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Scattering theory for Dirac fields in various spacetimes of the General Relativity

Thierry Daude

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THÈSE

présentée à

L'UNIVERSITÉ BORDEAUX I

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par Thierry DAUDÉ

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Soutenue le : 17 décembre 2004

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*À la mémoire de mon père,
à tous les miens.*

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Introduction

Les résultats présentés dans cette thèse concernent l'étude de la propagation d'ondes dans des géométries de type trou noir. Le concept de trou noir fut pensé pour la première fois au XVIIIème siècle dans le cadre de la mécanique newtonienne par Mitchell puis Laplace : intuitivement, il s'agit d'objets célestes extrêmement denses, exerçant une attraction gravitationnelle si forte sur les objets à leur surface, que même la lumière ne peut s'en échapper. De tels objets, lorsqu'ils sont vus de l'extérieur par des observateurs lointains, apparaissent alors forcément "noirs". Selon la théorie Newtonienne, l'existence de trous noirs nécessitait néanmoins, une concentration de matière exceptionnelle, jamais observée ni même imaginée, si bien que l'idée fut rapidement abandonnée. Il fallut attendre la découverte des lois de la Relativité Générale par Einstein en 1915 et ses premières prédictions pour que cette idée reprenne corps, mais dans un langage différent.

A la base des théories relativistes, il y a le constat suivant : la vitesse de la lumière semble être une limite supérieure à toutes les vitesses observées par l'expérience. Einstein érigea cette observation en postulat de sa nouvelle théorie. Les implications en sont profondes : les notions d'espace et de temps, pensées à la manière classique, sont en fait étroitement reliées et ne forment qu'une seule entité, *l'espace-temps*, fondamentalement quadri-dimensionnel. La géométrie de cet espace-temps, c'est à dire la façon de mesurer les distances entre deux "événements" différents, est d'un type nouveau par rapport à la traditionnelle géométrie euclidienne utilisée dans la mécanique classique de Newton : la géométrie qui décrit un espace-temps relativiste est désormais lorentzienne.

Commençons par l'exemple le plus simple d'espace-temps relativiste : l'espace-temps plat ou espace-temps de Minkowski. Le cadre de travail est alors une variété \mathcal{M} à 4 dimensions qui, dans un système de coordonnées bien choisi, est égal à

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_{x_1} \times \mathbb{R}_{x_2} \times \mathbb{R}_{x_3},$$

et la géométrie de cet espace-temps est caractérisée par la métrique lorentzienne

$$\eta = dt^2 - dx_1^2 - dx_2^2 - dx_3^2. \quad (1)$$

La métrique (1) a pour signature $(+ - - -)$. En conséquence, tout vecteur V de l'espace tangent $T_p\mathcal{M}$ au point $p \in \mathcal{M}$ entre dans l'une des trois catégories suivantes :

- temporel si $\eta(V, V) > 0$,
- isotrope ou nul si $\eta(V, V) = 0$,
- spatial si $\eta(V, V) < 0$.

En chaque point $p \in \mathcal{M}$, on appelle *cône d'onde* ou *cône de lumière* l'ensemble des vecteurs isotropes en ce point. Cette structure est la structure fondamentale de toute géométrie lorentzienne. Elle permet de définir les trajectoires physiquement admissibles pour tout ce qui vit dans l'espace-temps. Si l'on considère par exemple une particule et que l'on note $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ sa trajectoire, aussi appelée ligne

d'événements, alors le vecteur $\dot{\gamma}(s)$, tangent à cette trajectoire, doit être temporel ou isotrope pour tout $s \in \mathbb{R}$, c'est à dire que les vecteurs tangents doivent se trouver à l'intérieur des cônes de lumière en chaque point de la trajectoire. Ceci est la traduction mathématique du fait qu'aucune vitesse observée ne peut être supérieure à celle de la lumière.

Comment rendre compte des effets de la gravitation dans ce cadre relativiste ? L'idée d'Einstein fut de considérer la force de gravitation non plus comme une force au sens commun, agissant sur les objets massifs, mais comme une manifestation de la courbure de l'espace-temps. Plus précisément, l'espace-temps de la théorie de la Relativité Générale est alors décrit comme un couple (\mathcal{M}, g) où \mathcal{M} est une variété différentielle de dimension égale à 4 et g une métrique lorentzienne sur \mathcal{M} , *i.e.* une forme bilinéaire symétrique réelle de signature $(+ - - -)$.

La métrique g définit toujours une structure de cône d'onde en chaque point de l'espace-temps mais maintenant, ces cônes peuvent être déformés, plus ouverts, plus aigus, plus inclinés ou moins symétriques, selon les points où l'on se trouve. Les objets physiques se déplacent sur les trajectoires dont les vecteurs tangents se trouvent à l'intérieur des cônes de lumières en chaque point de la trajectoire. Plus généralement, les cônes de lumière déterminent la façon dont les ondes se propagent. Les directions isotropes, c'est à dire les vecteurs isotropes, correspondent aux directions caractéristiques des équations d'onde covariantes.

A l'origine de la courbure de l'espace-temps, il y a la distribution de matière, ou de manière équivalente d'énergie, présente dans l'univers. Ce lien est traduit par les équations d'Einstein :

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (2)$$

où $G_{\mu\nu}$ représente le tenseur d'Einstein (formé à partir du tenseur de courbure, lui-même formé à partir de la métrique et de ses dérivées premières et secondes) et $T_{\mu\nu}$ est le tenseur énergie-impulsion. Le membre de gauche décrit la géométrie de l'espace-temps, par exemple indique la forme des cônes de lumière en chaque point, alors que le membre de droite décrit la distribution d'énergie dans l'univers. Les équations (2) forment un système couplé d'équations aux dérivées partielles non linéaires et indiquent que la géométrie et la distribution d'énergie de l'espace-temps interagissent de manière complexe.

A cause de leur non-linéarité, trouver une solution aux équations d'Einstein dans toute leur généralité semble hors de portée, même si l'on suppose l'univers vide de toute forme d'énergie, *i.e.* $T_{\mu\nu} = 0$ dans (2). Aussi, ce fut avec étonnement qu'Einstein reçut et communiqua à la communauté scientifique, peu après sa propre découverte des fondements de la Relativité Générale, la première solution exacte de ses équations, distincte de l'espace-temps de Minkowski : une solution obtenue par K. Schwarzschild en 1916 et qui fournit maintenant le modèle le plus simple de trou noir¹.

La solution de Schwarzschild décrit un espace-temps à symétrie sphérique (situation idéalisée), solution des équations d'Einstein dans le vide. En coordonnées de Schwarzschild (t, r, θ, φ) , cet espace-temps est donné par

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r^+ \times S_{\theta, \varphi}^2,$$

muni de la métrique lorentzienne

$$g = F(r)dt^2 - \frac{1}{F(r)}dr^2 - r^2 d\omega^2, \quad (3)$$

où $F(r) = 1 - \frac{2M}{r}$, M est une constante et $d\omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ est la métrique usuelle sur la sphère S^2 . La constante M peut être interprétée comme la masse d'un astre, isolée dans l'univers et localisée

¹Einstein, en retour à la lettre où Schwarzschild lui anonçait sa solution, répondit à celui-ci : "I had not expected that the exact solution to the problem could be formulated. Your analytical treatment of the problem appears to me splendid."

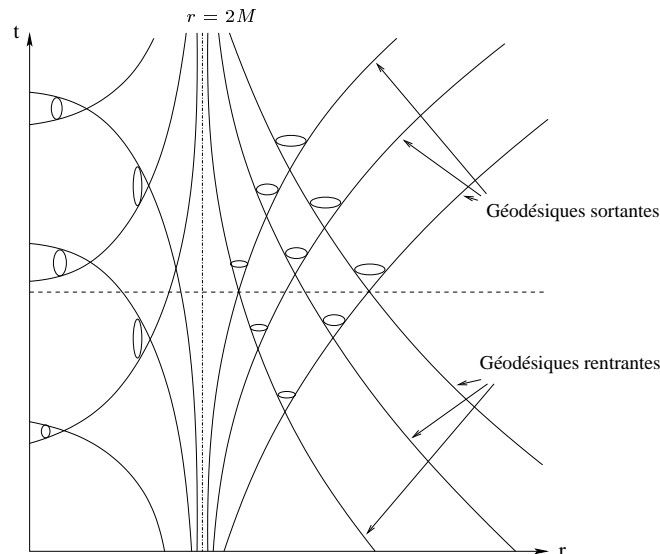


FIG. 1: Géodésiques isotropes radiales dans les coordonnées de Schwarzschild.

au point $r = 0$ (la densité de masse est alors infinie en ce point). La métrique (3) décrit alors la courbure de l'espace-temps générée par cette masse. Voici les principales propriétés de cette solution.

L'espace-temps est asymptotiquement plat, *i.e.* $g \rightarrow \eta$ quand $r \rightarrow +\infty$. Cela confirme notre interprétation que la masse M est isolée dans l'univers : loin de la source de gravitation, l'espace-temps apparaît plat. Plus important, la métrique (3) possède deux singularités : le point $\{r = 0\}$ qui est une véritable singularité de courbure, mais aussi la sphère $\{r = 2M\}$, appelée horizon des événements, qui est en fait une singularité de coordonnées. Cet horizon sépare l'espace-temps en deux régions connexes :

- L'extérieur du trou noir $\{r > 2M\}$. C'est un domaine *statique* : le champ de vecteurs $\frac{\partial}{\partial t}$ est de Killing (son flot est une isométrie), temporel et orthogonal à une famille d'hypersurfaces spatiales (les hypersurfaces de niveau de t).
- L'intérieur du trou noir $\{r < 2M\}$. Dans cette région, le champ de vecteurs $\frac{\partial}{\partial t}$ devient spatial alors que $\frac{\partial}{\partial r}$ est temporel. C'est une région *dynamique* où les objets sont entraînés sans résistance possible vers la singularité $\{r = 0\}$.

Afin de mieux visualiser les résultats précédents, on représente les cônes de lumière dans un diagramme espace-temps en deux dimensions (Figure 1). Ceci permet en outre de mieux comprendre la nature de la singularité à l'horizon des événements.

On voit que lorsque $r \rightarrow 2M$, les cônes deviennent plus aigus (resp. plus obtus) à l'extérieur (resp. à l'intérieur) de l'horizon. De plus, à l'extérieur du trou noir, les cônes futurs sont orientés dans le sens des t croissants tandis qu'à l'intérieur du trou noir, les cônes futurs sont orientés dans le sens des r décroissants : ce choix d'orientation temporelle est consistant avec une description de type trou noir de la géométrie ; une orientation opposée de l'intérieur correspondrait à un trou blanc. Ainsi, tout ce qui se trouve à l'intérieur du trou noir doit terminer son existence sur la singularité $\{r = 0\}$. Que se passe-t-il à l'horizon des événements ? Pour le comprendre, examinons le comportement des rayons lumineux. Puisque l'espace-temps est parfaitement sphérique, on regarde seulement les rayons lumineux radiaux dont les trajectoires sont contenues dans le plan engendré par (t, r) . Leurs trajectoires sont les *géodésiques isotropes radiales* de l'espace-temps. Ces trajectoires sont les chemins les plus rapides pour aller

vers l'horizon ou pour revenir depuis l'horizon. A l'extérieur du trou noir, les trajectoires des rayons lumineux radiaux sortants (resp. rentrants) sont données par les courbes intégrales des champs de vecteurs

$$V^\pm = \frac{\partial}{\partial t} \pm F(r) \frac{\partial}{\partial r},$$

et sont représentées sur la Figure 1. En particulier, on voit que le long des courbes sortantes (resp. rentrantes), la fonction t tend vers $+\infty$ (resp. $-\infty$) quand $r \rightarrow 2M$. Pour interpréter ce phénomène, il faut tout d'abord comprendre que t correspond à l'expérience du temps pour des observateurs statiques à l'infini, c'est à dire des observateurs représentés par le champ de vecteurs $\frac{\partial}{\partial t}$ et qui sont situés à l'infini spatial. Le système de coordonnées de Schwarzschild décrit donc l'espace-temps du point de vue de ce type d'observateurs. Les trajectoires des géodésiques isotropes radiales montrent qu'il faut un temps infini (mesuré par ces observateurs) pour que tout objet matériel ou sans masse, atteigne l'horizon ou en revienne. Pour eux, l'univers visible a la forme

$$\mathcal{M} = \mathbb{R}_t \times]2M, +\infty[\times S_{\theta, \varphi}^2,$$

et ils perçoivent l'horizon $\{r = 2M\}$ comme une *région asymptotique* de l'espace-temps.

Comme on l'a déjà dit, la singularité à l'horizon est une singularité de coordonnées. Finkelstein en 1958 proposa un nouveau système de coordonnées correspondant au point de vue d'observateurs plongeant dans le trou noir. L'idée est de remplacer la coordonnée t par une coordonnée u qui reste constante le long des rayons lumineux radiaux sortants. Un trou noir de Schwarzschild est alors représenté par la variété

$$\mathcal{M} = \mathbb{R}_u \times \mathbb{R}_r^+ \times S_{\theta, \varphi}^2,$$

munie de la métrique lorentzienne

$$g = F(r)du^2 - 2dudr - r^2d\omega^2. \quad (4)$$

La métrique (4) possède toujours une singularité en $\{r = 0\}$ mais reste régulière en $\{r = 2M\}$. En fait, $\{r = 2M\}$ apparaît comme une hypersurface régulière isotrope. On a représenté sur la Figure 2 un diagramme d'espace-temps avec la forme des cônes d'onde en différents points de l'espace-temps ainsi que quelques trajectoires radiales de rayons lumineux.

A l'aide de ce diagramme, on voit que rien n'empêche de s'approcher de l'horizon et on retrouve le fait qu'une fois à l'intérieur du trou noir, il n'est plus possible d'en ressortir.

Faut-il croire à tout ce que prédit la solution de Schwarzschild ? Grâce à un résultat de Birkhoff, on sait que toute solution des équations d'Einstein dans le vide, à symétrie sphérique, est donnée localement par la métrique (3). Ce résultat d'unicité permet entre autres d'utiliser cette solution pour décrire l'extérieur d'étoiles que l'on suppose isolées et à symétrie sphérique. Afin d'éviter toute discussion sur l'existence d'un horizon des événements et ses conséquences, il suffit de supposer que le rayon de l'étoile est supérieur à la valeur $2M$ (où M est la masse de l'étoile) : l'horizon est alors plongé à l'intérieur de l'étoile où l'espace-temps n'est plus décrit par la métrique de Schwarzschild (celle-ci est valable uniquement dans le vide).

Qu'advient-il pourtant lorsque l'étoile a épuisé tout son carburant et que les réactions nucléaires s'estompent en son sein ? La pression interne n'est alors plus suffisante pour compenser la force de gravitation qui tend à comprimer l'étoile. Les modèles d'effondrement gravitationnel dus à Oppenheimer et Snyder, puis à Wheeler et ses collaborateurs, ont clairement montré que le rayon d'une étoile suffisamment massive, en train de s'effondrer, doit passer en dessous du rayon de l'horizon des événements :

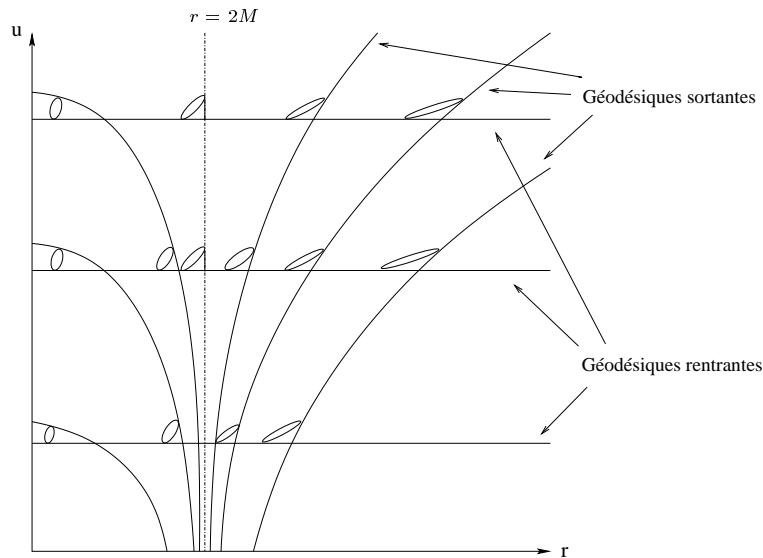


FIG. 2: Géodésiques isotropes radiales dans les coordonnées de Finkelstein.

l'effondrement engendre un trou noir². Les trous noirs de Schwarzschild, ou tout du moins leur région extérieure $\{r > 2M\}$, sont donc des modèles pertinents. La présence d'une singularité à l'intérieur du trou noir pose néanmoins plus de problèmes conceptuels. Les théorèmes de singularité de Hawking et Penrose ont pourtant montré que, dès qu'un horizon des événements est créé, alors une singularité de l'espace-temps doit se former à l'intérieur de l'horizon, ce qui renforce la validité de la solution de Schwarzschild. Notons que les phénomènes quantiques, de plus en plus importants au fur et à mesure que l'on s'approche de la singularité, pourraient empêcher la formation de celle-ci. En fait, une explication de ce qui se passe près de la singularité nécessiterait une théorie de la gravité quantique qui n'existe pas encore.

Il existe d'autres types de trous noirs que les trous noirs de Schwarzschild. En 1963, Roy Patrick Kerr, un mathématicien Néo-Zélandais, réussit à obtenir une solution exacte des équations d'Einstein dans le vide qui décrit des objets en rotation (symétrie cylindrique). Cette solution, appelée trou noir de Kerr, est particulièrement importante puisque tous les astres observés sont en rotation. Dans le système de coordonnées de Boyer-Lindquist, cette solution (\mathcal{M}, g) est donnée par

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r \times S_{\theta, \varphi}^2,$$

et

$$g = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4Mar \sin^2 \theta}{\rho^2} dt d\varphi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\sigma^2}{\rho^2} \sin^2 \theta d\varphi^2, \quad (5)$$

avec

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2 \theta, & \Delta &= r^2 + a^2 - 2Mr, \\ \sigma^2 &= (r^2 + a^2)\rho^2 + 2Mra^2 \sin^2 \theta = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \end{aligned}$$

où M est la masse du trou noir et a son moment angulaire par unité de masse. On observe tout d'abord que lorsqu'on prend $a = 0$ dans (5), on retrouve la métrique (3). Les trous noirs de Kerr généralisent

²Il existe toutefois deux états intermédiaires pour des étoiles moins massives : les naines blanches et les étoiles à neutrons

donc les trous noirs de Schwarzschild. La métrique (5) est indépendante de φ . Cela traduit la symétrie cylindrique de la solution. L'espace-temps est en fait en rotation par rapport à un axe passant par ses pôles Nord et Sud (ceci se voit à travers le terme $g_{t\varphi}$ non nul qui couple les variables t et φ). Enfin, l'espace-temps est de nouveau asymptotiquement plat : la solution de Kerr modélise donc un objet de masse M , en rotation et isolé dans l'univers.

La métrique (5) possède deux types de singularités : d'une part l'anneau $\{r = 0, \theta = \frac{\pi}{2}\}$ qui est une singularité de courbure³ et d'autre part, les lieux où la fonction Δ s'annule qui sont des horizons des événements. Selon les valeurs respectives de M et a , la fonction Δ peut avoir 0, 1 ou 2 racines. Il existe donc trois types d'espaces-temps de Kerr :

- Kerr lent lorsque $|a| < M$. La fonction Δ possède alors deux racines $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. Il existe donc deux horizons.
- Kerr extrême lorsque $|a| = M$. Δ possède dans ce cas une racine double $r_0 = M$ et il n'y a qu'un seul horizon.
- Kerr rapide lorsque $|a| > M$. Δ n'a pas de racine réelle. La singularité de courbure est nue.

On s'intéresse maintenant seulement au cas lent, qui est considéré comme le modèle générique de trou noir éternel en rotation (le cas extrême étant probablement instable). Les horizons séparent l'espace-temps en trois composantes connexes appelées blocs de Boyer-Lindquist :

Bloc I : C'est l'extérieur du trou noir $\{r > r_+\}$. C'est le bloc qui correspond le mieux à notre expérience usuelle de l'espace-temps. Les vecteurs $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ et $\frac{\partial}{\partial \varphi}$ sont spatiaux et pour $r \gg 1$, $\frac{\partial}{\partial t}$ est temporel. Cependant, le bloc I contient une région \mathcal{E} appelée ergosphère dans laquelle $g_t < 0$ et donc $\frac{\partial}{\partial t}$ y est spatial. \mathcal{E} est la région toroïdale entourant l'horizon :

$$\mathcal{E} = \{(t, r, \theta, \varphi); r_+ < r < M + \sqrt{M^2 - a^2 \cos^2 \theta}\}.$$

Dans l'ergosphère, les effets de la rotation sont extrêmes et le long de toute courbe non spatiale orientée vers le futur, la quantité $a\varphi$ est strictement croissante. L'existence de \mathcal{E} traduit le fait que le bloc I, comme tout bloc de Boyer-Lindquist, n'est pas stationnaire. Il n'existe pas de champ de vecteurs temporel globalement défini dont le flot soit une isométrie (champ de Killing). Cependant, l'extérieur de l'ergosphère est stationnaire et chaque point du bloc I admet un voisinage stationnaire.

Bloc II : C'est la région comprise entre l'horizon extérieur et l'horizon intérieur $\{r_- < r < r_+\}$. Le champ de vecteurs $\frac{\partial}{\partial r}$ y est temporel et $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \theta}$ et $\frac{\partial}{\partial \varphi}$ sont spatiaux. C'est une région dynamique où les objets sont entraînés sans résistance possible vers l'horizon interne.

Bloc III : C'est la région se situant au-delà de l'horizon interne $\{-\infty < r < r_-\}$. Ce bloc contient la singularité de courbure $\{r = 0, \theta = \frac{\pi}{2}\}$, une autre ergosphère

$$\mathcal{E}' = \{(t, r, \theta, \varphi); M - \sqrt{M^2 - a^2 \cos^2 \theta} < r < r_-\},$$

ainsi qu'une machine à remonter le temps (la seule région où $\frac{\partial}{\partial \varphi}$ est temporel) qui permet de relier deux points quelconques du bloc III par une courbe temporelle future.

On représente les résultats précédents sur les figures 3 et 4 adaptées des figures présentes dans [47].

Examinons plus précisément le bloc I. La fonction t des coordonnées de Boyer-Lindquist correspond de nouveau au temps propre d'observateurs positionnés à l'infini. Les géodésiques isotropes radiales

³Il faut remarquer ici que la sphère $r = 0$ n'est pas réduite à un point. En fait, la sphère de rayon r n'est réduite à un point pour aucune valeur de r , ce qui explique pourquoi on peut étendre la variable r à tout l'axe réel

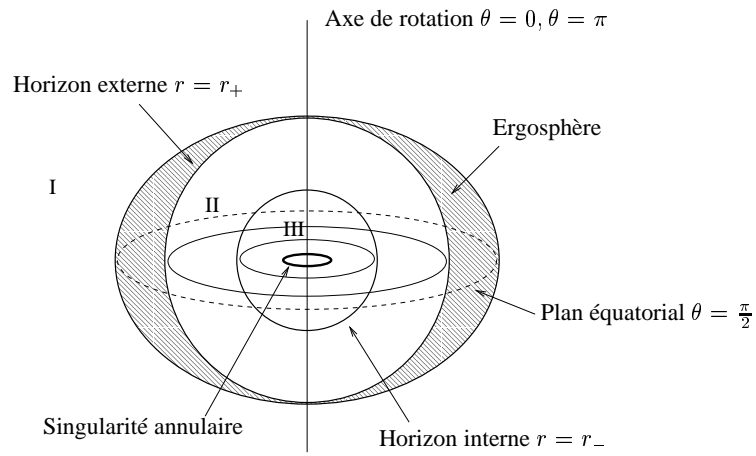


FIG. 3: Les blocs de Boyer-Lindquist et l'ergosphère d'un trou noir de Kerr.

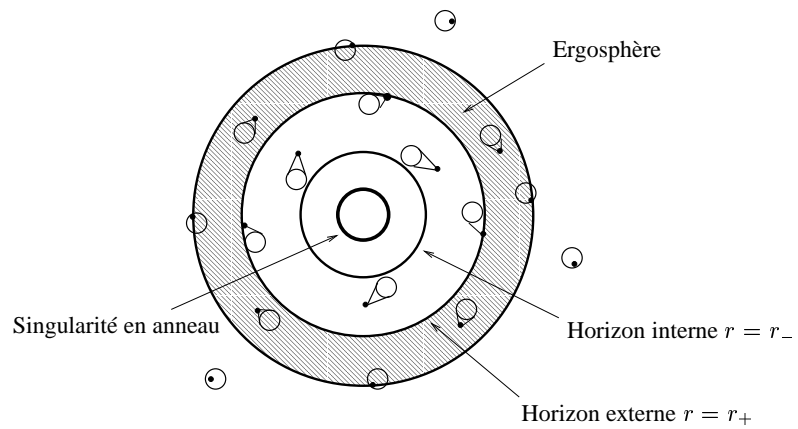


FIG. 4: Le plan équatorial d'un trou noir de Kerr : les cônes de lumière indiquent la rotation de l'espace-temps dans l'ergosphère

dans Schwarzschild sont remplacées par les “géodésiques isotropes principales”, courbes intégrales des champs de vecteurs

$$V^\pm = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} \pm \frac{\partial}{\partial r} + \frac{a}{\Delta} \frac{\partial}{\partial \varphi}.$$

Ces trajectoires correspondent aux rayons lumineux émis ou reçus en direction du trou noir par un observateur à l’infini. Le long de ces trajectoires, la fonction t tend vers $\pm\infty$ lorsque r tend vers r_+ . En conséquence, comme pour les trous noirs de Schwarzschild, l’horizon externe $\{r = r_+\}$ est perçu comme une région asymptotique de l’espace-temps pour des observateurs lointains⁴. L’univers visible a alors la forme

$$\mathcal{M} = \mathbb{R}_t \times]r_+, +\infty[\times S_{\theta, \varphi}^2.$$

L’existence d’une ergosphère dans le bloc I donne naissance à un phénomène nouveau et remarquable : la superradiance. A l’intérieur de l’ergosphère, les champs et les particules peuvent avoir une énergie négative. Penrose a montré par une expérience de pensée simple qu’il est possible d’utiliser cette propriété pour extraire une partie de l’énergie de rotation du trou noir : Une particule de spin 0 entre dans l’ergosphère et se décompose en deux photons dont un a une énergie négative ; celui-ci traverse l’horizon et l’autre ressort de l’ergosphère avec une énergie supérieure à celle de la particule initiale. Ceci est possible car l’ergosphère est située à l’extérieur de l’horizon externe et donc il est toujours possible de s’en échapper. Le phénomène analogue pour les champs de spin entier s’appelle la superradiance. Ce point sera détaillé plus loin.

Les espaces-temps de Minkowski, Schwarzschild et Kerr que l’on a considéré jusqu’ici sont des solutions exactes des équations d’Einstein dans le vide. Ce sont des exemples de géométrie où seule la gravité entre en jeu. Des phénomènes particulièrement intéressants apparaissent lorsqu’on couple les effets de la gravitation avec d’autres forces fondamentales de la physique. La découverte des trous noirs possédant une charge électrique est l’exemple le plus simple d’un tel couplage. Ce sont des solutions exactes des équations d’Einstein (2) couplées aux équations de Maxwell. En 1965, Newman *et al* obtinrent de cette manière une généralisation de la métrique de Kerr. Cette solution est donnée par

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r \times S_{\theta, \varphi}^2,$$

et

$$g = \left(1 + \frac{Q^2 - 2Mr}{\rho^2}\right) dt^2 + \frac{2a \sin^2 \theta (2Mr - Q^2)}{\rho^2} dt d\varphi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\sigma^2}{\rho^2} \sin^2 \theta d\varphi^2, \quad (6)$$

avec

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad \sigma^2 = (r^2 + a^2)\rho^2 + (2Mr - Q^2)a^2 \sin^2 \theta,$$

où M est la masse du trou noir, a son moment angulaire par unité de masse et Q sa charge. De plus, le vecteur potentiel, solution des équations de Maxwell, est égal à

$$A = -\frac{Qr}{\rho^2} (dt - a \sin^2 \theta d\varphi).$$

Ce type de solution est maintenant appelé trou noir de Kerr-Newman. Ce sont les trous noirs les plus généraux que l’on puisse considérer. Leurs caractéristiques essentielles ne sont pas fondamentalement

⁴Il s’agit encore une fois d’une simple singularité de coordonnées. Des observateurs tombant dans le trou noir n’ont aucun mal à franchir l’horizon.

différentes de celles d'un trou noir de Kerr. Lorsqu'on prend $a = 0$ dans (6), on obtient une solution, appelée trou noir de Reissner-Nordström, qui décrit un trou noir électriquement chargé et à symétrie sphérique, *i.e.*

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r^+ \times S_{\theta, \varphi}^2,$$

et

$$g = F(r)dt^2 - \frac{1}{F(r)}dr^2 - r^2d\omega^2, \quad (7)$$

où $F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

Plus généralement, l'étude de l'existence et des propriétés des solutions des équations d'Einstein (2) couplées à des équations de champs (Onde, Klein-Gordon, Maxwell, Dirac, perturbation gravitationnelle), notées génériquement

$$\mathcal{L}_g(x, \partial, \psi) = 0, \quad (8)$$

est un des problèmes fondamentaux de la relativité générale. Dans ce système, la métrique g intervient explicitement dans l'écriture de l'équation satisfaite par le champ. Celui-ci, en retour, apporte une contribution non nulle au tenseur énergie-impulsion des équations d'Einstein. Face aux difficultés majeures du système (2), (8), on fait l'hypothèse d'approximation linéaire, c'est à dire que l'on néglige l'influence du champ ψ sur la métrique g . Cela revient à étudier le champ sur une variété Einsteinienne fixe. Malgré cette approximation linéaire, on a espoir que ce type de résultats donne un bon aperçu de la physique des trous noirs et de leur interaction avec les autres champs de la physique. Plus particulièrement, cette approche permet d'étudier de manière rigoureuse certains aspects d'une théorie de la gravité quantique (encore inexistante) en traitant de manière classique le champ gravitationnel et de manière quantique les équations de champs. Ceci est à l'origine de la découverte fondamentale du rayonnement émis par les trous noirs : l'effet Hawking.

Dans cette thèse, on étudie successivement la propagation de champs de spin $\frac{1}{2}$ (ou champs de Dirac) dans l'espace-temps de Minkowski puis à l'extérieur de trous noirs de Reissner-Nordström et enfin Kerr-Newman. Ce qui nous intéresse plus spécialement ici est l'étude de la diffusion (ou diffraction) des champs de Dirac dans ces différentes géométries. Précisément, on veut comprendre comment des effets relativistes extrêmes (par exemple, l'existence d'horizons des événements) influent sur la propagation de ces champs.

La théorie de la diffusion est l'étude de la propagation de champs et de leur comportement asymptotique en temps. On s'intéresse uniquement à la partie du champ qui "diffuse", c'est à dire à la partie du champ qui ne reste pas confinée dans une région compacte mais plutôt, qui s'échappe vers les régions asymptotiques de l'espace-temps (l'horizon et l'infini spatial pour des trous noirs). Dans les régions asymptotiques, le champ vérifie souvent des équations plus simples. Le but d'une théorie de la diffusion est alors de caractériser ces équations simplifiées, en fonction des paramètres physiques du système (dans notre cas, en fonction de la géométrie des trous noirs).

En espace-temps plat, le cadre mathématique général de la théorie de la diffusion est le suivant. On suppose que l'équation de champ peut s'écrire sous la forme hamiltonienne

$$i\partial_t\psi(t) = H\psi(t), \quad (9)$$

où la fonction $\psi(t)$ prend ses valeurs dans un espace de Hilbert \mathcal{H} (par exemple, $\mathcal{H} = L^2(\mathbb{R}^3, dx)$) et H est un opérateur autoadjoint sur \mathcal{H} . Si l'on note e^{-itH} le propagateur associé à H , les solutions de (9) sont données par $\psi(t) = e^{-itH}\phi$ où $\phi = \psi(0)$ est la condition initiale du problème. On suppose de plus que l'opérateur H peut s'écrire sous la forme

$$H = H_0 + V(x),$$

où H_0 est un opérateur autoadjoint sur \mathcal{H} , appelé hamiltonien libre, et V un opérateur de multiplication sur \mathcal{H} . Le potentiel V doit être compris comme une perturbation de H_0 dont les effets sont négligeables dans les régions asymptotiques de l'espace. Précisément, V vérifie une certaine décroissance à l'infini du type $|V(x)| \leq C\langle x \rangle^{-\mu}$, $\mu > 0$. Intuitivement, on s'attend à ce que les solutions de (9) qui diffusent vers les régions asymptotiques $\{|x| \rightarrow +\infty\}$, puissent être approchées asymptotiquement par des solutions de l'équation libre

$$i\partial_t \psi_0(t) = H_0 \psi_0(t). \quad (10)$$

On cherche donc les solutions $\psi(t)$ de (9) pour lesquelles il existe une solution $\psi_0(t)$ de (10) telle que

$$\lim_{t \rightarrow \pm\infty} \|\psi(t) - \psi_0(t)\| = 0.$$

Si l'on écrit ces solutions en fonction de leur donnée initiale, *i.e.* $\psi(t) = e^{-itH} \phi$ et $\psi_0(t) = e^{-itH_0} \phi_0^\pm$, on a alors la relation suivante entre ϕ et ϕ_0^\pm

$$\phi = \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \phi_0^\pm.$$

Ceci incite à introduire les opérateurs d'onde $W^\pm : \phi_0^\pm \rightarrow \phi$ qui font le lien entre les données initiales ϕ_0^\pm et ϕ . Montrer l'existence de ces opérateurs (sur un sous-espace de \mathcal{H} bien choisi) est la première question fondamentale de la théorie de la diffusion.

Les opérateurs d'onde ont une chance d'exister si on les définit pour les données initiales ϕ_0^\pm qui diffusent vers l'infini. De manière générale, si $\psi(t) = e^{-itA} \phi$ est un champ qui satisfait l'équation $i\partial_t \psi(t) = A\psi(t)$ avec A un opérateur autoadjoint sur \mathcal{H} , cette propriété est intimement liée au type spectral auquel appartient la donnée initiale ϕ . Plus précisément, pour qu'il y ait diffusion, celle-ci doit appartenir à $\mathcal{H}_c(A)$, le sous-espace spectral⁵ de \mathcal{H} correspondant au spectre continu associé à A . En fait, on peut même se restreindre à $\mathcal{H}_{ac}(A)$ le sous-espace spectral correspondant au spectre absolument continu car dans la plupart des situations physiques, on a $\mathcal{H}_{sing}(A) = \emptyset$. Inversement, si la donnée initiale appartient à $\mathcal{H}_{pp}(A)$ alors l'énergie de la solution dans une région compacte de l'espace ne tend pas vers 0.

La théorie de la diffusion se résume alors dans les trois points suivants. On cherche tout d'abord à montrer l'existence des opérateurs d'onde directs, définis précisément par les limites fortes :

$$W^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \mathbf{1}^{ac}(H_0). \quad (11)$$

On note \mathcal{H}_\pm les images de W^\pm . Lorsque les opérateurs d'onde (11) existent, alors ils satisfont automatiquement les propriétés suivantes : W^\pm sont des isométries partielles vérifiant $(W^\pm)^{-1} W^\pm = \mathcal{H}_{ac}(H_0)$, $W^\pm (W^\pm)^{-1} = \mathcal{H}_\pm$ ainsi que les relations d'entrelacement

$$HW^\pm = W^\pm H_0. \quad (12)$$

En particulier, les parties absolument continues des opérateurs H et H_0 sont unitairement équivalentes et donc les espaces \mathcal{H}_\pm sont toujours inclus dans $\mathcal{H}_{ac}(H)$. On aimerait montrer que $\mathcal{H}_\pm = \mathcal{H}_{ac}(H)$. On dit

⁵Rappelons que le spectre de tout opérateur autoadjoint A sur \mathcal{H} peut se décomposer en $\sigma(A) = \sigma_{pp}(A) \cup \sigma_c(A)$ et $\sigma_c(A) = \sigma_{ac}(A) \cup \sigma_{sing}(A)$ où $\sigma_{pp}(A)$, $\sigma_c(A)$, $\sigma_{ac}(A)$ et $\sigma_{sing}(A)$ représentent respectivement le spectre purement ponctuel, le spectre continu, le spectre absolument continu et le spectre singulier de A . Ceci entraîne la décomposition suivante de l'espace de Hilbert \mathcal{H} :

$$\mathcal{H} = \mathcal{H}_{pp}(A) \oplus \mathcal{H}_c(A) \quad \text{et} \quad \mathcal{H}_c(A) = \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sing}(A).$$

que les opérateurs d'onde sont complets lorsque cette dernière propriété est satisfaite. Ce dernier point est équivalent à montrer l'existence des opérateurs d'onde inverses

$$\tilde{W}^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH} \mathbf{1}^{ac}(H). \quad (13)$$

Enfin, on dit que les opérateurs d'onde sont asymptotiquement complets si l'on montre aussi

$$\sigma_{sing}(H) = \emptyset. \quad (14)$$

Lorsque (11), (13) et (14) sont satisfaits alors on peut comparer toutes les solutions $\psi(t) = \bar{e}^{-itH} \phi$ de (9) qui diffusent à l'infini (*i.e.* $\phi \in \mathcal{H}_{ac}(H)$) avec des solutions plus simples $\psi_0(t) = e^{-itH_0} \phi_0^\pm$ satisfaisant (10).

La méthode standard pour prouver l'existence des opérateurs d'onde directs et inverses est la *méthode de Cook*. Celle-ci est basée sur l'observation suivante : si f est une fonction réelle, à valeurs dans un espace de Banach, différentiable, alors $\lim_{t \rightarrow \infty} f(t)$ existe lorsque sa dérivée $f'(t)$ appartient à $L^1(dt)$. En conséquence, il suffit simplement de montrer que

$$\frac{d}{dt} e^{itH} e^{-itH_0} \psi = e^{itH} iV e^{-itH_0} \psi \in L^1(dt), \quad (15)$$

$$\frac{d}{dt} e^{itH_0} e^{-itH} \psi = e^{itH_0} iV e^{-itH} \psi \in L^1(dt), \quad (16)$$

pour tout ψ dans un sous-espace dense de \mathcal{H} afin de prouver l'existence de (11), (13). En général, il est facile de montrer (15) car l'opérateur H_0 est simple et l'on a des formules explicites pour le manipuler. Par exemple, des techniques de phase stationnaire peuvent s'appliquer. Ce n'est pas le cas pour (16). Plusieurs méthodes ont été développées afin de pallier à cette dissymétrie du problème : les méthodes à trace (voir par exemple [69], Vol 3), les méthodes basées sur un principe d'absorption limite et la théorie des opérateurs H -smooth (voir [69], Vol 4 ou [78]) et enfin les méthodes complètement dépendantes du temps (voir [21]) que l'on a adoptées dans cette thèse et qui sont décrites plus loin. Toutes ces méthodes permettent de reformuler la question de l'existence d'opérateurs d'onde inverses sous une forme qui rend la méthode de Cook applicable.

Le formalisme décrit ci-dessus fonctionne pour des potentiels V à courte portée, *i.e.* si

$$|V(x)| \leq C \langle x \rangle^{-\mu}, \quad \mu > 1.$$

Dans ce cas, on peut comparer la dynamique physique e^{-itH} et la dynamique de comparaison e^{-itH_0} . Au contraire, si le potentiel est à longue portée, *i.e.* si

$$|V(x)| \leq C \langle x \rangle^{-\mu}, \quad 0 < \mu \leq 1,$$

alors les limites fortes (11), (13) n'existent plus. Il faut alors modifier la dynamique de comparaison e^{-itH_0} en lui ajoutant un terme de phase non local qui dépend du comportement asymptotique du potentiel V . Le choix du terme de phase n'est pas unique et dépend du type de décroissance du potentiel V (voir par exemple [21] et [78]). Dans le cas de potentiel à longue portée de type Coulomb ($\mu = 1$), on peut utiliser une modification à la Dollard [26]. Cette approche sera décrite plus loin.

En espace-temps plat, les résultats de diffusion pour des équations de champs sont nombreux. On se limite ici au cas d'une équation de Dirac avec potentiels. Pour des potentiels scalaires et matriciels à courte portée et singuliers, on peut citer les travaux de Boutet de Monvel-Berthier, Manda, Purice

[11] et Georgescu, Mântoiu [37] qui utilisent un principe d'absorption limite et la théorie des opérateurs H -smooth. Le cas de potentiels à longue portée de type Coulomb a été traité par Enss, Thaller [29] et Muthuramalingam, Sinha [59] en utilisant les techniques dépendantes du temps développées par Enss [27], [28]. Le cas de potentiels à longue portée plus généraux a été traité par Gâtel, Yafaev [34] aussi par un principe d'absorption limite.

On peut adapter le cadre mathématique général décrit ci-dessus pour étudier la théorie de la diffusion d'équations de champs par des trous noirs du point de vue d'observateurs lointains. Comme on l'a déjà dit, l'extérieur d'un trou noir est alors la seule partie visible pour de tels observateurs. Pour tous les trous noirs que l'on a cités, cette région extérieure est globalement hyperbolique : il existe une fonction temps t globalement définie telle que les hypersurfaces de niveaux $\Sigma_t = \{t = Cte\}$ soient un feuilletage de l'espace-temps par des hypersurfaces de Cauchy. En conséquence, on peut toujours écrire les équations de champ sous forme d'une équation d'évolution par rapport à la variable t sur une variété riemannienne fixe (par exemple, $\Sigma = \Sigma_0$). Si cette équation d'évolution peut s'écrire sous la forme hamiltonienne (9) alors on peut appliquer le formalisme précédent. Cela est toujours le cas si par exemple l'extérieur du trou noir est statique ou stationnaire. Une différence importante toutefois est l'existence de deux régions asymptotiques, l'horizon et l'infini spatial, aux propriétés géométriques très différentes. Il faut donc définir des opérateurs d'onde distincts, qui seront liés à la géométrie locale du trou noir, selon que l'on considère la partie des champs qui se propage vers l'horizon ou vers l'infini.

Pour les géométries de Schwarzschild et de Reissner-Nordström, la théorie de la diffusion est maintenant bien comprise pour de nombreuses équations de champs. L'extérieur de ces trous noirs est une région statique et entre donc dans le cadre précédent. De plus, la symétrie sphérique de ces espaces-temps permet de simplifier considérablement l'analyse en se ramenant à une équation d'évolution à une variable d'espace par une décomposition sur les harmoniques sphériques. Il est alors possible d'utiliser des méthodes de classe trace pour montrer l'existence et la complétude asymptotique des opérateurs d'onde. Pour des trous noirs de Schwarzschild, les premiers travaux ont été menés par Dimock [22] (existence des opérateurs d'onde pour une équation d'onde) puis Dimock, Kay [23] (existence des opérateurs d'onde modifiés pour des champs de Klein-Gordon). Ces travaux ont été poursuivis par Bachelot [3], [4] (existence et complétude asymptotique des opérateurs d'onde pour des champs de Maxwell et pour Klein-Gordon avec modification à l'infini) et par Nicolas [62] (existence et complétude asymptotique des opérateurs d'onde pour des champs de Dirac sans masse). Pour des trous noirs de Reissner-Nordström, Jin [51] et Melnyk [54], [55] ont établis l'existence et la complétude asymptotique des opérateurs d'onde (modifiés à l'infini) pour des champs de Dirac massifs et chargés.

Obtenir une théorie de la diffusion dans des espaces-temps de type trou noir est le premier pas vers une étude rigoureuse de phénomènes physiques comme l'effet Hawking ou l'étude des modes quasinormaux d'un trou noir. L'étude de l'effet Hawking, dans le cas de trous noirs éternels à symétrie sphérique, a été initiée par Dimock, Kay [24], [25]. Dans le cas d'une étoile en effondrement gravitationnel, l'existence de la radiation d'Hawking a été montrée par Bachelot [5], [6], [7] et Melnyk [56], [57]. L'étude des résonances⁶ de trous noirs de Schwarzschild a été menée par Bachelot, Motet-Bachelot [8] (équation de Regge-Wheeler) puis par Sá Barreto, Zworski [73] (équation des ondes).

L'étude de la théorie de la diffusion dans des trous noirs de Kerr ou de Kerr-Newman est bien plus compliquée. Le problème majeur provient de la non stationnarité de l'extérieur de ces trous noirs. Considérons, par exemple, l'équation des ondes dans Kerr.

$$\square_g f = 0,$$

⁶Les résonances sont les pôles complexes de l'extension analytique de l'opérateur de diffusion S , une notion reliée aux modes quasinormaux

où

$$\square_g = |g|^{-\frac{1}{2}} \sum_{\mu, \nu} \partial_\mu |g|^{\frac{1}{2}} g^{\mu\nu} \partial_\nu, \quad |g| = |\det g|,$$

est l'opérateur des ondes donné par la métrique (5). L'énergie mesurée par un observateur lointain, qui est conservée, s'écrit alors

$$\int_{\Sigma} \left\{ |\partial_t f|^2 + \frac{\Delta^2}{\sigma^2} |\partial_r f|^2 + \frac{\Delta}{\sigma^2} |\partial_\theta f|^2 + \frac{\rho^2 - 2Mr}{\sigma^2 \sin^2 \theta} |\partial_\varphi f|^2 \right\} \frac{\sigma^2}{\Delta} dr d\omega.$$

Le coefficient $\frac{\rho^2 - 2Mr}{\sigma^2 \sin^2 \theta}$ est négatif à l'intérieur de l'ergosphère et positif à l'extérieur. L'énergie conservée n'est donc pas toujours positive. Ceci entraîne notamment le phénomène de superradiance par lequel le champ peut extraire de l'énergie de l'ergosphère pour augmenter son énergie propre à l'extérieur de l'ergosphère. Plus généralement, pour des équations de spin entier, il n'existe pas de quantité positive de type énergie qui soit conservée au cours de l'évolution. En conséquence, l'équation qui décrit l'évolution du champ, s'écrit toujours sous la forme hamiltonienne (9) mais l'opérateur H n'est plus autoadjoint. Le formalisme détaillé plus haut n'est plus valable. Néanmoins, Häfner [44] a réussi à développer une théorie de la diffusion pour des champs de Klein-Gordon dans Kerr en limitant son étude aux modes non superradiants : ce sont des modes ayant des moments angulaires relativement faibles pour lesquels il existe une énergie positive conservée et le formalisme précédent s'applique donc.

C'est un fait remarquable que ce phénomène de superradiance n'existe pas pour des champs de Dirac ou plus généralement pour des champs de spin non entier. Il existe toujours dans ce cas une quantité positive qui est conservée au cours de l'évolution. L'équation qui régit cette évolution prend alors la forme voulue (9). Une théorie de la diffusion complète a ainsi été obtenue par Häfner, Nicolas [46] pour des champs de Dirac sans masse dans Kerr. Les difficultés essentielles de ce travail proviennent de la géométrie compliquée de l'espace-temps, notamment dans un voisinage de l'horizon, et de l'absence de symétrie sphérique. Afin d'écrire l'équation de Dirac sous une forme convenable pour une étude de la diffusion, c'est à dire une forme sans termes à longue portée artificiels à l'infini, il a fallu mieux comprendre les effets de la rotation du trou noir. Ceci a conduit à introduire une nouvelle tétrade de Newman-Penrose, adaptée à la rotation locale de l'espace-temps. L'absence de symétrie sphérique interdit quant à elle l'utilisation de méthodes à trace et impose de recourir à des méthodes plus avancées, estimation de Mourre, principe d'absorption limite et techniques de commutateurs, afin d'obtenir l'existence et la complétude des opérateurs d'onde. Du fait de la géométrie complexe de l'horizon, l'obtention d'une estimation de Mourre se révèle techniquement assez délicate.

D'autres types de résultats ont été obtenus concernant l'étude d'équations de champs dans les géométries de Schwarzschild, de Reissner-Nordström, de Kerr ou de Kerr-Newman. Le problème de Cauchy pour des champs de Dirac dans ces espaces-temps fait l'objet d'une étude précise par Nicolas [65]. Finster, Kamran, Smoller et Yau, ont aussi étudié les propriétés de ces mêmes champs à l'extérieur d'un trou noir de Kerr-Newman dans la série de travaux [30], [31], [32]. Ils montrent tout d'abord l'absence de solutions périodiques en temps, puis ils obtiennent une formule très explicite du propagateur associé à l'équation d'évolution grâce à laquelle ils établissent une estimation sur la vitesse de décroissance en temps dans L_{loc}^∞ . Nicolas [61], [64] examine l'existence de solutions pour une équation de Klein-Gordon non linéaire pour des trous noirs à symétrie sphérique généraux et aussi Kerr. Enfin, Häfner [45] est en train d'utiliser les résultats de [46] et [17] pour montrer rigoureusement l'existence d'une radiation d'Hawking dans un trou noir de Kerr-Newman.

Les théories de la diffusion développées dans cette thèse sont toutes obtenues par des méthodes complètement du temps. Initialement développées afin d'étudier la théorie de la diffusion de systèmes de

particules quantiques (l'équation qui décrit l'évolution d'un tel système est l'équation de Schrödinger), ces méthodes se révèlent particulièrement adaptées à l'étude d'équations relativistes. Elles sont en effet basées sur la structure fondamentale de la relativité : le cône de lumière. Ces méthodes sont par ailleurs extrêmement intéressantes pour traiter de problèmes à longue portée, notamment pour le choix de la modification dans les opérateurs d'onde. Elles ont été initiées par Enss [27], [28] puis reprises par Sigal, Soffer [71], [72], Graf [43], Dereziński [20] en vue de résoudre le problème à N-corps en mécanique quantique non relativiste. L'objet au coeur de ces méthodes est la dynamique physique e^{-itH} . Afin d'obtenir des informations sur cette dynamique, on s'intéresse en particulier au comportement asymptotique d'observables physiques comme les opérateurs de position ou de vitesse. La première étape consiste à obtenir des estimations de propagation faibles pour une solution. Celles-ci ont la forme suivante

$$\int_1^\infty \left\| B(t)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2, \quad (17)$$

où $B(t)$ est un opérateur autoadjoint dépendant du temps. Si l'on travaille par exemple en espace-temps plat, on prend généralement $B(t) = V(\frac{x}{t})$ avec V une fonction régulière à support compact. En changeant de place le support de V , on obtient une décroissance faible de l'énergie du champ dans différents cônes de l'espace-temps. On s'intéresse en particulier aux :

- *estimations de vitesse maximale* : Ces estimations signifient que l'énergie du champ tend vers 0 lorsque $t \rightarrow \infty$ dans le cône $\{|x| > t\}$. On retrouve asymptotiquement le fait que le champ ne peut se déplacer à une vitesse supérieure à la vitesse de la lumière (égale à 1 dans notre convention).
- *estimations de vitesse minimale* : Ces estimations fournissent une version faible du principe de Huygens. L'énergie du champ tend vers 0 lorsque $t \rightarrow \infty$ dans un cône $\{|x| \leq ct\}$ où la constante $c \neq 0$ est interprétée comme une vitesse minimale de propagation.
- *estimations de vitesse microlocale* : Dans la partie de l'espace-temps où l'énergie ne tend pas vers 0, on peut approcher asymptotiquement l'observable de position x par l'observable $t\mathcal{V}$ où \mathcal{V} est l'opérateur de vitesse classique.

Les méthodes pour obtenir des estimations du type (17) sont appelées méthodes de commutateur ou de la dérivée de Heisenberg positive. Pour les estimations de vitesse minimale, le point crucial est l'existence d'un opérateur conjugué au sens de la théorie de Mourre.

Une fois que l'on a obtenu ces estimations, on construit les opérateurs de vitesse asymptotique P^\pm . Toujours en espace-temps plat, ceux-ci sont définis par les limites fortes suivantes

$$g(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH} g\left(\frac{x}{t}\right) e^{-itH}, \quad (18)$$

où $g \in C_\infty(\mathbb{R}^3)$ l'espace des fonctions régulières qui tendent vers 0 à l'infini. L'intérêt de ces opérateurs est multiple. Tout d'abord, leurs spectres fournissent une information physiquement intéressante. Ceux-ci correspondent aux valeurs et aux directions possibles de la vitesse du champ dans les régions asymptotiques de l'espace-temps, perçues par un observateur lointain. Un point très important est le suivant : les états qui ont pour vitesse asymptotique 0, *i.e.* les éléments $\psi \in \mathcal{H}$, $\psi \neq 0$ tels que $P^\pm\psi = 0$ sont exactement les vecteurs propres de l'hamiltonien H . En d'autres termes, les solutions dont l'énergie ne tend pas vers 0 dans toute région compacte de l'espace ont aussi une vitesse asymptotique égale à 0. Ceci permet de simplifier considérablement la preuve de l'existence des opérateurs d'onde inverses. En effet, on peut remplacer dans la définition (13), la troncature $\mathbf{1}_{ac}(H)$ sur le spectre absolument continu de H par une troncature sur les valeurs différentes de 0 des spectres des opérateurs de vitesse asymptotique, *e.g.* $\mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm)$ en espace-temps plat. Ceci combiné aux estimations de propagation permet d'utiliser des arguments de type Cook pour prouver l'existence des opérateurs d'onde inverses.

Lorsqu'on étudie la propagation de champs à l'extérieur de trous noirs, les opérateurs de vitesse asymptotiques décrivent une vitesse radiale à une dimension. La projection sur leur spectre positif ou négatif fournit un moyen très simple de séparer la partie des champs qui diffuse vers l'horizon et celle qui diffuse vers l'infini spatial. Ceci nous évite notamment d'introduire des fonctions de troncature artificielles dans la définition des opérateurs d'onde.

Enfin, on construit les opérateurs d'onde modifiés. Comme les potentiels à longue portée sont de type Coulomb, on peut utiliser les idées de Dollard [26] pour définir la modification. Le terme de phase est génériquement noté $e^{-iX(t)}$ où $X(t)$ est un opérateur dépendant du temps qui doit commuter avec H_0 . Comment choisir cet opérateur ? Si l'on utilise la méthode de Cook, il faut alors montrer

$$\frac{d}{dt} e^{itH} e^{-itH_0} e^{-iX(t)} \psi = e^{itH} i(V(x) - X'(t)) e^{-itH_0} \psi \in L^1(dt).$$

D'après les estimations de vitesse microlocale, on sait que l'on peut approcher asymptotiquement l'observable de position par l'observable $t\mathcal{V}$. Comme \mathcal{V} commute avec H_0 , l'idée est donc de définir la phase $X(t)$ de telle sorte que $X'(t) = V(t\mathcal{V})$. On définit finalement l'opérateur de comparaison modifié par

$$U_D(t) = e^{-itH_0} T(e^{-i \int_0^t V(s\mathcal{V}) ds}),$$

où T représente l'ordonnement temporel. Comme on l'a déjà dit, le choix d'une modification n'est pas unique. Il est donc intéressant de posséder un critère, faisant intervenir des quantités physiques bien définies, afin de valider le choix d'une modification. Les relations d'entrelacement du type (12) fournissent un exemple d'un tel critère : les opérateurs d'onde, lorsqu'ils existent, doivent faire le lien entre les observables du système étudié et leurs équivalents asymptotiques. C'est le cas pour les opérateurs d'onde modifiés définis ci-dessus. En effet, ils vérifient des relations d'entrelacement entre les opérateurs de vitesse asymptotiques P^\pm et les opérateurs de vitesse classique $\pm\mathcal{V}$, *i.e.*

$$P^\pm = W^\pm(\pm\mathcal{V})(W^\pm)^{-1}.$$

On décrit maintenant plus en détails le contenu de chaque chapitre.

Résumé des chapitres :

Chapitre 1 : On établit une théorie de la diffusion pour des champs de Dirac massifs avec potentiels scalaires et matriciels à longue portée de type Coulomb. Une difficulté essentielle apparaît lorsqu'on veut définir les observables de la théorie, dont on veut étudier le comportement asymptotique. Notamment, l'observable de position standard x a la propriété désagréable de mélanger les énergies positives et négatives des champs. Ceci implique que l'observable de vitesse standard associée à x oscille autour de l'observable de vitesse classique \mathcal{V} : un phénomène connu sous le nom de Zitterbewegung. Les opérateurs de position et de vitesse standards ne constituent donc pas un bon choix d'observables physiques. Pour remédier à cette difficulté, on utilise comme observable de position l'opérateur de Newton-Wigner x_{nw} qui ne mélange pas les énergies positives et négatives mais qui malheureusement complique le détail des calculs. On établit ensuite les estimations de propagation et on construit les opérateurs de vitesse asymptotique. Pour obtenir les estimations de vitesse minimale, on introduit un nouvel opérateur conjugué au sens de la théorie de Mourre qui est particulièrement bien adapté aux opérateurs de Dirac. Les spectres des vitesses asymptotiques sont donnés exactement par la boule unité dans \mathbb{R} . On termine en prouvant l'existence et la complétude asymptotique des opérateurs d'onde modifiés à la Dollard.

Chapitre 2 : On étudie la diffusion de champs de Dirac sans masse avec potentiel scalaire à longue portée en espace-temps plat. En absence de masse, l'hamiltonien de Dirac libre s'écrit $H_0 = \Gamma.p$, où les matrices Γ sont les matrices de Dirac et $p = -i\nabla$. Les deux valeurs propres distinctes $\pm|p|$ de la matrice $\Gamma.p$ sont singulières en $p = 0$ et plus important, elles se confondent en ce point. Aussi, l'opérateur H_0 n'a pas de gap dans son spectre et donc n'est pas inversible (ce point particulier était essentiel dans le chapitre I). En raison des difficultés liées à l'absence de masse, les méthodes usuelles ne fonctionnent pas directement et l'étude de tels champs n'a jamais été traitée auparavant. On applique ici les méthodes précédentes en prenant soin de tronquer systématiquement les basses énergies afin d'obtenir les estimations de propagation et de construire les opérateurs de vitesse asymptotique. Les spectres de ces opérateurs sont cette fois donnés par la sphère unité dans \mathbb{R}^3 , *i.e.* les champs se propagent asymptotiquement à la vitesse de la lumière. Notons que le type de situation considérée dans cette partie se retrouve localement lorsque l'on étudie la propagation de champs de Dirac massifs à l'horizon de trous noirs : la gravitation extrême près de l'horizon empêche alors de voir les effets dus à la masse.

Chapitre 3 : On développe ici une théorie de la diffusion pour des champs de Dirac chargés, massifs ou non, à l'extérieur d'un trou noir de Reissner-Nordström. La symétrie sphérique du trou noir nous permet de simplifier le problème en travaillant sur chaque harmonique sphérique. Les méthodes précédentes s'appliquent alors facilement. Afin d'étudier à la fois le cas massif et le cas sans masse (ce dernier cas n'est pas traité dans [55]), on introduit un nouveau type d'opérateurs conjugués dans l'esprit de [37]. Ceux-ci permettent d'établir en outre les estimations de vitesse minimale sous une forme particulièrement agréable et précise. On construit ensuite les opérateurs de vitesse asymptotique P^\pm . Leurs spectres fournissent une information importante sur la propagation des champs dans les deux régions asymptotiques. Dans le cas massif par exemple, on montre que $\sigma(P^+) = \{-1\} \cup [0, 1]$ et $\sigma(P^-) = [-1, 0] \cup \{1\}$. Les champs se propagent donc asymptotiquement à la vitesse de la lumière lorsqu'ils se dirigent vers le trou noir alors que leur propagation est freinée par la masse à l'infini spatial. On termine enfin en prouvant l'existence et la complétude asymptotique des opérateurs d'onde classiques à l'horizon et modifiés à la Dollard à l'infini. L'aspect matriciel des potentiels à longue portée à l'infini pose problème pour la définition de la modification. En effet, les matrices qui entrent en jeu n'ont aucune raison de commuter avec la partie libre de l'équation et donc ne peuvent entrer dans la modification à la Dollard. On propose ici une nouvelle modification par rapport à [55] qui repose sur une réinterprétation d'un argument utilisé par Thaller [75] lorsque la matrice en question anticommute avec la partie libre de l'équation.

Chapitre 4 : On établit une théorie de la diffusion pour des champs de Dirac chargés, massifs ou non, à l'extérieur d'un trou noir de Kerr-Newman. Ce travail généralise dans deux directions les résultats de [46] qui établissent une théorie de la diffusion pour une équation de Dirac sans masse à l'extérieur d'un trou noir de Kerr : premièrement, on doit traiter ici de potentiels à longue portée dus à la masse et à l'interaction entre la charge du champ et la charge du trou noir ; deuxièmement, les méthodes que l'on applique sont complètement dépendantes du temps, contrairement à [46] où un principe d'absorption limite est utilisé. Dans un premier temps, on calcule l'équation à partir d'une nouvelle tétrade de Newman-Penrose (introduite dans [46]) qui suit la rotation locale de l'espace-temps. On obtient alors une écriture de l'équation sans perturbation à longue portée artificielle à l'infini. On observe ensuite, que sur chaque mode angulaire, l'hamiltonien physique peut être comparé avec un hamiltonien du type considéré dans le chapitre 3. La différence entre ces deux hamiltoniens est à courte portée. L'étape suivante consiste alors à prouver une théorie de la diffusion intermédiaire entre ces deux hamiltoniens puis à utiliser les résultats du chapitre 3 pour construire les opérateurs de vitesse asymptotique et les opérateurs d'onde complets (classiques à l'horizon et modifiés à la Dollard à l'infini). De manière générale, lors-

qu'on veut obtenir une théorie de la diffusion entre deux opérateurs dont la différence est à courte portée, il suffit d'établir les estimations de vitesse minimale et donc, il suffit d'obtenir un opérateur conjugué au sens de la théorie de Mourre. L'absence de symétrie sphérique combinée à la géométrie au voisinage de l'horizon rend toutefois techniquement plus complexe cette étape. On propose ici un nouvel opérateur conjugué, relativement simple, inspiré de nos résultats précédents et de [46].

Chaque chapitre de cette thèse a donné lieu à une publication [14] et à des prépublications [15], [16], [17]. Les chapitres peuvent être lus indépendamment. Quelques techniques importantes sont rappelées dans chacun d'eux. Enfin, ils comportent tous leur propre introduction.

