A Logical Investigation of Interaction Systems

Une investigation logique des systèmes d'interaction

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PhD Defense, December 12th 2005

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We use a notion of abstract games...

Games also give a computational interpretation to "topology"...

Where are we?

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- the first player is called the Angel;
- the second is called the Demon;
- moves alternate: first the Angel, then the Demon etc.

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We write $G_1 \leq G_2$ and say "G_2 simulates G_1".

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- potential executions of a program:
  - the Angel is the user,
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- a specification for a sequential / interactive program.

This was the starting intuition... (*cf.* Peter Hancock)

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no matter what the Demon plays...

(In particular, the Angel always has a move to play!)

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(Equivalently, an interaction system is a coalgebra for the monad $\mathcal{F}^2$ of "doubly iterated families" over the category **Set**.)

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$$\big(n_1(s_1, a_1 \quad), n_2(s_2, a_2 \quad)\big) \in R$$

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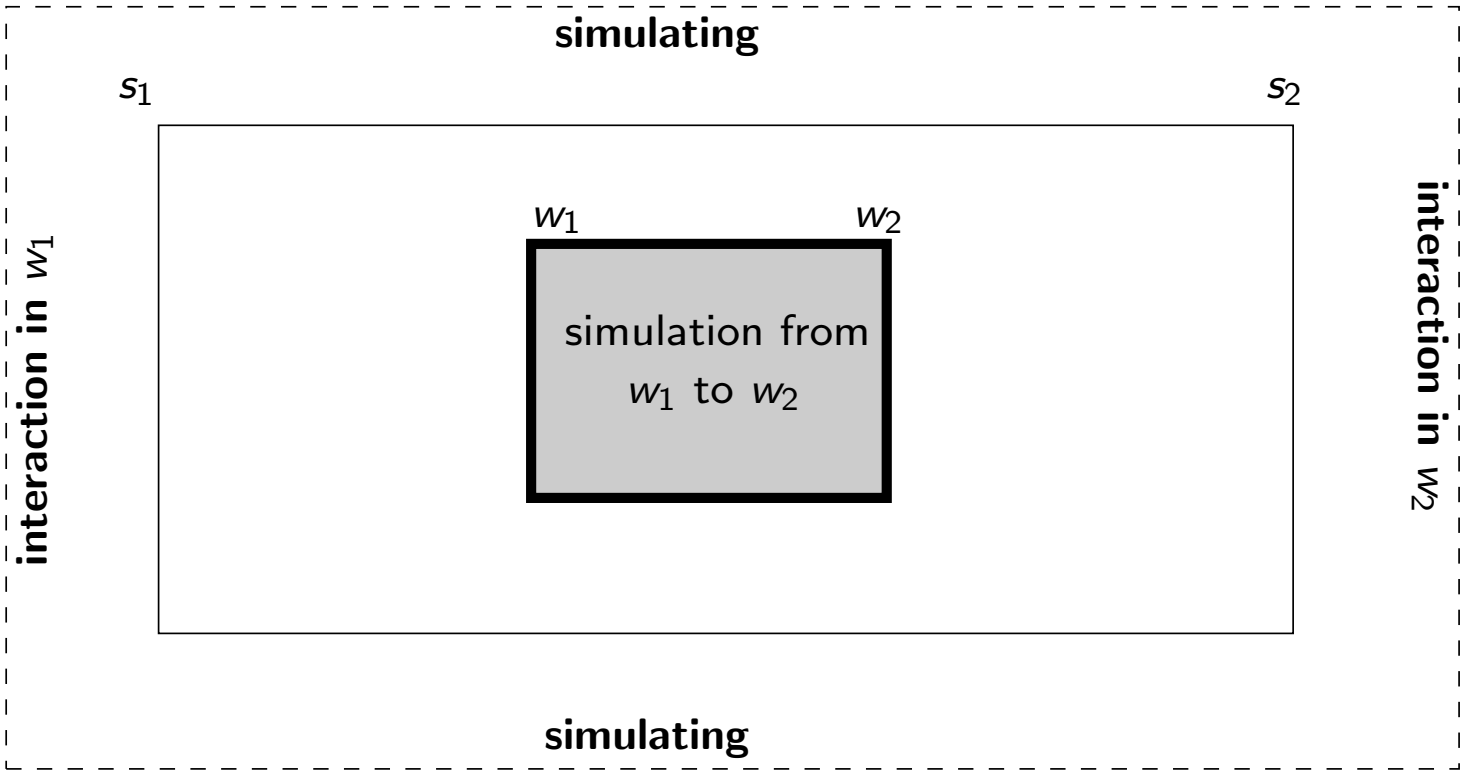
(This is **not** a morphism of coalgebras...)

