, \omega, \Omega, \|a_i\|_\infty, \|b_i\|_\infty, \|c_i\|_\infty)$  and  $0 < \sigma < p - 1$ .

**Step 4.**  $A^i(x, u, \nabla u), B(x, u, \nabla u) \in L^1_{\text{loc}}(\Omega)$  and

$$-\operatorname{div} A(x, u, \nabla u) \geq B(x, u, \nabla u) \quad \text{in } \mathcal{D}'(\Omega).$$

In view of (3.12) and the structure estimate (3.8),  $A^i(x, u, \nabla u) \in L^1_{\text{loc}}(\Omega)$ .

Given  $k > 0$ , let  $F_k \in C^\infty(\mathbb{R})$  be a non-increasing function such that  $F_k(t) = 1$  if  $t \leq k/2$ ,  $F_k(t) = 0$  if  $t \geq k$  and  $|F'_k| \leq 4/k$  in  $\mathbb{R}$ . Since  $F_k$  is non-increasing, we have in particular that  $0 \leq F_k \leq 1$  in  $\mathbb{R}$ .

Note that  $F_k \circ u = F_k \circ u_k$ . As a consequence of the first step, we thus have

$$F_k \circ u \in W^{1,p}_{\text{loc}}(\Omega) \quad \text{and} \quad \nabla(F_k \circ u) = F'_k(u) \nabla u \quad \text{a.e. in } \Omega.$$

Given  $\varphi \in C^\infty_0(\Omega \setminus \Sigma)$ ,  $\varphi \geq 0$  in  $\Omega$ , it follows from (3.28) applied to the function  $v = F_k(u) \varphi$  that

$$\begin{aligned} \int_{\Omega} B(x, u, \nabla u) F_k(u) \varphi &\leq \int_{\Omega} A^i(x, u, \nabla u) \partial_i u F'_k(u) \varphi + \int_{\Omega} A^i(x, u, \nabla u) \partial_i \varphi F_k(u) \\ &= \int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i u_k F'_k(u) \varphi + \int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i \varphi F_k(u), \end{aligned} \tag{3.50}$$

where we have used the fact that  $F_k(t) = F'_k(t) = 0$  for all  $t \geq k$ .

Given  $\psi \in C^\infty_0(\Omega)$ ,  $\psi \geq 0$  in  $\Omega$ , let  $(\psi_n)$  be a sequence of nonnegative functions in  $C^\infty_0(\Omega \setminus \Sigma)$  converging to  $\psi$  with respect to the  $W^{1,p}$ -norm and also a.e. in  $\Omega$ .

We first observe that, in view of (3.9) and (3.12), we have

$$B(x, u, \nabla u) \geq -b_0 |\nabla u|^{p-1} - b_1 u^{p-1} - f \in L^1_{\text{loc}}(\Omega).$$

It follows from Fatou's Lemma that

$$\int_{\Omega} B(x, u, \nabla u) F_k(u) \psi \leq \liminf_{n \rightarrow \infty} \int_{\Omega} B(x, u, \nabla u) F_k(u) \psi_n.$$

We now apply (3.50) with  $\varphi$  replaced by  $\psi_n$ . Since  $A^i(x, u_k, \nabla u_k) \in L^{p/(p-1)}_{\text{loc}}(\Omega)$ , we can take  $n \rightarrow \infty$  in the resulting inequality to get

$$\int_{\Omega} B(x, u, \nabla u) F_k(u) \psi \leq \int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i u_k F'_k(u) \psi + \int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i \psi F_k(u), \tag{3.51}$$

for every  $\psi \in C^\infty_0(\Omega)$  such that  $\psi \geq 0$  in  $\Omega$ .

We now let  $k \rightarrow \infty$  in the inequality above. By Fatou's Lemma,

$$\int_{\Omega} B(x, u, \nabla u) \psi \leq \liminf_{k \rightarrow \infty} \int_{\Omega} B(x, u, \nabla u) F_k(u) \psi. \quad (3.52)$$

Next, since

$$|A(x, u_k, \nabla u_k)| \leq a_0 |\nabla u_k|^{p-1} + a_1 u_k^{p-1} + g \leq a_0 |\nabla u|^{p-1} + a_1 u^{p-1} + g \in L^1_{\text{loc}}(\Omega),$$

it follows from the Dominated Convergence Theorem that

$$\lim_{k \rightarrow \infty} \int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i \psi F_k(u) = \int_{\Omega} A^i(x, u, \nabla u) \partial_i \psi. \quad (3.53)$$

Finally, recall that  $-4/k \leq F'_k \leq 0$  in  $\mathbb{R}$ . Using (3.10), we have

$$\begin{aligned} \int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i u_k F'_k(u) \psi &\leq \int_{\Omega} \left( |\nabla u_k|^p - c_1 u_k^p - c_2 \right) F'_k(u) \psi \\ &\leq \frac{4}{k} \int_{[\frac{k}{2} < u < k]} (|c_1| u_k^p + |c_2|) \psi. \end{aligned}$$

Since  $u_k^p/k \leq u_k^{p-1} \leq u^{p-1}$ , we get

$$\int_{\Omega} A^i(x, u_k, \nabla u_k) \partial_i u_k F'_k(u) \psi \leq 4 \int_{[\frac{k}{2} < u < k]} \left( |c_1| u^{p-1} + \frac{|c_2|}{k} \right) \psi \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.54)$$

It then follows from (3.51)–(3.54) that

$$\int_{\Omega} B(x, u, \nabla u) \psi \leq \int_{\Omega} A^i(x, u, \nabla u) \partial_i \psi, \quad \forall \psi \in C_0^\infty(\Omega), \psi \geq 0 \text{ in } \Omega.$$

In particular, since  $B(x, u, \nabla u)$  is bounded from below by an  $L^1_{\text{loc}}$ -function in  $\Omega$ , we must have  $B(x, u, \nabla u) \in L^1_{\text{loc}}(\Omega)$ .





# Chapitre 4

## An estimate in the spirit of Poincaré's inequality

### Sommaire

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### 4.1 Introduction<sup>5</sup>

Assume  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , is a bounded domain with Lipschitz boundary, and let  $1 \leq p < \infty$ . It is a well-known fact that there exists a constant  $A_0 = A_0(p, \Omega) > 0$  such that the following form of Poincaré's inequality holds :

$$\int_{\Omega} |f - f_{\Omega}|^p \leq A_0 \int_{\Omega} |Df|^p, \quad \forall f \in W^{1,p}(\Omega), \quad (4.1)$$

where  $f_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} f$ .

On the other hand, let  $(\rho_n) \subset L^1(\mathbb{R}^N)$  be a sequence of **radial** functions satisfying

$$\left\{ \begin{array}{l} \rho_n \geq 0 \quad \text{a.e. in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \rho_n = 1, \quad \forall n \geq 1, \\ \lim_{n \rightarrow \infty} \int_{|h| > \delta} \rho_n(h) dh = 0, \quad \forall \delta > 0. \end{array} \right. \quad (4.2)$$

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<sup>5</sup>Ce chapitre a été publié dans J. Eur. Math. Soc. (JEMS) **6** (2004), 1–15; une version résumée était déjà sortie dans C. R. Acad. Sci. Paris, Ser. I **337** (2003), 253–257.

Under these assumptions, the following pointwise limit was established by Bourgain, Brezis and Mironescu [20] :

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy = K_{p,N} \int_{\Omega} |Df|^p \quad (4.3)$$

for every  $f \in W^{1,p}(\Omega)$ , where  $K_{p,N} = \int_{S^{N-1}} |e_1 \cdot \sigma|^p d\mathcal{H}^{N-1}$ .

In the next chapter we shall prove several extensions of (4.3). Our aim here will be, rather, to show the following estimate related to (4.1) :

**Theorem 4.1** *Assume  $N \geq 2$ . Let  $(\rho_n) \subset L^1(\mathbb{R}^N)$  be a sequence of radial functions satisfying (4.2). Given  $\delta > 0$ , there exists  $n_0 \geq 1$  sufficiently large such that*

$$\int_{\Omega} |f - f_{\Omega}|^p \leq \left( \frac{A_0}{K_{p,N}} + \delta \right) \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy \quad (4.4)$$

for every  $f \in L^p(\Omega)$  and  $n \geq n_0$ .

The choice of  $n_0 \geq 1$  depends not only on  $\delta > 0$ , but also on  $p$ ,  $\Omega$  and on the sequence  $(\rho_n)_{n \geq 1}$ . Special cases of this inequality have been used in the study of problems related to the Ginzburg-Landau functional (see [21,22]; see also Corollaries 4.1–4.4 below).

We first point out that (4.4) is stronger than (4.1), in the sense that the right-hand side of (4.4) can be always estimated by  $\int_{\Omega} |Df|^p$ . In fact, given  $f \in W^{1,p}(\Omega)$ , we first extend  $f$  to  $\mathbb{R}^N$  so that  $f \in W^{1,p}(\mathbb{R}^N)$ . It is then easy to see that (see, e.g., [20, Theorem 1]; see also Lemma 5.1)

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy \leq \int_{\mathbb{R}^N} |Df|^p \leq C \int_{\Omega} |Df|^p. \quad (4.5)$$

If  $N = 1$ , then one can construct examples of sequences  $(\rho_n) \subset L^1(\mathbb{R})$  for which (4.4) fails (see [20, Counterexample 1]). In this case, we need to impose an additional condition on  $(\rho_n)$ ; see Theorem 4.3 below.

Theorem 4.1 will be deduced from the following compactness result :

**Theorem 4.2** *Assume  $N \geq 2$ . Let  $(\rho_n) \subset L^1(\mathbb{R}^N)$  be a sequence of radial functions satisfying (4.2). If  $(f_n) \subset L^p(\Omega)$  is a bounded sequence such that*

$$\int_{\Omega} \int_{\Omega} \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy \leq B, \quad \forall n \geq 1, \quad (4.6)$$

then  $(f_n)$  is relatively compact in  $L^p(\Omega)$ .

Assume that  $f_{n_j} \rightarrow f$  in  $L^p(\Omega)$ . Then,

- (a)  $f \in W^{1,p}(\Omega)$  if  $1 < p < \infty$ ;
- (b)  $f \in BV(\Omega)$  if  $p = 1$ .

In both cases, we have  $\int_{\Omega} |Df|^p \leq \frac{B}{K_{p,N}}$ , where  $B$  is given by (4.6).

This result was already known under the additional assumption that  $\rho_n$  is radially decreasing for every  $n \geq 1$  (see [20, Theorem 4]).

We now consider the case  $N = 1$ .

Given  $\rho_n \in L^1(\mathbb{R})$ , we shall assume that  $\rho_n$  is defined for every  $x \in \mathbb{R}$  in the following way

$$\rho_n(x) = \begin{cases} \lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} \rho_n & \text{if } x \text{ is a Lebesgue point of } \rho_n, \\ +\infty & \text{otherwise.} \end{cases}$$

Given  $\theta_0 \in (0, 1)$  we define

$$\rho_{n,\theta_0}(x) := \inf_{\theta_0 \leq \theta \leq 1} \rho_n(\theta x), \quad \forall x \in \mathbb{R}.$$

By construction,

$$\rho_{n,\theta_0}(x) \leq \rho_n(\theta x), \quad \forall x \in \mathbb{R}, \quad \forall \theta \in [\theta_0, 1]. \quad (4.7)$$

We then have the following result :

**Theorem 4.3** *Let  $(\rho_n) \subset L^1(\mathbb{R})$  be a sequence of functions satisfying (4.2). Assume there exist  $\theta_0 \in (0, 1)$  and  $\alpha_0 > 0$  such that*

$$\int_{\mathbb{R}} \rho_{n,\theta_0} \geq \alpha_0 > 0, \quad \forall n \geq 1. \quad (4.8)$$

*If  $(f_n) \subset L^p(0, 1)$  is a bounded sequence such that*

$$\int_0^1 \int_0^1 \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(x - y) dx dy \leq B, \quad \forall n \geq 1, \quad (4.9)$$

*then  $(f_n)$  is relatively compact in  $L^p(0, 1)$ .*

*Moreover, all the other statements of Theorems 4.1 and 4.2 are also valid. In particular, inequality (4.4) holds with  $\Omega = (0, 1)$ .*

## 4.2 Some examples

We now state some inequalities coming from Theorems 4.1 and 4.3. In all cases, condition (4.2) is satisfied for  $N \geq 1$ ; it is also easy to see that (4.8) holds when  $N = 1$ . Below, we denote by  $Q = (0, 1)^N$  the  $N$ -dimensional unit cube and by  $\sigma_N$  the  $(N - 1)$ -dimensional Hausdorff measure of  $S^{N-1}$ .

For every  $N \geq 1$  we then have the following corollaries :

**Corollary 4.1 (Bourgain-Brezis-Mironescu [21])**

$$\int_Q |f - f_Q|^p \leq C_{s_0}(1 - s)p \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}} dx dy, \quad \forall f \in L^p(Q),$$

for every  $0 < s_0 < s < 1$ .

**Proof.** We simply apply Theorem 4.1 (resp. Theorem 4.3) with

$$\rho_n(h) = \frac{p}{\sigma_N} \frac{1 - s_n}{|h|^{N-(1-s_n)p}}, \quad \forall h \in B_1,$$

where  $(s_n)$  is any sequence such that  $s_n \uparrow 1$ .

This inequality takes into account the correction factor  $(1 - s)^{1/p}$  we should put in front of the Gagliardo seminorm  $|f|_{W^{s,p}}$  as  $s \uparrow 1$ ; see (5.6) in the next chapter. In [21], the authors study related estimates arising from the Sobolev imbedding  $L^q \hookrightarrow W^{s,p}$  for the critical exponent  $\frac{1}{q} = \frac{1}{p} - \frac{s}{N}$ ; see also [68] for a more elementary approach.

**Corollary 4.2 (Bourgain-Brezis-Mironescu [22])**

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{1}{|\log \varepsilon|} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^p} \frac{dx dy}{(|x - y| + \varepsilon)^N}$$

for every  $f \in L^p(Q)$  and  $0 < \varepsilon < \varepsilon_0$ .

**Proof.** This follows from Theorem 4.1 (resp. Theorem 4.3) with

$$\rho_n(h) = \frac{1}{\sigma_N |\log \varepsilon_n|} \frac{1}{(|h| + \varepsilon_n)^N}, \quad \forall h \in B_1,$$

where  $\varepsilon_n \downarrow 0$ .

A stronger form of this inequality is the following

**Corollary 4.3**

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{1}{|\log \varepsilon|} \int_Q \int_{|x-y|>\varepsilon} \frac{|f(x) - f(y)|^p}{|x - y|^{N+p}} dx dy, \quad \forall f \in L^p(Q),$$

for every  $0 < \varepsilon < \varepsilon_0 \ll 1$ .

**Proof.** For any sequence  $\varepsilon_n \downarrow 0$ , we take

$$\rho_n(h) = \begin{cases} 0 & \text{if } |h| \leq \varepsilon_n, \\ \frac{1}{\sigma_N |\log \varepsilon_n|} \frac{1}{|h|^N} & \text{if } \varepsilon_n < |h| \leq 1. \end{cases}$$

We have been informed by H. Brezis that Bourgain and Brezis [16] have proved that

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{1}{|\log \varepsilon|} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{(|x - y| + \varepsilon)^{N+p}} dx dy, \quad \forall f \in L^p(Q),$$

for every  $0 < \varepsilon < \varepsilon_0$ , using a Paley-Littlewood decomposition of  $f$ . Note that this estimate can be deduced instead from the corollary above.

Here is another example :

**Corollary 4.4**

$$\int_Q |f - f_Q|^p \leq C_{\varepsilon_0} \frac{N+p}{\varepsilon^{N+p}} \int_Q \int_{\substack{Q \\ |x-y| < \varepsilon}} |f(x) - f(y)|^p dx dy, \quad \forall f \in L^p(Q),$$

for every  $0 < \varepsilon < \varepsilon_0$ .

**Proof.** We use Theorem 4.1 (resp. Theorem 4.3) applied to

$$\rho_n(h) = \begin{cases} \frac{1}{\sigma_N} \frac{N+p}{\varepsilon_n^{N+p}} |h|^p & \text{if } |h| < \varepsilon_n, \\ 0 & \text{if } |h| \geq \varepsilon_n. \end{cases}$$

Concerning the behavior of the constants, let  $A_0$  denote the best constant in (4.1). Then, in Corollary 4.1 the constant  $C_{s_0}$  can be chosen so that

$$C_{s_0} \rightarrow \frac{A_0}{K_{p,N}\sigma_N} \quad \text{as } s_0 \uparrow 1.$$

Similarly, in Corollaries 4.2–4.4 we have  $C_{\varepsilon_0}$  converging to the same limit as  $\varepsilon_0 \downarrow 0$ .

Applying Theorem 4.1 to  $p = 1$  and  $f = \chi_E$ , where  $E \subset Q$  is any measurable set, we get (see also [21] for related results) :

**Corollary 4.5** *Let  $N \geq 2$ . Given a sequence of radial functions  $(\rho_n) \subset L^1(\mathbb{R}^N)$  satisfying (4.2), then, for any  $C > A_0/K_{1,N}$ , there exists  $n_0 \geq 1$  such that*

$$|E||Q \setminus E| \leq C \int_E \int_{Q \setminus E} \frac{\rho_n(|x-y|)}{|x-y|} dx dy, \quad \forall E \subset Q \text{ measurable}, \quad \forall n \geq n_0.$$

## 4.3 Estimates in dimension $N = 1$

Given any  $g \in L^p(\mathbb{R})$ , let  $G_p : [0, \infty) \rightarrow [0, \infty)$  be the (continuous) function defined by

$$G_p(t) = \int_{\mathbb{R}} |g(x+t) - g(x)|^p dx, \quad \forall t \geq 0.$$

We start with the following

**Lemma 4.1** *Given  $0 < s < t$ , let  $k \in \mathbb{N}$  and  $\theta \in [0, 1)$  be such that  $\frac{t}{s} = k + \theta$ . Then, there exists  $C_p > 0$  such that for every  $g \in L^p(\mathbb{R})$  we have*

$$\frac{G_p(t)}{t^p} \leq C_p \left\{ \frac{G_p(s)}{s^p} + \frac{G_p(\theta s)}{t^p} \right\}. \quad (4.10)$$

**Proof.** Note that

$$\begin{aligned} |g(x+t) - g(x)|^p &= |g(x+ks+\theta s) - g(x)|^p \\ &\leq 2^{p-1} \left\{ |g(x+ks) - g(x)|^p + |g(x+ks+\theta s) - g(x+ks)|^p \right\} \\ &\leq 2^{p-1} k^{p-1} \sum_{j=0}^{k-1} |g(x+js+s) - g(x+js)|^p + \\ &\quad + 2^{p-1} |g(x+ks+\theta s) - g(x+ks)|^p. \end{aligned}$$

Integrating with respect to  $x \in \mathbb{R}$  and changing variables we get

$$G_p(t) \leq 2^{p-1} k^p G_p(s) + 2^{p-1} G_p(\theta s).$$

Recall that  $k \leq \frac{t}{s}$ . We then conclude that (4.10) holds with  $C_p = 2^{p-1}$ .

Another estimate we shall need is given by the lemma below :

**Lemma 4.2** *Let  $r > 0$ . There exists a constant  $C_p > 0$  so that the following holds : for every  $g \in L^p(0, 2r)$  such that  $g = 0$  a.e. in  $(r, 2r)$  we have*

$$\int_0^r |g|^p \leq C_p r^p \int_0^r \frac{|g(x+t) - g(x)|^p}{t^p} dx, \quad \forall t \in (0, r). \quad (4.11)$$

**Proof.** By a scaling argument, it suffices to prove the lemma for  $r = 1$ . We now extend  $g \in L^p(0, 2)$  to the entire half-line so that  $g = 0$  a.e. in  $(1, \infty)$ .

Given  $0 < t < 1$ , let  $k \geq 1$  be the first integer satisfying  $kt \geq 1$ . In particular, for  $x \in (0, 1)$  we have  $x + kt > 1$ ; thus,

$$|g(x)|^p = |g(x+kt) - g(x)|^p \leq k^{p-1} \sum_{j=0}^{k-1} |g(x+jt+t) - g(x+jt)|^p.$$

Integrating this inequality with respect to  $x$  we get

$$\begin{aligned} \int_0^1 |g|^p &\leq k^{p-1} \sum_{j=0}^{k-1} \int_0^\infty |g(x+jt+t) - g(x+jt)|^p dx \\ &\leq k^p \int_0^\infty |g(x+t) - g(x)|^p dx = k^p \int_0^1 |g(x+t) - g(x)|^p dx. \end{aligned}$$

Note however that  $k \leq \frac{2}{t}$ . The lemma now follows by taking  $C_p = 2^p$ .

## 4.4 Compactness in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $N \geq 2$

Given  $f \in L^p(\mathbb{R}^N)$ , we consider  $F_p : \mathbb{R}^N \rightarrow [0, \infty)$  defined by

$$F_p(h) = \int_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx, \quad \forall h \in \mathbb{R}^N.$$

This function is continuous and satisfies

$$F_p(h_1 + h_2) \leq 2^{p-1} [F_p(h_1) + F_p(h_2)], \quad \forall h_1, h_2 \in \mathbb{R}^N.$$

We have the following

**Lemma 4.3** *Assume  $N \geq 2$ . Then, there exists  $C_p > 0$  such that*

$$\int_{S^{N-1}} \frac{F_p(tv)}{t^p} d\sigma(v) \leq C_p \int_{S^{N-1}} \frac{F_p(sv)}{s^p} d\sigma(v) \quad \text{for every } 0 < s < t. \quad (4.12)$$

**Proof.** Let  $0 < s < t < \infty$ . Given  $v \in S^{N-1}$  and  $w \in (\mathbb{R}v)^\perp$ , we apply the one dimensional estimate in Lemma 4.1 to the function

$$g(\tau) = f(w + \tau v) \quad \text{for a.e. } \tau \geq 0.$$

Integrating the resulting expression with respect to  $w \in (\mathbb{R}v)^\perp$ , it follows that for every  $v \in S^{N-1}$  we have

$$\frac{F_p(tv)}{t^p} \leq C_p \left\{ \frac{F_p(sv)}{s^p} + \frac{F_p(\theta sv)}{t^p} \right\} \quad (4.13)$$

for some  $\theta \in [0, 1)$  (depending on  $s$  and  $t$ ). We now split the proof into two cases :

**Case 1.**  $N$  is even.

Let  $O \in O(N)$  be an orthogonal transformation such that  $\langle Ow, w \rangle = 0$  for every  $w \in \mathbb{R}^N$  (this is possible since  $N$  is even). We then consider

$$\begin{aligned} O_1 w &:= \frac{\theta}{2} w + \sqrt{1 - \frac{\theta^2}{4}} O w, \\ O_2 w &:= \frac{\theta}{2} w - \sqrt{1 - \frac{\theta^2}{4}} O w. \end{aligned}$$

Note that  $O_1, O_2 \in O(N)$  and

$$\theta w = O_1 w + O_2 w, \quad \forall w \in \mathbb{R}^N;$$

thus,

$$F_p(\theta sv) \leq 2^{p-1} \left\{ F_p(s O_1 v) + F_p(s O_2 v) \right\}.$$

Inserting this inequality into (4.13), we get

$$\frac{F_p(tv)}{t^p} \leq C_p \frac{F_p(sv) + F_p(s O_1 v) + F_p(s O_2 v)}{s^p}.$$

Integrating with respect to  $v \in S^{N-1}$ , we obtain (4.12).

**Case 2.**  $N$  is odd.

Let  $v \in S^{N-1}$ . We denote by  $S_v^{N-2}$  the  $(N-2)$ -sphere orthogonal to  $v$  :

$$S_v^{N-2} := S^{N-1} \cap (\mathbb{R}v)^\perp.$$



Reasoning as in the previous case, we see that

$$\int_{S_v^{N-2}} \frac{F_p(tw)}{t^p} d\mathcal{H}^{N-2} \leq C_p \int_{S_v^{N-2}} \frac{F_p(sw)}{s^p} d\mathcal{H}^{N-2}. \quad (4.14)$$

On  $S^{N-1}$ , we consider the measure  $\mu$  defined as

$$\mu(A) = \int_{S^{N-1}} \mathcal{H}^{N-2}(A \cap S_v^{N-2}) d\sigma(v) \quad \text{for every Borel set } A \subset S^{N-1}.$$

Note that  $\mu$  is invariant under orthogonal transformations, i.e.  $\mu(OA) = \mu(A)$  for every  $O \in O(N)$ , and  $\mu(S^{N-1}) = |S^{N-2}| |S^{N-1}|$ . It then follows that

$$\mu = |S^{N-2}| \mathcal{H}^{N-1} \llcorner_{S^{N-1}}.$$

We now integrate (4.14) with respect to  $v \in S^{N-1}$ . Using the observation above we get (4.12).

The lemma above implies the following compactness result :

**Proposition 4.1** *Assume  $N \geq 2$ . Let  $(f_n) \subset L^p(\mathbb{R}^N)$  be a bounded sequence of functions such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy \leq B, \quad \forall n \geq 1. \quad (4.15)$$

*Then,  $(f_n)$  is relatively compact in  $L^p_{\text{loc}}(\mathbb{R}^N)$ .*

**Proof.** Fix  $t_0 > 0$ . Let  $n_0 \geq 1$  be such that

$$\int_{B_{t_0}} \rho_n \geq \frac{1}{2}, \quad \forall n \geq n_0.$$

We first prove the following

**Claim.** There exists a constant  $C = C(p, N, B) > 0$  such that

$$\int_{S^{N-1}} F_{n,p}(tv) d\sigma(v) \leq Ct_0^p \quad (4.16)$$

for every  $0 < t < t_0$  and every  $n \geq n_0$ . ( $F_{n,p}$  denotes the function  $F_p$  associated to  $f_n$ .)

In fact, let  $s, \tau > 0$  be such that  $0 < s < t_0 \leq \tau$ . It follows from the previous lemma that

$$\int_{S^{N-1}} \frac{F_{n,p}(\tau v)}{\tau^p} d\sigma(v) \leq C_p \int_{S^{N-1}} \frac{F_{n,p}(sv)}{s^p} d\sigma(v).$$

We now multiply both sides by  $s^{N-1} \rho_n(s)$  and then integrate the resulting expression with respect to  $s$  running from 0 to  $t_0$ . We get

$$\begin{aligned} \frac{1}{2|S^{N-1}|} \int_{S^{N-1}} \frac{F_{n,p}(\tau v)}{\tau^p} d\sigma(v) &\leq \int_{S^{N-1}} \frac{F_{n,p}(\tau v)}{\tau^p} d\sigma(v) \int_0^{t_0} \rho_n(s) s^{N-1} ds \\ &\leq C \int_0^{t_0} \int_{S^{N-1}} \frac{F_{n,p}(sv)}{s^p} \rho_n(s) s^{N-1} d\sigma(v) ds \\ &\leq C \int_{\mathbb{R}^N} \frac{F_{n,p}(h)}{|h|^p} \rho_n(|h|) dh. \end{aligned}$$

Note that the last term is precisely the double integral in the left-hand side of (4.15). We then conclude that

$$\int_{S^{N-1}} F_{n,p}(\tau v) d\sigma(v) \leq C\tau^p, \quad \forall \tau \geq t_0, \quad \forall n \geq n_0.$$

We now let  $0 < t < t_0$ . Using the above estimate with  $\tau = t_0$  and  $\tau = t + t_0$ , we get

$$\begin{aligned} \int_{S^{N-1}} F_{n,p}(tv) d\sigma(v) &\leq 2^{p-1} \left\{ \int_{S^{N-1}} F_{n,p}(t_0v) d\sigma + \int_{S^{N-1}} F_{n,p}((t+t_0)v) d\sigma \right\} \\ &\leq 2^{p-1} C [t_0^p + (t+t_0)^p] \leq Ct_0^p \end{aligned}$$

for every  $n \geq n_0$ . This proves the claim.

Once we reach at this point, we can proceed as in [20].

We first set  $\Phi_\delta := \frac{1}{|B_\delta|} \chi_{B_\delta}$ . For any  $0 < \delta < t_0$ , it follows from the previous estimate that

$$\begin{aligned} \int_{\mathbb{R}^N} |\Phi_\delta * f_n(x) - f_n(x)|^p dx &= \int_{\mathbb{R}^N} \left| \int_{B_\delta} [f_n(x+h) - f_n(x)] dh \right|^p dx \\ &\leq \int_{\mathbb{R}^N} \int_{B_\delta} |f_n(x+h) - f_n(x)|^p dh dx \\ &= \frac{1}{|B_\delta|} \int_0^\delta \int_{S^{N-1}} F_{n,p}(tv) d\sigma(v) t^{N-1} dt \\ &\leq \frac{Ct_0^p}{|B_\delta|} \int_0^\delta t^{N-1} dt \leq Ct_0^p. \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^N} |\Phi_\delta * f_n(x) - f_n(x)|^p dx \leq Ct_0^p, \quad \forall n \geq n_0, \quad \forall \delta \in (0, t_0). \quad (4.17)$$

We now conclude the proof by applying a variant of the Fréchet-Kolmogorov Theorem. In fact, since  $(f_n)$  is bounded in  $L^p(\mathbb{R}^N)$ , then for every  $\delta > 0$  fixed the sequence  $(\Phi_\delta * f_n)$  is relatively compact in  $L^p_{\text{loc}}(\mathbb{R}^N)$  (see [24, Corollary IV.27]), hence it is totally bounded in  $L^p_{\text{loc}}(\mathbb{R}^N)$ . Using (4.17), it follows that  $(f_n)$  is also totally bounded in  $L^p_{\text{loc}}(\mathbb{R}^N)$ , which implies that  $(f_n)$  is relatively compact in  $L^p_{\text{loc}}(\mathbb{R}^N)$ .

## 4.5 An $L^p$ -estimate near the boundary of $\Omega$

In this section we shall prove the following

**Lemma 4.4** *Assume  $N \geq 2$ . Then, there exist constants  $r_0 > 0$  (depending on  $\Omega$  and on the sequence  $(\rho_n)_{n \geq 1}$ ) and  $C_1, C_2 > 0$  (depending on  $p, \Omega$  and  $N$ ) so that the following holds : given  $0 < r < r_0$  we can find  $n_0 \geq 1$  such that*

$$\int_{\Omega} |f|^p \leq C_1 \int_{\Omega_r} |f|^p + C_2 r^p \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy \quad (4.18)$$

for every  $f \in L^p(\Omega)$  and  $n \geq n_0$ .

Here,

$$\Omega_r := \left\{ x \in \Omega : d(x, \partial\Omega) > r \right\}.$$

**Proof.** Let  $x_0 \in \partial\Omega$ . Without loss of generality, we may assume that  $x_0 = 0$ . Take  $r_0 > 0$  sufficiently small such that (up to a rotation of  $\partial\Omega$ ) the set  $\partial\Omega \cap B_{4r_0}$  is the graph of a Lipschitz function  $\gamma$ . For simplicity, we can also assume that  $\gamma$  has Lipschitz constant at most  $1/2$ .

Given  $0 < r < r_0$ , we consider the graph of  $\gamma$  :

$$\Gamma_r := \left\{ x = (x', \gamma(x')) \in \mathbb{R}^N : x' \in B'_r \right\}.$$

Let  $\Lambda$  be the upper half cone

$$\Lambda := \left\{ x = (x', x_N) \in \mathbb{R}^N : |x'| \leq x_N \right\}.$$

Because  $\gamma$  has Lipschitz constant at most  $1/2$ , we have

$$\Omega \cap B_{r/2} \subset \Gamma_r + (\Lambda \cap B_r) \subset \Omega \cap B_{3r} \quad (4.19)$$

for every  $0 < r < r_0$ . We first prove the following

**Claim.** There exists  $n_0 \geq 1$ , depending on  $r \in (0, r_0)$ , such that if  $f \in L^p(\Omega)$  and  $f = 0$  a.e. in  $\Omega_r$ , then

$$\int_{\Omega \cap B_{r/2}} |f|^p \leq Cr^p \int_{\Omega \cap B_{4r}} \int_{\Omega \cap B_{4r}} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy$$

for every  $n \geq n_0$ .

In fact, given  $\xi \in \Gamma_r$  and  $v \in \Lambda \cap S^{N-1}$ , we consider the function

$$g(t) = f(\xi + tv) \quad \text{for a.e. } t \in (0, 2r).$$

Applying Lemma 4.2 to  $g$ , we get

$$\int_0^r |f(\xi + sv)|^p ds \leq Cr^p \int_0^r \frac{|f(\xi + sv + tv) - f(\xi + sv)|^p}{t^p} ds$$

for every  $0 < t < r$ .

Recall that  $\xi = (x', \gamma(x'))$  for some  $x' \in B'_r \subset \mathbb{R}^{N-1}$ . We first integrate the above estimate with respect to  $x' \in B'_r$ , and then we perform the change of coordinates

$$y = (x', \gamma(x')) + sv$$

with respect to the variables  $x'$  and  $s$ . Using (4.19) we then find

$$\begin{aligned} \int_{\Omega \cap B_{r/2}} |f|^p &\leq Cr^p \int_{\Gamma_r + (\Lambda \cap B_r)} \frac{|f(y + tv) - f(y)|^p}{t^p} dy \\ &\leq Cr^p \int_{\Omega \cap B_{3r}} \frac{|f(y + tv) - f(y)|^p}{t^p} dy. \end{aligned} \quad (4.20)$$

Take  $n_0 \geq 1$  sufficiently large so that

$$\int_{B_r} \rho_n \geq \frac{1}{2}, \quad \forall n \geq n_0.$$

Since each  $\rho_n$  is a radial function, there exists  $c > 0$  such that

$$\int_{\Lambda \cap B_r} \rho_n \geq c, \quad \forall n \geq n_0.$$

We now multiply (4.20) by  $\rho_n(t)t^{N-1}$ . Integrating the resulting expression with respect to  $t \in (0, r)$  and  $v \in \Lambda \cap S^{N-1}$ , we get

$$\begin{aligned} c \int_{\Omega \cap B_{r/2}} |f|^p &\leq Cr^p \int_{\Omega \cap B_{3r}} \int_{\Lambda \cap B_r} \frac{|f(y+h) - f(y)|^p}{|h|^p} \rho_n(|h|) dh dy \\ &\leq Cr^p \int_{\Omega \cap B_{4r}} \int_{\Omega \cap B_{4r}} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy. \end{aligned}$$

This completes the proof of the claim.

Using a standard covering argument, it follows from the claim above that there exists  $n_0 \geq 1$ , depending on  $r \in (0, r_0)$ , such that if  $f \in L^p(\Omega)$  and  $f = 0$  a.e. in  $\Omega_r$ , then

$$\int_{\Omega \setminus \Omega_{r/4}} |f|^p \leq Cr^p \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy \quad (4.21)$$

for every  $n \geq n_0$ , where the constant  $C > 0$  is independent of  $f$ ,  $r$  and  $n$ .

We now take  $f \in L^p(\Omega)$  arbitrary. In other words, we do not impose any restriction on the set  $\text{supp } f$ .

Let  $\zeta \in C^\infty(\Omega)$  be such that  $\zeta = 0$  on  $\Omega_r$ ,  $\zeta = 1$  on  $\Omega \setminus \Omega_{r/2}$ ,  $0 \leq \zeta \leq 1$  on  $\Omega$  and  $|\nabla \zeta| \leq C/r$  on  $\Omega$ . Applying (4.21) to the function  $\zeta f$  we get

$$\begin{aligned} \int_{\Omega \setminus \Omega_{r/4}} |f|^p &\leq Cr^p \int_{\Omega} \int_{\Omega} \frac{|\zeta(x)f(x) - \zeta(y)f(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy \\ &\leq 2^{p-1} Cr^p \left\{ \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy + \right. \\ &\quad \left. + \int_{\Omega} \int_{\Omega} |f(x)|^p \frac{|\zeta(x) - \zeta(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy \right\}. \end{aligned}$$

We now estimate the second double integral in the right-hand side. Since  $\zeta(x) = \zeta(y) = 1$  for every  $x, y \in \Omega \setminus \Omega_{r/2}$ , we have

$$\int_{\Omega} \int_{\Omega} |f(x)|^p \frac{|\zeta(x) - \zeta(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy = \iint_{\substack{x \in \Omega \setminus \Omega_{r/4} \\ y \in \Omega_{r/2}}} + \iint_{\substack{x \in \Omega_{r/4} \\ y \in \Omega}}.$$

Note that  $\text{dist}(\Omega \setminus \Omega_{r/4}, \Omega_{r/2}) = r/4$ , thus

$$\iint_{\substack{x \in \Omega \setminus \Omega_{r/4} \\ y \in \Omega_{r/2}}} \leq \frac{C}{r^p} \int_{|h| > \frac{r}{4}} \rho_n \cdot \int_{\Omega} |f|^p \quad \text{and} \quad \iint_{\substack{x \in \Omega_{r/4} \\ y \in \Omega}} \leq \frac{C}{r^p} \int_{\Omega_{r/4}} |f|^p.$$

We then conclude that

$$\begin{aligned} \int_{\Omega} |f|^p &= \int_{\Omega_{r/4}} |f|^p + \int_{\Omega \setminus \Omega_{r/4}} |f|^p \\ &\leq C \int_{\Omega_{r/4}} |f|^p + Cr^p \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy + \\ &\quad + C \int_{|h| > \frac{r}{4}} \rho_n \cdot \int_{\Omega} |f|^p. \end{aligned}$$

Taking  $n_0 \geq 1$  large enough so that

$$\int_{|h| > \frac{r}{4}} \rho_n \leq \frac{1}{2C}, \quad \forall n \geq n_0,$$

we see that (4.18) holds.

## 4.6 Proof of Theorems 4.1 and 4.2

**Proof of Theorem 4.2.** Given  $l \geq 1$ , we fix  $\varphi_l \in C_0^\infty(\Omega)$  such that  $\varphi_l = 1$  on  $\Omega_{1/l}$ . It is easy to see that the sequence  $(\varphi_l f_n)_{n \geq 1}$  satisfies the assumptions of Proposition 4.1. In particular,  $(f_n)$  is relatively compact in  $L^p(\Omega_l)$ . Applying a standard diagonalization argument, we can extract a subsequence  $(f_{n_j})$  such that  $f_{n_j} \rightarrow f$  in  $L^p_{\text{loc}}(\Omega)$ . Since the original sequence is bounded in  $L^p(\Omega)$ , we have  $f \in L^p(\Omega)$ .

**Claim.**  $f \in BV(\Omega)$  if  $p = 1$  and  $f \in W^{1,p}(\Omega)$  if  $1 < p < \infty$ ; moreover,

$$\int_{\Omega} |Df|^p \leq \frac{B}{K_{p,N}}.$$

Let  $\varphi \in C_0^\infty(B_1)$  be such that  $\varphi \geq 0$  and  $\int \varphi = 1$ . Given  $\delta > 0$ , we define

$$\varphi_\delta(x) := \frac{1}{\delta^N} \varphi\left(\frac{x}{\delta}\right), \quad \forall x \in \mathbb{R}^N.$$

It follows from Jensen's inequality and estimate (4.6) that

$$\int_{\Omega_\delta} \int_{\Omega_\delta} \frac{|\varphi_\delta * f_n(x) - \varphi_\delta * f_n(y)|^p}{|x - y|^p} \rho_n(|x - y|) dx dy \leq B, \quad \forall n \geq 1. \quad (4.22)$$

We now observe that for each  $\delta > 0$  fixed, the subsequence  $(\varphi_\delta * f_{n_j})_{j \geq 1}$  converges to  $\varphi_\delta * f$  in  $C^2(\overline{\Omega}_\delta)$ . Taking  $n_j \rightarrow \infty$  in (4.22) we get (see Remark 5.4)

$$K_{p,N} \int_{\Omega_\delta} |D(\varphi_\delta * f)|^p \leq B, \quad \forall \delta > 0.$$

The claim now follows by taking  $\delta \rightarrow 0$ .

We are left to prove that  $f_{n_j} \rightarrow f$  in  $L^p(\Omega)$ .

In order to show this, we apply (4.18) with  $f$  replaced by  $f_{n_j} - f$ . Using (4.5) and (4.6) we get

$$\int_{\Omega} |f_{n_j} - f|^p \leq C_1 \int_{\Omega_r} |f_{n_j} - f|^p + C_2 r^p 2^{p-1} \left( B + C \int_{\Omega} |Df|^p \right)$$

for every  $n_j \geq n_0(r)$ . For  $r > 0$  fixed we let  $j \rightarrow \infty$ . It follows that

$$\limsup_{j \rightarrow \infty} \int_{\Omega} |f_{n_j} - f|^p \leq C_2 r^p 2^{p-1} \left( B + C \int_{\Omega} |Df|^p \right).$$

Taking  $r \rightarrow 0$ , we conclude that  $f_{n_j} \rightarrow f$  in  $L^p(\Omega)$ .

As a corollary of Theorem 4.2, we have

**Proof of Theorem 4.1.** Let  $A_0 > 0$  be the best constant of the inequality (4.1). Assume by contradiction that there exists  $C > A_0/K_{p,N}$  for which (4.4) fails for every  $n \geq n_0$ . This means there exists a sequence  $(f_n)$  in  $L^p(\Omega)$  verifying the following properties :

$$\int_{\Omega} |f_n|^p = 1 \quad \text{and} \quad \int_{\Omega} f_n = 0, \quad (4.23)$$

$$\int_{\Omega} \int_{\Omega} \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dx \, dy < \frac{1}{C}. \quad (4.24)$$

Note that  $(f_n)$  satisfies the assumptions of Theorem 4.2. We can then extract a convergent subsequence  $f_{n_j} \rightarrow f$  in  $L^p(\Omega)$ . In particular, it follows from (4.23) that

$$\int_{\Omega} |f|^p = 1 \quad \text{and} \quad \int_{\Omega} f = 0.$$

On the other hand, from (4.24) we have

$$\int_{\Omega} |Df|^p \leq \frac{1}{K_{p,N}C}.$$

These two facts imply that  $1 \leq \frac{A_0}{K_{p,N}C}$ , a contradiction.

## 4.7 Proof of Theorem 4.3

We first observe that after replacing the sequence  $\rho_n$  by  $\frac{\rho_n(t) + \rho_n(-t)}{2}$ , we can always assume that each  $\rho_n$  is an even function. Note that (4.9) still holds with the same constant  $B$ .

To prove the theorem we shall follow the same steps as before. We start with a compactness lemma :

**Lemma 4.5** Assume there exists  $\theta_0 \in (0, 1)$  and  $\alpha_0 > 0$  such that (4.8) holds. If  $(f_n) \subset L^p(\mathbb{R})$  is a bounded sequence of functions such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f_n(x) - f_n(y)|^p}{|x - y|^p} \rho_n(x - y) dx dy \leq B, \quad \forall n \geq 1, \quad (4.25)$$

then  $(f_n)$  is relatively compact in  $L^p_{\text{loc}}(\mathbb{R})$ .

**Proof.** Let  $\ell_0 \geq 1$  be a fixed integer. We first prove the following

**Claim.** Estimate (4.10) still holds with  $\theta$  replaced by

$$\tilde{\theta} := 1 - \frac{\theta}{\ell_0} = 1 - \frac{1}{\ell_0} \left( \frac{t}{s} - k \right)$$

(with the constant  $C_p$  also depending on  $\ell_0$ ).

Indeed, it suffices to notice that

$$G_p(\theta s) \leq \ell_0^p G_p\left(\frac{\theta s}{\ell_0}\right) \leq 2^{p-1} \ell_0 \left\{ G_p(s) + G_p\left(s - \frac{\theta s}{\ell_0}\right) \right\}.$$

Inserting this inequality into (4.10), the claim follows.

Given  $\theta_0 \in (0, 1)$ , we take  $\ell_0 \geq 2$  sufficiently large so that  $1/\ell_0 < 1 - \theta_0$ ; in particular, we have  $\theta_0 < \tilde{\theta} \leq 1$ .

We now fix  $t_0 > 0$ . Take  $n_0 \geq 1$  sufficiently large so that

$$\int_0^{t_0} \rho_{n, \theta_0} \geq \frac{\alpha_0}{4}, \quad \forall n \geq n_0.$$

We know from our claim that

$$\frac{F_{n,p}(\tau)}{\tau^p} \leq C \left\{ \frac{F_{n,p}(s)}{s^p} + \frac{F_{n,p}(\tilde{\theta}s)}{\tau^p} \right\}$$

for every  $0 < s < t_0 \leq \tau$ . We multiply both sides of this inequality by  $\rho_{n, \theta_0}$ . Using (4.7) and integrating the resulting expression from 0 to  $t_0$  we get

$$\frac{\alpha_0}{4} \frac{F_{n,p}(\tau)}{\tau^p} \leq C \left\{ \int_0^\infty \frac{F_{n,p}(s)}{s^p} \rho_n(s) ds + \frac{1}{\tau^p} \int_0^{t_0} F_{n,p}(\tilde{\theta}s) \rho_n(\tilde{\theta}s) ds \right\} \quad (4.26)$$

for every  $\tau \geq t_0$  and  $n \geq n_0$ . We now estimate the second integral in the right-hand side of this inequality. We first observe that

$$\frac{1}{\tau^p} \int_0^{t_0} F_{n,p}(\tilde{\theta}s) \rho_n(\tilde{\theta}s) ds \leq \int_0^\tau \frac{F_{n,p}(\tilde{\theta}s)}{(\tilde{\theta}s)^p} \rho_n(\tilde{\theta}s) ds =: I.$$

We then make the change of variables  $h = \tilde{\theta}s$  (note that  $\tilde{\theta}$  is a function of  $s$  for fixed  $\tau$ ). Recall that, by definition,

$$\tilde{\theta}s = \left( \frac{k}{\ell_0} + 1 \right) s - \frac{\tau}{\ell_0} \quad \text{for } k \leq \frac{\tau}{s} < k + 1.$$

Thus,

$$\begin{aligned} I &= \sum_{k=1}^{\infty} \int_{\frac{\tau}{k+1}}^{\frac{\tau}{k}} \frac{F_{n,p}(\tilde{\theta}s)}{(\tilde{\theta}s)^p} \rho_n(\tilde{\theta}s) ds \\ &= \sum_{k=1}^{\infty} \int_{(1-\frac{1}{\ell_0})\frac{\tau}{k+1}}^{\frac{\tau}{k}} \frac{F_{n,p}(h)}{h^p} \rho_n(h) \frac{dh}{\frac{k}{\ell_0} + 1} \leq C \int_0^{\infty} \frac{F_{n,p}(h)}{h^p} \rho_n(h) dh. \end{aligned} \quad (4.27)$$

This last inequality comes from the fact that  $\frac{1}{k_0}$  belongs to at most  $Ck_0$  intervals of the form

$$\left( \left(1 - \frac{1}{\ell_0}\right) \frac{1}{k+1}; \frac{1}{k} \right) \quad \text{for } k \geq 1.$$

Inserting (4.27) into (4.26), and using (4.25), we conclude that

$$\frac{F_{n,p}(\tau)}{\tau^p} \leq \frac{C}{\alpha_0} \int_0^{\infty} \frac{F_{n,p}(s)}{s^p} \rho_n(s) ds \leq \frac{C}{\alpha_0} B$$

for every  $\tau \geq t_0$  and  $n \geq n_0$ . Proceeding as in the proof of (4.16), this implies that

$$\int_{\mathbb{R}} |f_n(x+t) - f_n(x)|^p dx \leq Ct_0^p, \quad \forall t \in (0, t_0), \quad \forall n \geq n_0.$$

In other words, the sequence  $(f_n)$  is relatively compact in  $L_{\text{loc}}^p(\mathbb{R})$  (see, for instance, [24, Theorem IV.25]).