Chapitre I

Propagation estimates for Dirac operators and application to scattering theory

I.1 Introduction

Time-dependent methods in scattering theory were introduced by Enss twenty five years ago in [27] and [28]. They were originally developed to solve the N-body problem in nonrelativistic quantum mechanics. This was achieved thanks to subsequent improvements due, among others, to Sigal and Soffer [71], Graf [43] and Dereziński [20] (in chronological order). A detailed and complete presentation of these methods can be found in the book by Dereziński and Gérard [21]. In the framework of relativistic quantum mechanics, such techniques provide an intuitive description of scattering, based on the essential structure of relativity : the light cone. In this work, we use such an approach to give a complete scattering theory for massive Dirac operators with long-range potentials in flat spacetime.

Similar results have been obtained first by Enss and Thaller [29] and Mutharamaligam and Sinha [59] using the Enss method and the RAGE theorem. Recently, Gâtel and Yafaev [34] improved these results for a large class of potentials by means of a stationary approach based on a limiting absorption principle and radiation estimates for time-independent observables. The novelty in our proof is the systematic use of time-dependent observables as proposed in [21]. In particular, this leads to propagation estimates for the Dirac fields which, in turn, will greatly simplify the construction of wave operators. The Mourre theory and commutator methods will be the basic tools in our study. Let us point out that these results can be used to develop scattering theories in General Relativity. For instance, we have in mind the works by Häfner and Nicolas [46], Melnyk [55] and Nicolas [62], on the scattering for Dirac fields on black hole spacetimes. In the case of Kerr black holes, it has been shown in [46] that a Mourre theory and time-dependent techniques are necessary.

We consider a massive Dirac hamiltonian denoted by H acting on the Hilbert space of physical states $\mathcal{H} = [L^2(\mathbb{R}^3)]^4$. The hamiltonian H is the sum of the usual free Dirac operator $H_0 = \Gamma \cdot p + m\Gamma^0$, $m > 0$, $p = -i\nabla$ where $\Gamma^0, \Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ are Dirac matrices and a potential $V(x)$ of Coulombian type at infinity which is the sum of a scalar and a matrix-valued multiplication operator. According to the Heisenberg description of quantum mechanics, we shall focus our attention on the unitary evolution e^{-itH} and on the behaviour of (time-dependent) observables along this evolution ; that is to say, if A_t denotes a time-dependent function with values in selfadjoint operators on \mathcal{H} then we are interested in studying the behaviour of operators of the following type

$$A_t(t) := e^{itH} A_t e^{-itH}.$$

Note that for $\psi \in \mathcal{H}$ such that the expectation value $(\psi, A_t(t)\psi)$ is well-defined, this quantity corresponds to the mean value of the results of many measurements which are all performed on systems identically prepared to be in the state ψ . Actually, our main objects of study will be *asymptotic observables* defined by

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} A_t e^{-itH},$$

when the limit exists. It was the essential idea of Enss [27], [28] to describe the evolution of asymptotic observables such as position and momentum and to use this information to obtain results in scattering theory. More precisely, in the case of Dirac operators, Enss and Thaller [29] proved the vanishing of the following limit

$$\lim_{t \rightarrow \pm\infty} e^{itH} \left(f\left(\frac{x}{t}\right) - f(\mathcal{V}) \right) e^{-itH} \psi, \quad \forall \psi \in \mathcal{H}^c(H), \quad (\text{I.1.1})$$

where $\mathcal{H}^c(H)$ denotes the continuous spectral subspace of H and $f \in C_\infty(\mathbb{R}^3)$, the space of smooth functions tending to 0 at infinity. Here x is the *standard position operator* and $\mathcal{V} = pH_0^{-1}$ is the *classical velocity operator*. This result can be interpreted as follows : there exists a correlation between the

localization of a scattering state at late times in a narrow cone (linearly increasing with time t) and its velocity, that is to say this result describes “propagation in phase space”. One consequence of this is that “scattering states have been incoming in the remote past and will be outgoing in the far future, moving away from the region of significant interactions” (quoted from [29]) which is already a weak version of scattering.

More recently, many authors turned their attention to the construction of such observables and their application to scattering. In particular, Sigal and Soffer [71], [72], Graf [43] and Dereziński and Gérard [21] improved the methods of Enss by using the method of *positive commutator* also called method of *positive Heisenberg derivative* due mainly to Mourre in [58] and refined, among others, by Amrein, Boutet-de-Monvel Berthier, Georgescu in [1]. A motivation of their work was related to one of the main problems in scattering theory : how to define wave operators when the interaction V is long-range (that is to say when the potential falls off no faster than $|x|^{-1}$ when $|x|$ tends to infinity). In such a case, the classical wave operators

$$\Omega^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \mathbf{1}^{ac}(H_0),$$

are no longer available. Instead it is necessary to replace the comparison dynamics e^{-itH_0} by a more complicated one, usually in the form $e^{-iS(t, \mathcal{V})}$ where the function S has to be well chosen. Unfortunately, this choice has no reason to be unique and thus it would be interesting to find some natural and uniquely defined objects that, in turn, entail a natural and unambiguous definition of the wave operators.

One example of uniquely defined construction associated to the dynamics e^{-itH} is the selfadjoint operator called *asymptotic velocity* denoted by P^\pm and defined by

$$P^\pm := s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH} \frac{x}{t} e^{-itH}, \quad (\text{I.1.2})$$

The notion of strong- C_∞ -limit is explained at the beginning of Section 5. The asymptotic velocity admits the other characterization in terms of the classical velocity operator \mathcal{V}

$$P^\pm := s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH} \mathcal{V} e^{-itH}, \quad (\text{I.1.3})$$

and we see that (I.1.2) together with (I.1.3) imply (I.1.1). For instance, Dereziński and Gérard succeeded in constructing wave operators of the form $\Omega^\pm = e^{itH} e^{-iS(t, \mathcal{V})}$ satisfying in particular the intertwining relations

$$\begin{aligned} \Omega^\pm H_0 (\Omega^\pm)^* &= H \mathbf{1}^c(H), \\ \Omega^\pm \mathcal{V} (\Omega^\pm)^* &= P^\pm. \end{aligned}$$

Let us emphasize that it is the essence of wave operators to make the link between the physical quantities associated to the system (here the energy H and the asymptotic velocity P^\pm) and simpler physical quantities which allow us to make some computations (here the energy of the free system H_0 and the classical velocity operator \mathcal{V}).

Although it is not obligatory to introduce such an observable in the case of Dirac operators, the asymptotic velocity turns out to be a relevant construction at least for two reasons. First it exists under rather weak conditions. It can be shown (see [21], Section 4.10) that for certain 2-body hamiltonians for which the asymptotic velocity exists, the wave operators fail to be complete. Thus, in this sense, it could serve to define a “weak” notion of scattering theory. Secondly the asymptotic velocity can be used as a very convenient tool to prove the existence and asymptotic completeness of wave operators. For instance, an important feature of P^\pm is

$$\mathbf{1}_{\{0\}}(P^\pm) = \mathbf{1}^{pp}(H), \quad (\text{I.1.4})$$

that is to say the states of zero asymptotic velocity coincide with the bound states of H . This property not only gives a first classification between the states in \mathcal{H} , which is the initial purpose of scattering, but will allow us to use the standard Cook method in the proof of asymptotic completeness.

Let us now briefly describe the content of each sections.

In Section I.2, we give an abstract framework for massive Dirac Hamiltonians and analyse some basic properties concerning their spectrum and problems of domain invariance. Next we study in details the Zitterbewegung phenomenon for the free Dirac operator H_0 which arises when one tries to define the velocity operator \mathcal{V} . Recall that, in the case of Schrödinger operators, the velocity operator is defined as the time derivative of the position operator and it turns out that it is independent of time. However, for Dirac operators, the time derivative of the position operator is time-dependent and oscillates around a mean value which is the classical velocity operator \mathcal{V} . This phenomenon will be the source of technical problems in the derivation of weak propagation estimates in Section 4. To overcome these future difficulties, we must introduce a new position observable, called the Newton-Wigner operator, whose time derivative is exactly \mathcal{V} .

Section I.3 is devoted to a short overview of Mourre theory as presented in the initial work of Mourre [58] but also revisited by Amrein, Boutet de Monvel Berthier, Georgescu in [2] and Georgescu and Gérard in [36]. In particular we define a new locally conjugate operator for Dirac Hamiltonians which turns out to be convenient for our purpose.

Section I.4 establishes the weak propagation estimates. *Large* and *minimal velocity estimates* give an important information on the probability to find the particle in certain cones in spacetime at late times. For instance, the minimal velocity estimates asserts that a particle having an energy strictly larger than its mass m has to escape from a narrow cone $|x| \leq c_m t$ asymptotically in time. On the other hand, *microlocal velocity estimates* are a slightly stronger version of (I.1.1) and indicate that we can approach the position operator x by $t\mathcal{V}$ at late times. All these estimates rely entirely on positive commutator methods and Mourre theory. In particular, we state a result ([39] and [49]) which shows how the minimal velocity estimates are intimately related to the existence of a locally conjugate operator. At last we state two results due to Cook and Kato which allow to make the link between weak propagation estimates and the existence of asymptotic observables.

In Section I.5, we prove that the asymptotic velocity P^\pm defined by (I.1.2) exists and can also be characterized by (I.1.3). Then we study its spectrum which, physically, corresponds to the asymptotic propagation velocity of the fields and we prove that

$$\sigma(P^\pm) = \overline{B}(0, 1),$$

$\overline{B}(0, 1)$ being the closed ball in \mathbb{R}^3 of center 0 and radius 1. Eventually, using only the minimal velocity estimates, we also prove property (I.1.4).

Section I.6 is devoted to the construction of wave operators Ω^\pm . In our case where potentials of coulombian types are considered, a Dollard modification is enough for the definition of these operators when combined with an idea due to Thaller [29], that allows us to define properly this modification and avoid problems caused by the matrix-valued potential. We will then make a crucial use of the asymptotic velocity operator P^\pm and property (I.1.4) to transform the problem into a time-dependent one for which Cook's method can be applied. It could seem strange to introduce such a time dependence in the proof but, in fact, one obtains an agreeable way of proving the existence and asymptotic completeness of the wave operators by handling only time-dependent quantities which are integrable along the evolution.

In Appendix I.A, we recall two well-known techniques used for the manipulation of functions of selfadjoint operators : for integrable functions, the Fourier transform can be used, but in the case of smooth and not necessarily integrable functions, the correct tool is the Helffer-Sjöstrand formula [48]. In

each case, we state a commutator expansion of $[T, f(A)]$ in terms of the multiple commutators $ad_A^k(T)$ for two selfadjoint operators A, T . In particular, the required assumptions on the operators A and T are carefully detailed.

Eventually, in Appendix I.B, we establish weak propagation estimates for time-dependent Dirac Hamiltonians and we construct the associated asymptotic velocity used in Section 6 for the construction of the wave operators.

I.2 Properties of Dirac operators

I.2.1 Abstract framework

In this paper, we shall denote by H_0 the free massive Dirac Hamiltonian on flat spacetime acting on $\mathcal{H} = [L^2(\mathbb{R}^3)]^4$ the Hilbert space of four component square integrable functions. Precisely, we consider the differential operator

$$H_0 = \Gamma \cdot p + m\Gamma^0 = \Gamma^1 p_1 + \Gamma^2 p_2 + \Gamma^3 p_3 + m\Gamma^0, \quad p_j = -i\partial_j,$$

m being the mass of the field. We shall assume that the mass m is strictly positive. H_0 is a selfadjoint operator on $D(H_0) = [H^1(\mathbb{R}^3)]^4$ where $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space of order one in \mathbb{R}^3 . Here Γ^μ correspond to the Dirac matrices satisfying the anti-commutation relations $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$ for every $\mu, \nu = 0, \dots, 3$ ($\delta^{\mu\nu}$ stands for the Kronecker symbol). We shall use the following usual representation for the Dirac matrices.

$$\Gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \Gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix},$$

where the Pauli matrices σ^i are given by

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

The free Hamiltonian H_0 will be perturbed by some external field. Let us consider two functions V_1 and V_2 belonging to the space

$$S^{-1}(\mathbb{R}^3) = \left\{ f \in C^\infty(\mathbb{R}^3) / \forall \alpha \in \mathbb{N}^3, |\partial_x^\alpha f(x)| \leq C \langle x \rangle^{-1-|\alpha|} \right\},$$

where $\langle x \rangle$ denotes the multiplication operator by $(1 + |x|^2)^{\frac{1}{2}}$ acting on each component of \mathcal{H} . The perturbation will be given by the matrix-valued multiplication operator $V(x) = V_1(x) + V_2(x)\Gamma^0$ acting on \mathcal{H} . The potential V_1 (resp. $V_2\Gamma^0$) is understood as an *electric potential* (resp. *scalar potential* i.e. $V_2\Gamma^0$ corresponds to an x-dependent rest mass). We refer to [75], chapter 4, for a presentation of the usual external fields for Dirac equation. Thus the interacting Dirac operator given by the sum of H_0 and V ,

$$H = H_0 + V(x), \tag{I.2.1}$$

is a self-adjoint operator on the domain $D(H) = D(H_0)$ by the Kato-Rellich theorem.

It is well known (see [75], chapter 1) that the spectrum of H_0 has the following structure

$$\sigma(H_0) = \sigma_{ac}(H_0) = (-\infty, -m] \cup [m, +\infty).$$

Moreover, the assumptions imposed on the interaction V imply that the difference of the resolvents $(H - z)^{-1} - (H_0 - z)^{-1} = -(H - z)^{-1}V(x)(H_0 - z)^{-1}$ is a compact operator in \mathcal{H} . Therefore by the Weyl theorem, the essential spectrum of H is the same as the essential spectrum of H_0 .

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = (-\infty, -m] \cup [m, +\infty).$$

However the operator H may have non-empty pure point spectrum. It will be crucial for the later analysis that the operator H be invertible. But assuming that $0 \in \sigma(H)$, it is then possible to find a smooth positive function e.g. $f \in \mathcal{S}(\mathbb{R}^3)$ the space of Schwartz functions such that the operator $H + f(x)$ be invertible. Thus, up to a smooth function, we can always consider that $0 \notin \sigma(H)$. For more details we refer to [10].

I.2.2 Domain invariance

In this section, we are interested in studying the invariance of the domain $D(\langle x \rangle^n)$, $n \in \mathbb{N}$ under the action of the unitary one-parameter group $\{e^{-itH}\}_{t \in \mathbb{R}}$. As a consequence, we shall also obtain some information on the invariance of $D(\langle x \rangle^n)$, $n \in \mathbb{N}$ under the action of the resolvent $(H - z)^{-1}$, $z \notin \sigma(H)$ and of any operator $\chi(H)$ with $\chi \in C_0^\infty(\mathbb{R})$. We state now the main result.

Theorem I.2.1 *Let H be the Dirac operator (I.2.1). Let x be the standard position observable. Then for any $n \in \mathbb{N}$,*

$$e^{-itH} D(\langle x \rangle^n) \subset D(\langle x \rangle^n), \quad (\text{I.2.2})$$

and there exists a constant C_n such that

$$\|\langle x \rangle^n e^{-itH} \psi\| \leq C_n (1 + |t|)^n \|\langle x \rangle^n \psi\|. \quad (\text{I.2.3})$$

Proof : We proceed by induction on n . The result is trivial for $n = 0$ and given $n > 0$ assume that (I.2.2) is satisfied for $n - 1$. The key of the proof is to approach $\langle x \rangle^n$ by a bounded operator X_λ for which estimate (I.2.2) is true uniformly in λ . We define

$$X_\lambda = \frac{\langle x \rangle^n}{1 + \lambda \langle x \rangle^n}.$$

Clearly we have

$$\lim_{\lambda \rightarrow 0} X_\lambda \psi = \langle x \rangle^n \psi, \quad \forall \psi \in D(\langle x \rangle^n),$$

and this operator is bounded as well as its derivative

$$[\partial_x, X_\lambda] = \frac{nx \langle x \rangle^{n-2}}{(1 + \lambda \langle x \rangle^n)^2}.$$

Let us compute the Heisenberg derivative of X_λ .

$$\begin{aligned} \frac{d}{dt} X_\lambda(t) &= e^{itH} i[H, X_\lambda] e^{-itH}, \\ &= e^{itH} \Gamma i[p, X_\lambda] e^{-itH}, \\ &= e^{itH} \Gamma \frac{n \frac{x}{\langle x \rangle} \langle x \rangle^{n-1}}{(1 + \lambda \langle x \rangle^n)^2} e^{-itH}. \end{aligned} \quad (\text{I.2.4})$$

Integrating (I.2.4) between 0 and t , we obtain $\forall \psi \in D(\langle x \rangle^n)$

$$\begin{aligned} \|X_\lambda e^{-itH} \psi\| &\leq \|X_\lambda \psi\| + \int_0^{|t|} \left\| \Gamma \frac{n \frac{x}{\langle x \rangle} \langle x \rangle^{n-1}}{(1 + \lambda \langle x \rangle^n)^2} e^{-isH} \psi \right\| ds, \\ &\leq \|X_\lambda \psi\| + n \int_0^{|t|} \left\| \frac{\langle x \rangle^{n-1}}{(1 + \lambda \langle x \rangle^n)^2} e^{-isH} \psi \right\| ds, \end{aligned} \quad (\text{I.2.5})$$

$$\leq \|X_\lambda \psi\| + n \int_0^{|t|} \|\langle x \rangle^{n-1} e^{-isH} \psi\| ds. \quad (\text{I.2.6})$$

As $D(\langle x \rangle^n) \subset D(\langle x \rangle^{n-1})$, the induction hypothesis implies

$$\|X_\lambda e^{-itH} \psi\| \leq \|X_\lambda \psi\| + n C_{n-1} \|\langle x \rangle^{n-1} \psi\| \int_0^{|t|} (1 + |s|)^{n-1} ds. \quad (\text{I.2.7})$$

The right-hand side of (I.2.7) is uniformly bounded in λ , as $\lambda \rightarrow 0$. Therefore, computing the last integral, we obtain

$$\begin{aligned} \|\langle x \rangle^n e^{-itH} \psi\| &\leq \|\langle x \rangle^n \psi\| + n C_{n-1} \|\langle x \rangle^{n-1} \psi\| (1 + |t|)^n, \\ &\leq C_n (1 + t)^n \|\langle x \rangle^n \psi\|, \end{aligned} \quad (\text{I.2.8})$$

which concludes the proof.

◇

Theorem I.2.1 has the following corollary, essential to derive the weak propagation estimates in Section 4.

Corollary I.2.1 *Let H a self-adjoint operator satisfying the conclusions of Theorem I.2.1. Let $z \in \mathbb{C}$ such that $\text{Im } z \neq 0$ and $n \in \mathbb{N}$. Then*

$$(H - z)^{-1} D(\langle x \rangle^n) \subset D(\langle x \rangle^n). \quad (\text{I.2.9})$$

Proof : It is an easy consequence of (I.2.2) and the resolvent formula

$$(H - z)^{-1} \psi = i\epsilon \int_0^\infty e^{i\epsilon z t} e^{-itH} \psi dt, \quad \forall \psi \in \mathcal{H}, \quad (\text{I.2.10})$$

where $\epsilon = \text{sgn}(\text{Im } z)$ and the integral converges in norm. As $\langle x \rangle^n$ is closed, we have $\forall \psi \in D(\langle x \rangle^n)$,

$$\begin{aligned} \|\langle x \rangle^n (H - z)^{-1} \psi\| &= \left\| \int_0^\infty e^{i\epsilon z t} \langle x \rangle^n e^{-itH} \psi dt \right\|, \\ &\leq C_n \|\langle x \rangle^n \psi\| \int_0^\infty (1 + t)^n e^{-\epsilon(\text{Im } z)t} dt. \end{aligned} \quad (\text{I.2.11})$$

Let us denote I_n this last integral. An integration by part yields

$$I_n = \frac{1}{\epsilon \text{Im } z} (1 + n I_{n-1}). \quad (\text{I.2.12})$$

We deduce by induction from (I.2.12) that

$$I_n \leq C_n \frac{\langle z \rangle^n}{|\text{Im } z|^{n+1}}, \quad (\text{I.2.13})$$

which is bounded for z fixed. This proves the corollary.

◇

I.2.3 Zitterbewegung and velocity operator

In this section, we give a short presentation of the *Zitterbewegung* phenomenon which naturally arises when one tries to define the “velocity” operator for the free Dirac operator H_0 . For more details, we refer to [75].

The velocity operator is usually defined as the time derivative of the position operator. The most natural choice is to consider the operator of multiplication by x acting on \mathcal{H} . Then we define

$$x(t) = e^{itH_0} x e^{-itH_0},$$

the time translated position operator. Formally we have

$$\begin{aligned} \frac{dx(t)}{dt} &= e^{itH_0} i[H_0, x] e^{-itH_0}, \\ &= e^{itH_0} \Gamma e^{-itH_0} =: \Gamma(t). \end{aligned} \quad (\text{I.2.14})$$

According to classical relativistic kinematics, we would have expected to obtain $\mathcal{V} = p H_0^{-1}$ i.e. the classical velocity operator instead of $\Gamma(t)$. Let us analyse the time dependence of $\Gamma(t)$.

$$\begin{aligned} \frac{d\Gamma(t)}{dt} &= e^{itH_0} i[H_0, \Gamma] e^{-itH_0}, \\ &= e^{itH_0} iH_0 [\Gamma - H_0^{-1} \Gamma H_0] e^{-itH_0}. \end{aligned}$$

An explicit short calculation shows that $\Gamma - H_0^{-1} \Gamma H_0 = 2(\Gamma - \mathcal{V})$. Thus

$$\frac{dx(t)}{dt} = 2iH_0 F(t), \quad (\text{I.2.15})$$

where $F = \Gamma - \mathcal{V}$. The operator F is one aspect of the *Zitterbewegung* phenomenon. One of its main features is that it anticommutes with H_0

$$F H_0 = -H_0 F. \quad (\text{I.2.16})$$

We conclude from (I.2.16) that $F(t) = e^{2itH_0} F$ and integrating (I.2.15) between 0 and t , we see that

$$\Gamma(t) = \mathcal{V} + e^{2itH_0} F.$$

Thus we see that the standard velocity oscillates without damping around the classical velocity operator \mathcal{V} and this oscillation is called *Zitterbewegung*. Integrating again, we obtain

$$x(t) = x + p H_0^{-1} t + \frac{1}{2iH_0} (e^{2itH_0} - 1) F.$$

All these formal results can be made rigorous and we have the following theorem (Thaller [75], thm 1.3, p 20)

Theorem I.2.2 *The domain $D(x)$ of the multiplication operator x is left invariant by the free evolution*

$$D(x(t)) \subset D(x),$$

and on this domain, we have

$$x(t) = x + p H_0^{-1} t + \frac{1}{2iH_0} (e^{2itH_0} - 1) F. \quad (\text{I.2.17})$$

The problem arising from the Dirac equation is that certain observables such as the position operator $x(t)$, mix positive and negative energies. To overcome this difficulty we can choose some other operators to be the position observables of the theory. In particular we are interested in the so-called *Newton-Wigner* observable. Let us first define the Foldy-Wouthuysen transformation U_{FW} . This transformation diagonalizes the Dirac operator H_0 in \mathcal{H} . Denoting \mathcal{F} the Fourier transform on \mathcal{H} we have

$$(\mathcal{F}H_0\mathcal{F}^{-1})(k) = \Gamma \cdot k + m\Gamma^0, \quad (\text{I.2.18})$$

and for each $k \in \mathbb{R}^3$, the right-hand side of (I.2.18) is a Hermitian 4×4 matrix which has the two eigenvalues $\{-\lambda(k), \lambda(k)\}$ where $\lambda(k) = \sqrt{k^2 + m^2}$ and both eigenvalues have multiplicity 2. Let us call $P(k)$ the unitary matrix such that

$$P(k)(\Gamma \cdot k + m\Gamma^0)P(k)^{-1} = \begin{pmatrix} \lambda(k) & 0 & 0 & 0 \\ 0 & \lambda(k) & 0 & 0 \\ 0 & 0 & -\lambda(k) & 0 \\ 0 & 0 & 0 & -\lambda(k) \end{pmatrix},$$

where

$$P(k) = \frac{(m + \lambda(k))\mathbf{1} + \Gamma^0\Gamma \cdot k}{\sqrt{2\lambda(k)(m + \lambda(k))}} = a_+(k)\mathbf{1} + a_-(k)\Gamma^0\frac{\Gamma \cdot k}{|k|}, \quad (\text{I.2.19})$$

and

$$a_{\pm} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{m}{\lambda(k)}}. \quad (\text{I.2.20})$$

We also have

$$P^{-1}(k) = \frac{(m + \lambda(k))\mathbf{1} - \Gamma^0\Gamma \cdot k}{\sqrt{2\lambda(k)(m + \lambda(k))}} = a_+(k)\mathbf{1} - a_-(k)\Gamma^0\frac{\Gamma \cdot k}{|k|}, \quad (\text{I.2.21})$$

We define the Foldy-Wouthuysen transformation $U_{FW} = P(p) = \mathcal{F}^{-1}P(k)\mathcal{F}$ acting from \mathcal{H} to \mathcal{H} . This transformation is clearly unitary on \mathcal{H} and H_0 conjugated by U_{FW} can be written as

$$U_{FW}H_0U_{FW}^{-1} = \begin{pmatrix} \sqrt{p^2 + m^2} & 0 \\ 0 & -\sqrt{p^2 + m^2} \end{pmatrix}.$$

Whence H_0 is unitarily equivalent to a pair of square root of Klein-Gordon hamiltonians.

Now we turn to the definition of the Newton-Wigner operator denoted x_{nw} . We set

$$x_{nw} = U_{FW}^{-1}xU_{FW}.$$

This operator has the following properties (see [75])

- It leaves invariant positive and negative energy subspaces of H_0 i.e. $[x_{nw}, \mathbf{1}_{\mathbb{R}^{\pm}}(H_0)] = 0$.
- On $D(x_{nw})$, the following equality holds

$$x_{nw}(t) = e^{itH_0}x_{nw}e^{-itH_0} = x_{nw} + \mathcal{V}t, \quad (\text{I.2.22})$$

or equivalently

$$i[H_0, x_{nw}] = \mathcal{V}. \quad (\text{I.2.23})$$

It is worth pointing out that the Zitterbewegung does not appear in the formula (I.2.22). This important feature of the Newton-Wigner operator will be helpful for the construction of the asymptotic velocity in the next sections. However the results we need to prove involve the standard position operator and so, we have to make the link between x and x_{nw} . Unfortunately we have the following complicated formula, see [75]

$$x_{nw} = x - \frac{\Gamma^0}{2i\lambda} \left(\Gamma - \frac{1}{\lambda(\lambda+m)} (\Gamma \cdot p) p \right) - \frac{1}{\lambda(\lambda+m)} S \wedge p, \quad (\text{I.2.24})$$

where S denotes the spin angular momentum and is defined by $S = -\frac{i}{4} \Gamma \wedge \Gamma$ and $\lambda = \sqrt{p^2 + m^2}$. The symbol $\Gamma \wedge \Gamma$ denotes the three matrices $\sum_{k,l} \epsilon_{jkl} \Gamma^k \Gamma^l$ where ϵ is the totally antisymmetric tensor. Observe that the spin angular momentum S is bounded, everywhere defined and self-adjoint. Concisely, we shall denote by Z the bounded operator on \mathcal{H} such that $x_{nw} = x + Z$. As the expression of Z is difficult to handle, the Newton-Wigner operator has not been used in previous works [11], [29] or [34] on Dirac's equation. However we shall need this operator for deriving the microlocal velocity estimates in Section 4 especially the formula (I.2.23) will be of great help to us. Eventually the only information on Z we shall need is given by the following lemma.

Lemma I.2.1 *The commutator $[x, Z]$ between x and Z is a bounded operator in \mathcal{H} .*

Proof : This result follows from the definition (I.2.24) of Z . According to this definition, we can view the operator Z as a matrix-valued function of p . Moreover we have

$$Z = (Z_{ij}(p))_{ij}, \text{ where } Z_{ij} \in S^{-1}(\mathbb{R}^3), \forall i, j = 1, 2, 3.$$

Therefore using the Helffer-Sjöstrand formula (I.A.29), we get for any $m = 1, 2, 3$

$$\begin{aligned} [x_m, Z] &= \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{Z}(z) \sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) [x_m, p_l] \left(\prod_{k_2=l}^3 R_{k_2} \right) dz \wedge d\bar{z}, \\ &= \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{Z}(z) \sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) i\delta_{lm} \left(\prod_{k_2=l}^3 R_{k_2} \right) dz \wedge d\bar{z}, \end{aligned}$$

where $R_k = (z_k - p_k)^{-1}$. Thus $[x_m, Z]$ is bounded by (I.A.22).

◇

I.3 Locally conjugate operator

The main idea of Mourre theory is to find an operator, usually denoted by A , which increases along the evolution e^{-itH} in a suitable sense. If we denote $A(t) = e^{itH} A e^{-itH}$, this means that the time derivative of $A(t)$ must be essentially positive or equivalently, the commutator between H and A has to be essentially positive. Precisely, can we find an open interval Δ of \mathbb{R} , a strictly positive constant ϵ and a compact operator K on \mathcal{H} such that

$$\mathbf{1}_\Delta(H) i[H, A] \mathbf{1}_\Delta(H) \geq \epsilon \mathbf{1}_\Delta^2(H) + \mathbf{1}_\Delta(H) K \mathbf{1}_\Delta(H). \quad (\text{I.3.1})$$

In the case of the Schrödinger operator, the usual generator of dilation $D = \frac{1}{2} \{x \cdot p + p \cdot x\}$ is a good candidate for a conjugate operator. It turns out that the same operator also satisfies (I.3.1), when H is the

Dirac operator previously defined, for a suitable open interval Δ in \mathbb{R} . However, there exist many other possible choices for A which may be more adapted to the Dirac equation and make the verification of the assumptions easier. For instance, see [11], [37] and [50]. We shall use a locally conjugate operator which is close to the choice made in [50].

Let us define the operator

$$A_0 = \frac{1}{2} \left\{ x.pH_0^{-1} + H_0^{-1}p.x \right\}.$$

Concisely, we write $A_0 = \frac{1}{2} \{x.\mathcal{V} + \mathcal{V}.x\}$ and commuting x and \mathcal{V} , it is easy to see that $A_0 = \mathcal{V}.x + B_0$ where $B_0 = \frac{i}{2}H_0^{-1}\Gamma.F$ is a bounded operator in \mathcal{H} . It is defined and essentially self-adjoint on $D(x)$ (see [50], Lemma 3.1). This operator has the important property

$$i[H_0, A_0] = \mathcal{V}^2, \quad (\text{I.3.2})$$

where the commutator is computed on a suitable domain e.g. $C_0^\infty(\mathbb{R}^3)$. Therefore the commutator between A_0 and H_0 is positive which seems to indicate that it could be another good candidate for being the operator A . Unfortunately, the addition of a matrix-valued potential V to H_0 prevents us from proving the Mourre estimate (I.3.1) for (H, A_0) .

Actually, it is better to consider the following operator

$$A = \frac{1}{2} \left\{ x.pH^{-1} + H^{-1}p.x \right\}, \quad (\text{I.3.3})$$

which is also defined and essentially self-adjoint on $D(x)$ (see below). It is easy to see that this operator is related to A_0 by a bounded operator. Precisely, using the resolvent identity, we have

$$A = A_0 - \frac{1}{2}x.pH^{-1}V(x)H_0^{-1} - \frac{1}{2}H_0^{-1}V(x)H^{-1}p.x = A_0 + B,$$

and B is bounded. Thus the Heisenberg derivative of A will be essentially \mathcal{V}^2 plus or minus some compact terms. Indeed,

$$\begin{aligned} i[H, A] &= \frac{i}{2} \left[H, x.pH^{-1} + H^{-1}p.x \right], \\ &= \frac{1}{2} \left(\Gamma.pH^{-1} + H^{-1}p.\Gamma \right) - \frac{1}{2} \left(x.\nabla V(x)H^{-1} + H^{-1}\nabla V(x).x \right). \end{aligned} \quad (\text{I.3.4})$$

The second term in the right-hand side of (I.3.4) is clearly compact in \mathcal{H} since $x.\nabla V(x)$ belongs to $S^{-1}(\mathbb{R}^3)$ and thus the standard compactness criterion applies.

To see that the first term in the right-hand side of (I.3.4) is essentially equal to \mathcal{V}^2 , observe that

$$\Gamma.pH_0^{-1} - \Gamma.pH^{-1} = \Gamma.pH_0^{-1}V(x)H^{-1} = L,$$

where the operator L is compact in \mathcal{H} by the standard compactness criterion. Thus this term is equal to

$$\frac{1}{2} \left(\Gamma.pH^{-1} + H^{-1}p.\Gamma \right) = \frac{1}{2} \left(\Gamma.\mathcal{V} + \mathcal{V}.\Gamma \right) - \frac{1}{2} \left(L + L^* \right).$$

Now we write $\Gamma = \mathcal{V} + F$ and noting that $\frac{1}{2}(\mathcal{V}.F + F.\mathcal{V}) = 0$, we eventually obtain

$$i[H, A] = \mathcal{V}^2 - \frac{1}{2} \left(L + L^* \right) - \frac{1}{2} \left(x.\nabla V(x)H^{-1} + H^{-1}\nabla V(x).x \right), \quad (\text{I.3.5})$$

or concisely, $i[H, A] = \mathcal{V}^2 + K$ where K is compact. We use this result to prove the following lemma

Lemma I.3.1 *Let Δ be an open interval of \mathbb{R} such that $\Delta \cap [-m, +m] = \emptyset$. Then there exists a strictly positive constant ϵ depending on Δ and a compact operator K such that (I.3.1) holds.*

Proof : Let $\chi \in C_0^\infty(\mathbb{R})$ a function such that $\text{supp } \chi \subset (-\infty, -m) \cup (+m, +\infty)$ and $\chi = 1$ on Δ . Let us compute the commutator

$$\begin{aligned} \chi(H)i[H, A]\chi(H) &= \chi(H)\mathcal{V}^2\chi(H) + K, \\ &= \chi(H_0)\mathcal{V}^2\chi(H_0) + K, \end{aligned}$$

because $\chi(H) - \chi(H_0)$ is compact. But if we diagonalize H_0 via the Foldy-Wouthuysen transformation, the first term can be written as

$$\begin{aligned} \chi(H_0)\mathcal{V}^2\chi(H_0) &= U_{FW}^{-1}\chi(\Gamma^0\lambda(p))U_{FW}\mathcal{V}^2U_{FW}^{-1}\chi(\Gamma^0\lambda(p))U_{FW}, \\ &= U_{FW}^{-1}\Gamma^0\chi(\lambda(p))\frac{p^2}{p^2+m^2}\Gamma^0\chi(\lambda(p))U_{FW}. \end{aligned}$$

Now provided the support of χ strictly avoids $[-m, +m]$, there exists a strictly positive constant $\tilde{\epsilon}$ such that $\chi(\lambda(p)) = 0$ for $p^2 < \tilde{\epsilon}$. Hence we get

$$U_{FW}^{-1}\Gamma^0\chi(\lambda(p))\frac{p^2}{p^2+m^2}\Gamma^0\chi(\lambda(p))U_{FW} \geq \frac{\tilde{\epsilon}}{\tilde{\epsilon}+m^2}U_{FW}^{-1}\Gamma^0\chi(\lambda(p))\Gamma^0\chi(\lambda(p))U_{FW},$$

or, if we set $\epsilon = \frac{\tilde{\epsilon}}{\tilde{\epsilon}+m^2}$,

$$\chi(H_0)\mathcal{V}^2\chi(H_0) \geq \epsilon\chi^2(H_0),$$

which concludes the proof of the lemma.

◇

Even though the Mourre estimate (I.3.1) is the crucial property of the a locally conjugate operator, some extra assumptions are needed to obtain a complete scattering theory. There exist several versions in the litterature depending on the degree of refinement required by the problem. A very complete account of the theory can be found in [1]. We shall use for the definition of a locally conjugate operator a certain notion of regularity between two self-adjoint operators introduced by Amrein, Boutet de Monvel-Berthier, Georgescu [1]. Precisely,

Definition I.3.1 *For a selfadjoint operator A , we say that another selfadjoint operator H belongs to $C^k(A)$, $k \in \mathbb{N}$, if and only if*

$$\text{(ABG)} \quad \exists z \in \mathbb{C} \setminus \sigma(H), s \longrightarrow e^{isA}(H-z)^{-1}e^{-isA} \in C^k(\mathbb{R}_s; \mathcal{B}(\mathcal{H})),$$

for the strong topology of $\mathcal{B}(\mathcal{H})$.

We give now the definition of a locally conjugate operator A and the main theorem we shall use.

Definition I.3.2 *Let H, A two self-adjoint operators on \mathcal{H} . Let $\Delta \subset \mathbb{R}$ an open interval. We shall say that A is a locally conjugate operator of H on Δ if it satisfies the following assumptions*

- (i) $H \in C^1(A)$.
- (ii) $i[H, A]$ defined as a quadratic form on $D(H) \cap D(A)$ extends to an element of $\mathcal{B}(D(H), \mathcal{H})$.

- (iii) $[A, [A, H]]$ well defined as a quadratic form on $D(H) \cap D(A)$ by (ii), extends to an element of $\mathcal{B}(D(H), D(H)^*)$.
- (iv) There exists a strictly positive constant ϵ and a compact operator K such that the Mourre estimate (I.3.1) holds.

Theorem I.3.1 *Let H, A two selfadjoint operators on \mathcal{H} . Assume that A is a locally conjugate operator of H on the interval Δ . Then H has no singular continuous spectrum in Δ and the number of eigenvalues of H in Δ is finite (counting multiplicity).*

The assumptions on the commutators are rather straightforward to check by a direct computation on a suitable dense domain e.g. $C_0^\infty(\mathbb{R}^3)$. On the other hand, the first assumption is somewhat more subtle. We have the following equivalent definitions for $C^1(A)$.

- (ABG') $\exists z \in \mathbb{C} \setminus \sigma(H)$, $|((H - z)^{-1}u, Au) - (Au, (H - z)^{-1}u)| \leq C\|u\|^2$, $\forall u \in D(A) \cap D(H)$.
- (ABG'') (i) $\exists z \in \mathbb{C} \setminus \sigma(H)$, $(H - z)^{-1}D(A) \subset D(A)$, $(H - \bar{z})^{-1}D(A) \subset D(A)$,
(ii) $|(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2)$, $\forall u \in D(H) \cap D(A)$.

Unfortunately, it is not easy to check $(H - z)^{-1}D(A) \subset D(A)$ in general without a better knowledge of $D(A)$. Therefore it is useful to consider another operator N called a *comparison operator* whose domain has to be well-known and that will allow to make the link between H and A . We shall use the following lemmata [38]

Lemma I.3.2 (Nelson) *Let $N \geq 1$ a self-adjoint operator on \mathcal{H} . Let A a symmetric operator on \mathcal{H} such that $D(N) \subset D(A)$. Assume that*

$$\begin{aligned} (i) \|Au\| &\leq C\|Nu\|, \quad \forall u \in D(N), \\ (ii) |(Au, Nu) - (Nu, Au)| &\leq C\|N^{\frac{1}{2}}u\|^2, \quad \forall u \in D(N). \end{aligned} \tag{I.3.6}$$

Then A is essentially self-adjoint on $D(N)$. Furthermore every core of N is also a core for A .

Lemma I.3.3 (Gérard, Laba) *Let H, H_0 and N three self-adjoint operators on \mathcal{H} satisfying $N \geq 1$, $D(H) = D(H_0)$ and $(H - z)^{-1}D(N) \subset D(N)$. Let A a symmetric operator on $D(N)$. Assume that H_0 and A satisfy the assumptions of Lemma I.3.2 and*

$$|(Au, Hu) - (Hu, Au)| \leq C(\|Hu\|^2 + \|u\|^2), \quad \forall u \in D(N). \tag{I.3.7}$$

Then we have

- $D(N)$ is dense in $D(A) \cap D(H)$ with the norm $\|Hu\| + \|Au\| + \|u\|$,
- the quadratic form $i[H, A]$ defined on $D(A) \cap D(H)$ is the unique extension of $i[H, A]$ on $D(N)$,
- $H \in C^1(A)$.

We now prove the assumptions of Theorem I.3.1 with H the Dirac operator and A the operator defined above. Let us define $N = p^2 + x^2 + 1 = p^2 + \langle x \rangle^2$ as the comparison operator. This operator is essentially selfadjoint on $[C_0^\infty(\mathbb{R}^3)]^4$ by Theorem X.28 in [69], Vol 3. We also denote by N its closure which is a selfadjoint operator on \mathcal{H} with domain $D(N) = \{u \in \mathcal{H}, Nu \in \mathcal{H}\}$. Moreover, it is easy to see that

$$\begin{aligned} \|x^2 u\|^2 &\leq \|Nu\|^2 + \|u\|^2, \\ \|p^2 u\|^2 &\leq \|Nu\|^2 + \|u\|^2, \end{aligned}$$

from which it follows that the domain $D(N)$ is characterized by ([69], chapter X, problem 23)

$$D(N) = [H^2(\mathbb{R}^3)]^4 \cap [D(x^2)]^4.$$

Now, we check the assumptions of Nelson's Lemma for H_0 and A .

- $D(N) \subset D(H_0)$, $D(N) \subset D(A)$ are obvious.
- For any $u \in D(N)$, we have

$$\begin{aligned} \|H_0 u\|^2 &= (u, p^2 u) + m^2 \|u\|^2, \\ &\leq C \|u\| (\|Nu\| + \|u\|), \\ &\leq C \|Nu\|^2. \end{aligned}$$

$$\begin{aligned} \|Au\| &= \frac{1}{2} \sum_j \left\| \left(x^j p_j H^{-1} + H^{-1} p_j x^j \right) u \right\|, \\ &= \frac{1}{2} \sum_j \left\{ \|p_j H^{-1} x^j u\| + \|H^{-1} p_j x^j u\| + \|[x^j, p_j H^{-1}]u\| \right\}, \\ &\leq \frac{1}{2} \sum_j \left\{ C \|x^j u\| + \|(iH^{-1} - ip_j H^{-1} \Gamma^j H^{-1})u\| \right\}, \\ &\leq C \sum_j \left\{ \|\langle x \rangle u\| + \|u\| \right\} \leq C \|Nu\|. \end{aligned}$$

where we used the fact that $H^{-1} p_j$ and $p_j H^{-1}$ are bounded on \mathcal{H} for any $j = 1, 2, 3$.

- It remains to see that for any $u \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} |(u, i[H_0, N]u)| &= |(u, i[H_0, \langle x \rangle^2]u)|, \\ &= |(u, 2\Gamma \cdot xu)|, \\ &\leq C \|u\| \|\langle x \rangle u\|, \\ &\leq C \|\langle x \rangle u\|^2, \\ &\leq C \|N^{\frac{1}{2}} u\|^2. \end{aligned}$$

We also have

$$|(u, [A, N]u)| = \frac{1}{2} |(u, [x \cdot p H^{-1} + H^{-1} p \cdot x, p^2 + x^2 + 1]u)|.$$

A straightforward calculation leads to

$$\begin{aligned} [x \cdot p H^{-1}, p^2 + x^2] &= 2ip^2 H^{-1} - ix \cdot p H^{-1} \left(\nabla V(x) \cdot p + p \cdot \nabla V(x) \right) H^{-1} + 2ix \cdot p H^{-1} \Gamma \cdot H^{-1} x \\ &\quad - 2ix \cdot H^{-1} x - 2x \cdot H^{-1} \Gamma H^{-1} + 2x \cdot p H^{-1} \Gamma \cdot H^{-1} \Gamma H^{-1}. \end{aligned}$$

Whence

$$\begin{aligned} |(u, [x \cdot p H^{-1}, p^2 + x^2]u)| &\leq C \left\{ \|\langle p \rangle u\| \|u\| + \|\langle x \rangle u\|^2 + \|\langle x \rangle u\| \|u\| \right\}, \\ &\leq C \|N^{\frac{1}{2}} u\|. \end{aligned}$$

The same estimate is true for the term $(u, [H^{-1} p \cdot x, p^2 + x^2]u)$. Thus the assumptions of Nelson's lemma are proved.

Moreover, the assumptions of the Gérard - Laba Lemma are entirely fulfilled since $(H - z)^{-1}D(\langle x \rangle^2) \subset D(\langle x \rangle^2)$ by domain invariance properties of Section 2 and since the commutator between H and A is bounded in \mathcal{H} by (I.3.5). Therefore, we have proved that $H \in C^1(A)$ as well as the second assumption of theorem I.3.1.

The hypothesis on the double commutator can easily be checked and it turns out that it is a bounded operator in \mathcal{H} . Recall that

$$[A, i[H, A]] = \left[A, \mathcal{V}^2 - \frac{1}{2}(L + L^*) - \frac{1}{2}(x \cdot \nabla V(x)H^{-1} + H^{-1}\nabla V(x) \cdot x) \right].$$

As the operator A is equal to $\mathcal{V} \cdot x + B$ where \mathcal{V}, B are bounded and as $i[H, A]$ is also bounded, we actually just need show that the commutator between $x \cdot \mathcal{V}$ and $i[H, A]$ remains bounded. We decompose the problem. First, $[x \cdot \mathcal{V}, \mathcal{V}^2] = [x, \mathcal{V}^2] \cdot \mathcal{V}$ is clearly bounded. According to the definition of $L = \Gamma \cdot p H_0^{-1} V(x) H^{-1}$, we see after some commutations that both terms in the following commutator

$$[x \cdot \mathcal{V}, L] = x \cdot \mathcal{V} L - L x \cdot \mathcal{V},$$

are bounded since $x \cdot V(x) \in L^\infty(\mathbb{R}^3)$. The fact that the remaining term is bounded follows immediately by the same procedure since $x \cdot (x \cdot \nabla V(x))$ also belongs to $L^\infty(\mathbb{R}^3)$.

We have thus proven the theorem

Theorem I.3.2 *Let H be the dirac operator defined above. Then the spectrum of H has no singular continuous spectrum. Moreover, $\sigma_{ac}(H) = (-\infty, -m) \cup (+m, +\infty)$ and in any compact interval contained in $(-\infty, -m) \cup (+m, +\infty)$, the number of eigenvalues is finite.*

We deduce from this theorem and $\sigma_{ess}(H) = (-\infty, -m] \cup [+m, +\infty)$ that $\sigma_{pp}(H) \subset [-m, +m]$.

Before we turn to the minimal velocity estimates, we make the following remark. The assumptions required in Theorem I.3.1 actually imply that the operator H belongs to the class $C^2(A)$. From [1] (Theorem 6.3.1), we know that if we assume the invariance of $D(H)$ under the action of the unitary one-parameter group e^{isA} , then the conditions $i[H, A]$ and $[A, [A, H]]$ bounded on \mathcal{H} entail that $H \in C^2(A)$. But the condition on the invariance follows from the following Lemma quoted in [36]

Lemma I.3.4 *Let H, A two selfadjoint operators such that $H \in C^1(A)$ and $i[H, A] \in \mathcal{B}(D(H), \mathcal{H})$ then $e^{isA}D(H) \subset D(H)$ for all $s \in \mathbb{R}$.*

I.4 Weak propagation estimates

The following weak propagation estimates denoted **WPE** are the main ingredients for constructing the asymptotic velocity. They take the general form

$$\int_1^\infty \left\| B(t) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H},$$

where $B(t)$ is a time-dependent self-adjoint operator on \mathcal{H} . These estimates give a very weak fall-off with respect to t of the function under the integral. We are mainly interested in the *maximal* and *minimal velocity estimates* which, roughly speaking, assert that given a state ψ in \mathcal{H} with bounded energy, there exist two constants c_m and c_M such that the ‘‘particle’’ can neither escape faster than c_M nor slower than c_m . The last type of estimates called *microlocal estimates* will help us to give another definition for the asymptotic velocity in terms of the classical velocity operator that will allow us to study its spectrum.

For the proof of the **WPE**, we shall use the following proposition given in [21], (p 384). We recall that $\mathbf{D} = \frac{d}{dt} + i[H, \cdot]$ is the Heisenberg derivative and satisfies

$$\frac{d}{dt} \left[e^{itH} B(t) e^{-itH} \right] = e^{itH} \mathbf{D} B(t) e^{-itH}.$$

Proposition I.4.1 *Let $\Phi(t)$ be a family of self-adjoint operators belonging to $W_{loc}^{1,1}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ i.e. there exists $B(t) \in L_{loc}^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ such that*

$$\Phi(t_1) - \Phi(t_2) = \int_{t_1}^{t_2} B(s) ds.$$

(i) *Assume $\mathbf{D}\Phi(t) \in L_{loc}^1(\mathbb{R}^+, \mathcal{B}(\mathcal{H}))$. Then*

$$\|\Phi(t)e^{-itH}\Psi\| \leq \|\Phi(0)\Psi\| + \int_0^t \|\mathbf{D}\Phi(s)e^{-isH}\Psi\| ds.$$

(ii) *Assume that $\Phi(t)$ is uniformly bounded and that there exists $C_0 > 0$ and some operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \dots, n$ such that*

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^n B_i^*(t)B_i(t),$$

with

$$\int_1^\infty \|B_i(t)e^{-itH}\psi\|^2 dt \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad i = 1, \dots, n.$$

Then there exists a constant C such that

$$\int_1^\infty \|B(t)e^{-itH}\psi\|^2 dt \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{I.4.1})$$

We stress the fact that the ideas of the proof are very simple, the essential step being to find an observable $\Phi(t)$ called *propagation observable* such that its Heisenberg derivative is essentially positive. Before we turn to the proof of the estimates, we briefly indicate how to make the link with the existence of asymptotic observables i.e. with observables taking the form

$$s - \lim_{t \rightarrow \infty} e^{itH} \Phi(t) e^{-itH},$$

where $\Phi(t)$ is a self-adjoint operator valued function. For this we shall use the following lemma given in [21] but which contains results initially due to Cook and Kato.

Lemma I.4.1 (Cook, Kato) *Let $\Phi(t)$ be a uniformly bounded function with values in self-adjoint operators, belonging to $W^{1,1}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$. Let \mathcal{D} a dense subspace of \mathcal{H} .*

(i) (Cook) *Assume that $\forall \psi \in \mathcal{D}$,*

$$\int_1^\infty \|\mathbf{D}\Phi(t)e^{-itH}\psi\| dt < \infty,$$

then there exists

$$s - \lim_{t \rightarrow \infty} e^{itH} \Phi(t) e^{-itH}. \quad (\text{I.4.2})$$

(ii) (Kato) Assume that

$$|(\psi_2, \mathbf{D}\Phi(t)\psi_1)| \leq \sum_{i=1}^n \|B_{2i}(t)\psi_2\| \|B_{1i}(t)\psi_1\|,$$

with

$$\begin{aligned} \int_1^\infty \|B_{2i}(t)e^{-itH}\psi\|^2 dt &\leq C\|\psi\|^2, \quad \psi \in \mathcal{H}, i = 1, \dots, n, \\ \int_1^\infty \|B_{1i}(t)e^{-itH}\psi\|^2 dt &\leq C\|\psi\|^2, \quad \psi \in \mathcal{D}, i = 1, \dots, n, \end{aligned}$$

then the limit (I.4.2) exists.

I.4.1 Minimal velocity estimates

It has been well-known since Ruelle's theorem [70] that the states ψ belonging to the continuous subspace \mathcal{H}^c of H tend to escape for large time t in a mean ergodic sense, that is to say

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \int_{|x| \leq R} \|e^{-itH}\psi\|^2 dx = 0,$$

for any finite R . Nevertheless this decay is not sufficient to prove precise results of scattering theory. We need more subtle estimates on how fast the states move away from the centre of the interaction. The following minimal velocity estimates improve the previous result in a very weak sense but which is enough for applications. These estimates ensure that the probability to find the “particle” in a narrow cone $|x| < \theta t$ goes to zero when $t \rightarrow \infty$ for θ small enough. The “particle” here simply refers to the wave function $\psi(t, x)$, i.e. the field. There is no second quantization involved. Exactly, we shall prove the following proposition

Proposition I.4.2 *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m, \sigma_{pp}(H)\} = \emptyset$. Then there exists a constant $\epsilon_0(\chi) > 0$ depending on χ such that*

$$\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_0]} \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{I.4.3})$$

Furthermore,

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[0, \epsilon_0]} \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{I.4.4})$$

Before we give the proof, we mention that these estimates first appeared in the paper [72] by Sigal and Soffer and have been intensively used to organize the proofs of asymptotic completeness for N -body problems in quantum mechanics. It appears that there exists a strong link between the notion of locally conjugate operator and minimal velocity estimates as stated in the next proposition (our proof will follow the result obtained by C. Gérard and F. Nier in [39])

Proposition I.4.3 (Gérard, Nier) *Let H, A two self-adjoint operators on \mathcal{H} . Assume that for $\epsilon > 0$, $H \in C^{1+\epsilon}(A)$, and the Mourre estimate*

$$\mathbf{1}_\Delta(H) i[H, A] \mathbf{1}_\Delta(H) \geq c_0 \mathbf{1}_\Delta^2(H), \quad (\text{I.4.5})$$

holds on an open interval Δ . Then

(i) $\forall g \in C_0^\infty(\mathbb{R}), \forall \chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } g \subset (-\infty, c_0)$ and $\text{supp } \chi \subset \Delta$,

$$\int_1^\infty \left\| g\left(\frac{A}{t}\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{I.4.6})$$

and

$$s - \lim_{t \rightarrow \infty} g\left(\frac{A}{t}\right) \chi(H) e^{-itH} = 0. \quad (\text{I.4.7})$$

(ii) Furthermore assume that there exists another self-adjoint operator A_1 which satisfies

$$\begin{aligned} D(A_1) &\subset D(A), \\ \pm A &\leq C A_1, \quad A_1 \geq 1, \end{aligned}$$

$$[A, A_1] A_1^{-1} \in \mathcal{B}(\mathcal{H}),$$

then, $\forall \chi \in C_0^\infty(\mathbb{R}), \text{supp } \chi \subset \Delta$, there exists ϵ_0 small enough such that

$$\int_1^\infty \left\| \mathbf{1}\left(\frac{A_1}{t} \leq \epsilon_0\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{I.4.8})$$

and

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[0, \epsilon_0)}\left(\frac{A_1}{t}\right) \chi(H) e^{-itH} = 0. \quad (\text{I.4.9})$$

Proof of Proposition I.4.2 : We already know from the previous section that the Dirac operator H belongs to the class $C^2(A)$. Since $\sigma(H) \cap (-m, +m) \subset \sigma_{pp}(H)$, we can assume that $\text{supp } \chi \subset (-\infty, -m) \cup (+m, +\infty)$. Let $\lambda \in (-\infty, -m) \cup (+m, +\infty)$. We can find a closed interval I containing λ on which Mourre estimate (I.3.1) holds

$$\mathbf{1}_I(H) i[H, A] \mathbf{1}_I(H) \geq \epsilon \mathbf{1}_I(H) + \mathbf{1}_I(H) K \mathbf{1}_I(H).$$

Since $\lambda \notin \sigma_{pp}(H)$, we have

$$s - \lim_{|I| \rightarrow 0} \mathbf{1}_I(H) = \mathbf{1}_{\{\lambda\}}(H) = 0.$$

Now, the fact that K is compact implies that $K \mathbf{1}_I(H)$ tends to 0 in norm when $|I|$ tends to 0. Thus, if we consider I with $\lambda \in I$ with $|I|$ small enough, there exists c_0 a strictly positive constant such that

$$\mathbf{1}_I(H) i[H, A] \mathbf{1}_I(H) \geq c_0 \mathbf{1}_I(H).$$

Therefore, for any χ such that $\text{supp } \chi \subset I$, minimal velocity estimates hold for the operator A according to (i) of Proposition I.4.3. Now we apply the second part of this proposition with $A_1 = \langle x \rangle$. We have to check the following assumptions.

- $D(\langle x \rangle) \subset D(A)$ has been seen in the previous section.
- For all u in $D(A)$, we have

$$|(u, Au)| \leq \frac{1}{2} \sum_j \left(|(u, x^j p_j H^{-1} u)| + |(u, H^{-1} p_j x^j u)| \right). \quad (\text{I.4.10})$$

But the first term in (I.4.10) can be estimated as follows

$$\begin{aligned} |(u, x^j p_j H^{-1} u)| &\leq |(\frac{x^j}{\langle x^j \rangle^{\frac{1}{2}}} u, p_j H^{-1} \langle x^j \rangle^{\frac{1}{2}} u)| + |(\frac{x^j}{\langle x^j \rangle^{\frac{1}{2}}} u, [\langle x^j \rangle^{\frac{1}{2}}, p_j H^{-1}] u)|, \\ &\leq C \left(\|\langle x^j \rangle^{\frac{1}{2}} u\|^2 + |(\frac{x^j}{\langle x^j \rangle^{\frac{1}{2}}} u, \frac{x^j}{\langle x^j \rangle^{\frac{3}{2}}} H^{-1} u)| + |(\frac{x^j}{\langle x^j \rangle^{\frac{1}{2}}} u, p_j H^{-1} \Gamma^j \frac{\langle x^j \rangle}{\langle x^j \rangle^{\frac{3}{2}}} H^{-1} u)| \right), \\ &\leq C (\|\langle x^j \rangle^{\frac{1}{2}} u\|^2 + \|u\|^2) \leq C \|\langle x \rangle^{\frac{1}{2}} u\|^2. \end{aligned}$$

We can prove the same estimate for the second term in (I.4.10) and we obtain

$$|(u, Au)| \leq C(u, \langle x \rangle u).$$

- Eventually $[A, \langle x \rangle] \langle x \rangle^{-1}$ is clearly bounded. Indeed it suffices to expand the commutator and use the fact that $[p, \langle x \rangle]$ and $[H^{-1}, \langle x \rangle]$ are bounded in \mathcal{H} .

To conclude the proof, observe that for t large enough, $\mathbf{1}_{[0, \frac{\epsilon_0}{2})}(\frac{|x|}{t}) \leq \mathbf{1}_{[0, \epsilon_0)}(\frac{\langle x \rangle}{t})$ therefore we can replace $\langle x \rangle$ by $|x|$ in the estimate. Thus proposition I.4.2 is proved for any χ with sufficiently small support. Let us prove the general case. Let χ be any function in $C_0^\infty(\mathbb{R})$ which satisfies the assumptions of proposition I.4.2. We can write, by a compactness argument,

$$\chi = \sum_{j=1}^N \chi_j,$$

where χ_j are $C_0^\infty(\mathbb{R})$ functions with sufficiently small supports, such that

$$\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_0)}\left(\frac{|x|}{t}\right) \chi_j(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad j = 1, \dots, N.$$

By the Schwarz inequality, we have

$$\begin{aligned} &\int_1^\infty \left(\mathbf{1}_{[0, \epsilon_0)}\left(\frac{|x|}{t}\right) \chi_i(H) e^{-itH} \psi, \mathbf{1}_{[0, \epsilon_0)}\left(\frac{|x|}{t}\right) \chi_j(H) e^{-itH} \psi \right) \frac{dt}{t} \\ &\leq \left(\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_0)}\left(\frac{|x|}{t}\right) \chi_i(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_0)}\left(\frac{|x|}{t}\right) \chi_j(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \\ &\leq C \|\psi\|^2. \end{aligned}$$

Hence,

$$\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_0)}\left(\frac{|x|}{t}\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq N^2 C \|\psi\|^2, \quad \forall \psi \in \mathcal{H},$$

which concludes the proof of the Proposition.

◇

I.4.2 Large velocity estimates

This estimate says that the charge of the field in the region $|x| \geq t$ tends to zero as t becomes large. Recall that the light velocity c is here taken to be 1, so this means that the particle does not travel faster than light. Precisely, we prove

Proposition I.4.4 *Let $1 < \theta_1 < \theta_2$. Let $\chi \in C_0^\infty(\mathbb{R})$. There exists a constant C such that*

$$(i) \int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{I.4.11})$$

Given any $g \in C^\infty(\mathbb{R})$ with $g' \in C_0^\infty(\mathbb{R})$ and $\text{supp } g \subset (1, +\infty)$, we have

$$(ii) s - \lim_{t \rightarrow +\infty} g \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{I.4.12})$$

Remark I.4.1 *We stress the fact that the cut-off function χ could be avoided in the previous proposition without any change, that is to say that the constant C does not depend on the support of χ if $\|\chi\|_{L^\infty} \leq 1$. Therefore, whatever the energy of a state ψ , it cannot escape to infinity faster than the light velocity.*

Proof : Let $1 < \theta_0 < \theta_1 < \theta_2$ and let $f \in C_0^\infty([\theta_0, +\infty))$ such that $f = 1$ on $[\theta_1, \theta_2]$ and $f \geq 0$. We define

$$F(s) = \int_{-\infty}^s f^2(u) du,$$

which is clearly bounded and continuous on \mathbb{R} . Let us define the propagation observable

$$\phi(t) = \chi(H) F \left(\frac{|x|}{t} \right) \chi(H).$$

$\phi(t)$ is a self-adjoint operator valued function uniformly bounded in t . We compute the Heisenberg derivative.

$$-\mathbf{D}\phi(t) = \frac{1}{t} \chi(H) f^2 \left(\frac{|x|}{t} \right) \frac{|x|}{t} \chi(H) - \chi(H) i[H, F \left(\frac{|x|}{t} \right)] \chi(H).$$

Now

$$\begin{aligned} i[H, F \left(\frac{|x|}{t} \right)] &= i[\Gamma \cdot p, F \left(\frac{|x|}{t} \right)], \\ &= \frac{1}{t} \sum_j \Gamma^j \cdot \frac{x_j}{|x|} f^2 \left(\frac{|x|}{t} \right) = \frac{1}{t} \Gamma \cdot \frac{x}{|x|} f^2 \left(\frac{|x|}{t} \right). \end{aligned}$$

Thus

$$-\mathbf{D}\phi(t) = \frac{1}{t} \chi(H) \left\{ f^2 \left(\frac{|x|}{t} \right) \frac{|x|}{t} - \Gamma \cdot \frac{x}{|x|} f^2 \left(\frac{|x|}{t} \right) \right\} \chi(H).$$

Now using the facts that $\Gamma \cdot \frac{x}{|x|} \leq 1_{4 \times 4}$ and $f \geq 0$, we have

$$-\mathbf{D}\phi(t) \geq \frac{1}{t} (\theta_0 - 1) \chi(H) \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) \chi(H).$$

As $(\theta_0 - 1) > 0$ by assumption, the assertion (i) follows from Proposition I.4.1.

