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# Dimensional reduction for heterogeneous, slit or cracked bodies

Jean-François Babadjian

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# THÈSE DE DOCTORAT

présentée par

**Jean-François Babadjian**

pour obtenir le titre de

**DOCTEUR DE L'UNIVERSITÉ PARIS NORD**

**Spécialité : Mathématiques**

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## RÉDUCTION DIMENSIONNELLE POUR DES MILIEUX HÉTÉROGÈNES, TROUÉS OU FISSURÉS

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Thèse soutenue le 14 octobre 2005 devant le jury composé de

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Thèse préparée au sein du  
Laboratoire des Propriétés Mécaniques et Thermodynamiques des Matériaux  
(UPR CNRS 9001), Université Paris Nord



*A ma Petite Claire*



*« On ne pouvait dire que ce fût une matérialisation ;  
strictement bidimensionnelle, elle ne méritait  
pas un aussi gros mot. »*

RAYMOND QUENEAU (Le chiendent)



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# RÉDUCTION DIMENSIONNELLE POUR DES MILIEUX HÉTÉROGÈNES, TROUÉS OU FISSURÉS

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## Résumé

Cette thèse traite de la justification de modèles de membranes comme limites de « comportements élastiques » non linéaires tridimensionnels (les guillemets ont trait à l'absence de l'hypothèse classique d'explosion de l'énergie lorsque le Jacobien de la transformation tend vers zéro). La réduction dimensionnelle est vue comme un problème de  $\Gamma$ -convergence sur l'énergie élastique, lorsque l'épaisseur tend vers zéro. Dans un premier temps, nous décrivons des hétérogénéités macroscopiques où les forces de surface peuvent engendrer une densité de moment fléchissant, produisant un vecteur de Cosserat. Puis nous considérerons des hétérogénéités microscopiques réparties périodiquement, donnant lieu à prendre en compte deux types de problèmes simultanés : la réduction de dimension et l'homogénéisation réitérée. Ensuite, des films minces possédant une microstructure dégénérée due à la présence de vide sur la surface moyenne seront étudiés dans le cas où l'épaisseur est beaucoup plus petite que la période de distribution des perforations. Enfin, nous envisagerons la possibilité de rupture et analyserons l'évolution quasistatique des fissures pour une énergie de surface de type Griffith.

**Mots clés :**  $\Gamma$ -convergence, relaxation, réduction de dimension, quasiconvexité, homogénéisation, domaines perforés, capacité non linéaire, fonctions à variation bornée, problèmes aux discontinuités libres, mécanique de la rupture.



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# DIMENSIONAL REDUCTION FOR HETEROGENEOUS, SLIT OR CRACKED BODIES

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## Abstract

This thesis is concerned with the justification of membrane models as zero-thickness limits of three dimensional nonlinear « elastic behavior » (the quotes refer to the absence of the usual requirement that the energy should blow up as the Jacobian of the transformation tends to zero). The dimensional reduction is viewed as a  $\Gamma$ -convergence problem for the elastic energy. We first consider macroscopic heterogeneities, also taking into account the case where the external loads induce a density of bending moment that produces a Cosserat vector. Then, we study periodic microscopic heterogeneities, which introduces two competing features : dimensional reduction and reiterated homogenization. Thin films with complex degenerate microstructure due to the presence of voids on the mid-surface are investigated when the thickness is much smaller than the period of distribution of the perforations. Finally, brittle thin films and their quasistatic crack evolution are presented for a Griffith type surface energy density.

**Key words :**  $\Gamma$ -convergence, relaxation, dimensional reduction, quasiconvexity, homogenization, perforated domains, nonlinear capacity, functions of bounded variation, free discontinuity problems, brittle fracture.



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# Notations

Excepté au chapitre 3, la dimension  $n$  sera toujours égale à 2 ou 3.

## Mesures

$\mathcal{L}^n$	mesure de Lebesgue $n$ -dimensionnelle dans $\mathbb{R}^n$
$\mathcal{H}^{n-1}$	mesure de Hausdorff $(n-1)$ -dimensionnelle
$\#$	mesure de dénombrement
$\mu _E$	$\mu$ restreinte à l'ensemble $E$
$\mu \otimes \nu$	mesure produit entre $\mu$ et $\nu$
$\text{Cap}_p$	$p$ -capacité

## Vecteurs et ensembles

$\mathbb{R}^{m \times n}$	ensemble des matrices réelles $m \times n$
$x_\alpha$	$(x_1, \dots, x_{n-1})$
$Q'_r(a)$	cube $(n-1)$ -dimensionnel de centre $a \in \mathbb{R}^{n-1}$ et de côté $r > 0$
$Q'$	$(0, 1)^{n-1}$
$Q$	$(0, 1)^n$
$B'_r(a)$	boule $(n-1)$ -dimensionnelle de centre $a \in \mathbb{R}^{n-1}$ et de rayon $r > 0$
$C_{1,N}$	$\{(x_\alpha, 0) \in \mathbb{R}^n : 1 \leq  x_\alpha  < N\}$
$C_{1,\infty}$	$\{(x_\alpha, 0) \in \mathbb{R}^n :  x_\alpha  \geq 1\}$
$\omega$	ouvert borné de $\mathbb{R}^{n-1}$
$I$	$(-1, 1)$
$\Omega$	$\omega \times I$
$\mathcal{A}(\omega)$	famille des ouverts contenus dans $\omega$
$\mathcal{A}_0$	familles des ouverts bornés contenus dans $\mathbb{R}^{n-1}$
$\mathcal{R}(\omega)$	sous-famille de $\mathcal{A}(\omega)$ obtenue en prenant toute union finie de cubes ouverts inclus dans $\omega$ , centrés en des points rationnels et de côtés rationnels
$\tilde{\subset}, \tilde{=}$	inclusion, égalité à un ensemble de $\mathcal{H}^{n-1}$ -mesure nulle près
$\llbracket a \rrbracket$	partie entière de $a$



## Notations fonctionnelles

$\int_E u d\mu$	$\mu(E)^{-1} \int_E u d\mu$ moyenne de $u$ sur $E$ par rapport à $\mu$
$\text{Supp}(u)$	support de $u$
$Du$	dérivée distributionnelle
$D^a u$	partie absolument continue de la dérivée
$D^j u$	partie saut de la dérivée
$D^c u$	partie Cantor de la dérivée
$S(u)$	ensemble des sauts (complémentaire des points de Lebesgue)
$\nu_u$	normale approchée à $S(u)$
$u^\pm$	limites approchées supérieures et inférieures
$\nabla u$	gradient (approché)
$\nabla_\alpha u$ (resp. $\nabla_n u$ )	gradient (approché) par rapport à $x_\alpha$ (resp. $x_n$ )

## Espaces fonctionnels

Soit  $U$  un ouvert de  $\mathbb{R}^n$ .

$\mathcal{C}_c^k(U; \mathbb{R}^m)$	$\{u : U \rightarrow \mathbb{R}^m \text{ } k\text{-fois continûment différentiables} : \text{Supp}(u) \subset U\}$
$\mathcal{C}_c^\infty(U; \mathbb{R}^m)$	$\bigcap_{k \in \mathbb{N}} \mathcal{C}_c^k(U; \mathbb{R}^m)$
$\mathcal{C}_c(U; \mathbb{R}^m)$	$\mathcal{C}_c^0(U; \mathbb{R}^m)$
$\mathcal{C}_0(U; \mathbb{R}^m)$	fermeture de $\mathcal{C}_c(U; \mathbb{R}^m)$ pour la convergence uniforme
$\mathcal{M}_b(U; \mathbb{R}^m)$	espace des mesures de Radon à valeur dans $\mathbb{R}^m$ , identifié à $[\mathcal{C}_0(U; \mathbb{R}^m)]'$
$L^p(U; \mathbb{R}^m)$	$\{u : U \rightarrow \mathbb{R}^m \text{ mesurables} : \int_U  u ^p dx < +\infty\}$ ( $1 \leq p < \infty$ )
$L^\infty(U; \mathbb{R}^m)$	$\{u : U \rightarrow \mathbb{R}^m \text{ mesurables} : \text{ess sup}_U  u  < +\infty\}$
$W^{1,p}(U; \mathbb{R}^m)$	$\{u \in L^p(U; \mathbb{R}^m) : Du = \nabla u \in L^p(U; \mathbb{R}^{m \times n})\}$ ( $1 \leq p \leq \infty$ )
$W_0^{1,p}(U; \mathbb{R}^m)$	fermeture de $\mathcal{C}_c^\infty(U; \mathbb{R}^m)$ dans $W^{1,p}(U; \mathbb{R}^m)$
$BV(U; \mathbb{R}^m)$	$\{u \in L^1(U; \mathbb{R}^m) : Du \in \mathcal{M}_b(U; \mathbb{R}^{m \times n})\}$
$SBV(U; \mathbb{R}^m)$	$\{u \in BV(U; \mathbb{R}^m) : D^c u = 0\}$
$SBV^p(U; \mathbb{R}^m)$	$\{u \in SBV(U; \mathbb{R}^m) : \nabla u \in L^p(U; \mathbb{R}^{m \times n}) \text{ et } \mathcal{H}^{n-1}(S(u)) < +\infty\}$

# Introduction générale

La déformation d'une surface élastique dans l'espace, soumise à des contraintes mécaniques, donne principalement lieu à deux types de phénomènes : l'étirement et la flexion. Une manière de modéliser de tels matériaux consiste à les assimiler à des objets tridimensionnels dont une des dimensions, que nous appellerons l'*épaisseur*, est beaucoup plus petite que les autres. Il s'agit alors de « passer à la limite » lorsque cette épaisseur, notée  $\varepsilon$ , tend vers zéro tout en essayant d'exhiber des propriétés inhérentes au modèle que nous souhaitons décrire. L'avantage d'une telle approche repose sur une bonne compréhension de l'élasticité non linéaire tridimensionnelle (voir CIARLET [39]) mais la difficulté consiste à donner un sens rigoureux à ce passage à la limite.

## Sur la modélisation des plaques élastiques

Une première méthode pour aborder ce type de problèmes est celle dite des *développements asymptotiques*. Cette méthode est basée sur l'Anzatz que les données et les solutions (éventuelles) du problème à  $\varepsilon$  fixé admettent un développement en puissance de  $\varepsilon$ . Cette analyse formelle, aussi utilisée en théorie de l'*homogénéisation*, a le mérite d'identifier efficacement les modèles limites mais ne se veut aucunement d'être rigoureuse. Elle a été utilisée dans le contexte de la réduction dimensionnelle de deux façons différentes pour un même matériau de Saint Venant Kirchhoff : d'une part par FOX, RAOULT & SIMO dans [62] qui, pour chaque ordre de grandeur en  $\varepsilon$ , résolvent une suite d'équations d'Euler-Lagrange, et d'autre part par PANTZ dans [78] (voir aussi [79, 80, 81]) qui résoud une suite de problèmes de minimisation. Ces résultats ont en commun qu'ils établissent formellement les ordres de grandeur des énergies de membrane et de flexion sous des hypothèses sur le chargement appliqué. En particulier, il se trouve que l'*énergie de membrane* apparaît à l'ordre  $\varepsilon$  du développement asymptotique lorsque les forces de surface sont d'ordre  $\varepsilon$  et les forces de volume d'ordre 1. Il est cependant suprenant de constater que leurs résultats diffèrent ; en effet, l'énergie de membrane obtenue par PANTZ n'est autre que la quasiconvexifiée de celle obtenue par FOX, RAOULT & SIMO à un terme près, qui est dû au fait que ces derniers sont contraints à restreindre leur ensemble de fonctions tests pour assurer l'existence d'une solution de chaque problème aux limites. Par ailleurs, l'*énergie de flexion* se comporte en  $\varepsilon^3$  à condition que les forces de surface soient d'ordre  $\varepsilon^3$  et les forces de volume d'ordre  $\varepsilon^2$ .

Ayant identifié des candidats pour décrire les modèles limites, il s'agit ensuite de justifier rigoureusement ces résultats formels grâce à un mode de passage à la limite approprié. Les problèmes d'élasticité non linéaire étant usuellement formulés comme des problèmes de minimisation portant sur l'énergie élastique, nous avons besoin d'un critère de convergence adapté au *Calcul des Variations*. Une notion adéquate de convergence variationnelle a été introduite par DE GIORGI & FRANZONI dans [51] sous le nom de  $\Gamma$ -convergence (voir les livres de DAL MASO [45] et BRAIDES [30] pour une étude approfondie) dont nous rappelons la définition :

**Définition 1.** Soit  $(X, d)$  un espace métrique et  $F_\varepsilon : X \rightarrow [0, +\infty]$ . On dit que  $F_\varepsilon$   $\Gamma$ -converge vers  $F$  si pour toute suite  $\{\varepsilon_j\} \searrow 0^+$  et pour tout  $u \in X$ ,

(i) pour toute suite  $u_j \rightarrow u$  dans  $X$

$$F(u) \leq \liminf_{j \rightarrow +\infty} F_{\varepsilon_j}(u_j);$$

(ii) il existe une suite  $\bar{u}_j \rightarrow u$  dans  $X$  telle que

$$F(u) = \lim_{j \rightarrow +\infty} F_{\varepsilon_j}(\bar{u}_j).$$

On dit alors que  $F$  est la  $\Gamma$ -limite de  $F_\varepsilon$ .

Cette notion est effectivement cohérente avec les problèmes de minimisation car elle assure, sous une hypothèse de compacité des suites minimisantes, la convergence des minimiseurs ainsi que de la valeur minimum. Cette propriété est traduite par le théorème suivant qui est le résultat fondamental de la théorie de la  $\Gamma$ -convergence :

**Théorème 2.** Soit  $(X, d)$  un espace métrique et  $F_\varepsilon : X \rightarrow [0, +\infty]$  une suite de fonctions qui  $\Gamma$ -converge vers  $F : X \rightarrow [0, +\infty]$ . S'il existe un compact  $K \subset X$  (indépendant de  $\varepsilon$ ) satisfaisant

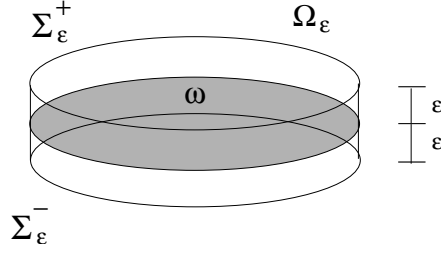
$$\inf_X F_\varepsilon = \inf_K F_\varepsilon,$$

alors

$$\exists \min_X F = \lim_{\varepsilon \rightarrow 0} \inf_X F_\varepsilon.$$

De plus, si  $\{u_\varepsilon\}$  est une suite telle que  $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \inf_X F_\varepsilon$ , alors tout point d'accumulation de  $\{u_\varepsilon\}$  est un minimiseur de  $F$ .

Il s'agit à présent d'appliquer ces méthodes aux problèmes de réduction de dimension. Pour ce faire, décrivons tout d'abord le modèle type qui sera, sous une forme ou une autre, le centre d'intérêt de cette thèse. Soit  $\omega \subset \mathbb{R}^2$  un ouvert borné à frontière Lipschitz et  $\varepsilon > 0$ ; notons  $\Omega_\varepsilon = \omega \times (-\varepsilon, \varepsilon)$  le cylindre occupé par un matériau hyperélastique dans sa configuration de référence (voir figure 1) et dont la densité d'énergie élastique est donnée par la fonction de Carathéodory  $W(\varepsilon) : \Omega_\varepsilon \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$  que nous supposons à croissance  $1 < p < \infty$ . Nous soumettons à ce matériau des forces de volume  $f(\varepsilon) \in L^{p'}(\Omega_\varepsilon; \mathbb{R}^3)$  et des forces de surface


 FIG. 1 – Cylindre d'épaisseur  $\varepsilon$ 

$g(\varepsilon) \in L^{p'}(\Sigma_\varepsilon; \mathbb{R}^3)$  sur les parois extrêmes  $\Sigma_\varepsilon = \omega \times \{-\varepsilon, \varepsilon\}$ , où  $p' = p/(p-1)$  désigne l'exposant conjugué de  $p$ . Nous adopterons par ailleurs une condition d'encastrement  $u(x) = x$  sur le bord latéral  $\partial\omega \times (-\varepsilon, \varepsilon)$  du cylindre. En l'absence de discontinuité du champ des déformations, le cadre fonctionnel naturel s'avère être celui des *Espaces de Sobolev*. Notons à titre de remarque que dans le Chapitre 4, ce cadre fonctionnel sera insuffisant pour décrire la présence de fissures et il sera nécessaire de travailler dans un sous espace des *Fonctions à Variation Bornée*. L'ensemble des déformations cinématiquement admissibles est alors donné par

$$\mathcal{C}(\varepsilon) := \{v \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) : v(x) = x \text{ sur } \partial\omega \times (-\varepsilon, \varepsilon)\}$$

et pour de telles déformations, l'énergie totale s'écrit comme l'énergie élastique privée du travail des forces extérieures, c'est-à-dire

$$E(\varepsilon)(u) = \int_{\Omega_\varepsilon} W(\varepsilon)(x; \nabla u) dx - \int_{\Omega_\varepsilon} f(\varepsilon) \cdot u dx - \int_{\Sigma_\varepsilon} g(\varepsilon) \cdot u d\mathcal{H}^2. \quad (1)$$

Le problème consiste alors à trouver la configuration d'équilibre de ce système mécanique, autrement dit de résoudre le problème de minimisation

$$\inf \{E(\varepsilon)(u) : u \in \mathcal{C}(\varepsilon)\}. \quad (2)$$

La première difficulté à laquelle nous sommes confrontés est liée à l'existence d'une solution à (2) pour  $\varepsilon$  fixé. Il se trouve qu'en l'absence d'hypothèses supplémentaires sur le potentiel élastique  $W(\varepsilon)$ , nous ne pouvons pas nous attendre, en général, à l'existence d'une telle solution. En effet, une condition suffisante pour que la fonctionnelle (1) admette un minimiseur sur  $\mathcal{C}(\varepsilon)$  est qu'elle soit semi-continue inférieurement pour la topologie faible de  $W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$ . Dans le cas vectoriel et sous une hypothèse de croissance d'ordre  $1 < p < \infty$ , ceci se traduit par une condition de *quasiconvexité* sur la fonction  $W(\varepsilon)$  (voir ACERBI & FUSCO [1], BALL & MURAT [18] ou DACOROGNA [44]). Une condition plus naturelle est celle dite de *polyconvexité*. Il s'agit d'une notion plus restrictive que la quasiconvexité mais qui a le mérite d'être compatible avec la propriété, requise en hyperélasticité, que l'énergie devrait exploser quand le Jacobien de la transformation

tend vers zéro, contrairement à la quasiconvexité où nous avons besoin d'imposer une borne supérieure. Sous cette hypothèse, BALL a démontré dans [17], l'existence d'un minimiseur pour (1) sur  $\mathcal{C}(\varepsilon)$  en tenant compte du fait que  $W(\varepsilon)(x; F) \rightarrow +\infty$  quand  $\det(F) \rightarrow 0$ . Cependant, nous ne ferons aucune de ces hypothèses sur  $W(\varepsilon)$  car cela nous interdirait de prendre en compte des matériaux importants de type Saint Venant Kirchhoff dont la densité d'énergie élastique n'est pas ni quasiconvexe ni polyconvexe (voir RAOULT [83]).

Bien que seul le modèle tridimensionnel ait un sens physique, nous ne nous attarderons pas à l'étudier en détail. En effet, selon notre approche, notre centre d'intérêt est le problème de minimisation associé à la  $\Gamma$ -limite qui lui admettra toujours une solution car la théorie de la  $\Gamma$ -convergence implique une *relaxation* de l'énergie. D'autre part, faute d'avoir l'existence d'un minimiseur pour (1), on peut toujours définir une suite de *quasi-minimiseurs*  $\{u_\varepsilon\} \subset \mathcal{C}(\varepsilon)$  de la façon suivante : pour toute suite  $h(\varepsilon) \searrow 0^+$ , il existe  $\{u_\varepsilon\} \subset \mathcal{C}(\varepsilon)$  telle que

$$E(\varepsilon)(u_\varepsilon) \leq \inf\{E(\varepsilon)(v) : v \in \mathcal{C}(\varepsilon)\} + h(\varepsilon).$$

Asymptotiquement, la suite  $\{u_\varepsilon\}$  se comporte exactement comme une suite de minimiseurs, en ce sens que, si elle converge vers une limite notée  $u$ , nécessairement  $u$  sera une solution du problème de minimisation associé à la  $\Gamma$ -limite.

La dépendance du domaine d'intégration en  $\varepsilon$ , et donc a fortiori de l'espace fonctionnel sur lequel est définie l'énergie, nous suggère d'effectuer un *changement d'échelle* (voir figure 2) afin d'étudier un problème équivalent formulé sur un domaine fixe  $\Omega := \omega \times (-1, 1)$ . Dans l'étude des problèmes non linéaires, il s'agit simplement de dilater la variable transverse  $x_3$ , contrairement au cas linéaire (voir CIARLET [40]) où la troisième composante du champ des déplacements doit aussi être modifiée. Nous renvoyons le lecteur à FONSECA & FRANCFORT [58] pour une discussion plus approfondie sur ces différences de *scaling* entre les problèmes linéaires et non linéaires. Pour tout  $u \in \mathcal{C}(\varepsilon)$ , on définit  $v \in \mathcal{C}_\varepsilon$  par  $u(x_\alpha, x_3) := v(x_\alpha, x_3/\varepsilon)$  où  $x_\alpha := (x_1, x_2) \in \omega$  désigne la variable du plan et

$$\mathcal{C}_\varepsilon := \{v \in W^{1,p}(\Omega; \mathbb{R}^3) : v(x) = (x_\alpha, \varepsilon x_3) \text{ sur } \partial\omega \times (-1, 1)\}.$$

En remplaçant  $u$  par  $v$  dans l'expression de l'énergie (1) nous obtenons que

$$E^\varepsilon(v) = E(\varepsilon)(u) = \varepsilon \int_\Omega W_\varepsilon \left( x; \nabla_\alpha v \middle| \frac{1}{\varepsilon} \nabla_3 v \right) dx - \varepsilon \int_\Omega f_\varepsilon \cdot v dx - \int_\Sigma g_\varepsilon \cdot v d\mathcal{H}^2.$$

où  $\Sigma := \omega \times \{-1, 1\}$  et

$$\begin{cases} W_\varepsilon(x; F) := W(\varepsilon)(x_\alpha, \varepsilon x_3; F), \\ g_\varepsilon(x) := g(\varepsilon)(x_\alpha, \varepsilon x_3), & g_\varepsilon \in L^{p'}(\Sigma; \mathbb{R}^3), \\ f_\varepsilon(x) := f(\varepsilon)(x_\alpha, \varepsilon x_3), & f_\varepsilon \in L^{p'}(\Omega; \mathbb{R}^3). \end{cases}$$

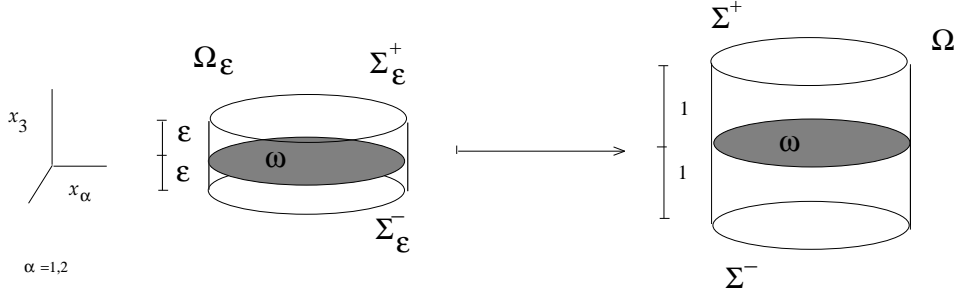


FIG. 2 – Changement d'échelle

Les résultats obtenus par développement asymptotique formel ont été justifiés rigoureusement, a posteriori, pour une plus large classe de matériaux homogènes au sens de la  $\Gamma$ -convergence. Étant donné que cette notion de convergence variationnelle est compatible avec le type de conditions limites que nous employons, nous prendrons le parti de les supprimer, étant entendu qu'elles ne modifient pas la forme de la  $\Gamma$ -limite (voir Theorem 21.1 dans DAL MASO [45]). Il en sera de même pour la forme linéaire correspondant au travail des forces extérieures (voir Proposition 6.21 dans DAL MASO [45]).

LE DRET & RAOULT ont obtenu un modèle de membrane dans [73, 74] en démontrant que la fonctionnelle  $E_m^\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  définie par

$$E_m^\varepsilon(u) := \begin{cases} \int_{\Omega} W\left(\nabla_{\alpha} u \middle| \frac{1}{\varepsilon} \nabla_3 u\right) dx & \text{si } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{sinon} \end{cases} \quad (3)$$

$\Gamma$ -converge pour la topologie forte de  $L^p(\Omega; \mathbb{R}^3)$  vers

$$E_m(u) := \begin{cases} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} & \text{si } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{sinon,} \end{cases}$$

où  $W_0 : \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  est définie par  $W_0(\overline{F}) := \inf\{W(\overline{F}|z) : z \in \mathbb{R}^3\}$  et  $\mathcal{Q}W_0$  est la quasiconvexifiée de  $W_0$ . Ils ont par ailleurs démontré dans l'étude de modèles avec directeurs dans [75], que les déformations tridimensionnelles affines en la variable transverse  $x_3$ , c'est-à-dire de la forme  $u_\varepsilon(x_\alpha, x_3) = u(x_\alpha) + \varepsilon x_3 d(x_\alpha)$  où  $d$  désigne le vecteur directeur, capturent totalement les effets membranaires. Ceci confirme ainsi la consistance des hypothèses cinématiques de Cosserat selon lesquelles les fibres orthogonales à la surface moyenne se déforment linéairement. Néanmoins, au vu de la condition de croissance imposée sur la densité d'énergie élastique, leur résultat ne permet pas de prendre en compte le fait que  $W(F)$  devrait tendre vers  $+\infty$  quand

$\det(F) \rightarrow 0$ . Ce type de comportement a été étudié par BEN BELGACEM dans [20, 21] qui a démontré que la  $\Gamma$ -limite de (3) est donnée dans ce cas par

$$E_m(u) := \begin{cases} 2 \int_{\omega} \mathcal{Q}RW_0(\nabla_{\alpha}u) dx_{\alpha} & \text{si } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{sinon,} \end{cases}$$

où  $\mathcal{R}$  désigne l'opération de *rang-1-convexification*. Comme l'ont remarqué LE DRET & RAOULT dans [73, 74] et BEN BELGACEM dans [20, 21], si  $W$  satisfait le *Principe de l'Indifférence Matérielle*, alors il en est de même pour  $\mathcal{Q}W_0$  et  $\mathcal{Q}RW_0$ . Autrement dit (voir CIARLET [39]),  $\mathcal{Q}W_0$  et  $\mathcal{Q}RW_0$  peuvent s'écrire comme des fonctions de la première forme fondamentale  $\nabla_{\alpha}u^T \nabla_{\alpha}u$  correspondant à la métrique de la surface déformée. C'est en ce sens qu'ils ont obtenu des modèles de membranes, mettant en évidence les phénomènes d'étirement dus au changement de métrique.

L'étude des modèles de plaques inextensibles a été entreprise par FRIESECKE, JAMES & MÜLLER dans [65, 66]. Notons par ailleurs que de tels modèles ont été partiellement justifiés par PANTZ dans [78, 80, 81] pour un matériau de Saint Venant Kirchhoff. Ceux-ci semblent plus délicats à obtenir dans la mesure où nous nous attendons à ce que les phénomènes de flexion fassent apparaître des dérivées d'ordre supérieur. En effet, comme toute solution correspond à des déformations isométriques  $\nabla_{\alpha}u^T \nabla_{\alpha}u = \text{Id}$ , l'énergie dépend cette fois de la seconde forme fondamentale qui mesure la courbure de la surface moyenne. Il s'avère alors naturel de supposer que  $W$  s'annule sur le groupe des rotations de  $\mathbb{R}^3$  dans  $\mathbb{R}^3$  noté  $\text{SO}(3)$ , annihilant ainsi les effets membranaires. De plus, étant donné que toute déformation à énergie finie est très proche d'une isométrie, nous supposons que  $W(F) \geq c \text{dist}(F, \text{SO}(3))^2$  impliquant que l'espace fonctionnel naturel est  $H^1(\Omega; \mathbb{R}^3)$ . Si  $W$  est de classe  $\mathcal{C}^2$  et satisfait le Principe de l'Indifférence Matérielle, FRIESECKE, JAMES & MÜLLER ont démontré que la fonctionnelle  $E_f^{\varepsilon} : H^1(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty)$  définie par

$$E_f^{\varepsilon}(u) := \frac{1}{\varepsilon^2} \int_{\Omega} W \left( \nabla_{\alpha}u \middle| \frac{1}{\varepsilon} \nabla_{\beta}u \right) dx$$

$\Gamma$ -converge pour la topologie faible ou forte de  $H^1(\Omega; \mathbb{R}^3)$  vers

$$E_f(u) := \begin{cases} \frac{1}{3} \int_{\omega} Q_2(\nabla_{\alpha}u^T \nabla_{\alpha}u) dx_{\alpha} & \text{si } u \in \mathcal{C}_f, \\ +\infty & \text{sinon,} \end{cases}$$

où  $b = \nabla_1 u \wedge \nabla_2 u$ ,

$$\mathcal{C}_f := \{u \in H^2(\omega; \mathbb{R}^3) : |\nabla_1 u| = |\nabla_2 u| = 1, \nabla_1 u \cdot \nabla_2 u = 0\}$$

et  $Q_2 : \mathbb{R}^{2 \times 2} \rightarrow [0, +\infty)$  est la forme quadratique définie par

$$Q_2(F) := \inf_{z \in \mathbb{R}^3} \frac{\partial^2 W}{\partial F^2}(\text{Id}) \left( \begin{pmatrix} F_{11} & F_{12} & z_1 \\ F_{21} & F_{22} & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix}, \begin{pmatrix} F_{11} & F_{12} & z_1 \\ F_{21} & F_{22} & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix} \right).$$

Ce résultat est basé sur un Lemme de rigidité (Theorem 3.1 dans [66]) qui établit que si une déformation est proche d'une rotation au sens de la norme  $L^2(\Omega; \mathbb{R}^{3 \times 3})$ , alors cette rotation est constante.

Pour un exposé plus exhaustif sur la hiérarchie de divers modèles de plaques, nous renvoyons le lecteur à [68].

## Les modèles de membrane

Dans cette thèse, nous nous concentrons exclusivement sur divers types de modèles membranaires dérivés par la méthode de  $\Gamma$ -convergence. Le point de départ repose sur un résultat abstrait d'existence et de représentation intégrale de  $\Gamma$ -limite démontré par BRAIDES, FONSECA & FRANCFORT dans [35]. Il est établi que si  $W_\varepsilon : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$  est une famille de fonctions de Carathéodory satisfaisant des conditions de croissance et de coercivité d'ordre  $1 < p < \infty$  uniformément en  $\varepsilon$  alors, pour une sous suite, la fonctionnelle  $E_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  définie par

$$E_\varepsilon(u) := \begin{cases} \int_{\Omega} W_\varepsilon \left( x; \nabla_\alpha u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx & \text{si } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{sinon} \end{cases}$$

$\Gamma$ -converge pour la topologie forte de  $L^p(\Omega; \mathbb{R}^3)$  vers

$$E(u) := \begin{cases} 2 \int_{\omega} \widehat{W}(x_\alpha; \nabla_\alpha u) dx_\alpha & \text{si } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{sinon,} \end{cases}$$

où  $\widehat{W} : \omega \times \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  est une fonction universelle. Par la suite, nous identifierons systématiquement les fonctions définies sur  $\omega$  avec celles définies sur  $\Omega$  indépendantes de  $x_3$ . A ce degré d'abstraction, il semble impossible de décrire plus précisément la forme de  $\widehat{W}$ . La question consiste alors à essayer d'identifier  $\widehat{W}$  dans certains cas particuliers. Une large littérature existe à ce sujet et nous nous proposons d'en donner un bref aperçu tout en soulignant les problèmes restés jusque là en suspens. Notons que dans la plupart des études à ce sujet, les auteurs imposent une condition de croissance sur l'énergie. Il ne s'agit donc pas rigoureusement d'hyperélasticité, car cela proscrit la contrainte usuelle que l'énergie devrait exploser quand le Jacobien de la transformation tend vers zéro, mais plutôt d'*élasticité non linéaire*.

Dans la Section 3 de [35], BRAIDES, FONSECA & FRANCFORT ont généralisé le résultat de LE DRET & RAOULT [74] au cas de matériaux hétérogènes dans la section, c'est-à-dire dont la



densité d'énergie élastique est donnée par  $W_\varepsilon(x; F) = W(x_3; F)$ . Ils ont démontré que

$$\widehat{W}(\overline{F}) := \inf_{L, \phi} \left\{ \int_{(0,1)^2 \times (-1,1)} W(x_3; \overline{F} + \nabla_\alpha \phi | L \nabla_3 \phi) dx_\alpha dx_3 : L > 0, \right. \\ \left. \phi \in W^{1,p}((0,1)^2 \times (-1,1); \mathbb{R}^3), \phi = 0 \text{ sur } \partial(0,1)^2 \times (-1,1) \right\} \quad (4)$$

et que si  $W$  est indépendant de  $x_3$ ,  $\widehat{W}$  coïncide exactement avec  $\mathcal{Q}W_0$ . Cependant, leurs arguments ne semblent pas suffire à traiter le cas d'une *hétérogénéité macroscopique* quelconque et indépendante de  $\varepsilon$ , c'est-à-dire quand  $W_\varepsilon(x; F) = W(x; F)$ . Ceci fait l'objet de la première partie du Chapitre 1 de cette thèse.

Ensuite, dans la Section 4 de [35], BRAIDES, FONSECA & FRANCFORT ont appliqué leur résultat abstrait à un problème typique d'homogénéisation, c'est-à-dire au cas d'un matériau possédant une *microstructure et un profil périodiquement oscillants* à une vitesse du même ordre de grandeur que l'épaisseur. Parallèlement, dans [8], ANSINI & BRAIDES ont traité le cas où la période d'oscillation de la frontière est beaucoup plus petite que l'épaisseur. Il s'agit alors d'un problème multi-échelle dans lequel on s'attend heuristiquement dans un premier temps à étudier un problème classique d'homogénéisation et ensuite de réduction de dimension portant sur la fonctionnelle homogénéisée. Par ailleurs, des problèmes d'*homogénéisation réitérée* ont été mis en oeuvre par SHU dans [88]. L'auteur a considéré des matériaux martensitiques dans lesquels sont mis en compétition d'une part un terme classique d'énergie élastique, et d'autre part un terme d'énergie interfaciale de type Van der Waals, quadratique par rapport aux dérivées d'ordre deux. Le cas général d'une énergie non linéaire dépendant du tenseur des dérivées d'ordre deux a été étudié de façon abstraite par SANTOS & ZAPPALE dans [87]. Les problèmes cités ci-dessus ne décrivent seulement que des phénomènes d'homogénéisation dans le plan, faisant intervenir une densité d'énergie du type  $W_\varepsilon(x; F) = W(x_\alpha/\varepsilon, x_3; F)$  où  $W(\cdot, x_3; F)$  est  $(0,1)^2$ -périodique. Dans ce cas, il a été démontré dans [35, 88] que

$$\widehat{W}(\overline{F}) := \inf_{T, \phi} \left\{ \int_{(0,T)^2 \times (-1,1)} W(x_\alpha, x_3; \overline{F} + \nabla_\alpha \phi | \nabla_3 \phi) dx_\alpha dx_3 : T > 0, \right. \\ \left. \phi \in W^{1,p}((0,T)^2 \times (-1,1); \mathbb{R}^3), \phi = 0 \text{ sur } \partial(0,T)^2 \times (-1,1) \right\}.$$

Cependant, l'homogénéisation dans la direction transverse  $x_3$ , c'est-à-dire la direction par rapport à laquelle on réduit la dimension, n'a pas été traitée. Cette situation correspond à un cas particulier des problèmes d'homogénéisation réitérée étudiés dans le Chapitre 2 de cette thèse.

Dans la Section 5 de [35], BRAIDES, FONSECA & FRANCFORT donnent une application aux problèmes d'*optimisation de forme*. Dans [57], FONSECA & FRANCFORT ont préalablement consi-

déré le cas de mixtures à géométrie cylindrique, c'est-à-dire dont la fonction caractéristique des phases ne dépend pas de  $x_3$ . Ce résultat a été généralisé dans [35] au cas où la distribution de matériau peut aussi dépendre de la variable transverse  $x_3$ . Le prix à payer est que les auteurs n'obtiennent pas exactement une caractérisation de la  $\Gamma$ -limite mais seulement une borne inférieure.

Des problèmes de *décollement* de films minces ont été traités par BHATTACHARYA, FONSECA & FRANCFORT dans [23] qui ont considéré la superposition de deux matériaux dont la possibilité de décollement est pénalisée par une *énergie interfaciale* payant le saut de déformation à l'interface. Autrement dit, ils ont étudié la  $\Gamma$ -convergence de fonctionnelles du type

$$\int_{\omega \times (0,1)} W^+ \left( \nabla_\alpha u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx + \int_{\omega \times (-1,0)} W^- \left( \nabla_\alpha u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx + \varepsilon^{\alpha-1} \int_{\omega} \Psi(u^+ - u^-) dx_\alpha$$

sur  $W^{1,p}(\omega \times (0,1); \mathbb{R}^3) \cup W^{1,p}(\omega \times (-1,0); \mathbb{R}^3)$  où  $u^\pm(x_\alpha) := u(x_\alpha, \pm 1)$ . Quand  $\alpha < 1$ , l'énergie interfaciale est forte et tend vers l'infini à moins que les déformations deviennent asymptotiquement continues à travers la surface moyenne. En revanche, lorsque  $\alpha \geq 1$ , l'énergie interfaciale est faible et la  $\Gamma$ -limite peut être finie même si les déformations sont discontinues à travers l'interface. En particulier, si  $\alpha = 1$ , on obtient un modèle contenant des énergies de volume et interfaciale alors que si  $\alpha > 1$ , l'énergie interfaciale disparaît à la limite. Une autre étude menée dans [6] par ANSINI dans le contexte du *Problème de la Passoire de Neumann*, donne aussi lieu à des phénomènes similaires. Contrairement à BHATTACHARYA, FONSECA & FRANCFORT, il n'y a pas d'énergie interfaciale présente au début de l'analyse et le décollement est dû à la présence d'une *zone de contact périodiquement distribuée* entre les deux films minces. Dans [6], ANSINI a considéré le cas où l'épaisseur des cylindres  $\varepsilon$  et la période de distribution des trous  $\delta$  sont du même ordre de grandeur et on peut remarquer que les mêmes arguments subsistent lorsque  $\delta \ll \varepsilon$ . Cependant, ils ne permettent pas de traiter le cas où  $\varepsilon \ll \delta$  qui fait l'objet du Chapitre 3 de cette thèse.

Dans [34], BRAIDES & FONSECA ont appliqué ces techniques de réduction dimensionnelle à des problèmes de *mécanique de la rupture*. Comme une déformation peut être discontinue à travers les fissures, le cadre fonctionnel des Espaces de Sobolev semble insuffisant. Dans [50], DE GIORGI, CARRIERO & LEACI ont suggéré, pour l'étude de la fonctionnelle de Mumford-Shah, d'utiliser une formulation faible de ce type de *Problèmes aux Discontinuités Libres* en remplaçant les fissures par l'ensemble des discontinuités du champ des déformations. On peut alors définir un cadre fonctionnel dans l'espace des fonctions *Spéciales à Variation Bornée*. BRAIDES & FONSECA ont obtenu un résultat de  $\Gamma$ -convergence dans [34], par la suite amélioré par BOUCHITTÉ, FONSECA, LEONI & MASCARENHAS dans [28] qui ont imposé moins de restrictions sur la densité d'énergie de surface. Néanmoins, dans les deux cas, les auteurs ont dû considérer des énergies de surface qui croissent linéairement par rapport au saut de déformation. Or, ce type de comportement n'est pas très réaliste car il proscrit des énergies de type Griffith ou Barenblatt

qui se comportent asymptotiquement comme une constante. BRAIDES & FONSECA ont toutefois suggéré une manière de traiter une énergie de type Griffith par perturbation singulière. Dans le Chapitre 4, nous présentons une démonstration alternative basée sur un argument direct. Notons que le problème étudié par BHATTACHARYA, FONSECA & FRANCFORT dans [23] peut être vu comme un problème de rupture en imposant à toute fissure de rester dans la section moyenne de la plaque.

Les problèmes décrits ci-dessus ont en commun qu'ils nécessitent une bonne compréhension de ce que nous appellerons le *gradient rescalé*. En effet, dans le cadre des Espaces de Sobolev, si nous nous donnons une suite minimisante  $\{u_\varepsilon\} \subset L^p(\Omega; \mathbb{R}^3)$  à énergie finie et convergeant fortement vers  $u$  dans  $L^p(\Omega; \mathbb{R}^3)$ , au vu de la condition de coercivité d'ordre  $1 < p < \infty$  satisfaite par  $W_\varepsilon$ , la borne

$$\sup_{\varepsilon > 0} \int_{\Omega} \left| \left( \nabla_\alpha u_\varepsilon \middle| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right) \right|^p dx < +\infty \quad (5)$$

apparaît naturellement. Ceci implique que  $u \in W^{1,p}(\omega; \mathbb{R}^3)$  et que (pour une sous suite)  $\frac{1}{\varepsilon} \nabla_3 u_\varepsilon \rightharpoonup b$  faiblement dans  $L^p(\Omega; \mathbb{R}^3)$ . BOCEA & FONSECA ont étudié la structure de telles suites dans [25]. Ils ont démontré que  $u_\varepsilon$  peut toujours se décomposer en une somme de deux suites  $w_\varepsilon$  et  $z_\varepsilon$ , où  $z_\varepsilon$  porte les effets de *concentration*, c'est-à-dire  $\{ |(\nabla_\alpha w_\varepsilon | \frac{1}{\varepsilon} \nabla_3 w_\varepsilon) |^p \}$  est équi-intégrable, et  $w_\varepsilon$  porte les *oscillations*, autrement dit  $z_\varepsilon \rightarrow 0$  en mesure. Plus précisément,

**Théorème 3.** *Soit  $\omega \subset \mathbb{R}^2$  un ouvert borné à frontière Lipschitz,  $\Omega := \omega \times (-1, 1)$  et  $\{u_\varepsilon\}$  une suite bornée dans  $W^{1,p}(\Omega; \mathbb{R}^3)$  satisfaisant (5). Supposons de plus que  $u_\varepsilon \rightharpoonup u$  faiblement dans  $W^{1,p}(\Omega; \mathbb{R}^3)$  et  $\frac{1}{\varepsilon} \nabla_3 u_\varepsilon \rightharpoonup b$  faiblement dans  $L^p(\Omega; \mathbb{R}^3)$ . Alors il existe une sous suite (toujours notée  $\{\varepsilon\}$ ) et une suite  $\{w_\varepsilon\} \subset W^{1,\infty}(\Omega; \mathbb{R}^3)$  telles que*

- (i)  $\mathcal{L}^3(\{x \in \Omega : u_\varepsilon(x) \neq w_\varepsilon(x)\}) \rightarrow 0$  ;
- (ii)  $\{ |(\nabla_\alpha w_\varepsilon | \frac{1}{\varepsilon} \nabla_3 w_\varepsilon) |^p \}$  est équi-intégrable ;
- (iii)  $w_\varepsilon \rightharpoonup u$  faiblement dans  $W^{1,p}(\Omega; \mathbb{R}^3)$  ;
- (iv)  $\frac{1}{\varepsilon} \nabla_3 w_\varepsilon \rightharpoonup b$  faiblement dans  $L^p(\Omega; \mathbb{R}^3)$ .

Ce résultat est à rapprocher du Lemme de Décomposition démontré par FONSECA, MÜLLER & PEDREGAL dans [61] (voir aussi FONSECA & LEONI [59]) établi pour des suites de gradients. Comme, a priori, le *vecteur de Cosserat*  $b$  est complètement indépendant de la déformation  $u$ , il semble naturel de le prendre en compte lors du processus de relaxation et de déterminer comment la fonctionnelle relaxée en dépend. Une première approche a été donnée par BOUCHITTÉ, FONSECA & MASCARENHAS dans [27]. En effet, le développement asymptotique mis en oeuvre par FOX, RAOULT & SIMO dans [62] ne proscrie pas totalement des forces de surface d'ordre 1. En fait, les forces de surface du type  $g_\varepsilon = g_0 + \varepsilon g$  sont admissibles à condition que la résultante du

terme d'ordre 1 soit nul :  $g_0(x_\alpha, 1) + g_0(x_\alpha, -1) = 0$ . Dans [74], LE DRET & RAOULT imposent implicitement à  $g_0$  d'être identiquement nul, ce qui se traduit physiquement par le fait que la plaque ne peut pas supporter une charge qui ne tend pas vers zéro avec l'épaisseur. Néanmoins, il est possible de prendre en compte le cas général comme l'ont fait BOUCHITTÉ, FONSECA & MASCARENHAS dans [27], faisant apparaître à la limite, en plus du champ des déformations  $u$ , une densité de moment fléchissant obtenue en faisant la moyenne dans l'épaisseur du vecteur de Cosserat  $b$ . Autrement dit, ils ont démontré que la fonctionnelle  $F_\varepsilon : L^p(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  définie par

$$F_\varepsilon(u, \bar{b}) := \begin{cases} \int_\Omega W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u\right) dx & \text{si } \begin{cases} u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ \frac{1}{\varepsilon} \int_{-1}^1 \nabla_3 u(\cdot, x_3) dx_3 = \bar{b}, \end{cases} \\ +\infty & \text{sinon,} \end{cases}$$

$\Gamma$ -converge vers

$$F(u, \bar{b}) = \begin{cases} 2 \int_\omega \mathcal{Q}^* W(\nabla_\alpha u | \bar{b}) dx_\alpha & \text{si } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{sinon,} \end{cases}$$

où, pour tout  $(\bar{F}|z) \in \mathbb{R}^{3 \times 2} \times \mathbb{R}^3$ ,

$$\begin{aligned} \mathcal{Q}^* W(\bar{F}|z) &:= \inf_{L > 0, \varphi} \left\{ \int_{(0,1)^2 \times (-1,1)} W(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \right. \\ &\quad \left. \varphi \in W^{1,p}((0,1)^2 \times (-1,1); \mathbb{R}^3), \varphi(\cdot, x_3) \text{ est } (0,1)^2\text{-périodique p.p. tout } x_3 \in (-1,1) \right. \\ &\quad \left. \text{et } L \int_{(0,1)^2 \times (-1,1)} \nabla_3 \varphi dx = z \right\}. \quad (6) \end{aligned}$$

Ce résultat a été généralisé dans la deuxième partie du Chapitre 1 de cette thèse au cas d'une hétérogénéité macroscopique quelconque, indépendante de  $\varepsilon$ . Il semble cependant difficile d'étendre ces résultats au cas plus général où l'on ne fait pas la moyenne dans la section. Dans [24], BOCEA a obtenu une représentation intégrale de la  $\Gamma$ -limite de la fonctionnelle  $G_\varepsilon : L^p(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  définie par

$$G_\varepsilon(u, b) := \begin{cases} \int_\Omega W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u\right) dx & \text{si } \begin{cases} u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ \frac{1}{\varepsilon} \nabla_3 u = b, \end{cases} \\ +\infty & \text{sinon,} \end{cases}$$

en terme de mesures d'Young engendrées par des gradients rescalés ( $\alpha$ -gradient  $p$ -Young measures). Cependant, il manque une bonne compréhension de ce que sont de telles mesures. Dans [26], BOCEA & FONSECA ont réussi à caractériser les mesures d'Young engendrées par des gradients rescalés lorsque le terme en dérivée par rapport à  $x_3$  est intégré dans la section (*bending Young measures*). Le cas général semble encore à l'heure actuelle hors de portée.

## Description des résultats

Le premier chapitre de cette thèse constitue un travail en collaboration avec GILLES FRANCFORT et a donné lieu à une publication [15] dans la revue *ESAIM : Control, Optimization and Calculus of Variations*. Le but est de généraliser les résultats de LE DRET & RAOULT [74], de BRAIDES, FONSECA & FRANCFORT [35] et de BOUCHITTÉ, FONSECA & MASCARENHAS [27] au cas d'une densité d'énergie élastique hétérogène de la forme  $W_\varepsilon(x; F) = W(x; F)$ , où  $W : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$  est une fonction de Carathéodory satisfaisant des conditions de croissance et de coercivité usuelles d'ordre  $1 < p < \infty$ . Plus précisément, il s'agit d'étudier le comportement asymptotique au sens de la  $\Gamma$ -convergence de la fonctionnelle

$$W^{1,p}(\Omega; \mathbb{R}^3) \ni u \mapsto \int_{\Omega} W \left( x; \nabla_{\alpha} u \left| \frac{1}{\varepsilon} \nabla_3 u \right. \right) dx$$

lorsque  $\varepsilon$  tend vers zéro. Comme toute suite minimisante à énergie finie admet une borne sur son gradient dans  $L^p(\Omega; \mathbb{R}^{3 \times 3})$ , le cadre fonctionnel naturel pour l'étude de tels problèmes est celui de l'espace de Sobolev  $W^{1,p}(\Omega; \mathbb{R}^3)$  muni de sa topologie faible. Dans la continuité des travaux effectués par LE DRET & RAOULT [74] et BRAIDES, FONSECA & FRANCFORT [35], nous donnons une expression de la  $\Gamma$ -limite généralisant ainsi leurs résultats au cas d'une membrane hétérogène. On démontre dans ce cas que la  $\Gamma$ -limite est donnée par

$$W^{1,p}(\omega; \mathbb{R}^3) \ni u \mapsto 2 \int_{\omega} \underline{W}(x_{\alpha}; \nabla_{\alpha} u) dx_{\alpha},$$

où la densité d'énergie élastique  $\underline{W} : \omega \times \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  est la fonction de Carathéodory définie par

$$\underline{W}(x_0; \bar{F}) := \inf_{L, \phi} \left\{ \int_{(0,1)^2 \times (-1,1)} W(x_0, x_3; \bar{F} + \nabla_{\alpha} \phi | L \nabla_3 \phi) dx_{\alpha} dx_3 : L > 0, \right. \\ \left. \phi \in W^{1,p}((0,1)^2 \times (-1,1); \mathbb{R}^3), \phi = 0 \text{ sur } \partial(0,1)^2 \times (-1,1) \right\}.$$

Il est trivial de constater que ce résultat étend effectivement (4) au cas hétérogène.

Comme nous l'avons souligné précédemment, si  $\{u_{\varepsilon}\} \subset W^{1,p}(\Omega; \mathbb{R}^3)$  est une suite minimisante à énergie finie, elle satisfait nécessairement (pour une sous suite)  $\frac{1}{\varepsilon} \nabla_3 u_{\varepsilon} \rightharpoonup b$  faiblement dans  $L^p(\Omega; \mathbb{R}^3)$ , faisant apparaître à la limite un vecteur de Cosserat. Il semble alors naturel de prendre en compte ce comportement lors de l'analyse par  $\Gamma$ -convergence. Tout comme dans BOUCHITTÉ, FONSECA & MASCARENHAS [27], faute de pouvoir considérer le cas général, nous traitons un problème simplifié qui consiste à prendre la moyenne dans la section de ce terme. Nous sommes alors amenés à étudier exactement la même fonctionnelle que précédemment mais cette fois-ci pour la topologie faible de  $W^{1,p}(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3)$ . La  $\Gamma$ -limite dépend alors de deux champs de vecteur : la déformation  $u$  et la moyenne dans la section du vecteur de Cosserat  $\bar{b}$ , interprétée

comme une densité de moment fléchissant. Elle est donnée par

$$W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \ni (u, \bar{b}) \mapsto 2 \int_{\omega} \mathcal{Q}^*W(x_{\alpha}; \nabla_{\alpha}u|\bar{b}) dx_{\alpha},$$

où la fonction  $\mathcal{Q}^*W : \omega \times \mathbb{R}^{3 \times 2} \times \mathbb{R}^3 \rightarrow [0, +\infty)$  est définie par

$$\mathcal{Q}^*W(x_0; \bar{F}|z) := \inf_{L>0, \varphi} \left\{ \int_{(0,1)^2 \times (-1,1)} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_{\alpha}\varphi|L\nabla_3\varphi) dx_{\alpha} dx_3 : \right. \\ \left. \varphi \in W^{1,p}((0,1)^2 \times (-1,1); \mathbb{R}^3), \varphi(\cdot, x_3) \text{ est } (0,1)^2\text{-périodique p.p. tout } x_3 \in (-1,1), \right. \\ \left. \text{et } L \int_{(0,1) \times (-1,1)} \nabla_3\varphi dx = z \right\}.$$

Contrairement à (6), il semble nécessaire de quasiconvexifier  $W$  dans la formule précédente. Ceci est dû au fait que nous avons besoin d'exploiter le caractère  $p$ -Lipschitz de l'intégrande qui n'est en général pas vérifié par  $W$  mais qui l'est toujours pour  $\mathcal{Q}W$  (voir DACOROGNA [44]). D'après la Proposition 1.1 dans BOUCHITTÉ, FONSECA & MASCARENHAS [27], nous constatons que cette dernière formule généralise effectivement (6).

Bien que nous étudions deux fois la même fonctionnelle, la différence des topologies nous conduit à deux modèles de membrane différents. Nous démontrons toutefois que le premier modèle est un cas particulier du deuxième via un problème de minimisation, c'est-à-dire

$$\int_{\omega} \underline{W}(x_{\alpha}; \nabla_{\alpha}u) dx_{\alpha} = \min_{\bar{b} \in L^p(\omega; \mathbb{R}^3)} \int_{\omega} \mathcal{Q}^*W(x_{\alpha}; \nabla_{\alpha}u|\bar{b}) dx_{\alpha}.$$

La démonstration de ces deux résultats repose sur le lemme d'équi-intégrabilité démontré par BOCEA & FONSECA dans [25] (voir Théorème 3) qui assure que toute suite atteignant la  $\Gamma$ -limite peut être choisie de telle sorte que son gradient rescalé est  $p$ -équi-intégrable. La stratégie consiste alors à se servir du résultat préalablement établi dans le cas homogène (pas de dépendance en  $x_{\alpha}$ ). Pour ce faire, nous introduisons une méthode de *dédoublément de variable* qui consiste à geler la variable macroscopique  $x_{\alpha}$ . Le Théorème de Scorza-Drăgani (voir EKELAND & TEMAM [54]) nous permet d'affirmer que la restriction de  $W$  sur un compact  $K \subset \Omega$ , dont le complémentaire a une mesure de Lebesgue arbitrairement petite, est continue. L'équi-intégrabilité du gradient rescalé assurera que l'énergie calculée sur  $\Omega \setminus K$  tend vers zéro. La difficulté qui apparaît alors est la différence de structure entre le problème de départ et le problème limite : comme le passage à la limite fait disparaître une dimension, nous ne pouvons pas, tout comme en relaxation classique, utiliser directement un argument de continuité uniforme. En effet, si  $x_0 \in \omega$  est fixé, pour tout  $(x_{\alpha}, x_3) \in K$ , nous devons assurer que  $(x_0, x_3) \in K$  ce qui n'est en général pas le cas car  $K$  n'est que compact. Une façon de remédier à ce problème consiste à étendre  $W$  en une fonction continue sur tout  $\Omega \times \mathbb{R}^{3 \times 3}$ . Ceci est rendu possible grâce à un théorème d'extension de fonctions continues en dehors d'un compact (voir Theorem 1, Section 1.2 dans EVANS & GARIEPY [55]). Nous avons toutefois besoin de nous restreindre à l'ensemble des points  $x \in \Omega$  tels

que le gradient rescalé reste dans une boule fermée  $B$  de  $\mathbb{R}^{3 \times 3}$  de rayon fixe mais arbitrairement grand. L'inégalité de Chebyshev implique alors que le complémentaire de cet ensemble a une mesure de Lebesgue arbitrairement petite, et donc, l'énergie calculée sur cet ensemble tend vers zéro. Comme  $W$  est continue sur  $K \times B$  et que  $K \times B$  est un compact de  $\mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$ , on peut étendre  $W$  en une fonction continue, notée  $\overline{W}$ , sur  $\mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$ . L'équi-intégrabilité et le fait que les « mauvais ensembles » ont une mesure de Lebesgue qui tend vers zéro, impliquent que l'énergie calculée sur  $\Omega \setminus K$  et sur l'ensemble des points  $x \in \Omega$  tels que le gradient rescalé n'appartient pas à  $B$  tend vers zéro. Nous pouvons alors remplacer  $W$  par  $\overline{W}$  sous le signe intégral. Ensuite, nous appliquons un argument de continuité uniforme à cette nouvelle fonction et nous nous servons de l'équi-intégrabilité du gradient rescalé pour revenir à  $W$  en remontant les étapes précédentes.

Le deuxième chapitre de cette thèse a fait l'objet d'un travail en collaboration avec MARGARIDA BAÍA et a donné lieu à deux articles : le premier [13] a été accepté dans la revue *Proceedings of the Royal Society of Edinburgh, Section A*, et l'autre [14] a été soumis dans la revue *Asymptotic Analysis*. Il s'agit toujours d'étudier des films minces hétérogènes, mais cette fois-ci, nous supposons que le matériau possède une microstructure périodique c'est-à-dire une hétérogénéité dépendant périodiquement de l'épaisseur  $\varepsilon$  (voir figure 3). Ceci nous conduit à considérer les

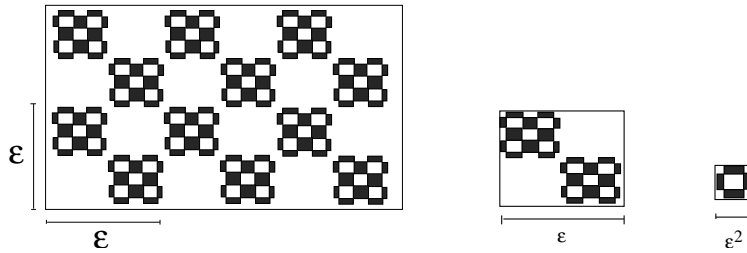


FIG. 3 – Exemple de domaine possédant une microstructure périodique d'ordre  $\varepsilon$  et  $\varepsilon^2$

deux phénomènes simultanés de réduction dimensionnelle et d'homogénéisation réitérée. Plus précisément, il s'agit d'étudier le comportement asymptotique de la fonctionnelle

$$W^{1,p}(\Omega; \mathbb{R}^3) \ni u \mapsto \int_{\Omega} W \left( x, \frac{x}{\varepsilon}, \frac{x_{\alpha}}{\varepsilon^2}; \nabla_{\alpha} u \left| \frac{1}{\varepsilon} \nabla_3 u \right. \right) dx$$

quand  $\varepsilon$  tend vers zéro, où  $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$  satisfait, comme précédemment, des conditions de croissance et de coercivité d'ordre  $1 < p < \infty$ . Dans un premier temps, il convient de discuter sous quelles hypothèses de régularité et de périodicité sur  $W$ , la fonctionnelle précédente est bien définie. Afin d'étendre le résultat établi dans la première partie du chapitre 1, nous avons pris le parti de supposer que  $W$  est mesurable en sa première variable  $x$  et continue par rapport aux autres arguments. Nous verrons que cela ne sera pas suffisant pour donner un sens au problème et nous préciserons ultérieurement ces hypothèses. Pour ce qui est de la

périodicité, revenons temporairement à la configuration physique en effectuant le changement d'échelle inverse : de façon équivalente, on est amené à étudier la fonctionnelle

$$W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) \ni v \mapsto \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W \left( x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}, \frac{x_3}{\varepsilon^2}, \frac{x_\alpha}{\varepsilon^2}; \nabla v \right) dx.$$

Comme  $\varepsilon^2 \ll \mathcal{L}^3(\Omega_\varepsilon)$ , heuristiquement on s'attend tout d'abord à étudier un problème d'homogénéisation pure (c'est-à-dire sans réduction dimensionnelle) dans lequel il est naturel de supposer que  $z \mapsto W(x_\alpha, y_3, y_\alpha, z_3, z_\alpha; F)$  est  $(0, 1)^3$ -périodique. Passant à la limite lorsque  $\varepsilon^2$  tend vers zéro, laissant  $\varepsilon$  fixé, ceci nous conduit à une fonctionnelle homogénéisée de la forme

$$W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) \ni v \mapsto \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W_{\text{hom}} \left( x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla v \right) dx, \quad (7)$$

où  $W_{\text{hom}} : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$  est définie par

$$W_{\text{hom}}(x_\alpha, y_3, y_\alpha; F) := \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^3} W(x_\alpha, y_3, y_\alpha, z_3, z_\alpha; F + \nabla \phi(z)) dz : \right. \\ \left. \phi \in W_0^{1,p}((0, T)^3; \mathbb{R}^3) \right\}. \quad (8)$$

L'analyse asymptotique de (7) révèle que les phénomènes d'homogénéisation et de réduction de dimension apparaissent en même temps à l'ordre  $\varepsilon$ . En revenant par changement d'échelle à la configuration « rescalée », il est équivalent d'étudier la fonctionnelle

$$W^{1,p}(\Omega; \mathbb{R}^3) \ni u \mapsto \int_{\Omega} W_{\text{hom}} \left( x, \frac{x_\alpha}{\varepsilon}; \nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u \right) dx.$$

Dans ce cas, on remarque que l'homogénéisation n'est effective que dans le plan et donc, nous supposons que  $y_\alpha \mapsto W_{\text{hom}}(x_\alpha, y_3, y_\alpha; F)$  est  $(0, 1)^2$ -périodique. En particulier, cette condition peut être obtenue à partir de la  $(0, 1)^2$ -périodicité de  $y_\alpha \mapsto W(x_\alpha, y_3, y_\alpha, z_3, z_\alpha; F)$ . Nous démontrons alors que cette dernière fonctionnelle  $\Gamma$ -converge vers

$$W^{1,p}(\omega; \mathbb{R}^3) \ni u \mapsto 2 \int_{\omega} \overline{W}_{\text{hom}}(x_\alpha; \nabla_\alpha u) dx_\alpha, \quad (9)$$

où  $\overline{W}_{\text{hom}} : \omega \times \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  est donnée par

$$\overline{W}_{\text{hom}}(x_\alpha; \overline{F}) := \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^2 \times (-1,1)} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)) dy : \right. \\ \left. \phi \in W^{1,p}((0, T)^2 \times (-1, 1); \mathbb{R}^3), \quad \phi = 0 \text{ sur } \partial(0, T)^2 \times (-1, 1) \right\}. \quad (10)$$

Au vu des formules (8) et (10), un critère de mesurabilité global sur  $W$  par rapport aux variables macroscopiques et oscillantes s'impose car le caractère Carathéodory, préalablement requis sur  $W$ , ne paraît pas se conserver lors du passage à la  $\Gamma$ -limite. En effet, comme  $W_{\text{hom}}$

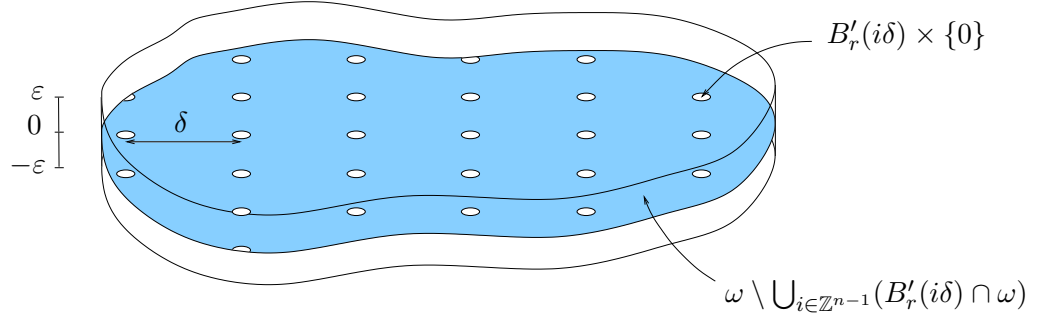


et  $\overline{W}_{\text{hom}}$  s'obtiennent en croisant les variables oscillantes, il est nécessaire de supposer que  $(x_\alpha, y, z) \mapsto W(x_\alpha, y_3, y_\alpha, z_3, z_\alpha; F)$  est  $\mathcal{L}^3 \otimes \mathcal{L}^3 \otimes \mathcal{L}^2$ -mesurable pour assurer, au final, que les intégrales dans (7), (8) (9) et (10) sont bien définies.

La technique utilisée dans la démonstration de ce résultat est très similaire à celle du chapitre 1. Dans un premier temps, nous prouvons le résultat dans le cas où la densité d'énergie élastique  $W$  ne dépend pas explicitement de la variable macroscopique  $x_\alpha$ . Il suffit alors d'adapter les arguments employés dans la démonstration du résultat d'homogénéisation réitérée (voir Theorem 22.1 dans BRAIDES & DEFRANCESCHI [32]) au cas de la réduction dimensionnelle. Ensuite nous étudions le cas général grâce à cette même technique de dédoublement de variable. Nous avons pu raffiner cette méthode grâce au théorème d'extension de Tietze (voir DiBENEDETTO [53]) qui permet d'étendre sur tout l'espace n'importe quelle fonction continue définie sur un fermé (au lieu d'un compact comme c'était le cas du Theorem 1, Section 1.2 dans EVANS & GARIEPY [55]). En utilisant le théorème de Scorza-Draconi, nous sommes alors à même d'établir que toute fonction de Carathéodory coïncide avec une fonction continue à l'extérieur d'un ensemble de mesure de Lebesgue arbitrairement petite. De plus, cette fonction continue hérite des propriétés de croissance et de périodicité faites sur  $W$ . Nous remplaçons alors  $W$  par cette fonction continue et nous procédons comme dans le chapitre 1 via un argument de continuité uniforme. A nouveau, les termes résiduels tendent vers zéro car il s'agit juste d'estimer l'énergie sur des ensembles de mesure arbitrairement petite. Le lemme d'équi-intégrabilité de BOCEA & FONSECA [25] rend cela possible car les suites minimisantes peuvent être choisies de façon à avoir un gradient rescalé  $p$ -équi-intégrable.

Le troisième chapitre a fait l'objet d'un travail en collaboration avec NADIA ANSINI et CATERINA IDA ZEPPIERI. Il s'agit d'étudier le comportement asymptotique de l'énergie élastique d'un matériau formé de deux couches minces dont l'interface est constituée de zones de connection périodiquement distribuées. Notons que l'un des deux films minces pourrait être remplacé par un substrat mince de telle sorte que notre problème s'interpréterait comme le décollement – dû à un défaut de l'interface – d'un film mince à partir de ce même substrat. Nous étudions ici le cas où l'épaisseur des films  $\varepsilon$  est beaucoup plus petite que la période de distribution  $\delta$  des zones de connection. Notre approche est très similaire de celle employée par ANSINI dans [6] qui a traité le cas où  $\varepsilon \sim \delta$ . Contrairement aux autres chapitres de cette thèse, nous ne nous placerons pas dans le cadre du passage 3D-2D mais dans celui plus général du passage  $nD-(n-1)D$  car ici, la dimension  $n$  de l'espace de départ joue un rôle particulièrement important. Une zone de connection est une boule  $(n-1)$ -dimensionnelle, notée  $B'_r(i\delta)$ , de centre  $i\delta$ , avec  $i \in \mathbb{Z}^{n-1}$ , et de rayon  $r > 0$  à déterminer de façon à obtenir un modèle limite non trivial. Si  $\omega \subset \mathbb{R}^{n-1}$  est un ouvert borné, le matériau que nous allons prendre en compte occupe dans sa configuration de référence l'ouvert  $\Omega_{\delta,r}^\varepsilon$  (voir figure 4) paramétré par

$$\Omega_{\delta,r}^\varepsilon := [\omega \times (-\varepsilon, 0)] \cup [\omega \times (0, \varepsilon)] \cup \left[ \bigcup_{i \in \mathbb{Z}^{n-1}} B'_r(i\delta) \cap \omega \times \{0\} \right].$$


 FIG. 4 – Le domaine  $\Omega_{\delta, r}^\varepsilon$ .

Tout comme dans Ansini [6], nous nous attendons à voir apparaître à la limite un terme d'énergie interfaciale d'ordre zéro pour des zones de connection ayant un rayon  $r$  bien déterminé. Ce problème ainsi posé met en jeu trois paramètres  $\varepsilon$ ,  $\delta$  et  $r$ . Il semble naturel de prendre en compte leurs taux de convergence lors de l'analyse asymptotique. Dans cette étude, nous supposons que  $\varepsilon \ll \delta$ . Par ailleurs, comme  $\omega \setminus \bigcup_{i \in \mathbb{Z}^{n-1}} (B'_r(i\delta) \cap \omega)$  est une partie de la frontière, toute déformation admissible peut être discontinue à travers cette partie du bord. On peut alors toujours supposer que  $r \ll \delta$  car sinon, il se pourrait que les zones de connection s'intersectent deux à deux, auquel cas toute déformation admissible serait nécessairement continue à l'interface, donnant lieu à une énergie interfaciale identiquement nulle. Reste à prendre en compte le dernier ratio  $r/\varepsilon$  mettant en évidence trois régimes différents. On se donne trois suites  $\{\varepsilon_j\}$ ,  $\{\delta_j\}$  et  $\{r_j\}$  qui tendent vers zéro, on note  $\Omega_j := \Omega_{\delta_j, r_j}^{\varepsilon_j}$  et nous supposons que

$$\lim_{j \rightarrow +\infty} \frac{r_j}{\delta_j} = \lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{\delta_j} = 0 \quad \text{et} \quad \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} =: \ell \in [0, +\infty].$$

Les trois régimes évoqués plus haut correspondent aux cas où  $\ell = 0$ ,  $0 < \ell < +\infty$  et  $\ell = +\infty$ .

Nous considérons des matériaux homogènes dont la densité d'énergie élastique est donnée par la fonction  $W : \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$  à croissance  $1 < p < n - 1$ . L'énergie élastique est alors définie par

$$W^{1,p}(\Omega_j; \mathbb{R}^m) \ni u \mapsto \frac{1}{\varepsilon_j} \int_{\Omega_j} W(\nabla u) dx.$$

On démontre que cette fonctionnelle  $\Gamma$ -converge vers

$$W^{1,p}(\omega; \mathbb{R}^m) \ni u^\pm \mapsto \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^+) dx_\alpha + \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^-) dx_\alpha + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-) dx_\alpha,$$

où  $\mathcal{Q}W_0$  est la même fonction que celle obtenue par LE DRET & RAOULT dans [74]. Les nombres  $0 < R^{(\ell)} < +\infty$  rendent compte de l'ordre de grandeur des rayons critiques selon le régime  $\ell$  :

$$R^{(0)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-p}}{\varepsilon_j \delta_j^{n-1}}, \quad R^{(\infty)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-p-1}}{\delta_j^{n-1}} \quad \text{et} \quad R^{(\ell)} = R^{(\infty)} = \ell^{-1} R^{(0)} \quad \text{si } \ell \in (0, +\infty).$$

En notant  $C_{1,\infty} := \{(x_\alpha, 0) \in \mathbb{R}^n : |x_\alpha| \geq 1\}$ , les fonctions  $\varphi^{(\ell)} : \mathbb{R}^m \rightarrow [0, +\infty)$  sont données par les formules de type capacités non linéaires suivantes :

$$\varphi^{(0)}(z) = \inf \left\{ \int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m), \nabla \zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \right. \\ \left. \zeta - z \in L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \text{ et } \zeta \in L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\},$$

si  $\ell \in (0, +\infty)$ ,

$$\varphi^{(\ell)}(z) := \inf \left\{ \int_{(\mathbb{R}^{n-1} \times (-1,1)) \setminus C_{1,\infty}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times (-1,1)) \setminus C_{1,\infty}; \mathbb{R}^m), \right. \\ \left. \nabla \zeta \in L^p((\mathbb{R}^{n-1} \times (-1,1)) \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \zeta - z \in L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right. \\ \left. \text{ et } \zeta \in L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\}$$

et enfin

$$\varphi^{(\infty)}(z) := \inf \left\{ \int_{\mathbb{R}^{n-1}} (\mathcal{Q}g_0(\nabla_\alpha \zeta^+) + \mathcal{Q}g_0(\nabla_\alpha \zeta^-)) dx_\alpha : \zeta^\pm \in W_{\text{loc}}^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m), \right. \\ \left. \zeta^+ = \zeta^- \text{ sur } B'_1(0), \nabla_\alpha \zeta^\pm \in L^p(\mathbb{R}^{n-1}; \mathbb{R}^{m \times (n-1)}), \right. \\ \left. \zeta^+ - z \text{ et } \zeta^- \in L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m) \right\}.$$

Dans les formules précédentes, nous avons noté  $g : \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$  et  $g_0 : \mathbb{R}^{m \times (n-1)} \rightarrow [0, +\infty)$  les fonctions définies par  $g(F) := \lim_{j \rightarrow +\infty} r_j^p \mathcal{Q}W(r_j^{-1}F)$  et  $g_0(\overline{F}) := \inf\{g(\overline{F}|z) : z \in \mathbb{R}^m\}$ ,  $p^* := (n-1)p/(n-1-p)$  désigne l'exposant de Sobolev en dimension  $n-1$ .

La démonstration, basée sur une méthode introduite par ANSINI & BRAIDES dans [9, 10], est très semblable à celle employée par ANSINI dans [6]. Elle consiste à séparer les contributions proches et lointaines des zones de connection en modifiant les suites minimisantes de façon à ce qu'elles aient une valeur constante sur la partie supérieure et inférieure de la frontière d'une sphère entourant chaque zone de connection. Malheureusement, par construction, le rayon de cette sphère est du même ordre de grandeur que  $\delta$ . Dans notre cas, comme  $\varepsilon \ll \delta$ , il se pourrait que cette sphère ne soit pas contenue entièrement dans l'épaisseur du cylindre. Pour remédier à cet inconvénient, nous changeons la géométrie du problème en modifiant les suites minimisantes, non pas sur une sphère, mais sur le bord latéral d'un cylindre. La valeur que nous imposons est exactement la moyenne de la suite sur la partie supérieure et inférieure du cylindre. Celle-ci est particulièrement intéressante car nous démontrons que le terme d'énergie interfaciale peut s'écrire comme la limite d'une sorte de somme de Riemann où la valeur de la fonction en un

point particulier est justement remplacée par cette moyenne. Le résultat de  $\Gamma$ -convergence est alors obtenu par un argument direct d'estimation des  $\Gamma$ -limites inférieure et supérieure.

Le quatrième chapitre de cette thèse est le contenu d'un article [12] accepté dans la revue *Calculus of Variations and Partial Differential Equations*. Il s'agit d'étudier des films minces cassables et l'évolution quasistatique des fissures au cours du temps. L'étude de la formulation faible du problème statique consiste à étudier le comportement asymptotique de la fonctionnelle

$$SBV^p(\Omega; \mathbb{R}^3) \ni u \mapsto \int_{\Omega} W \left( \nabla_{\alpha} u \Big|_{\varepsilon} \nabla_3 u \right) dx + \int_{S(u)} \left| \left( (\nu_u)_{\alpha} \Big|_{\varepsilon} (\nu_u)_3 \right) \right| d\mathcal{H}^2$$

lorsque  $\varepsilon$  tend vers zéro. Le problème majeur intervenant ici est le manque de compacité dans  $SBV^p(\Omega; \mathbb{R}^3)$  des suites minimisantes. Contrairement à BRAIDES & FONSECA dans [34] et BUCHITTÉ, FONSECA, LEONI & MASCARENHAS dans [28], le terme d'énergie de surface ne permet pas d'avoir une borne sur la partie saut du gradient des suites minimisantes. Pour remédier à ce problème, nous procédons comme FONSECA & FRANCFORT dans [56] en utilisant un argument de troncature. Nous pouvons alors démontrer que toute suite minimisante convergeant fortement dans  $L^1(\Omega; \mathbb{R}^3)$  vers une limite essentiellement bornée, est elle-même uniformément bornée dans  $L^{\infty}(\Omega; \mathbb{R}^3)$ . Cet argument nous permet d'établir le résultat de  $\Gamma$ -convergence pour des déformations appartenant à  $L^{\infty}(\Omega; \mathbb{R}^3)$ . Nous traitons ensuite le cas général par approximation en choisissant correctement la suite de fonctions de troncature. Nous démontrons alors que la  $\Gamma$ -limite est donnée par

$$SBV^p(\omega; \mathbb{R}^3) \ni u \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} + 2\mathcal{H}^1(S(u))$$

où  $\mathcal{Q}W_0$  est la même fonction que celle obtenue par LE DRET & RAOULT dans [74].

L'existence d'une évolution quasistatique du modèle de FRANCFORT & MARIGO [64] a été obtenu par DAL MASO & TOADER [48] dans le cas quadratique et antiplan, c'est-à-dire  $W(F) = |F|^2$  et  $u$  est à valeurs scalaires. Cependant, leur approche nécessite de se restreindre au cas bidimensionnel avec des restrictions topologiques sur les fissures : le nombre maximal de composantes connexes est a priori connu. Ce résultat a par la suite été généralisé par CHAMBOLLE dans [38] au cas de l'élasticité linéaire plane. Une autre approche utilisant une formulation faible sans  $SBV$  a permis à FRANCFORT & LARSEN dans [63] d'étendre ce résultat sans restriction aucune sur les fissures et sur la dimension. Le cas plus général de l'élasticité non linéaire a été par la suite traité par DAL MASO, FRANCFORT & TOADER dans [46, 47].

Nous nous donnons un intervalle de temps  $[0, T]$  pendant lequel nous imposons au matériau une condition limite dépendant de  $t$  sur le bord latéral  $\partial\omega \times (-1, 1)$ . Cette condition va être exprimée d'une façon peu habituelle dans le but de tenir compte des fissures pouvant apparaître sur le bord latéral. Pour ce faire, nous considérons le cylindre élargi  $\Omega' := \omega' \times (-1, 1)$ , où  $\bar{\omega} \subset \omega' \subset \mathbb{R}^2$ , et nous imposons la déformation  $g^{\varepsilon}(t) \in W^{1,p}(\Omega'; \mathbb{R}^3)$  sur  $\partial\omega \times (-1, 1)$  en exigeant que toute déformation cinématiquement admissible  $v \in SBV^p(\Omega'; \mathbb{R}^3)$  au temps  $t$ , coïncide avec

$g^\varepsilon(t)$  sur  $[\omega' \setminus \bar{\omega}] \times (-1, 1)$ . Notons que cette condition limite est le seul mécanisme qui fera croître les fissures au cours du chargement car le reste du bord  $\omega \times \{-1, 1\}$  est maintenu libre. Supposons qu'il n'y a pas de fissure préexistante et notons  $u_0^\varepsilon \in SBV^p(\Omega'; \mathbb{R}^3)$  une déformation initiale qui minimise l'énergie totale du système à  $t = 0$ , c'est-à-dire

$$v \mapsto \int_{\Omega} W \left( \nabla_{\alpha} v \Big|_{\frac{1}{\varepsilon}} \nabla_3 v \right) dx + \int_{S(v)} \left| \left( (\nu_v)_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_v)_3 \right) \right| d\mathcal{H}^2,$$

parmi  $\{v \in SBV^p(\Omega'; \mathbb{R}^3) : v = g^\varepsilon(0) \text{ p.p. sur } [\omega' \setminus \bar{\omega}] \times (-1, 1)\}$ . Tout comme dans DAL MASO, FRANCFORT & TOADER [47], nous prendrons le parti de supposer que toute suite impliquée est uniformément bornée dans  $L^\infty(\Omega'; \mathbb{R}^3)$ . Cette hypothèse, purement arbitraire, permet d'éviter des points techniques liés aux espaces  $GSBV$ . Cependant, elle trouve toute sa justification dans le cas scalaire par un argument de troncature à condition que  $g^\varepsilon(t)$  soit uniformément bornée dans  $L^\infty(\Omega')$  en  $\varepsilon$  et  $t$ . Le Theorem 2.1 dans [47] implique alors que pour tout  $t \in [0, T]$ , il existe un couple  $(u^\varepsilon(t), \Gamma^\varepsilon(t))$  tel que  $u^\varepsilon(0) = u_0^\varepsilon$  et  $\Gamma^\varepsilon(0) = S(u_0^\varepsilon)$  possédant les trois propriétés suivantes :

- *Irréversibilité* :  $\Gamma^\varepsilon(t_1) \subset \Gamma^\varepsilon(t_2)$  pour tout  $0 \leq t_1 \leq t_2 \leq T$  ;
- *Minimalité* : pour tout  $t \in [0, T]$ ,  $S(u^\varepsilon(t)) \subset \Gamma^\varepsilon(t)$  et  $u^\varepsilon(t)$  minimise

$$v \mapsto \int_{\Omega} W \left( \nabla_{\alpha} v \Big|_{\frac{1}{\varepsilon}} \nabla_3 v \right) dx + \int_{S(v) \setminus \Gamma^\varepsilon(t)} \left| \left( (\nu_v)_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_v)_3 \right) \right| d\mathcal{H}^2,$$

parmi  $\{v \in SBV^p(\Omega'; \mathbb{R}^3) : v = g^\varepsilon(t) \text{ p.p. sur } [\omega' \setminus \bar{\omega}] \times (-1, 1)\}$  ;

- *Conservation de l'énergie* : l'énergie totale

$$\mathcal{E}_\varepsilon(t) := \int_{\Omega} W \left( \nabla_{\alpha} u^\varepsilon(t) \Big|_{\frac{1}{\varepsilon}} \nabla_3 u^\varepsilon(t) \right) dx + \int_{\Gamma^\varepsilon(t)} \left| \left( (\nu_{\Gamma^\varepsilon(t)})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{\Gamma^\varepsilon(t)})_3 \right) \right| d\mathcal{H}^2$$

est absolument continue en temps et

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + \int_0^t \int_{\Omega} \partial W \left( \nabla_{\alpha} u^\varepsilon(\tau) \Big|_{\frac{1}{\varepsilon}} \nabla_3 u^\varepsilon(\tau) \right) \cdot \left( \nabla_{\alpha} \dot{g}^\varepsilon(\tau) \Big|_{\frac{1}{\varepsilon}} \nabla_3 \dot{g}^\varepsilon(\tau) \right) dx d\tau.$$

Signalons que le fait que  $\Gamma^\varepsilon(0) = S(u_0^\varepsilon)$  ne contredit pas l'hypothèse d'absence de fissure préexistante. Cela signifie simplement que la fissure créée à l'instant initial est l'ensemble des discontinuités de  $u_0^\varepsilon$ .