The analogue of Lemma 4.4 is the following

**Lemma 4.6** *There exist  $r_0 > 0$  (depending on  $(\rho_n)_{n \geq 1}$ ) and constants  $C_1, C_2 > 0$  (depending on  $p$ ) so that the following holds : given  $0 < r < r_0$  we can find  $n_0 \geq 1$  such that*

$$\int_0^1 |f|^p \leq C_1 \int_r^{1-r} |f|^p + C_2 r^p \int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_n(x - y) dx dy \quad (4.28)$$

for every  $f \in L^p(0, 1)$  and  $n \geq n_0$ .

**Proof.** We proceed exactly as in the proof of Lemma 4.4. Actually, this case is even simpler since the claim is essentially contained in Lemma 4.2. Note in particular that condition (4.8) is not needed here.

Theorem 4.3 can now be proved as in the previous section.





# Chapitre 5

## A new approach to Sobolev spaces and connections to $\Gamma$ -convergence

### Sommaire

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### 5.1 Introduction<sup>6</sup>

This chapter is motivated by some results of Bourgain, Brezis and Mironescu [20] who studied quantities of the type

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy, \quad (5.1)$$

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<sup>6</sup>Ce chapitre a été publié dans Calc. Var. Partial Differential Equations **19** (2004), 229–255.

and connected their limit, as  $\varepsilon \downarrow 0$ , to the  $W^{1,p}$ -norm of  $f$  (resp.  $BV$  for  $p = 1$ ). Here, we shall extend their work by replacing  $|\cdot|^p$  with a continuous function  $\omega$ ; the functions  $\rho_\varepsilon$  are no longer assumed to be radial.

Let  $\Omega \subset \mathbb{R}^N$  be an open set such that  $\partial\Omega$  is compact and Lipschitz. Given a function  $f \in L^1_{\text{loc}}(\Omega)$ , we consider the functional

$$\int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_\varepsilon(x - y) dx dy, \quad (5.2)$$

where  $\omega : [0, \infty) \rightarrow [0, \infty)$  is continuous and  $(\rho_\varepsilon) \subset L^1(\mathbb{R}^N)$  is a family of functions satisfying the following properties

$$\left\{ \begin{array}{l} \rho_\varepsilon \geq 0 \quad \text{a.e. in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \rho_\varepsilon = 1, \quad \forall \varepsilon > 0, \\ \lim_{\varepsilon \downarrow 0} \int_{|h| > \delta} \rho_\varepsilon(h) dh = 0, \quad \forall \delta > 0. \end{array} \right. \quad (5.3)$$

We show that, in general, there exists a sequence  $\varepsilon_j \downarrow 0$  for which the pointwise limit in (5.2) exists. Of course, the sequence  $(\varepsilon_j)$  will be chosen independently of the function  $f$  we start with. By imposing an extra condition on  $(\rho_\varepsilon)$ , we obtain new characterizations for the Sobolev spaces  $W^{1,p}$ , with  $1 \leq p < \infty$ , and  $BV$ . At the end we prove the  $\Gamma$ -convergence of (5.2). As we will see, our results can also be used to get some information about noncoercive functionals.

We have been inspired by the simplified proofs presented by Brezis [26], following a suggestion of E. Stein (see Lemma 5.4). Our approach unifies the proofs of some well-known results, including the  $BV$ -case, and also deals with more general families  $(\rho_\varepsilon) \subset L^1(\mathbb{R}^N)$  (cf. [18, 20, 26, 38, 55, 56, 67]).

### 5.1.1 The radial case

Before studying the case of an arbitrary family  $(\rho_\varepsilon)$ , let us first state some known results when  $\rho_\varepsilon$  is radial.

Assume that  $1 < p < \infty$ . In this case, Bourgain, Brezis and Mironescu [18] have proved the following

**Theorem 5.1** *Let  $1 < p < \infty$ . For any  $f \in W^{1,p}(\Omega)$ , we have*

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx dy = K_{p,N} \int_{\Omega} |Df|^p, \quad (5.4)$$

where  $K_{p,N} = \int_{S^{N-1}} |e_1 \cdot \sigma|^p d\mathcal{H}^{N-1}$ .

It is somewhat surprising here that the limit does not depend on the choice of the family  $(\rho_\varepsilon)$ . Let us consider, for instance,

$$\rho_\varepsilon(h) = \frac{p}{\sigma_N} \frac{\varepsilon}{|h|^{N-\varepsilon p}}, \quad \forall h \in B_1,$$

which we have already encountered in the previous chapter. Switching  $\varepsilon$  by  $1 - s$ , then (5.4) becomes :

$$\lim_{s \uparrow 1} (1 - s) \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{N+sp}} dx dy = \frac{\sigma_N K_{p,N}}{p} \int_{\Omega} |Df|^p. \quad (5.5)$$

Note that the double integral in the left-hand side of (5.5) is, by definition, (a power of) the Gagliardo seminorm  $|\cdot|_{W^{s,p}}$  of the fractional Sobolev spaces  $W^{s,p}(\Omega)$ , with  $0 < s < 1$ . This way, (5.5) can be rewritten as

$$\lim_{s \uparrow 1} (1 - s)^{1/p} |f|_{W^{s,p}} = \tilde{K} |f|_{W^{1,p}}, \quad \forall f \in W^{1,p}(\Omega), \quad (5.6)$$

which shows  $W^{1,p}$  as a “limit” of the spaces  $W^{s,p}$  as  $s \uparrow 1$ .

There is also a converse statement of Theorem 5.1, namely (see [20])

**Theorem 5.2** *Let  $1 < p < \infty$ . Assume that  $f \in L^p(\Omega)$  satisfies*

$$\liminf_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon}(|x - y|) dx dy < \infty.$$

*Then,  $f \in W^{1,p}(\Omega)$ . In particular, (5.4) holds.*

According to Theorems 5.1 and 5.2, we can actually characterize the space  $W^{1,p}(\Omega)$  just in terms of the quantities (5.1) as  $\varepsilon \downarrow 0$ .

The case  $p = 1$  is a little more delicate. Theorem 5.1 is still valid for  $p = 1$ , but Theorem 5.2 is no longer true. It turns out that the good approach is to look at functions  $f \in BV(\Omega)$ , of bounded variation. Here,

$$BV(\Omega) := \left\{ f \in L^1(\Omega) : Df \text{ is a Radon measure} \right\}.$$

We then consider  $BV(\Omega)$  equipped with the seminorm

$$|f|_{BV} := \sup \left\{ \int_{\Omega} f \operatorname{div} \Phi : \Phi \in C_0^{\infty}(\Omega; \mathbb{R}^N) \text{ and } \|\Phi\|_{L^{\infty}} \leq 1 \text{ in } \Omega \right\} = \int_{\Omega} |Df|.$$

The analogue of Theorem 5.1 for BV-functions was proved by Dávila [38] :

**Theorem 5.3** *For any  $f \in BV(\Omega)$ , we have*

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|} \rho_{\varepsilon}(|x - y|) dx dy = K_{1,N} \int_{\Omega} |Df|. \quad (5.7)$$

Its converse, however, had already been established in [20] :

**Theorem 5.4** *Assume that  $f \in L^1(\Omega)$  satisfies*

$$\liminf_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|} \rho_{\varepsilon}(|x - y|) dx dy < \infty.$$

*Then,  $f \in BV(\Omega)$ .*

Our next step will be to consider the general case of families of functions  $(\rho_\varepsilon)$  which are not necessarily radial. As we shall see below, the two main features will be : 1) the limit in (5.1) strongly depends on the choice of  $(\rho_\varepsilon)$ ; 2) by choosing  $\rho_\varepsilon$  concentrating along a certain direction, the analogues of Theorems 5.2 and 5.4 will no longer be true (see Corollary 5.2).

### 5.1.2 Construction of the subsequence $\varepsilon_j \downarrow 0$

We start with a family of functions  $(\rho_\varepsilon)$  in  $L^1(\mathbb{R}^N)$  satisfying (5.3). To each  $\varepsilon > 0$  we associate the positive Radon measure  $\mu_\varepsilon$  on  $S^{N-1}$  defined by

$$\mu_\varepsilon(E) := \int_{\mathbb{R}_+ E} \rho_\varepsilon \quad \text{for each Borel set } E \subset S^{N-1},$$

where  $\mathbb{R}_+ E := \{rx : r \geq 0 \text{ and } x \in E\}$  is the cone generated by  $E$  with respect to the origin.

The family  $(\mu_\varepsilon)$  is bounded in  $\mathcal{M}(S^{N-1})$  (the space of Radon measures on  $S^{N-1}$ ), so there exist a sequence  $\varepsilon_j \downarrow 0$  and  $\mu \in \mathcal{M}(S^{N-1})$  such that

$$\mu_{\varepsilon_j} \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(S^{N-1}). \quad (5.8)$$

In particular,  $\mu \geq 0$  on  $S^{N-1}$  and  $\mu(S^{N-1}) = 1$ .

In Section 5.2 we present some examples of admissible families  $(\rho_\varepsilon)$  for which the measure  $\mu$  can be written down explicitly.

### 5.1.3 The pointwise limit of (5.1) as $\varepsilon_j \downarrow 0$

Using the above notation, we have

**Theorem 5.5** *If  $f \in W^{1,p}(\Omega)$ ,  $p \geq 1$ , then there exists  $C > 0$  such that*

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy \leq C, \quad \forall \varepsilon > 0.$$

Moreover,

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon_j}(x - y) dx dy = \int_{\Omega} \left( \int_{S^{N-1}} |Df \cdot \sigma|^p d\mu(\sigma) \right).$$

The case  $p = 1$  can be further extended to include the case of  $BV$ -functions :

**Theorem 5.6** *If  $f \in BV(\Omega)$ , then there exists  $C > 0$  such that*

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(x - y) dx dy \leq C, \quad \forall \varepsilon > 0. \quad (5.9)$$

In addition, we have

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|} \rho_{\varepsilon_j}(x - y) dx dy = \int_{S^{N-1}} \left( \int_{\Omega} |Df \cdot \sigma| \right) d\mu(\sigma).$$

We point out that the right-hand side of the identity above is well-defined since the function  $\sigma \in S^{N-1} \mapsto \int_{\Omega} |Df \cdot \sigma|$  is continuous.

There are special choices of  $(\rho_\varepsilon)$  which give some very interesting expressions (see Sections 4.2 and 5.2). Taking, for instance,  $\rho_\varepsilon$  radial for each  $\varepsilon > 0$ , we can deduce Theorems 5.1 and 5.3 as corollaries of the above results. In fact, an easy computation shows that, in this case,  $\mu = \frac{1}{\sigma_N} \mathcal{H}^{N-1} \llcorner_{S^{N-1}}$ .

Another example is given by  $\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right)$ , for some fixed nonnegative function  $\rho \in L^1(\mathbb{R}^N)$ . We then obtain the following limit originally proved by Gobbino and Mora [56] :

**Corollary 5.1** *Let  $\rho \in L^1(\mathbb{R}^N)$ ,  $\rho \geq 0$  a.e. in  $\mathbb{R}^N$ .*

*If  $f \in W^{1,p}(\Omega)$ ,  $p > 1$ , or if  $f \in BV(\Omega)$  and  $p = 1$ , then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^N} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho\left(\frac{x - y}{\varepsilon}\right) dx dy = \int_{\mathbb{R}^N} \left( \int_{\Omega} \left| Df \cdot \frac{z}{|z|} \right|^p \right) \rho(z) dz.$$

In the special case where  $\Omega = \mathbb{R}^N$ , by a simple change of variables we may rewrite the above identity as

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x + \varepsilon h) - f(x)|^p}{|\varepsilon h|^p} \rho(h) dx dh = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \left| Df \cdot \frac{z}{|z|} \right|^p \right) \rho(z) dz.$$

We can also take families  $(\rho_\varepsilon)$  which privilege certain directions. Let, for instance,

$$\rho_\varepsilon := \frac{1}{2^N \varepsilon^{2N-1}} \chi_{(-\varepsilon, \varepsilon) \times (-\varepsilon^2, \varepsilon^2)^{(N-1)}};$$

we have (see Example 5.3) :

**Corollary 5.2** *If  $f \in W^{1,p}(\Omega)$ ,  $p > 1$ , or if  $f \in BV(\Omega)$  and  $p = 1$ , then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2N-1}} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} dx dy = 2^N \int_{\Omega} \left| \frac{\partial f}{\partial x_1} \right|^p.$$

$\begin{array}{l} |x_1 - y_1| < \varepsilon \\ |x_i - y_i| < \varepsilon^2 \\ i = 2, \dots, N \end{array}$

**Remark 5.1** The results in this section rely heavily on the Lipschitz regularity of  $\partial\Omega$ . In fact, take for instance  $N = 2$  and  $\Omega := B_1(0) \setminus \{(x_1, 0) : 0 \leq x_1 < 1\}$ . On  $\Omega$  one can easily construct a smooth function  $f \in W^{1,p}(\Omega)$  such that

$$\lim_{\substack{x_2 \downarrow 0 \\ \frac{1}{2} < x_1 < 1}} f(x_1, x_2) = 1 \quad \text{and} \quad \lim_{\substack{x_2 \uparrow 0 \\ \frac{1}{2} < x_1 < 1}} f(x_1, x_2) = 0.$$

However, taking  $(\rho_\varepsilon)$  radial we have

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx dy = +\infty.$$

See Theorem 5.2; note that  $f \notin W^{1,p}(B_1)$ , while the integral above is actually being computed on  $B_1 \times B_1$  (since  $\{(x_1, 0) : 0 \leq x_1 < 1\}$  is a null set in  $\mathbb{R}^2$ ). See also Remark 5.2 below.

### 5.1.4 Some new characterizations of $W^{1,p}$ for $1 < p < \infty$ , and $BV$

As we have already mentioned, if  $u \in W^{1,p}(\Omega)$  and  $1 < p < \infty$ , then

$$\limsup_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon}(x - y) dx dy < +\infty \quad (5.10)$$

(if  $p = 1$ , then  $W^{1,1}(\Omega)$  may be replaced by  $BV(\Omega)$ ). In order to prove the converse, we shall impose the following condition on the family  $(\rho_{\varepsilon})$ :

$$\left\{ \begin{array}{l} \text{there exist linearly independent vectors } v_1, \dots, v_N \in \mathbb{R}^N \\ \text{and } \delta > 0 \text{ such that} \\ C_{\delta}(v_i) \cap C_{\delta}(v_j) = \emptyset \quad \text{if } i \neq j, \\ \limsup_{\varepsilon \downarrow 0} \int_{C_{\delta}(v_i)} \rho_{\varepsilon} > 0, \quad \forall i = 1, \dots, N. \end{array} \right. \quad (5.11)$$

Here, for any  $v \in \mathbb{R}^N \setminus \{0\}$  and  $\delta > 0$ ,  $C_{\delta}(v)$  denotes the cone

$$C_{\delta}(v) := \left\{ w \in \mathbb{R}^N \setminus \{0\} : \frac{v}{|v|} \cdot \frac{w}{|w|} > (1 - \delta) \right\}.$$

We then have

**Theorem 5.7** *Let  $f \in L^p(\Omega)$ ,  $p \geq 1$ . Suppose*

$$\limsup_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon}(x - y) dx dy < \infty,$$

where  $(\rho_{\varepsilon})$  satisfies (5.3) and (5.11).

Then,  $f \in W^{1,p}(\Omega)$  if  $p > 1$ , and  $f \in BV(\Omega)$  if  $p = 1$ . Moreover, there exists  $\alpha > 0$  (depending only on  $(\rho_{\varepsilon})$ ) such that

$$\alpha \int_{\Omega} |Df|^p \leq \limsup_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon}(x - y) dx dy.$$

From Theorem 5.7, we deduce Theorems 5.2 and 5.4. Another corollary is the following (see [56]):

**Corollary 5.3** *Let  $\rho \in L^1(\mathbb{R}^N)$ ,  $\rho \geq 0$  a.e. in  $\mathbb{R}^N$ , be such that  $\int \rho > 0$ . Let  $f \in L^p(\Omega)$ , with  $p \geq 1$ . If*

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^N} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho\left(\frac{x - y}{\varepsilon}\right) dx dy < \infty,$$

then  $f \in W^{1,p}(\Omega)$  if  $p > 1$ , and  $f \in BV(\Omega)$  if  $p = 1$ .

We shall discuss in Section 5.10 what can be said about  $f$  without assuming (5.11).

We conclude this section with the following criterion to decide whether a function  $f$ , defined on an open connected set  $A \subset \mathbb{R}^N$ , is constant; this extends some of the results in [26]:

**Corollary 5.4** *Assume  $A \subset \mathbb{R}^N$  is an open connected set. Let  $f \in L^p(A)$  be such that*

$$\lim_{\varepsilon \downarrow 0} \int_A \int_A \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy = 0,$$

where  $p \geq 1$  and  $(\rho_\varepsilon)$  satisfies (5.3) and (5.11). Then,  $f = \text{const}$  a.e. in  $A$ .

Some variants of this corollary have been extensively studied by Ignat [60].

**Remark 5.2** A careful inspection in the proof of Theorem 5.7 shows that it still holds without any assumption on the regularity of  $\Omega$ . The converse statement, however, relies heavily on the smoothness of  $\partial\Omega$ . It would be interesting to find an expression similar to (5.10), which characterizes  $W^{1,p}(\Omega)$ , without any additional assumptions on  $\Omega$ . The example in Remark 5.1 suggests the following

**Open Problem 3** Suppose  $(\rho_\varepsilon)$  is a family of radial functions satisfying (5.3). Given  $1 < p < \infty$ , let  $f \in L^p(\Omega)$  be such that

$$\limsup_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(d_\Omega(x, y)) dx dy < +\infty,$$

where  $d_\Omega$  denotes the geodesic distance in  $\Omega$ . Can one conclude that  $u \in W^{1,p}(\Omega)$ , without assuming any regularity of  $\partial\Omega$ ?

The answer to this problem does not seem to be known even in the case of a disk without a line segment.

### 5.1.5 Extensions to continuous functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

We now consider the problem of determining the limit of (5.2) as  $\varepsilon_j \downarrow 0$ . Assuming that  $\omega$  is asymptotic linear at infinity, we obtain the following result which extends Theorem 5.6. Below, we denote by  $\omega_\mu$  the function

$$\omega_\mu(v) := \int_{S^{N-1}} \omega(|v \cdot \sigma|) d\mu(\sigma), \quad \forall v \in \mathbb{R}^N. \quad (5.12)$$

**Theorem 5.8** *Let  $\omega : [0, \infty) \rightarrow [0, \infty)$  be a continuous function satisfying*

$$\omega^\infty := \lim_{t \rightarrow \infty} \frac{\omega(t)}{t} \in [0, \infty).$$

*If  $\Omega \subset \mathbb{R}^N$  is unbounded, suppose in addition that there exists  $C > 0$  such that*

$$|\omega(t)| \leq Ct, \quad \forall t \geq 0.$$

*If  $f \in BV(\Omega)$ , then*

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_\Omega \int_\Omega \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy &= \\ &= \int_\Omega \omega_\mu(D^a f) + \omega^\infty \int_{S^{N-1}} \left( \int_\Omega |D^s f \cdot \sigma| \right) d\mu(\sigma), \end{aligned}$$

where  $Df = D^a f \mathcal{L}^N + D^s f$  is the Radon-Nikodym decomposition of  $Df$  with respect to the Lebesgue measure.



Another result in this direction is the following :

**Theorem 5.9** *Assume  $\tilde{\omega} : [0, \infty) \rightarrow [0, \infty)$  is convex and increasing, and let  $f \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$  be such that  $\tilde{\omega}(|Df|) \in L^1(\mathbb{R}^N)$ .*

*For any continuous function  $\omega$  satisfying*

$$0 \leq \omega(t) \leq \tilde{\omega}(t), \quad \forall t \geq 0, \quad (5.13)$$

*we have*

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy = \int_{\Omega} \omega_{\mu}(Df). \quad (5.14)$$

### 5.1.6 Remarks about the $\Gamma$ -convergence of (5.2)

We first recall the definition of  $\Gamma$ -lower and upper limits (with respect to the  $L^1(\Omega)$ -topology); see, for instance, the paper of De Giorgi and Franzoni [41] :

Given a bounded open set  $A \subset \mathbb{R}^N$ , let  $(F_j)$  be any sequence of functionals  $F_j : L^1(A) \rightarrow [0, +\infty]$ . For each  $f \in L^1(A)$ , we set

$$\Gamma_{L^1(A)}^{-} \text{-} \liminf F_j(f) := \min \left\{ \liminf_{j \rightarrow \infty} F_j(f_j) : f_j \rightarrow f \text{ in } L^1(A) \right\}, \quad (5.15)$$

$$\Gamma_{L^1(A)}^{-} \text{-} \limsup F_j(f) := \min \left\{ \limsup_{j \rightarrow \infty} F_j(f_j) : f_j \rightarrow f \text{ in } L^1(A) \right\}. \quad (5.16)$$

(A standard diagonalization argument shows that both minima are really attained.)

If both limits are equal at some point  $f \in L^1(A)$ , we say that the sequence  $(F_j)$   $\Gamma$ -converges at  $f$ , and we denote this common number by  $\Gamma_{L^1(A)}^{-} \text{-} \lim F_j(f)$ .

Given  $F : L^1(A) \rightarrow [0, +\infty]$ , the lower semicontinuous envelope of  $F$ ,  $\text{sc}_{L^1(A)}^{-} F$ , is the greatest  $L^1(A)$ -lower semicontinuous functional less than or equal to  $F$ . In terms of the  $\Gamma$ -convergence we have

$$\text{sc}_{L^1(A)}^{-} F(f) = \min \left\{ \liminf_{j \rightarrow \infty} F(f_j) : f_j \rightarrow f \text{ in } L^1(A) \right\}. \quad (5.17)$$

We recall that  $\omega^{**}$  denotes the convex lower semicontinuous envelope of  $\omega : [0, \infty) \rightarrow [0, \infty)$  (which in our case coincides with the greatest convex function less than or equal to  $\omega$ ).

**Theorem 5.10** *Assume  $\Omega \subset \mathbb{R}^N$  is a bounded open set with Lipschitz boundary and  $\omega : [0, \infty) \rightarrow [0, \infty)$  is continuous.*

*If*

$$\omega_{\mu}^{**} = (\omega^{**})_{\mu} \quad \text{in } \mathbb{R}^N, \quad (5.18)$$

*then*

$$\Gamma_{L^1(\Omega)}^{-} \text{-} \lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy = \text{sc}_{L^1(\Omega)}^{-} F(f)$$

for every  $f \in L^1(\Omega)$ , where  $F : L^1(\Omega) \rightarrow [0, +\infty]$  is given by

$$F(f) = \begin{cases} \int_{\Omega} \omega_{\mu}(Df) & \text{if } f \in C^1(\overline{\Omega}), \\ +\infty & \text{otherwise.} \end{cases} \quad (5.19)$$

The theorem above reduces the problem of studying the  $\Gamma$ -convergence of our functionals to a relaxation problem; namely, to determine the lower semicontinuous envelope of  $F$ . A very nice introduction to this subject can be found in the book of Buttazzo [35]. In view of Theorem 5.10, we have the following

**Corollary 5.5** *Under the assumptions of Theorem 5.10, we have*

$$\begin{aligned} \Gamma_{L^1(\Omega)}^- \lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy = \\ = \int_{\Omega} \omega_{\mu}^{**}(D^a f) dx + \int_{\Omega} (\omega_{\mu}^{**})^{\infty} \left( \frac{dD^s f}{|D^s f|} \right) d|D^s f| \end{aligned} \quad (5.20)$$

for every  $f \in BV(\Omega)$ , where  $(\omega_{\mu}^{**})^{\infty}$  is the recession function of  $\omega_{\mu}^{**}$  (see Section 5.11).

In Section 5.11, we present some examples of functions  $\omega$  and measures  $\mu$  for which (5.18) is satisfied. Note that equality (5.18) does not hold in general. In fact, take  $\mu = \frac{1}{\sigma_N} \mathcal{H}^{N-1} \llcorner_{S^{N-1}}$  (which corresponds to a family  $(\rho_{\varepsilon})$  of radial functions); then, one can construct a continuous function  $\omega$  which is not convex, while  $\omega_{\mu}$  is. We do not know whether condition (5.18) is necessary to prove the  $\Gamma$ -convergence of (5.2).

## 5.2 Determining the measure $\mu \in \mathcal{M}(S^{N-1})$

Before proceeding, we point out that the family  $(\mu_{\varepsilon})$  we defined in Section 5.1 is absolutely continuous with respect to the Hausdorff measure  $\mathcal{H}^{N-1} \llcorner_{S^{N-1}}$  in  $S^{N-1}$ , that is,  $\mu_{\varepsilon} \in L^1(S^{N-1})$  and it is given by

$$\mu_{\varepsilon}(\sigma) = \int_0^{\infty} \rho_{\varepsilon}(t\sigma) t^{N-1} dt \quad \text{for a.e. } \sigma \in S^{N-1}.$$

In particular,  $\mu_{\varepsilon} \geq 0$  a.e. in  $S^{N-1}$  and  $\int_{S^{N-1}} \mu_{\varepsilon} = 1$  for every  $\varepsilon > 0$ . Since  $\mu_{\varepsilon_j} \xrightarrow{*} \mu$  in  $\mathcal{M}(S^{N-1})$ , these properties imply that the Radon measure  $\mu$  itself is nonnegative and  $\int_{S^{N-1}} d\mu = 1$ .

**Example 5.1** Suppose that  $\rho_{\varepsilon}$  is radial for every  $\varepsilon > 0$ . Then,  $\mu_{\varepsilon} = \frac{1}{\sigma_N}$  for every  $\varepsilon > 0$ , and so  $\mu = \frac{1}{\sigma_N} \mathcal{H}^{N-1} \llcorner_{S^{N-1}}$ . Therefore,

$$\omega_{\mu}(v) = \int_{S^{N-1}} \omega(|v \cdot \sigma|) d\mathcal{H}^{N-1}(\sigma), \quad \forall v \in \mathbb{R}^N.$$

Taking in particular  $\omega(t) = t^p$ ,  $p > 0$ , and using the symmetry of  $S^{N-1}$  we have

$$\omega_\mu(v) = K_{p,N}|v|^p, \quad \forall v \in \mathbb{R}^N,$$

where

$$K_{p,N} = \int_{S^{N-1}} |e_1 \cdot \sigma|^p d\mathcal{H}^{N-1} = \frac{1}{\pi^{1/2}} \frac{\Gamma(\frac{N}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p}{2})}.$$

**Example 5.2** Let  $\rho \in L^1(\mathbb{R}^N)$ ,  $\rho \geq 0$  a.e. in  $\mathbb{R}^N$ , be such that  $\int \rho = 1$ . For each  $\varepsilon > 0$ , define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Therefore,

$$\mu(\sigma) = \mu_\varepsilon(\sigma) = \int_0^\infty \rho(t\sigma)t^{N-1} dt \quad \text{for a.e. } \sigma \in S^{N-1}.$$

The function  $\omega_\mu$  may be written in this case as

$$\omega_\mu(v) = \int_{\mathbb{R}^N} \omega\left(\left|v \cdot \frac{z}{|z|}\right|\right) \rho(z) dz, \quad \forall v \in \mathbb{R}^N.$$

**Example 5.3** Let

$$\rho_\varepsilon := \frac{1}{2^N \varepsilon^{2N-1}} \chi_{(-\varepsilon, \varepsilon) \times (-\varepsilon^2, \varepsilon^2)^{(N-1)}}.$$

It is easy to see that  $\mu = \frac{\delta_{e_1} + \delta_{-e_1}}{2}$ , whence

$$\omega_\mu(v) = \omega(|v_1|), \quad \forall v \in \mathbb{R}^N.$$

More generally, let  $1 \leq k \leq N$  be a fixed integer, and write  $\mathbb{R}^N = \mathbb{R}^k \oplus \mathbb{R}^{N-k}$ . We now define

$$\rho_\varepsilon := \frac{1}{|B_\varepsilon^k| \times |B_{\varepsilon^2}^{N-k}|} \chi_{B_\varepsilon^k \times B_{\varepsilon^2}^{N-k}}.$$

We observe that  $\text{supp } \mu \subset S^{k-1}$ ,  $\mu$  is uniform on  $S^{k-1}$  and  $\mu(S^{k-1}) = 1$ . We then conclude that  $\mu = \frac{1}{\sigma_k} \mathcal{H}^{k-1} \llcorner_{S^{k-1}}$ .

Taking in particular  $\omega(t) = t^p$ ,  $p > 0$ , we get

$$\omega_\mu(v) = K_{p,k}|v'|^p, \quad \forall v = (v', v'') \in \mathbb{R}^N,$$

where  $K_{p,k}$  is defined in Example 5.1.

In the next example, we show that given any nonnegative measure  $\mu \in \mathcal{M}(S^{N-1})$ , with  $\mu(S^{N-1}) = 1$ , one can find a family  $(\rho_\varepsilon)$  satisfying (5.3) for which

$$\mu_\varepsilon \xrightarrow{*} \mu \quad \text{in } \mathcal{M}(S^{N-1}). \quad (5.21)$$

**Example 5.4** Let  $\mu \in \mathcal{M}(S^{N-1})$ ,  $\mu \geq 0$ , be such that  $\mu(S^{N-1}) = 1$ . We define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{S^{N-1}} \eta\left(\frac{x}{\varepsilon} - y\right) d\mu(y), \quad \forall x \in \mathbb{R}^N,$$

where  $\eta \in C_0^\infty(\mathbb{R}^N)$  is a nonnegative function such that  $\int \eta = 1$  and  $\text{supp } \eta \subset B_1$ . Notice that  $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ ,  $\rho_\varepsilon \geq 0$  in  $\mathbb{R}^N$ ,  $\int \rho_\varepsilon = 1$  and  $\text{supp } \rho_\varepsilon \subset B_{2\varepsilon}$ . Thus,  $(\rho_\varepsilon)$  satisfies (5.3). In addition, one can easily check that (5.21) holds for such family.

We conclude this section with the following remark which will be useful in some of the proofs :

**Remark 5.3** Assume  $\theta \in C(S^{N-1})$ . For each  $\varepsilon > 0$  we have

$$\int_{\mathbb{R}^N} \theta\left(\frac{h}{|h|}\right) \rho_\varepsilon(h) dh = \int_{S^{N-1}} \theta(\sigma) d\mu_\varepsilon(\sigma).$$

In particular,

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \theta\left(\frac{h}{|h|}\right) \rho_{\varepsilon_j}(h) dh = \int_{S^{N-1}} \theta(\sigma) d\mu(\sigma).$$

## 5.3 The regular case

The next proposition implies that (5.2) always converges (up to the fixed subsequence  $\varepsilon_j \downarrow 0$  we have constructed) if  $\Omega$  is bounded and  $f$  is smooth. More precisely,

**Proposition 5.1** Assume  $\Omega \subset \mathbb{R}^N$  is bounded, and let  $\omega : [0, \infty) \rightarrow [0, \infty)$  be a continuous function.

If  $f \in C^2(\overline{\Omega})$ , then

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega\left(\frac{|f(x) - f(y)|}{|x - y|}\right) \rho_{\varepsilon_j}(x - y) dx dy = \int_{\Omega} \omega_\mu(Df). \quad (5.22)$$

**Proof.** For each  $f \in C^2(\overline{\Omega})$ , we set  $M_f := \|Df\|_{L^\infty}$ . Since  $\omega$  is uniformly continuous in  $[0, M_f]$ , given any  $\delta > 0$  there exists  $C_\delta > 0$  such that

$$|\omega(s) - \omega(t)| \leq C_\delta |s - t| + \delta, \quad \forall s, t \in [0, M_f].$$

In particular, we have

$$\begin{aligned} \left| \omega\left(\frac{|f(x) - f(y)|}{|x - y|}\right) - \omega\left(\left|Df(x) \cdot \frac{x - y}{|x - y|}\right|\right) \right| &\leq C_\delta \frac{|f(x) - f(y) - Df(x) \cdot (x - y)|}{|x - y|} + \delta \\ &\leq C_\delta |x - y| + \delta \end{aligned}$$

for every  $x, y \in \mathbb{R}^N$ , with  $x \neq y$ . Therefore,

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \left| \omega\left(\frac{|f(x) - f(y)|}{|x - y|}\right) - \omega\left(\left|Df(x) \cdot \frac{x - y}{|x - y|}\right|\right) \right| \rho_\varepsilon(x - y) dx dy &\leq \\ &\leq |\Omega| \left\{ C_\delta \int_{|h| \leq 1} |h| \rho_\varepsilon(h) dh + \delta + \max_{[0, M_f]} \omega \cdot \int_{|h| > 1} \rho_\varepsilon(h) dh \right\}. \end{aligned}$$

By (5.3), the first and the last terms in the right-hand side tend to zero as  $\varepsilon \downarrow 0$ , for every fixed  $\delta > 0$ . By taking  $\delta \downarrow 0$  in the resulting expression, we conclude that

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \left| \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) - \omega \left( \left| Df(x) \cdot \frac{x - y}{|x - y|} \right| \right) \right| \rho_{\varepsilon}(x - y) dx dy = 0.$$

In other words, to prove (5.22), it suffices to show that

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \left| Df(x) \cdot \frac{x - y}{|x - y|} \right| \right) \rho_{\varepsilon_j}(x - y) dx dy = \int_{\Omega} \omega_{\mu}(Df). \quad (5.23)$$

We first write

$$\begin{aligned} \int_{\Omega} \int_{\mathbb{R}^N} \omega \left( \left| Df(x) \cdot \frac{h}{|h|} \right| \right) \rho_{\varepsilon_j}(h) dx dh &= \\ &= \int_{\Omega} \int_{\Omega} \omega \left( \left| Df(x) \cdot \frac{x - y}{|x - y|} \right| \right) \rho_{\varepsilon_j}(x - y) dx dy + \\ &\quad + \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \omega \left( \left| Df(x) \cdot \frac{x - y}{|x - y|} \right| \right) \rho_{\varepsilon_j}(x - y) dx dy. \end{aligned} \quad (5.24)$$

To estimate the last term in (5.24), fix  $\lambda > 0$ . We have

$$\begin{aligned} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \omega \left( \left| Df(x) \cdot \frac{x - y}{|x - y|} \right| \right) \rho_{\varepsilon}(x - y) dx dy &\leq \\ &\leq \max_{[0, M_f]} \omega \cdot \left\{ |\Omega| \int_{|h| > \lambda} \rho_{\varepsilon}(h) dh + |\Omega \setminus \Omega_{\lambda}| \int_{|h| \leq \lambda} \rho_{\varepsilon}(h) dh \right\}. \end{aligned}$$

We first take  $\varepsilon \downarrow 0$  and then  $\lambda \downarrow 0$  to get

$$\lim_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \omega \left( \left| Df(x) \cdot \frac{x - y}{|x - y|} \right| \right) \rho_{\varepsilon}(x - y) dx dy = 0. \quad (5.25)$$

By Remark 5.3, (5.24) and (5.25), we conclude that (5.23) holds.

The next two remarks will be used in Section 5.11 to study the  $\Gamma$ -convergence of (5.2) :

**Remark 5.4** It follows from the proof of Proposition 5.1 that the convergence in (5.22) is uniform on the bounded subsets of  $C^2(\overline{\Omega})$ .

**Remark 5.5** A slight modification in the argument above shows that (5.22) still holds for any  $f \in C^1(\overline{\Omega})$ .

## 5.4 Some useful estimates

The following lemmas will be used throughout this paper. Since they have been extensively applied (see [20, 26, 38]), we shall only sketch their proofs.

**Lemma 5.1** *Assume  $\omega : [0, \infty) \rightarrow [0, \infty)$  is convex. If  $f \in W_{\text{loc}}^{1,1}(\mathbb{R}^N)$ , then*

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_\varepsilon(x - y) dx dy &\leq \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega \left( \left| Df(x) \cdot \frac{h}{|h|} \right| \right) \rho_\varepsilon(h) dx dh, \quad \forall \varepsilon > 0. \end{aligned} \quad (5.26)$$

**Proof.** Let  $\delta > 0$ . For any  $R > 0$ , it follows from a standard application of the Fundamental Theorem of Calculus and Jensen's inequality that

$$\begin{aligned} \int_{B_R} \int_{B_R} \omega \left( \frac{|f_\delta(x) - f_\delta(y)|}{|x - y|} \right) \rho_\varepsilon(x - y) dx dy &\leq \\ &\leq \int_{B_R} \int_{\mathbb{R}^N} \omega \left( \left| Df_\delta(x) \cdot \frac{h}{|h|} \right| \right) \rho_\varepsilon(h) dx dh \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega \left( \left| Df(x) \cdot \frac{h}{|h|} \right| \right) \rho_\varepsilon(h) dx dh. \end{aligned} \quad (5.27)$$

Taking  $\delta \downarrow 0$ , and then  $R \rightarrow \infty$ , we obtain (5.26).

**Lemma 5.2** *Assume  $f \in W^{1,p}(\Omega)$ , with  $p \geq 1$ . Let  $\bar{f} \in W^{1,p}(\mathbb{R}^N)$  be an extension of  $f$  in  $\mathbb{R}^N$ . For every  $r, \varepsilon > 0$  we have*

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy &\leq \\ &\leq \int_{N_r(\Omega)} \int_{|h| < r} \left| D\bar{f}(x) \cdot \frac{h}{|h|} \right|^p \rho_\varepsilon(h) dx dh + \frac{2^p \|f\|_{L^p}^p}{r^p} \int_{|h| \geq r} \rho_\varepsilon. \end{aligned} \quad (5.28)$$

**Proof.** For any  $\delta \in (0, r)$ , we have

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|\bar{f}_\delta(x) - \bar{f}_\delta(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy &= \\ &= \int_{\Omega} \int_{\Omega} \frac{|\bar{f}_\delta(x) - \bar{f}_\delta(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy \\ &\leq \int_{\Omega} \int_{\Omega} \frac{|\bar{f}_\delta(x) - \bar{f}_\delta(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy + \frac{2^p \|f\|_{L^p}^p}{r^p} \int_{|h| \geq r} \rho_\varepsilon. \end{aligned}$$

Proceeding as before to estimate the first term in the right-hand side of the inequality (note that if  $x, y \in \Omega$  and  $|x - y| < r$  then  $tx + (1 - t)y \in N_r(\Omega)$  for every  $t \in [0, 1]$ ), we obtain (5.28).

The next lemma can be proved exactly as above. Actually, applying Jensen's inequality as in the last estimate of (5.27), we do not make use of the weak convergence  $D\bar{f}_\delta \xrightarrow{*} Df$  in  $\mathcal{M}(\mathbb{R}^N)$ .

**Lemma 5.3** *Assume  $f \in BV(\Omega)$ . Let  $\bar{f} \in BV(\mathbb{R}^N)$  be an extension of  $f$  in  $\mathbb{R}^N$ . For every  $r, \varepsilon > 0$  we have*

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|} \rho_{\varepsilon}(x - y) dx dy &\leq \\ &\leq \int_{|h| < r} \left( \int_{N_r(\Omega)} \left| D\bar{f} \cdot \frac{h}{|h|} \right| \right) \rho_{\varepsilon}(h) dh + \frac{2\|f\|_{L^1}}{r} \int_{|h| \geq r} \rho_{\varepsilon}. \end{aligned}$$

The following lemma was pointed out by E. Stein. It comes from a simple application of Jensen's inequality and a change of variables.

**Lemma 5.4** *Assume  $\omega : [0, \infty) \rightarrow [0, \infty)$  is convex, and let  $f \in L^1_{\text{loc}}(\Omega)$ . For each  $r > 0$  we have*

$$\begin{aligned} \int_{\Omega_r} \int_{\Omega_r} \omega \left( \frac{|f_{\delta}(x) - f_{\delta}(y)|}{|x - y|} \right) \rho_{\varepsilon}(x - y) dx dy &\leq \\ &\leq \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon}(x - y) dx dy, \quad \forall \delta \in (0, r). \end{aligned}$$

## 5.5 Proof of Theorems 5.5 and 5.6

**Proof of Theorem 5.5.** Given  $f \in W^{1,p}(\Omega)$ , we take an extension  $\bar{f} \in W^{1,p}(\mathbb{R}^N)$  of  $f$ . For any  $g \in C_0^{\infty}(\mathbb{R}^N)$ , using the triangle inequality and Lemma 5.1 we have

$$\begin{aligned} &\left| \left( \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon_j}(x - y) dx dy \right)^{1/p} - \right. \\ &\quad \left. - \left( \int_{\Omega} \int_{\Omega} \frac{|g(x) - g(y)|^p}{|x - y|^p} \rho_{\varepsilon_j}(x - y) dx dy \right)^{1/p} \right| \leq \\ &\leq \left( \int_{\Omega} \int_{\Omega} \frac{|(f - g)(x) - (f - g)(y)|^p}{|x - y|^p} \rho_{\varepsilon_j}(x - y) dx dy \right)^{1/p} \\ &\leq \left( \int_{\mathbb{R}^N} |D\bar{f} - Dg|^p \right)^{1/p}. \end{aligned}$$

Let  $j \rightarrow \infty$ . We conclude the proof by using a variant of Proposition 5.1 for  $C_0^{\infty}$ -functions and the density of  $C_0^{\infty}(\mathbb{R}^N)$  in  $W^{1,p}(\mathbb{R}^N)$ .

**Proof of Theorem 5.6.** Given  $f \in BV(\Omega)$ , there exists an extension  $\bar{f} \in BV(\mathbb{R}^N)$  such that  $\int_{\partial\Omega} |D\bar{f}| = 0$  (see, e.g., [47]; this last property can be obtained by a local reflexion across the boundary). Applying Lemma 5.3, we see that (5.9) holds.

By Lemmas 5.3 and 5.4, we have for any  $0 < \delta < r$  that

$$\begin{aligned} \int_{\Omega_r \cap B_{1/r}} \int_{\Omega_r \cap B_{1/r}} \frac{|f_{\delta}(x) - f_{\delta}(y)|}{|x - y|} \rho_{\varepsilon_j}(x - y) dx dy &\leq \\ &\leq \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|} \rho_{\varepsilon_j}(x - y) dx dy \leq \\ &\leq \int_{\mathbb{R}^N} \left( \int_{N_r(\Omega)} \left| D\bar{f} \cdot \frac{h}{|h|} \right| \right) \rho_{\varepsilon_j}(h) dh + \frac{2\|f\|_{L^1}}{r} \int_{|h| \geq r} \rho_{\varepsilon_j}. \end{aligned} \tag{5.29}$$

We make the following remarks :

- (i) The function  $\sigma \in S^{N-1} \mapsto \int |D\bar{f} \cdot \sigma| \in \mathbb{R}$  is continuous;
- (ii) If  $\omega \subset \Omega$  is open and  $\int_{\partial\omega} |Df| = 0$ , then

$$\int_{\omega} |Df_{\delta} \cdot \sigma| \xrightarrow{\delta \downarrow 0} \int_{\omega} |Df \cdot \sigma|$$

uniformly with respect to  $\sigma \in S^{N-1}$ .

By (i) and Remark 5.3, we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \left( \int_{N_r(\Omega)} \left| D\bar{f} \cdot \frac{h}{|h|} \right| \right) \rho_{\varepsilon_j}(h) dh = \int_{S^{N-1}} \left( \int_{N_r(\Omega)} |D\bar{f} \cdot \sigma| \right) d\mu(\sigma). \quad (5.30)$$

Note that, by the outer regularity of Radon measures,

$$\int_{N_r(\Omega)} |D\bar{f} \cdot \sigma| \rightarrow \int_{\bar{\Omega}} |D\bar{f} \cdot \sigma| \quad \text{as } r \downarrow 0,$$

uniformly with respect to  $\sigma \in S^{N-1}$ ; hence,

$$\begin{aligned} \lim_{r \downarrow 0} \int_{S^{N-1}} \left( \int_{N_r(\Omega)} |D\bar{f} \cdot \sigma| \right) d\mu(\sigma) &= \int_{S^{N-1}} \left( \int_{\bar{\Omega}} |D\bar{f} \cdot \sigma| \right) d\mu(\sigma) \\ &= \int_{S^{N-1}} \left( \int_{\Omega} |Df \cdot \sigma| \right) d\mu(\sigma), \end{aligned} \quad (5.31)$$

where in the last step we used that  $\int_{\partial\Omega} |D\bar{f}| = 0$ .