Simulation, Visually



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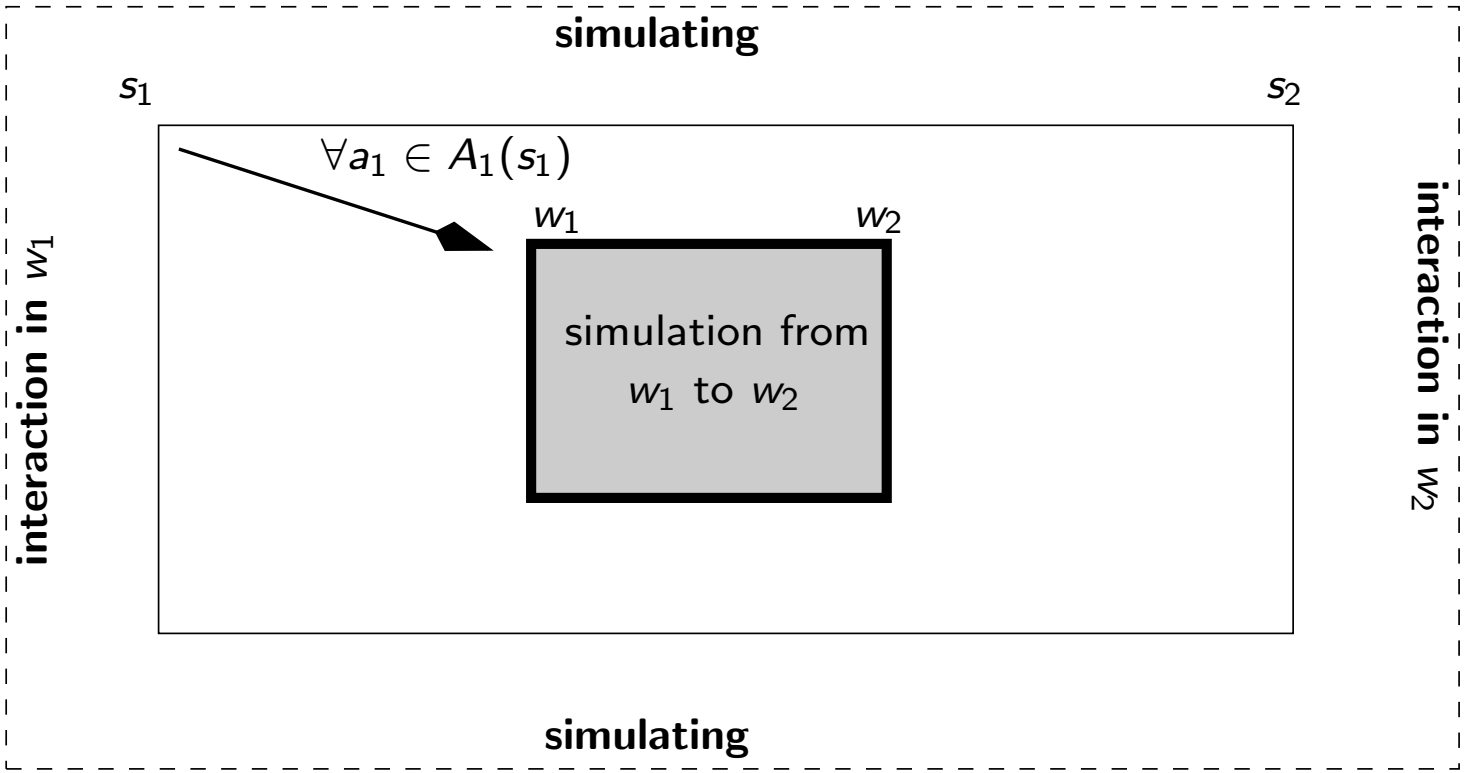