La question consiste ensuite à se demander si les processus de  $\Gamma$ -convergence et d'évolution quasistatique sont compatibles. Autrement dit, partant d'une évolution quasistatique du modèle tridimensionnel, il s'agit de savoir si elle converge en un certain sens vers une évolution quasistatique associée au modèle bidimensionnel limite. Notons  $g(t) \in W^{1,p}(\omega'; \mathbb{R}^3)$  la limite forte dans  $W^{1,p}(\Omega'; \mathbb{R}^3)$  de la suite  $\{g^\varepsilon(t)\}$ . Grâce à des estimations d'énergie nous pouvons déduire l'existence d'une déformation  $u(t) \in SBV^p(\omega'; \mathbb{R}^3)$  telle que  $u(t) = g(t)$  sur  $\omega' \setminus \bar{\omega}$  comme limite faible dans  $SBV^p(\Omega'; \mathbb{R}^3)$  de la suite  $\{u^\varepsilon(t)\}$ . Nous pouvons aussi définir une fissure  $\gamma(t) \subset \bar{\omega}$  comme une variante de la  $\sigma^p$ -limite, introduite par DAL MASO, FRANCFORT & TOADER dans [46, 47],

de la suite  $\{\Gamma^\varepsilon(t)\}$ . Il s'agit du plus grand ensemble de discontinuité de fonctions  $SBV^p(\omega')$  qui sont limites faibles dans  $SBV^p(\Omega')$  d'une suite de fonctions dont l'ensemble des discontinuités est contenu dans  $\Gamma^\varepsilon(t)$ , et dont le gradient approximé rescalé est uniformément borné dans  $L^p(\Omega'; \mathbb{R}^3)$ . Nous démontrons ensuite que le couple  $(u(t), \gamma(t))$  est une évolution quasistatique associée à la  $\Gamma$ -limite, c'est à dire  $\gamma(0) = S(u(0))$  et

- (i) *Irréversibilité* :  $\gamma(t_1) \subset \gamma(t_2)$  pour tout  $0 \leq t_1 \leq t_2 \leq T$  ;
- (ii) *Minimalité* : pour tout  $t \in [0, T]$ ,  $S(u(t)) \subset \gamma(t)$  et  $u(t)$  minimise

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)),$$

parmi  $\{v \in SBV^p(\omega'; \mathbb{R}^3) : v = g(t) \text{ p.p. sur } \omega' \setminus \bar{\omega}\}$  ;

- (iii) *Conservation de l'énergie* : l'énergie totale

$$\mathcal{E}(t) := 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(t))$$

est absolument continue en temps et

$$\mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha} u(\tau)) \cdot \nabla \dot{g}(\tau) dx_{\alpha} d\tau.$$

Le fait que  $\gamma(t)$  est croissante en temps est une propriété de la  $\sigma^p$ -convergence qui demeure dans notre cas malgré les changements effectués. La propriété de minimalité est établie à l'instant initial grâce à un argument de  $\Gamma$ -convergence avec condition de bord. Ceci est rendu possible car on suppose justement que l'ouvert ne contient pas de fissure préexistante. Nous devons par contre procéder autrement pour les temps suivants car l'énergie de surface diffère alors de celle utilisée dans le résultat de relaxation. Nous utilisons ici un théorème de transfert de saut qui est établi à partir du Theorem 2.1 de FRANCFORT & LARSEN dans [63] et par un argument de slicing. Finalement la conservation de l'énergie apparaît comme une conséquence, d'un côté du fait que l'on peut approcher une intégrale de Lebesgue par une somme de Riemann convenablement choisie (voir HAHN [71] et HENSTOCK [72]), et de l'autre de la convergence faible des contraintes  $\partial W(\nabla_{\alpha} u^\varepsilon(t) | \frac{1}{\varepsilon} \nabla_3 u^\varepsilon(t))$  vers  $(\partial(\mathcal{Q}W_0)(\nabla_{\alpha} u(t)) | 0)$  dans  $L^p(\Omega'; \mathbb{R}^{3 \times 3})$  (voir DAL MASO, FRANCFORT & TOADER [46]). Enfin, nous démontrons la convergence des énergies de volume et de surface tridimensionnelles vers leurs analogues bidimensionnelles respectivement :

$$\left\{ \begin{array}{l} \int_{\Omega} W \left( \nabla_{\alpha} u^\varepsilon(t) \middle| \frac{1}{\varepsilon} \nabla_3 u^\varepsilon(t) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha}, \\ \int_{\Gamma^\varepsilon(t)} \left| \left( (\nu_{\Gamma^\varepsilon(t)})_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_{\Gamma^\varepsilon(t)})_3 \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(t)). \end{array} \right.$$



## Chapitre 1

# Spatial heterogeneity in 3D-2D dimensional reduction





## 1.1 Introduction

The purpose of this article is to study the behavior of a thin elastic plate, as the thickness tends to zero. This approach renders more realistic the idealized view of a film as a thin plate. The originality of the work comes from the heterogeneity of the material under consideration. Previous results have been established in the homogeneous case; our aim here is to generalize those. As we will see, accounting for inhomogeneity leads to technical difficulties linked to the equi-integrable character of the scaled gradient. We will use a "classical" approach of the theory of dimension reduction. In recent years, the investigation of dimensional reduction has focussed on variational methods and used De Giorgi's  $\Gamma$ -convergence (see [32, 45]) as its main tool.

As far as 3D-2D asymptotic analysis is concerned, the seminal paper is [74], in which a membrane model is derived from three-dimensional hyperelasticity. In its footsteps several studies have derived or re-derived various membrane-like models in various settings; see in particular [35] and references therein; note that in Section 3 of that paper, a transversally inhomogeneous thin domain is studied, but that in-plane-homogeneity is imposed. Because of frame indifference, it may occur that the membrane effect is not excited by the loads: this is the case for example when the lateral boundary conditions on the thin domain are compressive (see e.g. Theorem 6.2 in [66]). Then the membrane energy, which results from a 3D-energy of the order of the thickness  $\varepsilon$ , is actually zero and lower energy modes are activated. In [62], a justification of classical nonlinear plate models for a homogeneous isotropic material is given by a formal asymptotic expansion. Recently, those results have been rigorously justified by means of variational methods for general homogeneous hyperelastic bodies. A Kirchhoff bending model in [65, 66], and a Föppl-von Kármán model in [67] have been obtained when the 3D-energy scales respectively like  $\varepsilon^3$  and  $\varepsilon^5$ .

The present study falls squarely within the membrane framework in the sense that, thanks to frame indifference, the stored energy function depends only on the first fundamental form of the deformed plate mid-surface. Our goal is to rigorously derive models for heterogeneous membranes from their heterogeneous thin 3D-counterparts. The paper is devoted to a generalization of the results established in [74], [35] and [27] to the case of a general inhomogeneity.

The key ingredient of this study is the equi-integrability theorem of [25] (Theorem 1.1 of that reference). An alternative proof of that theorem was also communicated to the authors [29]. This theorem shows that a sequence of scaled gradients  $\{(\nabla_\alpha u_\varepsilon | \frac{1}{\varepsilon} \nabla_3 u_\varepsilon)\}$ , which is bounded in  $L^p(\Omega; \mathbb{R}^{3 \times 3})$ , with  $p > 1$ , can be decomposed into the sum of two sequences  $\{w_\varepsilon\}$  and  $\{z_\varepsilon\}$  where  $\{ |(\nabla_\alpha w_\varepsilon | \frac{1}{\varepsilon} \nabla_3 w_\varepsilon) |^p \}$  is equi-integrable and  $z_\varepsilon \rightarrow 0$  in measure.

Let  $\omega$  be a bounded open subset of  $\mathbb{R}^2$ . Consider  $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ , the reference configuration of a hyperelastic heterogeneous thin film, with elastic energy density given by the  $\varepsilon$ -dependent Carathéodory function  $W_\varepsilon : \Omega_\varepsilon \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ . We will assume e.g. that the body is clamped on the lateral boundary  $\Gamma_\varepsilon := \partial\omega \times (-\varepsilon, \varepsilon)$  and that it is submitted to the action of surface traction densities on  $\Sigma_\varepsilon := \omega \times \{-\varepsilon, \varepsilon\}$ . The total energy of the system under a deformation  $u : \Omega_\varepsilon \rightarrow \mathbb{R}^3$

is given by

$$\mathcal{E}(\varepsilon)(u) = \int_{\Omega_\varepsilon} W_\varepsilon(x; \nabla u) dx - \int_{\Omega_\varepsilon} f_\varepsilon \cdot u dx - \int_{\Sigma_\varepsilon} g_\varepsilon \cdot u d\mathcal{H}^2,$$

where  $\mathcal{H}^2$  stands for the two-dimensional surface measure,  $f_\varepsilon \in L^{p'}(\Omega_\varepsilon; \mathbb{R}^3)$  denotes an appropriate dead load and  $g_\varepsilon \in L^{p'}(\Sigma_\varepsilon; \mathbb{R}^3)$  some surface traction densities ( $1/p + 1/p' = 1$ ). We denote by  $W_{\Gamma_\varepsilon}^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$  the space of kinematically admissible fields, that is the functions in  $W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$  with zero trace on  $\Gamma_\varepsilon$ . As is classical in hyperelasticity, the equilibrium problem is viewed as the minimization problem

$$\inf \left\{ \mathcal{E}(\varepsilon)(u) : u - x \in W_{\Gamma_\varepsilon}^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) \right\}.$$

Since the integration domain depends on  $\varepsilon$ , we reformulate the problem on a fixed domain through a  $1/\varepsilon$ -dilatation in the transverse direction  $x_3$ . Let  $x_\alpha$  the vector  $(x_1, x_2) \in \omega$ , we set  $v(x_\alpha, x_3/\varepsilon) := u(x_\alpha, x_3)$  and  $\mathcal{E}_\varepsilon(v) := \mathcal{E}(\varepsilon)(u)/\varepsilon$ , then

$$\mathcal{E}_\varepsilon(v) = \int_{\Omega} W_\varepsilon \left( x_\varepsilon; \nabla_\alpha v(x) \Big| \frac{1}{\varepsilon} \nabla_3 v(x) \right) dx - \int_{\Omega} f_\varepsilon(x_\varepsilon) \cdot v(x) dx - \frac{1}{\varepsilon} \int_{\Sigma} g_\varepsilon(x_\varepsilon) \cdot v(x) d\mathcal{H}^2,$$

where  $x_\varepsilon := (x_\alpha, \varepsilon x_3)$ . We set  $I := (-1, 1)$ ,  $\Omega := \omega \times I$ ,  $\Sigma := \omega \times \{-1, 1\}$ , denote by  $\nabla_\alpha v$  the  $3 \times 2$  matrix of partial derivatives  $\frac{\partial v_i}{\partial x_\alpha}$  ( $i \in \{1, 2, 3\}$ ,  $\alpha \in \{1, 2\}$ ) and by  $(\overline{F}|z)$ , the two first columns of which are those of the matrix  $\overline{F} \in \mathbb{R}^{3 \times 2}$ , while the last one is the vector  $z \in \mathbb{R}^3$ . A formal asymptotic expansion in [62] shows that the membrane theory arises if the body forces is of order 1 and the surfaces loadings is of order  $\varepsilon$ . We next assume that

$$\begin{cases} W_\varepsilon(x_\alpha, \varepsilon x_3; F) &= W(x_\alpha, x_3; F), \\ f_\varepsilon(x_\alpha, \varepsilon x_3) &= f(x_\alpha, x_3), \\ g_\varepsilon(x_\alpha, \varepsilon x_3) &= g_0(x_\alpha, x_3) + \varepsilon g(x_\alpha, x_3) \end{cases}$$

where  $f \in L^{p'}(\Omega; \mathbb{R}^3)$ ,  $g_0, g \in L^{p'}(\Sigma; \mathbb{R}^3)$  and  $W : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  is a Carathéodory function satisfying conditions of  $p$ -coercivity and  $p$ -growth : for some  $0 < \beta' \leq \beta < +\infty$  and some  $1 < p < \infty$ ,

$$\beta' |F|^p \leq W(x; F) \leq \beta (|F|^p + 1), \quad F \in \mathbb{R}^{3 \times 3}, \quad \text{for a.e. } x \in \Omega. \quad (1.1.1)$$

The usual Euclidian norm on the space  $\mathbb{R}^{m \times n}$  of real  $m \times n$  matrices is denoted by  $|F|$ . The minimisation problem becomes

$$\inf \left\{ \mathcal{E}_\varepsilon(v) : v - x_\varepsilon \in W_{\Gamma}^{1,p}(\Omega; \mathbb{R}^3) \right\}, \quad (1.1.2)$$

where  $W_{\Gamma}^{1,p}(\Omega; \mathbb{R}^3)$  stands for the functions in  $W^{1,p}(\Omega; \mathbb{R}^3)$  with zero trace on the lateral boundary  $\Gamma := \partial\omega \times I$ .

If we denote by  $g_0^\pm$  (resp.  $g^\pm$ ) the trace of  $g_0$  (resp.  $g$ ) on  $\omega \times \{\pm 1\}$ , in view of Remark 3.2.3 of [62], the loading vectors  $g_0^+$  and  $g_0^-$  must satisfy  $g_0^+ + g_0^- = 0$ . In the second section, we assume the stronger condition that  $g_0^+ = g_0^- = 0$ . The physical implication of this assumption is that the plate of thickness  $2\varepsilon$  cannot support a non vanishing resultant surface load as the thickness

$\varepsilon$  goes to zero. We generalize here the result of [74] and [35] to a general inhomogeneity. In the third section, we address the general case of admissible surface loadings. It deals with a similar problem, in which the class of surface forces generates a bending moment density as in [27]; the limit behavior is not solely characterized by the limit deformations (a  $\mathbb{R}^3$ -valued field defined on the mid-plane), but it also involves the average of the Cosserat vector also defined on the mid-plane. Once again, we generalize the result of [27] to the inhomogeneous case. The fourth and last section demonstrates that the classical membrane model can be seen as a particular case of the Cosserat model when the bending moment density is zero.

As for notation,  $\mathcal{A}(\omega)$  is the family of open subsets of  $\omega$ ;  $\mathcal{L}^n$  stands for the  $n$ -dimensional Lebesgue measure in  $\mathbb{R}^n$  (in the sequel,  $n$  will be equal to 2 or 3); for any measurable set  $A \subset \mathbb{R}^n$  and any measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we write  $\bar{f}_A := \mathcal{L}^n(A)^{-1} \int_A f(x) dx$  for the average of  $f$  over  $A$ ;  $\rightarrow$  always denotes strong convergence whereas  $\rightharpoonup$  (resp.  $\overset{*}{\rightharpoonup}$ ) denotes weak (resp. weak-\*) convergence. Finally, we loosely identify  $L^p(\omega; \mathbb{R}^3)$  (resp.  $W^{1,p}(\omega; \mathbb{R}^3)$ ) with those functions in  $L^p(\Omega; \mathbb{R}^3)$  (resp.  $W^{1,p}(\Omega; \mathbb{R}^3)$ ) that do not depend upon  $x_3$ .

## 1.2 Classical nonlinear membrane model

In this section, we assume that  $g_\varepsilon = \varepsilon g$  with  $g \in L^p(\Sigma; \mathbb{R}^3)$ . Thus, the minimization problem (1.1.2) becomes

$$\inf_{v-x_\varepsilon \in W_\Gamma^{1,p}(\Omega; \mathbb{R}^3)} \left\{ \int_\Omega W \left( x; \nabla_\alpha v \middle| \frac{1}{\varepsilon} \nabla_3 v \right) dx - \int_\Omega f \cdot v dx - \int_\Sigma g \cdot v d\mathcal{H}^2 \right\}.$$

Define for any  $(u; A) \in L^p(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$ ,

$$J_\varepsilon(u; A) := \begin{cases} \int_{A \times I} W \left( x; \nabla_\alpha u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx & \text{if } u \in W^{1,p}(A \times I; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

and for any sequence  $\{\varepsilon_j\} \searrow 0^+$

$$J_{\{\varepsilon_j\}}(u; A) := \inf_{\{u_j\}} \left\{ \liminf_{j \rightarrow +\infty} J_{\varepsilon_j}(u_j; A) : u_j \rightarrow u \text{ in } L^p(A \times I; \mathbb{R}^3) \right\}. \quad (1.2.1)$$

**Remark 1.2.1.** For any  $A \in \mathcal{A}(\omega)$ ,  $J_{\{\varepsilon_j\}}(u; A) = +\infty$  whenever  $u \in L^p(\Omega; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$ , as is easily seen in view of the definition of  $J_{\varepsilon_j}$ , together with the coercivity condition (1.1.1).

By virtue of Remark 1.2.1, together with Theorem 2.5 in [35], for all sequences  $\{\varepsilon_j\} \searrow 0^+$ , there exists a subsequence  $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$  such that  $J_{\{\varepsilon_n\}}(\cdot; A)$  defined in (1.2.1) is the  $\Gamma(L^p)$ -limit of  $J_{\varepsilon_n}(\cdot; A)$ . Further, there exists a Carathéodory function  $W_{\{\varepsilon_n\}} : \omega \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  such that, for all  $A \in \mathcal{A}(\omega)$  and all  $u \in W^{1,p}(A; \mathbb{R}^3)$

$$J_{\{\varepsilon_n\}}(u; A) = 2 \int_A W_{\{\varepsilon_n\}}(x_\alpha; \nabla_\alpha u) dx_\alpha.$$

**Remark 1.2.2.** Lemma 2.6 of [35] implies that  $J_{\{\varepsilon_n\}}(u; A)$  is unchanged if the approximating sequences  $\{u_n\}$  are constrained to match the lateral boundary condition of their target, i.e.  $u_n \equiv u$  on  $\partial A \times I$ .

From now onward, we will assume that  $\{\varepsilon_n\}$  denotes a subsequence of  $\{\varepsilon_j\}$  such that the  $\Gamma(L^p)$ -limit of  $J_{\varepsilon_n}(u; A)$  exists, in which case it coincides with  $J_{\{\varepsilon_n\}}(u; A)$ . Under the hypothesis that  $W$  is a homogeneous elastic energy density, it is proved in [74], Theorem 2, that  $J_{\{\varepsilon_n\}}(u; A)$  does not depend upon the choice of the sequence  $\{\varepsilon_n\}$ . It is given by

$$J_{\{\varepsilon_n\}}(u; A) = 2 \int_A \mathcal{Q}W_0(\nabla_\alpha u) dx_\alpha,$$

with for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ ,

$$W_0(\bar{F}) := \inf_{z \in \mathbb{R}^3} W(\bar{F}|z),$$

and,

$$\mathcal{Q}W_0(\bar{F}) := \inf_{\phi \in W_0^{1,p}(Q'; \mathbb{R}^3)} \int_{Q'} W_0(\bar{F} + \nabla_\alpha \phi) dx_\alpha,$$

where  $Q' := (0, 1)^2$ , and  $\mathcal{Q}W_0$  is the 2D-quasiconvexification of  $W_0$ . This result was extended to the case where  $W$  is also function of  $x_3$  in [35], Theorem 3.1. It is proved there that, in such a case,  $J_{\{\varepsilon_n\}}$  is given by

$$J_{\{\varepsilon_n\}}(u; A) = 2 \int_A \underline{W}(\nabla_\alpha u) dx_\alpha,$$

with for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ ,

$$\underline{W}(\bar{F}) := \inf_{L, \phi} \left\{ \int_{Q' \times I} W(x_3; \bar{F} + \nabla_\alpha \phi | L \nabla_3 \phi) dx_\alpha dx_3 : L > 0, \right. \\ \left. \phi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \phi = 0 \text{ on } \partial Q' \times I \right\}.$$

We wish to extend those results to the case where  $W$  is a function of *both*  $x_\alpha$  and  $x_3$ . We set, for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and for a.e.  $x_0 \in \omega$ ,

$$\underline{W}(x_0; \bar{F}) := \inf_{L, \phi} \left\{ \int_{Q' \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \phi | L \nabla_3 \phi) dx_\alpha dx_3 : L > 0, \right. \\ \left. \phi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \phi = 0 \text{ on } \partial Q' \times I \right\}. \quad (1.2.2)$$

The following theorem holds :

**Theorem 1.2.3.** *For any  $A \in \mathcal{A}(\omega)$ , the sequence  $J_\varepsilon(\cdot; A)$   $\Gamma(L^p)$ -converges to  $J_0(\cdot; A)$  where for all  $u \in L^p(\Omega; \mathbb{R}^3)$ ,*

$$J_0(u; A) = \begin{cases} 2 \int_A \underline{W}(x_\alpha; \nabla_\alpha u) dx_\alpha & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

The proof of this Theorem is a direct consequence of Lemmas 1.2.4, 1.2.5 below.

### 1.2.1 The lower bound

**Lemma 1.2.4.** *For all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and for a.e.  $x_0 \in \omega$ ,*

$$W_{\{\varepsilon_n\}}(x_0; \bar{F}) \geq \underline{W}(x_0; \bar{F}).$$

*Proof.* Let us fix  $\bar{F} \in \mathbb{R}^{3 \times 2}$ , we set  $u(x_\alpha) := \bar{F} \cdot x_\alpha$  and let  $x_0$  be a Lebesgue point of both  $W_{\{\varepsilon_n\}}(\cdot; \bar{F})$  and  $\underline{W}(\cdot; \bar{F})$ . We denote by  $Q'_r(x_0)$ , the cube of  $\mathbb{R}^2$  of center  $x_0$  and side length  $r$ , where  $r > 0$  is fixed and small enough so that  $Q'_r(x_0) \in \mathcal{A}(\omega)$ . According to the equi-integrability Theorem (Theorem 1.1 in [25]), there exists a subsequence of  $\{\varepsilon_n\}$  (not relabeled) and a sequence  $\{u_n\} \subset W^{1,p}(Q'_r(x_0) \times I; \mathbb{R}^3)$  such that

$$\left\{ \begin{array}{l} u_n \rightarrow 0 \text{ in } L^p(Q'_r(x_0) \times I; \mathbb{R}^3), \\ \left\{ \left| \left( \nabla_\alpha u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) \right|^p \right\} \text{ is equi-integrable,} \\ J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) = \lim_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W \left( x_\alpha, x_3; \bar{F} + \nabla_\alpha u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) dx_\alpha dx_3. \end{array} \right.$$

Set

$$F_n(x) := \left( \bar{F} + \nabla_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n(x) \right).$$

For any  $h \in \mathbb{N}$ , we cover  $Q'_r(x_0)$  with  $h^2$  disjoint cubes  $Q'_{i,h}$  of side length  $r/h$ . Thus  $Q'_r(x_0) = \bigcup_{i=1}^{h^2} Q'_{i,h}$  and

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) = \left( \limsup_{h \rightarrow +\infty} \right) \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h} \times I} W(x; F_n(x)) dx. \quad (1.2.3)$$

Since  $W$  is a Carathéodory integrand, Scorza-Dragoni's Theorem (see [54], Chapter VIII) implies the existence, for any  $\eta > 0$ , of a compact set  $K_\eta \subset \Omega$  such that

$$\mathcal{L}^3(\Omega \setminus K_\eta) < \eta, \quad (1.2.4)$$

and the restriction of  $W$  to  $K_\eta \times \mathbb{R}^{3 \times 3}$  is continuous. For any  $\lambda > 0$ , define

$$R_n^\lambda := \{x \in Q'_r(x_0) \times I : |F_n(x)| \leq \lambda\}.$$

By virtue of Chebyshev's inequality, there exists a constant  $C > 0$  – which does not depend on  $n$  or  $\lambda$  – such that

$$\mathcal{L}^3([Q'_r(x_0) \times I] \setminus R_n^\lambda) < \frac{C}{\lambda^p}. \quad (1.2.5)$$

Denoting by  $W^{\eta, \lambda}$  the continuous extension of  $W$  outside  $K_\eta \times \overline{B}(0, \lambda)$  (defined e.g. in Theorem 1, Section 1.2 in [55]),  $W^{\eta, \lambda}$  is continuous on  $\mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$  and satisfies the following bound

$$0 \leq W^{\eta, \lambda}(x; F) \leq \max_{K_\eta \times \overline{B}(0, \lambda)} W \leq \beta(1 + \lambda^p) \quad \text{for all } (x; F) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 3}. \quad (1.2.6)$$

In view of (1.2.3), we have

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \geq \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda \cap K_\eta} W^{\eta, \lambda}(x; F_n(x)) dx.$$

By virtue of (1.2.6) and (1.2.4),

$$\sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda \setminus K_\eta} W^{\eta, \lambda}(x; F_n(x)) dx \leq \beta(1 + \lambda^p) \eta \xrightarrow{\eta \rightarrow 0} 0,$$

uniformly in  $(n, h)$ . Therefore

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \geq \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda} W^{\eta, \lambda}(x; F_n(x)) dx.$$

Since  $W^{\eta, \lambda}$  is continuous, it is uniformly continuous on  $\overline{\Omega} \times \overline{B}(0, \lambda)$ . Thus there exists a continuous and increasing function  $\omega_{\eta, \lambda} : [0, +\infty) \rightarrow [0, +\infty)$  satisfying  $\omega_{\eta, \lambda}(0) = 0$  and such that

$$|W^{\eta, \lambda}(x; F_1) - W^{\eta, \lambda}(y; F_2)| \leq \omega_{\eta, \lambda}(|x - y| + |F_1 - F_2|), \quad \forall (x; F_1), (y; F_2) \in \overline{\Omega} \times \overline{B}(0, \lambda). \quad (1.2.7)$$

Consequently, for all  $(x_\alpha, x_3) \in [Q'_{i,h} \times I] \cap R_n^\lambda$  and all  $y_\alpha \in Q'_{i,h}$ ,

$$|W^{\eta, \lambda}(x_\alpha, x_3; F_n(x_\alpha, x_3)) - W^{\eta, \lambda}(y_\alpha, x_3; F_n(x_\alpha, x_3))| \leq \omega_{\eta, \lambda}(|x_\alpha - y_\alpha|) \leq \omega_{\eta, \lambda} \left( \frac{\sqrt{2}r}{h} \right).$$

We get, after integration in  $(x, y_\alpha)$  and summation,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{R_n^\lambda \cap [Q'_{i,h} \times I]} |W^{\eta, \lambda}(y_\alpha, x_3; F_n(x)) - W^{\eta, \lambda}(x_\alpha, x_3; F_n(x))| dx \right\} dy_\alpha \\ \leq 2r^2 \omega_{\eta, \lambda} \left( \frac{\sqrt{2}r}{h} \right) \xrightarrow{h \rightarrow +\infty} 0. \end{aligned}$$

Hence,

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \geq \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{[Q'_{i,h} \times I] \cap R_n^\lambda} W^{\eta, \lambda}(y_\alpha, x_3; F_n(x)) dx \right\} dy_\alpha.$$

Define the following sets which depend on all parameters  $(\eta, \lambda, i, h, n)$  :

$$\begin{aligned} E &:= \{(y_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (y_\alpha, x_3) \in K_\eta \text{ and } (x_\alpha, x_3) \in R_n^\lambda\}, \\ E_1 &:= \{(y_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (y_\alpha, x_3) \notin K_\eta \text{ and } (x_\alpha, x_3) \in R_n^\lambda\}, \\ E_2 &:= \{(y_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x_\alpha, x_3) \notin R_n^\lambda\}, \end{aligned}$$

and note that  $Q'_{i,h} \times Q'_{i,h} \times I = E \cup E_1 \cup E_2$ . Since  $W$  and  $W^{\eta, \lambda}$  coincide on  $K_\eta \times \overline{B}(0, \lambda)$ ,

$$\begin{aligned} J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) &\geq \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_E W^{\eta, \lambda}(y_\alpha, x_3; F_n(x)) dx dy_\alpha \\ &= \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_E W(y_\alpha, x_3; F_n(x)) dx dy_\alpha. \end{aligned} \quad (1.2.8)$$

We will prove that the corresponding terms over  $E_1$  and  $E_2$  are zero. Indeed, in view of (1.2.4) and the  $p$ -growth condition (1.1.1),

$$\begin{aligned} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{E_1} W(y_\alpha, x_3; F_n(x)) dx dy_\alpha &\leq \sum_{i=1}^{h^2} \frac{h^2}{r^2} \mathcal{L}^2(Q'_{i,h}) \mathcal{L}^3([Q'_{i,h} \times I] \setminus K_\eta) \beta(1 + \lambda^p) \\ &= \beta(1 + \lambda^p) \mathcal{L}^3([Q'_r(x_0) \times I] \setminus K_\eta) \\ &< \beta(1 + \lambda^p) \eta \xrightarrow{\eta \rightarrow 0} 0, \end{aligned} \quad (1.2.9)$$

uniformly in  $(n, h)$ . The bound from above in (1.1.1), the equi-integrability of  $\{|F_n|^p\}$  and (1.2.5) imply that

$$\begin{aligned} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{E_2} W(y_\alpha, x_3; F_n(x)) dx dy_\alpha &\leq \sum_{i=1}^{h^2} \frac{h^2}{r^2} \mathcal{L}^2(Q'_{i,h}) \beta \int_{[Q'_{i,h} \times I] \setminus R_n^\lambda} (1 + |F_n(x)|^p) dx \\ &= \beta \int_{[Q'_r(x_0) \times I] \setminus R_n^\lambda} (1 + |F_n(x)|^p) dx \xrightarrow{\lambda \rightarrow +\infty} 0, \end{aligned} \quad (1.2.10)$$



uniformly in  $(\eta, n, h)$ . Thus, in view of (1.2.8), (1.2.9), (1.2.10), Fatou's Lemma yields

$$\begin{aligned} J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) &\geq \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_n(x)) dx \right\} dy_\alpha \\ &\geq \limsup_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_n(x)) dx \right\} dy_\alpha \\ &\geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_n(x)) dx \right\} dy_\alpha, \end{aligned}$$

We apply, for a.e.  $y_\alpha \in Q'_{i,h}$ , Theorem 3.1 in [35] to the Carathéodory function  $(x_3; F) \mapsto W(y_\alpha, x_3; F)$ ; in particular

$$\liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_n(x)) dx \geq \frac{2r^2}{h^2} \underline{W}(y_\alpha; \bar{F}).$$

Thus

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \frac{2r^2}{h^2} \underline{W}(y_\alpha; \bar{F}) dy_\alpha = 2 \int_{Q'_r(x_0)} \underline{W}(y_\alpha; \bar{F}) dy_\alpha.$$

Dividing both sides of the previous inequality by  $r^2$  and passing to the limit when  $r \searrow 0^+$ , we obtain  $W_{\{\varepsilon_n\}}(x_0; \bar{F}) \geq \underline{W}(x_0; \bar{F})$ .  $\square$

## 1.2.2 The upper bound

**Lemma 1.2.5.** *For all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and for a.e.  $x_0 \in \omega$ ,*

$$W_{\{\varepsilon_n\}}(x_0; \bar{F}) \leq \underline{W}(x_0; \bar{F}).$$

*Proof.* For all  $k \geq 1$ , let  $L_k > 0$  and  $\varphi_k \in W^{1,\infty}(Q' \times I; \mathbb{R}^3)$  with  $\varphi_k = 0$  on  $\partial Q' \times I$  be such that

$$Z_k(x_0; \bar{F}) := \int_{Q' \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi_k | L_k \nabla_3 \varphi_k) dx_\alpha dx_3 \leq \underline{W}(x_0; \bar{F}) + \frac{1}{k}. \quad (1.2.11)$$

This is legitimate because of the density of  $W^{1,\infty}(Q' \times I; \mathbb{R}^3)$  into  $W^{1,p}(Q' \times I; \mathbb{R}^3)$  and the  $p$ -growth condition (1.1.1). We extend  $\varphi_k$  to  $\mathbb{R}^2 \times I$  by  $Q'$ -periodicity and set  $F_k(x) := (\bar{F} + \nabla_\alpha \varphi_k(x) | L_k \nabla_3 \varphi_k(x))$ . Then, there exists  $M_k > 0$  such that

$$\|F_k\|_{L^\infty(\mathbb{R}^2 \times I; \mathbb{R}^3)} \leq M_k. \quad (1.2.12)$$

Let  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and  $x_0$  be a Lebesgue point of  $\underline{W}(\cdot; \bar{F})$  and  $Z_k(\cdot; \bar{F})$  for all  $k \geq 1$ . We choose  $r > 0$  small enough such that  $Q'_r(x_0) \in \mathcal{A}(\omega)$ . Fix  $k \geq 1$  and set

$$\begin{cases} u(x_\alpha) := \bar{F} \cdot x_\alpha, \\ u_n^k(x_\alpha, x_3) := \bar{F} \cdot x_\alpha + L_k \varepsilon_n \varphi_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right). \end{cases}$$

Since  $u_n^k \xrightarrow{n \rightarrow +\infty} u$  in  $L^p(Q'_r(x_0) \times I; \mathbb{R}^3)$ ,

$$\begin{aligned} J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) &\leq \liminf_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W \left( x_\alpha, x_3; \nabla_\alpha u_n^k \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n^k \right) dx_\alpha dx_3 \\ &= \liminf_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W \left( x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) dx_\alpha dx_3. \end{aligned}$$

As before, we split  $Q'_r(x_0)$  into  $h^2$  disjoint cubes  $Q'_{i,h}$  of length  $r/h$ . Then,

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \leq \left( \liminf_{h \rightarrow +\infty} \right) \liminf_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h} \times I} W \left( x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) dx_\alpha dx_3.$$

Let  $K_\eta$  be like in Lemma 1.2.4 and  $W^{\eta,k}$  be a continuous extension of  $W$  outside  $K_\eta \times \overline{B}(0, M_k)$  which satisfies the analogue of (1.2.6) with  $M_k$  instead of  $\lambda$ . In view of the  $p$ -growth condition (1.1.1), (1.2.12) and (1.2.4), we get

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \setminus K_\eta} W \left( x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) dx_\alpha dx_3 \leq \beta(1 + M_k^p) \eta \xrightarrow{\eta \rightarrow 0} 0.$$

Thus,

$$\begin{aligned} J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) &\leq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap K_\eta} W^{\eta,k} \left( x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) dx_\alpha dx_3 \\ &\leq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h} \times I} W^{\eta,k} \left( x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) dx_\alpha dx_3. \end{aligned}$$

Since  $W^{\eta,k}$  is continuous, it is uniformly continuous on  $\overline{\Omega} \times \overline{B}(0, M_k)$ . Thus, there exists a continuous and increasing function  $\omega_{\eta,k} : [0, +\infty) \rightarrow [0, +\infty)$  satisfying  $\omega_{\eta,k}(0) = 0$  and the analogue of (1.2.7), replacing  $\lambda$  by  $M_k$ . Then, for every  $(x_\alpha, x_3) \in Q'_{i,h} \times I$  and every  $y_\alpha \in Q'_{i,h}$ ,

$$\begin{aligned} \left| W^{\eta,k} \left( x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) - W^{\eta,k} \left( y_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) \right) \right| &\leq \omega_{\eta,k}(|x_\alpha - y_\alpha|) \\ &\leq \omega_{\eta,k}(\sqrt{2}r/h). \end{aligned}$$

Integration and summation yield in turn

$$\begin{aligned} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{Q'_{i,h} \times I} \left| W^{\eta,k}(y_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right)) - W^{\eta,k}(x_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right)) \right| dx \right\} dy_\alpha \\ \leq 2r^2 \omega_{\eta,k}(\sqrt{2}r/h) \xrightarrow{h \rightarrow +\infty} 0, \end{aligned}$$

uniformly with respect to  $n \in \mathbb{N}$ . Hence,

$$\begin{aligned} & J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \\ & \leq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{Q'_{i,h} \times I} W^{\eta,k}(y_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right)) dx_\alpha dx_3 \right\} dy_\alpha. \end{aligned}$$

According to (1.2.6) and (1.2.4),

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{[Q'_{i,h} \times I] \setminus K_\eta} W^{\eta,k}(y_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right)) dy_\alpha dx_3 \right\} dx_\alpha \\ & \leq \beta(1 + M_k^p) \eta \xrightarrow{\eta \rightarrow 0} 0. \end{aligned}$$

Since  $W^{\eta,k}$  coincides with  $W$  on  $K_\eta \times \overline{B}(0, M_k)$ , we get

$$\begin{aligned} & J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \\ & \leq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{[Q'_{i,h} \times I] \cap K_\eta} W(y_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right)) dy_\alpha dx_3 \right\} dx_\alpha \\ & \leq \liminf_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \limsup_{n \rightarrow +\infty} \int_{Q'_{i,h}} \left\{ \int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right)) dy_\alpha dx_3 \right\} dx_\alpha. \end{aligned}$$

Riemann-Lebesgue's Lemma applied to the  $Q'$ -periodic function

$$\int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_k(\cdot, x_3)) dy_\alpha dx_3$$

implies that

$$J_{\{\varepsilon_n\}}(u; Q'_r(x_0)) \leq \liminf_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \frac{2r^2}{h^2} Z_k(y_\alpha; \overline{F}) dy_\alpha = 2 \int_{Q'_r(x_0)} Z_k(y_\alpha; \overline{F}) dy_\alpha.$$

dividing both sides of the inequality by  $r^2$  and letting  $r \searrow 0^+$ , we get in view of the definition of  $x_0$  and (1.2.11),

$$W_{\{\varepsilon_n\}}(x_0; \overline{F}) \leq Z_k(x_0; \overline{F}) \leq \underline{W}(x_0; \overline{F}) + \frac{1}{k}.$$

Passing to the limit when  $k \nearrow +\infty$  yields the desired result.  $\square$

*Proof of Theorem 1.2.3.* For a.e.  $x_0 \in \omega$  and for all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ ,  $W_{\{\varepsilon_n\}}(x_0; \overline{F}) = \underline{W}(x_0; \overline{F})$  thus for any  $A \in \mathcal{A}(\omega)$ , the sequence  $J_{\varepsilon_n}(\cdot; A)$   $\Gamma(L^p)$ -converges to  $J_0(\cdot; A)$ . Since the  $\Gamma(L^p)$ -limit does not depend upon the choice of sequence  $\{\varepsilon_n\}$ , appealing to Proposition 8.3 in [45] we conclude that the whole sequence  $J_\varepsilon(\cdot; A)$   $\Gamma(L^p)$ -converges to  $J_0(\cdot; A)$ .  $\square$

**Remark 1.2.6.** Proposition 1.4.1 gives another expression for the energy density  $\underline{W}$ .

**Remark 1.2.7.** By construction and thanks to Remark 3.3 of [35], Theorem 1.2.3 generalizes both Theorem 2 of [74] and Theorem 3.1 of [35].

### 1.3 Cosserat nonlinear membrane model

In this section, we assume as in [27] that  $g_\varepsilon := g_0 + \varepsilon g$  with  $g_0, g \in L^p(\Sigma; \mathbb{R}^3)$  and  $g_0^+ + g_0^- = 0$ . Thus, the minimization problem (1.1.2) reads as

$$\inf_{v-x_\varepsilon \in W_{\Gamma}^{1,p}(\Omega; \mathbb{R}^3)} \left\{ \int_{\Omega} W \left( x; \nabla_{\alpha} v \middle| \frac{1}{\varepsilon} \nabla_3 v \right) dx - L_{\varepsilon}(v) \right\},$$

with

$$L_{\varepsilon}(v) := \int_{\Omega} f \cdot v dx + \int_{\Sigma} g \cdot v d\mathcal{H}^2 + \int_{\omega} g_0^+ \cdot \left( \frac{v^+ - v^-}{\varepsilon} \right) dx_{\alpha}, \quad v^{\pm}(x_{\alpha}) := v(x_{\alpha}, \pm 1).$$

If  $v_{\varepsilon} \rightarrow v$  in  $L^p(\Omega; \mathbb{R}^3)$  is a minimizing sequence and if  $b_{\varepsilon} := \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon}$ , then

$$L_{\varepsilon}(v_{\varepsilon}) = \int_{\Omega} f \cdot v_{\varepsilon} dx + \int_{\Sigma} g \cdot v_{\varepsilon} d\mathcal{H}^2 + 2 \int_{\omega} g_0^+ \cdot \bar{b}_{\varepsilon} dx_{\alpha}, \quad \text{where } \bar{b}_{\varepsilon} = \int_{-1}^1 b_{\varepsilon}(\cdot, x_3) dx_3.$$

By virtue of the coercivity condition (1.1.1), we deduce that the sequence  $\{v_{\varepsilon}\}$  is uniformly bounded in  $W^{1,p}(\Omega; \mathbb{R}^3)$  and that, for a subsequence of  $\{\varepsilon\}$  still labeled  $\{\varepsilon\}$ ,  $v_{\varepsilon} \rightarrow v$  in  $W^{1,p}(\Omega; \mathbb{R}^3)$  and  $b_{\varepsilon} \rightarrow b$  in  $L^p(\Omega; \mathbb{R}^3)$  with  $v \in W^{1,p}(\omega; \mathbb{R}^3)$ . As previously,  $v$  is associated to the mid-plane deformation, whereas  $b$  is the Cosserat vector. In any case,  $L_{\varepsilon}(v_{\varepsilon}) \rightarrow L(v, \bar{b})$ , with

$$L(v, \bar{b}) := \int_{\omega} (2\bar{f} + g^+ + g^-) \cdot v dx_{\alpha} + 2 \int_{\omega} g_0^+ \cdot \bar{b} dx_{\alpha}, \quad (1.3.1)$$

where  $\bar{b}(x_{\alpha}) := \int_{-1}^1 b(x_{\alpha}, x_3) dx_3$  and  $\bar{f}(x_{\alpha}) := \int_{-1}^1 f(x_{\alpha}, x_3) dx_3$ . The desired membrane model should thus depend on the average  $\bar{b}$ , of  $b$  with respect to  $x_3$ . Once we establish our  $\Gamma$ -convergence result, we will be in a position to conclude that  $v$  and  $\bar{b}$  are truly independent and that the corresponding model is a Cosserat type membrane model.

To this end, we define, for all  $(u, \bar{b}; A) \in L^p(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$ ,

$$\mathcal{J}_{\varepsilon}(u, \bar{b}; A) := \begin{cases} \int_{A \times I} W \left( x; \nabla_{\alpha} u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx & \text{if } \begin{cases} u \in W^{1,p}(A \times I; \mathbb{R}^3), \\ \frac{1}{\varepsilon} \int_{-1}^1 \nabla_3 u(\cdot, x_3) dx_3 = \bar{b}, \end{cases} \\ +\infty & \text{otherwise,} \end{cases} \quad (1.3.2)$$

and for every sequence  $\{\varepsilon_j\} \searrow 0^+$

$$\mathcal{J}_{\{\varepsilon_j\}}(u, \bar{b}; A) := \inf_{\{u_j, \bar{b}_j\}} \left\{ \liminf_{j \rightarrow +\infty} \mathcal{J}_{\varepsilon_j}(u_j, \bar{b}_j; A) : u_j \rightarrow u \text{ in } L^p(A \times I; \mathbb{R}^3) \text{ and } \bar{b}_j \rightarrow \bar{b} \text{ in } L^p(A; \mathbb{R}^3) \right\}. \quad (1.3.3)$$

**Remark 1.3.1.** Let  $(u, \bar{b}; A) \in L^p(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$  and suppose that  $\mathcal{J}_{\{\varepsilon_j\}}(u, \bar{b}; A) < +\infty$ . Arguing as in Remark 1.2.1, we deduce that  $u \in W^{1,p}(A; \mathbb{R}^3)$ . Hence, if  $u \in L^p(\Omega; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$ , then  $\mathcal{J}_{\{\varepsilon_j\}}(u, \bar{b}; A) = +\infty$ .

**Remark 1.3.2.** Whenever  $u \in W^{1,p}(A; \mathbb{R}^3)$ , one has  $\mathcal{J}_{\{\varepsilon_j\}}(u, \bar{b}; A) < +\infty$ , which is easily obtained by considering the sequence  $\{u_j(x_\alpha, x_3) := u(x_\alpha) + \varepsilon_j x_3 \bar{b}_k(x_\alpha)\}$ , where  $\bar{b}_k \in C_c^\infty(A; \mathbb{R}^3)$  and  $\bar{b}_k \rightarrow \bar{b}$  strongly in  $L^p(A; \mathbb{R}^3)$ .

Theorem 1.2 in [27] shows that, if  $W$  is a homogeneous elastic energy density, then  $\mathcal{J}_{\{\varepsilon_j\}}$  is the  $\Gamma(L^p)$ -limit of  $\mathcal{J}_{\varepsilon_j}$ , by which we mean, from now onward, the  $\Gamma$ -limit with respect to, respectively, the strong topology of  $L^p(\Omega; \mathbb{R}^3)$ , and the weak topology of  $L^p(\omega; \mathbb{R}^3)$ . Furthermore, for all  $(u, \bar{b}; A) \in W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$ ,

$$\mathcal{J}_{\{\varepsilon_j\}}(u, \bar{b}; A) = 2 \int_A \mathcal{Q}^* W(\nabla_\alpha u | \bar{b}) dx_\alpha,$$

where, for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and  $z \in \mathbb{R}^3$ ,

$$\begin{aligned} \mathcal{Q}^* W(\bar{F} | z) := \inf_{L > 0, \varphi} \left\{ \int_{Q' \times I} W(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \right. \\ \left. \varphi(\cdot, x_3) \text{ } Q' \text{-periodic for a.e. } x_3 \in I \text{ and } L \int_{Q' \times I} \nabla_3 \varphi dx = z \right\}. \end{aligned}$$

We propose to extend this result to the heterogeneous case. We set, for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ ,  $z \in \mathbb{R}^3$  and a.e.  $x_0 \in \omega$ ,

$$\begin{aligned} \mathcal{Q}^* W(x_0; \bar{F} | z) := \inf_{L > 0, \varphi} \left\{ \int_{Q' \times I} \mathcal{Q} W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \right. \\ \left. \varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \varphi(\cdot, x_3) \text{ } Q' \text{-periodic for a.e. } x_3 \in I, \right. \\ \left. \text{and } L \int_{Q' \times I} \nabla_3 \varphi dx = z \right\} \end{aligned} \quad (1.3.4)$$

where, for a.e.  $x \in \Omega$  and all  $F \in \mathbb{R}^{3 \times 3}$ ,  $\mathcal{Q} W(x; \cdot)$ , the 3D-quasiconvexification of  $W(x; \cdot)$  is defined as

$$\mathcal{Q} W(x; F) = \inf_{\phi \in W_0^{1,p}(Q; \mathbb{R}^3)} \int_Q W(x; F + \nabla \phi(y)) dy$$

with  $Q := (0, 1)^3$ . Since  $\mathcal{Q} W(x; \cdot)$  is quasiconvex and satisfies a  $p$ -growth condition, for all  $F_1, F_2 \in \mathbb{R}^{3 \times 3}$  and for a.e.  $x \in \Omega$ ,

$$|\mathcal{Q} W(x; F_1) - \mathcal{Q} W(x; F_2)| \leq \beta(1 + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2| \quad (1.3.5)$$

(see [44], Lemma 2.2).

### 1.3.1 Some preliminary results

Elementary properties of  $\mathcal{Q}^* W$  are summarized in the following proposition :

**Proposition 1.3.3.** *i) For all  $(\bar{F}, z) \in \mathbb{R}^{3 \times 2} \times \mathbb{R}^3$  and a.e.  $x_0 \in \omega$ ,*

$$0 \leq \mathcal{Q}^*W(x_0; \bar{F}|z) \leq \beta(|\bar{F}|^p + |z|^p + 1). \quad (1.3.6)$$

*ii)  $\mathcal{Q}^*W$  is a Carathéodory function.*

*Proof.* Item i). We take  $\varphi(x) := zx_3/L$  as test function in (1.3.4) and use the  $p$ -growth condition (1.1.1).

Item ii). It suffices to show that  $\mathcal{Q}^*W(x_0; \cdot)$  is continuous for a.e.  $x_0 \in \omega$ . Let  $\bar{F}_n \rightarrow \bar{F}$  and  $z_n \rightarrow z$ . We first prove that  $\mathcal{Q}^*W(x_0; \cdot)$  is upper semicontinuous. For any  $\delta > 0$ , set  $L > 0$  and  $\varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3)$   $Q'$ -periodic satisfying  $L \int_{Q' \times I} \nabla_3 \varphi dx = z$  such that

$$\mathcal{Q}^*W(x_0; \bar{F}|z) \leq \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 \leq \mathcal{Q}^*W(x_0; \bar{F}|z) + \delta.$$

The sequence  $\{\varphi_n(x) := \varphi(x) + x_3(z_n - z)/L\}$  is in  $W^{1,p}(Q' \times I; \mathbb{R}^3)$  and it is  $Q'$ -periodic. Furthermore,  $\nabla_\alpha \varphi = \nabla_\alpha \varphi_n$  and  $\frac{L}{2} \int_{Q' \times I} \nabla_3 \varphi_n dx = z_n$ . Since

$$\|(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) - (\bar{F}_n + \nabla_\alpha \varphi_n | L \nabla_3 \varphi_n)\|_{L^\infty(Q' \times I; \mathbb{R}^{3 \times 3})} \leq |(\bar{F}|z) - (\bar{F}_n|z_n)| \rightarrow 0$$

while  $\{\nabla \varphi_n\}$  is bounded in  $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$ , (1.3.5) together with Hölder's inequality, yields

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \mathcal{Q}^*W(x_0; \bar{F}_n|z_n) - \mathcal{Q}^*W(x_0; \bar{F}|z) - \delta \\ & \leq \limsup_{n \rightarrow +\infty} \int_{Q' \times I} |\mathcal{Q}W(x_0, x_3; \bar{F}_n + \nabla_\alpha \varphi_n | L \nabla_3 \varphi_n) - \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi)| dx \\ & \leq \limsup_{n \rightarrow +\infty} \frac{\beta}{2} \int_{Q' \times I} (1 + |(\bar{F}_n + \nabla_\alpha \varphi_n | L \nabla_3 \varphi_n)|^{p-1} + |(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi)|^{p-1}) \\ & \quad \times |(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) - (\bar{F}_n + \nabla_\alpha \varphi_n | L \nabla_3 \varphi_n)| dx \\ & \leq \limsup_{n \rightarrow +\infty} C \left( 1 + \|\nabla \varphi\|_{L^p(Q' \times I; \mathbb{R}^{3 \times 3})}^{p-1} + \|\nabla \varphi_n\|_{L^p(Q' \times I; \mathbb{R}^{3 \times 3})}^{p-1} \right) \\ & \quad \times \|(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) - (\bar{F}_n + \nabla_\alpha \varphi_n | L \nabla_3 \varphi_n)\|_{L^p(Q' \times I; \mathbb{R}^{3 \times 3})} = 0 \end{aligned}$$

Passing to the limit when  $\delta \searrow 0^+$  yields the desired upper semicontinuity. Let us prove now that  $\mathcal{Q}^*W(x_0; \cdot)$  is lower semicontinuous. For every  $n \geq 1$ , choose  $L_n > 0$  and  $\varphi_n \in W^{1,p}(Q' \times I; \mathbb{R}^3)$   $Q'$ -periodic satisfying  $L_n \int_{Q' \times I} \nabla_3 \varphi_n dx = z_n$  such that

$$\int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F}_n + \nabla_\alpha \varphi_n | L_n \nabla_3 \varphi_n) dx_\alpha dx_3 \leq \mathcal{Q}^*W(x_0; \bar{F}_n|z_n) + \frac{1}{n}.$$

Set  $\tilde{\varphi}_n(x) := \varphi_n(x) + x_3(z - z_n)/L_n$ , then the function  $\tilde{\varphi}_n \in W^{1,p}(Q' \times I; \mathbb{R}^3)$  is  $Q'$ -periodic and satisfies  $\frac{L_n}{2} \int_{Q' \times I} \nabla_3 \tilde{\varphi}_n dx = z$ . Since

$$\|(\bar{F} + \nabla_\alpha \tilde{\varphi}_n | L_n \nabla_3 \tilde{\varphi}_n) - (\bar{F}_n + \nabla_\alpha \varphi_n | L_n \nabla_3 \varphi_n)\|_{L^\infty(Q' \times I; \mathbb{R}^{3 \times 3})} \leq |(\bar{F}|z) - (\bar{F}_n|z_n)| \rightarrow 0$$

while, in view of (1.1.1), the sequences  $\{(\nabla_\alpha \varphi_n | L_n \nabla_3 \varphi_n)\}$  and  $\{(\nabla_\alpha \tilde{\varphi}_n | L_n \nabla_3 \tilde{\varphi}_n)\}$  are bounded in  $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$  uniformly in  $n$ , (1.3.5) implies that

$$\begin{aligned} \mathcal{Q}^*W(x_0; \bar{F}|z) &\leq \liminf_{n \rightarrow +\infty} \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \tilde{\varphi}_n | L_n \nabla_3 \tilde{\varphi}_n) dx_\alpha dx_3 \\ &\leq \liminf_{n \rightarrow +\infty} \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F}_n + \nabla_\alpha \varphi_n | L_n \nabla_3 \varphi_n) dx_\alpha dx_3 \\ &\leq \liminf_{n \rightarrow +\infty} \mathcal{Q}^*W(x_0; \bar{F}_n | z_n). \end{aligned}$$

Thus  $\mathcal{Q}^*W(x_0; \cdot)$  is lower semicontinuous and the continuity follows.  $\square$

We propose to establish the following  $\Gamma$ -convergence result.

**Theorem 1.3.4.** *For all  $A \in \mathcal{A}(\omega)$ , the sequence  $\mathcal{J}_\varepsilon(\cdot, \cdot; A)$   $\Gamma(L^p)$ -converges to  $\mathcal{J}_0(\cdot, \cdot; A)$  defined for all  $(u, \bar{b}) \in L^p(\Omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3)$ , by*

$$\mathcal{J}_0(u, \bar{b}; A) = \begin{cases} 2 \int_A \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where  $\mathcal{Q}^*W$  is given by (1.3.4).

We first note, as in [35] p. 1374, that, if  $\mathcal{R}(\omega)$  is the countable family of all finite unions of open squares in  $\omega$  with faces parallel to the axes, centered at rational points and with rational edge lengths, then there exists a subsequence  $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$  of  $\{\varepsilon_j\}$  such that  $\mathcal{J}_{\{\varepsilon_n\}}(\cdot, \cdot; C)$  is, for all  $C \in \mathcal{R}(\omega)$ , the  $\Gamma(L^p)$ -limit of  $\mathcal{J}_{\varepsilon_n}(\cdot, \cdot; C)$ .

Then, the analogue of Step 2 in the proof of Theorem 2.5 of [35] holds, namely

**Lemma 1.3.5.** *For any  $A \in \mathcal{A}(\omega)$  and  $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ , there exists a sequence  $\{u_n\} \subset W^{1,p}(A \times I; \mathbb{R}^3)$  satisfying*

$$\begin{cases} u_n \rightarrow u \text{ in } L^p(A \times I; \mathbb{R}^3), \\ \bar{b}_n := \frac{1}{\varepsilon_n} \int_{-1}^1 \nabla_3 u_n(\cdot; x_3) dx_3 \rightarrow \bar{b} \text{ in } L^p(A; \mathbb{R}^3), \\ \lim_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon_n}(u_n, \bar{b}_n; A) = \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A). \end{cases} \quad (1.3.7)$$

*Proof.* The coercivity condition (1.1.1) implies that whenever  $u \in W^{1,p}(C; \mathbb{R}^3)$ , we can choose the attainment sequence  $\{u_n, \bar{b}_n\}$ , so that (1.3.7) holds true. Now let us fix  $\delta > 0$  and choose a subset  $C^\delta$  of  $A$  in  $\mathcal{R}(\omega)$  such that  $C^\delta \subset A$  and

$$\int_{A \setminus C^\delta} (1 + |\nabla_\alpha u|^p) dx_\alpha < \frac{\delta}{2\beta}.$$

Consider a sequence  $\{v_n^\delta\} \subset W^{1,p}(C^\delta \times I; \mathbb{R}^3)$  satisfying

$$\begin{cases} v_n^\delta \xrightarrow[n \rightarrow +\infty]{} u \text{ in } L^p(C^\delta \times I; \mathbb{R}^3), \\ \bar{b}_n^\delta := \frac{1}{\varepsilon_n} \int_{-1}^1 \nabla_3 v_n^\delta(\cdot, x_3) dx_3 \xrightarrow[n \rightarrow +\infty]{} \bar{b} \text{ in } L^p(C^\delta; \mathbb{R}^3), \\ \lim_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon_n}(v_n^\delta, \bar{b}_n^\delta; C^\delta) = \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; C^\delta). \end{cases} \quad (1.3.8)$$

In view of Lemma 2.2 in [27] (the proof in our context is identical to that of the homogeneous case), there is no loss of generality to assume, upon extracting a subsequence of  $\{\varepsilon_n\}$  (not relabeled), that  $v_n^\delta = u$  on a neighborhood of  $\partial C^\delta \times I$ . We extend  $v_n^\delta$  as  $u$  outside  $C^\delta$  (and correspondingly extend  $\bar{b}_n^\delta$  as 0). Since  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  is an increasing set function, we have  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; C^\delta) \leq \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A)$  and thus,

$$\begin{aligned} & \limsup_{\delta \rightarrow 0^+} \limsup_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon_n}(v_n^\delta, \bar{b}_n^\delta; A) \\ & \leq \limsup_{\delta \rightarrow 0^+} \left\{ \lim_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon_n}(v_n^\delta, \bar{b}_n^\delta; C^\delta) + 2\beta \int_{A \setminus C^\delta} (1 + |\nabla_\alpha u|^p) dx_\alpha \right\} \\ & = \limsup_{\delta \rightarrow 0^+} \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; C^\delta) \\ & \leq \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) \\ & \leq \liminf_{\delta \rightarrow 0^+} \liminf_{n \rightarrow +\infty} \mathcal{J}_{\varepsilon_n}(v_n^\delta, \bar{b}_n^\delta; A). \end{aligned}$$

Remark that (1.3.8), together with coercivity, implies that

$$\|\nabla_\alpha v_n^\delta\|_{L^p(A \times I; \mathbb{R}^{3 \times 2})} + \|\bar{b}_n^\delta\|_{L^p(A; \mathbb{R}^3)} \leq C,$$

independently of  $\delta, n$ ; in particular,  $\{\bar{b}_n^\delta\}$  lies in a subset of  $L^p(A; \mathbb{R}^3)$ , which is metrizable for the weak  $L^p$ -topology. A simple diagonalization lemma (Lemma 7.1 in [35]) permits to conclude the existence of a decreasing sequence  $\{\delta(n)\} \searrow 0^+$  such that the sequences  $\{u_n := v_n^{\delta(n)}\}$  satisfies (1.3.7).  $\square$

We now recall two results that will be of use in the proof of Lemma 1.3.10 below. Their proof can be found in [27] in the homogeneous case and the heterogeneity does not create any additional difficulty.

**Proposition 1.3.6.** *For any sequence  $\{\varepsilon_j\} \searrow 0^+$ , there exists a subsequence  $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$  such that, for any  $(u, \bar{b}) \in W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3)$ , the set function  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  defined in (1.3.3) is the trace on  $\mathcal{A}(\omega)$  of a Radon measure, which is absolutely continuous with respect to the two-dimensional Lebesgue measure.*



By virtue of Lemma 1.3.5 and Proposition 1.3.6, we will assume henceforth that  $\{\varepsilon_n\}$  denotes a subsequence of  $\{\varepsilon_j\}$  such that the  $\Gamma(L^p)$ -limit of  $\mathcal{J}_{\varepsilon_n}$  exists, in which case it coincides with  $\mathcal{J}_{\{\varepsilon_n\}}$ , and such that, for every  $(u, \bar{b}) \in W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3)$ , the set function  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  is the trace on  $\mathcal{A}(\omega)$  of a Radon measure, which is absolutely continuous with respect to the two-dimensional Lebesgue measure.

**Proposition 1.3.7.** *For all  $(u, \bar{b}; A) \in W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$ , the value of  $\mathcal{J}_{\{\varepsilon_j\}}(u, \bar{b}; A)$  is unchanged if  $W$  is replaced by  $\mathcal{Q}W$  in (1.3.2).*

**Remark 1.3.8.** If  $W$  does not depend on  $x_\alpha$ , we can show as in [27] that for all  $A \in \mathcal{A}(\omega)$ ,  $\mathcal{J}_\varepsilon(\cdot, \cdot; A)$   $\Gamma(L^p)$ -converges to  $\mathcal{J}_0(\cdot, \cdot; A)$  and

$$\mathcal{J}_0(u, \bar{b}; A) = 2 \int_A \mathcal{Q}^*W(\nabla_\alpha u | \bar{b}) dx_\alpha,$$

for every  $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3)$ , where  $\mathcal{Q}^*W$  is given by (1.3.4).

Theorem 1.3.4 is a direct consequence of the following two lemmas.

**Lemma 1.3.9.** *For all  $A \in \mathcal{A}(\omega)$  and for all  $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ ,*

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) \geq 2 \int_A \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha.$$

**Lemma 1.3.10.** *For all  $A \in \mathcal{A}(\omega)$  with  $A$  Lipschitz and for all  $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ ,*

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) \leq 2 \int_A \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha.$$

### 1.3.2 The lower bound

*Proof of Lemma 1.3.9.* Let  $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ . Take  $x_0 \in A$  to be a Lebesgue point of the Radon-Nikodym derivative of  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  with respect to the two-dimensional Lebesgue measure and of  $x_\alpha \mapsto \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha))$ , and choose  $r > 0$  small enough so that  $Q'_r(x_0) \subset A$ . According to the equi-integrability Theorem (Theorem 1.1 in [25]) together with Lemma 1.3.5, there exists a subsequence of  $\{\varepsilon_n\}$  (not relabeled) and a sequence  $\{u_n\} \subset W^{1,p}(Q'_r(x_0) \times I; \mathbb{R}^3)$  such that

$$\left\{ \begin{array}{l} u_n \rightarrow u \text{ in } L^p(Q'_r(x_0) \times I; \mathbb{R}^3), \\ \frac{1}{\varepsilon_n} \int_{-1}^1 \nabla_3 u_n(\cdot, x_3) dx_3 \rightarrow \bar{b} \text{ in } L^p(Q'_r(x_0); \mathbb{R}^3), \\ \left\{ \left| \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p \right\} \text{ is equi-integrable,} \\ \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) = \lim_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W \left( x_\alpha, x_3; \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx_\alpha dx_3. \end{array} \right.$$

We argue as in the proof of Lemma 1.2.4 with  $F_n(x) := \left( \nabla_\alpha u_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n(x) \right)$  and we obtain, since  $W \geq \mathcal{Q}W$

$$\begin{aligned} & \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) \\ & \geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \left\{ \liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W(y_\alpha, x_3; F_n(x_\alpha, x_3)) dx_\alpha dx_3 \right\} dy_\alpha \\ & \geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \left\{ \liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} \mathcal{Q}W(y_\alpha, x_3; F_n(x_\alpha, x_3)) dx_\alpha dx_3 \right\} dy_\alpha. \end{aligned}$$

Applying, for a.e.  $y_\alpha \in Q'_{i,h}$ , Remark 1.3.8 to the Carathéodory function  $(x_3; F) \mapsto W(y_\alpha, x_3; F)$ , we get

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) \geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \left\{ 2 \int_{Q'_{i,h}} \mathcal{Q}^*W(y_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) dx_\alpha \right\} dy_\alpha.$$

By Proposition 1.3.3 (ii),  $\mathcal{Q}^*W$  is a Carathéodory integrand, thus Scorza-Dragoni's Theorem implies the existence, for any  $\eta > 0$ , of a compact set  $C_\eta \subset A$ , such that

$$\mathcal{L}^2(A \setminus C_\eta) < \eta, \quad (1.3.9)$$

and  $\mathcal{Q}^*W$  is continuous on  $C_\eta \times \mathbb{R}^{3 \times 2}$ . Let  $S_\lambda := \{x_\alpha \in A : |(\nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha))| \leq \lambda\}$ , thanks to Chebyshev's inequality

$$\mathcal{L}^2(A \setminus S_\lambda) < \frac{C}{\lambda^p}. \quad (1.3.10)$$

Consequently

$$\begin{aligned} & \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) \geq \\ & \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h} \cap C_\eta} \left\{ 2 \int_{Q'_{i,h} \cap C_\eta \cap S_\lambda} \mathcal{Q}^*W(y_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) dx_\alpha \right\} dy_\alpha. \end{aligned}$$

Since  $\mathcal{Q}^*W$  is continuous on  $C_\eta \times \mathbb{R}^{3 \times 2}$ , it is uniformly continuous on  $C_\eta \times \bar{B}(0, \lambda)$  thus there exists a increasing and continuous function  $\omega_{\eta, \lambda} : [0, +\infty) \rightarrow [0, +\infty)$  satisfying  $\omega_{\eta, \lambda}(0) = 0$  and for every  $y_\alpha \in Q'_{i,h} \cap C_\eta$  and every  $x_\alpha \in Q'_{i,h} \cap C_\eta \cap S_\lambda$ ,

$$|\mathcal{Q}^*W(y_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) - \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha))| \leq \omega_{\eta, \lambda}(|x_\alpha - y_\alpha|) \leq \omega_{\eta, \lambda}(\sqrt{2}r/h).$$

Integration and summation yield

$$\begin{aligned} & \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h} \cap C_\eta} \int_{Q'_{i,h} \cap C_\eta \cap S_\lambda} |\mathcal{Q}^*W(y_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) \\ & \quad - \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha))| dx_\alpha dy_\alpha \leq r^2 \omega_{\eta, \lambda}(\sqrt{2}r/h) \xrightarrow{h \rightarrow +\infty} 0. \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) \\ & \geq \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{2\mathcal{L}^2(Q'_{i,h} \cap C_\eta)}{\mathcal{L}^2(Q'_{i,h})} \int_{Q'_{i,h} \cap C_\eta \cap S_\lambda} \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) dx_\alpha. \end{aligned}$$

By virtue of the  $p$ -growth condition (1.3.6) together with (1.3.9), we get

$$\begin{aligned} & \sum_{i=1}^{h^2} \frac{\mathcal{L}^2(Q'_{i,h} \setminus C_\eta)}{\mathcal{L}^2(Q'_{i,h})} \int_{Q'_{i,h} \cap C_\eta \cap S_\lambda} \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) dx_\alpha \\ & \leq \beta(1 + \lambda^p) \sum_{i=1}^{h^2} \mathcal{L}^2(Q'_{i,h} \setminus C_\eta) \\ & = \beta(1 + \lambda^p) \mathcal{L}^2(A \setminus C_\eta) \\ & < \beta(1 + \lambda^p) \eta \xrightarrow{\eta \rightarrow 0} 0. \end{aligned}$$

Thus, (1.3.9) and (1.3.10) yield

$$\begin{aligned} \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) & \geq 2 \limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \int_{Q'_r(x_0) \cap C_\eta \cap S_\lambda} \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) dx_\alpha \\ & = 2 \int_{Q'_r(x_0)} \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u(x_\alpha) | \bar{b}(x_\alpha)) dx_\alpha. \end{aligned}$$

The result follows after dividing the previous inequality by  $r^2$  and letting  $r \searrow 0^+$ .  $\square$

### 1.3.3 The upper bound

*Proof of Lemma 1.3.10.* The proof is divided into three steps. First, we address the case where  $u$  is affine and  $\bar{b}$  is constant; then, that where  $u$  is piecewise affine and continuous, and  $\bar{b}$  piecewise constant. Finally, we address the general case.

**Step 1.** Let  $A \in \mathcal{A}(\omega)$ , we assume that

$$\begin{cases} u(x_\alpha) = \bar{F} \cdot x_\alpha + c, & (\bar{F}, c) \in \mathbb{R}^{3 \times 2} \times \mathbb{R}^3, \\ \bar{b}(x_\alpha) = z, & z \in \mathbb{R}^3. \end{cases}$$

Thanks to the density of  $W^{1,\infty}(Q' \times I; \mathbb{R}^3)$  into  $W^{1,p}(Q' \times I; \mathbb{R}^3)$  and to the  $p$ -growth hypothesis (1.1.1), for any  $k \geq 1$ , there exists  $L_k > 0$  and  $\varphi_k \in W^{1,\infty}(Q' \times I; \mathbb{R}^3)$   $Q'$ -periodic satisfying  $\frac{L_k}{2} \int_{Q' \times I} \nabla_3 \varphi_k dx = z$  and such that

$$Z_k(x_0; \bar{F} | z) := \int_{Q' \times I} \mathcal{Q} W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi_k | L_k \nabla_3 \varphi_k) dx_\alpha dx_3 \leq \mathcal{Q}^* W(x_0; \bar{F} | z) + \frac{1}{k}.$$

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We extend  $\varphi_k$  to  $\mathbb{R}^2 \times I$  by  $Q'$ -periodicity. Choose  $r > 0$  small enough so that  $Q'_r(x_0) \subset A$  where  $x_0$  is a Lebesgue point of the Radon-Nikodym derivative of  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  with respect to the two-dimensional Lebesgue measure and of  $Z_k(\cdot; \bar{F}|z)$  for all  $k \geq 1$ . Fix  $k$  and set

$$u_n^k(x) := \bar{F} \cdot x_\alpha + c + L_k \varepsilon_n \varphi_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right).$$

Then,

$$u_n^k \xrightarrow{n \rightarrow +\infty} u \text{ in } L^p(Q'_r(x_0) \times I; \mathbb{R}^3),$$

and by virtue of Riemann-Lebesgue's Lemma,

$$\frac{1}{\varepsilon_n} \int_{-1}^1 \nabla_3 u_n^k dx_3 = L_k \int_{-1}^1 \nabla_3 \varphi_k \left( \frac{x_\alpha}{L_k \varepsilon_n}, x_3 \right) dx_3 \xrightarrow[n \rightarrow +\infty]{L^p(Q'_r(x_0); \mathbb{R}^3)} L_k \int_{Q' \times I} \nabla_3 \varphi_k dx = \bar{b}.$$

So  $\{u_n^k\}$  is admissible for  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0))$  and, thanks to Proposition 1.3.7,

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; Q'_r(x_0)) \leq \liminf_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} \mathcal{Q}W \left( x_\alpha, x_3; \nabla_\alpha u_n^k \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n^k \right) dx_\alpha dx_3.$$

Using an argument similar to that in the proof of Lemma 1.2.5, with  $\mathcal{Q}W$  instead of  $W$ , we get

$$\frac{d\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)}{d\mathcal{L}^2}(x_0) \leq 2\mathcal{Q}^*W(x_0; \bar{F}|z).$$

Thus, integration over  $A$  yields

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) \leq 2 \int_A \mathcal{Q}^*W(x_\alpha; \bar{F}|z) dx_\alpha.$$

**Step 2.** Assume that  $u$  is continuous and piecewise affine and  $\bar{b}$  is piecewise constant on  $A$ . There exists a partition  $A_1, \dots, A_N$  of  $A$  such that  $u(x_\alpha) = \bar{F}_i \cdot x_\alpha + c_i$  and  $\bar{b}(x_\alpha) = z_i$  on  $A_i$ , for all  $i = 1, \dots, N$ . Thanks to step 1, for all  $i = 1, \dots, N$ , we have

$$\mathcal{J}_{\{\varepsilon_n\}}(\bar{F}_i \cdot x_\alpha + c_i, z_i; A_i) \leq 2 \int_{A_i} \mathcal{Q}^*W(x_\alpha; \bar{F}_i|z_i) dx_\alpha.$$

In view of Proposition 1.3.6,  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  is a measure and we thus get

$$\begin{aligned} \mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) &= \sum_{i=1}^N \mathcal{J}_{\{\varepsilon_n\}}(\bar{F}_i \cdot x_\alpha + c_i, z_i; A_i) \\ &\leq 2 \sum_{i=1}^N \int_{A_i} \mathcal{Q}^*W(x_\alpha; \bar{F}_i|z_i) dx_\alpha \\ &= 2 \int_A \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha. \end{aligned}$$

**Step 3.** Consider  $A \in \mathcal{A}(\omega)$  with  $A$  Lipschitz and  $u \in W^{1,p}(A; \mathbb{R}^3)$ ,  $\bar{b} \in L^p(A; \mathbb{R}^3)$ . There exists a sequence  $\{u_n\}$  of continuous and piecewise affine functions in  $W^{1,p}(A; \mathbb{R}^3)$  and a sequence  $\{\bar{b}_n\}$  of piecewise constant functions in  $L^p(A; \mathbb{R}^3)$  such that  $u_n \rightarrow u$  in  $W^{1,p}(A; \mathbb{R}^3)$  and  $\bar{b}_n \rightarrow \bar{b}$  in  $L^p(A; \mathbb{R}^3)$ . Since  $\mathcal{J}_{\{\varepsilon_n\}}(\cdot, \cdot; A)$  is lower semicontinuous, we get, in view of the previous step,

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) \leq \liminf_{n \rightarrow +\infty} \mathcal{J}_{\{\varepsilon_n\}}(u_n, \bar{b}_n; A) \leq \liminf_{n \rightarrow +\infty} 2 \int_A \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u_n | \bar{b}_n) dx_\alpha. \quad (1.3.11)$$

By Proposition 1.3.3 and Lebesgue's Dominated Convergence Theorem,

$$\lim_{n \rightarrow +\infty} \int_A \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u_n | \bar{b}_n) dx_\alpha = \int_A \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha. \quad (1.3.12)$$

Thus (1.3.11) and (1.3.12) yield

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) \leq 2 \int_A \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha. \quad \square$$

*Proof of Theorem 1.3.4.* The two previous lemmas demonstrate that, provided  $A \in \mathcal{A}(\omega)$  is Lipschitz, then, for all  $(u, \bar{b}) \in W^{1,p}(A; \mathbb{R}^3) \times L^p(A; \mathbb{R}^3)$ ,  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) = \mathcal{J}_0(u, \bar{b}; A)$ . Whenever  $A \in \mathcal{A}(\omega)$  is an arbitrary open set, we define the nested sequence of Lipschitz open subsets  $A_k := \{x_\alpha \in A : \text{dist}(x_\alpha, \partial A) > 1/k\}$  of  $A$ , so that  $\bar{A}_k \subset A$  and  $\cup_{k \geq 1} A_k = A$ . Since  $(u, \bar{b}) \in W^{1,p}(A_k; \mathbb{R}^3) \times L^p(A_k; \mathbb{R}^3)$  and  $A_k$  is Lipschitz,

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A_k) = 2 \int_{A_k} \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha.$$

But  $\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; \cdot)$  is a measure, thus, letting  $k \nearrow +\infty$ ,

$$\mathcal{J}_{\{\varepsilon_n\}}(u, \bar{b}; A) = 2 \int_A \mathcal{Q}^* W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha.$$

In particular, the  $\Gamma(L^p)$ -limit does not depend upon the choice of sequence  $\{\varepsilon_n\}$  thus, in light of Proposition 8.3 in [45], the whole sequence  $\mathcal{J}_\varepsilon(u, \bar{b}; A)$   $\Gamma(L^p)$ -converges to  $\mathcal{J}_0(u, \bar{b}; A)$ . Then Remark 1.3.1 completes the proof of Theorem 1.3.4.  $\square$

**Remark 1.3.11.** If  $W$  does not depend upon  $x$ , Proposition 1.1 (iii) of [27] states that

$$\begin{aligned} \mathcal{Q}^* W(\bar{F} | z) &= \inf_{L > 0, \varphi} \left\{ \int_{Q' \times I} W(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \right. \\ &\quad \left. \varphi(\cdot, x_3) \text{ } Q' \text{-periodic for a.e. } x_3 \in I \text{ and } L \int_{Q' \times I} \nabla_3 \varphi dx = z \right\} \\ &= \inf_{L > 0, \varphi} \left\{ \int_{Q' \times I} \mathcal{Q} W(\bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \right. \\ &\quad \left. \varphi(\cdot, x_3) \text{ } Q' \text{-periodic for a.e. } x_3 \in I \text{ and } L \int_{Q' \times I} \nabla_3 \varphi dx = z \right\}. \end{aligned}$$

In other words, the result of [27] is recovered by Theorem 1.3.4.

**Remark 1.3.12.** Since  $\mathcal{Q}^*W$  is the integrand of the  $\Gamma(L^p)$ -limit of  $\mathcal{J}_\varepsilon$ , which satisfies a  $p$ -coercivity condition (see (1.1.1)), for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ , for all  $z \in \mathbb{R}^3$  and for a.e.  $x_0 \in \omega$ ,

$$\beta'(|\bar{F}|^p + |z|^p) \leq \mathcal{Q}^*W(x_0; \bar{F}|z). \quad (1.3.13)$$

**Remark 1.3.13.** Theorem 1.3.4 implies that the functional

$$(u, \bar{b}) \mapsto \int_{\omega} \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u | \bar{b}) dx_\alpha$$

is sequentially weakly lower semicontinuous on  $W^{1,p}(\omega; \mathbb{R}^3) \times L^p(\omega; \mathbb{R}^3)$ . Therefore, the function  $\mathcal{Q}^*W(x_0; \cdot|z)$  is quasiconvex and  $\mathcal{Q}^*W(x_0; \bar{F}|\cdot)$  is convex. Thanks to the  $p$ -growth condition (1.3.6),  $\mathcal{Q}^*W(x_0; \cdot|\cdot)$  is locally Lipschitz, because it is separately convex (see Theorem 2.3 in [44]).