Let us denote by  $I(r, \delta, j)$  the left-hand side of (5.29). According to Proposition 5.1, we have

$$I(r, \delta, \infty) = \lim_{j \rightarrow \infty} I(r, \delta, j) = \int_{S^{N-1}} \left( \int_{\Omega_r \cap B_{1/r}} |Df_{\delta} \cdot \sigma| \right) d\mu(\sigma).$$

Note that for a.e.  $r > 0$  we have

$$\int_{\partial(\Omega_r \cap B_{1/r})} |Df| = 0. \quad (5.32)$$

In particular, we can extract a sequence  $r_k \downarrow 0$  for which (5.32) holds for every  $r = r_k$ . As  $\delta \downarrow 0$ , it follows from (ii) that

$$I(r_k, 0, \infty) = \int_{S^{N-1}} \left( \int_{\Omega_{r_k} \cap B_{1/r_k}} |Df \cdot \sigma| \right) d\mu(\sigma), \quad \forall k \geq 1.$$

By the inner regularity of Radon measures, we finally get

$$I(0, 0, \infty) = \int_{S^{N-1}} \left( \int_{\Omega} |Df \cdot \sigma| \right) d\mu(\sigma). \quad (5.33)$$



It follows from (5.30), (5.31), and (5.33) that, as  $j \rightarrow \infty$ ,  $\delta \downarrow 0$ , and  $r_k \downarrow 0$ , the middle term in (5.29) stays bounded from below and from above by

$$\int_{S^{N-1}} \left( \int_{\Omega} |Df \cdot \sigma| \right) d\mu(\sigma).$$

This concludes the proof of Theorem 5.6.

## 5.6 Proof of Theorem 5.7

Let  $p \geq 1$  and  $f \in L^p(\Omega)$  be such that

$$\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon}(x - y) dx dy \leq C, \quad \forall \varepsilon > 0 \text{ small}, \quad (5.34)$$

for some  $C > 0$ . It follows from Lemma 5.4 that, for any  $0 < \delta < r$ , we have

$$\int_{\Omega_r \cap B_{1/r}} \int_{\Omega_r \cap B_{1/r}} \frac{|f_{\delta}(x) - f_{\delta}(y)|^p}{|x - y|^p} \rho_{\varepsilon}(x - y) dx dy \leq C, \quad \forall \varepsilon > 0 \text{ small}.$$

By Proposition 5.1 and Jensen's inequality (recall that  $\mu(S^{N-1}) = 1$ ), we get

$$\begin{aligned} \int_{\Omega_r \cap B_{1/r}} \left\{ \int_{S^{N-1}} |Df_{\delta}(x) \cdot \sigma| d\mu(\sigma) \right\}^p dx &\leq \\ &\leq \int_{\Omega_r \cap B_{1/r}} \int_{S^{N-1}} |Df_{\delta}(x) \cdot \sigma|^p dx d\mu(\sigma) \leq C, \end{aligned} \quad (5.35)$$

for every  $\delta \in (0, r)$ .

**Remark 5.6** In the special case of Examples 5.1 and 5.2, it is easy to see that the measure  $\mu$  satisfies the coercivity condition

$$\alpha |v| \leq \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma), \quad \forall v \in \mathbb{R}^N, \quad (5.36)$$

for some  $\alpha > 0$ . By (5.35) and (5.36), we conclude that

$$\int_{\Omega_r \cap B_{1/r}} |Df_{\delta}|^p \leq \frac{C}{\alpha^p}, \quad \forall \delta \in (0, r).$$

Therefore,  $f \in W^{1,p}(\Omega)$  if  $p > 1$ , and  $f \in BV(\Omega)$  if  $p = 1$ . In addition, the following estimate holds

$$\int_{\Omega} |Df|^p \leq \frac{C}{\alpha^p}.$$

This argument shows that, to characterize the elements in  $W^{1,p}(\Omega)$  for  $p > 1$ , or  $BV(\Omega)$  for  $p = 1$ , in terms of (5.34), it suffices to show that (5.36) holds.

Let  $I_\mu : S^{N-1} \rightarrow \mathbb{R}_+$  be the function given by

$$I_\mu(v) := \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma), \quad \forall v \in S^{N-1}.$$

Then,  $I_\mu$  is continuous; moreover, (5.36) holds if, and only if,  $I_\mu > 0$  in  $S^{N-1}$ . Conversely,  $I_\mu(v_0) = 0$  for some  $v_0 \in S^{N-1}$  if, and only if,  $v_0 \perp \text{supp } \mu$ , i.e.  $\text{supp } \mu$  is contained in an  $(N-1)$ -dimensional vector space. In other words, we have the following :

**Lemma 5.5** (5.36) holds if, and only if,  $\text{supp } \mu$  contains a basis of  $\mathbb{R}^N$ .

Theorem 5.7 is now an easy consequence of Lemma 5.5.

## 5.7 Proof of Theorem 5.8

Theorem 5.8 will be deduced from Theorem 5.6 and the following lemma applied to the function  $\beta(t) := \omega(t) - \omega^\infty t$ , where  $t \in [0, \infty)$ .

**Lemma 5.6** Let  $\beta : [0, \infty) \rightarrow \mathbb{R}$  be a continuous function such that

$$\lim_{t \rightarrow \infty} \frac{\beta(t)}{t} = 0. \quad (5.37)$$

If  $\Omega \subset \mathbb{R}^N$  is unbounded, suppose in addition that there exists  $C > 0$  such that

$$|\beta(t)| \leq Ct, \quad \forall t \geq 0. \quad (5.38)$$

If  $f \in BV(\Omega)$ , then

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \beta \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy = \int_{\Omega} \beta_\mu(D^a f), \quad (5.39)$$

where  $D^a f$  is the absolutely continuous part of  $Df$  with respect to the Lebesgue measure in  $\mathbb{R}^N$ , and  $\beta_\mu(v) := \int_{S^{N-1}} \beta(|v \cdot \sigma|) d\mu(\sigma)$  for every  $v \in \mathbb{R}^N$ .

In order to prove Lemma 5.6 we shall need the next two simple remarks :

**Remark 5.7** Let  $\nu_1, \nu_2 \in \mathcal{M}(\mathbb{R}^N)$  be such that  $\nu_1 \leq \nu_2$  in  $\mathbb{R}^N$ , then

$$\nu_1^a \leq \nu_2^a \quad \text{and} \quad \nu_1^s \leq \nu_2^s \quad \text{in } \mathbb{R}^N,$$

where  $\nu_i = \nu_i^a \mathcal{L}^N + \nu_i^s$  is the Radon-Nikodym decomposition of  $\nu_i$ ,  $i = 1, 2$ .

**Remark 5.8** If  $(\nu_j)$  is a sequence of nonnegative measures in  $\mathcal{M}(S^{N-1})$  such that  $\nu_j \xrightarrow{*} \nu$  in  $\mathcal{M}(S^{N-1})$ , then

$$\int_A d\nu \leq \liminf_{j \rightarrow \infty} \int_A d\nu_j \leq \limsup_{j \rightarrow \infty} \int_A d\nu_j \leq \int_{\bar{A}} d\nu \quad \forall A \subset S^{N-1} \text{ open.}$$

This is a simple consequence of the inner and outer regularity of Radon measures (see, e.g., [47]).

**Proof of Lemma 5.6.** Let  $f \in BV(\Omega)$ . After extending  $f$  to the whole space  $\mathbb{R}^N$  (take for instance  $f = 0$  in  $\mathbb{R}^N \setminus \Omega$ ), we may suppose that  $f \in BV(\mathbb{R}^N)$ . Motivated by the work of Dávila [38], we define

$$\nu_j(x) := \int_{\mathbb{R}^N} \beta \left( \frac{|f(x+h) - f(x)|}{|h|} \right) \rho_{\varepsilon_j}(h) dh \quad \text{for a.e. } x \in \mathbb{R}^N.$$

In particular,  $(\nu_j)$  is bounded in  $L^1_{\text{loc}}(\mathbb{R}^N)$  (by Lemma 5.3) so that, up to a subsequence, there exists  $\nu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$  such that

$$\nu_j \xrightarrow{*} \nu \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N).$$

We shall prove that  $\nu$  is absolutely continuous with respect to the Lebesgue measure, and  $\nu = \beta_\mu(D^a f)$  a.e. in  $\mathbb{R}^N$ .

**Step 1.**  $\nu^s = 0$  in  $\mathbb{R}^N$ .

By (5.37), for each  $\delta > 0$  there exists  $C_\delta > 0$  such that

$$|\beta(s)| \leq \delta s + C_\delta, \quad \forall s \geq 0. \quad (5.40)$$

We now take  $x_0 \in \mathbb{R}^N$  and  $R > 0$ . For  $r \in (0, R)$ , it follows from (5.40) and Lemma 5.3 that

$$\begin{aligned} \int_{B_{R-r}(x_0)} \nu_j &\leq \delta \int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \frac{|f(x+h) - f(x)|}{|h|} \rho_{\varepsilon_j}(h) dx dh + C_\delta |B_R| \\ &\leq \delta \int_{B_R(x_0)} |Df| + \frac{2\delta}{r} \|f\|_{L^1} \int_{|h|>r} \rho_{\varepsilon_j} + C_\delta |B_R|. \end{aligned}$$

Take  $j \rightarrow \infty$  and then  $r \downarrow 0$ ; Remark 5.8 implies that

$$\int_{B_R(x_0)} \nu \leq \delta \int_{B_R(x_0)} |Df| + C_\delta |B_R|, \quad \forall x_0 \in \mathbb{R}^N, \quad \forall R > 0.$$

In particular, by Remark 5.7,

$$0 \leq \nu^s \leq \delta |D^s f| \quad \text{in } \mathbb{R}^N, \quad \forall \delta > 0.$$

We now let  $\delta \downarrow 0$  to conclude that  $\nu^s = 0$  in  $\mathbb{R}^N$ .

**Step 2.**  $\nu^a = \beta_\mu(D^a f)$  a.e. in  $\mathbb{R}^N$ .

Let  $\delta > 0$ . By (5.37) and the continuity of  $\beta$  we have

$$|\beta(s) - \beta(t)| \leq C_\delta |s - t| + \delta(1 + s + t), \quad \forall s, t \geq 0, \quad (5.41)$$

for some  $C_\delta > 0$ .

Let  $x_0 \in \mathbb{R}^N$  and  $R > 0$ . For  $r \in (0, R)$  fixed, using (5.41) we can estimate

$$\int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \left| \beta \left( \frac{|f(x+h) - f(x)|}{|h|} \right) - \beta \left( \left| D^a f(x) \cdot \frac{h}{|h|} \right| \right) \right| \rho_{\varepsilon_j}(h) dx dh$$

by an expression of the form

$$C_\delta A_1 + \delta A_2,$$

where

$$A_1 := \int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \frac{|f(x+h) - f(x) - D^a f(x) \cdot h|}{|h|} \rho_{\varepsilon_j}(h) dx dh,$$

$$A_2 := \int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \left\{ 1 + \frac{|f(x+h) - f(x)|}{|h|} + |D^a f(x)| \right\} \rho_{\varepsilon_j}(h) dx dh.$$

Let us drop the  $x_0$  from  $B_{R-r}(x_0)$  for a moment, and simply write  $B_{R-r}$ . We first estimate the quantity

$$I := \int_{B_{R-r}} \frac{|f(x+h) - f(x) - D^a f(x) \cdot h|}{|h|} dx.$$

If  $|h| \geq r$ , we just bound  $I$  from above by

$$I \leq \int_{B_{R-r}} \left( \frac{|f(x+h)| + |f(x)|}{|h|} + |D^a f(x)| \right) dx \leq \frac{2}{r} \|f\|_{L^1} + \int_{B_R} |D^a f|.$$

If  $|h| < r$ , then we have

$$I \leq \int_{B_{R-r}} \left\{ \int_0^1 |D^a f(x+th) - D^a f(x)| dt \right\} dx + \int_0^1 \left\{ \int_{B_{R-r}(x_0+th)} |D^s f| \right\} dt$$

$$\leq \int_{B_{R-r}} \left\{ \int_0^1 |D^a f(x+th) - D^a f(x)| dt \right\} dx + \int_{B_R(x_0)} |D^s f|$$

Using these two inequalities, we can now estimate  $A_1$  as

$$A_1 \leq \int_{B_{R-r}} \int_{|h|<r} \left\{ \int_0^1 |D^a f(x+th) - D^a f(x)| dt \right\} \rho_{\varepsilon_j}(h) dx dh +$$

$$+ \int_{B_R} |D^s f| + \left( \frac{2\|f\|_{L^1}}{r} + \int_{B_R} |D^a f| \right) \int_{|h|\geq r} \rho_{\varepsilon_j} \leq$$

$$\leq \sup_{v \in B_r} \left\{ \int_{B_R} |D^a f(x+v) - D^a f(x)| dx \right\} +$$

$$+ \int_{B_R} |D^s f| + \left( \frac{2\|f\|_{L^1}}{r} + \int_{B_R} |D^a f| \right) \int_{|h|\geq r} \rho_{\varepsilon_j}.$$

We now consider  $A_2$ . Using Lemma 5.3, we have

$$A_2 \leq |B_R| + \int_{B_{R-r}} \int_{|h|<r} \frac{|f(x+h) - f(x)|}{|h|} \rho_{\varepsilon_j}(h) dx dh +$$

$$+ \frac{2\|f\|_{L^1}}{r} \int_{|h|\geq r} \rho_{\varepsilon_j} + \int_{B_R} |D^a f| \leq$$

$$\leq |B_R| + 2 \int_{B_R} |Df| + \frac{2\|f\|_{L^1}}{r} \int_{|h|\geq r} \rho_{\varepsilon_j}.$$

Taking  $j \rightarrow \infty$ , and then  $r \downarrow 0$ , it follows from Remarks 5.3 and 5.8, and the estimates above that

$$\left| \int_{B_R(x_0)} [\nu - \beta_\mu(D^a f)] \right| \leq C_\delta \int_{B_R(x_0)} |D^s f| + \delta \left( |B_R| + 2 \int_{B_R(x_0)} |Df| \right)$$

for all  $x_0 \in \mathbb{R}^N$  and  $R > 0$ .

In particular, by Remark 5.7,

$$|\nu^a - \beta_\mu(D^a f)| \leq \delta(1 + 2|D^a f|) \quad \text{a.e. in } \mathbb{R}^N, \quad \forall \delta > 0.$$

We let  $\delta \downarrow 0$  to conclude that  $\nu^a = \beta_\mu(D^a f)$  a.e. in  $\mathbb{R}^N$ .

**Step 3.** Proof of Lemma 5.6 completed.

It follows from Steps 1 and 2 that  $\nu = \beta_\mu(D^a f)$  a.e. in  $\mathbb{R}^N$ . We now prove (5.39). Given  $r > 0$ , we write

$$\begin{aligned} \int_{\Omega_r} \nu_j &\leq \int_{\Omega} \int_{\Omega} \beta \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy + \\ &\quad + \int_{\Omega_r} \int_{\mathbb{R}^N \setminus \Omega} \beta \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \leq \int_{\Omega} \nu_j. \end{aligned} \quad (5.42)$$

Observe that  $\text{dist}(\Omega_r, \mathbb{R}^N \setminus \Omega) = r > 0$ . Applying (5.40) (take, for instance,  $\delta = 1$ ) if  $\Omega$  is bounded, or (5.38) if not, it is easy to check that the term of the form  $\int_{\Omega_r} \int_{\mathbb{R}^N \setminus \Omega}$  in the expression above tends to 0 as  $j \rightarrow \infty$ . We obtain (5.39) by letting  $j \rightarrow \infty$ , and then  $r \downarrow 0$ , in (5.42).

## 5.8 Proof of Theorem 5.9

**Step 1.** (5.14) holds if  $\omega$  is convex.

For any  $\delta \in (0, r)$ ,  $r > 0$  fixed, it follows from Lemmas 5.1 and 5.4 that

$$\begin{aligned} \int_{\Omega_r \cap B_{1/r}} \int_{\Omega_r \cap B_{1/r}} \omega \left( \frac{|f_\delta(x) - f_\delta(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy &\leq \\ &\leq \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \leq \\ &\leq \int_{\Omega} \int_{\mathbb{R}^N} \omega \left( \left| Df(x) \cdot \frac{h}{|h|} \right| \right) \rho_{\varepsilon_j}(h) dx dh. \end{aligned} \quad (5.43)$$

Note that

$$\omega(|Df \cdot \sigma|) \leq \tilde{\omega}(|Df|) \in L^1(\mathbb{R}^N), \quad \forall \sigma \in S^{N-1}.$$

Thus,

$$\sigma \in S^{N-1} \longmapsto \int \omega(|Df(x) \cdot \sigma|) dx \in \mathbb{R} \quad \text{is continuous.}$$

We now let  $j \rightarrow \infty$ ,  $\delta \downarrow 0$ , and  $r \downarrow 0$  in (5.43). Applying Remark 5.3 and Proposition 5.1, we see that (5.14) holds in this case.

**Step 2.** (5.14) holds if  $\omega$  is convex on  $[R, \infty)$  for some  $R > 0$ . It suffices to write  $\omega$  as  $\omega = \omega_1 + \omega_2$  in  $[0, \infty)$ , where

$$\omega_1(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq R, \\ \omega(t) - \omega(R) & \text{if } t > R. \end{cases}$$

In particular,  $\omega_1$  is convex and  $\omega_2$  is bounded. Note that if  $\Omega$  is unbounded, then  $\tilde{\omega}(0) = 0$ ; thus,  $\omega_2(t) \leq \tilde{\omega}(t) \leq Ct$  for  $t \geq 0$  small. We now apply the previous step to  $\omega_1$  and Lemma 5.6 to  $\omega_2$ . This gives (5.14).

**Step 3.** Proof of Theorem 5.9 completed.

Let  $R > 0$  fixed. For an arbitrary continuous function  $\omega$  satisfying (5.13) we take two continuous functions  $0 \leq \underline{\omega} \leq \omega \leq \bar{\omega}$  such that  $\underline{\omega} = \omega = \bar{\omega}$  on  $[0, R]$ , and  $\underline{\omega} = 0$ ,  $\bar{\omega} = \tilde{\omega}$  on  $[R + 1, \infty)$ .

Applying Step 2 and Lemma 5.6 we conclude that

$$\begin{aligned} \int_{\Omega} \underline{\omega}_{\mu}(Df) &\leq \liminf_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \\ &\leq \limsup_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \leq \int_{\Omega} \bar{\omega}_{\mu}(Df). \end{aligned}$$

Taking  $R \rightarrow \infty$ , we obtain (5.14) from the Dominated Convergence Theorem.

## 5.9 Orlicz-Sobolev spaces and $W^{1,1}$

Let  $\omega : [0, \infty) \rightarrow [0, \infty)$  be a nondecreasing convex function such that  $\omega(0) = 0$  and satisfying the coercivity condition

$$\lim_{t \rightarrow \infty} \frac{\omega(t)}{t} = \infty. \quad (5.44)$$

The Orlicz spaces are defined as

$$L^{\omega}(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} \omega(\alpha|f|) < \infty \text{ for some } \alpha > 0 \right\}. \quad (5.45)$$

Analogously, we have the Orlicz-Sobolev spaces (see, e.g., [73])

$$W^{1,\omega}(\Omega) := \left\{ f \in L^{\omega}(\Omega) : |Df| \in L^{\omega}(\Omega) \right\}. \quad (5.46)$$

We have the following characterization for these spaces :

**Theorem 5.11** *Suppose that (5.3) and (5.11) hold.*

*Let  $f \in L^{\omega}(\Omega)$ . Then,  $f \in W^{1,\omega}(\Omega)$  if, and only if, there exists  $\beta > 0$  such that*

$$\limsup_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \omega \left( \beta \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon}(x - y) dx dy < \infty. \quad (5.47)$$

The description of the Sobolev space  $W^{1,1}(\Omega)$  is more delicate since it is not reflexive, and so bounded sequences do not necessarily converge weakly to an element in  $W^{1,1}(\Omega)$  (but they do converge weakly in  $BV(\Omega)$ ).

We first recall that given  $g \in L^1(\Omega)$  there exists a nondecreasing convex function  $\omega_g : [0, \infty) \rightarrow [0, \infty)$  such that  $g \in L^{\omega_g}(\Omega)$  (see, e.g., [42]). In particular,  $W^{1,1}(\Omega)$  can be written as the union of all Orlicz-Sobolev spaces. More precisely,

$$W^{1,1}(\Omega) = \bigcup_{\substack{\omega \text{ convex} \\ \text{and coercive}}} W^{1,\omega}(\Omega). \tag{5.48}$$

This gives an indirect characterization of  $W^{1,1}(\Omega)$ , by means of the Orlicz-Sobolev spaces, in terms of (5.47).

**Proof of Theorem 5.11.**

**Step 1.** If  $f \in W^{1,\omega}(\Omega)$ , then there exists  $\beta > 0$  such that (5.47) holds.

Using the Lipschitz regularity of  $\partial\Omega$ , we can extend  $f$  to the whole space  $\mathbb{R}^N$  so that  $f \in W^{1,\omega}(\mathbb{R}^N)$ . By the definition of the Orlicz-Sobolev spaces, there exists  $\beta > 0$  such that

$$\omega(\beta|Df|) \in L^1(\mathbb{R}^N).$$

Estimate (5.47) now follows immediately from Lemma 5.1 applied to the function  $\beta f$ .

**Step 2.** If (5.47) is satisfied, then  $f \in W^{1,\omega}(\Omega)$ .

Let  $\varepsilon_j \downarrow 0$  and  $\mu \in \mathcal{M}(S^{N-1})$  be as in (5.8). Let  $\alpha > 0$  be such that

$$\alpha|v| \leq \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma), \quad \forall v \in \mathbb{R}^N. \tag{5.49}$$

Take  $C > 0$  satisfying

$$\int_{\Omega} \int_{\Omega} \omega \left( \beta \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \leq C, \quad \forall j \geq 1.$$

Proceeding as in Section 5.6 and using (5.49) we have

$$\int_{\Omega_r \cap B_{1/r}} \omega(\alpha\beta|Df_\delta|) dx \leq C, \quad \forall \delta \in (0, r), \quad \forall r > 0.$$

In particular, we conclude that  $f \in BV(\Omega)$ . However, (5.44) implies that the family  $Df_\delta$  is equi-integrable on the compact subsets of  $\Omega$ . Therefore,  $Df \in L^1_{\text{loc}}(\Omega)$ , and so  $Df_\delta \rightarrow Df$  a.e. in  $\Omega$ . Letting  $\delta \downarrow 0$ , we conclude that  $\omega(\alpha\beta|Df|) \in L^1(\Omega)$ .

## 5.10 Some properties of $f$ under no additional assumptions on $(\rho_\varepsilon)$

Even without assumption (5.11), we can still derive some information about  $f$  just from (5.10). In order to simplify our notation, we state our results in the special case  $\Omega = \mathbb{R}^N$  :

**Theorem 5.12** *Let  $f \in L^p(\mathbb{R}^N)$ ,  $p \geq 1$ , be such that*

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy < \infty.$$

*Then, there exists a vector subspace  $E \subset \mathbb{R}^N$ , with  $\dim E \geq 1$ , such that*

$$\begin{aligned} f|_{E+w} &\in W^{1,p}(E+w) && \text{for a.e. } w \in E^\perp, && \text{if } p > 1; \\ f|_{E+w} &\in BV(E+w) && \text{for a.e. } w \in E^\perp, && \text{if } p = 1. \end{aligned}$$

*In addition, there exists  $\alpha > 0$  such that*

$$\alpha \int_{\mathbb{R}^N} |D_E f|^p \leq \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy.$$

In particular, we have

**Corollary 5.6** *Assume  $f \in L^p_{\text{loc}}(\mathbb{R}^N)$ ,  $p \geq 1$ , is such that*

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y) dx dy = 0.$$

*Then, there exist a vector space  $E \subset \mathbb{R}^N$ , with  $\dim E = k \geq 1$ , and a function  $\tilde{f} \in L^p_{\text{loc}}(\mathbb{R}^{N-k})$  such that*

$$f(v+w) = \tilde{f}(w) \quad \text{for a.e. } v \in E \text{ and a.e. } w \in E^\perp.$$

*In other words,  $f$  is a function of  $(N-k)$ -variables.*

Note that, by Corollary 5.2, this is the best we can expect from  $f$  in general.

**Proof of Theorem 5.12.** Let  $\varepsilon_j \downarrow 0$  and  $\mu \in \mathcal{M}(S^{N-1})$  be as in (5.8), and such that there exists  $C > 0$  satisfying

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon_j}(x - y) dx dy \leq C, \quad \forall j \in \mathbb{N} \text{ large enough.}$$

Arguing as in Section 5.6, we conclude that

$$\int_{\mathbb{R}^N} \left\{ \int_{S^{N-1}} |Df_\delta(x) \cdot \sigma| d\mu(\sigma) \right\}^p dx \leq C, \quad \forall \delta > 0.$$

Define

$$F := \left\{ w \in \mathbb{R}^N : \int_{S^{N-1}} |w \cdot \sigma| d\mu(\sigma) = 0 \right\}. \quad (5.50)$$

Note that  $F$  is a vector subspace properly contained in  $\mathbb{R}^N$ , since  $\mu \geq 0$  and  $\mu(S^{N-1}) = 1$ . Let  $k := \dim F^\perp \geq 1$ . Given  $v = v' + v'' \in F \oplus F^\perp = \mathbb{R}^N$ , we have

$$\begin{aligned} \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma) &= \int_{S^{N-1} \cap F^\perp} |v'' \cdot \sigma''| d\mu(\sigma) \\ &= \int_{S^{k-1}} |v'' \cdot \sigma''| d\mu(\sigma'') \geq \tilde{\alpha} |v''| \quad \text{for some } \tilde{\alpha} > 0. \end{aligned} \quad (5.51)$$



By (5.50) and (5.51), we conclude that

$$\tilde{\alpha}^p \int_{\mathbb{R}^N} |D_{F^\perp} f_\delta|^p \leq C, \quad \forall \delta > 0,$$

The theorem follows by letting  $\delta \downarrow 0$  and taking  $E = F^\perp$ .

## 5.11 Proof of Theorem 5.10

Throughout this section we shall assume that  $\Omega \subset \mathbb{R}^N$  is bounded.

For each  $j = 1, 2, \dots$  we take

$$F_j(f) := \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy, \quad \forall f \in L^1(\Omega).$$

Theorem 5.10 will be a consequence of the following two lemmas :

### Lemma 5.7

$$\Gamma_{L^1(\Omega)}^- \text{-lim sup}_{j \rightarrow \infty} F_j(f) \leq \text{sc}_{L^1(\Omega)}^- F(f), \quad \forall f \in L^1(\Omega), \quad (5.52)$$

where  $F$  is the functional given by (5.19).

**Proof.** Let  $f \in C^1(\overline{\Omega})$ . Taking the constant sequence  $f_j := f$  for each  $j \geq 1$  in (5.16), it follows from Remark 5.5 that

$$\Gamma_{L^1(\Omega)}^- \text{-lim sup}_{j \rightarrow \infty} F_j(f) \leq \lim_{j \rightarrow \infty} F_j(f) = \int_{\Omega} \omega_{\mu}(Df),$$

whence

$$\Gamma_{L^1(\Omega)}^- \text{-lim sup}_{j \rightarrow \infty} F_j(f) \leq F(f), \quad \forall f \in L^1(\Omega).$$

Since  $\Gamma_{L^1(\Omega)}^- \text{-lim sup}_{j \rightarrow \infty} F_j$  is lower semicontinuous in  $L^1(\Omega)$  (see [41]), (5.52) follows.

### Lemma 5.8

$$\Gamma_{L^1(\Omega)}^- \text{-lim inf}_{j \rightarrow \infty} F_j(f) \geq \text{sc}_{L^1(\Omega)}^- G_{\Omega}(f), \quad \forall f \in L^1(\Omega), \quad (5.53)$$

where, for each open set  $A \subset \mathbb{R}^N$ , the functional  $G_A$  is defined as

$$G_A(g) = \begin{cases} \int_A (\omega^{**})_{\mu}(Dg) & \text{if } g \in C^1(\overline{A}), \\ +\infty & \text{if } g \in L^1(A) \setminus C^1(\overline{A}). \end{cases} \quad (5.54)$$

**Proof.** Fix  $0 < \delta < r$ . Let  $f \in L^1(\Omega)$  and  $(f_j) \subset L^1(\Omega)$  be such that  $f_j \rightarrow f$  in  $L^1(\Omega)$ . Applying Lemma 5.4, for each  $j \geq 1$  we have

$$\begin{aligned} F_j(f_j) &\geq \int_{\Omega} \int_{\Omega} \omega^{**} \left( \frac{|f_j(x) - f_j(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \\ &\geq \int_{\Omega_r} \int_{\Omega_r} \omega^{**} \left( \frac{|f_{j,\delta}(x) - f_{j,\delta}(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy. \end{aligned}$$

Note that for each  $\delta > 0$  fixed we have  $f_{j,\delta} \rightarrow f_\delta$  in  $C^2(\overline{\Omega}_r)$ . It follows from Remark 5.4 that

$$\int_{\Omega_r} \int_{\Omega_r} \omega^{**} \left( \frac{|f_{j,\delta}(x) - f_{j,\delta}(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy \xrightarrow{j \rightarrow \infty} \int_{\Omega_r} (\omega^{**})_\mu(Df_\delta).$$

Therefore,

$$\liminf_{j \rightarrow \infty} F_j(f_j) \geq \int_{\Omega_r} (\omega^{**})_\mu(Df_\delta), \quad \forall \delta \in (0, r).$$

Given  $A \subset\subset \Omega$ , let  $r > 0$  sufficiently small so that  $A \subset \Omega_r$ . We have

$$\liminf_{j \rightarrow \infty} F_j(f_j) \geq G_A(f_\delta) \geq \text{sc}_{L^1(A)}^- G_A(f_\delta), \quad \forall \delta \in (0, r).$$

Letting  $\delta \downarrow 0$  and using the lower semicontinuity of  $\text{sc}_{L^1(A)}^- G_A$  in  $L^1(A)$ , we conclude that

$$\liminf_{j \rightarrow \infty} F_j(f_j) \geq \sup \left\{ \text{sc}_{L^1(A)}^- G_A(f) : A \subset\subset \Omega \right\}. \quad (5.55)$$

Since  $\Omega \subset \mathbb{R}^N$  is a bounded open set with Lipschitz boundary and  $(\omega^{**})_\mu$  is convex, we can apply Theorem 4.4 in [37] which implies that

$$\sup \left\{ \text{sc}_{L^1(A)}^- G_A(f) : A \subset\subset \Omega \right\} = \text{sc}_{L^1(\Omega)}^- G_\Omega(f). \quad (5.56)$$

Since the sequence  $f_j \rightarrow f$  in  $L^1(\Omega)$  was arbitrary, (5.53) follows from (5.55) and (5.56).

**Proof of Theorem 5.10.** Let

$$\tilde{G}(f) = \begin{cases} \int_{\Omega} \omega_\mu^{**}(Df) & \text{if } f \in C^1(\overline{\Omega}), \\ +\infty & \text{if } f \in L^1(\Omega) \setminus C^1(\overline{\Omega}). \end{cases}$$

Then,

$$\text{sc}_{L^1(\Omega)}^- F(f) = \text{sc}_{L^1(\Omega)}^- \tilde{G}(f), \quad \forall f \in L^1(\Omega). \quad (5.57)$$

(This follows from (5.17) and (5.58) below.)

By hypothesis,  $\omega_\mu^{**} = (\omega^{**})_\mu$ , so that  $G_\Omega(f) = \tilde{G}(f)$  for every  $f \in L^1(\Omega)$ . Theorem 5.10 now follows from (5.52), (5.53) and (5.57).

**Proof of Corollary 5.5.** By relaxation, we know that (see [35, Theorems 4.2.8 and 4.4.1])

$$\text{sc}_{L^1(\Omega)}^- F(f) = \int_{\Omega} \omega_\mu^{**}(Df), \quad \forall f \in C^1(\overline{\Omega}). \quad (5.58)$$

We now define the recession function  $(\omega_\mu^{**})^\infty : \mathbb{R}^N \rightarrow [0, +\infty]$  as

$$(\omega_\mu^{**})^\infty(v) := \lim_{t \rightarrow \infty} \frac{\omega_\mu^{**}(tv)}{t}, \quad \forall v \in \mathbb{R}^N.$$

(The limit above always exists in  $[0, +\infty]$  since  $\omega_\mu^{**}$  is convex.)

Applying Theorem 4.7 in [37] to (5.58), we get

$$\text{sc}_{L^1(\Omega)}^- F(f) = \int_{\Omega} \omega_{\mu}^{**}(D^a f) dx + \int_{\Omega} (\omega_{\mu}^{**})^{\infty} \left( \frac{dD^s f}{|dD^s f|} \right) d|D^s f| \quad (5.59)$$

for every  $f \in BV(\Omega)$ . Here,  $Df = D^a f \mathcal{L}^N + D^s f$  is the Radon-Nikodym decomposition of  $Df$ , and  $\frac{dD^s f}{|dD^s f|}$  denotes the Radon-Nikodym derivative of  $D^s f$  with respect to  $|D^s f|$ . This concludes the proof of the corollary.

**Remark 5.9** Let  $\nu$  be a vector-valued Radon measure in  $\mathbb{R}^N$  with values in  $\mathbb{R}^N$ . As in Goffman and Serrin [57], we define

$$\omega_{\mu}^{**} \nu(A) := \sup \left\{ \sum_i |A_i| \omega_{\mu}^{**} \left( \frac{\nu(A_i)}{|A_i|} \right) \right\}$$

for every Borel set  $A \subset \mathbb{R}^N$ , where the supremum is taken over all finite disjoint partitions  $A = \bigcup_i A_i$  in terms of Borel sets  $A_i$ , and  $|\cdot|$  denotes the Lebesgue measure in  $\mathbb{R}^N$  (if  $|A_i| = 0$ , then  $|A_i| \omega_{\mu}^{**} \left( \frac{\nu(A_i)}{|A_i|} \right)$  is to be understood as the limit  $(\omega_{\mu}^{**})^{\infty}(\nu(A_i))$ ). With such definition,  $\omega_{\mu}^{**} \nu$  is a positive measure in  $\mathbb{R}^N$  and (see [57, Theorem 2'])

$$\omega_{\mu}^{**} \nu(A) = \int_A \omega_{\mu}^{**}(\nu^a) dx + \int_A (\omega_{\mu}^{**})^{\infty} \left( \frac{d\nu^s}{|d\nu^s|} \right) d|\nu^s| \quad (5.60)$$

for any Borel set  $A \subset \mathbb{R}^N$ .

Applying (5.60) with  $\nu = Df$ ,  $f \in BV(\Omega)$ , and  $A = \Omega$ , we can rewrite (5.20) in the more elegant form

$$\Gamma_{L^1(\Omega)}^- \lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) dx dy = \int_{\Omega} \omega_{\mu}^{**} Df.$$

A question naturally arises from Theorem 5.10 : for what families of continuous functions  $\omega : [0, \infty) \rightarrow [0, \infty)$  and nonnegative Radon measures  $\mu \in \mathcal{M}(S^{N-1})$  do we have the equality

$$\omega_{\mu}^{**} = (\omega^{**})_{\mu} \quad \text{in } \mathbb{R}^N ? \quad (5.61)$$

Here are some examples :

**Example 5.5** If  $\omega$  is convex, then so is  $\omega_{\mu}$ . In this case, we have

$$\omega_{\mu}^{**} = (\omega^{**})_{\mu} = \omega_{\mu} \quad \text{in } \mathbb{R}^N.$$

In particular,

$$\Gamma_{L^1(\Omega)}^- \lim_{\varepsilon \downarrow 0} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon}(|x - y|) dx dy = K_{p,N} \int_{\Omega} |Df|^p,$$

for every  $f \in W^{1,p}(\Omega)$ . If  $p = 1$ , then the above limit still holds for  $f \in BV(\Omega)$ .

**Example 5.6** Assume that  $\omega : [0, \infty) \rightarrow [0, \infty)$  is concave. In particular,  $\omega$  is nondecreasing and the limit

$$\omega^\infty := \lim_{t \rightarrow \infty} \frac{\omega(t)}{t} \geq 0$$

exists. We then conclude that  $\omega^{**}(t) = \omega^\infty t + \omega(0)$  for every  $t \geq 0$ ; thus,

$$(\omega^{**})_\mu(v) = \omega^\infty \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma) + \omega(0), \quad \forall v \in \mathbb{R}^N.$$

We now observe that  $\omega_\mu$  is also concave. In addition,  $\omega_\mu(0) = \omega(0)$  and, for each  $v \in \mathbb{R}^N$ , we have

$$\lim_{t \rightarrow \infty} \frac{\omega_\mu(tv)}{t} = \omega^\infty \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma),$$

so that  $\omega_\mu^{**} \leq (\omega^{**})_\mu$  in  $\mathbb{R}^N$ . Since the reverse inequality is always true, we must have

$$\omega_\mu^{**}(v) = (\omega^{**})_\mu(v) = \omega^\infty \int_{S^{N-1}} |v \cdot \sigma| d\mu(\sigma) + \omega(0), \quad \forall v \in \mathbb{R}^N.$$

**Example 5.7** Suppose that  $\omega(0) = 0$  and

$$\lim_{t \rightarrow \infty} \frac{\omega(t)}{t} = 0.$$

In particular,  $\omega_\mu(0) = 0$  and, for each  $v \in \mathbb{R}^N$ ,

$$\lim_{t \rightarrow \infty} \frac{\omega_\mu(tv)}{t} = 0.$$

We conclude that

$$\omega_\mu^{**} = (\omega^{**})_\mu = 0 \quad \text{in } \mathbb{R}^N.$$

The next example shows that (5.61) holds for arbitrary continuous functions  $\omega : [0, \infty) \rightarrow [0, \infty)$  for a special class of measures  $\mu \in \mathcal{M}(S^{N-1})$ .

**Example 5.8** Assume that  $\mu = \sum_{i=1}^N \alpha_i \delta_{e_i}$ , where  $\alpha_i \geq 0$ ,  $\forall i = 1, \dots, N$ , and  $\sum_{i=1}^N \alpha_i = 1$ .

We shall prove that

$$\omega_\mu^{**}(v) = \sum_{i=1}^N \alpha_i \omega^{**}(|v_i|) = (\omega^{**})_\mu(v), \quad \forall v \in \mathbb{R}^N. \quad (5.62)$$

Let  $\xi \in \mathbb{R}^N$  and  $c \in \mathbb{R}$  be such that

$$\omega_\mu(v) = \sum_{i=1}^N \alpha_i \omega(|v_i|) \geq \xi \cdot v + c, \quad \forall v \in \mathbb{R}^N. \quad (5.63)$$

By a standard separation-of-variable argument, there exist  $a_1, \dots, a_{N-1} \in \mathbb{R}$  such that

$$\begin{aligned} \alpha_i \omega(|v_i|) &\geq \xi_i v_i + a_i, & \forall i = 1, \dots, N-1, \\ \alpha_N \omega(|v_N|) &\geq \xi_N v_N + c - (a_1 + \dots + a_{N-1}), \end{aligned}$$

for every  $v \in \mathbb{R}^N$ . From this, we get

$$(\omega^{**})_\mu(v) = \sum_{i=1}^N \alpha_i \omega^{**}(|v_i|) \geq \xi \cdot v + c, \quad \forall v \in \mathbb{R}^N. \quad (5.64)$$

Since (5.64) holds for any pair  $\xi \in \mathbb{R}^N$  and  $c \in \mathbb{R}$  such that (5.63) is satisfied, we conclude that  $(\omega^{**})_\mu \geq \omega_\mu^{**}$  in  $\mathbb{R}^N$ . This readily implies (5.62).

As a consequence of Example 5.8, we see that, for any  $\omega : [0, \infty) \rightarrow [0, \infty)$  continuous and for any  $f \in W^{1,1}(\Omega)$ , we have

$$\Gamma_{L^1(\Omega)}^- \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2N-1}} \int_{\Omega} \int_{\Omega} \omega \left( \frac{|f(x) - f(y)|}{|x - y|} \right) dx dy = 2^N \int_{\Omega} \omega^{**} \left( \left| \frac{\partial f}{\partial x_1} \right| \right).$$

$\begin{array}{l} |x_1 - y_1| < \varepsilon \\ |x_i - y_i| < \varepsilon^2 \\ i = 2, \dots, N \end{array}$

# Chapitre 6

## $W^{1,1}$ -maps with values into $S^1$

### Sommaire

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## 6.1 Introduction<sup>7</sup>

Let  $G \subset \mathbb{R}^3$  be a smooth bounded domain with  $\Omega = \partial G$  simply connected. In a recent paper, Bourgain, Brezis, and Mironescu [22] studied properties of

$$H^{1/2}(\Omega; S^1) = \left\{ g \in H^{1/2}(\Omega; \mathbb{R}^2) : |g| = 1 \text{ a.e. on } \Omega \right\}.$$

(In what follows, we identify  $\mathbb{R}^2$  with  $\mathbb{C}$ .)

The space  $W^{1,1} \cap L^\infty$  shares some properties with  $H^{1/2}$  and it is natural to investigate

$$W^{1,1}(\Omega; S^1) = \left\{ g \in W^{1,1}(\Omega; \mathbb{R}^2) : |g| = 1 \text{ a.e. on } \Omega \right\}.$$

---

<sup>7</sup>Ce chapitre a été écrit en collaboration avec H. Brezis et P. Mironescu ; le texte original est déjà sorti dans Geometric analysis of PDE and several complex variables (S. Chanillo, P. Cordaro, N. Hanges, J. Hounie et A. Meziani, eds.), Contemporary Mathematics, American Mathematical Society, 2004.

One of the issues that we shall discuss is the question of existence of a lifting and, more precisely, “optimal” liftings. If  $g \in W^{1,1}(\Omega; S^1) \cap C^0(\Omega; S^1)$ , then  $g$  admits a “canonical” lifting  $\varphi \in W^{1,1}(\Omega; \mathbb{R}) \cap C^0(\Omega; \mathbb{R})$  satisfying

$$\int_{\Omega} |\nabla \varphi| = \int_{\Omega} |\nabla g|. \quad (6.1)$$

(Since  $g \in C^0$  and  $\Omega$  is simply connected, there exists a  $\varphi \in C^0$  such that  $g = e^{i\varphi}$  and (6.1) holds for this  $\varphi$ .) However, if one removes the continuity assumption, then a general  $g \in W^{1,1}(\Omega; S^1)$  need not have a lifting  $\varphi$  in  $W^{1,1}(\Omega; \mathbb{R})$ . This obstruction phenomenon—which also holds for other Sobolev spaces—is due to topological singularities of  $g$  and has been extensively studied in [18]; see also earlier results of Schoen and Uhlenbeck [77] and Bethuel [9].

It has been established by Giaquinta, Modica, and Souček [54] that every map  $g \in W^{1,1}(\Omega; S^1)$  admits a lifting in  $BV(\Omega; \mathbb{R})$ . However, as we shall see below, for some maps  $g$  in  $W^{1,1}$  we may have

$$\min \left\{ \int_{\Omega} |D\varphi| : \varphi \in BV(\Omega; \mathbb{R}) \text{ and } g = e^{i\varphi} \text{ a.e.} \right\} > \int_{\Omega} |\nabla g|,$$

where the measure  $D\varphi$  is the distributional derivative of  $\varphi$ .

As we shall prove (see Corollary 6.6 below), there is always a  $\varphi \in BV(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi}$  and

$$\int_{\Omega} |D\varphi| \leq 2 \int_{\Omega} |\nabla g|. \quad (6.2)$$

The constant 2 in (6.2) is optimal (see Remark 6.2 below). Inequality (6.2) has been extended by Dávila and Ignat [39] to maps  $g \in BV(\Omega; S^1)$  (here,  $\Omega$  can be an arbitrary domain in  $\mathbb{R}^N$ ); the striking fact is that (6.2), with constant 2, holds in any dimension.

It is natural to study, for a given  $g \in W^{1,1}(\Omega; S^1)$ , the quantity

$$E(g) = \min \left\{ \int_{\Omega} |D\varphi| : \varphi \in BV(\Omega; \mathbb{R}) \text{ and } g = e^{i\varphi} \text{ a.e.} \right\}. \quad (6.3)$$

Another quantity which is commonly studied in the framework of Sobolev maps with values into manifolds (see [10], and also [54]) is the relaxed energy

$$E_{\text{rel}}(g) = \inf \left\{ \liminf_{n \rightarrow \infty} \int |\nabla g_n| : g_n \in C^\infty(\Omega; S^1) \text{ and } g_n \rightarrow g \text{ a.e.} \right\}. \quad (6.4)$$

It is not difficult to prove (see Proposition 6.2) that

$$E_{\text{rel}}(g) = E(g), \quad \forall g \in W^{1,1}(\Omega; S^1).$$

As we shall establish in Section 6.3, the gap

$$E(g) - \int_{\Omega} |\nabla g| \quad (6.5)$$

can be easily computed in terms of the minimal connection  $L(g)$  of the topological singularities of  $g$ . For example, if  $g \in C^\infty(\Omega \setminus \{P, N\}; S^1) \cap W^{1,1}$ ,  $\deg(g, P) = +1$ , and  $\deg(g, N) = -1$ , then  $L(g)$  is the geodesic distance in  $\Omega$  between  $N$  and  $P$ , and the gap (6.5) equals  $2\pi L(g)$ . For the definition of  $L(g)$  when  $g$  is an arbitrary element of  $W^{1,1}(\Omega; S^1)$ , see (6.9) below. The concept of a minimal connection connecting the topological singularities has its source in [28].