Let us prove (ii). It is enough to assume $g \geq 0$ and $g = 1$ for $x \geq R_0$. Let $\tilde{g} \in C_0^\infty(\mathbb{R})$ chosen such that $\text{supp } \tilde{g} \subset (1, +\infty)$ and $\tilde{g} = 1$ on $\text{supp } g'$. We define

$$\phi(t) = \chi(H) g \left(\frac{|x|}{t} \right) \chi(H),$$

and

$$\mathbf{D}\phi(t) = -\frac{1}{t} \chi(H) g' \left(\frac{|x|}{t} \right) \frac{|x|}{t} \chi(H) + \frac{1}{t} \chi(H) \Gamma \cdot \frac{x}{|x|} g' \left(\frac{|x|}{t} \right) \chi(H).$$

Using $\tilde{g}g' = g'$ we get

$$\mathbf{D}\phi(t) = \frac{1}{t}\chi(H)\tilde{g}\left(\frac{|x|}{t}\right)B(t)\tilde{g}\left(\frac{|x|}{t}\right)\chi(H), \quad (\text{I.4.13})$$

where $B(t) = -g'\left(\frac{|x|}{t}\right)\frac{|x|}{t} + \Gamma \cdot \frac{x}{|x|}g'\left(\frac{|x|}{t}\right)$ is uniformly bounded with respect to t . Now with (I.4.11) and (I.4.13), the existence of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) e^{-itH}, \quad (\text{I.4.14})$$

follows from Lemma I.4.1. Assume first that g has a compact support contained in $(1, +\infty)$ then (I.4.11) implies that

$$0 \leq \int_1^\infty (\psi, e^{itH} \phi(t) e^{-itH} \psi) \frac{dt}{t} \leq C \|\psi\|^2.$$

Hence the limit in (I.4.14) must be zero. Finally, to prove the general case, let us consider $g \in C^\infty(\mathbb{R})$ and $g \in C_0^\infty(\mathbb{R})$ such that $\text{supp } g_1 \subset (\theta_0, +\infty)$ with $\theta_0 > 1$, $g_1 \geq 0$, $g_1 = 1$ for $|x| \geq R_0$ and $g'_1 = g'^2$. We define the propagation observables by

$$\phi(t) = \chi(H)g_1\left(\frac{|x|}{t}\right)\chi(H),$$

and

$$\phi_R(t) = \chi(H)g_1\left(\frac{|x|}{Rt}\right)\chi(H),$$

where R is a positive real number. By the previous lemma, we know that $s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH}$ exists. Let us compute the Heisenberg derivative.

$$\begin{aligned} -\mathbf{D}\phi_R(t) &= \frac{1}{t}\chi(H)g^2\left(\frac{|x|}{Rt}\right)\frac{|x|}{Rt}\chi(H) - \frac{1}{Rt}\chi(H)\Gamma \cdot \frac{x}{|x|}g^2\left(\frac{|x|}{Rt}\right)\chi(H), \\ &\geq \frac{1}{t}\left(\theta_0 - \frac{1}{R}\right)\chi(H)g^2\left(\frac{x}{Rt}\right)\chi(H). \end{aligned} \quad (\text{I.4.15})$$

For $\theta_0 > 1$ and $R \geq 1$, we see that $(\theta_0 - \frac{1}{R})$ is strictly positive and thus $-\mathbf{D}\phi_R(t)$ is a positive operator. Now given some $t_0 \in \mathbb{R}$ fixed, we can write

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH} = e^{it_0H} \phi_R(t_0) e^{-it_0H} + \int_{t_0}^\infty e^{isH} \mathbf{D}\phi_R(s) e^{-isH} ds.$$

From the positivity of $-\mathbf{D}\phi_R(t)$, we deduce

$$0 \leq s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH} \leq e^{it_0H} \phi_R(t_0) e^{-it_0H}, \quad \forall R \geq 1. \quad (\text{I.4.16})$$

Now according to the definition of $\phi_R(t)$, observe that for t_0 fixed

$$s - \lim_{R \rightarrow +\infty} (e^{it_0H} \phi_R(t_0) e^{-it_0H}) = 0.$$

Hence we obtain from (I.4.16) that

$$s - \lim_{R \rightarrow +\infty} (s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH}) = 0. \quad (\text{I.4.17})$$

We conclude observing that $g_1\left(\frac{|x|}{t}\right) - g_1\left(\frac{|x|}{Rt}\right)$ has a compact support. Thus by the previous result (I.4.14), we get

$$s - \lim_{t \rightarrow +\infty} e^{itH} [\phi(t) - \phi_R(t)] e^{-itH} = 0.$$

Then if R tends to $+\infty$, we prove (ii) by (I.4.17).

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I.4.3 Microlocal velocity estimates

In this section, we shall prove the following proposition

Proposition I.4.5 *Let $0 < \theta_1 < \theta_2$ and let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m, \sigma_{pp}(H)\} = \emptyset$. Then*

$$(i) \quad \int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) (\mathcal{V} - \frac{x}{t}) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{I.4.18})$$

$$(ii) \quad s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) (\mathcal{V} - \frac{x}{t}) \chi(H) e^{-itH} = 0. \quad (\text{I.4.19})$$

Proof : Let $0 < \theta_0 < \theta_1 < \theta_2 < \theta_3$. Given two real numbers r_1, r_2 such that $0 < r_1 < r_2$, we denote by $C(r_1, r_2)$ the annulus $C(r_1, r_2) = \{x \in \mathbb{R}^3 : r_1 < |x| < r_2\}$. Let $J \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp } J \subset C(\frac{\theta_0}{2}, 2\theta_3)$ and $J = 1$ on $C(\theta_1, \theta_2)$. Let $\chi \in C_0^\infty(\mathbb{R})$ satisfying the above condition. We choose θ_0 and θ_3 such that $\theta_0 < \epsilon_\chi$ where ϵ_χ is defined according to the minimal velocity estimates and $\theta_3 > 1$.

We consider the following propagation observable

$$\phi(t) = \chi(H) J\left(\frac{x}{t}\right) \left\{ \frac{x_{nw}^2}{t^2} - 2\frac{A}{t} \right\} J\left(\frac{x}{t}\right) \chi(H),$$

which is uniformly bounded for $t \geq 1$. Indeed $J\left(\frac{x}{t}\right) \frac{x_{nw}^2}{t^2} J\left(\frac{x}{t}\right) = J\left(\frac{x}{t}\right) \left\{ \frac{x^2}{t^2} + \frac{x}{t} \cdot \frac{Z}{t} + \frac{Z}{t} \cdot \frac{x}{t} + \frac{Z^2}{t^2} \right\} J\left(\frac{x}{t}\right)$. This term is uniformly bounded since Z and $J\left(\frac{x}{t}\right) \frac{x}{t}$ are bounded. Furthermore,

$$J\left(\frac{x}{t}\right) \frac{A}{t} J\left(\frac{x}{t}\right) = \frac{1}{2} J\left(\frac{x}{t}\right) \left\{ \frac{x}{t} \cdot p H^{-1} + H^{-1} p \cdot \frac{x}{t} \right\} J\left(\frac{x}{t}\right),$$

is also uniformly bounded since pH^{-1} and $H^{-1}p$ are bounded. Let us compute its Heisenberg derivative.

$$\begin{aligned} -\mathbf{D}\phi(t) &= -\frac{1}{t} \chi(H) \left(\Gamma - \frac{x}{t} \right) \cdot \nabla J\left(\frac{x}{t}\right) \left\{ \frac{x_{nw}^2}{t^2} - 2\frac{A}{t} \right\} J\left(\frac{x}{t}\right) \chi(H) + hc, \\ &\quad + \frac{2}{t} \chi(H) J\left(\frac{x}{t}\right) \left\{ \frac{x_{nw}^2}{t^2} - \frac{A}{t} \right\} J\left(\frac{x}{t}\right) \chi(H), \\ &\quad - \frac{1}{t^2} \chi(H) J\left(\frac{x}{t}\right) i[H, x_{nw}^2] J\left(\frac{x}{t}\right) \chi(H), \\ &\quad + \frac{2}{t} \chi(H) J\left(\frac{x}{t}\right) i[H, A] J\left(\frac{x}{t}\right) \chi(H). \end{aligned}$$

Now we compute $i[H, x_{nw}^2]$ using $x_{nw} = x + Z$ and (I.2.23).

$$\begin{aligned} i[H, x_{nw}^2] &= i[H_0, x_{nw}^2] + i[V(x), x_{nw}^2], \\ &= (x_{nw} \mathcal{V} + \mathcal{V} x_{nw}) + (i[V(x), Z] x_{nw} + x_{nw} i[V(x), Z]). \end{aligned}$$

But the first term is equal to $2A_0$ plus a bounded operator in \mathcal{H} . And the second term is also a bounded operator in \mathcal{H} by Lemma I.2.1 and the fact that V belongs to $S^{-1}(\mathbb{R}^3)$. Furthermore, we have already seen that $i[H, A] = \mathcal{V}^2 + K$ and from the exact expression of K , it is easy to show that $J\left(\frac{x}{t}\right) K J\left(\frac{x}{t}\right) \in O(t^{-1})$. Thus, we obtain

$$\begin{aligned} -\mathbf{D}\phi(t) &= -\frac{1}{t} \chi(H) \left(\Gamma - \frac{x}{t} \right) \cdot \nabla J\left(\frac{x}{t}\right) \left\{ \frac{x_{nw}^2}{t^2} - 2\frac{A}{t} \right\} J\left(\frac{x}{t}\right) \chi(H) + hc, \\ &\quad + \frac{2}{t} \chi(H) J\left(\frac{x}{t}\right) \left\{ \frac{x_{nw}^2}{t^2} - \frac{A}{t} - \frac{A_0}{t} + \mathcal{V}^2 \right\} J\left(\frac{x}{t}\right) \chi(H), \\ &\quad + O(t^{-2}). \end{aligned}$$

Using that $A = A_0 + B$ and $x_{nw} = x + Z$ again, we can replace the second term by $\frac{2}{t}\chi(H)J(\frac{x}{t})\left\{\frac{x^2}{t^2} - 2\frac{A_0}{t} + \mathcal{V}^2\right\}J(\frac{x}{t})\chi(H) + O(t^{-2})$. Moreover, we have the following equality $\left\{\frac{x^2}{t^2} - 2\frac{A_0}{t} + \mathcal{V}^2\right\} = (\mathcal{V} - \frac{x}{t})^2$. Therefore the second term is equal to

$$\frac{2}{t}\chi(H)J(\frac{x}{t})(\mathcal{V} - \frac{x}{t})^2J(\frac{x}{t})\chi(H) + O(t^{-2}).$$

Finally, if we introduce $\tilde{J} \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \tilde{J} \subset [\frac{\theta_0}{2}, 2\theta_3]$ and $\tilde{J}(|x|)\nabla J(x) = \nabla J(x)$, then after some commutations the first term can be written as

$$\frac{1}{t}\chi(H)\tilde{J}(\frac{|x|}{t})B(t)\tilde{J}(\frac{|x|}{t})\chi(H) + O(t^{-2}),$$

where $B(t)$ is a uniformly bounded operator in t . Hence this term is integrable along the evolution by Propositions I.4.2 and I.4.4. Eventually, we obtain

$$-\mathbf{D}\phi(t) \geq \frac{2}{t}\chi(H)J(\frac{x}{t})(\mathcal{V} - \frac{x}{t})^2J(\frac{x}{t})\chi(H) + L^1(dt).$$

Commuting $J(\frac{x}{t})$ and $(\mathcal{V} - \frac{x}{t})$ and provided $J^2(\frac{x}{t}) - \mathbf{1}_{[\theta_1, \theta_2]}(\frac{|x|}{t}) \geq 0$, we get

$$-\mathbf{D}\phi(t) \geq \frac{2}{t}\chi(H)(\mathcal{V} - \frac{x}{t})\mathbf{1}_{[\theta_1, \theta_2]}(\frac{x}{t})(\mathcal{V} - \frac{x}{t})\chi(H) + L^1(dt).$$

We conclude the proof of (i) by Proposition I.4.1.

To prove (ii), let us consider the following propagation observable

$$\phi_0(t) = \chi(H)J(\frac{x}{t})\left\{\frac{x^2}{t^2} - 2\frac{A_0}{t} + \mathcal{V}^2\right\}J(\frac{x}{t})\chi(H),$$

and observe that it is equal to $\chi(H)J(\frac{x}{t})(\mathcal{V} - \frac{x}{t})^2J(\frac{x}{t})\chi(H)$. In particular it is a positive operator for any t . For technical reasons we shall approach $\phi_0(t)$ by another observable denoted $\phi(t)$ and given by

$$\phi(t) = \chi(H)J(\frac{x}{t})\left\{\frac{x_{nw}^2}{t^2} - 2\frac{A}{t} + \mathcal{V}^2\right\}J(\frac{x}{t})\chi(H).$$

It is easy to see that $\phi(t) = \phi_0(t) + O(t^{-1})$. Let us compute the Heisenberg derivative of $\phi(t)$. As shown in the previous calculation, we obtain

$$\begin{aligned} -\mathbf{D}\phi(t) &= \frac{2}{t}\chi(H)J(\frac{x}{t})(\mathcal{V} - \frac{x}{t})^2J(\frac{x}{t})\chi(H), \\ &\quad -\chi(H)J(\frac{x}{t})i[H, \mathcal{V}^2]J(\frac{x}{t})\chi(H), \\ &\quad +O(t^{-2}) + L^1(dt). \end{aligned}$$

But $i[H, \mathcal{V}^2] = i[V(x), \mathcal{V}^2]$ and from the exact expression of $\mathcal{V}^2 = \frac{p^2}{p^2+m^2}$, this last term leads to

$$\begin{aligned} [V(x), \mathcal{V}^2] &= [V(x), p^2]H_0^{-2} + p^2H_0^{-2}[H_0^2, V(x)]H_0^{-2}, \\ &= i\nabla V(x) \cdot p H_0^{-2} + ip \cdot \nabla V(x) H_0^{-2} - ip^2 H_0^{-2} \nabla V(x) \cdot p H_0^{-2} - ip^2 H_0^{-2} p \cdot \nabla V(x) H_0^{-2}. \end{aligned}$$

Therefore, after some commutations and since $\nabla V(x)$ belongs to $S^{-2}(\mathbb{R}^3)$, the second term in the Heisenberg derivative belongs to $O(t^{-2})$. Now applying Lemma I.4.1, we have proved that the following limit exists

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) e^{-itH}. \quad (\text{I.4.20})$$

Clearly, we can replace $\phi(t)$ by $\phi_0(t)$ in (I.4.20). But we also know by (i) that

$$0 \leq \int_1^{+\infty} (e^{-itH} \psi, \phi_0(t) e^{-itH} \psi) \frac{dt}{t} < +\infty,$$

Hence the limit (I.4.20) must vanish which concludes the proof.

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I.5 Asymptotic velocity P^\pm

I.5.1 Construction of P^\pm

In this section, we shall focus our attention on the construction of the asymptotic velocity P^\pm defined by

$$P^\pm := s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH} \frac{x}{t} e^{-itH}.$$

Here the convergence means that for any $\psi \in \mathcal{H}$ and any $J \in C_\infty(\mathbb{R}^3)$, the limit

$$J(P^\pm)\psi := \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH} \psi, \quad (\text{I.5.1})$$

exists. If (I.5.1) holds then the operators P^\pm are uniquely defined as vectors of (possibly non-densely defined) commuting self-adjoint operators. P^\pm are densely defined if, for some $g \in C_\infty(\mathbb{R}^3)$ such that $g(0) = 1$ we have $s - \lim_{R \rightarrow +\infty} (s - \lim_{t \rightarrow \pm\infty} e^{itH} g(\frac{x}{Rt}) e^{-itH}) = 1$ (see [21], Appendix B.2). The main tools will be Lemma I.4.1 and the *weak propagation estimates* defined in the previous section. We only treat the case $t \rightarrow +\infty$ and we construct P^+ , the construction is identical for P^- with $t \rightarrow -\infty$. Let us prove the theorem.

Theorem I.5.1 *Let H be the Dirac operator (I.2.1). Let $J \in C_\infty(\mathbb{R}^3)$. Then there exists the limit*

$$s - \lim_{t \rightarrow +\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH}. \quad (\text{I.5.2})$$

Moreover, if $J = 1$ on a neighbourhood of 0, then

$$s - \lim_{R \rightarrow +\infty} \left(s - \lim_{t \rightarrow +\infty} e^{itH} J\left(\frac{x}{Rt}\right) e^{-itH} \right) = 1. \quad (\text{I.5.3})$$

If we define P^+ by (I.5.1) then P^+ is a vector of commuting self-adjoint operators on \mathcal{H} defined on a dense subspace of \mathcal{H} and P^\pm commutes with H .

Proof : First, consider the case where ψ is an eigenvector of H . Then, there exists $E \in \mathbb{R}$ such that $H\psi = E\psi$. Let $J \in C_\infty(\mathbb{R}^3)$. We have

$$e^{itH} J\left(\frac{x}{t}\right) e^{-itH} \psi = J(0)\psi + e^{it(H-E)} \left\{ J\left(\frac{x}{t}\right) - J(0) \right\} \psi.$$

By Lebesgue's Theorem, it is immediate that $\lim_{t \rightarrow \infty} e^{it(H-E)} \{J(\frac{x}{t}) - J(0)\} \psi = 0$. Therefore

$$\lim_{t \rightarrow +\infty} e^{itH} J(\frac{x}{t}) e^{-itH} \psi = J(0) \psi, \quad (\text{I.5.4})$$

and we conclude that limit (I.5.2) exists on \mathcal{H}^{pp} the pure point subspace of H .

Now, let us assume that $\psi \in \mathcal{H}^c$ the continuous subspace of H . Our first task is to find a good propagation observable $\phi(t)$ in order to apply Lemma I.4.1. Since $\psi \in \mathcal{H}^c$, remark that by a density argument, the existence of (I.5.2) is equivalent to the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J(\frac{x}{t}) \chi(H) e^{-itH},$$

for any $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\text{supp } \chi \cap \{-m, +m, \sigma_{pp}(H)\} = \emptyset$ and any $J \in C_0^\infty(\mathbb{R}^3)$ such that J is constant on a neighbourhood of 0. Let us define $\phi(t) = \chi(H) \left\{ J(\frac{x}{t}) + (\mathcal{V} - \frac{x}{t}) \nabla J(\frac{x}{t}) \right\} \chi(H)$. By Proposition I.4.5, it is enough to prove the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) e^{-itH}.$$

Unfortunately, this propagation observable is not easy to work with. In order to avoid problems due to the Zitterbewegung phenomenon and the matrix-valued potential $V_2(x)\Gamma^0$, we need to approach $\phi(t)$ by another propagation observable which we denote by $\phi_a(t)$ and define as follows

$$\phi_a(t) = \chi(H) \left\{ J(\frac{x_{nw}}{t}) + (\mathcal{V}_a - \frac{x_{nw}}{t}) \cdot \nabla J(\frac{x_{nw}}{t}) \right\} \chi(H),$$

where $\mathcal{V}_a = \frac{1}{2}(pH^{-1} + H^{-1}p)$ is a bounded selfadjoint operator on \mathcal{H} . Clearly we have $\mathcal{V}_a = \mathcal{V} + B$ where B is bounded on \mathcal{H} . The next lemma will enable us to make the link between the operators $f(\frac{x}{t})$ and $f(\frac{x_{nw}}{t})$ and as a consequence give an estimate of $(\phi(t) - \phi_a(t))$.

Lemma I.5.1 *Let $f \in C_0^\infty(\mathbb{R}^3)$. Then*

$$(i) \quad f(\frac{x}{t}) - f(\frac{x_{nw}}{t}) \in O(t^{-1}).$$

Assume moreover that $f = 0$ on a neighbourhood of 0 and let $g \in S^{-\rho}(\mathbb{R}^3)$ where $\rho > 0$ is a real number. Then

$$(ii) \quad g(x) f(\frac{x_{nw}}{t}) \in O(t^{-\rho}),$$

$$(iii) \quad (\mathcal{V} - \mathcal{V}_a) f(\frac{x_{nw}}{t}) \in O(t^{-1}).$$

Then it follows that $\phi_a(t) - \phi(t) \in O(t^{-1})$ and thus it is enough to prove the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi_a(t) e^{-itH}. \quad (\text{I.5.5})$$

Let us compute its Heisenberg derivative.

$$\begin{aligned} \mathbf{D}\phi_a(t) &= \frac{1}{t} \chi(H) (\mathcal{V}_a - \frac{x_{nw}}{t}) \nabla^2 J(\frac{x_{nw}}{t}) (-\frac{x_{nw}}{t}) \chi(H) \\ &\quad + \chi(H) i [H, J(\frac{x_{nw}}{t}) + (\mathcal{V}_a - \frac{x_{nw}}{t}) \cdot \nabla J(\frac{x_{nw}}{t})] \chi(H). \end{aligned}$$

To compute the remaining commutator, we use the following lemma which we prove later

Lemma I.5.2 *Let $f \in C_0^\infty(\mathbb{R}^3)$ such that f is constant on a neighbourhood of 0. Then*

$$i[H, f(\frac{x_{nw}}{t})] = \frac{1}{t} \nabla f(\frac{x_{nw}}{t}) \cdot \mathcal{V} + O(t^{-2}). \quad (\text{I.5.6})$$

Therefore, we get

$$\begin{aligned} \mathbf{D}\phi_a(t) &= \frac{1}{t} \chi(H) (\mathcal{V}_a - \frac{x_{nw}}{t}) \nabla^2 J(\frac{x_{nw}}{t}) (\mathcal{V} - \frac{x_{nw}}{t}) \chi(H) \\ &\quad + \chi(H) i[H, \mathcal{V}_a] \cdot \nabla J(\frac{x_{nw}}{t}) \chi(H) \\ &\quad + O(t^{-2}). \end{aligned}$$

But $i[H, \mathcal{V}_a] \nabla J(\frac{x_{nw}}{t}) = -\frac{1}{2} (\nabla V(x) H^{-1} + hc) \nabla J(\frac{x_{nw}}{t}) \in O(t^{-2})$ by Lemma I.5.2 and the fact that $\nabla V(x) \nabla J(\frac{x_{nw}}{t}) \in O(t^{-2})$. Moreover, we can replace \mathcal{V}_a by \mathcal{V} as well as $\frac{x_{nw}}{t}$ by $\frac{x}{t}$ and we eventually obtain (using Lemma I.5.1)

$$\mathbf{D}\phi_a(t) = \frac{1}{t} \chi(H) (\mathcal{V} - \frac{x}{t}) \nabla^2 J(\frac{x}{t}) (\mathcal{V} - \frac{x}{t}) \chi(H) + O(t^{-2}).$$

Then by Proposition I.4.5 and by Lemma I.4.1, (I.5.5) exists.

The proof of (I.5.3) is a direct consequence of Proposition I.4.4, part (ii). Indeed, (I.5.3) is equivalent to $s - \lim_{R \rightarrow +\infty} \left(s - \lim_{t \rightarrow \pm\infty} e^{itH} J(\frac{x}{Rt}) e^{-itH} \right) = 0$, for $J \in C^\infty(\mathbb{R}^3)$ such that $\text{supp } J \cap \overline{B}(0, 1) = \emptyset$ and $\nabla J \in C_0^\infty(\mathbb{R}^3)$.

The fact that H commutes with P^+ follows from $[H, f(\frac{x}{t})] \in O(t^{-1})$.

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Proof : (of Lemma I.5.1) The first assertion follows readily from the Fourier transform. Indeed

$$\begin{aligned} f(\frac{x}{t}) - f(\frac{x_{nw}}{t}) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \hat{f}(\xi) \left\{ e^{i\xi \frac{x}{t}} - e^{i\xi \frac{x_{nw}}{t}} \right\} d\xi, \\ &= -\frac{1}{(2\pi)^{\frac{3}{2}} t} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i\xi \sigma \frac{x}{t}} Z e^{i\xi(1-\sigma) \frac{x_{nw}}{t}} d\sigma d\xi, \\ &\in O(t^{-1}), \end{aligned}$$

since Z is bounded in \mathcal{H} .

By the definition of the Newton-Wigner observable, we have

$$\begin{aligned} g(x) f(\frac{x_{nw}}{t}) &= g(x) U_{FW}^{-1} f(\frac{x}{t}) U_{FW}, \\ &= g(x) f(\frac{x}{t}) + g(x) [U_{FW}, f(\frac{x}{t})] U_{FW}. \end{aligned} \quad (\text{I.5.7})$$

Clearly the first term in (I.5.7) belongs to $O(t^{-\rho})$. Now recall that $U_{FW}^{-1} = P^{-1}(p)$ with P^{-1} given by (I.2.21). Thus each component of the matrix-valued function P^{-1} is in $S^0(\mathbb{R}^3)$ and we can use the Helffer-Sjöstrand formula (I.A.28) to estimate the commutator in the second term of (I.5.7). Hence

$$[P^{-1}(p), f(\frac{x}{t})] = \sum_{k=1}^N \frac{\nabla^k f(\frac{x}{t})}{t^k k!} a d_x^k(P^{-1}(p)) + O(t^{-(N+1)}), \quad (\text{I.5.8})$$

since $ad_x^k(P^{-1}(p))$ bounded for any $k \in \mathbb{N}$. To see this, we use once again the Helffer-Sjöstrand formula and we have for any $m = 1, 2, 3$

$$[P^{-1}(p), x_m] = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{P}^{-1}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) (-i\delta_{lm}) \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z},$$

where $R_k = (z_k - p_k)^{-1}$. Hence the first commutator is bounded. Noting that $[R_k, x_n] = R_k(-i\delta_{kn})R_k$, the multiple commutators are also bounded by induction. Finally, if we take $N \geq \rho$, then the assertion (ii) follows immediately from (I.5.7) and (I.5.8).

Now, using (i), we can write $(\mathcal{V} - \mathcal{V}_a)f(\frac{x_{nw}}{t})$ as $(\mathcal{V} - \mathcal{V}_a)f(\frac{x}{t}) + O(t^{-1})$. Moreover,

$$\begin{aligned} (\mathcal{V} - \mathcal{V}_a)f\left(\frac{x}{t}\right) &= \left(\frac{1}{2}pH^{-1}V(x)H_0^{-1} + hc\right)f\left(\frac{x}{t}\right), \\ &= \frac{1}{2}pH^{-1}V(x)f\left(\frac{x}{t}\right)H_0^{-1} - \frac{1}{2}pH^{-1}V(x)H_0^{-1}[H_0, f\left(\frac{x}{t}\right)]H_0^{-1} + hc, \\ &\in O(t^{-1}), \end{aligned}$$

since $V(x)f(\frac{x}{t}) \in O(t^{-1})$ and $[H_0, f(\frac{x}{t})] = -\frac{i}{t}\Gamma \cdot \nabla f(\frac{x}{t}) \in O(t^{-1})$ which concludes the proof of the lemma.

◇

Proof : (of Lemma I.5.2) Using the Fourier transform, we have the following formula (I.A.9).

$$i[H, f\left(\frac{x_{nw}}{t}\right)] = \frac{1}{(2\pi)^{\frac{3}{2}}t} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i(1-s)\xi \cdot \frac{x_{nw}}{t}} \left\{ \mathcal{V} + i[V(x), Z] \right\} e^{is\xi \cdot \frac{x_{nw}}{t}} ds d\xi.$$

Since the term $\mathcal{V} + i[V(x), Z]$ is bounded, the integral converges in the norm of \mathcal{H} . We then commute $\mathcal{V} + iV(x)Z$ with $e^{is\xi \cdot \frac{x_{nw}}{t}}$ and $-iZV(x)$ with $e^{-i(1-s)\xi \cdot \frac{x_{nw}}{t}}$. We obtain

$$\begin{aligned} i[H, f\left(\frac{x_{nw}}{t}\right)] &= \frac{1}{(2\pi)^{\frac{3}{2}}t} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \left(e^{i\xi \cdot \frac{x_{nw}}{t}} \left\{ \mathcal{V} + iV(x)Z \right\} - iZV(x)e^{i\xi \cdot \frac{x_{nw}}{t}} \right) d\xi \\ &\quad + \frac{1}{(2\pi)^{\frac{3}{2}}t} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i(1-s)\xi \cdot \frac{x_{nw}}{t}} [\mathcal{V} + iV(x)Z, e^{is\xi \cdot \frac{x_{nw}}{t}}] ds d\xi \\ &\quad - \frac{i}{(2\pi)^{\frac{3}{2}}t} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 [e^{i(1-s)\xi \cdot \frac{x_{nw}}{t}}, ZV(x)] e^{is\xi \cdot \frac{x_{nw}}{t}} ds d\xi. \end{aligned}$$

Now we use the formula (I.A.1) to estimate these last commutators. We have

$$[\mathcal{V}, e^{is\xi \cdot \frac{x_{nw}}{t}}] = \frac{is\xi}{t} \int_0^1 e^{i(1-\sigma)s\xi \cdot \frac{x_{nw}}{t}} [\mathcal{V}, x_{nw}] e^{i\sigma s\xi \cdot \frac{x_{nw}}{t}} d\sigma,$$

which clearly belongs to $O(\xi t^{-1})$ since $[\mathcal{V}, x_{nw}]$ is bounded in \mathcal{H} . We also have

$$[V(x)Z, e^{is\xi \cdot \frac{x_{nw}}{t}}] = \frac{is\xi}{t} \int_0^1 e^{i(1-\sigma)s\xi \cdot \frac{x_{nw}}{t}} [V(x)Z, x_{nw}] e^{i\sigma s\xi \cdot \frac{x_{nw}}{t}} d\sigma,$$

where $[V(x)Z, x_{nw}] = [V(x)Z, x] + [V(x)Z, Z] = V(x)[Z, x] + [V(x)Z, Z]$ is bounded by Lemma (I.2.1). Therefore, this term also belongs to $O(\xi t^{-1})$. We thus obtain

$$\begin{aligned} i[H, f(\frac{x_{nw}}{t})] &= \left(\frac{1}{(2\pi)^{\frac{3}{2}}t} \int_{\mathbb{R}^3} e^{i\xi \frac{x_{nw}}{t}} \widehat{\nabla} f(\xi) d\xi \right) \left\{ \mathcal{V} + iV(x)Z \right\} \\ &\quad - iZV(x) \left(\frac{1}{(2\pi)^{\frac{3}{2}}t} \int_{\mathbb{R}^3} e^{i\xi \frac{x_{nw}}{t}} \widehat{\nabla} f(\xi) d\xi \right) + O(t^{-2}), \\ &= \frac{1}{t} \nabla f(\frac{x_{nw}}{t}) \mathcal{V} + \frac{i}{t} \nabla f(\frac{x_{nw}}{t}) V(x)Z - \frac{i}{t} ZV(x) \nabla f(\frac{x_{nw}}{t}) + O(t^{-2}). \end{aligned}$$

Now using Lemma I.5.1, it is easy to see that $V(x) \nabla f(\frac{x_{nw}}{t}) \in O(t^{-1})$ which concludes the proof of Lemma I.5.2.

◇

In order to analyse the spectrum of P^\pm , we now give another characterization of the asymptotic velocity. Precisely, we make the link between P^\pm and the standard velocity operator \mathcal{V} .

Proposition I.5.1 *Let $J \in C_\infty(\mathbb{R}^3)$. Then*

$$\mathbf{1}_{\{0\}}(P^\pm) = \mathbf{1}^{pp}(H), \quad (\text{I.5.9})$$

and consequently,

$$\mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm) = \mathbf{1}^c(H), \quad (\text{I.5.10})$$

where $\mathbf{1}^{pp}(H)$ and $\mathbf{1}^c(H)$ denote respectively the projection onto the pure point subspace of H and onto the continuous subspace of \mathcal{H} . Furthermore, we have

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} J(\mathcal{V}) e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm) = J(P^\pm) \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm). \quad (\text{I.5.11})$$

Proof : The proof of (I.5.9) entirely relies on the minimal velocity estimates. We successively show that $\mathbf{1}_{\{0\}}(P^+) \geq \mathbf{1}^{pp}(H)$ and $\mathbf{1}_{\{0\}}(P^+) \leq \mathbf{1}^{pp}(H)$. Let $\psi \in \mathcal{H}$ such that $H\psi = E\psi$ and $J \in C_\infty(\mathbb{R}^3)$. We already saw in (I.5.4) that

$$\lim_{t \rightarrow +\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH} \psi = J(0)\psi.$$

This shows that $P^+\psi = 0$ and proves $\mathbf{1}_{\{0\}}(P^+) \geq \mathbf{1}^{pp}(H)$. Conversely, let us consider a function $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m, \sigma_{pp}(H)\} = \emptyset$. Let $J \in C_0^\infty(\mathbb{R}^3)$ such that $J(0) = 1$ and $\text{supp } J \subset \overline{B}(0, \epsilon_\chi)$ where $\epsilon_\chi > 0$ is defined by the minimal velocity estimates.

Then Theorem I.5.1 implies

$$\begin{aligned} s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J^2\left(\frac{x}{t}\right) \chi(H) e^{-itH} &= \chi^2(H) J^2(P^+), \\ &\geq \chi^2(H) \mathbf{1}_{\{0\}}(P^+). \end{aligned} \quad (\text{I.5.12})$$

But by Proposition I.4.2, the strong limit in (I.5.12) vanishes. Thus we have proved that

$$\mathbf{1}_{\{0\}}(P^+) \leq \mathbf{1}_{\overline{\{-m, +m, \sigma_{pp}(H)\}}}(H).$$

As the eigenvalues of H can only accumulate in $\{-m, +m\}$, we have $\mathbf{1}_{\overline{\{-m, +m, \sigma_{pp}(H)\}}}(H) = \mathbf{1}^{pp}(H)$ and the result holds.

Now, let us prove (I.5.11). We only treat the case $t \rightarrow +\infty$ and characterize P^+ . Using (I.5.10), by a density argument, it is enough to show

$$s - \lim_{t \rightarrow +\infty} e^{itH} (J(\mathcal{V}) - J(\frac{x}{t})) f(\frac{x}{t}) \chi(H) e^{-itH} = 0,$$

for any $J, f \in C_0^\infty(\mathbb{R}^3)$ such that $f = 0$ in a neighbourhood of 0 and $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\text{supp } \chi \cap \{-m, +m, \sigma_{pp}(H)\} = \emptyset$. By the Helffer-Sjöstrand formula (I.A.32), we have

$$J(\mathcal{V}) - J(\frac{x}{t}) = B_1(\mathcal{V} - \frac{x}{t}) + B_2(t),$$

where B_1 bounded and $B_2(t) \in O(t^{-1})$ since $[\mathcal{V}^j, x_k]$ bounded for any j, k . Then we have to prove that the following limit vanishes

$$s - \lim_{t \rightarrow +\infty} e^{itH} B_1(\mathcal{V} - \frac{x}{t}) f(\frac{x}{t}) \chi(H) e^{-itH}.$$

But this follows from Proposition I.4.5.

◇

I.5.2 Spectrum of P^\pm

This section is devoted to the analysis of the spectrum of P^\pm which corresponds to the physically relevant information given by the asymptotic velocity. We have already seen that

$$\mathbf{1}_{\{0\}}(P^\pm) = \mathbf{1}^{pp}(H),$$

which means that the states of zero asymptotic velocity coincide with the bound states of H . Now we are interested in the *scattering states* that is to say the states in $\mathcal{H} = \mathbf{1}^c(H)\mathcal{H}$ and we would like to classify them according to their asymptotic behaviours. We prove the proposition

Proposition I.5.2 *Let P^\pm be the asymptotic velocity defined in Theorem I.5.1. Then*

$$\sigma(P^\pm) = \overline{B}(0, 1).$$

Proof : As usual, we only give the proof for P^+ . Let us first prove that

$$\sigma(P^+) \subset \overline{B}(0, 1).$$

Let $\xi_0 \in \mathbb{R}^3 \setminus \overline{B}(0, 1)$ and let $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi = 1$ in a neighbourhood of ξ_0 and $\text{supp } \chi \cap \overline{B}(0, 1) = \emptyset$. We have to show that

$$\chi(P^+) = 0.$$

But by Proposition I.5.1, we have

$$\chi(P^+) = s - \lim_{t \rightarrow +\infty} e^{itH} \chi(\mathcal{V}) e^{-itH}.$$

Now it is easy to see that the spectrum of \mathcal{V} is equal to $\overline{B}(0, 1)$. Hence, $\chi(\mathcal{V}) = 0$ and $\chi(P^+) = 0$ which concludes the first part of the proof. Let us prove the reverse inclusion.

Let $\xi_0 \in \overline{B}(0, 1)$, $\xi_0 \neq 0$ and $|\xi_0| = r_2 > r_1 > r_0 > 0$. Let $g \in C_0^\infty(B(\xi_0, r_0))$ such that $g(\xi_0) \neq 0$. Here $B(\xi_0, r_0)$ denotes the ball centered at ξ_0 with radius r_0 . We want to show here that $g(P^+) \neq 0$. Let $J \in C^\infty(\mathbb{R})$ such that $J = 1$ on $(-\infty, r_1]$, $J = 0$ on $[r_2 - \frac{r_2 - r_1}{2}, +\infty)$ and $j = -J' \geq 0$. Clearly J and g satisfy the following relations

$$\bullet \quad J(|\xi - \xi_0|)g(\xi) = g(\xi). \quad (\text{I.5.13})$$

$$\bullet \quad j(|\xi - \xi_0|)|\xi - \xi_0| \geq r_1 j(|\xi - \xi_0|). \quad (\text{I.5.14})$$

$$\bullet \quad g(\xi)|\xi - \xi_0| \leq r_0 g(\xi). \quad (\text{I.5.15})$$

We define the propagation observable

$$\phi_a(t) = J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)g^2(\mathcal{V}_a)J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right),$$

where $\mathcal{V}_a = \frac{1}{2}(pH^{-1} + H^{-1}p)$ and by Proposition I.5.1 and Lemma I.5.1, we have

$$g^2(P^+) = s - \lim_{t \rightarrow +\infty} e^{itH} \phi_a(t) e^{-itH}. \quad (\text{I.5.16})$$

Let us compute its Heisenberg derivative.

$$\begin{aligned} \mathbf{D}\phi_a(t) &= \frac{1}{t}j\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)\frac{\frac{x_{nw}}{t} - \xi_0}{\left|\frac{x_{nw}}{t} - \xi_0\right|} \cdot (\xi_0 - \mathcal{V})g^2(\mathcal{V}_a)J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) + hc \\ &\quad + \frac{1}{t}j\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)\left|\frac{x_{nw}}{t} - \xi_0\right|g^2(\mathcal{V}_a)J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) + hc \\ &\quad + J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)i[H, g^2(\mathcal{V}_a)]J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) + O(t^{-2}). \end{aligned}$$

Now we commute the different terms in the last expression using the following result we shall prove later

$$[g(\mathcal{V}_a), f\left(\frac{x_{nw}}{t}\right)] \in O(t^{-1}). \quad (\text{I.5.17})$$

We obtain

$$\begin{aligned} \mathbf{D}\phi_a(t) &= \frac{1}{t}(Jj)^{\frac{1}{2}}\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)\frac{\frac{x_{nw}}{t} - \xi_0}{\left|\frac{x_{nw}}{t} - \xi_0\right|} \cdot (\xi_0 - \mathcal{V})g^2(\mathcal{V}_a)(Jj)^{\frac{1}{2}}\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) + hc \\ &\quad + \frac{1}{t}g(\mathcal{V}_a)j\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)\left|\frac{x_{nw}}{t} - \xi_0\right|J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)g(\mathcal{V}_a) + hc \\ &\quad + J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)i[H, g^2(\mathcal{V}_a)]J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) + O(t^{-2}). \end{aligned}$$

In the first two terms, we can replace \mathcal{V}_a by \mathcal{V} by Lemma I.5.1 again. Then by (I.5.14) and (I.5.15), we get

$$\begin{aligned} \mathbf{D}\phi_a(t) &\geq \frac{1}{t}(r_1 - r_0)g(\mathcal{V})(Jj)\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)g(\mathcal{V}) \\ &\quad + J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)i[H, g^2(\mathcal{V}_a)]J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) + O(t^{-2}). \end{aligned}$$

Moreover we claim and prove later that

$$J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)i[H, g^2(\mathcal{V}_a)]J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right) \in O(t^{-2}). \quad (\text{I.5.18})$$

This implies

$$\mathbf{D}\phi_a(t) \geq \frac{1}{t}(r_1 - r_0)g(\mathcal{V})(Jj)\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)g(\mathcal{V}) + R(t),$$

where $R(t) \in L^1(dt)$. As $g, J, j \geq 0$ and $r_1 - r_0 > 0$, we get

$$\mathbf{D}\phi(t) \geq R(t). \quad (\text{I.5.19})$$

We now conclude the proof. By (I.5.16) and (I.5.19), we have

$$\begin{aligned} g^2(P^+) &= e^{it_0H} \phi(t_0) e^{-it_0H} + \int_{t_0}^{+\infty} e^{itH} \mathbf{D}\phi(t) e^{-itH} dt, \\ &\geq e^{it_0H} \phi(t_0) e^{-it_0H} - \int_{t_0}^{+\infty} \|R(t)\| dt, \end{aligned} \quad (\text{I.5.20})$$

and allowing t_0 to tend to infinity, we can make the integral in (I.5.20) as small as we want. We claim that

$$\lim_{t_0 \rightarrow +\infty} \|e^{it_0H} \phi_a(t_0) e^{-it_0H}\|,$$

exists and is non-zero. Indeed first observe that using Lemma I.5.1, we have

$$\lim_{t_0 \rightarrow +\infty} \|e^{it_0H} \phi_a(t_0) e^{-it_0H}\| = \lim_{t_0 \rightarrow +\infty} \|e^{it_0H} \phi(t_0) e^{-it_0H}\|,$$

where $\phi(t) = J\left(\left|\frac{x}{t} - \xi_0\right|\right)g^2(\mathcal{V})J\left(\left|\frac{x}{t} - \xi_0\right|\right)$. Then we have

$$\begin{aligned} \|e^{it_0H} \phi(t_0) e^{-it_0H}\| &= \|e^{it_0\xi_0 \cdot p} \phi(t_0) e^{-it_0\xi_0 \cdot p}\|, \\ &= \|J\left(\frac{|x|}{t_0}\right)g^2(\mathcal{V})J\left(\frac{|x|}{t_0}\right)\|. \end{aligned}$$

Thus, $\lim_{t_0 \rightarrow +\infty} \|e^{it_0H} \phi(t_0) e^{-it_0H}\| = \|g^2(\mathcal{V})\| \neq 0$ and the result holds.

It remains to prove (I.5.17) and (I.5.18). First observe that we can replace $f\left(\frac{x_{nw}}{t}\right)$ by $f\left(\frac{x}{t}\right)$ in (I.5.17) using Lemma I.5.1. Now using the formula (I.A.2), we have

$$[g(\mathcal{V}_a), f\left(\frac{x}{t}\right)] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla} g(\xi) \int_0^1 e^{i(1-s)\xi \mathcal{V}_a} [\mathcal{V}_a, f\left(\frac{x}{t}\right)] e^{is\xi \mathcal{V}_a} ds d\xi, \quad (\text{I.5.21})$$

and the commutator in the right-hand-side of (I.5.21) is equal to

$$\frac{1}{2t} \left\{ p H^{-1} i \Gamma \cdot \nabla f\left(\frac{x}{t}\right) H^{-1} - i \nabla f\left(\frac{x}{t}\right) H^{-1} + hc \right\},$$

which clearly belongs to $O(t^{-1})$.

Let us show (I.5.18). Since $[H, \mathcal{V}_a]$ is bounded and \mathcal{V}_a bounded, we can use the Helffer-Sjöstrand formula (I.A.29) to estimate $i[H, g^2(\mathcal{V}_a)]$. If we denote $J\left(\left|\frac{x_{nw}}{t} - \xi_0\right|\right)$ by $J_{\xi_0}\left(\frac{x_{nw}}{t}\right)$ we obtain

$$\begin{aligned} J_{\xi_0}\left(\frac{x_{nw}}{t}\right) i[H, g^2(\mathcal{V}_a)] J_{\xi_0}\left(\frac{x_{nw}}{t}\right) &= \\ \frac{i}{2(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{g}^2 \sum_{l=1}^3 J_{\xi_0}\left(\frac{x_{nw}}{t}\right) \left(\prod_{k_1=1}^l R_{k_1} \right) (\partial_l V(x) H^{-1} + hc) \left(\prod_{k_2=l}^3 R_{k_2} \right) J_{\xi_0}\left(\frac{x_{nw}}{t}\right) dz \wedge d\bar{z}, \end{aligned} \quad (\text{I.5.22})$$

where $R_k = (z_k - \mathcal{V}_a^k)^{-1}$. Next we commute the operators $J_{\xi_0}(\frac{x_{nw}}{t})$ with $\prod_{k_1=1}^l R_{k_1}$ and $\prod_{k_2=1}^3 R_{k_2}$ under the integral (I.5.22). Furthermore note that, since $[R_k, x_{nw}]$ and $[[R_k, x_{nw}], x_{nw}]$ are bounded for any k , the Helffer-Sjöstrand formula (I.A.31) gives

$$[J_{\xi_0}(\frac{x_{nw}}{t}), \prod_{k_1=1}^l R_{k_1}] = \frac{1}{t}[x_{nw}, \prod_{k_1=1}^l R_{k_1}] \nabla J_{\xi_0}(\frac{x_{nw}}{t}) + O(t^{-2}),$$

and

$$[\prod_{k_2=1}^3 R_{k_2}, J_{\xi_0}(\frac{x_{nw}}{t})] = \frac{1}{t} \nabla J_{\xi_0}(\frac{x_{nw}}{t}) [x_{nw}, \prod_{k_2=1}^3 R_{k_2}] + O(t^{-2}).$$

Now using that $J_{\xi_0}(\frac{x_{nw}}{t}) \nabla V(x)$ and $\nabla J_{\xi_0}(\frac{x_{nw}}{t}) \nabla V(x)$ belong to $O(t^{-2})$ by Lemma I.5.1, we conclude that (I.5.22) also belongs to $O(t^{-2})$ which ends the proof of the proposition.

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I.6 Wave operators

We turn now to the construction of *wave operators* for the massive Dirac operator H in order to describe precisely the asymptotic behaviour of the field when t goes to infinity. As is well known, the presence of a long-range potential $V(x)$ prevents us from taking e^{-itH_0} for the comparison dynamics. In our case $V \in S^{-1}(\mathbb{R}^3)$, we can use the ideas of Dollard and Velo [26] and define the following comparison dynamics denoted by $U_0(t)$

$$U_0(t) = e^{-itH_0} T \left(e^{-i \int_0^t (V_1(s\mathcal{V}) + mH_0^{-1}V_2(s\mathcal{V})) ds} \right), \quad (\text{I.6.1})$$

where T denotes time ordering. In this definition of the Dollard modification, we add a phase (formally) denoted $e^{-iX(t)}$ to e^{-itH_0} . This phase must be chosen in such a way that the standard Cook method applies (see Lemma I.4.1). Thus it must satisfy two rules. First, it must commute with e^{-itH_0} . Second, the operator $H - H_0 - X'(t)$ must be “short-range”. We see from this last assumption that a good candidate for X would be $X'(t) = V(x)$ but it does not fulfill the first assumption. As it is suggested by the microlocal velocity estimates, we can approach (asymptotically) the position operator x by $t\mathcal{V}$ which commutes with e^{-itH_0} and define

$$X'(t) = V_1(t\mathcal{V}) + \Gamma^0 V_2(t\mathcal{V}).$$

Unfortunately, the matrix Γ^0 which appears in the potential also does not commute with e^{-itH_0} and this is why we replaced it in the definition of $U_0(t)$ by the operator mH_0^{-1} using ideas of Thaller [75]. Precisely, we use the following lemma

Lemma I.6.1 *Let H_0 be the free Dirac operator. Let us denote $G = \Gamma^0 - mH_0^{-1}$ and*

$$E(t) = \int_0^t e^{isH_0} G e^{-isH_0} ds.$$

Then $E(t)$ is a bounded operator uniformly in t . As a consequence, we have $\lim_{t \rightarrow \infty} \frac{1}{t} \|E(t)\| = 0$.

Proof : First, remark that G anticommutes with H_0 . Therefore, we have

$$e^{isH_0} G e^{-isH_0} = e^{i2sH_0} G, \quad \forall s \in \mathbb{R},$$

and we obtain by integration the following explicit form for $E(t)$

$$E(t) = (e^{i2tH_0} - 1)(2iH_0)^{-1}G.$$

Thus, $E(t)$ is uniformly bounded, in operator norm, with respect to t .

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Then we define $X'(t) = V_1(t\mathcal{V}) + mH_0^{-1}V_2(t\mathcal{V})$ which leads to the definition (I.6.1) of the Dollard modification. The main result of this part is given by the theorem.

Theorem I.6.1 *The wave operators defined by*

$$\Omega^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH} U_0(t), \quad (\text{I.6.2})$$

$$\tilde{\Omega}^\pm := s - \lim_{t \rightarrow \pm\infty} U_0^*(t) e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm), \quad (\text{I.6.3})$$

exist in \mathcal{H} . Furthermore, we have $\tilde{\Omega}^\pm \Omega^\pm = \mathbf{1}_{\mathcal{H}}$, $\Omega^\pm \tilde{\Omega}^\pm = \mathbf{1}^c(H)$, $(\Omega^\pm)^* = \tilde{\Omega}^\pm$ and \mathcal{V} and P^\pm satisfy the intertwining relation

$$(\Omega^\pm) \mathcal{V} (\Omega^\pm)^* = P^\pm. \quad (\text{I.6.4})$$

At last, we have

$$(\Omega^\pm) H_0 (\Omega^\pm)^* = H \mathbf{1}^c(H). \quad (\text{I.6.5})$$

Note that in the definition of the wave operator (I.6.3), we make a crucial use of the characterization (I.5.10) of the projection onto the continuous spectrum of H i.e.

$$\mathbf{1}^c(H) = \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm).$$

Proof : For the proof of this theorem, we shall follow the strategy used in [21] and appeal to some results for time-dependent Dirac hamiltonians given in Appendix B. This fully time-dependent approach avoids the use of a *limiting absorption principle* as well as a detailed study of the resolvent $(H - z)^{-1}$ and related estimates. Here, the central objects are time-dependent observables such as $J(\frac{x}{t})$ where J has a compact support, and propagation estimates obtained in Section 4. Let us now explain this strategy for the proof of (I.6.2). First, remark that by a density argument it is enough to prove the existence of

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} U_0(t) \chi(H_0) \mathbf{1}_\Theta(\mathcal{V}), \quad (\text{I.6.6})$$

where $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m, \sigma_{pp}(H)\} = \emptyset$ and Θ denotes a compact subset of $\mathbb{R}^3 \setminus \{0\}$ such that the annulus $C(\frac{\epsilon_0(\chi)}{2}, 1)$ is a subset of Θ (remember that $\epsilon_0(\chi)$ is defined by the minimal velocity estimates, see Proposition I.4.2). Now consider a function $J \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ such that $J = 1$ on a neighbourhood of Θ . Let us associate to any function $f \in S^{-1}(\mathbb{R}^3)$ the time-dependent function

$$f_J(t, x) = f(x) J\left(\frac{x}{t}\right),$$

defined for $t \geq 1$. Such a function obviously satisfies the properties

- For any y in a neighbourhood of Θ ,

$$f(ty) = f_J(t, ty) \quad (\text{I.6.7})$$

- For any $t \geq 1$ fixed, $x \rightarrow f_J(t, x) \in C_0^\infty(\mathbb{R}^3)$ and there is a constant M such that

$$\text{supp } f_J(t, \cdot) \subset B(0, Mt).$$

- The following estimates hold

$$|\partial_x^\alpha f_J(t, x)| \leq C \langle t \rangle^{-1-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^3, \quad (\text{I.6.8})$$

$$|\partial_t^k f_J(t, x)| \leq C \langle t \rangle^{-1-k}, \quad \forall k \in \mathbb{N}. \quad (\text{I.6.9})$$

We introduce now some notations. We call *effective time-dependent potential* the potential $V_J(t, x) = V_{1,J}(t, x) + V_{2,J}(t, x)\Gamma^0$. We denote by $H_J(t)$ the time-dependent hamiltonian $H_0 + V_J(t, x)$ and by $U_J(t, s)$ the associated dynamics (see [21], Appendix B.3, Proposition B.3.6). We also denote by $U_{0,J}(t)$ the following time-dependent Dollard modification

$$U_{0,J}(t) = e^{-itH_0} T \left(e^{-i \int_0^t (V_{1,J}(s, s\mathcal{V}) + mH_0^{-1}V_{2,J}(s, s\mathcal{V})) ds} \right),$$

We rewrite (I.6.6) as follows

$$\begin{aligned} s - \lim_{t \rightarrow \pm\infty} e^{itH} U_0(t) \chi(H_0) \mathbf{1}_\Theta(\mathcal{V}) &= \\ s - \lim_{t \rightarrow \pm\infty} e^{itH} \chi(H_0) U_J(t, 0) U_J(0, t) \mathbf{1}_\Theta(\mathcal{V}) U_J(t, 0) U_J(0, t) U_{0,J}(t). \end{aligned}$$

where we used the facts that $U_0(t) \mathbf{1}_\Theta(\mathcal{V}) = U_{0,J}(t) \mathbf{1}_\Theta(\mathcal{V}) = \mathbf{1}_\Theta(\mathcal{V}) U_{0,J}(t)$ by (I.6.7) and that H_0 commutes with $U_0(t)$. Now assume the existence of the limits

$$\begin{aligned} s - \lim_{t \rightarrow \pm\infty} U_J(0, t) U_{0,J}(t), \\ s - \lim_{t \rightarrow \pm\infty} U_J(0, t) \mathbf{1}_\Theta(\mathcal{V}) U_J(t, 0) =: \mathbf{1}_\Theta(\mathcal{V}_J^\pm), \\ s - \lim_{t \rightarrow \pm\infty} e^{itH} \chi(H_0) U_J(t, 0) \mathbf{1}_\Theta(\mathcal{V}_J^\pm), \end{aligned}$$

then the limit (I.6.6) will exist by the chain rule. Moreover, the situation is completely symmetric for the proof of (I.6.3). Indeed by a density argument it is enough to prove the existence of

$$s - \lim_{t \rightarrow \pm\infty} U_0^*(t) e^{-itH} \mathbf{1}_\Theta(P^\pm) \chi(H), \quad (\text{I.6.10})$$

for a compact subset Θ defined as above. But using the characterization (I.5.11) of the asymptotic velocity P^\pm , we see that

$$\begin{aligned} s - \lim_{t \rightarrow \pm\infty} U_0^*(t) e^{-itH} \mathbf{1}_\Theta(P^\pm) \chi(H) &= s - \lim_{t \rightarrow \pm\infty} U_0^*(t) \mathbf{1}_\Theta(\mathcal{V}) e^{-itH} \mathbf{1}_\Theta(P^\pm) \chi(H), \\ &= s - \lim_{t \rightarrow \pm\infty} \mathbf{1}_\Theta(\mathcal{V}) U_{0,J}^*(t) U_J(t, 0) U_J(0, t) e^{-itH} \mathbf{1}_\Theta(P^\pm) \chi(H), \end{aligned}$$

where we used $U_0^*(t) \mathbf{1}_\Theta(\mathcal{V}) = \mathbf{1}_\Theta(\mathcal{V}) U_{0,J}^*(t)$ by (I.6.7). Therefore if we prove the existence of the limits

$$\begin{aligned} s - \lim_{t \rightarrow \pm\infty} U_{0,J}^*(t) U_J(t, 0), \\ s - \lim_{t \rightarrow \pm\infty} U_J(0, t) e^{-itH} \mathbf{1}_\Theta(P^\pm) \chi(H), \end{aligned}$$

then the limit (I.6.10) will exist by the chain rule.

If we summarize the previous discussion, we see that we can divide the proof of Theorem I.6.1 into three steps. First, for time-dependent Dirac operators of the following form $H_J(t) = H_0 + V(t, x)$, we have to define the asymptotic velocity \mathcal{V}_J^\pm . This is done in Appendix B where we also obtain propagation estimates for time-dependent Hamiltonians (see Propositions I.B.1 and I.B.2). Next, we prove the existence and asymptotic completeness of wave operators for such hamiltonians. Exactly, we prove the lemma

Lemma I.6.2 *The limits*

$$s - \lim_{t \rightarrow \pm\infty} U_J(0, t)U_{0,J}(t), \quad (\text{I.6.11})$$

$$s - \lim_{t \rightarrow \pm\infty} U_{0,J}^*(t)U_J(t, 0), \quad (\text{I.6.12})$$

exist.

For the last step of the proof, we have to make the link between time-dependent and time-independent Hamiltonians. Precisely we show

Lemma I.6.3 *There exist the limits*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \chi(H_0)U_J(t, 0)\mathbf{1}_\Theta(\mathcal{V}_J^\pm), \quad (\text{I.6.13})$$

$$s - \lim_{t \rightarrow \pm\infty} U_J(0, t)e^{-itH} \mathbf{1}_\Theta(P^\pm)\chi(H). \quad (\text{I.6.14})$$

Proof (of Lemma I.6.2) : Since the proofs are identical, we only treat the case $t \rightarrow +\infty$ for (I.6.12). The basic tool to prove Lemma I.6.2 will be the Helffer-Sjöstrand formula with several variables presented in Appendix A and Cook's method (Lemma I.4.1). Let $\psi \in \mathcal{D}$ where \mathcal{D} is a dense subset of \mathcal{H} that we shall define precisely later. Let us compute the Heisenberg derivative of the expression $U_{0,J}^*(t)U_J(t, 0)\psi$. We obtain

$$\begin{aligned} \frac{d}{dt}U_{0,J}^*(t)U_J(t, 0)\psi &= U_{0,J}^*(t)i\left\{H_0 + V_J(t, t\mathcal{V}) - H_J(t)\right\}U_J(t, 0)\psi, \\ &= U_{0,J}^*(t)i\left\{(V_{1,J}(t, t\mathcal{V}) - V_{1,J}(t, x)) + \Gamma^0(V_{2,J}(t, t\mathcal{V}) - V_{2,J}(t, x))\right. \\ &\quad \left.+ (mH_0^{-1} - \Gamma^0)V_{2,J}(t, t\mathcal{V})\right\}U_J(t, 0)\psi, \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Let us prove that $I_1(t) = U_{0,J}^*(t)i(V_{1,J}(t, t\mathcal{V}) - V_{1,J}(t, x))U_J(t, 0)\psi$ belongs to $L^1(dt)$. Since the different commutators between the components of x and \mathcal{V} are bounded i.e.

$$[x_k, \mathcal{V}_l] = iH_0^{-1}\delta_{kl} - ip_lH_0^{-1}\Gamma^kH_0^{-1} \in B(\mathcal{H}),$$

we can apply the Helffer-Sjöstrand formula (I.A.33) and write $I_1(t)$ as

$$I_1(t) = U_{0,J}^*(t)C_1(t)(x - t\mathcal{V})U_J(t, 0)\psi + O(t^{-2}),$$

where $C_1(t)$ is a bounded operator satisfying $C_1(t) \leq Ct^{-2}$. Now observe that if we prove

$$(x - t\mathcal{V})U_J(t, 0)\langle x \rangle^{-1} \in O(t^\mu), \quad (\text{I.6.15})$$

with $0 < \mu < 1$ then $I_1(t)$ will belong to $L^1(dt)$. Here, we chose $\mathcal{D} = D(\langle x \rangle)$. Unfortunately, for the same technical reasons as in the proof of microlocal velocity estimates, it is not obvious to show (I.6.15) in such a form. Actually, we shall prove

$$(x_{nw} - t\mathcal{V}_a(t))U_J(t, 0)\langle x \rangle^{-1} \in O(t^\mu),$$

with $0 < \mu < 1$. Indeed, as x and x_{nw} are equal up to a bounded operator Z , we can replace x by x_{nw} in the previous expression without any change. Moreover, we also have to approach the classical velocity operator \mathcal{V} by $\mathcal{V}_a(t) = \frac{1}{2}(pH_J^{-1}(t) + H_J^{-1}(t)p)$. Note that for t large enough, the operator $H_J(t)$ is always invertible and since

$$\mathcal{V} - \mathcal{V}_a(t) = \frac{1}{2}pH_0^{-1}V_J(t, x)H_J^{-1}(t) + hc \in O(t^{-1}), \quad (\text{I.6.16})$$

we can also replace \mathcal{V} by $\mathcal{V}_a(t)$ in the previous expression. Finally, to prove (I.6.15), it is enough to show that

$$L(t) := \frac{1}{t^\mu} \frac{d}{dt} \left(U_J(0, t)(x_{nw} - t\mathcal{V}_a(t))U_J(t, 0)\langle x \rangle^{-1} \right) \in L^1((1, +\infty), dt).$$

But we have

$$\begin{aligned} L(t) &= \frac{1}{t^\mu} U_J(0, t) \left\{ i[H_J(t), x_{nw} - t\mathcal{V}_a(t)] - \mathcal{V}_a(t) - t \frac{d}{dt}(\mathcal{V}_a(t)) \right\} U_J(t, 0)\langle x \rangle^{-1}, \\ &= \frac{1}{t^\mu} U_J(0, t) \left\{ (\mathcal{V} - \mathcal{V}_a(t)) + i[V_J(t, x), Z] + it[H_J(t), \mathcal{V}_a(t)] - t \frac{d}{dt}(\mathcal{V}_a(t)) \right\} U_J(t, 0)\langle x \rangle^{-1}. \end{aligned}$$

It remains to show that all the terms between brackets belong to $O(t^{-1})$ for $t \geq 1$. It is obvious for $(\mathcal{V} - \mathcal{V}_a(t))$ and $[V_J(t, x), Z]$ since Z is bounded. Furthermore,

$$i[H_J(t), \mathcal{V}_a(t)] = -\frac{1}{2} \left(\nabla V_J(t, x)H_J^{-1}(t) + H_J(t)^{-1} \nabla V_J(t, x) \right) \in O(t^{-2}), \quad (\text{I.6.17})$$

since $\nabla V_J(t, x) \in O(t^{-2})$ by (I.6.8). For the last term, we use the fact that

$$\frac{d}{dt}(\mathcal{V}_a(t)) = \frac{1}{2} \left(pH_J^{-1}(t) \partial_t V_J(t, x) H_J^{-1}(t) + H_J^{-1}(t) \partial_t V_J(t, x) H_J^{-1}(t) p \right) \in O(t^{-2}), \quad (\text{I.6.18})$$

since $\partial_t V_J(t, x) \in O(t^{-2})$ by (I.6.9). This ends the proof for $I_1(t)$. The proof for $I_2(t)$ is identical.