## 1.4 Classical membrane model as a zero bending moment density

This section investigates the coherence of our results. In the absence of a bending moment density ( $g_0 = 0$ ), we show below that Theorem 1.3.4 boils down to Theorem 1.2.3. We first give another form of the energy density  $\underline{W}$  similar to the definition of  $\mathcal{Q}^*W$  (see (1.3.4)). Specifically,

**Proposition 1.4.1.** *For all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and for a.e.  $x_0 \in \omega$ ,*

$$\begin{aligned} \underline{W}(x_0; \bar{F}) &= \inf_{L, \varphi} \left\{ \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \right. \\ &\quad \left. L > 0, \varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \varphi(\cdot, x_3) \text{ } Q'\text{-periodic for a.e. } x_3 \in I \right\}. \end{aligned}$$

*Proof.* Set

$$\begin{aligned} W^*(x_0; \bar{F}) &:= \inf_{L, \varphi} \left\{ \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 : \right. \\ &\quad \left. L > 0, \varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3), \varphi(\cdot, x_3) \text{ } Q'\text{-periodic for a.e. } x_3 \in I \right\}. \end{aligned}$$

It is obvious that  $\underline{W}(x_0; \bar{F}) \geq W^*(x_0; \bar{F})$ . Conversely, for any  $\delta > 0$ , consider  $L > 0$  and  $\varphi \in W^{1,p}(Q' \times I; \mathbb{R}^3)$   $Q'$ -periodic, such that

$$\int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi | L \nabla_3 \varphi) dx_\alpha dx_3 \leq W^*(x_0; \bar{F}) + \delta.$$

We extend  $\varphi$  by  $Q'$ -periodicity and we set  $\varphi_n(x_\alpha, x_3) := \frac{1}{n}\varphi(nx_\alpha, x_3)$ . Then, Riemann-Lebesgue's Lemma applied to  $\int_{-1}^1 W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi(\cdot, x_3) | L \nabla_3 \varphi(\cdot, x_3)) dx_3$ , implies that

$$\lim_{n \rightarrow +\infty} \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi_n | L n \nabla_3 \varphi_n) dx_\alpha dx_3 \leq W^*(x_0; \bar{F}) + \delta. \quad (1.4.1)$$

For fixed  $n$ , the relaxation theorem of [1] (see Statement III.7 in [1]) – applied to  $Q' \times (-\frac{1}{Ln}, \frac{1}{Ln})$  and to  $\psi_n(x_\alpha, x_3) := \varphi_n(x_\alpha, Ln x_3)$  – yields a sequence

$$\varphi_{n,k} \xrightarrow[k \rightarrow +\infty]{} \varphi_n \text{ in } W^{1,p}(Q' \times I; \mathbb{R}^3)$$

such that,

$$\begin{aligned} & \int_{Q' \times I} \mathcal{Q}W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi_n | L n \nabla_3 \varphi_n) dx_\alpha dx_3 \\ &= \lim_{k \rightarrow +\infty} \int_{Q' \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi_{n,k} | L n \nabla_3 \varphi_{n,k}) dx_\alpha dx_3. \end{aligned} \quad (1.4.2)$$

Thus (1.4.1) together with (1.4.2) give

$$\lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{Q' \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \varphi_{n,k} | L n \nabla_3 \varphi_{n,k}) dx_\alpha dx_3 \leq W^*(x_0; \bar{F}) + \delta.$$

Furthermore, we have,

$$\lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \|\varphi_{n,k}\|_{L^p(Q' \times I; \mathbb{R}^3)} = 0.$$

By a standard diagonalization process, we can find an increasing sequence  $\{k(n)\}$ , with  $k(n) \nearrow +\infty$  such that, if we set  $\phi_n := \varphi_{n,k(n)}$ ,

$$\lim_{n \rightarrow +\infty} \int_{Q' \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \phi_n | L n \nabla_3 \phi_n) dx_\alpha dx_3 \leq W^*(x_0; \bar{F}) + \delta, \quad (1.4.3)$$

and  $\phi_n \rightarrow 0$  in  $L^p(Q' \times I; \mathbb{R}^3)$ . By virtue of the coercivity hypothesis (1.1.1),

$$\sup_{n \in \mathbb{N}} \|(\nabla_\alpha \phi_n | L n \nabla_3 \phi_n)\|_{L^p(Q' \times I; \mathbb{R}^{3 \times 3})} < +\infty.$$

We define the following sequence of non negative bounded Radon measures

$$\lambda_n := (1 + |(\nabla_\alpha \phi_n | L n \nabla_3 \phi_n)|^p) \chi_{Q' \times I} \mathcal{L}^3.$$

The coercive character (1.1.1) of  $W$  permits to assert that, up to a subsequence (not relabeled), there exists a non negative bounded Radon measure  $\lambda$  such that

$$\lambda_n \xrightarrow{*} \lambda \text{ in } \mathcal{M}_b(\mathbb{R}^3).$$

We cut  $\phi_n$  near the lateral boundary to obtain a sequence which vanishes on  $\partial Q' \times I$ . Let  $\theta_k \in C_c^\infty(Q')$  a cut-off function defined by

$$\begin{cases} \theta_k(x_\alpha) & := \begin{cases} 1 & \text{if } x_\alpha \in Q'_{1-1/k}(0), \\ 0 & \text{if } x_\alpha \notin Q'_{1-1/(k+1)}(0), \end{cases} \\ \|\nabla_\alpha \theta_k\|_{L^\infty(Q')} & \leq Ck^2, \end{cases} \quad (1.4.4)$$

We set  $\phi_n^k := \theta_k \phi_n$ , since  $\phi_n^k = 0$  on  $\partial Q' \times I$ , (1.4.3) together with (1.4.4) yields

$$W^*(x_0; \bar{F}) \quad (1.4.5)$$

$$\begin{aligned} & \geq \liminf_{k \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \int_{Q'_{1-1/k}(0) \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \phi_n^k |Ln \nabla_3 \phi_n^k) dx_\alpha dx_3 - \delta \\ & \geq \liminf_{k \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \int_{Q' \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \phi_n^k |Ln \nabla_3 \phi_n^k) dx_\alpha dx_3 \\ & \quad - \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{[Q'_{1-1/(k+1)}(0) \setminus Q'_{1-1/k}(0)] \times I} W(x_0, x_3; \bar{F} + \nabla_\alpha \phi_n^k |Ln \nabla_3 \phi_n^k) dx_\alpha dx_3 \\ & \quad - \beta(1 + |\bar{F}|^p) \limsup_{k \rightarrow +\infty} \mathcal{L}^2(Q' \setminus Q'_{1-1/(k+1)}(0)) - \delta \\ & \geq \underline{W}(x_0; \bar{F}) - \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \left\{ C\lambda_n \left( [Q'_{1-1/(k+1)}(0) \setminus Q'_{1-1/k}(0)] \times I \right) \right. \\ & \quad \left. + C'k^{2p} \int_{Q' \times I} |\phi_n|^p dx \right\} - \delta. \quad (1.4.6) \end{aligned}$$

Since  $Q'_{1-1/k}(0)$  is an increasing sequence of open sets, the union of which is  $Q'$ , we get

$$\begin{aligned} & \limsup_{k \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \lambda_n \left( [Q'_{1-1/(k+1)}(0) \setminus Q'_{1-1/k}(0)] \times I \right) \\ & \leq \limsup_{k \rightarrow +\infty} \lambda \left( \overline{[Q'_{1-1/(k+1)}(0) \setminus Q'_{1-1/k}(0)] \times I} \right) \\ & \leq \limsup_{k \rightarrow +\infty} \lambda \left( [Q' \setminus Q'_{1-1/(k-1)}(0)] \times [-1, 1] \right) = 0. \end{aligned}$$

Using the fact that  $\phi_n \rightarrow 0$  in  $L^p(Q' \times I; \mathbb{R}^3)$  and letting  $\delta$  tend to 0 in (1.4.6), we finally get

$$W^*(x_0; \bar{F}) \geq \underline{W}(x_0; \bar{F}).$$

□

Now that  $\underline{W}$  and  $Q^*W$  are expressed in near identical manner, Remarks 1.3.12 and 1.3.13 immediately imply that for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and for a.e.  $x_0 \in \omega$ , there exists  $b_0 \in \mathbb{R}^3$  such that

$$\underline{W}(x_0; \bar{F}) = \min_{z \in \mathbb{R}^3} Q^*W(x_0; \bar{F}|z) = Q^*W(x_0; \bar{F}|b_0).$$



In the absence of bending moments, the linear form  $L$  given by (1.3.1) does not depend upon  $\bar{b}$  and we may perform explicitly the minimum in  $\bar{b}$  in the limit minimization problem. For  $u \in W^{1,p}(\omega; \mathbb{R}^3)$ , a classical measurability selection criterion (see [54], Chapter VIII, Theorem 1.2), together with the coercivity condition (1.3.13), implies the existence of  $\bar{b}_0 \in L^p(\omega; \mathbb{R}^3)$  such that for a.e.  $x_0 \in \omega$ ,

$$\underline{W}(x_0; \nabla_\alpha u(x_0)) = \min_{z \in \mathbb{R}^3} \mathcal{Q}^*W(x_0; \nabla_\alpha u(x_0)|z) = \mathcal{Q}^*W(x_0; \nabla_\alpha u(x_0)|\bar{b}_0(x_0)).$$

Thus,

$$\begin{aligned} \inf_{\bar{b} \in L^p(\omega; \mathbb{R}^3)} \int_\omega \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u|\bar{b}) dx_\alpha &\leq \int_\omega \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u|\bar{b}_0) dx_\alpha \\ &= \int_\omega \underline{W}(x_\alpha; \nabla_\alpha u) dx_\alpha \\ &= \int_\omega \min_{z \in \mathbb{R}^3} \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u|z) dx_\alpha \\ &\leq \int_\omega \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u|\bar{b}) dx_\alpha, \end{aligned} \tag{1.4.7}$$

where the last inequality holds for all  $\bar{b} \in L^p(\omega; \mathbb{R}^3)$ . Taking the infimum in  $\bar{b}$  in the last term of (1.4.7), the inequalities become equalities thus

$$\inf_{\bar{b} \in L^p(\omega; \mathbb{R}^3)} \int_\omega \mathcal{Q}^*W(x_\alpha; \nabla_\alpha u|\bar{b}) dx_\alpha = \int_\omega \underline{W}(x_\alpha; \nabla_\alpha u) dx_\alpha.$$

This shows that Theorem 1.2.3 is recovered from Theorem 1.3.4.

## Chapitre 2

# Reiterated homogenization of thin films



## 2.1 Introduction

In this work we study the limit behavior of an elastic thin body whose microstructure is multi-periodic. The asymptotic analysis of media with multiple scale of homogenization is referred to as *Reiterated Homogenization*. Roughly speaking, we seek to approximate in a certain sense the microscopic behavior of such materials by a macroscopic, or average, description.

Let  $\Omega_\varepsilon$  the reference configuration of an elastic thin film that we assume to be a bounded and open subset of  $\mathbb{R}^3$ . Let us denote by  $\mathbb{R}^{m \times n}$  the space of real valued  $m \times n$  matrices. To take into account the periodic heterogeneity of this material, we suppose that its stored energy density, given by a function  $f : \Omega_\varepsilon \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ , satisfies suitable periodicity properties with respect to its second and third variables. Under the deformation  $u : \Omega_\varepsilon \rightarrow \mathbb{R}^3$  the elastic energy of this body turns out to be given by the functional

$$\int_{\Omega_\varepsilon} f \left( x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}; \nabla u(x) \right) dx. \quad (2.1.1)$$

When  $\Omega_\varepsilon = \Omega$ , this kind of asymptotic problems can be seen as a generalization of the *Iterated Homogenization Theorem* for linear integrands, proved in [22], in which the homogenized operator is derived by a formal two-scale asymptotic expansion method. This result has been recovered in several ways via other types of convergence such as  $H$ -convergence,  $G$ -convergence, and multiscale convergence (see [76] and references therein). In the framework of  $\Gamma$ -convergence (see e.g. [45, 30, 32]), the nonlinear setting has been investigated in [36, 32, 60, 16, 14]. All these results share the same property : The homogenized functional, so named in the literature on  $\Gamma$ -limits, is obtained by iterating twice the homogenization formula. The main difference remains in the assumptions on the elastic energy density  $f$ . Indeed, it is not clear what the natural regularity on  $f$  should be for the integral (2.1.1) to be well defined. Such problems have been discussed in [3], Section 5. In particular, for fixed  $F \in \mathbb{R}^{3 \times 3}$ , the measurability of the function  $x \mapsto f(x, x/\varepsilon, x/\varepsilon^2; F)$  is ensured whenever  $f$  is measurable in at most one variable and continuous in the other ones. The most realistic situation would consist in assuming measurability in one of the oscillating variable, as in [36, 32], allowing discontinuities in the microstructure ; This is relevant in the case of mixtures and composite materials. Nevertheless, for technical reasons, the authors find it necessary to assume a strong uniform continuity property in the other variables. On the other hand, in [60, 16, 14]), the authors choose to assume measurability with respect to the macroscopic variable and continuity in both oscillating variables. In particular, they are able to remove the uniform continuity condition but their analysis proscribes discontinuous microstructures. In what follows, we assume the latter and generalize previous results of [15, 13] already established in this spirit.

We will address the case where  $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$  is a three-dimensional cylindrical thin domain of basis  $\omega$ , a bounded open subset of  $\mathbb{R}^2$ , of thickness  $2\varepsilon$ , and whose microstructure is periodic of period  $\varepsilon^2$  in the bulk and also  $\varepsilon$  in the in-plane direction. Two simultaneous features occur in this case : reiterated homogenization and dimension reduction. Our main contribution

is that our arguments allow us to homogenize this material in the reducing direction. As usual, in order to study this problem as  $\varepsilon \rightarrow 0$  we rescale the  $\varepsilon$ -thin body into a reference domain of unit thickness (see e.g. [74, 35]), so that the resulting energy will be defined on a fixed body, while the dependence on  $\varepsilon$  turns out to be explicit in the transverse derivative. For this, we define  $v(x_\alpha, x_3/\varepsilon) = u(x_\alpha, x_3)$  on the rescaled cylinder  $\Omega := \omega \times I$ , where  $I := (-1, 1)$  and  $x_\alpha := (x_1, x_2)$  is the in-plane variable. In what follows we denote by  $\nabla_\alpha$  (resp.  $\nabla_3$ ) the gradient with respect to  $x_\alpha$  (resp.  $x_3$ ) and, for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and  $z \in \mathbb{R}^3$ ,  $(\bar{F}|z)$  stands for the  $3 \times 3$  matrix whose first two columns are those of  $\bar{F}$  and whose last one is  $z$ . Replacing  $v$  by  $u$  in (2.1.1), changing variables and dividing by  $\varepsilon$ , we are led to the study of the sequence of rescaled energies

$$\int_{\Omega} W \left( x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha v(x) \middle| \frac{1}{\varepsilon} \nabla_3 v(x) \right) dx, \quad (2.1.2)$$

where we set

$$f(x, y, z; F) = W(x_\alpha, y_3, y_\alpha, z_3, z_\alpha; F), \quad (2.1.3)$$

for some function  $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ . The motivation for studying such functionals comes from the work of [35] where an abstract dimensional reduction variational convergence result has been established in a general setting for a family of stored energies of the form  $W_\varepsilon(x; F)$ . Specific characterizations were derived for some particular cases. In Section 3 of that reference a heterogeneous nonlinear membrane model is derived by  $\Gamma$ -convergence, and heterogeneity in the transverse direction is considered. Precisely, the authors treat the case where the stored energy density is of the form  $W(x_3; F)$ , generalizing the previous work of [74] for homogeneous materials. These results have been recovered in [15] for energies with a general heterogeneity independent of the thickness, namely  $W(x; F)$ . Furthermore in Section 4 of [35] – and independently also in [88], Theorem 3 (i) – a 3D-2D analysis coupled with a homogenization in the in-plane direction is studied in the case where  $W_\varepsilon(x; F) = W(x_3, x_\alpha/\varepsilon; F)$ . Later, in [13], the author treated the case where the integrand  $W$  depends also on the macroscopic in-plane variable  $x_\alpha$  (see Theorem 1.2 in [13]). Finally, reiterated homogenization of thin films has been investigated in [88] (see Theorem 5) under different length scales for the film thickness and the material microstructure. Namely, functions of the form  $W(x_\alpha/\varepsilon^2; F)$  were considered. However, homogenization in the transverse direction  $x_3$  remained an open question.

As for notation, we will denote by  $\mathcal{L}^n$  the  $n$ -dimensional Lebesgue measure in  $\mathbb{R}^n$  (in the sequel,  $n$  will always be equal to 2 or 3); for any measurable set  $A \subset \mathbb{R}^n$  and any measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we write  $\underline{f}_A := \mathcal{L}^n(A)^{-1} \int_A f(x) dx$  for the average of  $f$  over  $A$ ;  $Q' := (0, 1)^2$  and  $Q := (0, 1)^3$  stand for the unit cubes in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  respectively and we will identify  $W^{1,p}(\omega; \mathbb{R}^3)$  (resp.  $L^p(\omega; \mathbb{R}^3)$ ) with those functions  $u \in W^{1,p}(\Omega; \mathbb{R}^3)$  (resp.  $L^p(\Omega; \mathbb{R}^3)$ ) such that  $D_3 u = 0$  in the sense of distributions.

Following the lines of [13] and [15] we assume that

- (A<sub>1</sub>)  $W(x, \cdot, \cdot; \cdot)$  is continuous on  $\mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$  for a.e.  $x \in \Omega$ ;
- (A<sub>2</sub>)  $W(\cdot, \cdot, \cdot; F)$  is  $\mathcal{L}^3 \otimes \mathcal{L}^3 \otimes \mathcal{L}^2$ -measurable for all  $F \in \mathbb{R}^{3 \times 3}$ ;

- (A<sub>3</sub>)  $\begin{cases} y_\alpha \mapsto W(x, y, z_\alpha; F) \text{ is } Q'\text{-periodic for all } (z_\alpha, y_3, F) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \text{ and a.e. } x \in \Omega; \\ (z_\alpha, y_3) \mapsto W(x, y, z_\alpha; F) \text{ is } Q\text{-periodic for all } (y_\alpha, F) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \text{ and a.e. } x \in \Omega; \end{cases}$
- (A<sub>4</sub>) there exists  $\beta > 0$  such that

$$\frac{1}{\beta}|F|^p - \beta \leq W(x, y, z_\alpha; F) \leq \beta(1 + |F|^p) \quad \text{for all } (y, z_\alpha, F) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \text{ and a.e. } x \in \Omega.$$

Let us formally justify the periodicity assumptions (A<sub>3</sub>) : Since the volume of  $\Omega_\varepsilon$  is of order  $\varepsilon$  and  $\varepsilon^2 \ll \varepsilon$ , in a first step, we can think of  $\varepsilon$  as being a fixed parameter and let  $\varepsilon^2$  tend to zero. Then at this point dimension reduction is not occurring and (2.1.1) can be seen as a single one-scale homogenization problem in which it is natural to assume  $f(x, y, \cdot; F)$  – or equivalently,  $(z_\alpha, y_3) \mapsto W(x, y_\alpha, y_3, z_\alpha; F)$  (see (2.1.3)) – to be  $Q$ -periodic. The homogenization formula for this case gives us a homogenized stored energy density  $W_{\text{hom}}(x, y_\alpha; F)$  that, in a second step, is used as the integrand of a similar problem as that treated in [13]. In particular, the required  $Q'$ -periodicity of  $W_{\text{hom}}(x, \cdot; F)$  can be obtained from the  $Q'$ -periodicity of  $y_\alpha \mapsto W(x, y_\alpha, y_3, z_\alpha; F)$ .

We prove the following theorem.

**Theorem 2.1.1.** *Let  $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  be a function satisfying (A<sub>1</sub>)-(A<sub>4</sub>). Then, the functional  $\mathcal{W}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  defined by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \int_\Omega W \left( x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \middle| \frac{1}{\varepsilon} \nabla_3 u(x) \right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise} \end{cases} \quad (2.1.4)$$

$\Gamma$ -converges for the strong  $L^p(\Omega; \mathbb{R}^3)$ -topology to  $\mathcal{W}_{\text{hom}} : L^p(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  given by

$$\mathcal{W}_{\text{hom}}(u) := \begin{cases} 2 \int_\omega \overline{W}_{\text{hom}}(x_\alpha; \nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\overline{W}_{\text{hom}}(x_\alpha; \overline{F}) := \inf_{T \in \mathbb{N}} \inf_\phi \left\{ \int_{(0,T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)) dy : \right. \\ \left. \phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3), \quad \phi = 0 \text{ on } \partial(0,T)^2 \times I \right\} \quad (2.1.5)$$

for a.e.  $x_\alpha \in \omega$  and all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ , and

$$W_{\text{hom}}(x, y_\alpha; F) := \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0, T)^3} W(x, y_\alpha, z_3, z_\alpha; F + \nabla \phi(z)) dz : \right. \\ \left. \phi \in W_0^{1,p}((0, T)^3; \mathbb{R}^3) \right\}, \quad (2.1.6)$$

for a.e.  $x \in \Omega$  and all  $(y_\alpha, F) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ .

We would also like to draw attention to assumption  $(A_2)$  : In the 3D-2D case, if we assume only  $W(\cdot, y, z_\alpha; F)$  to be  $\mathcal{L}^3$ -measurable for all  $(y, z_\alpha, F) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ , the functional (2.1.4) would be well defined but this would not be the case for the integrals in (2.1.5) and (2.1.6) (see also Remarks 1.1 and 2.2 in [13]) because their integrands would be only separately measurable. This global measurability assumption allows us to ensure measurability of the three integrands in (2.1.4), (2.1.5) and (2.1.6) obtained by crossing the variables (see Section 2.2 below).

We remark that most of the proofs presented are similar to those in [32] (Theorem 22.1 and Remark 22.8), and that our main contribution is to use arguments that allow us to weaken their uniform continuity hypothesis. Let us briefly describe how we proceed : The idea consists in proving the result for integrands which do not depend explicitly on  $x_\alpha$  (see Theorem 2.3.1 in Section 2.3), and then to treat the general case by freezing this macroscopic variable (Section 2.4). To do this, we need to extend the Carathéodory integrand  $W$  into a continuous function by means of Tietze's Extension Theorem (see e.g. [53]). This argument was already used in [13] and [15] where the authors used a weaker result (see Theorem 1 Section 1.2 in [55] and Lemma 4.1 in [13]). Since the problem at fixed  $\varepsilon$  and the asymptotic problem are of different nature (a three-dimensional problem becoming a two-dimensional one), a new difficulty appears. Indeed, arguing like in classical relaxation (see e.g. [14]), we would use the Scorza-Dragoni Theorem to find a compact set  $K \subset \Omega$ , the complement of which has arbitrarily small Lebesgue measure and such that  $W$  is continuous on  $K \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ . To localize our functional on small cubes so as to apply a uniform continuity argument, we fix  $x_0$  in  $\omega$  and for all  $(x_\alpha, x_3) \in K$  we must ensure that  $(x_0, x_3) \in K$ . But since  $K$  is compact, we cannot expect this to be true, and this argument fails. To overcome this difficulty (see also [15, 13]), we need to replace  $W$  by a function which is continuous everywhere. That is the aim of Proposition 2.4.1 which provides a continuous extension of Carathéodory functions.

## 2.2 Properties of the homogenized energy densities

In this section we prove properties of the stored energy densities  $W_{\text{hom}}$  and  $\overline{W}_{\text{hom}}$  that will be of use in the proof of Theorem 2.1.1.

**Remark 2.2.1.** Without loss of generality, we assume that  $W$  is non negative upon replacing  $W$  by  $W + \beta$  which is non negative in view of  $(A_4)$ .

### 2.2.1 Properties of $W_{\text{hom}}$

According to  $(A_1)$ ,  $(A_2)$ ,  $(A_4)$  and Fubini's Theorem it follows that the integral in (2.1.6) is well defined for  $\mathcal{L}^3 \otimes \mathcal{L}^2$ -a.e.  $(x, y_\alpha) \in \Omega \times \mathbb{R}^2$ . Moreover, the function

$$W_{\text{hom}}(\cdot, \cdot; F) \text{ is } \mathcal{L}^3 \otimes \mathcal{L}^2\text{-measurable for all } F \in \mathbb{R}^{3 \times 3}. \quad (2.2.1)$$

In fact, it can be proved that  $W_{\text{hom}}(x, \cdot; F)$  is upper semicontinuous (as the infimum of continuous functions) for a.e.  $x \in \Omega$  and all  $F \in \mathbb{R}^{3 \times 3}$ . Moreover (see e.g. [32])

$$W_{\text{hom}}(x, y_\alpha; \cdot) \text{ is continuous and quasiconvex for a.e. } x \in \Omega \text{ and all } y_\alpha \in \mathbb{R}^2. \quad (2.2.2)$$

By condition  $(A_3)$  it follows that

$$W_{\text{hom}}(x, \cdot; F) \text{ is } Q'\text{-periodic for a.e. } x \in \Omega \text{ and all } F \in \mathbb{R}^{3 \times 3}. \quad (2.2.3)$$

Finally,  $W_{\text{hom}}$  satisfies similar  $p$ -growth and  $p$ -coercivity conditions than  $W$  :

$$\frac{1}{\beta}|F|^p - \beta \leq W_{\text{hom}}(x, y_\alpha; F) \leq \beta(1 + |F|^p) \quad \text{for a.e. } x \in \Omega \text{ and all } (y_\alpha, F) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}. \quad (2.2.4)$$

In particular (see [44]) the following  $p$ -Lipschitz property holds :

$$|W_{\text{hom}}(x, y_\alpha; F_1) - W_{\text{hom}}(x, y_\alpha; F_2)| \leq \beta(1 + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \quad (2.2.5)$$

for all  $F_1$  and  $F_2 \in \mathbb{R}^{3 \times 3}$ , all  $y_\alpha \in \mathbb{R}^2$  and a.e.  $x \in \Omega$ .

### 2.2.2 Properties of $\overline{W}_{\text{hom}}$

In view of (2.2.1), (2.2.2) and (2.2.4), another application of Fubini's Theorem implies that the integral in (2.1.5) well defined for a.e.  $x_\alpha \in \omega$ . Thus so is  $\overline{W}_{\text{hom}}$  and the function  $\overline{W}_{\text{hom}}(\cdot; \overline{F})$  is measurable for all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ .

**Lemma 2.2.2.** *Let  $W$  satisfying  $(A_1)$ - $(A_4)$ , then  $\overline{W}_{\text{hom}}$  is a Carathéodory function.*

*Proof.* The measurability of  $\overline{W}_{\text{hom}}(\cdot; \overline{F})$  has already been established above. Let us now deal with the continuity of  $\overline{W}_{\text{hom}}(x_\alpha; \cdot)$ . Let  $\overline{F}$  and  $\overline{F}_n \in \mathbb{R}^{3 \times 2}$  be such that  $\overline{F}_n \rightarrow \overline{F}$ . From the definition of  $\overline{W}_{\text{hom}}(x_\alpha; \overline{F})$ , for fixed  $\delta > 0$  choose  $T \in \mathbb{N}$  and  $\varphi \in W^{1,p}((0, T)^2 \times I; \mathbb{R}^3)$ ,  $\varphi = 0$  on  $\partial(0, T)^2 \times I$ , such that

$$\overline{W}_{\text{hom}}(x_\alpha; \overline{F}) + \delta \geq \int_{(0, T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy_\alpha dy_3. \quad (2.2.6)$$

Therefore

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \overline{W}_{\text{hom}}(x_\alpha; \overline{F}_n) &\leq \limsup_{n \rightarrow +\infty} \int_{(0, T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F}_n + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy_\alpha dy_3 \\ &= \int_{(0, T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy_\alpha dy_3 \end{aligned}$$



due to the continuity (2.2.2) and the  $p$ -growth property (2.2.4) together with Lebesgue's Dominated Convergence Theorem. So by (2.2.6) and letting  $\delta \rightarrow 0$  we conclude that

$$\limsup_{n \rightarrow +\infty} \overline{W}_{\text{hom}}(x_\alpha; \overline{F}_n) \leq \overline{W}_{\text{hom}}(x_\alpha; \overline{F}). \quad (2.2.7)$$

Similarly, for each  $n \in \mathbb{N}$  consider  $T_n \in \mathbb{N}$  ( $T_n \nearrow +\infty$ ) and  $\varphi_n \in W^{1,p}((0, T_n)^2 \times I; \mathbb{R}^3)$ ,  $\varphi_n = 0$  on  $\partial(0, T_n)^2 \times I$ , such that

$$\begin{aligned} \overline{W}_{\text{hom}}(x_\alpha; \overline{F}_n) + \frac{1}{n} &\geq \int_{(0, T_n)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F}_n + \nabla_\alpha \varphi_n(y) | \nabla_3 \varphi_n(y)) dy_\alpha dy_3 \\ &= \int_{Q' \times I} W_{\text{hom}}(x_\alpha, y_3, T_n y_\alpha; \overline{F}_n + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y)) dy_\alpha dy_3, \end{aligned}$$

after a change of variables and where  $\psi_n(y) := \frac{1}{T_n} \varphi_n(T_n y_\alpha, y_3)$ . Clearly the function  $\psi_n$  belongs to  $W^{1,p}(Q' \times I; \mathbb{R}^3)$  and  $\psi_n = 0$  on  $\partial Q' \times I$ . By (2.2.4), the sequence  $\{(\nabla_\alpha \psi_n | T_n \nabla_3 \psi_n)\}$  is bounded in  $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$  uniformly in  $n \in \mathbb{N}$ . Hence, using (2.2.5), Hölder inequality and the fact that  $\overline{F}_n \rightarrow \overline{F}$ , we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{Q' \times I} |W_{\text{hom}}(x_\alpha, y_3, T_n y_\alpha; \overline{F}_n + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y)) \\ - W_{\text{hom}}(x_\alpha, y_3, T_n y_\alpha; \overline{F} + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y))| dy_\alpha dy_3 = 0. \end{aligned}$$

Consequently

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \overline{W}_{\text{hom}}(x_\alpha; \overline{F}_n) &\geq \liminf_{n \rightarrow +\infty} \int_{Q' \times I} W_{\text{hom}}(x_\alpha, y_3, T_n y_\alpha; \overline{F} + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y)) dy_\alpha dy_3 \\ &= \liminf_{n \rightarrow +\infty} \int_{(0, T_n)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi_n(y) | \nabla_3 \varphi_n(y)) dy_\alpha dy_3 \\ &\geq \overline{W}_{\text{hom}}(x_\alpha; \overline{F}). \end{aligned} \quad (2.2.8)$$

From (2.2.7) and (2.2.8), we conclude that  $\overline{W}_{\text{hom}}(x_\alpha; \cdot)$  is continuous.  $\square$

As a consequence of Lemma 2.2.2, the functional  $\mathcal{W}_{\text{hom}}$  makes sense on  $W^{1,p}(\Omega; \mathbb{R}^3)$ . Finally, the function  $\overline{W}_{\text{hom}}$  satisfies the same  $p$ -growth and  $p$ -coercivity condition as  $W$  and  $W_{\text{hom}}$  namely :

$$\frac{1}{\beta} |\overline{F}|^p - \beta \leq \overline{W}_{\text{hom}}(x_\alpha; \overline{F}) \leq \beta(1 + |\overline{F}|^p) \quad \text{for a.e. } x_\alpha \in \omega \text{ and all } \overline{F} \in \mathbb{R}^{3 \times 2}. \quad (2.2.9)$$

### 2.3 Independence on the in-plane macroscopic variable

In this section, we assume that  $W$  does not depend explicitly on  $x_\alpha$ , namely  $W : I \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ . For each  $\varepsilon > 0$ , consider the functional  $\mathcal{W}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  defined by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \int_{\Omega} W\left(x_3, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.3.1)$$

Our aim is to prove the following result.

**Theorem 2.3.1.** *Under assumptions (A<sub>1</sub>)-(A<sub>4</sub>) the functional  $\mathcal{W}_\varepsilon$   $\Gamma$ -converges for the strong  $L^p(\Omega; \mathbb{R}^3)$ -topology to  $\mathcal{W}_{\text{hom}} : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  defined by*

$$\mathcal{W}_{\text{hom}}(u) = \begin{cases} 2 \int_{\omega} \overline{W}_{\text{hom}}(\nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \overline{W}_{\text{hom}}(\overline{F}) &:= \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^2 \times I} W_{\text{hom}}(y_3, y_\alpha; \overline{F} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)) dy : \right. \\ &\quad \left. \phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3) \text{ and } \phi = 0 \text{ on } \partial(0,T)^2 \times I \right\}, \end{aligned} \quad (2.3.2)$$

for all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ , and

$$\begin{aligned} W_{\text{hom}}(y_3, y_\alpha; F) &:= \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^3} W(y_3, y_\alpha, z_3, z_\alpha; F + \nabla \phi(z)) dz : \right. \\ &\quad \left. \phi \in W_0^{1,p}((0,T)^3; \mathbb{R}^3) \right\}, \end{aligned} \quad (2.3.3)$$

for all  $(y, F) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$ .

We start by localizing the functionals given in (2.3.1) in order to highlight their dependence on the class of bounded and open subsets of  $\mathbb{R}^2$ , denoted by  $\mathcal{A}_0$ . As it will be clear from the proofs of Lemmas 2.3.6 and 2.3.7 below, it would not be sufficient to localize, as usual, on any open subset of  $\omega$ . Indeed, formulas (2.3.2) and (2.3.3) suggest to work in cubes of the type  $(0, T)^2$ , with  $T$  arbitrarily large, not necessarily contained in  $\omega$ . For each  $\varepsilon > 0$ , consider  $\mathcal{W}_\varepsilon : L^p(\mathbb{R}^2 \times I; \mathbb{R}^3) \times \mathcal{A}_0 \rightarrow [0, +\infty]$  defined by

$$\mathcal{W}_\varepsilon(u; A) := \begin{cases} \int_{A \times I} W\left(x_3, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A \times I; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.3.4)$$

### 2.3.1 Existence of $\Gamma$ -convergent subsequences

Given  $\{\varepsilon_j\} \searrow 0^+$  and  $A \in \mathcal{A}_0$ , consider the  $\Gamma$ -lim inf of  $\mathcal{W}_{\varepsilon_j}(\cdot; A)$  for the  $L^p(A \times I; \mathbb{R}^3)$ -topology, defined for  $u \in L^p(\mathbb{R}^2 \times I; \mathbb{R}^3)$ , by

$$\mathcal{W}_{\{\varepsilon_j\}}(u; A) := \inf_{\{u_j\}} \left\{ \liminf_{j \rightarrow +\infty} \mathcal{W}_{\varepsilon_j}(u_j; A) : u_j \rightarrow u \text{ in } L^p(A \times I; \mathbb{R}^3) \right\}.$$

In view of the  $p$ -coercivity condition ( $A_4$ ), for each  $A \in \mathcal{A}_0$  it follows that  $\mathcal{W}_{\{\varepsilon_j\}}(u; A)$  is infinite whenever  $u \in L^p(\mathbb{R}^2 \times I; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$ , so it suffices to study the case where  $u \in W^{1,p}(A; \mathbb{R}^3)$ .

**Remark 2.3.2.** Arguing similarly than the second step of the proof of Theorem 2.5 in [35], one can show that for every sequence  $\{\varepsilon_j\} \searrow 0^+$ , there exists a subsequence  $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$  such that for all  $A \in \mathcal{A}_0$  and all  $u \in W^{1,p}(A; \mathbb{R}^3)$ ,  $\mathcal{W}_{\{\varepsilon_n\}}(u; A)$  is the  $\Gamma$ -limit of  $\mathcal{W}_{\varepsilon_n}(u; A)$  for the strong  $L^p(A \times I; \mathbb{R}^3)$ -topology. The reason why we cannot use directly their result is because the authors were considering  $A$  as any open subset of a bounded domain while we are treating the case where the latter domain is the whole  $\mathbb{R}^2$ .

Our next objective is to show that for every  $A \in \mathcal{A}_0$  and every  $u \in W^{1,p}(A; \mathbb{R}^3)$ , then  $\mathcal{W}_{\{\varepsilon_n\}}(u; A) = \mathcal{W}_{\text{hom}}(u; A)$ , where  $\mathcal{W}_{\text{hom}} : L^p(\mathbb{R}^2 \times I; \mathbb{R}^3) \times \mathcal{A}_0 \rightarrow [0, +\infty]$  is given by

$$\mathcal{W}_{\text{hom}}(u; A) = \begin{cases} 2 \int_A \overline{W}_{\text{hom}}(\nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

### 2.3.2 Integral representation of the $\Gamma$ -limit

Following the proof of Lemma 2.5 in [35], it is possible to show that for each  $U \in \mathcal{A}_0$  and all  $u \in W^{1,p}(U; \mathbb{R}^3)$ , the restriction of  $\mathcal{W}_{\{\varepsilon_n\}}(u; \cdot)$  to  $\mathcal{A}(U)$  is a Radon measure, absolutely continuous with respect to the two-dimensional Lebesgue measure. But one has to ensure that the Integral Representation given by Theorem 4.3.2 in [37] is independent of the open set  $U \in \mathcal{A}_0$ . The following result, prevents this dependence from holding since it leads to an homogeneous integrand as will be seen in Lemma 2.3.4 below.

**Lemma 2.3.3.** *For all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ ,  $y_0$  and  $z_0 \in \mathbb{R}^2$ , and  $r > 0$*

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{F} \cdot; Q'_r(y_0)) = \mathcal{W}_{\{\varepsilon_n\}}(\overline{F} \cdot; Q'_r(z_0)).$$

*Proof.* Clearly, it suffices to establish the inequality

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{F} \cdot; Q'_r(y_0)) \geq \mathcal{W}_{\{\varepsilon_n\}}(\overline{F} \cdot; Q'_r(z_0)).$$

Let  $\{u_n\} \subset W^{1,p}(Q'_r(y_0) \times I; \mathbb{R}^3)$  with  $u_n \rightarrow 0$  in  $L^p(Q'_r(y_0) \times I; \mathbb{R}^3)$  be such that

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{F} \cdot; Q'_r(y_0)) = \lim_{n \rightarrow +\infty} \int_{Q'_r(y_0) \times I} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{F} + \nabla_\alpha u_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n(x) \right) dx. \quad (2.3.5)$$

By Theorem 1.1 in [25] and Lemma 2.6 in [35], we can assume without loss of generality that the sequence  $\{ |(\nabla_\alpha u_n|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n)|^p \}$  is equi-integrable and  $u_n = 0$  on  $\partial Q'_r(y_0) \times I$ . For all  $n \in \mathbb{N}$  we write

$$\frac{y_0 - z_0}{\varepsilon_n} = m_{\varepsilon_n} + s_{\varepsilon_n}$$

with  $m_{\varepsilon_n} \in \mathbb{Z}^2$  and  $s_{\varepsilon_n} \in [0, 1]^2$ ,

$$\frac{m_{\varepsilon_n}}{\varepsilon_n} = \theta_{\varepsilon_n} + l_{\varepsilon_n} \quad (2.3.6)$$

with  $\theta_{\varepsilon_n} \in \mathbb{Z}^2$  and  $l_{\varepsilon_n} \in [0, 1]^2$ , and we define

$$x_{\varepsilon_n} := m_{\varepsilon_n} \varepsilon_n - \varepsilon_n^2 l_{\varepsilon_n}. \quad (2.3.7)$$

Note that  $x_{\varepsilon_n} = y_0 - z_0 - \varepsilon_n s_{\varepsilon_n} - \varepsilon_n^2 l_{\varepsilon_n} \rightarrow y_0 - z_0$  as  $n \rightarrow +\infty$ . For all  $n \in \mathbb{N}$ , extend  $u_n$  by zero to the whole  $\mathbb{R}^2 \times I$  and set  $v_n(x_\alpha, x_3) = u_n(x_\alpha + x_{\varepsilon_n}, x_3)$  for  $(x_\alpha, x_3) \in Q'_r(z_0) \times I$ . Then  $\{v_n\} \subset W^{1,p}(Q'_r(z_0) \times I; \mathbb{R}^3)$  and  $v_n \rightarrow 0$  in  $L^p(Q'_r(z_0) \times I; \mathbb{R}^3)$  because

$$\begin{aligned} \int_{Q'_r(z_0) \times I} |v_n(x)|^p dx &= \int_{Q'_r(z_0) \times I} |u_n(x_\alpha + x_{\varepsilon_n}, x_3)|^p dx \\ &= \int_{Q'_r(z_0 + x_{\varepsilon_n}) \times I} |u_n(x)|^p dx \\ &\leq \int_{Q'_r(y_0) \times I} |u_n(x)|^p dx, \end{aligned} \quad (2.3.8)$$

since  $u_n \equiv 0$  on  $[\mathbb{R}^2 \setminus Q'_r(y_0)] \times I$ . We also remark that by the translation invariance of the Lebesgue measure, the sequence  $\{ |(\nabla_\alpha v_n|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n)|^p \}$  is still equi-integrable. In view of (2.3.5), (2.3.6), (2.3.7) and (A<sub>3</sub>),

$$\begin{aligned} &\mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; Q'_r(y_0)) \\ &\geq \limsup_{n \rightarrow +\infty} \int_{Q'_r(y_0 - x_{\varepsilon_n}) \times I} W \left( x_3, \frac{x_\alpha + x_{\varepsilon_n}}{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha + x_{\varepsilon_n}}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n(x) \right) dx \\ &= \limsup_{n \rightarrow +\infty} \int_{Q'_r(y_0 - x_{\varepsilon_n}) \times I} W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n(x) \right) dx \\ &\geq \limsup_{n \rightarrow +\infty} \int_{Q'_r(z_0) \times I} W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n(x) \right) dx \\ &\quad - \limsup_{n \rightarrow +\infty} \int_{[Q'_r(z_0) \setminus Q'_r(y_0 - x_{\varepsilon_n})] \times I} W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n(x) \right) dx. \end{aligned}$$

The  $p$ -growth condition (A<sub>4</sub>) and the fact that  $\mathcal{L}^2(Q'_r(z_0) \setminus Q'_r(y_0 - x_{\varepsilon_n})) \rightarrow 0$  yield

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} \int_{[Q'_r(z_0) \setminus Q'_r(y_0 - x_{\varepsilon_n})] \times I} W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n(x) \right) dx \\ &\leq \limsup_{n \rightarrow +\infty} 2\beta(1 + |\bar{F}|^p) \mathcal{L}^2(Q'_r(z_0) \setminus Q'_r(y_0 - x_{\varepsilon_n})) = 0, \end{aligned}$$

since  $v_n \equiv 0$  on  $[\mathbb{R}^2 \setminus Q'_r(y_0 - x_{\varepsilon_n})] \times I$ . Therefore

$$\begin{aligned} & \mathcal{W}_{\{\varepsilon_n\}}(\overline{F}; Q'_r(y_0)) \\ & \geq \limsup_{n \rightarrow +\infty} \int_{Q'_r(z_0) \times I} W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{F} + \nabla_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) dx. \end{aligned} \quad (2.3.9)$$

To eliminate the term  $\varepsilon_n l_{\varepsilon_n}$  in (2.3.9), and thus to recover  $\mathcal{W}_{\{\varepsilon_n\}}(\overline{F}; Q'_r(z_0))$ , we would like to apply a uniform continuity argument. Since for a.e.  $x_3 \in I$  the function  $W(x_3, \cdot, \cdot; \cdot)$  is continuous on  $\mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ , then  $(A_3)$  implies that it is uniformly continuous on  $\mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$  for any  $\lambda > 0$  and a.e.  $x_3 \in I$ , where  $\overline{B}(0, \lambda)$  stands for the closed ball of radius  $\lambda$  in  $\mathbb{R}^{3 \times 3}$ . We define

$$R_n^\lambda := \left\{ x \in Q'_r(z_0) \times I : \left| \left( \overline{F} + \nabla_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) \right| \leq \lambda \right\},$$

and we note that by Chebyshev's inequality

$$\mathcal{L}^3([Q'_r(z_0) \times I] \setminus R_n^\lambda) \leq C/\lambda^p, \quad (2.3.10)$$

for some constant  $C > 0$  independent of  $\lambda$  or  $n$ . Thus, in view of (2.3.9) and the fact that  $W$  is nonnegative,

$$\begin{aligned} & \mathcal{W}_{\{\varepsilon_n\}}(\overline{F}; Q'_r(y_0)) \\ & \geq \limsup_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{R_n^\lambda} W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{F} + \nabla_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) dx. \end{aligned}$$

Denoting by  $\omega_\lambda(x_3, \cdot) : [0, +\infty) \rightarrow [0, +\infty)$  the modulus of continuity of  $W(x_3, \cdot, \cdot; \cdot)$  on  $\mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$  defined by

$$\begin{aligned} \omega_\lambda(x_3, t) & := \sup \left\{ W(x_3, y, z_\alpha; F) - W(x_3, y', z'_\alpha; F') : y, y' \in \mathbb{R}^3, z_\alpha, z'_\alpha \in \mathbb{R}^2, \right. \\ & \quad \left. F, F' \in \overline{B}(0, \lambda) \text{ and } |y - y'| + |z_\alpha - z'_\alpha| + |F - F'| \leq t \right\}, \end{aligned}$$

we easily check that for a.e.  $x_3 \in I$ , the function  $t \mapsto \omega_\lambda(x_3, t)$  is continuous, increasing and satisfies  $\omega_\lambda(x_3, 0) = 0$ . Moreover, by continuity of  $W(x_3, \cdot, \cdot; \cdot)$  the previous supremum can be taken over rational points; Thus for all  $t \in [0, +\infty)$ , the function  $x_3 \mapsto \omega_\lambda(x_3, t)$  is measurable as the countable supremum of measurable functions. We get, for any  $x \in R_n^\lambda$

$$\begin{aligned} & \left| W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{F} + \nabla_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) \right. \\ & \quad \left. - W \left( x_3, \frac{x_\alpha}{\varepsilon_n} - \varepsilon_n l_{\varepsilon_n}, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{F} + \nabla_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) \right| \\ & \leq \omega_\lambda(x_3, \varepsilon_n l_{\varepsilon_n}). \end{aligned}$$

Then, the properties of  $\omega_\lambda$  and Beppo-Levi's Monotone Convergence Theorem yield

$$\begin{aligned} \mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; Q'_r(y_0)) &\geq \limsup_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \left\{ \int_{R_n^\lambda} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) dx \right. \\ &\quad \left. - r^2 \int_{-1}^1 \omega_\lambda(x_3, \varepsilon_n l_{\varepsilon_n}) dx_3 \right\} \\ &= \limsup_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{R_n^\lambda} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) dx. \end{aligned}$$

The equi-integrability of  $\{ |(\nabla_\alpha v_n | \frac{1}{\varepsilon_n} \nabla_3 v_n)|^p \}$ , the  $p$ -growth condition  $(A_4)$  and (2.3.10), imply that

$$\begin{aligned} \limsup_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{[Q'_r(z_0) \times I] \setminus R_n^\lambda} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) dx \\ \leq \beta \limsup_{\lambda \rightarrow +\infty} \sup_{n \in \mathbb{N}} \int_{[Q'_r(z_0) \times I] \setminus R_n^\lambda} \left( 1 + \left| \left( \nabla_\alpha v_n \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) \right|^p \right) dx = 0, \end{aligned}$$

and since  $v_n \rightarrow 0$  in  $L^p(Q'_r(z_0) \times I; \mathbb{R}^3)$ ,

$$\begin{aligned} \mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; Q'_r(y_0)) &\geq \limsup_{n \rightarrow +\infty} \int_{Q'_r(z_0) \times I} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha v_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) dx \\ &\geq \mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; Q'_r(z_0)). \end{aligned}$$

□

As a consequence of this lemma, we derive the following result.

**Lemma 2.3.4.** *There exists a continuous function  $W_{\{\varepsilon_n\}} : \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  such that for all  $A \in \mathcal{A}_0$  and all  $u \in W^{1,p}(A; \mathbb{R}^3)$ ,*

$$\mathcal{W}_{\{\varepsilon_n\}}(u; A) = 2 \int_A W_{\{\varepsilon_n\}}(\nabla_\alpha u(x_\alpha)) dx_\alpha.$$

*Proof.* Fix  $U \in \mathcal{A}_0$ . By Theorem 4.3.2 in [37], there exists a Carathéodory function  $W_{\{\varepsilon_n\}}^U : U \times \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  such that

$$\mathcal{W}_{\{\varepsilon_n\}}(u; A) = 2 \int_A W_{\{\varepsilon_n\}}^U(x_\alpha; \nabla_\alpha u(x_\alpha)) dx_\alpha$$

for all  $A \in \mathcal{A}(U)$  and all  $u \in W^{1,p}(U; \mathbb{R}^3)$ . Furthermore, for all  $x_\alpha \in U$  and all  $\bar{F} \in \mathbb{R}^{3 \times 2}$

$$W_{\{\varepsilon_n\}}^U(x_\alpha; \bar{F}) = \limsup_{r \rightarrow 0} \frac{\mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; Q'_r(x_\alpha))}{2r^2}.$$

Define  $W_{\{\varepsilon_n\}} : \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$  by

$$W_{\{\varepsilon_n\}}(\overline{F}) = \limsup_{r \rightarrow 0} \frac{\mathcal{W}_{\{\varepsilon_n\}}(\overline{F}; Q'_r(0))}{2r^2}.$$

As a consequence of Lemma 2.3.3,  $W_{\{\varepsilon_n\}}^U(x_\alpha; \overline{F}) = W_{\{\varepsilon_n\}}(\overline{F})$  for all  $x_\alpha \in U$  and for all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ . Taking  $A = U$ , it turns out that

$$\mathcal{W}_{\{\varepsilon_n\}}(u; A) = 2 \int_A W_{\{\varepsilon_n\}}(\nabla_\alpha u(x_\alpha)) dx_\alpha$$

holds for all  $u \in W^{1,p}(A; \mathbb{R}^3)$ . □

### 2.3.3 Characterization of the $\Gamma$ -limit

Our next objective is to show that  $\mathcal{W}_{\{\varepsilon_n\}}(u; A) = \mathcal{W}_{\text{hom}}(u; A)$  for any  $A \in \mathcal{A}_0$  and all  $u \in W^{1,p}(A; \mathbb{R}^3)$ . In view of Lemma 2.3.4, we only need to prove that  $\overline{W}_{\text{hom}}(\overline{F}) = W_{\{\varepsilon_n\}}(\overline{F})$  for all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ , and thus it suffices to work with affine functions instead of general Sobolev functions.

#### The lower bound

In order to estimate  $W_{\{\varepsilon_n\}}$  from below in terms of  $\overline{W}_{\text{hom}}$ , we will need the following result, close in spirit to Proposition 22.4 in [32].

**Proposition 2.3.5.** *Given  $M > 0$ ,  $\eta > 0$  and  $\varphi : [0, +\infty) \rightarrow [0, +\infty]$  a continuous and increasing function satisfying  $\varphi(t)/t \rightarrow +\infty$  as  $t \rightarrow +\infty$ , there exists  $\varepsilon_0 \equiv \varepsilon_0(M, \eta) > 0$  such that for every  $0 < \varepsilon < \varepsilon_0$ , every  $a \in \mathbb{R}^2$  and every  $u \in W^{1,p}((a + Q') \times I; \mathbb{R}^3)$  with*

$$\int_{(a+Q') \times I} \varphi(|\nabla u|^p) dx \leq M, \tag{2.3.11}$$

there exists  $v \in W_0^{1,p}((a + Q') \times I; \mathbb{R}^3)$  with  $\|v\|_{L^p((a+Q') \times I; \mathbb{R}^3)} \leq \eta$  satisfying

$$\int_{(a+Q') \times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u\right) dx \geq \int_{(a+Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u + \nabla v) dx - \eta.$$

*Proof.* The proof is divided into two steps. First, we prove this proposition under the additional hypothesis that  $a$  belong to a compact set of  $\mathbb{R}^2$ . Then, we conclude the result in its full generality replacing  $a$  by its decimal part  $a - \llbracket a \rrbracket$  and using the periodicity of the integrands  $W$  and  $W_{\text{hom}}$ .

**Step 1.** For  $a \in [-1, 1]^2$ , the claim of Proposition 2.3.5 holds. Indeed, if not then we may find  $\varphi$ ,  $M$  and  $\eta$  as above, and sequences  $\{\varepsilon_n\} \rightarrow 0^+$ ,  $\{a_n\} \subset [-1, 1]^2$  and  $\{u_n\} \subset W^{1,p}((a_n + Q') \times I; \mathbb{R}^3)$  with

$$\int_{(a_n+Q') \times I} \varphi(|\nabla u_n|^p) dx \leq M \tag{2.3.12}$$

such that, for every  $n \in \mathbb{N}$

$$\begin{aligned}
 & \int_{(a_n+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \nabla u_n \right) dx \\
 & < \inf \left\{ \int_{(a_n+Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u_n + \nabla v) dx : v \in W_0^{1,p}((a_n+Q') \times I; \mathbb{R}^3) \right. \\
 & \qquad \qquad \qquad \left. \|v\|_{L^p((a_n+Q') \times I; \mathbb{R}^3)} \leq \eta \right\} - \eta. \tag{2.3.13}
 \end{aligned}$$

From (2.3.12) and the Poincaré-Wirtinger Inequality, up to a translation argument, we can suppose that the sequence  $\{\|u_n\|_{W^{1,p}((a_n+Q') \times I; \mathbb{R}^3)}\}$  is uniformly bounded. From this fact and since the set  $(a_n+Q') \times I$  is an extension domain, there is no loss of generality in assuming that  $\{u_n\}$  is bounded in  $W^{1,p}(\mathbb{R}^3; \mathbb{R}^3)$  and that, due to (2.3.12),

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \varphi(|\nabla u_n|^p) dx \leq M_1 \tag{2.3.14}$$

for some constant  $M_1 > 0$  depending only on  $M$  (see the proof of Theorem 1, Section 4.4 in [55]). Passing to a subsequence, we can also assume that  $u_n \rightharpoonup u$  in  $W^{1,p}(\mathbb{R}^3; \mathbb{R}^3)$ . Let  $B$  a ball of sufficiently large radius so that  $a_n + Q' \subset B$  for all  $n \in \mathbb{N}$ . De La Vallée Poussin criterion (see e.g. Proposition 1.27 in [5]) and (2.3.14) guarantee that the sequence  $\{|\nabla u_n|^p\}$  is equi-integrable on  $B \times I$ . This implies that there exists  $\delta = \delta(\eta)$  such that

$$\sup_{n \in \mathbb{N}} \beta \int_E (1 + |\nabla u|^p + |\nabla u_n|^p) dx \leq \frac{\eta}{2} \tag{2.3.15}$$

whenever  $E$  is a measurable subset of  $B \times I$  satisfying  $\mathcal{L}^3(E) \leq \delta$ . As  $\{a_n\} \subset [-1, 1]^2$  we may suppose, without loss of generality, that  $a_n \rightarrow a \in [-1, 1]^2$ , and that for fixed  $0 < \rho < 1$ , with  $\rho^2 \ll \delta$ , the following hold for  $n$  large enough :

$$\begin{cases} a + (1 - \rho)Q' \subset a_n + Q' \subset a + (1 + \rho)Q', \\ \mathcal{L}^2(S_n) \leq \delta/2, \text{ where } S_n := [a_n + Q'] \setminus [a + (1 - \rho)Q'] \subset B, \\ \text{and } \|u_n - u\|_{L^p((a+(1+\rho)Q') \times I; \mathbb{R}^3)} \leq \eta. \end{cases} \tag{2.3.16}$$

Take now a sequence of cut-off functions  $\varphi_n \in C_c^\infty(\mathbb{R}^2; [0, 1])$  such that

$$\varphi_n = \begin{cases} 1 & \text{on } a + (1 - \rho)Q', \\ 0 & \text{outside } a_n + Q', \end{cases}$$

and  $\|\nabla \varphi_n\|_{L^\infty(\mathbb{R}^2)} \leq C/\rho$  for some constant  $C > 0$ . Let  $v_n = \varphi_n(u - u_n)$ , then  $v_n \in W_0^{1,p}((a_n + Q') \times I; \mathbb{R}^3)$  and



$$\int_{(a_n+Q') \times I} |v_n|^p dx \leq \int_{(a_n+Q') \times I} \varphi_n |u - u_n|^p dx \leq \int_{(a_n+Q') \times I} |u - u_n|^p dx \leq \eta^p.$$

Then, taking  $v_n$  as test function in (2.3.13), it follows from (2.3.15), (2.3.16) and (2.2.4) that

$$\begin{aligned} & \int_{(a_n+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \nabla u_n \right) dx \\ & < \int_{(a_n+Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u_n + \nabla v_n) dx - \eta \\ & \leq \int_{(a+(1-\rho)Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u) dx \\ & \quad + \beta \int_{S_n \times I} \left( 1 + |\nabla u|^p + |\nabla u_n|^p + \frac{C}{\rho^p} |u - u_n|^p \right) dx - \eta \\ & \leq \int_{(a+(1-\rho)Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u) dx - \frac{\eta}{2} + \frac{\beta C}{\rho^p} \int_{S_n \times I} |u - u_n|^p dx. \end{aligned} \quad (2.3.17)$$

Since  $u_n \rightarrow u$  in  $L^p(\mathbb{R}^3; \mathbb{R}^3)$ , by (2.3.16) and (2.3.17) we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{(a_n+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \nabla u_n \right) dx \\ \leq \int_{(a+(1-\rho)Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u) dx - \frac{\eta}{2}, \end{aligned} \quad (2.3.18)$$

and by Theorem 1.1 in [16] (with  $f(x, y; F) = W(x_3, x_\alpha, y_3, y_\alpha; F)$ ) and (2.3.18), we get

$$\begin{aligned} \int_{(a+(1-\rho)Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u) dx & \leq \liminf_{n \rightarrow +\infty} \int_{(a+(1-\rho)Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \nabla u_n \right) dx \\ & \leq \limsup_{n \rightarrow +\infty} \int_{(a_n+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \nabla u_n \right) dx \\ & \leq \int_{(a+(1-\rho)Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u) dx - \frac{\eta}{2} \end{aligned}$$

which is a contradiction.

**Step 2.** Let  $a \in \mathbb{R}^2$ , then  $a - \llbracket a \rrbracket \in [-1, 1]^2$ . Given  $u \in W^{1,p}((a+Q') \times I; \mathbb{R}^3)$ , set  $\tilde{u}(x_\alpha, x_3) := u(x_\alpha + \llbracket a \rrbracket, x_3)$  and thus  $\tilde{u} \in W^{1,p}((a - \llbracket a \rrbracket + Q') \times I; \mathbb{R}^3)$ . Applying Step 1 with  $\eta/3$ , we get the existence of  $0 < \varepsilon'_0 \equiv \varepsilon'_0(M, \eta)$  such that, for all  $0 < \varepsilon < \varepsilon'_0$ , there exist  $\tilde{v} \in W_0^{1,p}((a - \llbracket a \rrbracket + Q') \times I; \mathbb{R}^3)$  satisfying  $\|\tilde{v}\|_{L^p((a - \llbracket a \rrbracket + Q') \times I; \mathbb{R}^3)} \leq \eta/3$  and

$$\begin{aligned} & \int_{(a - \llbracket a \rrbracket + Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla \tilde{u}(x) \right) dx \\ & \geq \int_{(a - \llbracket a \rrbracket + Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla \tilde{u}(x) + \nabla \tilde{v}(x)) dx - \frac{\eta}{3}. \end{aligned}$$

Setting  $v(x_\alpha, x_3) := \tilde{v}(x_\alpha - \llbracket a \rrbracket, x_3)$ , then  $v \in W_0^{1,p}((a+Q') \times I; \mathbb{R}^3)$  and  $\|v\|_{L^p((a+Q') \times I; \mathbb{R}^3)} \leq \frac{\eta}{3} \leq \eta$ . Therefore, by a change of variables

$$\begin{aligned} \int_{(a+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha - \llbracket a \rrbracket}{\varepsilon}; \nabla u(x) \right) dx \\ \geq \int_{(a+Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u(x) + \nabla v(x)) dx - \frac{\eta}{3}, \end{aligned} \quad (2.3.19)$$

where we have used (A<sub>3</sub>) and (2.2.3). Writing

$$\frac{\llbracket a \rrbracket}{\varepsilon} =: m_\varepsilon + r_\varepsilon, \quad \text{with } m_\varepsilon \in \mathbb{Z}^2 \text{ and } |r_\varepsilon| < \sqrt{2}\varepsilon$$

and using again (A<sub>3</sub>), the inequality (2.3.19) reduces to

$$\begin{aligned} \int_{(a+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon} - r_\varepsilon; \nabla u(x) \right) dx \\ \geq \int_{(a+Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u(x) + \nabla v(x)) dx - \frac{\eta}{3}. \end{aligned} \quad (2.3.20)$$

Choose  $\lambda > 0$  large enough (depending on  $\eta$ ) so that

$$\beta \int_{\{|\nabla u| > \lambda\} \cap [(a+Q') \times I]} (1 + |\nabla u|^p) dx \leq \frac{\eta}{3}. \quad (2.3.21)$$

Then, from (2.3.21) and the  $p$ -growth condition (A<sub>4</sub>),

$$\begin{aligned} \int_{(a+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon} - r_\varepsilon; \nabla u(x) \right) dx \\ \leq \int_{\{|\nabla u| \leq \lambda\} \cap [(a+Q') \times I]} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon} - r_\varepsilon; \nabla u(x) \right) dx + \frac{\eta}{3}. \end{aligned} \quad (2.3.22)$$

Arguing as in Lemma 2.3.3, for a.e.  $x_3 \in I$ , the function  $W(x_3, \cdot, \cdot; \cdot)$  is uniformly continuous on  $\mathbb{R}^2 \times \mathbb{R}^3 \times \overline{B}(0, \lambda)$ . Denoting by  $\omega_\lambda(x_3, \cdot) : [0, +\infty) \rightarrow [0, +\infty)$  the modulus of continuity of  $W(x_3, \cdot, \cdot; \cdot)$  on  $\mathbb{R}^2 \times \mathbb{R}^3 \times \overline{B}(0, \lambda)$ , we get for every  $x \in \{|\nabla u| \leq \lambda\} \cap [(a+Q') \times I]$ ,

$$\left| W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon} - r_\varepsilon; \nabla u(x) \right) - W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u(x) \right) \right| < \omega_\lambda(x_3, r_\varepsilon).$$

As  $r_\varepsilon \rightarrow 0$ , by Beppo-Levi's Monotone Convergence Theorem, we get that

$$\int_{-1}^1 \omega_\lambda(x_3, r_\varepsilon) dx_3 \xrightarrow{\varepsilon \rightarrow 0} 0,$$

and thus, for any  $\eta > 0$ , there exists  $\varepsilon_0'' \equiv \varepsilon_0''(\eta) > 0$  such that for all  $\varepsilon < \varepsilon_0''$ ,

$$\begin{aligned} \int_{\{|\nabla u| \leq \lambda\} \cap [(a+Q') \times I]} \left| W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon} - r_\varepsilon; \nabla u(x) \right) \right. \\ \left. - W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u(x) \right) \right| dx < \frac{\eta}{3}. \end{aligned} \quad (2.3.23)$$

Consequently, by (2.3.22) and (2.3.23) we have

$$\begin{aligned} \int_{(a+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon} - r_\varepsilon; \nabla u(x) \right) dx \\ \leq \int_{\{|\nabla u| \leq \lambda\} \cap [(a+Q') \times I]} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u(x) \right) dx + \frac{2\eta}{3}. \end{aligned} \quad (2.3.24)$$

Thus, from (2.3.20), (2.3.24) and the fact that  $W$  is nonnegative, for all  $\varepsilon < \varepsilon_0 := \min\{\varepsilon'_0, \varepsilon''_0\}$  we have that

$$\int_{(a+Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u(x) \right) dx \geq \int_{(a+Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \nabla u(x) + \nabla v(x)) dx - \eta.$$

□

We now use this Proposition to derive the lower bound of the  $\Gamma$ -limit.

**Lemma 2.3.6.** *For all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ ,*

$$\bar{W}_{\text{hom}}(\bar{F}) \leq W_{\{\varepsilon_n\}}(\bar{F}).$$

*Proof.* Let  $\{w_n\} \subset W^{1,p}(Q' \times I; \mathbb{R}^3)$  be a sequence such that  $w_n \rightarrow 0$  in  $L^p(Q' \times I; \mathbb{R}^3)$ ,  $w_n = 0$  on  $\partial Q' \times I$  (see Lemma 2.6 in [35]),  $\{ |(\nabla_\alpha w_n|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n)|^p \}$  is equi-integrable (see Theorem 1.1 in [25]) and

$$W_{\{\varepsilon_n\}}(\bar{F}; Q') = \lim_{n \rightarrow +\infty} \int_{Q' \times I} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha w_n(x) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n(x) \right) dx.$$

From De La Vallée Poussin criterion (see e.g. Proposition 1.27 in [5]) there exists an increasing continuous function  $\varphi : [0, +\infty) \rightarrow [0, +\infty]$  satisfying  $\varphi(t)/t \rightarrow +\infty$  as  $t \rightarrow +\infty$  and such that

$$\sup_{n \in \mathbb{N}} \int_{Q' \times I} \varphi \left( \left| \left( \nabla_\alpha w_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n \right) \right|^p \right) dx \leq 1.$$

Changing variables it follows that

$$W_{\{\varepsilon_n\}}(\bar{F}) = \lim_{n \rightarrow +\infty} \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha z_n(x) \Big|_{\varepsilon_n} \nabla_3 z_n(x) \right) dx$$

and

$$\sup_{n \in \mathbb{N}} \frac{1}{T_n^2} \int_{(0, T_n)^2 \times I} \varphi(|\nabla z_n|^p) dx \leq 1, \quad (2.3.25)$$

where we set  $T_n := 1/\varepsilon_n$  and  $z_n(x) := T_n w_n(x_\alpha/T_n, x_3)$ ,  $z_n \in W^{1,p}((0, T_n)^2 \times I; \mathbb{R}^3)$  with  $z_n = 0$  on  $\partial(0, T_n)^2 \times I$ . For any  $n \in \mathbb{N}$  define  $I_n := \{1, \dots, \lfloor T_n \rfloor^2\}$ , and for  $i \in I_n$  take  $a_i^n \in \mathbb{Z}^2$  such that

$$\bigcup_{i \in I_n} (a_i^n + Q') \subseteq (0, T_n)^2. \quad (2.3.26)$$

Thus

$$W_{\{\varepsilon_n\}}(\bar{F}) \geq \limsup_{n \rightarrow +\infty} \frac{1}{2T_n^2} \sum_{i \in I_n} \int_{(a_i^n + Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha z_n(x) | \nabla_3 z_n(x) \right) dx. \quad (2.3.27)$$

Let  $M > 2$  and  $\eta > 0$ . For  $n \in \mathbb{N}$  define

$$I_n^M := \left\{ i \in I_n : \int_{(a_i^n + Q') \times I} \varphi(|\nabla z_n|^p) dx \leq M \right\}.$$

We note that for any  $M > 2$ , there exists  $n(M) \in \mathbb{N}$  such that for all  $n \geq n(M)$  sufficiently large so that  $T_n > M$ ,  $I_n^M \neq \emptyset$ . In fact, if not we may find  $M > 2$  and a subsequence  $n_k \in \mathbb{N}$  satisfying

$$\int_{(a_i^{n_k} + Q') \times I} \varphi(|\nabla z_{n_k}|^p) dx > M,$$

for all  $i \in I_{n_k}$ . Summation in  $i$  and (2.3.26) would yield to

$$\int_{(0, T_{n_k})^2 \times I} \varphi(|\nabla z_{n_k}|^p) dx > M \llbracket T_{n_k} \rrbracket^2$$

which is in contradiction with (2.3.25). We also note that in view of (2.3.25)

$$\#(I_n \setminus I_n^M) M \leq \sum_{i \in I_n \setminus I_n^M} \int_{(a_i^n + Q') \times I} \varphi(|\nabla z_n|^p) dx \leq \int_{(0, T_n)^2 \times I} \varphi(|\nabla z_n|^p) dx \leq T_n^2,$$

and so

$$\#(I_n \setminus I_n^M) \leq \frac{T_n^2}{M}. \quad (2.3.28)$$

By Proposition 2.3.5, there exists  $\varepsilon_0 \equiv \varepsilon_0(M, \eta)$  such that, for any  $n$  large enough satisfying  $0 \leq \varepsilon_n < \varepsilon_0$  and for any  $i \in I_n^M$ , we can find  $v_i^{n, M, \eta} \in W_0^{1,p}((a_i^n + Q') \times I; \mathbb{R}^3)$  with  $\|v_i^{n, M, \eta}\|_{L^p((a_i^n + Q') \times I; \mathbb{R}^3)} \leq \eta$  and

$$\begin{aligned} & \int_{(a_i^n + Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha z_n | \nabla_3 z_n \right) dx \\ & \geq \int_{(a_i^n + Q') \times I} W_{\text{hom}} \left( x_3, x_\alpha; \bar{F} + \nabla_\alpha (z_n + v_i^{n, M, \eta}) | \nabla_3 (z_n + v_i^{n, M, \eta}) \right) dx - \eta. \end{aligned}$$

Consequently, for  $n$  large enough

$$\begin{aligned}
 & \sum_{i \in I_n} \int_{(a_i^n + Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha z_n | \nabla_3 z_n \right) dx \\
 & \geq \sum_{i \in I_n^M} \int_{(a_i^n + Q') \times I} W_{\text{hom}} \left( x_3, x_\alpha; \bar{F} + \nabla_\alpha (z_n + v_i^{n,M,\eta}) | \nabla_3 (z_n + v_i^{n,M,\eta}) \right) dx \\
 & \quad - \eta \#(I_n^M).
 \end{aligned}$$

As  $\#(I_n^M) \leq \llbracket T_n \rrbracket^2$ , dividing by  $T_n^2$  and passing to the limit when  $n \rightarrow +\infty$  we obtain

$$\begin{aligned}
 & \limsup_{n \rightarrow +\infty} \frac{1}{T_n^2} \sum_{i \in I_n} \int_{(a_i^n + Q') \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha z_n | \nabla_3 z_n \right) dx + \eta \\
 & \geq \limsup_{n \rightarrow +\infty} \frac{1}{T_n^2} \sum_{i \in I_n^M} \int_{(a_i^n + Q') \times I} W_{\text{hom}} \left( x_3, x_\alpha; \bar{F} + \nabla_\alpha (z_n + v_i^{n,M,\eta}) | \nabla_3 (z_n + v_i^{n,M,\eta}) \right) dx. \quad (2.3.29)
 \end{aligned}$$

Hence, from (2.3.27) and (2.3.29),

$$\begin{aligned}
 & W_{\{\varepsilon_n\}}(\bar{F}) \geq \\
 & \limsup_{M \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{2T_n^2} \sum_{i \in I_n^M} \int_{(a_i^n + Q') \times I} W_{\text{hom}} \left( x_3, x_\alpha; \bar{F} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta} \right) dx \quad (2.3.30)
 \end{aligned}$$

where  $\phi^{n,M,\eta} \in W^{1,p}((0; T_n)^2 \times I; \mathbb{R}^3)$  is defined by

$$\phi^{n,M,\eta}(x) := \begin{cases} z_n(x) + v_i^{n,M,\eta}(x) & \text{if } x \in (a_i^n + Q') \times I \text{ and } i \in I_n^M, \\ z_n(x) & \text{otherwise} \end{cases}$$

and satisfies  $\phi^{n,M,\eta} = 0$  on  $\partial(0, T_n)^2 \times I$ . Now, in view of the definition of  $\phi^{n,M,\eta}$ , the  $p$ -growth condition (2.2.4) and (2.3.28),

$$\begin{aligned}
 & \frac{1}{T_n^2} \sum_{i \in I_n \setminus I_n^M} \int_{(a_i^n + Q') \times I} W_{\text{hom}} \left( x_3, x_\alpha; \bar{F} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta} \right) dx \\
 & = \frac{1}{T_n^2} \sum_{i \in I_n \setminus I_n^M} \int_{(a_i^n + Q') \times I} W_{\text{hom}} \left( x_3, x_\alpha; \bar{F} + \nabla_\alpha z_n | \nabla_3 z_n \right) dx \\
 & \leq \frac{C}{T_n^2} \sum_{i \in I_n \setminus I_n^M} \int_{(a_i^n + Q') \times I} (1 + |\nabla z_n|^p) dx \\
 & \leq \frac{C}{M} + \beta \int_{\bigcup_{i \in I_n \setminus I_n^M} \frac{1}{T_n} (a_i^n + Q') \times I} \left| \left( \nabla_\alpha w_n \Big| \frac{1}{\varepsilon_n} \nabla_3 w_n \right) \right|^p dx. \quad (2.3.31)
 \end{aligned}$$

By (2.3.28),  $\mathcal{L}^2\left(\bigcup_{i \in I_n \setminus I_n^M} \frac{1}{T_n}(a_i^n + Q')\right) \leq 1/M$ , consequently, in view of the equi-integrability of  $\left\{ \left| \left( \nabla_\alpha w_n \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n \right) \right|^p \right\}$ , we get

$$\sup_{n \in \mathbb{N}} \int_{\bigcup_{i \in I_n \setminus I_n^M} \frac{1}{T_n}(a_i^n + Q') \times I} \left| \left( \nabla_\alpha w_n \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n \right) \right|^p dx \xrightarrow{M \rightarrow +\infty} 0. \quad (2.3.32)$$

Therefore, (2.3.30), (2.3.31) and (2.3.32) imply

$$\begin{aligned} & W_{\{\varepsilon_n\}}(\overline{F}) \\ & \geq \limsup_{M \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{2T_n^2} \sum_{i \in I_n} \int_{(a_i^n + Q') \times I} W_{\text{hom}}(x_3, x_\alpha; \overline{F} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta}) dx \\ & = \limsup_{M \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} W_{\text{hom}}(x_3, x_\alpha; \overline{F} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta}) dx, \end{aligned} \quad (2.3.33)$$

because by definition of  $\phi^{n,M,\eta}$  and the  $p$ -growth property of  $W_{\text{hom}}$ , (2.2.4),

$$\begin{aligned} & \frac{1}{T_n^2} \int_{(0, T_n)^2 \setminus [\bigcup_{i \in I_n} (a_i^n + Q')] \times I} W_{\text{hom}}(x_3, x_\alpha; \overline{F} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta}) dx \\ & = \frac{1}{T_n^2} \int_{(0, T_n)^2 \setminus [\bigcup_{i \in I_n} (a_i^n + Q')] \times I} W_{\text{hom}}(x_3, x_\alpha; \overline{F} + \nabla_\alpha z_n | \nabla_3 z_n) dx \\ & \leq \frac{C}{T_n^2} \int_{(0, T_n)^2 \setminus [\bigcup_{i \in I_n} (a_i^n + Q')] \times I} (1 + |\nabla z_n|^p) dx \\ & = C \int_{Q' \setminus [\bigcup_{i \in I_n} \frac{1}{T_n}(a_i^n + Q')] \times I} \left( 1 + \left| \left( \nabla_\alpha w_n \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n \right) \right|^p \right) dx. \end{aligned}$$

and consequently, the equi-integrability of  $\left\{ \left| \left( \nabla_\alpha w_n \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n \right) \right|^p \right\}$  and the fact that

$$\mathcal{L}^2 \left( Q' \setminus \left[ \bigcup_{i \in I_n} \frac{1}{T_n}(a_i^n + Q') \right] \right) \rightarrow 0$$

as  $n \rightarrow +\infty$  yield

$$\limsup_{M \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{1}{T_n^2} \int_{(0, T_n)^2 \setminus [\bigcup_{i \in I_n} (a_i^n + Q')] \times I} W_{\text{hom}}(x_3, x_\alpha; \overline{F} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta}) dx = 0.$$

Hence by (2.3.33) and (2.3.2) we get that  $W_{\{\varepsilon_n\}}(\overline{F}) \geq \overline{W}_{\text{hom}}(\overline{F})$ .  $\square$

### The upper bound

Let us now prove the converse inequality.

**Lemma 2.3.7.** *For all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ ,*

$$\bar{W}_{\text{hom}}(\bar{F}) \geq W_{\{\varepsilon_n\}}(\bar{F}).$$

*Proof.* In view of (2.3.2), for  $\delta > 0$  fixed take  $T \equiv T_\delta \in \mathbb{N}$ , with  $T_\delta \rightarrow +\infty$  as  $\delta \rightarrow 0$ , and  $\phi \equiv \phi_\delta \in W^{1,p}((0, T)^2 \times I; \mathbb{R}^3)$  be such that  $\phi = 0$  on  $\partial(0, T)^2 \times I$  and

$$\bar{W}_{\text{hom}}(\bar{F}) + \delta \geq \int_{(0, T)^2 \times I} W_{\text{hom}}(x_3, x_\alpha; \bar{F} + \nabla_\alpha \phi(x) | \nabla_\alpha \phi(x)) dx. \quad (2.3.34)$$

By Theorem 1.1 in [16] (with  $f(y, z; F) = W(y_3, y_\alpha, z_3, z_\alpha; F)$ ) and, for instance, Theorem 21.1 in [45], there exists  $\{\phi_n\} \subset W_0^{1,p}((0, T)^2 \times I; \mathbb{R}^3)$  with  $\phi_n \rightarrow \phi$  in  $L^p((0, T)^2 \times I; \mathbb{R}^3)$  such that

$$\begin{aligned} & \int_{(0, T)^2 \times I} W_{\text{hom}}(x_3, x_\alpha; \bar{F} + \nabla_\alpha \phi(x) | \nabla_3 \phi(x)) dx \\ &= \lim_{n \rightarrow +\infty} \int_{(0, T)^2 \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx. \end{aligned} \quad (2.3.35)$$

Further, in view of the Decomposition Lemma (see [61]), we can assume – upon extracting a subsequence –  $\{|\nabla \phi_n|^p\}$  to be equi-integrable. Fix  $n \in \mathbb{N}$  such that  $\varepsilon_n \ll 1$ . For all  $i \in \mathbb{Z}^2$  let  $a_i^n \in \varepsilon_n \mathbb{Z}^2 \cap (i(T+1) + [0, \varepsilon_n]^2)$  (uniquely defined). In particular, the cubes  $a_i^n + (0, T)^2$  are not overlapping because if  $i, j \in \mathbb{Z}^2$  with  $i \neq j$ , then  $|i - j| \geq 1$  and thus  $|a_i^n - a_j^n| > T$ . Set

$$\tilde{\phi}_n(x) := \begin{cases} \phi_n(x_\alpha - a_i^n, x_3) & \text{if } x \in (a_i^n + (0, T)^2) \times I \text{ and } i \in \mathbb{Z}^2, \\ 0 & \text{otherwise,} \end{cases}$$

then  $\tilde{\phi}_n \in W^{1,p}(\mathbb{R}^2 \times I; \mathbb{R}^3)$ . Let  $I_n := \{i \in \mathbb{Z}^2 : (0, T/\varepsilon_n)^2 \cap (a_i^n + (0, T)^2) \neq \emptyset\}$ . Note that

$$\#(I_n) \leq \left( \left\lceil \frac{1}{\varepsilon_n} \right\rceil + 1 \right)^2. \quad (2.3.36)$$

If  $\psi_n(x) := \varepsilon_n \tilde{\phi}_n(x_\alpha/\varepsilon_n, x_3)$  then  $\psi_n \rightarrow 0$  in  $L^p((0, T)^2 \times I; \mathbb{R}^3)$ , as  $n \rightarrow +\infty$ , because

$$\begin{aligned}
 \int_{(0, T)^2 \times I} |\psi_n(x)|^p dx &= \varepsilon_n^p \int_{(0, T)^2 \times I} \left| \tilde{\phi}_n \left( \frac{x_\alpha}{\varepsilon_n}, x_3 \right) \right|^p dx \\
 &= \varepsilon_n^{p+2} \int_{(0, T/\varepsilon_n)^2 \times I} |\tilde{\phi}_n(x)|^p dx \\
 &\leq \varepsilon_n^{p+2} \sum_{i \in I_n} \int_{(a_i^n + (0, T)^2) \times I} |\phi_n(x_\alpha - a_i^n, x_3)|^p dx \\
 &= \varepsilon_n^{p+2} \#(I_n) \int_{(0, T)^2 \times I} |\phi_n(x)|^p dx \\
 &\leq \varepsilon_n^{p+2} \left( \left\lfloor \frac{1}{\varepsilon_n} \right\rfloor + 1 \right)^2 \int_{(0, T)^2 \times I} |\phi_n(x)|^p dx \rightarrow 0,
 \end{aligned}$$

where we have used the fact that  $\tilde{\phi}_n \equiv 0$  on  $[(0, T/\varepsilon_n)^2 \setminus \bigcup_{i \in I_n} (a_i^n + (0, T)^2)] \times I$  and that the sequence  $\{\phi_n\}$  is uniformly bounded  $L^p((0, T)^2 \times I; \mathbb{R}^3)$ . Consequently, the  $p$ -growth condition  $(A_4)$  implies that

$$\begin{aligned}
 &\mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; (0, T)^2) \\
 &\leq \liminf_{n \rightarrow +\infty} \int_{(0, T)^2 \times I} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha \psi_n(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 \psi_n(x) \right) dx \\
 &= \liminf_{n \rightarrow +\infty} \int_{(0, T)^2 \times I} W \left( x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha \tilde{\phi}_n \left( \frac{x_\alpha}{\varepsilon_n}, x_3 \right) \middle| \nabla_3 \tilde{\phi}_n \left( \frac{x_\alpha}{\varepsilon_n}, x_3 \right) \right) dx \\
 &= \liminf_{n \rightarrow +\infty} \varepsilon_n^2 \int_{(0, T/\varepsilon_n)^2 \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha \tilde{\phi}_n(x) \middle| \nabla_3 \tilde{\phi}_n(x) \right) dx \\
 &\leq \liminf_{n \rightarrow +\infty} \varepsilon_n^2 \left[ \sum_{i \in I_n} \int_{(a_i^n + (0, T)^2) \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha \tilde{\phi}_n(x) \middle| \nabla_3 \tilde{\phi}_n(x) \right) dx \right. \\
 &\quad \left. + 2\beta(1 + |\bar{F}|^p) \mathcal{L}^2 \left( \left(0, \frac{T}{\varepsilon_n}\right)^2 \setminus \bigcup_{i \in I_n} (a_i^n + (0, T)^2) \right) \right]. \quad (2.3.37)
 \end{aligned}$$

Since  $a_i^n \in i(T+1) + [0, \varepsilon_n)^2$ , for  $n$  large enough it follows that  $a_i^n + (0, T)^2 \subset i(T+1) + (0, T+1)^2$ . Hence

$$\begin{aligned}
 &\# \left( \left\{ i \in I_n : a_i^n + (0, T)^2 \subset \left(0, \frac{T}{\varepsilon_n}\right)^2 \right\} \right) \\
 &\geq \# \left( \left\{ i \in I_n : i(T+1) + (0, T+1)^2 \subset \left(0, \frac{T}{\varepsilon_n}\right)^2 \right\} \right) \\
 &= \left\lfloor \frac{T}{\varepsilon_n(T+1)} \right\rfloor^2.
 \end{aligned}$$



Consequently, we can estimate the Lebesgue measure in (2.3.37) by

$$\begin{aligned}
 & \mathcal{L}^2 \left( \left(0, \frac{T}{\varepsilon_n}\right)^2 \setminus \bigcup_{i \in I_n} (a_i^n + (0, T)^2) \right) \\
 & \leq \mathcal{L}^2 \left( \left(0, \frac{T}{\varepsilon_n}\right)^2 \setminus \bigcup_{i \in I_n: a_i^n + (0, T)^2 \subset (0, T/\varepsilon_n)^2} (a_i^n + (0, T)^2) \right) \\
 & = \frac{T^2}{\varepsilon_n^2} - \# \left( \left\{ i \in I_n : a_i^n + (0, T)^2 \subset \left(0, \frac{T}{\varepsilon_n}\right)^2 \right\} \right) T^2 \\
 & \leq \frac{T^2}{\varepsilon_n^2} \left( 1 - \varepsilon_n^2 \left\lfloor \frac{T}{\varepsilon_n(T+1)} \right\rfloor^2 \right).
 \end{aligned}$$

Letting  $n$  tend to  $+\infty$  yields that

$$\begin{aligned}
 & \mathcal{W}_{\{\varepsilon_n\}}(\bar{F}; (0, T)^2) \\
 & \leq \liminf_{n \rightarrow +\infty} \varepsilon_n^2 \sum_{i \in I_n} \int_{(a_i^n + (0, T)^2) \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha \tilde{\phi}_n(x) | \nabla_3 \tilde{\phi}_n(x) \right) dx \\
 & \quad + 2\beta(1 + |\bar{F}|^p) T^2 \left( 1 - \left( \frac{T}{T+1} \right)^2 \right), \tag{2.3.38}
 \end{aligned}$$

By a change of variables, for all  $i \in I_n$

$$\begin{aligned}
 & \int_{(a_i^n + (0, T)^2) \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha \tilde{\phi}_n(x) | \nabla_3 \tilde{\phi}_n(x) \right) dx \\
 & = \int_{(0, T)^2 \times I} W \left( x_3, x_\alpha + a_i^n, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha + a_i^n}{\varepsilon_n}; \bar{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx \\
 & = \int_{(0, T)^2 \times I} W \left( x_3, x_\alpha + a_i^n - i(T+1), \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx, \tag{2.3.39}
 \end{aligned}$$

where we have used  $(A_3)$ , the fact that  $T \in \mathbb{N}$  and  $a_i^n/\varepsilon_n \in \mathbb{Z}^2$ . In order to apply a uniform continuity argument and recover (2.3.35) we define  $R_n^\lambda := \{x \in (0, T)^2 \times I : |(\bar{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x))| \leq \lambda\}$  and we observe that, according to Chebyshev's inequality,

$$\mathcal{L}^3([(0, T)^2 \times I] \setminus R_n^\lambda) \leq C/\lambda^p,$$

for some constant  $C > 0$  which does not depend on  $n$  and  $\lambda$ . Then by (2.3.38) and (2.3.39)

$$\begin{aligned}
 & \mathcal{W}_{\{\varepsilon_n\}}(\overline{F}; (0, T)^2) \\
 & \leq \liminf_{\lambda \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i \in I_n} \varepsilon_n^2 \int_{R_n^\lambda} W \left( x_3, x_\alpha + a_i^n - i(T+1), \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx \\
 & \quad + 2\beta(1 + |\overline{F}|^p) T^2 \left( 1 - \left( \frac{T}{T+1} \right)^2 \right). \tag{2.3.40}
 \end{aligned}$$

Indeed, the  $p$ -growth condition (A<sub>4</sub>), the equi-integrability of  $\{|\nabla \phi_n|^p\}$  and (2.3.36) imply that

$$\begin{aligned}
 & \limsup_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i \in I_n} \varepsilon_n^2 \int_{[(0, T)^2 \times I] \setminus R_n^\lambda} W \left( x_3, x_\alpha + a_i^n - i(T+1), \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx \\
 & \leq \limsup_{\lambda \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \beta \varepsilon_n^2 \#(I_n) \int_{[(0, T)^2 \times I] \setminus R_n^\lambda} (1 + |(\overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x))|^p) dx = 0.
 \end{aligned}$$

Arguing as in the proof of Lemma 2.3.3, for a.e.  $x_3 \in I$ , the function  $W(x_3, \cdot, \cdot; \cdot)$  is uniformly continuous on  $\mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$ , thus for all  $x \in R_n^\lambda$ ,

$$\begin{aligned}
 & \left| W \left( x_3, x_\alpha + a_i^n - i(T+1), \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) \right. \\
 & \quad \left. - W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) \right| \\
 & \leq \omega_\lambda(x_3, |a_i^n - i(T+1)|) \leq \omega_\lambda(x_3, \varepsilon_n). \tag{2.3.41}
 \end{aligned}$$

In view of (2.3.41), (2.3.36), and Beppo-Levi's Monotone Convergence Theorem we get

$$\begin{aligned}
 & \liminf_{\lambda \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \varepsilon_n^2 \sum_{i \in I_n} \int_{R_n^\lambda} W \left( x_3, x_\alpha + a_i^n - i(T+1), \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx \\
 & \leq \liminf_{\lambda \rightarrow +\infty} \liminf_{n \rightarrow +\infty} (1 + \varepsilon_n^2) \left\{ \int_{R_n^\lambda} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx \right. \\
 & \quad \left. + T^2 \int_{-1}^1 \omega_\lambda(x_3, \varepsilon_n) dx_3 \right\} \\
 & \leq \liminf_{\lambda \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \int_{R_n^\lambda} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx \\
 & \leq \liminf_{n \rightarrow +\infty} \int_{(0, T)^2 \times I} W \left( x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \overline{F} + \nabla_\alpha \phi_n(x) | \nabla_3 \phi_n(x) \right) dx.
 \end{aligned}$$

Consequently by (2.3.34), (2.3.35), (2.3.40) and Lemma 2.3.4

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{F}) \leq \overline{W}_{\text{hom}}(\overline{F}) + \delta + 2\beta(1 + |\overline{F}|^p) \left( 1 - \left( \frac{T}{T+1} \right)^2 \right).$$

The result follows by letting  $\delta$  tend to zero.  $\square$

*Proof of Theorem 2.3.1.* From Lemma 2.3.6 and Lemma 2.3.7, we conclude that  $\overline{W}_{\text{hom}}(\overline{F}) = W_{\{\varepsilon_n\}}(\overline{F})$  for all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ . As a consequence,  $\mathcal{W}_{\{\varepsilon_n\}}(u; A) = \mathcal{W}_{\text{hom}}(u; A)$  for all  $A \in \mathcal{A}_0$  and all  $u \in W^{1,p}(A; \mathbb{R}^3)$ . Since the  $\Gamma$ -limit does not depend upon the extracted subsequence, Proposition 8.3 in [45] implies that the whole sequence  $\mathcal{W}_\varepsilon(\cdot; A)$   $\Gamma$ -converges to  $\mathcal{W}_{\text{hom}}(\cdot; A)$  for the strong  $L^p(A \times I; \mathbb{R}^3)$ -topology.  $\square$

## 2.4 The general case

### 2.4.1 Existence and integral representation of the $\Gamma$ -limit

To prove Theorem 2.1.1 it is convenient to localize the functionals  $\mathcal{W}_\varepsilon$  in (2.1.4) on the class of all bounded open subsets of  $\omega$ , denoted by  $\mathcal{A}(\omega)$ . For each  $\varepsilon > 0$  we consider the family of functionals  $\mathcal{W}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega) \rightarrow [0, +\infty]$  defined by

$$\mathcal{W}_\varepsilon(u; A) := \begin{cases} \int_{A \times I} W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A \times I; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.4.1)$$

Given  $\{\varepsilon_j\} \searrow 0^+$  and  $A \in \mathcal{A}(\omega)$  we define the  $\Gamma$ -lower limit of  $\mathcal{W}_{\varepsilon_j}(\cdot; A)$  with respect to the  $L^p(A \times I; \mathbb{R}^3)$ -topology by

$$\mathcal{W}_{\{\varepsilon_j\}}(u; A) := \inf_{\{u_j\}} \left\{ \liminf_{j \rightarrow +\infty} \mathcal{W}_{\varepsilon_j}(u_j; A) : u_j \rightarrow u \text{ in } L^p(A \times I; \mathbb{R}^3) \right\}$$

for all  $u \in L^p(\Omega; \mathbb{R}^3)$ . Our main objective is to show that

$$\mathcal{W}_{\{\varepsilon_j\}} = \mathcal{W}_{\text{hom}} \quad (2.4.2)$$

where  $\mathcal{W}_{\text{hom}} : L^p(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega) \rightarrow [0, +\infty]$  is given by

$$\mathcal{W}_{\text{hom}}(u; A) = \begin{cases} 2 \int_A \overline{W}_{\text{hom}}(x_\alpha; \nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

By hypothesis  $(A_4)$  it follows that  $\mathcal{W}_{\{\varepsilon_j\}}(u; A) = +\infty$  for each  $A \in \mathcal{A}(\omega)$  whenever  $u \in L^p(\Omega; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$ . As a consequence of Theorem 2.5 in [35], given  $\{\varepsilon_j\} \searrow 0^+$  there exists a subsequence  $\{\varepsilon_{j_n}\} \equiv \{\varepsilon_n\}$  of  $\{\varepsilon_j\}$  such that, for each  $A \in \mathcal{A}(\omega)$ , the functional  $\mathcal{W}_{\{\varepsilon_n\}}(\cdot; A)$  is the  $\Gamma$ -limit of  $\mathcal{W}_{\varepsilon_n}(\cdot; A)$  for the strong  $L^p(A \times I; \mathbb{R}^3)$ . Moreover given  $u \in W^{1,p}(A; \mathbb{R}^3)$

$$\mathcal{W}_{\{\varepsilon_n\}}(u; A) = 2 \int_A W_{\{\varepsilon_n\}}(x_\alpha; \nabla_\alpha u(x_\alpha)) dx_\alpha,$$

for some Carathéodory function  $W_{\{\varepsilon_n\}} : \omega \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ .

Accordingly, to prove equality (2.4.2) it suffices to show that  $W_{\{\varepsilon_n\}}(x_\alpha; \overline{F}) = \overline{W}_{\text{hom}}(x_\alpha; \overline{F})$  for a.e.  $x_\alpha \in \omega$  and all  $\overline{F} \in \mathbb{R}^{3 \times 2}$ , which allow us to work with affine functions instead of with general Sobolev functions.

The following proposition, that is of use in the sequel, allows us to extend continuously Carathéodory integrands. It relies on Scorza-Dragoni's Theorem (see [54]) and on Tietze's Extension Theorem (see Theorem 3.1 in [53]).

**Proposition 2.4.1.** *Let  $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  satisfying  $(A_1)$ - $(A_4)$ . Then for any  $m \in \mathbb{N}$ , there exists a compact set  $C_m \subset \Omega$  and a continuous function  $W^m : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  such that  $W^m(x, \cdot, \cdot; \cdot) = W(x, \cdot, \cdot; \cdot)$  for all  $x \in C_m$  and*

$$\mathcal{L}^3(\Omega \setminus C_m) < \frac{1}{m}. \quad (2.4.3)$$

Moreover,

- $y_\alpha \mapsto W^m(x, y_\alpha, y_3, z_\alpha; F)$  is  $Q'$ -periodic for all  $(z_\alpha, y_3, F) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 3}$  and a.e.  $x \in \Omega$ ,
  - $(z_\alpha, y_3) \mapsto W^m(x, y_\alpha, y_3, z_\alpha; F)$  is  $Q$ -periodic for all  $(y_\alpha, F) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$  and a.e.  $x \in \Omega$ ;
- and for some  $\beta > 0$ , we have

$$-\beta \leq W^m(x, y, z_\alpha; F) \leq \beta(1 + |F|^p) \quad \text{for all } (y, z_\alpha, F) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \text{ and a.e. } x \in \Omega. \quad (2.4.4)$$

*Proof.* By Scorza-Dragoni's Theorem for any  $m \in \mathbb{N}$  there exists a compact set  $C_m \subset \Omega$  with  $\mathcal{L}^3(\Omega \setminus C_m) < 1/m$  such that  $W$  is continuous on  $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ . Since  $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$  is a closed set, according to Tietze's Extension Theorem one can extend  $W$  into a continuous function  $W^m$  outside  $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ . By the construction of  $W^m$  it can be seen that it satisfies the same periodicity and growth condition than  $W$  and that it is bounded from below by  $-\beta$ .  $\square$

Note that the above result improve Lemma 4.1 in [13] in which only a separately continuous function was obtained.

### 2.4.2 Characterization of the $\Gamma$ -limit

Let  $\overline{F} \in \mathbb{R}^{3 \times 2}$  and  $x_0 \in \omega$  be a Lebesgue point of both  $W_{\{\varepsilon_n\}}(\cdot; \overline{F})$  and  $\overline{W}_{\text{hom}}(\cdot; \overline{F})$ . Consider  $r > 0$  small enough so that  $Q'_r(x_0) \in \mathcal{A}(\omega)$ .

#### The lower bound

**Lemma 2.4.2.** *For all  $\overline{F} \in \mathbb{R}^{3 \times 2}$  and a.e.  $x_0 \in \omega$ ,*

$$W_{\{\varepsilon_n\}}(x_0; \overline{F}) \geq \overline{W}_{\text{hom}}(x_0; \overline{F}).$$

*Proof.* By Theorem 1.1 in [25] we can find a sequence  $\{u_n\} \subset W^{1,p}(Q'_r(x_0) \times I; \mathbb{R}^3)$  with  $u_n \rightarrow 0$  in  $L^p(Q'_r(x_0) \times I; \mathbb{R}^3)$ , such that the sequence of scaled gradients  $\{(\nabla_\alpha u_n | \frac{1}{\varepsilon_n} \nabla_3 u_n)\}$  is  $p$ -equi-integrable and

$$\begin{aligned} \mathcal{W}_{\{\varepsilon_n\}}(\bar{F} \cdot; Q'_r(x_0)) &= 2 \int_{Q'_r(x_0)} W_{\{\varepsilon_n\}}(x_\alpha; \bar{F}) dx_\alpha \\ &= \lim_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n(x) \right) dx. \end{aligned}$$

To simplify notation, set

$$F_n(x) := \left( \bar{F} + \nabla_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n(x) \right).$$

Given  $m \in \mathbb{N}$  let  $C_m$  and  $W^m$  be given by Proposition 2.4.1. Then since  $W \geq 0$  and  $W = W^m$  on  $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$  we get

$$\mathcal{W}_{\{\varepsilon_n\}}(\bar{F} \cdot; Q'_r(x_0)) \geq \limsup_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{[Q'_r(x_0) \times I] \cap C_m} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx.$$

By the  $p$ -growth condition (2.4.4), the equi-integrability of  $\{|F_n|^p\}$  and relation (2.4.3), we obtain

$$\int_{[Q'_r(x_0) \times I] \setminus C_m} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx \leq \beta \int_{[Q'_r(x_0) \times I] \setminus C_m} (1 + |F_n(x)|^p) dx \xrightarrow{m \rightarrow +\infty} 0,$$

uniformly with respect to  $n \in \mathbb{N}$ . Then, we get that

$$\mathcal{W}_{\{\varepsilon_n\}}(\bar{F} \cdot; Q'_r(x_0)) \geq \limsup_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx.$$

For any  $h \in \mathbb{N}$ , we split  $Q'_r(x_0)$  into  $h^2$  disjoint cubes  $Q'_{i,h}$  of side length  $r/h$  so that

$$Q'_r(x_0) = \bigcup_{i=1}^{h^2} Q'_{i,h}$$

and

$$\begin{aligned} \mathcal{W}_{\{\varepsilon_n\}}(\bar{F} \cdot; Q'_r(x_0)) &\geq \limsup_{m \rightarrow +\infty} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h} \times I} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx \\ &\geq \limsup_{\lambda \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx \end{aligned}$$

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where, given  $\lambda > 0$ , we define  $R_n^\lambda := \{x \in Q'_r(x_0) \times I : |F_n(x)| \leq \lambda\}$ . Note that thanks to Chebyshev's inequality,

$$\mathcal{L}^3([Q'_r(x_0) \times I] \setminus R_n^\lambda) \leq C/\lambda^p, \quad (2.4.5)$$

for some constant independent of  $\lambda$  and  $n$ . Since  $W^m$  is continuous and separately periodic it is in particular uniformly continuous on  $\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$ . Let  $\omega_{m,\lambda} : [0, +\infty) \rightarrow [0, +\infty)$  the modulus of continuity of  $W^m$  on  $\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$ . Then, for every  $(x_\alpha, x_3) \in [Q'_{i,h} \times I] \cap R_n^\lambda$  and every  $x'_\alpha \in Q'_{i,h}$ ,

$$\left| W^m \left( x_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) - W^m \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) \right| \leq \omega_{m,\lambda} \left( \frac{\sqrt{2}r}{h} \right).$$

We get, after integration in  $(x_\alpha, x_3, x'_\alpha)$  and summation,

$$\begin{aligned} & \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \left\{ \int_{R_n^\lambda \cap [Q'_{i,h} \times I]} \left| W^m \left( x_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) \right. \right. \\ & \qquad \qquad \qquad \left. \left. - W^m \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) \right| dx \right\} dx'_\alpha \\ & \leq 2r^2 \omega_{m,\lambda} \left( \frac{\sqrt{2}r}{h} \right) \xrightarrow{h \rightarrow +\infty} 0, \end{aligned}$$

uniformly with respect to  $n \in \mathbb{N}$ . Hence,

$$\begin{aligned} & \mathcal{W}_{\{\varepsilon_n\}}(\overline{F}; Q'_r(x_0)) \geq \\ & \limsup_{\lambda \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{Q'_{i,h}} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda} W^m \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) dx dx'_\alpha. \end{aligned}$$

Define the following sets which depend on all parameters  $(m, \lambda, i, h, n)$  :

$$\begin{cases} E := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x'_\alpha, x_3) \in C_m \text{ and } (x_\alpha, x_3) \in R_n^\lambda\}, \\ E_1 := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x'_\alpha, x_3) \notin C_m \text{ and } (x_\alpha, x_3) \in R_n^\lambda\}, \\ E_2 := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x_\alpha, x_3) \notin R_n^\lambda\}, \end{cases}$$

and note that  $Q'_{i,h} \times Q'_{i,h} \times I = E \cup E_1 \cup E_2$ . Since  $W$  and  $W^m$  coincide on  $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$ , we have

$$\begin{aligned}
 & \mathcal{W}_{\{\varepsilon_n\}}(\overline{F}\cdot; Q'_r(x_0)) \\
 & \geq \limsup_{\lambda \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_E W^m \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) dx dx'_\alpha \\
 & = \limsup_{\lambda \rightarrow +\infty} \limsup_{m \rightarrow +\infty} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_E W \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) dx dx'_\alpha. \quad (2.4.6)
 \end{aligned}$$

We will prove that the corresponding terms over  $E_1$  and  $E_2$  are zero. Indeed, in view of (2.4.3) and the  $p$ -growth condition  $(A_4)$ ,

$$\begin{aligned}
 & \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{E_1} W \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) dx dx'_\alpha \\
 & \leq \sum_{i=1}^{h^2} \frac{h^2}{r^2} \mathcal{L}^2(Q'_{i,h}) \mathcal{L}^3([Q'_{i,h} \times I] \setminus C_m) \beta(1 + \lambda^p) \\
 & < \frac{\beta(1 + \lambda^p)}{m} \xrightarrow{m \rightarrow +\infty} 0, \quad (2.4.7)
 \end{aligned}$$

uniformly in  $(n, h)$ . The bound from above in  $(A_4)$ , the equi-integrability of  $\{|F_n|^p\}$  and (2.4.5) imply that

$$\begin{aligned}
 & \sum_{i=1}^{h^2} \frac{h^2}{r^2} \int_{E_2} W \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x_\alpha, x_3) \right) dx dx'_\alpha \\
 & \leq \sum_{i=1}^{h^2} \frac{h^2}{r^2} \mathcal{L}^2(Q'_{i,h}) \beta \int_{[Q'_{i,h} \times I] \setminus R_n^\lambda} (1 + |F_n|^p) dx \\
 & = \beta \int_{[Q'_r(x_0) \times I] \setminus R_n^\lambda} (1 + |F_n|^p) dx \xrightarrow{\lambda \rightarrow +\infty} 0, \quad (2.4.8)
 \end{aligned}$$

uniformly in  $(m, n, h)$ . Thus, in view of (2.4.6), (2.4.7), (2.4.8), Fatou's Lemma yields

$$\begin{aligned}
 & \mathcal{W}_{\{\varepsilon_n\}}(\overline{F}\cdot; Q'_r(x_0)) \\
 & \geq \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{h^2}{r^2} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \int_{Q'_{i,h} \times I} W \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx dx'_\alpha \\
 & \geq \limsup_{h \rightarrow +\infty} \frac{h^2}{r^2} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n(x) \right) dx dx'_\alpha. \quad (2.4.9)
 \end{aligned}$$

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We now fix  $x'_\alpha \in Q'_{i,h}$  such that  $\overline{W}_{\text{hom}}(x'_\alpha, x_3, y, z_\alpha; \overline{F})$  is well defined, then by Theorem 2.3.1 we get that

$$\liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W \left( x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{F} + \nabla_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n} \nabla_3 u_n(x)} \right) dx \geq 2 \frac{r^2}{h^2} \overline{W}_{\text{hom}}(x'_\alpha; \overline{F}). \quad (2.4.10)$$

Gathering (2.4.9) and (2.4.10), we get

$$\int_{Q'_r(x_0)} W_{\{\varepsilon_n\}}(x_\alpha; \overline{F}) dx_\alpha \geq \int_{Q'_r(x_0)} \overline{W}_{\text{hom}}(x'_\alpha; \overline{F}) dx'_\alpha.$$

As a consequence the claim follows by the choice of  $x_0$ , after dividing the previous inequality by  $r^2$  and letting  $r \rightarrow 0$ .  $\square$

### The upper bound

**Lemma 2.4.3.** *For all  $\overline{F} \in \mathbb{R}^{3 \times 2}$  and a.e.  $x_0 \in \omega$ ,*

$$W_{\{\varepsilon_n\}}(x_0; \overline{F}) \leq \overline{W}_{\text{hom}}(x_0; \overline{F}).$$

*Proof.* For every  $m \in \mathbb{N}$ , consider the set  $C_m$  and the function  $W^m$  given by Proposition 2.4.1, and define  $\overline{(W^m)}_{\text{hom}}$  and  $(W^m)_{\text{hom}}$  as (2.1.5) and (2.1.6), with  $W^m$  in place of  $W$ . Applying Lemma 2.2.2 with still  $W^m$  instead of  $W$ , it follows that  $\overline{(W^m)}_{\text{hom}}$  is a Carathéodory integrand. Thus, by Scorza-Dragoni's Theorem, for fixed  $\eta > 0$  and any  $m \in \mathbb{N}$  there exists a compact subset  $K_\eta^m$  of  $\omega$  with

$$\mathcal{L}^2(\omega \setminus K_\eta^m) \leq \eta \quad (2.4.11)$$

and such that  $\overline{(W^m)}_{\text{hom}} : K_\eta^m \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  is continuous.

**Step 1.** We first state that,

$$2 \liminf_{m \rightarrow +\infty} \int_{Q'_r(x_0)} \overline{(W^m)}_{\text{hom}}(x_\alpha; \overline{F}) dx_\alpha \geq 2 \int_{Q'_r(x_0)} W_{\{\varepsilon_n\}}(x_\alpha, \overline{F}) dx_\alpha. \quad (2.4.12)$$

To show this inequality we first decompose  $Q'_r(x_0)$  into  $h^2$  small disjoint cubes  $Q'_{i,h}$  such that

$$Q'_r(x_0) = \bigcup_{i=1}^{h^2} Q'_{i,h} \quad \text{and} \quad \mathcal{L}^2(Q'_{i,h}) = (r/h)^2.$$

Let

$$I_{h,\eta}^m := \{i \in \{1, \dots, h^2\} : K_\eta^m \cap Q'_{i,h} \neq \emptyset\}.$$

For  $i \in I_{h,\eta}^m$  choose  $x_i^{h,\eta,m} \in K_\eta^m \cap Q'_{i,h}$ . By Theorem 2.3.1 together with Lemma 2.6 in [35] there exists a sequence  $\{u_i^{n,h,\eta,m}\} \subset W^{1,p}(Q'_{i,h} \times I; \mathbb{R}^3)$  with  $u_i^{n,h,\eta,m} = 0$  on  $\partial Q'_{i,h} \times I$ ,  $u_i^{n,h,\eta,m} \xrightarrow[n \rightarrow +\infty]{} 0$  in  $L^p(Q'_{i,h} \times I; \mathbb{R}^3)$ , and such that



$$\begin{aligned}
 & 2 \int_{Q'_{i,h}} \overline{(W^m)_{\text{hom}}(x_i^{h,\eta,m}; \bar{F})} dx_\alpha \\
 &= \lim_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W^m \left( x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha u_i^{n,h,\eta,m} \Big| \frac{1}{\varepsilon_n} \nabla_3 u_i^{n,h,\eta,m} \right) dx.
 \end{aligned}$$

Setting

$$u_n^{\eta,m}(x) := \begin{cases} u_i^{n,h,\eta,m}(x) & \text{if } x_\alpha \in Q'_{i,h} \text{ and } i \in I_{h,\eta}^m, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that  $\{u_n^{\eta,m}\} \subset W^{1,p}(Q'_r(x_0) \times I; \mathbb{R}^3)$  and  $u_n^{\eta,m} \xrightarrow{n \rightarrow +\infty} 0$  in  $L^p(Q'_r(x_0) \times I; \mathbb{R}^3)$ . Thus

$$\begin{aligned}
 & 2 \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{Q'_{i,h}} \overline{(W^m)_{\text{hom}}(x_i^{h,\eta,m}; \bar{F})} dx_\alpha \\
 & \geq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{Q'_{i,h} \times I} W^m \left( x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{F} + \nabla_\alpha u_n^{\eta,m} \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n^{\eta,m} \right) dx. \quad (2.4.13)
 \end{aligned}$$

In view of (2.2.4) and (2.4.11) we have

$$\sup_{h \in \mathbb{N}} \sum_{i \in I_{h,\eta}^m} \int_{Q'_{i,h} \setminus K_\eta^m} \overline{(W^m)_{\text{hom}}(x_i^{h,\eta,m}; \bar{F})} dx_\alpha \leq \beta(1 + |\bar{F}|^p) \mathcal{L}^2(Q'_r(x_0) \setminus K_\eta^m) \xrightarrow{\eta \rightarrow 0} 0. \quad (2.4.14)$$

To simplify notation we set

$$F_n^{\eta,m}(x) := \left( \bar{F} + \nabla_\alpha u_n^{\eta,m}(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n^{\eta,m}(x) \right).$$

From (2.4.13) and (2.4.14) it comes

$$\begin{aligned}
 & 2 \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{Q'_{i,h} \cap K_\eta^m} \overline{(W^m)_{\text{hom}}(x_i^{h,\eta,m}; \bar{F})} dx_\alpha \\
 & \geq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{[Q'_{i,h} \cap K_\eta^m] \times I} W^m \left( x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m} \right) dx. \quad (2.4.15)
 \end{aligned}$$

Since  $\overline{(W^m)_{\text{hom}}(\cdot; \bar{F})}$  is continuous on  $K_\eta^m$ , it is uniformly continuous. Thus, denoting by  $\omega_\eta^m$  its modulus of continuity on  $K_\eta^m$ , we have for all  $x_\alpha \in Q'_{i,h} \cap K_\eta^m$ ,

$$|\overline{(W^m)_{\text{hom}}}(x_\alpha; \bar{F}) - \overline{(W^m)_{\text{hom}}}(x_i^{h,\eta,m}; \bar{F})| \leq \omega_\eta^m(|x_\alpha - x_i^{h,\eta,m}|) \leq \omega_\eta^m \left( \frac{\sqrt{2}r}{h} \right) \xrightarrow{h \rightarrow +\infty} 0. \quad (2.4.16)$$

## 2.4. THE GENERAL CASE

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In view of (2.4.11), (2.4.16) and (2.4.15), we get since  $Q'_{i,h} \cap K_\eta^m = \emptyset$  for  $i \notin I_{h,\eta}^m$ ,

$$\begin{aligned}
& 2 \int_{Q'_r(x_0)} \overline{(W^m)}_{\text{hom}}(x_\alpha; \overline{F}) dx_\alpha \\
&= 2 \lim_{\eta \rightarrow 0} \int_{K_\eta^m} \overline{(W^m)}_{\text{hom}}(x_\alpha; \overline{F}) dx_\alpha \\
&= 2 \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{Q'_{i,h} \cap K_\eta^m} \overline{(W^m)}_{\text{hom}}(x_\alpha; \overline{F}) dx_\alpha \\
&= 2 \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{Q'_{i,h} \cap K_\eta^m} \overline{(W^m)}_{\text{hom}}(x_i^{h,\eta,m}; \overline{F}) dx_\alpha \\
&\geq \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{(Q'_{i,h} \cap K_\eta^m) \times I} W^m \left( x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m} \right) dx. \quad (2.4.17)
\end{aligned}$$

Let  $R_{n,\eta,m}^\lambda := \{x \in Q'_r(x_0) \times I : |F_n^{\eta,m}(x)| \leq \lambda\}$ . From (2.4.13) and the fact that  $u_n^{\eta,m} \equiv 0$  on  $[Q'_r(x_0) \times I] \setminus \bigcup_{i \in I_{h,\eta}^m} Q_{i,h}$ , we get

$$\sup_{n,m,\eta} \int_{Q'_r(x_0) \times I} \left| \left( \nabla_\alpha u_n^{\eta,m} \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n^{\eta,m} \right) \right|^p dx < +\infty. \quad (2.4.18)$$

In particular, according to Chebyshev's inequality, we have

$$\mathcal{L}^3([Q'_r(x_0) \times I] \setminus R_{n,\eta,m}^\lambda) \leq \frac{C}{\lambda^p}, \quad (2.4.19)$$

for some constant  $C > 0$  independent of  $n, \eta, m$  and  $\lambda$ . Since  $W^m$  is continuous on  $\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$  and separately periodic in its second and third variable, it is uniformly continuous on  $\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$ . Thus, denoting by  $\omega_{m,\lambda}$  its modulus of continuity on  $\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$ , we have for all  $x \in [(Q'_{i,h} \cap K_\eta^m) \times I] \cap R_{n,\eta,m}^\lambda$ ,

$$\begin{aligned}
& \left| W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m}(x) \right) - W^m \left( x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m}(x) \right) \right| \\
&\leq \omega_{m,\lambda}(|x_\alpha - x_i^{h,\eta,m}|) \\
&\leq \omega_{m,\lambda} \left( \frac{\sqrt{2}r}{h} \right) \xrightarrow{h \rightarrow +\infty} 0,
\end{aligned}$$

uniformly with respect to  $n \in \mathbb{N}$ . Then, according to (2.4.17) and the fact that  $Q'_{i,h} \cap K_\eta^m = \emptyset$  for  $i \notin I_{h,\eta}^m$ ,

$$\begin{aligned}
 & 2 \int_{Q'_r(x_0)} \overline{(W^m)_{\text{hom}}(x_\alpha; \overline{F})} dx_\alpha \\
 & \geq \liminf_{\lambda \rightarrow +\infty} \liminf_{\eta \rightarrow 0} \liminf_{h \rightarrow +\infty} \liminf_{n \rightarrow +\infty} \sum_{i \in I_{h,\eta}^m} \int_{[(Q'_{i,h} \cap K_\eta^m) \times I] \cap R_{n,\eta,m}^\lambda} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m} \right) dx, \\
 & = \liminf_{\lambda \rightarrow +\infty} \liminf_{\eta \rightarrow 0} \liminf_{n \rightarrow +\infty} \int_{[K_\eta^m \times I] \cap R_{n,\eta,m}^\lambda} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m} \right) dx.
 \end{aligned}$$

In view of the  $p$ -growth condition (2.4.4), (2.4.11) and the definition of  $R_{n,\eta}^\lambda$ ,

$$\sup_{n \in \mathbb{N}} \int_{R_{n,\eta,m}^\lambda \setminus [K_\eta^m \times I]} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m} \right) dx \leq 2\beta(1 + \lambda^p)\eta \xrightarrow{\eta \rightarrow 0} 0,$$

then

$$2 \int_{Q'_r(x_0)} \overline{(W^m)_{\text{hom}}(x_\alpha; \overline{F})} dx_\alpha \geq \liminf_{\lambda \rightarrow +\infty} \liminf_{\eta \rightarrow 0} \liminf_{n \rightarrow +\infty} \int_{R_{n,\eta,m}^\lambda} W^m \left( x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; F_n^{\eta,m} \right) dx.$$

By a diagonalization argument, given  $\lambda_m \nearrow +\infty$ , and  $\eta_m \searrow 0^+$  there exists  $n_m \nearrow +\infty$  such that

$$2 \liminf_{m \rightarrow +\infty} \int_{Q'_r(x_0)} \overline{(W^m)_{\text{hom}}(x_\alpha; \overline{F})} dx_\alpha \geq \liminf_{m \rightarrow +\infty} \int_{R_m} W^m \left( x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{F} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m \right) dx,$$

where  $v_m := u_{n_m}^{\eta_m, m} \in W^{1,p}(Q'_r(x_0) \times I; \mathbb{R}^3)$  with  $v_m \rightarrow 0$  in  $L^p(Q'_r(x_0) \times I; \mathbb{R}^3)$ , and where  $R_m := R_{n_m, \eta_m, m}^{\lambda_m}$ . Using Theorem 1.1 in [25], we can assume, without loss of generality, that the sequence  $\{ |(\nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m)|^p \}$  is equi-integrable. Then, since  $W^m = W$  on  $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{aligned}
 & 2 \liminf_{m \rightarrow +\infty} \int_{Q'_r(x_0)} \overline{(W^m)_{\text{hom}}(x_\alpha; \overline{F})} dx_\alpha \\
 & \geq \liminf_{m \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W^m \left( x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{F} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m \right) dx \\
 & \geq \liminf_{m \rightarrow +\infty} \int_{[Q'_r(x_0) \times I] \cap C_m} W \left( x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{F} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m \right) dx \\
 & = \liminf_{m \rightarrow +\infty} \int_{Q'_r(x_0) \times I} W \left( x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{F} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m \right) dx
 \end{aligned}$$

where we have used the growth conditions on  $W$ , the  $p$ -equi-integrability of the above sequence of scaled gradients, and (2.4.4) and (2.4.3). As a result we get inequality (2.4.12).

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**Step 2.** Fixed  $\delta > 0$ , let  $T \in \mathbb{N}$  and  $\varphi \in W^{1,p}((0, T)^2 \times I; \mathbb{R}^3)$  with  $\varphi = 0$  on  $\partial(0, T)^2 \times I$  be such that

$$\overline{W}_{\text{hom}}(x_0; \overline{F}) + \delta \geq \int_{(0, T)^2 \times I} W_{\text{hom}}(x_0, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy. \quad (2.4.20)$$

Taking  $(T, \varphi)$  in the definition of  $\overline{(W^m)}_{\text{hom}}$  it follows that

$$\begin{aligned} & \int_{Q'_r(x_0)} \overline{(W^m)}_{\text{hom}}(x_\alpha; \overline{F}) dx_\alpha \\ & \leq \frac{1}{2T^2} \int_{Q'_r(x_0)} \int_{(0, T)^2 \times I} (W^m)_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha. \end{aligned} \quad (2.4.21)$$

Define  $E_m := \{(x_\alpha, y_\alpha, y_3) \in Q'_r(x_0) \times (0, T)^2 \times I : (x_\alpha, y_3) \in C_m\}$ . From (2.4.3) it follows that

$$\begin{aligned} \mathcal{L}^2 \otimes \mathcal{L}^3([Q'_r(x_0) \times (0, T)^2 \times I] \setminus E_m) &= T^2 \mathcal{L}^3([Q'_r(x_0) \times I] \setminus C_m) \\ &\leq T^2/m. \end{aligned} \quad (2.4.22)$$

Since  $(W^m)_{\text{hom}} = W_{\text{hom}}$  on  $C_m \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$

$$\begin{aligned} & \int_{Q'_r(x_0)} \int_{(0, T)^2 \times I} (W^m)_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha \\ &= \int_{E_m} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha \\ & \quad + \int_{[Q'_r(x_0) \times (0, T)^2 \times I] \setminus E_m} (W^m)_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha \\ &\leq \int_{Q'_r(x_0) \times (0, T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha \\ & \quad + C \int_{[Q'_r(x_0) \times (0, T)^2 \times I] \setminus E_m} (1 + |\nabla \varphi(y)|^p) dy dx_\alpha \end{aligned} \quad (2.4.23)$$

by property (2.2.4) with  $W^m$  in place of  $W$ . Passing to the limit as  $m \rightarrow +\infty$ , relations (2.4.21), (2.4.22) and (2.4.23) yield to

$$\begin{aligned} & \limsup_{m \rightarrow +\infty} \int_{Q'_r(x_0)} \overline{(W^m)}_{\text{hom}}(x_\alpha; \overline{F}) dx_\alpha \\ & \leq \frac{1}{2T^2} \int_{Q'_r(x_0)} \int_{(0, T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha. \end{aligned}$$

Hence by (2.4.12) we obtain

$$\int_{Q'_r(x_0)} W_{\{\varepsilon_n\}}(x_\alpha; \bar{F}) dx_\alpha \leq \frac{1}{2T^2} \int_{Q'_r(x_0)} \int_{(0,T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \bar{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy dx_\alpha.$$

Choose  $x_0 \in \omega$  to be also a Lebesgue point of the function

$$x_\alpha \mapsto \int_{(0,T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \bar{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy.$$

Dividing the previous inequality by  $r^2$  and letting  $r \rightarrow 0$ , we get according to (2.4.20) that

$$\begin{aligned} W_{\{\varepsilon_n\}}(x_0; \bar{F}) &\leq \frac{1}{2T^2} \int_{(0,T)^2 \times I} W_{\text{hom}}(x_0, y_3, y_\alpha; \bar{F} + \nabla_\alpha \varphi(y) | \nabla_3 \varphi(y)) dy \\ &\leq \bar{W}_{\text{hom}}(x_0; \bar{F}) + \delta. \end{aligned}$$

The result follows by letting  $\delta \rightarrow 0$ .  $\square$

*Proof of Theorem 2.1.1.* As a consequence of Lemmas 2.4.3 and 2.4.2, we have  $\bar{W}_{\text{hom}}(x_\alpha; \bar{F}) = W_{\{\varepsilon_n\}}(x_\alpha; \bar{F})$  for a.e.  $x_\alpha \in \omega$  and all  $\bar{F} \in \mathbb{R}^{3 \times 2}$ . Therefore, we have  $\mathcal{W}_{\{\varepsilon_n\}}(u; A) = \mathcal{W}_{\text{hom}}(u; A)$  for all  $A \in \mathcal{A}(\omega)$  and all  $u \in W^{1,p}(A; \mathbb{R}^3)$ . Since the result does not depend upon the specific choice of the subsequence, we conclude thanks to Proposition 8.3 in [45] that the whole sequence  $\mathcal{W}_\varepsilon(\cdot; A)$   $\Gamma$ -converges to  $\mathcal{W}_{\text{hom}}(\cdot; A)$  for the  $L^p(A \times I; \mathbb{R}^3)$ -topology.  $\square$

To conclude, let us state an interesting consequence of Theorem 2.1.1.

**Corollary 2.4.4.** *Let  $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  be a continuous function satisfying  $(A_1)$ ,  $(A_2)$  and  $(A_4)$ , and such that  $W(x, \cdot; F)$  is  $Q$ -periodic for all  $F \in \mathbb{R}^{3 \times 3}$  and a.e.  $x \in \Omega$ . Then, the functional  $\mathcal{W}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  defined by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \int_{\Omega} W\left(x, \frac{x}{\varepsilon}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

$\Gamma$ -converges for the strong  $L^p(\Omega; \mathbb{R}^3)$ -topology toward  $\mathcal{W}_{\text{hom}} : L^p(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$  given by

$$\mathcal{W}_{\text{hom}}(u) := \begin{cases} 2 \int_{\omega} \bar{W}_{\text{hom}}(x_\alpha, \nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \bar{W}_{\text{hom}}(x_\alpha, \bar{F}) &:= \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^2 \times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \bar{F} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)) dy : \right. \\ &\quad \left. \phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3), \quad \phi = 0 \text{ on } \partial(0,T)^2 \times I \right\}, \end{aligned}$$

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for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  and a.e.  $x_\alpha \in \omega$  and

$$W_{\text{hom}}(x, y_\alpha; F) := \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^3} W(x, y_\alpha, z_3; F + \nabla \phi(z)) dz : \phi \in W_0^{1,p}((0,T)^3; \mathbb{R}^3) \right\},$$

for all  $(y_\alpha, F) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$  and a.e.  $x \in \Omega$ .



## Chapitre 3

# The Neumann sieve problem in dimensional reduction





### 3.1 Introduction

This paper deals with a bilayer thin film connected through a separation hyperplane ("sieve plane") by  $\delta$ -periodically distributed "contact zones". Replacing one of the two thin films by a thin substrate, one can think of the problem as modeling the debonding of a film from this substrate. Specifically, we consider nonlinear elastic bodies where the thickness is described by the parameter  $\varepsilon$  and the connecting zones are  $(n-1)$ -dimensional balls  $B'_r(x_i^\delta)$  with center  $x_i^\delta := i\delta$ ,  $i \in \mathbb{Z}^{n-1}$ , and radius  $r > 0$ . Then, the investigated domain occupies the reference configuration parametrized as

$$\Omega_{\delta,r}^\varepsilon := \omega^{+\varepsilon} \cup \omega^{-\varepsilon} \cup (\omega_{\delta,r} \times \{0\})$$

where  $\omega$  is a bounded open subset of  $\mathbb{R}^{n-1}$ ,  $\omega^{+\varepsilon} := \omega \times (0, \varepsilon)$ ,  $\omega^{-\varepsilon} := \omega \times (-\varepsilon, 0)$  and  $\omega_{\delta,r} := \bigcup_{i \in \mathbb{Z}^{n-1}} B'_r(x_i^\delta) \cap \omega$ .

In the nonlinear membrane setting the elastic energy of such materials is given by the functional

$$\frac{1}{\varepsilon} \int_{\Omega_{\delta,r}^\varepsilon} W(\nabla u) \, dx,$$

where  $u : \Omega_{\delta,r}^\varepsilon \rightarrow \mathbb{R}^m$  is the deformation field and  $W$  stands for the stored energy density, which is a Borel function satisfying standard coercivity and growth conditions of order  $p$ . In this paper, we study the multiple scales asymptotic behavior of the previous functional as  $\varepsilon$ ,  $\delta$  and  $r$  tend to zero at the same time. The notion of  $\Gamma$ -convergence (see [30, 32, 45]) has proved to be successful in order to obtain limit models (see e.g. [35] and references therein). It is natural to expect limit behaviors that depend on the mutual vanishing rates of convergence of  $\varepsilon$ ,  $\delta$  and  $r$ .

Since, at fixed  $\varepsilon$ ,  $\delta$  and  $r$ ,  $(\omega \setminus \omega_{\delta,r}) \times \{0\}$  is a part of the boundary of  $\Omega_{\delta,r}^\varepsilon$ , admissible deformations may be discontinuous across this part of the mid-section. This failure of the interface may asymptotically create a debonding which costs a possible interfacial energy of order zero, penalizing the jump of the deformation. This new energy describes, by a nonlinear capacity type formula, the contribution near each balls  $B'_r(x_i^\delta)$  and its value seems to be closely related to the scaling of the radius of the connecting zones with respect to the thickness and the period of distribution of the zones. Indeed, if the radius is too large, this interfacial energy will blow up and thus, deformations with finite energy will be continuous across the mid-section. Consequently, the effective energy would be exactly the same as that where the body is perfect. On the other hand, if the radius is too small, it yields to a zero interfacial energy leading to two decoupled problems. Although the discontinuities remain present, we lose the interaction energy between the two regions. In other words, in order to recover non trivial models, by which we mean limit models remembering the presence of the zones, we need the right assumptions on the mutual rates of convergence of the three small parameters  $\varepsilon$ ,  $\delta$  and  $r$ .

The cases  $\varepsilon = 1$  and  $\varepsilon = \delta$  have been studied in [6], which investigates in addition the interaction through the separating surface of two different energies, possibly satisfying different growth conditions. It is proved there that, in order to recover a non trivial limit model, the

meaningful radii (critical size) of the contact zones must be of order of  $\delta^{(n-1)/(n-p)}$  and  $\delta^{n/(n-p)}$ , respectively. In fact, as noticed above, if the radius  $r_\delta$  tends to zero faster than the critical size then, in the limit, we obtain two decoupled problems, while if  $r_\delta$  tends to zero slower than the critical size, we get exactly the same result as that obtained in the absence of connecting zones in the mid-section.

The proof of [6] is based on a technical result (see Lemma 3.4 of that reference) that allows to modify a sequence on a suitable  $n$ -dimensional annulus surrounding the  $(n-1)$ -dimensional balls  $B'_{r_\delta}(x_i^\delta)$ , and to study the behavior of the family of energies along the modified sequence. It gives rise to three terms in the  $\Gamma$ -limit. The first two terms represent the contribution of the new sequence "far" from  $B'_{r_\delta}(x_i^\delta)$ ; more precisely, they are the  $\Gamma$ -limits of two uncoupled problems defined separately on the upper part and lower part (with respect to the "sieve plane") of the considered domain. The third one describes, by the nonlinear capacity formula  $\varphi$ , the contribution "near"  $B'_{r_\delta}(x_i^\delta)$ . Note that the function  $\varphi$  is the same in both cases  $\varepsilon = 1$  and  $\varepsilon = \delta$ , since the annulus is contained in a strip of thickness  $c\delta$ , with  $c < 1/2$  arbitrarily small independently of  $\delta$ . The cases  $\delta \sim \varepsilon$  and  $\delta \ll \varepsilon$  can be treated in the same way. This approach follows the method introduced in [9, 10] where the asymptotic behavior of periodically-perforated nonlinear domains has been studied; in particular, Lemma 3.4 in [6] is a suitable variant, for the sieve problem, of Lemma 3.1 in [9, 10]. For related problems to this subject we also refer to [49, 77, 82], [11], [41, 42, 43], [52] and [84, 85, 86].

In this work, we focus on the case  $\varepsilon \ll \delta$ . Namely, for every sequence  $\{\delta_j\}$  converging to 0, we assume that  $\varepsilon_j = o(\delta_j)$ . We expect, as in [6], the existence of some meaningful radius  $r_j := r(\varepsilon_j, \delta_j) \ll \delta_j$  for which the limit model is not trivial and we expect also different regimes which may depend on the behavior of  $r_j$  with respect to  $\varepsilon_j$ . The method introduced in [6] can not be applied in our case since, by construction, the annuli surrounding the balls  $B'_{r_\delta}(x_i^\delta)$  are well contained in a strip of thickness  $c\delta$  but not necessarily in that of thickness  $2\varepsilon$  since  $\varepsilon \ll \delta$  and  $c$  is supposed to be independent of  $\varepsilon$  and  $\delta$ . However, we may adapt the method in [6] by considering, instead of annuli, suitable cylinders surrounding the balls  $B'_{r_\delta}(x_i^\delta)$  with radii of order of  $\delta$  (but strictly less than  $\delta/2$ ) and thickness  $2\varepsilon$  (see Lemmas 3.4.2 and 3.4.3). Hence, let  $\Omega_j := \Omega_{\delta_j, r_j}^{\varepsilon_j}$ , if we define

$$\mathcal{F}_j(u) := \begin{cases} \frac{1}{\varepsilon_j} \int_{\Omega_j} W(\nabla u) dx & \text{if } u \in W^{1,p}(\Omega_j; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

then, denoting

$$\ell := \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} \in [0, +\infty],$$

there exists a meaningful radius  $r_j$  such that, up to subsequences,  $\mathcal{F}_j$   $\Gamma$ -converges to

$$\mathcal{F}^{(\ell)}(u^+, u^-) = \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u^+) dx_\alpha + \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u^-) dx_\alpha + R^{(\ell)} \int_{\omega} \varphi^{(\ell)}(u^+ - u^-) dx_\alpha$$

on  $W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$  with respect to the convergence introduced in Definition 3.3.1, where  $\mathcal{Q}W_0$  denotes the  $(n-1)$ -quasiconvexification of  $W_0$  and  $W_0(\bar{F}) := \inf\{W(\bar{F}|z) : z \in \mathbb{R}^m\}$ . For each regime  $\ell \in [0, +\infty]$ , the numbers  $R^{(\ell)}$  account for the scaling, that is the behavior of  $r_j$  with respect to  $\varepsilon_j$  and/or  $\delta_j$  (see Theorem 3.3.3), while the interfacial energy densities  $\varphi^{(\ell)}$  are given by nonlinear capacitary type formulas where boundary conditions are only imposed on the lateral part of the boundary (see Theorem 3.3.5). Relations (3.5.6) and (3.5.17) show that whatever the regime, this new energy corresponds to a cohesive interface since  $\varphi^{(\ell)}$  increases continuously from zero with the jump in the deformation across the interface.

Let us briefly describe each regime :

(1) The case  $\ell = +\infty$  deals with very thin films because  $\varepsilon \ll r \ll \delta$  and thus, we expect  $r = r(\delta)$  to depend only on  $\delta$ . For fixed  $r$  and  $\delta$ , passing to the limit when  $\varepsilon$  tends to zero, we have two pure dimensional reduction problems stated on the upper and lower cylinders respectively. It leads to a limit functional consisting in two terms, each of them being a copy of the functional obtained in [74] on the upper and lower part of the domain, respectively. Since the corresponding limit deformations  $u^+$  and  $u^-$  must match inside the connecting zones, both terms are not completely decoupled. We are then in a situation quite similar to that of [9, 10], except that here, both periodically perforated  $(n-1)$ -dimensional bodies are linked to each other inside the perforations. It is thus coherent to expect some meaningful radii of order  $\delta^{(n-1)/(n-1-p)}$ . This strong separation between both phenomena of reduction and perforation leads to break the isotropy. This is quite clear from the point of view of the scaling since an inspection of the proofs of Lemmas 3.6.2 and 3.6.4 shows that the extra interfacial term appears thanks to suitable dilatations which have not the same scaling in the in-plane and transverse directions. Note that the formula leading to  $\varphi^{(\infty)}$  (see Lemma 3.3.5) involves a "Le Dret-Raoult type" functional, underlining the importance of the reduction.

(2) On the other hand, the case  $\ell = 0$  describes larger thin films because  $r \ll \varepsilon \ll \delta$ . Here the meaningful radii  $r = r(\varepsilon, \delta)$  should thus depend on both  $\varepsilon$  and  $\delta$ . Indeed, we obtain that  $r$  must be of order  $\varepsilon^{1/(n-p)}\delta^{(n-1)/(n-p)}$  and we get a better estimate of its rate of convergence since  $r \ll \delta^{n/(n-p)}$ . Note that this latter coefficient is exactly that obtained in [6] for the case  $\delta = \varepsilon$ . Contrary to the previous case ( $\ell = +\infty$ ), the isotropy seems to be preserved because here, dimensional reduction and perforation are not completely decoupled : the reduction parameter  $\varepsilon$  is sandwiched between both perforation parameters  $r$  and  $\delta$ . Mathematically, this can be seen by noticing that the dilatations leading to the interfacial energy are identical in every directions (see for instance the end of the proof of Lemma 3.6.4). The price to pay is that  $\varphi^{(0)}$  is obtained as the limit of a minimization problem stated on larger and larger cylinders in the in-plane as well as in the transverse variable. This is more difficult to treat since as usual in dimensional reduction problems, we have no information on the value of the admissible test functions on the extreme parts of the boundary. The formula describing  $\varphi^{(0)}$  in Lemma 3.3.5 does not underline explicitly the reduction procedure which is only characterized by boundary conditions expressed on the lateral part of the boundary.

(3) In the intermediate case  $\ell \in (0, +\infty)$ , then  $r \sim \varepsilon \ll \delta$  and it turns out that the previous scalings are equivalent because  $R^{(0)} = \ell R^{(\infty)}$ . From the formulas obtained in Lemma 3.3.5, we find that the interfacial energy is continuous with respect to  $\ell$  in the extreme regimes since  $R^{(\ell)}\varphi^{(\ell)}(z) \rightarrow R^{(\infty)}\varphi^{(\infty)}(z)$  as  $\ell \rightarrow +\infty$  and  $R^{(\ell)}\varphi^{(\ell)}(z) \rightarrow R^{(0)}\varphi^{(0)}(z)$  as  $\ell \rightarrow 0$ . As in the previous case ( $\ell = 0$ ), the lateral boundary conditions are the only mean to describe the dimensional reduction in the formula which gives  $\varphi^{(\ell)}$  in Lemma 3.3.5.

The paper is organized as follows : after recalling some useful notation in Section 3.2, we state the main result, Theorems 3.3.3 and 3.3.5, in Section 3.3. Then, we establish in Section 3.4 some auxiliary results like rescaled Poincaré type inequalities and joining lemmas. Section 3.5 is devoted to a definition of the interfacial energy density as a limit of minimum problems ; hence, in Section 3.6 we prove the  $\Gamma$ -convergence result (Theorem 3.3.3). It is only in Section 3.7 that we give an explicit description in terms of a "capacitary type" formula of the interfacial energy density (Theorem 3.3.5).

## 3.2 Notation

Given  $x \in \mathbb{R}^n$ , we set  $x_\alpha := (x_1, \dots, x_{n-1})$  for the in-plane variable and  $\nabla_\alpha = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}} \right)$  (resp.  $\nabla_n$ ) for the derivative with respect to  $x_\alpha$  (resp.  $x_n$ ).

The notation  $\mathbb{R}^{m \times n}$  stands for the set of  $m \times n$  matrices. Given a matrix  $F \in \mathbb{R}^{m \times n}$ , we write  $F = (\overline{F} | F_n)$  where  $\overline{F} = (F_1, \dots, F_{n-1})$  and  $F_i$  denotes the  $i$ -th column of  $F$ ,  $1 \leq i \leq n$  and  $\overline{F} \in \mathbb{R}^{m \times (n-1)}$ .

The Lebesgue measure in  $\mathbb{R}^k$  will be denoted by  $\mathcal{L}^k$  (in the sequel,  $k$  will always be equal to  $n$  or  $n-1$ ). We use standard notation for Lebesgue and Sobolev spaces  $L^s(A; \mathbb{R}^m)$  and  $W^{1,s}(A; \mathbb{R}^m)$ .

Let  $\omega$  be a bounded open subset of  $\mathbb{R}^{n-1}$  and  $I = (-1, 1)$ , we define  $\Omega := \omega \times I$ . In the sequel, we will identify  $L^s(\omega; \mathbb{R}^m)$  (resp.  $W^{1,s}(\omega; \mathbb{R}^m)$ ) with the space of functions  $v \in L^s(\Omega; \mathbb{R}^m)$  (resp.  $W^{1,s}(\Omega; \mathbb{R}^m)$ ) such that  $\nabla_n v = 0$  in the sense of distributions.

Let  $a \in \mathbb{R}^{n-1}$  and  $\rho > 0$ , we denote by  $B'_\rho(a)$  the open ball of  $\mathbb{R}^{n-1}$  of center  $a$  and radius  $\rho$  and by  $Q'_\rho(a)$  the open cube of  $\mathbb{R}^{n-1}$  with center  $a$  and length side  $\rho$ . We write  $B'_\rho$  instead of  $B'_\rho(0)$  in order not to overburden notation. Let  $x_i^\delta = i\delta$  with  $i \in \mathbb{Z}^{n-1}$ , we set  $Q'_{i,\delta} := Q'_\delta(x_i^\delta)$ .

We denote  $U^{+a} = U \times (0, a)$  and  $U^{-a} = U \times (-a, 0)$  with  $U \subseteq \mathbb{R}^{n-1}$  and  $a > 0$ , while if  $a = 1$ , then  $U^+ = U^{+1}$  and  $U^- = U^{-1}$ . We set  $C_{1,\infty} := \{(x_\alpha, 0) \in \mathbb{R}^n : 1 \leq |x_\alpha|\}$  and  $C_{1,N} := \{(x_\alpha, 0) \in \mathbb{R}^n : 1 \leq |x_\alpha| < N\}$  for every  $N > 1$ .

Let  $p \geq 1$  and  $k = n$  or  $n-1$ . We denote the  $p$ -capacity of  $B'_1$  with respect to an open set  $A \subset \mathbb{R}^k$  by

$$\text{Cap}_p(B'_1; A) = \inf \left\{ \int_A |\nabla \psi|^p dx : \psi \in W_0^{1,p}(A) \text{ and } \psi = 1 \text{ on } B'_1 \right\}$$

### 3.3. STATEMENT OF THE MAIN RESULT

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and the  $p$ -capacity of  $B'_1$  with respect to  $\mathbb{R}^k$  by

$$\text{Cap}_p(B'_1; \mathbb{R}^k) = \inf \left\{ \int_{\mathbb{R}^k} |\nabla \psi|^p dx : \psi \in W^{1,p}(\mathbb{R}^k) \text{ and } \psi = 1 \text{ on } B'_1 \right\}.$$

The letter  $c$  will stand for an arbitrary fixed strictly-positive constant.

### 3.3 Statement of the main result

Since we are going to work with varying domains, we have to precise the meaning of "converging sequences".

**Definition 3.3.1.** *Let  $\Omega_j = \omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j} \cup (\omega_{\delta_j, r_j} \times \{0\})$ . Given a sequence  $\{u_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$ , we define  $\hat{u}_j(x_\alpha, x_n) := u_j(x_\alpha, \varepsilon_j x_n)$ . We say that  $\{u_j\}$  converges (resp. converges weakly) to  $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$  if we have*

$$\begin{aligned} \hat{u}_j^+ &:= \hat{u}_j|_{\omega^+} \rightarrow u^+ \text{ in } L^p(\omega^+; \mathbb{R}^m) \quad (\text{resp. weakly in } W^{1,p}(\omega^+; \mathbb{R}^m)), \\ \hat{u}_j^- &:= \hat{u}_j|_{\omega^-} \rightarrow u^- \text{ in } L^p(\omega^-; \mathbb{R}^m) \quad (\text{resp. weakly in } W^{1,p}(\omega^-; \mathbb{R}^m)). \end{aligned}$$

Moreover, we say that the sequence  $\{|\nabla u_j|^p / \varepsilon_j\}$  is equi-integrable on  $\omega^{\pm\varepsilon_j}$  if  $\{|\left(\nabla_\alpha \hat{u}_j \middle| \frac{1}{\varepsilon_j} \nabla_n \hat{u}_j\right)^p\}$  is equi-integrable on  $\omega^\pm$ .

**Remark 3.3.2.** By virtue of Definition 3.3.1, a sequence  $\{u_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$  converges to  $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$  if and only if

$$\lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |u_j - u^\pm|^p dx = 0, \quad (3.3.1)$$

while we have weak convergence provided (3.3.1) and

$$\sup_{j \in \mathbb{N}} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx = \sup_{j \in \mathbb{N}} \int_{\omega^\pm} \left| \left( \nabla_\alpha \hat{u}_j \middle| \frac{1}{\varepsilon_j} \nabla_n \hat{u}_j \right) \right|^p dx < +\infty \quad (3.3.2)$$

hold true.

Note that Definition 3.3.1 and Remark 3.3.2 are still valid if we consider the domain  $\omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j}$  instead of  $\Omega_j$ .

The main result of this paper is the following

**Theorem 3.3.3.** *Let  $1 < p < n - 1$ . Let  $\omega$  be a bounded open subset of  $\mathbb{R}^{n-1}$  satisfying  $\mathcal{L}^{n-1}(\partial\omega) = 0$  and  $W : \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$  be a Borel function such that  $W(0) = 0$  and satisfying a growth condition of order  $p$  : there exists a constant  $\beta > 0$  such that*

$$|F|^p - 1 \leq W(F) \leq \beta(|F|^p + 1), \quad \text{for every } F \in \mathbb{R}^{m \times n}. \quad (3.3.3)$$

Let  $\{\delta_j\}$ ,  $\{\varepsilon_j\}$  and  $\{r_j\}$  be sequences of strictly positive numbers converging to zero such that

$$\lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{\delta_j} = 0$$

and

$$\text{if } \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} = \ell \in (0, +\infty], \quad \text{then } 0 < R^{(\ell)} := \lim_{j \rightarrow +\infty} \frac{r_j^{n-1-p}}{\delta_j^{n-1}} < +\infty$$

or

$$\text{if } \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} = 0, \quad \text{then } 0 < R^{(0)} := \lim_{j \rightarrow +\infty} \frac{r_j^{n-p}}{\varepsilon_j \delta_j^{n-1}} < +\infty.$$

Then, upon extracting a subsequence, the sequence of functionals  $\mathcal{F}_j : L^p(\Omega_j; \mathbb{R}^m) \rightarrow [0, +\infty]$  defined by

$$\mathcal{F}_j(u) := \begin{cases} \frac{1}{\varepsilon_j} \int_{\Omega_j} W(\nabla u) dx & \text{if } u \in W^{1,p}(\Omega_j; \mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases}$$

$\Gamma$ -converges to

$$\mathcal{F}^{(\ell)}(u^+, u^-) = \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^+) dx_{\alpha} + \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^-) dx_{\alpha} + R^{(\ell)} \int_{\omega} \varphi^{(\ell)}(u^+ - u^-) dx_{\alpha}$$

on  $W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$  with respect to the convergence introduced in Definition 3.3.1, where  $\mathcal{Q}W_0$  denotes the  $(n-1)$ -quasiconvexification of  $W_0$  and  $W_0(\bar{F}) := \inf\{W(\bar{F}|z) : z \in \mathbb{R}^m\}$ .

**Remark 3.3.4.** Theorem 3.3.3 still holds true when  $R^{(\ell)} = 0$  or  $+\infty$ . However, if  $R^{(\ell)} = 0$ , we loose the limit interfacial energy leading to two decoupled problems. Further, if  $R^{(\ell)} = +\infty$ , limit deformations with finite energy must be continuous across the mid-section ( $u^+ = u^-$ ) and we obtain the same result than in [74].

The following result provides a characterization of the interfacial energy density  $\varphi^{(\ell)}$  according to each regimes  $\ell \in [0, +\infty]$ .

**Theorem 3.3.5.** Let  $p^* = (n-1)p/(n-1-p)$  be the Sobolev exponent in dimension  $(n-1)$ . For the same subsequence than that of Theorem 3.3.3, there exists the limit

$$g(F) := \lim_{j \rightarrow +\infty} r_j^p \mathcal{Q}W(r_j^{-1}F),$$

for all  $F \in \mathbb{R}^{m \times n}$ , where  $\mathcal{Q}W$  denotes the  $n$ -quasiconvexification of  $W$ , and the functions  $\varphi^{(\ell)} : \mathbb{R}^m \rightarrow [0, +\infty)$  are described by the following formulas : for every  $z \in \mathbb{R}^m$ , if  $\ell \in (0, +\infty)$ ,

$$\varphi^{(\ell)}(z) := \inf \left\{ \int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} g(\nabla_{\alpha} \zeta | \ell \nabla_n \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m), \right. \\ \left. \begin{aligned} \nabla \zeta \in L^p((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \quad \zeta - z \in L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \\ \zeta \in L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \end{aligned} \right\},$$

### 3.4. PRELIMINARY RESULTS

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if  $\ell = +\infty$

$$\varphi^{(\infty)}(z) := \inf \left\{ \int_{\mathbb{R}^{n-1}} \left( \mathcal{Q}g_0(\nabla_\alpha \zeta^+) + \mathcal{Q}g_0(\nabla_\alpha \zeta^-) \right) dx_\alpha : \zeta^\pm \in W_{\text{loc}}^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m), \right. \\ \left. \zeta^+ = \zeta^- \text{ in } B_1', \quad \nabla_\alpha \zeta^\pm \in L^p(\mathbb{R}^{n-1}; \mathbb{R}^{m \times (n-1)}), \right. \\ \left. (\zeta^+ - z), \zeta^- \in L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m) \right\},$$

where  $\mathcal{Q}g_0$  denotes the  $(n-1)$ -quasiconvexification of  $g_0$  and  $g_0(\overline{F}) := \inf\{g(\overline{F}|z) : z \in \mathbb{R}^m\}$ , and if  $\ell = 0$

$$\varphi^{(0)}(z) = \inf \left\{ \int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m), \nabla \zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \right. \\ \left. \zeta - z \in L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \zeta \in L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\}.$$

**Remark 3.3.6.** Without loss of generality we may assume that  $W$  is quasiconvex (upon first relax the energy); hence, by (3.3.3),  $W$  satisfies the following  $p$ -Lipschitz condition (See e.g. [44]) :

$$|W(F_1) - W(F_2)| \leq c(1 + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \quad \text{for all } F_1, F_2 \in \mathbb{R}^{m \times n}. \quad (3.3.4)$$

## 3.4 Preliminary results

### 3.4.1 Some rescaled Poincaré Inequalities

Since we deal with varying domains, depending on different parameters, it is useful to note how the constant in Poincaré type inequalities rescale with respect to such parameters.

**Lemma 3.4.1.** *Let  $A$  be an open bounded and connected subset of  $\mathbb{R}^{n-1}$  with Lipschitz boundary and let  $A_\rho := \rho A$  for  $\rho > 0$ .*

(i) *There exists a constant  $c > 0$  (depending only on  $(A, n, p)$ ) such that for every  $\rho, \varepsilon > 0$*

$$\int_{A_\rho^{\pm\varepsilon}} |u - \bar{u}_{A_\rho^{\pm\varepsilon}}|^p dx \leq c \int_{A_\rho^{\pm\varepsilon}} (\rho^p |\nabla_\alpha u|^p + \varepsilon^p |\nabla_n u|^p) dx,$$

*for every  $u \in W^{1,p}(A_\rho^{\pm\varepsilon}; \mathbb{R}^m)$  where  $\bar{u}_{A_\rho^{\pm\varepsilon}} = \int_{A_\rho^{\pm\varepsilon}} u dx$ .*

(ii) *If  $B$  is an open and connected subset of  $A$  with Lipschitz boundary and  $B_\rho := \rho B$  then there exists a constant  $c > 0$  (depending only on  $(A, B, n, p)$ ) such that for every  $\rho, \varepsilon > 0$*

$$\int_{A_\rho^{\pm\varepsilon}} |u - \bar{u}_{B_\rho^{\pm\varepsilon}}|^p dx \leq c \int_{A_\rho^{\pm\varepsilon}} (\rho^p |\nabla_\alpha u|^p + \varepsilon^p |\nabla_n u|^p) dx,$$

*for every  $u \in W^{1,p}(A_\rho^{\pm\varepsilon}; \mathbb{R}^m)$  where  $\bar{u}_{B_\rho^{\pm\varepsilon}} = \int_{B_\rho^{\pm\varepsilon}} u dx$ .*



*Proof.* Let us define  $v(x_\alpha, x_n) := u(\rho x_\alpha, \varepsilon x_n)$  then  $v \in W^{1,p}(A^\pm; \mathbb{R}^m)$ . By a change of variable, we get that  $\bar{u}_{A_\rho^\pm \varepsilon} = \bar{v}_{A^\pm}$ . Moreover, by the Poincaré Inequality, there exists a constant  $c = c(A, n, p) > 0$  such that

$$\begin{aligned} \int_{A_\rho^\pm \varepsilon} |u - \bar{u}_{A_\rho^\pm \varepsilon}|^p dx &= \varepsilon \rho^{n-1} \int_{A^\pm} |v - \bar{v}_{A^\pm}|^p dy \\ &\leq c \varepsilon \rho^{n-1} \int_{A^\pm} |\nabla v|^p dy \\ &= c \int_{A_\rho^\pm \varepsilon} (\rho^p |\nabla_\alpha u|^p + \varepsilon^p |\nabla_n u|^p) dx \end{aligned}$$

and it completes the proof of (i). Now, if  $B_\rho \subset A_\rho$ , we get that

$$\begin{aligned} &\int_{A_\rho^\pm \varepsilon} |u - \bar{u}_{B_\rho^\pm \varepsilon}|^p dx \\ &\leq c \left( \int_{A_\rho^\pm \varepsilon} |u - \bar{u}_{A_\rho^\pm \varepsilon}|^p dx + \varepsilon \rho^{n-1} \mathcal{L}^{n-1}(A) |\bar{u}_{A_\rho^\pm \varepsilon} - \bar{u}_{B_\rho^\pm \varepsilon}|^p \right) \\ &\leq c \int_{A_\rho^\pm \varepsilon} |u - \bar{u}_{A_\rho^\pm \varepsilon}|^p dx + c \frac{\mathcal{L}^{n-1}(A)}{\mathcal{L}^{n-1}(B)} \left( \int_{B_\rho^\pm \varepsilon} |u - \bar{u}_{A_\rho^\pm \varepsilon}|^p dx + \int_{B_\rho^\pm \varepsilon} |u - \bar{u}_{B_\rho^\pm \varepsilon}|^p dx \right) \\ &\leq c \int_{A_\rho^\pm \varepsilon} (\rho^p |\nabla_\alpha u|^p + \varepsilon^p |\nabla_n u|^p) dx. \end{aligned}$$

□

### 3.4.2 A joining lemma

The following lemma, which is the key point of the proof of our main result, is very close to Lemma 3.4 in [6]. However, [6] does not apply to our case since the domain  $\Omega_j$  has thickness  $\varepsilon_j \ll \delta_j$ ; for this reason we make a different construction introducing cylinders surrounding the connecting zones (instead of balls as in [6]).

**Lemma 3.4.2.** *Let  $\{u_j\} \subset W^{1,p}(\omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j}; \mathbb{R}^m)$  be a sequence weakly converging to  $(u^+, u^-) \in W^{1,p}(\omega; \mathbb{R}^m) \times W^{1,p}(\omega; \mathbb{R}^m)$  satisfying*

$$\sup_{j \in \mathbb{N}} \frac{1}{\varepsilon_j} \int_{\omega^{\pm \varepsilon_j}} W(\nabla u_j) dx < +\infty. \quad (3.4.1)$$

*Let  $k \in \mathbb{N}$  and  $\{\delta_j, \varepsilon_j\}$  be sequences of strictly positive numbers converging to 0 such that  $\varepsilon_j \ll \delta_j$ . Set  $\rho_j = \gamma \delta_j$  with  $\gamma < 1/2$  and*

$$Z_j := \{i \in \mathbb{Z}^{n-1} : \text{dist}(x_i^{\delta_j}, \mathbb{R}^{n-1} \setminus \omega) > \delta_j\}.$$

*For all  $i \in Z_j$ , there exists  $k_i \in \{0, \dots, k-1\}$  such that having set*

$$C_j^i := \left\{ x_\alpha \in \omega : 2^{-k_i-1} \rho_j < |x_\alpha - x_i^{\delta_j}| < 2^{-k_i} \rho_j \right\},$$

### 3.4. PRELIMINARY RESULTS

$$u_j^{i\pm} := \int_{(C_j^i)^{\pm\varepsilon_j}} u_j dx \quad (3.4.2)$$

and

$$\rho_j^i := \frac{3}{4}2^{-k_i}\rho_j,$$

there exists a sequence  $\{w_j\} \subset W^{1,p}(\omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j}; \mathbb{R}^m)$  still weakly converging to  $(u^+, u^-)$  such that

$$w_j = u_j \text{ in } \left(\omega \setminus \bigcup_{i \in Z_j} C_j^i\right)^{\pm\varepsilon_j}, \quad (3.4.3)$$

$$w_j = u_j^{i\pm} \text{ on } (\partial B'_{\rho_j^i}(x_i^{\delta_j}))^{\pm\varepsilon_j} \quad (3.4.4)$$

and satisfying

$$\limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |W(\nabla w_j) - W(\nabla u_j)| dx \leq \frac{c}{k}. \quad (3.4.5)$$

*Proof.* For all  $j \in \mathbb{N}$ ,  $i \in Z_j$ ,  $k \in \mathbb{N}$  and  $h \in \{0, \dots, k-1\}$ , we define

$$C_j^{i,h} := \left\{ x_\alpha \in \omega : 2^{-h-1}\rho_j < |x_\alpha - x_i^{\delta_j}| < 2^{-h}\rho_j \right\},$$

$$(u_j^{i,h})^\pm := \int_{(C_j^{i,h})^{\pm\varepsilon_j}} u_j dx$$

and

$$\rho_j^{i,h} := \frac{3}{4}2^{-h}\rho_j. \quad (3.4.6)$$

Let  $\phi \equiv \phi_j^{i,h} \in \mathcal{C}_c^\infty(C_j^{i,h}; [0, 1])$  be a cut-off function depending only on  $x_\alpha$  such that  $\phi = 1$  on  $\partial B'_{\rho_j^{i,h}}(x_i^{\delta_j})$  and  $|\nabla_\alpha \phi| \leq c/\rho_j^{i,h}$ . For every  $x \in (C_j^{i,h})^{\pm\varepsilon_j}$ , we set

$$w_j^{i,h}(x) := \phi(x_\alpha)(u_j^{i,h})^\pm + (1 - \phi(x_\alpha))u_j(x),$$

then

$$\begin{aligned} \int_{(C_j^{i,h})^{\pm\varepsilon_j}} |\nabla w_j^{i,h}|^p dx &\leq c \int_{(C_j^{i,h})^{\pm\varepsilon_j}} \left( |\nabla_\alpha \phi|^p |u_j - (u_j^{i,h})^\pm|^p + |\nabla u_j|^p \right) dx \\ &\leq c \int_{(C_j^{i,h})^{\pm\varepsilon_j}} \left( \frac{|u_j - (u_j^{i,h})^\pm|^p}{(\rho_j^{i,h})^p} + |\nabla u_j|^p \right) dx. \end{aligned}$$

Since  $\varepsilon_j \ll \delta_j$ , by (3.4.6) and Lemma 3.4.1 (i), with  $\rho = \rho_j^{i,h}$  and  $A_\rho = C_j^{i,h}$ , we have that

$$\begin{aligned} &\int_{(C_j^{i,h})^{\pm\varepsilon_j}} |\nabla w_j^{i,h}|^p dx \\ &\leq c \int_{(C_j^{i,h})^{\pm\varepsilon_j}} \left( |\nabla_\alpha u_j|^p + \left(\frac{\varepsilon_j}{\rho_j^{i,h}}\right)^p |\nabla_n u_j|^p \right) dx + c \int_{(C_j^{i,h})^{\pm\varepsilon_j}} |\nabla u_j|^p dx \\ &\leq m_j(k, \gamma) c \int_{(C_j^{i,h})^{\pm\varepsilon_j}} |\nabla u_j|^p dx, \end{aligned} \quad (3.4.7)$$

where,

$$m_j(k, \gamma) := \max\left\{1, \left(\frac{2^{k+1}}{3\gamma}\right)^p \left(\frac{\varepsilon_j}{\delta_j}\right)^p\right\} \xrightarrow{j \rightarrow +\infty} 1.$$

As

$$\sum_{h=0}^{k-1} \int_{(C_j^{i,h})^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx \leq \int_{B'_{\rho_j}(x_i^{\delta_j})^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx,$$

there exists  $k_i \in \{0, \dots, k-1\}$  such that, having set  $C_j^i := C_j^{i,k_i}$ , we get

$$\int_{(C_j^i)^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx \leq \frac{1}{k} \int_{B'_{\rho_j}(x_i^{\delta_j})^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx. \quad (3.4.8)$$

Hence, if we define the sequence

$$w_j(x) := \begin{cases} w_j^{i,k_i}(x) & \text{if } x \in (C_j^i)^{\pm\varepsilon_j} \text{ and } i \in Z_j \\ u_j(x) & \text{otherwise,} \end{cases}$$

by the  $p$ -growth condition (3.3.3), (3.4.7) and (3.4.8), we have

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |W(\nabla w_j) - W(\nabla u_j)| dx &= \sum_{i \in Z_j} \frac{1}{\varepsilon_j} \int_{(C_j^i)^{\pm\varepsilon_j}} |W(\nabla w_j^{i,k_i}) - W(\nabla u_j)| dx \\ &\leq \frac{c}{k} m_j(k, \gamma) \sum_{i \in Z_j} \frac{1}{\varepsilon_j} \int_{B'_{\rho_j}(x_i^{\delta_j})^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx \\ &\leq \frac{c}{k} m_j(k, \gamma) \left(1 + \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx\right). \end{aligned}$$

Then, (3.4.1) concludes the proof of (3.4.5). Note that, by construction,  $\{w_j\}$  satisfies (3.4.3) and (3.4.4) and it converges weakly to  $(u^+, u^-)$ . In fact,

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |w_j - u^\pm|^p dx &= \frac{1}{\varepsilon_j} \sum_{i \in Z_j} \int_{(C_j^i)^{\pm\varepsilon_j}} |\phi u_j^{i,\pm} + (1 - \phi)u_j - u^\pm|^p dx \\ &\quad + \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j} \setminus \bigcup_{i \in Z_j} (C_j^i)^{\pm\varepsilon_j}} |u_j - u^\pm|^p dx \\ &\leq \frac{c}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |u_j - u^\pm|^p dx + \frac{c}{\varepsilon_j} \sum_{i \in Z_j} \int_{(C_j^i)^{\pm\varepsilon_j}} |u_j - u_j^{i,\pm}|^p dx. \end{aligned}$$

Since  $\varepsilon_j \ll \delta_j$  and  $\rho_j^i \leq \delta_j$ , by Lemma 3.4.1 (i) applied with  $\rho = \rho_j^i$ , we get

$$\frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |w_j - u^\pm|^p dx \leq \frac{c}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |u_j - u^\pm|^p dx + c\delta_j^p \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx; \quad (3.4.9)$$

### 3.4. PRELIMINARY RESULTS

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moreover, by (3.4.7), we have

$$\frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla w_j|^p dx \leq \frac{c}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx. \quad (3.4.10)$$

Hence, by (3.4.9), (3.4.10), (3.4.1) and Remark 3.3.2 we get also the weak convergence of  $\{w_j\}$ .  $\square$

If the sequence  $\{|\nabla u_j|^p/\varepsilon_j\}$  is equi-integrable in  $\omega^{\pm\varepsilon_j}$  (see Definition 3.3.1), then we do not have to choose for every  $i \in Z_j$  a suitable cylinder  $(C_j^i)^{\pm\varepsilon_j}$  but we may consider the same radius independently of  $i$  as the following lemma shows.

**Lemma 3.4.3.** *Let  $\{u_j\}$ ,  $\{\delta_j\}$ ,  $\{\varepsilon_j\}$ ,  $\{\rho_j\}$  and  $Z_j$  be as in Lemma 3.4.2 and such that  $\{|\nabla u_j|^p/\varepsilon_j\}$  is equi-integrable in  $\omega^{\pm\varepsilon_j}$ . Set*

$$C_j^i := \left\{ x_\alpha \in \omega : \frac{2}{3}\rho_j < |x_\alpha - x_i^{\delta_j}| < \frac{4}{3}\rho_j \right\} \quad \text{and} \quad u_j^{i\pm} := \int_{(C_j^i)^{\pm\varepsilon_j}} u_j dx$$

for every  $i \in Z_j$ . Then, there exists a sequence  $\{w_j\} \subset W^{1,p}(\omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j}; \mathbb{R}^m)$  weakly converging to  $(u^+, u^-)$  such that

$$w_j = u_j \quad \text{in} \quad \left( \omega \setminus \bigcup_{i \in Z_j} C_j^i \right)^{\pm\varepsilon_j}, \quad (3.4.11)$$

$$w_j = u_j^{i\pm} \quad \text{on} \quad (\partial B'_{\rho_j}(x_i^{\delta_j}))^{\pm\varepsilon_j} \quad (3.4.12)$$

and

$$\limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |W(\nabla w_j) - W(\nabla u_j)| dx \leq o(1) \quad \text{as} \quad \gamma \rightarrow 0. \quad (3.4.13)$$

Moreover, the sequence  $\{|\nabla w_j|^p/\varepsilon_j\}$  is also equi-integrable in  $\omega^{\pm\varepsilon_j}$ .

*Proof.* Let  $\phi \equiv \phi_j^i \in C_c^\infty(C_j^i; [0, 1])$  be a cut-off function depending only on  $x_\alpha$  such that  $\phi = 1$  on  $\partial B'_{\rho_j}(x_i^{\delta_j})$  and  $|\nabla_\alpha \phi| \leq c/\rho_j$ . For every  $x \in (C_j^i)^{\pm\varepsilon_j}$ , we define

$$w_j^i(x) := \phi(x_\alpha) u_j^{i\pm} + (1 - \phi(x_\alpha)) u_j(x).$$

Then, reasoning as in the proof of Lemma 3.4.2, we have that

$$\int_{(C_j^i)^{\pm\varepsilon_j}} W(\nabla w_j^i) dx \leq c \int_{(C_j^i)^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx.$$

Hence, if we define

$$w_j(x) := \begin{cases} w_j^i(x) & \text{if } x \in (C_j^i)^{\pm\varepsilon_j} \text{ and } i \in Z_j, \\ u_j(x) & \text{otherwise.} \end{cases}$$

it satisfies (3.4.11) and (3.4.12). Moreover,

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |W(\nabla w_j) - W(\nabla u_j)| dx &\leq \sum_{i \in Z_j} \frac{1}{\varepsilon_j} \int_{(C_j^i)^{\pm\varepsilon_j}} |W(\nabla w_j^i) - W(\nabla u_j)| dx \\ &\leq c \sum_{i \in Z_j} \frac{1}{\varepsilon_j} \int_{(B'_{4\rho_j/3}(x_i^{\delta_j}) \cap \omega)^{\pm\varepsilon_j}} (1 + |\nabla u_j|^p) dx. \end{aligned}$$

Since  $\#(Z_j) \leq c/\delta_j^{n-1}$ , we get that

$$\mathcal{L}^{n-1} \left( \bigcup_{i \in Z_j} (B'_{4\rho_j/3}(x_i^{\delta_j}) \cap \omega) \right) \leq c\gamma^{n-1}$$

and by the equi-integrability of  $\{|\nabla u_j|^p/\varepsilon_j\}$ , we obtain (3.4.13). Finally, the weak convergence of  $\{w_j\}$  can be proved as in Lemma 3.4.2 while the equi-integrability of  $\{|\nabla w_j|^p/\varepsilon_j\}$  is just a consequence of the definition of  $\{w_j\}$ .  $\square$

### 3.5 Energy contribution close to the "connecting zones"

Let us consider the domain  $\Omega_j = \omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j} \cup (\omega_{\delta_j, r_j} \times \{0\})$  where  $\omega_{\delta_j, r_j} := \bigcup_{i \in Z^{n-1}} B'_{r_j}(x_i^{\delta_j}) \cap \omega$ . Our  $\Gamma$ -convergence analysis deals with the case where the thickness  $\varepsilon_j$  of  $\Omega_j$  is much smaller than the period  $\delta_j$  of distribution of the connecting zones  $B'_{r_j}(x_i^{\delta_j}) \times \{0\}$  i.e.

$$\lim_{j \rightarrow +\infty} \frac{\varepsilon_j}{\delta_j} = 0.$$

The aim of this section is to study the contribution close to the connecting zones. We may exclude that the radius  $r_j \geq \delta_j/2$  otherwise the zones may overlap. More precisely, we assume that  $r_j \ll \delta_j$  i.e.

$$\lim_{j \rightarrow +\infty} \frac{r_j}{\delta_j} = 0. \quad (3.5.1)$$

This choice will be justify a posteriori since it will be the only admissible assumption to get a meaningful  $\Gamma$ -convergence result.