One of our main results is

**Theorem 6.1** *Let  $g \in W^{1,1}(\Omega; S^1)$ . We have*

$$E(g) - \int_{\Omega} |\nabla g| = 2\pi L(g). \quad (6.6)$$

The first result of this kind (see [10]) concerned the Dirichlet integral  $\int |\nabla g|^2$  and maps  $g$  from a 3-d domain into  $S^2$ . Inequality  $\leq$  in (6.6) has been known for some time (see [44] and [54]); it relies on the dipole construction introduced in [28]. The exact lower bound for the relaxed energy is always a more delicate issue. It can presumably be proved using the theory of Cartesian currents of [54]; however, the precise relationship between the formalism of [54] and (6.6) is yet to be clarified. We call the attention of the reader to the fact that, in the  $H^{1/2}$ -setting studied in [22], the analogue of Theorem 6.1 is open; we only have

$$E_{\text{rel}}(g) - |g|_{H^{1/2}}^2 \sim L(g).$$

There is another interpretation of  $L(g)$  as the “ $L^1$ -distance” of  $g \wedge \nabla g$  to the class of gradient maps. More precisely, given  $g \in W^{1,1}(\Omega; \mathbb{R}^2)$ , consider the vector field  $g \wedge \nabla g$  defined in a local frame by

$$g \wedge \nabla g = (g \wedge g_x, g \wedge g_y).$$

When  $g$  is smooth with values into  $S^1$ ,  $g \wedge \nabla g$  is a gradient map since we may always write  $g = e^{i\varphi}$ , so that  $g \wedge \nabla g = \nabla \varphi$ . However, if  $g \in W^{1,1}(\Omega; S^1)$ , then  $g \wedge \nabla g$  is an  $L^1$ -vector field which need not be a gradient map, e.g., when  $g(x) \sim (x - a)/|x - a|$  near a point  $a \in \Omega$ , then  $g \wedge \nabla g$  is not a gradient map since

$$(g \wedge g_x)_y \neq (g \wedge g_y)_x \quad \text{in } \mathcal{D}'(\Omega).$$

Actually, the following holds :

**Theorem 6.2** *For every  $g \in W^{1,1}(\Omega; S^1)$ , we have*

$$L(g) = \frac{1}{2\pi} \inf_{\psi \in C^\infty(\Omega; \mathbb{R})} \int_{\Omega} |g \wedge \nabla g - \nabla \psi| = \frac{1}{2\pi} \min_{\psi \in BV(\Omega; \mathbb{R})} \int_{\Omega} |g \wedge \nabla g - D\psi|. \quad (6.7)$$

*There are many minimizers  $\psi$  in (6.7); however, at least one of them satisfies  $g = e^{i\psi}$  a.e. in  $\Omega$ .*

Let  $g \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$ . Following the ideas of Brezis, Coron, and Lieb [28] (or, more specifically, of Demengel and Hadiji [44] for this particular setting), we introduce the distribution  $T(g) \in \mathcal{D}'(\Omega; \mathbb{R})$ , defined by its action on  $\text{Lip}(\Omega; \mathbb{R})$  through the formula

$$\langle T(g), \zeta \rangle = \int_{\Omega} (g \wedge \nabla g) \cdot \nabla^\perp \zeta, \quad (6.8)$$



where  $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$ . In other words,

$$T(g) = -(g \wedge g_x)_y + (g \wedge g_y)_x = 2 \text{Det}(\nabla g),$$

where  $\text{Det}(\nabla g)$  denotes the distributional Jacobian of  $g$ . We then set

$$L(g) = \frac{1}{2\pi} \max_{\|\nabla \zeta\|_{L^\infty} \leq 1} \langle T(g), \zeta \rangle. \quad (6.9)$$

We first state some analogues of the results in [22] :

**Theorem 6.3** *Assume  $g \in W^{1,1}(\Omega; S^1)$ . There exist two sequences  $(P_i), (N_i)$  in  $\Omega$  such that  $\sum_i |P_i - N_i| < \infty$  and*

$$T(g) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega). \quad (6.10)$$

Moreover,

$$L(g) = \inf \sum_i d(P_i, N_i), \quad (6.11)$$

where  $d$  denotes the geodesic distance on  $\Omega$ , and the infimum is taken over all possible sequences  $(P_i), (N_i)$  satisfying (6.10).

As it was already pointed out in [22, Lemma 20], we have

$$\langle T(g), \zeta \rangle = 2\pi \int_{\mathbb{R}} \text{deg}(g, \Gamma_\lambda) d\lambda,$$

where  $\Gamma_\lambda = \{x \in \Omega; \zeta(x) = \lambda\}$  is equipped with the appropriate orientation (Lemma 20 in [22] is stated for  $g \in H^{1/2}$ , but the proof also covers the case where  $g \in W^{1,1}$ ). Here is a new property :

**Theorem 6.4** *Assume  $g \in W^{1,1}(\Omega; S^1)$ , and let  $\zeta \in \text{Lip}(\Omega; \mathbb{R})$  with  $\|\nabla \zeta\|_{L^\infty} \leq 1$ . Then,*

$$\int_{\mathbb{R}} |\text{deg}(g, \Gamma_\lambda)| d\lambda \leq L(g). \quad (6.12)$$

In particular, if  $\zeta$  is a maximizer in (6.9), then

$$\text{deg}(g, \Gamma_\lambda) \geq 0 \quad \text{for a.e. } \lambda. \quad (6.13)$$

Finally, we study a notion of relaxed Jacobian determinants in the spirit of Fonseca, Fusco, and Marcellini [50], and also Giaquinta, Modica, and Souček [52]. Given  $g \in W^{1,1}(\Omega; S^1)$ , we set (using the same notation as in [50])

$$TV(g) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |g_{nx} \wedge g_{ny}| : g_n \in C^\infty(\Omega; \mathbb{R}^2) \text{ and } g_n \rightarrow g \text{ in } W^{1,1} \right\}. \quad (6.14)$$

Of course this number is possibly infinite. The following is a far-reaching extension of some results in [50] :

**Theorem 6.5** *Let  $g \in W^{1,1}(\Omega; S^1)$ . Then,*

$$TV(g) < \infty \iff \text{Det}(\nabla g) \text{ is a measure.}$$

*In this case, we have*

$$\text{Det}(\nabla g) = \pi \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega)$$

*and*

$$TV(g) = |\text{Det}(\nabla g)|_{\mathcal{M}}.$$

*In particular,  $\frac{1}{\pi}TV(g)$  is an integer which equals the number of topological singularities of  $g$  (counting their multiplicities).*

**Remark 6.1** The conclusion of Theorem 6.5 still holds if one replaces the strong  $W^{1,1}$ -convergence in (6.14) by the weak  $W^{1,1}$ -convergence. There are numerous variants and extensions of Theorem 6.5 in Sections 6.4 and 6.5 below.

## 6.2 Properties of $W^{1,1}(S^1; S^1)$

Even though the core of this chapter deals with maps from a two dimensional manifold  $\Omega$  with values into  $S^1$ , it is illuminating to start with the study of  $W^{1,1}$ -maps from  $S^1$  into itself.

Let  $g \in W^{1,1}(S^1; S^1)$ . There are two natural quantities associated with  $g$ ; namely,

$$E(g) = \min \left\{ |\varphi|_{BV} : \varphi \in BV(S^1; \mathbb{R}), g = e^{i\varphi} \text{ a.e.} \right\}$$

and

$$E_{\text{rel}}(g) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| : g_n \in C^\infty(S^1; S^1), \deg g_n = 0, g_n \rightarrow g \text{ a.e.} \right\}.$$

It turns out that the two quantities are equal and that they can be easily computed in terms of  $g$ :

**Theorem 6.6** *Let  $g \in W^{1,1}(S^1; S^1)$ . Then,*

$$E_{\text{rel}}(g) = E(g) = \int_{S^1} |\dot{g}| + 2\pi |\deg g|. \quad (6.15)$$

**Proof.** First equality in (6.15) : “ $\geq$ ”. Let  $(g_n) \subset C^\infty(S^1; S^1)$  be such that  $\deg g_n = 0$  and  $g_n \rightarrow g$  a.e. Then, we may write  $g_n = e^{i\psi_n}$ , with  $\psi_n \in C^\infty(S^1; \mathbb{R})$  and

$$\int_{S^1} |\dot{\psi}_n| = \int_{S^1} |\dot{g}_n|.$$

Subtracting a suitable integer multiple of  $2\pi$ , we may assume that  $(\psi_n)$  is bounded in  $W^{1,1}(S^1; \mathbb{R})$ . After passing to a subsequence, we may further assume that  $\psi_n \rightarrow \psi$  a.e. for some  $\psi \in BV(S^1; \mathbb{R})$ . Therefore,

$$\liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| = \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{\psi}_n| \geq \int_{S^1} |\dot{\psi}|$$

and, clearly,  $e^{i\psi} = g$  a.e.

“ $\leq$ ”. Let  $\psi \in BV(S^1; \mathbb{R})$  be such that

$$|\psi|_{BV} = \min \{ |\varphi|_{BV} : g = e^{i\varphi} \text{ a.e.} \}.$$

Consider a sequence  $(\psi_n) \subset C^\infty(S^1; \mathbb{R})$  such that  $\psi_n \rightarrow \psi$  a.e. and  $\int_{S^1} |\dot{\psi}_n| \rightarrow |\psi|_{BV}$ . If we set  $g_n = e^{i\psi_n}$ , then clearly  $g_n \in C^\infty(S^1; S^1)$ ,  $\deg g_n = 0$ , and  $g_n \rightarrow g$  a.e. Moreover,

$$\lim_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| = \lim_{n \rightarrow \infty} \int_{S^1} |\dot{\psi}_n| = |\psi|_{BV}.$$

Second equality in (6.15) : “ $\geq$ ”. This assertion has been established under slightly more general assumptions in [22].

Here is an alternative approach. Let  $g \in W^{1,1}(S^1; S^1)$ . We prove that, if  $\varphi \in BV(S^1; \mathbb{R})$  satisfies  $g = e^{i\varphi}$  a.e., then

$$|\varphi|_{BV} \geq \int_{S^1} |\dot{g}| + 2\pi |\deg g|. \quad (6.16)$$

The main ingredient will be the chain rule formula for  $BV$ -maps, due to Vol’pert ; see [87], and also [1].

**Chain rule.** Let  $\varphi \in BV(S^1; \mathbb{R})$ . Recall that there is a representative  $\varphi_0$  of  $\varphi$  which is continuous except at (at most) countably many points  $a_n \in S^1$ ; in the sequel, we take  $\varphi$  to be  $\varphi_0$  itself. Moreover, at the points  $a_n$ ,  $\varphi$  admits limits from the “right” and from the “left”, say  $\varphi(a_n+)$  and  $\varphi(a_n-)$ .

Let  $\dot{\varphi}$  be the distributional derivative of  $\varphi$ , which is a Borel measure. The diffuse part of  $\dot{\varphi}$  is

$$\dot{\varphi}_d = \dot{\varphi} - \sum_n [\varphi(a_n+) - \varphi(a_n-)] \delta_{a_n}.$$

Vol’pert’s chain rule for  $BV$ -maps on a bounded interval (or a closed curve) asserts that, if  $F \in C^1(\mathbb{R}; \mathbb{R})$ , then

$$\overline{F \circ \varphi}^\cdot = F'(\varphi) \dot{\varphi}_d + \sum_n [F(\varphi(a_n+)) - F(\varphi(a_n-))] \delta_{a_n}.$$

A more general version of the chain rule, which is valid in  $\mathbb{R}^N$ , is stated and explained in the proof of Lemma 6.5 in Section 6.3 below.

We now return to the proof of (6.16). By the chain rule formula, we have

$$\dot{g} = ie^{i\varphi} \dot{\varphi}_d + \sum_n (e^{i\varphi(a_n+)} - e^{i\varphi(a_n-)}) \delta_{a_n}.$$

Using the continuity of  $g$ , we have  $g(a_n) = e^{i\varphi(a_n+)} = e^{i\varphi(a_n-)}$  for each  $n$ . Hence,

$$\dot{g} = ie^{i\varphi}\dot{\varphi}_d.$$

Since  $\dot{g} \in L^1$  and  $e^{i\varphi} = g$  a.e., we thus find that

$$g \wedge \dot{g} = \frac{1}{ig}\dot{g} = \dot{\varphi}_d.$$

Consequently,

$$|\dot{\varphi}|_{\mathcal{M}} = |\dot{\varphi}_d|_{\mathcal{M}} + |\dot{\varphi} - \dot{\varphi}_d|_{\mathcal{M}} = |g \wedge \dot{g}|_{\mathcal{M}} + |g \wedge \dot{g} - \dot{\varphi}|_{\mathcal{M}} = \int_{S^1} |\dot{g}| + |g \wedge \dot{g} - \dot{\varphi}|_{\mathcal{M}}. \quad (6.17)$$

On the other hand,

$$|g \wedge \dot{g} - \dot{\varphi}|_{\mathcal{M}} \geq |\langle g \wedge \dot{g} - \dot{\varphi}, 1 \rangle| = |\langle g \wedge \dot{g}, 1 \rangle| = 2\pi |\deg g|. \quad (6.18)$$

(The last equality is clear when  $g$  is smooth; the case of a general  $W^{1,1}$ -map follows by approximation.)

Finally, by combining (6.17) and (6.18) we find that

$$|\varphi|_{BV} \geq \int_{S^1} |\dot{g}| + 2\pi |\deg g|,$$

as claimed.

Second equality in (6.15) : “ $\leq$ ”. Since  $S^1 \setminus \{1\}$  is simply connected, we may write  $g = e^{i\varphi}$  on  $S^1 \setminus \{1\}$ , for some  $\varphi \in W^{1,1}(S^1 \setminus \{1\}; \mathbb{R})$  such that  $|\dot{\varphi}| = |\dot{g}|$  in  $S^1 \setminus \{1\}$ . Since  $\varphi$  is continuous, we have

$$\varphi(1+) - \varphi(1-) = 2\pi \deg g.$$

Passing to the full  $S^1$ , we have

$$|\varphi|_{BV} = \int_{S^1 \setminus \{1\}} |\dot{\varphi}| + |\varphi(1+) - \varphi(1-)| = \int_{S^1} |\dot{g}| + 2\pi |\deg g|.$$

As a consequence of Theorem 6.6, we have

**Corollary 6.1** For every  $g \in W^{1,1}(S^1; S^1)$ ,

$$E(g) \leq 2|g|_{W^{1,1}}. \quad (6.19)$$

**Remark 6.2** The constant 2 in (6.19) is optimal. Indeed, for  $g = \text{Id}$ , we have  $|g|_{W^{1,1}} = 2\pi$ , while  $E(g) = 4\pi$  by Theorem 6.6.

It is easy to see from the definition of the relaxed energy that  $E_{\text{rel}}$  is lower semicontinuous with respect to the pointwise a.e. convergence in  $S^1$ . In view of Theorem 6.6, we have the following :

**Corollary 6.2** *Let  $(g_n) \subset W^{1,1}(S^1; S^1)$  be such that  $g_n \rightarrow g$  a.e. for some map  $g \in W^{1,1}(S^1; S^1)$ . Then,*

$$\int_{S^1} |\dot{g}| + 2\pi |\deg g| \leq \liminf_{n \rightarrow \infty} \left( \int_{S^1} |\dot{g}_n| + 2\pi |\deg g_n| \right). \quad (6.20)$$

**Remark 6.3** The constant  $2\pi$  in (6.20) cannot be improved. In fact, assume that (6.20) holds with  $2\pi$  replaced by some  $C$ . In particular, for any sequence  $(g_n) \subset C^\infty(S^1; S^1)$  such that  $\deg g_n = 0$  and  $g_n \rightarrow \text{Id}$  a.e., we have

$$2\pi + C = \int_{S^1} |\dot{g}| + C |\deg g| \leq \liminf_{n \rightarrow \infty} \left( \int_{S^1} |\dot{g}_n| + C |\deg g_n| \right) = \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n|. \quad (6.21)$$

On the other hand, according to Theorem 6.6, the sequence  $(g_n)$  can be chosen so that

$$\lim_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| = \int_{S^1} |\dot{g}| + 2\pi |\deg g| = 4\pi. \quad (6.22)$$

A comparison between (6.21) and (6.22) implies  $C \leq 2\pi$ .

Inequality (6.20) still holds if one replaces  $|\deg g|$  and  $|\deg g_n|$  by  $\deg g$  and  $\deg g_n$ , under the additional assumption that the sequence  $(g_n)$  is **bounded** in  $W^{1,1}$ ; this assumption is essential, see Remark 6.4 below. More precisely, we have

**Proposition 6.1 (Bourgain-Brezis-Mironescu [22])** *Given  $g_n, g \in W^{1,1}(S^1; S^1)$  satisfying  $g_n \rightarrow g$  a.e and*

$$\sup_n |g_n|_{BV} < \infty,$$

*then*

$$\int_{S^1} |\dot{g}| + 2\pi \deg g \leq \liminf_{n \rightarrow \infty} \left( \int_{S^1} |\dot{g}_n| + 2\pi \deg g_n \right). \quad (6.23)$$

We present here an alternative approach based on Corollary 6.2.

**Proof.** Assume  $|g_n|_{BV} \leq C, \forall n$ . In particular,

$$|\deg g_n| \leq \frac{1}{2\pi} \int_{S^1} |\dot{g}_n| \leq \frac{C}{2\pi}.$$

Since  $\deg g_n$  takes only integer values, after passing to a subsequence, we can assume that  $d = \deg g_n, \forall n$ . Given  $\varepsilon > 0$ , let  $h \in C^\infty(S^1; S^1)$  be such that  $\deg h = -d$  and  $h(x) = 1, \forall x \in S^1 \setminus B_\varepsilon(1)$ . Clearly,

$$hg_n \rightarrow hg \quad \text{a.e. in } S^1 \quad \text{and} \quad \deg hg_n = 0, \quad \forall n.$$

It follows from Corollary 6.2 that

$$\int_{S^1} |\dot{g}h + g\dot{h}| + 2\pi(\deg g - d) \leq \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n h + g_n \dot{h}| \leq \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| + \int_{S^1} |\dot{h}|. \quad (6.24)$$

On the other hand, since  $h(x) = 1$  for  $x \in S^1 \setminus B_\varepsilon(1)$ , we have

$$\begin{aligned} \int_{S^1} |\dot{g}h + g\dot{h}| &= \int_{S^1 \setminus B_\varepsilon(1)} |\dot{g}| + \int_{S^1 \cap B_\varepsilon(1)} |\dot{g}h + g\dot{h}| \\ &\geq \int_{S^1 \setminus B_\varepsilon(1)} |\dot{g}| - \int_{S^1 \cap B_\varepsilon(1)} |\dot{g}| + \int_{S^1 \cap B_\varepsilon(1)} |\dot{h}| \\ &= \int_{S^1} |\dot{g}| - 2 \int_{S^1 \cap B_\varepsilon(1)} |\dot{g}| + \int_{S^1} |\dot{h}|. \end{aligned} \quad (6.25)$$

Comparison between (6.24) and (6.25) yields

$$\int_{S^1} |\dot{g}| - 2 \int_{S^1 \cap B_\varepsilon(1)} |\dot{g}| + 2\pi(\deg g - d) \leq \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n|$$

Taking  $\varepsilon \rightarrow 0$ , we obtain (6.23).

An immediate consequence of Proposition 6.1 is

**Corollary 6.3** *Under the assumptions of Proposition 6.1, we have*

$$\int_{S^1} |\dot{g}| \leq \liminf_{n \rightarrow \infty} \left( \int_{S^1} |\dot{g}_n| - 2\pi |\deg g_n - \deg g| \right).$$

**Remark 6.4** Proposition 6.1 (or, equivalently, Corollary 6.3) is **false** without the assumption  $\sup_n |g_n|_{BV} < \infty$ . Here is an example. Let  $n \geq 1$  be a fixed integer. Given  $0 \leq j \leq n-1$ , let  $a_{j,n} = \frac{2\pi j}{n}$  and  $I_{j,n} = [a_{j,n}, a_{j+1,n} - \frac{1}{2n}] \subset \mathbb{R}$ . On each interval  $I_{j,n}$ , we define  $f_n(t) = 2\pi j - a_{j,n}$ . We then extend  $f_n$  continuously to  $[0, 2\pi]$ , so that  $f_n$  is affine linear outside the set  $\bigcup_j I_{j,n}$ , and  $f_n(2\pi) = 2\pi(n-1)$ .

By construction,  $f_n$  is Lipschitz, nondecreasing, and  $f_n(2\pi) - f_n(0) \in 2\pi\mathbb{Z}$ . Note that

$$\begin{aligned} d(f_n(t), -t + 2\pi\mathbb{Z}) &\leq |a_{j+1,n} - a_{j,n}| = \frac{2\pi}{n}, \quad \forall t \in \bigcup_j I_{j,n}; \\ |[0, 2\pi] \setminus \bigcup_j I_{j,n}| &= \frac{n}{2^n}. \end{aligned}$$

Set  $g_n(\theta) = e^{-if_n(\theta)}$ . Then, we have  $g_n \rightarrow g$  a.e., where  $g = \text{Id}$ ; however,

$$\int_{S^1} |\dot{g}| + 2\pi \deg g = 4\pi,$$

while

$$\int_{S^1} |\dot{g}_n| + 2\pi \deg g_n = 0, \quad \forall n.$$

### 6.3 Properties of $W^{1,1}(\Omega; S^1)$

We start with the rigorous definitions of  $T(g)$  and of the class  $\text{Lip}$  mentioned in the Introduction. If  $g \in W^{1,1}(\Omega; \mathbb{R}^2)$ , we set

$$|\nabla g| = \left[ \left( \frac{\partial g_1}{\partial x} \right)^2 + \left( \frac{\partial g_1}{\partial y} \right)^2 + \left( \frac{\partial g_2}{\partial x} \right)^2 + \left( \frac{\partial g_2}{\partial y} \right)^2 \right]^{1/2},$$

where  $(x, y)$  is any orthonormal frame at some point on  $\Omega$ , and we let

$$|g|_{W^{1,1}} = \int_{\Omega} |\nabla g|.$$

We now recall the definition of  $T(g)$  :

$$\langle T(g), \zeta \rangle = \int_{\Omega} [(g \wedge g_x)\zeta_y - (g \wedge g_y)\zeta_x], \quad \forall \zeta \in \text{Lip}(\Omega; \mathbb{R}).$$

Here,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1,$$

and the integrand is computed in any orthonormal frame  $(x, y)$  such that  $(x, y, n)$  is direct, where  $n$  is the outward normal to  $G$ . (This integrand is frame invariant.) The class of testing functions,  $\text{Lip}(\Omega; \mathbb{R})$ , is the set of functions which are Lipschitz with respect to the geodesic distance  $d$  in  $\Omega$ . For such a map, we set

$$|\zeta|_{\text{Lip}} = \sup_{x \neq y} \frac{|\zeta(x) - \zeta(y)|}{d(x, y)} = \|\nabla \zeta\|_{L^\infty}.$$

We next collect some straightforward properties of  $T(g)$  and  $L(g)$  :

**Lemma 6.1** *We have*

- a)  $T(\bar{g}) = -T(g), \forall g \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$  ;
- b)  $T(gh) = T(g) + T(h), \forall g, h \in W^{1,1}(\Omega; S^1)$  ;
- c)  $L(g) \leq \frac{1}{2\pi} |g|_{W^{1,1}} \|g\|_{L^\infty}, \forall g \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$  ;
- d) *if  $g_n, g \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$  are such that  $g_n \rightarrow g$  in  $W^{1,1}$  and  $\|g_n\|_{L^\infty} \leq C$ , then  $L(g_n) \rightarrow L(g)$ .*

**Proof.** The only property that requires some proof is d). Since

$$|\langle T(g_n), \zeta \rangle - \langle T(g), \zeta \rangle| \leq \int_{\Omega} |g_n| |\nabla(g_n - g)| |\nabla \zeta| + \int_{\Omega} |g_n - g| |\nabla g| |\nabla \zeta|,$$

we have

$$|L(g_n) - L(g)| \leq C |g_n - g|_{W^{1,1}} + \|(g_n - g)\nabla g\|_{L^1},$$

and d) follows by dominated convergence.

Recall the following density result of Bethuel and Zheng [12] :

**Lemma 6.2** *The class*

$$\mathcal{R} = \left\{ g \in W^{1,1}(\Omega; S^1) : g \in C^\infty(\Omega \setminus A; S^1), \text{ where } A \text{ is some finite set} \right\}$$

*is dense in  $W^{1,1}(\Omega; S^1)$ .*

When  $g \in \mathcal{R}$ , a straightforward adaptation of the proof of Lemma 2 in [22] yields the following :

**Lemma 6.3** *If  $g \in W^{1,1}(\Omega; S^1)$ ,  $g \in C^\infty(\Omega \setminus \{a_1, \dots, a_k\}; S^1)$ , then*

$$T(g) = 2\pi \sum_{j=1}^k d_j \delta_{a_j} \quad \text{in } \mathcal{D}'(\Omega).$$

Here,  $d_j = \deg(g, a_j)$  is the topological degree of  $g$  restricted to any small circle around  $a_j$ , positively oriented with respect to the outward normal. Moreover,  $L(g)$  is the length of the minimal connection associated to the configuration  $(a_j, d_j)$ , with respect to the geodesic distance on  $\Omega$  (see Remark 6.5 below).

**Remark 6.5** By the definition of  $T(g)$ , we have  $\langle T(g), 1 \rangle = 0$ . Thus, by Lemma 6.3 we have  $\sum_{j=1}^k d_j = 0$ . Therefore, we may write the collection of points  $(a_j)$  (repeated with multiplicity  $|d_j|$ ) as

$$(P_1, \dots, P_\ell, N_1, \dots, N_\ell),$$

where  $\ell = \frac{1}{2} \sum_{j=1}^k |d_j|$ ; the points of degree 0 do not appear in this list,  $a_j$  is counted among the points  $P_i$  if  $d_j > 0$ , and among the points  $N_i$  otherwise. Then,

$$L(g) = \min_{\sigma \in S_\ell} \sum_{j=1}^{\ell} d(P_j, N_{\sigma(j)}).$$

This formula first appeared in the context of  $S^2$ -valued maps; see [28].

Using the density of  $\mathcal{R}$  in  $W^{1,1}(\Omega; S^1)$ , one can easily obtain Theorem 6.3 from Lemma 6.3. The analogue of Theorem 6.3 for  $H^{1/2}(\Omega; S^1)$  was proved in [22], and the arguments there also apply to our case.

A converse to Theorem 6.3 is also true. Namely, for any sequence of points  $(P_i), (N_i)$  satisfying  $\sum_i |P_i - N_i| < \infty$ , one can find  $g \in W^{1,1}(\Omega; S^1)$  such that (6.10) holds; see [22]. Motivated by this, we state the following :

**Open Problem 4** Let  $1 < p < 2$ . Given  $g \in W^{1,p}(\Omega; S^1)$ , can one find  $(P_i), (N_i)$  such that  $\sum_i |P_i - N_i|^{2/p-1} < \infty$  and (6.10) holds?

**Open Problem 5** Given two sequences  $(P_i), (N_i)$  such that  $\sum_i |P_i - N_i|^{2/p-1} < \infty$  for some  $1 < p < 2$ , does there exist some  $g \in W^{1,p}(\Omega; S^1)$  such that (6.10) holds? If the answer is negative (as we suspect), what is the right condition on the points  $P_i, N_i$  (in terms of the  $p$ -capacity?) which guarantees the existence of  $g$ ?



We now consider the following class

$$Y = \overline{C^\infty(\Omega; S^1)}^{W^{1,1}};$$

this class is properly contained in  $W^{1,1}(\Omega; S^1)$  (see Remark 6.7 below).

It turns out that maps in  $Y$  can be characterized in terms of their distribution  $T(g)$  :

**Theorem 6.7** *Let  $g \in W^{1,1}(\Omega; S^1)$ . Then, the following properties are equivalent :*

- a)  $g \in Y$  ;
- b)  $T(g) = 0$  ;
- c) there exists  $\varphi \in W^{1,1}(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi}$  a.e. in  $\Omega$ .

**Remark 6.6** The analogue of this result for maps in  $B_1 \subset \mathbb{R}^3$  with values into  $S^2$  was proved by Bethuel [8]. When  $\Omega$  is a smooth bounded open set in  $\mathbb{R}^2$ , the equivalence a)  $\Leftrightarrow$  b) was established by Demengel [43]. We could adapt the argument in [43] to our case, but we present below a different approach, based on an idea of Carbou [36].

**Remark 6.7** Using Theorem 6.7, it is easy to construct maps in  $W^{1,1}(\Omega; S^1) \setminus Y$ . Assume, e.g., that  $\Omega = S^2$ , and let  $g(x, y, z) = \frac{(x, y)}{|(x, y)|}$ . By Lemma 6.3, we have  $T(g) = 2\pi(\delta_N - \delta_S)$ , where  $N, S$  are the North and South pole of  $S^2$ . By Theorem 6.7, this implies that  $g \notin Y$ .

**Proof of Theorem 6.7.**

a)  $\Rightarrow$  b) By Lemma 6.3, we have  $T(g) = 0$  if  $g \in C^\infty(\Omega; S^1)$ . By Lemma 6.1,  $g \mapsto T(g)$  is continuous with respect to  $W^{1,1}$ -convergence, and thus  $T(g) = 0, \forall g \in Y$ .

b)  $\Rightarrow$  c) We argue as in [36] ; see also [18]. Let  $x_0 \in \Omega$  and assume that  $\Omega \subset \mathbb{R}^2$  near  $x_0$ . Since  $T(g) = 0$ , the  $L^1$ -vector field

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} g \wedge g_x \\ g \wedge g_y \end{pmatrix}$$

satisfies, near  $x_0$ ,  $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$  in the sense of distributions. By a variant of the Poincaré

Lemma (see [18]), we may find a neighborhood  $\omega$  of  $x_0$  and a function  $\psi \in W^{1,1}(\omega; \mathbb{R})$  such that  $g = e^{i(\psi+C)}$  in  $\omega$ , for some constant  $C$ .

Consider a finite covering of  $\Omega$  with open sets  $\omega_j$  such that

- (i) in each  $\omega_j$  we may write  $g = e^{i\varphi_j}$  for some  $\varphi_j \in W^{1,1}(\omega_j; \mathbb{R})$  ;
- (ii)  $\omega_j \cap \omega_k$  is connected,  $\forall j, \forall k$ .

In  $\omega_j \cap \omega_k$ , the map  $\varphi_j - \varphi_k$  belongs to  $W^{1,1}$  and is  $2\pi\mathbb{Z}$ -valued ; thus, it has to be constant a.e. Since  $\Omega$  is simply connected, we may therefore find a map  $\varphi$  in  $W^{1,1}(\Omega; \mathbb{R})$  such that  $\varphi - \varphi_j$  is, a.e. in  $\omega_j$ , a constant integer multiple of  $2\pi$ . In particular,  $g = e^{i\varphi}$  in  $\Omega$ .

c)  $\Rightarrow$  a) Let  $(\varphi_n) \subset C^\infty(\Omega; \mathbb{R})$  be such that  $\varphi_n \rightarrow \varphi$  in  $W^{1,1}$ . Set  $g_n = e^{i\varphi_n}$ . Then, clearly,  $g_n \in C^\infty(\Omega; S^1)$  and  $g_n \rightarrow g$  in  $W^{1,1}$ .

**Remark 6.8** It follows from Theorem 6.7 that, given a map  $g \in W^{1,1}(\Omega; S^1)$ , in general we may **not** write  $g = e^{i\varphi}$  for some  $\varphi \in W^{1,1}(\Omega; \mathbb{R})$  ; consider, for example, the map  $g$  in Remark 6.7. However, it follows from Theorem 6.2 that we may write  $g = e^{i\varphi}$  for some  $\varphi \in BV(\Omega; \mathbb{R})$ . This conclusion still holds for maps  $g \in BV(\Omega; S^1)$  ; see [54], and also [39].

**Remark 6.9** In view of Theorem 6.2, the equivalence  $L(g) = 0 \Leftrightarrow \exists \varphi \in W^{1,1}(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi}$  becomes transparent. Indeed, if  $L(g) = 0$ , then, by Theorem 6.2, there is some  $\varphi \in BV(\Omega; \mathbb{R})$  such that  $D\varphi = g \wedge \nabla g \in L^1$ . Thus,  $\varphi \in W^{1,1}$  and it is straightforward that

$$\nabla(ge^{-i\varphi}) = e^{-i\varphi}(\nabla g - ig\nabla\varphi) = ige^{-i\varphi}(g \wedge \nabla g - \nabla\varphi) = 0,$$

so that  $g = e^{i(\varphi+C)}$  for some constant  $C$ . On the other hand, if  $g = e^{i\varphi}$  for some  $\varphi \in W^{1,1}$ , then  $\nabla\varphi = g \wedge \nabla g$  (as above), and thus  $L(g) = 0$ .

Before starting the proof of Theorem 6.2, we recall the “generalized dipole” construction presented in [22] :

**Lemma 6.4** *Let  $g \in W^{1,1}(\Omega; S^1)$ . Then, for each  $\varepsilon > 0$ , there is some  $h = h_\varepsilon \in W^{1,1}(\Omega; S^1)$  such that*

$$(i) \quad |h|_{W^{1,1}} \leq 2\pi L(g) + \varepsilon;$$

$$(ii) \quad T(h) = T(g);$$

(iii) *there is a function  $\psi = \psi_\varepsilon \in BV(\Omega; \mathbb{R})$  such that  $h = e^{i\psi}$  a.e. and*

$$|\psi|_{BV} \leq 4\pi L(g) + \varepsilon;$$

$$(iv) \quad \text{meas}(\text{supp } \psi) = \text{meas}(\text{supp}(h - 1)) < \varepsilon.$$

**Proof of Theorem 6.2.** Let  $\psi \in BV(\Omega; \mathbb{R})$  and  $\zeta \in \text{Lip}(\Omega; \mathbb{R})$  be such that  $|\nabla\zeta| \leq 1$ . Then,

$$|g \wedge \nabla g - D\psi|_{\mathcal{M}(\Omega)} \geq \int_{\Omega} (g \wedge \nabla g) \cdot \nabla^\perp \zeta - \int_{\Omega} D\psi \cdot \nabla^\perp \zeta = \langle T(g), \zeta \rangle,$$

so that

$$\frac{1}{2\pi} |g \wedge \nabla g - D\psi|_{\mathcal{M}(\Omega)} \geq L(g),$$

by taking the supremum over  $\zeta$ .

It thus remains to construct, for each  $\varepsilon > 0$ , a map  $\psi \in C^\infty(\Omega; \mathbb{R})$  such that

$$\int_{\Omega} |g \wedge \nabla g - \nabla\psi| \leq 2\pi L(g) + \varepsilon.$$

Recall that, by Lemma 6.4, we may find some  $h \in W^{1,1}(\Omega; S^1)$  such that  $T(h) = T(g)$  and

$$\int_{\Omega} |\nabla h| \leq 2\pi L(g) + \varepsilon/2.$$

Set  $k := g\bar{h}$ , so that  $k \in Y$ , by Lemma 6.1 and Theorem 6.7. Write  $k = e^{i\varphi}$  for some  $\varphi \in W^{1,1}$ , and let  $\psi \in C^\infty(\Omega; \mathbb{R})$  be such that  $\int_{\Omega} |\nabla\varphi - \nabla\psi| < \frac{\varepsilon}{2}$ .

Then,

$$\begin{aligned} \int_{\Omega} |g \wedge \nabla g - \nabla \psi| &= \int_{\Omega} |(hk) \wedge \nabla(hk) - \nabla \psi| = \int_{\Omega} |h \wedge \nabla h + k \wedge \nabla k - \nabla \psi| \\ &= \int_{\Omega} |h \wedge \nabla h + \nabla \varphi - \nabla \psi| \leq \int_{\Omega} |h \wedge \nabla h| + \int_{\Omega} |\nabla \varphi - \nabla \psi| \\ &\leq \int_{\Omega} |\nabla h| + \frac{\varepsilon}{2} \leq 2\pi L(g) + \varepsilon. \end{aligned}$$

In order to conclude the proof of Theorem 6.2, we only need to show the following

**Claim.** Given  $g \in W^{1,1}(\Omega; S^1)$ , there exists some  $\varphi \in BV(\Omega; \mathbb{R})$  such that

$$g = e^{i\varphi} \quad \text{a.e. in } \Omega \quad (6.26)$$

and

$$|g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} = 2\pi L(g). \quad (6.27)$$

In other words, in (6.7), one may restrict the minimization to the class of functions  $\psi \in BV(\Omega; \mathbb{R})$  such that  $g = e^{i\psi}$ .

Using the same argument as above, we can write  $g$  as

$$g = h_n e^{i\varphi_n} \quad \text{in } \Omega, \quad (6.28)$$

where  $\varphi_n \in W^{1,1}(\Omega; \mathbb{R})$ ,  $h_n \in W^{1,1}(\Omega; S^1)$ , and

$$|h_n|_{W^{1,1}} \leq 2\pi L(g) + \frac{1}{n}.$$

Moreover, in view of (iv) in Lemma 6.4, we can also assume that  $h_n \rightarrow 1$  a.e.

Note that

$$\int_{\Omega} |g \wedge \nabla g - \nabla \varphi_n| = \int_{\Omega} |h_n \wedge \nabla h_n| = \int_{\Omega} |\nabla h_n| \leq 2\pi L(g) + \frac{1}{n}. \quad (6.29)$$

Subtracting a suitable integer multiple of  $2\pi$  from  $\varphi_n$ , we may assume that  $(\varphi_n)$  is bounded in  $W^{1,1}(\Omega; \mathbb{R})$ . After passing to a subsequence if necessary, we can find  $\varphi \in BV(\Omega; \mathbb{R})$  such that

$$\varphi_n \rightarrow \varphi \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla \varphi_n \xrightarrow{*} D\varphi \quad \text{in } \mathcal{M}(\Omega).$$

Since  $h_n \rightarrow 1$  a.e. in  $\Omega$ , it follows from (6.28) that  $g = e^{i\varphi}$  a.e. in  $\Omega$ . Letting  $n \rightarrow \infty$  in (6.29), we obtain

$$\int_{\Omega} |g \wedge \nabla g - D\varphi| \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |g \wedge \nabla g - \nabla \varphi_n| \leq 2\pi L(g).$$

This establishes “ $\leq$ ” in (6.27). The reverse inequality follows trivially from (6.7).

**Remark 6.10** Here is an example which shows that a minimizing function  $\psi$  in (6.7) is not necessarily a lifting of  $g$  (modulo constants). Assume for simplicity  $\Omega$  is flat and consider a map  $g$  having four singular points, say  $P_1 = (0, 0)$ ,  $P_2 = (1, 1)$ ,  $N_1 = (1, 0)$  and  $N_2 = (0, 1)$  in  $\Omega$ . In other words,  $S = P_1N_1P_2N_2$  is a square. We may write  $g = e^{i\psi_1} = e^{i\psi_2}$ , where

$$\psi_1 \in C^\infty(\Omega \setminus ([P_1, N_1] \cup [P_2, N_2])) \quad \text{and} \quad \psi_2 \in C^\infty(\Omega \setminus ([P_1, N_2] \cup [P_2, N_1])).$$

Then,  $|g \wedge \nabla g - D\psi_1| = 2\pi\nu_1$  (resp.  $|g \wedge \nabla g - D\psi_2| = 2\pi\nu_2$ ), where  $\nu_1$  (resp.  $\nu_2$ ) denotes the 1-dimensional Hausdorff measure on  $[P_1, N_1] \cup [P_2, N_2]$  (resp.  $[P_1, N_2] \cup [P_2, N_1]$ ).

It follows from Theorem 6.2 that  $\psi_1, \psi_2$  are minimizers in (6.7). Moreover, we may assume that  $\psi_1 = \psi_2$  in the square  $S$ . By convexity, the function  $\psi = (\psi_1 + \psi_2)/2$  is also a minimizer. Outside  $\bar{S}$ ,  $\psi$  is smooth and, clearly,  $g = \alpha e^{i\psi}$  in  $\Omega \setminus \bar{S}$  for some  $\alpha \in S^1$ . One may check that  $\alpha = -1$ , and thus

$$e^{i\psi} = \begin{cases} g & \text{in } S, \\ -g & \text{in } \Omega \setminus \bar{S}, \end{cases}$$

so that  $\psi$  is not a lifting of  $g$ .

Going back to the general situation, let  $K$  be the set of minimizers of the problem

$$\min_{\psi \in BV} \int |g \wedge \nabla g - D\psi|$$

satisfying  $\int \psi = 0$ . Clearly,  $K$  is convex and compact in  $L^1(\Omega; \mathbb{R})$ .

**Open Problem 6** Is it true that

$$\psi \text{ is an extreme point of } K \iff g = e^{i(\psi+C)} \text{ for some constant } C?$$

Another result, strongly related to Theorem 6.1, is the following :

**Theorem 6.8** Let  $g \in W^{1,1}(\Omega; S^1)$ . Then,

$$\inf \left\{ |\varphi_2|_{BV} : g = e^{i(\varphi_1 + \varphi_2)}, \varphi_1 \in W^{1,1}(\Omega; \mathbb{R}), \varphi_2 \in BV(\Omega; \mathbb{R}) \right\} = 4\pi L(g). \quad (6.30)$$

The analogue of Theorem 6.8 for the space  $H^{1/2}(\Omega; S^1)$  was established in [22], and the arguments there can be adapted to our case. The proof we present below for “ $\geq$ ” in (6.30) is however different.

**Proof of Theorem 6.8.**

**Proof of “ $\leq$ ”.** With  $\varepsilon > 0$  fixed and  $h$  given by Lemma 6.4, we write  $g = hk$ , where  $k = g\bar{h}$ . By Lemma 6.1 a), b), we have  $T(k) = 0$ . Therefore, by Theorem 6.7 we may write  $k = e^{i\varphi}$  for some  $\varphi \in W^{1,1}(\Omega; \mathbb{R})$ . It follows that  $g = e^{i(\varphi + \psi)}$ , with  $\psi$  given by Lemma 6.4. Inequality “ $\leq$ ” in (6.30) follows from (iii) in Lemma 6.4.

**Proof of “ $\geq$ ”.** We rely on the following

**Lemma 6.5** *Let  $\varphi \in BV(\Omega; \mathbb{R})$  be such that  $g = e^{i\varphi} \in W^{1,1}(\Omega; B^1)$ . Then,*

$$|D\varphi|_{\mathcal{M}(\Omega)} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)}.$$

**Proof.** We split the measure  $D\varphi$  as

$$D\varphi = D^a\varphi + D^c\varphi + D^j\varphi, \quad (6.31)$$

where a, c, and j stand respectively for the absolutely continuous, Cantor, and jump part. Applying Vol'pert's chain rule to the composition  $f(\varphi)$ , where  $f(t) = e^{it}$ , we obtain

$$Dg = D(f \circ \varphi) = f'(\varphi)D^a\varphi + f'(\varphi)D^c\varphi + \frac{f(\varphi^+) - f(\varphi^-)}{\varphi^+ - \varphi^-} D^j\varphi. \quad (6.32)$$

The meaning of this identity is the following : recall that, for every function  $\varphi \in BV(\Omega)$ , the Lebesgue set of  $\varphi$  is the complement of a set of  $\sigma$ -finite  $\mathcal{H}^1$ -measure. We may assume that  $\varphi$  coincides with its precise representative on the Lebesgue set of  $\varphi$ . Since  $|D^a\varphi|(A) = |D^c\varphi|(A) = 0$  whenever  $\mathcal{H}^1(A) < \infty$ , the first two terms in the right-hand side of (6.32) are well-defined (i.e. independently of the choice of the representative of  $\varphi$ ). The last term in (6.32) is to be understood as follows : the jump set  $J$  of  $\varphi$  is a countable union of Lipschitz curves  $\mathcal{C}_i$  and, at  $\mathcal{H}^1$ -a.e. point  $x$  of  $\mathcal{C}_i$ ,  $\mathcal{C}_i$  has a normal vector and  $\varphi$  has one-sided limits at  $x$  along the normal direction ; the quantities  $\varphi^+$  and  $\varphi^-$  stand for the two one-sided limits. We refer the reader to [1] for a proof of (6.32). Since  $g \in W^{1,1}$ , it follows that  $D^c g = D^j g = 0$ , so that  $D^c \varphi = 0$  and

$$\nabla g = f'(\varphi)D^a\varphi = igD^a\varphi. \quad (6.33)$$

From (6.33), we obtain that

$$g \wedge \nabla g = \frac{1}{ig} \nabla g = D^a\varphi.$$

Thus,

$$D^j\varphi = D\varphi - g \wedge \nabla g.$$

Since the decomposition (6.31) consists of mutually orthogonal measures, we have

$$\begin{aligned} |D\varphi| &= |D^a\varphi| + |D^j\varphi| = |ig \nabla g|_{\mathcal{M}(\Omega)} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} \\ &= |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)}. \end{aligned}$$

**Proof of Theorem 6.8 completed.** Write  $g = e^{i(\varphi_1 + \varphi_2)}$ , with  $\varphi_1 \in W^{1,1}$ ,  $\varphi_2 \in BV$ . Then, with  $h = ge^{-i\varphi_1}$ , we have  $h = e^{i\varphi_2}$ ,  $h \in W^{1,1}$  and  $T(h) = T(g)$ . Theorem 6.2 and Lemma 6.5 yield

$$\begin{aligned} |D\varphi_2|_{\mathcal{M}(\Omega)} &= |h|_{W^{1,1}} + |h \wedge \nabla h - D\varphi_2|_{\mathcal{M}(\Omega)} \\ &\geq |h|_{W^{1,1}} + 2\pi L(h) \geq 4\pi L(h) = 4\pi L(g), \end{aligned}$$

since  $2\pi L(h) \leq |h|_{W^{1,1}}$ , by Lemma 6.1.

Maps in  $W^{1,1}(\Omega; S^1)$  need not belong to  $H^{1/2}(\Omega; S^1)$ . Nevertheless, we have the following link between  $W^{1,1}$  and  $H^{1/2}$  :

**Theorem 6.9** *Let  $g \in W^{1,1}(\Omega; S^1)$ . Then, there exist  $h \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$  and  $\varphi \in W^{1,1}(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi}h$  a.e. in  $\Omega$ .*

The analogue of Theorem 6.9 for  $H^{1/2}(\Omega; S^1)$  was established in [22].

**Proof.** We rely on the following additional property of the maps  $h = h_\varepsilon$  constructed in Lemma 6.4 (see [22]) :

(v)  $h \in H^{1/2}(\Omega; S^1)$ .

Pick any of the maps  $h$  as in Lemma 6.4. Then,  $T(g\bar{h}) = 0$ , so that, by Theorem 6.7, we may write  $g\bar{h} = e^{i\varphi}$  for some  $\varphi \in W^{1,1}(\Omega; \mathbb{R})$ . The decomposition  $g = e^{i\varphi}h$  has all the required properties.

From Theorem 6.2, we have

**Corollary 6.4** *For each  $g \in W^{1,1}(\Omega; S^1)$ , there exists  $\varphi \in BV(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi}$  a.e. in  $\Omega$ .*

**Corollary 6.5 (Giaquinta-Modica-Souček [54])** *Given  $g \in W^{1,1}(\Omega; S^1)$ , we can find a sequence  $(g_n) \subset C^\infty(\Omega; S^1)$ , bounded in  $W^{1,1}$ , such that  $g_n \rightarrow g$  a.e.*

We now establish the

**Proposition 6.2** *For each  $g \in W^{1,1}(\Omega; S^1)$ , we have*

$$E_{\text{rel}}(g) = E(g).$$

**Proof.** “ $\leq$ ”. Let  $\varphi \in BV(\Omega; \mathbb{R})$  be such that  $g = e^{i\varphi}$ . Let  $(\varphi_n) \subset C^\infty(\Omega; \mathbb{R})$  be such that  $\varphi_n \rightarrow \varphi$  a.e. and  $\int_\Omega |\nabla \varphi_n| \rightarrow |\varphi|_{BV}$ . We define  $g_n = e^{i\varphi_n} \in C^\infty(\Omega; S^1)$ . Then,

$$g_n \rightarrow g \quad \text{a.e.} \quad \text{and} \quad \int_\Omega |\nabla g_n| = \int_\Omega |\nabla \varphi_n| \rightarrow |\varphi|_{BV},$$

so that “ $\leq$ ” follows.