Now let us treat the term $I_3(t)$. Let us denote concisely $T \left(e^{-i \int_0^t (V_{1,J}(s, s\mathcal{V}) + mH_0^{-1}V_{2,J}(s, s\mathcal{V})) ds} \right)$ by $e^{-iX(t)}$. To apply Cook's Lemma, it is enough to show that

$$I_{s,u} := \left\| \int_s^u e^{iX(t)} (\partial_t E(t)) e^{itH_0} V_{2,J}(t, t\mathcal{V}) U_J(t, 0) \psi dt \right\|, \quad (\text{I.6.19})$$

tends to 0 when $s, u \rightarrow \infty$. From the definition of $E(t)$, it is clear that it anticommutes with H_0 and with \mathcal{V} . Thus we can write $e^{-iX(t)} (\partial_t E(t)) = (\partial_t E(t)) e^{-i\tilde{X}(t)}$ where

$$e^{-i\tilde{X}(t)} = T \left(e^{-i \int_0^t (V_{1,J}(s, -s\mathcal{V}) - mH_0^{-1}V_{2,J}(s, -s\mathcal{V})) ds} \right).$$

By an integration by part in (I.6.19), we obtain

$$\begin{aligned} I_{s,u} \leq & \left\| E(u)e^{-i\tilde{X}(u)}e^{iuH_0}V_{2,J}(u,u\mathcal{V})U_J(u,0)\psi \right\| \\ & + \left\| E(s)e^{-i\tilde{X}(s)}e^{isH_0}V_{2,J}(s,s\mathcal{V})U_J(s,0)\psi \right\| \\ & + \left\| \int_s^u E(t)\partial_t(e^{-i\tilde{X}(t)}e^{itH_0}V_{2,J}(t,t\mathcal{V})U_J(t,0)\psi) dt \right\|. \end{aligned}$$

Using $V_{2,J}(t,t\mathcal{V}) \in O(t^{-1})$ and Lemma I.6.1, the first two terms tend to 0 as $s, u \rightarrow \infty$. If we compute the derivative in the last term, we find

$$\begin{aligned} \partial_t(e^{-i\tilde{X}(t)}e^{itH_0}V_{2,J}(t,t\mathcal{V})U_J(t,0)\psi) = & \\ & -i(V_{1,J}(t,-t\mathcal{V}) - mH_0^{-1}V_{2,J}(t,-t\mathcal{V}))e^{-i\tilde{X}(t)}e^{itH_0}V_{2,J}(t,t\mathcal{V})U_J(t,0)\psi \\ & + e^{-i\tilde{X}(t)}e^{itH_0}(\partial_t V_{2,J}(t,t\mathcal{V}) + \mathcal{V}\nabla V_{2,J}(t,t\mathcal{V}))U_J(t,0)\psi \\ & - ie^{-i\tilde{X}(t)}e^{itH_0}V_{2,J}(t,t\mathcal{V})(V_{1,J}(t,x) + V_{2,J}(t,x)\Gamma^0)U_J(t,0)\psi. \end{aligned}$$

All these terms belong to $O(t^{-2})$ by (I.6.8) and (I.6.9). Since $E(t)$ is bounded, the last integral also tends to 0 which concludes the proof of the lemma.

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Proof (of Lemma I.6.3) : Since the proofs are identical, we only treat the case $t \rightarrow +\infty$. Let us consider a function $g \in C_0^\infty(\Theta)$ such that $g = 1$ on the annulus $C(\epsilon_0(\chi), 1)$. It is enough to prove the existence of (I.6.13) replacing $\mathbf{1}_\Theta(\mathcal{V}_J^+)$ by $g(\mathcal{V}_J^+)$. Let us define two other functions g_1 and g_2 belonging to $C_0^\infty(\Theta)$ such that $g_1g = g$ and $g_2g_1 = g_1$. Then using Proposition I.B.3, we have

$$\begin{aligned} g(\mathcal{V}_J^+) &= s - \lim_{t \rightarrow +\infty} U_J(0,t)g_1\left(\frac{x}{t}\right)g(\mathcal{V})U_J(t,0), \\ &= s - \lim_{t \rightarrow +\infty} U_J(0,t)g_2\left(\frac{x}{t}\right)g_1\left(\frac{x}{t}\right)g_2\left(\frac{x}{t}\right)g(\mathcal{V})U_J(t,0). \end{aligned}$$

Now we introduce the operator $M(t) = g_1\left(\frac{x}{t}\right) + (\mathcal{V} - \frac{x}{t}) \cdot \nabla g_1\left(\frac{x}{t}\right)$ and using Proposition I.B.4, we have

$$g(\mathcal{V}_J^+) = s - \lim_{t \rightarrow +\infty} U_J(0,t)g_2\left(\frac{x}{t}\right)M(t)g_2\left(\frac{x}{t}\right)g(\mathcal{V})U_J(t,0).$$

For technical reasons, we approach the operator $M(t)$ by $M_a(t) = g_1\left(\frac{x_n w}{t}\right) + (\mathcal{V}_a - \frac{x_n w}{t}) \cdot \nabla g_1\left(\frac{x_n w}{t}\right)$ where $\mathcal{V}_a = \frac{1}{2}(pH^{-1} + H^{-1}p)$ and also the operator $g(\mathcal{V})$ by $g(\mathcal{V}_a(t))$ where $\mathcal{V}_a(t)$ defined as in Lemma I.6.2. Using the Helffer-Sjöstrand formula and (I.6.16), we have

$$g(\mathcal{V}) - g(\mathcal{V}_a(t)) \in O(t^{-1}). \quad (\text{I.6.20})$$

Hence, by the same arguments as in the proof of Proposition I.4.5, we obtain

$$g(\mathcal{V}_J^+) = s - \lim_{t \rightarrow +\infty} U_J(0,t)g_2\left(\frac{x}{t}\right)M_a(t)g_2\left(\frac{x}{t}\right)g(\mathcal{V}_a(t))U_J(t,0).$$

Then we define the propagation observable

$$\phi_a(t) = \chi(H)g_2\left(\frac{x}{t}\right)M_a(t)g_2\left(\frac{x}{t}\right)g(\mathcal{V}_a(t)),$$

and it is easy to see that

$$(I.6.13) = s - \lim_{t \rightarrow +\infty} e^{itH} \phi_a(t) U_J(t, 0).$$

Let us compute the time derivative of this quantity. We obtain

$$\begin{aligned} \frac{d}{dt} \phi_a(t) + iH \phi_a(t) - i\phi_a(t) H_J(t) = \\ -\frac{1}{t} \chi(H) \left(\nabla g_2\left(\frac{x}{t}\right) \frac{x}{t} M_a(t) g_2\left(\frac{x}{t}\right) + hc \right) g(\mathcal{V}_a(t)) \end{aligned} \quad (I.6.21)$$

$$-\frac{1}{t} \chi(H) g_2\left(\frac{x}{t}\right) \left(\mathcal{V}_a - \frac{x_{nw}}{t} \right) \nabla^2 g_1\left(\frac{x_{nw}}{t}\right) \frac{x_{nw}}{t} g_2\left(\frac{x}{t}\right) g(\mathcal{V}_a(t)) \quad (I.6.22)$$

$$+ \chi(H) g_2\left(\frac{x}{t}\right) M_a(t) g_2\left(\frac{x}{t}\right) \frac{d}{dt} (g(\mathcal{V}_a(t))) \quad (I.6.23)$$

$$+ \chi(H) (i[H_0, g_2\left(\frac{x}{t}\right)] M_a(t) g_2\left(\frac{x}{t}\right) + hc) g(\mathcal{V}_a(t)) \quad (I.6.24)$$

$$+ \chi(H) g_2\left(\frac{x}{t}\right) i(H M_a(t) - M_a(t) H_J(t)) g_2\left(\frac{x}{t}\right) g(\mathcal{V}_a(t)) \quad (I.6.25)$$

$$+ \chi(H) g_2\left(\frac{x}{t}\right) M_a(t) g_2\left(\frac{x}{t}\right) i[H_J(t), g(\mathcal{V}_a(t))]. \quad (I.6.26)$$

Observing that $i[H_0, g_2\left(\frac{x}{t}\right)] = \frac{1}{t} \Gamma \cdot \nabla g_2\left(\frac{x}{t}\right)$, the terms (I.6.21) and (I.6.24) can be written after commutation operations and using (I.6.20) as

$$\frac{1}{t} \chi(H) \eta\left(\frac{x}{t}\right) B(t) \eta\left(\frac{x}{t}\right) g(\mathcal{V}) + O(t^{-2}),$$

where $B(t)$ is a uniformly bounded operator in t and $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp } \eta \cap C(\epsilon_0(\chi), 1) = \emptyset$ and $\text{supp } \eta \cap \text{supp } g = \emptyset$. Thus these terms are integrable along the evolution using Propositions I.4.2, I.4.4 and I.B.3.

Moreover, combining the Helffer-Sjöstrand formula and using (I.6.17) and (I.6.18), the following estimates hold

$$i[H_J(t), g(\mathcal{V}_a(t))] \in O(t^{-2}), \quad (I.6.27)$$

$$\frac{d}{dt} g(\mathcal{V}_a(t)) \in O(t^{-2}). \quad (I.6.28)$$

Therefore, the terms (I.6.23) and (I.6.26) belong to $O(t^{-2})$ and are integrable in norm.

Now note that $V_J(t, x) g_2\left(\frac{x}{t}\right) = V(x) g_2\left(\frac{x}{t}\right)$. Hence we have the following equality

$$g_2\left(\frac{x}{t}\right) i(H M_a(t) - M_a(t) H_J(t)) g_2\left(\frac{x}{t}\right) = g_2\left(\frac{x}{t}\right) i[H, M_a(t)] g_2\left(\frac{x}{t}\right),$$

and using Lemma I.5.2, the term (I.6.25) becomes

$$\begin{aligned} \frac{1}{t} \chi(H) g_2\left(\frac{x}{t}\right) \left(\mathcal{V}_a - \frac{x_{nw}}{t} \right) \nabla^2 g_1\left(\frac{x_{nw}}{t}\right) \mathcal{V} g_2\left(\frac{x}{t}\right) g(\mathcal{V}_a(t)) \\ + \chi(H) g_2\left(\frac{x}{t}\right) i[H, \mathcal{V}_a] \nabla g_1\left(\frac{x_{nw}}{t}\right) g_2\left(\frac{x}{t}\right) g(\mathcal{V}_a(t)) + O(t^{-2}). \end{aligned}$$

Repeating the proof of Proposition I.4.5, the sum of (I.6.22) and (I.6.25) is equal to

$$\frac{1}{t} \chi(H) g_2\left(\frac{x}{t}\right) \left(\mathcal{V} - \frac{x}{t} \right) \nabla^2 g_1\left(\frac{x}{t}\right) \left(\mathcal{V} - \frac{x}{t} \right) g_2\left(\frac{x}{t}\right) g(\mathcal{V}_a(t)) + O(t^{-2}).$$

Eventually, using (I.6.20) once again, we obtain

$$\begin{aligned} \frac{d}{dt}\phi_a(t) + iH\phi_a(t) - i\phi_a(t)H_J(t) &= \frac{1}{t}\chi(H)g_2\left(\frac{x}{t}\right)\left(\mathcal{V} - \frac{x}{t}\right)\nabla^2 g_1\left(\frac{x}{t}\right)\left(\mathcal{V} - \frac{x}{t}\right)g_2\left(\frac{x}{t}\right)g(\mathcal{V}) \\ &\quad + L^1(dt) + O(t^{-2}). \end{aligned}$$

Thus using Propositions I.4.5 and I.B.4, this term is integrable along the evolution and we conclude the proof of (I.6.13) by Lemma I.4.1.

The proof of (I.6.14) is essentially the same as the previous one. We omit it.

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Therefore, we have constructed the Dollard-modified wave operators Ω^\pm and $\tilde{\Omega}^\pm$. The fact that $(\Omega^\pm)^* = \tilde{\Omega}^\pm$ follows from [21], Lemma B.5.1. It remains to prove the intertwining relations (I.6.4) and (I.6.5). Using Proposition I.5.1, we see that

$$\begin{aligned} P^\pm \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm) &= s - \lim_{t \rightarrow \pm\infty} e^{itH} \mathcal{V} e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm), \\ &= s - \lim_{t \rightarrow \pm\infty} e^{itH} U_0(t) \mathcal{V} U_0^*(t) e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm), \\ &= \Omega^\pm \mathcal{V} (\Omega^\pm)^*, \end{aligned}$$

which proves (I.6.4). Now (I.6.5) is equivalent to

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} (H - H_0) e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm) = 0.$$

But using Theorem I.5.1, it is enough to show that

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} V(x) J\left(\frac{x}{t}\right) e^{-itH} = 0,$$

for any $J \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. As $V \in S^{-1}(\mathbb{R}^3)$, $V(x)J\left(\frac{x}{t}\right) \in O(t^{-1})$ and the result holds.

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APPENDIX

I.A Functions of selfadjoint operators and applications

In this appendix we give some useful formulae to study functions of selfadjoint operators $f(A)$ and commutators of the following form $[T, f(A)]$. Here, A and T denote two (possibly vectors of) commuting selfadjoint operators acting on a Hilbert space \mathcal{H} . We first consider the case of a function $f \in \mathcal{L}(\mathbb{R}^3)$; in this case, the Fourier transform is enough to obtain the formulae. For smooth functions $f \in C^\infty(\mathbb{R}^3)$ which are not necessarily integrable, we shall need the Helffer-Sjöstrand formula presented Section I.A.2.

I.A.1 Fourier transform

General setting : Let A a vector of commuting selfadjoint operators in \mathcal{H} and let us denote by $W(\xi) = \{e^{-i\xi A}\}$ the induced unitary representation of \mathbb{R}^3 in \mathcal{H} . We need the function space $\mathcal{F}^{1,N}(\mathbb{R}^3) = \{f \in L^1(\mathbb{R}^3) / (1 + |\xi|^2)^{\frac{N}{2}} \hat{f}(\xi) \in L^1(\mathbb{R}^3)\}$ for $N \in \mathbb{N}$. Equivalently we have $\mathcal{F}^{1,N}(\mathbb{R}^3) = \{f \in$

$L^1(\mathbb{R}^3)/\widehat{\nabla^k f} \in L^1(\mathbb{R}^3)$, $k = 1, \dots, N$. As a first case we consider a bounded operator T in \mathcal{H} and a function $f \in W^{1,1}(\mathbb{R}^3)$. We now define the class $C^1(A; \mathcal{H})$ of regular bounded operators with respect to A by

$$C^1(A; \mathcal{H}) = \left\{ T \in \mathcal{B}(\mathcal{H}) / s \longrightarrow e^{is\xi A} T e^{-is\xi A} \in C^1(\mathbb{R}_s; \mathcal{B}(\mathcal{H})) \right\}.$$

Then we have the following equivalence (see [1], Proposition 5.2)

$$T \in C^1(A; \mathcal{H}) \iff [T, A] \in \mathcal{B}(\mathcal{H}).$$

In particular, for T a bounded operator belonging to $C^1(A; \mathcal{H})$, we can write

$$\begin{aligned} [T, e^{i\xi A}] &= \int_0^1 \frac{d}{ds} \left(e^{i(1-s)\xi A} T e^{is\xi A} \right) ds, \\ &= (i\xi) \int_0^1 e^{i(1-s)\xi A} [T, A] e^{is\xi A} ds. \end{aligned} \quad (\text{I.A.1})$$

Now assume that $f \in \mathcal{F}^{1,1}(\mathbb{R}^3)$. Using the Fourier transform of f denoted by \hat{f} and (I.A.1), we have

$$\begin{aligned} i[T, f(A)] &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \hat{f}(\xi) i[T, e^{i\xi A}] d\xi, \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla f}(\xi) \int_0^1 e^{i(1-s)\xi A} i[T, A] e^{is\xi A} ds d\xi, \end{aligned} \quad (\text{I.A.2})$$

where the right-hand-sides in the two equalities of (I.A.2) make sense on $\mathcal{B}(\mathcal{H})$ since $\hat{f} \in L^1(\mathbb{R}^3)$ and $\widehat{\nabla f} \in L^1(\mathbb{R}^3)$ by hypothesis and $[T, A]$ bounded on \mathcal{H} .

Now we consider the case of an unbounded operator T in \mathcal{H} . Let us introduce some definitions and basic results. Let H be another selfadjoint operator in a Hilbert space \mathcal{H} . Assume that the family of unitary operators $\{W(\xi)\}_{\xi \in \mathbb{R}^3}$ leaves invariant the domain $D(H)$ of H . We denote by \mathcal{G}^1 this domain and by \mathcal{G}^s , $0 \leq s \leq 1$, the interpolation space between $\mathcal{G}^0 = \mathcal{H}$ and $\mathcal{G}^1 = D(H)$. We also denote by $\mathcal{G}^{-s} = (\mathcal{G}^s)^*$ the dual Hilbert space of \mathcal{G}^s . We make the usual identification $\mathcal{H}^* = \mathcal{H}$ and thus we have

$$\mathcal{G}^s \hookrightarrow \mathcal{G}^t \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{G}^{-t} \hookrightarrow \mathcal{G}^{-s}, \quad 0 \leq t \leq s \leq 1.$$

At last we denote by $\mathcal{B}(\mathcal{G}^s, \mathcal{G}^t)$ the set of bounded operators between \mathcal{G}^s and \mathcal{G}^t . Then we have the following result (see [10], Proposition 1).

Proposition I.A.1 *Let H, T two selfadjoint operators on \mathcal{H} satisfying $T \in \mathcal{B}(\mathcal{G}^1, \mathcal{G}^{-1})$. Let A a vector of commuting selfadjoint operators on \mathcal{H} . Let $-1 \leq t \leq s \leq 1$. Assume that $e^{i\xi A} \mathcal{G}^1 \subset \mathcal{G}^1$ and $[T, A] \in \mathcal{B}(\mathcal{G}^s, \mathcal{G}^t)$. Then one has*

$$i[T, e^{i\xi A}] = i\xi \int_0^1 e^{i(1-s)\xi A} i[T, A] e^{is\xi A} ds, \quad (\text{I.A.3})$$

where the right-hand-side is well defined as a strong integral on $\mathcal{B}(\mathcal{G}^s, \mathcal{G}^t)$ i.e. the integral is strongly convergent. As a direct application, we have for all $f \in \mathcal{F}^{1,1}(\mathbb{R}^3)$

$$i[T, f(A)] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla f}(\xi) \int_0^1 e^{i(1-s)\xi A} i[T, A] e^{is\xi A} ds d\xi. \quad (\text{I.A.4})$$

Application (Commutator expansion) : Commutator expansions of two selfadjoint operators T, A are formulae which relate the commutator $[T, f(A)]$ with the successive commutators $ad_A^k(T)$ defined by recurrence by $ad_A^1(T) = [T, A]$ and for any $k \geq 1$, $ad_A^k(T) = [ad_A^{k-1}(T), A]$.

Proposition I.A.2 *Let H, T two selfadjoint operators on \mathcal{H} such that $T \in \mathcal{B}(\mathcal{G}^1, \mathcal{G}^{-1})$. Let A a vector of commuting selfadjoint operators on \mathcal{H} . Let $f \in \mathcal{F}^{1, N+1}(\mathbb{R}^3)$. Assume that*

(H1) $e^{i\xi A} \mathcal{G}^1 \subset \mathcal{G}^1$;

(H2) $ad_A^k(T) \in \mathcal{B}(\mathcal{H})$ for any $k \leq N + 1$.

Then we have

$$[T, f(A)] = \sum_{k=0}^N \frac{\nabla^k f(A)}{k!} ad_A^k(T) + R_N, \quad (\text{I.A.5})$$

where $\|R_N\| \leq c_N(f) \|ad_A^{N+1}(T)\|$. If we replace the operator A by $\frac{A}{t}$ we have the following useful consequence of (I.A.5)

$$[T, f(\frac{A}{t})] = \sum_{k=0}^N \frac{\nabla^k f(\frac{A}{t})}{t^k k!} ad_A^k(T) + O(t^{-N-1}). \quad (\text{I.A.6})$$

Proof : In formula (I.A.4), we commute $[T, A]$ with $e^{is\xi A}$ to obtain

$$[T, f(A)] = \nabla f(A)[T, A] + \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i(1-s)\xi A} [[T, A], e^{is\xi A}] ds d\xi.$$

Now we use formula (I.A.3) in this last integral. We repeat the procedure N times to obtain (I.A.5).

◇

The assumption on the stability of the domain $D(H)$ under the action of the unitary group $e^{i\xi A}$ is not easy to prove in general. We already saw that, if $[H, A] \in \mathcal{B}(D(H), \mathcal{H})$ then it suffices to show that $(H - z)^{-1} D(A) \subset D(A)$ for $z \notin \sigma(H)$ (see Lemma I.3.4). Actually, when the commutator between $[H, A]$ is bounded, we have the following equivalences

Lemma I.A.1 *Let H, A two selfadjoint operators on \mathcal{H} . Assume that $i[H, A]$ defined as a quadratic form on $D(H) \cap D(A)$ extends to a bounded operator on \mathcal{H} . Then the following assertions are equivalent*

- (i) $(H - z)^{-1} D(A) \subset D(A)$,
- (ii) $e^{i\xi A} D(H) \subset D(H)$,
- (iii) $(A - z)^{-1} D(H) \subset D(H)$,
- (iv) $e^{isH} D(A) \subset D(A)$.

In particular, if H or A is bounded on \mathcal{H} then the four assertions are automatically satisfied.

Proof : According to Lemma I.3.4, we only need to prove that (ii) implies (iii) and (iv) implies (i). These two implications follow from the same argument. Assume (ii) and $[H, A]$ bounded. Let $\psi \in D(H)$. We have to show that $\|H(A - z)^{-1}\psi\| < \infty$. For we use the resolvent formula

$$H(A - z)^{-1}\psi = i \int_0^\infty e^{izt} H e^{-itA} \psi dt, \quad \text{Im } z > 0.$$

Then we commute H and e^{-itA} under the integral and we use formula (I.A.3). We obtain

$$H(A-z)^{-1}\psi = i \int_0^\infty e^{izt} e^{-itA} H\psi dt + i \int_0^\infty e^{izt} (-it) \int_0^1 e^{-i(1-s)tA} [H, A] e^{-istA} \psi ds dt.$$

Finally one has

$$\|H(A-z)^{-1}\psi\| \leq \|H\psi\| \int_0^\infty e^{-Imzt} dt + C\|\psi\| \int_0^\infty e^{-Imzt}|t| dt,$$

since $[H, A]$ is bounded. Both integrals converge then the result holds.

◇

We now give some examples needed in the previous sections. H is the massive Dirac operator defined in section 2. In this case, the domain \mathcal{G}^1 of H is equal to $[H^1(\mathbb{R}^3)]^4$.

Example 1 Let $T = H$ and A the locally conjugate operator defined in section 3. Then the assumptions of Proposition I.A.1 are obviously satisfied since $e^{i\xi A} \mathcal{G}^1 \subset \mathcal{G}^1$ and $i[H, A] \in \mathcal{B}(\mathcal{H})$. Then we have

$$i[H, f(A)] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i(1-s)\xi A} (\mathcal{V}^2 + K) e^{is\xi A} ds d\xi, \quad (\text{I.A.7})$$

where K is compact on \mathcal{H} . The right hand side is a bounded operator on \mathcal{H} .

Example 2 Let $T = H$ and let A be $\mathcal{V}_a = \frac{1}{2}(pH^{-1} + H^{-1}p)$ the approached classical velocity operator defined in section 6. Clearly the commutator $[H, \mathcal{V}_a] = \frac{i}{2}(\nabla V H^{-1} + H^{-1} \nabla V)$ is bounded. Then $e^{i\xi \mathcal{V}_a} \mathcal{G}^1 \subset \mathcal{G}^1$ is satisfied by Lemma I.A.1 since \mathcal{V}_a is also bounded. Thus we have

$$i[H, f(\mathcal{V}_a)] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i(1-s)\xi \mathcal{V}_a} \frac{i}{2} (\nabla V(x) H^{-1} + H^{-1} \nabla V(x)) e^{is\xi \mathcal{V}_a} ds d\xi. \quad (\text{I.A.8})$$

The right hand side is a bounded operator on \mathcal{H} .

Example 3 Let $T = H$ and let $A = x_{nw}$ the Newton-Wigner operator defined in Section 2. Again we have to check that $e^{i\xi x_{nw}} \mathcal{G}^1 \subset \mathcal{G}^1$. First observe that $[H, x_{nw}] = \mathcal{V} + [V, Z]$ is bounded on \mathcal{H} . Then it is equivalent to show that $(H-z)^{-1} D(x_{nw}) \subset D(x_{nw})$ by Lemma I.A.1. But $D(x_{nw}) = D(x)$ and the result follows from the domain invariance property proved in Section 2. Finally we have

$$i[H, f(x_{nw})] = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \widehat{\nabla} f(\xi) \int_0^1 e^{i(1-s)\xi x_{nw}} \frac{i}{2} (\mathcal{V} + i[V(x), Z]) e^{is\xi x_{nw}} ds d\xi. \quad (\text{I.A.9})$$

The right hand side is a bounded operator on \mathcal{H} .

I.A.2 Helffer-Sjöstrand formula

In this appendix, we give a brief review of the Helffer-Sjöstrand formula which first appeared in [48] and which is useful to estimate functions of selfadjoint operators $f(A)$ for functions f which are not integrable. We follow the presentation given by Davies in [18] and we state a version needed in Section 6.

First, consider a function f belonging to the class of smooth real-functions

$$S^{-\rho}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : |f^{(k)}(x)| \leq C\langle x \rangle^{-\rho-k}, \forall k \in \mathbb{N}\}, \quad \rho > 0.$$

We call *almost-analytic extension* of f and we denote it by \tilde{f} the following function

$$\tilde{f}(z) = \tilde{f}(x + iy) := \left\{ \sum_{k=0}^n f^{(k)}(x) \frac{(iy)^k}{k!} \right\} \tau\left(\frac{y}{\langle x \rangle}\right), \quad (\text{I.A.10})$$

where n is an integer larger than 1 and $\tau \in C_0^\infty(\mathbb{R})$ such that $\tau(x) = 1$ for $|x| \leq 1$ and $\tau(x) = 0$ for $|x| \geq 2$. This function satisfies the properties (see [18])

$$\bullet \quad \tilde{f}|_{\mathbb{R}} = f, \quad (\text{I.A.11})$$

$$\bullet \quad \text{supp } \tilde{f} \subset \{x + iy : |y| \leq C\langle x \rangle\}, \quad (\text{I.A.12})$$

$$\bullet \quad |\partial_{\bar{z}} \tilde{f}(z)| \leq C\langle x \rangle^{-\rho-1-n} |y|^n, \quad (\text{I.A.13})$$

$$\bullet \quad f(x) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) (z - x)^{-1} dz \wedge d\bar{z}. \quad (\text{I.A.14})$$

Now, given a self-adjoint operator A in a Hilbert space \mathcal{H} , we can define the operator $f(A)$ as follows

$$f(A) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z) (z - A)^{-1} dz \wedge d\bar{z}, \quad (\text{I.A.15})$$

where the integral converges in operator norm in \mathcal{H} by (I.A.13) and since $\rho > 0$. Moreover, we stress the fact that the operator $f(A)$ does not depend of the choices of $n \geq 1$ and τ (see Lemma 2.2.4 in [18]).

For the applications, we use this formula for time-dependent potentials of the following form. Let $J \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Given a function $f \in S^{-\rho}(\mathbb{R})$, we denote by $f(t, x)$ the time-dependent function defined by

$$f(t, x) = f(x) J\left(\frac{x}{t}\right).$$

With respect to the variable x , this function belongs to $C_0^\infty(K_t) \subset S^{-\rho}(\mathbb{R})$ where K_t denotes a compact subset of \mathbb{R} such that $K_t \subset [-Mt, Mt]$, for M large enough and independent of t . Thus we can define the almost-analytic extension of f with respect to x by

$$\tilde{f}(t, z) := \left\{ \sum_{k=0}^n f^{(k)}(t, x) \frac{(iy)^k}{k!} \right\} \tau\left(\frac{y}{\langle x \rangle}\right). \quad (\text{I.A.16})$$

This function satisfies the following properties

$$\bullet \quad \text{supp } \tilde{f} \subset [-Mt, Mt]^2, \quad (\text{I.A.17})$$

$$\bullet \quad |\partial_{\bar{z}} \tilde{f}(z)| \leq Ct^{-\rho-1-n} |y|^n, \quad (\text{I.A.18})$$

$$\bullet \quad f(t, x) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(t, z) (z - x)^{-1} dz \wedge d\bar{z}. \quad (\text{I.A.19})$$

We can define time-dependent functions of self-adjoint operators A as previously by

$$f(t, A) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(t, z) (z - A)^{-1} dz \wedge d\bar{z}. \quad (\text{I.A.20})$$

Now we are interested in extending this last formula to functions f of several variables. Let us consider f belonging to the class of functions

$$S^{-\rho}(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3) : |\partial_x^\alpha f(x)| \leq C\langle x \rangle^{-\rho-|\alpha|}, \forall \alpha \in \mathbb{N}^3\}, \quad \rho > 0.$$

We can define the Helffer-Sjöstrand formula for this function with respect to the first variable. If $x = (x_1, x_2, x_3)$, we have

$$\tilde{f}(x_1 + iy_1, x_2, x_3) = \left\{ \sum_{\alpha_1=0}^{n_1} \partial_{x_1}^{\alpha_1} f(x) \frac{(iy_1)^{\alpha_1}}{\alpha_1!} \right\} \tau\left(\frac{y_1}{\langle x_1 \rangle}\right).$$

Using (I.A.13), we see that the functions $x_2 \rightarrow \tilde{f}(x_1 + iy_1, x_2, x_3)$ (resp. $x_3 \rightarrow \tilde{f}(x_1 + iy_1, x_2, x_3)$) also belong to the class $S^{-\rho}(\mathbb{R})$ uniformly with respect to the other variables. That is to say we have

$$|\partial_{x_j}^{\alpha_j} \tilde{f}(x_1 + iy_1, x_2, x_3)| \leq C \langle x \rangle^{-\rho}, \quad \forall \alpha_j \in \mathbb{N}, \quad j = 2, 3,$$

and the constant C does not depend on the variables x_1, y_1, x_2 (resp. x_1, y_1, x_3). Thus by induction we define the complete almost-analytic extension

$$\tilde{f}(z_1, z_2, z_3) = \left\{ \sum_{\alpha \leq \beta} \partial_x^\alpha f(x) \frac{(iy_1)^{\alpha_1}}{\alpha_1!} \frac{(iy_2)^{\alpha_2}}{\alpha_2!} \frac{(iy_3)^{\alpha_3}}{\alpha_3!} \right\} \tau\left(\frac{y_1}{\langle x_1 \rangle}\right) \tau\left(\frac{y_2}{\langle x_2 \rangle}\right) \tau\left(\frac{y_3}{\langle x_3 \rangle}\right),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (n_1, n_2, n_3)$ and $n_j \geq 1$, for all $j = 1, 2, 3$. This function satisfies the properties.

- $\text{supp } \tilde{f} \subset \{(x, y) \in \mathbb{R}^6 / |y_j| \leq C \langle x_j \rangle, j = 1, 2, 3\},$ (I.A.21)

- $|\partial_{\bar{z}} \tilde{f}(z)| \leq C \langle x \rangle^{-\rho-3-|\alpha|} |y_1|^{n_1} |y_2|^{n_2} |y_3|^{n_3},$ (I.A.22)

- $f(x) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) (z_1 - x_1)^{-1} (z_2 - x_2)^{-1} (z_3 - x_3)^{-1} dz \wedge d\bar{z}.$ (I.A.23)

Here, we denoted $z = (z_1, z_2, z_3) = (x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$. Moreover, $\partial_{\bar{z}}$ stands for $\partial_{\bar{z}_1} \partial_{\bar{z}_2} \partial_{\bar{z}_3}$ and $dz \wedge d\bar{z}$ for $(dz_1 \wedge d\bar{z}_1) \wedge (dz_2 \wedge d\bar{z}_2) \wedge (dz_3 \wedge d\bar{z}_3)$. The formulae (I.A.22) and (I.A.23) follow from (I.A.13) and (I.A.14).

Eventually, if A denotes a vector of commuting self-adjoint operators in \mathcal{H} , we can define $f(A)$ as follows

$$f(A) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) (z_1 - A_1)^{-1} (z_2 - A_2)^{-1} (z_3 - A_3)^{-1} dz \wedge d\bar{z}. \quad (\text{I.A.24})$$

Of course, we have a time dependent version of this last formula. Let us consider a function $J \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Define the time-dependent function $f(t, x) = f(x)J(\frac{x}{t})$ as previously. Thus the almost-analytic extension

$$\tilde{f}(t, z_1, z_2, z_3) = \left\{ \sum_{\alpha} \partial_x^\alpha f(t, x) \frac{(iy_1)^{\alpha_1}}{\alpha_1!} \frac{(iy_2)^{\alpha_2}}{\alpha_2!} \frac{(iy_3)^{\alpha_3}}{\alpha_3!} \right\} \tau\left(\frac{y_1}{\langle x_1 \rangle}\right) \tau\left(\frac{y_2}{\langle x_2 \rangle}\right) \tau\left(\frac{y_3}{\langle x_3 \rangle}\right),$$

satisfies the properties

- $\text{supp } \tilde{f} \subset [-Mt, Mt]_{z_1}^2 \times [-Mt, Mt]_{z_2}^2 \times [-Mt, Mt]_{z_3}^2,$ (I.A.25)

- $|\partial_{\bar{z}} \tilde{f}(t, z)| \leq C t^{-\rho-3-|\alpha|} |y_1|^{\alpha_1} |y_2|^{\alpha_2} |y_3|^{\alpha_3},$ (I.A.26)

- $f(t, x) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(t, z) (z_1 - x_1)^{-1} (z_2 - x_2)^{-1} (z_3 - x_3)^{-1} dz \wedge d\bar{z}.$ (I.A.27)

Given A a vector of commuting selfadjoint operators on \mathcal{H} , we define

$$f(t, A) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(t, z) (z_1 - A_1)^{-1} (z_2 - A_2)^{-1} (z_3 - A_3)^{-1} dz \wedge d\bar{z}. \quad (\text{I.A.28})$$

Before to give commutator expansion formula, let us make the following observation. The assumption $f \in S^{-\rho}(\mathbb{R}^3)$ with $\rho > 0$ is only needed to ensure the convergence of the integral in operator norm. Actually, formulae (I.A.15), (I.A.20), (I.A.24), (I.A.28) hold if we assume $\rho \in \mathbb{R}$ and the almost-analytic extensions satisfy the same properties. When $\rho \leq 0$, the integrals converge for the strong convergence.

Application 1 (Commutator expansion) : Let A a vector of commuting selfadjoint operators on \mathcal{H} and T selfadjoint on \mathcal{H} . Let $f \in S^\rho(\mathbb{R}^3)$ with $\rho \in \mathbb{R}$. Assume that

(H3) $A \in C^1(T)$.

(H4) $ad_A^k(T) \in \mathcal{B}(\mathcal{H})$, $k = 1, 2$.

Then we have

$$[T, f(A)] = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) [T, R_1 R_2 R_3] dz \wedge d\bar{z},$$

where $R_j = (z_j - A_j)^{-1}$. The assumption **(H3)** allows us to expand the commutator under the integral and one obtains

$$[T, f(A)] = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) [T, A_l] \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z}. \quad (\text{I.A.29})$$

Provided $[T, A_j]$ bounded for $l = 1, 2, 3$, the integral in (I.A.29) converges in operator norm for $\rho < 1$ by (I.A.22). Now commuting $[T, A_l]$ with $\prod_{k_2=l}^3 R_{k_2}$ and noting that

$$\frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z} = \nabla f(A) \cdot [T, A],$$

we get

$$[T, f(A)] = \nabla f(A) \cdot [T, A] + \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) \left[[T, A_l], \prod_{k_2=l}^3 R_{k_2} \right] \right) dz \wedge d\bar{z}.$$

Since $ad_A^2(T)$ bounded on \mathcal{H} , this last integral converges in norm of operator for $\rho < 2$. Eventually we have proven that for any $f \in S^\rho(\mathbb{R}^3)$ with $\rho < 2$,

$$[T, f(A)] = \nabla f(A) \cdot [T, A] + B, \quad (\text{I.A.30})$$

where B bounded on \mathcal{H} .

In many cases we have to deal with functions of time-dependent Hamiltonians of the form $f(\frac{A}{t})$. Therefore under the same hypotheses on f, T, A , we have

$$[T, f(\frac{A}{t})] = \frac{1}{t} \nabla f(\frac{A}{t}) \cdot [T, A] + O(t^{-2}). \quad (\text{I.A.31})$$

Application 2 : Given two vectors of selfadjoint operators A and T on \mathcal{H} and a function f belonging to the space $S^\rho(\mathbb{R}^3)$, we would like to express the difference between $f(A)$ and $f(T)$ as the product of an operator C and $A - T$. We apply formula (I.A.24)

$$f(A) - f(T) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(R_1^A R_2^A R_3^A - R_1^T R_2^T R_3^T \right) dz \wedge d\bar{z},$$

where $R_j^A = (z_j - A_j)^{-1}$ and $R_j^T = (z_j - T_j)^{-1}$. Using the resolvent identity $R_j^A - R_j^T = R_j^T(A_j - T_j)R_j^A$ which makes sense on $\mathcal{B}(\mathcal{H})$, one obtains

$$f(A) - f(T) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^T \right) (A_l - T_l) \left(\prod_{k_2=l}^3 R_{k_2}^A \right) \right) dz \wedge d\bar{z}.$$

Now we want to commute $A_l - T_l$ with $\prod_{k_2=l}^3 R_{k_2}^A$ under the integral. Thus we have to assume that $A \in C^1(T)$. In this case, we get

$$f(A) - f(T) = C.(A - T) + B, \quad (\text{I.A.32})$$

where

$$C = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^T \right) \left(\prod_{k_2=l}^3 R_{k_2}^A \right) \right) dz \wedge d\bar{z},$$

is a bounded operator on \mathcal{H} if $\rho < 1$ according to (I.A.22). Moreover if we suppose $[T, A_j]$ bounded then

$$B = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^T \right) \left[(A_l - T_l), \prod_{k_2=l}^3 R_{k_2}^A \right] \right) dz \wedge d\bar{z},$$

is bounded if $\rho < 2$ by (I.A.22) since $A \in C^1(T)$ implies $[T, R_j^A] = R_j^A[T, A_j]R_j^A$.

We end this application by giving a time-dependent version useful for Section 6. Given two self-adjoint operators A and T in a Hilbert space \mathcal{H} and given two functions $f \in \mathcal{S}(\mathbb{R}^3)$ and $J \in C_0^\infty(\mathbb{R}^3)$, we want to express the difference between the functions of operators $f(t, A)$ and $f(t, tT)$ as the product of one operator $C(t)$ and $A - tT$. Moreover, we want to obtain good estimates of $C(t)$ with respect to t when t goes to infinity. Assume that $A \in C^1(T)$ and $[A, T]$ bounded on \mathcal{H} then using formula (I.A.32), we get

$$f(t, A) - f(t, tT) = C(t).(A - tT) + B(t). \quad (\text{I.A.33})$$

From the exact expressions of $C(t)$ and $B(t)$, by (I.A.26) and (I.A.25), if we assume $\rho < 1$, we obtain the following estimates

$$\begin{aligned} C(t) &\in O(t^{\rho-1}), \\ B(t) &\in O(t^{\rho-1}). \end{aligned}$$

I.B Time-dependent Dirac operator and asymptotic velocity

In this appendix, we study time-dependent Dirac operators of the form

$$H(t) = H_0 + V(t, x),$$

where $V(t, x) = V_1(t, x) + V_2(t, x)\Gamma^0$ is the sum of a scalar and a matrix-valued time-dependent potentials. The main assumptions on the time decay of $V(t, x)$ will be

$$|\partial_t^k V(t, x)| \leq C \langle t \rangle^{-1-k}, \quad k \in \mathbb{N} \quad (\text{I.B.1})$$

$$|\partial_x^\alpha V(t, x)| \leq C \langle t \rangle^{-1-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^3. \quad (\text{I.B.2})$$

For such Hamiltonians, it is possible to define an associated unitary dynamics (see [21], appendix B.3, Proposition B.3.6) which we will denote by $U(t, s)$ and which satisfies

- The map $(t, s) \longrightarrow U(t, s)$ is strongly continuous with values in unitary operators in \mathcal{H} such that

$$U(t, t) = 1, \quad U(t, u)U(u, s) = U(t, s).$$

- If we denote $B = (H_0^2 + 1)^{\frac{1}{2}}$, we have

$$\begin{aligned} \partial_s U(t, s)B^{-1} &= U(t, s)iH(s)B^{-1}, \\ \partial_t B^{-1}U(t, s) &= -iB^{-1}H(t)U(t, s). \end{aligned}$$

We wish to define the asymptotic velocity and to describe some of its properties. Let us first prove the proposition

Proposition I.B.1 *Under the previous assumptions, the limit*

$$s - C_\infty - \lim_{t \rightarrow \pm\infty} U(0, t)\mathcal{V}U(t, 0) =: \mathcal{V}^\pm, \quad (\text{I.B.3})$$

exists.

Proof : By a density argument, it is enough to show that

$$\lim_{t \rightarrow \pm\infty} U(0, t)g(\mathcal{V})U(t, 0)\psi,$$

exists for any $\psi \in \mathcal{H}$ and any $g \in C_0^\infty(\mathbb{R}^3)$. As in Lemma I.6.2, let us introduce the operator

$$\mathcal{V}_a(t) = \frac{1}{2}(pH(t)^{-1} + H(t)^{-1}p).$$

Using (I.6.20), it is enough to show the existence of $\lim_{t \rightarrow \pm\infty} U(0, t)g(\mathcal{V}_a(t))U(t, 0)\psi$. But we have

$$\frac{d}{dt}U(0, t)g(\mathcal{V}_a(t))U(t, 0)\psi = U(0, t)\left(\frac{d}{dt}g(\mathcal{V}_a(t)) + i[H(t), g(\mathcal{V}_a(t))]\right)U(t, 0)\psi.$$

By (I.6.27) and (I.6.28), this term is integrable in norm. Hence the result follows from Proposition I.4.1. \diamond

As in the case of time-independent Hamiltonians, we can give an alternative definition of \mathcal{V}^\pm in terms of the position observable. Exactly, we prove the following proposition

Proposition I.B.2 *The asymptotic velocity \mathcal{V}^\pm is characterized by*

$$\mathcal{V}^\pm = s - C_\infty - \lim_{t \rightarrow \pm\infty} U(0, t)\frac{x}{t}U(t, 0). \quad (\text{I.B.4})$$

The first step is to obtain propagation estimates for solutions $U(t, 0)\psi$. The next proposition summarizes the large and minimal velocity estimates in this case as obtained for time-independent hamiltonians.

Proposition I.B.3 *Suppose that $j, g \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } j \cap \text{supp } g = \emptyset$. Then*

$$\int_1^\infty \left\| j\left(\frac{x}{t}\right)g(\mathcal{V})U(t, 0)\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{I.B.5})$$

Furthermore, if $J \in C_0^\infty(\mathbb{R}^3)$ such that $J = 1$ on a neighbourhood of $\text{supp } g$, then

$$s - \lim_{t \rightarrow \pm\infty} U(0, t)J\left(\frac{x}{t}\right)g(\mathcal{V})U(t, 0) = g(\mathcal{V}^\pm). \quad (\text{I.B.6})$$

Proof : Assume first that $0 \notin \text{supp } g$. Since the intersection of the supports of j and g is empty, by a covering argument we may assume that there exists $v \in \mathbb{R}^3$, $v \neq 0$ such that

$$\text{supp } g \subset \{x : \langle v, x \rangle > \theta_2\}, \quad (\text{I.B.7})$$

$$\text{supp } j \subset \{x : \langle v, x \rangle < \theta_1\}, \quad (\text{I.B.8})$$

where $0 < \theta_1 < \theta_2$. Let us choose a function $\tilde{J} \in C^\infty(\mathbb{R})$ such that $\tilde{J}' \in C_0^\infty(\mathbb{R})$, $\tilde{J}'(x) = 0$ when $x \geq \theta_1$ and

$$\tilde{J}'(\langle v, x \rangle) \geq j^2(x), \quad (\text{I.B.9})$$

Now we define $J(x) := \tilde{J}(\langle v, x \rangle)$ and the propagation observable

$$\phi(t) = g(\mathcal{V}_a(t))J\left(\frac{x_{nw}}{t}\right)g(\mathcal{V}_a(t)),$$

where $\mathcal{V}_a(t) = \frac{1}{2}(pH(t)^{-1} + H(t)^{-1}p)$. We compute its Heisenberg derivative using Lemma I.5.2

$$\begin{aligned} \mathbf{D}\phi(t) &= \left(\frac{d}{dt}g(\mathcal{V}_a(t))\right)J\left(\frac{x_{nw}}{t}\right)g(\mathcal{V}_a(t)) + hc \\ &\quad + i[H(t), g(\mathcal{V}_a(t))]J\left(\frac{x_{nw}}{t}\right)g(\mathcal{V}_a(t)) + hc \\ &\quad + \frac{1}{t}g(\mathcal{V}_a(t))\left(\mathcal{V} - \frac{x_{nw}}{t}\right)\nabla J\left(\frac{x_{nw}}{t}\right)g(\mathcal{V}_a(t)) + O(t^{-2}). \end{aligned}$$

The first two terms are integrable in norm by (I.6.27) and (I.6.28). Moreover, we can replace the Newton-Wigner variable x_{nw} by x by Lemma I.5.1 and also $g(\mathcal{V}_a(t))$ by $g(\mathcal{V})$ thanks to (I.6.20). Now we claim that there exists a constant C_0 strictly positive such that

$$\frac{1}{t}g(\mathcal{V})\left(\mathcal{V} - \frac{x}{t}\right)\nabla J\left(\frac{x}{t}\right)g(\mathcal{V}) \geq \frac{C_0}{t}g(\mathcal{V})j^2\left(\frac{x}{t}\right)g(\mathcal{V}).$$

Indeed, observe that if we commute certain terms in this last expression, we obtain, using (I.B.7), (I.B.8), (I.B.9) and the fact that $[\mathcal{V}, f(\frac{x}{t})] \in O(t^{-1})$

$$\begin{aligned} \frac{1}{t}g(\mathcal{V})\left(\mathcal{V} - \frac{x}{t}\right)\nabla J\left(\frac{x}{t}\right)g(\mathcal{V}) &= \frac{1}{t}g(\mathcal{V})\langle v, \mathcal{V} - \frac{x}{t} \rangle \tilde{J}'\left(\langle v, \frac{x}{t} \rangle\right)g(\mathcal{V}), \\ &\geq \frac{1}{t}\left(\tilde{J}'\left(\langle v, \frac{x}{t} \rangle\right)\right)^{\frac{1}{2}}g(\mathcal{V})\langle v, \mathcal{V} \rangle g(\mathcal{V})\left(\tilde{J}'\left(\langle v, \frac{x}{t} \rangle\right)\right)^{\frac{1}{2}} \\ &\quad - \frac{1}{t}g(\mathcal{V})\left(\tilde{J}'\left(\langle v, \frac{x}{t} \rangle\right)\right)^{\frac{1}{2}}\langle v, \frac{x}{t} \rangle \left(\tilde{J}'\left(\langle v, \frac{x}{t} \rangle\right)\right)^{\frac{1}{2}}g(\mathcal{V}) + O(t^{-2}), \\ &\geq \frac{1}{t}(\theta_2 - \theta_1)g(\mathcal{V})j^2\left(\frac{x}{t}\right)g(\mathcal{V}) + O(t^{-2}). \end{aligned}$$

The constant C_0 is strictly positive since $\theta_2 > \theta_1$. Eventually we have

$$\mathbf{D}\phi(t) \geq \frac{C_0}{t}g(\mathcal{V})j^2\left(\frac{x}{t}\right)g(\mathcal{V}) + O(t^{-2}).$$

Thus we can conclude the proof of (I.B.5) using Proposition I.4.1.

If $0 \in \text{supp } g$ then one can find $v \in \mathbb{R}^3$ such that

$$\text{supp } j \subset \{x : \langle v, x \rangle > \theta_2\},$$

$$\text{supp } g \subset \{x : \langle v, x \rangle < \theta_1\},$$

where $0 < \theta_1 < \theta_2$. Now we make the same computations with $-\mathbf{D}\phi(t)$ to get the result.

To show (I.B.6), we consider a function $J \in C^\infty(\mathbb{R}^3)$ such that $\nabla J \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } \nabla J \cap \text{supp } g = \emptyset$. We claim that the following limit exists and is equal to 0

$$s - \lim_{t \rightarrow \infty} U(0, t) J\left(\frac{x}{t}\right) g(\mathcal{V}) U(t, 0), \quad (\text{I.B.10})$$

which proves (I.B.6). Let us denote $\phi(t) = J\left(\frac{x}{t}\right) g(\mathcal{V}_a(t))$. Then, using (I.6.27), (I.6.28) and (I.6.20), we obtain

$$|(\psi, \mathbf{D}\phi(t)\psi)| \leq \frac{C}{t} \|\tilde{j}\left(\frac{x}{t}\right) \tilde{g}(\mathcal{V})\psi\|^2 + O(t^{-2}),$$

where $\tilde{j}, \tilde{g} \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp } \tilde{j} \cap \text{supp } \tilde{g} = \emptyset$ and $\tilde{j}\nabla J = \nabla J$ and $\tilde{g}g = g$. Thus the following limit

$$s - \lim_{t \rightarrow \infty} U(0, t) \phi(t) U(t, 0), \quad (\text{I.B.11})$$

exists by Lemma I.4.1 and (I.B.5). This implies that the limit (I.B.10) exists by (I.6.20). Now, assume that $J \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } J \cap \text{supp } g = \emptyset$. Then we also know that

$$\int_1^\infty \left\| J\left(\frac{x}{t}\right) g(\mathcal{V}) U(t, 0) \psi \right\|^2 \frac{dt}{t} < \infty, \quad \forall \psi \in \mathcal{H}.$$

So in this case the limit (I.B.10) is zero. To conclude the proof, we need show that there is no propagation for large $\frac{x}{t}$. But this follows by the same limit procedure used in Proposition I.4.4. We omit the details.

◇

We prove now the ‘‘microlocal velocity estimate’’.

Proposition I.B.4 *Suppose that $g, J \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } g \cap \text{supp } \nabla J = \emptyset$. Then*

$$\int_1^\infty \left\| \left(\frac{x}{t} - \mathcal{V}\right) J\left(\frac{x}{t}\right) g(\mathcal{V}) U(t, 0) \psi \right\|^2 \frac{dt}{t} < \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{I.B.12})$$

Moreover,

$$s - \lim_{t \rightarrow \infty} \left(\frac{x}{t} - \mathcal{V}\right) J\left(\frac{x}{t}\right) g(\mathcal{V}) U(t, 0) = 0. \quad (\text{I.B.13})$$

Proof : Let us define the following propagation observables

$$\phi_0(t) = g(\mathcal{V}) J\left(\frac{x}{t}\right) \left(\frac{x}{t} - \mathcal{V}\right)^2 J\left(\frac{x}{t}\right) g(\mathcal{V}),$$

and

$$\phi_a(t) = g(\mathcal{V}_a(t)) J\left(\frac{x}{t}\right) \left\{ \frac{x_{nw}^2}{t^2} - 2\frac{A(t)}{t} + \mathcal{V}^2 \right\} J\left(\frac{x}{t}\right) g(\mathcal{V}_a(t)),$$

where $A(t) = \frac{1}{2} \left(x \cdot p H^{-1}(t) + H^{-1}(t) p \cdot x \right)$ which is well-defined for t large enough. By the same arguments as in Proposition I.4.5 and using (I.6.20), we have $\phi_a(t) = \phi_0(t) + O(t^{-1})$. The Heisenberg derivative of $\phi_a(t)$ equals

$$\mathbf{D}\phi_a(t) = \frac{2}{t} g(\mathcal{V}) J\left(\frac{x}{t}\right) \left(\frac{x}{t} - \mathcal{V}\right)^2 J\left(\frac{x}{t}\right) g(\mathcal{V}) + O(t^{-2}).$$

Thus (I.B.12) holds by Proposition I.4.1. To show (I.B.13), first observe that it is equivalent to prove that

$$s - \lim_{t \rightarrow \infty} U(0, t) \phi_0(t) U(t, 0) = 0. \quad (\text{I.B.14})$$

But, by Lemma I.4.1 and the previous computation of the Heisenberg derivative of $\phi_a(t)$, we know that the limit

$$s - \lim_{t \rightarrow \infty} U(0, t) \phi_a(t) U(t, 0),$$

exists. Hence, since $\phi_a(t) = \phi_0(t) + O(t^{-1})$, this proves the existence of the limit in (I.B.14). Moreover, $\phi_0(t) \geq 0$ and thus

$$0 \leq \int_1^\infty (\psi, U(0, t) \phi_0(t) U(t, 0) \psi) \frac{dt}{t} < \infty,$$

by (I.B.12). Therefore the limit (I.B.14) is zero.

◇

We finally give the proof of Proposition I.B.2. It is enough to show that for any $f, g, J \in C_0^\infty(\mathbb{R}^3)$ such that $J = 1$ on $\text{supp } g$,

$$s - \lim_{t \rightarrow \infty} U(0, t) \left(f\left(\frac{x}{t}\right) - f(\mathcal{V}) \right) J\left(\frac{x}{t}\right) g(\mathcal{V}) U(t, 0) = 0.$$

But we already saw in Proposition I.5.1 that

$$f\left(\frac{x}{t}\right) - f(\mathcal{V}) = B\left(\frac{x}{t} - \mathcal{V}\right) + O(t^{-1}),$$

where B is a bounded operator. Thus we conclude the proof using Proposition I.B.4.

◇

Chapitre II

Scattering theory for massless Dirac fields with long-range potentials

II.1 Introduction

Since the last twenty years, the scattering theory for *massive* Dirac Hamiltonians with long-range potentials has been precisely studied in [14], [29], [34], [59]. However, all the methods used in these papers fail when we want to extend them to the case of *massless* Dirac Hamiltonians. In this work, we use the same fully time-dependent approach, developed in [14], to establish a complete scattering theory for such Hamiltonians with scalar long-range potentials. This can be understood as the study of the asymptotic behaviour of *massless charged* Dirac fields. Let us point out that such situations also appear when we consider, for instance, the propagation of massive Dirac fields in certain curved spacetimes such as extreme Reissner-Nordström black holes. In this case, the particular geometry near the event horizon tends to make the effect of the mass disappear and leads (locally) to a situation of the above type.

The analytical difficulties involved by the absence of mass are twofold. First, massless Dirac Hamiltonians have no gap in their spectrum and thus, are not invertible. Consequently, the methods used in [14], [29] and [59] do not work directly. The second (related) difficulty is due to the singularity at 0 of the scalar symbol locally associated to massless Dirac Hamiltonians. This forces us to be particularly cautious when we consider what happens for the low energies. Precisely, we shall need to systematically cut-off low energies in the course of the computations. It also prevents us from using the same pseudodifferential techniques as in [34].

The time-dependent methods developed by Dereziński and Gérard in [21] for Schrödinger operators are well adapted to relativistic equations as showed in [14]. The foundation of these time-dependent methods is the analysis of the asymptotic behaviour of physically relevant observables such as position and velocity. They are based on weak propagation estimates. The basic tools to obtain such estimates are commutator methods and Mourre’s theory. We call them “weak” since these estimates only guarantee that the energy contained in certain cones of the spacetime tends to 0 when $t \rightarrow \infty$ but they provide no information on the decay rate. Nevertheless, they give a natural and visual interpretation of the scattering results we require and they suffice to construct Dollard-modified wave operators. Let us emphasize the role played by the *minimal velocity estimate* whose meaning is that the fields must escape at late times from any fixed compact region with a small but strictly positive velocity : a weak form of the Huygens principle. This will enable us to use standard Cook’s method in the construction of the wave operators.

Next, we use these weak propagation estimates to construct the asymptotic velocity operators P^\pm and study their spectra. There are several reasons which motivate these constructions. First, the spectra of P^\pm give the admissible values of the speed as well as the direction of propagation of fields : an information of physical significance. Besides, it allows us to classify physical states according to their asymptotic behaviours, which can be viewed as a weak version of scattering. For instance, the states with zero asymptotic velocity correspond to the *bound states* of our Dirac Hamiltonian whereas the states with non-zero asymptotic velocity correspond to the *scattering states* (i.e. the states whose energy does not remain trapped in a compact region of the spacetime). Such a classification allows to considerably simplify the structure of the proof of existence and asymptotic completeness of wave operators. Eventually, they can serve to choose, among many possibilities, a particular Dollard modification such that the associated Dollard-modified wave operators satisfy natural intertwining relations between the physical observables corresponding to local velocities $\pm\mathcal{V}$ and asymptotic velocities P^\pm .

Let us briefly describe the contents of this paper. We first present some properties of massless Dirac Hamiltonians with scalar long-range potential. In defining the observables of the theory such as position and velocity, it turns out that there is some freedom in the choice of these observables. This is due to the so-called Zitterbewegung phenomenon. A precise description of Zitterbewegung as well as the definitions of the “good” observables of the theory (the Newton-Wigner operator x_{nw} for the position

and the classical velocity operator \mathcal{V} for the velocity) are the objects of Section II.2. In Section II.3, we present the basics of Mourre's theory in an abstract setting. We introduce here a new locally conjugate operator which proves useful for obtaining the minimal velocity estimate in an optimal form. Section II.4 is devoted to the proof of the weak propagation estimates. In particular, we show how the minimal velocity estimate is related to the existence of a locally conjugate operator. In Section II.5, we construct the asymptotic velocity operators P^\pm and obtain a precise characterization of their spectra. Eventually, we apply all the previous results to the proof of existence and asymptotic completeness of Dollard-modified wave operators in Section II.6. In Appendix II.A, we recall basic results on the Helffer-Sjöstrand formula, useful to manipulate functions of selfadjoint operators. Appendix II.B contains two abstract propositions on the commutator methods used throughout this paper.

II.2 Dirac's equation

II.2.1 Analytic framework

Let us consider a massless Dirac operator $H = H_0 + V$ acting on the Hilbert space of physical states $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$. Here, $H_0 = \Gamma \cdot p$, with $p = -i\nabla$, denotes the free massless Dirac operator and $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ are the Dirac matrices. Together with the matrix Γ^0 defined below, the Γ 's satisfy the anticommutation relations $\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta_{ij}$ for any $i, j = 0, 1, 2, 3$. We shall use the following representation for the Dirac matrices. Set

$$\Gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \Gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix},$$

where the Pauli matrices σ^i are given by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The long-range potential V is assumed to be a scalar regular function with a certain decay at infinity. Let us fix $\rho > 0$. We assume that

$$V \in S^{-\rho}(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3) : |\partial^\alpha f(x)| \leq C \langle x \rangle^{-\rho-|\alpha|}, \forall \alpha \in \mathbb{N}^3\}.$$

Under this assumption, it follows by the Kato-Rellich Theorem that H is a selfadjoint operator on \mathcal{H} and its domain $D(H)$ is equal to $[H^1(\mathbb{R}^3)]^4$.

Let us denote e^{-itH} the unitary evolution associated to H . For any initial data $\psi_0 \in \mathcal{H}$, there exists a unique solution $\psi(t) \in C^0(\mathbb{R}_t; \mathcal{H})$ satisfying

$$i\partial_t \psi(t) = H\psi(t), \quad \psi(0) = \psi_0. \quad (\text{II.2.1})$$

This solution is given by $\psi(t) = e^{-itH} \psi_0$. Since e^{-itH} is a unitary operator on \mathcal{H} , the energy is conserved along the evolution i.e. $\|\psi(t)\| = \|\psi_0\|$, for all $t \in \mathbb{R}$.

In this paper, we are interested in studying the asymptotic behaviour of solutions of (II.2.1). It is well known that it critically depends on the spectral properties of the Hamiltonian H . A first important remark is that

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \mathbb{R}, \quad (\text{II.2.2})$$

by the standard Weyl Theorem. Note that, contrary to the massive case, the massless Dirac operator has no gap in its spectrum and is not invertible. Invertibility was one of the essential features in [14] to obtain a complete scattering theory. Despite this, we shall see that the methods used in [14] can be extended in a natural way to the massless case. The spectrum of H will be further explored in Section II.3 by means of Mourre's theory. We turn now to a brief overview of some important properties satisfied by massless Dirac Hamiltonians.