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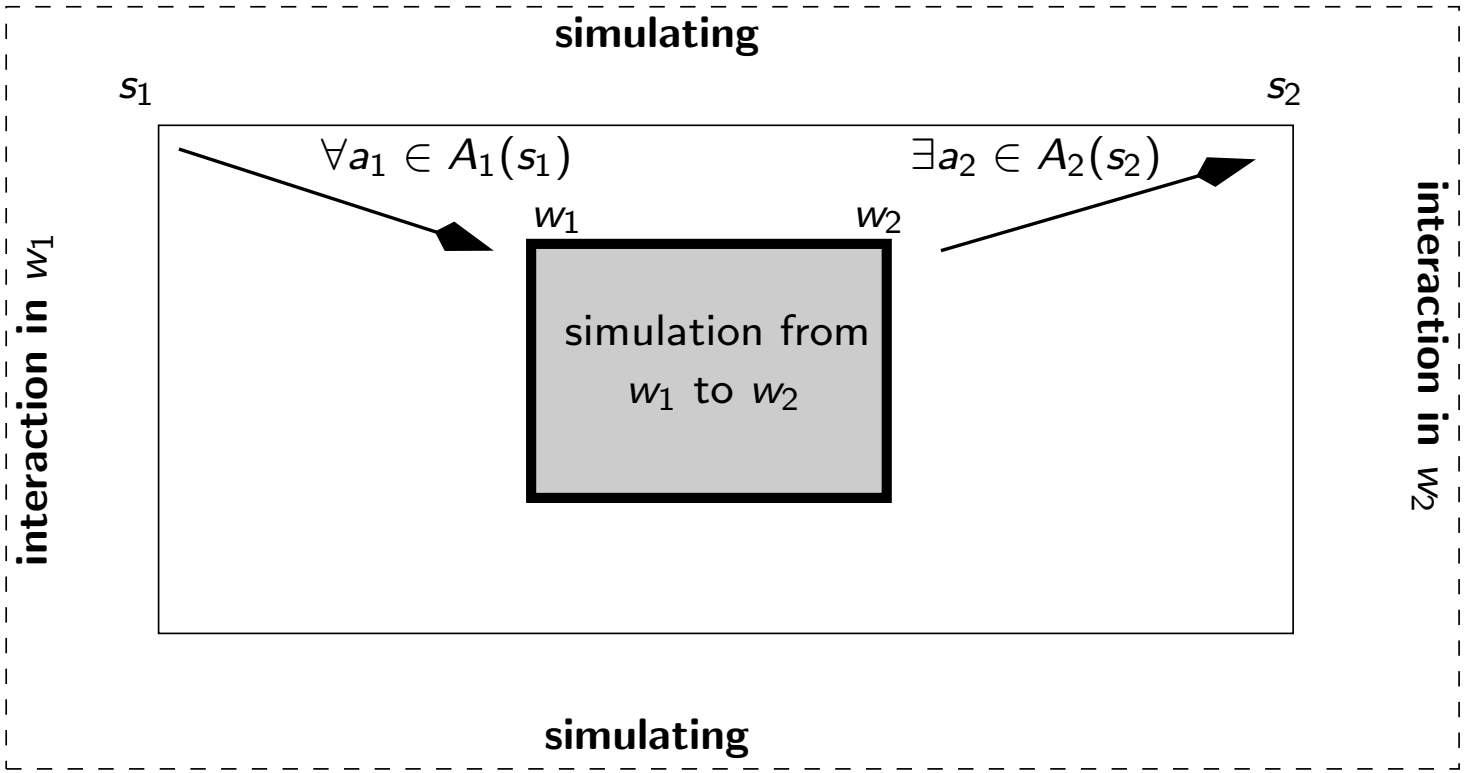