“ $\geq$ ”. Let  $(g_n) \subset C^\infty(\Omega; S^1)$  be such that  $g_n \rightarrow g$  a.e. and  $\int_\Omega |\nabla g_n| \rightarrow E_{\text{rel}}(g)$ . Since  $\Omega$  is simply connected, we may write  $g_n = e^{i\varphi_n}$ , with  $\varphi_n \in C^\infty(\Omega; \mathbb{R})$ . Since  $\int_\Omega |\nabla g_n| = \int_\Omega |\nabla \varphi_n|$ , we may find some  $\varphi \in BV(\Omega; \mathbb{R})$  such that, after subtracting an integer multiple of  $2\pi$  from  $\varphi_n$  and up to some subsequence,  $\varphi_n \rightarrow \varphi$  a.e.; we then conclude that

$$|\varphi|_{BV} \leq \liminf_{n \rightarrow \infty} \int_\Omega |\nabla \varphi_n| = E_{\text{rel}}(g).$$

The relaxed energy is also related to the minimal connection  $L(g)$ . This is the content of Theorem 6.1 :

$$E_{\text{rel}}(g) = \int_\Omega |\nabla g| + 2\pi L(g), \quad \forall g \in W^{1,1}(\Omega; S^1). \quad (6.34)$$

**Proof of Theorem 6.1.** Inequality “ $\leq$ ” in (6.34) was proved in [44] when  $\Omega$  is a smooth bounded open set in  $\mathbb{R}^2$ , and their argument could be easily adapted to our situation. Here is another way. By Proposition 6.2, we know that

$$E_{\text{rel}}(g) = |D\varphi_0|_{\mathcal{M}}$$

for some  $\varphi_0 \in BV(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi_0}$ . By Lemma 6.5 and Theorem 6.2, we have

$$|D\varphi_0|_{\mathcal{M}} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi_0|_{\mathcal{M}} \geq |g|_{W^{1,1}} + 2\pi L(g).$$

For the reverse inequality “ $\geq$ ” in (6.34), we argue as follows. By Theorem 6.2, we may find some  $\varphi_1 \in BV$  such that  $g = e^{i\varphi_1}$  and

$$|g \wedge \nabla g - D\varphi_1|_{\mathcal{M}} = 2\pi L(g).$$

Combining with Lemma 6.5 yields

$$|D\varphi_1|_{\mathcal{M}} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi_1|_{\mathcal{M}} = |g|_{W^{1,1}} + 2\pi L(g).$$

By Proposition 6.2, we finally get

$$E_{\text{rel}}(g) \geq |D\varphi_1|_{\mathcal{M}} = |g|_{W^{1,1}} + 2\pi L(g).$$

**Corollary 6.6** *For each  $g \in W^{1,1}(\Omega; S^1)$ , there is some  $\varphi \in BV(\Omega; \mathbb{R})$  such that  $g = e^{i\varphi}$  a.e. and  $|\varphi|_{BV} \leq 2|g|_{W^{1,1}}$ .*

Corollary 6.6 is a special case of a much more general result of Dávila and Ignat [39] which asserts that the same conclusion holds for maps  $g \in BV(\Omega; S^1)$ .

**Proof.** The corollary follows from Proposition 6.2, Theorem 6.1, and the inequality

$$L(g) \leq \frac{1}{2\pi} |g|_{W^{1,1}}, \quad \forall g \in W^{1,1}(\Omega; S^1).$$

We now present a coarea type formula proved in [22], which relates the quantity  $\langle T(g), \zeta \rangle$  and the degree of  $g \in W^{1,1}(\Omega; S^1)$  with respect to the level sets of  $\zeta$ . More precisely, let  $\zeta \in C^\infty(\Omega; \mathbb{R})$ . If  $\lambda \in \mathbb{R}$  is a regular value of  $\zeta$ , let

$$\Gamma_\lambda = \{x \in \Omega : \zeta(x) = \lambda\}.$$

We orient  $\Gamma_\lambda$  such that, for each  $x \in \Gamma_\lambda$ , the basis  $(\tau(x), \nabla\zeta(x), n(x))$  is direct, where  $n(x)$  denotes the outward normal to  $\Omega$  at  $x$ .

Given  $g \in W^{1,1}(\Omega; S^1)$ , the restriction of  $g$  to the level set  $\Gamma_\lambda$  belongs to  $W^{1,1} \subset C^0$  for a.e.  $\lambda$ ; this follows from the coarea formula. Therefore,  $\deg(g; \Gamma_\lambda)$  makes sense for a.e.  $\lambda$ , and  $\Gamma_\lambda$  is a union of simple curves, say  $\Gamma_\lambda = \bigcup \gamma_j$ . Then, we set

$$\deg(g; \Gamma_\lambda) = \sum_j \deg(g; \gamma_j).$$

In [22], the authors proved that for every  $g \in W^{1,1}(\Omega; S^1)$  we have

$$\langle T(g), \zeta \rangle = 2\pi \int_{\mathbb{R}} \deg(g; \Gamma_\lambda) d\lambda.$$

We point out that this formula still holds if  $\zeta \in \text{Lip}(\Omega; \mathbb{R})$ . If we assume in addition that  $|\zeta|_{\text{Lip}} \leq 1$ , then a simple corollary of (6.35) is the inequality :

$$\left| \int_{\mathbb{R}} \text{deg}(g; \Gamma_\lambda) d\lambda \right| \leq L(g). \quad (6.35)$$

The novelty in Theorem 6.4 is that this estimate remains true if one replaces  $\text{deg}(g; \Gamma_\lambda)$  by its absolute value inside the integral in (6.35).

**Proof of Theorem 6.4.** We shall first establish (6.12) for functions  $g$  in the class  $\mathcal{R}$ , and then we argue by density.

Let  $g \in \mathcal{R}$  and  $\zeta \in \text{Lip}(\Omega; \mathbb{R})$ , with  $|\zeta|_{\text{Lip}} \leq 1$ . By Lemma 6.3, we can find finitely many points  $P_i, N_i$  such that

$$T(g) = 2\pi \sum_{i=1}^k (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega).$$

Let  $\lambda \in \mathbb{R}$  be a regular value of  $\zeta$  such that  $\lambda \neq \zeta(P_i), \zeta(N_i)$  for any  $i \in \{1, \dots, k\}$ . Then, we have

$$\text{deg}(g; \Gamma_\lambda) = \text{card} \{i : \zeta(P_i) > \lambda\} - \text{card} \{i : \zeta(N_i) > \lambda\},$$

so that

$$\text{deg}(g; \Gamma_\lambda) = \frac{1}{2} \sum_{i=1}^k \left\{ \text{sign} [\zeta(P_i) - \zeta] - \text{sign} [\zeta(N_i) - \zeta] \right\}.$$

After relabeling the negative points  $N_i$  if necessary, we can assume that

$$L(g) = \sum_{i=1}^k d(P_i, N_i).$$

Let  $\gamma_i$  be a geodesic arc in  $\Omega$  connecting  $P_i$  to  $N_i$ . Clearly,

$$\frac{1}{2} \left| \text{sign} [\zeta(P_i) - \zeta] - \text{sign} [\zeta(N_i) - \zeta] \right| \leq \text{card} \{x \in \gamma_i : \zeta(x) = \lambda\}.$$

Using the area formula, we obtain

$$\int_{\mathbb{R}} |\text{deg}(g; \Gamma_\lambda)| d\lambda \leq \sum_{i=1}^k \int_{\mathbb{R}} \text{card} \{x \in \gamma_i : \zeta(x) = \lambda\} d\lambda = \sum_{i=1}^k \int_{\gamma_i} \left| \frac{\partial \zeta}{\partial \tau} \right| \leq L(g).$$

This establishes (6.12) for maps  $g \in \mathcal{R}$ .

For a general  $g \in W^{1,1}(\Omega; S^1)$ , it follows from Lemma 6.2 that we can find a sequence  $(g_n) \subset \mathcal{R}$  such that  $g_n \rightarrow g$  strongly in  $W^{1,1}$ . In particular, by Lemma 6.1 d) we have

$$L(g_n) \rightarrow L(g).$$

Passing to a subsequence, we may assume that  $u_n|_{\Gamma_\lambda}$  converges to  $u|_{\Gamma_\lambda}$  in  $W^{1,1}$ , and hence uniformly, for a.e.  $\lambda$ . Thus,

$$\text{deg}(g_n; \Gamma_\lambda) \rightarrow \text{deg}(g; \Gamma_\lambda) \quad \text{for a.e. } \lambda.$$



Applying Fatou's Lemma, we find

$$\int_{\mathbb{R}} |\deg(g; \Gamma_\lambda)| d\lambda \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} |\deg(g_n; \Gamma_\lambda)| d\lambda \leq \lim_{n \rightarrow \infty} L(g_n) = L(g).$$

This proves (6.12). Note that (6.13) follows immediately from (6.12). In fact, if  $\zeta$  maximizes (6.9), then

$$L(g) = \int_{\mathbb{R}} \deg(g; \Gamma_\lambda) d\lambda \leq \int_{\mathbb{R}} |\deg(g; \Gamma_\lambda)| d\lambda \leq L(g).$$

Therefore,  $\deg(g; \Gamma_\lambda) = |\deg(g; \Gamma_\lambda)| \geq 0$  for a.e.  $\lambda$ .

Given two (infinite) sequences of points  $(P_i)$  and  $(N_i)$  in  $\Omega$  such that

$$\sum_{i=1}^{\infty} d(P_i, N_i) < \infty, \quad (6.36)$$

we may introduce the distribution

$$T = 2\pi \sum_{i=1}^{\infty} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega),$$

and the number

$$L = \frac{1}{2\pi} \max_{|\zeta|_{\text{Lip}} \leq 1} \langle T, \zeta \rangle,$$

where the best Lipschitz constant  $|\zeta|_{\text{Lip}}$  refers to the geodesic distance  $d$  on  $\Omega$ . The distribution  $T$  admits many representations, and it has been proved in [22, Lemma 12'] (see also Proposition 7.2) that

$$L = \inf \left\{ \sum_j d(\tilde{P}_j, \tilde{N}_j) : \sum_j (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) = \sum_i (\delta_{P_i} - \delta_{N_i}) \text{ in } \mathcal{D}'(\Omega) \right\}.$$

We also recall that if the sequences  $(P_i), (N_i)$  consist of a **finite** number of points, say  $P_1, P_2, \dots, P_k, N_1, N_2, \dots, N_k$ , then

$$L = \min_{\sigma} \sum_{i=1}^k d(P_i, N_{\sigma(i)}), \quad (6.37)$$

where the minimum in (6.37) is taken over all permutations of the integers  $\{1, 2, \dots, k\}$ .

In our next result, we are **given** points  $(P_i), (N_i)$  satisfying (6.36), and we ask what is the least " $W^{1,1}$ -energy" needed to produce singularities of degree +1 at the points  $P_i$ , and degree -1 at the points  $N_i$ ; more precisely, we consider the class of all maps  $g$  in  $W^{1,1}(\Omega; S^1)$  such that

$$T(g) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}). \quad (6.38)$$

[We know (see Lemma 16 in [22]) that such class of maps  $g$  is not empty.]

The answer is given by

**Theorem 6.10** *Let  $P_i, N_i \in \Omega$  be such that  $\sum_i d(P_i, N_i) < \infty$ . Then,*

$$\inf \left\{ \int_{\Omega} |\nabla g| : g \in W^{1,1}(\Omega; S^1) \text{ satisfying (6.38)} \right\} = 2\pi L. \quad (6.39)$$

*In particular,*

$$\begin{aligned} d(P, N) &= \frac{1}{2\pi} \inf \left\{ \int_{\Omega} |\nabla g| : g \in W^{1,1}(\Omega; S^1), T(g) = 2\pi(\delta_P - \delta_N) \right\} \\ &= \frac{1}{2\pi} \inf \left\{ \int_{\Omega} |\nabla g| \left| \begin{array}{l} g \in W_{\text{loc}}^{1,\infty}(\Omega \setminus \{P, N\}; S^1), \\ \deg(g, P) = +1 \text{ and } \deg(g, N) = -1 \end{array} \right. \right\}. \end{aligned} \quad (6.40)$$

**Proof.** Given  $P_i, N_i$  as above, we fix some  $g_0 \in W^{1,1}(\Omega; S^1)$  such that

$$T(g_0) = T = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}).$$

By Lemma 6.4, for each  $\varepsilon > 0$  we may find a map  $h \in W^{1,1}(\Omega; S^1)$  such that  $T(h) = T(g_0) = T$  and

$$\int_{\Omega} |\nabla h| \leq 2\pi L(g_0) + \varepsilon = 2\pi L + \varepsilon,$$

which implies “ $\leq$ ” in (6.39). Inequality “ $\geq$ ” in (6.39) follows from Lemma 6.1 c). To prove the second equality in (6.40), it suffices to apply Lemma 15 in [22].

In view of Theorem 6.10, it is natural to define, for every  $P, N \in \Omega$ ,

$$\rho(P, N) := \frac{1}{2\pi} \inf \left\{ [g]_{W^{1,1}} : g \in W^{1,1}(\Omega; S^1), T(g) = 2\pi(\delta_P - \delta_N) \right\}.$$

Here,  $[ ]_{W^{1,1}}$  is a general given semi-norm on  $W^{1,1}(\Omega; \mathbb{C})$  equivalent to  $| |_{W^{1,1}}$ . Of course,  $\rho$  depends on the choice of  $[ ]_{W^{1,1}}$ . We require from  $[ ]_{W^{1,1}}$  some structural properties :

- (P1)  $[\alpha g]_{W^{1,1}} = [g]_{W^{1,1}}, \forall g \in W^{1,1}(\Omega; \mathbb{C}), \forall \alpha \in S^1$ ;
- (P2)  $[\bar{g}]_{W^{1,1}} = [g]_{W^{1,1}}, \forall g \in W^{1,1}(\Omega; \mathbb{C})$ ;
- (P3)  $[gh]_{W^{1,1}} \leq \|g\|_{L^\infty} [h]_{W^{1,1}} + \|h\|_{L^\infty} [g]_{W^{1,1}}, \forall g, h \in W^{1,1}(\Omega; \mathbb{C}) \cap L^\infty$ .

It follows from (P3) that  $\rho$  is a distance.

For each  $g \in W^{1,1}(\Omega; S^1)$ , we may define a new relaxed energy associated to  $[ ]_{W^{1,1}}$  by setting

$$\tilde{E}_{\text{rel}}(g) = \inf \left\{ \liminf_{n \rightarrow \infty} [g_n]_{W^{1,1}} : g_n \in C^\infty(\Omega; S^1), g_n \rightarrow g \text{ a.e.} \right\}.$$

Let

$$\tilde{L}(g) = \frac{1}{2\pi} \sup \left\{ \langle T(g), \zeta \rangle : |\zeta(x) - \zeta(y)| \leq \rho(x, y), \forall x, y \in \Omega \right\}.$$

We end this section with the following

**Open Problem 7** Given  $g \in W^{1,1}(\Omega; S^1)$ , is it true that

$$\tilde{E}_{\text{rel}}(g) = [g]_{W^{1,1}} + 2\pi \tilde{L}(g) ?$$

## 6.4 $W^{1,1}(\Omega; S^1)$ and relaxed Jacobians

Given any function  $g \in W^{1,p}(\Omega; \mathbb{R}^2)$ , with  $p \geq 1$ , a natural concept associated to  $g$  is the following

$$TV_\tau(g) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |g_{nx} \wedge g_{ny}| : g_n \in C^\infty(\Omega; \mathbb{R}^2), g_n \rightarrow g \text{ with respect to } \tau \right\},$$

for some topology  $\tau$ .

There are several topologies  $\tau$  of interest. For example, given  $1 \leq p < 2$  and  $g \in W^{1,p}(\Omega; \mathbb{R}^2)$ , we consider

$$\begin{aligned} TV_{p,s}(g) &= TV \text{ computed with respect to the strong } W^{1,p}\text{-topology,} \\ TV_{p,w}(g) &= TV \text{ computed with respect to the weak } W^{1,p}\text{-topology.} \end{aligned}$$

In the case  $p = 1$ , for every  $g \in W^{1,1}(\Omega; \mathbb{R}^2)$ , we also define

$$TV_{1,w^*}(g) = TV \text{ computed with respect to the weak* } W^{1,1}\text{-topology.}$$

In what follows, we are going to work with the weak  $W^{1,1}$ -topology and simply write  $TV$  for the total variation  $TV_{1,w}$ . But we will also state results for  $TV_{p,w}$  and  $TV_{p,s}$  for every  $1 \leq p < 2$ , and for  $TV_{1,w^*}$ ; see Remarks 6.11 and 6.13 below.

Let us start with a simple

**Proposition 6.3** *Assume  $g \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$  and  $TV(g) < \infty$ . Then,  $\text{Det}(\nabla g) \in \mathcal{M}(\Omega)$  and*

$$|\text{Det}(\nabla g)|_{\mathcal{M}} \leq TV(g). \quad (6.41)$$

**Proof.** Since  $TV(g) < \infty$ , there exists a sequence  $(g_n) \subset C^\infty(\Omega; \mathbb{R}^2)$  such that

$$g_n \rightharpoonup g \quad \text{weakly in } W^{1,1}, \quad (6.42)$$

$$\int_{\Omega} |g_{nx} \wedge g_{ny}| \leq TV(g) + \frac{1}{n}. \quad (6.43)$$

Let  $M = \|g\|_{L^\infty}$  and  $P : \mathbb{R}^2 \rightarrow B_M$  be the orthogonal projection onto  $B_M$ . Set  $\tilde{g}_n = P g_n$ . It is easy to see (using Dunford-Pettis' theorem) that  $\tilde{g}_n$  satisfies (6.42) and (6.43). Moreover, by a standard regularization argument, we may assume that the functions  $\tilde{g}_n$  are smooth. In what follows, we will denote  $\tilde{g}_n$  by  $g_n$ , and so we also have

$$\|g_n\|_{L^\infty} \leq \|g\|_{L^\infty}. \quad (6.44)$$

We claim that

$$g_n \wedge \nabla g_n \rightharpoonup g \wedge \nabla g \quad \text{weakly in } L^1.$$

In fact, it suffices to notice that

$$\int_{\Omega} |g_n - g| |\nabla g_n| \rightarrow 0,$$

which follows from Egorov's and Dunford-Pettis' theorems. Hence,

$$g_{nx} \wedge g_{ny} = \frac{1}{2} \left[ (g_n \wedge g_{ny})_x + (g_{nx} \wedge g_n)_y \right]$$

converges to  $\text{Det}(\nabla g)$  in the sense of distributions. We deduce from (6.43) that  $\text{Det}(\nabla g)$  belongs to  $\mathcal{M}(\Omega)$  and that (6.41) holds.

**Remark 6.11** The conclusion of Proposition 6.3 is no longer true if we compute the total variation of  $g$  with respect to the weak\*-topology of  $W^{1,1}$ ,  $TV_{1,w^*}(g)$ . In fact, assume  $g \in W^{1,1}(\Omega; S^1)$ . It follows from Corollary 6.2 that there exists  $(g_n) \subset C^\infty(\Omega; S^1)$  such that  $g_n \xrightarrow{*} g$  in  $W^{1,1}$ . Since  $g_{nx} \wedge g_{ny} = 0$  for each  $n$ , we conclude that  $TV_{1,w^*}(g) = 0$ . On the other hand, for some maps  $g$  in  $W^{1,1}(\Omega; S^1)$  we have  $\text{Det}(\nabla g) \neq 0$ ; see Theorem 6.11 below. *A fortiori*, the conclusion of Proposition 3 fails if  $\tau$  is the strong  $L^1$ -topology (or the convergence pointwise a.e.).

In general, the inequality in (6.41) is strict. This fact was pointed out by an example in [66]; see also [52]. There, the map  $g \in W^{1,1}(\Omega; \mathbb{R}^2)$  takes its values in an eight-shaped curve and satisfies  $\text{Deg}(\nabla g) = 0$  in the sense of distributions, while  $TV(g) > 0$ . It is therefore remarkable that equality in (6.41) holds whenever the map  $g$  takes its values in  $S^1$ . This is the content of our next result, which extends Theorem 6.5 :

**Theorem 6.11** *Assume  $g \in W^{1,p}(\Omega; S^1)$ ,  $1 \leq p < 2$ , is such that  $\text{Det}(\nabla g) \in \mathcal{M}$ . Then, there exists a sequence  $(g_n) \subset C^\infty(\Omega; \mathbb{R}^2)$  such that*

$$g_n \rightarrow g \quad \text{strongly in } W^{1,p}$$

and

$$TV(g) = \lim_{n \rightarrow \infty} \int_{\Omega} |g_{nx} \wedge g_{ny}| = |\text{Det}(\nabla g)|_{\mathcal{M}}.$$

Moreover, in this case,

$$\text{Det}(\nabla g) = \pi \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega).$$

In particular,  $\frac{1}{\pi} |\text{Det}(\nabla g)|_{\mathcal{M}}$  equals the number of topological singularities of  $g$ , taking into account their multiplicities.

**Remark 6.12** Theorem 6.11 extends and clarifies some of the results of [50]. Although in their case  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^2$ , the above results, stated for  $\Omega = \partial G$ , adapt easily to bounded domains; see Section 6.5.2.

**Proof of Theorem 6.11.** The fact that

$$\text{Det}(\nabla g) \text{ measure} \implies \text{Det}(\nabla g) = \pi \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i})$$

is a consequence of Theorem 6.5 and of Theorem 7.5, which will be established in the next chapter; see also [80]. Let us assume, for simplicity, that  $\text{Det}(\nabla g) = \pi(\delta_P - \delta_N)$ ;

the argument below still applies to the general case. Suppose, in addition, that  $\Omega$  is flat and horizontal near  $P$  and  $N$ . We start by defining, near  $P$  and  $N$ , a map  $h$  by setting

$$h(x) = \left( \frac{x - P}{|x - P|} \right)^{\pm 1} \quad \text{near } P, \quad h(x) = \left( \frac{x - N}{|x - N|} \right)^{\mp 1} \quad \text{near } N.$$

For appropriate choices of  $\pm$ , we have  $\deg(h, P) = +1$  and  $\deg(h, N) = -1$ . Then,  $h$  extends to a map in  $C^\infty(\Omega \setminus \{P, N\}; S^1) \cap W^{1,p}(\Omega; S^1)$ ,  $1 \leq p < 2$ . Set

$$h_n(x) = \begin{cases} h(x) & \text{if } d(x, P) \geq 1/n \text{ and } d(x, N) \geq 1/n, \\ n d(x, P) h(x) & \text{if } d(x, P) < 1/n, \\ n d(x, N) h(x) & \text{if } d(x, N) < 1/n. \end{cases}$$

Clearly,  $h_n \rightarrow h$  in  $W^{1,p}$  and

$$\int_{\Omega} |h_{nx} \wedge h_{ny}| = 2\pi.$$

Let  $k := g\bar{h}$ . Since  $T(k) = 0$ , we may write  $k = e^{i\varphi}$  for some  $\varphi \in W^{1,1}$  (see Theorem 6.7). Moreover,  $g, h \in W^{1,p} \cap L^\infty$  implies  $k \in W^{1,p}$ . From this, we easily conclude that  $\varphi \in W^{1,p}$ . Let  $(\varphi_n) \subset C^\infty(\Omega; \mathbb{R})$  be such that  $\varphi_n \rightarrow \varphi$  in  $W^{1,p}$ . Since a point has zero  $H^1$ -capacity (see Lemma 3.1), we may also assume that  $\varphi_n(x) = 0$  if  $d(x, P) \leq 1/n$  or  $d(x, N) \leq 1/n$ . Clearly,  $g_n = h_n e^{i\varphi_n}$  belongs to  $C^\infty(\Omega; \mathbb{R}^2)$  and  $g_n \rightarrow g$  in  $W^{1,p}$ . Since  $g_{nx} \wedge g_{ny} = h_{nx} \wedge h_{ny}$ , we obtain

$$\int_{\Omega} |g_{nx} \wedge g_{ny}| = 2\pi = |\text{Det}(\nabla g)|_{\mathcal{M}},$$

which shows that

$$TV(g) \leq |\text{Det}(\nabla g)|_{\mathcal{M}}.$$

The reverse inequality follows from Proposition 6.3.

**Remark 6.13** Theorem 6.11 and Proposition 6.3 imply that, for every  $p \in [1, 2)$ ,

$$TV_{p,w}(g) = TV_{p,s}(g) = TV(g), \quad \forall g \in W^{1,p}(\Omega; S^1).$$

We do not know whether the same holds without assuming that  $g$  is  $S^1$ -valued :

**Open Problem 8** Let  $g \in W^{1,1}(\Omega; \mathbb{R}^2)$ . Is it true that

$$TV_{1,w}(g) = TV_{1,s}(g) ?$$

Assume in addition that  $g \in W^{1,p}(\Omega; \mathbb{R}^2)$  for some  $1 < p < 2$ . Does one have

$$TV_{1,w}(g) = TV_{1,s}(g) = TV_{p,w}(g) = TV_{p,s}(g) ?$$

**Remark 6.14** The analogue of Remark 6.13 for  $p \geq 2$  is true, but uninteresting. Indeed, every  $g \in W^{1,p}(\Omega; S^1)$ , with  $p \geq 2$ , is a strong limit in  $W^{1,p}$  of a sequence  $(g_n)$  in  $C^\infty(\Omega; S^1)$  (see, e.g., [12]). Thus,  $TV(g) = 0$  and  $TV_{p,w}(g) = TV_{p,s}(g) = 0$  for every  $g \in W^{1,p}(\Omega; S^1)$ .

## 6.5 Further directions and open problems

### 6.5.1 Some examples of $BV$ -functions with jumps

It is natural to try to extend the above (or part of the above) results to the class of maps  $g$  in  $BV(\Omega; S^1)$ , where  $\Omega = \partial G$ ,  $G \subset \mathbb{R}^3$  as in the Introduction. Every  $g \in BV(\Omega; S^1)$  admits a lifting  $\varphi \in BV(\Omega; \mathbb{R})$  (see [54] and also [39]). Hence, we may define the two quantities  $E(g)$  and  $E_{\text{rel}}(g)$  as in (6.3) and (6.4), and we always have  $E(g) = E_{\text{rel}}(g)$ . The difficulty starts when we try to find a simple formula for  $E$  as in Theorem 6.1. To illustrate the heart of the matter, it is worthwhile to start, as in Section 6.2, with the simpler case  $BV(S^1; S^1)$ .

Clearly, every  $g \in BV(S^1; S^1)$  admits a lifting  $\varphi \in BV(S^1; \mathbb{R})$ . Hence, we may define the two quantities  $E(g)$  and  $E_{\text{rel}}(g)$  as in Section 6.2, and we always have  $E(g) = E_{\text{rel}}(g)$ . A major obstruction appears when trying to find an explicit formula for them. First, there are two natural ways of defining the  $BV$ -norm of  $g$  :

$$|g|_{BV} = \int_{S^1} |\dot{g}|$$

and

$$|g|_{BV S^1} = \int_{S^1} (|\dot{g}_a| + |\dot{g}_c|) + \sum_n d_{S^1}(g(a_n+), g(a_n-)),$$

where  $d_{S^1}$  denotes the geodesic distance on  $S^1$ . It is easy to see that

$$\begin{aligned} |g|_{BV} &= \inf \left\{ \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| : g_n \in C^\infty(S^1; \mathbb{R}^2) \text{ and } g_n \rightarrow g \text{ a.e.} \right\}, \\ |g|_{BV S^1} &= \inf \left\{ \liminf_{n \rightarrow \infty} \int_{S^1} |\dot{g}_n| : g_n \in C^\infty(S^1; S^1) \text{ and } g_n \rightarrow g \text{ a.e.} \right\}. \end{aligned}$$

We also have, for every  $g \in BV(S^1; S^1)$ ,

$$E(g) \geq |g|_{BV S^1} \geq |g|_{BV}.$$

Moreover,

$$E(g) - |g|_{BV} = 0 \quad \iff \quad g \in C^0 \text{ and } \deg g = 0.$$

An interesting estimate for  $E(g)$  when  $g \in BV$  is the following

**Theorem 6.12** *For every  $g \in BV(S^1; S^1)$ , we have*

$$E(g) \leq 2|g|_{BV}.$$

The above result is a variant of a nice theorem of [39] which asserts that if  $u \in BV(U; S^1)$ , where  $U$  is a domain in  $\mathbb{R}^N$ , then  $u = e^{i\varphi}$  for some  $\varphi \in BV(U; \mathbb{R})$  with  $|\varphi|_{BV} \leq 2|g|_{BV}$ . The proof of Theorem 6.12 is a straightforward adaptation of the ingenious method in [39]. As we have already pointed out in Remark 6.2, the constant 2 in Theorem 6.12 is optimal in  $W^{1,1}$ . A less intuitive fact is that the constant 2 is also optimal for piecewise constant functions. Here is an example :

**Example 6.1** Fix an integer  $k \geq 1$  and set

$$g(\theta) = e^{i2\pi j/k} \quad \text{for } \frac{2\pi j}{k} < \theta < \frac{2\pi(j+1)}{k}, \quad j = 0, 1, \dots, k-1.$$

Then,

$$|g|_{BV} = 2k \sin \frac{\pi}{k} \quad \text{and} \quad E(g) = 4\pi - \frac{4\pi}{k}.$$

The inequality

$$E(g) \leq 4\pi - \frac{4\pi}{k}$$

is straightforward; however, the reverse inequality is more delicate and relies on the following lemma whose proof is left to the reader :

**Lemma 6.6** *For every choice of  $\alpha_1, \dots, \alpha_k \in \mathbb{Z}$  with  $\sum_j \alpha_j = 1$ , we have*

$$\sum_{j=1}^k \left| \frac{1}{k} - \alpha_j \right| \geq 2 - \frac{2}{k}.$$

A striking difference with formula (6.15) is that neither

$$\frac{1}{2\pi}(E(g) - |g|_{BV}) \quad \text{nor} \quad \frac{1}{2\pi}(E(g) - |g|_{BVS^1})$$

is necessarily an integer. Here is an example :

**Example 6.2** Let

$$g(\theta) = \begin{cases} 1 & \text{for } 0 < \theta < 2\pi/3, \\ e^{i2\pi/3} & \text{for } 2\pi/3 < \theta < 4\pi/3, \\ e^{i4\pi/3} & \text{for } 4\pi/3 < \theta < 2\pi. \end{cases}$$

An easy computation shows that

$$E(g) = \frac{8\pi}{3}, \quad |g|_{BV} = 3\sqrt{3}, \quad \text{and} \quad |g|_{BVS^1} = 2\pi.$$

In fact, it seems hopeless to have an analogue of Theorem 6.6 since there is no reasonable notion of degree for maps in  $BV(S^1; S^1)$ . This is a consequence of

**Theorem 6.13** *The space  $BV(S^1; S^1)$  is path-connected.*

**Proof.** Let  $\varphi \in BV(S^1; \mathbb{R})$  be such that  $g = e^{i\varphi}$ . We claim that the map

$$F : t \in [0, 1] \longmapsto e^{it\varphi} \in BV(S^1; S^1) \tag{6.45}$$

is strongly continuous. This implies, in particular, that every map in  $BV(S^1; S^1)$  can be connected to 1.

The continuity of  $F$  in (6.45) follows from

**Lemma 6.7** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be such that :*

- (i)  $t \mapsto f(t, x)$  is continuous,  $\forall x \in \mathbb{R}$ ;
- (ii)  $f_x$  is continuous and bounded.

*Then, for every  $\varphi \in BV(\Omega; \mathbb{R})$ , the map*

$$t \longmapsto f(t, \varphi) \in BV(\Omega; \mathbb{R})$$

*is continuous.*

**Proof.** It suffices to establish continuity at  $t = 0$ . Set  $F(t) = f(t, \varphi)$ . For every  $t$ , we have  $F(t) \in BV(\Omega; \mathbb{R})$ . Let  $C > 0$  be such that  $|f_x(t, x)| \leq C, \forall t, \forall x$ .

Since

$$|f(t, x)| \leq |f(t, 0)| + C|x|,$$

we find that  $F(t) \rightarrow F(0)$  in  $L^1(\Omega)$  as  $t \rightarrow 0$ . Therefore, it suffices to prove that  $DF(t) \rightarrow DF(0)$  in  $\mathcal{M}(\Omega)$ . By the chain rule, we have

$$DF(t) = f_x(t, \varphi(x))D^d\varphi + \frac{f(t, \varphi(x+)) - f(t, \varphi(x-))}{\varphi(x+) - \varphi(x-)}D^j\varphi.$$

Thus,  $|DF(t)| \leq C|D\varphi|, \forall t$ . On the other hand,  $f_x(t, \varphi(x)) \rightarrow f_x(0, \varphi(x))$  a.e. with respect to  $D^d\varphi$ . Moreover,

$$\frac{f(t, \varphi(x+)) - f(t, \varphi(x-))}{\varphi(x+) - \varphi(x-)} \rightarrow \frac{f(0, \varphi(x+)) - f(0, \varphi(x-))}{\varphi(x+) - \varphi(x-)}$$

a.e. with respect to  $D^j\varphi$ . Therefore,

$$|D\varphi(t) - D\varphi(0)|_{\mathcal{M}} \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

by dominated convergence.

There is however an interesting concept of multivalued degree which associates to every  $g \in BV(S^1; S^1)$  a bounded subset of  $\mathbb{Z}$ . The starting point is the following

**Definition 6.1** *Let  $g \in BV(I; S^1)$ , where  $I$  is an interval. A canonical lifting of  $g$  is any map  $\varphi \in BV(I; \mathbb{R})$  such that*

$$g = e^{i\varphi} \quad \text{in } I \quad \text{and} \quad E(g) = |D\varphi|_{\mathcal{M}(I)}.$$

The structure of canonical liftings is quite rigid. In fact, the following holds :

**Theorem 6.14** *If  $\varphi_1$  and  $\varphi_2$  are two canonical liftings of the same map  $g$ , then*

$$\dot{\varphi}_1 - \dot{\varphi}_2 = \pi \sum_{\text{finite}} \pm \delta_{a_i}.$$

*Moreover, if  $g \in BV \cap C^0$ , then the canonical lifting is uniquely determined modulo  $2\pi$ , and coincides with a continuous lifting.*



Using canonical liftings, we may define a multivalued degree for maps in  $BV(S^1; S^1)$  :

**Definition 6.2** *Let  $g \in BV(S^1; S^1)$ . Assume  $g$  is continuous at  $z \in S^1$ . We let*

$$\text{Deg}_1 g = \left\{ \frac{\varphi(z+) - \varphi(z-)}{2\pi} : \varphi \text{ is a canonical lifting of } g \text{ in } S^1 \setminus \{z\} \right\}.$$

Since, clearly, for each canonical lifting we have

$$\left| \frac{\varphi(z+) - \varphi(z-)}{2\pi} \right| \leq \frac{1}{2\pi} \int_{S^1} |\dot{\varphi}|,$$

the set  $\text{Deg}_1 g$  is bounded. It follows from the second part of Theorem 6.14 that  $\text{Deg}_1 g = \{\text{deg } g\}$  if  $g \in BV \cap C^0$ . As another example, let

$$g(\theta) = \begin{cases} 1 & \text{if } 0 < \theta < \pi, \\ -1 & \text{if } \pi < \theta < 2\pi. \end{cases}$$

Then, it is easy to see that  $\text{Deg}_1 g = \{-1, 0, 1\}$ .

We collect below some properties of  $\text{Deg}_1$  :

**Theorem 6.15** *Assume  $g \in BV(S^1; S^1)$ . Then,*

- (a)  $\text{Deg}_1 g$  is a finite set of successive integers ;
- (b)  $\text{Deg}_1 g$  is independent of the choice of  $z$ .

Another possible definition of a multivalued degree is the following :

**Definition 6.3** *Given  $g \in BV(S^1; S^1)$ , we set*

$$\text{Deg}_2 g = \left\{ d : \exists (g_n) \subset C^\infty(S^1; S^1) \text{ s.t. } g_n \rightarrow g \text{ a.e., } \int |\dot{g}_n| \rightarrow \int |\dot{g}|, \text{deg } g_n = d \right\}.$$

Actually, both definitions yield the same degree :

**Theorem 6.16** *We have*

$$\text{Deg} := \text{Deg}_1 = \text{Deg}_2.$$

*Moreover, the function  $g \mapsto \text{Deg } g$  is continuous in the multivalued sense.*

A final interesting property of  $\text{Deg}$  is that it is “almost always” single-valued :

**Theorem 6.17** *Let*

$$\mathcal{U} = \left\{ g \in BV(S^1; S^1) : \text{Deg } g \text{ is single-valued} \right\}.$$

*Then,  $\mathcal{U}$  is a dense open subset of  $BV(S^1; S^1)$ .*

We omit the proofs of Theorems 6.14–6.17 and we refer the reader to [31] for details.

### 6.5.2 Some analogues of Theorems 6.1, 6.3, and 6.5 for domains in $\mathbb{R}^2$

Most of the above results admit counterparts in the case where the 2-d manifold  $\Omega$  is replaced by a bounded, simply connected domain in  $\mathbb{R}^2$  with smooth boundary. To illustrate this, we state the analogues of the main results; namely, Theorems 6.1, 6.3, and 6.5.

Let  $g \in W^{1,1}(\Omega; S^1)$ , and consider the distribution

$$\langle T(g), \zeta \rangle = \int_{\Omega} (g \wedge \nabla g) \cdot \nabla^{\perp} \zeta, \quad \forall \zeta \in W_0^{1,\infty}(\Omega; S^1).$$

A natural (pseudo-) metric on  $\bar{\Omega}$  is given by

$$d_{\Omega}(x, y) = \min \{ |x - y|, d(x, \partial\Omega) + d(y, \partial\Omega) \}.$$

Note that if  $\zeta \in W_0^{1,\infty}(\Omega)$ , then

$$|\zeta(x) - \zeta(y)| \leq \|\nabla \zeta\|_{L^{\infty}} d_{\Omega}(x, y), \quad \forall x, y \in \bar{\Omega}.$$

We also set

$$L(g) = \frac{1}{2\pi} \max_{\substack{\zeta \in W_0^{\infty}(\Omega) \\ \|\nabla \zeta\|_{L^{\infty}} \leq 1}} \langle T(g), \zeta \rangle.$$

We then have the following

**Theorem 6.18** *There exist sequences  $(P_i), (N_i)$  in  $\bar{\Omega}$  such that  $\sum_i d_{\Omega}(P_i, N_i) < \infty$  and*

$$T(g) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}) \quad \text{in } [W_0^{1,\infty}(\Omega)]^*.$$

Moreover,

$$L(g) = \inf \sum_i d_{\Omega}(P_i, N_i),$$

where the infimum is taken over all possible representations of  $T(g)$ .

With  $E(g)$  defined exactly as in (6.3), and  $E_{\text{rel}}(g)$  as in (6.4) (where  $\Omega$  is replaced by  $\bar{\Omega}$ ), we have

**Theorem 6.19** *For every  $g \in W^{1,1}(\Omega; S^1)$ ,*

$$E(g) = E_{\text{rel}}(g) = \int_{\Omega} |\nabla g| + 2\pi L(g).$$

Similarly, defining  $TV(g)$  as in (6.14) (with  $\Omega$  replaced by  $\bar{\Omega}$ ), we also have

**Theorem 6.20** *Let  $g \in W^{1,1}(\Omega; S^1)$ . Then,*

$$TV(g) < \infty \iff \text{Det}(\nabla g) \in \mathcal{M}(\Omega) = [C_0(\overline{\Omega})]^*.$$

*In this case, there exists a finite number of points  $a_i \in \Omega$  and integers  $d_i \in \mathbb{Z} \setminus \{0\}$  such that*

$$\text{Det}(\nabla g) = \pi \sum_{i=1}^k d_i \delta_{a_i}$$

and

$$TV(g) = |\text{Det}(\nabla g)|_{\mathcal{M}} = \pi \sum_{i=1}^k |d_i|.$$

Theorems 6.19, 6.18, and 6.20 are established in [31].

### 6.5.3 Extensions of Theorems 6.1, 6.2, and 6.3 to higher dimensions

Let  $G \subset \mathbb{R}^{N+1}$  be a smooth bounded domain and let  $\Omega := \partial G$ . Given a map  $u$  in  $W^{1,N-1}(\Omega; S^{N-1})$ , we define the  $L^1$ -vector field

$$D(u) = (D_1, \dots, D_N),$$

where

$$D_j = \det(u_{x_1}, \dots, u_{x_{j-1}}, u, u_{x_{j+1}}, \dots, u_{x_N}).$$

We then associate to  $u$  the distribution

$$T(u) = \text{div} D(u) = N \text{Det}(\nabla u).$$

Set

$$L(u) = \frac{1}{\sigma_N} \max_{\|\nabla \zeta\|_{L^\infty} \leq 1} \langle T(u), \zeta \rangle.$$

Here, we denote by  $\sigma_N$  the  $(N-1)$ -Hausdorff measure of  $S^{N-1}$ . The relaxed energy is defined by

$$E_{\text{rel}}(u) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{N-1} : u_n \in C^\infty(\Omega; S^{N-1}) \text{ and } u_n \rightarrow u \text{ a.e.} \right\},$$

where  $|\cdot|$  denotes the Euclidean norm.

We then have the following analogues of Theorems 6.1–6.3 :

**Theorem 6.21** *For every  $u \in W^{1,N-1}(\Omega; S^{N-1})$ ,*

$$E_{\text{rel}}(u) = \int_{\Omega} |\nabla u|^{N-1} + \sigma_N L(u).$$

**Theorem 6.22** For every  $u \in W^{1,N-1}(\Omega; S^{N-1})$ ,

$$\inf_{v \in C^\infty(\Omega; S^{N-1})} \int_{\Omega} |D(u) - D(v)| = \sigma_N L(u).$$

**Theorem 6.23** For every  $u \in W^{1,N-1}(\Omega; S^{N-1})$ , there exist sequences  $(P_i), (N_i)$  in  $\Omega$  such that  $\sum_i |P_i - N_i| < \infty$  and

$$T(u) = \sigma_N \sum_i (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega).$$

For the proofs, we refer the reader to [31].

### 6.5.4 Extension of $TV$ to higher dimensions and to fractional Sobolev spaces

Let  $\Omega$  and  $u$  be as in Section 6.5.3. Set

$$TV(u) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |\det \nabla u_n| : u_n \in C^\infty(\Omega; \mathbb{R}^N) \text{ and } u_n \rightarrow u \text{ in } W^{1,N-1} \right\}.$$

The analogue of Theorem 6.5 becomes

**Theorem 6.24** Let  $u \in W^{1,N-1}(\Omega; S^{N-1})$ . Then,

$$TV(u) < \infty \iff \text{Det}(\nabla u) \text{ is a measure.}$$

In this case, we have

$$\text{Det}(\nabla u) = \frac{\sigma_N}{N} \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \mathcal{D}'(\Omega)$$

and

$$TV(u) = |\text{Det}(\nabla u)|_{\mathcal{M}}.$$

**Remark 6.15** In the definition of  $TV$  given above, one cannot replace the strong convergence in  $W^{1,N-1}$  by the weak convergence when  $N \geq 3$ . Indeed, we point out that every map  $u \in W^{1,N-1}(\Omega; S^{N-1})$  is a weak limit in  $W^{1,N-1}$  of a sequence  $(u_n) \subset C^\infty(\Omega; S^{N-1})$ , when  $N \geq 3$ . However, one can replace the strong convergence of  $u_n$  in  $W^{1,N-1}$  by the weak convergence of  $u_n$  in  $W^{1,N-1}$  **and** the equi-integrability of  $|\nabla u_n|^{N-1}$  (see [31] for details).

We may go one step further. Let  $N - 1 < p < \infty$ . In [17], the authors have defined the distribution  $\text{Det}(\nabla u)$  for maps  $u \in W^{(N-1)/p,p}(\Omega; S^{N-1})$ . By analogy with the above definitions of  $TV$ , set

$$TV(u) = \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |\det \nabla u_n| : u_n \in C^\infty(\Omega; \mathbb{R}^N), u_n \rightarrow u \text{ in } W^{(N-1)/p,p} \right\}.$$

We have the following

**Theorem 6.25** *Let  $N - 1 < p \leq N$  and  $u \in W^{(N-1)/p,p}(\Omega; S^{N-1})$ . Then,*

$$TV(u) < \infty \iff \text{Det}(\nabla u) \text{ is a measure}$$

and the conclusions of Theorem 6.24 hold.

We refer to [31] for the proofs of Theorems 6.24 and 6.25. The case  $p > N$  in Theorem 6.25 is still open :

**Open Problem 9** Does the assertion of Theorem 6.24 hold when  $p > N$ ?

### 6.5.5 Extension of Theorem 6.3 to maps with values into a curve

Let  $G \subset \mathbb{R}^3$  be a smooth bounded domain with  $\Omega = \partial G$  simply connected. Assume  $\Gamma \subset \mathbb{R}^2$  is a smooth curve, with finitely many self-intersections. We then define

$$W^{1,1}(\Omega; \Gamma) = \left\{ g \in W^{1,1}(\Omega; \mathbb{R}^2) : g(x) \in \Gamma \text{ for a.e. } x \in \Omega \right\}.$$

Given a map  $g \in W^{1,1}(\Omega; \Gamma)$ , we define the distribution  $T(g)$  exactly as in (6.8).