II.2.2 Domain invariance

Dirac Hamiltonians have the following interesting property : the domain $D(\langle x \rangle^n)$, $n \in \mathbb{N}$, of the selfadjoint operator $\langle x \rangle^n = (1 + x^2)^{\frac{n}{2}}$ on \mathcal{H} is invariant under the action of the unitary evolution $\{e^{-itH}\}_{t \in \mathbb{R}}$. Furthermore, this implies that the domain $D(\langle x \rangle^n)$, $n \in \mathbb{N}$, is also invariant under the action of the resolvent $(H - z)^{-1}$, $z \notin \sigma(H)$. Precisely, we have (see [75])

Theorem II.2.1 *Let H be the Dirac operator defined above. Then for any $n \in \mathbb{N}$,*

$$e^{-itH} D(\langle x \rangle^n) \subset D(\langle x \rangle^n),$$

and there exists a constant C_n such that

$$\|\langle x \rangle^n e^{-itH} \psi\| \leq C_n (1 + |t|)^n \|\langle x \rangle^n \psi\|.$$

Consequently, for any $z \in \mathbb{C} \setminus \sigma(H)$, we have

$$(H - z)^{-1} D(\langle x \rangle^n) \subset D(\langle x \rangle^n).$$

II.2.3 Velocity operator and Zitterbewegung

The velocity operator $V(t)$ is usually defined as the time derivative of the position operator denoted $X(t) = e^{itH} X e^{-itH}$. In Dirac's theory, the notions of position operator, and thus of velocity operator, are not uniquely defined and each choice for X leads to different difficulties of interpretation, see [75], Chapter 1. However, we can be guided by the principle of correspondence between classical and quantum mechanics to choose the more adequate observables. In the case of massive Dirac fields, this principle leads to defining the *classical velocity operator* \mathcal{V}_m as

$$\mathcal{V}_m = \frac{p}{H_0^m}, \quad H_0^m = \Gamma \cdot p + m\Gamma^0.$$

Although H_0 is not invertible, this expression has a limit (in the strong sense) when $m \rightarrow 0$. Indeed, we can write \mathcal{V}_m as

$$\mathcal{V}_m = H_0^m \frac{p}{\lambda^2(p)}, \quad \lambda(p) = \sqrt{p^2 + m^2}.$$

This last expression tends strongly to the classical velocity operator

$$\mathcal{V} = H_0 \frac{p}{p^2} = (\Gamma \cdot p) \frac{p}{p^2},$$

which is obviously a well defined bounded operator on \mathcal{H} . The observable \mathcal{V} satisfies the following properties : (1) \mathcal{V} commutes with H_0 , (2) $H_0 \mathcal{V} = \mathcal{V} H_0 = p$, (3) \mathcal{V} can be viewed as a vector of matrix-valued functions of the variable p and $\mathcal{V} = \mathcal{V}(p)$ belongs to the space $C_b^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$ of bounded

continuous functions on \mathbb{R}^3 . Note that $\mathcal{V}(p)$ do not belong to the space $S^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$ but only to the space $S^0(\mathbb{R}_p^3 \setminus \{0\}; (\mathcal{M}_4(\mathbb{C}))^3)$. When viewed as a function of the variable p , the operator \mathcal{V} is singular at 0. This is the origin of the problems encountered in this paper. In particular, we cannot use directly the Helffer-Sjöstrand formula or pseudodifferential calculus on the function $\mathcal{V}(p)$.

Let us now make the link between the classical velocity operator \mathcal{V} and the time derivative of a position observable. First consider the *standard position observable* $x(t) = e^{itH_0} x e^{-itH_0}$ for the free Dirac operator H_0 . This is a vector of commuting selfadjoint operators on \mathcal{H} . By Theorem II.2.1, its natural domain contains $D(x)$. Let us compute its time derivative. We obtain

$$\begin{aligned} \frac{dx(t)}{dt} &= e^{itH_0} i[H_0, x] e^{-itH_0}, \\ &= e^{itH_0} \Gamma e^{-itH_0} =: \Gamma(t). \end{aligned}$$

We call $\Gamma(t)$ the *standard velocity operator* associated to H_0 . If we compute its time derivative again, we obtain

$$\begin{aligned} \frac{d\Gamma(t)}{dt} &= e^{itH_0} i[H_0, \Gamma] e^{-itH_0}, \\ &= 2iH_0 e^{itH_0} (\Gamma - \mathcal{V}) e^{-itH_0}, \end{aligned} \tag{II.2.3}$$

where we used $p = H_0 \mathcal{V}$. The operator $\Gamma - \mathcal{V}$, which we denote F , anticommutes with H_0 i.e. $F H_0 = -H_0 F$. Hence, we have $F(t) := e^{itH_0} F e^{-itH_0} = e^{2itH_0} F$ and if we integrate (II.2.3) between 0 and t , we get

$$\Gamma(t) = \mathcal{V} + e^{2itH_0} F.$$

Thus, the standard velocity operator $\Gamma(t)$ oscillates without damping around the classical velocity operator \mathcal{V} and this oscillation is called the Zitterbewegung phenomenon. For the proof of the weak propagation estimates, we need to find a position observable $X(t)$ which satisfies

$$\frac{d}{dt} X(t) = \mathcal{V}. \tag{II.2.4}$$

It is clearly not the case for the standard position observable x because of the Zitterbewegung. We describe now another position observable called Newton-Wigner observable satisfying (II.2.4).

II.2.4 The Newton-Wigner observable

First, we diagonalize the free Dirac operator using the Foldy-Wouthuysen transformation. Denoting \mathcal{F} the Fourier transform on \mathcal{H} , we have

$$(\mathcal{F} H_0 \mathcal{F}^{-1})(k) = \Gamma \cdot k, \tag{II.2.5}$$

and for each $k \in \mathbb{R}^3$, the right-hand side of (II.2.5) is a Hermitian 4×4 matrix which has two eigenvalues $\{-|k|, |k|\}$ and both have multiplicity 2. Let us call $U(k)$ the unitary matrix such that

$$U(k)(\Gamma \cdot k)U(k)^{-1} = \Gamma^0 |k|.$$

It is given by

$$U(k) = \frac{1}{\sqrt{2}} \left(1 + \Gamma^0 \Gamma \cdot \frac{k}{|k|} \right), \quad U^*(k) = \frac{1}{\sqrt{2}} \left(1 - \Gamma^0 \Gamma \cdot \frac{k}{|k|} \right). \tag{II.2.6}$$

We define the Foldy-Wouthuysen transformation $U_{FW} = U(p) = \mathcal{F}^{-1}U(k)\mathcal{F}$ acting from \mathcal{H} to \mathcal{H} . This transformation is clearly unitary on \mathcal{H} and H_0 conjugated by U_{FW} can be written as

$$U_{FW}H_0U_{FW}^{-1} = \Gamma^0|p|.$$

Now, the Newton-Wigner observable x_{nw} is defined by

$$x_{nw} = U_{FW}^{-1}xU_{FW}.$$

Obviously, it is a vector of commuting selfadjoint operators on \mathcal{H} unitarily equivalent to the standard position observable. Its domain is given by $D(x_{nw}) = U_{FW}^{-1}D(x)$. Furthermore, it is straightforward to check that the Newton-Wigner observable satisfies (II.2.4) or equivalently

$$i[H_0, x_{nw}] = \mathcal{V}. \quad (\text{II.2.7})$$

We shall use the Newton-Wigner observable as an intermediate position operator in Section II.4 to prove the microlocal velocity estimate. Nevertheless, we prefer to state this estimate using the standard position observable since it is easier to interpret. Thus, we have to make the link between these two observables. Writing $x_{nw} = x + U_{FW}^{-1}[x, U_{FW}]$, we get

$$x_{nw} = x - \frac{\Gamma^0}{2i|p|} \left(\Gamma - \frac{1}{|p|^2}(\Gamma \cdot p)p \right) - \frac{1}{|p|^2}S \wedge p = x + Z,$$

where S denotes the spin angular momentum and is defined by $S = -\frac{i}{4}\Gamma \wedge \Gamma$. The symbol $\Gamma \wedge \Gamma$ denotes the three matrices $\sum_{k,l} \epsilon_{jkl} \Gamma^k \Gamma^l$ where ϵ is the totally antisymmetric tensor. The spin angular momentum S is bounded, everywhere defined and selfadjoint. Note that, the operator Z can also be viewed as a vector of matrix-valued functions of the variable p . Like \mathcal{V} , $Z = Z(p)$ belongs to the space $S^0(\mathbb{R}_p^3 \setminus \{0\}; (\mathcal{M}_4(\mathbb{C}))^3)$ and is singular at 0.

However, we saw in [14] that the manipulation of the operators \mathcal{V} , X_{nw} and Z as well as functions of these, is crucial for the proof of the propagation estimates. To avoid problems due to the singularity at 0, we need to cut-off the low energies. This is done as follows. Let a function $\eta \in C_b^\infty(\mathbb{R}^3)$ such that $\|\eta\|_\infty \leq 1$, $\eta(x) = 0$ for $|x| \leq \frac{\epsilon}{2}$ and $\eta(x) = 1$ for $|x| \geq \epsilon$. We define the cut-off operator $\eta_p = \eta(p)$ which is bounded on \mathcal{H} . We denote $\eta^c = 1 - \eta$, $\text{supp } \eta^c \subset B(0, \epsilon)$ and $\eta_p^c = \eta^c(p)$.

Now, we define the *modified* Newton-Wigner observable

$$X_{nw} = \eta_p x_{nw} \eta_p.$$

Lemma II.2.1 *The operator X_{nw} is essentially selfadjoint on $D(x)$.*

Proof : We can write $X_{nw} = x_p + Z_p$ where $x_p = \eta_p x \eta_p$ and $Z_p = \eta_p Z \eta_p$. First remark that the operator $Z_p = Z_\eta(p)$ belongs to the space $S^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$ since the values of p around 0 have been cut-off thanks to η_p . Therefore, Z_p is bounded on \mathcal{H} . Using the Kato-Rellich Theorem, we only have to prove that the operator x_p is essentially selfadjoint on $D(x)$. Let us check the assumptions of the Nelson Lemma (lemma II.3.2 below) with $N = x^2 + 1$. Clearly, N is a selfadjoint operator on \mathcal{H} with domain $D(N) = D(x^2)$ and $N \geq 1$. The operator x_p is symmetric and its domain contains $D(x)$ and thus $D(N)$. Indeed, for any $u \in D(x)$, we have

$$x_p u = \eta_p^2 x u + \eta_p [x, \eta_p] u. \quad (\text{II.2.8})$$

Using the Fourier transform, the commutator

$$[x, \eta_p] = i\nabla\eta(p) \in B(\mathcal{H}). \quad (\text{II.2.9})$$

is bounded on \mathcal{H} since $\eta \in S^0(\mathbb{R}^3)$. It follows that $D(x_p)$ contains $D(x)$. Assume now that $u \in D(N)$. We deduce from (II.2.8) that $\|x_p u\| \leq C\|Nu\|$. It remains to check that $|(u, [x_p, N]u)| \leq C(u, Nu)$. From (II.2.9), we get

$$\begin{aligned} [x_p, N] &= [\eta_p, x^2]x\eta_p + hc, \\ &= -i(\nabla\eta(p) \cdot x + x \cdot \nabla\eta(p))x\eta_p + hc. \end{aligned}$$

Therefore, using (II.2.9) again, we obtain $|(u, [x_p, N]u)| \leq C(\|xu\|^2 + \|u\|^2) \leq C(u, Nu)$ which concludes the proof of the lemma.

◇

We also introduce the *modified* classical velocity operator

$$\mathcal{V}_p = \eta_p \mathcal{V} \eta_p.$$

Viewed as a vector of matrix-valued functions of p , this operator belongs to $S^0(R_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$. We denote $\mathcal{V}_p = \mathcal{V}_\eta(p)$. Finally, from (II.2.7), we have the following relation between X_{nw} and \mathcal{V}_p .

$$i[H_0, X_{nw}] = \mathcal{V}_p. \quad (\text{II.2.10})$$

II.3 Mourre theory

II.3.1 Abstract theory

Mourre's theory [58] is a powerful tool for studying the spectrum of selfadjoint operators. Especially, it provides criteria for a selfadjoint operator H on \mathcal{H} to have empty singular continuous spectrum : a fundamental prerequisite in scattering theory.

The main idea is to introduce another selfadjoint operator A , called locally conjugate operator for H , which increases along the evolution. This means that the Heisenberg derivative of the operator A must be essentially positive. Precisely, we look for an operator A such that there exist a strictly positive constant ϵ and a compact operator K which satisfy

$$\mathbf{1}_\Delta(H) i[H, A] \mathbf{1}_\Delta(H) \geq \epsilon \mathbf{1}_\Delta^2(H) + \mathbf{1}_\Delta(H) K \mathbf{1}_\Delta(H), \quad (\text{II.3.1})$$

on an open interval Δ of \mathbb{R} . In this case, we say that (H, A) satisfy a Mourre estimate on Δ . Although this last assumption is the key ingredient of Mourre's theory, we need some extra technical hypotheses to define properly what is a locally conjugate operator for a selfadjoint operator H . We do not state here the original version proposed by Mourre but a refined version inspired by Amrein, Boutet de Monvel-Berthier, Georgescu [1]. We first introduce a notion of regularity between two selfadjoint operators H, A given in [1].

Definition II.3.1 *For a selfadjoint operator A , we say that another selfadjoint operator H belongs to $C^k(A)$, $k \in \mathbb{N}$, if and only if $\exists z \in \mathbb{C} \setminus \sigma(H), s \rightarrow e^{isA}(H - z)^{-1}e^{-isA} \in C^k(\mathbb{R}_s; \mathcal{B}(\mathcal{H}))$, for the strong topology of $\mathcal{B}(\mathcal{H})$.*

Then one has the following definition

Definition II.3.2 *Let H, A two self-adjoint operators on \mathcal{H} . Let $\Delta \subset \mathbb{R}$ an open interval. We shall say that A is a locally conjugate operator for H on Δ if it satisfies the following assumptions*

- (i) $H \in C^1(A)$.
- (ii) $i[H, A]$ defined as a quadratic form on $D(H) \cap D(A)$ extends to an element of $\mathcal{B}(D(H), \mathcal{H})$.
- (iii) $[A, [A, H]]$ well defined as a quadratic form on $D(H) \cap D(A)$ by (ii) extends to an element of $\mathcal{B}(D(H), D(H)^*)$.
- (iv) There exists a strictly positive constant ϵ and a compact operator K such that the Mourre estimate (II.3.1) holds.

Let us make a few remarks about the conditions stated above. Firstly, there exist weaker versions depending on the properties of the Hamiltonian we consider. For instance, the assumptions (i) and (ii) could be replaced by (see [1])

- (i)' $e^{isA}D(H) \subset D(H)$.
- (ii)' $i[H, A]$ extends to an element of $\mathcal{B}(D(H), D(H)^*)$.

Actually, the conditions (i) and (ii) we propose in the definition imply the conditions (i)' and (ii)'. Indeed we have the following result due to Gérard, Georgescu [36]

Lemma II.3.1 *Let H, A two self-adjoint operators such that $H \in C^1(A)$ and $i[H, A] \in \mathcal{B}(D(H), \mathcal{H})$ then $e^{isA}D(H) \subset D(H)$ for all $s \in \mathbb{R}$.*

The assumption $H \in C^1(A)$ has the advantage of being easier to prove than $e^{isA}D(H) \subset D(H)$, thanks to general criteria, see below Nelson's Lemma and an extended version due to Gérard, Laba [38]. Moreover, assumptions (i) to (iii) together with Lemma II.3.1 imply that $H \in C^2(A)$, see Theorem 6.3.1, [1]. This point will be particularly useful to obtain the minimal velocity estimates in Section 4. At last, we mention that, in the case of Hamiltonians having a spectral gap, the condition (iii) on the double commutator could be slightly weakened, see Theorem 7.3.5, [1]. However, contrary to the massive case, massless Dirac Hamiltonians have no gap in their spectrum and this refinement is not available.

We now state the main result of this section, due to Mourre

Theorem II.3.1 *Let H, A two selfadjoint operators on \mathcal{H} . Assume that A is a locally conjugate operator for H on an interval Δ . Then H has no singular continuous spectrum in Δ . Moreover, the number of eigenvalues of H in Δ is finite (counting multiplicity).*

A subtle aspect of Mourre's theory is the manipulation of commutators of unbounded selfadjoint operators. Such commutators $[H, A]$ are naturally defined as quadratic forms on the intersection of the domains $D(H) \cap D(A)$ and one needs to be cautious when extending them. We have the following equivalent properties for the class of operators $C^1(A)$, [1].

- (1) $\exists z \in \mathbb{C} \setminus \sigma(H)$, $|((H - z)^{-1}u, Au) - (Au, (H - z)^{-1}u)| \leq C\|u\|^2$, $\forall u \in D(A) \cap D(H)$.
- (2) (i) $\exists z \in \mathbb{C} \setminus \sigma(H)$, $(H - z)^{-1}D(A) \subset D(A)$, $(H - \bar{z})^{-1}D(A) \subset D(A)$,
(ii) $|(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2)$, $\forall u \in D(H) \cap D(A)$.

In general, the domain $D(A)$ is not explicitly known and thus it is not easy to check the condition $(H - z)^{-1}D(A) \subset D(A)$. To overcome this difficulty, it is useful to consider another operator N called *comparison operator* whose domain is well-known and that allows us to make the link between H and A . Precisely, we shall use the following lemmata [38]

Lemma II.3.2 (Nelson) *Let $N \geq 1$ a self-adjoint operator on \mathcal{H} . Let A a symmetric operator on \mathcal{H} such that $D(N) \subset D(A)$. Assume that*

$$\begin{aligned} (i) \|Au\| &\leq C\|Nu\|, \quad \forall u \in D(N), \\ (ii) |(Au, Nu) - (Nu, Au)| &\leq C\|N^{\frac{1}{2}}u\|^2, \quad \forall u \in D(N). \end{aligned} \quad (\text{II.3.2})$$

Then A is essentially self-adjoint on $D(N)$. Furthermore every core for N is also a core for A .

Lemma II.3.3 (Gérard, Laba) *Let H, H_0 and N three self-adjoint operators on \mathcal{H} satisfying $N \geq 1$, $D(H) = D(H_0)$ and $(H - z)^{-1}D(N) \subset D(N)$. Let A a symmetric operator on $D(N)$. Assume that H_0 and A satisfy the assumptions of Lemma II.3.2 and*

$$|(Au, Hu) - (Hu, Au)| \leq C(\|Hu\|^2 + \|u\|^2), \quad \forall u \in D(N). \quad (\text{II.3.3})$$

Then we have

- $D(N)$ is dense in $D(A) \cap D(H)$ with the norm $\|Hu\| + \|Au\| + \|u\|$,
- the quadratic form $i[H, A]$ defined on $D(A) \cap D(H)$ is the unique extension of $i[H, A]$ on $D(N)$,
- $H \in C^1(A)$.

II.3.2 Locally conjugate operator

Not only the existence of a locally conjugate operator for H provides an important information on the spectrum $\sigma(H)$ but it also enables to obtain the minimal velocity estimate in a simple way. As we shall see in Section II.4, it is preferable to use a locally conjugate operator whose domain contains $D(x)$ to derive this estimate more directly. Moreover, the constant ϵ in the Mourre estimate (II.3.1) is physically relevant since it gives the upper bound of the minimal velocity with which the fields escape to infinity (see Proposition II.4.1 for the exact statement of this assertion). Intuitively, massless fields should propagate with a speed equal to the light velocity (equal to 1 in our convention). Therefore, we expect to obtain the value 1 for the constant ϵ in (II.3.1).

As suggested by the massive case [14], a good candidate for a locally conjugate operator seems to be $\frac{1}{2}(x \cdot \mathcal{V} + \mathcal{V} \cdot x)$. However, it is not easy to manipulate commutators involving this operator since \mathcal{V} is singular at 0 when viewed as a function of p . Therefore, we prefer to consider the operator

$$A = \frac{1}{2}(x \cdot \mathcal{V}_p + \mathcal{V}_p \cdot x). \quad (\text{II.3.4})$$

Lemma II.3.4 *The operator A is essentially selfadjoint on $D(x)$.*

Proof : We apply Nelson's Lemma II.3.2 with $N = \langle x \rangle$. Let $u \in D(x)$. We can write $Au = \mathcal{V}_p \cdot xu + \frac{1}{2}[x, \mathcal{V}_p]u$ and we have

$$[x, \mathcal{V}_p] = i\nabla \cdot \mathcal{V}_\eta(p) \in \mathcal{B}(\mathcal{H}), \quad (\text{II.3.5})$$

since $\mathcal{V}_\eta \in S^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$. Hence, $D(x)$ is contained in $D(A)$ and $\|Au\| \leq C\|Nu\|$. It remains to check that $|(u, [A, N]u)| \leq C(u, Nu)$. We have $[A, N] = \frac{1}{2}(x \cdot [\mathcal{V}_p, \langle x \rangle] + hc)$. But using the Helffer-Sjöstrand formula, we have

$$[\mathcal{V}_p, \langle x \rangle] = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{\mathcal{V}}_\eta(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) \left(-i \frac{x_l}{\langle x \rangle} \right) \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z},$$

where $R_k = (z - p_k)^{-1}$. Therefore, the commutator $[\mathcal{V}_p, \langle x \rangle]$ is bounded by (II.A.2) and we have $|(u, [A, N]u)| \leq C\|xu\| \|u\| \leq C(u, Nu)$.

◇

Lemma II.3.5 *The massless Dirac operator H belongs to the class $C^1(A)$. Furthermore, the commutators $[H, A]$ and $[[H, A], A]$ are bounded on \mathcal{H} . In consequence, $H \in C^2(A)$.*

Proof : We define the comparison operator $N = p^2 + x^2 + 1$. It is well known that the domain of N is equal to $D(N) = H^2(\mathbb{R}^3) \cap D(x^2)$. Thus $D(N) \subset D(x) \subset D(A)$. We know that $\|Au\| \leq C(\|xu\| + \|u\|) \leq C\|Nu\|$. It remains to see that $|(u, [A, N]u)| \leq C\|N^{\frac{1}{2}}u\|$. We have

$$\begin{aligned} [A, N] &= \frac{1}{2} \left([x, p^2] \cdot \mathcal{V}_p + \mathcal{V}_p \cdot [x, p^2] \right) + \frac{1}{2} \left(x \cdot [\mathcal{V}_p, x^2] + [\mathcal{V}_p, x^2] \cdot x \right), \\ &= ip \cdot \mathcal{V}_p - ix \cdot \nabla \mathcal{V}_\eta(p) \cdot x, \end{aligned}$$

where $x \cdot \nabla \mathcal{V}_\eta(p) \cdot x = \sum_{i,j} x_i (\partial_j \mathcal{V}_\eta^i)(p) x_j$. Therefore, we get

$$|(u, [A, N]u)| \leq C(\|pu\| \|u\| + \|xu\|^2) \leq C(u, Nu),$$

and (A, N) satisfies the assumptions of Nelson's Lemma II.3.2.

The fact that (H_0, N) also satisfies Lemma II.3.2 is immediate. Moreover, $(H - z)^{-1}D(N) \subset D(N)$ by Theorem II.2.1. To apply Lemma II.3.3, it remains to check whether $|(u, [H, A]u)| \leq C(\|Hu\| + \|u\|)$, for any $u \in D(N)$. We have

$$i[H, A] = \frac{1}{2}(\Gamma \cdot \mathcal{V}_p + \mathcal{V}_p \cdot \Gamma) + \frac{i}{2}(x \cdot [V, \mathcal{V}_p] + [V, \mathcal{V}_p] \cdot x).$$

The first term is clearly bounded on \mathcal{H} . Using the equality $x \cdot [V(x), \mathcal{V}_p] = [xV(x), \mathcal{V}_p] - i \nabla \cdot \mathcal{V}_\eta(p) V(x)$ and (II.3.5), we rewrite the second term as

$$i[xV(x), \mathcal{V}_p] + \frac{1}{2}(\nabla \cdot \mathcal{V}_\eta(p) V(x) + V(x) \nabla \cdot \mathcal{V}_\eta(p)).$$

Since $\nabla \cdot \mathcal{V}_\eta(p) V(x)$ is bounded on \mathcal{H} , we only have to prove that $[V(x)x, \mathcal{V}_p]$ is bounded. By the Helffer-Sjöstrand formula, we get

$$[xV(x), \mathcal{V}_p] = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{\mathcal{V}}_\eta^j(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) [x^j V(x), p_l] \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z}, \quad (\text{II.3.6})$$

Since $[x^j V(x), p_l] = i \partial_l (x^j V(x)) \in S^{-\rho}(\mathbb{R}^3)$, this term is bounded. Therefore, we finally have

$$i[H, A] = \frac{1}{2}(\Gamma \cdot \mathcal{V}_p + \mathcal{V}_p \cdot \Gamma) + i[xV(x), \mathcal{V}_p] + \frac{1}{2}(\nabla \cdot \mathcal{V}_\eta(p) V(x) + V(x) \nabla \cdot \mathcal{V}_\eta(p)) \in \mathcal{B}(\mathcal{H}), \quad (\text{II.3.7})$$

and $H \in C^1(A)$.

It remains to check the assertion concerning the double commutator. A long but straightforward computation using several times the Helffer-Sjöstrand formula, the condition $V \in S^{-\rho}(\mathbb{R}^3)$ with $\rho > 0$ and (II.2.9), (II.3.5) shows that

$$[i[H, A], A] \in \mathcal{B}(\mathcal{H}).$$

Eventually, $H \in C^2(A)$ by Lemma II.3.1 and the discussion below it. This concludes the proof of the Lemma.

◇

Let us now check the Mourre estimate (II.3.1). Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. First observe that

$$\frac{1}{2}(\Gamma \cdot \mathcal{V}_p + \mathcal{V}_p \cdot \Gamma) = \eta_p \mathcal{V}^2 \eta_p + \frac{1}{2} \eta_p (F \cdot \mathcal{V} + \mathcal{V} \cdot F) \eta_p,$$

where $F = \Gamma - \mathcal{V}$. Since $\mathcal{V}^2 = 1$ and F anticommutes with \mathcal{V} , we have $\frac{1}{2}(\Gamma \cdot \mathcal{V}_p + \mathcal{V}_p \cdot \Gamma) = \eta_p^2$ and we can rewrite (II.3.7) as

$$i[H, A] = \eta_p^2 + i[xV(x), \mathcal{V}_p] + \frac{1}{2}(\nabla \cdot \mathcal{V}_\eta(p)V(x) + V(x)\nabla \cdot \mathcal{V}_\eta(p)). \quad (\text{II.3.8})$$

Since $V \in S^{-\rho}(\mathbb{R}^3)$, the term $\frac{1}{2}(\nabla \cdot \mathcal{V}_\eta(p)V(x) + V(x)\nabla \cdot \mathcal{V}_\eta(p))$ is compact by the standard compactness criterion. Similarly, using (II.3.6) and the fact that $[x^j V(x), p_l] = i\partial_l(x^j V(x)) \in S^{-\rho}(\mathbb{R}^3)$, the term $[xV(x), \mathcal{V}_p]$ is also compact. Eventually we obtain

$$\chi(H)i[H, A]\chi(H) = \chi(H)\eta_p^2\chi(H) + \chi(H)K\chi(H), \quad (\text{II.3.9})$$

with K a compact operator on \mathcal{H} . Now, we can choose the function η such that $\chi(|x|)\eta^c(x) = 0$. In this case, we have $\chi(H_0)\eta_p^c = 0$. Since $\chi(H) - \chi(H_0)$ is a compact operator on \mathcal{H} , we have

$$\chi(H)\eta_p^2\chi(H) = \chi(H_0)\eta_p^2\chi(H_0) + K,$$

where K is compact. We write $\eta_p^2 = 1 - (\eta^2)_p^c$ and we finally obtain

$$\chi(H)\eta_p^2\chi(H) = \chi(H_0)(1 - (\eta^2)_p^c)\chi(H_0) + K = \chi^2(H_0) + K = \chi^2(H) + K,$$

where K denotes different compact operators on \mathcal{H} . Finally, for any $\chi \in C_0^\infty(\mathbb{R})$ equal to 0 on a neighbourhood of 0, the Mourre estimate (II.3.1) holds with the constant ϵ equal to 1.

As a consequence of Theorem II.3.1 and $\sigma_{ess}(H) = \mathbb{R}$, we have proved

Theorem II.3.2 *The singular continuous spectrum $\sigma_{sing}(H)$ of the Dirac operator H is empty. Moreover, H can only have an eigenvalue of infinite multiplicity at 0.*

II.4 Weak propagation estimates

In what follows, we shall use constantly the results given in appendix II.B. We define by $\mathbf{D} = \frac{d}{dt} + i[H, \cdot]$ the Heisenberg derivative which acts on time-dependent selfadjoint observables $B(t)$. The Heisenberg derivative satisfies

$$\frac{d}{dt} \left[e^{itH} B(t) e^{-itH} \right] = e^{itH} \mathbf{D} B(t) e^{-itH}.$$

II.4.1 Minimal velocity estimate

The minimal velocity estimate takes the following form

Proposition II.4.1 *For any constant $\delta \in]0, 1[$ and for any $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[0, 1-\delta]} \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C_\delta \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{II.4.1})$$

Furthermore

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[0, \delta]} \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{II.4.2})$$

In order to prove Proposition II.4.1, we first recall an abstract proposition, given in [39], which shows how the minimal velocity estimate is closely related to the existence of a locally conjugate operator A for H in the sense of Mourre theory.

Proposition II.4.2 (Gérard, Nier) *Let H, A two selfadjoint operators on \mathcal{H} . Assume that for $\epsilon > 0$, $H \in C^{1+\epsilon}(A)$, and assume that there exists $c_0 > 0$ such that the Mourre estimate*

$$\mathbf{1}_\Delta(H) i[H, A] \mathbf{1}_\Delta(H) \geq c_0 \mathbf{1}_\Delta^2(H), \quad (\text{II.4.3})$$

holds on an open interval Δ . Then

(i) $\forall g \in C^\infty(\mathbb{R}), \forall \chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } g \subset (-\infty, c_0)$, $g = 1$ on $(-\infty, 0)$ and $\text{supp } \chi \subset \Delta$,

$$\int_1^\infty \left\| g \left(\frac{A}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{II.4.4})$$

and

$$s - \lim_{t \rightarrow \infty} g \left(\frac{A}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{II.4.5})$$

(ii) Furthermore, assume that there exists another selfadjoint operator $B \geq 1$ which satisfies

$$D(B) \subset D(A), \quad \pm A \leq B, \quad [A, B] B^{-1} \in \mathcal{B}(\mathcal{H}),$$

then, $\forall \chi \in C_0^\infty(\mathbb{R})$, $\text{supp } \chi \subset \Delta$ and for any $\delta \in]0, c_0[$, we have

$$\int_1^\infty \left\| \mathbf{1}_{[0, c_0-\delta]} \left(\frac{B}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C_\delta \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{II.4.6})$$

and

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[0, c_0-\delta]} \left(\frac{B}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{II.4.7})$$

The first part of the proposition clearly indicates that the minimal velocity estimate holds in the spectral representation of the locally conjugate operator A . We would like to obtain a more physical interpretation of the estimate by replacing A by the position observable x . That is what the second part of the Proposition allows us to do.

Proof (of Proposition II.4.1) : Let us apply Proposition II.4.2. We already know from Lemma II.3.5 that $H \in C^2(A)$. Let Δ an open interval included in $\mathbb{R} \setminus \{0\}$ and $\delta \in]0, 1[$. Then there exists a compact operator K such that the Mourre estimate

$$\mathbf{1}_\Delta(H)i[H, A]\mathbf{1}_\Delta(H) \geq \mathbf{1}_\Delta(H) + \mathbf{1}_\Delta(H)K\mathbf{1}_\Delta(H),$$

holds. Since the only possible eigenvalue for H is 0 and since Δ avoids this value, $s - \lim \mathbf{1}_\Delta(H) = 0$ when $|\Delta|$ tends to 0. Moreover, since K is compact, $K\mathbf{1}_\Delta(H)$ tends to 0 in operator norm when $|\Delta|$ tends to 0. Therefore, for $|\Delta|$ small enough, we have

$$\mathbf{1}_\Delta(H)i[H, A]\mathbf{1}_\Delta(H) \geq \left(1 - \frac{\delta}{2}\right)\mathbf{1}_\Delta(H).$$

Hence, if we choose $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \subset \Delta$, (II.4.4) holds with $c_0 = 1 - \frac{\delta}{2}$. Now, we apply the second part of Proposition II.4.2 with the operator $B = \langle x \rangle_\mu = (x^2 + \mu)^{\frac{1}{2}}$ and $\mu > 0$. We must check that

- $D(B) \subset D(A)$. This follows from $D(B) = D(x)$ and the fact that $D(x) \subset D(A)$ (see Section II.3).
- $\pm A \leq B$. Let $u \in D(B)$. We have

$$\begin{aligned} |(u, Au)| &= \left| \frac{1}{2}(u, (x\mathcal{V}_p + \mathcal{V}_p \cdot x)u) \right|, \\ &\leq |(\langle x \rangle^{\frac{1}{2}}u, \langle x \rangle^{-\frac{1}{2}}x\mathcal{V}_p u)|, \\ &\leq |(\langle x \rangle^{\frac{1}{2}}u, \langle x \rangle^{-1}x\mathcal{V}_p \langle x \rangle^{\frac{1}{2}}u)| + |(u, \langle x \rangle^{-\frac{1}{2}}x[\langle x \rangle^{\frac{1}{2}}, \mathcal{V}_p]u)|. \end{aligned} \quad (\text{II.4.8})$$

Since $\|\frac{x}{\langle x \rangle}\| \leq 1$ and $\|\mathcal{V}_p\| \leq 1$, by Cauchy-Schwartz, the first term in (II.4.8) is bounded by $\|\langle x \rangle^{\frac{1}{2}}u\|^2 = (u, \langle x \rangle u)$. To estimate the commutator in the second term, we use the Helffer-Sjöstrand formula. We obtain

$$\frac{x}{\langle x \rangle^{\frac{1}{2}}} \cdot [\langle x \rangle^{\frac{1}{2}}, \mathcal{V}_p] = \left(\frac{i}{2\pi}\right)^3 \sum_j \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{\mathcal{V}}_\eta^j(z) \frac{x_j}{\langle x \rangle^{\frac{1}{2}}} \left(\sum_{l=1}^3 \left(\prod_{k=1}^l R_k \right) \left(\frac{i}{2} \frac{x_l}{\langle x \rangle^{\frac{3}{2}}} \right) \left(\prod_{k'=l}^3 R_{k'} \right) \right) dz \wedge d\bar{z},$$

where $R_k = (z_k - p_k)^{-1}$. We commute $\langle x \rangle^{-\frac{1}{2}}$ with the resolvents on its left and noting that $[R_k, \langle x \rangle^{-\frac{1}{2}}]$ is bounded, we obtain

$$\frac{x}{\langle x \rangle^{\frac{1}{2}}} \cdot [\langle x \rangle^{\frac{1}{2}}, \mathcal{V}_p] = \left(\frac{i}{2\pi}\right)^3 \sum_j \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{\mathcal{V}}_\eta^j(z) \frac{x_j}{\langle x \rangle} \left(\sum_{l=1}^3 \left(\prod_{k=1}^l R_k \right) \left(\frac{i}{2} \frac{x_l}{\langle x \rangle} \right) \left(\prod_{k'=l}^3 R_{k'} \right) \right) dz \wedge d\bar{z} + B_1,$$

where B_1 is bounded on \mathcal{H} by (II.A.2). Since $\frac{x_j}{\langle x \rangle}$ is bounded for any $l = 1, 2, 3$, this last integral converges in operator norm and is bounded on \mathcal{H} by (II.A.2). Eventually, we have

$$\begin{aligned} |(u, Au)| &\leq (u, \langle x \rangle u) + C(u, u), \\ &\leq (u, \langle x \rangle_\mu u), \end{aligned}$$

for $\mu \geq C + 1$.

– $[A, \langle x \rangle_\mu] \langle x \rangle_\mu^{-1}$ bounded on \mathcal{H} . Indeed we have

$$[A, \langle x \rangle_\mu] \langle x \rangle_\mu^{-1} = \frac{1}{2} \left(x. [\mathcal{V}_p, \langle x \rangle_\mu] + [\mathcal{V}_p, \langle x \rangle_\mu]. x \right) \langle x \rangle_\mu^{-1}. \quad (\text{II.4.9})$$

But using the Helffer-Sjöstrand formula, we get

$$x. [\mathcal{V}_p, \langle x \rangle_\mu] = \left(\frac{i}{2\pi} \right)^3 \sum_j \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{\mathcal{V}}_\eta^j(z) x_j \left(\sum_{l=1}^3 \left(\prod_{k=1}^l R_k \right) \left(-i \frac{x_l}{\langle x \rangle_\mu} \right) \left(\prod_{k'=l}^3 R_{k'} \right) \right) dz \wedge d\bar{z}.$$

Then if we commute x_j with the product of resolvents on its right and since $[x_j, R_k]$ is bounded for any $k = 1, 2, 3$, we obtain $x. [\mathcal{V}_p, \langle x \rangle_\mu] = B_1.x + B_2$ where B_1 and B_2 are bounded thanks to estimate (II.A.2). In particular, (II.4.9) is bounded on \mathcal{H} .

Hence, we have established (II.4.6) with the operator $B = \langle x \rangle_\mu$, $\text{supp } \chi \subset \Delta$ and any $\delta \in]0, 1[$. Now, observe that $\mathbf{1}_{[0, \delta]} \left(\frac{\langle x \rangle_\mu}{t} \right) - \mathbf{1}_{[0, \delta]} \left(\frac{|x|}{t} \right) \in L^1(dt)$. Thus, (II.4.1) and (II.4.2) hold for any χ with sufficiently small support. It remains to treat the general case. Note that by local compactness, we can write any $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 0$ on a neighbourhood of 0 as a finite sum

$$\chi = \sum_{j=1}^N \chi_j,$$

where the χ_j 's have sufficiently small support and satisfy (II.4.1). By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \int_1^\infty \left(\mathbf{1}_{[0, 1-\delta]} \left(\frac{|x|}{t} \right) \chi_i(H) e^{-itH} \psi, \mathbf{1}_{[0, 1-\delta]} \left(\frac{|x|}{t} \right) \chi_j(H) e^{-itH} \psi \right) \frac{dt}{t} \\ & \leq \left(\int_1^\infty \left\| \mathbf{1}_{[0, 1-\delta]} \left(\frac{|x|}{t} \right) \chi_i(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_1^\infty \left\| \mathbf{1}_{[0, 1-\delta]} \left(\frac{|x|}{t} \right) \chi_j(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, we obtain

$$\int_1^\infty \left\| \mathbf{1}_{[0, 1-\delta]} \left(\frac{|x|}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq CN^2 \|\psi\|^2,$$

which concludes the proof of the proposition.

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II.4.2 Maximal velocity estimate

The maximal velocity estimate states in a weak sense that massless Dirac fields do not propagate faster than the light velocity (equal to 1 in our convention). More precisely, we prove in the following proposition that the energy contained in $\mathcal{C} = \{|x| \geq t\}$ tends to 0 when $t \rightarrow \infty$.

Proposition II.4.3 *For any $1 < \theta_1 < \theta_2$, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \psi \in \mathcal{H}. \quad (\text{II.4.10})$$

Given any $g \in C^\infty(\mathbb{R})$ with $g' \in C_0^\infty(\mathbb{R})$ and $\text{supp } g \in (1, +\infty)$, we have

$$s - \lim_{t \rightarrow +\infty} g \left(\frac{|x|}{t} \right) e^{-itH} = 0. \quad (\text{II.4.11})$$

Proof : The proof is identical to the proof given in the massive case [14]. We omit it.

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II.4.3 Microlocal velocity estimate

Roughly speaking, the microlocal velocity estimate asserts that, in the region of the spacetime where the energy does not decrease with time, we can approach the position observable x by the operator $t\mathcal{V}$ in a weak sense. Although the strategy used to prove the microlocal velocity estimate remains identical to the one given in [14], the low energy case requires caution since we want to use the Newton-Wigner observable $x_{nw} = x + Z$ where Z is an unbounded operator of domain $D(|p|^{-1})$. The idea is to use the operators X_{nw} instead of x_{nw} . We have

Proposition II.4.4 *For any $0 < \theta_1 < 1 < \theta_2$, for any $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) \left(\mathcal{V} - \frac{x}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{II.4.12})$$

Furthermore

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|x|}{t} \right) \left(\mathcal{V} - \frac{x}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{II.4.13})$$

In the proof of Proposition II.4.4, we need several technical lemmata. The first one permits to make the link between the operators x and X_{nw} . More precisely, it says that a cut-off function $f(\frac{x}{t})$ is almost the same as a cut-off function $f(\frac{X_{nw}}{t})$ up to a term that is small for large time.

Lemma II.4.1 *Let $f \in C_0^\infty(\mathbb{R}^3)$ such that $f = 0$ on a neighbourhood of 0. Let $g \in S^{-\mu}(\mathbb{R}^3)$ with $0 \leq \mu \leq 2$. Then*

$$g(x) f\left(\frac{X_{nw}}{t}\right) \in O(t^{-\mu}). \quad (\text{II.4.14})$$

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. If we choose the function η in the definition of X_{nw} such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$, we have

$$\chi(H) \left(f\left(\frac{X_{nw}}{t}\right) - f\left(\frac{x}{t}\right) \right) \in O(t^{-\min(1, \rho)}), \quad (\text{II.4.15})$$

where ρ is the decay rate of the potential V (see subsection II.2.1).

Proof : Assume first that $0 \leq \mu \leq 1$. We rewrite (II.4.14) as follows $g(x) f\left(\frac{X_{nw}}{t}\right) = g(x) \left(f\left(\frac{X_{nw}}{t}\right) - f\left(\frac{x}{t}\right) \right) + g(x) f\left(\frac{x}{t}\right)$. Clearly, it suffices to prove that

$$g(x) \left(f\left(\frac{X_{nw}}{t}\right) - f\left(\frac{x}{t}\right) \right) \in O(t^{-\mu}). \quad (\text{II.4.16})$$

Using the Helffer-Sjöstrand formula, we have

$$g(x) \left(f\left(\frac{X_{nw}}{t}\right) - f\left(\frac{x}{t}\right) \right) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^x \right) g(x) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) \left(\prod_{k_2=l}^3 R_{k_2}^{X_{nw}} \right) \right) dz \wedge d\bar{z},$$

where $R_{k_1}^x = (z_{k_1} - \frac{x^{k_1}}{t})^{-1}$ and $R_{k_2}^{X_{nw}} = (z_{k_2} - \frac{X_{nw}^{k_2}}{t})^{-1}$. Now, remark that $X_{nw} - x = x(\eta_p^2 - 1) + [\eta_p, x]\eta_p$. Using (II.2.9), we can write more concisely, $X_{nw} - x = xB_1 + B_2$ where B_1, B_2 are bounded. Therefore, we have

$$g(x)(f(\frac{X_{nw}}{t}) - f(\frac{x}{t})) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^x \right) g(x) \left(\frac{x^l}{t} B_1 - \frac{B_2}{t} \right) \left(\prod_{k_2=l}^3 R_{k_2}^{X_{nw}} \right) \right) dz \wedge d\bar{z}.$$

It is immediate that the term involving $\frac{B_2}{t}$ belongs to $O(t^{-1})$ by (II.A.2). Let us estimate the remaining term. We claim that

$$\|g(x) \frac{x^l}{t} R_l^x\| \leq \frac{C}{t^\mu} \frac{\langle z^l \rangle}{|Im z^l|}. \quad (\text{II.4.17})$$

Indeed, we first write $g(x) \frac{x^l}{t} R_l^x = \frac{1}{t^\mu} sgn(x^l) g(x) |x^l|^\mu \left(\frac{|x^l|}{t}\right)^{1-\mu} R_l^x$. Now, since $0 \leq \mu \leq 1$, we use that $\|(\frac{|x^l|}{t})^{1-\mu} R_l^x\| \leq C \frac{\langle z^l \rangle}{|Im z^l|}$ and $g(x) |x^l|^\mu$ bounded on \mathcal{H} , to prove that (II.4.17) holds. Hence, we conclude the proof of (II.4.16) using (II.A.2).

We assume now that $1 \leq \mu \leq 2$ and we denote $\mu_0 = \mu - 1$. Clearly, we have $0 \leq \mu_0 \leq 1$. We also consider a function $f_1 \in C_0^\infty(\mathbb{R}^3)$ such that $f_1 = 0$ on a neighbourhood of 0 and $f f_1 = f$. We have

$$\begin{aligned} g(x) f(\frac{X_{nw}}{t}) &= g(x) f(\frac{X_{nw}}{t}) f_1(\frac{X_{nw}}{t}), \\ &= g(x) \langle x \rangle f(\frac{X_{nw}}{t}) \langle x \rangle^{-1} f_1(\frac{X_{nw}}{t}) + g(x) \langle x \rangle [\langle x \rangle^{-1}, f(\frac{X_{nw}}{t})] f_1(\frac{X_{nw}}{t}) = L_1 + L_2. \end{aligned}$$

The first term L_1 obviously belongs to $O(t^{-\mu})$ by (II.4.16) since $g(x) \langle x \rangle \in S^{-\mu_0}(\mathbb{R}^3)$. For the second term L_2 , we use the Helffer-Sjöstrand formula (II.A.10) to get

$$g(x) \langle x \rangle [\langle x \rangle^{-1}, f(\frac{X_{nw}}{t})] f_1(\frac{X_{nw}}{t}) = \frac{1}{t} g(x) \langle x \rangle \nabla f(\frac{X_{nw}}{t}) \cdot [\langle x \rangle^{-1}, X_{nw}] f_1(\frac{X_{nw}}{t}) + O(t^{-2}).$$

Note that we used the fact that $[\langle x \rangle^{-1}, X_{nw}]$ and $[[\langle x \rangle^{-1}, X_{nw}], X_{nw}]$ are bounded on \mathcal{H} . Thus, we deduce from (II.4.16) that $L_2 \in O(t^{-\mu})$ which concludes the first part of the Lemma.

To prove the second part, we first rewrite (II.4.15) as

$$\chi(H) (f(\frac{X_{nw}}{t}) - f(\frac{x}{t})) = \chi(H_0) (f(\frac{X_{nw}}{t}) - f(\frac{x}{t})) + (\chi(H) - \chi(H_0)) (f(\frac{X_{nw}}{t}) - f(\frac{x}{t})) = I_1 + I_2.$$

We show that the term I_1 belongs to $O(t^{-1})$ and that the term I_2 belongs to $O(t^{-\min(1, \rho)})$. By the Helffer-Sjöstrand formula, we have

$$I_1 = \chi(H_0) \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^x \right) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) \left(\prod_{k_2=l}^3 R_{k_2}^{X_{nw}} \right) \right) dz \wedge d\bar{z}.$$

Since $\|R_l^x \frac{x^l}{t}\| \leq C \frac{\langle z_l \rangle}{|Im z_l|}$ and $\|R_l^{X_{nw}} \frac{x^l}{t}\| \leq C \frac{\langle z_l \rangle}{|Im z_l|}$, the integral converges in norm by (II.A.2). Now we commute $\chi(H_0)$ with $\prod_{k_1=1}^l R_{k_1}^x$. Since for any $k = 1, 2, 3$, $[\chi(H_0), R_k^x] = R_k^x [\chi(H_0), \frac{x^k}{t}] R_k^x$ and $[\chi(H_0), \frac{x^k}{t}] \in O(t^{-1})$ by the Helffer-Sjöstrand formula, we have

$$I_1 = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^x \right) \chi(H_0) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) \left(\prod_{k_2=l}^3 R_{k_2}^{X_{nw}} \right) \right) dz \wedge d\bar{z} + O(t^{-1}).$$

Using $\eta_p + \eta_p^c = 1$, we write

$$\chi(H_0) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) = \chi(H_0) \left(\eta_p^c \frac{x^l}{t} (\eta_p^c - 2) - \left[\frac{x^l}{t}, \eta_p^c \right] + \frac{Z_p^l}{t} \right).$$

But the assumptions on the supports of η and χ imply that $\chi(H_0)\eta_p^c = 0$. Then, since Z_p^l is bounded, we conclude that

$$\chi(H_0) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) \in O(t^{-1}),$$

by (II.2.9). Finally, $I_1 \in O(t^{-1})$.

It remains to study the term I_2 . Using the Helffer-Sjöstrand formula, we have

$$\begin{aligned} I_2 &= \frac{1}{(2\pi)^4} \int_{\mathbb{C}^4} \partial_{\bar{w}} \tilde{\chi}(w) \partial_{\bar{z}} \tilde{f}(z) (w - H)^{-1} V(x) (w - H_0)^{-1} \\ &\quad \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^x \right) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) \left(\prod_{k_2=l}^3 R_{k_2}^{X_{nw}} \right) \right) dw \wedge d\bar{w} \wedge dz \wedge d\bar{z}. \end{aligned}$$

We would like to commute $(w - H_0)^{-1}$ with the terms on its right. Since for any $k = 1, 2, 3$,

$$[(w - H_0)^{-1}, R_k^x] = R_k^x (w - H_0)^{-1} [H_0, \frac{x^k}{t}] (w - H_0)^{-1} R_k^x \in O(t^{-1}),$$

$$[(w - H_0)^{-1}, \frac{X_{nw}^l}{t} - \frac{x^l}{t}] = -\frac{i}{t} (w - H_0)^{-1} (\mathcal{V}_p^l - \Gamma^l) (w - H_0)^{-1} \in O(t^{-1}),$$

we obtain

$$\begin{aligned} I_2 &= \frac{1}{(2\pi)^4} \int_{\mathbb{C}^4} \partial_{\bar{w}} \tilde{\chi}(w) \partial_{\bar{z}} \tilde{f}(z) (w - H)^{-1} \\ &\quad \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1}^x \right) V(x) \left(\frac{X_{nw}^l}{t} - \frac{x^l}{t} \right) (w - H_0)^{-1} \left(\prod_{k_2=l}^3 R_{k_2}^{X_{nw}} \right) \right) dw \wedge d\bar{w} \wedge dz \wedge d\bar{z} + O(t^{-1}). \end{aligned}$$

But, since $V \in S^{-\rho}(\mathbb{R}^3)$, we already saw in the first part of the Lemma that this term belongs to $O(t^{-\rho})$ by (II.4.17). Hence $L_2 \in O(t^{-\min(1, \rho)})$ and the Lemma is proved.

◇

We prove now another technical lemma.

Lemma II.4.2 *Let $J \in C_0^\infty(\mathbb{R}^3)$, $J = 0$ on a neighbourhood of 0. Then,*

$$i[H, J(\frac{X_{nw}}{t})] = \frac{1}{t} \nabla J(\frac{X_{nw}}{t}) \cdot \mathcal{V}_p + L^1(dt).$$

Proof : We apply the commutator expansion (II.A.9) given in the appendix. We have to check 1) $H \in C^1(X_{nw})$, 2) $[H, X_{nw}]$ and $[[H, X_{nw}], X_{nw}]$ bounded on \mathcal{H} . To prove the first point, we use Lemmata II.3.2 and II.3.3 with the comparison operator $N = p^2 + x^2 + 1$. Recall that $D(N) = H^2(\mathbb{R}^3) \cap D(x^2)$.

The operator X_{nw} is symmetric and its domain contains $D(N)$. Let $u \in D(N)$. We already saw that $\|X_{nw}u\| \leq C\|x|u\| \leq C\|Nu\|$. Moreover, as a quadratic form on $D(N)$, we have

$$\begin{aligned} [X_{nw}, N] &= \eta_p[x, p^2]\eta_p + ([\eta_p, x^2]x\eta_p + hc) + [Z_p, x^2], \\ &= 2i\eta_p p\eta_p - i((\nabla\eta(p)).x + x.\nabla\eta(p))x\eta_p + hc - i(\nabla Z_\eta(p)).x + x.\nabla Z_\eta(p). \end{aligned}$$

Since each component of $Z_\eta(p)$ and $\eta(p)$ belong to $S^0(\mathbb{R}_p^3; \mathbb{C})$, using (II.2.9), we can estimate this commutator as follows

$$|(u, [X_{nw}, N]u)| \leq C(\|p|u\| \|u\| + \|x|u\|^2 + \|x|u\| \|u\|) \leq C\|N^{\frac{1}{2}}u\|^2,$$

and thus, (X_{nw}, N) satisfies Nelson's Lemma II.3.2. (H_0, N) also satisfies this Lemma and by Theorem II.2.1, we know that $(H - z)^{-1}D(N) \subset D(N)$. It remains to check the assumption on the commutator $[H, X_{nw}]$. As a quadratic form on $D(N)$, we have

$$i[H, X_{nw}] = i[H_0, X_{nw}] + i([V(x), \eta_p]x\eta_p + hc) + [V(x), Z_p], \quad (\text{II.4.18})$$

By (II.2.10), the first term in (II.4.18) is equal to \mathcal{V}_p . Using $[V(x), \eta_p]x = [V(x)x, \eta_p] - iV(x)\nabla\eta(p)$, the second term in (II.4.18) can be written as

$$i([V(x), \eta_p]x\eta_p + hc) = (i[V(x)x, \eta_p]\eta_p + V(x)\nabla\eta(p)\eta_p + hc).$$

Since $V \in S^{-\rho}(\mathbb{R}^3)$ and using the Helffer-Sjöstrand formula, all these terms are bounded operators on \mathcal{H} . Finally, using the fact that $Z_\eta(p) \in S^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$, the third term in (II.4.18) is also bounded by the same argument. Thus we have

$$|(u, i[H, X_{nw}]u)| \leq C\|u\|^2.$$

Now, straightforward computations using the Helffer-Sjöstrand formula show that the double commutator $[[H, X_{nw}], X_{nw}]$ is also bounded on \mathcal{H} .

Therefore, we can use the commutator expansion (II.A.9) and we get

$$\begin{aligned} i[H, X_{nw}] &= \frac{1}{t}\nabla J\left(\frac{X_{nw}}{t}\right)i[H, X_{nw}] + O(t^{-2}), \\ &= \frac{1}{t}\nabla J\left(\frac{X_{nw}}{t}\right).\mathcal{V}_p + \frac{1}{t}\nabla J\left(\frac{X_{nw}}{t}\right).i[V(x), X_{nw}] + O(t^{-2}). \end{aligned}$$

Now, we claim that

$$\nabla J\left(\frac{X_{nw}}{t}\right).i[V(x), X_{nw}] \in O(t^{-\rho}), \quad (\text{II.4.19})$$

which entails Lemma II.4.2. Using Lemma II.4.1, it suffices to show that $\langle x \rangle^\rho i[V(x), X_{nw}]$ is bounded. We have

$$\langle x \rangle^\rho i[V(x), X_{nw}] = \langle x \rangle^\rho (i[V(x)x, \eta_p]\eta_p + V(x)\nabla\eta(p)\eta_p + hc) + \langle x \rangle^\rho i[V(x), Z_p].$$

Since $V \in S^{-\rho}(\mathbb{R}^3)$, the term $\langle x \rangle^\rho V(x)\nabla\eta(p)\eta_p$ is clearly bounded. Let us analyse $\langle x \rangle^\rho i[V(x), Z_p]$. By the Helffer-Sjöstrand formula, we have

$$\langle x \rangle^\rho i[V(x), Z_p] = \langle x \rangle^\rho \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{Z}_\eta(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l R_{k_1} \right) (-i\partial_l V(x)) \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z}.$$

If we commute $\langle x \rangle^\rho$ with the resolvents on its right and since $\langle x \rangle^\rho \partial_l V(x)$ is bounded, we obtain

$$\langle x \rangle^\rho i[V(x), Z_p] = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \widetilde{Z}_\eta(z) \left(\sum_{l=1}^3 \left(\prod_{k_1=1}^l [\langle x \rangle^\rho, R_{k_1}] \right) (-i \partial_l V(x)) \left(\prod_{k_2=l}^3 R_{k_2} \right) \right) dz \wedge d\bar{z} + B,$$

where B is bounded by (II.A.2). Now, note that $[\langle x \rangle^\rho, R_k] = \rho R_k x^k \langle x \rangle^{\rho-2} R_k$. Therefore, if this term is bounded, we have proved that $\langle x \rangle^\rho i[V(x), Z_p]$ is bounded and in the other case, we commute $x^k \langle x \rangle^{\rho-2}$ with the resolvent on its right. By induction, we finally prove that $\langle x \rangle^\rho i[V(x), Z_p]$ is bounded.

By the same argument, the remaining term $\langle x \rangle^\rho i[V(x), \eta_p] \eta_p$ is also bounded on \mathcal{H} which concludes the proof of the Lemma.

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Eventually, we shall need the Lemma

Lemma II.4.3 *Let $f, g \in C_0^\infty(\mathbb{R}^3)$. Then,*

$$\left[f\left(\frac{X_{nw}}{t}\right), \mathcal{V}_p \right] \in O(t^{-1}), \quad \left[f\left(\frac{X_{nw}}{t}\right), g(\mathcal{V}_p) \right] \in O(t^{-1}).$$

Proof : Using the Helffer-Sjöstrand, it suffices to show that $[X_{nw}, \mathcal{V}_p]$ is bounded on \mathcal{H} . We have

$$\begin{aligned} [X_{nw}, \mathcal{V}_p] &= [x_p, \mathcal{V}_p] + [Z_p, \mathcal{V}_p], \\ &= -\eta_p \nabla \mathcal{V}_\eta(p) \eta_p + [Z_p, \mathcal{V}_p], \end{aligned}$$

which is bounded since $\mathcal{V}_\eta(p) \in S^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$.

◇

Proof (of Proposition II.4.4) : For technical reasons, we need to introduce an approximation of the generator of dilations A . We define

$$\mathcal{A} = \frac{1}{2} (X_{nw} \cdot \mathcal{V}_p + \mathcal{V}_p \cdot X_{nw}).$$

\mathcal{A} is essentially selfadjoint on $D(x)$. Indeed, since $X_{nw} = x_p + Z_p$, we have $\mathcal{A} = A_p + B$ where B is a symmetric bounded operator and $A_p = \frac{1}{2} (x_p \cdot \mathcal{V}_p + \mathcal{V}_p \cdot x_p)$. Thus, if we prove that A_p is essentially selfadjoint on the domain $D(x)$, so is \mathcal{A} by the Kato-Rellich Theorem. Commuting η_p with x , we rewrite A_p as follows

$$\begin{aligned} A_p &= \frac{1}{2} (x \cdot \eta_p^2 \mathcal{V}_\eta^2 + \eta_p^2 \mathcal{V}_\eta^2 \cdot x) + B, \\ &= A_1 + B, \end{aligned}$$

where B is bounded by (II.2.9). Again we only have to prove that A_1 is essentially selfadjoint on $D(x)$. This follows using the same proof as in Lemma II.3.4.

Let $0 < \theta_0 < \theta_1 < 1 < \theta_2 < \theta_3$. Let $J \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp } J \subset \{\theta_0 \leq |x| \leq \theta_3\}$ and $J = 1$ on $\{\theta_1 \leq |x| \leq \theta_2\}$. Let us consider the propagation observable

$$\phi(t) = \chi(H) J\left(\frac{X_{nw}}{t}\right) \left\{ \frac{X_{nw}^2}{t^2} - 2 \frac{\mathcal{A}}{t} \right\} J\left(\frac{X_{nw}}{t}\right) \chi(H),$$

which is a selfadjoint operator-valued function uniformly bounded with respect to t . Let us compute its Heisenberg derivative

$$\begin{aligned} -\mathbf{D}\phi(t) &= -\frac{1}{t}\chi(H)(\mathbf{D}J(\frac{X_{nw}}{t}))\{\frac{X_{nw}^2}{t^2} - 2\frac{\mathcal{A}}{t}\}J(\frac{X_{nw}}{t})\chi(H) + hc \\ &\quad + \frac{2}{t}\chi(H)J(\frac{X_{nw}}{t})\{\frac{X_{nw}^2}{t^2} - \frac{\mathcal{A}}{t}\}J(\frac{X_{nw}}{t})\chi(H) \\ &\quad - \chi(H)J(\frac{X_{nw}}{t})i[H, \frac{X_{nw}^2}{t^2} - 2\frac{\mathcal{A}}{t}]J(\frac{X_{nw}}{t})\chi(H). \end{aligned}$$

From Lemma II.4.2, we get

$$\mathbf{D}J(\frac{X_{nw}}{t}) = -\frac{1}{t}\nabla J(\frac{X_{nw}}{t}) \cdot (\mathcal{V}_p - \frac{X_{nw}}{t}) + L^1(dt). \quad (\text{II.4.20})$$

Now we claim and we shall prove later that

$$J(\frac{X_{nw}}{t})i[H, \frac{X_{nw}^2}{t^2}]J(\frac{X_{nw}}{t}) = \frac{2}{t}J(\frac{X_{nw}}{t})\frac{\mathcal{A}}{t}J(\frac{X_{nw}}{t}) + L^1(dt), \quad (\text{II.4.21})$$

$$J(\frac{X_{nw}}{t})i[H, \frac{\mathcal{A}}{t}]J(\frac{X_{nw}}{t}) = \frac{1}{t}J(\frac{X_{nw}}{t})\mathcal{V}_p^2J(\frac{X_{nw}}{t}) + L^1(dt). \quad (\text{II.4.22})$$

Thus, using (II.4.20), (II.4.21) and (II.4.22), we obtain the following expression for $-\mathbf{D}\phi(t)$

$$\begin{aligned} -\mathbf{D}\phi(t) &= -\frac{1}{t}\chi(H)\nabla J(\frac{X_{nw}}{t}) \cdot (\mathcal{V}_p - \frac{X_{nw}}{t})\{\frac{X_{nw}^2}{t^2} - 2\frac{\mathcal{A}}{t}\}J(\frac{X_{nw}}{t})\chi(H) + hc \\ &\quad + \frac{2}{t}\chi(H)J(\frac{X_{nw}}{t})\{\frac{X_{nw}^2}{t^2} - 2\frac{\mathcal{A}}{t} + \mathcal{V}_p^2\}J(\frac{X_{nw}}{t})\chi(H) + L^1(dt). \end{aligned} \quad (\text{II.4.23})$$

Let us consider a function $j \in C_0^\infty(\mathbb{R}^3)$ such that $\text{supp } j \cap \{\theta_1 \leq |x| \leq \theta_2\} = \emptyset$ and $j\nabla J = \nabla J$. Then, using Lemma II.4.3, the first term of (II.4.23) can be written as

$$\chi(H)j(\frac{X_{nw}}{t})B(t)j(\frac{X_{nw}}{t})\chi(H) + L^1(dt),$$

where $B(t)$ is operator-valued function uniformly bounded in t . Since the support of j is disjoint from a neighbourhood of the unit sphere S^2 , this term is integrable along the evolution by Propositions II.4.1 and II.4.3.