Finally, it remains to fix the behavior of  $r_j$  with respect to  $\varepsilon_j$ . Let us denote

$$\ell := \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j}.$$

It yields us to consider all the possible scenario, namely to distinguish the cases where  $\ell$  is finite, infinite or zero.

#### 3.5.1 The case $\ell \in (0, +\infty]$

Setting  $N_j = \delta_j/r_j$ , we define the space

$$X_j^\gamma(z) := \left\{ \zeta \in W^{1,p}((B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}; \mathbb{R}^m) : \zeta = z \text{ on } (\partial B'_{\gamma N_j})^+, \zeta = 0 \text{ on } (\partial B'_{\gamma N_j})^- \right\},$$

we consider the following minimum problem

$$\varphi_{\gamma,j}^{(\ell)}(z) := \inf \left\{ \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p W \left( r_j^{-1} \nabla_\alpha \zeta \middle| \varepsilon_j^{-1} \nabla_n \zeta \right) dx : \zeta \in X_j^\gamma(z) \right\}. \quad (3.5.2)$$

In the next proposition we study the behavior of  $\{\varphi_{\gamma,j}^{(\ell)}\}$  as  $j \rightarrow +\infty$  and  $\gamma \rightarrow 0^+$ .

**Proposition 3.5.1.** *Let  $\ell \in (0, +\infty]$ . If*

$$0 < R^{(\ell)} := \lim_{j \rightarrow +\infty} \frac{r_j^{n-1-p}}{\delta_j^{n-1}} < +\infty \quad (3.5.3)$$

then,

(i) *there exists a constant  $c > 0$  (independent of  $j$  and  $\gamma$ ) such that*

$$0 \leq \varphi_{\gamma,j}^{(\ell)}(z) \leq c(|z|^p + \gamma^{n-1})$$

for all  $z \in \mathbb{R}^m$ ,  $j \in \mathbb{N}$  and  $\gamma > 0$ ;

(ii) *there exists a constant  $c > 0$  (independent of  $j$  and  $\gamma$ ) such that*

$$|\varphi_{\gamma,j}^{(\ell)}(z) - \varphi_{\gamma,j}^{(\ell)}(w)| \leq c|z - w| \left( \gamma^{(n-1)(p-1)/p} + r_j^{p-1} + |z|^{p-1} + |w|^{p-1} \right) \quad (3.5.4)$$

for every  $z, w \in \mathbb{R}^m$ ,  $j \in \mathbb{N}$  and  $\gamma > 0$ ;

(iii) *for every fixed  $\gamma > 0$ , up to subsequences,  $\varphi_{\gamma,j}^{(\ell)}$  converges locally uniformly on  $\mathbb{R}^m$  to  $\varphi_\gamma^{(\ell)}$  as  $j \rightarrow +\infty$  and*

$$|\varphi_\gamma^{(\ell)}(z) - \varphi_\gamma^{(\ell)}(w)| \leq c|z - w| \left( \gamma^{(n-1)(p-1)/p} + |z|^{p-1} + |w|^{p-1} \right) \quad (3.5.5)$$

for every  $z, w \in \mathbb{R}^m$ ;

(iv) *up to subsequences,  $\varphi_\gamma^{(\ell)}$  converges locally uniformly on  $\mathbb{R}^m$ , as  $\gamma \rightarrow 0$ , to a continuous function  $\varphi^{(\ell)} : \mathbb{R}^m \rightarrow [0, +\infty)$  satisfying*

$$0 \leq \varphi^{(\ell)}(z) \leq c|z|^p, \quad |\varphi^{(\ell)}(z) - \varphi^{(\ell)}(w)| \leq c|z - w|(|z|^{p-1} + |w|^{p-1}) \quad (3.5.6)$$

for every  $z, w \in \mathbb{R}^m$ .

*Proof.* Fix  $\gamma > 0$ , then  $\gamma N_j > 2$  for  $j$  large enough.

(i) According to the  $p$ -growth condition (3.3.3),

$$0 \leq \varphi_{\gamma,j}^{(\ell)}(z) \leq \beta \left( C_{\gamma,j}(z) + \mathcal{L}^{n-1}(B'_1) \gamma^{n-1} \frac{\delta_j^{n-1}}{r_j^{n-1-p}} \right), \quad (3.5.7)$$

where

$$C_{\gamma,j}(z) := \inf \left\{ \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left| \left( \nabla_\alpha \zeta \middle| \frac{r_j}{\varepsilon_j} \nabla_n \zeta \right) \right|^p dx : \zeta \in X_j^\gamma(z) \right\}.$$

Since  $\mathcal{C}_{\gamma,j}(z)$  is invariant by rotations, reasoning as in [6] Section 4.1, we can consider the minimization problem with respect to a particular class of scalar test functions as follows

$$\begin{aligned} \frac{\mathcal{C}_{\gamma,j}(z)}{|z|^p} &= \inf \left\{ \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left| \left( \nabla_\alpha \psi \left| \frac{r_j}{\varepsilon_j} \nabla_n \psi \right) \right|^p dx : \psi \in W^{1,p}((B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}), \right. \\ &\quad \left. \psi = 1 \text{ on } (\partial B'_{\gamma N_j})^+ \text{ and } \psi = 0 \text{ on } (\partial B'_{\gamma N_j})^- \right\} \\ &\leq \inf \left\{ \int_{B'_{\gamma N_j}} (|\nabla_\alpha \psi^+|^p + |\nabla_\alpha \psi^-|^p) dx : (\psi^+ - 1), \psi^- \in W_0^{1,p}(B'_{\gamma N_j}) \right. \\ &\quad \left. \text{and } \psi^+ = \psi^- \text{ in } B'_1 \right\}. \end{aligned} \quad (3.5.8)$$

Let  $\psi_1^\pm$  be the unique minimizer of the last strictly convex minimization problem in (3.5.8). It turns out that  $\psi_2^\pm := 1 - \psi_1^\mp$  is also a minimizer. Thus by uniqueness,  $\psi_1^\pm = \psi_2^\pm$  and in particular,  $\psi_1^\pm = 1/2$  in  $B'_1$ . Hence,

$$\begin{aligned} \mathcal{C}_{\gamma,j}(z) &\leq |z|^p \inf \left\{ \int_{B'_{\gamma N_j}} (|\nabla_\alpha \psi^+|^p + |\nabla_\alpha \psi^-|^p) dx_\alpha : (\psi^+ - 1), \psi^- \in W_0^{1,p}(B'_{\gamma N_j}), \right. \\ &\quad \left. \text{and } \psi^+ = \psi^- = \frac{1}{2} \text{ in } B'_1 \right\} \\ &= 2|z|^p \inf \left\{ \int_{B'_{\gamma N_j}} |\nabla_\alpha \psi|^p dx_\alpha : \psi \in W_0^{1,p}(B'_{\gamma N_j}) \text{ and } \psi = \frac{1}{2} \text{ in } B'_1 \right\} \\ &= \frac{|z|^p}{2^{p-1}} \inf \left\{ \int_{B'_{\gamma N_j}} |\nabla_\alpha \psi|^p dx_\alpha : \psi \in W_0^{1,p}(B'_{\gamma N_j}) \text{ and } \psi = 1 \text{ in } B'_1 \right\} \\ &= \frac{|z|^p}{2^{p-1}} \text{Cap}_p(B'_1; B'_{\gamma N_j}). \end{aligned} \quad (3.5.9)$$

Since

$$\lim_{j \rightarrow +\infty} \text{Cap}_p(B'_1; B'_{\gamma N_j}) = \text{Cap}_p(B'_1; \mathbb{R}^{n-1}) < +\infty;$$

hence, by (3.5.3), (3.5.7) and (3.5.9) we conclude the proof of (i).

(ii) For every  $\eta > 0$ , there exists  $\zeta_{\gamma,j} \in X_j^\gamma(z)$  such that

$$\int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p W \left( r_j^{-1} \nabla_\alpha \zeta_{\gamma,j} \left| \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j} \right) \right) dx \leq \varphi_{\gamma,j}^{(\ell)}(z) + \eta. \quad (3.5.10)$$

We want to modify  $\zeta_{\gamma,j}$  in order to get an admissible test function for  $\varphi_{\gamma,j}^{(\ell)}(w)$ . More precisely, we just have to modify  $\zeta_{\gamma,j}$  on a neighborhood of  $(\partial B'_{\gamma N_j})^+$  to change the boundary condition  $z$

### 3.5. ENERGY CONTRIBUTION CLOSE TO THE "CONNECTING ZONES"

into  $w$ . To this aim we introduce a cut-off function  $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1}; [0, 1])$ , independent of  $x_n$ , such that

$$\theta(x_\alpha) = \begin{cases} 1 & \text{if } x_\alpha \in B'_1, \\ 0 & \text{if } x_\alpha \notin B'_2 \end{cases} \quad \text{and} \quad |\nabla_\alpha \theta| \leq c.$$

Hence, we define  $\tilde{\zeta}_{\gamma,j} \in X_j^\gamma(w)$  as follows

$$\tilde{\zeta}_{\gamma,j}(x) = \begin{cases} \zeta_{\gamma,j}(x) + (1 - \theta(x_\alpha))(w - z) & \text{in } (B'_{\gamma N_j})^+ \\ \zeta_{\gamma,j}(x) & \text{in } (B'_{\gamma N_j})^- \cup (B'_1 \times \{0\}). \end{cases}$$

By (3.5.10), since  $\zeta_{\gamma,j} = \tilde{\zeta}_{\gamma,j}$  in  $(B'_{\gamma N_j})^-$ , we have that

$$\begin{aligned} & \varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) \\ & \leq r_j^p \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left( W(r_j^{-1} \nabla_\alpha \tilde{\zeta}_{\gamma,j} | \varepsilon_j^{-1} \nabla_n \tilde{\zeta}_{\gamma,j}) - W(r_j^{-1} \nabla_\alpha \zeta_{\gamma,j} | \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j}) \right) dx + \eta \\ & = r_j^p \int_{(B'_{\gamma N_j})^+} \left( W(r_j^{-1} \nabla_\alpha \tilde{\zeta}_{\gamma,j} | \varepsilon_j^{-1} \nabla_n \tilde{\zeta}_{\gamma,j}) - W(r_j^{-1} \nabla_\alpha \zeta_{\gamma,j} | \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j}) \right) dx + \eta. \end{aligned}$$

By (3.3.4) and Hölder's Inequality, we obtain that

$$\begin{aligned} & \varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) - \eta \\ & \leq c \int_{(B'_{\gamma N_j})^+} \left( r_j^{p-1} + \left| \left( \nabla_\alpha \zeta_{\gamma,j} \middle| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) \right|^{p-1} + \left| \left( \nabla_\alpha \tilde{\zeta}_{\gamma,j} \middle| \frac{r_j}{\varepsilon_j} \nabla_n \tilde{\zeta}_{\gamma,j} \right) \right|^{p-1} \right) \\ & \quad \times \left| \left( \nabla_\alpha \tilde{\zeta}_{\gamma,j} - \nabla_\alpha \zeta_{\gamma,j} \middle| \frac{r_j}{\varepsilon_j} (\nabla_n \tilde{\zeta}_{\gamma,j} - \nabla_n \zeta_{\gamma,j}) \right) \right| dx \\ & \leq c \int_{(B'_{\gamma N_j})^+} \left( r_j^{p-1} + 2 \left| \left( \nabla_\alpha \zeta_{\gamma,j} \middle| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) \right|^{p-1} + |\nabla_\alpha \theta|^{p-1} |w - z|^{p-1} \right) |\nabla_\alpha \theta| |w - z| dx \\ & \leq c |z - w|^p \int_{B'_{\gamma N_j}} |\nabla_\alpha \theta|^p dx_\alpha + c r_j^{p-1} |z - w| \int_{B'_{\gamma N_j}} |\nabla_\alpha \theta| dx_\alpha \\ & \quad + 2c |z - w| \|\nabla_\alpha \theta\|_{L^p(B'_{\gamma N_j}; \mathbb{R}^{n-1})} \left\| \left( \nabla_\alpha \zeta_{\gamma,j} \middle| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) \right\|_{L^p((B'_{\gamma N_j})^+; \mathbb{R}^{m \times n})}^{p-1}. \end{aligned}$$

Since  $\gamma N_j > 2$  and  $\text{Supp}(\theta) \subset B'_2$ , we obtain that

$$\begin{aligned} & \varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) \\ & \leq c |z - w| \left( |z - w|^{p-1} + r_j^{p-1} + \left\| \left( \nabla_\alpha \zeta_{\gamma,j} \middle| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) \right\|_{L^p((B'_{\gamma N_j})^+; \mathbb{R}^{m \times n})}^{p-1} \right) + \eta. \end{aligned} \quad (3.5.11)$$



By the  $p$ -growth condition (3.3.3), (3.5.10) and (i), we have that

$$\begin{aligned}
 & \int_{(B'_{\gamma N_j})^+} \left| \left( \nabla_\alpha \zeta_{\gamma,j} \left| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) \right|^p dx \\
 & \leq \int_{(B'_{\gamma N_j})^+} r_j^p W \left( r_j^{-1} \nabla_\alpha \zeta_{\gamma,j} \left| \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j} \right) \right) dx + r_j^p \mathcal{L}^{n-1}(B'_{\gamma N_j}) \\
 & \leq \varphi_{\gamma,j}^{(\ell)}(z) + \eta + c\gamma^{n-1} \frac{\delta_j^{n-1}}{r_j^{n-1-p}} \\
 & \leq c(|z|^p + \gamma^{n-1}) + \eta + c\gamma^{n-1} \frac{\delta_j^{n-1}}{r_j^{n-1-p}}.
 \end{aligned} \tag{3.5.12}$$

Hence, by (3.5.11), (3.5.12) and (3.5.3) we have that

$$\varphi_{\gamma,j}^{(\ell)}(w) - \varphi_{\gamma,j}^{(\ell)}(z) \leq c|z - w| \left( |z|^{p-1} + |w|^{p-1} + r_j^{p-1} + \gamma^{(n-1)(p-1)/p} + \eta^{(p-1)/p} \right) + \eta$$

and (3.5.4) follows by the arbitrariness of  $\eta$ .

By (ii) and Ascoli-Arzelà's Theorem we have that, up to subsequences,  $\varphi_{\gamma,j}^{(\ell)}$  converges uniformly on compact sets of  $\mathbb{R}^m$  to  $\varphi_\gamma^{(\ell)}$  as  $j \rightarrow +\infty$ . Moreover, passing to the limit in (3.5.4) as  $j \rightarrow +\infty$  we get

$$|\varphi_\gamma^{(\ell)}(w) - \varphi_\gamma^{(\ell)}(z)| \leq c|z - w| \left( |z|^{p-1} + |w|^{p-1} + \gamma^{(n-1)(p-1)/p} \right).$$

Hence, we can apply again Ascoli-Arzelà's Theorem to conclude that, up to subsequences,  $\varphi_\gamma^{(\ell)}$  converges uniformly on compact sets of  $\mathbb{R}^m$  to  $\varphi^{(\ell)}$  as  $\gamma \rightarrow 0^+$ . In particular,  $\varphi^{(\ell)} : \mathbb{R}^m \rightarrow [0, +\infty)$  is a continuous function and

$$0 \leq \varphi^{(\ell)}(z) \leq c|z|^p, \quad |\varphi^{(\ell)}(z) - \varphi^{(\ell)}(w)| \leq c(|z|^{p-1} + |w|^{p-1})|z - w|$$

for every  $z, w \in \mathbb{R}^m$ . □

### 3.5.2 The case $\ell = 0$

If  $\ell = 0$  we expect that the contribution due to the connecting zones is obtained studying the behavior, as  $j \rightarrow +\infty$  and  $\gamma \rightarrow 0^+$ , of the sequence  $\varphi_{\gamma,j}^{(0)}$  defined as follows

$$\begin{aligned}
 \varphi_{\gamma,j}^{(0)}(z) & := \frac{\varepsilon_j}{r_j} \inf \left\{ \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p W \left( r_j^{-1} \nabla_\alpha \zeta \left| \varepsilon_j^{-1} \nabla_n \zeta \right) \right) dx : \zeta \in X_j^\gamma(z) \right\} \\
 & = \inf \left\{ \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} r_j^p W(r_j^{-1} \nabla \zeta) dx : \zeta \in Y_j^\gamma(z) \right\}
 \end{aligned} \tag{3.5.13}$$

where the second equality has been obtained changing variable,  $I_j := (-\varepsilon_j/r_j, \varepsilon_j/r_j)$  and

$$Y_j^\gamma(z) = \left\{ \zeta \in W^{1,p}((B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}; \mathbb{R}^m) : \begin{aligned} \zeta &= z \text{ on } (\partial B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}, \\ \zeta &= 0 \text{ on } (\partial B'_{\gamma N_j})^{-(\varepsilon_j/r_j)} \end{aligned} \right\}.$$

Arguing as in the proof of Proposition 3.5.1, we want to prove the same properties for  $\varphi_{\gamma,j}^{(0)}$ .

**Proposition 3.5.2.** *Let  $\ell = 0$ . If*

$$0 < R^{(0)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-p}}{\delta_j^{n-1} \varepsilon_j} < +\infty \quad (3.5.14)$$

then,

(i) *there exists a constant  $c > 0$  (independent of  $j$  and  $\gamma$ ) such that*

$$0 \leq \varphi_{\gamma,j}^{(0)}(z) \leq c(|z|^p + \gamma^{n-1})$$

for all  $z \in \mathbb{R}^m$ ,  $j \in \mathbb{N}$  and  $\gamma > 0$ ;

(ii) *there exists a constant  $c > 0$  (independent of  $j$  and  $\gamma$ ) such that*

$$|\varphi_{\gamma,j}^{(0)}(z) - \varphi_{\gamma,j}^{(0)}(w)| \leq c|z - w| \left( \gamma^{(n-1)(p-1)/p} + r_j^{n-1} + |z|^{p-1} + |w|^{p-1} \right) \quad (3.5.15)$$

for every  $z, w \in \mathbb{R}^m$ ,  $j \in \mathbb{N}$  and  $\gamma > 0$ ;

(iii) *for every fixed  $\gamma > 0$ , up to subsequences,  $\varphi_{\gamma,j}^{(0)}$  converges locally uniformly in  $\mathbb{R}^m$  to  $\varphi_\gamma^{(0)}$  as  $j \rightarrow +\infty$ , and*

$$|\varphi_\gamma^{(0)}(w) - \varphi_\gamma^{(0)}(z)| \leq c|z - w| \left( \gamma^{(n-1)(p-1)/p} + |z|^{p-1} + |w|^{p-1} \right) \quad (3.5.16)$$

for every  $z, w \in \mathbb{R}^m$ ;

(iv) *up to subsequences,  $\varphi_\gamma^{(0)}$  converges locally uniformly in  $\mathbb{R}^m$ , as  $\gamma \rightarrow 0^+$ , to a continuous function  $\varphi^{(0)} : \mathbb{R}^m \rightarrow [0, +\infty)$  satisfying*

$$0 \leq \varphi^{(0)}(z) \leq c|z|^p, \quad |\varphi^{(0)}(z) - \varphi^{(0)}(w)| \leq c|z - w| (|z|^{p-1} + |w|^{p-1}) \quad (3.5.17)$$

for every  $z, w \in \mathbb{R}^m$ .

*Proof.* Fix  $\gamma > 0$ , then  $\gamma N_j > 2$  and  $\varepsilon_j/r_j > 2$  for  $j$  large enough.

(i) According to the  $p$ -growth condition (3.3.3),

$$0 \leq \varphi_{\gamma,j}^{(0)}(z) \leq \beta \left( \mathcal{C}_{\gamma,j}(z) + 2\mathcal{L}^{n-1}(B'_1) \gamma^{n-1} \frac{\varepsilon_j \delta_j^{n-1}}{r_j^{n-p}} \right), \quad (3.5.18)$$

where

$$\mathcal{C}_{\gamma,j}(z) = \inf \left\{ \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} |\nabla \zeta|^p dx : \zeta \in Y_j^\gamma(z) \right\}.$$

Arguing similarly as in the proof of Proposition 3.5.1, we can rewrite

$$\begin{aligned} \frac{\mathcal{C}_{\gamma,j}(z)}{|z|^p} &= \inf \left\{ \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} |\nabla \psi|^p dx : \psi \in W^{1,p}((B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}), \right. \\ &\quad \left. \psi = 1 \text{ on } (\partial B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}, \quad \psi = 0 \text{ on } (\partial B'_{\gamma N_j})^{-(\varepsilon_j/r_j)} \right\}. \end{aligned} \quad (3.5.19)$$

Let  $\psi_1$  be the unique minimizer of the strictly convex minimization problem (3.5.19). It turns out that  $\psi_2(x_\alpha, x_n) := 1 - \psi_1(x_\alpha, -x_n)$  is also a minimizer. Thus by uniqueness,  $\psi_1 = \psi_2$  and in particular,  $\psi_1 = \psi_2 = 1/2$  on  $B'_1 \times \{0\}$ . Thus

$$\begin{aligned} \mathcal{C}_{\gamma,j}(z) &= 2|z|^p \inf \left\{ \int_{(B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}} |\nabla \psi|^p dx : \psi \in W^{1,p}((B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}), \right. \\ &\quad \left. \psi = 0 \text{ on } (\partial B'_{\gamma N_j})^{+(\varepsilon_j/r_j)} \text{ and } \psi = \frac{1}{2} \text{ on } B'_1 \times \{0\} \right\} \\ &= \frac{|z|^p}{2^{p-1}} \inf \left\{ \int_{(B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}} |\nabla \psi|^p dx : \psi \in W^{1,p}((B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}), \right. \\ &\quad \left. \psi = 0 \text{ on } (\partial B'_{\gamma N_j})^{+(\varepsilon_j/r_j)} \text{ and } \psi = 1 \text{ on } B'_1 \times \{0\} \right\} \\ &\leq \frac{|z|^p}{2^p} \text{Cap}_p(B'_1; B'_{\gamma N_j} \times I_j). \end{aligned} \quad (3.5.20)$$

Since

$$\lim_{j \rightarrow +\infty} \text{Cap}_p(B'_1; B'_{\gamma N_j} \times I_j) = \text{Cap}_p(B'_1; \mathbb{R}^n) < +\infty;$$

hence, by (3.5.14), (3.5.18) and (3.5.20) we conclude the proof of (i).

(ii) We can proceed as in the proof of Proposition 3.5.1 (ii) using a different cut-off function also depending on  $x_n$ . Namely, let  $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^n; [0, 1])$  be such that

$$\theta(x_\alpha, x_n) = \begin{cases} 1 & \text{if } (x_\alpha, x_n) \in B'_1 \times (-1, 1), \\ 0 & \text{if } (x_\alpha, x_n) \notin B'_2 \times (-2, 2) \end{cases} \quad \text{and} \quad |\nabla \theta| \leq c.$$

Hence, if  $\zeta_{\gamma,j} \in Y_j^\gamma(z)$  is a sequence which almost attains the infimum value  $\varphi_{\gamma,j}^{(0)}$ , we define  $\tilde{\zeta}_{\gamma,j} \in Y_j^\gamma(w)$  as follows

$$\tilde{\zeta}_{\gamma,j}(x) = \begin{cases} \zeta_{\gamma,j}(x) + (1 - \theta(x))(w - z) & \text{in } (B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}, \\ \zeta_{\gamma,j}(x) & \text{in } ((B'_{\gamma N_j})^{-(\varepsilon_j/r_j)}) \cup (B'_1 \times \{0\}). \end{cases}$$

By (3.5.14) we conclude the proof of (ii) by similar computations as in the proof of Proposition 3.5.1 (ii).

The proof of (iii) and (iv) follows as in the proof of Proposition 3.5.1 (iii) and (iv).  $\square$

### 3.5.3 Discrete approximation for the interfacial energy

We are now able to describe the contribution close to the connecting zones as  $j \rightarrow +\infty$  and  $\gamma \rightarrow 0^+$ .

**Proposition 3.5.3.** *Let  $\ell \in [0, +\infty]$ . Let  $\{u_j\}$  be a sequence weakly converging to  $(u^+, u^-)$  in the sense of Definition 3.3.1 such that  $\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\Omega_j; \mathbb{R}^m)} < +\infty$ , and let  $u_j^{i\pm}$  be as in (3.4.2). Then,*

$$\lim_{\gamma \rightarrow 0^+} \lim_{j \rightarrow +\infty} \int_{\omega} \left| \sum_{i \in Z_j} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right| dx_\alpha = 0. \quad (3.5.21)$$

*Proof.* Since  $\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\Omega_j; \mathbb{R}^m)} < +\infty$ , by Propositions 3.5.1 or 3.5.2, we have that

$$\begin{aligned} & \limsup_{j \rightarrow +\infty} \int_{\omega} \left| \sum_{i \in Z_j} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right| dx_\alpha \\ & \leq \limsup_{j \rightarrow +\infty} \int_{\omega} \sum_{i \in Z_j} \left| \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) - \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \right| \chi_{Q'_{i, \delta_j}} dx_\alpha \\ & \quad + \limsup_{j \rightarrow +\infty} \int_{\omega} \left| \sum_{i \in Z_j} \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right| dx_\alpha \\ & \leq o(1) + \limsup_{j \rightarrow +\infty} \int_{\omega} \left| \sum_{i \in Z_j} \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right| dx_\alpha, \end{aligned}$$

as  $\gamma \rightarrow 0^+$ . By (3.5.6) or (3.5.17) and Hölder's Inequality we have that

$$\begin{aligned} & \limsup_{j \rightarrow +\infty} \int_{\omega} \left| \sum_{i \in Z_j} \varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right| dx_\alpha \\ & \leq \limsup_{j \rightarrow +\infty} \sum_{i \in Z_j} \int_{Q'_{i, \delta_j}} |\varphi^{(\ell)}(u_j^{i+} - u_j^{i-}) - \varphi^{(\ell)}(u^+ - u^-)| dx_\alpha \\ & \leq c \limsup_{j \rightarrow +\infty} \left( \sum_{i \in Z_j} \int_{Q'_{i, \delta_j}} (|u_j^{i+} - u^+|^p + |u_j^{i-} - u^-|^p) dx_\alpha \right)^{1/p}. \end{aligned}$$

Hence, it remains to prove that

$$\limsup_{j \rightarrow +\infty} \sum_{i \in Z_j} \int_{Q'_{i, \delta_j}} |u^\pm - u_j^{i\pm}|^p dx_\alpha = 0. \quad (3.5.22)$$

By Lemma 3.4.1 (ii) applied with  $\rho = \delta_j$ ,  $B_\rho = C_j^i$  and  $A_\rho = Q'_{i,\delta_j}$ , we have since  $\varepsilon_j \ll \delta_j$

$$\begin{aligned} \int_{Q'_{i,\delta_j}} |u^\pm - u_j^{i\pm}|^p dx_\alpha &\leq \frac{c}{\varepsilon_j} \left( \int_{(Q'_{i,\delta_j})^\pm} |u_j - u^\pm|^p dx + \int_{(Q'_{i,\delta_j})^\pm} |u_j - u_j^{i\pm}|^p dx \right) \\ &\leq \frac{c}{\varepsilon_j} \int_{(Q'_{i,\delta_j})^\pm} |u_j - u^\pm|^p dx + \frac{c\delta_j^p}{\varepsilon_j} \int_{(Q'_{i,\delta_j})^\pm} |\nabla u_j|^p dx, \end{aligned} \quad (3.5.23)$$

for all  $i \in Z_j$ ; hence, summing up on  $i \in Z_j$ , we get

$$\sum_{i \in Z_j} \int_{Q'_{i,\delta_j}} |u^\pm - u_j^{i\pm}|^p dx_\alpha \leq \frac{c}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |u_j - u^\pm|^p dx + \frac{c\delta_j^p}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx,$$

and, passing into the limit as  $j \rightarrow +\infty$ , by Remark 3.3.2 we get (3.5.22) which concludes the proof of (3.5.21).  $\square$

**Remark 3.5.4.** If  $\ell \in (0, +\infty)$  then

$$0 < R^{(\ell)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-1-p}}{\delta_j^{n-1}} < +\infty \quad \text{if and only if} \quad 0 < \lim_{j \rightarrow +\infty} \frac{r_j^{n-p}}{\delta_j^{n-1}\varepsilon_j} < +\infty;$$

hence, both describe equivalently the meaningful radius.

## 3.6 $\Gamma$ -convergence result

### 3.6.1 The liminf inequality

Let  $\{u_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m) \cap L^\infty(\Omega_j; \mathbb{R}^m)$  be a sequence converging to  $(u^+, u^-) \in W^{1,p}(\omega, \mathbb{R}^m) \times W^{1,p}(\omega, \mathbb{R}^m)$ , in the sense of Definition 3.3.1, such that  $\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\Omega_j; \mathbb{R}^m)} < +\infty$  and

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j) < +\infty.$$

By Lemma 3.4.2, for every fixed  $k \in \mathbb{N}$ , there exists a sequence  $\{w_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m) \cap L^\infty(\Omega_j; \mathbb{R}^m)$  weakly converging to  $(u^+, u^-)$  satisfying (3.4.3), (3.4.4) and such that

$$\begin{aligned} &\liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{\omega^{+\varepsilon_j}} W(\nabla u_j) dx + \int_{\omega^{-\varepsilon_j}} W(\nabla u_j) dx \right) \\ &\geq \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{\omega^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{\omega^{-\varepsilon_j}} W(\nabla w_j) dx \right) - \frac{c}{k} \\ &\geq \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{(\omega \setminus E_j)^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{(\omega \setminus E_j)^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\ &\quad + \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{E_j^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{E_j^{-\varepsilon_j}} W(\nabla w_j) dx \right) - \frac{c}{k}, \end{aligned} \quad (3.6.1)$$

### 3.6. $\Gamma$ -CONVERGENCE RESULT

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where  $E_j := \bigcup_{i \in Z_j} B'_{\rho_j^i}(x_i^{\delta_j})$ .

We first treat the contribution far from the connecting zones. In this case, we just extend the sequence  $\{w_j\}$  inside each cylinder surrounding the contact zones by the value  $u_j^{i+}$  and  $u_j^{i-}$  in the upper and lower part, respectively. Then, we apply the classical result of dimensional reduction proved in [74] to the upper and lower part of the cylinder, separately.

**Proposition 3.6.1.** *We have*

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{(\omega \setminus E_j)^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{(\omega \setminus E_j)^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\ \geq \int_{\omega} (\mathcal{Q}W_0(\nabla_{\alpha} u^+) + \mathcal{Q}W_0(\nabla_{\alpha} u^-)) dx_{\alpha}. \end{aligned}$$

*Proof.* We define

$$v_j := \begin{cases} w_j & \text{in } (\omega \setminus E_j)^{\pm\varepsilon_j}, \\ u_j^{i\pm} & \text{in } B'_{\rho_j^i}(x_i^{\delta_j})^{\pm\varepsilon_j} \text{ if } i \in Z_j. \end{cases} \quad (3.6.2)$$

Then  $\{v_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$  converges weakly to  $(u^+, u^-)$ . In fact,

$$\sup_{j \in \mathbb{N}} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla v_j|^p dx \leq \sup_{j \in \mathbb{N}} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx < +\infty. \quad (3.6.3)$$

Moreover, since  $\rho_j^i < \rho_j < \delta_j/2$ , then  $B'_{\rho_j^i}(x_i^{\delta_j}) \subset Q'_{i,\delta_j}$ ; hence,

$$\int_{\omega^{\pm\varepsilon_j}} |v_j - u^{\pm}|^p dx \leq \int_{(\omega \setminus E_j)^{\pm\varepsilon_j}} |w_j - u^{\pm}|^p dx + \sum_{i \in Z_j} \int_{(Q'_{i,\delta_j})^{\pm\varepsilon_j}} |u^{\pm} - u_j^{i\pm}|^p dx$$

and arguing as in (3.5.23), we obtain that

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |v_j - u^{\pm}|^p dx &\leq \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |w_j - u^{\pm}|^p dx + \frac{c}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |u_j - u^{\pm}|^p dx \\ &\quad + \frac{c \delta_j^p}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla u_j|^p dx. \end{aligned} \quad (3.6.4)$$

Passing to the limit as  $j \rightarrow +\infty$  in (3.6.4), by (3.6.3) and Remark 3.3.2 we get that  $\{v_j\}$  converges weakly to  $(u^+, u^-)$ .

Since  $W(0) = 0$ , by (3.6.2) and [74] Theorem 2, we have

$$\begin{aligned}
 & \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{(\omega \setminus E_j)^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{(\omega \setminus E_j)^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\
 &= \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{(\omega \setminus E_j)^{+\varepsilon_j}} W(\nabla v_j) dx + \int_{(\omega \setminus E_j)^{-\varepsilon_j}} W(\nabla v_j) dx \right) \\
 &= \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{\omega^{+\varepsilon_j}} W(\nabla v_j) dx + \int_{\omega^{-\varepsilon_j}} W(\nabla v_j) dx \right) \\
 &\geq \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^+) dx_{\alpha} + \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^-) dx_{\alpha}.
 \end{aligned}$$

□

Let us now deal with the contribution near the connecting zones. We want to modify  $\{w_j\}$  in each surrounding cylinder in order to get an admissible test function for the minimum problems (3.5.2) or (3.5.13).

**Proposition 3.6.2.** *Let  $\ell \in [0, +\infty]$ . Then*

$$\liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{E_j^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{E_j^{-\varepsilon_j}} W(\nabla w_j) dx \right) \geq R^{(\ell)} \int_{\omega} \varphi^{(\ell)}(u^+ - u^-) dx_{\alpha} + o(1),$$

as  $\gamma \rightarrow 0^+$ .

*Proof.* Let  $\ell \in (0, +\infty]$ , the case  $\ell = 0$  can be treated similarly. Let  $i \in Z_j$ . Since  $\rho_j^i < \gamma \delta_j$ , we can define

$$\zeta_j^i(y_{\alpha}, y_n) := \begin{cases} w_j(x_i^{\delta_j} + r_j y_{\alpha}, \varepsilon_j y_n) - u_j^{i-} & \text{in } (B'_{\rho_j^i/r_j} \times I) \setminus C_{1, \rho_j^i/r_j}, \\ (u_j^{i+} - u_j^{i-}) & \text{in } (B'_{\gamma N_j} \setminus B'_{\rho_j^i/r_j})^+, \\ 0 & \text{in } (B'_{\gamma N_j} \setminus B'_{\rho_j^i/r_j})^- \end{cases}$$

where  $N_j = \delta_j/r_j$ . Then  $\zeta_j^i \in W^{1,p}((B'_{\gamma N_j} \times I) \setminus C_{1, \gamma N_j}; \mathbb{R}^m)$ ,  $\zeta_j^i = (u_j^{i+} - u_j^{i-})$  on  $(\partial B'_{\gamma N_j})^+$  and

$\zeta_j^i = 0$  on  $(\partial B'_{\gamma N_j})^-$ . Since  $W(0) = 0$ , changing variable, by (3.5.2) we get

$$\begin{aligned}
 & \frac{1}{\varepsilon_j} \left( \int_{B'_{\rho_j^i}(x_i^j)^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{B'_{\rho_j^i}(x_i^j)^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\
 &= r_j^{n-1} \left( \int_{(B'_{\rho_j^i/r_j})^+} W(r_j^{-1} \nabla_\alpha \zeta_j^i | \varepsilon_j^{-1} \nabla_n \zeta_j^i) dy + \int_{(B'_{\rho_j^i/r_j})^-} W(r_j^{-1} \nabla_\alpha \zeta_j^i | \varepsilon_j^{-1} \nabla_n \zeta_j^i) dy \right) \\
 &= r_j^{n-1} \int_{(B'_{\gamma N_j} \times I) \setminus C_{1, \gamma N_j}} W(r_j^{-1} \nabla_\alpha \zeta_j^i | \varepsilon_j^{-1} \nabla_n \zeta_j^i) dy \\
 &\geq r_j^{n-1-p} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}). \tag{3.6.5}
 \end{aligned}$$

Summing up in (3.6.5), for  $i \in Z_j$ , we get that

$$\begin{aligned}
 & \frac{1}{\varepsilon_j} \left( \int_{E_j^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{E_j^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\
 &= \sum_{i \in Z_j} \frac{1}{\varepsilon_j} \left( \int_{B'_{\rho_j^i}(x_i^j)^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{B'_{\rho_j^i}(x_i^j)^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\
 &\geq r_j^{n-1-p} \sum_{i \in Z_j} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) = \frac{r_j^{n-1-p}}{\delta_j^{n-1}} \sum_{i \in Z_j} \delta_j^{n-1} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}).
 \end{aligned}$$

Passing to the limit as  $j \rightarrow +\infty$  we get, by (3.5.3) and Proposition 3.5.3, that

$$\begin{aligned}
 & \liminf_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{E_j^{+\varepsilon_j}} W(\nabla w_j) dx + \int_{E_j^{-\varepsilon_j}} W(\nabla w_j) dx \right) \\
 &\geq R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-) dx_\alpha \\
 &\quad + R^{(\ell)} \liminf_{j \rightarrow +\infty} \int_\omega \left( \sum_{i \in Z_j} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right) dx_\alpha \\
 &= R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-) dx_\alpha + o(1),
 \end{aligned}$$

as  $\gamma \rightarrow 0^+$ , which completes the proof.  $\square$

We now prove the liminf inequality for any arbitrary converging sequence.

**Lemma 3.6.3.** *Let  $\ell \in [0, +\infty]$ . Then for every sequence  $\{u_j\}$  converging to  $(u^+, u^-)$  we have*

$$\begin{aligned}
 \liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j) &\geq \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^+) dx_\alpha + \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^-) dx_\alpha \\
 &\quad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-) dx_\alpha.
 \end{aligned}$$



*Proof.* Let  $u_j \rightarrow (u^+, u^-)$  such that  $\liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j) < +\infty$ . Reasoning as in [6] Proposition 5.2, by [33] Lemma 3.5, upon passing to a subsequence, we have for every  $M > 0$  and  $\eta > 0$ , the existence of  $R_M > M$  and of a Lipschitz function  $\Phi_M \in \mathcal{C}_c^1(\mathbb{R}^m; \mathbb{R}^m)$  with  $\text{Lip}(\Phi_M) = 1$  such that

$$\Phi_M(z) = \begin{cases} z & \text{if } |z| < R_M, \\ 0 & \text{if } |z| > 2R_M \end{cases}$$

and

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j) \geq \liminf_{j \rightarrow +\infty} \mathcal{F}_j(\Phi_M(u_j)) - \eta. \quad (3.6.6)$$

Note that  $\Phi_M(u_j) \in W^{1,p}(\Omega_j; \mathbb{R}^m) \cap L^\infty(\Omega_j; \mathbb{R}^m)$ ,  $\sup_{j \in \mathbb{N}} \|\Phi_M(u_j)\|_{L^\infty(\Omega_j; \mathbb{R}^m)} < R_M$  and it converges to  $(\Phi_M(u^+), \Phi_M(u^-))$  as  $j \rightarrow +\infty$ . Hence, if we apply (3.6.1), Propositions 3.6.1 and 3.6.2 to  $\{\Phi_M(u_j)\}$  in place of  $\{u_j\}$ , we get, letting  $k \rightarrow +\infty$  and  $\gamma \rightarrow 0$ , that

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \mathcal{F}_j(\Phi_M(u_j)) &\geq \int_\omega \mathcal{Q}W_0(\nabla_\alpha \Phi_M(u^+)) dx_\alpha + \int_\omega \mathcal{Q}W_0(\nabla_\alpha \Phi_M(u^-)) dx_\alpha \\ &\quad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(\Phi_M(u^+) - \Phi_M(u^-)) dx_\alpha. \end{aligned} \quad (3.6.7)$$

Note that  $\Phi_M(u^\pm) \rightharpoonup u^\pm$  weakly in  $W^{1,p}(\omega; \mathbb{R}^m)$  as  $M \rightarrow +\infty$ ; hence, by (3.6.6), (3.6.7), (3.5.6) and the lower semicontinuity of  $\int_\omega \mathcal{Q}W_0(\nabla_\alpha \zeta) dx_\alpha$  with respect to the weak  $W^{1,p}(\omega; \mathbb{R}^m)$ -convergence (see [1]), we have that

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \mathcal{F}_j(u_j) &\geq \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^+) dx_\alpha + \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^-) dx_\alpha \\ &\quad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-) dx_\alpha. \end{aligned}$$

by the arbitrariness of  $\eta$ . □

### 3.6.2 The limsup inequality

For every  $(u^+, u^-) \in W^{1,p}(\omega, \mathbb{R}^m) \times W^{1,p}(\omega, \mathbb{R}^m)$  the limsup inequality is obtained by suitably modifying the recovery sequences  $\{u_j^\pm\}$  for the  $\Gamma$ -limits of

$$\frac{1}{\varepsilon_j} \int_{\omega^{+\varepsilon_j}} W(\nabla u^+) dx \quad \text{and} \quad \frac{1}{\varepsilon_j} \int_{\omega^{-\varepsilon_j}} W(\nabla u^-) dx.$$

**Lemma 3.6.4.** *Let  $\ell \in [0, +\infty]$  and let  $\omega$  be an open bounded subset of  $\mathbb{R}^{n-1}$  such that  $\mathcal{L}^{n-1}(\partial\omega) = 0$ . Then, for all  $(u^+, u^-) \in W^{1,p}(\omega, \mathbb{R}^m) \times W^{1,p}(\omega, \mathbb{R}^m)$  and for all  $\eta > 0$  there exists a sequence  $\{\bar{u}_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$  converging to  $(u^+, u^-)$  such that*

$$\begin{aligned} \limsup_{j \rightarrow +\infty} \mathcal{F}_j(\bar{u}_j) &\leq \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^+) dx_\alpha + \int_\omega \mathcal{Q}W_0(\nabla_\alpha u^-) dx_\alpha \\ &\quad + R^{(\ell)} \int_\omega \varphi^{(\ell)}(u^+ - u^-) dx_\alpha + \eta R^{(\ell)} \mathcal{L}^{n-1}(\omega). \end{aligned}$$

*Proof.* The proof of the upper bound is divided into three steps. We first construct a sequence  $\{\bar{u}_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$  that we expect to be a recovery sequence. In the second step we prove that  $\{\bar{u}_j\}$  converges to  $(u^+, u^-)$  in the sense of Definition 3.3.1. Finally, we prove that it satisfies the lim sup inequality. We first deal with the case  $\ell \in (0, +\infty]$ .

**Step 1 : Definition of a recovery sequence.** Let  $u^\pm \in W^{1,p}(\omega; \mathbb{R}^m) \cap L^\infty(\omega; \mathbb{R}^m)$ . According to [74] Theorem 2 and [25] Theorem 1.1, there exist two sequences  $\{u_j^\pm\} \subset W^{1,p}(\omega^{\pm\varepsilon_j}; \mathbb{R}^m)$  such that  $u_j^\pm \rightarrow u^\pm$ , with respect to the convergence as in Definition 3.3.1, the sequences of gradients  $\{|\nabla u_j^\pm|^p/\varepsilon_j\}$  are equi-integrable on  $\omega^{\pm\varepsilon_j}$ , respectively, and

$$\lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} W(\nabla u_j^\pm) dx = \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u^\pm) dx_\alpha. \quad (3.6.8)$$

Moreover, using a truncation argument (as in [9] Lemma 6.1, Step 2), we may assume without loss of generality that

$$\sup_{j \in \mathbb{N}} \|u_j^\pm\|_{L^\infty(\omega^{\pm\varepsilon_j}; \mathbb{R}^m)} < +\infty.$$

Let  $u_j := u_j^+ \chi_{\omega^{+\varepsilon_j}} + u_j^- \chi_{\omega^{-\varepsilon_j}} \in W^{1,p}(\omega^{+\varepsilon_j} \cup \omega^{-\varepsilon_j}; \mathbb{R}^m)$  and let  $\{w_j\}$  be the sequence obtained from  $\{u_j\}$  as in Lemma 3.4.3, then  $\sup_{j \in \mathbb{N}} \|w_j\|_{L^\infty(\omega^{\pm\varepsilon_j}; \mathbb{R}^m)} < +\infty$ .

We first define  $\bar{u}_j$  far from the connecting zones i.e.

$$\bar{u}_j(x) := w_j(x) \text{ for every } x \in \left( \omega \setminus \bigcup_{i \in \mathbb{Z}^{n-1}} B'_{\rho_j}(x_i^{\delta_j}) \right)^{\pm\varepsilon_j}. \quad (3.6.9)$$

Then we pass to define  $\bar{u}_j$  on each  $B'_{\rho_j}(x_i^{\delta_j})^{\pm\varepsilon_j}$  making a distinction between  $i \in Z_j$  and  $i \in \mathbb{Z}^{n-1} \setminus Z_j$ .

If  $i \in Z_j$ , by (3.5.2), for every  $\eta > 0$  there exists  $\zeta_{\gamma,j}^i \in X_j^\gamma(u_j^{i+} - u_j^{i-})$  such that

$$\int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p W \left( r_j^{-1} \nabla_\alpha \zeta_{\gamma,j}^i | \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j}^i \right) dx \leq \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) + \eta. \quad (3.6.10)$$

Then, we define

$$\bar{u}_j(x_\alpha, x_n) := \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) + u_j^{i-} \text{ for every } x \in B'_{\rho_j}(x_i^{\delta_j})^{\pm\varepsilon_j}, \quad i \in Z_j. \quad (3.6.11)$$

In particular,  $\bar{u}_j = u_j^{i\pm} = w_j$  on  $(\partial B'_{\rho_j}(x_i^{\delta_j}))^{\pm\varepsilon_j}$ .

Let us now treat the contact zones not well contained in  $\omega$ , i.e. when  $i \notin Z_j$ . For fixed  $\gamma > 0$  and  $j$  large enough we have that  $\gamma N_j > 2$ . Let  $\psi \in W^{1,p}(B'_2; [0, 1])$  be such that  $\psi = 1$  on  $\partial B'_2$  and  $\psi = 0$  in  $B'_1$  and define

$$\psi_{\gamma,j}(x) := \begin{cases} 0 & \text{in } (B'_{\gamma N_j})^-, \\ \psi(x_\alpha) & \text{in } (B'_2)^+, \\ 1 & \text{in } (B'_{\gamma N_j} \setminus B'_2)^+. \end{cases}$$

Then  $\psi_{\gamma,j} \in W^{1,p}((B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}; [0, 1])$ ,  $\psi_{\gamma,j} = 1$  on  $(\partial B'_{\gamma N_j})^+$  and  $\psi_{\gamma,j} = 0$  on  $(\partial B'_{\gamma N_j})^-$ . Let  $w_j^\pm = w_j \chi_{\omega^{\pm \varepsilon_j}}$ , we extend them to the whole  $\omega \times (-\varepsilon_j, \varepsilon_j)$  by reflection i.e. we define  $\tilde{w}_j^\pm(x_\alpha, x_n) = w_j^\pm(x_\alpha, -x_n)$  for every  $x \in \omega^{\mp \varepsilon_j}$  and  $\tilde{w}_j^\pm(x) = w_j^\pm(x)$  for every  $x \in \omega^{\pm \varepsilon_j}$ . Hence, we define

$$\bar{u}_j(x_\alpha, x_n) := \psi_{\gamma,j} \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \tilde{w}_j^+(x) + \left( 1 - \psi_{\gamma,j} \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \right) \tilde{w}_j^-(x) \quad (3.6.12)$$

for every  $x \in (B'_{\rho_j}(x_i^{\delta_j}) \times (-\varepsilon_j, \varepsilon_j)) \cap \Omega_j$  and  $i \in \mathbb{Z}^{n-1} \setminus Z_j$ . In particular, we have that  $\bar{u}_j = w_j$  on  $(\partial B'_{\rho_j}(x_i^{\delta_j}) \times (-\varepsilon_j, \varepsilon_j)) \cap \Omega_j$ . We have thus define a sequence  $\{\bar{u}_j\} \subset W^{1,p}(\Omega_j; \mathbb{R}^m)$ .

**Step 2 : The sequence  $\{\bar{u}_j\}$  weakly converges to  $(u^+, u^-)$ .** Let us check (3.3.1) and (3.3.2). We will only treat the upper cylinder  $\omega^{+\varepsilon_j}$ , the lower part being analogous. First

$$\begin{aligned} & \frac{1}{\varepsilon_j} \int_{\omega^{+\varepsilon_j}} |\bar{u}_j - u^+|^p dx \\ &= \frac{1}{\varepsilon_j} \int_{(\omega \setminus \bigcup_{i \in \mathbb{Z}^{n-1}} B'_{\rho_j}(x_i^{\delta_j}))^{+\varepsilon_j}} |w_j^+ - u^+|^p dx \\ & \quad + \frac{1}{\varepsilon_j} \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j})^{+\varepsilon_j}} \left| \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) + u_j^{i-} - u^+ \right|^p dx \\ & \quad + \frac{1}{\varepsilon_j} \sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j} \int_{(\omega \cap B'_{\rho_j}(x_i^{\delta_j}))^{+\varepsilon_j}} \left| \psi_{\gamma,j} \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) (w_j^+ - \tilde{w}_j^-) + \tilde{w}_j^- - u^+ \right|^p dx \\ & \leq \frac{1}{\varepsilon_j} \int_{\omega^{+\varepsilon_j}} |w_j - u^+|^p dx + c \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j})} |u^+ - u_j^{i+}|^p dx_\alpha \\ & \quad + \frac{c}{\varepsilon_j} \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j})^{+\varepsilon_j}} \left| \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) - (u_j^{i+} - u_j^{i-}) \right|^p dx \\ & \quad + \frac{c}{\varepsilon_j} \int_{(\omega \cap \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}))^{+\varepsilon_j}} (|w_j^+|^p + |\tilde{w}_j^-|^p + |u^+|^p) dx. \end{aligned} \quad (3.6.13)$$

Since  $\lim_{j \rightarrow +\infty} \mathcal{L}^{n-1}(\omega \cap \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j})) = 0$  and  $\sup_{j \in \mathbb{N}} \|w_j^\pm\|_{L^\infty(\omega^{\pm \varepsilon_j}; \mathbb{R}^m)} < +\infty$ , then we have that

$$\lim_{j \rightarrow +\infty} \frac{c}{\varepsilon_j} \int_{(\omega \cap \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}))^{+\varepsilon_j}} (|w_j^+|^p + |\tilde{w}_j^-|^p + |u^+|^p) dx = 0. \quad (3.6.14)$$

Moreover, reasoning as in the proof of Proposition 3.5.3 (see relation (3.5.23)), we have that

$$\lim_{j \rightarrow +\infty} \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j})} |u^+ - u_j^{i+}|^p dx_\alpha = 0, \quad (3.6.15)$$

### 3.6. $\Gamma$ -CONVERGENCE RESULT

and, by the convergence  $w_j \rightarrow (u^+, u^-)$ , it remains only to prove that

$$\lim_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j}) + \varepsilon_j} \left| \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) - (u_j^{i+} - u_j^{i-}) \right|^p dx = 0. \quad (3.6.16)$$

In fact, changing variable, we get that

$$\begin{aligned} \frac{1}{\varepsilon_j} \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j}) + \varepsilon_j} \left| \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) - (u_j^{i+} - u_j^{i-}) \right|^p dx \\ = r_j^{n-1} \sum_{i \in Z_j} \int_{(B'_{\gamma N_j})^+} \left| \zeta_{\gamma,j}^i(x) - (u_j^{i+} - u_j^{i-}) \right|^p dx, \end{aligned}$$

and by, Poincaré's Inequality

$$\int_{B'_{\gamma N_j}} \left| \zeta_{\gamma,j}^i(x_\alpha, x_n) - (u_j^{i+} - u_j^{i-}) \right|^p dx_\alpha \leq c (\gamma N_j)^p \int_{B'_{\gamma N_j}} |\nabla_\alpha \zeta_{\gamma,j}^i(x_\alpha, x_n)|^p dx_\alpha$$

for a.e.  $x_n \in (0, 1)$ . Hence, by the  $p$ -growth condition (3.3.3) and (3.6.10) if we integrate with respect to  $x_n$  and sum up in  $i \in Z_j$ , we get that

$$\begin{aligned} \frac{1}{\varepsilon_j} \sum_{i \in Z_j} \int_{B'_{\rho_j}(x_i^{\delta_j}) + \varepsilon_j} \left| \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) - (u_j^{i+} - u_j^{i-}) \right|^p dx \\ \leq c r_j^{n-1} \gamma^p N_j^p \sum_{i \in Z_j} \int_{(B'_{\gamma N_j})^+} |\nabla_\alpha \zeta_{\gamma,j}^i|^p dx \\ \leq c r_j^{n-1} \gamma^p N_j^p \sum_{i \in Z_j} \int_{(B'_{\gamma N_j})^+} \left| \left( \nabla_\alpha \zeta_{\gamma,j}^i \middle| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j}^i \right) \right|^p dx \\ \leq c r_j^{n-1} \gamma^p N_j^p \sum_{i \in Z_j} \left( \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) + \eta + r_j^p \mathcal{L}^{n-1}(B'_{\gamma N_j}) \right) \\ \leq c \gamma^p \delta_j^p \frac{r_j^{n-1-p}}{\delta_j^{n-1}} \left( \left( \eta + c \gamma^{n-1} \frac{\delta_j^{n-1}}{r_j^{n-1-p}} \right) \mathcal{L}^{n-1}(\omega) + \sum_{i \in Z_j} \delta_j^{n-1} \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \right) \end{aligned} \quad (3.6.17)$$

By Proposition 3.5.3 and (3.5.3), passing into the limit as  $j \rightarrow +\infty$  in (3.6.17), we get (3.6.16).

It remains to prove that (3.3.2) holds. In fact,

$$\begin{aligned} \frac{1}{\varepsilon_j} \int_{\omega + \varepsilon_j} |\nabla \bar{u}_j|^p dx \\ = \frac{1}{\varepsilon_j} \int_{\left( \omega \cup \bigcup_{i \in \mathbb{Z}^{n-1}} B'_{\rho_j}(x_i^{\delta_j}) \right) + \varepsilon_j} |\nabla w_j|^p dx \\ + \frac{1}{\varepsilon_j} \int_{\bigcup_{i \in Z_j} B'_{\rho_j}(x_i^{\delta_j}) + \varepsilon_j} \left| \left( r_j^{-1} \nabla_\alpha \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \middle| \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \right) \right|^p dx \\ + \frac{1}{\varepsilon_j} \int_{\left( \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}) \cap \omega \right) + \varepsilon_j} |\nabla \bar{u}_j|^p dx. \end{aligned} \quad (3.6.18)$$

We can be easily shown that

$$\begin{aligned} & \frac{1}{\varepsilon_j} \int_{\bigcup_{i \in Z_j} B'_{\rho_j}(x_i^{\delta_j})^{+\varepsilon_j}} \left| \left( r_j^{-1} \nabla_\alpha \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \Big| \varepsilon_j^{-1} \nabla_n \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \right) \right|^p dx \\ & \leq \frac{r_j^{n-1-p}}{\delta_j^{n-1}} \left( \sum_{i \in Z_j} \delta_j^{n-1} \varphi_{\gamma,j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \right) + \mathcal{L}^{n-1}(\omega) \left( \eta \frac{r_j^{n-1-p}}{\delta_j^{n-1}} + \gamma^{n-1} \right); \end{aligned} \quad (3.6.19)$$

while,

$$\begin{aligned} & \frac{1}{\varepsilon_j} \int_{\left( \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}) \cap \omega \right)^{+\varepsilon_j}} |\nabla \bar{u}_j|^p dx \\ & \leq c \sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j} \left( \frac{1}{r_j^p \varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{+\varepsilon_j}} \left| \nabla_\alpha \psi_{\gamma,j} \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{\varepsilon_j} \right) \right|^p (|w_j^+|^p + |\tilde{w}_j^-|^p) dx \right. \\ & \quad \left. + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{+\varepsilon_j}} (|\nabla w_j^+|^p + |\nabla \tilde{w}_j^-|^p) dx \right) \\ & \leq c \sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j} \left( r_j^{n-1-p} \int_{B_2} |\nabla_\alpha \psi|^p dx_\alpha + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{+\varepsilon_j}} |\nabla w_j^+|^p dx \right. \\ & \quad \left. + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{-\varepsilon_j}} |\nabla w_j^-|^p dx \right) \\ & \leq c \sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j} \left( \frac{r_j^{n-1-p}}{\delta_j^{n-1}} \mathcal{L}^{n-1}(Q'_{i,\delta_j}) + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{+\varepsilon_j}} |\nabla w_j^+|^p dx \right. \\ & \quad \left. + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{-\varepsilon_j}} |\nabla w_j^-|^p dx \right). \end{aligned} \quad (3.6.20)$$

Note that the previous sum can be computed over all  $i \in \mathbb{Z}^{n-1} \setminus Z_j$  such that  $Q'_{i,\delta_j} \cap \omega \neq \emptyset$ . Let

$$\omega'_j := \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j, Q'_{i,\delta_j} \cap \omega \neq \emptyset} Q'_{i,\delta_j},$$

then

$$\sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j, Q'_{i,\delta_j} \cap \omega \neq \emptyset} \mathcal{L}^{n-1}(Q'_{i,\delta_j}) = \mathcal{L}^{n-1}(\omega'_j) \rightarrow \mathcal{L}^{n-1}(\partial\omega) = 0. \quad (3.6.21)$$

Moreover, by Lemma 3.4.3 we have that  $\sup_j \frac{1}{\varepsilon_j} \int_{\omega^{\pm\varepsilon_j}} |\nabla w_j^\pm|^p dx < +\infty$ ; hence, by Proposition 3.5.3, (3.5.3), (3.6.18), (3.6.19) and (3.6.20) we get (3.3.2).

**Step 3 : The sequence  $\{\bar{u}_j\}$  is a recovery sequence.** We now prove the lim sup inequality.

$$\begin{aligned}
 & \limsup_{j \rightarrow +\infty} \int_{\omega^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx \\
 &= \limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{(\omega \setminus \bigcup_{i \in \mathbb{Z}^{n-1}} B'_{\rho_j}(x_i^{\delta_j}))^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx + \int_{\bigcup_{i \in \mathbb{Z}^j} B'_{\rho_j}(x_i^{\delta_j})^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx \right. \\
 & \quad \left. + \int_{(\omega \cap \bigcup_{i \in \mathbb{Z}^{n-1} \setminus \mathbb{Z}^j} B'_{\rho_j}(x_i^{\delta_j}))^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx \right). \tag{3.6.22}
 \end{aligned}$$

We deal with the first term in (3.6.22). By definition of  $\bar{u}_j$  (3.6.9), Lemma 3.4.3 and (3.6.8), we have that

$$\begin{aligned}
 & \limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{(\omega \setminus \bigcup_{i \in \mathbb{Z}^{n-1}} B'_{\rho_j}(x_i^{\delta_j}))^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx \\
 &= \limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{(\omega \setminus \bigcup_{i \in \mathbb{Z}^{n-1}} B'_{\rho_j}(x_i^{\delta_j}))^{\pm \varepsilon_j}} W(\nabla w_j) dx \\
 &\leq \limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{\omega^{\pm \varepsilon_j}} W(\nabla u_j^{\pm}) dx + o(1) \\
 &= \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^{\pm}) dx_{\alpha} + o(1), \tag{3.6.23}
 \end{aligned}$$

as  $\gamma \rightarrow 0^+$ . For every  $i \in \mathbb{Z}^j$ , by (3.6.11) and (3.6.10) we get that

$$\begin{aligned}
 & \frac{1}{\varepsilon_j} \left( \int_{B'_{\rho_j}(x_i^{\delta_j})^{+\varepsilon_j}} W(\nabla \bar{u}_j) dx + \int_{B'_{\rho_j}(x_i^{\delta_j})^{-\varepsilon_j}} W(\nabla \bar{u}_j) dx \right) \\
 &= r_j^{n-1} \int_{(B'_{\gamma N_j} \times I) \setminus C_{1, \gamma N_j}} W \left( r_j^{-1} \nabla_{\alpha} \zeta_{\gamma, j}^i | \varepsilon_j^{-1} \nabla_n \zeta_{\gamma, j}^i \right) dx \\
 &\leq r_j^{n-1-p} \left( \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) + \eta \right);
 \end{aligned}$$

hence, by (3.5.3) and Proposition 3.5.3 we get

$$\begin{aligned}
 & \limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \left( \int_{\bigcup_{i \in \mathbb{Z}^j} B'_{\rho_j}(x_i^{\delta_j})^{+\varepsilon_j}} W(\nabla \bar{u}_j) dx + \int_{\bigcup_{i \in \mathbb{Z}^j} B'_{\rho_j}(x_i^{\delta_j})^{-\varepsilon_j}} W(\nabla \bar{u}_j) dx \right) \\
 &\leq R^{(\ell)} \int_{\omega} \varphi^{(\ell)}(u^+ - u^-) dx_{\alpha} + R^{(\ell)} \mathcal{L}^{n-1}(\omega) \eta \\
 & \quad + \limsup_{j \rightarrow +\infty} \int_{\omega} \left| \sum_{i \in \mathbb{Z}^j} \varphi_{\gamma, j}^{(\ell)}(u_j^{i+} - u_j^{i-}) \chi_{Q'_{i, \delta_j}} - \varphi^{(\ell)}(u^+ - u^-) \right| dx_{\alpha} \\
 &= R^{(\ell)} \int_{\omega} \varphi^{(\ell)}(u^+ - u^-) dx_{\alpha} + R^{(\ell)} \mathcal{L}^{n-1}(\omega) \eta + o(1), \tag{3.6.24}
 \end{aligned}$$

as  $\gamma \rightarrow 0^+$ . Finally, if  $i \notin Z_j$ , by the  $p$ -growth condition (3.3.3) and (3.6.20), we obtain that

$$\begin{aligned}
 & \frac{1}{\varepsilon_j} \left( \int_{\left( \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}) \cap \omega \right)^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx \right) \\
 & \leq \sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j} \frac{\beta}{\varepsilon_j} \left( \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{\pm \varepsilon_j}} (1 + |\nabla \bar{u}_j|^p) dx \right) \\
 & \leq c \mathcal{L}^{n-1} \left( \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}) \cap \omega \right) \\
 & \quad + c \sum_{i \in \mathbb{Z}^{n-1} \setminus Z_j} \left( \frac{r_j^{n-1-p}}{\delta_j^{n-1}} \mathcal{L}^{n-1}(Q'_{i, \delta_j}) + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{+\varepsilon_j}} |\nabla w_j^+|^p dx \right. \\
 & \quad \left. + \frac{1}{\varepsilon_j} \int_{(B'_{\rho_j}(x_i^{\delta_j}) \cap \omega)^{-\varepsilon_j}} |\nabla w_j^-|^p dx \right).
 \end{aligned}$$

Since

$$\lim_{j \rightarrow +\infty} \mathcal{L}^{n-1} \left( \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}) \cap \omega \right) = 0;$$

by (3.5.3), the equi-integrability of  $\{|\nabla w_j^\pm|^p / \varepsilon_j\}$  on  $\omega^{\pm \varepsilon_j}$  and (3.6.21), we have that

$$\limsup_{j \rightarrow +\infty} \frac{1}{\varepsilon_j} \int_{\left( \omega \cap \bigcup_{i \in \mathbb{Z}^{n-1} \setminus Z_j} B'_{\rho_j}(x_i^{\delta_j}) \right)^{\pm \varepsilon_j}} W(\nabla \bar{u}_j) dx = 0. \quad (3.6.25)$$

Gathering (3.6.22)-(3.6.25) and passing to the limit as  $\gamma \rightarrow 0$  we get the limsup inequality for every  $u^\pm \in W^{1,p}(\omega; \mathbb{R}^m) \cap L^\infty(\omega; \mathbb{R}^m)$ .

We now remove the boundedness assumption simply noting that any arbitrary  $W^{1,p}(\omega; \mathbb{R}^m)$  function can be approximated by a sequence of functions belonging to  $W^{1,p}(\omega; \mathbb{R}^m) \cap L^\infty(\omega; \mathbb{R}^m)$  with respect to the strong convergence of  $W^{1,p}(\omega; \mathbb{R}^m)$ . Then, by the lower semicontinuity of the  $\Gamma$ -lim sup and the continuity of

$$(v^+, v^-) \mapsto \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha v^+) dx_\alpha + \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha v^-) dx_\alpha + R^{(\ell)} \int_{\omega} \varphi^{(\ell)}(v^+ - v^-) dx_\alpha$$

with respect to the strong  $W^{1,p}(\omega; \mathbb{R}^m)$ -convergence we get the thesis for  $\ell \in (0, +\infty]$ .

If  $\ell = 0$ , we may repeat the proof for the case  $\ell \in (0, +\infty]$  with slight changes. Let us start by dealing with Step 1. First, we have to notice that for the definition of  $\{\bar{u}_j\}$  on  $B'_{\rho_j}(x_i^{\delta_j})^{\pm \varepsilon_j}$ , for  $i \in Z_j$ , we have to consider, for any  $\eta > 0$ , a function  $\zeta_{\gamma,j} \in Y_j^\gamma(z)$  such that

$$\int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1, \gamma N_j}} r_j^p W \left( r_j^{-1} \nabla \zeta_{\gamma,j} \right) dx \leq \varphi_{\gamma,j}^{(0)}(z) + \eta;$$

hence,

$$\bar{u}_j(x_\alpha, x_n) := \zeta_{\gamma,j}^i \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{r_j} \right) + u_j^{i-} \text{ for every } x \in B'_{\rho_j}(x_i^{\delta_j})^{\pm \varepsilon_j}, \quad i \in Z_j.$$

While for the definition of  $\{\bar{u}_j\}$  on  $B'_{\rho_j}(x_i^{\delta_j})^{\pm \varepsilon_j}$ , for  $i \in \mathbb{Z}^{n-1} \setminus Z_j$ , we have to introduce a suitable function  $\psi_{\gamma,j}$  different from that one used in (3.6.12). In fact, for a fixed  $\gamma > 0$  and  $j$  large enough we can always assume that  $\gamma N_j > 2$  and  $\varepsilon_j/r_j > 2$ . Let  $\psi \in W^{1,p}(B'_2 \times (0, 2); [0, 1])$  such that  $\psi = 0$  on  $B'_1 \times \{0\}$  and  $\psi = 1$  on  $\partial B'_2 \times (0, 2)$ . We then define

$$\psi_{\gamma,j}(x) := \begin{cases} 0 & \text{in } (B'_{\gamma N_j})^{-(\varepsilon_j/r_j)}, \\ \psi(x_\alpha, x_n) & \text{in } (B'_2)^{+2}, \\ 1 & \text{in } (B'_{\gamma N_j})^{+(\varepsilon_j/r_j)} \setminus (B'_2)^{+2}. \end{cases}$$

The functions  $\psi_{\gamma,j} \in W^{1,p}((B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}; [0, 1])$  and satisfy  $\psi_{\gamma,j} = 1$  on  $(\partial B'_{\gamma N_j})^{+(\varepsilon_j/r_j)}$  and  $\psi_{\gamma,j} = 0$  in  $(B'_{\gamma N_j})^{-(\varepsilon_j/r_j)}$ . Hence, we define

$$\bar{u}_j(x) := \psi_{\gamma,j} \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{r_j} \right) \tilde{w}_j^+(x) + \left( 1 - \psi_{\gamma,j} \left( \frac{x_\alpha - x_i^{\delta_j}}{r_j}, \frac{x_n}{r_j} \right) \right) \tilde{w}_j^-(x)$$

for every  $x \in (B'_{\rho_j}(x_i^{\delta_j}) \times (-\varepsilon_j, \varepsilon_j)) \cap \Omega_j$  and  $i \in \mathbb{Z}^{n-1} \setminus Z_j$ . In particular, we have that  $\bar{u}_j = w_j$  on  $(\partial B'_{\rho_j}(x_i^{\delta_j}) \times (-\varepsilon_j, \varepsilon_j)) \cap \Omega_j$ .

Taking into account the definition of  $\{\bar{u}_j\}$  we can proceed as in Steps 2 and 3 also in the case  $\ell = 0$ .  $\square$

### 3.7 Representation formula for the interfacial energy density

This section is devoted to describe explicitly the interfacial energy density  $\varphi^{(\ell)}$  for  $\ell \in [0, +\infty]$ . As in [6], we expect to find a "capacitary type" formula for each regime  $\ell \in (0, +\infty)$ ,  $\ell = +\infty$  and  $\ell = 0$ .

We recall that  $\varphi^{(\ell)}$  can be obtained as pointwise limit of the sequence  $\{\varphi_{\gamma,j}^{(\ell)}\}$  as  $j \rightarrow +\infty$  and  $\gamma \rightarrow 0^+$  where for  $\ell \in (0, +\infty]$

$$\varphi_{\gamma,j}^{(\ell)}(z) = \inf \left\{ \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p W \left( r_j^{-1} \left( \nabla_\alpha \zeta \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta \right) \right) dx : \quad \zeta \in X_j^\gamma(z) \right\}$$

while if  $\ell = 0$ ,

$$\varphi_{\gamma,j}^{(0)}(z) = \inf \left\{ \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} r_j^p W(r_j^{-1} \nabla \zeta) dx : \quad \zeta \in Y_j^\gamma(z) \right\}$$



(see Section 3.5). Then the main difficulty occurring in the description of  $\varphi^{(\ell)}$  is due to the fact that the minimum problems are stated on (increasingly) varying domains.

We first introduce convenient notation for the sequel. Let  $g_j : \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$  be the sequence of functions given by

$$g_j(F) := r_j^p W(r_j^{-1}F)$$

for every  $F \in \mathbb{R}^{m \times n}$ . By (3.3.3) and (3.3.4) it follows that

$$|F|^p - r_j^p \leq g_j(F) \leq \beta(r_j^p + |F|^p), \quad \text{for all } F \in \mathbb{R}^{m \times n} \quad (3.7.1)$$

and the following  $p$ -Lipschitz condition holds :

$$|g_j(F_1) - g_j(F_2)| \leq c(r_j^{p-1} + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \quad \text{for all } F_1, F_2 \in \mathbb{R}^{m \times n}.$$

Then, according to the Ascoli-Arzelà Theorem, up to subsequences,  $g_j$  converges locally uniformly in  $\mathbb{R}^{m \times n}$  to a function  $g$  that satisfies the conditions :

$$|F|^p \leq g(F) \leq \beta|F|^p, \quad \text{for all } F \in \mathbb{R}^{m \times n} \quad (3.7.2)$$

and

$$|g(F_1) - g(F_2)| \leq c(|F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \quad \text{for all } F_1, F_2 \in \mathbb{R}^{m \times n}. \quad (3.7.3)$$

The proof of the representation formula for the energy density  $\varphi^{(\ell)}$  will be performed in three main steps : we first prove an auxiliary  $\Gamma$ -convergence result for a suitable sequence of energies stated on a fixed domain, then we describe the limit functional space occurring in the limit "capacitary" formula, finally, we prove that  $\varphi^{(\ell)}$  is described by a representation formula of "capacitary type".

### 3.7.1 The case $\ell \in (0, +\infty)$

We define

$$X_N(z) := \left\{ \zeta \in W^{1,p}((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m) : \begin{array}{l} \zeta = z \text{ on } (\partial B'_N)^+ \\ \text{and } \zeta = 0 \text{ on } (\partial B'_N)^- \end{array} \right\}$$

for  $N > 1$ . We recall the following  $\Gamma$ -convergence result.

**Proposition 3.7.1.** *Let*

$$\ell = \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} \in (0, +\infty),$$

then the sequence of functionals  $G_j^{(\ell)} : L^p((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m) \rightarrow [0, +\infty]$ , defined by

$$G_j^{(\ell)}(\zeta) := \begin{cases} \int_{(B'_N \times I) \setminus C_{1,N}} g_j \left( \nabla_\alpha \zeta \Big| \frac{r_j}{\varepsilon_j} \nabla_n \zeta \right) dx & \text{if } \zeta \in X_N(z) \\ +\infty & \text{otherwise,} \end{cases}$$

$\Gamma$ -converges, with respect to the  $L^p$ -convergence, to

$$G^{(\ell)}(\zeta) := \begin{cases} \int_{(B'_N \times I) \setminus C_{1,N}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx & \text{if } \zeta \in X_N(z) \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* Since  $\ell = \lim_{j \rightarrow +\infty} (r_j/\varepsilon_j) \in (0, +\infty)$ , by the local uniform convergence of  $g_j$  to  $g$  we have that the sequence of quasiconvex functions  $F \mapsto g_j(\overline{F}|(r_j/\varepsilon_j)F_n)$  pointwise converges to  $F \mapsto g(\overline{F}|\ell F_n)$ . Hence the conclusion comes from [32] Propositions 12.8 and 11.7.  $\square$

**Remark 3.7.2.** We denote by  $p^*$  the Sobolev exponent in dimension  $(n-1)$  i.e.

$$p^* := \frac{(n-1)p}{n-1-p}.$$

We recall that if  $(a, b) \subset \mathbb{R}$ , the space  $L^p(a, b; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$  is a reflexible and separable Banach space (see e.g. [2] or [89]). Hence, by the Banach-Alaoglu-Bourbaki Theorem, any bounded sequence admits a weakly converging subsequence.

**Proposition 3.7.3** (Limit space). *Let*

$$\ell = \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} \in (0, +\infty), \quad 0 < R^{(\ell)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-1-p}}{\delta_j^{n-1}} < +\infty \quad (3.7.4)$$

and let  $\{\zeta_{\gamma,j}\} \subset X_j^\gamma(z)$  such that, for every fixed  $\gamma > 0$ ,

$$\sup_{j \in \mathbb{N}} \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} g_j \left( \nabla_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) dx \leq c. \quad (3.7.5)$$

Then, there exists a sequence  $\{\tilde{\zeta}_j\} \subset W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m)$  such that

$$\tilde{\zeta}_j = \zeta_{\gamma,j} \quad \text{on} \quad (B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}$$

and such that, up to subsequences, it converges weakly to  $\zeta$  in  $W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m)$ .

Moreover, the function  $\zeta$  satisfies the following properties

$$\left\{ \begin{array}{l} \zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m), \\ \nabla \zeta \in L^p((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \\ \zeta - z \in L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \\ \zeta \in L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)). \end{array} \right. \quad (3.7.6)$$

*Proof.* By (3.7.1), (3.7.4) and (3.7.5) we deduce that, for every fixed  $\gamma > 0$ ,

$$\sup_{j \in \mathbb{N}} \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left| \left( \nabla_{\alpha} \zeta_{\gamma,j} \left| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right. \right) \right|^p dx \leq c. \quad (3.7.7)$$

We now define

$$\tilde{\zeta}_j := \begin{cases} z & \text{in } (\mathbb{R}^{n-1} \setminus B'_{\gamma N_j})^+, \\ \zeta_{\gamma,j} & \text{in } (B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}, \\ 0 & \text{in } (\mathbb{R}^{n-1} \setminus B'_{\gamma N_j})^-; \end{cases}$$

by (3.7.7), we get that

$$\int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} \left| \left( \nabla_{\alpha} \tilde{\zeta}_j \left| \frac{r_j}{\varepsilon_j} \nabla_n \tilde{\zeta}_j \right. \right) \right|^p dx = \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} \left| \left( \nabla_{\alpha} \zeta_{\gamma,j} \left| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right. \right) \right|^p dx \leq c. \quad (3.7.8)$$

Hence,

$$\tilde{\zeta}_j(\cdot, x_n) - z \in W^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m) \quad \text{for a.e. } x_n \in (0, 1)$$

and

$$\tilde{\zeta}_j(\cdot, x_n) \in W^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m) \quad \text{for a.e. } x_n \in (-1, 0).$$

Since  $p < n - 1$ , according to the Sobolev Inequality (see e.g. [2]), there exists a constant  $c = c(n, p) > 0$  (independent of  $x_n$ ) such that

$$\left( \int_{\mathbb{R}^{n-1}} |\tilde{\zeta}_j(x_{\alpha}, x_n) - z|^{p^*} dx_{\alpha} \right)^{p/p^*} \leq c \int_{\mathbb{R}^{n-1}} |\nabla_{\alpha} \tilde{\zeta}_j(x_{\alpha}, x_n)|^p dx_{\alpha} \quad (3.7.9)$$

for a.e.  $x_n \in (0, 1)$ , and

$$\left( \int_{\mathbb{R}^{n-1}} |\tilde{\zeta}_j(x_{\alpha}, x_n)|^{p^*} dx_{\alpha} \right)^{p/p^*} \leq c \int_{\mathbb{R}^{n-1}} |\nabla_{\alpha} \tilde{\zeta}_j(x_{\alpha}, x_n)|^p dx_{\alpha} \quad (3.7.10)$$

### 3.7. REPRESENTATION FORMULA FOR THE INTERFACIAL ENERGY DENSITY

for a.e.  $x_n \in (-1, 0)$ . If we integrate (3.7.9) and (3.7.10) with respect to  $x_n$ , by (3.7.8) and Remark 3.7.2, we get that there exist  $\zeta_1 \in L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$  and  $\zeta_2 \in L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$  such that, up to subsequences,

$$\begin{cases} \tilde{\zeta}_j - z \rightharpoonup \zeta_1 & \text{in } L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \\ \tilde{\zeta}_j \rightharpoonup \zeta_2 & \text{in } L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \\ \nabla \tilde{\zeta}_j \rightharpoonup \nabla \zeta_1 & \text{in } L^p((\mathbb{R}^{n-1})^+; \mathbb{R}^{m \times n}), \\ \nabla \tilde{\zeta}_j \rightharpoonup \nabla \zeta_2 & \text{in } L^p((\mathbb{R}^{n-1})^-; \mathbb{R}^{m \times n}). \end{cases}$$

In particular, we have that

$$\begin{cases} \tilde{\zeta}_j \rightharpoonup \zeta_1 + z & \text{in } W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1})^+; \mathbb{R}^m), \\ \tilde{\zeta}_j \rightharpoonup \zeta_2 & \text{in } W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1})^-; \mathbb{R}^m). \end{cases}$$

Then, since  $\zeta_1 + z = \zeta_2$  on  $B'_1$  in the sense of traces, we can define

$$\zeta = \begin{cases} \zeta_1 + z & \text{in } (\mathbb{R}^{n-1})^+ \\ \zeta_2 & \text{in } (\mathbb{R}^{n-1})^- \cup (B'_1 \times \{0\}), \end{cases}$$

which satisfies (3.7.6). □

Now we are able to describe the interfacial energy density  $\varphi^{(\ell)}$  as the following nonlinear capacity formula.

**Proposition 3.7.4** (Representation formula). *We have*

$$\varphi^{(\ell)}(z) = \inf \left\{ \int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m), \right. \\ \left. \nabla \zeta \in L^p((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \zeta - z \in L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right. \\ \left. \text{and } \zeta \in L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\}$$

for every  $z \in \mathbb{R}^m$ .

*Proof.* We define

$$\psi^{(\ell)}(z) := \inf \left\{ \int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m), \right. \\ \left. \nabla \zeta \in L^p((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \zeta - z \in L^p(0, 1; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right. \\ \left. \text{and } \zeta \in L^p(-1, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\}$$

for every  $z \in \mathbb{R}^m$ , we want to prove that  $\varphi^{(\ell)}(z) = \psi^{(\ell)}(z)$ . For every fixed  $\eta > 0$ , by definition of  $\varphi_{\gamma,j}^{(\ell)}(z)$  (see (3.5.2)), there exists  $\{\zeta_{\gamma,j}\} \subset X_j^\gamma(z)$  such that

$$\int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} g_j \left( \nabla_\alpha \zeta_{\gamma,j} \Big| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_{\gamma,j} \right) dx \leq \varphi_{\gamma,j}^{(\ell)}(z) + \eta.$$

By Proposition 3.5.1(i) we have that (3.7.5) is satisfied then by Propositions 3.7.3 and 3.7.1 we get

$$\begin{aligned} \lim_{j \rightarrow +\infty} \varphi_{\gamma,j}^{(\ell)}(z) + \eta &\geq \liminf_{j \rightarrow +\infty} \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} g_j \left( \nabla_\alpha \tilde{\zeta}_j \Big| \frac{r_j}{\varepsilon_j} \nabla_n \tilde{\zeta}_j \right) dx \\ &\geq \liminf_{j \rightarrow +\infty} \int_{(B'_N \times I) \setminus C_{1,N}} g_j \left( \nabla_\alpha \tilde{\zeta}_j \Big| \frac{r_j}{\varepsilon_j} \nabla_n \tilde{\zeta}_j \right) dx \\ &\geq \int_{(B'_N \times I) \setminus C_{1,N}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx \end{aligned}$$

where  $\zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m)$  satisfies (3.7.6). Note that for every fixed  $\gamma > 0$  and  $j$  large enough we can always assume that  $\gamma N_j > N$  for some fixed  $N > 2$ . Hence, passing to the limit as  $N \rightarrow +\infty$  and  $\gamma \rightarrow 0^+$ , we obtain

$$\varphi^{(\ell)}(z) + \eta \geq \int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx \geq \psi^{(\ell)}(z) \quad (3.7.11)$$

and by the arbitrariness of  $\eta$  we get the first inequality.