We denote by  $A_1, \dots, A_k$  the bounded connected components of  $\mathbb{R}^2 \setminus \Gamma$ . We then have (see [31]) :

**Theorem 6.26** *Given  $g \in W^{1,1}(\Omega; \Gamma)$ , there exist sequences  $(P_{i,j}), (N_{i,j})$  in  $\Omega$ , with  $j = 1, \dots, k$ , such that  $\sum_{i,j} |A_j| d(P_{i,j}, N_{i,j}) < \infty$  and*

$$T(g) = 2 \sum_{j=1}^k |A_j| \sum_i (\delta_{P_{i,j}} - \delta_{N_{i,j}}). \quad (6.46)$$

There are many open directions here :

- 1) Does Theorem 6.26 remain valid for any smooth (or even rectifiable) curve, without assuming that the number of self-intersections of  $\Gamma$  is finite?
- 2) What are the counterparts of Theorems 6.1, 6.2, and 6.5 in this general setting?

# Chapitre 7

## On the distributions of the form

$$\sum_i (\delta_{p_i} - \delta_{n_i})$$

### Sommaire

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## 7.1 Introduction<sup>8</sup>

Given a complete metric space  $(X, d)$  and two sequences of points  $(p_i), (n_i) \subset X$  such that  $\sum_i d(p_i, n_i) < \infty$ , we consider the following linear functional in  $[\text{Lip}(X)]^*$  :

$$T := \sum_i (\delta_{p_i} - \delta_{n_i}); \tag{7.1}$$

more precisely,  $T$  is given by

$$\langle T, \zeta \rangle = \sum_i [\zeta(p_i) - \zeta(n_i)], \quad \forall \zeta \in \text{Lip}(X). \tag{7.2}$$

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<sup>8</sup>Ce chapitre a été publié dans J. Funct. Anal. **210** (2004), 391–435; une version résumée était déjà sortie dans C. R. Acad. Sci. Paris, Ser. I **336** (2003), 571–576.

Note that  $\sum_i d(p_i, n_i) < \infty$  implies that  $T$  is well-defined and continuous in  $\text{Lip}(X)$ .

We shall present in this chapter some properties satisfied by  $T$ . Our proofs rely on the existence of irreducible representations of  $T$ , a notion which we introduce below; see Definition 7.3.

In applications,  $T$  describes the location and the topological degree of singularities of maps  $u$  defined on  $X$  with values into a sphere  $S^k$ .

As we have already seen, this is the case for maps  $u \in W^{1,1}(S^2; S^1)$ . We could also have considered maps  $u \in H^{1/2}(S^2; S^1)$ . The way we define  $T(u)$  in this setting, however, is more involved. We refer the reader to [22] for details (see also [19]). These spaces come from the study of the Ginzburg-Landau model in 3-d.

Another example, but now arising from liquid crystals in  $\mathbb{R}^3$ , is when we take  $X = \mathbb{R}^3$  and  $k = 2$ . We then consider

$$H^1(\mathbb{R}^3; S^2) = \left\{ u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \int_{\mathbb{R}^3} |\nabla u|^2 < \infty \text{ and } |u| = 1 \text{ a.e. in } \mathbb{R}^3 \right\}.$$

Note that, for any  $u \in H^1(\mathbb{R}^3; S^2)$ , we have

$$D(u) := (u \cdot u_y \wedge u_z, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y) \in L^1(\mathbb{R}^3; \mathbb{R}^3).$$

In particular, the distribution  $\text{div } D(u)$  is well-defined in  $\mathbb{R}^3$ ; moreover, one can show there exist sequences of points  $(r_i), (q_i) \subset \mathbb{R}^3$  such that (see [27])

$$\begin{aligned} \sum_{i=1}^{\infty} d(r_i, q_i) &< \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla u|^2, \\ \text{div } D(u) &= 4\pi \sum_{i=1}^{\infty} (\delta_{r_i} - \delta_{q_i}) \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \end{aligned}$$

Let  $p_1, \dots, p_k, n_1, \dots, n_k$  be finitely many points (not necessarily distinct) in  $X$ . The length of the minimal connection between these points is given by

$$L := \min_{\sigma \in S_k} \sum_{i=1}^k d(p_i, n_{\sigma(i)}), \quad (7.3)$$

where  $S_k$  denotes the group of permutations of  $\{1, \dots, k\}$ . It has been shown by Brezis, Coron, and Lieb [28] that the number  $L$  satisfies

$$L = \sup_{|\zeta|_{\text{Lip}} \leq 1} \sum_{i=1}^k [\zeta(p_i) - \zeta(n_i)], \quad (7.4)$$

where  $|\zeta|_{\text{Lip}}$  denotes the best Lipschitz constant of  $\zeta$  (see Brezis [25] for an elementary proof; a third proof of the same result has been recently found by Sandier [76]). It is easy to see that the supremum in (7.4) is actually achieved.

More generally, consider two sequences  $(p_i), (n_i) \subset X$  such that

$$\sum_i d(p_i, n_i) < \infty. \quad (7.5)$$

(By abuse of notation, we allow sequences indexed on a finite subset of  $\mathbb{N}$ , which includes the previous case.)

Motivated by (7.4), we define the length of the minimal connection between these points as

$$\|T\| := \sup_{\substack{\zeta \in \text{Lip}(X) \\ |\zeta|_{\text{Lip}} \leq 1}} \langle T, \zeta \rangle = \sup_{|\zeta|_{\text{Lip}} \leq 1} \sum_i [\zeta(p_i) - \zeta(n_i)], \quad (7.6)$$

where  $T$  is the linear functional given by (7.1). We point out that the supremum is still achieved in this case; see Proposition 7.2. In Section 7.2, we compare this number with some alternative definitions.

Let

$$\mathcal{Z} := \left\{ T \in [\text{Lip}(X)]^* \mid \begin{array}{l} T \text{ can be written in the form (7.1) for some} \\ (p_i), (n_i) \subset X \text{ such that } \sum_i d(p_i, n_i) < \infty \end{array} \right\}.$$

Note that if  $T \in \mathcal{Z}$  then  $-T \in \mathcal{Z}$ , and  $T_1 + T_2 \in \mathcal{Z}$  whenever  $T_1, T_2 \in \mathcal{Z}$ . As we shall see in Section 7.12,  $\mathcal{Z}$  is a complete metric space with respect to the distance induced by  $\|\cdot\|$ .

We also introduce the notion of support of  $T$  :

**Definition 7.1** *Let  $(\omega_i)_{i \in I}$  be the family of all open subsets of  $X$  such that, for each  $i \in I$ , the following holds : if  $\zeta \in [\text{Lip}(X)]^*$  and  $\zeta \equiv 0$  on  $X \setminus \omega_i$ , then  $\langle T, \zeta \rangle = 0$ . We set  $\text{supp } T := X \setminus \bigcup_{i \in I} \omega_i$ .*

Clearly,  $\text{supp } T \subset \overline{\bigcup_i \{p_i\} \cup \bigcup_i \{n_i\}}$ , although the strictly inequality can actually occur; see, however, Theorem 7.4 below. As we have already mentioned in the previous chapter, there are several possible representations of  $T$  as a sum of the form (7.1). Moreover, such representations need not be equivalent modulo a permutation of points. In fact, if  $(q_i)$  is a sequence rapidly converging to  $p$  in  $X$  (in the sense that  $\sum_i d(q_i, q_{i+1}) < \infty$ ), then we can write  $\delta_p - \delta_n = \sum_{i=1}^{\infty} (\delta_{q_{i+1}} - \delta_{q_i})$  in  $[\text{Lip}(X)]^*$ , where  $n := q_1$ .

The next proposition is the counterpart of (7.3) in the general setting (see [22, Lemma 12'], and also Proposition 7.2 below)

**Proposition 7.1** *For any  $T \in \mathcal{Z}$ , we have*

$$\|T\| = \inf_{\substack{(\tilde{p}_i) \\ (\tilde{n}_i)}} \left\{ \sum_i d(\tilde{p}_i, \tilde{n}_i) : T = \sum_i (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i}) \text{ in } [\text{Lip}(X)]^* \right\}. \quad (7.7)$$

In contrast with the case of a finite number of points, the infimum above need not be achieved in general; see Example 7.1 below. Here is a case where it is still attained :

**Theorem 7.1** *If  $\mathcal{H}^1(\text{supp } T) = 0$ , then the infimum in (7.7) is attained. In other words, there exist  $(\tilde{p}_i), (\tilde{n}_i)$  in  $X$  such that*

$$\|T\| = \sum_i d(\tilde{p}_i, \tilde{n}_i) \quad \text{and} \quad T = \sum_i (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i}) \text{ in } [\text{Lip}(X)]^*.$$



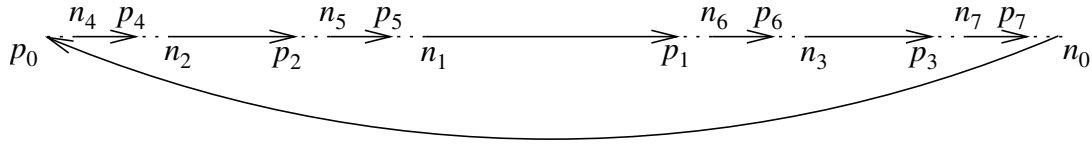


FIG. 7.1 – Dipoles  $\delta_{p_i} - \delta_{n_i}$  in Example 7.1

Above,  $\mathcal{H}^1$  denotes the 1-dimensional Hausdorff measure. In particular, if we assume the set  $\overline{\bigcup_i \{p_i\} \cup \bigcup_i \{n_i\}}$  is countable, then Theorem 7.1 holds.

In any case, it is always possible to decompose  $T$  in terms of simpler functionals, taking into account the length of its minimal connection. But let us first introduce a definition :

**Definition 7.2**  $T \in \mathcal{Z}$  is said to be regular in  $X$  if there exist  $(\tilde{p}_i), (\tilde{n}_i) \subset X$  such that

$$\|T\| = \sum_i d(\tilde{p}_i, \tilde{n}_i) \quad \text{and} \quad T = \sum_i (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i}) \quad \text{in } [\text{Lip}(X)]^*.$$

$T \in \mathcal{Z}$  is singular in  $X$  if, whenever  $T = T_1 + T_2$ ,  $\|T\| = \|T_1\| + \|T_2\|$ , and  $T_1$  is regular in  $X$ , then  $T_1 = 0$ .

Here is an example of  $T \in \mathcal{Z}$  which is singular :

**Example 7.1** Let  $X = [0, 1]$  and  $C_\alpha \subset [0, 1]$  be a Cantor-type set with Lebesgue measure  $\alpha > 0$ . We denote by  $(J_k)_{k \geq 1}$ ,  $J_k = (n_k, p_k)$ , the sequence of disjoint open intervals which are removed from  $[0, 1]$  in the construction of  $C_\alpha$ . We then take  $p_0 = 0$  and  $n_0 = 1$ . In Section 7.6, we show that  $T = \sum_{i \geq 0} (\delta_{p_i} - \delta_{n_i})$  is singular and  $\|T\| = \alpha$ . For descriptive purposes we can think of representing each dipole  $\delta_{p_i} - \delta_{n_i}$  as an arrow pointing from  $n_i$  to  $p_i$ . In Figure 7.1, we represent  $T$  geometrically according to this convention.

We have the following

**Theorem 7.2** For any  $T \in \mathcal{Z}$  there exist  $T_{\text{reg}}, T_{\text{sing}} \in \mathcal{Z}$  such that  $T_{\text{reg}}$  is regular,  $T_{\text{sing}}$  is singular,

$$T = T_{\text{reg}} + T_{\text{sing}}, \quad \text{and} \quad \|T\| = \|T_{\text{reg}}\| + \|T_{\text{sing}}\|. \tag{7.8}$$

Moreover, there exists  $(T_j) \subset \mathcal{Z}$  such that

$$T_{\text{sing}} = \sum_j T_j, \quad \|T_{\text{sing}}\| = \sum_j \|T_j\|, \quad \text{and} \quad \|T_j\| = \mathcal{H}^1(\text{supp } T_j), \quad \forall j. \tag{7.9}$$

In addition, each set  $\text{supp } T_j$  is homeomorphic to the Cantor set in  $\mathbb{R}$ .

The decomposition of  $T$  in terms of a regular and a singular part, as in (7.8), need not be unique; see Example 7.9.

We point out that Theorem 7.1 is a special case of the above. In fact, it follows from the proof of Theorem 7.2 that  $T_{\text{reg}}, T_{\text{sing}}$ , and  $(T_j)$  can be chosen so that

$$\text{supp } T = \text{supp } T_{\text{reg}} \cup \text{supp } T_{\text{sing}} \quad \text{and} \quad \bigcup_j \text{supp } T_j \subset \text{supp } T_{\text{sing}}.$$

Therefore, if  $\mathcal{H}^1(\text{supp } T) = 0$ , then  $\|T_j\| = \mathcal{H}^1(\text{supp } T_j) = 0$  for each  $j$ . We conclude that  $T_{\text{sing}} = \sum_j T_j = 0$  in  $[\text{Lip}(X)]^*$ , and so  $T = T_{\text{reg}}$  is regular in  $X$ .

A natural question regarding  $T \in \mathcal{Z}$  is whether it has a “simplest” representation in the following sense :

**Definition 7.3** *The representation  $\sum_i (\delta_{p_i} - \delta_{n_i})$  is reducible if there exist  $\mathbb{N}_1 \subset \mathbb{N}_2 \subset \mathbb{N}$ , with  $\text{card } \mathbb{N}_1 < \text{card } \mathbb{N}_2$ , and points  $r_i, q_i \in X$ ,  $i \in \mathbb{N}_1$ , such that*

$$\sum_{i \in \mathbb{N}_2} (\delta_{p_i} - \delta_{n_i}) = \sum_{i \in \mathbb{N}_1} (\delta_{r_i} - \delta_{q_i}) \quad \text{in } [\text{Lip}(X)]^*.$$

$\sum_i (\delta_{p_i} - \delta_{n_i})$  will be called irreducible if it is not reducible.

The next result states that one can always find an irreducible representation of  $T$  :

**Theorem 7.3** *Any linear functional  $T \in \mathcal{Z}$  has an irreducible representation. More precisely, there exist sequences  $(\hat{p}_i), (\hat{n}_i)$  in  $X$ , satisfying (7.5), such that*

$$T = \sum_i (\delta_{\hat{p}_i} - \delta_{\hat{n}_i}) \quad \text{in } [\text{Lip}(X)]^*, \quad (7.10)$$

and so that this representation is irreducible.

Our proof of Theorem 7.3 relies on the notion of maximal paths; see Section 7.5. This approach requires the following interesting lemma :

**Lemma 7.1** *If*

$$\sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) = (\delta_{r_1} - \delta_{q_1}) + (\delta_{r_2} - \delta_{q_2}) \quad \text{in } [\text{Lip}(X)]^*$$

for some  $r_1, q_1, r_2, q_2 \in X$ , then there exists  $\tilde{\mathbb{N}} \subset \mathbb{N}$  such that

$$\sum_{i \in \tilde{\mathbb{N}}} (\delta_{p_i} - \delta_{n_i}) \quad \text{equals} \quad (\delta_{r_1} - \delta_{q_1}) \quad \text{or} \quad (\delta_{r_1} - \delta_{q_2}) \quad \text{in } [\text{Lip}(X)]^*.$$

As a corollary of this lemma, we can now give a simpler characterization of irreducible representations (see also Proposition 7.8) :

**Corollary 7.1**  $\sum_i (\delta_{p_i} - \delta_{n_i})$  is reducible if, and only if, one of the following conditions holds :

- (a)  $p_i = n_j$  for some  $i, j \geq 1$ ;
- (b) there exists an infinite subset  $\tilde{\mathbb{N}} \subset \mathbb{N}$  such that

$$\sum_{i \in \tilde{\mathbb{N}}} (\delta_{p_i} - \delta_{n_i}) = \delta_r - \delta_q \quad \text{in } [\text{Lip}(X)]^*$$

for some  $r, q \in X$ .

If  $T$  can be written as a finite sum of dipoles of the form  $\delta_p - \delta_n$ , then the irreducible representation of  $T$  is unique (modulo a permutation of the points). This need not be the case in general. Assume, for example, that  $X = [0, 1]$ , and let  $(p_i), (n_i)$  be two sequences converging to 0, such that  $p_i > n_i > p_{i+1}$  for every  $i \geq 1$ . Then,

$$\begin{aligned} & \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}), \\ & (\delta_{p_1} - \delta_0) + \sum_{i=1}^{\infty} (\delta_{p_{i+1}} - \delta_{n_i}), \\ & (\delta_{p_1} - \delta_0) + (\delta_{p_2} - \delta_0) + \sum_{i=1}^{\infty} (\delta_{p_{i+2}} - \delta_{n_i}), \quad \dots \end{aligned}$$

are all irreducible representations of the same operator in  $[\text{Lip}[0, 1]]^*$ .

However, we have the following

**Theorem 7.4** *Assume (7.10) is an irreducible representation of  $T$ . Then,*

$$\text{supp } T = \overline{\bigcup_i \{\hat{p}_i\} \cup \bigcup_i \{\hat{n}_i\}}.$$

*In particular, if  $\zeta \in \text{Lip}(X)$  and  $\zeta = 0$  on  $\text{supp } T$ , then  $\langle T, \zeta \rangle = 0$ .*

A simple consequence of Theorem 7.4 is the corollary below :

**Corollary 7.2** *Let  $T \in \mathcal{Z}$ . If  $\text{supp } T$  is finite, then there exist finitely many  $\hat{p}_1, \dots, \hat{p}_{k_0}, \hat{n}_1, \dots, \hat{n}_{k_0} \in X$  such that*

$$T = \sum_{i=1}^{k_0} (\delta_{\hat{p}_i} - \delta_{\hat{n}_i}) \quad \text{in } [\text{Lip}(X)]^*. \quad (7.11)$$

Another result in this direction is the theorem below which completely solves an open problem raised by H. Brezis. We denote by  $\text{BLip}(X)$  the subspace of bounded Lipschitz functions :

**Theorem 7.5** *Let  $T \in \mathcal{Z}$ . Assume that*

$$|\langle T, \zeta \rangle| \leq C \|\zeta\|_{L^\infty}, \quad \forall \zeta \in \text{BLip}(X) \quad (7.12)$$

*for some  $C > 0$ . Then, there exist points  $a_1, \dots, a_k$  and integers  $d_1, \dots, d_k, \sum_i d_i = 0$ , such that*

$$T = \sum_{i=1}^k d_i \delta_{a_i} \quad \text{in } [\text{Lip}(X)]^*.$$

We point out that (7.11) is equivalent to saying that (7.12) holds (since  $\sum_i d_i = 0$ ). Theorem 7.5 has been proved by Smets [80] (using the Riesz Representation Theorem) under the additional assumption that  $X$  is locally compact. Our proof, instead, makes use of the existence of irreducible representations of  $T$ , which only requires  $X$  to be complete. Very simple examples show that Theorem 7.5 is no longer true without this assumption on  $X$ .

Ambrosio and Kirchheim [2, 3] have recently extended the theory of currents and rectifiable sets to more general complete metric spaces. Since our functionals  $T$  can be seen as a current in their setting, a natural question is whether some of our theorems could be deduced from their results. Unfortunately, we have not been able to establish the relation between our approach and their formalism.

We conclude this section by explaining the notion of indecomposable functionals taken from Federer [48]. Given  $T \in \mathcal{Z}$ , we define

$$m(T) := \sup_{\|\zeta\|_{L^\infty}=1} \langle T, \zeta \rangle, \quad \forall \zeta \in \text{BLip}(X).$$

Let

$$\mathcal{I} := \{T \in \mathcal{Z} : m(T) < \infty\}.$$

It follows from Theorem 7.5 that  $T \in \mathcal{I}$  if, and only if,  $T$  can be written in terms of finitely many dipoles. In fact, we have  $m(T) = 2k_0$ , where  $k_0 \geq 0$  is the smallest integer such that (7.11) holds. Moreover,

$$m(T_1 + T_2) \leq m(T_1) + m(T_2), \quad \forall T_1, T_2 \in \mathcal{I}.$$

We now consider  $\mathcal{I}$  equipped with the norm

$$N(T) := \|T\| + m(T), \quad \forall T \in \mathcal{I}.$$

As in Federer [48, §4.2.25], we say that  $T \in \mathcal{I}$  is *indecomposable* if there exists no  $S \in \mathcal{I}$  with

$$S \neq 0 \neq T - S \quad \text{and} \quad N(T) = N(S) + N(T - S).$$

It is then easy to see that  $T \in \mathcal{I}$  is indecomposable if, and only if, there exist  $r, q \in X$  such that  $T = (\delta_r - \delta_q)$  in  $[\text{Lip}(X)]^*$ . Thus, every element in  $\mathcal{I}$  can be written as a finite sum of indecomposable parts, which is none other than a connection of  $T$ . Note, however, that this notion is restricted to the subspace  $\mathcal{I} \subsetneq \mathcal{Z}$ .

## 7.2 Alternative definitions of minimal connections

Throughout this chapter, we shall always assume that the sequences  $(p_i)$  and  $(n_i)$  in  $X$  satisfy  $\sum_i d(p_i, n_i) < \infty$ .

Let  $T := \sum_i (\delta_{p_i} - \delta_{n_i})$  in  $[\text{Lip}(X)]^*$ . There are several alternative ways of defining the length of the minimal connection between  $(p_i)$  and  $(n_i)$  :

**Definition 7.4**

$$L_1 := \inf_{\substack{\sigma: \mathbb{N} \rightarrow \mathbb{N} \\ \text{bijection}}} \sum_{i=1}^{\infty} d(p_i, n_{\sigma(i)}). \quad (7.13)$$

**Definition 7.5**

$$L_2 := \lim_{k \rightarrow \infty} \min_{\sigma \in S_k} \sum_{i=1}^k d(p_i, n_{\sigma(i)}).$$

**Definition 7.6**

$$L_3 := \inf_{(\tilde{n}_i)} \left\{ \sum_{i=1}^{\infty} d(p_i, \tilde{n}_i) : T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{\tilde{n}_i}) \text{ in } [\text{Lip}(X)]^* \right\}.$$

**Definition 7.7**

$$L_4 := \inf_{\substack{(\tilde{p}_i) \\ (\tilde{n}_i)}} \left\{ \sum_{i=1}^{\infty} d(\tilde{p}_i, \tilde{n}_i) : T = \sum_{i=1}^{\infty} (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i}) \text{ in } [\text{Lip}(X)]^* \right\}.$$

Clearly, we have

$$L_1 \leq L_2 \quad \text{and} \quad \|T\| \leq L_4 \leq L_3. \quad (7.14)$$

Using (7.4) and  $\sum_i d(p_i, n_i) < \infty$ , we can actually prove the following (see also [22])

**Proposition 7.2**

$$L_1 \leq L_2 = L_3 = L_4 = \|T\|. \quad (7.15)$$

Moreover, the supremum in (7.6) is achieved.

**Proof.**

**Step 1.**  $L_3 \leq L_2$ .

Given  $k \geq 1$  and  $\sigma \in S_k$ , we extend  $\sigma$  to  $\mathbb{N}$  so that  $\sigma(i) = i$  for every  $i > k$ . In particular,  $T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_{\sigma(i)}})$  in  $[\text{Lip}(X)]^*$ . By definition, we have

$$L_3 \leq \sum_{i=1}^{\infty} d(p_i, n_{\sigma(i)}) = \sum_{i=1}^k d(p_i, n_{\sigma(i)}) + \sum_{i>k} d(p_i, n_i).$$

Since  $\sigma \in S_k$  is arbitrary, we conclude that

$$L_3 \leq \min_{\sigma \in S_k} \sum_{i=1}^k d(p_i, n_{\sigma(i)}) + \sum_{i>k} d(p_i, n_i).$$

Letting  $k \rightarrow \infty$ , we get  $L_3 \leq L_2$ .

**Step 2.**  $L_2 \leq \|T\|$ .

Given  $\varepsilon > 0$ , we fix  $k \geq 1$  large enough so that  $\sum_{i>k} d(p_i, n_i) < \varepsilon$ . Let  $\sigma \in S_k$  and  $\zeta \in \text{Lip}(X)$ ,  $|\zeta|_{\text{Lip}} \leq 1$ , be such that

$$\sum_{i=1}^k d(p_i, n_{\sigma(i)}) = \sum_{i=1}^k [\zeta(p_i) - \zeta(n_i)].$$

Thus,

$$L_2 - \varepsilon \leq \sum_{i=1}^k d(p_i, n_{\sigma(i)}) \leq \sum_{i=1}^{\infty} [\zeta(p_i) - \zeta(n_i)] + \varepsilon \leq \|T\| + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we must have  $L_2 \leq \|T\|$ .

In view of (7.14), (7.15) follows the two previous steps.

**Step 3.** The supremum in (7.6) is attained.

For each  $k \geq 1$ , let  $\zeta_k \in \text{Lip}(X)$ ,  $|\zeta_k|_{\text{Lip}} \leq 1$ , be such that

$$\sum_{i=1}^k [\zeta_k(p_i) - \zeta_k(n_i)] = \min_{\sigma \in S_k} \sum_{i=1}^k d(p_i, n_{\sigma(i)}).$$

For the sake of normalization, we may assume that  $\zeta_k(x_0) = 0$  for some fixed  $x_0 \in X$ . In particular, for each  $i \geq 1$  the sequences  $(\zeta_k(x_i))_k$ , where  $x_i = p_i$  or  $n_i$ , are bounded. Passing to a subsequence if necessary, we may assume that all the limits

$$\tilde{\zeta}(x_i) := \lim_{k \rightarrow \infty} \zeta_k(x_i), \quad x_i = p_i, n_i,$$

exist. It is easy to see that  $\tilde{\zeta}$ , defined on  $A := \bigcup_i \{p_i\} \cup \bigcup_i \{n_i\}$ , satisfies

$$L = \sum_i [\tilde{\zeta}(p_i) - \tilde{\zeta}(n_i)]$$

(we use here that  $\sum_i d(p_i, n_i) < \infty$ ).

On the other hand, since  $|\tilde{\zeta}|_{\text{Lip}(A)} \leq 1$ , we can extend  $\tilde{\zeta}$  to  $X$  without increasing its Lipschitz constant (take for instance  $\zeta(x) := \inf_{a \in A} \{\tilde{\zeta}(a) + d(x, a)\}$ ). We conclude the supremum in (7.6) is achieved.

**Remark 7.1** The strict inequality  $L_1 < \|T\|$  may actually occur in (7.15). In fact, take  $(a_i)_{i \in \mathbb{Z}}$  such that  $\sum_i d(a_i, a_{i+1}) < \infty$ . In particular, both limits

$$r := \lim_{i \rightarrow +\infty} a_i \quad \text{and} \quad q := \lim_{i \rightarrow -\infty} a_i$$

exist (since  $X$  is complete). Thus,

$$T = \sum_{i=-\infty}^{+\infty} (\delta_{a_{i+1}} - \delta_{a_i}) = \delta_r - \delta_q \quad \text{in } [\text{Lip}(X)]^*.$$

Note that  $\|T\| = d(r, q)$ , but  $L_1 = 0$ .

**Remark 7.2** The infimum in (7.13) need not be achieved in general. Consider the sequence of points  $(p_i)_{i \geq 1}$  and  $(n_i)_{i \geq 1}$  given in Example 7.1. We claim that  $L_1 = 0$ , even though  $p_i \neq n_j, \forall i, j$ . In fact, given  $\varepsilon > 0$  we can find  $i_1, j_1 \in \mathbb{N}$ ,  $i_1, j_1 \neq 1$ , such that  $|p_1 - n_{i_1}| + |n_1 - p_{j_1}| < \frac{\varepsilon}{2}$ . We set  $\sigma(1) := i_1$  and  $\sigma(j_1) := 1$ . Proceeding by induction, at each step  $k > 1$  we can extend this bijection  $\sigma$  so that

$$\sigma : \{1, \dots, k\} \cup \{j_1, \dots, j_k\} \longrightarrow \{i_1, \dots, i_k\} \cup \{1, \dots, k\}$$

satisfies

$$\sum_{l=1}^k |p_l - n_{\sigma(l)}| < \varepsilon, \quad \forall k \geq 1.$$

At the end, we conclude that  $L_1 \leq \varepsilon$ . Since  $\varepsilon > 0$  was arbitrary, the claim follows.

### 7.3 Cycles

As we have already mentioned in Example 7.1, we can think of identifying each dipole  $\delta_{p_i} - \delta_{n_i}$  with an arrow pointing from  $n_i$  to  $p_i$ . In order to make a clear distinction between all the dipoles, we shall usually indicate each  $\delta_{p_i} - \delta_{n_i}$  by its index  $i$ . This way we will be able to distinguish equal dipoles arising from different indices.

Our strategy to deal with the linear functional  $T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i})$  will be to equip the set of arrows  $i$  with a suitable order relation. The motivation of this approach comes from elementary concepts in Geometry, as it will soon become clear.

We start with the following :

**Definition 7.8** A chain  $(\Lambda, \leq)$  is a set of indices  $\Lambda \subset \mathbb{N}$  equipped with a partial order relation  $\leq$ .

In general, we shall call  $\Lambda$  itself a chain,  $\leq$  being implicitly understood. The order  $\leq$  induces an orientation in the set of dipoles  $(\delta_{p_i} - \delta_{n_i})_{i \in \Lambda}$ .

We shall usually be interested in the order relation  $\leq$  modulo cyclic permutations of the elements in  $\Lambda$ . In order to make this precise, we start with an auxiliary notion :

**Definition 7.9** A subchain  $\Lambda_1 \subset \Lambda$  (equipped with the order relation induced from  $\Lambda$ ) is called a segment if whenever  $\lambda_1 \leq \lambda \leq \lambda_2$  in  $\Lambda$  and  $\lambda_1, \lambda_2 \in \Lambda_1$ , then  $\lambda \in \Lambda_1$ .

We now introduce the notion of a cycle :

**Definition 7.10** Given two chains  $\Lambda, \tilde{\Lambda}$ , we write  $\Lambda \sim \tilde{\Lambda}$  if

- (i)  $\Lambda = \tilde{\Lambda}$  (as sets);
- (ii) there exist two disjoint segments  $\Lambda_1, \Lambda_2 \subset \Lambda$  such that  $\Lambda = \Lambda_1 \cup \Lambda_2$  and the inclusions  $\Lambda_1, \Lambda_2 \subset \tilde{\Lambda}$  are order preserving.

It is easy to see that  $\sim$  defines an equivalence relation in the class of all chains. The equivalence class  $[\Lambda]$  of  $\Lambda$  induced by  $\sim$  will be called a cycle.

Assume  $\Lambda$  is the finite chain containing  $\lambda_1 \leq \dots \leq \lambda_k$ , which we denote as  $(\lambda_1 \dots \lambda_k)$ . In this special case,  $[\Lambda]$  will be the union of all cyclic permutations of  $\Lambda$ , namely

$$[\Lambda] = \left\{ (\lambda_1 \dots \lambda_k), (\lambda_2 \dots \lambda_k \lambda_1), \dots, (\lambda_k \lambda_1 \dots \lambda_{k-1}) \right\}.$$

Since any representative of  $[\Lambda]$  ( $\Lambda$  now being finite or infinite) contains the same set of indices, we can actually think of  $[\Lambda]$  as being the set of indices  $i \in \Lambda$  itself. Moreover,  $[\Lambda]$  has a well-defined orientation, induced by the order of any of its representatives  $\tilde{\Lambda} \in [\Lambda]$ .

We now define

$$\begin{aligned} T_{[\Lambda]} &:= \sum_{\lambda \in \Lambda} (\delta_{p_\lambda} - \delta_{n_\lambda}) \quad \text{in } [\text{Lip}(X)]^*, \\ \ell_{[\Lambda]} &:= \sum_{\lambda \in \Lambda} d(p_\lambda, n_\lambda), \\ L_{[\Lambda]} &:= \|T_{[\Lambda]}\|. \end{aligned}$$

We call  $\ell_{[\Lambda]}$  the *length* of  $[\Lambda]$ .

Given  $\varepsilon > 0$ , an  $\varepsilon$ -chain  $\Lambda_\varepsilon = (\lambda_1 \cdots \lambda_k)$  is a finite subchain of  $\Lambda$  such that if  $i \in \Lambda$  and  $i \in \{1, \dots, \lfloor \frac{1}{\varepsilon} \rfloor\}$ , then  $i \in \Lambda_\varepsilon$ . Note that if  $\Lambda$  is infinite, then it has an infinite number of  $\varepsilon$ -chains (for an  $\varepsilon > 0$  fixed), since one can always add to  $\Lambda_\varepsilon$  indices in  $\Lambda$  outside  $\{1, \dots, \lfloor \frac{1}{\varepsilon} \rfloor\}$ .

The *co-length* of  $[\Lambda_\varepsilon]$  is the number

$$\ell_{[\Lambda_\varepsilon]}^* := d(p_{\lambda_1}, n_{\lambda_2}) + \cdots + d(p_{\lambda_{k-1}}, n_{\lambda_k}) + d(p_{\lambda_k}, n_{\lambda_1}).$$

It measures the total jump from one dipole to the next one as we travel along  $[\Lambda_\varepsilon]$ .

**Lemma 7.2** *If  $\Lambda_{\varepsilon_1} \subset \Lambda_{\varepsilon_2}$ , then*

$$\ell_{[\Lambda_{\varepsilon_1}]} + \ell_{[\Lambda_{\varepsilon_1}]}^* \leq \ell_{[\Lambda_{\varepsilon_2}]} + \ell_{[\Lambda_{\varepsilon_2}]}^*. \quad (7.16)$$

**Proof.** It suffices to check (7.16) when  $\Lambda_{\varepsilon_2}$  differs from  $\Lambda_{\varepsilon_1}$  by exactly one index and then argue by induction. In order to add an index  $i_2$  between  $i_1$  and  $i_3$ , we just need to apply the triangle inequality to get

$$d(p_{i_1}, n_{i_3}) \leq d(p_{i_1}, n_{i_2}) + d(p_{i_2}, n_{i_2}) + d(p_{i_2}, n_{i_3}).$$

Notice that the second term in the right-hand side enters in the definition of the length  $\ell_{[\Lambda_{\varepsilon_2}]}$ , while the other two appear in the definition of the co-length  $\ell_{[\Lambda_{\varepsilon_2}]}^*$ . This proves the lemma.

A simple consequence of (7.16) is the equality below :

**Proposition 7.3**

$$\ell_{[\Lambda]}^* := \lim_{\varepsilon \downarrow 0} \left( \inf_{\Lambda_\varepsilon} \ell_{[\Lambda_\varepsilon]}^* \right) = \lim_{\varepsilon \downarrow 0} \left( \sup_{\Lambda_\varepsilon} \ell_{[\Lambda_\varepsilon]}^* \right), \quad (7.17)$$

where both the infimum and the supremum are taken over the class of all  $\varepsilon$ -chains of  $\Lambda$ . We define the common number  $\ell_{[\Lambda]}^*$  in (7.17) to be the co-length of  $[\Lambda]$ .

**Proof.** We denote by  $\ell^*$  the limit in the right-hand side of (7.17) (note that it is well-defined, but may be infinite). Given  $m < \ell^*$ , let  $\tilde{\Lambda}_\varepsilon$  be an  $\varepsilon$ -chain of  $\Lambda$  such that  $m < \ell_{[\tilde{\Lambda}_\varepsilon]}^*$ . We now take a sequence of  $\varepsilon_j$ -chains  $\Lambda_{\varepsilon_j}$ , where  $\varepsilon_j \downarrow 0$ , such that

$$\lim_{j \rightarrow \infty} \ell_{[\Lambda_{\varepsilon_j}]}^* = \lim_{\varepsilon \downarrow 0} \left( \inf_{\Lambda_\varepsilon} \ell_{[\Lambda_\varepsilon]}^* \right).$$

Since  $\tilde{\Lambda}_\varepsilon$  is finite, there exists  $j_0 \geq 1$  sufficiently large so that  $\Lambda_{\varepsilon_j} \supset \tilde{\Lambda}_\varepsilon$  for every  $j \geq j_0$ .



Applying (7.16) we get

$$m < \ell_{[\tilde{\Lambda}_\varepsilon]}^* \leq \ell_{[\Lambda_{\varepsilon_j}]}^* + (\ell_{[\Lambda_{\varepsilon_j}]} - \ell_{[\tilde{\Lambda}_\varepsilon]}), \quad \forall j \geq j_0.$$

Taking  $j \rightarrow \infty$  and then  $\varepsilon \downarrow 0$ , we conclude that

$$m \leq \lim_{\varepsilon \downarrow 0} \left( \inf_{\Lambda_\varepsilon} \ell_{[\Lambda_\varepsilon]}^* \right),$$

from which (7.17) follows.

Combining Lemma 7.2 and Proposition 7.3, we get

**Corollary 7.3** *Given a chain  $\Lambda$ , for any subchain  $\tilde{\Lambda} \subset \Lambda$  we have*

$$\ell_{[\tilde{\Lambda}]} + \ell_{[\tilde{\Lambda}]}^* \leq \ell_{[\Lambda]} + \ell_{[\Lambda]}^*.$$

**Corollary 7.4** *Assume  $\Lambda$  is a chain. If  $\Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda$  is an increasing sequence of subchains such that  $\Lambda = \bigcup_k \Lambda_k$ , then*

$$\ell_{[\Lambda]}^* = \lim_{k \rightarrow \infty} \ell_{[\Lambda_k]}^*.$$

Note that, for every  $\Lambda_\varepsilon$ , we have

$$L_{[\Lambda_\varepsilon]} \leq \min \{ \ell_{[\Lambda_\varepsilon]}, \ell_{[\Lambda_\varepsilon]}^* \},$$

since both  $\ell_{[\Lambda_\varepsilon]}$  and  $\ell_{[\Lambda_\varepsilon]}^*$  correspond to special choices of permutations in (7.3).

Taking  $\varepsilon \downarrow 0$ , we conclude that

$$L_{[\Lambda]} \leq \min \{ \ell_{[\Lambda]}, \ell_{[\Lambda]}^* \}.$$

There are three cases of interest when the equality holds :

**Definition 7.11** *Assume  $[\Lambda]$  is a cycle.*

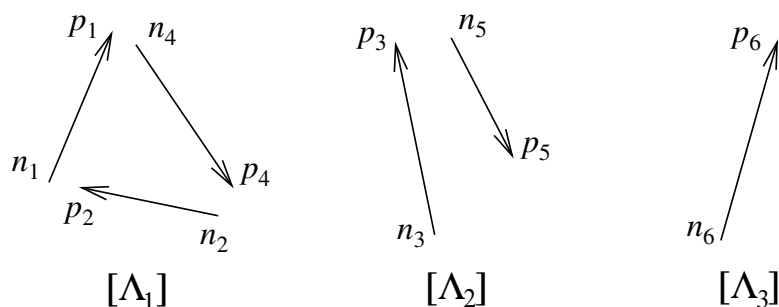
- (a)  $[\Lambda]$  is a minimal cycle if  $L_{[\Lambda]} = \ell_{[\Lambda]}$  ;
- (b)  $[\Lambda]$  is a co-minimal cycle if  $L_{[\Lambda]} = \ell_{[\Lambda]}^*$  ;
- (c)  $[\Lambda]$  is a loop if  $\ell_{[\Lambda]}^* = 0$  (this is a special case of (b)) ; in particular,

$$T_{[\Lambda]} = 0 \quad \text{in } [\text{Lip}(X)]^*. \quad (7.18)$$

Here are some examples :

**Example 7.2** Assume  $\Lambda = (1 \ 2 \ \dots \ k)$  ; that is, consider the dipoles  $\delta_{p_1} - \delta_{n_1}, \dots, \delta_{p_k} - \delta_{n_k}$ , oriented in this order. We have :

- (i) If  $L_{[\Lambda]} = \ell_{[\Lambda]}$ , then the pairs  $[p_1, n_1], \dots, [p_k, n_k]$  form a minimal connection.
- (ii) If  $L_{[\Lambda]} = \ell_{[\Lambda]}^*$ , then a minimal connection is given by  $[p_1, n_2], \dots, [p_{k-1}, n_k], [p_k, n_1]$ .

FIG. 7.2 – Decomposition of  $[\Lambda]$  in terms of three co-minimal cycles

(iii) More generally, let  $\sigma \in S_k$  be a permutation which minimizes (7.3). Recall that  $\sigma$  can be written as a composition of disjoint cycles (in the algebraic sense), say  $\sigma_1, \dots, \sigma_j$ . Note, however, that each  $\sigma_l$  induces in a natural way a cycle  $[\Lambda_l]$  (in the sense of Definition 7.10). For instance, if

$$\sigma_1 : 1 \mapsto i_1 \mapsto \dots \mapsto i_\alpha \mapsto 1,$$

then  $\Lambda_1 = (1 \ i_1 \ \dots \ i_\alpha)$ . This way, we can write  $\{1, \dots, k\} = \Lambda_1 \cup \dots \cup \Lambda_j$  so that

$$L_{[\Lambda]} = \sum_{i=1}^k d(p_i, n_{\sigma(i)}) = \sum_{l=1}^j \sum_{i \in \Lambda_l} d(p_i, n_{\sigma(i)}) = \sum_{l=1}^j \ell_{[\Lambda_l]}^*. \quad (7.19)$$

Figure 7.2 shows such a decomposition with  $k = 6$ ,  $\Lambda_1 = (1 \ 4 \ 2)$ ,  $\Lambda_2 = (3 \ 5)$  and  $\Lambda_3 = (6)$ . Proposition 7.4 extends this construction to the case of an infinite number of points.

**Example 7.3** Let  $X = [0, 1]$  and  $p_i, n_i \in [0, 1]$  be as in Example 7.1. We consider  $\Lambda_0 = \mathbb{N} \cup \{0\}$  oriented clockwise with respect to Figure 7.1. Using the equality  $L_2 = \|T\|$  in Proposition 7.2, it is easy to see that

$$L_{[\Lambda_0]} = \alpha = \ell_{[\Lambda_0]}^*,$$

where  $\alpha$  is the Lebesgue measure of  $C_\alpha$ . In other words,  $[\Lambda_0]$  is a co-minimal cycle. Note that if we consider the cycle  $[\Lambda_0]_{\text{anti}}$  oriented in the opposite direction (i.e. counter-clockwise with respect to Figure 7.1), then

$$\ell_{[\Lambda_0]_{\text{anti}}}^* = \ell_{[\Lambda_0]} + \ell_{[\Lambda_0]}^* = 2.$$

The proposition below extends (7.19) to the case of infinitely many points :

**Proposition 7.4** *Let*

$$T := \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{in } [\text{Lip}(X)]^*.$$

There exists a sequence of disjoint co-minimal cycles  $[\Lambda_j]$  such that  $\mathbb{N} = \bigcup_j \Lambda_j$  and

$$\|T\| = \sum_j \ell_{[\Lambda_j]}^*. \quad (7.20)$$

**Proof.** For each  $k \geq 1$ , let  $\sigma \in S_k$  be such that

$$\sum_{i=1}^k d(p_i, n_{\sigma(i)}) = \min_{\bar{\sigma} \in S_k} \sum_{i=1}^k d(p_i, n_{\bar{\sigma}(i)}). \quad (7.21)$$

It follows from Example 7.2 (iii), that we can write  $\{1, \dots, k\} = \Lambda_1 \cup \dots \cup \Lambda_j$  in terms of disjoint chains (this decomposition actually depends on  $k$ ) so that (7.19) holds. For  $i > k$ , we let  $\Lambda_i = (i)$ .

We now relabel  $\Lambda_1, \dots, \Lambda_j, \Lambda_{k+1}, \dots$  as

$$\Lambda_{1,k}, \Lambda_{2,k}, \Lambda_{3,k}, \dots,$$

in such a way that the smallest integer in  $\Lambda_{j_1,k}$  is less than the smallest integer in  $\Lambda_{j_2,k}$  whenever  $j_1 < j_2$ .

By construction,  $1 \in \Lambda_{1,k}$  for every  $k \geq 1$ .

Let  $\alpha_k$  be the smallest integer in  $\Lambda_{1,k}$  greater than 1. If  $\alpha_k \rightarrow \infty$  as  $k \rightarrow \infty$ , then we set  $\Lambda_1 := (1)$ . Otherwise,  $(\alpha_k)$  has a convergent subsequence  $\alpha_{k_l} \rightarrow a_1$ ; since  $\alpha_{k_l}$  is an integer, we actually have  $\alpha_{k_l} = a_1$  for all  $l$  sufficiently large.

Let  $\beta_l$  be the smallest integer in  $\Lambda_{1,k_l}$  greater than  $a_1$ . If  $\beta_l \rightarrow \infty$ , then we set  $\Lambda_1 := (1 \ a_1)$ . Otherwise, passing to a further subsequence if necessary, we may assume that  $\beta_l = b_1$ , for all  $l$  large enough; moreover, we can also assume that one of the following inclusions is order preserving :

$$(1 \ a_1 \ b_1) \subset \Lambda_{1,k_l}, \quad \forall l \text{ large}, \quad \text{or} \quad (1 \ b_1 \ a_1) \subset \Lambda_{1,k_l}, \quad \forall l \text{ large}.$$

Using a standard diagonalization argument, we can construct a subsequence  $(k_l)$  (not necessarily the same as the one above) and a chain  $\Lambda_1$ , containing 1, such that the following holds :

- (a) given an  $\varepsilon$ -chain  $\Lambda_{1,\varepsilon} \subset \Lambda_1$ , we can find  $N = N(\Lambda_{1,\varepsilon}) \geq 1$  sufficiently large so that  $\Lambda_{1,\varepsilon} \subset \Lambda_{1,k_l}$  for every  $l \geq N$ , and this inclusion is order preserving.

We now repeat the same construction with  $\Lambda_{2,k_l}$  and so on (the only difference here is that we should start with the smallest integer in the set  $\mathbb{N} \setminus \Lambda_1$ , which necessarily belongs to  $\Lambda_{2,k_l}$  for  $l$  sufficiently large). This way we can construct disjoint chains  $\Lambda_2, \Lambda_3, \dots$  and a **universal** subsequence  $(k_l)$  (here we apply once again a diagonalization argument) so that

- (b)  $\mathbb{N} = \bigcup_{j \geq 1} \Lambda_j$ ;  
(c) property (a) holds for every  $\Lambda_j$ , after replacing  $\Lambda_1$  by  $\Lambda_j$ .