For the second term in (II.4.23), note that $\{\frac{X_{nw}^2}{t^2} - 2\frac{\mathcal{A}}{t} + \mathcal{V}_p^2\} = (\frac{X_{nw}}{t} - \mathcal{V}_p)^2$. Thus we obtain

$$-\mathbf{D}\phi(t) = \frac{2}{t}\chi(H)J(\frac{X_{nw}}{t})(\frac{X_{nw}}{t} - \mathcal{V}_p)^2J(\frac{X_{nw}}{t})\chi(H) + L^1(dt).$$

By Lemma II.4.1, one can replace X_{nw} by x in the equality above. Furthermore, one can also replace \mathcal{V}_p by \mathcal{V} thanks to the following result

$$\mathcal{V}_p J(\frac{x}{t})\chi(H) = \mathcal{V}J(\frac{x}{t})\chi(H) + O(t^{-\rho}). \quad (\text{II.4.24})$$

To prove (II.4.24), let us write $\mathcal{V}_p J(\frac{x}{t})\chi(H) = \mathcal{V}_p \chi(H_0)J(\frac{x}{t}) + \mathcal{V}_p [J(\frac{x}{t}), \chi(H_0)] + \mathcal{V}_p J(\frac{x}{t})(\chi(H) - \chi(H_0))$. By the Helffer-Sjöstrand formula, one has $(\chi(H) - \chi(H_0))J(\frac{x}{t}) \in O(t^{-\rho})$ and $[J(\frac{x}{t}), \chi(H_0)] \in$

$O(t^{-1})$. Now recall that we chose η such that $\eta^c(x)\chi(|x|) = 0$. Hence, one has $\chi(H_0)\eta_p^c = 0$. Now using that $\eta_p + \eta_p^c = 1$, we obtain $\mathcal{V}_p\chi(H_0) = \mathcal{V}\chi(H_0)$ which proves (II.4.24). Finally, we have

$$-\mathbf{D}\phi(t) = \frac{2}{t}\chi(H)J\left(\frac{x}{t}\right)\left(\frac{x}{t} - \mathcal{V}\right)^2 J\left(\frac{x}{t}\right)\chi(H) + L^1(dt).$$

Thus, (II.4.12) holds by Proposition II.B.1. It only remains to prove (II.4.21) and (II.4.22). We have

$$\begin{aligned} J\left(\frac{X_{nw}}{t}\right)i\left[H, \frac{X_{nw}^2}{t^2}\right]J\left(\frac{X_{nw}}{t}\right) &= \frac{1}{t}J\left(\frac{X_{nw}}{t}\right)\left(\frac{X_{nw}}{t}.i[H, X_{nw}] + hc\right)J\left(\frac{X_{nw}}{t}\right), \\ &= \frac{2}{t}J\left(\frac{X_{nw}}{t}\right)\mathcal{A}J\left(\frac{X_{nw}}{t}\right) + \frac{1}{t}J\left(\frac{X_{nw}}{t}\right)\left(\frac{X_{nw}}{t}.i[V(x), X_{nw}] + hc\right)J\left(\frac{X_{nw}}{t}\right), \end{aligned}$$

by (II.2.10). Moreover, the second term belongs to $O(t^{-1-\rho})$ by (II.4.19) which implies (II.4.21). Eventually, we have

$$\begin{aligned} J\left(\frac{X_{nw}}{t}\right)i\left[H, \frac{\mathcal{A}}{t}\right]J\left(\frac{X_{nw}}{t}\right) &= \frac{1}{t}J\left(\frac{X_{nw}}{t}\right)\mathcal{V}_p^2 J\left(\frac{X_{nw}}{t}\right) + \frac{1}{2t}J\left(\frac{X_{nw}}{t}\right)\left(i[V(x), X_{nw}].\mathcal{V}_p + hc\right)J\left(\frac{X_{nw}}{t}\right) \\ &\quad + \frac{1}{2}J\left(\frac{X_{nw}}{t}\right)\left(\frac{X_{nw}}{t}.i[V(x), \mathcal{V}_p] + hc\right)J\left(\frac{X_{nw}}{t}\right). \end{aligned} \quad (\text{II.4.25})$$

The second term in (II.4.25) clearly belongs to $O(t^{-1-\rho})$ by (II.4.19). Moreover, we have

$$J\left(\frac{X_{nw}}{t}\right)[V(x), \mathcal{V}_p] \in O(t^{-1-\rho}), \quad (\text{II.4.26})$$

by the same argument as for (II.4.19). Therefore, the third term in (II.4.18) also belongs to $O(t^{-1-\rho})$. Hence, (II.4.22) holds which concludes the proof of (II.4.12).

To prove (II.4.13), let us consider the following propagation observable

$$\begin{aligned} \phi(t) &= \chi(H)J\left(\frac{x}{t}\right)(\mathcal{V} - \frac{x}{t})^2 J\left(\frac{x}{t}\right)\chi(H), \\ &= \chi(H)J\left(\frac{x}{t}\right)\left\{\frac{x^2}{t^2} - 2\frac{\mathcal{A}}{t} + \mathcal{V}^2\right\}J\left(\frac{x}{t}\right)\chi(H), \end{aligned}$$

In particular, note that $\phi(t)$ is a positive operator for any t . Let us prove that

$$\lim_{t \rightarrow \infty} (e^{-itH}\psi, \phi(t)e^{-itH}\psi) = \lim_{t \rightarrow \infty} \left\| (\mathcal{V} - \frac{x}{t})J\left(\frac{x}{t}\right)\chi(H)e^{-itH}\psi \right\|^2 = 0,$$

which proves (II.4.13). For technical reasons, we need to modify $\phi(t)$ slightly. Let us define the modified propagation observable

$$\phi_m(t) = \chi(H)J\left(\frac{X_{nw}}{t}\right)\left\{\frac{X_{nw}^2}{t} - 2\frac{\mathcal{A}}{t} + \mathcal{V}_p^2\right\}J\left(\frac{X_{nw}}{t}\right)\chi(H).$$

where the function η in the definition of X_{nw} and \mathcal{V}_p satisfies the property $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$. By Lemma II.4.1, we have the following relation between $\phi_m(t)$ and $\phi(t)$.

$$\phi_m(t) = \phi(t) + O(t^{-\rho}).$$

Thus it is sufficient to show

$$\lim_{t \rightarrow \infty} (e^{-itH}\psi, \phi_m(t)e^{-itH}\psi) = 0.$$

Let us compute the Heisenberg derivative of $\phi_m(t)$. As shown in the computation of the first part of the proposition and using again Lemma II.4.1, we obtain

$$\begin{aligned} -\mathbf{D}\phi_m(t) &= \frac{2}{t}\chi(H)J\left(\frac{X_{nw}}{t}\right)\left(\mathcal{V} - \frac{x}{t}\right)^2 J\left(\frac{x}{t}\right)\chi(H), \\ &\quad -\chi(H)J\left(\frac{X_{nw}}{t}\right)i[H, \mathcal{V}_p^2]J\left(\frac{X_{nw}}{t}\right)\chi(H), \\ &\quad +L^1(dt). \end{aligned}$$

The term $J\left(\frac{X_{nw}}{t}\right)i[H, \mathcal{V}_p^2]J\left(\frac{X_{nw}}{t}\right) = J\left(\frac{X_{nw}}{t}\right)i[V(x), \eta_p^2]J\left(\frac{X_{nw}}{t}\right)$ belongs to $L^1(dt)$ by Lemma II.4.1. Thus

$$-\mathbf{D}\phi_m(t) = \frac{2}{t}\chi(H)J\left(\frac{X_{nw}}{t}\right)\left(\mathcal{V} - \frac{x}{t}\right)^2 J\left(\frac{x}{t}\right)\chi(H) + L^1(dt).$$

Hence, by (II.4.12), we can apply Lemma II.B.1 and the following limit exists

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi_m(t) e^{-itH}. \quad (\text{II.4.27})$$

Clearly, we can replace $\phi_m(t)$ by $\phi(t)$ in (II.4.27) without changing the limit. Now, we also know by (II.4.12) that

$$0 \leq \int_1^{+\infty} (e^{-itH} \psi, \phi(t) e^{-itH} \psi) \frac{dt}{t} < +\infty,$$

Hence the limit (II.4.27) vanishes which concludes the proof.

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II.5 Asymptotic velocity

In this section, we use the previous propagation estimates to construct the asymptotic velocity operators. We follow the exposition given in [21] with slight changes for Dirac operators already introduced in [14]. We recall that the construction of the asymptotic velocity is valid under the weak assumption : $V \in S^{-\rho}(\mathbb{R}^3)$ with $\rho > 0$. In the remaining of this paper, we will only give the proofs of the results for $t \rightarrow +\infty$ since the proofs for $t \rightarrow -\infty$ are identical.

II.5.1 Existence of P^\pm

The main theorem of this section is

Theorem II.5.1 *Let H be the massless Dirac operator. Let $J \in C_\infty(\mathbb{R}^3)$. Then, the limits*

$$J(P^\pm) := s - \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH}, \quad (\text{II.5.1})$$

exist. The operators P^\pm defined by (II.5.1) are vectors of commuting selfadjoint operators on \mathcal{H} defined on a dense subspace of \mathcal{H} and P^\pm commute with H . Furthermore, the states having zero asymptotic velocity are the bound states of H , i.e.

$$\mathbf{1}_{\{0\}}(P^\pm) = \mathbf{1}^{pp}(H) \text{ or equivalently } \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm) = \mathbf{1}^c(H) \quad (\text{II.5.2})$$

where $\mathbf{1}^{pp}(H)$ and $\mathbf{1}^c(H)$ denote respectively the projections onto the pure point and continuous spectral subspaces of H . We can also characterize the asymptotic velocity P^\pm in terms of the classical velocity operator \mathcal{V} by

$$J(P^\pm)\mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH} J(\pm\mathcal{V})e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm). \quad (\text{II.5.3})$$

Proof : First consider $\psi \in \mathcal{H}^{pp}$, $\psi \neq 0$. Since the only possible eigenvalue for H is 0, we have $H\psi = 0$. Let $J \in C_\infty(\mathbb{R}^3)$. Then

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{x}{t}\right)e^{-itH} \psi &= J(0)\psi + \lim_{t \rightarrow \pm\infty} e^{itH} \left\{ J\left(\frac{x}{t}\right) - J(0) \right\} \psi, \\ &= J(0)\psi. \end{aligned}$$

Therefore, the limit (II.5.1) exists on \mathcal{H}^{pp} . Now, assume that $\psi \in \mathcal{H}^c$. Let $J \in C_0^\infty(\mathbb{R}^3)$ and $\chi \in C_0^\infty(\mathbb{R})$ such that J is constant on a neighbourhood of 0 and $\chi = 0$ on a neighbourhood of 0. By a density argument and since $[\chi(H), J(\frac{x}{t})] \in O(t^{-1})$, the existence of (II.5.1) is equivalent to that of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J\left(\frac{x}{t}\right) \chi(H) e^{-itH}. \quad (\text{II.5.4})$$

Let us introduce the propagation observable $\phi(t) = \chi(H) \left\{ J\left(\frac{x}{t}\right) + (\mathcal{V} - \frac{x}{t}) \cdot \nabla J\left(\frac{x}{t}\right) \right\} \chi(H)$. Clearly, $\phi(t)$ is a selfadjoint operator-valued function uniformly bounded in t . By Proposition II.4.4, the existence of (II.5.4) is equivalent to that of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) e^{-itH}. \quad (\text{II.5.5})$$

For technical reasons, we need to introduce the modified propagation observable

$$\phi_m(t) = \chi(H) \left\{ J\left(\frac{X_{nw}}{t}\right) + (\mathcal{V}_p - \frac{X_{nw}}{t}) \cdot \nabla J\left(\frac{X_{nw}}{t}\right) \right\} \chi(H),$$

where the function η (in the definition of the Newton-Wigner observable X_{nw} and the classical velocity operator \mathcal{V}_p) is chosen such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$. By Lemma II.4.1 and the proof of Proposition II.4.4, it is enough to prove the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi_m(t) e^{-itH}. \quad (\text{II.5.6})$$

Let us compute the Heisenberg derivative of $\phi_m(t)$. Using Lemma II.4.2, we obtain

$$\begin{aligned} \mathbf{D}\phi_m(t) &= -\frac{1}{t} \chi(H) \left\langle (\mathcal{V}_p - \frac{X_{nw}}{t}), \nabla^2 J\left(\frac{X_{nw}}{t}\right) \cdot \frac{X_{nw}}{t} \right\rangle \chi(H) \\ &\quad + \chi(H) i \left[H, J\left(\frac{X_{nw}}{t}\right) + (\mathcal{V}_p - \frac{X_{nw}}{t}) \cdot \nabla J\left(\frac{X_{nw}}{t}\right) \right] \chi(H), \\ &= \frac{1}{t} \chi(H) \left\langle (\mathcal{V}_p - \frac{X_{nw}}{t}), \nabla^2 J\left(\frac{X_{nw}}{t}\right) \cdot (\mathcal{V}_p - \frac{X_{nw}}{t}) \right\rangle \chi(H) \\ &\quad + \chi(H) i [V(x), \mathcal{V}_p] \cdot \nabla J\left(\frac{X_{nw}}{t}\right) \chi(H) + L^1(dt). \end{aligned} \quad (\text{II.5.7})$$

Now, the second term in (II.5.7) is integrable along the evolution by (II.4.26). Whence we get

$$\mathbf{D}\phi_m(t) = \frac{1}{t} \chi(H) \left\langle (\mathcal{V}_p - \frac{X_{nw}}{t}), \nabla^2 J\left(\frac{X_{nw}}{t}\right) \cdot (\mathcal{V}_p - \frac{X_{nw}}{t}) \right\rangle \chi(H) + L^1(dt).$$

By Propositions II.B.1 and II.4.4, the limit (II.5.6) exists and hence so does (II.5.1).

In order to prove that the operators P^\pm defined by (II.5.1) are vectors of selfadjoint operators on \mathcal{H} with dense domain, it suffices to show that

$$s - \lim_{R \rightarrow +\infty} \left(s - \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{x}{Rt}\right) e^{-itH} \right) = 1, \quad (\text{II.5.8})$$

for a given $J \in C_\infty(\mathbb{R}^3)$ with $J(0) = 1$ (see Proposition B.2.1 in [21]). The proof of (II.5.8) is identical to that given in [14]. We omit it. Eventually, using the definition of P^\pm and $[H, \frac{x}{t}] \in O(t^{-1})$, it is immediate that P^\pm commute with H .

Now, we prove that the states of zero asymptotic velocity correspond to the (possible) bound states of H . This result only requires the minimal velocity estimate. We already saw that, for any $\psi \in \mathcal{H}^p$, we have

$$\lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{x}{t}\right) e^{-itH} \psi = J(0)\psi.$$

This entails that $P^\pm \psi = 0$ and proves $\mathbf{1}_{\{0\}}(P^\pm) \geq \mathbf{1}^{pp}(H)$. Let us prove the converse inequality. We consider a function $J \in C_0^\infty(\mathbb{R}^3)$ such that $J(0) = 1$, $J \geq 0$ and $\text{supp } J \subset B(0, \epsilon)$ with $0 < \epsilon < 1$ and a function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Then (II.5.1) implies

$$\begin{aligned} s - \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{x}{t}\right) \chi^2(H) e^{-itH} &= J(P^\pm) \chi^2(H), \\ &\geq \mathbf{1}_{\{0\}}(P^\pm) \chi^2(H). \end{aligned}$$

Note that the left hand side is equal to 0 by Proposition II.4.1. Therefore, we have $\mathbf{1}_{\{0\}}(P^\pm) \leq \mathbf{1}_{\{0\}}(H)$. But the only possible eigenvalue of H is precisely 0. Thus $\mathbf{1}_{\{0\}}(P^\pm) \leq \mathbf{1}^{pp}(H)$ and we have proved (II.5.2).

It remains to give another characterization of P^+ in term of the classical velocity operator \mathcal{V} . By definition of the asymptotic velocity P^+ and by a density argument, we simply need to prove

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \left(J(\mathcal{V}) - J\left(\frac{x}{t}\right) \right) f\left(\frac{x}{t}\right) \chi(H) e^{-itH} = 0, \quad (\text{II.5.9})$$

where $J, f \in C_0^\infty(\mathbb{R}^3)$ such that $f = 0$ on a neighbourhood of 0 and $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi = 0$ on a neighbourhood of 0. Now using the Helffer-Sjöstrand formula (II.A.11), we see that the left hand side of (II.5.9) is equal to

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \left(C(\mathcal{V} - \frac{x}{t}) + B(t) \right) f\left(\frac{x}{t}\right) \chi(H) e^{-itH},$$

where C is a bounded on \mathcal{H} and $B(t) \in O(t^{-1})$. Now, this limit vanishes thanks to Proposition II.4.4.

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II.5.2 Spectrum of P^\pm

This section is devoted to the study of the spectrum of P^\pm . We recall that in the case of a massive Dirac equation, the spectrum of the corresponding asymptotic velocity P_m^\pm was the unit closed ball i.e. $\sigma(P_m^\pm) = \bar{B}(0, 1)$. Intuitively, when the Dirac field is massive, the modulus of the asymptotic velocity can take any value between 0 and 1 and no space-like direction is preferred. Moreover, the states corresponding to the zero asymptotic velocity are the bound states of the massive Hamiltonian. We do not expect such a result in the case of a massless Dirac equation since there is no mass to slow the field down. Actually, we have

Theorem II.5.2 *Let P^\pm be the asymptotic velocity defined in Theorem II.5.1. If S^2 denotes the unit 2-sphere, then*

$$\begin{aligned}\sigma(P^\pm) &= \{0\} \cup S^2, & \text{if } \sigma_{pp}(H) \neq \emptyset, \\ \sigma(P^\pm) &= S^2, & \text{otherwise.}\end{aligned}$$

Proof : Since we know that $\{0\}$ corresponds to the bound states of H , it is enough to show that $\sigma(P^\pm \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm)) = S^2$. Let us first prove $\sigma(P^\pm \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm)) \subset S^2$. We shall use the characterization (II.5.3) of the asymptotic velocity P^\pm and the fact that

$$\sigma(\mathcal{V}) = S^2. \quad (\text{II.5.10})$$

Let $\xi_0 \in \mathbb{R}^3 \setminus S^2$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ chosen such that $\chi(\xi_0) = 1$ and $\text{supp } \chi \cap S^2 = \emptyset$. It suffices to prove $\chi(P^\pm) = 0$ to prove the above inclusion, since in this case, ξ_0 belongs to the resolvent set of P^\pm . But, from (II.5.3), one has

$$\chi(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH} \chi(\mathcal{V}) e^{-itH}.$$

This term is equal to 0 by (II.5.10).

Let us prove the converse inclusion $\sigma(P^+ \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^+)) \supset S^2$. Let $\xi_0 \in S^2$ and choose two real numbers r_1, r_0 such that $|\xi_0| = 1 > r_1 > r_0 > 0$. Let $g \in C_0^\infty(B(\xi_0, r_0))$ such that $g(\xi_0) = 1$. This time we want to show that $g(P^+) \neq 0$. Let $J \in C^\infty(\mathbb{R})$ such that $J = 1$ on $(-\infty, r_1]$, $J = 0$ on $[1 - \frac{1-r_1}{2}, +\infty)$ and $j = -J' \geq 0$. Clearly J and g satisfy the following relations

$$\bullet \quad J(|\xi - \xi_0|)g(\xi) = g(\xi). \quad (\text{II.5.11})$$

$$\bullet \quad j(|\xi - \xi_0|)|\xi - \xi_0| \geq r_1 j(|\xi - \xi_0|). \quad (\text{II.5.12})$$

$$\bullet \quad g(\xi)|\xi - \xi_0| \leq r_0 g(\xi). \quad (\text{II.5.13})$$

By Theorem II.5.1 and (II.5.11), we have

$$g^2(P^+) = s - \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\left|\frac{x}{t} - \xi_0\right|\right) g^2(\mathcal{V}) J\left(\left|\frac{x}{t} - \xi_0\right|\right) e^{-itH}.$$

Now let us consider a function $\chi \in C_0^\infty(\mathbb{R})$, $\chi \neq 0$, such that $\chi = 0$ on a neighbourhood of 0. It suffices to prove that for any such function χ , we have

$$\chi(H)g^2(P^+)\chi(H) \neq 0.$$

If we introduce the propagation observable $\phi(t) = \chi(H)J\left(\left|\frac{x}{t} - \xi_0\right|\right)g^2(\mathcal{V})J\left(\left|\frac{x}{t} - \xi_0\right|\right)\chi(H)$, it is the same to prove

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \phi(t) e^{-itH} \neq 0. \quad (\text{II.5.14})$$

For technical reasons, we introduce the modified propagation observable

$$\phi_m(t) = \chi(H)J\left(\left|\frac{X_{nw}}{t} - \xi_0\right|\right)g^2(\mathcal{V}_p)J\left(\left|\frac{X_{nw}}{t} - \xi_0\right|\right)\chi(H),$$

where the function η in the definition of X_{nw} and \mathcal{V}_p is chosen such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$. By Lemma II.4.1 and (II.4.24), we see that

$$\phi_m(t) = \phi(t) + O(t^{-\rho}),$$

whence it is enough to show that

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \phi_m(t) e^{-itH} \neq 0. \quad (\text{II.5.15})$$

Let us compute the Heisenberg derivative of $\phi_m(t)$. Using Lemma II.4.2, we obtain

$$\begin{aligned} \mathbf{D}\phi_m(t) &= \frac{1}{t} \chi(H) j \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \frac{\frac{X_{nw}}{t} - \xi_0}{\left| \frac{X_{nw}}{t} - \xi_0 \right|} \cdot \frac{X_{nw}}{t} g^2 \left(\frac{X_{nw}}{t} \right) J \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \chi(H) + hc \\ &\quad - \frac{1}{t} \chi(H) j \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \frac{\frac{X_{nw}}{t} - \xi_0}{\left| \frac{X_{nw}}{t} - \xi_0 \right|} \cdot \mathcal{V}_p g^2(\mathcal{V}_p) J \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \chi(H) + hc \\ &\quad + \chi(H) J \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) i[V(x), g^2(\mathcal{V}_p)] J \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \chi(H) \\ &\quad + L^1(dt). \end{aligned} \quad (\text{II.5.16})$$

Now, the Helffer-Sjöstrand formula together with (II.4.26) imply that the third term in (II.5.16) belongs to $L^1(dt)$. Eventually we obtain

$$\begin{aligned} \mathbf{D}\phi_m(t) &= \frac{1}{t} \chi(H) j \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \left| \frac{X_{nw}}{t} - \xi_0 \right| g^2 \left(\frac{X_{nw}}{t} \right) J \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \chi(H) + hc \\ &\quad + \frac{1}{t} \chi(H) j \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \frac{\frac{X_{nw}}{t} - \xi_0}{\left| \frac{X_{nw}}{t} - \xi_0 \right|} \cdot (\xi_0 - \mathcal{V}_p) g^2(\mathcal{V}_p) J \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) \chi(H) \\ &\quad + L^1(dt). \end{aligned} \quad (\text{II.5.17})$$

Then commuting certain terms in (II.5.17) and using (II.5.12), (II.5.13) and Lemma II.4.3, we get

$$\mathbf{D}\phi_m(t) \geq \frac{1}{t} (r_1 - r_0) g(\mathcal{V}_p) (Jj) \left(\left| \frac{X_{nw}}{t} - \xi_0 \right| \right) g(\mathcal{V}_p) + L^1(dt).$$

Therefore, as $r_1 - r_0 > 0$, one has $\mathbf{D}\phi_m(t) \geq R(t)$ with $R(t) \in L^1(dt)$.

We can conclude the proof of the proposition in the following way. Let us write

$$\begin{aligned} \chi(H) g^2(P^+) \chi(H) &= e^{it_0 H} \phi_m(t_0) e^{-it_0 H} + \int_{t_0}^{\infty} e^{itH} \mathbf{D}\phi_m(t) e^{-itH} dt, \\ &\geq e^{it_0 H} \phi_m(t_0) e^{-it_0 H} - \int_{t_0}^{\infty} \|R(t)\| dt. \end{aligned}$$

Observe that the quantity $\int_{t_0}^{\infty} \|R(t)\| dt$ becomes as small as we want when t_0 tends to infinity. Thus it suffices to show that the limit

$$\lim_{t_0 \rightarrow \infty} \left\| e^{it_0 H} \phi_m(t_0) e^{-it_0 H} \right\|,$$

exists and is nonzero to prove (II.5.15). Equivalently, we prove that

$$\lim_{t_0 \rightarrow \infty} \left\| e^{it_0 H} \phi(t_0) e^{-it_0 H} \right\| \neq 0. \quad (\text{II.5.18})$$

We have

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} \left\| e^{it_0 H} \phi(t_0) e^{-it_0 H} \right\| &= \lim_{t_0 \rightarrow \infty} \left\| e^{it_0 \xi_0 \cdot p} \phi(t_0) e^{-it_0 \xi_0 \cdot p} \right\|, \\ &= \lim_{t_0 \rightarrow \infty} \left\| \chi(H_0 + V(x + t_0 \xi_0)) J \left(\frac{|x|}{t_0} \right) g^2(\mathcal{V}) J \left(\frac{|x|}{t_0} \right) \chi(H_0 + V(x + t_0 \xi_0)) \right\|. \end{aligned}$$

Since $\chi(H_0 + V(x + t_0\xi_0))J(\frac{|x|}{t_0})g^2(\mathcal{V})J(\frac{|x|}{t_0})\chi(H_0 + V(x + t_0\xi_0))$ goes strongly to $\chi(H_0)g^2(\mathcal{V})\chi(H_0)$ which is a non-zero operator, (II.5.18) holds. Hence $\chi(H)g^2(P^+)\chi(H) \neq 0$ which concludes the proof of the inclusion $\sigma(P^+ \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^+)) \supset S^2$.

◇

II.6 Wave operators

This section is devoted to the construction of Dollard modified wave operators. We assume here that $V \in S^{-\rho}(\mathbb{R}^3)$ with $\rho > \frac{1}{2}$. We shall see that the asymptotic velocity operators P^\pm enable to transform the problem into a time-dependent problem. The great advantage is that we then manipulate quantities which are directly integrable along the evolution. In particular, we will not need stationary phase estimates or radiation estimates as used in [34].

II.6.1 Time-dependent version

The main objects of study in this subsection are time-dependent massless Dirac Hamiltonians of the following type $H(t) = H_0 + V(t, x)$ where $V(t, x)$ is a scalar time-dependent potential belonging to $L^1_{loc}(\mathbb{R}_t; \mathcal{B}(\mathcal{H}))$. We aim at constructing time-dependent wave operators for this kind of Hamiltonians. Naturally, we need a certain decay of the potential V : we assume this decay only in the variable t . Precisely, for a given $\rho > 0$,

$$|\partial_x^\alpha V(t, x)| \leq C \langle t \rangle^{-\rho - |\alpha|}, \quad \forall \alpha \in \mathbb{N}^3, \quad \forall x \in \mathbb{R}^3, \quad (\text{II.6.1})$$

We can define an associated unitary dynamics $U(t, s)$ for such time-dependent Hamiltonians (see [21], appendix B.3, Proposition B.3.6, for a detailed presentation). In particular, this unitary dynamics satisfies

- The map $(t, s) \longrightarrow U(t, s)$ is strongly continuous with values in unitary operators on \mathcal{H} and satisfies

$$U(t, t) = 1, \quad U(t, u)U(u, s) = U(t, s).$$

- If we denote $B = (H_0^2 + 1)^{\frac{1}{2}}$, we have

$$\begin{aligned} \partial_s U(t, s) B^{-1} &= U(t, s) i H(s) B^{-1}, \\ \partial_t B^{-1} U(t, s) &= -i B^{-1} H(t) U(t, s). \end{aligned}$$

We would like to prove the existence of the asymptotic velocity operators for the unitary dynamics $U(t, 0)$. However, contrary to the massive case [14], this can be done only for high energies. Precisely, we have

Proposition II.6.1 *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Then, for any $J \in C_\infty(\mathbb{R}^3)$, the limits*

$$s - \lim_{t \rightarrow \pm\infty} U(0, t) J(\pm\mathcal{V}) \chi(H_0) U(t, 0) =: \mathcal{P}_{J, \chi}, \quad (\text{II.6.2})$$

exist.

Proof : Let us choose a function η such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$. We have $g(\mathcal{V})\chi(H_0) = g(\mathcal{V}_p)\chi(H_0)$. Thus, we can replace \mathcal{V} by \mathcal{V}_p in (II.6.2). We compute the time derivative of this expression. We obtain

$$\begin{aligned} \frac{d}{dt}U(0, t)g(\mathcal{V}_p)\chi(H_0)U(t, 0)\psi &= U(0, t)i[V(t, x), g(\mathcal{V}_p)]\chi(H_0)U(t, 0)\psi \\ &\quad + U(0, t)g(\mathcal{V}_p)i[V(t, x), \chi(H_0)]U(t, 0)\psi. \end{aligned} \quad (\text{II.6.3})$$

Using the Helffer-Sjöstrand formula and (II.6.1), we have

$$i[V(t, x), \chi(H_0)] = -\frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z)(z - H_0)^{-1} \Gamma \cdot \nabla V(t, x)(z - H_0)^{-1} dz \wedge d\bar{z} \in L^1(dt). \quad (\text{II.6.4})$$

Similarly, since $\mathcal{V}_p = \mathcal{V}_\eta(p)$ belongs to $S^0(\mathbb{R}_p^3; (\mathcal{M}_4(\mathbb{C}))^3)$, we have

$$[V(t, x), g(\mathcal{V}_p)] \in L^1(dt). \quad (\text{II.6.5})$$

Hence, (II.6.3) is integrable along the evolution by (II.6.4) and (II.6.5) and the limit (II.6.2) exists using Lemma II.B.1.

◇

Beside these ‘‘asymptotic velocity’’ operators, we also need to establish weak propagation estimates of the same type as in the time-independent case. First, we prove

Proposition II.6.2 *Suppose that $j, g \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } j \cap \text{supp } g = \emptyset$. Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 0$ on a neighbourhood of 0. Then*

$$\int_1^\infty \left\| j\left(\frac{x}{t}\right)g(\mathcal{V})\chi(H_0)U(t, 0)\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{II.6.6})$$

Furthermore, if $J \in C_0^\infty(\mathbb{R}^3)$ such that $J = 1$ on a neighbourhood of $\text{supp } g$, then

$$s - \lim_{t \rightarrow \pm\infty} U(0, t)J\left(\frac{x}{t}\right)g(\mathcal{V})\chi(H_0)U(t, 0) = \mathcal{P}_{g, \chi}. \quad (\text{II.6.7})$$

Proof : Since the intersection of the supports of j and g is empty, by a covering argument, we may assume that there exists $v \in \mathbb{R}^3$, $v \neq 0$ such that

$$\text{supp } g \subset \{x : \langle v, x \rangle > \theta_2\}, \quad (\text{II.6.8})$$

$$\text{supp } j \subset \{x : \langle v, x \rangle < \theta_1\}, \quad (\text{II.6.9})$$

where $0 < \theta_1 < \theta_2$. Let us choose a function $\tilde{J} \in C^\infty(\mathbb{R})$ such that $\tilde{J}' \in C_0^\infty(\mathbb{R})$, $\tilde{J}'(x) = 0$ when $x \geq \theta_1$ and

$$\tilde{J}'(\langle v, x \rangle) \geq j^2(x). \quad (\text{II.6.10})$$

Now we define $J(x) := \tilde{J}(\langle v, x \rangle)$. We consider the propagation observable

$$\phi(t) = \chi(H_0)g(\mathcal{V})J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V})\chi(H_0).$$

If we choose the function η in the definition of \mathcal{V}_p and X_{nw} such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$, then we have

$$\phi(t) = \chi(H_0)g(\mathcal{V}_p)J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V}_p)\chi(H_0).$$

We also call the ‘‘Heisenberg derivative’’ of $\phi(t)$ the following operator $\mathbf{D}\phi(t) = \frac{d}{dt}\phi(t) + i[H(t), \phi(t)]$. Then we get

$$\begin{aligned} \mathbf{D}\phi(t) &= i[V(t, x), \chi(H_0)]g(\mathcal{V}_p)J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V}_p)\chi(H_0) + hc \\ &\quad + \chi(H_0)i[V(t, x), g(\mathcal{V}_p)]J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V}_p)\chi(H_0) + hc \\ &\quad - \frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\frac{X_{nw}}{t}\nabla J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V}_p)\chi(H_0) \\ &\quad + \chi(H_0)g(\mathcal{V}_p)i[H(t), J\left(\frac{X_{nw}}{t}\right)]g(\mathcal{V}_p)\chi(H_0). \end{aligned} \quad (\text{II.6.11})$$

Clearly, the first two terms in (II.6.11) are integrable in norm by (II.6.4) and (II.6.5). Using the same proof as in Lemma II.4.2, we also have

$$i[H(t), J\left(\frac{X_{nw}}{t}\right)] = \frac{1}{t}\nabla J\left(\frac{X_{nw}}{t}\right) \cdot \mathcal{V}_p + L^1(dt). \quad (\text{II.6.12})$$

Eventually, we get

$$\mathbf{D}\phi(t) = \frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\nabla J\left(\frac{X_{nw}}{t}\right) \cdot (\mathcal{V}_p - \frac{X_{nw}}{t})g(\mathcal{V}_p)\chi(H_0) + L^1(dt).$$

Using Lemma II.4.1, we can replace $J\left(\frac{X_{nw}}{t}\right)$ by $J\left(\frac{x}{t}\right)$ in the above expression and we get

$$\mathbf{D}\phi(t) = \frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\nabla J\left(\frac{x}{t}\right) \cdot (\mathcal{V}_p - \frac{x}{t})g(\mathcal{V}_p)\chi(H_0) + L^1(dt).$$

Now, we claim that there exists a constant C_0 strictly positive such that

$$\frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\nabla J\left(\frac{x}{t}\right) \cdot (\mathcal{V}_p - \frac{x}{t})g(\mathcal{V}_p)\chi(H_0) \geq \frac{C_0}{t}\chi(H_0)g(\mathcal{V}_p)j^2\left(\frac{x}{t}\right)g(\mathcal{V}_p)\chi(H_0). \quad (\text{II.6.13})$$

To see this, we use that $[\mathcal{V}_p, f\left(\frac{x}{t}\right)] \in O(t^{-1})$ to rewrite (II.6.13) as

$$\begin{aligned} \frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\nabla J\left(\frac{x}{t}\right) \cdot (\mathcal{V}_p - \frac{x}{t})g(\mathcal{V}_p)\chi(H_0) &= \frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\bar{J}'(\langle v, \frac{x}{t} \rangle)\langle v, \mathcal{V}_p - \frac{x}{t} \rangle g(\mathcal{V}_p)\chi(H_0), \\ &\geq \frac{1}{t}\chi(H_0)(\bar{J}'(\langle v, \frac{x}{t} \rangle))^{\frac{1}{2}}g(\mathcal{V}_p)\langle v, \mathcal{V}_p \rangle g(\mathcal{V}_p)(\bar{J}'(\langle v, \frac{x}{t} \rangle))^{\frac{1}{2}}\chi(H_0) \\ &\quad - \frac{1}{t}\chi(H_0)g(\mathcal{V}_p)(\bar{J}'(\langle v, \frac{x}{t} \rangle))^{\frac{1}{2}}\langle v, \frac{x}{t} \rangle (\bar{J}'(\langle v, \frac{x}{t} \rangle))^{\frac{1}{2}}g(\mathcal{V}_p)\chi(H_0) + L^1(dt). \end{aligned}$$

Using (II.6.8), (II.6.9) and (II.6.10), we obtain

$$\frac{1}{t}\chi(H_0)g(\mathcal{V}_p)\nabla J\left(\frac{x}{t}\right) \cdot (\mathcal{V}_p - \frac{x}{t})g(\mathcal{V}_p)\chi(H_0) \geq \frac{1}{t}(\theta_2 - \theta_1)\chi(H_0)g(\mathcal{V}_p)j^2\left(\frac{x}{t}\right)g(\mathcal{V}_p)\chi(H_0).$$

and the constant $C_0 = \theta_2 - \theta_1$ is strictly positive. Eventually, we have

$$\mathbf{D}\phi(t) \geq \frac{C_0}{t}g(\mathcal{V})j^2\left(\frac{x}{t}\right)g(\mathcal{V}) + L^1(dt).$$

We conclude the proof of (II.6.6) using Proposition II.B.1.

We now prove (II.6.7). We assume without loss of generality that $g = g_1^2$. Let us consider a function $J \in C^\infty(\mathbb{R}^3)$ such that $\nabla J \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } J \cap \text{supp } g_1 = \emptyset$. We also introduce a function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$. We have $g(\mathcal{V})\chi(H_0) = g(\mathcal{V}_p)\chi(H_0)$. Now, using Proposition II.6.1, it is enough to prove that

$$s - \lim_{t \rightarrow +\infty} U(0, t) J\left(\frac{x}{t}\right) g_1^2(\mathcal{V}_p) \chi^2(H_0) U(t, 0) = 0.$$

After some commutations, this is equivalent to proving that

$$s - \lim_{t \rightarrow \infty} U(0, t) \phi(t) U(t, 0) = 0, \quad (\text{II.6.14})$$

where $\phi(t) = \chi(H_0) g_1(\mathcal{V}_p) J\left(\frac{x}{t}\right) g_1(\mathcal{V}_p) \chi(H_0)$. We compute its Heisenberg derivative.

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t} \chi(H_0) g_1(\mathcal{V}_p) \nabla J\left(\frac{x}{t}\right) \cdot \left(\Gamma - \frac{x}{t}\right) g_1(\mathcal{V}_p) \chi(H_0) \\ &\quad + i[V(t, x), \chi(H_0)] g_1(\mathcal{V}_p) J\left(\frac{x}{t}\right) g_1(\mathcal{V}_p) \chi(H_0) + hc \\ &\quad + \chi(H_0) i[V(t, x), g_1(\mathcal{V}_p)] J\left(\frac{x}{t}\right) g_1(\mathcal{V}_p) \chi(H_0) + hc. \end{aligned}$$

Using (II.6.4), (II.6.5), we have

$$\mathbf{D}\phi(t) = \frac{1}{t} \chi(H_0) g_1(\mathcal{V}_p) \tilde{j}\left(\frac{x}{t}\right) B(t) \tilde{j}\left(\frac{x}{t}\right) g_1(\mathcal{V}_p) \chi(H_0) + L^1(dt),$$

where $B(t)$ is uniformly bounded and $\tilde{j} \in C_0^\infty(\mathbb{R}^3)$ satisfies $\text{supp } \tilde{j} \cap \text{supp } g_1 = \emptyset$ and $\tilde{j} \nabla J = \nabla J$. This expression is integrable along the evolution by (II.6.6) and thus the limit in (II.6.14) exists by Lemma II.B.1. Now, assume furthermore that the function J is compactly supported i.e. $J \in C_0^\infty(\mathbb{R}^3)$. Then, since $\text{supp } J \cap \text{supp } g = \emptyset$ by hypothesis, using (II.6.6) again, the limit in (II.6.14) is equal to 0.

To conclude the proof, we need to show that (II.6.14) remains true for general J , not compactly supported. But this follows by the same limiting procedure used in [14]. We omit the details.

◇

Let us now give a time-dependent version of the microlocal velocity estimate for high energies.

Proposition II.6.3 *Suppose that $g, J \in C_0^\infty(\mathbb{R}^3)$ and $\text{supp } g \cap \text{supp } \nabla J = \emptyset$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Then*

$$\int_1^\infty \left\| \left(\frac{x}{t} - \mathcal{V}\right) J\left(\frac{x}{t}\right) g(\mathcal{V}) \chi(H_0) U(t, 0) \psi \right\|^2 \frac{dt}{t} < C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{II.6.15})$$

Moreover,

$$s - \lim_{t \rightarrow \infty} \left(\frac{x}{t} - \mathcal{V}\right) J\left(\frac{x}{t}\right) g(\mathcal{V}) \chi(H_0) U(t, 0) = 0. \quad (\text{II.6.16})$$

Proof : Let us define the uniformly bounded propagation observable

$$\phi(t) = \chi(H_0) g(\mathcal{V}) J\left(\frac{x}{t}\right) \left(\frac{x}{t} - \mathcal{V}\right)^2 J\left(\frac{x}{t}\right) g(\mathcal{V}) \chi(H_0).$$

We choose a function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$ and we define

$$\begin{aligned}\phi_m(t) &= \chi(H_0)g(\mathcal{V}_p)J\left(\frac{X_{nw}}{t}\right)\left(\frac{X_{nw}}{t} - \mathcal{V}_p\right)^2 J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V}_p)\chi(H_0), \\ &= \chi(H_0)g(\mathcal{V}_p)J\left(\frac{X_{nw}}{t}\right)\left(\frac{X_{nw}^2}{t^2} - 2\frac{\mathcal{A}}{t} + \mathcal{V}_p^2\right)J\left(\frac{X_{nw}}{t}\right)g(\mathcal{V}_p)\chi(H_0).\end{aligned}$$

Using Lemma II.4.1 and $\chi(H_0)g(\mathcal{V}) = \chi(H_0)g(\mathcal{V}_p)$, it is immediate that

$$\phi_m(t) = \phi(t) + O(t^{-\min(1,\rho)}). \quad (\text{II.6.17})$$

Now, thanks to (II.6.4), (II.6.5) and (II.6.12), the Heisenberg derivative of $\phi_m(t)$ is

$$\begin{aligned}\mathbf{D}\phi_m(t) &= \frac{2}{t}\chi(H_0)g(\mathcal{V}_p)J\left(\frac{X_{nw}}{t}\right)\left(\frac{X_{nw}}{t} - \mathcal{V}_p\right)^2 J\left(\frac{x}{t}\right)g(\mathcal{V}_p)\chi(H_0) + L^1(dt), \\ &= \frac{2}{t}\phi_m(t) + L^1(dt), \\ &= \frac{2}{t}\phi(t) + L^1(dt),\end{aligned} \quad (\text{II.6.18})$$

where, for the last equality, we use (II.6.17). Thus, (II.6.15) holds by Proposition II.B.1.

In order to prove (II.6.16), it is enough to show that

$$s - \lim_{t \rightarrow \infty} U(0, t)\phi(t)U(t, 0) = 0.$$

First, observe that, from (II.6.18) and Lemma II.B.1, the limit

$$s - \lim_{t \rightarrow \infty} U(0, t)\phi_m(t)U(t, 0),$$

exists. Therefore, by (II.6.17), the limit

$$s - \lim_{t \rightarrow \infty} U(0, t)\phi(t)U(t, 0), \quad (\text{II.6.19})$$

also exists. Now, noting that $\phi(t) \geq 0$, we have

$$0 \leq \int_1^\infty (\psi, U(0, t)\phi(t)U(t, 0)\psi) \frac{dt}{t} < \infty,$$

using (II.6.15) again. Therefore the limit (II.6.19) is zero which concludes the proof of the proposition.

◇

We end this section with the proof of the existence and asymptotic completeness of the Dollard modified wave operators with energy cut-off for time-dependent massless Dirac Hamiltonians. From now on, we assume that the potential $V(t, x)$ satisfies (II.6.1) with $\rho > \frac{1}{2}$. We define the Dollard modified dynamics (see [26]) by

$$U_0(t) = e^{-itH_0}T\left(e^{-i\int_0^t V(s, \mathcal{V})ds}\right),$$

where T denotes time-ordering. The main theorem of this subsection is

Theorem II.6.1 *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Then the following limits*

$$W_\chi^\pm = s - \lim_{t \rightarrow \pm\infty} U(0, t)\chi(H_0)U_0(t), \quad (\text{II.6.20})$$

$$\widetilde{W}_\chi^\pm = s - \lim_{t \rightarrow \pm\infty} U_0^*(t)\chi(H_0)U(t, 0), \quad (\text{II.6.21})$$

exist.

Proof : As the proofs are identical, we only prove (II.6.21). We want to use Cook's method (Lemma II.B.1). Let $\psi \in D(\langle x \rangle)$. We compute

$$\begin{aligned} \frac{d}{dt}(U_0^*(t)\chi(H_0)U(t, 0)\psi) &= U_0^*(t)i(V(t, t\mathcal{V})\chi(H_0) - \chi(H_0)V(t, x))U(t, 0)\psi, \\ &= U_0^*(t)i(V(t, t\mathcal{V}) - V(t, x))\chi(H_0)U(t, 0)\psi + L^1(dt), \end{aligned}$$

where we used (II.6.4) in the second equality. Now, we use the Helffer-Sjöstrand formula (II.A.12) to estimate this last quantity. We have

$$V(t, t\mathcal{V}) - V(t, x) = C(t) \cdot (t\mathcal{V} - x) + B(t),$$

with $B(t), C(t) \in O(t^{-\rho-1})$. Since $\rho > \frac{1}{2}$, it is enough to show that

$$(t\mathcal{V} - x)\chi(H_0)U(t, 0)\psi \in O(t^{\frac{1}{2}}), \quad (\text{II.6.22})$$

to prove (II.6.21). By the usual argument, we choose a function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$ and thus, (II.6.22) is equivalent to proving

$$(t\mathcal{V}_p - X_{nw})\chi(H_0)U(t, 0)\psi \in O(t^{\frac{1}{2}}). \quad (\text{II.6.23})$$

Since $\psi \in D(\langle x \rangle)$ and $\rho > \frac{1}{2}$, it is enough to show that

$$L(t) := \frac{1}{t^{\frac{1}{2}}} \frac{d}{dt} \left(U(0, t)(t\mathcal{V}_p - X_{nw})\chi(H_0)U(t, 0)\langle x \rangle^{-1} \right) \in L^1(dt), \quad (\text{II.6.24})$$

in order to prove (II.6.23). Using (II.6.4) and (II.6.5), we have

$$L(t) = \frac{1}{t^{\frac{1}{2}}} U(0, t) \left(\mathcal{V}_p - [H(t), X_{nw}] \right) \chi(H_0) U(t, 0) \langle x \rangle^{-1} + L^1(dt). \quad (\text{II.6.25})$$

By (II.6.1) and the same proof as in Lemma II.4.2, we have $i[H(t), X_{nw}] = \mathcal{V}_p + O(t^{-\rho})$. Therefore, we finally obtain

$$L(t) \in L^1(dt),$$

which concludes the proof of (II.6.21) and the proof of the Theorem.

◇

II.6.2 Dollard modified wave operators

Theorem II.6.2 *The Dollard modified wave operators defined by*

$$\Omega^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH} U_0(t), \quad (\text{II.6.26})$$

$$\tilde{\Omega}^\pm := s - \lim_{t \rightarrow \pm\infty} U_0^*(t) e^{-itH} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm), \quad (\text{II.6.27})$$

exist on \mathcal{H} . Furthermore, we have $\tilde{\Omega}^\pm \Omega^\pm = \mathbf{1}_{\mathcal{H}}$, $\Omega^\pm \tilde{\Omega}^\pm = \mathbf{1}^c(H)$, $(\Omega^\pm)^* = \tilde{\Omega}^\pm$ and $\pm\mathcal{V}$ and P^\pm satisfy the intertwining relation

$$(\Omega^\pm)(\pm\mathcal{V})(\Omega^\pm)^* = P^\pm \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^\pm), \quad (\text{II.6.28})$$

and

$$(\Omega^\pm)H_0(\Omega^\pm)^* = H. \quad (\text{II.6.29})$$

Proof : We transform the time-independent problem into a time-dependent one to which we apply the previous results. First consider a function $g \in C_0^\infty(\mathbb{R}^3)$ such that $0 \notin \text{supp } g$ and $S^2 \subset \text{supp } g$. Therefore we have $g(\mathcal{V}) = \mathbf{1}_{\mathcal{H}}$ and $g(P^+) = \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P^+)$ by Theorem II.5.2. Now, let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Using $\mathbf{1}_{\{0\}}(P^+) = \mathbf{1}^{pp}(H)$ (Theorem II.5.1) and a density argument, the existence of the limits (II.6.26) and (II.6.27) is equivalent to that of the limits

$$s - \lim_{t \rightarrow +\infty} e^{itH} U_0(t) g^2(\mathcal{V}) \chi^3(H_0), \quad (\text{II.6.30})$$

$$s - \lim_{t \rightarrow +\infty} U_0^*(t) e^{-itH} g^2(P^+) \chi^3(H), \quad (\text{II.6.31})$$

Let us introduce a function $J \in C_0^\infty(\mathbb{R}^3)$ such that $J = 0$ on a neighbourhood of 0 and $J = 1$ on a neighbourhood of $\text{supp } g$, denoted Θ . We associate to the potential $V(x)$ the time-dependent potential $V_J(t, x)$ defined by

$$V_J(t, x) = V(x) J\left(\frac{x}{t}\right).$$

This potential satisfies the properties

- For any y in Θ ,

$$V(ty) = V_J(t, ty) \quad (\text{II.6.32})$$

- For any $t \geq 1$ fixed, $x \rightarrow V_J(t, x) \in C_0^\infty(\mathbb{R}^3)$ and there is a constant M such that

$$\text{supp } V_J(t, \cdot) \subset B(0, Mt).$$

- The following estimates hold

$$|\partial_t^k V_J(t, x)| \leq C \langle t \rangle^{-\rho-k}, \quad \forall k \in \mathbb{N}, \quad \forall x \in \mathbb{R}^3, \quad (\text{II.6.33})$$

$$|\partial_x^\alpha V_J(t, x)| \leq C \langle t \rangle^{-\rho-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^3, \quad \forall x \in \mathbb{R}^3. \quad (\text{II.6.34})$$

From (II.6.32), we call the potential $V_J(t, x)$ *effective time-dependent potential* on $\text{supp } g$. This property allows us to make the link with the time-dependent results of the previous subsection.

We introduce the time-dependent massless Dirac Hamiltonian

$$H_J(t) = H_0 + V_J(t, x),$$

and we denote $U_J(t, s)$ the unitary dynamics associated to $H_J(t)$. Since the potential $V_J(t, x)$ satisfies the property (II.6.1), the results of the previous subsection hold. In particular, to any function $f \in C_\infty(\mathbb{R}^3)$, we define the ‘‘asymptotic velocity’’ operators associated to $H_J(t)$, i.e. the following limit

$$\mathcal{P}_{f, \chi} = s - \lim_{t \rightarrow \pm\infty} U_J(0, t) f(\pm \mathcal{V}) \chi(H_0) U_J(t, 0),$$

exists. Eventually, we introduce the time-dependent Dollard dynamics

$$U_{0, J}(t) = e^{-itH_0} T \left(e^{-i \int_0^t V_J(s, s\mathcal{V}) ds} \right),$$

where T denotes time ordering.

Let us now begin the proof of (II.6.30) and (II.6.31). First, using the unitarity of $U_J(t, s)$, we have

$$(II.6.30) = s - \lim_{t \rightarrow +\infty} e^{itH} U_J(t, 0) U_J(0, t) g(\mathcal{V}) \chi^2(H_0) U_J(t, 0) U_J(0, t) \chi(H_0) U_0(t) g(\mathcal{V}),$$

since $g(\mathcal{V})$ and $\chi(H_0)$ commute with $U_0(t)$. But, observe that $U_0(t)g(\mathcal{V}) = U_{0, J}(t)g(\mathcal{V})$ by property (II.6.32). Hence, we obtain

$$(II.6.30) = s - \lim_{t \rightarrow +\infty} \left(e^{itH} U_J(t, 0) \right) \left(U_J(0, t) g(\mathcal{V}) \chi^2(H_0) U_J(t, 0) \right) \left(U_J(0, t) \chi(H_0) U_{0, J}(t) g(\mathcal{V}) \right). \quad (II.6.35)$$

By Proposition II.6.1 and Theorem II.6.1, we already know that the limits

$$\begin{aligned} & s - \lim_{t \rightarrow +\infty} U_J(0, t) \chi(H_0) U_{0, J}(t), \\ & s - \lim_{t \rightarrow +\infty} U_J(0, t) g(\mathcal{V}) \chi^2(H_0) U_J(t, 0) = \mathcal{P}_{g, \chi^2}^+, \end{aligned}$$

exist. Therefore, by the chain rule, it is enough to prove the existence of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} U_J(t, 0) \mathcal{P}_{g, \chi^2}^+, \quad (II.6.36)$$

in order to prove (II.6.30).

The proof of (II.6.31) is almost identical to that of (II.6.30). Indeed, using the characterization (II.5.3) of the asymptotic velocity P^+ , we have

$$\begin{aligned} g^2(P^+) \chi^3(H) &= s - \lim_{t \rightarrow +\infty} e^{itH} g(\mathcal{V}) \chi(H_0) e^{-itH} g(P^+) \chi^2(H) \\ &\quad + s - \lim_{t \rightarrow +\infty} e^{itH} g(\mathcal{V}) (\chi(H) - \chi(H_0)) e^{-itH} g(P^+) \chi^2(H). \end{aligned}$$

The second term can be written as

$$s - \lim_{t \rightarrow +\infty} e^{itH} g(\mathcal{V}) (\chi(H) - \chi(H_0)) g\left(\frac{x}{t}\right) e^{-itH} \chi^2(H),$$

and thus vanishes since $(\chi(H) - \chi(H_0))g\left(\frac{x}{t}\right) \in O(t^{-\rho})$. Therefore we obtain

$$(II.6.31) = s - \lim_{t \rightarrow +\infty} \left(g(\mathcal{V}) U_{0, J}^*(t) \chi(H_0) U_J(t, 0) \right) \left(U_J(0, t) e^{-itH} g(P^+) \chi^2(H) \right),$$

where we used the fact that $g(\mathcal{V}) U_0^*(t) = g(\mathcal{V}) U_{0, J}^*(t)$ by (II.6.32). But we know that the limit

$$s - \lim_{t \rightarrow +\infty} U_{0, J}^*(t) \chi(H_0) U_J(t, 0),$$

exists by Theorem II.6.1. Hence, using the chain rule, it suffices to prove the existence of the limit

$$s - \lim_{t \rightarrow +\infty} U_J(0, t) e^{-itH} g(P^+) \chi^2(H), \quad (\text{II.6.37})$$

in order to prove (II.6.31).

Since the proofs of (II.6.36) and (II.6.37) are almost identical, we only show the latter. Let us introduce two functions $g_1, g_2 \in C_0^\infty(\Theta)$ such that $g_1 g = g$ and $g_2 g_1 = g_1$. By Theorem II.5.1, we have

$$\begin{aligned} s - \lim_{t \rightarrow +\infty} U_J(0, t) e^{-itH} g(P^+) \chi^2(H) &= s - \lim_{t \rightarrow +\infty} U_J(0, t) e^{-itH} g(P^+) g_1(P^+) \chi^2(H), \\ &= s - \lim_{t \rightarrow +\infty} U_J(0, t) \chi(H_0) g(\mathcal{V}) g_2\left(\frac{x}{t}\right) g_1\left(\frac{x}{t}\right) g_2\left(\frac{x}{t}\right) \chi(H) e^{-itH}. \end{aligned} \quad (\text{II.6.38})$$

Let us define the uniformly bounded propagation observable

$$\phi(t) = \chi(H_0) g(\mathcal{V}) g_2\left(\frac{x}{t}\right) \left(g_1\left(\frac{x}{t}\right) + \left(\mathcal{V} - \frac{x}{t}\right) \cdot \nabla g_1\left(\frac{x}{t}\right) \right) g_2\left(\frac{x}{t}\right) \chi(H).$$

By Proposition II.4.4, we have

$$(\text{II.6.38}) = s - \lim_{t \rightarrow +\infty} U_J(0, t) \phi(t) e^{-itH}.$$

For technical reasons, we introduce the modified propagation observable

$$\begin{aligned} \phi_m(t) &= \chi(H_0) g(\mathcal{V}_p) g_2\left(\frac{x}{t}\right) \left(g_1\left(\frac{X_{nw}}{t}\right) + \left(\mathcal{V}_p - \frac{X_{nw}}{t}\right) \cdot \nabla g_1\left(\frac{X_{nw}}{t}\right) \right) g_2\left(\frac{x}{t}\right) \chi(H), \\ &= g(\mathcal{V}_p) g_2\left(\frac{x}{t}\right) M(t) g_2\left(\frac{x}{t}\right) \chi(H), \end{aligned}$$

where the function η in the definition of X_{nw} and \mathcal{V}_p is chosen such that $\text{supp } \eta^c \cap \text{supp } \chi = \emptyset$. Obviously $\phi_m(t)$ is also uniformly bounded in t . By Lemma II.4.1 and (II.4.24), we have $\phi_m(t) = \phi(t) + O(t^{-\rho})$ and therefore, it is enough to prove the existence of the limit

$$s - \lim_{t \rightarrow +\infty} U_J(0, t) \phi_m(t) e^{-itH}, \quad (\text{II.6.39})$$

in order to prove (II.6.31). Let us compute the time derivative of the function in (II.6.39). We obtain

$$\begin{aligned} \frac{d}{dt} U_J(0, t) \phi_m(t) e^{-itH} &= U_J(0, t) \left(\frac{d}{dt} \phi_m(t) + i(H_J(t) \phi_m(t) - \phi_m(t) H) \right) e^{-itH}, \\ &= -\frac{1}{t} U_J(0, t) \chi(H_0) g(\mathcal{V}_p) \left(\nabla g_2\left(\frac{x}{t}\right) M(t) g_2\left(\frac{x}{t}\right) + hc \right) \chi(H) e^{-itH} \\ &\quad - \frac{1}{t} U_J(0, t) \chi(H_0) g(\mathcal{V}_p) g_2\left(\frac{x}{t}\right) \left\langle \left(\mathcal{V}_p - \frac{X_{nw}}{t}\right), \nabla^2 g_1\left(\frac{X_{nw}}{t}\right) \cdot \frac{X_{nw}}{t} \right\rangle g_2\left(\frac{x}{t}\right) \chi(H) e^{-itH} \\ &\quad + U_J(0, t) i[H_J(t), \chi(H_0) g(\mathcal{V}_p)] g_2\left(\frac{x}{t}\right) M(t) g_2\left(\frac{x}{t}\right) \chi(H) e^{-itH} \\ &\quad + \frac{1}{t} U_J(0, t) \chi(H_0) g(\mathcal{V}_p) \left(\Gamma \cdot \nabla g_2\left(\frac{x}{t}\right) M(t) g_2\left(\frac{x}{t}\right) + hc \right) \chi(H) e^{-itH} \\ &\quad + U_J(0, t) \chi(H_0) g(\mathcal{V}_p) g_2\left(\frac{x}{t}\right) i(H_J(t) M(t) - M(t) H) g_2\left(\frac{x}{t}\right) \chi(H) e^{-itH}, \\ &= L_1 + L_2 + L_3 + L_4 + L_5. \end{aligned}$$

Since $[H_J(t), \chi(H_0) g(\mathcal{V}_p)] = [V_J(t, x), \chi(H_0)] g(\mathcal{V}_p) + \chi(H_0) [V_J(t, x), g(\mathcal{V}_p)]$, the term L_3 belongs to $L^1(dt)$ by (II.6.4) and (II.6.5).

The terms L_1 and L_4 can be written after some commutations as

$$U_J(0, t)\chi(H_0)g(\mathcal{V})\tilde{g}\left(\frac{x}{t}\right)B(t)\tilde{g}\left(\frac{x}{t}\right)\chi(H)e^{-itH} + L^1(dt),$$

where $B(t)$ is an operator-valued function uniformly bounded in t and the function $\tilde{g} \in C_0^\infty(\Theta)$ satisfies the three conditions : 1) $\text{supp } g \cap \text{supp } \tilde{g} = \emptyset$, 2) $\tilde{g}\nabla g_2\left(\frac{x}{t}\right) = \nabla g_2\left(\frac{x}{t}\right)$, 3) $\text{supp } \tilde{g} \cap S^2 = \emptyset$. This last condition is reasonable since, by definition of g_2 , one has $\nabla g_2 = 0$ on a neighbourhood of S^2 . Hence, the terms L_1 and L_4 are integrable along the evolution by Propositions II.4.1, II.4.3 and II.6.2.

Using Lemma II.4.2, we see that the term L_5 is equal to

$$\begin{aligned} L_5 &= \frac{1}{t}U_J(0, t)\chi(H_0)g(\mathcal{V}_p)g_2\left(\frac{x}{t}\right)\left\langle\left(\mathcal{V}_p - \frac{X_{nw}}{t}\right), \nabla^2 g_1\left(\frac{X_{nw}}{t}\right) \cdot \mathcal{V}_p\right\rangle g_2\left(\frac{x}{t}\right)\chi(H)e^{-itH} \\ &\quad + U_J(0, t)\chi(H_0)g(\mathcal{V}_p)g_2\left(\frac{x}{t}\right)i[V_J(t, x)M(t) - M(t)V(x)]g_2\left(\frac{x}{t}\right)\chi(H)e^{-itH}. \end{aligned} \quad (\text{II.6.40})$$

But, since $\text{supp } g_2 \subset \Theta$, we have $V(x)g_2\left(\frac{x}{t}\right) = V_J(t, x)g_2\left(\frac{x}{t}\right)$ by (II.6.32). Thus, the second term in (II.6.40) is equal to

$$U_J(0, t)\chi(H_0)g(\mathcal{V}_p)g_2\left(\frac{x}{t}\right)i[V_J(t, x), M(t)]g_2\left(\frac{x}{t}\right)\chi(H)e^{-itH}.$$

Since we have $[V_J(t, x), \mathcal{V}_p] \in L^1(dt)$ by the same proof as (II.6.5) and since $[V_J(t, x), f\left(\frac{X_{nw}}{t}\right)] \in L^1(dt)$ for any $f \in C_0^\infty(\mathbb{R}^3)$, by the same proof as in Lemma II.4.2, the term L_5 is integrable along the evolution.

Eventually, if we put together these results, we obtain

$$\begin{aligned} \frac{d}{dt}U_J(0, t)\phi_m(t)e^{-itH} &= \\ \frac{1}{t}U_J(0, t)\chi(H_0)g(\mathcal{V}_p)g_2\left(\frac{x}{t}\right)\left\langle\left(\mathcal{V}_p - \frac{X_{nw}}{t}\right), \nabla^2 g_1\left(\frac{X_{nw}}{t}\right) \cdot \left(\mathcal{V}_p - \frac{X_{nw}}{t}\right)\right\rangle g_2\left(\frac{x}{t}\right)\chi(H)e^{-itH} &+ L^1(dt). \end{aligned}$$

By the usual argument, we can replace X_{nw} by x and \mathcal{V}_p by \mathcal{V} in the equation above. We conclude that the time derivative $\frac{d}{dt}U_J(0, t)\phi_m(t)e^{-itH}$ belongs to $L^1(dt)$ by Propositions II.4.4 and II.6.3. Hence, the limit (II.6.39) exists by Lemma II.B.1.