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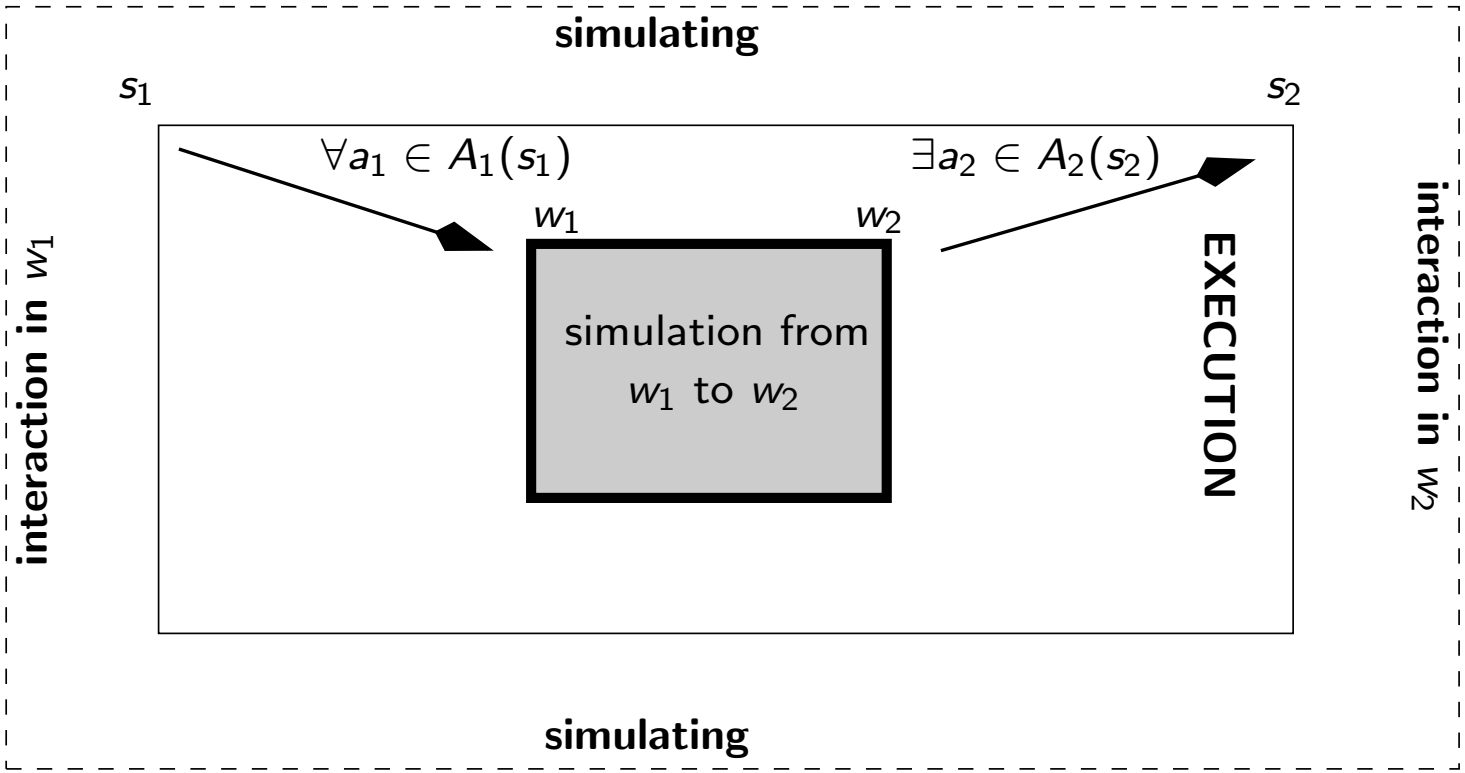