By (b), we have

$$T = \sum_j T_{[\Lambda_j]} \quad \text{in } [\text{Lip}(X)]^*. \quad (7.22)$$

Moreover, (c) implies that

$$\ell_{[\Lambda_j]} = \lim_{l \rightarrow \infty} \ell_{[\Lambda_j, k_l]}, \quad \forall j.$$

On the other hand, it follows from (c) and (7.16) that

$$\ell_{[\Lambda_j, \varepsilon]} + \ell_{[\Lambda_j, \varepsilon]}^* \leq \ell_{[\Lambda_j, k_l]} + \ell_{[\Lambda_j, k_l]}^*, \quad \forall l \geq N.$$

Thus,

$$\ell_{[\Lambda_j]}^* \leq \liminf_{l \rightarrow \infty} \ell_{[\Lambda_j, k_l]}^*. \quad (7.23)$$

We now rewrite (7.21) as

$$\sum_j \ell_{[\Lambda_j, k]}^* = \min_{\bar{\sigma} \in S_k} \sum_{i=1}^k d(p_i, n_{\bar{\sigma}(i)}) + \sum_{i>k} d(p_i, n_i)$$

Applying Proposition 7.2 we obtain

$$\lim_{l \rightarrow \infty} \sum_j \ell_{[\Lambda_j, k_l]}^* = \|T\|. \quad (7.24)$$

Combining (7.22), (7.23) and (7.24) we get

$$\|T\| \leq \sum_j \|T_{[\Lambda_j]}\| \leq \sum_j \ell_{[\Lambda_j]}^* \leq \sum_j \liminf_{l \rightarrow \infty} \ell_{[\Lambda_j, k_l]}^* \leq \lim_{l \rightarrow \infty} \sum_j \ell_{[\Lambda_j, k_l]}^* = \|T\|.$$

Therefore, we must have equality everywhere. In particular,

$$\|T\| = \sum_j \|T_{[\Lambda_j]}\| \quad \text{and} \quad \|T_{[\Lambda_j]}\| = \ell_{[\Lambda_j]}^*, \quad \forall j,$$

which is precisely (7.20).

We now present some properties of  $[\Lambda]$  when  $\ell_{[\Lambda]}^* < \infty$ . Let us first introduce some notation

**Definition 7.12** *Let  $\Lambda$  be a chain. Given a family of points  $(x_\lambda)_{\lambda \in \Lambda}$ , we say that the limit*

$$a := \lim_{\lambda \in \Lambda \uparrow} x_\lambda$$

*exists if, given  $\varepsilon > 0$ , there exists  $\lambda_0 \in \Lambda$  such that  $d(x_\lambda, a) < \varepsilon$ ,  $\forall \lambda \geq \lambda_0$ .*

*The limit  $a := \lim_{\lambda \in \Lambda \downarrow} x_\lambda$  is defined similarly, after replacing  $\lambda \geq \lambda_0$  by  $\lambda \leq \lambda_0$ .*

**Proposition 7.5** *If  $\ell_{[\Lambda]}^* < \infty$ , then the following limits exist*

$$\lim_{\lambda \in \Lambda \uparrow} p_\lambda, \quad \lim_{\lambda \in \Lambda \downarrow} p_\lambda, \quad \lim_{\lambda \in \Lambda \uparrow} n_\lambda, \quad \text{and} \quad \lim_{\lambda \in \Lambda \downarrow} n_\lambda.$$

**Proof.** It suffices to show the first limit exists (since all the others can be derived from this case). Moreover, because  $\Lambda$  is countable, we only need to show that for every increasing sequence  $(\lambda_j)_{j \geq 1}$  in  $\Lambda$ ,  $(p_{\lambda_j})$  converges.

We have

$$\sum_{j=1}^{\infty} d(p_{\lambda_j}, p_{\lambda_{j+1}}) \leq \sum_{j=1}^{\infty} \left\{ d(p_{\lambda_j}, n_{\lambda_j}) + d(n_{\lambda_j}, p_{\lambda_{j+1}}) \right\} \leq \ell_{[\Lambda]} + \ell_{[\Lambda]}^*.$$

Therefore,  $(p_{\lambda_j})$  is Cauchy, so it converges.

**Corollary 7.5** *If  $\ell_{[\Lambda]}^* < \infty$ , then  $\bigcup_{\lambda \in \Lambda} \{p_\lambda\}$  and  $\bigcup_{\lambda \in \Lambda} \{n_\lambda\}$  are relatively compact in  $X$ . In particular,  $\text{supp } T_{[\Lambda]}$  is compact.*

**Remark 7.3** Let  $\Lambda_1$  and  $\Lambda_2$  be two disjoint chains such that  $\ell_{[\Lambda_1]}^* + \ell_{[\Lambda_2]}^* < \infty$ . We take  $\Lambda := \Lambda_1 \cup \Lambda_2$  with the order induced from each  $\Lambda_i$  and such that  $\Lambda_1 \leq \Lambda_2$ . In view of Proposition 7.5 we can define

$$r_i := \lim_{\lambda \in \Lambda_i \uparrow} p_\lambda \quad \text{and} \quad q_i := \lim_{\lambda \in \Lambda_i \downarrow} n_\lambda \quad \text{for } i = 1, 2.$$

Clearly, we have

$$\begin{aligned} L_{[\Lambda]} &\leq L_{[\Lambda_1]} + L_{[\Lambda_2]}, \\ \ell_{[\Lambda]} &= \ell_{[\Lambda_1]} + \ell_{[\Lambda_2]}, \\ \ell_{[\Lambda]}^* &= \left( \ell_{[\Lambda_1]}^* - d(r_1, q_1) \right) + d(r_1, q_2) + \left( \ell_{[\Lambda_2]}^* - d(r_2, q_2) \right) + d(r_2, q_1). \end{aligned} \tag{7.25}$$

In particular,

$$\ell_{[\Lambda]}^* \geq \left( \ell_{[\Lambda_1]}^* - d(r_1, q_1) \right) + \left( \ell_{[\Lambda_2]}^* - d(r_2, q_2) \right). \tag{7.26}$$

## 7.4 Simple cycles

Throughout this section, we shall assume that  $[\Lambda]$  is a nonempty cycle such that  $\ell_{[\Lambda]}^* < \infty$ . Recall that

$$T_{[\Lambda]} = \sum_{\lambda \in \Lambda} (\delta_{p_\lambda} - \delta_{n_\lambda}) \quad \text{in } [\text{Lip}(X)]^*.$$

We define the *gap* of  $[\Lambda]$  to be the number given by

$$\text{gap } [\Lambda] := \sup_{\bar{\Lambda} \in [\Lambda]} \left\{ \text{dist} \left( \lim_{\lambda \in \bar{\Lambda} \uparrow} p_\lambda, \lim_{\lambda \in \bar{\Lambda} \downarrow} n_\lambda \right) \right\}. \tag{7.27}$$

Roughly speaking,  $\text{gap } [\Lambda]$  measures the jump of  $[\Lambda]$  across two adjacent dipoles, while the co-length  $\ell_{[\Lambda]}^*$  measures the total jump along  $[\Lambda]$ . We point out that, since  $\ell_{[\Lambda]}^* < \infty$ , the supremum in (7.27) is actually achieved.

**Example 7.4** Assume  $\Lambda$  is finite, say  $\Lambda = (1 \cdots k)$ . In this case, we have

$$\text{gap} [\Lambda] = \max \left\{ d(p_1, n_2), \dots, d(p_{k-1}, n_k), d(p_k, n_1) \right\}.$$

In particular, if  $\text{gap} [\Lambda] = 0$ , then  $T_{[\Lambda]} = 0$  in  $[\text{Lip}(X)]^*$ . This need not be the case in general. In fact, in Example 7.3 we have  $\text{gap} [\Lambda_0] = 0$ , even though  $L_{[\Lambda_0]} = \alpha > 0$ .

We now consider the following

**Definition 7.13**  $[\Lambda]$  is a closed cycle if  $\text{gap} [\Lambda] = 0$ .

For example, we have

**Lemma 7.3** If  $[\Lambda]$  is a co-minimal cycle and  $T_{[\Lambda]}$  is singular in  $X$ , then  $[\Lambda]$  is a closed cycle.

**Proof.** Let  $[\Lambda]$  be a co-minimal cycle such that  $\text{gap} [\Lambda] > 0$ . Without loss of generality, we may assume that the supremum in (7.27) is achieved by  $\Lambda$  itself :

$$d(n_0, p_0) > 0, \quad \text{where } n_0 := \lim_{\lambda \in \Lambda \uparrow} p_\lambda \text{ and } p_0 := \lim_{\lambda \in \Lambda \downarrow} n_\lambda.$$

We define the chain  $\Lambda_0 := \Lambda \cup \{0\}$  oriented in such a way that 0 is the largest element in  $\Lambda_0$ . Applying Remark 7.3 with  $\Lambda_1 := \Lambda$ ,  $\Lambda_2 := \{0\}$ ,  $r_1 = q_2 := n_0$  and  $q_1 = r_2 := p_0$ , we get

$$\|T_{[\Lambda_0]}\| \leq \ell_{[\Lambda_0]}^* = \ell_{[\Lambda]}^* - d(n_0, p_0) = \|T_{[\Lambda]}\| - d(n_0, p_0) \leq \|T_{[\Lambda_0]}\|$$

(we use the triangle inequality to obtain the last estimate). Thus,

$$T_{[\Lambda]} = (\delta_{n_0} - \delta_{p_0}) + T_{[\Lambda_0]} \quad \text{and} \quad \|T_{[\Lambda]}\| = d(n_0, p_0) + \|T_{[\Lambda_0]}\|.$$

We conclude that  $T_{[\Lambda]}$  is not singular.

In order to introduce the notion of simple cycles, we shall need an auxiliary

**Definition 7.14** A subchain  $\Lambda_1 \subset \Lambda$  is a segment of  $[\Lambda]$  if  $\Lambda_1$  is a segment of some representative  $\tilde{\Lambda} \in [\Lambda]$  (see Definition 7.9). Equivalently,  $\Lambda_1 \subset \Lambda$  is a segment of  $[\Lambda]$  if either  $\Lambda_1$  or  $\Lambda \setminus \Lambda_1$  is a segment of  $\Lambda$ .

A simple cycle will be defined as follows :

**Definition 7.15**  $[\Lambda]$  is a simple cycle if

- (i)  $[\Lambda]$  is a closed cycle ;
- (ii) if  $\Lambda_1$  is a segment of  $[\Lambda]$  such that  $[\Lambda_1]$  is a closed cycle, then  $\Lambda_1 = \Lambda$ .

Since  $\text{gap} [\Lambda] = 0$ , condition (ii) in the definition above is equivalent to saying that

- (ii') if  $\Lambda_1 \subsetneq \Lambda$  is a segment, then  $\lim_{\lambda \in \Lambda_1 \uparrow} p_\lambda \neq \lim_{\lambda \in \Lambda_1 \downarrow} n_\lambda$ .

Note that  $[\Lambda_0]$  given by Example 7.3 is a simple cycle.

The orientation of a simple cycle  $[\Lambda]$  is compatible with the topology induced by  $X$  on the set  $\bigcup_{\lambda \in \Lambda} \{p_\lambda, n_\lambda\}$  in the following sense :

**Lemma 7.4** Assume  $[\Lambda]$  is a simple cycle. Given

- (i) a sequence  $(\lambda_k)_{k \geq 1}$  in  $\Lambda$  such that either  $p_{\lambda_k} \rightarrow p_{\lambda_0}$  or  $n_{\lambda_k} \rightarrow p_{\lambda_0}$ ,
- (ii) two indices  $\mu_1, \mu_2 \in \Lambda$  such that  $\mu_1 < \lambda_0 < \mu_2$  with respect to some representative  $\tilde{\Lambda} \in [\Lambda]$ ,

then there exists  $k_0 \geq 1$  such that

$$\mu_1 < \lambda_k < \mu_2 \quad \text{in } \tilde{\Lambda}, \quad \forall k \geq k_0. \quad (7.28)$$

**Proof.** Assume by contradiction there exist  $p_{\lambda_k} \rightarrow p_{\lambda_0}$  and  $\mu_1 < \lambda_0 < \mu_2$  in  $\Lambda$  such that (7.28) does not hold. (The case where  $n_{\lambda_k} \rightarrow p_{\lambda_0}$  can be dealt with in a similar way.)

Replacing  $\Lambda$  by another representative of  $[\Lambda]$ , we can assume that

$$\mu_1 < \lambda_0 < \mu_2 \leq \lambda_k \quad \text{in } \Lambda, \quad \forall k \geq k_0. \quad (7.29)$$

Moreover, passing to a subsequence if necessary, we can also assume that  $(\lambda_k)_{k \geq 1}$  is either nondecreasing or non-increasing in  $\Lambda$ . We consider each one of these possibilities separately :

**Case 1.**  $(\lambda_k)_{k \geq 1}$  is nondecreasing in  $\Lambda$ .

Let

$$\Lambda_1 := \bigcup_{k=1}^{\infty} (\lambda_0 < \lambda \leq \lambda_k).$$

Note that  $\Lambda_1$  is a segment of  $\Lambda$ . We claim that  $\text{gap}[\Lambda_1] = 0$ . In order to see this, it suffices to show that

$$\lim_{\lambda \in \Lambda_1 \downarrow} n_\lambda = p_{\lambda_0} = \lim_{\lambda \in \Lambda_1 \uparrow} p_\lambda. \quad (7.30)$$

The first equality holds because  $\text{gap}[\Lambda] = 0$ , while the second one follows from  $p_{\lambda_k} \rightarrow p_{\lambda_0}$ . Therefore, we have constructed a closed segment  $[\Lambda_1]$  strictly contained in  $[\Lambda]$ , which is a contradiction.

**Case 2.**  $(\lambda_k)_{k \geq 1}$  is non-increasing in  $\Lambda$ .

In this case, we take

$$\Lambda_1 := \bigcap_{k=1}^{\infty} (\lambda_0 < \lambda \leq \lambda_k).$$

We first observe that  $\Lambda_1$  is nonempty since  $\mu_2 \in \Lambda_1$ . In order to get a contradiction, it suffices to show that the second equality in (7.30) holds, and then argue as before.

If  $\lambda_k = \tilde{\lambda}$  for all  $k \geq 1$  sufficiently large, then  $\Lambda_1 = (\lambda_0 < \lambda \leq \tilde{\lambda})$  and we are done. On the other hand, if  $(\lambda_k)_{k \geq 1}$  has infinitely many distinct terms, then we have  $d(p_{\lambda_k}, n_{\lambda_k}) \rightarrow 0$ . Thus,

$$\lim_{\lambda \in \Lambda_1 \uparrow} p_\lambda = \lim_{k \rightarrow \infty} n_{\lambda_k} = \lim_{k \rightarrow \infty} p_{\lambda_k} = p_{\lambda_0}.$$

(The first equality follows from  $\text{gap}[\Lambda] = 0$ .) As we explained before, this gives a contradiction.

Using the same ideas we can prove a slightly more general result :

**Lemma 7.5** *Let  $[\Lambda]$  be a simple cycle. Given  $\mu_1 < \nu_2 \leq \nu_2 < \mu_2$  in  $\Lambda$ , let  $q$  be a point in the closure of the set*

$$\bigcup_{\nu_1 \leq \lambda \leq \nu_2} \{p_\lambda\} \cup \{n_\lambda\}.$$

*If  $(\lambda_k)_{k \geq 1}$  is a sequence in  $\Lambda$  such that  $p_{\lambda_k} \rightarrow q$ , then there exists  $k_0 \geq 1$  such that*

$$\mu_1 < \lambda_k < \mu_2, \quad \forall k \geq k_0.$$

This lemma will be used to prove our next result :

**Proposition 7.6** *Assume  $[\Lambda]$  is a simple cycle. Then,*

$$\ell_{[\Lambda]}^* = \mathcal{H}^1(S_{[\Lambda]}),$$

where

$$S_{[\Lambda]} := \overline{\bigcup_{\lambda \in \Lambda} \{p_\lambda\} \cup \{n_\lambda\}}.$$

**Proof.** We split the proof into three steps :

**Step 1.** Given  $\mu_1 < \mu_2$  in  $\Lambda$ , we consider the segment  $\tilde{\Lambda} := (\mu_1 < \lambda < \mu_2)$ . Then, we have

$$\text{diam } S_{[\tilde{\Lambda}]} \leq \ell_{[\tilde{\Lambda}]} + \ell_{[\tilde{\Lambda}]}^* - d(p_{\mu_1}, n_{\mu_2}). \quad (7.31)$$

Since  $S_{[\tilde{\Lambda}]}$  is compact, for any  $\eta > 0$  we can find an  $\varepsilon$ -chain

$$\tilde{\Lambda}_\varepsilon = (\tilde{\lambda}_1 \cdots \tilde{\lambda}_k) \subset \tilde{\Lambda}$$

such that

$$S_{[\tilde{\Lambda}]} \subset \bigcup_{i=1}^k [B_\eta(p_{\tilde{\lambda}_i}) \cup B_\eta(n_{\tilde{\lambda}_i})].$$

Thus,

$$\begin{aligned} \text{diam } S_{[\tilde{\Lambda}]} &\leq d(n_{\tilde{\lambda}_1}, p_{\tilde{\lambda}_1}) + d(p_{\tilde{\lambda}_1}, n_{\tilde{\lambda}_2}) + \cdots + d(n_{\tilde{\lambda}_k}, p_{\tilde{\lambda}_k}) + 2\eta \\ &= \ell_{[\tilde{\Lambda}_\varepsilon]} + \ell_{[\tilde{\Lambda}_\varepsilon]}^* - d(n_{\tilde{\lambda}_1}, p_{\tilde{\lambda}_k}) + 2\eta \\ &\leq \ell_{[\tilde{\Lambda}]} + \ell_{[\tilde{\Lambda}]}^* - d(n_{\tilde{\lambda}_1}, p_{\tilde{\lambda}_k}) + 2\eta. \end{aligned}$$

We first let  $\varepsilon \downarrow 0$ . Then,  $\text{gap } [\Lambda] = 0$  implies that  $n_{\tilde{\lambda}_1} \rightarrow p_{\mu_1}$  and  $p_{\tilde{\lambda}_k} \rightarrow n_{\mu_2}$ . Since  $\eta > 0$  was arbitrary, (7.31) follows.

**Step 2.**

$$\mathcal{H}^1(S_{[\Lambda]}) \leq \ell_{[\Lambda]}^*.$$

(This inequality holds even if  $[\Lambda]$  is just a closed cycle, not necessarily simple.)

Let  $\delta > 0$  fixed. Given an  $\varepsilon$ -chain  $\Lambda_\varepsilon = (\lambda_1 \cdots \lambda_k) \subset \Lambda$ , we define the segments

$$\Lambda_i := (\lambda_i < \lambda < \lambda_{i+1}), \quad i = 1, \dots, k \quad (7.32)$$

(we use the convention that  $\lambda_{k+1} := \lambda_1$ ).



By taking  $\varepsilon > 0$  sufficiently small, we can assume that

$$\text{diam } S_{[\Lambda_i]} \leq \delta, \quad \forall i = 1, \dots, k.$$

Since  $\text{gap } [\Lambda] = 0$ , we have

$$S_{[\Lambda]} = \bigcup_{i=1}^k S_{[\Lambda_i]}.$$

It follows from the previous step and (7.26) that

$$\begin{aligned} \mathcal{H}_\delta^1(S_{[\Lambda]}) &\leq \sum_{i=1}^k \text{diam } S_{[\Lambda_i]} \\ &\leq \sum_{i=1}^k \{ \ell_{[\Lambda_i]} + (\ell_{[\Lambda_i]}^* - d(p_{\lambda_i}, n_{\lambda_{i+1}})) \} \\ &\leq \sum_{i \notin \Lambda_\varepsilon} d(p_i, n_i) + \ell_{[\Lambda]}^*. \end{aligned}$$

Taking  $\varepsilon \downarrow 0$ , we conclude that

$$\mathcal{H}^1(S_{[\Lambda]}) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^1(S_{[\Lambda]}) \leq \ell_{[\Lambda]}^*.$$

**Step 3.**

$$\ell_{[\Lambda]}^* \leq \mathcal{H}^1(S_{[\Lambda]}). \quad (7.33)$$

Given an  $\varepsilon$ -chain  $\Lambda_\varepsilon = (\lambda_1 \cdots \lambda_k) \subset \Lambda$ , we consider the segments  $\Lambda_i$  given by (7.32). Since  $[\Lambda]$  is simple, the sets  $S_{[\Lambda_i]}$  are disjoint (see Lemma 7.5). Let  $\delta > 0$  be such that

$$d(S_{[\Lambda_{i_1}]}, S_{[\Lambda_{i_2}]}) > 2\delta, \quad \forall i_1 \neq i_2. \quad (7.34)$$

Take  $(B_j)_{j \in J}$  to be a finite open covering of  $S_{[\Lambda]}$  so that  $\text{diam } B_j < \delta$  for every  $j \in J$ .

We claim we can select

- (i) a new  $\varepsilon$ -chain  $\tilde{\Lambda}_\varepsilon = (\tilde{\lambda}_1 \cdots \tilde{\lambda}_l)$  containing  $\Lambda_\varepsilon$ ;
- (ii)  $l$  distinct elements from the family  $(B_j)_{j \in J}$ , say  $\tilde{B}_1, \dots, \tilde{B}_l$ ;

such that

$$p_{\tilde{\lambda}_i}, n_{\tilde{\lambda}_{i+1}} \in \tilde{B}_i, \quad \forall i = 1, \dots, l. \quad (7.35)$$

We proceed as follows :

We first define the segments

$$\Gamma_j := \left\{ \lambda \in \Lambda : \mu_1 \leq \lambda \leq \mu_2 \text{ for some } \mu_1 < \mu_2 \text{ such that } p_{\mu_1}, n_{\mu_2} \in B_j \right\}.$$

Note that if  $B_j \cap S_{[\Lambda]} \neq \emptyset$ , then  $\Gamma_j \neq \emptyset$ . In fact, assume for instance that  $p_\mu \in B_j$ . Since  $\text{gap } [\Lambda] = 0$ , then either there exists  $\mu_2 > \mu$  such that  $n_{\mu_2} = p_\mu$  or we can find a decreasing sequence  $\mu_j \downarrow \mu$  such that  $n_{\mu_j} \rightarrow p_\mu$ . The set  $B_j$  being open, we conclude that  $n_{\mu_{j_0}} \in B_j$  for some  $\mu_{j_0} > \mu$ . Thus, in both case we have  $\Gamma_j \neq \emptyset$ . Moreover, (7.34) implies that  $\Gamma_j$  is contained in some segment  $(\lambda_i \leq \lambda \leq \lambda_{i+1})$ .

We also define

$$r_j := \lim_{\lambda \in \Gamma_j \uparrow} p_\lambda$$

to be the upper endpoint of  $\Gamma_j$ .

Let  $\tilde{\lambda}_1 := \lambda_1$  and  $\tilde{B}_1$  be an element of the family  $(B_j)_{j \in J}$  containing  $p_{\lambda_1}$ . By abuse of notation, we denote by  $\Gamma_1$  the segment  $\Gamma_j$  corresponding to  $\tilde{B}_1$ . We have two possibilities for  $\Gamma_1$  :

(a<sub>1</sub>)  $\Gamma_1$  has a largest element  $\tilde{\lambda}_2$  : in this case, we have

$$n_{\tilde{\lambda}_2} \in \tilde{B}_1 \quad \text{and} \quad p_{\tilde{\lambda}_2} \notin \tilde{B}_1$$

(since  $\tilde{B}_1$  is open and  $\text{gap}[\Lambda] = 0$ ) ; we then choose  $\tilde{B}_2 \in (B_j)_{j \in J}$  such that  $p_{\tilde{\lambda}_2} \in \tilde{B}_2$  ;

(b<sub>1</sub>)  $\Gamma_1$  does not have a largest element : this implies the existence of an increasing sequence  $(\mu_j)$  in  $\Gamma_1$  such that

$$n_{\mu_j} \in \tilde{B}_1, \quad \forall j \geq 1, \quad \text{and} \quad p_{\mu_j} \rightarrow r_1;$$

moreover,  $d(p_{\mu_j}, n_{\mu_j}) \rightarrow 0$ .

Let  $\tilde{B}_2 \in (B_j)_{j \in J}$  be such that  $r_1 \in \tilde{B}_2$ . Since  $\tilde{B}_1$  and  $\tilde{B}_2$  are both open, we can find  $j_0 \geq 1$  sufficiently large so that

$$n_{\mu_{j_0}} \in \tilde{B}_1 \cap \tilde{B}_2 \quad \text{and} \quad p_{\mu_{j_0}} \in \tilde{B}_2.$$

We then take  $\tilde{\lambda}_2 := \mu_{j_0}$ .

Note that in both cases we have

$$\tilde{B}_1 \neq \tilde{B}_2, \quad n_{\tilde{\lambda}_2} \in \tilde{B}_1, \quad \text{and} \quad p_{\tilde{\lambda}_2} \in \tilde{B}_2.$$

We can now repeat the construction above with  $\tilde{\lambda}_2$  and  $\tilde{B}_2$ , and so on until we get  $n_{\lambda_l} \in \tilde{B}_l$ . In order to see this will be indeed the case, it suffices to prove the following :

**Claim.**  $\tilde{\Lambda}_\varepsilon := (\tilde{\lambda}_1 \cdots \tilde{\lambda}_l) \supset \Lambda_\varepsilon$ .

Let us check for instance that  $\lambda_2 \in \tilde{\Lambda}_\varepsilon$  (the general case follows by induction) : let  $1 < l_1 < l$  be such that  $\tilde{\lambda}_{l_1} < \lambda_2 \leq \tilde{\lambda}_{l_1+1}$ . Since

$$p_{\tilde{\lambda}_{l_1}}, n_{\tilde{\lambda}_{l_1+1}} \in \tilde{B}_{l_1} \quad \text{and} \quad p_{\tilde{\lambda}_{l_1}} \in S_{[\Lambda_1]},$$

we have  $n_{\tilde{\lambda}_{l_1+1}} \in S_{[\Lambda_1]}$ . On the other hand,  $\lambda_2 \leq \tilde{\lambda}_{l_1+1}$  implies that  $p_{\tilde{\lambda}_{l_1+1}} \notin S_{[\Lambda_1]}$ . In particular,

$$d(p_{\tilde{\lambda}_{l_1+1}}, n_{\tilde{\lambda}_{l_1+1}}) > 2\delta,$$

from which we conclude that  $\tilde{\lambda}_{l_1+1} = \lambda_2$ . This establishes the claim.

By construction, the sets  $\tilde{B}_1, \dots, \tilde{B}_l$  are all distinct and (7.35) holds. It follows from (7.35) that

$$d(p_{\tilde{\lambda}_i}, n_{\tilde{\lambda}_{i+1}}) \leq \text{diam } \tilde{B}_i.$$

Thus,

$$\ell_{[\tilde{\Lambda}_\varepsilon]}^* = \sum_{i=1}^l d(p_{\tilde{\lambda}_i}, n_{\tilde{\lambda}_{i+1}}) \leq \sum_{i=1}^l \text{diam } \tilde{B}_i \leq \sum_{j \in J} \text{diam } B_j.$$

In particular,

$$\inf_{\Lambda_\varepsilon} \ell_{[\Lambda_\varepsilon]}^* \leq \sum_{j \in J} \text{diam } B_j.$$

where the infimum is taken over all  $\varepsilon$ -chains of  $[\Lambda]$ . We now take the infimum with respect to all (finite) open coverings  $(B_j)_{j \in J}$  of  $S_{[\Lambda]}$ , with  $\text{diam } B_j < \delta$  for all  $j \in J$ . We get

$$\inf_{\Lambda_\varepsilon} \ell_{[\Lambda_\varepsilon]}^* \leq \mathcal{H}_\delta^1(S_{[\Lambda]}).$$

Note that this estimate holds for  $\varepsilon > 0$  fixed and every  $\delta > 0$  sufficiently small. Taking  $\delta \downarrow 0$  and then  $\varepsilon \downarrow 0$ , we obtain (7.33).

We conclude this section with the following result which will be used in the proof of Theorem 7.2 :

**Proposition 7.7** *Let  $[\Lambda]$  be a co-minimal cycle. If  $T_{[\Lambda]}$  is singular, then we can write  $\Lambda = \bigcup_j \Lambda_j$  as a disjoint union of subchains, where each  $[\Lambda_j]$  is a simple cycle and*

$$\ell_{[\Lambda]}^* = \sum_j \ell_{[\Lambda_j]}^*. \quad (7.36)$$

*In particular,  $[\Lambda_j]$  is a co-minimal cycle and  $T_{[\Lambda_j]}$  is singular for every  $j$ .*

**Proof.** Consider the family

$$\mathcal{F} := \left\{ (\Lambda_k)_{k \in \Lambda} \left| \begin{array}{l} \Lambda_k \text{ is a subchain of } [\Lambda] \text{ and } k \in \Lambda_k, \quad \forall k \in \Lambda; \\ \text{if } \Lambda_{k_1} \cap \Lambda_{k_2} \neq \emptyset, \text{ then } \Lambda_{k_1} = \Lambda_{k_2}; \\ \sum_{\Lambda_k} \ell_{[\Lambda_k]}^* \leq \ell_{[\Lambda]}^* \end{array} \right. \right\}. \quad (7.37)$$

(The sum  $\sum_{\Lambda_k}$  is taken over all disjoint components of  $(\Lambda_k)_{k \in \Lambda}$ .)

Since  $(\Lambda_k)_{k \in \Lambda} \in \mathcal{F}$  (i.e. we take  $\Lambda_k = \Lambda$  for each  $k$ ),  $\mathcal{F}$  is nonempty. We consider the order relation  $\leq$  in  $\mathcal{F}$  given by  $(\tilde{\Lambda}_k)_k \leq (\hat{\Lambda}_k)_k$  iff  $\tilde{\Lambda}_k \supset \hat{\Lambda}_k$  for every  $k \in \Lambda$ .

**Step 1.** If  $(\Lambda_k)_{k \in \Lambda} \in \mathcal{F}$ , then  $[\Lambda_k]$  is a co-minimal cycle and  $T_{[\Lambda_k]}$  is singular for every  $k \in \Lambda$ . Moreover,

$$\ell_{[\Lambda]}^* = \sum_{\Lambda_k} \ell_{[\Lambda_k]}^*.$$

Since  $[\Lambda]$  is co-minimal cycle, it follows from the triangle inequality applied to  $T_{[\Lambda]} = \sum_{\Lambda_k} T_{[\Lambda_k]}$  that

$$L_{[\Lambda]} \leq \sum_{\Lambda_k} L_{[\Lambda_k]} \leq \sum_{\Lambda_k} \ell_{[\Lambda_k]}^* \leq \ell_{[\Lambda]}^* = L_{[\Lambda]}. \quad (7.38)$$

Therefore, equality holds everywhere in (7.38), and we have

$$\ell_{[\Lambda]}^* = \sum_{\Lambda_k} \ell_{[\Lambda_k]}^* \quad \text{and} \quad L_{[\Lambda_k]} = \ell_{[\Lambda_k]}^*, \quad \forall k \in \Lambda.$$

In particular,  $[\Lambda_k]$  is a co-minimal cycle. Since  $L_{[\Lambda]} = \sum_{\Lambda_k} L_{[\Lambda_k]}$  and  $T_\Lambda$  is singular, we conclude that every  $T_{[\Lambda_k]}$  is singular (see Remark 7.8 (b)).

**Step 2.**  $\mathcal{F}$  has a maximal element.

By Zorn's Lemma, it suffices to show that if  $((\Lambda_{k,\alpha})_{k \in \Lambda})_\alpha$  is a linearly ordered family in  $\mathcal{F}$ , then it has an upper bound.

For each  $k \in \Lambda$ , we set  $\Lambda_k := \bigcap_\alpha \Lambda_{k,\alpha}$ .

Clearly, the first two properties in (7.37) are satisfied by  $(\Lambda_k)_{k \in \Lambda}$ . We now check the last one.

Let  $\Lambda_{k_1}, \dots, \Lambda_{k_l}$  be the first  $l$  disjoint subchains in  $(\Lambda_k)_{k \in \Lambda}$ . Take  $\alpha_0$  sufficiently large so that  $\Lambda_{k_1, \alpha_0}, \dots, \Lambda_{k_l, \alpha_0}$  are disjoint. Applying Corollary 7.3, we get

$$\begin{aligned} \sum_{i=1}^l \ell_{[\Lambda_{k_i}]}^* &\leq \sum_{i=1}^l \ell_{[\Lambda_{k_i, \alpha_0}]}^* + \sum_{i=1}^l (\ell_{[\Lambda_{k_i, \alpha_0}]} - \ell_{[\Lambda_{k_i}]}) \\ &\leq \ell_{[\Lambda]}^* + \ell_{[\Lambda]} - \ell_{[\Lambda_{k_1} \cup \dots \cup \Lambda_{k_l}]}. \end{aligned}$$

Since  $l$  was arbitrary and  $\Lambda = \bigcup_k \Lambda_k$ , we conclude that  $\sum_{\Lambda_k} \ell_{[\Lambda_k]}^* \leq \ell_{[\Lambda]}^*$ . Thus,  $(\Lambda_k)_{k \in \Lambda} \in \mathcal{F}$ .

We can now invoke Zorn's Lemma to conclude that  $\mathcal{F}$  has a maximal element.

**Step 3.** Proof of the proposition completed.

Let  $(\Lambda_k)_{k \in \Lambda}$  be a maximal element of  $\mathcal{F}$ . We claim that  $[\Lambda_k]$  is simple for every  $k \in \Lambda$ .

Suppose by contradiction that  $\Lambda_k$  is not simple. By definition, we can split  $\Lambda_k = \Lambda_{k,1} \cup \Lambda_{k,2}$  so that both  $\Lambda_{k,1}$  and  $\Lambda_{k,2}$  are segments of  $[\Lambda_k]$  and  $\text{gap}[\Lambda_{k,1}] = 0$ . Since  $\text{gap}[\Lambda] = 0$ , we also have  $\text{gap}[\Lambda_{k,2}] = 0$ . It follows from Remark 7.3 that

$$\ell_{[\Lambda_k]}^* = \ell_{[\Lambda_{k,1}]}^* + \ell_{[\Lambda_{k,2}]}^*,$$

but this contradicts the maximality of  $(\Lambda_k)_{k \in \Lambda}$  in  $\mathcal{F}$ .

The proposition follows from Step 1 after relabeling and removing the repeated components of  $(\Lambda_k)_{k \in \Lambda}$ .

## 7.5 Paths and loops

Let  $\Gamma$  be a chain such that  $\ell_{[\Gamma]}^* < \infty$ . We know from Proposition 7.5 that both limits

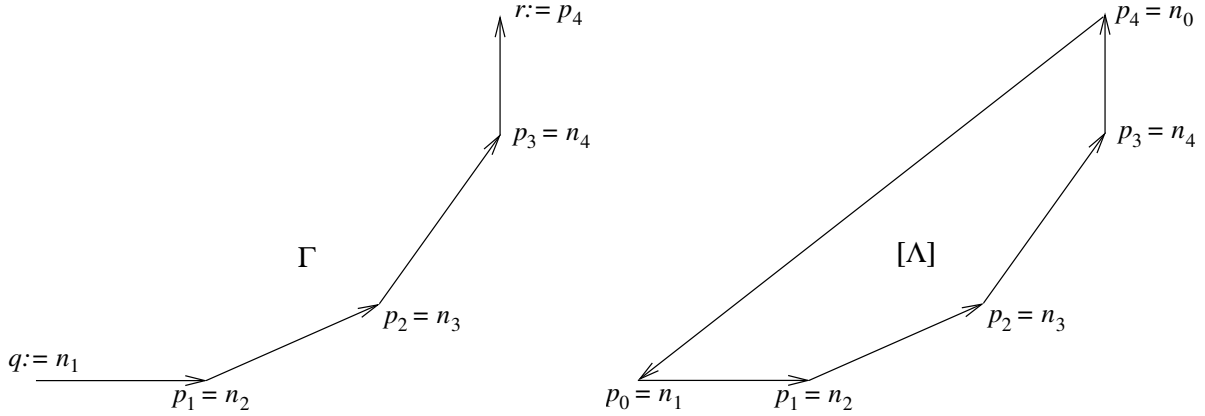
$$r := \lim_{\lambda \in \Gamma \uparrow} p_\lambda \quad \text{and} \quad q := \lim_{\lambda \in \Gamma \downarrow} n_\lambda$$

exist. Clearly, we have  $\ell_{[\Gamma]}^* \geq d(r, q)$ .

**Definition 7.16**  $\Gamma$  is a path from  $q$  to  $r$  if

$$\ell_{[\Gamma]}^* = d(r, q).$$

This definition can be rephrased in terms of loops (see Definition 7.11 (c)). In fact, let  $p_0 := q$  and  $n_0 := r$ . We consider the chain  $\Lambda := \Gamma \cup \{0\}$ , where 0 is the largest element in  $\Lambda$ . Then,  $\Gamma$  is a path from  $q$  to  $r$  iff  $[\Lambda]$  is a loop. In particular, all results for loops can be translated in terms of paths, and conversely.

FIG. 7.3 – A finite path  $\Gamma$  from  $q$  to  $r$  and the cycle  $[\Lambda]$  associated to  $\Gamma$ 

**Example 7.5** Assume  $\Gamma = (1 \cdots k)$  is a finite path from  $q$  to  $r$ . Then  $q = n_1$  and  $r = p_k$ ; moreover, we have  $p_i = n_{i+1}$  for every  $i = 1, \dots, k - 1$ . Figure 7.3 shows a finite path  $\Gamma$  (with  $k = 4$ ) and the cycle  $[\Lambda]$  associated to it.

A less trivial example is given by Example 7.1 with  $\alpha = 0$ . In this case, we take  $\Gamma = \mathbb{N}$  oriented from left to right in Figure 7.1. It is easy to see that  $\Gamma$  is a path from 0 to 1.

**Remark 7.4** If  $\Gamma$  is a path from  $q$  to  $r$ , then it follows from (7.18) that

$$T_{[\Gamma]} = \delta_r - \delta_q \quad \text{in } [\text{Lip}(X)]^*. \quad (7.39)$$

In particular,

$$L_{[\Gamma]} = \ell_{[\Gamma]}^* = d(r, q) \leq \ell_{[\Lambda]}. \quad (7.40)$$

**Remark 7.5** Assume  $\Gamma_1$  is a segment of  $\Gamma$ , and let  $r_1$  and  $q_1$  be the corresponding endpoints. We claim that  $\Gamma_1$  is a path from  $q_1$  to  $r_1$ .

Suppose for simplicity that  $\Gamma_2 := \Gamma \setminus \Gamma_1$  is also a segment and  $\Gamma_1 \leq \Gamma_2$ . Note that  $q_1 = q$  and  $r_2 = r$ . Applying (7.25) with  $\Lambda$  replaced by  $\Gamma$  we get

$$\left( \ell_{[\Gamma_1]}^* - d(r_1, q_1) \right) + d(r_1, q_2) + \left( \ell_{[\Gamma_2]}^* - d(r_2, q_2) \right) = 0.$$

Since each one of these terms is nonnegative, we must have  $\ell_{[\Gamma_1]}^* = d(r_1, q_1)$  (note also that  $d(r_1, q_2) = 0$ ).

The general case follows from the above since  $\Gamma \setminus \Gamma_1$  is a union of at most two segments.

A simple consequence of Proposition 7.4 is the following :

**Corollary 7.6** Assume

$$\sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) = 0 \quad \text{in } [\text{Lip}(X)]^*.$$

Then, we can write  $\mathbb{N} = \bigcup_j \Lambda_j$  as a disjoint union, where each  $[\Lambda_j]$  is a loop.

The corollary below is just a restatement of the previous one in terms of paths :

**Corollary 7.7** *If*

$$\sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) = \delta_r - \delta_q \quad \text{in } [\text{Lip}(X)]^*$$

for some  $r, q \in X$ ,  $r \neq q$ , then there exists a path  $\Gamma$  from  $q$  to  $r$ .

As a consequence, we can now prove Lemma 7.1 :

**Proof of Lemma 7.1.** Assume that  $q_2$  cannot be connected to  $r_1$  by any path. We write

$$(\delta_{p_0} - \delta_{n_0}) + \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) = \delta_{r_1} - \delta_{q_1} \quad \text{in } [\text{Lip}(X)]^*,$$

where  $p_0 := q_2$  and  $n_0 := r_2$ . It follows from Corollary 7.7 that there exists a path  $\Gamma$  from  $q_1$  to  $r_1$ . We claim that  $0 \notin \Gamma$ . In fact, otherwise the segment  $[\lambda > 0] \subset \Gamma$  would be a path from  $p_0 = q_2$  to  $r_1$ , which cannot be the case by assumption. The result now follows directly from (7.39).

Combining Corollaries 7.1 and 7.7, we get the proposition below. This is especially suitable to study irreducible representations (see, e.g., Lemma 7.7).

**Proposition 7.8** *Assume  $\sum_i (\delta_{p_i} - \delta_{n_i})$  is reducible and  $p_i \neq n_j$  for every  $i, j$ . Then, there exists an infinite path  $\Gamma$  from  $q$  to  $r$ , for some  $r, q \in X$ .*

The following lemma will be used in the proof of Theorem 7.3 :

**Lemma 7.6** *Let  $\tilde{\mathbb{N}} \subset \mathbb{N}$ . For each  $j \in \tilde{\mathbb{N}}$ , there exists a path  $\Gamma_{\tilde{\mathbb{N}}, j}$  which is maximal among all paths in  $\tilde{\mathbb{N}}$  containing  $j$ .*

**Proof.** This is a simple application of Zorn's Lemma. In fact, note that  $(j)$  is a path containing  $j$ . Moreover, if  $(\Gamma_\alpha)$  is a linearly ordered set of paths containing  $j$ , then we define  $\Gamma := \bigcup_\alpha \Gamma_\alpha$ , equipped with the order relation induced from each  $\Gamma_\alpha$ . We claim that  $\Gamma$  is a path.

In fact, let  $(\alpha_j)$  be an increasing sequence such that  $\Gamma = \bigcup_j \Gamma_{\alpha_j}$ . On the one hand, Corollary 7.4 says that

$$\ell_{[\Gamma]}^* = \lim_{j \rightarrow \infty} \ell_{[\Gamma_{\alpha_j}]}^*. \quad (7.41)$$

In particular,  $\ell_{[\Gamma]}^* \leq \ell_{[\Gamma]} < \infty$ . We conclude from Proposition 7.5 that both limits

$$r := \lim_{\lambda \in \Gamma \uparrow} p_\lambda \quad \text{and} \quad q := \lim_{\lambda \in \Gamma \downarrow} n_\lambda \quad (7.42)$$

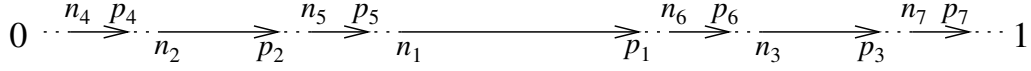
exist.

On the other hand, each  $\Gamma_{\alpha_j}$  is a path from some  $q_j$  to some  $r_j$ . It is easy to see from (7.42) that  $q_j \rightarrow q$  and  $r_j \rightarrow r$ . Recall that

$$\ell_{[\Gamma_{\alpha_j}]}^* = d(r_j, q_j), \quad \forall j.$$

As  $j \rightarrow \infty$ , we get  $\ell_{[\Gamma]}^* = d(r, q)$ ; thus,  $\Gamma$  is a path.

The statement now follows from Zorn's Lemma.

FIG. 7.4 – Dipoles  $\delta_{p_i} - \delta_{n_i}$  in Example 7.6

## 7.6 Examples

We shall use the same notation as in Examples 7.1 and 7.3 throughout this section.

The example below shows that the converse to Theorem 7.1 does not hold in general. Namely,  $T \in \mathcal{Z}$  may be regular and yet we can have  $\mathcal{H}^1(\text{supp } T) > 0$ .

**Example 7.6** Assume  $X = [0, 1]$ . We consider the chain  $\Lambda := \mathbb{N}$  oriented so that  $k_1 \leq k_2$  iff  $p_{k_1} \leq p_{k_2}$  in  $[0, 1]$  (see Figure 7.4). We claim that

$$T_{[\Lambda]} = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{is irreducible.}$$

In view of Proposition 7.8, it suffices to show that if  $\Gamma$  is a path containing  $i_0 \geq 1$ , then  $\Gamma = (i_0)$ . Let  $r, q \in C_\alpha$ ,  $q \leq r$ , be the endpoints of  $\Gamma$ . It is easy to see that the inclusion  $\Gamma \subset \Lambda$  is order preserving and that  $\Gamma$  is a segment of  $\Lambda$ . Thus,

$$\ell_{[\Gamma]}^* = d(r, q) + |[q, r] \setminus \bigcup_{i \in \Gamma} J_i| = d(r, q) + |C_\alpha \cap [q, r]|.$$

Since  $\Gamma$  is a path, the second term in the right-hand side has to vanish. In other words, we must have  $(q, r) \subset J_{i_0}$ , which implies that  $\Gamma = (i_0)$ . This proves the claim.

According to Proposition 7.2, we have

$$L_{[\Lambda]} = \sum_{i=1}^{\infty} d(p_i, n_i) = 1 - \alpha.$$

In particular,  $[\Lambda]$  is a minimal cycle and  $T_{[\Lambda]}$  is regular.

In the next example we show that  $T_{[\Lambda]} + (\delta_0 - \delta_1)$  is singular.

**Example 7.7** As in Example 7.3, we consider the chain  $\Lambda_0 := \Lambda \cup \{0\}$  so that  $[\Lambda_0]$  is oriented clockwise with respect to Figure 7.1.

According to the previous example,

$$T_{[\Lambda_0]} = \sum_{i=0}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{is irreducible.}$$

Moreover, it follows from Proposition 7.2 that

$$L_{[\Lambda_0]} = \ell_{[\Lambda_0]}^* = \alpha.$$

In particular,  $[\Lambda_0]$  is a co-minimal cycle.

We claim that  $T_{[\Lambda_0]}$  is singular.

Let  $\zeta_k$  be the Lipschitz function such that  $\zeta_k(t) = 0$  if  $t \leq n_k$ ,  $\zeta_k(t) = d(p_k, n_k)$  if  $t \geq p_k$ , and  $\zeta_k$  is affine linear on  $J_k$ . We define

$$\zeta(t) := t - \sum_{k=1}^{\infty} \zeta_k(t)$$

(by construction,  $\zeta$  is constant on each  $J_k$ ). Note that  $|\zeta|_{\text{Lip}} \leq 1$  and

$$L_{[\Lambda_0]} = \alpha = \sum_{i=0}^{\infty} [\zeta(p_i) - \zeta(n_i)]$$

In other words,  $\zeta$  is a function which achieves the supremum in (7.6).