Therefore, we have constructed the Dollard-modified wave operators Ω^\pm and $\tilde{\Omega}^\pm$. The fact that $\tilde{\Omega}^\pm\Omega^\pm = \mathbf{1}_{\mathcal{H}}$, $\Omega^\pm\tilde{\Omega}^\pm = \mathbf{1}^c(H)$, $(\Omega^\pm)^* = \tilde{\Omega}^\pm$ follows from Lemma B.5.1 in [21]. It remains to prove the intertwining relations (II.6.28) and (II.6.29). Using (II.5.3), we see that

$$P^+ = s - \lim_{t \rightarrow +\infty} e^{itH}\mathcal{V}e^{-itH} = s - \lim_{t \rightarrow +\infty} e^{itH}U_0(t)\mathcal{V}U_0^*(t)e^{-itH} = \Omega^+\mathcal{V}(\Omega^+)^*,$$

which proves (II.6.28). Now, (II.6.29) is equivalent to

$$s - \lim_{t \rightarrow +\infty} e^{itH}(H - H_0)e^{-itH}g(P^+) = 0.$$

But, using Theorem II.5.1, we have to show that

$$s - \lim_{t \rightarrow +\infty} e^{itH}V(x)g\left(\frac{x}{t}\right)e^{-itH} = 0.$$

Since $V \in S^{-\rho}(\mathbb{R}^3)$, $V(x)g\left(\frac{x}{t}\right) \in O(t^{-\rho})$ and the result holds.

◇

APPENDIX

II.A Helffer-Sjöstrand formula

In this appendix, we give a brief review of the Helffer-Sjöstrand formula introduced in [48]. Let us consider a function f belonging to the function space

$$S^{-\rho}(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3) : |\partial_x^\alpha f(x)| \leq C\langle x \rangle^{-\rho-|\alpha|}, \forall \alpha \in \mathbb{N}^3\}, \quad \rho > 0.$$

Let $\tau \in C_0^\infty(\mathbb{R})$ satisfying $\tau(x) = 1$ for $|x| \leq 1$ and $\tau(x) = 0$ for $|x| \geq 2$. We define the almost-analytic extension of f in the following way

$$\tilde{f}(z_1, z_2, z_3) = \left\{ \sum_{\alpha \leq \beta} \partial_x^\alpha f(x) \frac{(iy_1)^{\alpha_1}}{\alpha_1!} \frac{(iy_2)^{\alpha_2}}{\alpha_2!} \frac{(iy_3)^{\alpha_3}}{\alpha_3!} \right\} \tau\left(\frac{y_1}{\langle x_1 \rangle}\right) \tau\left(\frac{y_2}{\langle x_2 \rangle}\right) \tau\left(\frac{y_3}{\langle x_3 \rangle}\right),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (n_1, n_2, n_3)$ and $n_j \geq 1$, for all $j = 1, 2, 3$. This function satisfies the properties.

$$\bullet \quad \text{supp } \tilde{f} \subset \{(x, y) \in \mathbb{R}^6 / |y_j| \leq C\langle x_j \rangle, j = 1, 2, 3\}, \quad (\text{II.A.1})$$

$$\bullet \quad |\partial_{\bar{z}} \tilde{f}(z)| \leq C\langle x \rangle^{-\rho-3-|\alpha|} |y_1|^{n_1} |y_2|^{n_2} |y_3|^{n_3}, \quad (\text{II.A.2})$$

$$\bullet \quad f(x) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) (z_1 - x_1)^{-1} (z_2 - x_2)^{-1} (z_3 - x_3)^{-1} dz \wedge d\bar{z}. \quad (\text{II.A.3})$$

Here, we have denoted $z = (z_1, z_2, z_3) = (x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)$. Moreover, $\partial_{\bar{z}}$ stands for $\partial_{\bar{z}_1} \partial_{\bar{z}_2} \partial_{\bar{z}_3}$ and $dz \wedge d\bar{z}$ for $(dz_1 \wedge d\bar{z}_1) \wedge (dz_2 \wedge d\bar{z}_2) \wedge (dz_3 \wedge d\bar{z}_3)$. Eventually, if A denotes a vector of commuting self-adjoint operators on \mathcal{H} , we can define $f(A)$ by

$$f(A) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(z) (z_1 - A_1)^{-1} (z_2 - A_2)^{-1} (z_3 - A_3)^{-1} dz \wedge d\bar{z}. \quad (\text{II.A.4})$$

Note that, since we assumed $f \in S^{-\rho}(\mathbb{R}^3)$ with $\rho > 0$, the integral in (II.A.4) converges in operator norm. Nevertheless, the formula (II.A.4) remains true when we consider functions $f \in S^{-\rho}(\mathbb{R}^3)$ with $\rho \leq 0$. In this case, the integral only converges strongly.

We need a time dependent version of the Helffer-Sjöstrand formula. Let us consider a function $J \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$. Define the time-dependent function $f(t, x) = f(x)J(\frac{x}{t})$. The almost-analytic extension is given by

$$\tilde{f}(t, z_1, z_2, z_3) = \left\{ \sum_{\alpha} \partial_x^\alpha f(t, x) \frac{(iy_1)^{\alpha_1}}{\alpha_1!} \frac{(iy_2)^{\alpha_2}}{\alpha_2!} \frac{(iy_3)^{\alpha_3}}{\alpha_3!} \right\} \tau\left(\frac{y_1}{\langle x_1 \rangle}\right) \tau\left(\frac{y_2}{\langle x_2 \rangle}\right) \tau\left(\frac{y_3}{\langle x_3 \rangle}\right),$$

and satisfies the properties

$$\bullet \quad \text{supp } \tilde{f} \subset [-Mt, Mt]_{z_1}^2 \times [-Mt, Mt]_{z_2}^2 \times [-Mt, Mt]_{z_3}^2, \quad (\text{II.A.5})$$

$$\bullet \quad |\partial_{\bar{z}} \tilde{f}(t, z)| \leq Ct^{-\rho-3-|\alpha|} |y_1|^{\alpha_1} |y_2|^{\alpha_2} |y_3|^{\alpha_3}, \quad (\text{II.A.6})$$

$$\bullet \quad f(t, x) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(t, z) (z_1 - x_1)^{-1} (z_2 - x_2)^{-1} (z_3 - x_3)^{-1} dz \wedge d\bar{z}. \quad (\text{II.A.7})$$

Given A a vector of commuting selfadjoint operators on \mathcal{H} , we define

$$f(t, A) = \frac{-i}{(2\pi)^3} \int_{\mathbb{C}^3} \partial_{\bar{z}} \tilde{f}(t, z) (z_1 - A_1)^{-1} (z_2 - A_2)^{-1} (z_3 - A_3)^{-1} dz \wedge d\bar{z}. \quad (\text{II.A.8})$$

Application 1 (Commutator expansion) : Let A a vector of commuting selfadjoint operators on \mathcal{H} and T a selfadjoint operator on \mathcal{H} . Let $f \in S^{-\rho}(\mathbb{R}^3)$ with $\rho \in \mathbb{R}$. We want a formula to express the commutator $[T, f(A)]$ in function of the commutator $[T, A]$.

Assume that

(H1) $A \in C^1(T)$ if T is unbounded.

(H2) $[T, A]$ and $[[T, A], A]$ are bounded on \mathcal{H} .

Then we have

$$[T, f(A)] = \nabla f(A) \cdot [T, A] + B, \quad (\text{II.A.9})$$

where B is bounded on \mathcal{H} if $\rho > -2$. In many cases, we have to deal with functions of time-dependent Hamiltonians of the form $f(\frac{A}{t})$. Therefore, under the same hypotheses on f, T, A , we have

$$[T, f(\frac{A}{t})] = \frac{1}{t} \nabla f(\frac{A}{t}) \cdot [T, A] + O(t^{-2}). \quad (\text{II.A.10})$$

Application 2 : Given two vectors of selfadjoint operators A and T on \mathcal{H} and a function f belonging to the space $S^{-\rho}(\mathbb{R}^3)$, we would like to express the difference between $f(A)$ and $f(T)$ as the product of an operator B and $A - T$. Assume that $A \in C^1(T)$ and $[T, A]$ is bounded on \mathcal{H} , then

$$f(A) - f(T) = B_1 \cdot (A - T) + B_2, \quad (\text{II.A.11})$$

where B_i , $i = 1, 2$ are bounded if $\rho > -1$.

We end this application by giving a time-dependent version useful for Section II.6. Given two self-adjoint operators A and T in a Hilbert space \mathcal{H} and given two functions $f \in S^{-\rho}(\mathbb{R}^3)$ and $J \in C_0^\infty(\mathbb{R}^3)$, we want to express the difference between the functions of operators $f(t, A)$ and $f(t, tT)$ as the product of one operator $B_1(t)$ and $A - tT$. Assume that $A \in C^1(T)$ and $[A, T]$ bounded on \mathcal{H} then, we have

$$f(t, A) - f(t, tT) = C_1(t) \cdot (A - tT) + B_2(t). \quad (\text{II.A.12})$$

where $B_1(t), B_2(t) \in O(t^{-\rho-1})$.

II.B Propagation estimates

In this appendix, we give the fundamental criteria used throughout this paper to prove the weak propagation estimates and the existence of asymptotic observables of sections II.4, II.5 and II.6. We refer to [21], appendix B, for a more detailed presentation.

Let $U(t, s)$ be the unitary evolution generated by a time-dependent Hamiltonian $H(t)$. Assume that there exists a positive invertible operator B such that $U(t, s)$ be B -regularly generated (see [21], Appendix B.3 for these definitions). For simplicity, we write $U(t) = U(t, 0)$. Note that $U(t)$ satisfies

$$\frac{d}{dt} U(t) \psi = -iH(t)U(t)\psi, \quad \psi \in D(B^{\frac{1}{2}}), \quad U(0) = 1.$$

Let $A(t)$ a function with values in selfadjoint operators. We denote $\mathbf{D}A(t)$ the Heisenberg derivative associated with $A(t)$, i.e.

$$\mathbf{D}A(t) = \frac{d}{dt} A(t) + i[H(t), A(t)].$$

We say that $A(t) \in W_{loc}^{1,1}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ if there exists $B(t) \in L_{loc}^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ such that, for any $t_1, t_2 \in \mathbb{R}$,

$$A(t_1) - A(t_2) = \int_{t_1}^{t_2} B(s) ds.$$

Proposition II.B.1 *Suppose that $\Phi(t)$ is a family of selfadjoint operators satisfying the conditions :*

a) $\Phi(t) \in W_{loc}^{1,1}(\mathbb{R}^+, \mathcal{B}(\mathcal{H}))$,

b) $\|B^{\frac{1}{2}}\Phi(t)B^{-\frac{1}{2}}\| \leq C$ for almost all t ,

c) *The time-dependent operator $[H(t), \Phi(t)]$ originally defined as a quadratic form on $D(B^{\frac{1}{2}})$ extends to an element of $L_{loc}^1(\mathbb{R}^+, \mathcal{B}(\mathcal{H}))$.*

(i) *Assume that $\mathbf{D}\Phi(t) \in L_{loc}^1(\mathbb{R}^+, \mathcal{B}(\mathcal{H}))$. Then*

$$\|\Phi(t)U(t)\psi\| \leq \|\Phi(0)\psi\| + \int_0^t \|\mathbf{D}\Phi(s)U(s)\psi\| ds.$$

(ii) *Assume that $\Phi(t)$ is uniformly bounded and that there exists $C_0 > 0$ and some operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \dots, n$ such that*

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t)B(t) - \sum_{i=1}^n B_i^*(t)B_i(t),$$

with

$$\int_1^\infty \|B_i(t)U(t)\psi\|^2 dt \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad i = 1, \dots, n.$$

Then there exists a constant C such that

$$\int_1^\infty \|B(t)U(t)\psi\|^2 dt \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{II.B.1})$$

Weak propagation estimates of the type of (II.B.1) allow to prove the existence of asymptotic observables. Precisely, we have the following Lemma which summarizes previous results due to Cook and Kato

Lemma II.B.1 (Cook, Kato) *Let $\Phi(t)$ be a uniformly bounded function with values in selfadjoint operators satisfying the conditions a), b), c) of Proposition II.B.1. Let \mathcal{D} a dense subspace of \mathcal{H} .*

(i) *(Cook) Assume that for $\psi \in \mathcal{D}$,*

$$\int_1^\infty \|\mathbf{D}\Phi(t)U(t)\psi\| dt < \infty,$$

then there exists

$$s - \lim_{t \rightarrow \infty} U^*(t)\Phi(t)U(t). \quad (\text{II.B.2})$$

(ii) *(Kato) Assume that*

$$|(\psi_2, \mathbf{D}\Phi(t)\psi_1)| \leq \sum_{i=1}^n \|B_{2i}(t)\psi_2\| \|B_{1i}(t)\psi_1\|,$$

with

$$\begin{aligned} \int_1^\infty \|B_{2i}(t)U(t)\psi\|^2 dt &\leq C\|\psi\|^2, \quad \psi \in \mathcal{D}, \quad i = 1, \dots, n, \\ \int_1^\infty \|B_{1i}(t)U(t)\psi\|^2 dt &\leq C\|\psi\|^2, \quad \psi \in \mathcal{D}, \quad i = 1, \dots, n, \end{aligned}$$

then the limit (II.B.2) exists.

Chapitre III

Scattering of charged Dirac fields by a Reissner-Nordström black hole

III.1 Introduction

Scattering theory for field equations in black hole spacetimes has been a subject of intense research in the course of the last twenty years. Complete results have been obtained for *Schwarzschild black holes*, which describe asymptotically flat spacetimes containing nothing but a static, spherically symmetric, uncharged black hole, by Bachelot [3], [4], Dimock [22], Dimock and Kay [23] and Nicolas [62], for *Reissner-Nordström black holes*, a slight generalization of Schwarzschild black holes, describing a static, spherically symmetric, charged black hole, by Jin [51] and Melnyk [54], [55] and recently, for the more general family of *Kerr black holes*, describing eternal, uncharged, rotating spacetimes, by Häfner [44] and Häfner, Nicolas [46]. Such studies have been strongly motivated by the discovery of striking phenomena in the 1960's and 1970's : the Hawking effect and superradiance. We refer to Bachelot [5], [6] and Melnyk [56] for an application of scattering results to a detailed study of the Hawking effect.

In this paper, we establish a complete time-dependent scattering theory for charged, massive or not, Dirac fields outside a Reissner-Nordström black hole. This work extends in three directions some analogous results obtained in [54], [55]. Firstly, the method we adopt is based on intermediate results interesting for their own sake : the construction of asymptotic velocity operators and the study of their spectra ; this is new in this context. Secondly, we give a new definition of the Dollard modified wave operators thanks to a reinterpretation of a classical argument due to Thaller [75]. Finally, we obtain a complete scattering theory for massless charged Dirac fields, a case not covered in [55].

A peculiarity of scattering theories in black hole spacetimes is that we have to deal with two asymptotic regions. Indeed, when studying the propagation of fields in the outer region of a black hole, the point of view commonly adopted is that of an observer static at spacelike infinity. Such an observer perceives the event horizon as an asymptotic region : this entails a description of the exterior of the Reissner-Nordström black hole as a spherically symmetric spacetime with two asymptotic ends (the horizon and infinity) with *very different geometrical structures*. After a decomposition into spin-weighted spherical harmonics, Dirac's equation takes the form of a one-dimensional transport equation perturbed by scalar and matrix-valued potentials. The scalar potential describes the interaction of the charges of the black hole and that of the field whereas the matrix-valued potentials correspond to the effects of the mass of the field and its angular momentum. The behaviour of the potentials reflects the geometry of the black hole. At the horizon, the matrix-valued potentials are exponentially decreasing, and thus short-range, whereas the scalar potential tends to a non-zero constant. At infinity, all potentials are long-range, of Coulombian type.

Recall that the object of scattering theory is to prove that the energy does not remain trapped in compact sets but rather, escapes towards the asymptotic regions where fields obey simpler equations. These properties are encoded in the notion of wave operators (see [69], Vol 3, for an introduction to the subject). To prove the existence and asymptotic completeness of (Dollard modified at infinity) wave operators, we follow the formalism developed by Dereziński and Gérard [21] in the study of the n -body problem in non-relativistic quantum mechanics and extended to Dirac's equation in Minkowski spacetime in our previous papers [14] and [15]. This formalism puts forward the study of the asymptotic behaviour of physical observables such as position and velocity. Precisely, it consists in 1) the proofs of several propagation estimates, 2) the construction of the asymptotic velocity operators and the analysis of their spectra, 3) a natural definition of the modified wave operators, inherited from the results of the first two steps, and the proof of their existence and asymptotic completeness.

This formalism provides at each step intermediate results that are both intuitive and physically relevant. The propagation estimates give weak informations on the decay of the density of probability of presence of the fields in different cones of the spacetime. For instance, a particularly important estimate

is the minimal velocity estimate whose meaning is the following : even though the Huygens principle is not valid, the solution propagates locally faster than a small but positive velocity, in a weak sense. Note here that this estimate is crucial since it allows to use Cook's method for the construction of both direct and inverse wave operators. Similarly, the asymptotic velocity operators are also physically relevant constructions whose spectra correspond to the velocity of Dirac fields in the asymptotic regions as perceived by an observer static at infinity. Moreover, these operators turn out to be a good tool to organise the proof of existence and asymptotic completeness of wave operators. In this paper, we would like to emphasize the fact that the propagation estimates and the asymptotic velocity operators are almost more essential than the constructions of the wave operators themselves : they express in a natural and visual way all the scattering information that we require.

Another important advantage of this method is that it helps us to choose the Dollard modification at infinity. Recall first that the essence of wave operators is to make the link between the observables of the theory and their asymptotic equivalents. An example is the classical intertwining relations between the pair of operators (H, H_0) corresponding to the full Hamiltonian and the "free" Hamiltonian. Now, let us denote $(\pm\mathcal{V}, P^\pm)$ the pair given by the classical velocity operators and the asymptotic velocity operators. We can choose a modification, and thus define the modified wave operators, in such a way that they also satisfy classical intertwining relations between $\pm\mathcal{V}$ and P^\pm .

Eventually, in the context of black hole spacetimes, this approach turns out to be very convenient to take into account the existence of two asymptotic regions. Without additional technical difficulties in the proofs, the propagation estimates give naturally different informations according to which asymptotic region we consider. They also allow to characterize differently the asymptotic velocity operators in these regions. Another major advantage of this method appears in the construction of wave operators since we can use the asymptotic velocity operators to separate the incoming and outgoing parts of a solution without introducing arbitrary cut-off functions.

The main tools used throughout this paper are the *commutator method* and *Mourre theory* which we briefly present in the course of the text. We also point out the strong link between the existence of a conjugate operator in the sense of Mourre's theory and the minimal velocity estimate. Such a link suggests to define a new conjugate operator, inspired by [37], adapted to the geometry of the black hole and particularly useful to obtain this estimate in an optimal form.

An important point in this paper is the extension of an argument, initially used by Thaller [75] to deal with a mass potential, in the definition of a Dollard modification. Roughly speaking, we show that when a potential can be written as the product of a scalar long-range function times an operator which anticommutes with the free Hamiltonian, then this "apparently" long-range perturbation is artificial and does not enter in the Dollard modification.

Let us now briefly describe the contents of this paper. The Reissner-Nordström black hole, Dirac's equation as well as a precise description of the point of view of an observer static at infinity are the objects of section III.2. Some fundamental properties of Dirac Hamiltonians on such a background are given in section III.3 : domain invariance, Zitterbewegung and absence of eigenvalues. Section III.4 is devoted to a brief presentation of Mourre's theory. In particular, we introduce there our new conjugate operator. In section III.5, we present the commutator method and we establish the different propagation estimates. In section III.6, we construct the asymptotic velocity operators P^\pm and analyse their spectra. Finally, section III.7 is devoted to the construction of the (Dollard-modified at infinity) wave operators. At the horizon, we compare the full dynamics with a simple transport equation along the principal null geodesic flow. At spacelike infinity, we compare the full dynamics with a dynamics corresponding to flat spacetime Dirac radial dynamics plus a Dollard modification due to long-range potentials coming from the mass and the charge of Dirac fields. This definition is new and avoids the comparison with a full

flat spacetime Dirac dynamics as in [55]. Eventually, we use our previous results [14], [15] to give an alternative definition of the wave operators at infinity in the same spirit as [55].

III.2 Reissner-Nordström black hole and Dirac equation

A Reissner-Nordström black hole is described in Schwarzschild coordinates by the four dimensional smooth manifold

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r^+ \times S_\omega^2,$$

equipped with the lorentzian metric

$$g = g_{\mu\nu} dx^\mu dx^\nu = F(r) dt^2 - (F(r))^{-1} dr^2 - r^2 d\omega^2, \quad (\text{III.2.1})$$

where

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

and $d\omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the euclidean metric on the sphere S^2 . The Reissner-Nordström metric is a *static, spherically symmetric, charged* exact solution of the Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Here, $G_{\mu\nu}$ denotes the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor,

$$T_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}),$$

where $F_{\mu\nu}$ is the electromagnetic 2-form, solution of Maxwell's equations $\nabla^\mu F_{\nu\rho} = 0$, $\nabla_{[\mu} F_{\nu\rho]} = 0$ and given here in terms of a global electromagnetic vector potential $F_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$, $A_\nu dx^\nu = -\frac{Q}{r} dt$. The quantities M and Q are interpreted as the mass and charge of the Reissner-Nordström black hole.

The metric (III.2.1) obviously has two types of singularities. Firstly, the point $\{r = 0\}$. This is a true singularity or *curvature singularity*. It means that certain scalars obtained by contracting the Riemann tensor blow up when $r \rightarrow 0$. Secondly, the spheres whose radii are the roots of the function F . The number of these roots depends on the respective values of the constants M and Q .

- If $M^2 > Q^2$ then F vanishes twice at the values $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The spheres $\{r = r_+\}$ and $\{r = r_-\}$ are called the *exterior* and *interior horizons* of the Reissner-Nordström black hole.
- If $M^2 = Q^2$ then F has a unique double root $r_0 = r_- = r_+$ and the exterior and interior horizons coincide. This spacetime is called extreme Reissner-Nordström black hole.
- If $M^2 < Q^2$ then F has no root. In this case, we say that the curvature singularity $\{r = 0\}$ is a naked singularity.

Despite appearances, both exterior and interior horizons (when they exist) are not true singularities in the sense given for $\{r = 0\}$. They are, in fact, due to the choice of the Schwarzschild coordinates. Using appropriate coordinate systems, these horizons are understood as regular null hypersurfaces that can be crossed one way but would require speeds greater than that of light to be crossed the other way. Hence their name : event horizons.

In this paper, we only consider the case $M^2 > Q^2$. Our purpose is to describe the scattering of linear charged Dirac fields in the region of the Reissner-Nordström spacetime lying outside the black hole. We denote $\mathcal{B}_I = \mathbb{R}_t \times]r_+, +\infty[\times S^2$ this region. The first step is to choose a way of describing the geometry of the black hole. In other words, we have to choose an adequate system of coordinates convenient for

the later analysis and physically relevant according to the information we want to obtain. In our case, we shall be guided by the static structure of the exterior of a Reissner-Nordström black hole and its related system of coordinates. Recall that \mathcal{B}_I is static ; that is to say, it possesses a globally defined, nonvanishing, timelike, Killing vector field ξ^a whose integral curves are orthogonal to a given spacelike hypersurface Σ . Obviously, $\frac{\partial}{\partial t}$ and $\Sigma_0 = \{0\} \times]r_+, +\infty[\times S^2$ are the mentioned Killing vector field and hypersurface. The Schwarzschild coordinates are the natural choice of coordinates since we use the Killing parameter along the integral curves of ξ^a as time coordinate (as seen by the relation $\xi^a = \frac{\partial}{\partial t}$). In particular, the coefficients of the metric do not depend on t . Two important consequences are the following :

- Dirac's equation can be viewed as an evolution equation with respect to the coordinate t . This is due to the global hyperbolicity of \mathcal{B}_I (immediately given by the static structure of \mathcal{B}_I). We refer to [65] for a more detailed study of this point.
- The coefficients of the Dirac equation will not depend on the coordinate t (an easy consequence of the time-independence of the coefficients of the metric).

Since time-dependent scattering theories rely heavily on the existence of a unitary propagator on a fixed Hilbert space, the choice of Schwarzschild coordinates is almost compulsory.

There is, however, another important reason to choose this coordinate system. Indeed, it corresponds to the natural notion of observers static at infinity. These observers are located far from the exterior horizon and their velocity 4-vector is given by

$$U = \frac{1}{\sqrt{2}} F^{-\frac{1}{2}}(r) \frac{\partial}{\partial t}.$$

The time coordinate t describes their experience of time (i.e. t corresponds to their proper time, see [66]). When describing the exterior of the black hole using Schwarzschild coordinates, we implicitly take the point of view of such observers. Typically we can think of a telescope on earth aimed in the direction of the black hole. This choice is natural with the idea of scattering experiments we have in mind.

Summarizing the previous discussion, we see that, for observers static at infinity, the Dirac equation takes the form of an evolution equation with coefficients independent of t . Do we need to impose boundary conditions at the horizon ? The answer is no. Indeed, the exterior of the Reissner-Nordström black hole is the only visible part of the spacetime for such observers. This stems from the fact that the horizon appears to them as a singularity of the metric. To make this point clearer, we consider the incoming and outgoing null radial geodesics i.e. the trajectories followed by light-rays aimed radially at the black hole or at infinity. They are given by the integral curves of the vector fields $\frac{\partial}{\partial t} \pm F(r) \frac{\partial}{\partial r}$. Let us introduce a new radial coordinate r_* , called the Regge-Wheeler coordinate, which has the property of straightening the null radial geodesics. It is given implicitly by the relation

$$\frac{dr}{dr_*} = F(r). \quad (\text{III.2.2})$$

From (III.2.2), we obtain the explicit expression

$$r_* = r + M \log \Delta(r) + \frac{r_+^2 + r_-^2}{2(r_+ - r_-)} \log \left(\frac{r - r_+}{r - r_-} \right) + R_0, \quad (\text{III.2.3})$$

where $\Delta(r) = r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-)$ and R_0 is any constant of integration. Note that, in the coordinate system (t, r_*, ω) , the horizon $\{r = r_+\}$ is pushed away to $\{r_* = -\infty\}$ and the metric takes the form

$$g = F(r)(dt^2 - dr_*^2) - r^2 d\omega^2. \quad (\text{III.2.4})$$

Thus, the incoming and outgoing null radial geodesics are now generated by the vector fields $\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r_*}$ and take the simple form

$$\gamma^\pm(s) = (s, r_0 \pm s, \omega_0), \quad s \in \mathbb{R}, \quad (\text{III.2.5})$$

where (r_0, ω_0) are fixed. From (III.2.5), we conclude that the horizon is never reached by a light-ray, in a finite time t . Hence, it corresponds to an asymptotic region of the spacetime and no boundary conditions on the horizon is needed.

A nice way to compute the Dirac equation is for instance, to use the 2-spinor approach and the Newman-Penrose formalism. We refer to [68] for a detailed presentation and [65] for an application to the computation of a Dirac equation. We will not give the details here. From [55] or [65], the charged Dirac equation on \mathcal{B}_I takes the following form

$$\left(F^{-\frac{1}{2}} \gamma^0 (\partial_t + i \frac{qQ}{r}) + F^{\frac{1}{2}} \gamma^1 (\partial_{r_*} + \frac{1}{r} + \frac{F'}{4F}) + \frac{1}{r} \gamma^2 (\partial_\theta + \frac{1}{2} \cot \theta) + \frac{1}{r \sin \theta} \gamma^3 \partial_\varphi \right) \phi = -im\phi. \quad (\text{III.2.6})$$

The matrices γ^μ , $\mu = 1, \dots, 4$ are the Dirac matrices and they satisfy the relations of anticommutation $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the inverse of the Minkowski metric. A possible representation in terms of Pauli matrices σ^μ is the following. Recall that

$$\begin{aligned} \sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma^3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned}$$

and thus, we define

$$\gamma^0 = i \begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^k = i \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}.$$

The Hilbert space of finite energy associated to equation (III.2.6) is $[L^2(\mathbb{R}_{r_*} \times S_\omega^2, r^2 F^{\frac{1}{2}}(r) dr_* d\omega)]^4$. As shown in [62], we can get rid of some long-range terms in (III.2.6) if we work with the density spinor $\psi = r F^{\frac{1}{4}} \phi$. Therefore, the charged Dirac equation under Hamiltonian form is

$$i\partial_t \psi = H\psi, \quad (\text{III.2.7})$$

where the Hamiltonian H is given by

$$H = \Gamma^1 D_{r_*} - i \frac{F^{\frac{1}{2}}(r)}{r} \Gamma^2 (\partial_\theta + \frac{\cot \theta}{2}) - i \frac{F^{\frac{1}{2}}(r)}{r \sin \theta} \Gamma^3 \partial_\varphi + m F^{\frac{1}{2}}(r) \Gamma^0 + \frac{qQ}{r}. \quad (\text{III.2.8})$$

Here, we denoted $\Gamma^1 = \gamma^0 \gamma^1$, $\Gamma^2 = \gamma^0 \gamma^2$, $\Gamma^3 = \gamma^0 \gamma^3$ and $\Gamma^0 = \gamma^0$. The operator H acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R} \times S^2, dr_* d\omega; \mathbb{C}^4)$. Finally, observe that the operator

$$H_{S^2} = -i\Gamma^2 (\partial_\theta + \frac{\cot \theta}{2}) - \frac{i}{\sin \theta} \Gamma^3 \partial_\varphi,$$

is the Dirac operator acting on the sphere S^2 . Therefore, if we denote $a(r_*) = \frac{F^{\frac{1}{2}}(r)}{r}$, $b_m(r_*) = m F^{\frac{1}{2}}(r)$ and $c_q(r_*) = \frac{qQ}{r}$, the Dirac Hamiltonian takes the following simple form

$$H = \Gamma^1 D_{r_*} + a(r_*) H_{S^2} + b_m(r_*) \Gamma^0 + c_q(r_*). \quad (\text{III.2.9})$$

In what follows, the geometry of the Reissner-Nordström black hole, especially the existence of two asymptotic regions having very distinct features, is encoded in the r_* -decay properties of the potentials in (III.2.9). Of course, this difference will appear in the lack of symmetry of our constructions between $r_* \rightarrow -\infty$ and $r_* \rightarrow +\infty$.

III.3 Abstract analytic framework and fundamental properties of Dirac Hamiltonians

In this Section, we first diagonalize the Dirac operator on the 2-sphere, H_{S^2} , using a decomposition on spin-weighted spherical harmonics. This allows us to reduce our 3 + 1 dimensional evolution equation (III.2.7), (III.2.9) into a 1 + 1 dimensional one. We then define an analytic framework adapted to the study of this simplified Dirac equation. In particular, we carefully define the classes of symbols to which the potentials belong. We solve the Cauchy problem and we state some fundamental properties of such Dirac Hamiltonians for later use : domain invariance properties, Zitterbewegung phenomenon and absence of eigenvalues.

III.3.1 Spin weighted harmonics

The operator H_{S^2} has compact resolvent and hence can be diagonalized into an infinite sum of matrix-valued multiplication operators. For this, we introduce a generalization of usual spherical harmonics called *spin-weighted harmonics*. We refer to I.M. Gel'Fand and Z.Y. Sapiro [35] for a detailed presentation.

For each spinorial weight m such that $2m \in \mathbb{Z}$, for each $l - |m| \in \mathbb{N}$ and $l - |n| \in \mathbb{N}$, we define the functions W_{mn}^l by

$$W_{mn}^l(\theta, \varphi) = e^{in\varphi} u_{mn}^l(\theta) \in L^2(S^2), \quad (\text{III.3.10})$$

with

$$\frac{d^2 u_{mn}^l}{d\theta^2} + \cot \theta \frac{du_{mn}^l}{d\theta} + \left[l(l+1) - \frac{n^2 - 2mn \cos \theta + m^2}{\sin^2 \theta} \right] u_{mn}^l = 0, \quad (\text{III.3.11})$$

and

$$\frac{du_{mn}^l}{d\theta} - \frac{m - n \cos \theta}{\sin \theta} u_{mn}^l = \begin{cases} -i[(l+n)(l+n-1)]^{\frac{1}{2}} u_{m,n-1}^l = 0, & n > -l, \\ 0, & n = l. \end{cases} \quad (\text{III.3.12})$$

$$\frac{du_{mn}^l}{d\theta} + \frac{m - n \cos \theta}{\sin \theta} u_{mn}^l = \begin{cases} -i[(l+n)(l+n-1)]^{\frac{1}{2}} u_{m,n+1}^l = 0, & n < l, \\ 0, & n = l. \end{cases} \quad (\text{III.3.13})$$

The functions u_{mn}^l are normalized by

$$\int_0^\pi |u_{mn}^l(\theta)|^2 \sin \theta d\theta = \frac{1}{2\pi},$$

and they satisfy the relations of induction

$$\frac{du_{mn}^l}{d\theta} \mp \frac{n - m \cos \theta}{\sin \theta} u_{mn}^l = -i[(l \pm m)(l \mp m + 1)]^{\frac{1}{2}} u_{m \mp 1, n}^l.$$

We set $\mathcal{I}_m = \{(l, n), l - |m| \in \mathbb{N}, l - |n| \in \mathbb{N}\}$. The family $\{W_{mn}^l(\theta, \varphi)\}_{(l,n) \in \mathcal{I}_m}$ forms a Hilbert basis of $L^2(S^2)$. Thus, any function in $L^2(\mathbb{R}_{r_*} \times S^2)$ can be decomposed into an infinite sum

$$u(r_*, \theta, \varphi) = \sum_{(l,n) \in \mathcal{I}_m} u_{ln}(r_*) W_{mn}^l(\theta, \varphi),$$

where $u_{ln}(r_*) \in L^2(\mathbb{R})$.

Now, let us consider the Hilbert basis of $L^2(S^2; \mathbb{C}^4)$ given by $F_n^l = {}^t \left\{ W_{-\frac{1}{2}, n}^l, W_{\frac{1}{2}, n}^l, W_{-\frac{1}{2}, n}^l, W_{\frac{1}{2}, n}^l \right\}$, for any index $(l, n) \in \mathcal{I}_{\frac{1}{2}}$. Thus \mathcal{H} is decomposed into the infinite direct sum

$$\mathcal{H} = \bigoplus_{(l, n) \in \mathcal{I}_{\frac{1}{2}}} \left[L^2(\mathbb{R}_{r_*})^4 \otimes F_n^l \right] := \bigoplus_{(l, n) \in \mathcal{I}_{\frac{1}{2}}} \mathcal{H}_{ln}.$$

The relations satisfied by the spin-weighted harmonics imply that the Hamiltonian H can be decomposed into

$$H = \bigoplus_{(l, n) \in \mathcal{I}_{\frac{1}{2}}} H^{ln},$$

with

$$H^{ln} := H|_{\mathcal{H}_{ln}} = \Gamma^1 D_{r_*} - a(r_*) \left(l + \frac{1}{2} \right) \Gamma^2 + b_m(r_*) \Gamma^0 + c_q(r_*). \quad (\text{III.3.14})$$

We denote $a_l(r_*) = -a(r_*) \left(l + \frac{1}{2} \right)$. We can identify the operator (III.3.14) acting on $L^2(\mathbb{R}_{r_*} \times S^2; \mathbb{C}^4)$ with its restriction to $L^2(\mathbb{R}_{r_*}; \mathbb{C}^4)$ which we still denote H^{ln} . Eventually, we have

$$H^{ln} = \Gamma^1 D_{r_*} + a_l(r_*) \Gamma^2 + b_m(r_*) \Gamma^0 + c_q(r_*). \quad (\text{III.3.15})$$

According to the simple form of the matrix $\Gamma^1 = \text{diag}(-1, 1, 1, -1)$, the Dirac equation restricted to any spin-weighted harmonics can be viewed as a system of transport equations perturbed by three (possibly matrix-valued) potentials depending on the mass m , the charge q and the harmonics l .

III.3.2 Symbol classes

We introduce here some classes of functions which measure the r_* -decay of the potentials on the asymptotic regions. As a motivation, we first analyse the asymptotic behaviour of the function $F(r(r_*))$ which enters in the definition of two potentials. Recall that

$$F(r) = \frac{\Delta(r)}{r^2}, \text{ where } \Delta(r) = (r - r_+)(r - r_-).$$

From (III.2.3), it is easy to see that $r_* \simeq r(r_*)$ as $r_* \rightarrow +\infty$. Therefore, the function $F(r)$ is bounded at spacelike infinity. On the other hand, taking $R_0 = 0$ in (III.2.3), we obtain the expression

$$r - r_+ = (r - r_-)^{\frac{r_+^2}{r_-^2}} e^{2\kappa_+(r_* - r)}, \quad (\text{III.3.16})$$

where the constant $\kappa_+ = \frac{r_+ - r_-}{2r_+^2}$ is called the surface gravity at the black hole horizon. Since $r(r_*) \rightarrow r_+$ as $r_* \rightarrow -\infty$, the function $r(r_*) - r_+ \simeq e^{2\kappa_+ r_*}$ is exponentially decreasing at the horizon. Since $r - r_-$ and $\frac{1}{r^2}$ are bounded when $r_* \rightarrow -\infty$, the same holds for $\Delta(r)$ and $F(r)$.

This leads us to define the following symbol classes as subsets of $C^\infty(\mathbb{R}_{r_*})$. Let $\eta > 0$.

- $f \in \mathbf{T}^{m, n}$ if and only if $\forall \alpha \in \mathbb{N}$, $\partial_{r_*}^\alpha f \in \begin{cases} O(\langle r_* \rangle^{m-\alpha}), & r_* \rightarrow +\infty, \\ O(e^{n\eta|r_*|}), & r_* \rightarrow -\infty. \end{cases}$

To make the link with more usual symbol classes i.e. symbols having power-decay behaviour at the infinities, and in order to be able to use standard pseudodifferential calculus later, we also introduce the following sets.

- $f \in T^{m, n}$ if and only if $\forall \alpha \in \mathbb{N}$, $\partial_{r_*}^\alpha f \in \begin{cases} O(\langle r_* \rangle^{m-\alpha}), & r_* \rightarrow +\infty, \\ O(\langle r_* \rangle^{n-\alpha}), & r_* \rightarrow -\infty. \end{cases}$

If one or both of the indices $m, n = \pm\infty$, we define (for instance $T^{m, -\infty}$)

$$\bullet f \in T^{m, -\infty} \text{ if and only if } \forall \alpha, n \in \mathbb{N}, \partial_{r_*}^\alpha f \in \begin{cases} O(\langle r_* \rangle^{m-\alpha}), & r_* \longrightarrow +\infty, \\ O(\langle r_* \rangle^{-n}), & r_* \longrightarrow -\infty. \end{cases}$$

The other possibilities are straightforward. At last, we simply denote by T^m the set $T^{m, m}$. We list some properties of these spaces.

$$\begin{aligned} (i) \quad & \mathbf{T}^{m, n} \times \mathbf{T}^{p, q} \subset \mathbf{T}^{m+p, n+q}, \\ (ii) \quad & T^{m, n} \times T^{p, q} \subset T^{m+p, n+q}, \\ (iii) \quad & \partial_{r_*}^\alpha : \mathbf{T}^{m, n} \longrightarrow \mathbf{T}^{m-\alpha, n}, \forall \alpha \in \mathbb{N}, \\ (iv) \quad & \partial_{r_*}^\alpha : T^{m, n} \longrightarrow T^{m-\alpha, n-\alpha}, \forall \alpha \in \mathbb{N}, \\ (v) \quad & T^{m, n} \subset T^{\max(m, n)}. \end{aligned}$$

We recall from Appendix III.A the class $S^{m, n}$ of pseudodifferential symbols

$$\bullet f \in S^{m, n} \text{ if and only if } f \in C^\infty(\mathbb{R}_{r_*} \times \mathbb{R}_\xi) \text{ and } \forall \alpha, \beta \in \mathbb{N}, \partial_{r_*}^\alpha \partial_\xi^\beta f \in O(\langle r_* \rangle^{m-\alpha} \langle \xi \rangle^{n-\beta}), r_* \longrightarrow \infty.$$

Obviously, if $n < 0$, we have the inclusions

$$\mathbf{T}^{m, n} \subset T^{m, -\infty} \subset T^m \subset S^{m, 0}.$$

Let us set $\eta = \kappa_+$ in the definition of the above spaces. The potentials a, b_m, c_q satisfy the following Proposition.

Proposition III.3.1 *The potentials a, b_m, c_q belong to the following symbol classes*

$$\begin{aligned} (i) \quad & a \in \mathbf{T}^{-1, -1}, \\ (ii) \quad & b_m \in \mathbf{T}^{0, -1}, b_m - m \in \mathbf{T}^{-1, 0}, b'_m \in \mathbf{T}^{-2, -1}, \\ (ii) \quad & c_q \in \mathbf{T}^{-1, 0}, c_q - c_0 \in \mathbf{T}^{0, -2}, c'_q \in \mathbf{T}^{-2, -2}. \end{aligned}$$

Proof : To prove these assertions, we need to make the link between the respective asymptotic behaviours of functions $f(r)$ and $f(r(r_*))$. This is done in the following Lemma. Let us first introduce the class of functions

$$f(r) \in \Pi^m \text{ if and only if } \forall \alpha \in \mathbb{N}, \partial_r^\alpha f(r) \in \begin{cases} O(\langle r \rangle^{m-\alpha}), & r \longrightarrow +\infty, \\ O(1), & r \longrightarrow r_+. \end{cases}$$

Then, from [44], Lemme 9.7.1, we have

Lemma III.3.1 (i) $f(r) \in \Pi^m \implies f(r(r_*)) \in \mathbf{T}^{m, 0}$ and $\forall \alpha \in \mathbb{N}^*, \partial_{r_*}^\alpha f(r(r_*)) \in \mathbf{T}^{m-\alpha, -2}$,
(ii) $f(r_*) \in \mathbf{T}^{m, n}, g(r) \in \Pi^k \implies f(r_*)g(r(r_*)) \in \mathbf{T}^{m+k, n}$.

From (III.2.3), (III.3.16) and Lemma III.3.1, it is immediate that

$$\begin{aligned} \bullet \quad & \Delta \in \mathbf{T}^{2, -2}, \\ \bullet \quad & \sqrt{\Delta} \in \mathbf{T}^{1, -1}, \\ \bullet \quad & F^{\frac{1}{2}} \in \mathbf{T}^{0, -1}. \end{aligned}$$

Recall that $a(r_*) = \frac{F^{\frac{1}{2}}}{r}$. Thus, since $\frac{1}{r} \in \Pi^{-1}$, Lemma III.3.1 implies that $a \in \mathbf{T}^{-1,-1}$.

We also have $b'_m(r) = m\sqrt{\Delta} \frac{Mr-Q^2}{r^4}$. Obviously, the function $\frac{Mr-Q^2}{r^4} \in \Pi^{-3}$. Then, Lemma III.3.1 implies that $b'_m \in \mathbf{T}^{-2,-1}$.

As $b_m(r) = mF^{\frac{1}{2}}(r)$, we have $b_m \in \mathbf{T}^{0,-1}$. Moreover, it is straightforward to see that $b_m - m \in \Pi^{-1}$. Hence, Lemma III.3.1 implies that $b_m - m \in \mathbf{T}^{-1,0}$.

At last, we have $c'_q(r) = -\frac{qQ\Delta}{r^4}$. Since $\frac{qQ}{r^4}$ belongs to Π^{-4} , we conclude $c'_q \in \mathbf{T}^{-2,-2}$ by Lemma III.3.1.

Trivially, $c_q(r) = \frac{qQ}{r}$ belongs to Π^{-1} . Hence, $c_q \in \mathbf{T}^{-1,0}$.

Eventually, we have $c_q(r) - c_0 = -\frac{qQ}{rr_+}(r - r_+)$. Since $\frac{qQ}{rr_+} \in \Pi^{-1}$ and $r - r_+ \in \mathbf{T}^{1,-2}$, Lemma III.3.1 implies $c_q - c_0 \in \mathbf{T}^{0,-2}$.

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III.3.3 Global Cauchy problem

If we work on a fixed spin-weighted spherical harmonics, it is immediate by the Kato-Rellich Theorem that H^{ln} is selfadjoint on \mathcal{H}_{ln} and $D(H^{ln}) \simeq H^1(\mathbb{R}, \mathbb{C}^4)$. Therefore, we conclude that H is selfadjoint on \mathcal{H} and its domain is given by

$$D(H) = \left\{ \psi = \bigoplus_{(l,n) \in \mathcal{I}_{\frac{1}{2}}} \psi_{ln}, \psi_{ln} \in D(H^{ln}), \sum_{(l,n) \in \mathcal{I}_{\frac{1}{2}}} \|H^{ln} \psi_{ln}\|^2 < \infty \right\}.$$

We associate to the selfadjoint operator H the unitary evolution e^{-itH} by the spectral theorem. Then, Stone's Theorem (see [69]) implies that, given any $\psi_0 \in \mathcal{H}$, there exists a unique solution $\psi(t) \in C(\mathbb{R}_t; \mathcal{H})$ satisfying

$$\begin{cases} i\partial_t \psi(t) &= H\psi(t), \\ \psi(0) &= \psi_0, \end{cases} \quad (\text{III.3.17})$$

and this solution is given by $\psi(t) = e^{-itH} \psi_0$. Moreover, the energy is conserved along the evolution

$$\|\psi(t)\|_{\mathcal{H}} = \|\psi_0\|_{\mathcal{H}}, \quad \forall t \in \mathbb{R}. \quad (\text{III.3.18})$$

III.3.4 Domain invariance

In this Section, we describe a remarkable property of Dirac operators which is false in non-relativistic quantum mechanics. Precisely, the domain $D(\langle r_* \rangle^n)$, $n \in \mathbb{N}$ is stable under the actions of the unitary evolution $\{e^{-itH}\}_{t \in \mathbb{R}}$ and the resolvent $(H - z)^{-1}$, $z \notin \sigma(H)$. This property will be used in Section III.4 to prove the existence of a locally conjugate operator for H . The proof of the following result can be found in [14].

Proposition III.3.2 *Let H be the Dirac operator (III.2.9). Let r_* be the standard radial position observable. Then $\langle r_* \rangle$ is selfadjoint on \mathcal{H} with its natural domain $D(\langle r_* \rangle) = \{\psi \in \mathcal{H}, \langle r_* \rangle \psi \in \mathcal{H}\}$ and for all $n \in \mathbb{N}$, we have*

$$e^{-itH} D(\langle r_* \rangle^n) \subset D(\langle r_* \rangle^n), \quad (\text{III.3.19})$$

and for any $z \notin \sigma(H)$,

$$(H - z)^{-1} D(\langle r_* \rangle^n) \subset D(\langle r_* \rangle^n). \quad (\text{III.3.20})$$

III.3.5 Definition of the velocity operator and Zitterbewegung

In this Section, we review the usual definitions of the *velocity operator* in the case of Dirac Hamiltonians. Naturally, this will lead us to present the so-called *Zitterbewegung* phenomenon. For more details, we refer to [75]. In the remaining of this Section, we will work with the free Dirac operator $H_0^m = \Gamma^1 D_{r_*} + m\Gamma^0$.

The velocity operator is usually defined as the time derivative of the position operator. Let us consider the standard position operator

$$r_*(t) = e^{itH_0^m} r_* e^{-itH_0^m}.$$

If we derive this expression with respect to t , we obtain

$$\begin{aligned} \frac{dr_*(t)}{dt} &= e^{itH_0^m} i[H_0^m, r_*] e^{-itH_0^m}, \\ &= e^{itH_0} \Gamma^1 e^{-itH_0} =: \Gamma^1(t). \end{aligned} \quad (\text{III.3.21})$$

$\Gamma^1(t)$ is called the *standard velocity operator*. Let us assume that $m \neq 0$. According to classical relativistic kinematics, we would have expected the operator $\mathcal{V}_m = D_{r_*} (H_0^m)^{-1}$ to be the correct velocity operator instead of $\Gamma^1(t)$. The operator \mathcal{V}_m is called the *classical velocity operator*. Let us see what is the relation between these two distinct definitions. For this purpose, we analyse the time dependence of $\Gamma^1(t)$. We have

$$\begin{aligned} \frac{d\Gamma^1(t)}{dt} &= e^{itH_0^m} i[H_0^m, \Gamma^1] e^{-itH_0^m}, \\ &= e^{itH_0^m} iH_0^m [\Gamma^1 - (H_0^m)^{-1} \Gamma^1 H_0^m] e^{-itH_0^m}. \end{aligned}$$

An explicit computation shows that $\Gamma^1 - (H_0^m)^{-1} \Gamma^1 H_0^m = 2(\Gamma^1 - \mathcal{V}_m)$. Hence, if we define the operator $F := \Gamma^1 - \mathcal{V}_m$, we obtain

$$\frac{d\Gamma^1(t)}{dt} = 2iH_0^m F(t), \quad (\text{III.3.22})$$

where $F(t) = e^{itH_0^m} F e^{-itH_0^m}$. This operator F has the important property that it anticommutes with H_0^m i.e.

$$F H_0^m = -H_0^m F. \quad (\text{III.3.23})$$

Thus (III.3.23) implies that $F(t) = e^{2itH_0^m} F$. Now, if we integrate (III.3.22) between 0 and t , we get

$$\Gamma^1(t) = \mathcal{V}_m + e^{2itH_0^m} F. \quad (\text{III.3.24})$$

Therefore, the standard velocity operator oscillates without damping around the classical velocity operator and these oscillations are called *Zitterbewegung phenomenon*. From (III.3.23), it is immediate to understand the origin of the *Zitterbewegung*. It is due to the fact that the standard position operator \mathfrak{r} mixes the negative and positive energy states.

Integrating (III.3.24) again, we obtain

$$r_*(t) = r_* + \mathcal{V}_m t + \frac{1}{2iH_0^m} (e^{2itH_0^m} - 1) F.$$

Remark that, in the case of a massless Dirac Hamiltonian, H_0^0 in our notations, these two notions of velocity operators coincide. Indeed, since Γ^1 commutes with H_0^0 , (III.3.21) leads immediately to

$$\frac{dr_*(t)}{dt} = \Gamma^1.$$

Integrating once again, we obtain

$$r_*(t) = r_* + \Gamma^1 t.$$

Hence, by analogy, we set

$$\mathcal{V}_0 = \Gamma^1.$$

The fact that the standard and classical velocity operators do not coincide when $m \neq 0$ is problematic for using commutator methods and for proving the microlocal velocity estimates in Section III.5.3 below. In particular, we will need to find a position operator R satisfying

$$i[H_0^m, R] = \mathcal{V}_m.$$

This suggests to change the standard observables of the theory. There are basically two possibilities for the choice of a new position operator in Dirac theory. In [14], [15], we chose the Newton-Wigner operator, see [75] for a definition. However, this choice led to technical problems due to the matrix-valued nature of the Newton-Wigner observable. In this paper, we prefer to use the fact that the Zitterbewegung phenomenon is due to the possibility for the operator r_* to mix negative and positive energies. Thus, we could restrict the problem to these subspaces separately. Let us define

$$\tilde{r}_* = \mathbf{1}_{\mathbb{R}^-} (H_0^m) r_* \mathbf{1}_{\mathbb{R}^-} (H_0^m) + \mathbf{1}_{\mathbb{R}^+} (H_0^m) r_* \mathbf{1}_{\mathbb{R}^+} (H_0^m).$$

Then we have

$$\tilde{r}_*(t) = \tilde{r}_* + \mathcal{V}_m t,$$

or equivalently

$$i[H_0^m, \tilde{r}_*] = \mathcal{V}_m. \quad (\text{III.3.25})$$

Notice that (III.3.25) can be rewritten in the following way

$$\mathbf{1}_{\mathbb{R}^-} (H_0^m) i[H_0^m, r_*] \mathbf{1}_{\mathbb{R}^-} (H_0^m) = \mathcal{V}_m \mathbf{1}_{\mathbb{R}^-} (H_0^m), \quad (\text{III.3.26})$$

and

$$\mathbf{1}_{\mathbb{R}^+} (H_0^m) i[H_0^m, r_*] \mathbf{1}_{\mathbb{R}^+} (H_0^m) = \mathcal{V}_m \mathbf{1}_{\mathbb{R}^+} (H_0^m). \quad (\text{III.3.27})$$

Therefore, in practice, we will never really use the position operator \tilde{r}_* but, instead, we will use (III.3.26) and (III.3.27) which allows us to work with the standard position observable r_* . The great advantage of (III.3.26) and (III.3.27) is that we now only need to manipulate scalar differential operators instead of matrix-valued differential operators. Indeed, observe first that the free Dirac operator H_0^m can be decomposed into

$$H_0^m = \lambda_m(D_{r_*}) \mathbf{1}_{\mathbb{R}^+} (H_0^m) - \lambda_m(D_{r_*}) \mathbf{1}_{\mathbb{R}^-} (H_0^m), \quad \text{where } \lambda_m(D_{r_*}) = \sqrt{D_{r_*}^2 + m^2}. \quad (\text{III.3.28})$$

Then, using the decomposition (III.3.28), (III.3.26) and (III.3.27) become respectively

$$\mathbf{1}_{\mathbb{R}^-} (H_0^m) i[-\lambda_m(D_{r_*}), r_*] \mathbf{1}_{\mathbb{R}^-} (H_0^m) = -\lambda'_m(D_{r_*}) \mathbf{1}_{\mathbb{R}^-} (H_0^m), \quad (\text{III.3.29})$$

and

$$\mathbf{1}_{\mathbb{R}^+} (H_0^m) i[\lambda_m(D_{r_*}), r_*] \mathbf{1}_{\mathbb{R}^+} (H_0^m) = \lambda'_m(D_{r_*}) \mathbf{1}_{\mathbb{R}^+} (H_0^m). \quad (\text{III.3.30})$$

The passage from matrix-valued differential operators to scalar differential operators will enable us to use standard results on pseudodifferential calculus stated in Appendix III.A

III.3.6 Absence of eigenvalues

Contrary to the case of Dirac Hamiltonians in flat spacetime with electromagnetic potential, the Dirac operator outside a Reissner-Nordström black hole has no eigenvalue. This is essentially due to the exponential fall off of the potential at the horizon.

Proposition III.3.3 *The pure point spectrum of H is empty i.e.*

$$\sigma_{pp}(H) = \emptyset.$$

Proof : It is sufficient to prove that $H - c_0$ has no eigenvalue on any spin-weighted harmonics. Assume that $u \in L^2(\mathbb{R}; \mathbb{C}^4)$ is a non-zero eigenvector of $H^{ln} - c_0$ with eigenvalue $\lambda \in \mathbb{R}$. Clearly, $u \in H^1(\mathbb{R}; \mathbb{C}^4)$. We define $w(r_*) = e^{-i\lambda\Gamma^1 r_*} u(r_*)$. This function also belongs to $H^1(\mathbb{R}; \mathbb{C}^4)$ and thus, $\lim_{r_* \rightarrow -\infty} w(r_*) = 0$. Moreover, $w(r_*)$ satisfies the following equation

$$\frac{dw(r_*)}{dr_*} = e^{-i\lambda\Gamma^1 r_*} (-i\Gamma^1) V(r_*) e^{i\lambda\Gamma^1 r_*} w(r_*),$$

where $V(r_*) = a_l(r_*)\Gamma^2 + b_m(r_*)\Gamma^0 + (c_q(r_*) - c_0)$. Integrating this expression between $-\infty$ and r_* , we obtain

$$|w(r_*)| \leq \int_{-\infty}^{r_*} |V(s)| |w(s)| ds.$$

Since $\int_{-\infty}^{r_*} |V(s)| ds < \infty$, we conclude by Gronwall's Lemma that $w = 0$.

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III.4 Mourre theory for Dirac Hamiltonians

III.4.1 Abstract theory

Mourre theory [58] is a powerful tool for studying the spectral properties of a given selfadjoint operator H . Among other things, it gives some criteria to conclude to the absence of singular continuous spectrum for H , a typically useful information for scattering theory. Roughly speaking, to establish a Mourre theory, it suffices to find another selfadjoint operator A which, among other technical properties, satisfies the *Mourre estimate* : there exist a strictly positive constant ϵ and a compact operator K on \mathcal{H} such that

$$E_I(H) i[H, A] E_I(H) \geq \epsilon E_I(H) + K, \quad (\text{III.4.1})$$

where $E_I(H)$ denotes the spectral projection on I associated to H . In other words, the main point is to find an operator which “increases” along the evolution. The other property required for A is a certain notion of regularity between the operator H and the unitary evolution e^{-itA} . We follow here the presentation given in [1]. We begin by some definitions.

Definition III.4.1 *For a selfadjoint operator A , we say that another selfadjoint operator H belongs to $C^k(A)$, $k \in \mathbb{N}$, if and only if*

$$(\text{ABG}) \quad \exists z \in \mathbb{C} \setminus \sigma(H), s \longrightarrow e^{isA} (H - z)^{-1} e^{-isA} \in C^k(\mathbb{R}_s; \mathcal{B}(\mathcal{H})),$$

for the strong topology of $\mathcal{B}(\mathcal{H})$.

We define now the notion of locally conjugate operator A as follows.

Definition III.4.2 Let H, A two self-adjoint operators on \mathcal{H} . Let $I \subset \mathbb{R}$ an open interval. We say that A is a locally conjugate operator for H on I if (H, A) satisfy the following assumptions

- (i) $H \in C^1(A)$.
- (ii) $i[H, A]$ defined as a quadratic form on $D(H) \cap D(A)$ extends to an element of $\mathcal{B}(D(H), \mathcal{H})$.
- (iii) $[A, [A, H]]$ well defined as a quadratic form on $D(H) \cap D(A)$ by (ii), extends to an element of $\mathcal{B}(D(H), D(H)^*)$.
- (iv) There exists a strictly positive constant ϵ and a compact operator K such that the Mourre estimate (III.4.1) holds.

If we find such an operator, we have the fundamental Theorem.

Theorem III.4.1 Let H, A two selfadjoint operators on \mathcal{H} . Assume that A is a locally conjugate operator for H on the interval I . Then H has no singular continuous spectrum in I . Furthermore, the number of eigenvalues of H in I is finite and such eigenvalues have finite multiplicity.

One of the difficulties in Mourre theory consists in working with commutators $i[H, A]$ between unbounded selfadjoint operators. We have to be careful to define correctly such quantities since $D(H)$ and $D(A)$ can be unknown or have an intersection which is not even dense in \mathcal{H} . Another subtle feature of the theory is the description of the class $C^1(A)$. From [1], (ABG) is equivalent to the following assertions

$$\text{(ABG')} \quad \exists z \in \mathbb{C} \setminus \sigma(H), \quad |((H - z)^{-1}u, Au) - (Au, (H - z)^{-1}u)| \leq C\|u\|^2, \quad \forall u \in D(A) \cap D(H).$$

$$\text{(ABG'')} \quad \begin{aligned} & (i) \exists z \in \mathbb{C} \setminus \sigma(H), (H - z)^{-1}D(A) \subset D(A), (H - \bar{z})^{-1}D(A) \subset D(A), \\ & (ii) |(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2), \quad \forall u \in D(H) \cap D(A). \end{aligned}$$

Both conditions (i), (ii) of (ABG'') are difficult to check when the domains of H and A are not explicitly known. One possibility to check in such a case the condition $H \in C^1(A)$ consists in finding first a common core for H and A . This procedure is described in [38]. Let us recall Nelson's Lemma

Lemma III.4.1 (Nelson) Let $N \geq 1$ a self-adjoint operator on \mathcal{H} . Let A a symmetric operator on \mathcal{H} such that $D(N) \subset D(A)$. Assume that

$$\begin{aligned} & (i) \|Au\| \leq C\|Nu\|, \quad \forall u \in D(N), \\ & (ii) |(Au, Nu) - (Nu, Au)| \leq C\|N^{\frac{1}{2}}u\|^2, \quad \forall u \in D(N). \end{aligned} \tag{III.4.2}$$

Then A is essentially self-adjoint on $D(N)$. Furthermore every core of N is also a core for A .

Then we have the following criterion [38]

Lemma III.4.2 (Gérard, Laba) Let H, H_0 and N three self-adjoint operators on \mathcal{H} satisfying $N \geq 1$, $D(H) = D(H_0)$ and $(H - z)^{-1}D(N) \subset D(N)$. Let A a symmetric operator on $D(N)$. Assume that H_0 and A satisfy the assumptions of Lemma III.4.1 and

$$|(Au, Hu) - (Hu, Au)| \leq C(\|Hu\|^2 + \|u\|^2), \quad \forall u \in D(N). \tag{III.4.3}$$

Then we have

- $D(N)$ is dense in $D(A) \cap D(H)$ with the norm $\|Hu\| + \|Au\| + \|u\|$,

- the quadratic form $i[H, A]$ defined on $D(A) \cap D(H)$ is the unique extension of $i[H, A]$ on $D(N)$,
- $H \in C^1(A)$.

The operator N above is called a *comparison operator*. When the assumptions of Lemma III.4.2 are satisfied, it is enough to compute the commutator $i[H, A]$ as a quadratic form on the common core $D(N)$ or on any core of N .

Remark III.4.1 *To prove the minimal velocity estimate in Section III.5, we will need to prove that $H \in C^2(A)$. But, notice that the assumptions required in Theorem III.4.1 actually imply that H belongs to this class. Indeed, from [1], Theorem 6.3.4, we know that, if we assume the stability property*

$$e^{-isA}D(H) \subset D(H), \quad (\text{III.4.4})$$

then the conditions $i[H, A] \in \mathcal{B}(D(H), \mathcal{H})$ together with $[A, [A, H]] \in \mathcal{B}(D(H), D(H)^)$ imply that $H \in C^2(A)$. Now the proof of (III.4.4) follows directly from the following Lemma [36]*

Lemma III.4.3 (Georgescu, Gérard) *Let H and A two self-adjoint operators such that $H \in C^1(A)$ and $i[H, A] \in \mathcal{B}(D(H), \mathcal{H})$ then $e^{isA}D(H) \subset D(H)$ for all $s \in \mathbb{R}$.*

III.4.2 Locally conjugate operator for H

In general, there exist many possibilities for the choice of a locally conjugate operator A but no notion of “adapted conjugate operator” has been put forward yet. In the case of Dirac operators in flat spacetime, at least four examples have been introduced, see [11], [14], [34], [37]. However, as quoted in [37], a good choice of A can allow to impose a minimum number of restrictions on the Hamiltonian H we consider and may serve to greatly simplify the proofs.