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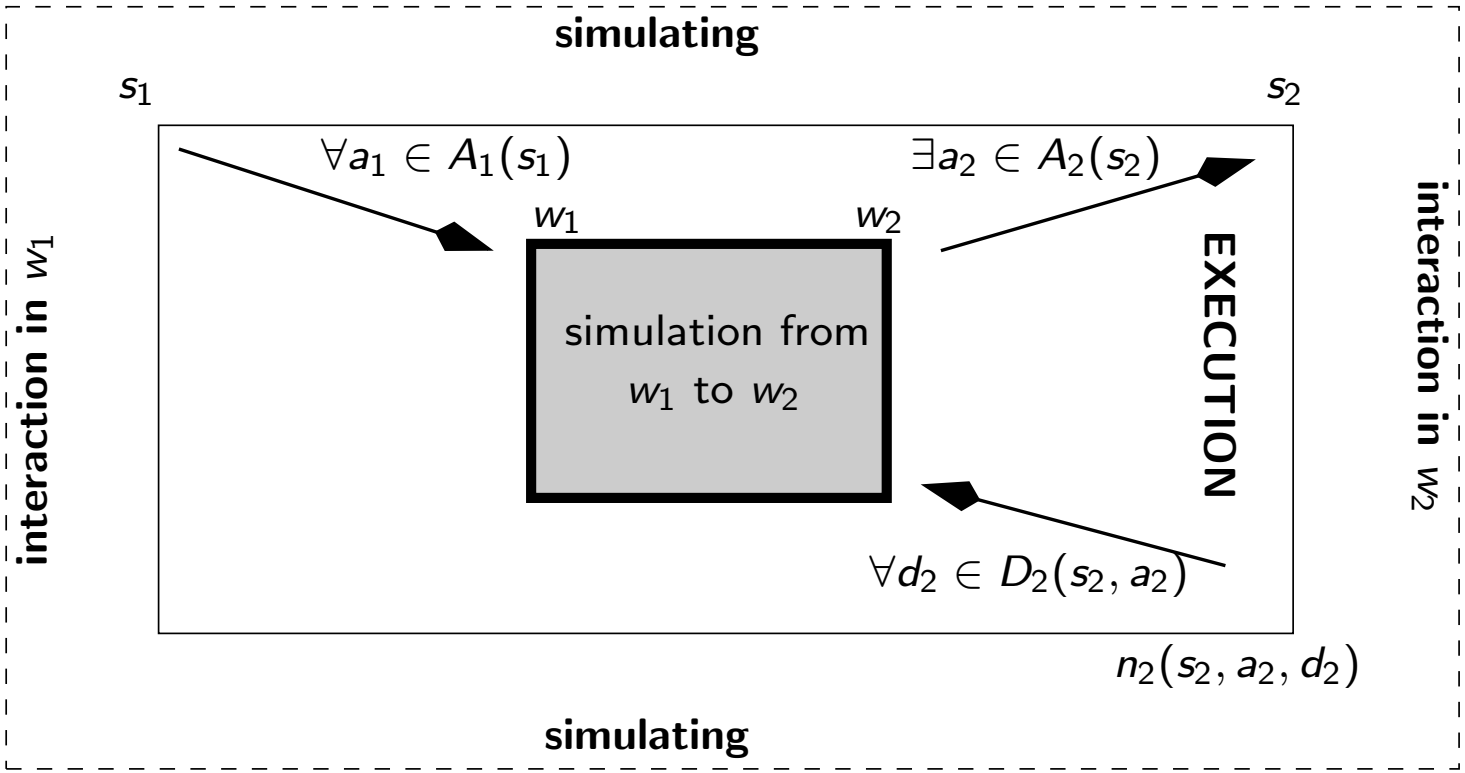

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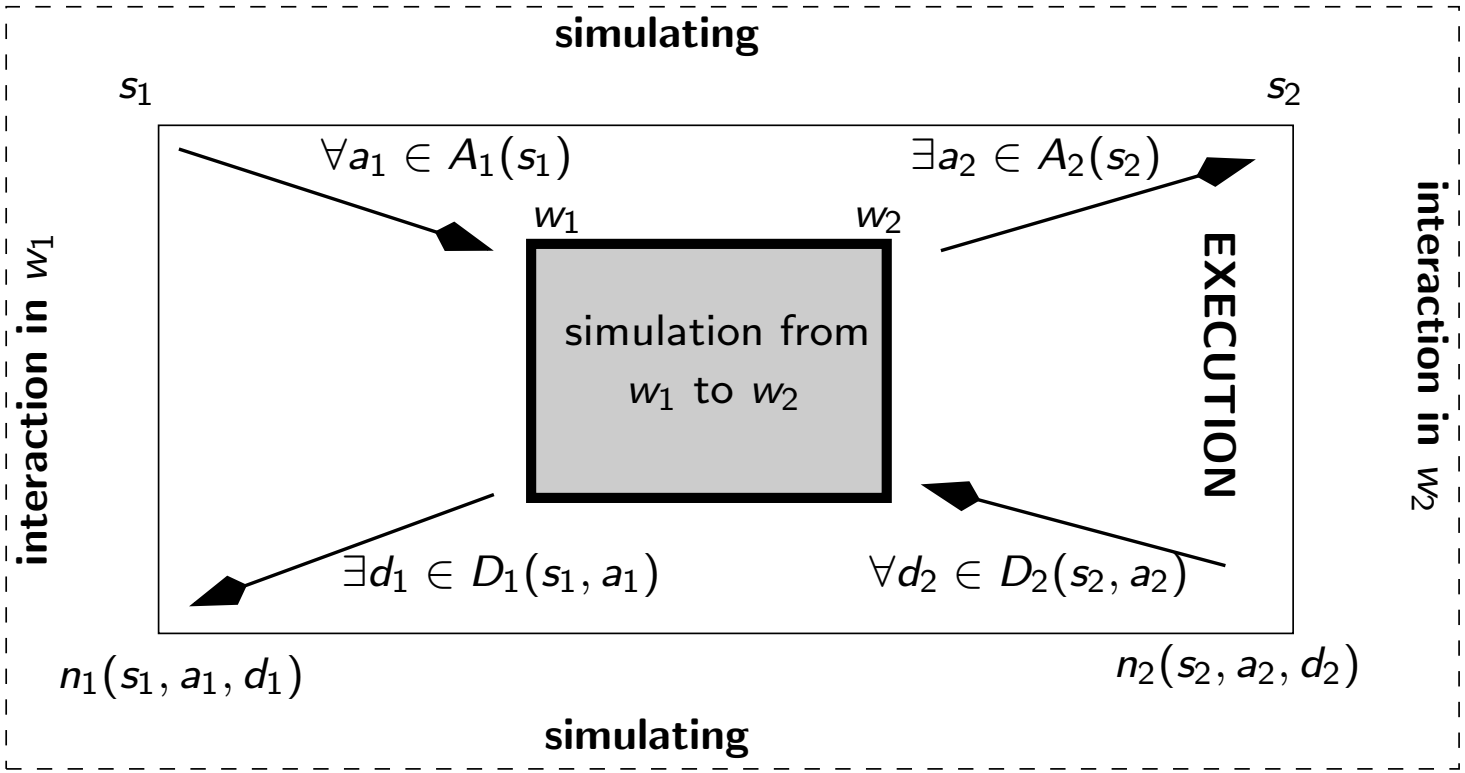