We now prove the converse inequality. For every fixed  $\eta > 0$ , there exists  $\zeta \in W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m)$  satisfying (3.7.6) such that

$$\int_{(\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx \leq \psi^{(\ell)}(z) + \eta. \quad (3.7.12)$$

Let  $N > 2$ , and choose  $j$  large enough so that  $\gamma N_j > N$ . We consider a cut-off function  $\theta_N \in \mathcal{C}_c^\infty(B'_N; [0, 1])$  such that  $\theta_N = 1$  in  $B'_{N/2}$ ,  $|\nabla_\alpha \theta_N| \leq c/N$  and we define

$$\zeta_N(x) := \begin{cases} \theta_N(x_\alpha) \zeta(x) + (1 - \theta_N(x_\alpha)) z & \text{in } (B'_N)^+, \\ \theta_N(x_\alpha) \zeta(x) & \text{in } (B'_N)^- \cup (B'_1 \times \{0\}) \end{cases}$$

so that  $\zeta_N \in X_N(z)$ . By Proposition 3.7.1, there exists a sequence  $\{\zeta_N^j\} \subset X_N(z)$  strongly converging to  $\zeta_N$  in  $L^p((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m)$  such that

$$\int_{(B'_N \times I) \setminus C_{1,N}} g(\nabla_\alpha \zeta_N | \ell \nabla_n \zeta_N) dx = \lim_{j \rightarrow +\infty} \int_{(B'_N \times I) \setminus C_{1,N}} g_j \left( \nabla_\alpha \zeta_N^j \Big| \frac{r_j}{\varepsilon_j} \nabla_n \zeta_N^j \right) dx \quad (3.7.13)$$

Let us define  $\zeta_{\gamma,j} \in X_j^\gamma(z)$  by

$$\zeta_{\gamma,j}(x) := \begin{cases} \zeta_N^j & \text{in } (B'_N \times I) \setminus C_{1,N}, \\ z & \text{in } (B'_{\gamma N_j} \setminus B'_N)^+, \\ 0 & \text{in } (B'_{\gamma N_j} \setminus B'_N)^-. \end{cases}$$

Consequently,  $\zeta_{\gamma,j}$  is an admissible test function for (3.5.2) and we get, since  $g_j(0) = 0$ , that

$$\begin{aligned} \varphi_{\gamma,j}^{(\ell)}(z) &\leq \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} g_j \left( \nabla_\alpha \zeta_{\gamma,j} \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_{\gamma,j} \right) dx \\ &= \int_{(B'_N \times I) \setminus C_{1,N}} g_j \left( \nabla_\alpha \zeta_N^j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_N^j \right) dx. \end{aligned}$$

Taking the limit as  $j \rightarrow +\infty$  and using (3.7.13) and (3.7.2), we obtain

$$\begin{aligned} \lim_{j \rightarrow +\infty} \varphi_{\gamma,j}^{(\ell)}(z) &\leq \int_{(B'_N \times I) \setminus C_{1,N}} g(\nabla_\alpha \zeta_N | \ell \nabla_n \zeta_N) dx \\ &\leq \int_{(B'_{N/2} \times I) \setminus C_{1,N/2}} g(\nabla_\alpha \zeta | \ell \nabla_n \zeta) dx + c \int_{(B'_N \setminus B'_{N/2})^+} |\nabla \zeta_N|^p dx \\ &\quad + c \int_{(B'_N \setminus B'_{N/2})^-} |\nabla \zeta_N|^p dx. \end{aligned} \tag{3.7.14}$$

Let us examine the contribution of the gradient in (3.7.14),

$$\begin{aligned} &\int_{(B'_N \setminus B'_{N/2})^+} |\nabla \zeta_N|^p dx + \int_{(B'_N \setminus B'_{N/2})^-} |\nabla \zeta_N|^p dx \\ &\leq c \int_{(B'_N \setminus B'_{N/2})^+} (|\nabla_\alpha \theta_N|^p |\zeta - z|^p + |\nabla \zeta|^p) dx \\ &\quad + c \int_{(B'_N \setminus B'_{N/2})^-} (|\nabla_\alpha \theta_N|^p |\zeta|^p + |\nabla \zeta|^p) dx \\ &\leq c \left( \int_{(\mathbb{R}^{n-1} \setminus B'_{N/2})^+} |\nabla \zeta|^p dx + \int_{(\mathbb{R}^{n-1} \setminus B'_{N/2})^-} |\nabla \zeta|^p dx \right) \\ &\quad + \frac{c}{N^p} \left( \int_{(B'_N \setminus B'_{N/2})^+} |\zeta - z|^p dx + \int_{(B'_N \setminus B'_{N/2})^-} |\zeta|^p dx \right). \end{aligned} \tag{3.7.15}$$

Since  $p^* > p$  we can apply the Hölder Inequality with  $q = p^*/p$ ; hence,

$$\begin{aligned}
 & \frac{c}{N^p} \left( \int_{(B'_N \setminus B'_{N/2})^+} |\zeta - z|^p dx + \int_{(B'_N \setminus B'_{N/2})^-} |\zeta|^p \right) \\
 & \leq c \left( \int_0^1 \left( \int_{B'_N \setminus B'_{N/2}} |\zeta - z|^{p^*} dx_\alpha \right)^{p/p^*} dx_n \right. \\
 & \quad \left. + \int_{-1}^0 \left( \int_{B'_N \setminus B'_{N/2}} |\zeta|^{p^*} dx_\alpha \right)^{p/p^*} dx_n \right) \\
 & \leq c \left( \int_0^1 \left( \int_{\mathbb{R}^{n-1} \setminus B'_{N/2}} |\zeta - z|^{p^*} dx_\alpha \right)^{p/p^*} dx_n \right. \\
 & \quad \left. + \int_{-1}^0 \left( \int_{\mathbb{R}^{n-1} \setminus B'_{N/2}} |\zeta|^{p^*} dx_\alpha \right)^{p/p^*} dx_n \right). \quad (3.7.16)
 \end{aligned}$$

Hence by (3.7.6), (3.7.15) and (3.7.16) we have that, for every fixed  $\gamma > 0$ ,

$$\lim_{N \rightarrow +\infty} \int_{(B'_N \setminus B'_{N/2})^\pm} |\nabla \zeta_N|^p dx = 0$$

which implies thanks to (3.7.12) and (3.7.14) that

$$\lim_{j \rightarrow +\infty} \varphi_{\gamma,j}^{(\ell)}(z) \leq \psi^{(\ell)}(z) + \eta.$$

Then we get the converse inequality by letting  $\gamma \rightarrow 0$  and by the arbitrariness of  $\eta$ .  $\square$

### 3.7.2 The case $\ell = +\infty$

In this case the description of  $\varphi^{(\infty)}$  involves a dimensional reduction type problem stated on varying domains. We find more convenient to first deal with a fixed domain, applying some well-known  $\Gamma$ -convergence and integral representation results proved in [74] and [35].

Let  $G_j^\pm : L^p((B'_N)^\pm; \mathbb{R}^m) \rightarrow [0, +\infty]$  be defined by

$$G_j^+(\zeta) := \begin{cases} \int_{(B'_N)^+} g_j \left( \nabla_\alpha \zeta \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta \right) dx & \text{if } \begin{cases} \zeta \in W^{1,p}((B'_N)^+; \mathbb{R}^m) \\ \zeta = z \text{ on } (\partial B'_N)^+ \end{cases} \\ +\infty & \text{otherwise} \end{cases}$$

and

$$G_j^-(\zeta) := \begin{cases} \int_{(B'_N)^-} g_j \left( \nabla_\alpha \zeta \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta \right) dx & \text{if } \begin{cases} \zeta \in W^{1,p}((B'_N)^-; \mathbb{R}^m) \\ \zeta = 0 \text{ on } (\partial B'_N)^- \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

**Proposition 3.7.5.** *Let*

$$\ell = \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} = +\infty,$$

*then, the sequences of functionals  $G_j^\pm$   $\Gamma$ -converge, with respect to the  $L^p$ -convergence, to*

$$G^+(\zeta) := \begin{cases} \int_{B'_N} \mathcal{Q}g_0(\nabla_\alpha \zeta) dx_\alpha & \text{if } \zeta - z \in W_0^{1,p}(B'_N; \mathbb{R}^m) \\ +\infty & \text{otherwise} \end{cases}$$

*and*

$$G^-(\zeta) := \begin{cases} \int_{B'_N} \mathcal{Q}g_0(\nabla_\alpha \zeta) dx_\alpha & \text{if } \zeta \in W_0^{1,p}(B'_N; \mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

*respectively, where  $g_0(\bar{F}) = \inf\{g(\bar{F}|F_n) : F_n \in \mathbb{R}^m\}$  for every  $\bar{F} \in \mathbb{R}^{m \times (n-1)}$ .*

*Proof.* We prove the  $\Gamma$ -convergence result only for  $G_j^+$ , the other one being analogous. According to [35] Theorem 2.5 and Lemma 2.6 there exists a continuous function  $\hat{g} : \mathbb{R}^{m \times (n-1)} \rightarrow [0, +\infty)$ , such that, up to a subsequence,  $G_j^+$   $\Gamma$ -converges to

$$G^+(\zeta) := \begin{cases} \int_{B'_N} \hat{g}(\nabla_\alpha \zeta) dx_\alpha & \text{if } \zeta - z \in W_0^{1,p}(B'_N; \mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases}$$

Hence, it remains to show that  $\hat{g} = \mathcal{Q}g_0$ . By [35] Lemma 2.6, it is enough to prove it for  $W^{1,p}$ -function without boundary condition; hence, it is not restrictive to take  $\zeta$  an affine function. Let  $\zeta(x_\alpha) := \bar{F} \cdot x_\alpha$ , by [35] Theorem 2.5, there exists a sequence  $\{\zeta_j\} \subset W^{1,p}((B'_N)^+; \mathbb{R}^m)$  (the so-called recovery sequence) converging to  $\zeta$  in  $L^p((B'_N)^+; \mathbb{R}^m)$ , such that

$$\hat{g}(\bar{F}) c_N = G^+(\zeta) = \lim_{j \rightarrow +\infty} \int_{(B'_N)^+} g_j \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx \quad (3.7.17)$$

where  $c_N = \mathcal{L}^{n-1}(B'_N)$ . Moreover, by [25] Theorem 1.1, we can assume, without loss of generality, that the sequence  $\left\{ \left| \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) \right|^p \right\}$  is equi-integrable. By (3.7.17) and (3.7.1), we have that

$$\sup_{j \in \mathbb{N}} \int_{(B'_N)^+} \left| \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) \right|^p dx \leq c;$$

hence, for every fixed  $M > 0$ , if we define

$$A_j^M := \left\{ x \in (B'_N)^+ : \left| \left( \nabla_\alpha \zeta_j(x) \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j(x) \right) \right| \leq M \right\},$$



we get that  $\mathcal{L}^n((B'_N)^+ \setminus A_j^M) \leq c/M^p$  for some constant  $c > 0$  independent of  $j$  and  $M$ . Fix  $M > 0$ , by (3.7.17), we have

$$\hat{g}(\overline{F}) c_N \geq \limsup_{j \rightarrow +\infty} \int_{A_j^M} g_j \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx. \quad (3.7.18)$$

Moreover, for all  $x \in A_j^M$ ,

$$\left| g_j \left( \nabla_\alpha \zeta_j(x) \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j(x) \right) - g \left( \nabla_\alpha \zeta_j(x) \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j(x) \right) \right| \leq \sup_{|F| \leq M} |g_j(F) - g(F)|,$$

and then,

$$\int_{A_j^M} \left| g_j \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) - g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) \right| dx \leq c_N \sup_{|F| \leq M} |g_j(F) - g(F)|.$$

Hence, by the uniform convergence of  $g_j$  to  $g$  on compact sets, we have that

$$\lim_{j \rightarrow +\infty} \int_{A_j^M} \left( g_j \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) - g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) \right) dx = 0.$$

By (3.7.18), we get

$$\hat{g}(\overline{F}) c_N \geq \limsup_{j \rightarrow +\infty} \int_{A_j^M} g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx. \quad (3.7.19)$$

Note that, since  $\mathcal{L}^n((B'_N)^+ \setminus A_j^M) \rightarrow 0$  as  $M \rightarrow +\infty$ , by the  $p$ -growth condition (3.7.2) and the equi-integrability, we get that

$$\limsup_{j \rightarrow +\infty} \int_{(B'_N)^+ \setminus A_j^M} g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx = o(1), \quad \text{as } M \rightarrow +\infty. \quad (3.7.20)$$

Consequently, (3.7.19) and (3.7.20) imply that

$$\hat{g}(\overline{F}) c_N \geq \limsup_{j \rightarrow +\infty} \int_{(B'_N)^+} g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx. \quad (3.7.21)$$

Finally, from [74] Theorem 2, we know that

$$\liminf_{j \rightarrow +\infty} \int_{(B'_N)^+} g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx \geq \mathcal{Q}g_0(\overline{F}) c_N;$$

hence, by (3.7.21) we obtain that  $\hat{g}(\overline{F}) \geq \mathcal{Q}g_0(\overline{F})$ .

We now prove the converse inequality. By [74] Theorem 2, there exists a sequence  $\{\zeta_j\} \subset W^{1,p}((B'_N)^+; \mathbb{R}^m)$  converging to  $\zeta$  in  $L^p((B'_N)^+; \mathbb{R}^m)$  such that

$$\mathcal{Q}g_0(\overline{F}) c_N = \lim_{j \rightarrow +\infty} \int_{(B'_N)^+} g \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx. \quad (3.7.22)$$

Without loss of generality, we can still assume that the sequence  $\{ |(\nabla_\alpha \zeta_j|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j)|^p \}$  is equi-integrable. Thus, reasoning as above to prove (3.7.21) assuming (3.7.17), we can obtain, as a consequence of (3.7.22), that

$$\mathcal{Q}g_0(\overline{F}) c_N \geq \limsup_{j \rightarrow +\infty} \int_{(B'_N)^+} g_j \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx. \quad (3.7.23)$$

Now, by [35] Theorem 2.5, we have that

$$\liminf_{j \rightarrow +\infty} \int_{(B'_N)^+} g_j \left( \nabla_\alpha \zeta_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j \right) dx \geq \hat{g}(\overline{F}) c_N;$$

hence,  $\mathcal{Q}g_0(\overline{F}) \geq \hat{g}(\overline{F})$ , which concludes the proof of the proposition.  $\square$

**Remark 3.7.6.** By [74] Theorem 2 for every  $\zeta \in W^{1,p}(B'_N; \mathbb{R}^m)$  the recovery sequence is given by  $\zeta_j(x_\alpha, x_n) := \zeta(x_\alpha) + (\varepsilon_j/r_j) x_n b_j(x_\alpha)$  for a suitable sequence of functions  $b_j \in \mathcal{C}_c^\infty(B'_N; \mathbb{R}^m)$ . Note that by definition  $\{\zeta_j\}$  keeps the boundary conditions of  $\zeta$ . Reasoning as in the the proof of Proposition 3.7.5 we can observed that  $\{\zeta_j\}$  is also a recovery sequence for  $G_j^+$  (see e.g. (3.7.23)). The same remark holds for  $G_j^-$ .

**Proposition 3.7.7.** *Let*

$$\ell = \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} = +\infty,$$

*then the sequence of functionals  $G_j^{(\infty)} : L^p((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m) \rightarrow [0, +\infty]$  defined by*

$$G_j^{(\infty)}(\zeta) := \begin{cases} \int_{(B'_N \times I) \setminus C_{1,N}} g_j \left( \nabla_\alpha \zeta \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta \right) dx & \text{if } \zeta \in X_N(z) \\ +\infty & \text{otherwise} \end{cases}$$

*$\Gamma$ -converges, with respect to the  $L^p$ -convergence, to*

$$G^{(\infty)}(\zeta) := \begin{cases} \int_{(B'_N \times I) \setminus C_{1,N}} \mathcal{Q}g_0(\nabla_\alpha \zeta) dx & \text{if } \zeta \in X_N(z) \text{ and } \nabla_n \zeta = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* The lim inf inequality is a straightforward consequence of Proposition 3.7.5.

Dealing with the lim sup inequality, let us consider  $\zeta \in X_N(z)$  with  $\nabla_n \zeta = 0$ . We denote by  $\zeta^\pm \in W^{1,p}(B'_N; \mathbb{R}^m)$  the restriction of  $\zeta$  to  $(B'_N)^+$  and  $(B'_N)^-$ , respectively. By Proposition 3.7.5 and Remark 3.7.6, there exist two sequences  $\{\zeta_j^\pm\}$  such that

$$\begin{aligned} \zeta_j^+ &\rightarrow \zeta^+ \text{ in } L^p((B'_N)^+; \mathbb{R}^m), & \zeta_j^+ &= z \text{ on } (\partial B'_N)^+ \\ \zeta_j^- &\rightarrow \zeta^- \text{ in } L^p((B'_N)^-; \mathbb{R}^m), & \zeta_j^- &= 0 \text{ on } (\partial B'_N)^- \end{aligned} \quad (3.7.24)$$

and

$$\begin{aligned} \lim_{j \rightarrow +\infty} \int_{(B'_N)^+} g_j \left( \nabla_\alpha \zeta_j^+ \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j^+ \right) dx &= \int_{B'_N} \mathcal{Q}g_0(\nabla_\alpha \zeta^+) dx_\alpha \\ \lim_{j \rightarrow +\infty} \int_{(B'_N)^-} g_j \left( \nabla_\alpha \zeta_j^- \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_j^- \right) dx &= \int_{B'_N} \mathcal{Q}g_0(\nabla_\alpha \zeta^-) dx_\alpha. \end{aligned} \quad (3.7.25)$$

Moreover, since  $\zeta \in W^{1,p}((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m)$ , by Remark 3.7.6,  $\zeta_j^+$  and  $\zeta_j^-$  have the same trace on  $B'_1 \times \{0\}$ ; hence,  $\zeta_j^+ = \zeta_j^- = \zeta$  on  $B'_1 \times \{0\}$ . Then we can define

$$\bar{\zeta}_j := \begin{cases} \zeta_j^+ & \text{in } (B'_N)^+, \\ \zeta & \text{on } B'_1 \times \{0\}, \\ \zeta_j^- & \text{in } (B'_N)^- \end{cases}$$

with  $\bar{\zeta}_j \in W^{1,p}((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m)$ . In particular, by (3.7.24) we have that  $\{\bar{\zeta}_j\} \subset X_N(z)$  and  $\bar{\zeta}_j \rightarrow \zeta$  in  $L^p((B'_N \times I) \setminus C_{1,N}; \mathbb{R}^m)$ . Finally, by (3.7.25), we have

$$\begin{aligned} \lim_{j \rightarrow +\infty} G_j^{(\infty)}(\bar{\zeta}_j) &= \lim_{j \rightarrow +\infty} \int_{(B'_N \times I) \setminus C_{1,N}} g_j \left( \nabla_\alpha \bar{\zeta}_j \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \bar{\zeta}_j \right) dx \\ &= \int_{B'_N} \mathcal{Q}g_0(\nabla_\alpha \zeta^+) dx_\alpha + \int_{B'_N} \mathcal{Q}g_0(\nabla_\alpha \zeta^-) dx_\alpha \\ &= \int_{(B'_N \times I) \setminus C_{1,N}} \mathcal{Q}g_0(\nabla_\alpha \zeta) dx \end{aligned}$$

which completes the proof of the lim sup inequality.  $\square$

**Proposition 3.7.8** (Limit space). *Let*

$$\ell = \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} = +\infty, \quad 0 < R^{(\infty)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-1-p}}{\delta_j^{n-1}} < +\infty$$

and let  $\{\zeta_{\gamma,j}\} \subset X_j^\gamma(z)$  such that, for every fixed  $\gamma > 0$ ,

$$\sup_{j \in \mathbb{N}} \int_{(B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}} g_j \left( \nabla_\alpha \zeta_{\gamma,j} \Big|_{\frac{r_j}{\varepsilon_j}} \nabla_n \zeta_{\gamma,j} \right) dx \leq c. \quad (3.7.26)$$

Then, there exists a sequence  $\{\tilde{\zeta}_j\} \subset W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1} \times I) \setminus C_{1,\infty}; \mathbb{R}^m)$  such that

$$\tilde{\zeta}_j = \zeta_{\gamma,j} \quad \text{on } (B'_{\gamma N_j} \times I) \setminus C_{1,\gamma N_j}$$

and such that, up to subsequences, it converges weakly to  $\zeta^+$  in  $W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1})^+; \mathbb{R}^m)$  and to  $\zeta^-$  in

$W_{\text{loc}}^{1,p}((\mathbb{R}^{n-1})^-; \mathbb{R}^m)$ . Moreover, the functions  $\zeta^\pm$  satisfy the following properties

$$\left\{ \begin{array}{l} \zeta^\pm \in W_{\text{loc}}^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m) \\ \zeta^+ = \zeta^- \quad \text{in } B'_1 \\ \nabla_\alpha \zeta^\pm \in L^p(\mathbb{R}^{n-1}; \mathbb{R}^{m \times (n-1)}) \\ (\zeta^+ - z) \text{ and } \zeta^- \in L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m). \end{array} \right.$$

*Proof.* We can reason as in Proposition 3.7.3 using the fact that, by (3.7.26),

$$\int_{(\mathbb{R}^{n-1})^\pm} |\nabla_n \tilde{\zeta}_j|^p dx \leq c \left( \frac{\varepsilon_j}{r_j} \right)^p;$$

hence, in the limit we have that  $\nabla_n \zeta = 0$  a.e. in  $(\mathbb{R}^{n-1})^\pm$ .  $\square$

**Proposition 3.7.9** (Representation formula). *We have*

$$\varphi^{(\infty)}(z) = \inf \left\{ \int_{\mathbb{R}^{n-1}} (\mathcal{Q}g_0(\nabla_\alpha \zeta^+) + \mathcal{Q}g_0(\nabla_\alpha \zeta^-)) dx_\alpha : \zeta^\pm \in W_{\text{loc}}^{1,p}(\mathbb{R}^{n-1}; \mathbb{R}^m), \right. \\ \left. \begin{array}{l} \zeta^+ = \zeta^- \text{ in } B'_1, \quad \nabla_\alpha \zeta^\pm \in L^p(\mathbb{R}^{n-1}; \mathbb{R}^{m \times (n-1)}), \\ (\zeta^+ - z) \text{ and } \zeta^- \in L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m) \end{array} \right\}$$

for every  $z \in \mathbb{R}^m$ .

*Proof.* Reasoning as in the proof of Proposition 3.7.4, by Propositions 3.7.7 and 3.7.8 we get the representation formula for  $\varphi^{(\infty)}$ .  $\square$

### 3.7.3 The case $\ell = 0$

We first recall the following  $\Gamma$ -convergence result.

**Proposition 3.7.10.** *The sequence of functionals  $G_j^{(0)} : L^p((B'_N \times (-N, N)) \setminus C_{1,N}; \mathbb{R}^m) \rightarrow [0, +\infty]$ , defined by*

$$G_j^{(0)}(\zeta) := \begin{cases} \int_{(B'_N \times (-N, N)) \setminus C_{1,N}} g_j(\nabla \zeta) dx & \text{if } \zeta \in W^{1,p}((B'_N \times (-N, N)) \setminus C_{1,N}; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

$\Gamma$ -converges, with respect to the  $L^p$ -convergence, to

$$G^{(0)}(\zeta) := \begin{cases} \int_{(B'_N \times (-N, N)) \setminus C_{1,N}} g(\nabla \zeta) dx & \text{if } \zeta \in W^{1,p}((B'_N \times (-N, N)) \setminus C_{1,N}; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* The result is an immediate consequence of the pointwise convergence of the sequence of quasiconvex functions  $g_j$  towards  $g$  together with Proposition 12.8 in [32].  $\square$

**Proposition 3.7.11** (Limit space). *Let*

$$\ell = \lim_{j \rightarrow +\infty} \frac{r_j}{\varepsilon_j} = 0, \quad 0 < R^{(0)} = \lim_{j \rightarrow +\infty} \frac{r_j^{n-p}}{\delta_j^{n-1} \varepsilon_j} < +\infty \quad (3.7.27)$$

and let  $\{\zeta_{\gamma,j}\} \subset Y_j^\gamma(z)$  such that, for every fixed  $\gamma > 0$ ,

$$\sup_{j \in \mathbb{N}} \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} g_j(\nabla \zeta_{\gamma,j}) dx \leq c. \quad (3.7.28)$$

Then, there exists a sequence  $\{\tilde{\zeta}_j\} \subset W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m)$  such that

$$\tilde{\zeta}_j = \zeta_{\gamma,j} \quad \text{on} \quad (B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}$$

and such that, up to subsequences, it converges weakly to  $\zeta$  in  $W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m)$ . Moreover, the function  $\zeta$  satisfies the following properties

$$\left\{ \begin{array}{l} \zeta \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m), \\ \nabla \zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \\ \zeta - z \in L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)), \\ \zeta \in L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)). \end{array} \right. \quad (3.7.29)$$

*Proof.* By (3.7.28), (3.7.1) and (3.7.27), we deduce that, for every fixed  $\gamma > 0$ ,

$$\sup_{j \in \mathbb{N}} \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} |\nabla \zeta_{\gamma,j}|^p dx \leq c. \quad (3.7.30)$$

Let us extend  $\zeta_{\gamma,j}$  by reflection

$$\bar{\zeta}_{\gamma,j}(x) = \begin{cases} \zeta_{\gamma,j}(x_\alpha, 2\frac{\varepsilon_j}{r_j} - x_n) & \text{if } x_\alpha \in B'_{\gamma N_j} \text{ and } x_n \in (\varepsilon_j/r_j, 2\varepsilon_j/r_j), \\ \zeta_{\gamma,j}(x_\alpha, -2\frac{\varepsilon_j}{r_j} - x_n) & \text{if } x_\alpha \in B'_{\gamma N_j} \text{ and } x_n \in (-2\varepsilon_j/r_j, -\varepsilon_j/r_j) \end{cases} \quad (3.7.31)$$

and then, we extend it by  $(2\varepsilon_j/r_j)$ -periodicity in the  $x_n$  direction. The resulting sequence, still denoted by  $\{\bar{\zeta}_{\gamma,j}\}$ , is defined in  $(B'_{\gamma N_j} \times \mathbb{R}) \setminus C_{1,\gamma N_j}$ . Hence, we define on  $\mathbb{R}^n \setminus C_{1,\infty}$ ,

$$\bar{\zeta}_j(x) := \begin{cases} z & \text{in } (\mathbb{R}^{n-1} \setminus B'_{\gamma N_j}) \times (0, +\infty), \\ \bar{\zeta}_{\gamma,j}(x) & \text{in } (B'_{\gamma N_j} \times \mathbb{R}) \setminus C_{1,\gamma N_j}, \\ 0 & \text{in } (\mathbb{R}^{n-1} \setminus B'_{\gamma N_j}) \times (-\infty, 0). \end{cases} \quad (3.7.32)$$

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Let us now introduce the cut-off functions  $\phi_j \in C_c^\infty(-2\varepsilon_j/r_j, 2\varepsilon_j/r_j; [0, 1])$  such that  $\phi_j(x_n) = 1$  if  $|x_n| \leq \varepsilon_j/r_j$ ,  $\phi_j(x_n) = 0$  if  $|x_n| \geq 2\varepsilon_j/r_j$  and  $|\nabla_n \phi_j| \leq c(r_j/\varepsilon_j)$ . Then, we introduce our last sequence,

$$\tilde{\zeta}_j(x_\alpha, x_n) := \begin{cases} \bar{\phi}_j(x_n)\zeta_j(x_\alpha, x_n) + (1 - \phi_j(x_n))z & \text{if } (x_\alpha, x_n) \in \mathbb{R}^{n-1} \times (0, +\infty), \\ \phi_j(x_n)\bar{\zeta}_j(x_\alpha, x_n) & \text{if } (x_\alpha, x_n) \in \mathbb{R}^{n-1} \times (-\infty, 0). \end{cases}$$

Note that

$$\tilde{\zeta}_j = \zeta_{\gamma,j} \quad \text{on} \quad (B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}. \quad (3.7.33)$$

Moreover, by (3.7.30)-(3.7.33) we have that

$$\sup_{j \in \mathbb{N}} \int_{\mathbb{R}^n \setminus C_{1,\infty}} |\nabla_\alpha \tilde{\zeta}_j|^p dx \leq c, \quad (3.7.34)$$

while, for every  $(a, b) \subset \mathbb{R}$ , with  $-\infty < a < b < +\infty$ , we have

$$\int_{(\mathbb{R}^{n-1} \times (a,b)) \setminus C_{1,\infty}} |\nabla_n \tilde{\zeta}_j|^p dx \leq c, \quad (3.7.35)$$

for  $j$  large enough and  $c$  independent of  $(a, b)$ . Reasoning as in Proposition 3.7.3, with  $(0, +\infty)$  and  $(-\infty, 0)$  in place of  $(0, 1)$  and  $(-1, 0)$ , respectively, we can conclude that there exist  $\zeta_1 \in L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$  and  $\zeta_2 \in L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$  such that, up to subsequences,

$$\tilde{\zeta}_j - z \rightharpoonup \zeta_1 \quad \text{in} \quad L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m))$$

and

$$\tilde{\zeta}_j \rightharpoonup \zeta_2 \quad \text{in} \quad L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)).$$

Moreover, by (3.7.34) and (3.7.35), we have that, up to subsequences,  $\tilde{\zeta}_j$  converges weakly to  $\zeta$  in  $W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m)$  where

$$\zeta = \begin{cases} \zeta_1 + z & \text{in } \mathbb{R}^{n-1} \times (0, +\infty) \\ \zeta_2 & \text{in } (\mathbb{R}^{n-1} \times (-\infty, 0)) \cup (B'_1 \times \{0\}). \end{cases}$$

In particular, for every compact set  $K \subset \mathbb{R}^n \setminus C_{1,\infty}$ , we have that

$$\int_K |\nabla \zeta|^p dx \leq \liminf_{j \rightarrow +\infty} \int_K |\nabla \tilde{\zeta}_j|^p dx \leq c$$

for some constant  $c$  independent of  $K$ ; hence, we get that  $\nabla \zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m \times n})$  which concludes the description of the limit function  $\zeta$ .  $\square$

**Proposition 3.7.12** (Representation formula). *We have*

$$\varphi^{(0)}(z) = \inf \left\{ \int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m), \nabla \zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \right. \\ \left. \zeta - z \in L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \text{ and } \zeta \in L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\}$$

for every  $z \in \mathbb{R}^m$ .

*Proof.* We define

$$\psi^{(0)}(z) := \inf \left\{ \int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx : \zeta \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m), \nabla \zeta \in L^p(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^{m \times n}), \right. \\ \left. \zeta - z \in L^p(0, +\infty; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \text{ and } \zeta \in L^p(-\infty, 0; L^{p^*}(\mathbb{R}^{n-1}; \mathbb{R}^m)) \right\}$$

and let us prove that  $\varphi^{(0)}(z) = \psi^{(0)}(z)$ .

By definition of  $\varphi_{\gamma,j}^{(0)}$  (see (3.5.13)), for every fixed  $\eta > 0$ , there exists  $\zeta_{\gamma,j} \in Y_j^\gamma(z)$  such that

$$\int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} g_j(\nabla \zeta_{\gamma,j}) dx \leq \varphi_{\gamma,j}^{(0)}(z) + \eta; \quad (3.7.36)$$

hence, by Proposition 3.5.2 (i), (3.7.28) is satisfied. Then by Propositions 3.7.10 and 3.7.11 we get that

$$\begin{aligned} \lim_{j \rightarrow +\infty} \varphi_{\gamma,j}^{(0)}(z) + \eta &\geq \liminf_{j \rightarrow +\infty} \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} g_j(\nabla \tilde{\zeta}_j) dx \\ &\geq \liminf_{j \rightarrow +\infty} \int_{(B'_N \times (-N, N)) \setminus C_{1,N}} g_j(\nabla \tilde{\zeta}_j) dx \\ &\geq \int_{(B'_N \times (-N, N)) \setminus C_{1,N}} g(\nabla \zeta) dx \end{aligned} \quad (3.7.37)$$

for some fixed  $N > 1$ , where  $\zeta$  satisfies (3.7.29). Thus, passing to the limit in (3.7.37) as  $N \rightarrow +\infty$  and  $\gamma \rightarrow 0^+$ , it follows that

$$\varphi^{(0)}(z) \geq \int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx \geq \psi^{(0)}(z).$$

Let us prove now the converse inequality. For any fixed  $\eta > 0$ , let  $\zeta \in W_{\text{loc}}^{1,p}(\mathbb{R}^n \setminus C_{1,\infty}; \mathbb{R}^m)$  be as in (3.7.29) and satisfying

$$\int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx \leq \psi^{(0)}(z) + \eta. \quad (3.7.38)$$

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For every  $j \in \mathbb{N}$  and  $\gamma > 0$ , we consider a cut-off function  $\theta_{\gamma,j} \in \mathcal{C}_c^\infty(B'_{\gamma N_j}; [0, 1])$  such that  $\theta_{\gamma,j} = 1$  in  $B'_{(\gamma N_j)/2}$ ,  $|\nabla_\alpha \theta_{\gamma,j}| \leq c/\gamma N_j$  and we define  $\zeta_{\gamma,j} \in Y_j^\gamma(z)$  by

$$\zeta_{\gamma,j}(x) := \begin{cases} \theta_{\gamma,j}(x_\alpha)\zeta(x) + (1 - \theta_{\gamma,j}(x_\alpha))z & \text{in } (B'_{\gamma N_j})^{+(\varepsilon_j/r_j)} \\ \theta_{\gamma,j}(x_\alpha)\zeta(x) & \text{in } (B'_{\gamma N_j})^{-(\varepsilon_j/r_j)} \cup (B'_1 \times \{0\}). \end{cases}$$

Consequently,  $\zeta_{\gamma,j}$  is an admissible test function for (3.5.13) and we get that

$$\varphi_{\gamma,j}^{(0)}(z) \leq \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} g_j(\nabla \zeta_{\gamma,j}) dx.$$

The same kind of computations as those already employed in the proof of Lemma 3.7.4 (see (3.7.14)-(3.7.16)) show that

$$\lim_{j \rightarrow +\infty} \varphi_{\gamma,j}^{(0)}(z) \leq \limsup_{j \rightarrow +\infty} \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} g_j(\nabla \zeta) dx + o(1), \quad \text{as } \gamma \rightarrow 0^+.$$

On the other hand, Fatou's Lemma implies that

$$\limsup_{j \rightarrow +\infty} \int_{(B'_{\gamma N_j} \times I_j) \setminus C_{1,\gamma N_j}} g_j(\nabla \zeta) dx \leq \int_{\mathbb{R}^n \setminus C_{1,\infty}} g(\nabla \zeta) dx + o(1), \quad \text{as } \gamma \rightarrow 0^+.$$

Hence by (3.7.38), passing to the limit as  $\gamma \rightarrow 0^+$ , we get that

$$\varphi^{(0)}(z) \leq \psi^{(0)}(z) + \eta.$$

By the arbitrariness of  $\eta$  we conclude the proof.  $\square$





## Chapitre 4

# Quasistatic evolution of a brittle thin film



## 4.1 Introduction

Following Griffith's theory of brittle fracture, the variational model of quasistatic crack evolution proposed in [64] is based on the competition between the elastic energy and a surface energy which is necessary to produce a new crack or extending a preexisting one. The classical model was plagued by a few defects, being unable to initiate a crack, or to predict its path during the propagation; the authors were able to overcome these weaknesses, assuming neither a preexisting crack, nor a pre-defined crack path. In their formulation, the time-continuous growth of the cracks is seen as a limit of a discrete time evolution as the time step tends to zero. The first precise mathematical justification of this limit process in the scalar-valued case was given in [63] in the framework of *Special Functions of Bounded Variation (SBV)*. It was subsequently generalized to the vector-valued case in [46, 47] (see references therein) in the framework of *Generalized Special Functions of Bounded Variation (GSBV)*. To obtain compactness, the authors had to either add some conservative body and surface loadings with appropriate coerciveness as in [46], or to impose an empirical  $L^\infty$ -boundness hypothesis on every minimizing sequence as in [47]. This latter assumption permits to work in the space  $SBV^p$  of all  $SBV$ -functions with  $p$ -integrable approximate gradient and whose jump set has finite area, in lieu of  $GSBV$  (see Definition 4.1.1 below). The limit crack was defined through a new notion of convergence of rectifiable sets, called  $\sigma^p$ -convergence, related to the notion of jump sets of  $SBV$  functions and based on the weak convergence in  $SBV^p$ .

Before dealing with quasistatic evolutions, let us briefly describe the static model. Let  $U$  an open subset of  $\mathbb{R}^n$  representing the reference configuration of a homogeneous elastic material with cracks, whose stored energy density is given by the function  $W : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ , where  $\mathbb{R}^{m \times n}$  stands for the set of real  $m \times n$  matrices. According to Griffith's theory, the total energy under the deformation  $u : U \setminus K \rightarrow \mathbb{R}^m$  is given by

$$\mathcal{E}(u, K) := \int_{U \setminus K} W(\nabla u) d\mathcal{L}^n + \mathcal{H}^{n-1}(U \cap K),$$

where  $K$  is an unknown crack. Throughout the paper,  $\mathcal{L}^n$  and  $\mathcal{H}^{n-1}$  denote the  $n$ -dimensional Lebesgue measure and the  $(n-1)$ -dimensional Hausdorff measure in  $\mathbb{R}^n$  respectively (in the sequel,  $n$  will always be equal to 2 or 3). It is often convenient to use the weak formulation of this problem in the framework of Special Functions of Bounded Variation, replacing the crack  $K$  by the jump set  $S(u)$  of the deformation  $u \in SBV(U; \mathbb{R}^m)$  and where  $\nabla u$  is now the approximate gradient of  $u$ . We refer to [5, 55] for the definitions and basic properties of Functions with Bounded Variation. It is also usual to impose polynomial growth of order  $1 < p < \infty$  on  $W$ . Thus, a natural space is the space  $SBV^p(U; \mathbb{R}^m)$  of functions  $u \in SBV(U; \mathbb{R}^m)$  such that  $\mathcal{H}^{n-1}(S(u) \cap U) < +\infty$  and  $\nabla u \in L^p(U; \mathbb{R}^{m \times n})$ ; for such functions, the energy becomes

$$\mathcal{E}(u) := \mathcal{E}(u, S(u)) = \int_U W(\nabla u) d\mathcal{L}^n + \mathcal{H}^{n-1}(S(u) \cap U).$$

Let us now return to the quasistatic evolution. Adding some appropriate boundary conditions

and possibly some body and surface loadings, at each time  $t \in [0, T]$ , we seek to minimize the total energy  $(u, K) \mapsto \mathcal{E}(u, K)$  among all legal competitors. Namely,  $(u(t), K(t))$  is a minimum energy configuration provided that

$$\mathcal{E}(u(t), K(t)) \leq \mathcal{E}(u, K)$$

for every crack  $K$  containing  $K(t)$  and every deformation  $u$ , possibly discontinuous across  $K$  and satisfying the boundary conditions. More precisely, an irreversible quasistatic evolution of minimum energy configuration is an application  $[0, T] \ni t \mapsto (u(t), K(t))$  satisfying the following conditions :

- (i) *Irreversibility* :  $K(t)$  increases with  $t$  ;
- (ii) *Static equilibrium* : for every  $t \in [0, T]$ , the pair  $(u(t), K(t))$  is a minimum energy configuration ;
- (iii) *Nondissipativity* : the function  $t \mapsto \mathcal{E}(u(t), K(t))$  is absolutely continuous.

Condition (iii) express the conservation of the energy in the sense that the derivative of the internal energy  $\mathcal{E}(u(t), K(t))$  is equal to the power of the applied forces.

Sometimes a natural small parameter, denoted by  $\varepsilon > 0$ , is involved in the model and one should look at the behavior of the energy and of the quasistatic evolution  $(u^\varepsilon(t), K^\varepsilon(t))$  when  $\varepsilon$  tends to zero. The notion of  $\Gamma$ -convergence – see [45] for a complete treatment on that subject – has proved useful in investigating the variational convergence for static problems ; in the present context of variational evolution, it remains to see whether such a process is compatible with the evolution. A problem of this type have been studied in [70] where the authors proved a stability result of variational models of quasistatic crack evolution under  $\Gamma$ -convergence in the antiplanar case. Indeed, they have shown that  $(u^\varepsilon(t), K^\varepsilon(t))$  converges, in a certain sense, to a quasistatic evolution of the relaxed model. They had to define a new variational notion of convergence for rectifiable sets that they called  $\sigma$ -convergence. The limit crack is thus seen as the  $\sigma$ -limit of  $K^\varepsilon(t)$  while the limit deformation is nothing but the weak  $SBV^p(U)$ -limit of  $u^\varepsilon(t)$ . In the same spirit, [69] investigates a notion of quasistatic evolution for the elliptic approximation of the Mumford-Shah functional and proves its convergence to a quasistatic growth of brittle fractures in linearly elastic bodies.

In this paper we treat cylindrical bodies whose thickness becomes arbitrarily small and ask how the quasistatic evolution behaves when the small parameter, the thickness of the cylinder, tends to zero. It leads us to perform a 3D-2D asymptotic analysis of a thin film, taking into account the possibility of fracture. An abundant literature exists on dimensional reduction problems and we point the reader to e.g. [23, 34, 35, 74] and references therein.

Let  $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$  be a three-dimensional cylinder of basis  $\omega$ , a bounded open subset of  $\mathbb{R}^2$  and of thickness  $2\varepsilon$ , representing the reference configuration of a homogeneous elastic body containing some cracks. The study of the static problem relies on the computation of the  $\Gamma$ -limit of the sequence of functionals associated to the internal energy, as  $\varepsilon$  tends to zero. Namely, we

want to look at the behavior of the functional

$$SBV^p(\Omega_\varepsilon; \mathbb{R}^3) \ni u \mapsto \int_{\Omega_\varepsilon} W(\nabla u) d\mathcal{L}^3 + \mathcal{H}^2(S(u) \cap \Omega_\varepsilon)$$

when  $\varepsilon \rightarrow 0$ , in the sense of  $\Gamma$ -convergence. It is convenient to rescale the problem on the unit cylinder  $\Omega := \Omega_1$ . Denoting by  $x_\alpha := (x_1, x_2)$  the in-plane variable, we set  $v(x_\alpha, x_3/\varepsilon) = u(x_\alpha, x_3)$ . Replacing  $u$  by  $v$  in the previous energy, changing variables and dividing by  $\varepsilon$ , we are led to studying the equivalent sequence of functionals

$$SBV^p(\Omega; \mathbb{R}^3) \ni v \mapsto \int_{\Omega} W \left( \nabla_\alpha v \middle| \frac{1}{\varepsilon} \nabla_3 v \right) d\mathcal{L}^3 + \int_{S(v) \cap \Omega} \left| \left( (\nu_v)_\alpha \middle| \frac{1}{\varepsilon} (\nu_v)_3 \right) \right| d\mathcal{H}^2,$$

where, from now on,  $\nabla_\alpha$  (resp.  $\nabla_3$ ) will stand for the (approximate) differential with respect to  $x_\alpha$  (resp.  $x_3$ ),  $F = (F_\alpha | F_3)$  for some matrix  $F \in \mathbb{R}^{3 \times 3}$  and  $z = (z_\alpha | z_3)$  for some vector  $z \in \mathbb{R}^3$ . Such studies have been performed in [23] where the authors imposed to the crack to live on the mid-surface of the plate and also in [28, 34]. Unlike the latest references, we are working with materials satisfying Griffith's principle. This was not previously taken into account because the authors forced the surface energy to grow linearly with respect to the jump of the deformation. This restriction was due to the application of Integral Representation Theorems in  $SBV(\Omega; \mathbb{R}^3)$  (see e.g. Theorem 2.4 in [31] or Theorem 1 in [28]) which hold for such surface energies. Furthermore, it was necessary to obtain a bound in  $BV(\Omega; \mathbb{R}^3)$  of any minimizing sequence. We are convinced that such a behavior is not very realistic because it does not permit to take into account neither Griffith nor Barenblatt theory in which the surface energy behaves asymptotically like a constant : the toughness. However, in [34], Example 2.10, the authors suggest a way to recover a brittle elastic material obeying Griffith's law by a singular perturbation approach (see also [33], Section 8).

The originality of our work comes from the fact that we are directly dealing with Griffith surface energy, which was proscribed in [28, 34] as explained above. We propose here an alternative proof of this result, deriving the  $\Gamma$ -limit by an argument very closed to that used in the proof of Theorem 2.1 in [56]. Indeed, to overcome the lack of compactness in  $BV(\Omega; \mathbb{R}^3)$ , we use a regular truncation function. It permits us to show that for a deformation belonging to  $L^\infty(\Omega; \mathbb{R}^3)$ , there is no loss of generality in requiring that any minimizing sequence is uniformly bounded in  $L^\infty(\Omega; \mathbb{R}^3)$  and thus, weak  $SBV^p(\Omega; \mathbb{R}^3)$ -compactness follows. The two-dimensional surface energy remains of Griffith type whereas the bulk energy follows the one obtained in [28, 34] : the stored energy density is given by the function  $\mathcal{Q}W_0 : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  where  $W_0(\overline{F}) := \inf \{ W(\overline{F}|z) : z \in \mathbb{R}^3 \}$  and  $\mathcal{Q}W_0$  is the 2D-quasiconvexification of  $W_0$ . Note that we obtain exactly the same energy density as in [74], in which the authors were treating healthy materials, that is without any crack.

Then, it remains to pass to the limit in the quasistatic evolution. We need a notion of convergence of rectifiable sets, in the spirit of  $\sigma^p$ -convergence introduced in [46], but better adapted to the dimensional reduction problems. Intuitively, any limit crack should be a one-dimensional set, that is a two-dimensional set which is invariant by translation in the  $x_3$  direction. It seems

that  $\sigma^p$ -convergence do not permit to obtain such limit cracks. This is why we are led to defining a new notion of convergence (see Definition 4.4.1) similar to that of  $\sigma^p$ -convergence (see Definition 4.1 in [46]), but for the fact that we impose to any sequence of test functions to have an approximate scaled gradient uniformly bounded in  $L^p$ . Nevertheless it is not sufficient to pass to the limit in the surface term because we need to have compactness for sequences of cracks with bounded scaled surface energy. This difficulty is overcome thanks to Proposition 4.4.3. The last tool is a Jump Transfer Theorem (see Theorem 2.1 in [63]) stated in a rescaled version in Theorem 4.4.4 and whose proof uses a De Giorgi type slicing argument.

As in [47], we will assume that any minimizing sequence involved in the quasistatic evolution, remains bounded in  $L^\infty(\Omega; \mathbb{R}^3)$  uniformly in  $\varepsilon > 0$  and in time  $t \in [0, T]$  (see (4.2.8) and the proof of Lemma 4.5.1). We do not attempt to justify this hypothesis which appears naturally in the scalar case by a truncation argument, whenever the prescribed boundary deformation is also bounded in  $L^\infty(\Omega \times [0, T]; \mathbb{R})$ , uniformly in  $\varepsilon > 0$ . It will yield weak  $SBV^p(\Omega; \mathbb{R}^3)$ -compactness and thus allow to define a limit deformation field. Note that we could also have taken the path to add appropriate body and surface loadings, as in [46], with the right order of magnitude, in which case we should be working in a suitable subspace of  $GSBV(\Omega; \mathbb{R}^3)$  instead of  $SBV^p(\Omega; \mathbb{R}^3)$ . We insist on the fact that this boundedness hypothesis is completely empirical but has the advantage of avoiding technical difficulties connected with the space  $GSBV(\Omega; \mathbb{R}^3)$ .

As mentioned before, we adopt the following

**Definition 4.1.1.** *For any open set  $U \subset \mathbb{R}^n$ ,*

$$SBV^p(U; \mathbb{R}^m) := \{w \in SBV(U; \mathbb{R}^m) : \nabla w \in L^p(U; \mathbb{R}^{m \times n}) \text{ and } \mathcal{H}^{n-1}(S(w) \cap U) < \infty\}.$$

Throughout the text, letters as  $C$  stand for generic constants which may vary from line to line. We will denote by  $\tilde{\subset}$  and  $\tilde{=}$  inclusion and equality of sets up to a set of zero  $\mathcal{H}^{n-1}$ -measure respectively. As for notation,  $\mathcal{M}_b(U)$  stands for the space of signed Radon measures with finite total variation. If  $\mu \in \mathcal{M}_b(U)$  and  $E$  is a Borel subset of  $\mathbb{R}^n$ , we will denote by  $\mu|_E$  the restriction of the measure  $\mu$  to  $E$  that is, for every Borel subset  $B$  of  $\mathbb{R}^n$ ,  $\mu|_E(B) = \mu(E \cap B)$ . For any  $\mu$ -measurable set  $A \subset \mathbb{R}^n$  and any  $\mu$ -measurable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we write  $\bar{f}_A := \mu(A)^{-1} \int_A f d\mu$  for the average of  $f$  over  $A$ . For the remainder of the paper, strong convergence will always be denoted by  $\rightarrow$ , whereas weak (resp. weak-\*) convergence will be denoted by  $\rightharpoonup$  (resp.  $\overset{*}{\rightharpoonup}$ ). We recall that a sequence  $\{u_k\} \subset SBV^p(U; \mathbb{R}^m)$  is said to converge weakly to some

$u \in SBV^p(U; \mathbb{R}^m)$ , and we write  $u_k \rightharpoonup u$  in  $SBV^p(U; \mathbb{R}^m)$ , if

$$\left\{ \begin{array}{l} u_k \rightarrow u \text{ in } L^1(U; \mathbb{R}^m), \\ u_k \overset{*}{\rightharpoonup} u \text{ in } L^\infty(U; \mathbb{R}^m), \\ \nabla u_k \rightharpoonup \nabla u \text{ in } L^p(U; \mathbb{R}^{m \times n}), \\ (u_k^+ - u_k^-) \otimes \nu_{u_k} \mathcal{H}_{[S(u_k)]}^{n-1} \overset{*}{\rightharpoonup} (u^+ - u^-) \otimes \nu_u \mathcal{H}_{[S(u)]}^{n-1} \text{ in } \mathcal{M}_b(U; \mathbb{R}^{m \times n}). \end{array} \right.$$

## 4.2 Formulation of the problem

### 4.2.1 The physical configuration

**Reference configuration :** Let  $\omega$  be a bounded open subset of  $\mathbb{R}^2$  with Lipschitz boundary and define  $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ . The cylinder  $\overline{\Omega}_\varepsilon$  denotes the reference configuration of a homogeneous elastic body with cracks, whose stored energy density is given by the function  $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ . We assume  $W$  to be a  $\mathcal{C}^1$  and quasiconvex function satisfying standard  $p$ -growth and  $p$ -coercivity conditions ( $1 < p < \infty$ ) : there exists  $\beta > 0$  such that

$$\frac{1}{\beta} |F|^p - \beta \leq W(F) \leq \beta(1 + |F|^p), \text{ for all } F \in \mathbb{R}^{3 \times 3}. \quad (4.2.1)$$

Let  $p' = (p-1)/p$  be the conjugate exponent of  $p$ . In particular (see [44]), the derivative of  $W$ , denoted by  $\partial W$ , satisfies some  $(p-1)$ -growth condition, namely

$$|\partial W(F)| \leq \beta(1 + |F|^{p-1}), \text{ for all } F \in \mathbb{R}^{3 \times 3}. \quad (4.2.2)$$

**Boundary conditions :** Let  $\omega' \subset \mathbb{R}^2$  be such that  $\overline{\omega} \subset \omega'$  and  $\Omega'_\varepsilon := \omega' \times (-\varepsilon, \varepsilon)$  be the enlarged cylinder. We submit the body to a “smooth” given deformation

$$G^\varepsilon \in W^{1,1}([0, T]; W^{1,p}(\Omega'_\varepsilon; \mathbb{R}^3))$$

on the lateral boundary  $\partial\omega \times (-\varepsilon, \varepsilon)$  of the cylinder  $\Omega_\varepsilon$ . For homogeneity, we assume that  $G^\varepsilon$  satisfies

$$\|G^\varepsilon\|_{W^{1,1}([0, T]; W^{1,p}(\Omega'_\varepsilon; \mathbb{R}^3))} \leq C\varepsilon^{1/p}. \quad (4.2.3)$$

As in [47, 63], we express the lateral boundary condition at time  $t$  by requiring that  $U = G^\varepsilon(t)$  a.e. on  $[\omega' \setminus \overline{\omega}] \times (-\varepsilon, \varepsilon)$  whenever  $U \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3)$  is a kinematically admissible deformation field. Since the Dirichlet boundary condition is only prescribed on the lateral boundary, we do not need, as in [47, 63], to extend the whole cylinder but only its lateral part, which can be trivially done by extending the base domain  $\omega$  to  $\omega'$ . As a consequence, any admissible crack will necessarily be contained in  $\overline{\omega} \times (-\varepsilon, \varepsilon)$ . We will assume that the remainder of the boundary,  $\omega \times \{-\varepsilon, \varepsilon\}$ , is traction free so that the prescribed boundary deformation is the only driving



mechanism. Since the admissible cracks never run into  $\omega \times \{-\varepsilon, \varepsilon\}$ , there is no need, in contrast to [47, 63], to remove the part  $\omega \times \{-\varepsilon, \varepsilon\}$  from the surface energy of the crack, so that the surface energy of a crack associated to a test function  $V \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3)$  will be exactly  $\mathcal{H}^2(S(V))$ .

**Initial conditions :** As in [63], we consider a body without any preexisting crack (see Remark 4.5.4). If  $U_0^\varepsilon \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3)$  is a given initial deformation satisfying  $U_0^\varepsilon = G^\varepsilon(0)$  a.e. on  $[\omega' \setminus \bar{\omega}] \times (-\varepsilon, \varepsilon)$ , we suppose that the Griffith equilibrium condition is satisfied, that is  $U_0^\varepsilon$  minimizes

$$V \mapsto \int_{\Omega_\varepsilon} W(\nabla V) dx + \mathcal{H}^2(S(V))$$

among  $\{V \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3) : V = G^\varepsilon(0) \text{ a.e. on } [\omega' \setminus \bar{\omega}] \times (-\varepsilon, \varepsilon)\}$ .

**Quasistatic evolution :** If we follow word for word the arguments developed in [46, 47], applying them to  $\omega' \times (-\varepsilon, \varepsilon)$  in place of what is used in those references, namely  $\Omega''_\varepsilon$ , a Lipschitz extension of  $\omega \times (-\varepsilon, \varepsilon)$  with  $\bar{\omega} \times [-\varepsilon, \varepsilon] \subset \Omega''_\varepsilon$ , we get the existence of a crack  $K^\varepsilon(t) \subset \bar{\omega} \times (-\varepsilon, \varepsilon)$  increasing in time and a deformation field  $U^\varepsilon(t) \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3)$  such that

- $U^\varepsilon(0) = U_0^\varepsilon$ ,  $K^\varepsilon(0) \cong S(U_0^\varepsilon)$  and  $U^\varepsilon(0)$  minimizes

$$V \mapsto \int_{\Omega_\varepsilon} W(\nabla V) dx + \mathcal{H}^2(S(V))$$

- among  $\{V \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3) : V = G^\varepsilon(0) \text{ a.e. on } [\omega' \setminus \bar{\omega}] \times (-\varepsilon, \varepsilon)\}$ ;
- for any  $t \in (0, T]$ ,  $S(U^\varepsilon(t)) \tilde{\subset} K^\varepsilon(t)$  and  $U^\varepsilon(t)$  minimizes

$$V \mapsto \int_{\Omega_\varepsilon} W(\nabla V) dx + \mathcal{H}^2(S(V) \setminus K^\varepsilon(t))$$

- among  $\{V \in SBV^p(\Omega'_\varepsilon; \mathbb{R}^3) : V = G^\varepsilon(t) \text{ a.e. on } [\omega' \setminus \bar{\omega}] \times (-\varepsilon, \varepsilon)\}$ ;
- the total energy

$$E_\varepsilon(t) := \int_{\Omega_\varepsilon} W(\nabla U^\varepsilon(t)) dx + \mathcal{H}^2(K^\varepsilon(t))$$

is absolutely continuous with respect to the time  $t$  and

$$E_\varepsilon(t) = E_\varepsilon(0) + \int_0^t \int_{\Omega_\varepsilon} \partial W(\nabla U^\varepsilon(\tau)) \cdot \nabla \dot{G}^\varepsilon(\tau) dx d\tau.$$

#### 4.2.2 The rescaled configuration

As usual in dimension reduction, we perform a scaling so as to study an equivalent problem stated on a cylinder of unit thickness  $\Omega := \omega \times I$ , where  $I := (-1, 1)$ . Let  $\Omega' := \omega' \times I$  be the enlarged rescaled cylinder.

**Boundary conditions (rescaled) :** Let  $g^\varepsilon(t, x_\alpha, x_3) = G^\varepsilon(t, x_\alpha, \varepsilon x_3)$  be the rescaled boundary deformation. Changing variables, (4.2.3) implies that

$$\|g^\varepsilon\|_{W^{1,1}([0,T];L^p(\Omega';\mathbb{R}^3))} + \left\| \left( \nabla_\alpha g^\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 g^\varepsilon \right) \right\|_{W^{1,1}([0,T];L^p(\Omega';\mathbb{R}^{3 \times 3}))} \leq C. \quad (4.2.4)$$

In fact, we will further restrict  $g^\varepsilon$  in (4.2.7) by requiring the strong convergence of both  $g^\varepsilon$  and its scaled gradient together with a uniform bound in  $L^\infty(\Omega' \times [0, T]; \mathbb{R}^3)$ . In particular, the limit  $g \in W^{1,1}([0, T]; W^{1,p}(\Omega'; \mathbb{R}^3))$  satisfies  $\nabla_3 g = 0$ . In the sequel, we will identify the space  $SBVP(\omega; \mathbb{R}^3)$  (resp.  $W^{1,p}(\omega; \mathbb{R}^3)$ ,  $L^p(\omega; \mathbb{R}^3)$ ) with those functions  $v \in SBVP(\Omega; \mathbb{R}^3)$  (resp.  $W^{1,p}(\Omega; \mathbb{R}^3)$ ,  $L^p(\Omega; \mathbb{R}^3)$ ) such that  $D_3 v = 0$  in the sense of distributions. As a consequence  $g \in W^{1,1}([0, T]; W^{1,p}(\omega'; \mathbb{R}^3))$ .

**Initial conditions (rescaled) :** Let  $u_0^\varepsilon(x_\alpha, x_3) = U_0^\varepsilon(x_\alpha, \varepsilon x_3)$  the rescaled initial deformation, then  $u_0^\varepsilon \in SBVP(\Omega'; \mathbb{R}^3)$ ,  $u_0^\varepsilon = g^\varepsilon(0)$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$  and  $u_0^\varepsilon$  minimizes

$$v \mapsto \int_\Omega W \left( \nabla_\alpha v \Big| \frac{1}{\varepsilon} \nabla_3 v \right) dx + \int_{S(v)} \left| \left( (\nu_v)_\alpha \Big| \frac{1}{\varepsilon} (\nu_v)_3 \right) \right| d\mathcal{H}^2,$$

among  $\{v \in SBVP(\Omega'; \mathbb{R}^3) : v = g^\varepsilon(0) \text{ a.e. on } [\omega' \setminus \bar{\omega}] \times I\}$ .

**Quasistatic evolution (rescaled) :** We set

$$u^\varepsilon(t, x_\alpha, x_3) = U^\varepsilon(t, x_\alpha, \varepsilon x_3) \text{ and } \Gamma^\varepsilon(t) = \{x \in \bar{\omega} \times I : (x_\alpha, \varepsilon x_3) \in K^\varepsilon(t)\}.$$

Then,  $u^\varepsilon(t) \in SBVP(\Omega'; \mathbb{R}^3)$  and  $\Gamma^\varepsilon(t) \subset \bar{\omega} \times I$  is increasing in time. Moreover,

- $u^\varepsilon(0) = u_0^\varepsilon$ ,  $\Gamma^\varepsilon(0) \cong S(u_0^\varepsilon)$ ;
- for all  $t \in (0, T]$ ,  $S(u^\varepsilon(t)) \tilde{\subset} \Gamma^\varepsilon(t)$  and  $u^\varepsilon(t)$  minimizes

$$v \mapsto \int_\Omega W \left( \nabla_\alpha v \Big| \frac{1}{\varepsilon} \nabla_3 v \right) dx + \int_{S(v) \setminus \Gamma^\varepsilon(t)} \left| \left( (\nu_v)_\alpha \Big| \frac{1}{\varepsilon} (\nu_v)_3 \right) \right| d\mathcal{H}^2, \quad (4.2.5)$$

- among  $\{v \in SBVP(\Omega'; \mathbb{R}^3) : v = g^\varepsilon(t) \text{ a.e. on } [\omega' \setminus \bar{\omega}] \times I\}$ ;
- the total energy

$$\mathcal{E}_\varepsilon(t) := \int_\Omega W \left( \nabla_\alpha u^\varepsilon(t) \Big| \frac{1}{\varepsilon} \nabla_3 u^\varepsilon(t) \right) dx + \int_{\Gamma^\varepsilon(t)} \left| \left( (\nu_{\Gamma^\varepsilon(t)})_\alpha \Big| \frac{1}{\varepsilon} (\nu_{\Gamma^\varepsilon(t)})_3 \right) \right| d\mathcal{H}^2$$

is absolutely continuous in time and

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon(0) + \int_0^t \int_\Omega \partial W \left( \nabla_\alpha u^\varepsilon(\tau) \Big| \frac{1}{\varepsilon} \nabla_3 u^\varepsilon(\tau) \right) \cdot \left( \nabla_\alpha \dot{g}^\varepsilon(\tau) \Big| \frac{1}{\varepsilon} \nabla_3 \dot{g}^\varepsilon(\tau) \right) dx d\tau. \quad (4.2.6)$$

We would like to perform an asymptotic analysis of this quasistatic evolution when the thickness  $\varepsilon$  tends to zero. To this end, we start by stating a  $\Gamma$ -convergence result in order to guess

how the energy is behaving through the dimensional reduction. In fact, we will prove in Section 4.3 that the functional

$$SBV^p(\Omega; \mathbb{R}^3) \ni v \mapsto \int_{\Omega} W \left( \nabla_{\alpha} v \left| \frac{1}{\varepsilon} \nabla_3 v \right. \right) dx + \int_{S(v)} \left| \left( (\nu_v)_{\alpha} \left| \frac{1}{\varepsilon} (\nu_v)_3 \right. \right) \right| d\mathcal{H}^2$$

$\Gamma$ -converges for the strong  $L^1(\Omega; \mathbb{R}^3)$ -topology to

$$SBV^p(\omega; \mathbb{R}^3) \ni v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v)),$$

where  $W_0 : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  is defined by  $W_0(\bar{F}) := \inf\{W(\bar{F}|z) : z \in \mathbb{R}^3\}$ ,

$$\mathcal{Q}W_0(\bar{F}) := \inf_{\varphi \in W_0^{1,\infty}(Q'; \mathbb{R}^3)} \int_{Q'} W_0(\bar{F} + \nabla_{\alpha} \varphi(x_{\alpha})) dx_{\alpha}$$

is the 2D-quasiconvexification of  $W_0$  and  $Q' := (0, 1)^2$  is the unit square of  $\mathbb{R}^2$ . A two-dimensional quasistatic evolution relative to the boundary data  $g(t)$  for the relaxed model is an application  $[0, T] \ni t \mapsto (u(t), \gamma(t))$  such that  $u(t) \in SBV^p(\omega'; \mathbb{R}^3)$ ,  $u(t) = g(t)$  a.e. on  $\omega' \setminus \bar{\omega}$ ,  $\gamma(t) \subset \bar{\omega}$  and the three following properties hold :

- (i) *Irreversibility* :  $\gamma(t_1) \tilde{\subset} \gamma(t_2)$ , for every  $0 \leq t_1 \leq t_2 \leq T$  ;
- (ii) *Minimality* :  $S(u(0)) \cong \gamma(0)$ ,  $u(0)$  minimizes

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v)),$$

among  $\{v \in SBV^p(\omega'; \mathbb{R}^3) : v = g(0) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$  and, for every  $t \in (0, T]$ ,  $S(u(t)) \tilde{\subset} \gamma(t)$  and  $u(t)$  minimizes

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)),$$

among  $\{v \in SBV^p(\omega'; \mathbb{R}^3) : v = g(t) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$  ;

- (iii) *Nondissipativity* : The total energy

$$\mathcal{E}(t) := 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(t))$$

is absolutely continuous in time and

$$\mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha} u(\tau)) \cdot \nabla \dot{g}(\tau) dx_{\alpha} d\tau.$$

Note that the previous equality makes sense because we will prove in Proposition 4.4.7 that  $\mathcal{Q}W_0$  is of class  $\mathcal{C}^1$  provided  $W$  is also of class  $\mathcal{C}^1$  (see also the Appendix for an alternative proof). The main result of this paper is that the three-dimensional quasistatic evolution converges, in the sense detailed below, towards a two-dimensional quasistatic evolution associated with the  $\Gamma$ -limit model. This is formally expressed as the following

## 4.2. FORMULATION OF THE PROBLEM

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**Theorem 4.2.1.** *For all  $\varepsilon > 0$ , let  $[0, T] \ni t \mapsto (u^\varepsilon(t), \Gamma^\varepsilon(t))$  be a three-dimensional (rescaled) quasistatic evolution relative to the boundary data  $g^\varepsilon(t)$ . Assume that*

$$\left\{ \begin{array}{l} \sup_{\varepsilon > 0} \|g^\varepsilon\|_{L^\infty(\Omega' \times [0, T]; \mathbb{R}^3)} < +\infty, \\ g^\varepsilon \rightarrow g \text{ in } W^{1,1}([0, T]; W^{1,p}(\Omega'; \mathbb{R}^3)), \quad \frac{1}{\varepsilon} \nabla_3 g^\varepsilon \rightarrow H \text{ in } W^{1,1}([0, T]; L^p(\Omega'; \mathbb{R}^3)) \end{array} \right. \quad (4.2.7)$$

and that

$$\sup_{\varepsilon > 0} \|u^\varepsilon(t)\|_{L^\infty(\Omega'; \mathbb{R}^3)} < +\infty \quad (4.2.8)$$

uniformly with respect to  $t \in [0, T]$ . Then, there exists a two-dimensional quasistatic evolution  $[0, T] \ni t \mapsto (u(t), \gamma(t))$  relative to the boundary data  $g(t)$  for the relaxed model and a sequence  $\{\varepsilon_n\} \searrow 0^+$  such that for every  $t \in [0, T]$ ,

- $\Gamma^{\varepsilon_n}(t)$  converges to  $\gamma(t)$  in the sense of Definition 4.4.1,  $u^{\varepsilon_n}(0) \rightharpoonup u(0)$  in  $SBV^p(\Omega'; \mathbb{R}^3)$ ;
- $u^{\varepsilon_{n_i}}(t) \rightharpoonup u(t)$  in  $SBV^p(\Omega'; \mathbb{R}^3)$ , for some  $t$ -dependent subsequence  $\{\varepsilon_{n_i}\} \subset \{\varepsilon_n\}$ ;
- the total energy  $\mathcal{E}_{\varepsilon_n}(t)$  converges to  $\mathcal{E}(t)$  and more precisely,

$$\left\{ \begin{array}{l} \int_{\Omega} W \left( \nabla_\alpha u^{\varepsilon_n}(t) \left| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(t) \right. \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u(t)) dx_\alpha, \\ \int_{\Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon_n}(t)})_\alpha \left| \frac{1}{\varepsilon_n} (\nu_{\Gamma^{\varepsilon_n}(t)})_3 \right. \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(t)). \end{array} \right.$$

As it has been discussed in the introduction, it seems that the right functional setting for this kind of problems is that of *Generalized Special Functions of Bounded Variation* (see [46]) adding appropriate conservative surface and body forces. Here, as in [47], we will only deal with *Special Functions of Bounded Variation*, imposing, without any justification, that the minimizing fields are bounded in  $L^\infty(\Omega; \mathbb{R}^3)$ , uniformly in time (see (4.2.8) and the proof of Lemma 4.5.1). Note that this assumption is automatically satisfied in the scalar case by a truncation argument, provided that  $g^\varepsilon$  is also bounded in  $L^\infty(\Omega' \times [0, T]; \mathbb{R})$ . This will let us get a bound in  $BV(\Omega; \mathbb{R}^3)$  for any sequence with bounded energy and thus apply Ambrosio's Compactness Theorem in  $SBV$  (Theorem 4.8 in [5]). In a  $GSBV$  context, we could state a result similar to that in [46] upon adding some appropriate conservative body and surface loadings with the right order of magnitude. Adequate coerciveness assumptions would allow us to get rid of the empirical  $L^\infty$ -bound (4.2.8) because we would have natural compactness in a suitable subspace of  $GSBV(\Omega; \mathbb{R}^3)$  and we would then obtain a membrane limit model.

Let us first state a compactness result in  $SBV^p(\Omega; \mathbb{R}^3)$  which ensures that any limit deformation field does not depend on the  $x_3$  variable.

**Lemma 4.2.2.** *Let  $\{\varepsilon_n\} \searrow 0^+$  and  $\{u_n\} \subset SBV^p(\Omega; \mathbb{R}^3)$  such that*

$$\sup_{n \in \mathbb{N}} \left\{ \|u_n\|_{L^\infty(\Omega; \mathbb{R}^3)} + \int_{\Omega} \left| \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx + \int_{S(u_n)} \left| \left( (\nu_{u_n})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{u_n})_3 \right) \right| d\mathcal{H}^2 \right\} < +\infty. \quad (4.2.9)$$

*Then, there exists a subsequence  $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$  and a function  $u \in SBV^p(\omega; \mathbb{R}^3)$  such that  $u_{n_k} \rightharpoonup u$  in  $SBV^p(\Omega; \mathbb{R}^3)$ .*

*Proof.* In view of (4.2.9), we have in particular

$$\sup_{n \in \mathbb{N}} \left\{ \|u_n\|_{L^\infty(\Omega; \mathbb{R}^3)} + \|\nabla u_n\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})} + \mathcal{H}^2(S(u_n)) \right\} < +\infty.$$

Thus, according to Ambrosio's Compactness Theorem (Theorem 4.8 in [5]), we can find a subsequence  $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$  and a function  $u \in SBV^p(\Omega; \mathbb{R}^3)$  such that  $u_{n_k} \rightharpoonup u$  in  $SBV^p(\Omega; \mathbb{R}^3)$ . Let us show that  $u$  does not depend on  $x_3$ . Indeed, (4.2.9) implies that

$$\frac{1}{\varepsilon_{n_k}} \left[ \int_{\Omega} |\nabla_3 u_{n_k}|^p dx + \int_{S(u_{n_k})} |(\nu_{u_{n_k}})_3| d\mathcal{H}^2 \right] \leq C.$$

By lower semicontinuity of

$$v \mapsto \int_{\Omega} |\nabla_3 v|^p dx + \int_{S(v)} |(\nu_v)_3| d\mathcal{H}^2$$

with respect to the  $SBV^p(\Omega; \mathbb{R}^3)$ -convergence (see [28, 34]), we deduce that  $\nabla_3 u = 0$   $\mathcal{L}^3$ -a.e. in  $\Omega$  and that  $(\nu_u)_3 = 0$   $\mathcal{H}^2$ -a.e. in  $S(u)$ . Since  $u \in SBV^p(\Omega; \mathbb{R}^3)$ , it implies that  $D_3 u = 0$  in the sense of Radon measures and thus  $u \in SBV^p(\omega; \mathbb{R}^3)$ .  $\square$

Note that the  $L^\infty$ -bound in (4.2.9) will follow from assumption (4.2.8), an assumption that we do not attempt to justify, while the two other bounds will appear naturally in the energy estimates.

### 4.3 A $\Gamma$ -convergence result

This section is devoted to the study of the static problem. Let us define  $\mathcal{I}_\varepsilon : BV(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \int_{\Omega} W \left( \nabla_{\alpha} u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx + \int_{S(u)} \left| \left( (\nu_u)_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_u)_3 \right) \right| d\mathcal{H}^2 & \text{if } u \in SBV^p(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise} \end{cases}$$

Then, the following  $\Gamma$ -convergence result holds :

**Theorem 4.3.1.** *Let  $\omega$  be a bounded open subset of  $\mathbb{R}^2$  and  $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  be a continuous function satisfying (4.2.1). Then the functional  $\mathcal{I}_\varepsilon$   $\Gamma$ -converges for the strong  $L^1(\Omega; \mathbb{R}^3)$ -topology towards  $\mathcal{J} : BV(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  defined by*

$$\mathcal{J}(u) := \begin{cases} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} + 2\mathcal{H}^1(S(u)) & \text{if } u \in SBV^p(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

**Remark 4.3.2.** It has been noted in [74] p. 556 (see also [34] p. 306) that the function  $\mathbb{R}^{3 \times 3} \ni (\overline{F}|F_3) \mapsto \mathcal{Q}W_0(\overline{F})$  is quasiconvex : for any  $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^3)$ ,

$$\mathcal{Q}W_0(\overline{F}) \leq \int_Q \mathcal{Q}W_0(\overline{F} + \nabla_{\alpha} \varphi(y)) dy,$$

where  $Q := (0, 1)^3$  is the unit square of  $\mathbb{R}^3$ .

We first localize our functionals on  $\mathcal{A}(\omega)$ , the family of open subsets of  $\omega$ . Define  $\mathcal{I}_\varepsilon$  and  $\mathcal{J} : BV(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega) \rightarrow \overline{\mathbb{R}}$  by

$$\mathcal{I}_\varepsilon(u; A) := \begin{cases} \int_{A \times I} W \left( \nabla_{\alpha} u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx \\ \quad + \int_{S(u) \cap [A \times I]} \left| \left( (\nu_u)_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_u)_3 \right) \right| d\mathcal{H}^2 & \text{if } u \in SBV^p(A \times I; \mathbb{R}^3), \\ +\infty & \text{otherwise} \end{cases} \quad (4.3.1)$$

and

$$\mathcal{J}(u; A) := \begin{cases} 2 \int_A \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} + 2\mathcal{H}^1(S(u) \cap A) & \text{if } u \in SBV^p(A; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \quad (4.3.2)$$

For every sequence  $\{\varepsilon_j\} \searrow 0^+$  and all  $(u; A) \in BV(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$ , we define the  $\Gamma$ -lower limit by

$$\mathcal{I}(u; A) := \inf_{\{u_j\}} \left\{ \liminf_{j \rightarrow +\infty} \mathcal{I}_{\varepsilon_j}(u_j; A) : u_j \rightarrow u \text{ in } L^1(A \times I; \mathbb{R}^3) \right\}. \quad (4.3.3)$$

Let  $\mathcal{R}(\omega)$  be the countable subfamily of  $\mathcal{A}(\omega)$  obtained by taking every finite union of open cubes of  $\omega$ , centered at rational points and with rational edge length. Theorem 8.5 in [45] together with a diagonalization argument imply the existence of a subsequence  $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$  such that, for any  $C \in \mathcal{R}(\omega)$  (or  $C = \omega$ ),  $\mathcal{I}(\cdot; C)$  is the  $\Gamma$ -limit of  $\mathcal{I}_{\varepsilon_n}(\cdot; C)$  for the strong  $L^1(C \times I; \mathbb{R}^3)$ -topology. To prove Theorem 4.3.1, it is enough to show that  $\mathcal{I}(u; \omega) = \mathcal{J}(u; \omega)$ .

### 4.3.1 A truncation argument

The main problem with definition (4.3.3) of  $\mathcal{I}$  is that any minimizing sequence is not necessarily bounded in  $BV(\Omega; \mathbb{R}^3)$  and thus, not necessarily weakly convergent in this space. What is missing is either a bound on the jump part of the derivative (this bound was appearing naturally in [34, 28] because the authors were considering a surface energy density with linear growth with respect to the jump of the deformation), or an  $L^\infty$ -bound on the minimizing sequences. In the spirit of [56], we define for all  $(u; A) \in BV(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega)$

$$\mathcal{I}_\infty(u; A) := \inf_{\{u_n\}} \left\{ \liminf_{n \rightarrow +\infty} \mathcal{I}_{\varepsilon_n}(u_n; A) : u_n \rightarrow u \text{ in } L^1(A \times I; \mathbb{R}^3), \sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(A \times I; \mathbb{R}^3)} < +\infty \right\}.$$

Obviously, we have that  $\mathcal{I}(u; A) \leq \mathcal{I}_\infty(u; A)$ . In fact, we will prove that both functionals coincide if the deformation belongs to  $BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$  in Lemma 4.3.3 below. Thus for a deformation  $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ , strong  $L^1(\Omega; \mathbb{R}^3)$ -convergence and weak  $BV(\Omega; \mathbb{R}^3)$ -convergence are, in a sense, equivalent for the computation of the  $\Gamma$ -limit. This will permit us to prove in Lemma 4.3.6 that for such deformations  $\mathcal{I}_\infty(u; \cdot)$  is a Radon measure on  $\omega$ , absolutely continuous with respect to sum of the Lebesgue measure and of the restriction of the Hausdorff measure to  $S(u)$ . By Lebesgue's Decomposition Theorem, it will be enough to identify the Radon-Nikodym derivatives of  $\mathcal{I}_\infty(u; \cdot)$  with respect to  $\mathcal{L}^2$  and  $\mathcal{H}^1_{|S(u)}$ . This will be done for the proof of the upper bound in Lemma 4.3.7 by a blow up argument. The general case will be treated in Lemma 4.3.8 thanks to a truncation argument. The lower bound will be proved in Lemma 4.3.9 using the quasiconvexity properties of  $\mathcal{Q}W_0$  (see Remark 4.3.2) and a lower semicontinuity result in  $SBVP(\Omega; \mathbb{R}^3)$ .

**Lemma 4.3.3.** *For any  $C \in \mathcal{R}(\omega)$  (or  $C = \omega$ ) and all  $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ ,  $\mathcal{I}(u; C) = \mathcal{I}_\infty(u; C)$ .*

*Proof.* It is enough to show that  $\mathcal{I}(u; C) \geq \mathcal{I}_\infty(u; C)$  for all  $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ . If  $\mathcal{I}(u; C) = +\infty$ , the result is obvious, thus there is no loss of generality in assuming that  $\mathcal{I}(u; C) < +\infty$ . By the very definition of the  $\Gamma$ -limit, there exists a sequence  $u_n \rightarrow u$  in  $L^1(C \times I; \mathbb{R}^3)$  such that

$$\mathcal{I}(u; C) = \lim_{n \rightarrow +\infty} \mathcal{I}_{\varepsilon_n}(u_n; C). \quad (4.3.4)$$

Since  $\mathcal{I}(u; C) < +\infty$ , in view of (4.3.1), we deduce that, for  $n$  large enough,  $u_n \in SBVP(C \times I; \mathbb{R}^3)$ . We consider only those  $n$ 's.