Given  $r, q \in [0, 1]$ ,  $r \neq q$ , we have  $|\zeta(r) - \zeta(q)| < d(r, q)$ . Thus,

$$\|T_{[\Lambda_0]} - (\delta_r - \delta_q)\| \geq \sum_{i=0}^{\infty} [\zeta(p_i) - \zeta(n_i)] - [\zeta(r) - \zeta(q)] > \|T_{[\Lambda_0]}\| - d(r, q).$$

This proves our claim (see Lemma 7.8).

We now combine somewhat Examples 7.6 and 7.7 :

**Example 7.8** Let  $X = S^1$  equipped with its geodesic distance. We shall identify  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$ . Using the same notation as above, we define

$$r_k := e^{2\pi p_k i} \quad \text{and} \quad q_k := e^{2\pi n_k i}, \quad \forall k \geq 1.$$

We consider the chain  $\Lambda = \mathbb{N}$  oriented anticlockwise with respect to Figure 7.5.

Note that

$$\ell_{[\Lambda]} = 2\pi(1 - \alpha) \quad \text{and} \quad \ell_{[\Lambda]}^* = 2\pi\alpha.$$

Applying Proposition 7.2, we get

$$L_{[\Lambda]} = 2\pi \min \{ \alpha, 1 - \alpha \}.$$

Thus,

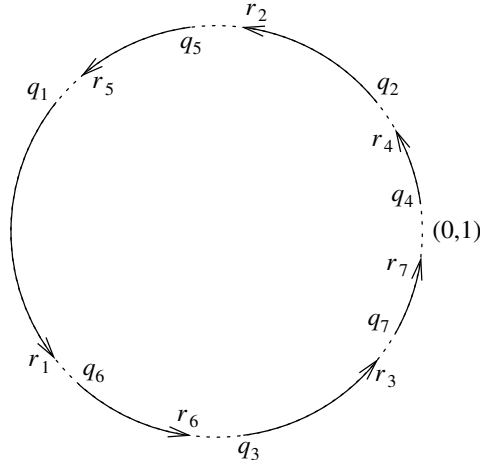
- (a) if  $0 < \alpha < \frac{1}{2}$ , then  $[\Lambda]$  is a co-minimal cycle and  $T_{[\Lambda]}$  is singular (we proceed as in the previous example);
- (b) if  $\frac{1}{2} \leq \alpha < 1$ , then  $[\Lambda]$  is a minimal cycle and  $T_{[\Lambda]}$  is regular.

## 7.7 Proof of Theorem 7.3

It suffices to consider the case where

$$T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{in} \quad [\text{Lip}(X)]^*$$



FIG. 7.5 – Dipoles  $\delta_{r_k} - \delta_{q_k}$  in Example 7.8

is an infinite sum of dipoles. The strategy will be to construct a sequence of disjoint paths  $\Gamma_1, \Gamma_2, \dots$  and sets  $\mathbb{N} =: \mathbb{N}_1 \supset \mathbb{N}_2 \supset \dots$  inductively as follows.

Let  $\Gamma_1$  be a maximal path containing 1 (such a path exists by Lemma 7.6). Set  $\mathbb{N}_2 := \{j \in \mathbb{N} : j \notin \Gamma_1\}$ .

Given  $k \geq 2$  such that  $\mathbb{N}_k \neq \emptyset$ , let  $j_k$  be the smallest integer in  $\mathbb{N}_k$  and let  $\Gamma_k$  be a maximal path among those in  $\mathbb{N}_k$  containing  $j_k$ . Set  $\mathbb{N}_{k+1} := \{j \in \mathbb{N}_k : j \notin \Gamma_k\}$ .

By construction, each  $\Gamma_k$  is a path from some  $\hat{n}_k$  to some  $\hat{p}_k$ , and these paths are disjoint from each other. It follows from (7.39) that

$$T = \sum_{k \in \mathbb{N}'} T_{[\Gamma_k]} = \sum_{k \in \mathbb{N}'} (\delta_{\hat{p}_k} - \delta_{\hat{n}_k}) \quad \text{in } [\text{Lip}(X)]^*,$$

where  $\mathbb{N}' := \{k : \hat{p}_k \neq \hat{n}_k\}$ .

We claim this representation is irreducible.

Suppose by contradiction it is reducible. By maximality, we must have  $\hat{p}_i \neq \hat{n}_j$  for all  $i, j \in \mathbb{N}'$ . Then, according to Proposition 7.8, we can find an infinite path  $\Gamma'$  from  $q$  to  $r$  ( $\Gamma'$  is a path with respect to the dipoles  $\delta_{\hat{p}_k} - \delta_{\hat{n}_k}$ ). In particular,

$$\sum_{k \in \Gamma'} (\delta_{\hat{p}_k} - \delta_{\hat{n}_k}) = \delta_r - \delta_q \quad \text{in } [\text{Lip}(X)]^*.$$

Consider  $\Gamma := \bigcup_{k \in \Gamma'} \Gamma_k$  with the order induced by  $\Gamma'$ , i.e.  $\lambda_1 \leq \lambda_2$  in  $\Gamma$  iff one of the following conditions holds :

- (i)  $\lambda_1, \lambda_2 \in \Gamma_k$  for some  $k \in \Gamma'$  and  $\lambda_1 \leq \lambda_2$  in  $\Gamma_k$ ;
- (ii)  $\lambda_1 \in \Gamma_{k_1}, \lambda_2 \in \Gamma_{k_2}$  and  $k_1 < k_2$  in  $\Gamma'$ .

Then, one can easily check that  $\Gamma$  is a path from  $q$  to  $r$  (associated to the dipoles  $\delta_{p_k} - \delta_{n_k}$ ). But this contradicts the maximality of  $\Gamma_{k_0}$ , where  $k_0$  is the smallest integer in  $\Gamma$ . This concludes the proof of the theorem.

**Remark 7.6** Since  $d(\hat{p}_k, \hat{n}_k) \leq \sum_{j \in \mathbb{N}_k} d(p_j, n_j)$  for every  $k$ , we conclude that the points  $\hat{p}_k, \hat{n}_k \in X$  constructed above also satisfy the estimate

$$\sum_k d(\hat{p}_k, \hat{n}_k) \leq \sum_i d(p_i, n_i).$$

## 7.8 A lemma on irreducible representations

**Lemma 7.7** Assume  $T \in \mathcal{Z}$  and  $T \neq 0$  in  $[\text{Lip}(X)]^*$ . Let

$$T = \sum_i (\delta_{p_i} - \delta_{n_i}) \tag{7.43}$$

be an irreducible representation of  $T$ .

Then, given any  $\delta > 0$  and  $i_0 \geq 1$ , there exists  $\zeta \in \text{BLip}(X)$  such that

$$\|\zeta\|_{L^\infty} \leq 1, \quad \text{supp } \zeta \subset \overline{B_\delta(p_{i_0})}, \quad \text{and} \quad \langle T, \zeta \rangle \geq \frac{1}{4}. \tag{7.44}$$

**Proof.** If the representation in (7.43) is a finite sum of Dirac masses, then we are done. Therefore, we can assume that

$$T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{in } [\text{Lip}(X)]^*$$

and  $i_0 = 1$ .

Let  $A := X \setminus B_\delta(p_1)$ . We consider the quotient space  $X/A$  endowed with the metric

$$d(\bar{x}, \bar{y}) := \min \left\{ d(x, y), d(x, A) + d(y, A) \right\}, \quad \forall x, y \in X.$$

The quotient map  $\pi : X \rightarrow X/A$  induces the linear functional

$$\bar{T} = \sum_{i=1}^{\infty} (\delta_{\bar{p}_i} - \delta_{\bar{n}_i}) \quad \text{in } [\text{Lip}(X/A)]^*.$$

Since the representation in (7.43) is irreducible, we have  $\bar{T} \neq 0$  in  $[\text{Lip}(X/A)]^*$ .

In fact, suppose by contradiction that  $\bar{T} = 0$ . Applying Corollary 7.7 to the identity

$$\sum_{i=2}^{\infty} (\delta_{\bar{p}_i} - \delta_{\bar{n}_i}) = \delta_{\bar{n}_1} - \delta_{\bar{p}_1} \quad \text{in } [\text{Lip}(X/A)]^*,$$

we can find a path  $\Gamma$  starting at  $\bar{p}_1$ . Since  $\bar{p}_1 \neq \bar{n}_j$  for every  $j \geq 1$ ,  $\Gamma$  has no smallest index. In particular, the path  $(\lambda \leq \lambda_0) \subset \Gamma$  contains infinitely many indices for all  $\lambda_0 \in \Gamma$ . Choosing  $\lambda_0$  appropriately, we can assume that

$$\ell_{[\lambda \leq \lambda_0]} = \sum_{i \in (\lambda \leq \lambda_0)} d(\bar{p}_i, \bar{n}_i) < \frac{r}{2}.$$

Therefore, after replacing  $\Gamma$  by  $(\lambda \leq \lambda_0)$  if necessary, we may assume that  $\ell_{[\Gamma]} < \frac{r}{2}$ . In particular,  $\bar{p}_i, \bar{n}_i \in B_{r/2}(\bar{p}_1)$  for every  $i \in \Gamma$ . Since the restriction of the quotient map  $\pi : B_{r/2}(p_1) \rightarrow B_{r/2}(\bar{p}_1)$  is an isometry,  $\Gamma$  induces a path in  $X$  starting at  $p_1$ , in the family of dipoles  $\delta_{p_i} - \delta_{n_i}$ . But this contradicts the fact that the representation of  $T$  is irreducible. We conclude that  $\bar{T} \neq 0$  in  $[\text{Lip}(X/A)]^*$ .

Let  $\bar{L} > 0$  be the length of the minimal connection of  $\bar{T}$ . By Proposition 7.2, there exist  $k \geq 1$ ,  $\sigma \in S_k$ , and  $\tilde{\zeta} \in \text{Lip}(X/A)$ ,  $|\tilde{\zeta}|_{\text{Lip}} \leq 1$ , such that

$$\frac{2\bar{L}}{3} \leq \sum_{i=1}^k d(\bar{p}_i, \bar{n}_{\sigma(i)}) = \sum_{i=1}^k [\tilde{\zeta}(\bar{p}_i) - \tilde{\zeta}(\bar{n}_i)] \leq \frac{4\bar{L}}{3}. \quad (7.45)$$

By taking  $k$  large enough, we may also assume that

$$\sum_{i \geq k+1} d(\bar{p}_i, \bar{n}_i) \leq \frac{\bar{L}}{3}.$$

For the sake of normalization, we set  $\tilde{\zeta}(A) = 0$ . We claim that  $\tilde{\zeta}$  can be chosen so that

$$\|\tilde{\zeta}\|_{L^\infty} \leq \frac{4\bar{L}}{3}.$$

In fact, for each  $i = 1, \dots, k$ , we define the intervals

$$J_i := [\tilde{\zeta}(\bar{n}_{\sigma(i)}), \tilde{\zeta}(\bar{p}_i)] \subset \mathbb{R}.$$

(Note that  $\tilde{\zeta}(\bar{p}_i) \geq \tilde{\zeta}(\bar{n}_{\sigma(i)})$  by (7.45).)

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  continuous such that  $h(0) = 0$ ,  $h$  is constant outside  $\bigcup_i J_i$  and  $h$  is affine linear with slope 1 on each  $J_i$ . It is easy to see that  $h \circ \tilde{\zeta}$  satisfies  $|h \circ \tilde{\zeta}|_{\text{Lip}} \leq 1$  and (7.45). Moreover, since  $\sum_i |J_i| \leq \frac{4}{3}\bar{L}$ , we have  $\|h \circ \tilde{\zeta}\|_\infty \leq \frac{4}{3}\bar{L}$ . This proves our claim.

We now let  $\bar{\zeta} := \frac{3}{4\bar{L}}\tilde{\zeta}$ . Then,  $\|\bar{\zeta}\|_\infty \leq 1$  and

$$\langle \bar{T}, \bar{\zeta} \rangle = \sum_{i=1}^k [\bar{\zeta}(\bar{p}_i) - \bar{\zeta}(\bar{n}_i)] + \sum_{i \geq k+1} [\bar{\zeta}(\bar{p}_i) - \bar{\zeta}(\bar{n}_i)] \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

The lemma now follows by taking the pull-back of  $\bar{\zeta}$ , namely  $\zeta := \bar{\zeta} \circ \pi$ .

**Remark 7.7** An inspection of the proof shows that one can construct  $\zeta$  so that (7.44) holds with  $1/4$  replaced by any number  $\theta \in (0, 1)$ .

## 7.9 Proof of Theorems 7.4 and 7.5

Theorems 7.4 and 7.5 can now be immediately derived from Lemma 7.7 :

**Proof of Theorem 7.4.** As we have already pointed out, we always have

$$\operatorname{supp} T \subset \overline{\bigcup_i \{\hat{p}_i\} \cup \bigcup_i \{\hat{n}_i\}},$$

even if the representation is not irreducible. To prove the reverse inclusion, let  $B \subset X$  be an open set containing some  $\hat{p}_{i_0}$  or some  $\hat{n}_{i_0}$ . Using the previous lemma we can find  $\zeta \in \operatorname{Lip}(X)$  such that  $\langle T, \zeta \rangle > 0$  and  $\operatorname{supp} \zeta \subset B$ . In other words,  $B \cap \operatorname{supp} T \neq \emptyset$ .

**Proof of Theorem 7.5.** Assume the irreducible representation of  $T$  has infinitely many terms. We shall show that there is no  $C > 0$  for which (7.12) is true.

Without loss of generality, we may assume there are infinitely many distinct points  $p_i$ , say  $\tilde{p}_1, \tilde{p}_2, \dots$ . Given  $M > 0$ , let  $\delta > 0$  be such that the balls  $B_\delta(\tilde{p}_i)$  are disjoint from each other for  $i = 1, \dots, M$ . Applying the lemma above to these balls, then, for each  $i_0$ , we get a bounded Lipschitz function  $\zeta_{i_0}$  satisfying (7.44). The function  $\zeta := \sum_{i=1}^M \zeta_i$  satisfies

$$\|\zeta\|_\infty \leq 1 \quad \text{and} \quad \langle T, \zeta \rangle \geq \frac{M}{4}.$$

Since  $M$  can be chosen arbitrarily large, the theorem follows.

## 7.10 Some comments about Definition 7.2

In this section we present some properties related to regular and singular functionals in  $\mathcal{Z}$  (in the sense of Definition 7.2). At the end, we shall prove that every  $T \in \mathcal{Z}$  can be decomposed in terms of a regular and singular part.

We first show that Definition 7.2 is intrinsic in the sense that it does not depend on the ambient space  $X$ . More precisely, we have

**Proposition 7.9** *Let  $T \in \mathcal{Z}$ . Then,*

- (a)  *$T$  is regular in  $X$  iff  $T$  is regular in  $\operatorname{supp} T$ ;*
- (b)  *$T$  is singular in  $X$  iff  $T$  is singular in  $\operatorname{supp} T$ .*

*In particular, the minimization problem (7.7) has a solution in  $X$  if, and only if, it has a solution in  $\operatorname{supp} T$ .*

**Proof.**

**Step 1.** Proof of (a).

Assume  $T$  is regular in  $\operatorname{supp} T$ . By definition, there exist  $(p_i), (n_i) \subset \operatorname{supp} T$  such that

$$\|T\| = \sum_i d(p_i, n_i) \quad \text{and} \quad T = \sum_i (\delta_{p_i} - \delta_{n_i}) \quad \text{in} \quad [\operatorname{Lip}(\operatorname{supp} X)]^*.$$

Since the number  $\|T\|$  is the same, whether we compute it using  $X$  or  $\operatorname{supp} T$  as the ambient space, we conclude that  $T$  is regular in  $X$ .

Suppose now that  $T$  is regular in  $X$ . Then, we can find sequences  $(p_i), (n_i) \subset X$  such that

$$\|T\| = \sum_i d(p_i, n_i) \quad \text{and} \quad T = \sum_i (\delta_{p_i} - \delta_{n_i}) \quad \text{in} \quad [\operatorname{Lip}(X)]^*.$$

It follows from Remark 7.6 that we can construct an irreducible representation of  $T$  :

$$T = \sum_j (\delta_{\hat{p}_j} - \delta_{\hat{n}_j}) \quad \text{in } [\text{Lip}(X)]^*,$$

so that

$$\|T\| \leq \sum_i d(\hat{p}_i, \hat{n}_i) \leq \sum_i d(p_i, n_i) = \|T\|.$$

Since  $\hat{p}_j, \hat{n}_j \in \text{supp } T$  for every  $j$  (see Theorem 7.4), we conclude that  $T$  is regular in  $\text{supp } T$ .

**Step 2.** Proof of (b).

Assume  $T$  is not singular in  $\text{supp } T$ . Then, we can find  $T_1, T_2 \in \mathcal{Z}(\text{supp } T)$  such that  $T = T_1 + T_2$ ,  $\|T\| = \|T_1\| + \|T_2\|$ , and  $T_1 \neq 0$  is regular in  $\text{supp } T$ . By (a),  $T_1$  is also regular in  $X$ . We conclude that  $T$  is not singular in  $X$ .

The converse statement is a trivial consequence of the following lemma :

**Lemma 7.8** *If  $T \in \mathcal{Z}$  is not singular in  $X$ , then there exist  $r, q \in \text{supp } T$ ,  $r \neq q$ , such that*

$$\|T\| = d(r, q) + \|T - (\delta_r - \delta_q)\|. \quad (7.46)$$

**Proof.**

**Step 1.** (7.46) holds for some  $r, q \in X$ , with  $r \neq q$ .

Let  $T_1 = \sum_i (\delta_{r_i} - \delta_{q_i}) \in \mathcal{Z}$  be regular and nonzero such that

$$\|T\| = \|T_1\| + \|T - T_1\| = \sum_i d(r_i, q_i) + \|T - T_1\|.$$

Without loss of generality, we may assume that  $r_1 \neq q_1$ . Then, applying the triangle inequality we have

$$\begin{aligned} \|T\| &\leq \|(\delta_{r_1} - \delta_{q_1})\| + \|T - (\delta_{r_1} - \delta_{q_1})\| \\ &\leq \|(\delta_{r_1} - \delta_{q_1})\| + \left\| \sum_{i \neq 1} (\delta_{r_i} - \delta_{q_i}) \right\| + \left\| T - \sum_i (\delta_{r_i} - \delta_{q_i}) \right\| \\ &\leq \sum_i d(r_i, q_i) + \|T - T_1\| = \|T\|. \end{aligned}$$

Therefore, equality must hold everywhere. Since  $d(r_1, q_1) = \|(\delta_{r_1} - \delta_{q_1})\|$ , we conclude that (7.46) holds with  $r := r_1$  and  $q := q_1$ .

**Step 2.** Proof of the lemma completed.

Let  $r := n_0$  and  $q := p_0$  be two distinct points in  $X$  for which (7.46) holds, and let  $T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i})$  in  $[\text{Lip}(X)]^*$  be an irreducible representation of  $T$ .

Applying Proposition 7.4 to

$$\tilde{T} := T - (\delta_{n_0} - \delta_{p_0}) = \sum_{i=0}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{in } [\text{Lip}(X)]^*,$$

we can decompose  $\tilde{T}$  in terms of disjoint cycles :

$$\tilde{T} = \sum_j \tilde{T}_{[\Lambda_j]} \quad \text{in } [\text{Lip}(X)]^*$$

such that

$$\|\tilde{T}\| = \sum_j \ell_{[\Lambda_j]}^*. \quad (7.47)$$

Without loss of generality, we may assume that  $0 \in \Lambda_1$ .

We claim that  $\Lambda_1 \setminus \{0\}$  is nonempty. In fact, assume by contradiction that  $\Lambda_1 = (0)$ . Then, we would have

$$\|T\| = d(p_0, n_0) + \|\tilde{T}\| = 2d(p_0, n_0) + \sum_{j \neq 1} \ell_{[\Lambda_j]}^* \geq 2d(p_0, n_0) + \|T\|.$$

(In the last step we use that  $T = \sum_{j \neq 1} T_{[\Lambda_j]}$ , and so

$$\|T\| \leq \sum_{j \neq 1} \|T_{[\Lambda_j]}\| \leq \sum_{j \neq 1} \ell_{[\Lambda_j]}^*.)$$

Therefore, we must have  $p_0 = n_0$ , which is a contradiction.

By taking another representative of  $[\Lambda_1]$  if necessary, we may assume that 0 is the largest element in  $\Lambda_1$ . Let

$$\tilde{r} := \lim_{\lambda \in \Lambda_1 \setminus \{0\} \uparrow} p_\lambda \quad \text{and} \quad \tilde{q} := \lim_{\lambda \in \Lambda_1 \setminus \{0\} \downarrow} n_\lambda.$$

Since  $p_\lambda, n_\lambda \in \text{supp } T$  for every  $\lambda \neq 0$ , we have  $\tilde{r}, \tilde{q} \in \text{supp } T$ .

Note that  $\tilde{r} \neq \tilde{q}$ . In fact, if  $\tilde{r} = \tilde{q}$ , then we would have

$$\ell_{[\Lambda]}^* \geq \ell_{[0]}^* + \ell_{[\Lambda \setminus \{0\}]}^*.$$

In other words, (0) could have been taken as one of the cycles in (7.47). As we have already seen, this cannot be the case; thus,  $\tilde{r} \neq \tilde{q}$ .

We claim that (7.46) holds with  $\tilde{r}$  and  $\tilde{q}$ .

By a slight abuse of notation, let us denote by  $[\Lambda_1]_{\text{new}}$  the cycle  $[\Lambda_1]$  where  $\delta_{p_0} - \delta_{n_0}$  is replaced by the dipole  $\delta_{\tilde{q}} - \delta_{\tilde{r}}$ . It is easy to see that (see, e.g., Remark 7.3)

$$\ell_{[\Lambda_1]}^* = \ell_{[\Lambda_1]_{\text{new}}}^* + d(\tilde{r}, n_0) + d(p_0, \tilde{q}).$$

We then have

$$\begin{aligned} \|T\| &= d(p_0, n_0) + \|\tilde{T}\| \\ &= d(p_0, n_0) + d(\tilde{r}, n_0) + d(p_0, \tilde{q}) + \ell_{[\Lambda_1]_{\text{new}}}^* + \sum_{j \neq 1} \ell_{[\Lambda_j]}^* \\ &\geq d(\tilde{r}, \tilde{q}) + \|T + (\delta_{\tilde{q}} - \delta_{\tilde{r}})\|. \end{aligned}$$

Thus,

$$\|T\| \geq d(\tilde{r}, \tilde{q}) + \|T - (\delta_{\tilde{r}} - \delta_{\tilde{q}})\| \geq \|T\|.$$

This establishes the lemma.

A simple consequence of Lemma 7.8 is the following : assume we are given some  $T \in \mathcal{Z}$ , and we want to show that  $T$  is singular ; it suffices to show that if  $r, q \in \text{supp } T$  satisfies (7.46), then  $r = q$ . We have already used this fact in Section 7.6.

We now state some properties related to Definition 7.2 :

**Remark 7.8** Assume  $T = T_1 + T_2$ , where  $T_1, T_2 \in \mathcal{Z}$  and  $\|T\| = \|T_1\| + \|T_2\|$ . Then, we have :

- (a) if  $T_1$  and  $T_2$  are both regular, then so is  $T$  ;
  - (b) if  $T$  is singular, then  $T_1$  and  $T_2$  are singular as well.
- (a) and (b) follow immediately from Definition 7.2 (in fact, they still hold in the case of infinite sums). Note, however, that
- (c) if we know that  $T$  and  $T_1$  are regular, then we cannot conclude that  $T_2$  is regular.
- Take, for instance,

$$T_1 := \sum_{i \geq 1} (\delta_{p_i} - \delta_{n_i}) \quad \text{and} \quad T_2 := - \sum_{i \geq 0} (\delta_{p_i} - \delta_{n_i}),$$

where  $p_i$  and  $n_i$  are given by Example 7.1. Then,  $T_1$  and  $T = \delta_1 - \delta_0$  are regular, but  $T_2$  is singular. Also note that

$$\text{supp } T = \{0, 1\} \subsetneq C_\alpha = \text{supp } T_1 \cup \text{supp } T_2.$$

- (d) it is possible to construct  $T_1, T_2 \in \mathcal{Z}$ , both singular, such that  $T = T_1 + T_2$  is regular. Let  $X = S^1$  equipped with its geodesic distance. We consider two sequences in  $S^1$  (see Figure 7.6) :

$$\begin{aligned} r_k &:= e^{\pi p_k i} & \text{and} & & q_k &:= e^{\pi n_k i} & \forall k > 0, \\ r_k &:= e^{\pi(p_k+1)i} & \text{and} & & q_k &:= e^{\pi(n_k+1)i} & \forall k < 0, \end{aligned}$$

where  $p_k$  and  $n_k$  belong to the Cantor set  $C_\alpha$  as before. Then,

$$\begin{aligned} T_1 &:= \sum_{k \geq 1} (\delta_{r_k} - \delta_{q_k}) + (\delta_{(1,0)} - \delta_{(-1,0)}), \\ T_2 &:= \sum_{k \leq -1} (\delta_{r_k} - \delta_{q_k}) + (\delta_{(-1,0)} - \delta_{(1,0)}) \end{aligned}$$

are both singular for every  $\alpha \in (0, 1)$  and  $\|T_1\| = \|T_2\| = \alpha\pi$ .

We now take  $\alpha = \frac{1}{2}$ . Then, (see, e.g., Example 7.8)

$$T = T_1 + T_2 \text{ is regular} \quad \text{and} \quad \|T\| = \pi = \|T_1\| + \|T_2\|.$$

The proposition below gives the first part of Theorem 7.2 :

**Proposition 7.10** *Given  $T \in \mathcal{Z}$ , there exist  $T_{\text{reg}}, T_{\text{sing}} \in \mathcal{Z}$  such that  $T_{\text{reg}}$  is regular,  $T_{\text{sing}}$  is singular,*

$$T = T_{\text{reg}} + T_{\text{sing}}, \quad \text{and} \quad \|T\| = \|T_{\text{reg}}\| + \|T_{\text{sing}}\|.$$

Moreover,  $T_{\text{reg}}$  and  $T_{\text{sing}}$  can be chosen so that

$$\text{supp } T = \text{supp } T_{\text{reg}} \cup \text{supp } T_{\text{sing}}. \tag{7.48}$$

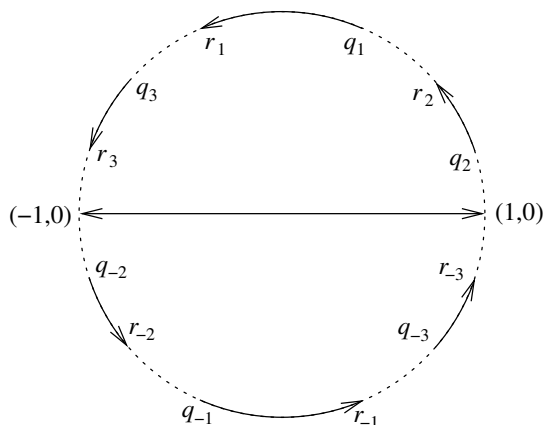


FIG. 7.6 – Dipoles in Remark 7.8 (d)

**Proof.** In view of Proposition 7.9, it suffices to prove the result for  $X = \text{supp } T$ , in which case (7.48) is automatically satisfied.

We proceed by transfinite induction.

Let  $T_0 := T$  and  $T_{0,1} := 0$ .

Let  $\alpha \geq 1$  be a nonlimit countable ordinal. If  $T_{\alpha-1}$  is not singular, then we can find  $T_\alpha, T_{\alpha,1} \in \mathcal{Z}$  such that  $T_{\alpha,1}$  is regular,  $T_{\alpha,1} \neq 0$  in  $[\text{Lip}(X)]^*$ ,

$$T_{\alpha-1} = T_{\alpha,1} + T_\alpha \quad \text{and} \quad \|T_{\alpha-1}\| = \|T_{\alpha,1}\| + \|T_\alpha\|. \tag{7.49}$$

Assume, for instance, that  $\alpha = k \in \mathbb{N}$ . Summing (7.49) over  $\alpha$  replaced by  $j$  we get

$$T = \sum_{j=1}^k T_{j,1} + T_k \quad \text{and} \quad \|T\| = \sum_{j=1}^k \|T_{j,1}\| + \|T_k\|.$$

If  $\alpha$  is a limit countable ordinal, then we take

$$T_\alpha := T - \sum_{\beta < \alpha} T_{\beta,1} \quad \text{and} \quad T_{\alpha,1} = 0.$$

By construction, for every  $\alpha$  we have

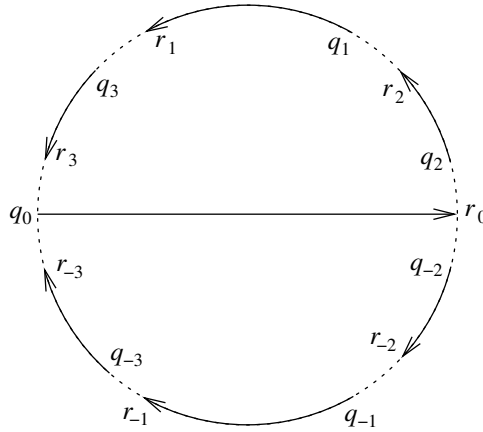
$$T = \sum_{\beta \leq \alpha} T_{\beta,1} + T_\alpha \quad \text{and} \quad \|T\| = \sum_{\beta \leq \alpha} \|T_{\beta,1}\| + \|T_\alpha\|.$$

In particular, (see Remark 7.8 (a))

$$\sum_{\beta \leq \alpha} T_{\beta,1} \quad \text{is regular.}$$

On the other hand, note that if  $T_\alpha$  is not singular, then we have the strict inequality  $\|T_\alpha\| > \|T_{\alpha+1}\|$ . In other words, the family  $(\|T_\alpha\|)_\alpha$  is strictly decreasing, so it can only



FIG. 7.7 – Dipoles  $\delta_{r_k} - \delta_{q_k}$  in Example 7.9

have countably many terms. Therefore, our construction has to stop at some countable ordinal  $\alpha_0$ , which means that  $T_{\alpha_0}$  is singular. Thus,

$$T = T_{\text{reg}} + T_{\text{sing}}, \quad \text{where} \quad T_{\text{reg}} := \sum_{\beta \leq \alpha_0} T_{\beta,1} \quad \text{and} \quad T_{\text{sing}} := T_{\alpha_0}.$$

We now show that the decomposition of  $T$  in terms of a regular and a singular part need not be unique.

**Example 7.9** Let  $X = S^1$  equipped with its geodesic distance as before. We consider two sequences  $(r_k)_{k \in \mathbb{Z}}, (q_k)_{k \in \mathbb{Z}} \subset S^1$  given by (see Figure 7.7)

$$\begin{aligned} r_k &:= e^{\pi p_k i} & \text{and} & & q_k &:= e^{\pi n_k i}, & \forall k \geq 0, \\ r_k &:= e^{-\pi p_k i} & \text{and} & & q_k &:= e^{-\pi n_k i}, & \forall k < 0. \end{aligned}$$

Then,

$$T := \sum_{k=-\infty}^{\infty} (\delta_{r_k} - \delta_{q_k}) \text{ is irreducible} \quad \text{and} \quad \|T\| = \pi.$$

Moreover,

$$T = \sum_{k < 0} (\delta_{r_k} - \delta_{q_k}) + \sum_{k \geq 0} (\delta_{r_k} - \delta_{q_k}) \quad \text{in} \quad [\text{Lip}(X)]^*$$

is a decomposition of  $T$  in terms of a regular and a singular part. By symmetry, we also have a second decomposition; namely,  $T = \sum_{k > 0} + \sum_{k \leq 0}$ .

## 7.11 Proof of Theorem 7.2 completed

In view of Proposition 7.10, we are left to show that (7.9) holds, where each  $\text{supp } T_i$  is homeomorphic to the Cantor set in  $\mathbb{R}$ .

Without loss of generality, we may assume that  $T$  is singular. Let

$$T = \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \quad \text{in } [\text{Lip}(X)]^*$$

be an irreducible representation of  $T$ .

Applying Proposition 7.4, we can find a sequence of disjoint co-minimal cycles  $([\Lambda_j])$  such that

$$T = \sum_j T_{[\Lambda_j]} \quad \text{and} \quad \|T\| = \sum_j \|T_{[\Lambda_j]}\|. \quad (7.50)$$

Since  $T$  is singular, so is  $T_{[\Lambda_j]}$  for each  $j$ . Moreover, Proposition 7.7 implies that we can further split each  $[\Lambda_j]$  in terms of simple cycles so that (7.36) holds. Therefore, we can assume that each  $[\Lambda_j]$  is a simple cycle.

Since the representation

$$T_j := T_{[\Lambda_j]} = \sum_{\lambda \in \Lambda_j} (\delta_{p_\lambda} - \delta_{n_\lambda}) \quad \text{in } [\text{Lip}(X)]^* \quad (7.51)$$

is still irreducible, we have  $S_{[\Lambda_j]} = \text{supp } T_j$ . In particular, we conclude from Proposition 7.6 that

$$\|T_j\| = \mathcal{H}^1(\text{supp } T_j). \quad (7.52)$$

Assertion (7.9) is an immediate consequence of (7.50)–(7.52). Note also that

$$\text{supp } T = \overline{\bigcup_j \text{supp } T_j}.$$

The last part of the theorem follows from the proposition below :

**Proposition 7.11** *Assume that  $[\Lambda]$  is a simple cycle and*

$$T_{[\Lambda]} = \sum_{\lambda \in \Lambda} (\delta_{p_\lambda} - \delta_{n_\lambda}) \quad \text{in } [\text{Lip}(X)]^*$$

*is an irreducible representation of  $T_{[\Lambda]}$ . Then,  $\text{supp } T_{[\Lambda]}$  is homeomorphic to the Cantor set in  $\mathbb{R}$ .*

**Proof.** Since  $\Lambda$  is infinite, we can assume that  $\Lambda = \mathbb{N} \cup \{0\}$ , and 0 is its largest element. The fact that the representation of  $T_{[\Lambda]}$  is irreducible and  $\text{gap } [\Lambda] = 0$  implies that  $\Lambda$  cannot have a smallest element. We now take  $\Lambda_1 := \Lambda \setminus \{0\}$ .

Let  $C \subset [0, 1]$  be the standard Cantor set in  $\mathbb{R}$ . We denote by  $(J_k)_{k \geq 1}$ ,  $J_k = (a_k, b_k)$ , the sequence of disjoint open intervals which are removed from  $[0, 1]$  in the construction of  $C$ . We define  $\Omega := \mathbb{N}$  to be an ordered set so that  $k_1 \leq k_2$  iff  $a_{k_1} \leq a_{k_2}$ .

We claim there exists a bijection  $\sigma : \Omega \rightarrow \Lambda_1$  which preserves the order of  $\Omega$ , i.e. if  $k_1 \leq k_2$  in  $\Omega$ , then  $\sigma(k_1) \leq \sigma(k_2)$  in  $\Lambda_1$ .

In fact, let  $\sigma(1) := 1$ . We next define

$$\begin{aligned} \sigma(2) &:= \text{smallest integer in } \{\lambda \in \Lambda_1 : \lambda < 1\}, \\ \sigma(3) &:= \text{smallest integer in } \{\lambda \in \Lambda_1 : \lambda > 1\}. \end{aligned}$$

Note that  $\sigma(2) < \sigma(1) < \sigma(3)$ . Moreover, we can keep this construction indefinitely since each of the sets of the form

$$\{\lambda \in \Lambda_1 : \lambda < \lambda_0\}, \quad \{\lambda \in \Lambda_1 : \lambda > \lambda_0\}, \quad \text{and} \quad \{\lambda \in \Lambda_1 : \lambda_0 < \lambda < \lambda_1\}$$

has no smallest nor largest element. This proves our claim.

We define the map

$$\begin{array}{ccc} h : \bigcup_k \{a_k\} \cup \{b_k\} & \longrightarrow & \text{supp } T_{[\Lambda]} \\ a_k & \longmapsto & n_{\sigma(k)} \\ b_k & \longmapsto & p_{\sigma(k)} \end{array} .$$

Note that  $h$  is uniformly continuous (since  $\ell_{[\Lambda]} + \ell_{[\Lambda]}^* < \infty$ ), and so it can be extended by continuity as a map  $h : C \rightarrow \text{supp } T_{[\Lambda]}$ . It is easy to see that  $h$  is surjective.

We claim that  $h$  is injective. Suppose by contradiction  $h$  is not injective. We can find  $c < d$  in  $C$  such that  $h(c) = h(d)$ . Let  $\Omega_1 := \{k : J_k \subset (c, d)\}$ . Then  $\sigma(\Omega_1)$  is a segment of  $\Lambda_1 \subsetneq \Lambda$  such that  $\text{gap } \sigma(\Omega_1) = 0$ . In other words,  $[\sigma(\Omega_1)]$  is a closed cycle, which is a contradiction.

We conclude that  $h$  is a continuous bijection between  $C$  and  $\text{supp } T_{[\Lambda]}$ . Since  $C$  is compact,  $h$  is a homeomorphism.

## 7.12 Compact subsets of $\mathcal{Z}$

We start with the following (see [22])

**Proposition 7.12**  *$\mathcal{Z}$  is a complete metric space.*

**Proof.** It suffices to show that if the series  $T := \sum_k T_k$  converges absolutely and  $T_k \in \mathcal{Z}$  for each  $k \geq 1$ , then  $T \in \mathcal{Z}$ .

For  $k \geq 1$  fixed, it follows from Proposition 7.2 that we can find sequences  $(p_i^k)_i$  and  $(n_i^k)_i$  in  $X$  such that

$$\begin{aligned} T_k &= \sum_i (\delta_{p_i^k} - \delta_{n_i^k}) \quad \text{in } [\text{Lip}(X)]^*, \\ \sum_i d(p_i^k, n_i^k) &\leq \|T_k\| + \frac{1}{2^k}. \end{aligned}$$

Thus,

$$\sum_k \sum_i d(p_i^k, n_i^k) \leq \sum_k \|T_k\| + 1 < \infty,$$

from which we conclude that

$$T = \sum_k \sum_i (\delta_{p_i^k} - \delta_{n_i^k}) \in \mathcal{Z}.$$

In order to describe the compact subsets of  $\mathcal{Z}$ , we first introduce a definition :

**Definition 7.17**  $\mathcal{A} \subset \mathcal{Z}$  is equisummable if, and only if,  $\mathcal{A}$  is bounded and, for each  $\varepsilon > 0$ , there exist  $k_\varepsilon \geq 1$  and  $K_\varepsilon \subset X$  compact such that the following holds : for every  $T \in \mathcal{A}$  we can find  $T_1, T_2 \in \mathcal{Z}$ ,  $T = T_1 + T_2$  in  $\mathcal{Z}$ , where

- (i)  $T_1$  can be written as a sum of at most  $k_\varepsilon$  dipoles supported in  $K_\varepsilon$ ;
- (ii)  $\|T_2\| < \varepsilon$ .

Note that this definition is satisfied if  $\mathcal{A}$  is finite. More generally, we have the following

**Theorem 7.6**  $\mathcal{A} \subset \mathcal{Z}$  is relatively compact if, and only if,  $\mathcal{A}$  is equisummable.

**Proof.** Assume  $\mathcal{A}$  is relatively compact in  $\mathcal{Z}$ , and let  $(T_k)$  in  $\mathcal{A}$  be such that  $T_k \rightarrow T \in \mathcal{Z}$ . It suffices to show that  $(T_k)$  is equisummable. Without loss of generality, we may assume that  $T = 0$ ; in other words,  $\|T_k\| \rightarrow 0$ . Given  $\varepsilon > 0$ , let  $k_0 \geq 1$  be such that  $\|T_k\| < \varepsilon$  for every  $k \geq k_0$ . On the other hand, Definition 7.17 clearly holds for the finite set  $\{T_1, \dots, T_{k_0-1}\}$ . We conclude that  $(T_k)$  is equisummable.

We now prove the converse statement. By assumption, given  $\varepsilon > 0$  there exist  $k_\varepsilon \geq 1$  and a compact set  $K_\varepsilon \subset X$  such that for each  $T \in \mathcal{A}$  we can write  $T = T_1 + T_2$  in  $\mathcal{Z}$ , where

$$T_1 = \sum_{i=1}^{k_\varepsilon} (\delta_{p_i} - \delta_{n_i}), \quad p_i, n_i \in K_\varepsilon, \quad \text{and} \quad \|T_2\| \leq \varepsilon. \quad (7.53)$$

Since  $\mathcal{A}$  is bounded and  $K_\varepsilon$  is compact,  $\{T_1\}_{T \in \mathcal{A}}$  is relatively compact in  $\mathcal{Z}$ . In particular, there exists a finite number of balls  $B_\varepsilon(S_1), \dots, B_\varepsilon(S_n)$  in  $\mathcal{Z}$  which cover  $\{T_1\}_{T \in \mathcal{A}}$ . By (7.53), the balls  $B_{2\varepsilon}(S_1), \dots, B_{2\varepsilon}(S_n)$  cover  $\mathcal{A}$ , which means that  $\mathcal{A}$  is totally bounded. Since  $X$  is complete,  $\mathcal{A}$  is relatively compact.

In contrast with Proposition 7.12, the example below shows that  $\mathcal{Z}$  is not closed in  $[\text{Lip}(X)]^*$  with respect to the weak\* topology :

**Example 7.10** Let  $X = [0, 1]$ . For each  $k \geq 1$  we define

$$T_k := \sum_{\substack{j=0 \\ j \text{ even}}}^{2^k-2} (\delta_{\frac{j+1}{2^k}} - \delta_{\frac{j}{2^k}}).$$

It is easy to see that

$$T_k \xrightarrow{*} \frac{1}{2}(\delta_1 - \delta_0) \notin \mathcal{Z}.$$

Recall that, in general,  $\text{Lip}(X)$  is not separable; this implies that the unit ball in  $[\text{Lip}(X)]^*$  is not metrizable with respect to the weak\* topology. The example below shows that bounded sequences in  $\mathcal{Z}$  do not necessarily converge in the weak\* topology of  $[\text{Lip}(X)]^*$ .

**Example 7.11** We take  $X = [0, 1] \subset \mathbb{R}$ . Let

$$T_k = k(\delta_{1/k} - \delta_0).$$

If  $\zeta \in \text{Lip}(X)$  has a derivative at 0, then we have

$$\langle T_k, \zeta \rangle = \frac{\zeta(1/k) - \zeta(0)}{1/k} \rightarrow \zeta'(0).$$

It is then clear that  $(T_k)$  has no subsequence converging in  $[\text{Lip}(X)]^*$ . However, because  $\|T_k\| = 1$ , we can find a subnet  $(T_{k_\alpha})_{\alpha \in A}$  such that

$$T_{k_\alpha} \xrightarrow{*} T$$

for some  $T \in \overline{\text{conv}}(\mathcal{Z})$ . Since  $\text{supp } T = \{0\}$ , it follows from Corollary 7.2 that  $T \notin \mathcal{Z}$  (otherwise  $T$  would be expressed in terms of finitely many dipoles, which clearly cannot be the case).

An alternative approach to show that  $T \notin \mathcal{Z}$  (without making use of irreducible representations) is the following. Assume by contradiction that  $T \in \mathcal{Z}$ . Then, given  $\varepsilon > 0$ , there exists  $M_\varepsilon > 0$  such that

$$|\langle T, \zeta \rangle| \leq M_\varepsilon \|\zeta\|_{L^\infty} + \varepsilon |\zeta|_{\text{Lip}}, \quad \forall \zeta \in \text{Lip}(X).$$

This estimate implies that given any sequence  $(\zeta_j)_{j \geq 1}$  in  $\text{Lip}(X)$ , such that  $|\zeta_j|_{\text{Lip}} \leq 1$ ,  $\forall j \geq 1$ , and  $\zeta_j \rightarrow 0$  uniformly in  $X$ , then

$$\lim_{j \rightarrow \infty} \langle T, \zeta_j \rangle = 0;$$

but this property contradicts the fact that  $\langle T, \zeta \rangle = \zeta'(0)$  for every  $\zeta \in C^1[0, 1]$ .

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## Résumé

Dans cette thèse, nous étudions d'abord le problème des singularités éliminables des EDP elliptiques du second ordre ; le cas modèle étant  $-\Delta u + cu \geq f$  sur  $\Omega \setminus K$ , avec  $u \geq 0$  et  $\text{cap}_2(K) = 0$ . Nous démontrons aussi un principe du maximum fort pour l'opérateur  $-\Delta + a(x)$ , avec un potentiel  $a \in L^1$ . Ces deux résultats utilisent plusieurs formulations de l'inégalité de Kato classique.

Nous présentons ensuite quelques variantes de l'inégalité de Poincaré, motivés par une nouvelle caractérisation des espaces de Sobolev.

Puis, nous nous intéressons aux singularités topologiques des fonctions dans l'espace  $W^{1,1}(S^2; S^1)$ . À cet effet, nous étudions leur énergie relaxée et la variation totale du jacobien. Finalement, nous considérons plusieurs propriétés des distributions de la forme  $\sum_j (\delta_{p_j} - \delta_{n_j})$ , définies sur un espace métrique complet.

**Mots-clés:** Inégalité de Kato, singularités éliminables, inégalité de Poincaré, espaces de Sobolev, singularités topologiques, énergie relaxée, connexion minimale

## Abstract

In this dissertation, we first study removable singularity results for second order elliptic PDEs, the model case being the following :  $-\Delta u + cu \geq f$  in  $\Omega \setminus K$ , with  $u \geq 0$  and  $\text{cap}_2(K) = 0$ . We also prove a strong maximum principle for the operator  $-\Delta + a(x)$ , with a potential  $a \in L^1$ . These results rely on some variants of the standard Kato's inequality.

We next present an estimate in the spirit of Poincaré's inequality. The motivation for our result comes from a new characterization of the Sobolev spaces.

We also study topological singularities of maps  $g \in W^{1,1}(S^2; S^1)$ ; we compute, for instance, the relaxed energy and the total variation of the Jacobian of  $g$ . Finally, we consider several properties of the distributions of the form  $\sum_j (\delta_{p_j} - \delta_{n_j})$ , defined on a complete metric space.

**Keywords:** Kato's inequality, removable singularities, Poincaré's inequality, Sobolev spaces, topological singularities, relaxed energy, minimal connection