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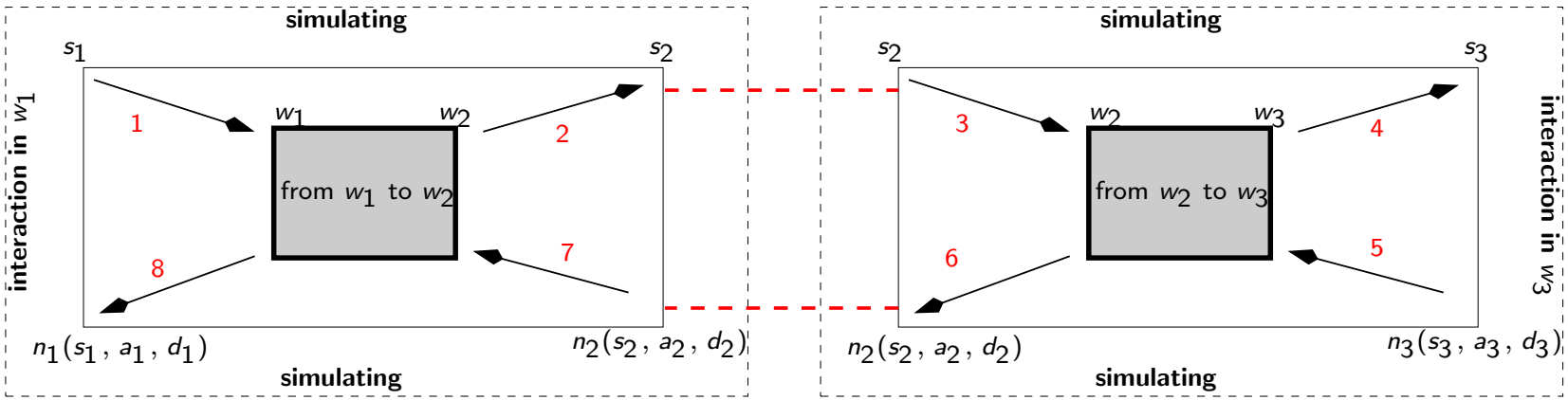

(flow of interaction)

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- an Angel action in w^* is a strategy to play several times in w;
- a Demon reaction is a sequence of responses.