Let us define a smooth truncation function  $\varphi_i \in C_c^1(\mathbb{R}^3; \mathbb{R}^3)$  satisfying

$$\varphi_i(z) = \begin{cases} z & \text{if } |z| < e^i, \\ 0 & \text{if } |z| \geq e^{i+1} \end{cases} \quad \text{and} \quad |\nabla \varphi_i(z)| \leq 1. \quad (4.3.5)$$

Let  $w_n^i := \varphi_i(u_n)$ , thanks to the Chain Rule formula, Theorem 3.96 in [5],  $w_n^i \in SBVP(C \times I; \mathbb{R}^3)$

and

$$\begin{cases} \|w_n^i\|_{L^\infty(C \times I; \mathbb{R}^3)} \leq e^i, \\ S(w_n^i) \subset S(u_n), \\ \nabla w_n^i(x) = \nabla \varphi_i(u_n(x)) \circ \nabla u_n(x) \quad \mathcal{L}^3\text{-a.e. on } C \times I. \end{cases} \quad (4.3.6)$$

Since  $u \in L^\infty(\Omega; \mathbb{R}^3)$ , we can choose  $i \geq M_0$  large enough so that  $u = \varphi_i(u)$ , thus according to (4.3.5)

$$\|w_n^i - u\|_{L^1(C \times I; \mathbb{R}^3)} = \|\varphi_i(u_n) - \varphi_i(u)\|_{L^1(C \times I; \mathbb{R}^3)} \leq \|u_n - u\|_{L^1(C \times I; \mathbb{R}^3)}. \quad (4.3.7)$$

The growth condition (4.2.1), (4.3.5) and (4.3.6) imply that

$$\begin{aligned} & \int_{C \times I} W \left( \nabla_\alpha w_n^i \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n^i \right) dx \\ &= \int_{\{u_n < e^i\}} W \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx \\ & \quad + \int_{\{e^i \leq u_n < e^{i+1}\}} W \left( \nabla \varphi_i(u_n) \circ \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla \varphi_i(u_n) \circ \nabla_3 u_n \right) dx \\ & \quad + W(0) \mathcal{L}^3(\{u_n \geq e^{i+1}\}) \\ & \leq \int_{C \times I} W \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx + \beta \mathcal{L}^3(\{u_n \geq e^i\}) \\ & \quad + \beta \int_{\{e^i \leq u_n < e^{i+1}\}} \left| \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx \\ & \leq \int_{C \times I} W \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx + \frac{\beta}{e^i} \|u_n\|_{L^1(C \times I; \mathbb{R}^3)} \\ & \quad + \beta \int_{\{e^i \leq u_n < e^{i+1}\}} \left| \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx, \end{aligned}$$

where we have used Chebyshev's inequality. Let  $M \in \mathbb{N}$ , a summation for  $i = M_0 + 1$  to  $M$  implies using (4.3.6) and the fact that  $\nu_{w_n^i}(x) = \pm \nu_{u_n}(x)$  for  $\mathcal{H}^2$ -a.e.  $x \in S(w_n^i)$ ,

$$\begin{aligned} & \frac{1}{M} \sum_{i=M_0+1}^M \left[ \int_{C \times I} W \left( \nabla_\alpha w_n^i \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n^i \right) dx + \int_{S(w_n^i) \cap [C \times I]} \left| \left( (\nu_{w_n^i})_\alpha \middle| \frac{1}{\varepsilon_n} (\nu_{w_n^i})_3 \right) \right| d\mathcal{H}^2 \right] \\ & \leq \int_{C \times I} W \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx + \int_{S(u_n) \cap [C \times I]} \left| \left( (\nu_{u_n})_\alpha \middle| \frac{1}{\varepsilon_n} (\nu_{u_n})_3 \right) \right| d\mathcal{H}^2 + \frac{c}{M}, \end{aligned}$$

where

$$c = \beta \sup_{n \in \mathbb{N}} \|u_n\|_{L^1(C \times I; \mathbb{R}^3)} \sum_{i \geq 1} \frac{1}{e^i} + \beta \sup_{n \in \mathbb{N}} \left\| \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right\|_{L^p(C \times I; \mathbb{R}^{3 \times 3})}^p < +\infty.$$



We may find some  $i_n \in \{M_0 + 1, \dots, M\}$  such that

$$\begin{aligned} & \int_{C \times I} W \left( \nabla_\alpha w_n^{i_n} \Big| \frac{1}{\varepsilon_n} \nabla_3 w_n^{i_n} \right) dx + \int_{S(w_n^{i_n}) \cap [C \times I]} \left| \left( (\nu_{w_n^{i_n}})_\alpha \Big| \frac{1}{\varepsilon_n} (\nu_{w_n^{i_n}})_3 \right) \right| d\mathcal{H}^2 \\ & \leq \int_{C \times I} W \left( \nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx + \int_{S(u_n) \cap [C \times I]} \left| \left( (\nu_{u_n})_\alpha \Big| \frac{1}{\varepsilon_n} (\nu_{u_n})_3 \right) \right| d\mathcal{H}^2 + \frac{c}{M}. \end{aligned} \quad (4.3.8)$$

Set  $v_n := w_n^{i_n}$ , thus in view of (4.3.7),  $v_n \rightarrow u$  in  $L^1(C \times I; \mathbb{R}^3)$ . Moreover, (4.3.6) implies that

$$\|v_n\|_{L^\infty(C \times I; \mathbb{R}^3)} \leq e^{i_n} \leq e^M.$$

Finally, by virtue of (4.3.4) and (4.3.8),

$$\mathcal{I}(u; C) + \frac{c}{M} \geq \liminf_{n \rightarrow +\infty} \mathcal{I}_{\varepsilon_n}(v_n; C) \geq \mathcal{I}_\infty(u; C).$$

The proof is achieved upon letting  $M$  tend to  $+\infty$  □

To prove the upper bound in Lemma 4.3.7 below, we need a little bit more than the only continuity condition imposed on  $W$ , namely a  $p$ -Lipschitz condition. If  $W$  was quasiconvex, this property would be immediate. Since we do not want to restrict too much the stored energy density, we will show that there is no loss of generality in assuming  $W$  to be quasiconvex. Let  $\mathcal{Q}W$  be the 3D-quasiconvexification of  $W$  defined by

$$\mathcal{Q}W(F) := \inf_{\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^3)} \int_Q W(F + \nabla \varphi(x)) dx \quad \text{for all } F \in \mathbb{R}^{3 \times 3}.$$

**Lemma 4.3.4.** *For all  $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ , the value of  $\mathcal{I}_\infty(u; \omega)$  does not change if we replace  $W$  by  $\mathcal{Q}W$  in (4.3.1).*

*Proof.* We denote  $\mathcal{I}_\varepsilon^\mathcal{Q}$  (resp.  $\mathcal{I}^\mathcal{Q}$ ,  $\mathcal{I}_\infty^\mathcal{Q}$ ), the value of  $\mathcal{I}_\varepsilon$  (resp.  $\mathcal{I}$ ,  $\mathcal{I}_\infty$ ) with  $\mathcal{Q}W$  instead of  $W$  in (4.3.1). By the same arguments as above, we may assume that  $\{\varepsilon_n\}$  is a subsequence of  $\{\varepsilon_j\}$  such that  $\mathcal{I}^\mathcal{Q}(u; \omega) = \mathcal{I}_\infty^\mathcal{Q}(u; \omega)$  is the  $\Gamma$ -limit of  $\mathcal{I}_{\varepsilon_n}^\mathcal{Q}(u; \omega)$ , for every  $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ .

Let  $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ , since  $W \geq \mathcal{Q}W$ , we obviously have  $\mathcal{I}_\infty(u; \omega) \geq \mathcal{I}_\infty^\mathcal{Q}(u; \omega)$ . Let us prove the converse inequality. By the definition of the  $\Gamma$ -limit, we may find a sequence  $\{u_n\} \subset SBV^p(\Omega; \mathbb{R}^3)$  such that  $u_n \rightarrow u$  in  $L^1(\Omega; \mathbb{R}^3)$ ,  $\sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(\Omega; \mathbb{R}^3)} < +\infty$  and

$$\mathcal{I}_\infty^\mathcal{Q}(u; \omega) = \lim_{n \rightarrow +\infty} \left\{ \int_\Omega \mathcal{Q}W \left( \nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx + \int_{S(u_n)} \left| \left( (\nu_{u_n})_\alpha \Big| \frac{1}{\varepsilon_n} (\nu_{u_n})_3 \right) \right| d\mathcal{H}^2 \right\}.$$

We undo the scaling by letting  $v_n(x_\alpha, x_3) := u_n(x_\alpha, x_3/\varepsilon_n)$ . Then  $v_n \in SBV^p(\Omega_{\varepsilon_n}; \mathbb{R}^3)$ ,

$$\frac{1}{\varepsilon_n} \int_{\Omega_{\varepsilon_n}} |v_n - u| dx \rightarrow 0, \quad \sup_{n \in \mathbb{N}} \|v_n\|_{L^\infty(\Omega_{\varepsilon_n}; \mathbb{R}^3)} < +\infty$$

and

$$\mathcal{I}_\infty^{\mathcal{Q}}(u; \omega) = \lim_{n \rightarrow +\infty} \frac{1}{\varepsilon_n} \left\{ \int_{\Omega_{\varepsilon_n}} \mathcal{Q}W(\nabla v_n) dx + \mathcal{H}^2(S(v_n)) \right\}. \quad (4.3.9)$$

For all  $n \in \mathbb{N}$ , Theorem 8.1 in [31] and Proposition 2.8 in [56] yield the existence of a sequence  $\{v_{n,k}\}_{k \in \mathbb{N}} \subset SBVP(\Omega_{\varepsilon_n}; \mathbb{R}^3)$  satisfying  $v_{n,k} \rightarrow v_n$  in  $L^1(\Omega_{\varepsilon_n}; \mathbb{R}^3)$  as  $k \rightarrow +\infty$ ,

$$\int_{\Omega_{\varepsilon_n}} \mathcal{Q}W(\nabla v_n) dx + \mathcal{H}^2(S(v_n)) = \lim_{k \rightarrow +\infty} \left\{ \int_{\Omega_{\varepsilon_n}} W(\nabla v_{n,k}) dx + \mathcal{H}^2(S(v_{n,k})) \right\} \quad (4.3.10)$$

and  $\sup_{k \in \mathbb{N}} \|v_{n,k}\|_{L^\infty(\Omega_{\varepsilon_n}; \mathbb{R}^3)} < +\infty$ . Since the previous bound is of the form  $e^M$  (see [56] p. 417), for some constant  $M > 0$  independent of  $n$ , this last relation holds uniformly with respect to  $n \in \mathbb{N}$ . Gathering (4.3.9) and (4.3.10), we get

$$\mathcal{I}_\infty^{\mathcal{Q}}(u; \omega) = \lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \frac{1}{\varepsilon_n} \left\{ \int_{\Omega_{\varepsilon_n}} W(\nabla v_{n,k}) dx + \mathcal{H}^2(S(v_{n,k})) \right\}.$$

Let  $u_{n,k}(x_\alpha, x_3) := v_{n,k}(x_\alpha, \varepsilon_n x_3)$ , then  $\{u_{n,k}\} \subset SBVP(\Omega; \mathbb{R}^3)$ ,

$$\lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{\Omega} |u_{n,k} - u| dx = 0, \quad \sup_{n,k \in \mathbb{N}} \|u_{n,k}\|_{L^\infty(\Omega; \mathbb{R}^3)} < +\infty$$

and

$$\begin{aligned} \mathcal{I}_\infty^{\mathcal{Q}}(u; \omega) &= \lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \left\{ \int_{\Omega} W \left( \nabla_\alpha u_{n,k} \middle| \frac{1}{\varepsilon_n} \nabla_3 u_{n,k} \right) dx \right. \\ &\quad \left. + \int_{S(u_{n,k})} \left| \left( (\nu_{u_{n,k}})_\alpha \middle| \frac{1}{\varepsilon_n} (\nu_{u_{n,k}})_3 \right) \right| d\mathcal{H}^2 \right\}. \end{aligned}$$

By a diagonalization argument, we can find a sequence  $k_n \nearrow +\infty$  such that upon denoting  $w_n := u_{n,k_n}$ , then  $w_n \in SBVP(\Omega; \mathbb{R}^3)$ ,  $w_n \rightarrow u$  in  $L^1(\Omega; \mathbb{R}^3)$ ,  $\sup_{n \in \mathbb{N}} \|w_n\|_{L^\infty(\Omega; \mathbb{R}^3)} < +\infty$  and

$$\begin{aligned} \mathcal{I}_\infty^{\mathcal{Q}}(u; \omega) &= \lim_{n \rightarrow +\infty} \left\{ \int_{\Omega} W \left( \nabla_\alpha w_n \middle| \frac{1}{\varepsilon_n} \nabla_3 w_n \right) dx + \int_{S(w_n)} \left| \left( (\nu_{w_n})_\alpha \middle| \frac{1}{\varepsilon_n} (\nu_{w_n})_3 \right) \right| d\mathcal{H}^2 \right\} \\ &\geq \mathcal{I}_\infty(u; \omega). \end{aligned}$$

□

**Remark 4.3.5.** From Comments on Theorem 8. (iii) p. 560 in [74], we always have  $\mathcal{Q}W_0 = \mathcal{Q}((\mathcal{Q}W)_0)$ . As a consequence, by Lemma 4.3.4, we may assume without loss of generality, upon replacing  $W$  by  $\mathcal{Q}W$ , that  $W$  is quasiconvex. In particular (see [44]), the following  $p$ -Lipschitz condition holds,

$$|W(F_1) - W(F_2)| \leq \beta(1 + |F_1|^{p-1} + |F_2|^{p-1})|F_1 - F_2|, \quad \text{for all } F_1, F_2 \in \mathbb{R}^{3 \times 3}. \quad (4.3.11)$$

### 4.3.2 Integral Representation of the $\Gamma$ -limit

Lemma 4.3.3 is essential for the proof of the following result because it allows us to replace strong  $L^1(\Omega; \mathbb{R}^3)$ -convergence of any minimizing sequence by strong  $L^p(\Omega; \mathbb{R}^3)$ -convergence, where  $1 < p < \infty$  is the same exponent as in (4.2.1).

**Lemma 4.3.6.** *For all  $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ ,  $\mathcal{I}_\infty(u; \cdot)$  is the restriction to  $\mathcal{A}(\omega)$  of a Radon measure absolutely continuous with respect to  $\mathcal{L}^2 + \mathcal{H}^1_{[S(u)]}$ .*

*Proof.* Let  $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ . The  $p$ -growth condition (4.2.1) implies that

$$\mathcal{I}_\infty(u; A) \leq 2\beta \int_A (1 + |\nabla_\alpha u|^p) dx_\alpha + 2\mathcal{H}^1(S(u) \cap A). \quad (4.3.12)$$

Thus, thanks to e.g. Lemma 7.3 in [35], it is enough to show the existence of a Radon measure  $\hat{\mu}$  on  $\mathbb{R}^2$  such that for every  $A, B$  and  $C \in \mathcal{A}(\omega)$ ,

- (i)  $\mathcal{I}_\infty(u; A) \leq \mathcal{I}_\infty(u; A \setminus \overline{C}) + \mathcal{I}_\infty(u; B)$  if  $\overline{C} \subset B \subset A$ ;
- (ii) for any  $\delta > 0$ , there exists  $C_\delta \in \mathcal{A}(\omega)$  such that  $\overline{C}_\delta \subset A$  and  $\mathcal{I}_\infty(u; A \setminus \overline{C}_\delta) \leq \delta$ ;
- (iii)  $\mathcal{I}_\infty(u; \omega) \geq \hat{\mu}(\mathbb{R}^2)$ ;
- (iv)  $\mathcal{I}_\infty(u; A) \leq \hat{\mu}(\overline{A})$ .

Since  $u \in SBV^p(\omega; \mathbb{R}^3)$ ,  $\mathcal{H}^1(S(u)) < +\infty$ , thus  $\mathcal{H}^1_{[S(u)]}$  is a Radon measure. Then, for any  $\delta > 0$ , there exists  $C_\delta \in \mathcal{A}(\omega)$  such that  $\overline{C}_\delta \subset A$  and

$$2\beta \int_{A \setminus \overline{C}_\delta} (1 + |\nabla_\alpha u|^p) dx_\alpha + 2\mathcal{H}^1(S(u) \cap [A \setminus \overline{C}_\delta]) \leq \delta.$$

Thus, thanks to the growth condition (4.2.1), we have  $\mathcal{I}_\infty(u; A \setminus \overline{C}_\delta) \leq \delta$  and item (ii) holds true. Furthermore, by Lemma 4.3.3 and the definition of the  $\Gamma$ -limit, there exists a sequence  $\{u_n\} \subset SBV^p(\Omega; \mathbb{R}^3)$  such that  $u_n \rightarrow u$  in  $L^1(\Omega; \mathbb{R}^3)$  and  $\mathcal{I}_{\varepsilon_n}(u_n; \omega) \rightarrow \mathcal{I}_\infty(u; \omega)$ . Denoting by

$$\mu_n := W \left( \nabla_\alpha u_n \Big|_{\frac{1}{\varepsilon_n} \nabla_3 u_n} \right) \mathcal{L}^3_{[\Omega]} + \left| \left( (\nu_{u_n})_\alpha \Big|_{\frac{1}{\varepsilon_n} (\nu_{u_n})_3} \right) \right| \mathcal{H}^2_{[S(u_n) \cap \Omega]},$$

for a subsequence of  $\{\varepsilon_n\}$  (not relabeled), there exists a Radon measure  $\mu$  such that  $\mu_n \xrightarrow{*} \mu$  in  $\mathcal{M}_b(\mathbb{R}^3)$ . Let  $\hat{\mu}(B) := \mu(B \times [-1, 1])$ , for every Borel set  $B \subset \mathbb{R}^2$ . Thus,  $\hat{\mu}(\mathbb{R}^2) \leq \mathcal{I}_\infty(u; \omega)$  and item (iii) follows. Moreover, for every  $A \in \mathcal{A}(\omega)$ ,

$$\mathcal{I}_\infty(u; A) \leq \liminf_{n \rightarrow +\infty} \mathcal{I}_{\varepsilon_n}(u_n; A) \leq \limsup_{n \rightarrow +\infty} \mu_n(\overline{A} \times [-1, 1]) \leq \mu(\overline{A} \times [-1, 1]) = \hat{\mu}(\overline{A})$$

which establishes item (iv). We now show the subadditivity condition expressed in item (i). For any  $\eta > 0$ , we can find a sequence  $\{v_n\} \subset SBV^p([A \setminus \overline{C}] \times I; \mathbb{R}^3)$  such that  $v_n \rightarrow u$  in  $L^1([A \setminus \overline{C}] \times I; \mathbb{R}^3)$ ,

$$\sup_{n \in \mathbb{N}} \|v_n\|_{L^\infty([A \setminus \overline{C}] \times I; \mathbb{R}^3)} < +\infty \quad (4.3.13)$$

### 4.3. A $\Gamma$ -CONVERGENCE RESULT

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and

$$\liminf_{n \rightarrow +\infty} \mathcal{I}_{\varepsilon_n}(v_n; A \setminus \overline{C}) \leq \mathcal{I}_\infty(u; A \setminus \overline{C}) + \eta.$$

In particular,  $v_n \rightarrow u$  in  $L^p([A \setminus \overline{C}] \times I; \mathbb{R}^3)$  and we may extract a subsequence  $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$  for which

$$\lim_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_{n_k}}(v_{n_k}; A \setminus \overline{C}) \leq \mathcal{I}_\infty(u; A \setminus \overline{C}) + \eta. \quad (4.3.14)$$

Let  $R_0 \in \mathcal{R}(\omega)$  satisfy  $C \subset\subset R_0 \subset\subset B$ , thus, since  $\mathcal{I}(u; R_0)$  is the  $\Gamma$ -limit of  $\mathcal{I}_{\varepsilon_{n_k}}(u; R_0)$ , thanks to Lemma 4.3.3, there exists a sequence  $\{u_k\} \subset SBV^p(R_0 \times I; \mathbb{R}^3)$  such that  $u_k \rightarrow u$  in  $L^1(R_0 \times I; \mathbb{R}^3)$ ,

$$\sup_{k \in \mathbb{N}} \|u_k\|_{L^\infty(R_0 \times I; \mathbb{R}^3)} < +\infty \quad (4.3.15)$$

and

$$\mathcal{I}_{\varepsilon_{n_k}}(u_k; R_0) \rightarrow \mathcal{I}_\infty(u; R_0). \quad (4.3.16)$$

In particular, we have  $u_k \rightarrow u$  in  $L^p(R_0 \times I; \mathbb{R}^3)$ . According to the  $p$ -coercivity condition (4.2.1) the following sequence of Radon measures

$$\begin{aligned} \lambda_k := & \left(1 + \left| \left( \nabla_\alpha v_{n_k} \Big|_{\frac{1}{\varepsilon_{n_k}} \nabla_3 v_{n_k}} \right) \right|^p \right) \mathcal{L}^3_{\lfloor (R_0 \setminus \overline{C}) \times I} \\ & + \left| \left( (\nu_{v_{n_k}})_\alpha \Big|_{\frac{1}{\varepsilon_{n_k}} (\nu_{v_{n_k}})_3} \right) \right| \mathcal{H}^2_{[S(v_{n_k}) \cap ((R_0 \setminus \overline{C}) \times I)} \\ & + \left(1 + \left| \left( \nabla_\alpha u_k \Big|_{\frac{1}{\varepsilon_{n_k}} \nabla_3 u_k} \right) \right|^p \right) \mathcal{L}^3_{\lfloor (R_0 \setminus \overline{C}) \times I} \\ & + \left| \left( (\nu_{u_k})_\alpha \Big|_{\frac{1}{\varepsilon_{n_k}} (\nu_{u_k})_3} \right) \right| \mathcal{H}^2_{[S(u_k) \cap ((R_0 \setminus \overline{C}) \times I)} \end{aligned}$$

is uniformly bounded, and thus, for a subsequence that will not be relabeled, there exists a positive Radon measure  $\lambda$  such that  $\lambda_k \xrightarrow{*} \lambda$  in  $\mathcal{M}_b(R_0 \setminus \overline{C})$ .

Let  $t > 0$ , define  $R_t := \{x_\alpha \in R_0 : \text{dist}(x_\alpha, \partial R_0) > t\}$  and for any  $0 < \delta < \eta$ ,  $L_\delta := R_{\eta-\delta} \setminus \overline{R_{\eta+\delta}}$ . Since we are localizing in  $\mathbb{R}^2$ , we consider a cut-off function  $\varphi_\delta \in \mathcal{C}_c^\infty(R_{\eta-\delta}; [0, 1])$  depending only on  $x_\alpha$  and satisfying  $\varphi_\delta = 1$  on  $R_\eta$  and  $\|\varphi_\delta\|_{L^\infty(R_{\eta-\delta})} \leq C/\delta$ . Define

$$w_k(x) := u_k(x)\varphi_\delta(x_\alpha) + v_{n_k}(x)(1 - \varphi_\delta(x_\alpha)).$$

Then,  $w_k \in SBV^p(A \times I; \mathbb{R}^3)$ ,  $w_k \rightarrow u$  in  $L^p(A \times I; \mathbb{R}^3)$  and in view of (4.3.13) and (4.3.15)

$$\sup_{k \in \mathbb{N}} \|w_k\|_{L^\infty(A \times I; \mathbb{R}^3)} \leq \sup_{k \in \mathbb{N}} \|u_k\|_{L^\infty(R_0 \times I; \mathbb{R}^3)} + \sup_{k \in \mathbb{N}} \|v_{n_k}\|_{L^\infty([A \setminus \overline{C}] \times I; \mathbb{R}^3)} < +\infty.$$

From (4.3.3), (4.3.14) and (4.3.16), we deduce that

$$\begin{aligned}
 \mathcal{I}_\infty(u; A) &\leq \liminf_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_{n_k}}(w_k; A) \\
 &\leq \liminf_{k \rightarrow +\infty} \left\{ \mathcal{I}_{\varepsilon_{n_k}}(u_k; R_{\eta+\delta}) + \mathcal{I}_{\varepsilon_{n_k}}(v_{n_k}; A \setminus \overline{R}_{\eta-\delta}) \right. \\
 &\quad \left. + C\lambda_k(L_\delta) + \frac{C}{\delta^p} \int_{L_\delta \times I} |u_k - v_{n_k}|^p dx. \right\} \\
 &\leq \mathcal{I}_\infty(u; R_0) + \mathcal{I}_\infty(u; A \setminus \overline{C}) + \eta + C\lambda(\overline{L}_\delta) \\
 &\leq \mathcal{I}_\infty(u; B) + \mathcal{I}_\infty(u; A \setminus \overline{C}) + \eta + C\lambda(\overline{L}_\delta),
 \end{aligned}$$

where we have used the fact that  $\mathcal{I}_\infty(u; \cdot)$  is an increasing set function. Note that the previous computation would not hold if we had considered  $\mathcal{I}$  instead of  $\mathcal{I}_\infty$  because the minimizing sequences would only converge in  $L^1$ . Letting  $\delta$  tend to zero, we obtain,

$$\mathcal{I}_\infty(u; A) \leq \mathcal{I}_\infty(u; B) + \mathcal{I}_\infty(u; A \setminus \overline{C}) + \eta + C\lambda(\partial R_\eta).$$

Now choose a sequence  $\eta_h \rightarrow 0$  such that  $\lambda(\partial R_{\eta_h}) = 0$ . Letting  $h \nearrow +\infty$  yields

$$\mathcal{I}_\infty(u; A) \leq \mathcal{I}_\infty(u; B) + \mathcal{I}_\infty(u; A \setminus \overline{C})$$

which completes the proof of item (i). Thus, according to (4.3.12) and Lemma 7.3 in [35],  $\mathcal{I}_\infty(u; \cdot)$  is the restriction to  $\mathcal{A}(\omega)$  of the Radon measure  $\hat{\mu}$  which is absolutely continuous with respect to  $\mathcal{L}^2 + \mathcal{H}^1_{[S(u)]}$ .  $\square$

As a consequence of Lemma 4.3.6 and Lebesgue's Decomposition Theorem, there exists a  $\mathcal{L}^2$ -measurable function  $h$  and a  $\mathcal{H}^1_{[S(u)]}$ -measurable function  $g$  such that for every  $A \in \mathcal{A}(\omega)$ ,

$$\mathcal{I}_\infty(u; A) = \int_A h d\mathcal{L}^2 + \int_{A \cap S(u)} g d\mathcal{H}^1. \quad (4.3.17)$$

We denote by  $Q'_\rho(x_0)$  the open cube of  $\mathbb{R}^2$  centered at  $x_0 \in \omega$  and of side length  $\rho > 0$ , where  $\rho$  is small enough so that  $Q'_\rho(x_0) \in \mathcal{A}(\omega)$ . Since the measures  $\mathcal{L}^2$  and  $\mathcal{H}^1_{[S(u)]}$  are mutually singular,  $h$  is the Radon-Nikodym derivative of  $\mathcal{I}_\infty(u; \cdot)$  with respect to  $\mathcal{L}^2$ ,

$$h(x_0) = \lim_{\rho \rightarrow 0} \frac{\mathcal{I}_\infty(u; Q'_\rho(x_0))}{\rho^2}, \quad \text{for } \mathcal{L}^2\text{-a.e. } x_0 \in \omega$$

and  $g$  is the Radon-Nikodym derivative of  $\mathcal{I}_\infty(u; \cdot)$  with respect to  $\mathcal{H}^1_{[S(u)]}$ ,

$$g(x_0) = \lim_{\rho \rightarrow 0} \frac{\mathcal{I}_\infty(u; Q'_\rho(x_0))}{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))}, \quad \text{for } \mathcal{H}^1\text{-a.e. } x_0 \in S(u).$$

Now we would like to identify both densities  $g$  and  $h$ . Note that we cannot use classical Integral Representation Theorems in *SBV* (see e.g. Theorem 2.4 in [31] or Theorem 1 in [28]) because the term of surface energy does not grow linearly in the deformation jump.

### 4.3.3 Characterization of the $\Gamma$ -limit

#### The upper bound

We will proceed in two steps to prove the upper bound. Firstly, we will show that the inequality holds for deformations belonging to  $L^\infty(\Omega; \mathbb{R}^3)$  (see Lemma 4.3.7 below). Indeed, for those, we will use the integral representation proved above. In fact, we will show, with the help of a blow up argument, that the inequality holds separately for the surface and the bulk terms. Then, we will prove the inequality in its full generality in Lemma 4.3.8, using a truncation argument as in the proof of Lemma 4.3.3.

**Lemma 4.3.7.** *For all  $u \in BV(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ ,  $\mathcal{I}_\infty(u; \omega) \leq \mathcal{J}(u; \omega)$ .*

*Proof.* It is enough to consider the case where  $\mathcal{J}(u; \omega) < +\infty$  and thus  $u \in SBV^p(\omega; \mathbb{R}^3)$ . Let  $u \in L^\infty(\omega; \mathbb{R}^3) \cap SBV^p(\omega; \mathbb{R}^3)$ , according to (4.3.17) and (4.3.2), we must show that  $h(x_0) \leq 2QW_0(\nabla_\alpha u(x_0))$  for  $\mathcal{L}^2$ -a.e.  $x_0 \in \omega$  and  $g(x_0) \leq 2$  for  $\mathcal{H}^1$ -a.e.  $x_0 \in S(u)$ .

Let us first treat the surface term. We have for  $\mathcal{H}^1$ -a.e.  $x_0 \in S(u)$ ,

$$\begin{aligned} g(x_0) &= \lim_{\rho \rightarrow 0} \frac{\mathcal{I}_\infty(u; Q'_\rho(x_0))}{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))} \\ &\leq \limsup_{\rho \rightarrow 0} \frac{1}{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))} \left[ 2 \int_{Q'_\rho(x_0)} W(\nabla_\alpha u | 0) dx_\alpha + 2\mathcal{H}^1(S(u) \cap Q'_\rho(x_0)) \right] \\ &= \limsup_{\rho \rightarrow 0} \frac{\mu(Q'_\rho(x_0))}{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))} + 2, \end{aligned}$$

where we set  $\mu := 2W(\nabla_\alpha u | 0)\mathcal{L}^2$ . But since  $\mu$  and  $\mathcal{H}^1_{[S(u)}$  are mutually singular, we have for  $\mathcal{H}^1$ -a.e.  $x_0 \in S(u)$

$$\lim_{\rho \rightarrow 0} \frac{\mu(Q'_\rho(x_0))}{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))} = 0.$$

This shows that  $g(x_0) \leq 2$  for  $\mathcal{H}^1$ -a.e.  $x_0 \in S(u)$ .

Concerning the bulk term, choose  $x_0 \in \omega$  such that

$$\lim_{\rho \rightarrow 0} \int_{Q'_\rho(x_0)} |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)|^p dx_\alpha = 0. \quad (4.3.18)$$

and

$$\lim_{\rho \rightarrow 0} \frac{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))}{\rho^2} = 0. \quad (4.3.19)$$

Since  $\nabla_\alpha u \in L^p(\omega; \mathbb{R}^{3 \times 2})$  and the measures  $\mathcal{L}^2$  and  $\mathcal{H}^1_{[S(u)}$  are mutually singular, it follows that  $\mathcal{L}^2$ -a.e.  $x_0 \in \omega$  satisfies (4.3.18) and (4.3.19). For every  $\rho > 0$ , Theorem 2 in [74] implies

the existence of a sequence  $\{v_n^\rho\} \subset W^{1,p}(Q'_\rho(x_0) \times I; \mathbb{R}^3)$  such that  $v_n^\rho \rightarrow \nabla_\alpha u(x_0) \cdot x_\alpha$  in  $L^p(Q'_\rho(x_0) \times I; \mathbb{R}^3)$  (thus a fortiori in  $L^1(Q'_\rho(x_0) \times I; \mathbb{R}^3)$ ) and

$$\int_{Q'_\rho(x_0) \times I} W \left( \nabla_\alpha v_n^\rho \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n^\rho \right) dx \rightarrow 2\rho^2 \mathcal{Q}W_0(\nabla_\alpha u(x_0)). \quad (4.3.20)$$

Moreover, by the construction of this recovery sequence (see [74]), there is no loss of generality in assuming that  $\{v_n^\rho\}$  further satisfies

$$\sup_{\rho>0, n \in \mathbb{N}} \|v_n^\rho\|_{L^\infty(Q'_\rho(x_0) \times I; \mathbb{R}^3)} < +\infty.$$

From the coercivity condition (4.2.1), we get

$$M := \sup_{\rho>0, n \in \mathbb{N}} \int_{Q'_\rho(x_0) \times I} \left| \left( \nabla_\alpha v_n^\rho \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n^\rho \right) \right|^p dx < +\infty. \quad (4.3.21)$$

Define  $u_n^\rho(x) := u(x_\alpha) + v_n^\rho(x_\alpha, x_3) - \nabla_\alpha u(x_0) \cdot x_\alpha$ . Then,

$$u_n^\rho \rightarrow u \text{ in } L^1(Q'_\rho(x_0) \times I; \mathbb{R}^3) \text{ as } n \rightarrow +\infty, \quad \sup_{\rho>0, n \in \mathbb{N}} \|u_n^\rho\|_{L^\infty(Q'_\rho(x_0) \times I; \mathbb{R}^3)} < +\infty$$

and  $S(u_n^\rho) \cap [Q'_\rho(x_0) \times I] = [S(u) \cap Q'_\rho(x_0)] \times I$ . Thus,

$$\begin{aligned} & \frac{\mathcal{I}_\infty(u; Q'_\rho(x_0))}{\rho^2} \\ & \leq \liminf_{n \rightarrow +\infty} \frac{1}{\rho^2} \left\{ \int_{Q'_\rho(x_0) \times I} W \left( \nabla_\alpha u_n^\rho \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n^\rho \right) dx \right. \\ & \quad \left. + \int_{S(u_n^\rho) \cap [Q'_\rho(x_0) \times I]} \left| \left( (\nu_{u_n^\rho})_\alpha \middle| \frac{1}{\varepsilon_n} (\nu_{u_n^\rho})_3 \right) \right| d\mathcal{H}^2 \right\} \\ & \leq \liminf_{n \rightarrow +\infty} \frac{1}{\rho^2} \int_{Q'_\rho(x_0) \times I} W \left( \nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0) + \nabla_\alpha v_n^\rho(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n^\rho(x) \right) dx \\ & \quad + 2 \frac{\mathcal{H}^1(S(u) \cap Q'_\rho(x_0))}{\rho^2}. \end{aligned}$$

Thus from (4.3.19), we obtain

$$h(x_0) \leq \liminf_{\rho \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{\rho^2} \int_{Q'_\rho(x_0) \times I} W \left( \nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0) + \nabla_\alpha v_n^\rho(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n^\rho(x) \right) dx.$$

Relations (4.3.11), (4.3.20), (4.3.21) and Hölder's inequality yield

$$\begin{aligned}
 h(x_0) &\leq \liminf_{\rho \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{\rho^2} \left\{ \int_{Q'_\rho(x_0) \times I} W \left( \nabla_\alpha v_n^\rho \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n^\rho \right) dx \right. \\
 &\quad \left. + C \int_{Q'_\rho(x_0) \times I} \left( 1 + |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)|^{p-1} \right. \right. \\
 &\quad \left. \left. + \left| \left( \nabla_\alpha v_n^\rho(x) \middle| \frac{1}{\varepsilon_n} \nabla_3 v_n^\rho(x) \right) \right|^{p-1} \right) |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)| dx \right\} \\
 &\leq 2\mathcal{Q}W_0(\nabla_\alpha u(x_0)) + C \limsup_{\rho \rightarrow 0} \left( \int_{Q'_\rho(x_0)} |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)|^p dx_\alpha \right) \\
 &\quad + C(1 + M^{(p-1)/p}) \limsup_{\rho \rightarrow 0} \left( \int_{Q'_\rho(x_0)} |\nabla_\alpha u(x_\alpha) - \nabla_\alpha u(x_0)|^p dx_\alpha \right)^{1/p}.
 \end{aligned}$$

Thanks to (4.3.18), we conclude that  $h(x_0) \leq 2\mathcal{Q}W_0(\nabla_\alpha u(x_0))$  for  $\mathcal{L}^2$ -a.e.  $x_0 \in \omega$ .  $\square$

Let us now turn back to the general case.

**Lemma 4.3.8.** *For all  $u \in BV(\Omega; \mathbb{R}^3)$ ,  $\mathcal{I}(u; \omega) \leq \mathcal{J}(u; \omega)$ .*

*Proof.* As in the proof of Lemma 4.3.7, we can assume without loss of generality that  $\mathcal{J}(u; \omega) < +\infty$  and thus that  $u \in SBV^p(\omega; \mathbb{R}^3)$ . In particular, it implies that  $\mathcal{I}(u; \omega) < +\infty$ . Let  $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^3; \mathbb{R}^3)$  be the truncation function introduced in Lemma 4.3.3 and defined by (4.3.5). The Chain Rule formula, Theorem 3.96 in [5], implies that  $\varphi_i(u) \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$  and

$$\begin{cases} \|\varphi_i(u)\|_{L^\infty(\omega; \mathbb{R}^3)} \leq e^i, \\ S(\varphi_i(u)) \subset S(u), \\ \nabla_\alpha(\varphi_i(u(x_\alpha))) = \nabla \varphi_i(u(x_\alpha)) \circ \nabla_\alpha u(x_\alpha) \quad \mathcal{L}^2\text{-a.e. in } \omega \end{cases} \quad (4.3.22)$$

and  $\varphi_i(u) \rightarrow u$  in  $L^1(\omega; \mathbb{R}^3)$  as  $i \rightarrow +\infty$ . Since  $\varphi_i(u) \in L^\infty(\omega; \mathbb{R}^3) \cap SBV^p(\omega; \mathbb{R}^3)$ , it follows from Lemmas 4.3.3 and 4.3.7 that

$$\mathcal{I}(\varphi_i(u); \omega) = \mathcal{I}_\infty(\varphi_i(u); \omega) \leq \mathcal{J}(\varphi_i(u); \omega)$$

and by lower semicontinuity of  $\mathcal{I}(\cdot; \omega)$  with respect to the strong  $L^1(\omega; \mathbb{R}^3)$ -convergence, we have

$$\mathcal{I}(u; \omega) \leq \liminf_{i \rightarrow +\infty} \mathcal{I}(\varphi_i(u); \omega) \leq \limsup_{i \rightarrow +\infty} \mathcal{J}(\varphi_i(u); \omega). \quad (4.3.23)$$

But, in view of (4.3.22),  $\mathcal{H}^1(S(\varphi_i(u))) \leq \mathcal{H}^1(S(u))$  and, thanks to (4.3.5),

$$\int_\omega \mathcal{Q}W_0(\nabla_\alpha(\varphi_i(u))) dx_\alpha \leq \int_{\{|u| < e^i\}} \mathcal{Q}W_0(\nabla_\alpha u) dx_\alpha + \beta \int_{\{|u| \geq e^i\}} (1 + |\nabla_\alpha u|^p) dx_\alpha.$$



Thus,

$$\begin{aligned}
 & \limsup_{i \rightarrow +\infty} \mathcal{J}(\varphi_i(u); \omega) \\
 & \leq \limsup_{i \rightarrow +\infty} \left\{ 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} + 2\beta \int_{\{|u| \geq e^i\}} (1 + |\nabla_{\alpha} u|^p) dx_{\alpha} + 2\mathcal{H}^1(S(u)) \right\} \\
 & \leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha} + 2\mathcal{H}^1(S(u)) \\
 & = \mathcal{J}(u; \omega),
 \end{aligned} \tag{4.3.24}$$

where we have used the fact that, by Chebyshev's inequality,  $\mathcal{L}^2(\{|u| \geq e^i\}) \leq \|u\|_{L^1(\omega; \mathbb{R}^3)}/e^i \rightarrow 0$  as  $i \rightarrow +\infty$ . Gathering (4.3.23) and (4.3.24), we deduce that  $\mathcal{I}(u; \omega) \leq \mathcal{J}(u; \omega)$  and this completes the proof of the Lemma.  $\square$

### The lower bound

Let us now prove the lower bound. The proof is essentially based on a lower semicontinuity result in  $SBV^p(\Omega; \mathbb{R}^3)$ . The main difficulty remains to show that any deformation  $u \in BV(\Omega; \mathbb{R}^3)$  satisfying  $\mathcal{I}(u; \omega) < +\infty$  belongs in fact to  $SBV^p(\omega; \mathbb{R}^3)$ .

**Lemma 4.3.9.** *For all  $u \in BV(\Omega; \mathbb{R}^3)$ ,  $\mathcal{I}(u; \omega) \geq \mathcal{J}(u; \omega)$ .*

*Proof.* It is not restrictive to assume that  $\mathcal{I}(u; \omega) < +\infty$ . By  $\Gamma$ -convergence, there exists a sequence  $\{u_n\} \subset SBV^p(\Omega; \mathbb{R}^3)$  such that  $u_n \rightarrow u$  in  $L^1(\Omega; \mathbb{R}^3)$  and

$$\lim_{n \rightarrow +\infty} \left[ \int_{\Omega} W \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) dx + \int_{S(u_n)} \left| \left( (\nu_{u_n})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{u_n})_3 \right) \right| d\mathcal{H}^2 \right] = \mathcal{I}(u; \omega). \tag{4.3.25}$$

Let us show that  $u \in SBV(\omega; \mathbb{R}^3)$ . We will use the truncation function  $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^3; \mathbb{R}^3)$  defined in (4.3.5). The Chain Rule formula, Theorem 3.96 in [5], implies that  $\varphi_i(u_n) \in SBV^p(\Omega; \mathbb{R}^3)$  and

$$\begin{cases} \|\varphi_i(u_n)\|_{L^{\infty}(\Omega; \mathbb{R}^3)} \leq e^i, \\ S(\varphi_i(u_n)) \subset S(u_n), \\ \nabla(\varphi_i(u_n)(x)) = \nabla \varphi_i(u_n(x)) \circ \nabla u_n(x) \quad \mathcal{L}^3\text{-a.e. in } \Omega. \end{cases}$$

As  $\nu_{u_n}(x) = \pm \nu_{\varphi_i(u_n)}(x)$  for  $\mathcal{H}^2$ -a.e.  $x \in S(\varphi_i(u_n))$ , we get

$$\begin{aligned}
 & \sup_{n \in \mathbb{N}} \left[ \int_{\Omega} \left| \left( \nabla_{\alpha}(\varphi_i(u_n)) \middle| \frac{1}{\varepsilon_n} \nabla_3(\varphi_i(u_n)) \right) \right|^p dx + \int_{S(\varphi_i(u_n))} \left| \left( (\nu_{\varphi_i(u_n)})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{\varphi_i(u_n)})_3 \right) \right| d\mathcal{H}^2 \right] \\
 & \leq \sup_{n \in \mathbb{N}} \left[ \int_{\Omega} \left| \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx + \int_{S(u_n)} \left| \left( (\nu_{u_n})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{u_n})_3 \right) \right| d\mathcal{H}^2 \right] < +\infty,
 \end{aligned}$$

where we used (4.3.25) together with the coercivity condition (4.2.1). Lemma 4.2.2 and a diagonalization argument yield the existence of a subsequence (still denoted by  $\{\varepsilon_n\}$ ), and a function  $v_i \in SBV^p(\omega; \mathbb{R}^3)$  such that  $\varphi_i(u_n) \rightarrow v_i$  in  $SBV^p(\Omega; \mathbb{R}^3)$  as  $n \rightarrow +\infty$ . But, since  $u_n \rightarrow u$  and  $\varphi_i(u_n) \rightarrow \varphi_i(u)$  in  $L^1(\Omega; \mathbb{R}^3)$  as  $n \rightarrow +\infty$ , we deduce that  $v_i = \varphi_i(u) \in SBV^p(\omega; \mathbb{R}^3)$  for every  $i \in \mathbb{N}$ . By virtue of Theorem 3.96 in [5],

$$0 = D^c v_i = \nabla \varphi_i(\tilde{u}) \circ D^c u \quad \text{in } \omega \setminus S(u),$$

where  $\tilde{u}$  denotes the approximate limit of  $u$  at  $x_\alpha \in \omega \setminus S(u)$ . Define  $E_i := \{x_\alpha \in \omega \setminus S(u) : |\tilde{u}(x_\alpha)| < e^i\}$ , since  $\tilde{u}$  is a Borel function and  $S(u)$  is a Borel set (see Proposition 3.64 (a) in [5]),  $E_i$  is a Borel set. Moreover, as  $\{E_i\}$  is an increasing sequence of sets whose union is  $\omega \setminus S(u)$ , we get

$$\begin{aligned} |D^c u|(\omega) &= |D^c u|(\omega \setminus S(u)) = \lim_{i \rightarrow +\infty} |D^c u|(E_i) = \lim_{i \rightarrow +\infty} |\nabla \varphi_i(\tilde{u}) \circ D^c u|(E_i) \\ &= \lim_{i \rightarrow +\infty} |D^c v_i|(E_i) = 0, \end{aligned}$$

where we have used the fact that  $\nabla \varphi_i(\tilde{u}(x_\alpha)) = \text{Id}$  for all  $x_\alpha \in E_i$ . Thus  $u \in SBV(\omega; \mathbb{R}^3)$  and by (4.3.25), Remark 4.3.2, Theorem 5.29 in [5] and Theorem 3.7 in [4]

$$\begin{aligned} \mathcal{I}(u; \omega) &\geq \liminf_{n \rightarrow +\infty} \left[ \int_{\Omega} \mathcal{Q}W_0(\nabla_\alpha u_n) dx + \mathcal{H}^2(S(u_n)) \right] \\ &\geq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u) dx_\alpha + 2\mathcal{H}^1(S(u)). \end{aligned}$$

In particular, the  $p$ -coercivity of  $\mathcal{Q}W_0$  implies that  $u \in SBV^p(\omega; \mathbb{R}^3)$  and thus, according to (4.3.2), that  $\mathcal{I}(u; \omega) \geq \mathcal{J}(u, \omega)$ .  $\square$

*Proof of Theorem 4.3.1.* We have shown that for any sequence  $\{\varepsilon_j\} \searrow 0^+$ , there exists a further subsequence  $\{\varepsilon_{j_n}\} \equiv \{\varepsilon_n\}$  such that  $\mathcal{I}_{\varepsilon_n}(\cdot; \omega)$   $\Gamma$ -converges to  $\mathcal{I}(\cdot; \omega)$  for the strong  $L^1(\Omega; \mathbb{R}^3)$ -topology. By virtue of Lemmas 4.3.8 and 4.3.9, we have  $\mathcal{I}(\cdot; \omega) = \mathcal{J}(\cdot; \omega)$ . Since the  $\Gamma$ -limit does not depend upon the extracted subsequence, we deduce, in the light of Proposition 8.3 in [45], that the whole sequence  $\mathcal{I}_\varepsilon(\cdot; \omega)$   $\Gamma$ -converges to  $\mathcal{J}(\cdot; \omega)$ .  $\square$

#### 4.3.4 Boundary conditions

Let us now deal with boundary condition constraints that will be of use in Lemmas 4.5.2 and 4.5.5 in order to prove the minimality property of the limit quasistatic evolution. Indeed, it will allow to extend functions on the enlarged cylinder  $\Omega'$  by the value of the boundary condition. The following result, very close in spirit to Lemma 2.6 in [35], relies on De Giorgi's slicing argument together with the fact that we can consider cut-off functions depending only on  $x_\alpha$  (see also the proof of Lemma 4.3.6). It is established that any recovery sequence can be chosen so as to match the lateral boundary condition of its target.

We recall that  $\omega'$  is a bounded open subset of  $\mathbb{R}^2$  containing  $\omega$  and that  $\Omega' = \omega' \times I$ . In all that follows, if  $v \in SBV^p(\Omega'; \mathbb{R}^3)$ , we will denote by  $v^-$  (resp.  $v^+$ ) the inner (resp. outer) trace of  $v$  on  $\partial\omega \times I$ .

**Lemma 4.3.10.** *For every  $u \in SBV^p(\omega; \mathbb{R}^3) \cap L^\infty(\omega; \mathbb{R}^3)$ , there exists a sequence  $\{\bar{u}_\varepsilon\} \subset SBV^p(\Omega; \mathbb{R}^3)$  such that  $\bar{u}_\varepsilon \rightarrow u$  in  $L^p(\Omega; \mathbb{R}^3)$ ,  $\bar{u}_\varepsilon = u$  in a neighborhood of  $\partial\omega \times I$  and*

$$\mathcal{J}(u) = \lim_{\varepsilon \rightarrow 0} \left[ \int_{\Omega} W \left( \nabla_{\alpha} \bar{u}_{\varepsilon} \Big|_{\frac{1}{\varepsilon}} \nabla_3 \bar{u}_{\varepsilon} \right) dx + \int_{S(\bar{u}_{\varepsilon})} \left| \left( (\nu_{\bar{u}_{\varepsilon}})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{\bar{u}_{\varepsilon}})_3 \right) \right| d\mathcal{H}^2 \right].$$

*Proof.* According to Theorem 4.3.1 and Lemma 4.3.3, there exists  $\{u_\varepsilon\} \subset SBV^p(\Omega; \mathbb{R}^3)$  strongly converging to  $u$  in  $L^1(\Omega; \mathbb{R}^3)$ , satisfying  $\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^3)} < +\infty$  and

$$\mathcal{J}(u) = \lim_{\varepsilon \rightarrow 0} \left[ \int_{\Omega} W \left( \nabla_{\alpha} u_{\varepsilon} \Big|_{\frac{1}{\varepsilon}} \nabla_3 u_{\varepsilon} \right) dx + \int_{S(u_{\varepsilon})} \left| \left( (\nu_{u_{\varepsilon}})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{u_{\varepsilon}})_3 \right) \right| d\mathcal{H}^2 \right].$$

In particular,  $u_\varepsilon \rightarrow u$  in  $L^p(\Omega; \mathbb{R}^3)$  and from the  $p$ -coercivity condition (4.2.1), it follows that

$$C := \sup_{\varepsilon > 0} \left[ \int_{\Omega} \left( 1 + \left| \left( \nabla_{\alpha} u_{\varepsilon} \Big|_{\frac{1}{\varepsilon}} \nabla_3 u_{\varepsilon} \right) \right|^p \right) dx + \int_{S(u_{\varepsilon})} \left| \left( (\nu_{u_{\varepsilon}})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{u_{\varepsilon}})_3 \right) \right| d\mathcal{H}^2 \right] < +\infty. \quad (4.3.26)$$

Set

$$K_{\varepsilon} := \left\lfloor \frac{1}{\|u_{\varepsilon} - u\|_{L^p(\Omega; \mathbb{R}^3)}^{1/2}} \right\rfloor, \quad M_{\varepsilon} := \left\lfloor \sqrt{K_{\varepsilon}} \right\rfloor$$

and denote

$$\omega(\varepsilon) := \left\{ x_{\alpha} \in \omega : \text{dist}(x_{\alpha}, \partial\omega) < \frac{M_{\varepsilon}}{K_{\varepsilon}} \right\} \quad \text{and} \quad \omega_i^{\varepsilon} := \left\{ x_{\alpha} \in \omega : \text{dist}(x_{\alpha}, \partial\omega) \in \left[ \frac{i}{K_{\varepsilon}}, \frac{i+1}{K_{\varepsilon}} \right) \right\},$$

for all  $i \in \{0, \dots, M_{\varepsilon} - 1\}$ . From (4.3.26), we get the existence of a  $i(\varepsilon) \in \{0, \dots, M_{\varepsilon} - 1\}$  such that

$$\int_{\omega_{i(\varepsilon)}^{\varepsilon} \times I} \left( 1 + \left| \left( \nabla_{\alpha} u_{\varepsilon} \Big|_{\frac{1}{\varepsilon}} \nabla_3 u_{\varepsilon} \right) \right|^p \right) dx + \int_{S(u_{\varepsilon}) \cap [\omega_{i(\varepsilon)}^{\varepsilon} \times I]} \left| \left( (\nu_{u_{\varepsilon}})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{u_{\varepsilon}})_3 \right) \right| d\mathcal{H}^2 \leq \frac{C}{M_{\varepsilon}}. \quad (4.3.27)$$

Let us now consider a cut-off function  $\phi_{\varepsilon} \in \mathcal{C}_c^{\infty}(\omega; [0, 1])$  independent of  $x_3$  and satisfying

$$\phi_{\varepsilon}(x_{\alpha}) = \begin{cases} 1 & \text{if } \text{dist}(x_{\alpha}, \partial\omega) > \frac{i(\varepsilon) + 1}{K_{\varepsilon}}, \\ 0 & \text{if } \text{dist}(x_{\alpha}, \partial\omega) \leq \frac{i(\varepsilon)}{K_{\varepsilon}} \end{cases} \quad \text{and} \quad \|\nabla_{\alpha} \phi_{\varepsilon}\|_{L^{\infty}(\omega)} \leq 2K_{\varepsilon}.$$

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Define  $\bar{u}_\varepsilon(x) := \phi_\varepsilon(x_\alpha)u_\varepsilon(x) + (1 - \phi_\varepsilon(x_\alpha))u(x_\alpha)$ ; then  $\bar{u}_\varepsilon \in SBV^p(\Omega; \mathbb{R}^3)$ ,  $\bar{u}_\varepsilon \rightarrow u$  in  $L^p(\Omega; \mathbb{R}^3)$ ,  $\bar{u}_\varepsilon = u$  in a neighborhood of  $\partial\omega \times I$  and  $S(\bar{u}_\varepsilon) \subset S(u_\varepsilon) \cup (S(u) \times I)$ . The  $p$ -growth condition (4.2.1) implies that

$$\begin{aligned}
\mathcal{J}(u) &\geq \limsup_{\varepsilon \rightarrow 0} \left[ \int_{\{x_\alpha \in \omega: \text{dist}(x_\alpha, \partial\omega) > \frac{i(\varepsilon)+1}{K_\varepsilon}\} \times I} W \left( \nabla_\alpha \bar{u}_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 \bar{u}_\varepsilon \right) dx \right. \\
&\quad \left. + \int_{S(\bar{u}_\varepsilon) \cap [\{x_\alpha \in \omega: \text{dist}(x_\alpha, \partial\omega) > \frac{i(\varepsilon)+1}{K_\varepsilon}\} \times I]} \left| \left( (\nu_{\bar{u}_\varepsilon})_\alpha \Big| \frac{1}{\varepsilon} (\nu_{\bar{u}_\varepsilon})_3 \right) \right| d\mathcal{H}^2 \right] \\
&\geq \limsup_{\varepsilon \rightarrow 0} \left[ \int_\Omega W \left( \nabla_\alpha \bar{u}_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 \bar{u}_\varepsilon \right) dx + \int_{S(\bar{u}_\varepsilon)} \left| \left( (\nu_{\bar{u}_\varepsilon})_\alpha \Big| \frac{1}{\varepsilon} (\nu_{\bar{u}_\varepsilon})_3 \right) \right| d\mathcal{H}^2 \right. \\
&\quad \left. - 2\beta \int_{\{x_\alpha \in \omega: \text{dist}(x_\alpha, \partial\omega) \leq \frac{i(\varepsilon)}{K_\varepsilon}\}} (1 + |\nabla_\alpha u|^p) dx_\alpha \right. \\
&\quad \left. - 2\mathcal{H}^1 \left( S(u) \cap \left\{ x_\alpha \in \omega : \text{dist}(x_\alpha, \partial\omega) \leq \frac{i(\varepsilon)}{K_\varepsilon} \right\} \right) \right. \\
&\quad \left. - c_1 K_\varepsilon^p \int_{\omega_{i(\varepsilon)}^\varepsilon \times I} |u_\varepsilon - u|^p dx - c_2 \int_{\omega_{i(\varepsilon)}^\varepsilon \times I} \left( 1 + |\nabla_\alpha u|^p + \left| \left( \nabla_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right) \right|^p \right) dx \right. \\
&\quad \left. - c_3 \int_{S(u_\varepsilon) \cap [\omega_{i(\varepsilon)}^\varepsilon \times I]} \left| \left( (\nu_{u_\varepsilon})_\alpha \Big| \frac{1}{\varepsilon} (\nu_{u_\varepsilon})_3 \right) \right| d\mathcal{H}^2 - c_4 \mathcal{H}^1(S(u) \cap \omega_{i(\varepsilon)}^\varepsilon) \right] \\
&\geq \limsup_{\varepsilon \rightarrow 0} \left[ \int_\Omega W \left( \nabla_\alpha \bar{u}_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 \bar{u}_\varepsilon \right) dx + \int_{S(\bar{u}_\varepsilon)} \left| \left( (\nu_{\bar{u}_\varepsilon})_\alpha \Big| \frac{1}{\varepsilon} (\nu_{\bar{u}_\varepsilon})_3 \right) \right| d\mathcal{H}^2 \right. \\
&\quad \left. - c \left( \int_{\omega(\varepsilon)} (1 + |\nabla_\alpha u|^p) dx_\alpha + \mathcal{H}^1(S(u) \cap \omega(\varepsilon)) + \|u_\varepsilon - u\|_{L^p(\Omega; \mathbb{R}^3)}^{p/2} + \frac{1}{M_\varepsilon} \right) \right],
\end{aligned}$$

where we have used (4.3.27) in the last inequality. Thus, since  $M_\varepsilon \rightarrow +\infty$ ,  $M_\varepsilon/K_\varepsilon \rightarrow 0$  and  $\mathcal{H}^1(S(u)) < +\infty$ , we obtain from the previous relation and the  $\Gamma$ -lim inf inequality

$$\mathcal{J}(u) = \lim_{\varepsilon \rightarrow 0} \left[ \int_\Omega W \left( \nabla_\alpha \bar{u}_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 \bar{u}_\varepsilon \right) dx + \int_{S(\bar{u}_\varepsilon)} \left| \left( (\nu_{\bar{u}_\varepsilon})_\alpha \Big| \frac{1}{\varepsilon} (\nu_{\bar{u}_\varepsilon})_3 \right) \right| d\mathcal{H}^2 \right].$$

□

Let us now state a  $\Gamma$ -convergence result involving the boundary conditions. Consider a sequence of boundary conditions  $\{g_\varepsilon\} \subset W^{1,p}(\Omega'; \mathbb{R}^3)$ , and let  $g \in W^{1,p}(\omega'; \mathbb{R}^3)$  and  $H \in L^p(\Omega'; \mathbb{R}^3)$  be such that

$$\begin{cases} \sup_{\varepsilon > 0} \|g_\varepsilon\|_{L^\infty(\Omega'; \mathbb{R}^3)} < +\infty, \\ g_\varepsilon \rightarrow g \text{ in } W^{1,p}(\Omega'; \mathbb{R}^3), \quad \frac{1}{\varepsilon} \nabla_3 g_\varepsilon \rightarrow H \text{ in } L^p(\Omega'; \mathbb{R}^3). \end{cases} \quad (4.3.28)$$

Then,

**Corollary 4.3.11.** *The functional  $\mathcal{I}_\varepsilon^{g_\varepsilon} : BV(\Omega'; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  defined by*

$$\mathcal{I}_\varepsilon^{g_\varepsilon}(u) := \begin{cases} \int_{\Omega} W \left( \nabla_\alpha u \middle| \frac{1}{\varepsilon} \nabla_3 u \right) dx \\ \quad + \int_{S(u)} \left| \left( (\nu_u)_\alpha \middle| \frac{1}{\varepsilon} (\nu_u)_3 \right) \right| d\mathcal{H}^2 & \text{if } \begin{cases} u \in SBVP(\Omega'; \mathbb{R}^3), \\ u = g_\varepsilon \text{ on } [\omega' \setminus \overline{\omega}] \times I, \end{cases} \\ +\infty & \text{otherwise} \end{cases}$$

$\Gamma$ -converges for the strong  $L^1(\Omega'; \mathbb{R}^3)$ -topology towards  $\mathcal{J}^g : BV(\Omega'; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$  defined by

$$\mathcal{J}^g(u) := \begin{cases} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u) dx_\alpha + 2\mathcal{H}^1(S(u)) & \text{if } \begin{cases} u \in SBVP(\omega'; \mathbb{R}^3), \\ u = g \text{ on } \omega' \setminus \overline{\omega}, \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

**Remark 4.3.12.** Note that in the statement of the previous Corollary, the bulk integrals are still computed over  $\Omega$  (resp.  $\omega$ ) as in Theorem 4.3.1, however, since the jump set of the deformations can now reach the lateral boundary  $\partial\omega \times I$ , the surface integrals are implicitly computed over  $\overline{\omega} \times I$  (resp.  $\overline{\omega}$ ) or equivalently  $\Omega'$  (resp.  $\omega'$ ).

*Proof.* Let us first prove the  $\Gamma$ -lim inf inequality. Consider a sequence  $\{u_\varepsilon\} \subset L^1(\Omega'; \mathbb{R}^3)$  strongly converging to  $u$  in  $L^1(\Omega'; \mathbb{R}^3)$ . It is not restrictive to assume that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{g_\varepsilon}(u_\varepsilon) < +\infty.$$

Then, for a (not relabeled) subsequence,  $u_\varepsilon \in SBVP(\Omega'; \mathbb{R}^3)$ ,  $u_\varepsilon = g_\varepsilon$  on  $[\omega' \setminus \overline{\omega}] \times I$  and arguing as in the proof of Lemma 4.3.9, we get that  $u \in SBVP(\omega'; \mathbb{R}^3)$ . Consequently, since  $u = g$  on  $\omega' \setminus \overline{\omega}$ , we get from Theorem 4.3.1 and the definition of  $\mathcal{J}^g$  that

$$\mathcal{J}^g(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{g_\varepsilon}(u_\varepsilon).$$

Let  $u \in SBVP(\omega'; \mathbb{R}^3)$  satisfying  $u = g$  on  $\omega' \setminus \overline{\omega}$ . It remains to construct a recovery sequence. We first assume that  $u \in L^\infty(\omega'; \mathbb{R}^3)$ . Then, by virtue of Lemma 4.3.10, there exists a sequence  $\{\bar{u}_\varepsilon\} \subset SBVP(\Omega; \mathbb{R}^3)$  satisfying  $\bar{u}_\varepsilon \rightarrow u$  in  $L^1(\Omega; \mathbb{R}^3)$ ,  $\bar{u}_\varepsilon = u$  in a neighborhood of  $\partial\omega \times I$  and

$$\begin{aligned} & 2 \int_{\omega} \mathcal{Q}W_0(\nabla_\alpha u) dx_\alpha + 2\mathcal{H}^1(S(u) \cap \omega) \\ &= \lim_{\varepsilon \rightarrow 0} \left[ \int_{\Omega} W \left( \nabla_\alpha \bar{u}_\varepsilon \middle| \frac{1}{\varepsilon} \nabla_3 \bar{u}_\varepsilon \right) dx + \int_{S(\bar{u}_\varepsilon) \cap \Omega} \left| \left( (\nu_{\bar{u}_\varepsilon})_\alpha \middle| \frac{1}{\varepsilon} (\nu_{\bar{u}_\varepsilon})_3 \right) \right| d\mathcal{H}^2 \right]. \end{aligned} \quad (4.3.29)$$

Since  $g \in W^{1,p}(\omega' \setminus \overline{\omega}; \mathbb{R}^3)$ , by Corollary 3.89 in [5], the function

$$v_\varepsilon := \bar{u}_\varepsilon \chi_\Omega + g \chi_{[\omega' \setminus \overline{\omega}] \times I} \in BV(\Omega'; \mathbb{R}^3)$$

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and, viewing  $D\bar{u}_\varepsilon$  (resp.  $Dg$ ) as measures on all  $\mathbb{R}^3$  and concentrated on  $\Omega$  (resp.  $[\omega' \setminus \bar{\omega}] \times I$ ), we get as  $v_\varepsilon^- = \bar{u}_\varepsilon^- = u^-$  and  $v_\varepsilon^+ = g^+$  on  $\partial\omega \times I$

$$Dv_\varepsilon = D\bar{u}_\varepsilon + (g^+ - \bar{u}^-) \otimes \nu_{\partial\omega \times I} + Dg.$$

In particular, we observe that  $v_\varepsilon \in SBVP(\Omega'; \mathbb{R}^3)$  and  $v_\varepsilon \rightarrow u$  in  $L^1(\Omega'; \mathbb{R}^3)$  but we may have created some additional jump set on  $\partial\omega \times I$ . However,  $S(v_\varepsilon) \cap [\partial\omega \times I] \cong [S(u) \cap \partial\omega] \times I$ , and since  $\nu_{\partial\omega \times I} = \nu_{v_\varepsilon}$   $\mathcal{H}^2$ -a.e. in  $S(v_\varepsilon) \cap [\partial\omega \times I]$  and  $(\nu_{\partial\omega \times I})_3 = 0$ ,

$$\int_{S(v_\varepsilon) \cap [\partial\omega \times I]} \left| \left( (\nu_{v_\varepsilon})_\alpha \middle| \frac{1}{\varepsilon} (\nu_{v_\varepsilon})_3 \right) \right| d\mathcal{H}^2 = \mathcal{H}^2(S(v_\varepsilon) \cap [\partial\omega \times I]) = 2\mathcal{H}^1(S(u) \cap \partial\omega).$$

Replacing in (4.3.29), it yields

$$\begin{aligned} \mathcal{J}^g(u) &= 2 \int_\omega \mathcal{Q}W_0(\nabla_\alpha u) dx_\alpha + 2\mathcal{H}^1(S(u)) \\ &= \lim_{\varepsilon \rightarrow 0} \left[ \int_\Omega W \left( \nabla_\alpha v_\varepsilon \middle| \frac{1}{\varepsilon} \nabla_3 v_\varepsilon \right) dx + \int_{S(v_\varepsilon)} \left| \left( (\nu_{v_\varepsilon})_\alpha \middle| \frac{1}{\varepsilon} (\nu_{v_\varepsilon})_3 \right) \right| d\mathcal{H}^2 \right]. \end{aligned} \quad (4.3.30)$$

Actually,  $v_\varepsilon = g$  on  $[\omega' \setminus \bar{\omega}] \times I$  so that we need to modify  $v_\varepsilon$  in order it to have the value  $g_\varepsilon$  instead of  $g$  on the enlarged part of the domain. Let  $H_j \in \mathcal{C}_c^\infty(\Omega; \mathbb{R}^3)$  be a sequence strongly converging to  $H$  in  $L^p(\Omega; \mathbb{R}^3)$  and extended by the value zero on  $[\omega' \setminus \bar{\omega}] \times I$ , and set  $b_j(x_\alpha, x_3) := \int_{-1}^{x_3} H_j(x_\alpha, s) ds$ . We now define

$$u_\varepsilon^j(x) := v_\varepsilon(x) - g(x_\alpha) + g_\varepsilon(x) - \varepsilon b_j(x).$$

It follows that  $u_\varepsilon^j \in SBVP(\Omega'; \mathbb{R}^3)$ ,  $u_\varepsilon^j \rightarrow u$  in  $L^1(\Omega'; \mathbb{R}^3)$  as  $\varepsilon \rightarrow 0$ ,  $u_\varepsilon^j = g_\varepsilon$  on  $[\omega' \setminus \bar{\omega}] \times I$ . Furthermore, since  $u_\varepsilon^j$  is a smooth perturbation of  $v_\varepsilon$  on the whole domain  $\Omega'$ , both sequences have the same jump set, namely  $S(u_\varepsilon^j) \cong S(u_\varepsilon)$ , and consequently, the surface energy is not affected,

$$\int_{S(u_\varepsilon^j)} \left| \left( (\nu_{u_\varepsilon^j})_\alpha \middle| \frac{1}{\varepsilon} (\nu_{u_\varepsilon^j})_3 \right) \right| d\mathcal{H}^2 = \int_{S(v_\varepsilon)} \left| \left( (\nu_{v_\varepsilon})_\alpha \middle| \frac{1}{\varepsilon} (\nu_{v_\varepsilon})_3 \right) \right| d\mathcal{H}^2. \quad (4.3.31)$$

Let us treat now the bulk energy. According to Remark 4.3.5, (4.3.11) and Hölder's Inequality,

we have

$$\begin{aligned}
 & \int_{\Omega} W \left( \nabla_{\alpha} u_{\varepsilon}^j \middle| \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon}^j \right) dx \\
 &= \int_{\Omega} W \left( \nabla_{\alpha} v_{\varepsilon} - \nabla_{\alpha} g + \nabla_{\alpha} g_{\varepsilon} - \varepsilon \nabla_{\alpha} b_j \middle| \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon} + \frac{1}{\varepsilon} \nabla_3 g_{\varepsilon} - H_j \right) dx \\
 &\leq \int_{\Omega} W \left( \nabla_{\alpha} v_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon} \right) dx + \beta \int_{\Omega} \left( 1 + \left| \left( \nabla_{\alpha} v_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon} \right) \right|^{p-1} + \left| \left( \nabla_{\alpha} g_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 g_{\varepsilon} \right) \right|^{p-1} \right. \\
 &\quad \left. + |\nabla_{\alpha} g|^{p-1} + |(\varepsilon \nabla_{\alpha} b_j | H_j)|^{p-1} \right) \left| \left( \nabla_{\alpha} g_{\varepsilon} - \nabla_{\alpha} g - \varepsilon \nabla_{\alpha} b_j \middle| \frac{1}{\varepsilon} \nabla_3 g_{\varepsilon} - H_j \right) \right| dx \\
 &\leq \int_{\Omega} W \left( \nabla_{\alpha} v_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon} \right) dx + c \left( 1 + \left\| \left( \nabla_{\alpha} v_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon} \right) \right\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})}^{p-1} \right. \\
 &\quad \left. + \left\| \left( \nabla_{\alpha} g_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 g_{\varepsilon} \right) \right\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})}^{p-1} + \|\nabla_{\alpha} g\|_{L^p(\Omega; \mathbb{R}^{3 \times 2})}^{p-1} + \|(\varepsilon \nabla_{\alpha} b_j | H_j)\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})}^{p-1} \right) \\
 &\quad \times \left\| \left( \nabla_{\alpha} g_{\varepsilon} - \nabla_{\alpha} g - \varepsilon \nabla_{\alpha} b_j \middle| \frac{1}{\varepsilon} \nabla_3 g_{\varepsilon} - H_j \right) \right\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})}.
 \end{aligned}$$

Passing to the limit when  $\varepsilon \rightarrow 0$ , (4.3.28) yields

$$\begin{aligned}
 \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} W \left( \nabla_{\alpha} u_{\varepsilon}^j \middle| \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon}^j \right) dx \\
 \leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} W \left( \nabla_{\alpha} v_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 v_{\varepsilon} \right) dx + c' \|H - H_j\|_{L^p(\Omega; \mathbb{R}^3)}. \quad (4.3.32)
 \end{aligned}$$

Gathering (4.3.30), (4.3.31), (4.3.32) and remembering that  $H_j \rightarrow H$  in  $L^p(\Omega; \mathbb{R}^3)$ , we get that

$$\mathcal{J}^g(u) = \lim_{j \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \left[ \int_{\Omega} W \left( \nabla_{\alpha} u_{\varepsilon}^j \middle| \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon}^j \right) dx + \int_{S(u_{\varepsilon}^j)} \left| \left( (\nu_{u_{\varepsilon}^j})_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_{u_{\varepsilon}^j})_3 \right) \right| d\mathcal{H}^2 \right] \quad (4.3.33)$$

where we have also used the  $\Gamma$ -liminf inequality. A standard diagonalization procedure (see e.g. Lemma 7.1 in [35]) implies the existence of a sequence  $j_{\varepsilon} \nearrow +\infty$  as  $\varepsilon \rightarrow +\infty$  such that  $u_{\varepsilon} := u_{\varepsilon}^{j_{\varepsilon}} \in SBVP(\Omega'; \mathbb{R}^3)$ ,  $u_{\varepsilon} \rightarrow u$  in  $L^1(\Omega'; \mathbb{R}^3)$ ,  $u_{\varepsilon} = g_{\varepsilon}$  on  $[\omega' \setminus \bar{\omega}] \times I$  and

$$\mathcal{J}^g(u) = \lim_{\varepsilon \rightarrow 0} \left[ \int_{\Omega} W \left( \nabla_{\alpha} u_{\varepsilon} \middle| \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon} \right) dx + \int_{S(u_{\varepsilon})} \left| \left( (\nu_{u_{\varepsilon}})_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_{u_{\varepsilon}})_3 \right) \right| d\mathcal{H}^2 \right].$$

If  $u$  does not belong to  $L^{\infty}(\omega'; \mathbb{R}^3)$ , we can consider  $\varphi_i(u) \in SBVP(\omega'; \mathbb{R}^3) \cap L^{\infty}(\omega'; \mathbb{R}^3)$  where  $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^3; \mathbb{R}^3)$  is the truncation function defined in (4.3.5) and  $i$  is large enough (independently of  $\varepsilon$ ) so that  $e^i > \|g\|_{L^{\infty}(\omega'; \mathbb{R}^3)}$ . In particular,  $\varphi_i(u) = \varphi_i(g) = g$  on  $\omega' \setminus \bar{\omega}$  and we can apply the previous case. It implies, for each  $i \in \mathbb{N}$ , the existence of a sequence  $\{u_{\varepsilon}^i\} \subset SBVP(\Omega'; \mathbb{R}^3)$  strongly converging to  $\varphi_i(u)$  in  $L^1(\Omega'; \mathbb{R}^3)$  satisfying  $u_{\varepsilon}^i = g_{\varepsilon}$  on  $[\omega' \setminus \bar{\omega}] \times I$  and

$$\mathcal{J}(\varphi_i(u)) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}^{g_{\varepsilon}}(u_{\varepsilon}^i).$$

Since  $\varphi_i(u) \rightarrow u$  in  $L^1(\omega'; \mathbb{R}^3)$  we get that

$$\lim_{i \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon^i - u\|_{L^1(\Omega'; \mathbb{R}^3)} = 0. \quad (4.3.34)$$

Furthermore, by (4.3.24) together with the lower semicontinuity of  $\mathcal{J}$  with respect to the strong  $L^1(\omega'; \mathbb{R}^3)$ -convergence, we obtain that

$$\mathcal{J}^g(u) = \lim_{i \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{g_\varepsilon}(u_\varepsilon^i). \quad (4.3.35)$$

A standard diagonalization argument (see e.g. Lemma 7.1 in [35]) applied to (4.3.34) and (4.3.35) yields the existence of a sequence  $i_\varepsilon \nearrow +\infty$  as  $\varepsilon \rightarrow 0$  such that  $u_\varepsilon := u_\varepsilon^{i_\varepsilon} \in SBV^p(\Omega', \mathbb{R}^3)$ ,  $u_\varepsilon = g_\varepsilon$  on  $[\omega' \setminus \bar{\omega}] \times I$  and

$$\mathcal{J}^g(u) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{g_\varepsilon}(u_\varepsilon).$$

□

## 4.4 A few tools

### 4.4.1 Convergence of sets

The notion of  $\sigma^p$ -convergence introduced in [46, 47] does not seem to naturally provide a one dimensional limit crack. Indeed, let  $\Gamma_n \subset \Omega'$  be a sequence of  $\mathcal{H}^2$ -rectifiable sets; we denote by  $\nu_{\Gamma_n}$  its generalized normal defined  $\mathcal{H}^2$ -a.e. on  $\Gamma_n$ . We assume that there is an a priori bound on the scaled surface energy associated with this sequence of cracks i.e.

$$\sup_{n \in \mathbb{N}} \int_{\Gamma_n} \left| \left( (\nu_{\Gamma_n})_\alpha \middle| \frac{1}{\varepsilon_n} (\nu_{\Gamma_n})_3 \right) \right| d\mathcal{H}^2 < +\infty. \quad (4.4.1)$$

Note that this bound will appear naturally in the energy estimates. Intuitively, we expect that any limit crack of  $\Gamma_n$  will be a subset of  $\omega'$  of Hausdorff dimension equal to one. But, the sequences of test functions taken in the definition of the  $\sigma^p$ -convergence do not contain enough information in order for this to be true. Indeed, (4.4.1) implies in particular that  $\mathcal{H}^2(\Gamma_n) \leq C$ , thus according to Lemma 4.7 in [46], we have (for a subsequence) that  $\Gamma_n$   $\sigma^p$ -converges in  $\Omega'$  to some  $\mathcal{H}^2$ -rectifiable set  $\Gamma \subset \Omega'$ . We would like to be able to state that  $\Gamma$  is of the form  $\gamma \times I$  for some  $\mathcal{H}^1$ -rectifiable set  $\gamma \subset \omega'$ . By lower semicontinuity of

$$v \mapsto \int_{S(v)} |(\nu_v)_3| d\mathcal{H}^2$$

for the weak  $SBV^p(\Omega')$ -convergence, we have, according to Lemma 4.3 in [46] and (4.4.1), that  $(\nu_\Gamma)_3(x) = 0$   $\mathcal{H}^2$ -a.e.  $x \in \Gamma$ . But this does not tell us that  $\Gamma \cong \gamma \times I$ . We know, by the very definition of the  $\sigma^p$ -convergence, that there exists a function  $u \in SBV^p(\Omega')$  and a sequence  $u_n \rightharpoonup u$  in  $SBV^p(\Omega')$  such that  $S(u_n) \tilde{\subset} \Gamma_n$  and  $\Gamma \cong S(u)$ . To prove that  $\Gamma \cong \gamma \times I$ , it would be enough to show that  $D_3 u = 0$  in the sense of Radon measures. This would be immediate if



the approximate scaled gradient of  $u_n$  was bounded in  $L^p(\Omega'; \mathbb{R}^3)$ . Since, in the sequel, we will only be interested in minimizing sequences satisfying this property, it prompts us to redefine the notion of  $\sigma^p$ -convergence in a 3D-2D dimensional reduction setting.

**Definition 4.4.1.** Let  $\{\varepsilon_n\} \searrow 0^+$  and  $\Gamma_n \subset \Omega'$  be a sequence of  $\mathcal{H}^2$ -rectifiable sets. We say that  $\Gamma_n$  converges towards  $\gamma$  in  $\Omega'$  if  $\gamma \subset \omega'$ , (4.4.1) holds and

(a) if  $u_k \rightharpoonup u$  in  $SBVP(\Omega')$ ,  $S(u_k) \tilde{\subset} \Gamma_{n_k}$  and

$$\sup_{k \in \mathbb{N}} \int_{\Omega'} \left| \left( \nabla_{\alpha} u_k \Big|_{\frac{1}{\varepsilon_{n_k}} \nabla_3 u_k} \right) \right|^p dx < +\infty,$$

for a subsequence  $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$ , then  $u \in SBVP(\omega')$  and  $S(u) \tilde{\subset} \gamma$ ;

(b) there exists a function  $u \in SBVP(\omega')$  and a sequence  $u_n \in SBVP(\Omega')$  such that  $u_n \rightharpoonup u$  in  $SBVP(\Omega')$ ,  $S(u_n) \tilde{\subset} \Gamma_n$ ,

$$\sup_{n \in \mathbb{N}} \int_{\Omega'} \left| \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n} \nabla_3 u_n} \right) \right|^p dx < +\infty$$

and  $S(u) \cong \gamma$ .

According to property (b) of Definition 4.4.1,  $\gamma$  is necessarily a  $\mathcal{H}^1$ -rectifiable set. In the following Remark, we state few properties of this kind of convergence as lower semicontinuity with respect to the Hausdorff measure and stability with respect to the inclusion.

**Remark 4.4.2.** Let  $\Gamma_n \rightarrow \gamma$  in the sense of Definition 4.4.1, then

1. for every Borel set  $E \subset \omega'$  such that  $\mathcal{H}^1(E) < +\infty$  (or  $E$  a compact set),

$$2\mathcal{H}^1(\gamma \setminus E) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^2(\Gamma_n \setminus (E \times [-1, 1]));$$

2. if  $\Gamma_n \tilde{\subset} \Gamma'_n$  and  $\Gamma'_n \rightarrow \gamma'$  in the sense of Definition 4.4.1, then  $\gamma \tilde{\subset} \gamma'$ ;

3. if  $\Gamma_n \xrightarrow{\sigma^p} \Gamma$ , then  $\gamma \times I \tilde{\subset} \Gamma$ .

Replacing every approximate gradients by approximate scaled gradients and using Lemma 4.2.2 instead of Ambrosio's Compactness Theorem, the exact analogues of the proofs of Lemma 4.5 and Proposition 4.6 in [46] would demonstrate that any sequence of  $\mathcal{H}^2$ -rectifiable sets  $\Gamma_n \subset \Omega'$  satisfying (4.4.1) admits a convergent subsequence in the sense of Definition (4.4.1). But this compactness result will not be sufficient because, in the proof of Theorem 4.2.1, we will deal with sequence of  $\mathcal{H}^2$ -rectifiable sets which are increasing with respect to the time parameter  $t$ . The following Proposition, which is the analogue of Lemma 4.8 in [46], states a version of Helly's Theorem for a sequence of increasing  $\mathcal{H}^2$ -rectifiable sets.

**Proposition 4.4.3.** Let  $[0, T] \ni t \mapsto \Gamma_n(t)$  a sequence of  $\mathcal{H}^2$ -rectifiable sets of  $\Omega'$  that increases with  $t$ , i.e.

$$\Gamma_n(s) \tilde{\subset} \Gamma_n(t) \subset \Omega', \quad \text{for every } s, t \in [0, T] \text{ with } s < t.$$

Assume that

$$\sup_{n \in \mathbb{N}} \int_{\Gamma_n(t)} \left| \left( (\nu_{\Gamma_n(t)})_\alpha \left| \frac{1}{\varepsilon_n} (\nu_{\Gamma_n(t)})_3 \right) \right| d\mathcal{H}^2 < +\infty,$$

uniformly in  $t$ . Then, there exists a subsequence  $\Gamma_{n_k}(t)$  and a  $t$ -increasing  $\mathcal{H}^1$ -rectifiable set  $\gamma(t) \subset \omega'$  such that for every  $t \in [0, T]$ ,  $\Gamma_{n_k}(t)$  converges to  $\gamma(t)$  in the sense of Definition 4.4.1.

#### 4.4.2 Transfer of jump sets

We now state a transfer of jump sets result in a rescaled version. It permits, under weak  $SBV^p(\Omega'; \mathbb{R}^3)$ -convergence assumptions of a sequence  $\{u_n\}$  – with associated bounded scaled bulk energy – toward its limit  $u$ , the transfer of the part of the jump set of a 2D admissible deformation that lies in the jump set of  $u$  onto that of the sequence  $\{u_n\}$ . The proof relies on De Giorgi's slicing argument.

**Theorem 4.4.4** (Jump Transfer). *Let  $\{u_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  and  $u \in SBV^p(\omega'; \mathbb{R}^3)$  such that  $S(u_n) \subset \bar{\omega} \times I$ ,  $u_n \rightarrow u$  in  $L^1(\Omega'; \mathbb{R}^3)$  and*

$$M := \sup_{n \in \mathbb{N}} \int_{\Omega'} \left| \left( \nabla_\alpha u_n \left| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx < +\infty.$$

Then, for all  $\phi \in SBV^p(\omega'; \mathbb{R}^3)$ , there exists  $\{\phi_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  such that

- $\phi_n = \phi$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ ,
- $\phi_n \rightarrow \phi$  in  $L^1(\Omega'; \mathbb{R}^3)$ ,
- $\left( \nabla_\alpha \phi_n \left| \frac{1}{\varepsilon_n} \nabla_3 \phi_n \right) \rightarrow (\nabla_\alpha \phi | 0)$  in  $L^p(\Omega'; \mathbb{R}^{3 \times 3})$ ,
- $\int_{[S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]} \left| \left( (\nu_{\phi_n})_\alpha \left| \frac{1}{\varepsilon_n} (\nu_{\phi_n})_3 \right) \right| d\mathcal{H}^2 \rightarrow 0$ .

*Proof.* We first undo the scaling, coming back to the cylinder of thickness  $2\varepsilon_n$ . Then, we extend periodically the function in the transverse direction. Note that the periodic extension may generate some additional jump at the interface of each slice of thickness  $2\varepsilon_n$ . Despite this new discontinuities, we can still apply the classical Jump Transfer Theorem (Theorem 2.1 in [63]) and, by contradiction, we show that we can choose a slice of thickness  $2\varepsilon_n$  that satisfies good estimations. Finally, we observe that, after translation and dilation, the restriction of the function to this particular slice satisfies the conclusion of Theorem 4.4.4.