This operation satisfies w^* is "least" s.t. $w^* \simeq \mathsf{skip} \cup w; w^*$,

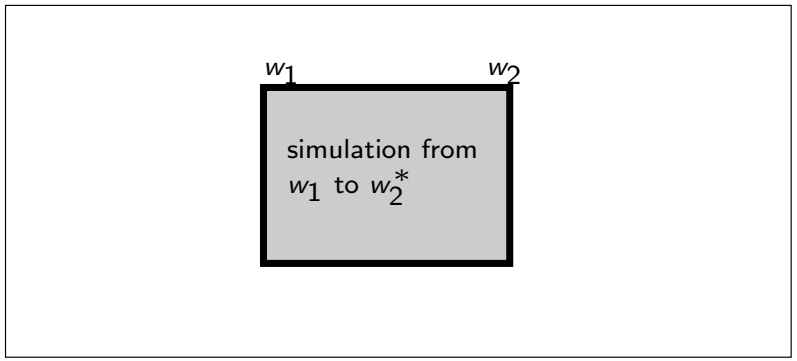
where $w_1; w_2$ is the game

"one move in w_1 and then one move in w_2."

Programming Interpretation

A simulation from w_1 to w_2^* is a relation R s.t. if $(s_1, s_2) \in R$:

- for every action from s_1, there is a "list" of actions from s_2;
- s.t. for any "list" of reactions, there is a reaction;
- s.t. the simulation can be sustained from the new states.



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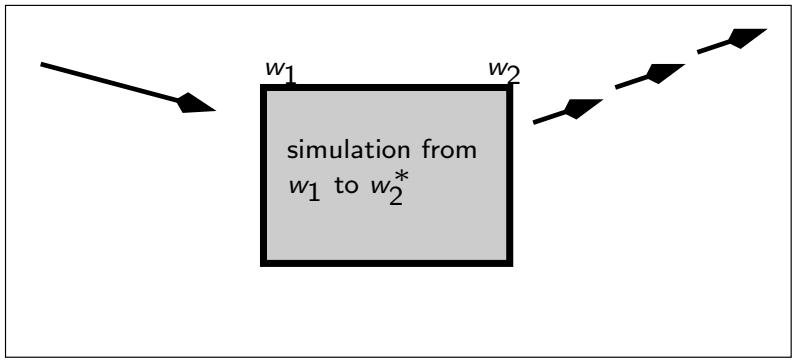

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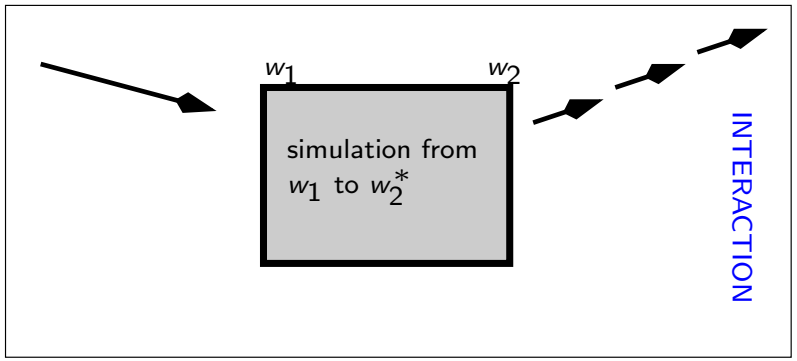

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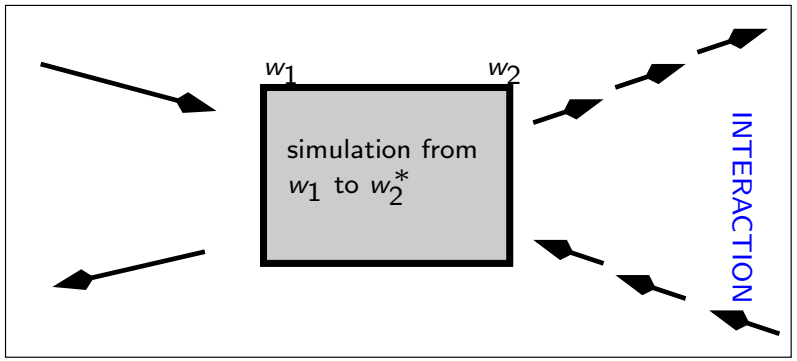

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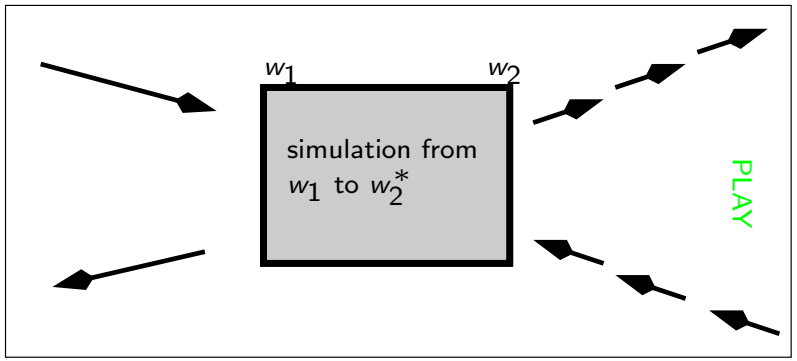

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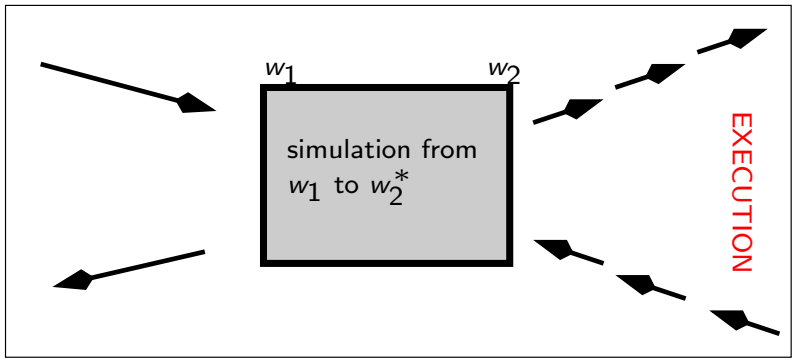

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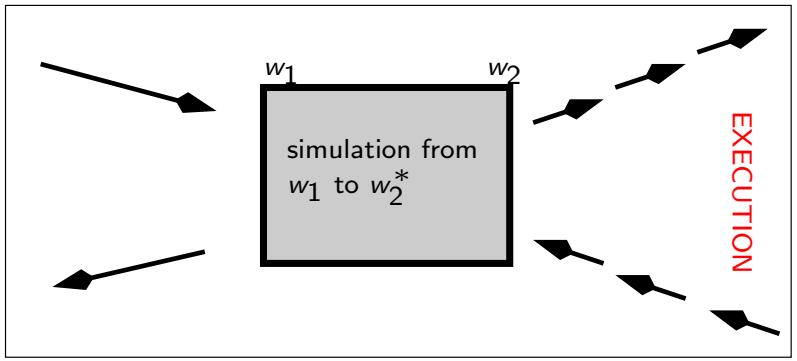

This is just a program implementing w_1 in terms of w_2!

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- some simple operations look complicated.

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$$\begin{array}{rcccl} w^\circ & : & \mathcal{P}(S) & \to & \mathcal{P}(S) \\ & & U & \mapsto & \left\{ s \mid (\exists a)(\forall d)\, n(s,a,d) \in U \right\} \end{array}$$

i.e. $s \in w^\circ(U)$ iff "the Angel can reach U from s in exactly one interaction".

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This defines an equivalence of categories **PT** $\simeq$ **Sim**!

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This gives a self-dual symmetric monoidal category.

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Those correspond to concrete operations on interaction systems...

Monoidal closure

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The adjoint to $\otimes$ is given by

$$P_1 \multimap P_2 : \mathcal{P}(S_1 \times S_2) \to \mathcal{P}(S_1 \times S_2)$$

Monoidal closure

We can extend this to a self-dual monoidal closed category.

The adjoint to $\otimes$ is given by

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with

$$(s_1, s_2) \in \big(P_1 \multimap P_2\big)(r)$$
$$\text{iff}$$
$$(\forall x_1 \subseteq S_1)\, s_1 \in P_1(x_1) \Rightarrow s_2 \in P_2\big(r(x_1)\big)$$

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A proof/term becomes a safety property,

i.e. a subset $x \subseteq S$ s.t. $x \subseteq P(x)$.

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We get a simple, non-trivial model for the differential λ-calculus!

Where are we?

Achievements

- a new category of games and simulations;
- an intuitive/informal model for "real-life" programming;
- giving a computational interpretation of "basic topologies";
- concrete example of interaction system to give a (complete) topological semantics to "linear geometric theories";
- this category is a denotational model for full linear logic;
- and the differential (typed) λ-calculus;
- which can extended to second order.

Future Work

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- generalize in the spirit of "containers";
- study concrete example of interfaces.

...

Et voilà !