**Step 1.** We come back to the non rescaled cylinder  $\Omega'_{\varepsilon_n}$  of thickness  $2\varepsilon_n$ . We set  $v_n(x_\alpha, x_3) := u_n(x_\alpha, x_3/\varepsilon_n)$ . Thus  $v_n \in SBV^p(\Omega'_{\varepsilon_n}; \mathbb{R}^3)$  and  $S(v_n) \subset \bar{\omega} \times (-\varepsilon_n, \varepsilon_n)$ . Moreover,

$$\begin{cases} \frac{1}{\varepsilon_n} \int_{\Omega'_{\varepsilon_n}} |v_n - u| dx = \int_{\Omega'} |u_n - u| dx, \\ \frac{1}{\varepsilon_n} \int_{\Omega'_{\varepsilon_n}} |\nabla v_n|^p dx = \int_{\Omega'} \left| \left( \nabla_\alpha u_n \left| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx. \end{cases} \quad (4.4.2)$$

We now to extend  $v_n$  by periodicity in the  $x_3$  direction. The discontinuities of the resulting function will be those inherited from the discontinuities of  $v_n$  and from additional jumps that may occur at the interface of each slice. Let

$$N_n := \begin{cases} \frac{1}{2\varepsilon_n} - \frac{1}{2} & \text{if } \frac{1}{2\varepsilon_n} + \frac{1}{2} \in \mathbb{N}, \\ \left\lfloor \frac{1}{2\varepsilon_n} + \frac{1}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

For every  $i \in \{-N_n, \dots, N_n\}$ , we set  $I_{i,n} := ((2i-1)\varepsilon_n, (2i+1)\varepsilon_n)$  and  $\Omega'_{i,n} := \omega' \times I_{i,n}$ . Note that  $N_n$  is the smaller integer such that  $\Omega' \cap \Omega'_{i,n} \neq \emptyset$  for every  $i \in \{-N_n, \dots, N_n\}$ . We define the function  $w_n$  on  $\Omega'(n) := \omega' \times (-(2N_n+1)\varepsilon_n, (2N_n+1)\varepsilon_n)$  by extending  $v_n$  by periodicity in the  $x_3$  direction on  $\Omega'(n)$  :

$$w_n(x_\alpha, x_3) = v_n(x_\alpha, x_3 - 2i\varepsilon_n) \text{ if } x_3 \in I_{i,n}.$$

Since  $\Omega' \subset \Omega'(n)$ ,  $w_n$  is a fortiori defined on  $\Omega'$ ,  $w_n \in SBV^p(\Omega'; \mathbb{R}^3)$  and  $S(w_n) \cap \Omega' \subset \bar{\omega} \times I$ .

**Step 2.** We would like to apply the classical Jump Transfer Theorem (Theorem 2.1 in [63]) to the function  $w_n$ . From (4.4.2), we have that

$$\begin{aligned} \int_{\Omega'} |w_n - u| dx &= \sum_{i=-N_n}^{N_n} \int_{\Omega'_{i,n} \cap \Omega'} |v_n(x_\alpha, x_3 - 2i\varepsilon_n) - u(x_\alpha)| dx \\ &\leq (2N_n + 1) \int_{\Omega'_{\varepsilon_n}} |v_n - u| dx \\ &= \varepsilon_n (2N_n + 1) \int_{\Omega'} |u_n - u| dx \\ &\leq (1 + 2\varepsilon_n) \int_{\Omega'} |u_n - u| dx \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega'} |\nabla w_n|^p dx &= \sum_{i=-N_n}^{N_n} \int_{\Omega'_{i,n} \cap \Omega'} |\nabla v_n(x_\alpha, x_3 - 2i\varepsilon_n)|^p dx \\ &\leq (2N_n + 1) \int_{\Omega'_{\varepsilon_n}} |\nabla v_n|^p dx \\ &= \varepsilon_n (2N_n + 1) \int_{\Omega'} \left| \left( \nabla_\alpha u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx \\ &\leq (1 + 2\varepsilon_n) M, \end{aligned}$$

which implies, thanks to De La Vallée Poussin criterion (see Proposition 1.27 in [5]), that the sequence  $\{|\nabla w_n|\}$  is equi-integrable. We are now in position to apply Theorem 2.1 in [63] to

the sequence  $\{w_n\}$ . Indeed, an inspection of the proof of this result shows that the weak  $L^1$ -convergence required by  $\{|\nabla w_n|\}$  can be replaced, without passing to a subsequence, by its equi-integrability (see p. 1477 in [63]). Thus, for all  $\phi \in SBV^p(\omega'; \mathbb{R}^3)$ , we get the existence of a sequence  $\{\psi_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  such that

- $\psi_n = \phi$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ ,
- $\psi_n \rightarrow \phi$  in  $L^1(\Omega'; \mathbb{R}^3)$ ,
- $\nabla \psi_n \rightarrow (\nabla_\alpha \phi | 0)$  in  $L^p(\Omega'; \mathbb{R}^{3 \times 3})$ ,
- $\mathcal{H}^2([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]) \rightarrow 0$ .

**Step 3.** Since  $\bigcup_{i=-N_n+1}^{N_n-1} \Omega'_{i,n} \subset \Omega'$ , we may find a  $i_n \in \{-N_n+1, \dots, N_n-1\}$  such that

$$\begin{aligned} & (2N_n - 1) \left\{ \int_{\Omega'_{i_n,n}} |\psi_n - \phi| dx + \int_{\Omega'_{i_n,n}} |\nabla \psi_n - (\nabla_\alpha \phi | 0)|^p dx \right. \\ & \qquad \qquad \qquad \left. + \mathcal{H}^2_{[\Omega'_{i_n,n}]}([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]) \right\} \\ & \leq \int_{\Omega'} |\psi_n - \phi| dx + \int_{\Omega'} |\nabla \psi_n - (\nabla_\alpha \phi | 0)|^p dx + \mathcal{H}^2([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]). \end{aligned}$$

Since  $2N_n - 1 \geq 1/\varepsilon_n - 2$ , we have

$$\begin{aligned} & \frac{1}{\varepsilon_n} \left\{ \int_{\Omega'_{i_n,n}} |\psi_n - \phi| dx + \int_{\Omega'_{i_n,n}} |\nabla \psi_n - (\nabla_\alpha \phi | 0)|^p dx \right. \\ & \qquad \qquad \qquad \left. + \mathcal{H}^2_{[\Omega'_{i_n,n}]}([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]) \right\} \\ & \leq 3 \int_{\Omega'} |\psi_n - \phi| dx + 3 \int_{\Omega'} |\nabla \psi_n - (\nabla_\alpha \phi | 0)|^p dx \\ & \qquad \qquad \qquad + 3\mathcal{H}^2([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]). \end{aligned}$$

**Step 4.** We will show that, after a translation and a dilation,  $\psi_n|_{\Omega'_{i_n,n}}$  is the right candidate for Theorem 4.4.4. Let us come back to the cylinder  $\Omega'_{\varepsilon_n} = \omega' \times (-\varepsilon_n, \varepsilon_n)$ ; letting

$$\varphi_n(x_\alpha, x_3) := \psi_n|_{\Omega'_{i_n,n}}(x_\alpha, x_3 + 2i_n\varepsilon_n) \text{ if } x_3 \in (-\varepsilon_n, \varepsilon_n),$$

then  $\varphi_n \in SBV^p(\Omega'_{\varepsilon_n}; \mathbb{R}^3)$ ,  $\varphi_n = \phi$  a.e. on  $[\omega' \setminus \bar{\omega}] \times (-\varepsilon_n, \varepsilon_n)$  and

$$\begin{aligned} & \frac{1}{\varepsilon_n} \left\{ \int_{\Omega'_{\varepsilon_n}} |\varphi_n - \phi| dx + \int_{\Omega'_{\varepsilon_n}} |\nabla \varphi_n - (\nabla_\alpha \phi | 0)|^p dx \right. \\ & \qquad \qquad \qquad \left. + \mathcal{H}^2_{[\Omega'_{\varepsilon_n}]}([S(\varphi_n) \setminus S(v_n)] \setminus [S(\phi) \setminus S(u)]) \right\} \\ & \leq 3 \int_{\Omega'} |\psi_n - \phi| dx + 3 \int_{\Omega'} |\nabla \psi_n - (\nabla_\alpha \phi | 0)|^p dx \\ & \qquad \qquad \qquad + 3\mathcal{H}^2([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]). \end{aligned}$$

Performing the scaling so as to come back to the unit cylinder, we get, upon setting  $\phi_n(x_\alpha, x_3) := \varphi_n(x_\alpha, \varepsilon_n x_3)$ , that  $\phi_n \in SBV^p(\Omega'; \mathbb{R}^3)$ ,  $\phi_n = \phi$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$  and

$$\begin{aligned} & \int_{\Omega'} |\phi_n - \phi| dx + \int_{\Omega'} \left| \left( \nabla_\alpha \phi_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 \phi_n \right) - (\nabla_\alpha \phi | 0) \right|^p dx \\ & \quad + \int_{[S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]} \left| \left( (\nu_{\phi_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{\phi_n})_3 \right) \right| d\mathcal{H}^2 \\ & \leq 3 \int_{\Omega'} |\psi_n - \phi| dx + 3 \int_{\Omega'} |\nabla \psi_n - (\nabla_\alpha \phi | 0)|^p dx \\ & \quad + 3\mathcal{H}^2([S(\psi_n) \setminus S(w_n)] \setminus [S(\phi) \setminus S(u)]) \rightarrow 0. \end{aligned}$$

□

**Remark 4.4.5.** Since for  $\mathcal{H}^2$ -a.e.  $x \in S(\phi_n) \cap S(\phi)$ ,  $\nu_{\phi_n}(x) = \pm \nu_\phi(x)$ , we have

$$\begin{aligned} & \int_{[S(\phi_n) \setminus S(u_n)] \setminus [S(\phi) \setminus S(u)]} \left| \left( (\nu_{\phi_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{\phi_n})_3 \right) \right| d\mathcal{H}^2 \\ & \geq \int_{S(\phi_n) \setminus S(u_n)} \left| \left( (\nu_{\phi_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{\phi_n})_3 \right) \right| d\mathcal{H}^2 - \int_{S(\phi) \setminus S(u)} |((\nu_\phi)_\alpha | 0)| d\mathcal{H}^2 \\ & = \int_{S(\phi_n) \setminus S(u_n)} \left| \left( (\nu_{\phi_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{\phi_n})_3 \right) \right| d\mathcal{H}^2 - 2\mathcal{H}^1(S(\phi) \setminus S(u)), \end{aligned}$$

thus

$$\limsup_{n \rightarrow +\infty} \int_{S(\phi_n) \setminus S(u_n)} \left| \left( (\nu_{\phi_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{\phi_n})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S(\phi) \setminus S(u)).$$

The following Theorem establishes a link between the convergence in the sense of Definition 4.4.1 and the Jump Transfer Theorem. It will allow to pass to the limit in the surface energy.

**Theorem 4.4.6.** *Let  $\Gamma_n \subset \Omega'$  be a sequence of  $\mathcal{H}^2$ -rectifiable sets converging towards  $\gamma$  in the sense of Definition 4.4.1. Then, for every  $v \in SBV^p(\omega'; \mathbb{R}^3)$ , there exists  $\{v_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  such that  $v_n = v$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ ,*

- $v_n \rightarrow v$  in  $L^1(\Omega'; \mathbb{R}^3)$ ,
- $\left( \nabla_\alpha v_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n \right) \rightarrow (\nabla_\alpha v | 0)$  in  $L^p(\Omega'; \mathbb{R}^{3 \times 3})$ ,
- $\limsup_{n \rightarrow +\infty} \int_{S(v_n) \setminus \Gamma_n} \left| \left( (\nu_{v_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{v_n})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S(v) \setminus \gamma)$ .

*Proof.* According to Definition 4.4.1 (b), there exists a function  $u \in SBV^p(\omega'; \mathbb{R}^3)$  and a sequence  $\{u_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  such that  $u_n \rightarrow u$  in  $SBV^p(\Omega'; \mathbb{R}^3)$ ,  $S(u_n) \tilde{\subset} \Gamma_n$ ,  $S(u) \cong \gamma$  and

$$\sup_{n \in \mathbb{N}} \int_{\Omega'} \left| \left( \nabla_\alpha u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) \right|^p dx < +\infty.$$

Theorem 4.4.4 and Remark 4.4.5 yield, for any  $v \in SBV^p(\omega'; \mathbb{R}^3)$ , the existence of a sequence  $\{v_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  such that  $v_n = v$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ ,

- $v_n \rightarrow v$  in  $L^1(\Omega'; \mathbb{R}^3)$ ,
- $\left( \nabla_\alpha v_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 v_n \right) \rightarrow (\nabla_\alpha v | 0)$  in  $L^p(\Omega'; \mathbb{R}^{3 \times 3})$ ,
- $\limsup_{n \rightarrow +\infty} \int_{S(v_n) \setminus S(u_n)} \left| \left( (\nu_{v_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{v_n})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S(v) \setminus S(u))$ .

As  $S(u_n) \tilde{\subset} \Gamma_n$  and  $S(u) \cong \gamma$ , we get

$$\limsup_{n \rightarrow +\infty} \int_{S(v_n) \setminus \Gamma_n} \left| \left( (\nu_{v_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{v_n})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S(v) \setminus \gamma).$$

□

#### 4.4.3 Convergence of the stresses

The energy conservation involves the derivative of the stored energy density. Thus, we have to ensure that the  $\mathcal{C}^1$  character of  $W$  is preserved by passing to the  $\Gamma$ -limit. The following Proposition provides an answer to this question. In the Appendix, we will present an alternative proof which could also apply in a more general setting.

**Proposition 4.4.7.** *Let  $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  function satisfying (4.2.1), then the function  $\mathcal{Q}W_0 : \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^1$ .*

*Proof.* According to [74], the function  $W_0$  is continuous and satisfies

$$\frac{1}{\beta} |\bar{F}|^p - \beta \leq W_0(\bar{F}) \leq \beta(1 + |\bar{F}|^p)$$

for every  $\bar{F} \in \mathbb{R}^{3 \times 2}$ . As a consequence, since  $p - 1 > 0$ ,

$$\liminf_{|\bar{F}| \rightarrow +\infty} \frac{W_0(\bar{F})}{|\bar{F}|^{p-1}} = +\infty \quad \text{and} \quad \limsup_{|\bar{F}| \rightarrow +\infty} \frac{W_0(\bar{F})}{|\bar{F}|^p} = \beta < +\infty.$$

Furthermore, for all  $\bar{F} \in \mathbb{R}^{3 \times 2}$  there exists  $F_3 \in \mathbb{R}^3$  such that  $W_0(\bar{F}) = W(\bar{F}|F_3)$ . Since  $W$  is differentiable,

$$\begin{aligned} & \limsup_{|\bar{G}| \rightarrow 0} \frac{W_0(\bar{F} + \bar{G}) - W_0(\bar{F}) - d \cdot \bar{G}}{|\bar{G}|} \\ & \leq \limsup_{|\bar{G}| \rightarrow 0} \frac{W((\bar{F}|F_3) + (\bar{G}|0)) - W(\bar{F}|F_3) - \partial W(\bar{F}|F_3) \cdot (\bar{G}|0)}{|(\bar{G}|0)|} = 0, \end{aligned}$$

where  $d \in \mathbb{R}^{3 \times 2}$  is defined by  $d_{ij} := (\partial W(\bar{F}|F_3))_{ij}$  for all  $i \in \{1, 2, 3\}$  and all  $j \in \{1, 2\}$ . It yields that  $W_0$  is upper semidifferentiable and the thesis follows from Theorem B in [19]. □

From the previous Lemma, the function  $\mathcal{Q}W_0$  is of class  $\mathcal{C}^1$  and we denote by  $\partial(\mathcal{Q}W_0)$  its differential. The following result is the analogue of Lemma 4.11 in [46] in a 3D-2D dimensional

reduction setting. It asserts, under assumptions of weak  $SBVP(\Omega; \mathbb{R}^3)$ -convergence of the deformations together with the convergence of the bulk energy, the weak  $L^{p'}(\Omega; \mathbb{R}^{3 \times 3})$ -convergence the stresses.

**Lemma 4.4.8.** *Let  $\{u_n\} \subset SBVP(\Omega; \mathbb{R}^3)$  and  $u \in SBVP(\omega; \mathbb{R}^3)$  be such that  $u_n \rightharpoonup u$  in  $SBVP(\Omega; \mathbb{R}^3)$  and*

$$\int_{\Omega} W \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u) dx_{\alpha}. \quad (4.4.3)$$

Then,

$$\partial W \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) \rightharpoonup (\partial(\mathcal{Q}W_0)(\nabla_{\alpha} u)|0) \quad \text{in } L^{p'}(\Omega; \mathbb{R}^{3 \times 3}).$$

*Proof.* Let  $\Psi \in L^p(\Omega; \mathbb{R}^{3 \times 3})$ , we denote by  $\bar{\Psi} \in L^p(\Omega; \mathbb{R}^{3 \times 2})$  the restriction of  $\Psi$  to  $\mathbb{R}^{3 \times 2}$  i.e.  $\bar{\Psi}_{ij} = \Psi_{ij}$  if  $i \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$ . It is enough to show that

$$\begin{aligned} \int_{\Omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha} u) \cdot \bar{\Psi} dx &= \int_{\Omega} (\partial(\mathcal{Q}W_0)(\nabla_{\alpha} u)|0) \cdot \Psi dx \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \partial W \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) \cdot \Psi dx. \end{aligned}$$

Let  $h_k \searrow 0^+$ , according to Remark 4.3.2, Theorem 5.29 in [5] we have

$$\begin{aligned} \int_{\Omega} \mathcal{Q}W_0(\nabla_{\alpha} u + h_k \bar{\Psi}) dx &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \mathcal{Q}W_0(\nabla_{\alpha} u_n + h_k \bar{\Psi}) dx \\ &\leq \liminf_{n \rightarrow +\infty} \int_{\Omega} W \left( \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) + h_k \Psi \right) dx. \end{aligned}$$

As a consequence, from (4.4.3) we get that

$$\begin{aligned} \int_{\Omega} \frac{\mathcal{Q}W_0(\nabla_{\alpha} u + h_k \bar{\Psi}) - \mathcal{Q}W_0(\nabla_{\alpha} u)}{h_k} dx \\ \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{1}{h_k} \left[ W \left( \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) + h_k \Psi \right) - W \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) \right] dx. \end{aligned}$$

We may find a  $n_k \in \mathbb{N}$  such that

$$\begin{aligned} \int_{\Omega} \frac{\mathcal{Q}W_0(\nabla_{\alpha} u + h_k \bar{\Psi}) - \mathcal{Q}W_0(\nabla_{\alpha} u)}{h_k} dx - \frac{1}{k} \\ \leq \int_{\Omega} \frac{1}{h_k} \left[ W \left( \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) + h_k \Psi \right) - W \left( \nabla_{\alpha} u_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u_n \right) \right] dx \end{aligned}$$

for all  $n \geq n_k$ . We define  $\eta_n = h_k$  if  $n_k \leq n \leq n_{k+1}$  and pass to the limit when  $n \rightarrow +\infty$ . Since  $\Phi \mapsto \int_{\Omega} W(\Phi) dx$  is a  $\mathcal{C}^1$ -map from  $L^p(\Omega; \mathbb{R}^{3 \times 3})$  to  $\mathbb{R}$  with differential  $\Psi \mapsto \int_{\Omega} \partial W(\Phi) \cdot \Psi dx$ , it follows that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\Omega} \frac{\mathcal{Q}W_0(\nabla_{\alpha} u + \eta_n \bar{\Psi}) - \mathcal{Q}W_0(\nabla_{\alpha} u)}{\eta_n} dx \\ & \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \frac{1}{\eta_n} \left[ W \left( \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) + \eta_n \Psi \right) - W \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right] dx \\ & \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \partial W \left( \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) + \tau_n \Psi \right) \cdot \Psi dx, \end{aligned}$$

for some  $\tau_n \in [0, \eta_n]$ . Lebesgue's Dominated Convergence Theorem in the left hand side, together with Lemma 4.9 of [46] in the right hand side yield

$$\begin{aligned} \int_{\Omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha} u) \cdot \bar{\Psi} dx & = \lim_{n \rightarrow +\infty} \int_{\Omega} \frac{\mathcal{Q}W_0(\nabla_{\alpha} u + \eta_n \bar{\Psi}) - \mathcal{Q}W_0(\nabla_{\alpha} u)}{\eta_n} dx \\ & \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \partial W \left( \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) + \tau_n \Psi \right) \cdot \Psi dx \\ & = \liminf_{n \rightarrow +\infty} \int_{\Omega} \partial W \left( \nabla_{\alpha} u_n \middle| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \cdot \Psi dx. \end{aligned}$$

□

## 4.5 Convergence of the quasistatic evolution

The first step of the analysis consists in defining a limit deformation field and a crack. This is done in Lemma 4.5.1 by means of energy estimates which are possible, thanks to the  $L^{\infty}$ -boundness assumption (4.2.8) and to the bound of the prescribed boundary deformation (4.2.4). The limit deformation  $u(t)$  turns out to be the weak  $SBVP(\Omega'; \mathbb{R}^3)$ -limit of  $u^{\varepsilon}(t)$  while the limit crack  $\gamma(t)$  is obtained through the convergence of  $\Gamma^{\varepsilon}(t)$  in the sense of Definition 4.4.1. Then, we derive a minimality property for  $u(t)$ . At time  $t = 0$  in Lemma 4.5.2, we use a  $\Gamma$ -convergence argument. This is possible because, in the absence of preexisting cracks, the surface term of the energy at time 0 is precisely that introduced in the  $\Gamma$ -limit analysis of Section 4.3. Nevertheless, we cannot proceed in this way for the next times in Lemma 4.5.5 because of the presence of  $\Gamma^{\varepsilon}(t)$  in the surface term. We need here to construct directly a sequence thanks to the Jump Transfer Theorem, Theorem 4.4.4. Then, we show that  $(u(t), \gamma(t))$  is a quasistatic evolution for the relaxed model by proving that the energy conservation holds. To do this, we use, on the one hand, the approximation of the Lebesgue integral by Riemann sums in Lemma 4.5.7, and, on the other hand, the convergence of the total energy at the initial time (that can be proved directly) together with the weak convergence of the stresses and the strong convergence assumption (4.2.7) in Lemma 4.5.8. Finally, in Lemma 4.5.9, we show the convergence of the total energy at any time.



### 4.5.1 Energy estimates and compactness

**Lemma 4.5.1.** *There exists a subsequence  $\{\varepsilon_n\} \searrow 0^+$ , a deformation field  $u(t) \in SBV^p(\omega'; \mathbb{R}^3)$  satisfying  $u(t) = g(t)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$ , and a time-increasing crack  $\gamma(t) \subset \bar{\omega}$  such that, for every  $t \in [0, T]$ ,  $S(u(t)) \tilde{\subset} \gamma(t)$ . Moreover,  $\Gamma^{\varepsilon_n}(t)$  converges to  $\gamma(t)$  in the sense of Definition 4.4.1,  $u^{\varepsilon_n}(0) \rightharpoonup u(0)$  in  $SBV^p(\Omega'; \mathbb{R}^3)$  and, for every  $t \in (0, T]$ , there exists a  $t$ -dependent subsequence  $\{\varepsilon_{n_t}\} \subset \{\varepsilon_n\}$  such that  $u^{\varepsilon_{n_t}}(t) \rightharpoonup u(t)$  in  $SBV^p(\Omega'; \mathbb{R}^3)$ .*

*Proof.* Firstly, at time  $t = 0$ , we test the minimality of  $u^\varepsilon(0)$  with  $v = g^\varepsilon(0)$ . Since we have  $S(u^\varepsilon(0)) \cong \Gamma^\varepsilon(0)$ , we deduce by (4.2.4) and the growth condition (4.2.1) that  $\mathcal{E}_\varepsilon(0) \leq C$ .

Then, we take  $v = g^\varepsilon(t)$  as test function in (4.2.5) at time  $t$ . As  $S(u^\varepsilon(t)) \tilde{\subset} \Gamma^\varepsilon(t)$ , it follows from (4.2.4) together with the growth condition (4.2.1) that

$$\int_{\Omega} W \left( \nabla_{\alpha} u^\varepsilon(t) \Big|_{\frac{1}{\varepsilon}} \nabla_3 u^\varepsilon(t) \right) dx \leq \int_{\Omega} W \left( \nabla_{\alpha} g^\varepsilon(t) \Big|_{\frac{1}{\varepsilon}} \nabla_3 g^\varepsilon(t) \right) dx \leq C. \quad (4.5.1)$$

Thus, Hölder's inequality, (4.2.2), (4.2.6) and (4.5.1) imply the existence of a constant  $C > 0$ , independent of  $t$  and  $\varepsilon$  such that for every  $t \in [0, T]$ ,  $\mathcal{E}_\varepsilon(t) \leq C$ . Hence, by the coercivity condition (4.2.1),

$$\int_{\Omega} \left| \left( \nabla_{\alpha} u^\varepsilon(t) \Big|_{\frac{1}{\varepsilon}} \nabla_3 u^\varepsilon(t) \right) \right|^p dx + \int_{\Gamma^{\varepsilon}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon}(t)})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{\Gamma^{\varepsilon}(t)})_3 \right) \right| d\mathcal{H}^2 \leq C. \quad (4.5.2)$$

In view of Proposition 4.4.3, we may find a subsequence  $\{\varepsilon_n\} \searrow 0^+$  and an  $\mathcal{H}^1$ -rectifiable set  $\gamma(t) \subset \omega'$ , increasing in time, such that  $\Gamma^{\varepsilon_n}(t)$  converges to  $\gamma(t)$  in the sense of Definition 4.4.1. According to Remark 4.4.2-1, since  $\bar{\omega}$  is compact,  $\gamma(t) \tilde{\subset} \bar{\omega}$  and

$$2\mathcal{H}^1(\gamma(t)) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^2(\Gamma^{\varepsilon_n}(t)) \leq \liminf_{n \rightarrow +\infty} \int_{\Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon_n}(t)})_{\alpha} \Big|_{\frac{1}{\varepsilon_n}} (\nu_{\Gamma^{\varepsilon_n}(t)})_3 \right) \right| d\mathcal{H}^2. \quad (4.5.3)$$

As  $S(u^\varepsilon(t)) \tilde{\subset} \Gamma^\varepsilon(t)$  and  $u^\varepsilon(t) = g^\varepsilon(t)$   $\mathcal{L}^3$ -a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ , we have by (4.2.8) and (4.5.2)

$$\begin{aligned} \|u^\varepsilon(t)\|_{L^\infty(\Omega'; \mathbb{R}^3)} + \int_{\Omega'} \left| \left( \nabla_{\alpha} u^\varepsilon(t) \Big|_{\frac{1}{\varepsilon}} \nabla_3 u^\varepsilon(t) \right) \right|^p dx \\ + \int_{S(u^\varepsilon(t))} \left| \left( (\nu_{u^\varepsilon(t)})_{\alpha} \Big|_{\frac{1}{\varepsilon}} (\nu_{u^\varepsilon(t)})_3 \right) \right| d\mathcal{H}^2 \leq C, \end{aligned}$$

for some constant  $C > 0$  independent of  $\varepsilon > 0$  and  $t \in [0, T]$ . We insist, once again, on the fact that we do not try to justify the boundness assumption on  $u^\varepsilon(t)$ . In view of Lemma 4.2.2, there exists a further subsequence of  $\{\varepsilon_n\}$  (still denoted by  $\{\varepsilon_n\}$ ) and  $u(0) \in SBV^p(\omega'; \mathbb{R}^3)$  such that  $u^{\varepsilon_n}(0) \rightharpoonup u(0)$  in  $SBV^p(\Omega'; \mathbb{R}^3)$ . Moreover, as  $u^{\varepsilon_n}(0) = g^{\varepsilon_n}(0)$   $\mathcal{L}^3$ -a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ , from (4.2.7) we get  $u(0) = g(0)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$ . Thanks to condition (a) of Definition 4.4.1, we deduce that  $S(u(0)) \tilde{\subset} \gamma(0)$ .

We set for a.e.  $t \in [0, T]$ ,

$$\begin{cases} \theta_n(t) & := \int_{\Omega} \partial W \left( \nabla_{\alpha} u^{\varepsilon_n}(t) \Big| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(t) \right) \cdot \left( \nabla_{\alpha} g^{\varepsilon_n}(t) \Big| \frac{1}{\varepsilon_n} \nabla_3 g^{\varepsilon_n}(t) \right) dx, \\ \theta(t) & := \limsup_{n \rightarrow +\infty} \theta_n(t). \end{cases} \quad (4.5.4)$$

From (4.2.2), (4.2.4), (4.5.2),  $\theta \in L^1(0, T)$  and by virtue of Fatou's Lemma

$$\limsup_{n \rightarrow +\infty} \int_0^t \theta_n(s) ds \leq \int_0^t \theta(s) ds. \quad (4.5.5)$$

For a.e.  $t \in [0, T]$ , we extract a  $t$ -dependent subsequence  $\{n_t\}$  such that

$$\theta(t) = \lim_{n_t \rightarrow +\infty} \theta_{n_t}(t). \quad (4.5.6)$$

Lemma 4.2.2 implies that for every  $t \in (0, T]$ , upon extracting a further subsequence (not relabeled),  $u^{\varepsilon_{n_t}}(t) \rightharpoonup u(t)$  in  $SBV^p(\Omega'; \mathbb{R}^3)$  for some  $u(t) \in SBV^p(\omega'; \mathbb{R}^3)$ . Moreover, as  $u^{\varepsilon_{n_t}}(t) = g^{\varepsilon_{n_t}}(t)$   $\mathcal{L}^3$ -a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ , from (4.2.7), we get  $u(t) = g(t)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$ . By condition (a) of Definition 4.4.1, we get that  $S(u(t)) \tilde{\subset} \gamma(t)$ .  $\square$

### 4.5.2 Minimality property

For all  $t \in [0, T]$ , we define the limit energy by

$$\mathcal{E}(t) := 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(t)). \quad (4.5.7)$$

Our goal is to show that  $u(t)$  satisfies some minimality property inherited from that of  $u^{\varepsilon}(t)$ . We will distinguish the initial time from the subsequent times. At time  $t = 0$ , we will further show the convergence of the bulk and the surface energy to their two-dimensional counterpart respectively, for the subsequence  $\{\varepsilon_n\}$ . Concerning the next times, we will only be able to prove the convergence of the bulk energy toward its two-dimensional analogue, for the  $t$ -dependent subsequence  $\{\varepsilon_{n_t}\}$ . The convergence of the total energy, or equivalently of the surface energy, will be established later in Lemma 4.5.9 for a subsequence independent of the time.

**Lemma 4.5.2.** *At time  $t = 0$ ,  $u(0)$  minimizes*

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v)),$$

among  $\{v \in SBV^p(\omega'; \mathbb{R}^3) : v = g(0) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$ . Moreover,  $\gamma(0) \cong S(u(0))$  and we have

$$\begin{cases} \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(0) \Big| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(0) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(0)) dx_{\alpha}, \\ \int_{\Gamma^{\varepsilon_n}(0)} \left| \left( \nu_{\Gamma^{\varepsilon_n}(0)} \Big|_{\alpha} \frac{1}{\varepsilon_n} \left( \nu_{\Gamma^{\varepsilon_n}(0)} \Big|_3 \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(0)). \end{cases}$$

In particular,  $\mathcal{E}_{\varepsilon_n}(0) \rightarrow \mathcal{E}(0)$  and

$$\partial W \left( \nabla_\alpha u^{\varepsilon_n}(0) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(0) \right) \rightarrow (\partial(\mathcal{Q}W_0)(\nabla_\alpha u(0))|0) \quad \text{in } L^p(\Omega; \mathbb{R}^{3 \times 3}).$$

*Proof.* Let  $v \in SBVP(\omega'; \mathbb{R}^3)$  such that  $v = g(0)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$ . By virtue of Corollary 4.3.11, there exists a sequence  $\{w_n\} \subset SBVP(\Omega'; \mathbb{R}^3)$  satisfying  $w_n = g^{\varepsilon_n}(0)$   $\mathcal{L}^3$ -a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ ,  $w_n \rightarrow v$  in  $L^1(\Omega'; \mathbb{R}^3)$  and

$$\begin{aligned} & 2 \int_\omega \mathcal{Q}W_0(\nabla_\alpha v) dx_\alpha + 2\mathcal{H}^1(S(v)) \\ &= \lim_{n \rightarrow +\infty} \left[ \int_\Omega W \left( \nabla_\alpha w_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n \right) dx + \int_{S(w_n)} \left| \left( (\nu_{w_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{w_n})_3 \right) \right| d\mathcal{H}^2 \right]. \end{aligned} \quad (4.5.8)$$

Taking  $w_n$  as test function in the minimality condition for  $u^{\varepsilon_n}(0)$  we get

$$\begin{aligned} & \int_\Omega W \left( \nabla_\alpha u^{\varepsilon_n}(0) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(0) \right) dx + \int_{S(u^{\varepsilon_n}(0))} \left| \left( (\nu_{u^{\varepsilon_n}(0)})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{u^{\varepsilon_n}(0)})_3 \right) \right| d\mathcal{H}^2 \\ & \leq \int_\Omega W \left( \nabla_\alpha w_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n \right) dx + \int_{S(w_n)} \left| \left( (\nu_{w_n})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{w_n})_3 \right) \right| d\mathcal{H}^2. \end{aligned} \quad (4.5.9)$$

Remark 4.3.2 and Theorem 5.29 in [5] yield

$$\begin{aligned} 2 \int_\omega \mathcal{Q}W_0(\nabla_\alpha u(0)) dx_\alpha & \leq \liminf_{n \rightarrow +\infty} \int_\Omega \mathcal{Q}W_0(\nabla_\alpha u^{\varepsilon_n}(0)) dx \\ & \leq \liminf_{n \rightarrow +\infty} \int_\Omega W \left( \nabla_\alpha u^{\varepsilon_n}(0) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(0) \right) dx \end{aligned} \quad (4.5.10)$$

and thanks to (4.5.3) together with the fact that  $\Gamma^{\varepsilon_n}(0) \cong S(u^{\varepsilon_n}(0))$ ,

$$\begin{aligned} 2\mathcal{H}^1(\gamma(0)) & \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^2(\Gamma^{\varepsilon_n}(0)) \\ & = \liminf_{n \rightarrow +\infty} \mathcal{H}^2(S(u^{\varepsilon_n}(0))) \\ & \leq \liminf_{n \rightarrow +\infty} \int_{S(u^{\varepsilon_n}(0))} \left| \left( (\nu_{u^{\varepsilon_n}(0)})_\alpha \Big|_{\frac{1}{\varepsilon_n}} (\nu_{u^{\varepsilon_n}(0)})_3 \right) \right| d\mathcal{H}^2. \end{aligned} \quad (4.5.11)$$

Finally, from (4.5.8), (4.5.9), (4.5.10) and (4.5.11) we get by letting  $n \rightarrow +\infty$ ,

$$2 \int_\omega \mathcal{Q}W_0(\nabla_\alpha u(0)) dx_\alpha + 2\mathcal{H}^1(\gamma(0)) \leq 2 \int_\omega \mathcal{Q}W_0(\nabla_\alpha v) dx_\alpha + 2\mathcal{H}^1(S(v)).$$

Taking  $v = u(0)$  in the previous inequality, we observe that  $\mathcal{H}^1(\gamma(0)) \leq \mathcal{H}^1(S(u(0)))$ , which implies, as  $S(u(0)) \tilde{\subset} \gamma(0)$ , that  $S(u(0)) \cong \gamma(0)$ . It establishes the minimality property satisfied by  $u(0)$ . Taking still  $v = u(0)$ , (4.5.9) and (4.5.11) give

$$\limsup_{n \rightarrow +\infty} \int_\Omega W \left( \nabla_\alpha u^{\varepsilon_n}(0) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(0) \right) dx \leq 2 \int_\omega \mathcal{Q}W_0(\nabla_\alpha u(0)) dx_\alpha.$$

and this shows with (4.5.10) that

$$\int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(0) \middle| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(0) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(0)) dx_{\alpha}.$$

We report in (4.5.9) and obtain

$$\limsup_{n \rightarrow +\infty} \int_{S(u^{\varepsilon_n}(0))} \left| \left( (\nu_{u^{\varepsilon_n}(0)})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{u^{\varepsilon_n}(0)})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S(u(0))),$$

which implies together with (4.5.11) that

$$\int_{S(u^{\varepsilon_n}(0))} \left| \left( (\nu_{u^{\varepsilon_n}(0)})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{u^{\varepsilon_n}(0)})_3 \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(S(u(0))).$$

This yields that  $\mathcal{E}_{\varepsilon_n}(0) \rightarrow \mathcal{E}(0)$  and the convergence of the stresses follows from Lemma 4.4.8.  $\square$

**Remark 4.5.3.** It is immediate from the previous lemma that  $u(0)$  minimizes

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(0)),$$

among  $\{v \in SBV^p(\omega'; \mathbb{R}^3) : v = g(0) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$ .

**Remark 4.5.4.** Note that the previous result holds because we did not allow the body to contain a preexisting crack. Indeed, in this case, since the energy we are minimizing at the initial time is the same as the functional involved in the  $\Gamma$ -limit analysis, we can take as competitor in the minimization a recovery sequence. It permits us to show the convergence of the total energy at time  $t = 0$ ; that is essential if one is to prove that it still holds true at subsequent times in Lemma 4.5.8. If we had considered a body containing a preexisting crack, we would be unable to obtain such a convergence, but only a convergence of the bulk energy. Indeed, if  $\Gamma_0^{\varepsilon} \subset \bar{\omega} \times I$  denoted a preexisting (rescaled) crack with bounded scaled surface energy, then, according to the formulation in [46],  $(u_0^{\varepsilon}, \Gamma_0^{\varepsilon})$  would have to minimize

$$(v, \Gamma) \mapsto \int_{\Omega} W \left( \nabla_{\alpha} v \middle| \frac{1}{\varepsilon} \nabla_3 v \right) dx + \int_{\Gamma} \left| \left( (\nu_{\Gamma})_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_{\Gamma})_3 \right) \right| d\mathcal{H}^2$$

among every  $\mathcal{H}^2$ -rectifiable crack  $\Gamma \subset \bar{\omega} \times I$  with  $\Gamma_0^{\varepsilon} \tilde{\subset} \Gamma$ , and all deformation  $v \in SBV^p(\Omega'; \mathbb{R}^3)$  such that  $v = g^{\varepsilon}(t)$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$  and  $S(v) \tilde{\subset} \Gamma$ . In particular, setting  $\Gamma := \Gamma_0^{\varepsilon} \cup S(v)$  for all  $v \in SBV^p(\Omega'; \mathbb{R}^3)$  satisfying  $v = g^{\varepsilon}(0)$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ , we would get that  $u_0^{\varepsilon}$  must minimize

$$v \mapsto \int_{\Omega} W \left( \nabla_{\alpha} v \middle| \frac{1}{\varepsilon} \nabla_3 v \right) dx + \int_{S(v) \setminus \Gamma_0^{\varepsilon}} \left| \left( (\nu_v)_{\alpha} \middle| \frac{1}{\varepsilon} (\nu_v)_3 \right) \right| d\mathcal{H}^2$$

among such  $v$ 's. Hence, since by Theorem 3.15 in [46] (or Theorem 2.1 in [47])  $(u^{\varepsilon}(0), \Gamma^{\varepsilon}(0)) = (u_0^{\varepsilon}, \Gamma_0^{\varepsilon})$ , the argument used in the proof of Lemma 4.5.2 would not hold anymore. We would only be able to state, as in the following Lemma 4.5.5, the convergence of the bulk energy. Unfortunately, the convergence of the surface energy would then remain an open question.

We are now going to state a minimality property satisfied by  $u(t)$  for  $t \in (0, T]$ . The following result ensures the convergence of the three-dimensional bulk energy to its two-dimensional counterpart for a  $t$ -dependent subsequence. But the convergence of the total energy, or equivalently of the surface energy, cannot be established at this stage in a manner similar to that used in Lemma 4.5.2 at the initial time.

**Lemma 4.5.5.** *For every  $t \in (0, T]$ ,  $u(t)$  minimizes*

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)),$$

among  $\{v \in SBV^p(\omega'; \mathbb{R}^3) : v = g(t) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$ . Moreover, we have

$$\int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_{n_t}}(t) \Big|_{\frac{1}{\varepsilon_{n_t}}} \nabla_3 u^{\varepsilon_{n_t}}(t) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha}.$$

In particular,

$$\partial W \left( \nabla_{\alpha} u^{\varepsilon_{n_t}}(t) \Big|_{\frac{1}{\varepsilon_{n_t}}} \nabla_3 u^{\varepsilon_{n_t}}(t) \right) \rightarrow (\partial(\mathcal{Q}W_0)(\nabla_{\alpha}u(t)|0)) \quad \text{in } L^{p'}(\Omega; \mathbb{R}^{3 \times 3})$$

and thus, for a.e.  $t \in [0, T]$ ,

$$\theta(t) = 2 \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha}u(t)) \cdot \nabla_{\alpha} \dot{g}(t) dx_{\alpha}. \quad (4.5.12)$$

*Proof.* We first prove the minimality property. In this case, unlike Lemma 4.5.2, we cannot use a  $\Gamma$ -convergence argument because of the presence of an  $\varepsilon$ -dependent crack in the surface term. We will construct a minimizing sequence with the help of the Jump Transfer Theorem.

Let  $w \in SBV^p(\omega'; \mathbb{R}^3)$  such that  $w = g(t)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$ . Since  $\Gamma^{\varepsilon_n}(t)$  converges to  $\gamma(t)$  in the sense of Definition 4.4.1, from Theorem 4.4.6, there exists a sequence  $\{w_n\} \subset SBV^p(\Omega'; \mathbb{R}^3)$  satisfying  $w_n = w = g(t)$  a.e. on  $[\omega' \setminus \bar{\omega}] \times I$ ,  $w_n \rightarrow w$  in  $L^1(\Omega'; \mathbb{R}^3)$  and

$$\begin{cases} \left( \nabla_{\alpha} w_n \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n \right) \rightarrow (\nabla_{\alpha} w | 0) \text{ in } L^p(\Omega'; \mathbb{R}^{3 \times 3}), \\ \limsup_{n \rightarrow +\infty} \int_{S(w_n) \setminus \Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{w_n})_{\alpha} \Big|_{\frac{1}{\varepsilon_n}} (\nu_{w_n})_3 \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(S(w) \setminus \gamma(t)). \end{cases} \quad (4.5.13)$$

A measurable selection criterion (see e.g. [54]) together with the coercivity condition (4.2.1) imply the existence of  $z \in L^p(\omega; \mathbb{R}^3)$  such that  $W_0(\nabla_{\alpha}w) = W(\nabla_{\alpha}w|z)$   $\mathcal{L}^2$ -a.e. in  $\omega$ . By density, there exists a sequence  $z_j \in C_c^{\infty}(\Omega; \mathbb{R}^3)$  such that  $z_j \rightarrow H(t) - z$  in  $L^p(\Omega; \mathbb{R}^3)$  where  $H(t)$  is defined in (4.2.7). Denoting by  $b_j := \int_{-1}^{x_3} z_j(\cdot, s) ds$ , we take  $w_n + g^{\varepsilon_n}(t) - g(t) - \varepsilon_n b_j$  as test function in (4.2.5) and we get

$$\begin{aligned} & \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(t) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(t) \right) dx \\ & \leq \int_{\Omega} W \left( \nabla_{\alpha} w_n + \nabla_{\alpha} g^{\varepsilon_n}(t) - \nabla_{\alpha} g(t) - \varepsilon_n \nabla_{\alpha} b_j \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 w_n + \frac{1}{\varepsilon_n} \nabla_3 g^{\varepsilon_n}(t) - z_j \right) dx \\ & \quad + \int_{S(w_n) \setminus \Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{w_n})_{\alpha} \Big|_{\frac{1}{\varepsilon_n}} (\nu_{w_n})_3 \right) \right| d\mathcal{H}^2. \end{aligned}$$

We replace  $n$  by  $n_t$  (see Lemma 4.5.1) and pass to the limit when  $n_t$  tends to  $+\infty$ . In view of Theorem 5.29 in [5] and of Remark 4.3.2,

$$\begin{aligned} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} &\leq \liminf_{n_t \rightarrow +\infty} \int_{\Omega} \mathcal{Q}W_0(\nabla_{\alpha}u^{\varepsilon_{n_t}}(t)) dx \\ &\leq \liminf_{n_t \rightarrow +\infty} \int_{\Omega} W \left( \nabla_{\alpha}u^{\varepsilon_{n_t}}(t) \middle| \frac{1}{\varepsilon_{n_t}} \nabla_3 u^{\varepsilon_{n_t}}(t) \right) dx. \end{aligned} \quad (4.5.14)$$

Thus, using (4.2.7) and (4.5.13) in the right hand side, we get

$$2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} \leq \int_{\Omega} W(\nabla_{\alpha}w|H(t) - z_j) dx + 2\mathcal{H}^1(S(w) \setminus \gamma(t)).$$

Passing to the limit when  $j \rightarrow +\infty$  we obtain

$$\begin{aligned} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} &\leq \int_{\Omega} W(\nabla_{\alpha}w|z) dx + 2\mathcal{H}^1(S(w) \setminus \gamma(t)) \\ &= 2 \int_{\omega} W_0(\nabla_{\alpha}w) dx_{\alpha} + 2\mathcal{H}^1(S(w) \setminus \gamma(t)). \end{aligned}$$

We would like to replace  $W_0$  by its quasiconvexification in the previous relation. To this end, we use a relaxation argument. First of all, we approach  $\gamma(t)$  from inside by a compact set, so as to work on an open subset of  $\omega$ . This is possible because, since  $\mathcal{H}^1(\gamma(t)) < +\infty$ , then  $\mathcal{H}^1_{\lfloor \gamma(t)}$  is a Radon measure. Thus, for any  $\eta > 0$ , there exists a compact set  $K_t^{\eta} \subset \gamma(t)$  such that  $\mathcal{H}^1(\gamma(t) \setminus K_t^{\eta}) \leq \eta$ . In particular,

$$2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} \leq 2 \int_{\omega} W_0(\nabla_{\alpha}w) dx_{\alpha} + 2\mathcal{H}^1(S(w) \setminus K_t^{\eta}). \quad (4.5.15)$$

Let  $v \in SBVP(\omega'; \mathbb{R}^3)$  satisfying  $v = g(t)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$ . In view of Theorem 8.1 together with Remark 8.2 in [31] and arguing as in the proof of Lemma 4.3.10 and Corollary 4.3.11, it is easily deduced that there exists a sequence  $\{w_k\} \subset SBVP(\omega'; \mathbb{R}^3)$  such that  $w_k \rightarrow v$  in  $L^1(\omega'; \mathbb{R}^3)$ ,  $w_k = g(t)$   $\mathcal{L}^2$ -a.e. on  $\omega' \setminus \bar{\omega}$  and

$$\int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}v) dx_{\alpha} + \mathcal{H}^1(S(v) \setminus K_t^{\eta}) = \lim_{k \rightarrow +\infty} \left[ \int_{\omega} W_0(\nabla_{\alpha}w_k) dx_{\alpha} + \mathcal{H}^1(S(w_k) \setminus K_t^{\eta}) \right].$$

In (4.5.15), we replace  $w$  by  $w_k$  and we pass to the limit when  $k \rightarrow +\infty$ ; we get

$$\begin{aligned} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} &\leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus K_t^{\eta}) \\ &\leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)) + 2\eta. \end{aligned}$$

The minimality property follows after letting  $\eta \rightarrow 0$ .

Concerning the convergence of the bulk energy, the previous calculation with  $v = u(t)$  and the fact that  $S(u(t)) \tilde{\subset} \gamma(t)$  yield

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(t) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(t) \right) dx \leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha}. \quad (4.5.16)$$

Note that (4.5.16) holds for the sequence  $\{\varepsilon_n\}$  which is independent of the time. Thus, from (4.5.14), we deduce that

$$\int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_{n_t}}(t) \Big|_{\frac{1}{\varepsilon_{n_t}}} \nabla_3 u^{\varepsilon_{n_t}}(t) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha}.$$

In particular, Lemma 4.4.8 implies the convergence of the stresses and thanks to (4.2.7), (4.5.4) and (4.5.6), we have for a.e.  $t \in [0, T]$ ,

$$\begin{aligned} \theta(t) &= \lim_{n_t \rightarrow +\infty} \int_{\Omega} \partial W \left( \nabla_{\alpha} u^{\varepsilon_{n_t}}(t) \Big|_{\frac{1}{\varepsilon_{n_t}}} \nabla_3 u^{\varepsilon_{n_t}}(t) \right) \cdot \left( \nabla_{\alpha} \dot{g}^{\varepsilon_{n_t}}(t) \Big|_{\frac{1}{\varepsilon_{n_t}}} \nabla_3 \dot{g}^{\varepsilon_{n_t}}(t) \right) dx \\ &= \int_{\Omega} (\partial(\mathcal{Q}W_0)(\nabla_{\alpha} u(t))|0) \cdot (\nabla_{\alpha} \dot{g}(t)|\dot{H}(t)) dx \\ &= 2 \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha} u(t)) \cdot \nabla_{\alpha} \dot{g}(t) dx_{\alpha}. \end{aligned}$$

□

**Remark 4.5.6.** According to Remark 4.5.3 and Lemma 4.5.5, for every  $t \in [0, T]$ , the function  $u(t)$  minimizes

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)), \quad (4.5.17)$$

among  $\{v \in SBVP(\omega'; \mathbb{R}^3) : v = g(t) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$ . Equivalently, the pair  $(u(t), \gamma(t))$  satisfies the following unilateral minimality property :

$$2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(t)) \leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(\gamma'), \quad (4.5.18)$$

for every  $\mathcal{H}^1$ -rectifiable set  $\gamma' \subset \bar{\omega}$  such that  $\gamma(t) \tilde{\subset} \gamma'$  and every  $v \in SBVP(\omega'; \mathbb{R}^3)$  satisfying  $v = g(t)$  a.e. on  $\omega' \setminus \bar{\omega}$  and  $S(v) \tilde{\subset} \gamma'$ . Indeed, for such pairs  $(v, \gamma')$ , from (4.5.17) we get that

$$\begin{aligned} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha} &\leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)) \\ &\leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(\gamma' \setminus \gamma(t)) \\ &= 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(\gamma') - 2\mathcal{H}^1(\gamma(t)) \end{aligned}$$

where the second inequality holds since  $S(v) \tilde{\subset} \gamma'$  and the last equality because  $\gamma(t) \tilde{\subset} \gamma'$ . On the other hand, (4.5.17) follows from (4.5.18) by taking  $\gamma' := S(v) \cup \gamma(t)$ .

### 4.5.3 Energy conservation

The last step in proving that  $(u(t), \gamma(t))$  is a quasistatic evolution relative to the boundary data  $g(t)$  consists in showing that the two-dimensional total energy  $\mathcal{E}(t)$  defined in (4.5.7) is absolutely continuous in time. This is the aim of Lemmas 4.5.7 and 4.5.8 that follow.

**Lemma 4.5.7.** *For every  $t \in [0, T]$ ,*

$$\mathcal{E}(t) \geq \mathcal{E}(0) + 2 \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha}u(\tau)) \cdot \nabla_{\alpha}\dot{g}(\tau) dx_{\alpha} d\tau.$$

*Proof.* We proceed as in [47] by approximation of the Lebesgue integral by Riemann sums. Let  $s < t$ , at time  $s$  we test the minimality of  $u(s)$  against  $u(t) + g(s) - g(t)$ . By Lemma 4.5.5,

$$2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(s)) dx_{\alpha} \leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t) + \nabla_{\alpha}g(s) - \nabla_{\alpha}g(t)) dx_{\alpha} + 2\mathcal{H}^1(S(u(t)) \setminus \gamma(s)).$$

Thus, since  $S(u(t)) \tilde{\subset} \gamma(t)$  and  $\gamma(s) \subset \gamma(t)$ ,

$$\begin{aligned} \mathcal{E}(s) &= 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(s)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(s)) \\ &\leq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t) + \nabla_{\alpha}g(s) - \nabla_{\alpha}g(t)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(t)) \\ &= 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t) + \nabla_{\alpha}g(s) - \nabla_{\alpha}g(t)) dx_{\alpha} - 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} + \mathcal{E}(t). \end{aligned}$$

It implies that for some  $\rho(s, t) \in [0, 1]$ ,

$$\mathcal{E}(t) - \mathcal{E}(s) \geq 2 \int_{\omega} \left[ \partial(\mathcal{Q}W_0) \left( \nabla_{\alpha}u(t) + \rho(s, t) \int_s^t \nabla_{\alpha}\dot{g}(\tau) d\tau \right) \cdot \int_s^t \nabla_{\alpha}\dot{g}(\tau) d\tau \right] dx. \quad (4.5.19)$$

Fix  $t \in [0, T]$ , thanks to Lemma 4.12 in [46], there exists a subdivision  $0 \leq s_0^n \leq s_1^n \leq \dots \leq s_{k(n)}^n = t$  such that

$$\lim_{n \rightarrow +\infty} \sup_{1 \leq i \leq k(n)} (s_i^n - s_{i-1}^n) = 0$$

and

$$\begin{cases} \lim_{n \rightarrow +\infty} \sum_{i=1}^{k(n)} \left\| (s_i^n - s_{i-1}^n) \nabla_{\alpha}\dot{g}(s_i^n) - \int_{s_{i-1}^n}^{s_i^n} \nabla_{\alpha}\dot{g}(\tau) d\tau \right\|_{L^p(\omega; \mathbb{R}^{3 \times 2})} = 0 \\ \lim_{n \rightarrow +\infty} \sum_{i=1}^{k(n)} \left| (s_i^n - s_{i-1}^n) \theta(s_i^n) - \int_{s_{i-1}^n}^{s_i^n} \theta(\tau) d\tau \right| = 0. \end{cases} \quad (4.5.20)$$

For all  $s \in (s_i^n, s_{i+1}^n]$ , we define

$$u_n(s) := u(s_{i+1}^n), \quad \text{and} \quad \Psi_n(s) := \rho(s_i^n, s_{i+1}^n) \int_{s_i^n}^{s_{i+1}^n} \nabla_{\alpha}\dot{g}(\tau) d\tau.$$



As  $\nabla_\alpha \dot{g} \in L^1(0, T; L^p(\omega'; \mathbb{R}^{3 \times 2}))$ , we have

$$\|\Psi_n(s)\|_{L^p(\omega'; \mathbb{R}^{3 \times 2})} \rightarrow 0, \quad (4.5.21)$$

uniformly with respect to  $s \in [0, t]$ . In (4.5.19), we replace  $s$  by  $s_i^n$  and  $t$  by  $s_{i+1}^n$ , then a summation for  $i = 0$  to  $k(n) - 1$  yields

$$\mathcal{E}(t) - \mathcal{E}(0) \geq 2 \int_0^t \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u_n(\tau) + \Psi_n(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha d\tau.$$

From (4.5.21) and Lemma 4.9 in [46], we have for a.e.  $\tau \in (0, t)$ ,

$$\left| \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u_n(\tau) + \Psi_n(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha - \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u_n(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha \right| \rightarrow 0.$$

Thus, according to (4.2.2) together with Lebesgue's Dominated Convergence Theorem,

$$\int_0^t \left| \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u_n(\tau) + \Psi_n(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha - \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u_n(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha \right| d\tau \rightarrow 0.$$

Thus,

$$\mathcal{E}(t) - \mathcal{E}(0) \geq \limsup_{n \rightarrow +\infty} 2 \int_0^t \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u_n(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha d\tau.$$

But in view of (4.5.20), (4.2.2) and Hölder's inequality,

$$\begin{aligned} & \sum_{i=1}^{k(n)} \left| \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u(s_i^n)) \cdot \left( (s_i^n - s_{i-1}^n) \nabla_\alpha \dot{g}(s_i^n) - \int_{s_{i-1}^n}^{s_i^n} \nabla_\alpha \dot{g}(\tau) d\tau \right) dx_\alpha \right| \\ & \leq C \left( 1 + \|\nabla_\alpha u\|_{L^\infty(0, t; L^p(\omega; \mathbb{R}^{3 \times 2}))}^{p-1} \right) \sum_{i=1}^{k(n)} \left\| (s_i^n - s_{i-1}^n) \nabla_\alpha \dot{g}(s_i^n) - \int_{s_{i-1}^n}^{s_i^n} \nabla_\alpha \dot{g}(\tau) d\tau \right\|_{L^p(\omega; \mathbb{R}^{3 \times 2})} \\ & \rightarrow 0, \end{aligned}$$

thus, using again (4.5.20) and (4.5.12),

$$\begin{aligned} \mathcal{E}(t) - \mathcal{E}(0) & \geq 2 \limsup_{n \rightarrow +\infty} \sum_{i=1}^{k(n)} (s_i^n - s_{i-1}^n) \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u(s_i^n)) \cdot \nabla_\alpha \dot{g}(s_i^n) dx_\alpha \\ & = 2 \int_0^t \int_\omega \partial(\mathcal{Q}W_0)(\nabla_\alpha u(\tau)) \cdot \nabla_\alpha \dot{g}(\tau) dx_\alpha d\tau. \end{aligned}$$

□

It now remains to show that the inequality proved in Lemma 4.5.7 is actually an equality. This is the object of the following Lemma.

**Lemma 4.5.8.** *For every  $t \in [0, T]$ ,*

$$\mathcal{E}(t) \leq \mathcal{E}(0) + 2 \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha}u(\tau)) \cdot \nabla_{\alpha}\dot{g}(\tau) dx_{\alpha} d\tau.$$

*Proof.* According to (4.5.3) and (4.5.14),

$$\begin{aligned} \liminf_{n_t \rightarrow +\infty} \mathcal{E}_{\varepsilon_{n_t}}(t) &= \liminf_{n_t \rightarrow +\infty} \left[ \int_{\Omega} W \left( \nabla_{\alpha}u^{\varepsilon_{n_t}}(t) \middle| \frac{1}{\varepsilon_{n_t}} \nabla_3 u^{\varepsilon_{n_t}}(t) \right) dx \right. \\ &\quad \left. + \int_{\Gamma^{\varepsilon_{n_t}}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon_{n_t}}(t)})_{\alpha} \middle| \frac{1}{\varepsilon_{n_t}} (\nu_{\Gamma^{\varepsilon_{n_t}}(t)})_3 \right) \right| d\mathcal{H}^2 \right] \\ &\geq 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha} + 2\mathcal{H}^1(\gamma(t)) \\ &= \mathcal{E}(t). \end{aligned} \tag{4.5.22}$$

On the other hand, by Lemma 4.5.2, (4.5.4), (4.5.5) and (4.5.12) we have

$$\begin{aligned} \limsup_{n_t \rightarrow +\infty} \mathcal{E}_{\varepsilon_{n_t}}(t) &\leq \limsup_{n \rightarrow +\infty} \mathcal{E}_{\varepsilon_n}(t) \\ &\leq \lim_{n \rightarrow +\infty} \mathcal{E}_{\varepsilon_n}(0) + \limsup_{n \rightarrow +\infty} \int_0^t \theta_n(\tau) d\tau \\ &= \mathcal{E}(0) + \int_0^t \theta(\tau) d\tau \\ &= \mathcal{E}(0) + \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha}u(\tau)) \cdot \nabla_{\alpha}\dot{g}(\tau) dx_{\alpha} d\tau. \end{aligned} \tag{4.5.23}$$

Accordingly, relations (4.5.22) and (4.5.23) complete the proof of the Lemma.  $\square$

By virtue of Lemmas 4.5.7 and 4.5.8, the two-dimensional total energy  $\mathcal{E}(t)$  is absolutely continuous with respect to the time  $t$  and

$$\mathcal{E}(t) = \mathcal{E}(0) + 2 \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha}u(\tau)) \cdot \nabla_{\alpha}\dot{g}(\tau) dx_{\alpha} d\tau$$

hence,  $(u(t), \gamma(t))$  is a quasistatic evolution relative to the boundary data  $g(t)$ . Let us show now that the three-dimensional bulk and surface energies are converging towards the two-dimensional bulk and surface energies respectively. Note that the following convergence result holds for a subsequence  $\{\varepsilon_n\}$  independent of  $t$  unlike in Lemma 4.5.5 where we stated the convergence of the volume energy for a  $t$ -dependent subsequence  $\{\varepsilon_{n_t}\}$ .

**Lemma 4.5.9.** *For every  $t \in [0, T]$ ,*

$$\begin{cases} \int_{\Omega} W \left( \nabla_{\alpha}u^{\varepsilon_n}(t) \middle| \frac{1}{\varepsilon_n} \nabla_3 u^{\varepsilon_n}(t) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha}u(t)) dx_{\alpha}, \\ \int_{\Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon_n}(t)})_{\alpha} \middle| \frac{1}{\varepsilon_n} (\nu_{\Gamma^{\varepsilon_n}(t)})_3 \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(t)). \end{cases}$$

In particular,  $\mathcal{E}_{\varepsilon_n}(t) \rightarrow \mathcal{E}(t)$ .

*Proof.* For  $t = 0$ , the result is already proved in Lemma 4.5.2. Assume now that  $t \in (0, T]$  and let  $\{n_j\}$  be a  $t$ -dependent subsequence such that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(t) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(t) \right) dx \\ = \lim_{j \rightarrow +\infty} \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_{n_j}}(t) \Big|_{\frac{1}{\varepsilon_{n_j}}} \nabla_3 u^{\varepsilon_{n_j}}(t) \right) dx. \end{aligned} \quad (4.5.24)$$

Arguing as in the proofs of Lemmas 4.5.1 and 4.5.5, we can suppose that, for a subsequence of  $n_j$  (still denoted by  $n_j$ ),  $u^{\varepsilon_{n_j}}(t) \rightarrow u^*(t)$  in  $SBVP(\Omega'; \mathbb{R}^3)$  for some  $u^*(t) \in SBVP(\omega'; \mathbb{R}^3)$  with  $u^*(t) = g(t)$  a.e. on  $\omega' \setminus \bar{\omega}$ ,  $S(u^*(t)) \tilde{\subset} \gamma(t)$ , and which is also a minimizer of

$$v \mapsto 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} v) dx_{\alpha} + 2\mathcal{H}^1(S(v) \setminus \gamma(t)),$$

among  $\{v \in SBVP(\omega'; \mathbb{R}^3) : v = g(t) \text{ a.e. on } \omega' \setminus \bar{\omega}\}$ . Hence,

$$\int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^*(t)) dx_{\alpha} = \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha}. \quad (4.5.25)$$

According to Remark 4.3.2 and Theorem 5.29 in [5],

$$\begin{aligned} 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u^*(t)) dx_{\alpha} &\leq \liminf_{j \rightarrow +\infty} \int_{\Omega} \mathcal{Q}W_0(\nabla_{\alpha} u^{\varepsilon_{n_j}}(t)) dx \\ &\leq \lim_{j \rightarrow +\infty} \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_{n_j}}(t) \Big|_{\frac{1}{\varepsilon_{n_j}}} \nabla_3 u^{\varepsilon_{n_j}}(t) \right) dx. \end{aligned}$$

Thus, (4.5.24) and (4.5.25) imply that

$$2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha} \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(t) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(t) \right) dx,$$

which ensure together with (4.5.16) the convergence of the bulk energy i.e.

$$\int_{\Omega} W \left( \nabla_{\alpha} u^{\varepsilon_n}(t) \Big|_{\frac{1}{\varepsilon_n}} \nabla_3 u^{\varepsilon_n}(t) \right) dx \rightarrow 2 \int_{\omega} \mathcal{Q}W_0(\nabla_{\alpha} u(t)) dx_{\alpha}. \quad (4.5.26)$$

But in view of (4.2.6), (4.5.5) and Lemmas 4.5.2 and 4.5.5,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathcal{E}_{\varepsilon_n}(t) &= \limsup_{n \rightarrow +\infty} \mathcal{E}_{\varepsilon_n}(0) + \limsup_{n \rightarrow +\infty} \int_0^t \theta_n(\tau) d\tau \\ &\leq \mathcal{E}(0) + \int_0^t \theta(\tau) d\tau \\ &= \mathcal{E}(0) + 2 \int_0^t \int_{\omega} \partial(\mathcal{Q}W_0)(\nabla_{\alpha} u(\tau)) \cdot \nabla_{\alpha} \dot{g}(\tau) dx_{\alpha} d\tau \\ &= \mathcal{E}(t). \end{aligned} \quad (4.5.27)$$

Thus (4.5.26) and (4.5.27) yield

$$\limsup_{n \rightarrow +\infty} \int_{\Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon_n}(t)})_\alpha \left| \frac{1}{\varepsilon_n} (\nu_{\Gamma^{\varepsilon_n}(t)})_3 \right. \right) \right| d\mathcal{H}^2 \leq 2\mathcal{H}^1(\gamma(t))$$

which, together with (4.5.3), gives the convergence of the surface term

$$\int_{\Gamma^{\varepsilon_n}(t)} \left| \left( (\nu_{\Gamma^{\varepsilon_n}(t)})_\alpha \left| \frac{1}{\varepsilon_n} (\nu_{\Gamma^{\varepsilon_n}(t)})_3 \right. \right) \right| d\mathcal{H}^2 \rightarrow 2\mathcal{H}^1(\gamma(t)).$$

□

## 4.6 Appendix

For the convenience of the reader, we propose an alternative proof of Proposition 4.4.7. We feel it interesting because the same one could be used to show that, in the Integral Representation Theorem, Theorem 2.5 in [35] and at least in the homogeneous case, the integrand of the  $\Gamma$ -limit is of class  $\mathcal{C}^1$ .

*Proof of Proposition 4.4.7.* The proof is divided into two steps. We first prove that  $\mathcal{Q}W_0$  admits partial derivatives and then we show that they are continuous which is sufficient to state that  $\mathcal{Q}W_0$  is a  $\mathcal{C}^1$  function.

**Step 1 :  $\mathcal{Q}W_0$  admits partial derivatives.** Let  $\bar{F}, \bar{G} \in \mathbb{R}^{3 \times 2}$  and  $h_n \searrow 0^+$ . According to Theorem 2 in [74], there exists a sequence  $\{u_\varepsilon\} \subset W^{1,p}(Q' \times I; \mathbb{R}^3)$  such that  $u_\varepsilon \rightarrow 0$  in  $L^p(Q' \times I; \mathbb{R}^3)$  and

$$2\mathcal{Q}W_0(\bar{F}) = \lim_{\varepsilon \rightarrow 0} \int_{Q' \times I} W \left( \bar{F} + \nabla_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right. \right) dx. \quad (4.6.1)$$

The coercivity condition (4.2.1) implies that the sequence  $\{(\bar{F} + \nabla_\alpha u_\varepsilon | \frac{1}{\varepsilon} \nabla_3 u_\varepsilon)\}$  is bounded in  $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$  uniformly with respect to  $\varepsilon > 0$ . Thus, Hölder's inequality together with (4.2.2) imply that, for a subsequence,  $\{\partial W(\bar{F} + \nabla_\alpha u_\varepsilon | \frac{1}{\varepsilon} \nabla_3 u_\varepsilon)\}$  is weakly convergent in  $L^{p'}(Q' \times I; \mathbb{R}^{3 \times 3})$ . In particular, there exists  $\xi^* \in \mathbb{R}^{3 \times 3}$  such that

$$\int_{Q' \times I} \partial W \left( \bar{F} + \nabla_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right. \right) dx \rightarrow \xi^*. \quad (4.6.2)$$

Since  $\Phi \mapsto \int_{Q' \times I} W(\Phi) dx$  is a  $\mathcal{C}^1$ -map from  $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$  into  $\mathbb{R}$  with differential given by  $\Psi \mapsto \int_{Q' \times I} \partial W(\Phi) \cdot \Psi dx$ , we get using the  $\Gamma$ -lim inf inequality and (4.6.1)

$$\begin{aligned} & \frac{\mathcal{Q}W_0(\bar{F} + h_n \bar{G}) - \mathcal{Q}W_0(\bar{F})}{h_n} \\ & \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{h_n} \int_{Q' \times I} \left[ W \left( \bar{F} + h_n \bar{G} + \nabla_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right. \right) - W \left( \bar{F} + \nabla_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right. \right) \right] dx \\ & \leq \liminf_{\varepsilon \rightarrow 0} \int_{Q' \times I} \partial W \left( \bar{F} + t_n^\varepsilon \bar{G} + \nabla_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right. \right) dx \cdot (\bar{G}|0), \end{aligned}$$

for some  $t_n^\varepsilon \in [0, h_n]$ . Consider  $\varepsilon_n > 0$ , with  $\varepsilon_n \xrightarrow{n} 0$ , such that

$$\frac{\mathcal{Q}W_0(\overline{F} + h_n \overline{G}) - \mathcal{Q}W_0(\overline{F})}{h_n} - \frac{1}{n} \leq \int_{Q' \times I} \partial W \left( \overline{F} + t_n^{\varepsilon_n} \overline{G} + \nabla_\alpha u_{\varepsilon_n} \Big| \frac{1}{\varepsilon_n} \nabla_3 u_{\varepsilon_n} \right) dx \cdot (\overline{G}|0).$$

Thus, from Lemma 4.9 in [46] and (4.6.2), we get

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{\mathcal{Q}W_0(\overline{F} + h_n \overline{G}) - \mathcal{Q}W_0(\overline{F})}{h_n} & \\ & \leq \limsup_{n \rightarrow +\infty} \int_{Q' \times I} \partial W \left( \overline{F} + t_n^{\varepsilon_n} \overline{G} + \nabla_\alpha u_{\varepsilon_n} \Big| \frac{1}{\varepsilon_n} \nabla_3 u_{\varepsilon_n} \right) dx \cdot (\overline{G}|0) \\ & = \limsup_{n \rightarrow +\infty} \int_{Q' \times I} \partial W \left( \overline{F} + \nabla_\alpha u_{\varepsilon_n} \Big| \frac{1}{\varepsilon_n} \nabla_3 u_{\varepsilon_n} \right) dx \cdot (\overline{G}|0). \\ & = \xi^* \cdot (\overline{G}|0). \end{aligned}$$

Let  $i \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$ , we choose  $\overline{G} = t e_i \otimes e_j$ , where  $\{e_1, e_2, e_3\}$  denotes the canonical basis of  $\mathbb{R}^3$  and  $t \in \mathbb{R}$ . Then

$$\limsup_{n \rightarrow +\infty} \frac{\mathcal{Q}W_0(\overline{F} + h_n t e_i \otimes e_j) - \mathcal{Q}W_0(\overline{F})}{h_n} \leq t \xi_{ij}^*. \quad (4.6.3)$$

Since  $\mathcal{Q}W_0$  is quasiconvex, it is rank one convex (see [44]) thus  $t \mapsto f_{ij}(t) := \mathcal{Q}W_0(\overline{F} + t e_i \otimes e_j)$  is convex. Let  $F_{ij}^* \in \partial f_{ij}(0)$  (the subdifferential of  $f_{ij}$  in 0), we have

$$f_{ij}(h_n t) \geq f_{ij}(0) + F_{ij}^* h_n t$$

hence

$$t F_{ij}^* \leq \liminf_{n \rightarrow +\infty} \frac{\mathcal{Q}W_0(\overline{F} + h_n t e_i \otimes e_j) - \mathcal{Q}W_0(\overline{F})}{h_n} \quad (4.6.4)$$

In view of (4.6.3) and (4.6.4), we deduce that

$$F_{ij}^* = \xi_{ij}^* = \lim_{n \rightarrow +\infty} \frac{\mathcal{Q}W_0(\overline{F} + h_n e_i \otimes e_j) - \mathcal{Q}W_0(\overline{F})}{h_n}.$$

Thus, the subdifferential of  $f_{ij}$  in 0 is reduced to a point, which shows that  $\mathcal{Q}W_0$  admits partial derivatives, denoted by  $\partial_{ij}(\mathcal{Q}W_0)$ ; in particular, applying the same argument to another recovery sequence, we deduce that the value of  $\xi_{ij}^*$  does not depend upon the sequence  $\{u_\varepsilon\}$  satisfying (4.6.1). From this and from (4.6.2), we conclude that, for any sequence  $\{u_\varepsilon\}$  satisfying (4.6.1),

$$\partial_{ij}(\mathcal{Q}W_0)(\overline{F}) = \lim_{\varepsilon \rightarrow 0} \int_{Q' \times I} \partial_{ij} W \left( \overline{F} + \nabla_\alpha u_\varepsilon \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon \right) dx, \quad (4.6.5)$$

where  $\partial_{ij} W$  are the partial derivatives of  $W$ .

**Step 2 : The partial derivatives of  $\mathcal{Q}W_0$  are continuous.** Let  $\bar{F}_n \rightarrow \bar{F}$ , our aim is to show that  $\partial_{ij}(\mathcal{Q}W_0)(\bar{F}_n) \rightarrow \partial_{ij}(\mathcal{Q}W_0)(\bar{F})$ . To this end, we will construct a common recovery sequence for the computation of the  $\Gamma$ -limit at both  $w$  and  $w_n$ , where  $w(x) := \bar{F} \cdot x_\alpha$  and  $w_n(x) := \bar{F}_n \cdot x_\alpha$ . Thanks to Theorem 2 in [74], there exists a sequence  $\{u_\varepsilon^n\} \subset W^{1,p}(Q' \times I; \mathbb{R}^3)$  such that  $u_\varepsilon^n \rightarrow 0$  in  $L^p(Q' \times I; \mathbb{R}^3)$  as  $\varepsilon \rightarrow 0$  and

$$\mathcal{Q}W_0(\bar{F}_n) = \lim_{\varepsilon \rightarrow 0} \int_{Q' \times I} W \left( \bar{F}_n + \nabla_\alpha u_\varepsilon^n \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon^n \right) dx.$$

The continuity of  $\mathcal{Q}W_0$  yield that

$$\mathcal{Q}W_0(\bar{F}) = \lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{Q' \times I} W \left( \bar{F}_n + \nabla_\alpha u_\varepsilon^n \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon^n \right) dx.$$

By virtue of the coercivity condition (4.2.1), the sequence  $\{(\nabla_\alpha u_\varepsilon^n \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon^n)\}$  is bounded in  $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$  uniformly with respect to  $n \in \mathbb{N}$  and  $\varepsilon > 0$ . Furthermore, from (4.6.5), we get

$$\partial_{ij}(\mathcal{Q}W_0)(\bar{F}_n) = \lim_{\varepsilon \rightarrow 0} \int_{Q' \times I} \partial_{ij} W \left( \bar{F}_n + \nabla_\alpha u_\varepsilon^n \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon^n \right) dx.$$

It implies, according to (4.2.2) that for a (not relabeled) subsequence, the limit of  $\partial_{ij}(\mathcal{Q}W_0)(\bar{F}_n)$  exists as  $n \rightarrow +\infty$  and thus

$$\lim_{n \rightarrow +\infty} \partial_{ij}(\mathcal{Q}W_0)(\bar{F}_n) = \lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{Q' \times I} \partial_{ij} W \left( \bar{F}_n + \nabla_\alpha u_\varepsilon^n \Big| \frac{1}{\varepsilon} \nabla_3 u_\varepsilon^n \right) dx.$$

By a standard diagonalization process, we may find a sequence  $\{\varepsilon_n\} \searrow 0^+$  such that letting  $v_n := u_{\varepsilon_n}^n$ , then  $v_n \rightarrow 0$  in  $L^p(Q' \times I; \mathbb{R}^3)$ ,

$$\mathcal{Q}W_0(\bar{F}) = \lim_{n \rightarrow +\infty} \int_{Q' \times I} W \left( \bar{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx \quad (4.6.6)$$

and

$$\lim_{n \rightarrow +\infty} \partial_{ij}(\mathcal{Q}W_0)(\bar{F}_n) = \lim_{n \rightarrow +\infty} \int_{Q' \times I} \partial_{ij} W \left( \bar{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx. \quad (4.6.7)$$

Let us show that  $\{v_n\}$  is a recovery sequence for the computation of the  $\Gamma$ -limit at  $w$ . We need to prove that we can replace  $\bar{F}_n$  by  $\bar{F}$  in (4.6.6). To this end, we are going to use a uniform continuity argument. Since  $W$  is continuous, it is uniformly continuous on any compact subset of  $\mathbb{R}^{3 \times 3}$ . We thus restrict ourselves to those  $x$ 's such that the scaled gradient of  $v_n$  lives in a ball of  $\mathbb{R}^{3 \times 3}$  of arbitrarily large radius. Indeed, let  $M > 0$ , we define

$$A_M := \left\{ x \in Q' \times I : \left| \left( \bar{F}_n + \nabla_\alpha v_n(x) \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n(x) \right) \right| \leq M \right\}. \quad (4.6.8)$$

Chebyshev's inequality yield that  $\mathcal{L}^3([Q' \times I] \setminus A_M) \leq C/M^p$ , where  $C > 0$  is a constant independant of  $n$ . Moreover, according to Theorem 1.1 in [25], there is no loss of generality to

assume that the sequence of scaled gradients  $\{|\left(\nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n\right)^p\}$  is equi-integrable. Hence by (4.2.1), we have that

$$\begin{aligned} & \left| \int_{[Q' \times I] \setminus A_M} W \left( \bar{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx \right| \\ & \leq \beta \int_{[Q' \times I] \setminus A_M} \left( 1 + \left| \left( \bar{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) \right|^p \right) dx \xrightarrow{M \rightarrow +\infty} 0 \end{aligned} \quad (4.6.9)$$

uniformly with respect to  $n \in \mathbb{N}$ , then

$$\mathcal{Q}W_0(\bar{F}) = \lim_{M \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{2} \int_{A_M} W \left( \bar{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx.$$

Let us choose  $n$  large enough (independently of  $M$ ) such that  $|\bar{F}_n - \bar{F}| \ll 1$ . Denoting by  $\omega_M : [0, +\infty) \rightarrow [0, +\infty)$  the modulus of continuity of  $W$  on  $\bar{B}(0, M+1)$ , we get for every  $M > 0$ ,

$$\int_{A_M} \left| W \left( \bar{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) - W \left( \bar{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) \right| dx \leq 2\omega_M(|\bar{F}_n - \bar{F}|) \xrightarrow{n \rightarrow +\infty} 0.$$

Consequently, by the same argument than for (4.6.9), we obtain that

$$\begin{aligned} \mathcal{Q}W_0(\bar{F}) &= \lim_{M \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{2} \int_{A_M} W \left( \bar{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx \\ &= \lim_{n \rightarrow +\infty} \int_{Q' \times I} W \left( \bar{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx, \end{aligned}$$

hence, thanks to (4.6.5),

$$\partial_{ij}(\mathcal{Q}W_0)(\bar{F}) = \lim_{n \rightarrow +\infty} \int_{Q' \times I} \partial_{ij} W \left( \bar{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx.$$

To show the continuity of  $\partial_{ij}(\mathcal{Q}W_0)$ , it is enough to show that we can replace  $\bar{F}$  by  $\bar{F}_n$  in the previous equality and then, relation (4.6.7) will complete the proof of the Lemma. Since

$$\begin{aligned} & \left| \int_{[Q' \times I] \setminus A_M} \partial_{ij} W \left( \bar{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx \right| \\ & \leq \beta \int_{[Q' \times I] \setminus A_M} \left( 1 + \left| \left( \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) \right|^{p-1} \right) dx \\ & \leq \frac{C\beta}{M^p} + \frac{C^{1/p}}{M} \left\| \left( \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) \right\|_{L^p(Q' \times I; \mathbb{R}^3 \times \mathbb{R}^3)}^{p-1} \xrightarrow{M \rightarrow +\infty} 0 \end{aligned}$$

uniformly with respect to  $n \in \mathbb{N}$ , then

$$\partial_{ij}(\mathcal{Q}W_0)(\bar{F}) = \lim_{M \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{2} \int_{A_M} \partial_{ij} W \left( \bar{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx.$$

We are going to use the same uniform continuity argument than before to  $\partial_{ij}W$ . Indeed, as  $\partial_{ij}W$  is continuous, it is also uniformly continuous on  $\overline{B}(0, M+1)$  for all  $M > 0$  and

$$\int_{A_M} \left| \partial_{ij}W \left( \overline{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) - \partial_{ij}W \left( \overline{F} + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) \right| dx \leq 2\omega^M(|\overline{F}_n - \overline{F}|) \xrightarrow{n \rightarrow +\infty} 0,$$

where  $\omega^M : [0, +\infty) \rightarrow [0, +\infty)$  denotes the modulus of continuity of  $\partial_{ij}W$  on  $\overline{B}(0, M+1)$ . Hence, we get that

$$\partial_{ij}(\mathcal{Q}W_0)(\overline{F}) = \lim_{M \rightarrow +\infty} \lim_{n \rightarrow +\infty} \frac{1}{2} \int_{A_M} \partial_{ij}W \left( \overline{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx$$

and arguing as above,

$$\partial_{ij}(\mathcal{Q}W_0)(\overline{F}) = \lim_{n \rightarrow +\infty} \int_{Q' \times I} \partial_{ij}W \left( \overline{F}_n + \nabla_\alpha v_n \Big| \frac{1}{\varepsilon_n} \nabla_3 v_n \right) dx.$$

According to (4.6.7)

$$\partial_{ij}(\mathcal{Q}W_0)(\overline{F}) = \lim_{n \rightarrow +\infty} \partial_{ij}(\mathcal{Q}W_0)(\overline{F}_n),$$

which shows the continuity of the partial derivatives of  $\mathcal{Q}W_0$ . □





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