In our case, we shall be guided by the following two requirements :

- As showed in Proposition III.5.2 below, the minimal velocity estimates are intimately related to the existence of a conjugate operator A . Thus, the assumptions needed in this proposition should be satisfied very directly. In particular, it is convenient to find an operator A such that its domain $D(A)$ contains the domain $D(r_*)$ of the position observable and satisfying $\|Au\| \leq \|\langle r_* \rangle u\|$. If these points are satisfied then the strictly positive constant ϵ entering in the Mourre estimate (III.4.1) is of physical interest since it gives, in some sense, the exact minimal velocity with which Dirac fields escape from the centre of the interaction. The operator A must be chosen such that this constant be optimal.
- If possible, the conjugate operator A must be scalar in order to avoid the problems due to the non commutativity of the Dirac matrices. This point is important if we want to work with the minimal hypotheses on the regularity and decay at infinity of the potentials V but also if we are interested in higher order regularity properties of the boundary values of the resolvents.

The usual generator of dilations $D = \frac{1}{2}(r_* \cdot D_{r_*} + D_{r_*} \cdot r_*)$ used in [34] does not satisfy the first requirement since, obviously, its domain does not contain $D(r_*)$. The conjugate operators used in [11] and [14] were matrix-valued operators. Although they could be used in our case, because the assumptions needed on the potential V are not too strong, we prefer to follow the presentation given in [37] which presents the advantage to cumulate the two requirements above. Furthermore, it allows us to treat massive and massless Dirac fields in a unified setting. We shall also see that the choice of conjugate operator in [37] is the exact scalar equivalent to the operator introduced in [14].

The conjugate operators constructed in [37] work for Hamiltonians of the following form : $L = L_0 + V = l(D_{r_*}) + V$ with $l : \mathbb{R} \rightarrow M_4(\mathbb{C})$ a continuous function. The potential V is chosen such

that $(L + i)^{-1} - (L_0 + i)^{-1}$ is compact. Note already that this last assumption will not be satisfied by H because of the presence of two distinct asymptotic regions. To solve this new technical difficulty, we will need to use cut off functions to separate the two different ends. The operators $L_0 = l(D_{r_*})$ are *locally scalar* in the sense given by Georgescu, Mântoiu in [37].

Definition III.4.3 Let $l : \mathbb{R} \rightarrow M_4(\mathbb{C})$ a continuous function with values in the set of hermitian matrices of dimension 4. We say that the selfadjoint operator $L_0 = l(D_{r_*})$ on $\mathcal{H} = L^2(\mathbb{R}; M_4(\mathbb{C}))$ is scalar on an open real set I if

$$L_0 E_I(L_0) = \mu(D_{r_*}) E_I(L_0), \quad (\text{III.4.5})$$

where $E_I(L_0)$ is the spectral projection on I associated to the operator L_0 and $\mu : \mathbb{R} \rightarrow \mathbb{R}$ is some Borel function. L_0 is called scalar at a point $\lambda \in \mathbb{R}$ if λ has an open neighbourhood on which L_0 is scalar. The largest open set on which L_0 is scalar is denoted by $\Xi(L_0)$. We then say that L_0 is locally scalar on $\Xi(L_0)$

Let us first introduce some notations and basic results, given in [37], about this class of operators. Recall that the spectrum of the selfadjoint operator $L_0 = l(D_{r_*})$ is given by $\cup_{p \in \mathbb{R}} \sigma(l(p))$ where $\sigma(l(p))$ is the spectrum of the operator $l(p)$ acting on \mathbb{C}^4 . For any subset I of \mathbb{R} , we define

$$\Omega(I) = \{p \in \mathbb{R} \mid \sigma(l(p)) \cap I \neq \emptyset\}.$$

Using the Fourier transform, we see that $\mathcal{F} E_I(L_0) \mathcal{F}^* = \mathcal{F} E_I(l(D_{r_*})) \mathcal{F}^*$ is the operator of multiplication by the matrix-valued function $p \rightarrow E_I(l(p))$. Clearly, we have the following equivalence

$$p \in \Omega(I) \iff E_I(l(p)) \neq 0. \quad (\text{III.4.6})$$

Note that the condition $\Omega(I) = \emptyset$ corresponds to intervals I included in the resolvent set of L_0 i.e. $I \subset \mathbb{R} \setminus \sigma(L_0)$. Using (III.4.6), we also obtain the fundamental relation

$$E_I(L_0) = E_{\Omega(I)}(D_{r_*}) E_I(L_0) = E_I(L_0) E_{\Omega(I)}(D_{r_*}), \quad (\text{III.4.7})$$

which points out the connection between our matrix-valued operator and its scalar representation. It follows from (III.4.7) that the function μ in (III.4.5) could be chosen arbitrarily in $\mathbb{R} \setminus \Omega(I)$ whereas its restriction to $\Omega(I)$ is uniquely defined. We shall denote μ_I this restriction. We have the following criterion for an operator L_0 to be locally scalar on I . A function μ satisfying (III.4.5) exists if and only if for each $p \in \Omega(I)$, the matrix $l(p)$ has exactly one eigenvalue in I . This eigenvalue is just $\mu(p) \equiv \mu_I(p)$. Of course, when $\Omega(I) = \emptyset$, condition (III.4.5) is trivially satisfied for any function μ . Lemma 2.3 in [37] gives a simple method of checking that $L_0 = l(D_{r_*})$ is locally scalar on an open interval I such that $\Omega(I) \neq \emptyset$. We set

$$\delta(S) = \min\{|\lambda - \lambda'|; \lambda, \lambda' \in \sigma(S), \lambda \neq \lambda'\},$$

where S denotes a hermitian matrix of dimension 4.

Lemma III.4.4 If

$$\inf_{p \in \Omega(I)} \delta(l(p)) > 0,$$

then L_0 is locally scalar on I .

Let us see how these results apply to the Dirac Hamiltonian H . Because of the spherical symmetry of the equation, we only need to construct a conjugate operator on each spin-weighted harmonics. We recall that

$$H = \Gamma^1 D_{r_*} + a(r_*)\Gamma^2 + b(r_*)\Gamma^0 + c(r_*),$$

where we dropped the indices for H^{ln} and the potentials a_l, b_m, c_q . The potentials have very distinct behaviours at the horizon and at infinity. As a consequence, we are lead to decomposing H into a simplified dynamics and a perturbation in two different ways. At spacelike infinity, we put

$$H = H_0^m + V_0,$$

where $H_0^m = \Gamma^1 D_{r_*} + m\Gamma^0$ and $V_0(r_*) = a(r_*)\Gamma^2 + (b(r_*) - m)\Gamma^0 + c(r_*)$. On the other hand, at the horizon,

$$H = H_1 + V_1,$$

where $H_1 = \Gamma^1 D_{r_*} + c_0$ and $V_1(r_*) = a(r_*)\Gamma^2 + b(r_*)\Gamma^0 + (c(r_*) - c_0)$. It is easy to see that $V_0 \in T^{-1,0}(\mathbb{R})$ and $V_1 \in T^{0,-\infty}(\mathbb{R})$. The operators H_0^m and H_1 can be written as $h_0^m(D_{r_*})$ and $h_1(D_{r_*})$ where $h_0^m, h_1 : \mathbb{R} \rightarrow M_4(\mathbb{C})$ are infinitely differentiable functions on \mathbb{R} .

Let us prove that H_0^m, H_1 are locally scalar on suitable domains. The operator H_0^m can be decomposed onto its positive and negative spectral subspaces by

$$H_0^m = \lambda_m(D_{r_*})\mathbf{1}_{\mathbb{R}^+}(H_0^m) - \lambda_m(D_{r_*})\mathbf{1}_{\mathbb{R}^-}(H_0^m),$$

where $\pm\lambda_m(p) = \pm\sqrt{p^2 + m^2}$ are the eigenvalues of the operator $h_0^m(p)$. We denote $\Omega_0(I) = \{p \in \mathbb{R} \mid \sigma(h_0^m(p)) \cap I \neq \emptyset\}$. Obviously, for a given I , we have $\Omega_0(I) = \lambda_m^{-1}(I) \cup (-\lambda_m)^{-1}(I)$ and remark that $\Omega_0(I) = \emptyset$ if $I \subset (-m, +m)$. When $m \neq 0$, we have $\delta_{\mathbb{R} \setminus [-m, +m]}(h_0^m) \geq 2m > 0$ since $\lambda_m(p) \geq m$ and $-\lambda_m(p) \leq -m$ for all $p \in \mathbb{R}$. Thus H_0^m obviously satisfies Lemma III.4.4 on $\mathbb{R} \setminus [-m, +m]$. Since h_0^m is trivially scalar on $(-m, +m)$, we get $\Xi(H_0^m) = \mathbb{R}$. When $m = 0$, Lemma III.4.4 is satisfied on any closed interval I not containing $\{0\}$. Thus we have $\Xi(H_0^0) = \mathbb{R} \setminus \{0\}$.

According to the simple form of the Dirac matrix $\Gamma^1 = \text{diag}(-1, 1, 1, -1)$, we easily see that $h_1(p)$ has two distinct eigenvalues : $\pm p + c_0$ and thus, H_1 can be decomposed as follows.

$$H_1 = \nu_1(D_{r_*})\mathbf{1}_{]c_0, +\infty[}(H_1) + \nu_2(D_{r_*})\mathbf{1}_{]-\infty, c_0[}(H_1),$$

where $\nu_1(p) = |p| + c_0$ and $\nu_2(p) = -|p| + c_0$. Once again, H_1 trivially satisfies Lemma III.4.4 on any closed interval not containing the value $\{c_0\}$. Hence, we get $\Xi(H_1) = \mathbb{R} \setminus \{c_0\}$. We denote $\Omega_1(I) = \{p \in \mathbb{R} \mid \sigma(h_1(p)) \cap I \neq \emptyset\}$. For a given $I \subset \Xi(H_1)$, we have $\Omega_1(I) = \nu_1^{-1}(I) \cup \nu_2^{-1}(I)$.

Let us return to the construction of a conjugate operator for a operator L_0 which is locally scalar on $\Xi(L_0)$. For such a class of operators, there exists a quite natural way of defining a conjugate operator A on suitable open real sets I . However, the condition $I \subset \Xi(L_0)$ is not sufficient. There exist certain values $\lambda \in \Xi(L_0)$, called threshold values, where a Mourre estimate does not hold. A definition of the threshold set, denoted $\tau(L_0)$, has been proposed in [37], Definition 2.5. We state here a characterization of these values (see [37], Proposition 2.8).

Proposition III.4.1 *Assume that L_0 is scalar on the open real set I . Let μ be as in (III.4.5). Then a real number $\lambda \in I$ is not a threshold value of L_0 if and only if λ has a compact neighbourhood $J \subset I$ such that the set $\mu_J^{-1}(J) = \mu^{-1}(J) \cap \Omega(I)$ is compact and $\mu'(p) \neq 0$ on it. In particular, if $\Omega(I) = \emptyset$, this condition is trivially satisfied for any $\lambda \in I$. Note moreover that $\tau(L_0) \cap \Xi(L_0)$ is a closed subset of $\Xi(L_0)$.*

Let us apply this proposition to H_0^m, H_1 . Choose an interval $I \subset \Xi(H_0^m)$ such that $H_0^m E_I(H_0^m) = \pm \lambda_m(D_{r_*}) E_I(H_0^m)$. We can assume that $\Omega_0(I) \neq \emptyset$. Let J be a compact set included in I . It is clear that $\pm \lambda_m^{-1}(J)$ are also compact since $\pm \lambda_m^{-1}$ are C^∞ on I . From the exact expression of $\lambda_m(p)$, we obtain that $\pm \lambda_m(p) = 0$ if and only if $p = 0$. Once again, we have to separate the massive and the massless cases. If $m \neq 0$, since $\lambda_m(0) = m$, we conclude that the threshold set of H_0^m is $\tau(H_0^m) = \{-m, +m\}$. When $m = 0$, the threshold set as defined previously is empty but we see that, in a certain sense, it coincides with the set $\{0\}$ where H_0^m is not scalar.

In the same way, the threshold set for H_1 is empty according to our definition but it formally coincides with $\{c_0\}$ the set where H_1 is not scalar.

Georgescu, Mântoiu gave a formal argument to justify their choice of conjugate operator for operators $L_0 = l(D_{r_*})$ which are locally scalar on a certain domain, [37], Section [2.2]. We reproduce it here. Let us assume that an open interval I has been chosen such that (III.4.5) holds. Since the only relevant values of μ are the values taken on the set $\Omega(I)$, we assume that $\Omega(I) \neq \emptyset$ and that μ has been extended into a C^∞ function from \mathbb{R} to \mathbb{R} and its restriction to $\Omega(I)$ is exactly μ_I (this extension has no other aim than to define properly the first derivative of μ and use it in the next computation). Let J be the compact subset of $I \setminus \tau(L)$ defined in proposition III.4.1. Then the set $\Omega(J)$, equal to $\mu_I^{-1}(J) = \mu^{-1}(J) \cap \Omega(I)$, is a compact subset of $\Omega(I)$. Therefore, using the continuity of the function μ and using Proposition III.4.1 again, we have

$$\min_{p \in \Omega(J)} |\mu'(p)| > 0.$$

Let us denote $\epsilon_J > 0$ this minimum and thus,

$$|\mu'_{|\Omega(J)}| \geq \epsilon_J. \quad (\text{III.4.8})$$

Now, using (III.4.5) and (III.4.7), we can write the left-hand-side of (III.4.1) as

$$\begin{aligned} E_J(L_0) i[L_0, A] E_J(L_0) &= E_J(L_0) i L_0 A E_J(L_0) - E_J(L_0) i A L_0 E_J(L_0), \\ &= E_J(L_0) E_{\Omega(I)}(D_{r_*}) i[\mu(D_{r_*}), A] E_{\Omega(I)}(D_{r_*}) E_J(L_0). \end{aligned} \quad (\text{III.4.9})$$

Let us consider an operator $A = \frac{1}{2} \{F(D_{r_*}) r_* + r_* F(D_{r_*})\}$ where F is a C^∞ function from \mathbb{R} to \mathbb{R} we shall define later. It is immediate that

$$i[\mu(D_{r_*}), A] = F(D_{r_*}) \mu'(D_{r_*}). \quad (\text{III.4.10})$$

Now, (III.4.9) and (III.4.10) give

$$E_J(L_0) i[L_0, A] E_J(L_0) = E_J(L_0) E_{D_{r_*}}(\Omega(J)) F(D_{r_*}) \mu'(D_{r_*}) E_{D_{r_*}}(\Omega(J)) E_J(L_0).$$

Therefore, in order to obtain a Mourre estimate, it suffices to choose a function F such that

$$F_{|\Omega(J)}(D_{r_*}) \mu'_{|\Omega(J)}(D_{r_*}) \geq \epsilon > 0.$$

In [37], Georgescu and Mantoiu chose a function F such that $F \in C_0^\infty(\Omega(I))$ and satisfying $F_{|\Omega(J)} = \mu'_{|\Omega(J)}$. Therefore, from (III.4.8), they obtained

$$E_L(J) i[L, A] E_L(J) \geq \epsilon_J^2 E_L(J).$$

Intuitively, we would have expected to obtain ϵ_J in the Mourre estimate since ϵ_J corresponds to the minimal value taken by the scalar velocity operator μ' . Now observe that, if we consider a function $F \in C_0^\infty(\Omega(I))$ such that $F = \frac{\mu'}{|\mu'|}$ on $\Omega(J)$, we obtain

$$E_J(L_0)i[L_0, A]E_J(L_0) \geq \epsilon_J E_J(L_0).$$

Let us apply this result in our case. We first introduce two cut-off functions in order to separate the two asymptotic regions. Let $j_-, j_+ \in C^\infty(\mathbb{R})$ be two positive functions such that $j_-(r_*) = 1$ for $r_* \leq 0$ and $j_-(r_*) = 0$ for $r_* \geq 1$, $j_+(r_*) = 1$ for $r_* \geq 0$ and $j_+(r_*) = 0$ for $r_* \leq -1$. We introduce the operators

$$R_-(r_*) = j_-^2(r_*)r_*, \quad R_+(r_*) = j_+^2(r_*)r_*.$$

Let us consider an open interval I included in one of the following intervals $(-\infty, -m)$, $(-m, q_0)$, $(c_0, +m)$, or $(+m, +\infty)$. Here, we assumed without loss of generality that $q_0 \in (-m, +m)$. Note that, on such an interval I , both H_0^m and H_1 are scalar and

$$H_0^m E_I(H_0^m) = \lambda(D_{r_*})E_I(H_0^m), \quad (\text{III.4.11})$$

$$H_1 E_I(H_1) = \nu(D_{r_*})E_I(H_1). \quad (\text{III.4.12})$$

The function λ (resp. ν) is equal to $\pm\lambda_m$ or 0 (resp. ν_1 or ν_2), depending on I . Let J be a compact included in I . Then, we define two functions F, G in the following way. $F \in C_0^\infty(\Omega_1(I))$ and satisfies

$$F(p) = \frac{\nu'(p)}{|\nu'(p)|}, \quad \forall p \in \Omega_1(J). \quad (\text{III.4.13})$$

Remark that the set $\Omega_1(I)$ is never empty since there is no gap in the spectrum of H_1 and (III.4.13) is always well defined. Since this is not the case for $\Omega_0(I)$, we introduce two different definitions for G . If the set $\Omega_0(I) = \emptyset$ (which occurs when $I \subset (-m, +m)$), then we define $G = 0$. If $\Omega_0(I) \neq \emptyset$, then $G \in C_0^\infty(\Omega_0(I))$ and satisfies

$$G(p) = \frac{\lambda'(p)}{|\lambda'(p)|}, \quad \forall p \in \Omega_0(J). \quad (\text{III.4.14})$$

We introduce the operators

$$A_- = \frac{1}{2}(F(D_{r_*})R_-(r_*) + R_-(r_*)F(D_{r_*})),$$

and

$$A_+ = \frac{1}{2}(G(D_{r_*})R_+(r_*) + R_+(r_*)G(D_{r_*})).$$

We denote $A = A_- + A_+$. Let us state some simple properties of the conjugate operator A . Obviously, A is a symmetric operator on \mathcal{H} . Moreover, A can also be viewed as a pseudodifferential operator. Since $R_\pm(r_*) \in \Psi^{1,0}$ and $F(D_{r_*}), G(D_{r_*}) \in \Psi^{0,-\infty}$, we have $A \in \Psi^{1,-\infty}$ by Theorem III.A.2. As a consequence, according to Theorem III.A.1, the symmetric operator A is well defined on $\dot{H}^{1,-\infty} \supset H^{1,0} = D(r_*)$. If we apply Nelson's Lemma III.4.1 with $N = \langle r_* \rangle$, we conclude that A is essentially selfadjoint on $D(r_*)$.

Let us prove that $H \in C^1(A)$. We have to check the assertions of Lemma III.4.2. Let us define the comparison operator $N = D_{r_*}^2 + \langle r_* \rangle^2$. From [55], we know that N is a selfadjoint operator on \mathcal{H} with domain $D(N) = H^2(\mathbb{R}) \cap D(\langle r_* \rangle^2)$. It is immediate to see that $\Gamma^1 D_{r_*}$ satisfies the assumptions of

Lemma III.4.1. From [1], Proposition 7.6.3., the operator A also satisfies these assumptions. Moreover, we already saw in Proposition III.3.2 that

$$(H - z)^{-1}D(N) \subset D(N).$$

Thus, it remains to prove that

$$|(u, i[H, A]u)| \leq C(\|Hu\|^2 + \|u\|^2).$$

Actually, we shall prove that the commutator $i[H, A]$ belongs to $\mathcal{B}(\mathcal{H})$ which, of course, implies this last assertion. We have

$$\begin{aligned} i[H, A_-] &= i[H_1, A_-] + i[V_1, A_-], \\ &= \frac{1}{2}(F(D_{r_*})R'_-(r_*) + R'_-(r_*)F(D_{r_*}))\Gamma^1 + i[V_1, A_-]. \end{aligned}$$

In order to analyse the term $i[V_1, A_-]$, we use the standard results for pseudodifferential operators presented in Appendix III.A. Let $f \in T^{m,n}$. Then, we have

$$\begin{aligned} i[f(r_*), A_-] &= \frac{1}{2}(i[f(r_*), F(D_{r_*})]R_-(r_*) + hc), \\ &= \frac{1}{2}(i[f(r_*)j_-^2(r_*), F(D_{r_*})]r_* - f(r_*)i[j_-^2(r_*), F(D_{r_*})]r_* + hc). \end{aligned}$$

First observe that $f(r_*)j_-^2(r_*) \in T^{-\infty,n} \subset \Psi^{n,0}$. Since $F(D_{r_*}) \in \Psi^{0,-\infty}$, we have

$$[f(r_*)j_-^2(r_*), F(D_{r_*})] \in \Psi^{n-1,-\infty},$$

by Theorem III.A.2. Hence, we get

$$i[f(r_*)j_-^2(r_*), F(D_{r_*})]r_* \in \Psi^{n,-\infty}.$$

Furthermore, the term $[j_-^2(r_*), F(D_{r_*})]$ belongs to $\Psi^{-\infty,-\infty}$ by Proposition III.A.1. Thus, we also have

$$f(r_*)i[j_-^2(r_*), F(D_{r_*})]r_* \in \Psi^{-\infty,-\infty},$$

by Theorem III.A.2.

Now, recall that $a \in T^{-1,-\infty}$, $b \in T^{0,-\infty}$ and $c - c_0 \in T^{0,-\infty}$. Therefore, applying the previous results to the potential $V_1 = a(r_*)\Gamma^2 + b(r_*)\Gamma^0 + c(r_*) - c_0$, we obtain

$$i[V_1, A_-] = a_-^w(r_*, D_{r_*})\Gamma^2 + b_-^w(r_*, D_{r_*})\Gamma^0 + c_-^w(r_*, D_{r_*}),$$

where all the symbols a_- , b_- , c_- belong to the class $S^{-\infty,-\infty}$.

Since $R'_-(r_*) \in T^{-\infty,0} \subset \Psi^{0,0}$, the operator $\frac{1}{2}(F(D_{r_*})R'_-(r_*) + R'_-(r_*)F(D_{r_*}))$ is also a pseudodifferential operator with symbol in $S^{0,-\infty}$. We denote it $d_-^w(r_*, D_{r_*})$. Hence, we obtain

$$i[H, A_-] = d_-^w(r_*, D_{r_*})\Gamma^1 + a_-^w(r_*, D_{r_*})\Gamma^2 + b_-^w(r_*, D_{r_*})\Gamma^0 + c_-^w(r_*, D_{r_*}), \quad (\text{III.4.15})$$

and this commutator is bounded on \mathcal{H} by Theorem III.A.1.

By the same procedure, we also prove that

$$i[H, A_+] = d_+^w(r_*, D_{r_*})\Gamma^1 + a_+^w(r_*, D_{r_*})\Gamma^2 + b_+^w(r_*, D_{r_*})\Gamma^0 + c_+^w(r_*, D_{r_*}), \quad (\text{III.4.16})$$

where the symbols a_+, b_+, c_+ belong to the class $S^{-1, -\infty}$ and $d_+ \in S^{0, -\infty}$. Thus this commutator is also bounded on \mathcal{H} by Theorem III.A.1.

Eventually, applying Lemma III.4.2, we conclude that $H \in C^1(A)$.

Let us prove that the double commutator $[[H, A], A]$ is bounded on \mathcal{H} . As already mentioned, the operator A can be viewed as a pseudodifferential operator with symbol in $S^{0, -\infty}$. Thus, using (III.4.15), (III.4.16) and Theorem III.A.2, it is immediate to see that

$$[[H, A], A] = D^w(r_*, D_{r_*})\Gamma^1 + A^w(r_*, D_{r_*})\Gamma^2 + B^w(r_*, D_{r_*})\Gamma^0 + C^w(r_*, D_{r_*}),$$

where $D^w \in \Psi^{-1, -\infty}$ and $A^w, B^w, C^w \in \Psi^{-2, -\infty}$. Therefore, $[[H, A], A]$ is bounded on \mathcal{H} by Theorem III.A.2.

It remains to prove the Mourre estimate. We begin by a compactness result.

Lemma III.4.5 *Let $\chi \in C_0^\infty(\mathbb{R})$. Then,*

- (i) $j_+^2(r_*)(\chi(H) - \chi(H_0^m))$ is compact,
- (ii) $j_-^2(r_*)(\chi(H) - \chi(H_1))$ is compact.

Proof : As the proofs are identical, we only show (i). Using the Helffer-Sjöstrand formula, we can write

$$\chi(H) - \chi(H_0^m) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H)^{-1} V_0(r_*) (z - H_0^m)^{-1} dz \wedge d\bar{z}.$$

Since the integral converges in operator norm, we have

$$j_+^2(r_*)(\chi(H) - \chi(H_0^m)) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) j_+^2(r_*) (z - H)^{-1} V_0(r_*) (z - H_0^m)^{-1} dz \wedge d\bar{z}.$$

Using $[j_+^2(r_*), (z - H)^{-1}] = (z - H)^{-1} i\Gamma^1 (j_+^2)'(r_*) (z - H)^{-1}$, we obtain

$$\begin{aligned} j_+^2(r_*)(\chi(H) - \chi(H_0^m)) &= \\ &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H)^{-1} i\Gamma^1 (j_+^2)'(r_*) (z - H)^{-1} V_0(r_*) (z - H_0^m)^{-1} dz \wedge d\bar{z} \\ &+ \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H)^{-1} j_+^2(r_*) V_0(r_*) (z - H_0^m)^{-1} dz \wedge d\bar{z}. \end{aligned}$$

Let us recall the standard compactness criterion : if $f, g \in C_\infty(\mathbb{R})$ and H is a selfadjoint operator on \mathcal{H} with domain $D(H) = H^1(\mathbb{R})$ then

$$f(r_*)g(H) \text{ is compact.} \quad (\text{III.4.17})$$

Now, observe that $(j_+^2)'' \in C_0^\infty(\mathbb{R})$ and $j_+^2 V_0 \in T^{-1, -\infty} \subset C_\infty(\mathbb{R})$. Therefore, the terms $(j_+^2)''(r_*) (z - H)^{-1}$ and $j_+^2(r_*) V_0(r_*) (z - H_0^m)^{-1}$ are compact. Since both integrals converge in operator norm, the Lemma is proved.

◇

Let us choose a function $\chi \in C_0^\infty(J)$. We have

$$\chi(H)i[H, A_-]\chi(H) = \chi(H)i[H_1, A_-]\chi(H) + \chi(H)i[V_1, A_+]\chi(H).$$

We already saw that $i[V_1, A_-]$ is a pseudodifferential operator belonging to the class $\Psi^{-1, -\infty}$. Therefore, it is a compact operator by theorem III.A.3 and so is $\chi(H)i[V_1, A_-]\chi(H)$. Furthermore, we can write

$$\begin{aligned}\chi(H)i[H_1, A_-]\chi(H) &= \chi(H_1)i[H_1, A_-]\chi(H_1) \\ &\quad + (\chi(H) - \chi(H_1))i[H_1, A_-]\chi(H_1) + hc \\ &\quad + (\chi(H) - \chi(H_1))i[H_1, A_-](\chi(H) - \chi(H_1)).\end{aligned}\tag{III.4.18}$$

The explicit expression for $i[H_1, A_-]$ is $\frac{1}{2}(F(D_{r_*})R'_-(r_*) + R'_-(r_*)F(D_{r_*}))\Gamma^1$. Using Theorem III.A.2 and Theorem III.A.3, we also have

$$i[H_1, A_-] = F(D_{r_*})R'_-(r_*)\Gamma^1 + K = R'_-(r_*)F(D_{r_*})\Gamma^1 + K,$$

where K denotes a compact operator. Now, the second term in (III.4.18) can be written as

$$\begin{aligned}(\chi(H) - \chi(H_1))i[H_1, A_-]\chi(H_1) &= (\chi(H) - \chi(H_1))R'_-F(D_{r_*})\chi(H_1) + K, \\ &= (\chi(H) - \chi(H_1))j_-^2(r_*)F(D_{r_*})\chi(H_1) \\ &\quad + (\chi(H) - \chi(H_1))(j_-^2)'(r_*)r_*F(D_{r_*})\chi(H_1) + K.\end{aligned}$$

The first term is compact by Lemma III.4.5 and since $(j_-^2)'(r_*)r_* \in C_0^\infty(\mathbb{R})$, the second term is compact by (III.4.17). The other terms in (III.4.18) involving $\chi(H) - \chi(H_1)$ are treated in the same way. Therefore, we obtain

$$\chi(H)i[H_1, A_-]\chi(H) = \chi(H_1)i[H_1, A_-]\chi(H_1) + K,$$

where K is compact. We are almost in the situation described at the beginning of this part. Since $\text{supp } \chi \subset J$, using (III.4.11), we have

$$\begin{aligned}\chi(H_1)i[H_1, A_-]\chi(H_1) &= \chi(H_1)i[\nu(D_{r_*}), A_-]\chi(H_1), \\ &= \chi(H_1)\frac{1}{2}(F(D_{r_*})i[\nu(D_{r_*}), r_*]j_-^2(r_*) + hc)\chi(H_1) \tag{III.4.19} \\ &\quad + \chi(H_1)\frac{1}{2}(F(D_{r_*})r_*i[\nu(D_{r_*}), j_-^2(r_*)] + hc)\chi(H_1). \tag{III.4.20}\end{aligned}$$

According to Proposition III.A.1, the commutator $[\nu(D_{r_*}), j_-^2(r_*)]$ is a pseudodifferential operator in the class $\Psi^{-\infty, 0}$ since $\nu(D_{r_*}) \in \Psi^{0, 1}$. Then, the pseudodifferential operator $F(D_{r_*})r_*i[\nu(D_{r_*}), j_-^2(r_*)]$ belongs to the class $\psi^{-\infty, -\infty}$ by Theorem III.A.2. We conclude that the term (III.4.20) is compact by theorem III.A.3.

At last, the term (III.4.19) is equal to

$$\chi(H_1)\frac{1}{2}(F(D_{r_*})i[\nu(D_{r_*}), r_*]j_-^2(r_*) + hc)\chi(H_1) = \chi(H_1)\frac{1}{2}(F(D_{r_*})\nu'(D_{r_*})j_-^2(r_*) + hc)\chi(H_1).$$

After certain commutations, we can rewrite this term as

$$\chi(H_1)\frac{1}{2}(F(D_{r_*})i[\nu(D_{r_*}), r_*]j_-^2(r_*) + hc)\chi(H_1) = j_-(r_*)\chi(H_1)F(D_{r_*})\nu'(D_{r_*})\chi(H_1)j_-(r_*) + K,$$

where K compact. Since $\text{supp } \chi \subset J$, we have $\chi(H_1) = E_{\Omega_0(J)}(D_{r_*})\chi(H_1) = \chi(H_1)E_{\Omega_0(J)}(D_{r_*})$ by (III.4.7). Then, from the definition (III.4.14) of F , there exists a strictly positive constant ϵ_J such that

$$\chi(H_1)F(D_{r_*})\nu'(D_{r_*})\chi(H_1) \geq \epsilon_J\chi^2(H_1).$$

Therefore, we have

$$\chi(H_1)i[H_1, A_-]\chi(H_1) \geq \epsilon_J \chi(H_1)j_-^2(r_*)\chi(H_1) + K,$$

and using Lemma III.4.5 once more, we obtain

$$\chi(H_1)i[H_1, A_-]\chi(H_1) \geq \epsilon_J \chi(H)j_-^2(r_*)\chi(H) + K.$$

Putting together all the previous results, we get

$$\chi(H)i[H, A_-]\chi(H) \geq \epsilon_J \chi(H)j_-^2(r_*)\chi(H) + K. \quad (\text{III.4.21})$$

In a similar way, for any compact set J included in $(-\infty, -m)$ or $(+m, +\infty)$, there exists a strictly positive constant ϵ'_J such that

$$\chi(H)i[H, A_+]\chi(H) \geq \epsilon'_J \chi(H)j_+^2(r_*)\chi(H) + K. \quad (\text{III.4.22})$$

Therefore, adding (III.4.21) and (III.4.22), we get

$$\chi(H)i[H, A]\chi(H) \geq \min(\epsilon_J, \epsilon'_J)\chi(H)(j_-^2(r_*) + j_+^2(r_*))\chi(H) + K.$$

Since $(j_-^2 + j_+^2) - 1 \in C_0^\infty(\mathbb{R})$, we have $\chi(H)(j_-^2 + j_+^2)\chi(H) = \chi^2(H) + K$ where K is compact on \mathcal{H} by the standard compactness criterion. Therefore, we obtain the Mourre estimate

$$\chi(H)i[H, A]\chi(H) \geq \min(\epsilon_J, \epsilon'_J)\chi^2(H) + K. \quad (\text{III.4.23})$$

When $J \subset (-m, +m)$, we have $i[H, A_+] = 0$ since $A_+ = 0$ in this case. Whence

$$\chi(H)i[H, A]\chi(H) \geq \epsilon_J \chi(H)j_-^2(r_*)\chi(H) + K.$$

Since $\chi(H)j_+^2(r_*)\chi(H) = \chi(H)j_+^2(r_*)(\chi(H) - \chi(H_0^m))$ is compact by Lemma III.4.5, we have

$$\chi(H)(j_-^2)\chi(H) = \chi(H)(j_-^2 + j_+^2)\chi(H) + K.$$

By the preceding argument, we obtain the Mourre estimate

$$\chi(H)i[H, A_+]\chi(H) \geq \epsilon_J \chi^2(H) + K. \quad (\text{III.4.24})$$

Remark III.4.2 *The constants ϵ_J and ϵ'_J are respectively equal to*

$$\epsilon_J = \min_{p \in \Omega(J)} |\nu'(p)| = 1,$$

and

$$\epsilon'_J = \min_{p \in \Omega(J)} |\lambda'_m(p)| = \lambda'_m(d_J) \leq 1,$$

where $d_J = \text{dist}(\Omega(J), 0)$. Let us denote by ϵ_χ the strictly positive constant which enters in the Mourre estimate

$$\chi(H)i[H, A]\chi(H) \geq \epsilon_\chi \chi^2(H) + K,$$

for a given $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m, c_0\} = \emptyset$. Then

– If $m \neq 0$ and $\text{supp } \chi \subset (-m, +m)$ then $\epsilon_\chi = 1$.

- If $m \neq 0$ and $\text{supp } \chi \cap (\mathbb{R} \setminus (-m, +m)) \neq \emptyset$ then $\epsilon_\chi = \lambda'_m(\delta_{\text{supp } \chi}) < 1$.
- If $m = 0$ then $\epsilon_\chi = 1$.

Using Theorem III.4.1 and Proposition III.3.3, we thus have proved

Theorem III.4.2 *The spectrum of the Dirac Hamiltonian H is purely absolutely continuous, i.e.*

$$\sigma(H) = \sigma_{ac}(H) = \mathbb{R}.$$

In particular, we retrieve the fact that the energy never remains trapped in a compact set as shown in [31], [32]. In other words, the fields must either escape to infinity or dive into the horizon of the black hole.

III.5 Weak propagation estimates

In this Section, we prove various propagation estimates which will be the key ingredients to construct and analyse the asymptotic velocity in Section III.6. The main tool used to establish these estimates is the method of *positive commutator*. We refer to [21] for a detailed presentation.

We denote $\mathbf{D} = \frac{d}{dt} + i[H, \cdot]$ the Heisenberg derivative which acts on time-dependent selfadjoint observables $B(t)$ and satisfies the following property

$$\frac{d}{dt} \left[e^{itH} B(t) e^{-itH} \right] = e^{itH} \mathbf{D} B(t) e^{-itH}.$$

We shall use constantly the following fundamental criterion.

Proposition III.5.1 *Let $\Phi(t)$ be a family of selfadjoint operators belonging to $W_{loc}^{1,1}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ i.e. there exists $B(t) \in L_{loc}^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ such that*

$$\Phi(t_1) - \Phi(t_2) = \int_{t_1}^{t_2} B(s) ds.$$

(i) *Assume that $\mathbf{D}\Phi(t) \in L_{loc}^1(\mathbb{R}^+, \mathcal{B}(\mathcal{H}))$. Then*

$$\|\Phi(t) e^{-itH} \psi\| \leq \|\Phi(0) \psi\| + \int_0^t \|\mathbf{D}\Phi(s) e^{-isH} \psi\| ds.$$

(ii) *Assume that $\Phi(t)$ is uniformly bounded and that there exists $C_0 > 0$ and some operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \dots, n$ such that*

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t) B(t) - \sum_{i=1}^n B_i^*(t) B_i(t),$$

with

$$\int_1^\infty \|B_i(t) e^{-itH} \psi\|^2 dt \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad i = 1, \dots, n.$$

Then there exists a constant C such that

$$\int_1^\infty \|B(t) e^{-itH} \psi\|^2 dt \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.1})$$

Estimates taking the form of (III.5.1) are called *weak propagation estimates* since the only information we have concerning the time-decay of $\frac{1}{t}\|B(t)e^{-itH}\psi\|$ is that this quantity belongs to $L^1(dt)$. In particular, we cannot conclude that $\|B(t)e^{-itH}\psi\| \in O(t^{-\rho})$ for any $\rho > 0$. Nevertheless, this weak information is enough to prove the existence of *asymptotic observables*. By this, we mean the following : if $B(t)$ is a time-dependent uniformly bounded observable, the asymptotic observable related to $B(t)$ along the evolution e^{-itH} is given by the limit

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} B(t) e^{-itH}. \quad (\text{III.5.2})$$

We state here two criteria, due to Cook and Kato, to prove the existence of (III.5.2) in a slightly more general form that we will need later. We refer to [21], Lemma B.4.2, for the proof of the following result.

Lemma III.5.1 (Cook, Kato) *Let $\mathcal{H}_1, \mathcal{H}_2$ two Hilbert spaces and H_1 (resp. H_2) two selfadjoint operators on \mathcal{H}_1 (resp. \mathcal{H}_2). We denote $\|\cdot\|_1$ and $\|\cdot\|_2$ the norms on \mathcal{H}_1 and \mathcal{H}_2 respectively. Let $\phi(t) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a uniformly bounded function belonging to $W_{loc}^{1,1}(\mathbb{R}; \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))$. We suppose that*

$$H_2\phi(t) - \phi(t)H_1,$$

originally defined as a quadratic form on $D(H_2) \times D(H_1)$ extends to an element of $L_{loc}^1(\mathbb{R}; \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))$. We denote

$$\mathbf{D}_2\phi(t) = \frac{d}{dt}\phi(t) + i(H_2\phi(t) - \phi(t)H_1).$$

Let \mathcal{D} a dense subspace of \mathcal{H}_1 .

(i) (Cook) *Assume that for $\psi \in \mathcal{D}$,*

$$\int_1^\infty \|\mathbf{D}_2\Phi(t)e^{-itH_1}\psi\|_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)} dt < \infty,$$

then there exists

$$s - \lim_{t \rightarrow \infty} e^{itH_2}\Phi(t)e^{-itH_1}. \quad (\text{III.5.3})$$

(ii) (Kato) *Assume that*

$$|(\psi_2, \mathbf{D}_2\Phi(t)\psi_1)| \leq \sum_{i=1}^n \|B_{2i}(t)\psi_2\|_2 \|B_{1i}(t)\psi_1\|_1,$$

with

$$\begin{aligned} \int_1^\infty \|B_{2i}(t)e^{-itH}\psi\|_2^2 dt &\leq C\|\psi\|_2^2, & \psi \in \mathcal{H}_2, i = 1, \dots, n, \\ \int_1^\infty \|B_{1i}(t)e^{-itH}\psi\|_1^2 dt &\leq C\|\psi\|_1^2, & \psi \in \mathcal{D}, i = 1, \dots, n, \end{aligned}$$

then the limit (III.5.3) exists.

III.5.1 Large velocity estimates

In General Relativity, a signal cannot travel faster than the speed of light (equal to 1 in the radial direction with our choice of variables). Hence, although the solutions of the Dirac equation (III.2.7) spread out in all the spacetime, we expect that they will not live, “in a certain sense”, in the region $\{|r_*| > t\}$. The following estimates give a rigorous meaning of this expectation.

Proposition III.5.2 *Let $1 < \theta_1 < \theta_2$. Let $\chi \in C_0^\infty(\mathbb{R})$. Then, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{|r_*|}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.4})$$

Given any $g \in C^\infty(\mathbb{R})$ with $g' \in C_0^\infty(\mathbb{R})$ and $\text{supp } g \subset (1, +\infty)$, we have

$$s - \lim_{t \rightarrow \pm\infty} g \left(\frac{|r_*|}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{III.5.5})$$

Proof : Let $1 < \theta_0 < \theta_1 < \theta_2$ and let $f \in C_0^\infty([\theta_0, +\infty[)$ such that $f = 1$ on $[\theta_1, \theta_2]$ and $f \geq 0$. We define

$$F(r_*) = \int_{-\infty}^{r_*} f^2(u) du,$$

which is clearly bounded and continuous on \mathbb{R} . Let us choose the propagation observable

$$\phi(t) = \chi(H) F \left(\frac{|r_*|}{t} \right) \chi(H).$$

The selfadjoint operator-valued function $\Phi(t)$ is uniformly bounded in t . The computation of its Heisenberg derivative leads to

$$\begin{aligned} -\mathbf{D}\phi(t) &= \chi(H) \mathbf{D}F \left(\frac{|r_*|}{t} \right) \chi(H), \\ &= \frac{1}{t} \chi(H) f^2 \left(\frac{|r_*|}{t} \right) \frac{|r_*|}{t} \chi(H) - \frac{1}{t} \chi(H) \Gamma^1 f^2 \left(\frac{|r_*|}{t} \right) \frac{r_*}{|r_*|} \chi(H). \end{aligned} \quad (\text{III.5.6})$$

Now, the assumptions required on f and $\|\Gamma^1 \frac{r_*}{|r_*|}\| \leq 1$ yield

$$-\mathbf{D}\phi(t) \geq \frac{1}{t} (\theta_0 - 1) \chi(H) \mathbf{1}_{[\theta_1, \theta_2]}^2 \left(\frac{|r_*|}{t} \right) \chi(H).$$

As $(\theta_0 - 1) > 0$, (III.5.4) follows from Proposition III.5.1.

To prove (III.5.5), it is enough to assume $g \geq 0$ and $g(r_*) = 1$ for $r_* \geq R_0$. We begin by considering the case where $g \in C_0^\infty(\mathbb{R})$. Let $\tilde{g} \in C_0^\infty((1, +\infty))$ chosen such that $\tilde{g} = 1$ on $\text{supp } g'$. We define

$$\phi(t) = \chi(H) g \left(\frac{|r_*|}{t} \right) \chi(H).$$

Computing its Heisenberg derivative, we obtain

$$\mathbf{D}\phi(t) = -\frac{1}{t} \chi(H) g' \left(\frac{|r_*|}{t} \right) \frac{|r_*|}{t} \chi(H) + \frac{1}{t} \chi(H) \Gamma^1 g' \left(\frac{|r_*|}{t} \right) \frac{r_*}{|r_*|} \chi(H).$$

Using $\tilde{g} g' = g'$, we get

$$\mathbf{D}\phi(t) = \frac{1}{t} \chi(H) \tilde{g} \left(\frac{|r_*|}{t} \right) B(t) \tilde{g} \left(\frac{|r_*|}{t} \right) \chi(H), \quad (\text{III.5.7})$$

where $B(t) = -g' \left(\frac{|r_*|}{t} \right) \frac{|r_*|}{t} + \Gamma^1 g' \left(\frac{|r_*|}{t} \right) \frac{r_*}{|r_*|}$ is uniformly bounded. Now, the existence of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) e^{-itH}, \quad (\text{III.5.8})$$

follows from (III.5.7), (III.5.4) and Lemma III.5.1. Moreover as $\text{supp } g \subset (1, +\infty)$, (III.5.4) implies that the limit in (III.5.8) is zero.

Finally, we prove the general case. Let us assume that there exists $g_1 \in C_0^\infty(\mathbb{R})$ such that $\text{supp } g_1 \subset (1, +\infty)$, $g_1' = g_1^2$. We introduce the propagation observable

$$\phi_R(t) = \chi(H)g\left(\frac{|r_*|}{Rt}\right)\chi(H),$$

where R is a positive number. Let us compute its Heisenberg derivative. We obtain

$$\begin{aligned} -\mathbf{D}\phi_R(t) &= \frac{1}{t}\chi(H)g_1^2\left(\frac{|r_*|}{Rt}\right)\frac{|r_*|}{Rt}\chi(H) - \frac{1}{Rt}\chi(H)g_1^2\left(\frac{|r_*|}{Rt}\right)\frac{r_*}{|r_*|}\chi(H), \\ &\geq \frac{1}{t}\left(\theta_0 - \frac{1}{R}\right)\chi(H)g_1^2\left(\frac{|r_*|}{Rt}\right)\chi(H). \end{aligned}$$

For $\theta_0 > 1$ and $R \geq 1$, we see that $(\theta_0 - \frac{1}{R})$ is positive. Thus, under these conditions,

$$\mathbf{D}\phi_R(t) \leq 0. \quad (\text{III.5.9})$$

Now, if we fix $t_0 \in \mathbb{R}$, we have

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH} = e^{it_0H} \phi_R(t_0) e^{-it_0H} + \int_{t_0}^{\infty} e^{isH} \mathbf{D}\phi_R(s) e^{-isH} ds,$$

and we deduce from (III.5.9) and the positivity of g that

$$0 \leq s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH} \leq e^{it_0H} \phi_R(t_0) e^{-it_0H}, \quad \forall R \geq 1. \quad (\text{III.5.10})$$

But remark that by definition of $\phi_R(t)$, we have

$$s - \lim_{R \rightarrow +\infty} (e^{it_0H} \phi_R(t_0) e^{-it_0H}) = 0.$$

Hence, we obtain from (III.5.10) that

$$s - \lim_{R \rightarrow +\infty} (s - \lim_{t \rightarrow +\infty} e^{itH} \phi_R(t) e^{-itH}) = 0. \quad (\text{III.5.11})$$

At last, observe that $g\left(\frac{|r_*|}{Rt}\right) - g\left(\frac{|r_*|}{t}\right)$ has compact support. Therefore, using (III.5.8), we have

$$s - \lim_{t \rightarrow +\infty} e^{itH} (\phi(t) - \phi_R(t)) e^{-itH} = 0.$$

Then, if we let R to tend to $+\infty$, (III.5.5) holds by (III.5.11).

◇

It is possible to be more precise concerning what happens at spacelike infinity. We can show that the maximal velocity in estimate (III.5.4) depends on the energy of the solution. We proved that, in the general case, this velocity in the radial direction is 1. But, if we consider states whose energy lies in $[-m, +m]$ then the maximal velocity is in fact zero. In other words, such states do not propagate towards infinity but instead, fall into the black hole. Precisely, we have

Proposition III.5.3 *Let $0 < \theta$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \subset [-m, +m]$. Then, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[\theta, +\infty)}\left(\frac{r_*}{t}\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.12})$$

Moreover, we have

$$s - \lim_{t \rightarrow \pm\infty} \mathbf{1}_{[\theta, +\infty)}\left(\frac{r_*}{t}\right) \chi(H) e^{-itH} = 0. \quad (\text{III.5.13})$$

Proof : Remark that, since $\chi(H_0^m) = 0$, it suffices to prove that

$$J\left(\frac{r_*}{t}\right) (\chi(H) - \chi(H_0^m)) \in O(t^{-1}), \quad (\text{III.5.14})$$

for a function $J \in C^\infty(\mathbb{R})$ such that $J(r_*) = 1$ for $r_* \geq \theta$ and $J(r_*) = 0$ for $r_* \leq \frac{\theta}{2}$. But this follows from the Helffer-Sjöstrand formula and the fact that $J\left(\frac{r_*}{t}\right) V_0(r_*) \in O(t^{-1})$.

◇

III.5.2 Minimal velocity estimates

As mentioned in the Introduction, the minimal velocity estimates tend to replace the well known Huygens principle. They have the following interpretation : at late times, the energy of a solution of (III.2.7) located in a narrow cone $\mathcal{C} = \{|r_*| \leq ct\}$ tends to 0 in a very weak sense. In other words, Dirac fields must escape from \mathcal{C} when t tends to infinity and the constant c thus corresponds to a lower bound of its *minimal velocity*. In our case, even though it is not necessary for the later constructions (asymptotic velocity and wave operators), it would be interesting to distinguish the minimal velocity estimates at the horizon and infinity since the exact values of these velocities have no reason to be the same in the two asymptotic regions.

It is well known that this type of estimate is closely related to the existence of a conjugate operator for H (see for instance [49]). We state here a Proposition, originally due to Sigal and Soffer in [72], with a slight modification useful to separate the two asymptotic regions. Precisely, we have

Proposition III.5.4 *Let H, A_\pm three selfadjoint operators on \mathcal{H} . Assume that $H \in C^2(A_\pm)$, and there exist strictly positive constants $c_{\pm, I}$ such that the Mourre estimates*

$$E_I(H) i[H, A_\pm] E_I(H) \geq c_{\pm, I} E_I(H) j_\pm^2(r_*) E_I(H), \quad (\text{III.5.15})$$

hold on an open interval I . Here, the functions $j_\pm \in C^\infty(\mathbb{R})$ are the positive functions defined in Section III.4. Assume moreover that $(1 - j_\pm^2(r_))A_\pm$ bounded on \mathcal{H} . Then, $\forall g_\pm \in C^\infty(\mathbb{R}), \forall \chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } g_\pm \subset (-\infty, c_{\pm, I}), g_\pm = 1$ on $(-\infty, 0)$ and $\text{supp } \chi \subset I$,*

$$\int_1^\infty \left\| \mathbf{1}_{\mathbb{R}^\pm}(r_*) g_\pm\left(\frac{A_\pm}{t}\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{III.5.16})$$

and

$$s - \lim_{t \rightarrow \infty} \mathbf{1}_{\mathbb{R}^\pm}(r_*) g_\pm\left(\frac{A_\pm}{t}\right) \chi(H) e^{-itH} = 0. \quad (\text{III.5.17})$$

Gérard and Nier in [39] proved that it was possible to replace the operators A_\pm by physically more meaningful operators such as the position operator.

Lemma III.5.2 (Gérard, Nier) *Let A_{\pm} be locally conjugate operators for H on the interval I satisfying (III.5.15). Assume that there exists another selfadjoint operator B which satisfies*

- (i) $D(B) \subset D(A_{\pm})$,
- (ii) $\pm A_{\pm} \leq B$, $B \geq 1$,
- (iii) $[A_{\pm}, B]B^{-1} \in \mathcal{B}(\mathcal{H})$,

then for any $c_0 < c_{\pm, I}$, we have

$$\|\mathbf{1}(\frac{B}{t} \leq c_0)\mathbf{1}(\frac{A_{\pm}}{t} \geq c_{\pm, I})\| \in O(t^{-1}). \quad (\text{III.5.18})$$

Consequently, $\forall \chi \in C_0^{\infty}(\mathbb{R})$, $\text{supp } \chi \subset I$, we have

$$\int_1^{\infty} \left\| \mathbf{1}_{\mathbb{R}^{\pm}}(r_*)\mathbf{1}_{[0, c_0]}(\frac{B}{t})\chi(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{III.5.19})$$

and

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{\mathbb{R}^{\pm}}(r_*)\mathbf{1}_{[0, c_0]}(\frac{B}{t})\chi(H)e^{-itH} = 0. \quad (\text{III.5.20})$$

Let us apply these results. We prove the following Proposition.

Proposition III.5.5 *Let $\chi \in C_0^{\infty}(\mathbb{R})$ such that $\text{supp } \chi \subset \mathbb{R} \setminus \{-m, +m, c_0\}$. Then there exists a strictly positive constant ϵ_{χ} such that for all $\delta \in]0, \epsilon_{\chi}[$, we have*

$$\int_1^{\infty} \left\| \mathbf{1}_{(-1+\delta, \epsilon_{\chi}-\delta)}(\frac{r_*}{t})\chi(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C_{\delta}\|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.21})$$

Furthermore,

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{(-1+\delta, \epsilon_{\chi}-\delta)}(\frac{r_*}{t})\chi(H)e^{-itH} = 0. \quad (\text{III.5.22})$$

Proof : We prove (III.5.21). Let J be a closed interval included in $\mathbb{R} \setminus \{-m, +m, c_0\}$. For such an interval J , there exist conjugate operators A_{\pm} according to Section III.4.1 and these operators satisfy the following Mourre estimates

$$E_J(H)i[H, A_{\pm}]E_J(H) \geq \epsilon_{\pm, J}E_J(H)j_{\pm}^2(r_*)E_J(H) + E_J(H)KE_J(H),$$

where K is compact. Since $\sigma_p(H) = \emptyset$, the operator $E_J(H)KE_J(H)$ tends to 0 in norm if $|J|$ tends to 0. Therefore, if we let $|J|$ tend to 0, we can find $\epsilon_{\pm, J} > \delta_J$ with $\delta_J > 0$ as small as we want such that

$$E_J(H)i[H, A_{\pm}]E_J(H) \geq (\epsilon_{\pm, J} - \delta_J)E_J(H)j_{\pm}^2(r_*)E_J(H).$$

Let $\delta \in]0, \min(\epsilon_{-, J}, \epsilon_{+, J})[$. We take $\delta_J = \frac{\delta}{2}$. Now, let us apply Lemma III.5.2 with the operator $B = \langle r_* \rangle_{\mu} = (r_*^2 + \mu^2)^{\frac{1}{2}}$.

We already showed that $D(\langle r_* \rangle_{\mu}) = D(r_*) \subset D(A_{\pm})$. To prove (ii), we estimate for any $u \in D(r_*)$,

$$\begin{aligned} |(u, A_{-}u)| &\leq |(u, F(D_{r_*})R_{-}(r_*)u)|, \\ &\leq |(\langle r_* \rangle^{\frac{1}{2}}u, \langle r_* \rangle^{-\frac{1}{2}}F(D_{r_*})j_{-}^2(r_*)r_*\langle r_* \rangle^{-\frac{1}{2}}\langle r_* \rangle^{\frac{1}{2}}u)|. \end{aligned}$$

After a certain number of commutations and using standard results of Appendix III.A, we obtain

$$|(u, A_- u)| \leq |(j_-(r_*) \langle r_* \rangle^{\frac{1}{2}} u, F(D_{r_*}) r_* \langle r_* \rangle^{-1} j_-(r_*) \langle r_* \rangle^{\frac{1}{2}} u)| + C \|u\|^2.$$

Since the functions F, G entering in the definition of A satisfy $\|F\|_\infty \leq 1$ and $\|G\|_\infty \leq 1$, we have in particular $\|F(D_{r_*})\| \leq 1$. As $\|r_* \langle r_* \rangle^{-1} j_-(r_*)\| \leq 1$, we get

$$\begin{aligned} |(u, A_- u)| &\leq \|\langle r_* \rangle^{\frac{1}{2}} u\|^2 + C \|u\|^2, \\ &\leq (u, \langle r_* \rangle_\mu u), \end{aligned} \quad (\text{III.5.23})$$

for μ large enough. Similarly, we have

$$|(u, A_+ u)| \leq (u, \langle r_* \rangle_\mu u). \quad (\text{III.5.24})$$

The assumption (iii) of Lemma III.5.2 follows immediatly from theorem III.A.2. Therefore, by Lemma III.5.2, for any $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \subset J$, we obtain

$$\int_1^\infty \left\| \mathbf{1}_{\mathbb{R}^\pm}(r_*) \mathbf{1}_{[0, \epsilon_\pm, J-\delta)} \left(\frac{\langle r_* \rangle_\mu}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.25})$$

Let us denote $\epsilon_\chi = \epsilon_{+, J}$. Now, observe that for t large enough, we have

$$\left\| \mathbf{1}_{\mathbb{R}^+}(r_*) \mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{\langle r_* \rangle_\mu}{t} \right) - \mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \right\|^2 = \int_{\sqrt{t^2(\epsilon_\chi - \delta)^2 - \mu^2}}^{t(\epsilon_\chi - \delta)} dr_* = \frac{\mu^2}{2t(\epsilon_\chi - \delta)} + O(t^{-3}).$$

Hence, we have

$$\int_1^\infty \left\| \left(\mathbf{1}_{\mathbb{R}^\pm}(r_*) \mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{\langle r_* \rangle_\mu}{t} \right) - \mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2.$$

Thus we can replace $\mathbf{1}_{\mathbb{R}^\pm}(r_*) \mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{\langle r_* \rangle_\mu}{t} \right)$ by $\mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right)$ in (III.5.25) and we get

$$\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.26})$$

Similarly, at the horizon, since $\epsilon_{-, J} = 1$, we obtain

$$\int_1^\infty \left\| \mathbf{1}_{(-1+\delta, 0]} \left(\frac{r_*}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.27})$$

Therefore, (III.5.26) and (III.5.27) imply (III.5.21) for any χ with sufficiently small support. It only remains to prove (III.5.21), (III.5.22) for general χ such that $\text{supp } \chi \cap \{-m, +m, 0\} = \emptyset$. But such a function can be written as $\chi = \sum_{j=1}^N \chi_j$ where $\text{supp } \chi_j$ are small enough such that

$$\int_1^\infty \left\| \mathbf{1}_{(-1+\delta, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi_j(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.28})$$

is satisfied for any $j = 1, \dots, N$. Then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\int_1^\infty \left(\mathbf{1}_{(-1+\delta, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi_i(H) e^{-itH} \psi, \mathbf{1}_{(-1+\delta, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi_j(H) e^{-itH} \psi \right) \frac{dt}{t} \\ &\leq \left(\int_1^\infty \left\| \mathbf{1}_{(-1+\delta, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi_i(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_1^\infty \left\| \mathbf{1}_{(-1+\delta, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi_j(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq C \|\psi\|^2. \end{aligned}$$

Hence

$$\int_1^\infty \left\| \mathbf{1}_{(-1+\delta, \epsilon_\chi - \delta)} \left(\frac{r_*}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq N^2 C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{III.5.29})$$

which concludes the proof of the proposition.

◇

Having proved both maximal and minimal velocity estimates, we can provide a first picture of the propagation of Dirac fields outside Reissner-Nordström black holes. Let $a, b \in]0, 1[$. We define the domains $\mathcal{C}_{a,b}^\pm = \{(t, r_*) \in \mathbb{R} \times \mathbb{R}^\pm; (1-a)t \leq |r_*| \leq (1+b)t\}$.

- If $m \neq 0$ and $\text{supp } \chi \subset (-m, +m)$, the energy is concentrated in the domain $\mathcal{C}_{\delta,\delta}^-$ for any $\delta \in]0, 1[$.
- If $m \neq 0$ and $\text{supp } \chi \cap (\mathbb{R} \setminus (-m, +m)) \neq \emptyset$, the energy is concentrated in the domain $\mathcal{C}_{\delta,\delta}^- \cup \mathcal{C}_{1-\epsilon,\delta}^+$ for any $\delta \in]0, 1[$ and a fixed $\epsilon > 0$ depending on the distance between the threshold set $\{-m, +m\}$ and the energy of the field.
- If $m = 0$ then the energy is concentrated in the domain $\mathcal{C}_{\delta,\delta}^- \cup \mathcal{C}_{\delta,\delta}^+$ for any $\delta \in]0, 1[$.

III.5.3 Microlocal velocity estimates

In both previous propagation estimates, we have no information on what happens in the regions $\mathcal{C}_{\delta,\delta}^-$ and $\mathcal{C}_{1-\epsilon,\delta}^+$ where the energy of the Dirac field is concentrated at late times. The microlocal velocity estimates state that, in these regions, we can approach the position operator r_* by the operator $t\mathcal{V}$. Here, \mathcal{V} denotes the classical velocity operator. As we saw in Section III.3.5, it must be chosen according to the asymptotic region we consider.

Microlocal velocity estimates at the horizon

At the horizon, the mass of the Dirac field is not perceived. Henceforth, the classical velocity observable will be $\mathcal{V} = \Gamma^1$. We prove the following estimates.

Proposition III.5.6 *Let $0 < \theta_1 < \theta_2$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, c_0, m\} = \emptyset$. Then, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[-\theta_2, -\theta_1]} \left(\frac{r_*}{t} \right) (\Gamma^1 - \frac{r_*}{t}) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{III.5.30})$$

and

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[-\theta_2, -\theta_1]} \left(\frac{r_*}{t} \right) (\Gamma^1 - \frac{r_*}{t}) \chi(H) e^{-itH} = 0. \quad (\text{III.5.31})$$

Proof : By Propositions III.5.2 and III.5.5, we can assume that $\theta_1 < 1$ and $\theta_2 > 1$. Let $J \in C_0^\infty(\mathbb{R}^-)$ such that $J = 1$ on a neighbourhood of $[-\theta_2, -\theta_1]$. Let $R \in C_0^\infty(\mathbb{R})$ such that $R' = 0$ on a neighbourhood of 0 and $R(r_*) = \frac{r_*^2}{2}$ on $\text{supp } J$. We define the propagation observable

$$\phi(t) = \chi(H) J \left(\frac{r_*}{t} \right) M(t) J \left(\frac{r_*}{t} \right) \chi(H),$$

where

$$M(t) = R \left(\frac{r_*}{t} \right) + (\Gamma^1 - \frac{r_*}{t}) R' \left(\frac{r_*}{t} \right).$$

Clearly, $\phi(t)$ is an uniformly bounded selfadjoint operator-valued function of t . Let us compute its Heisenberg derivative. We obtain

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t}\chi(H)(\Gamma^1 - \frac{r_*}{t})J'(\frac{r_*}{t})M(t)J(\frac{r_*}{t})\chi(H) + hc \\ &\quad + \frac{1}{t}\chi(H)J(\frac{r_*}{t})(\Gamma^1 - \frac{r_*}{t})R''(\frac{r_*}{t})(\Gamma^1 - \frac{r_*}{t})J(\frac{r_*}{t})\chi(H) \\ &\quad + \chi(H)J(\frac{r_*}{t})\{ia(r_*)[\Gamma^2, \Gamma^1] + ib(r_*)[\Gamma^0, \Gamma^1]\}J(\frac{r_*}{t})\chi(H). \end{aligned} \quad (\text{III.5.32})$$

Now, denote by \tilde{J} a function in $C_0^\infty(\mathbb{R}^-)$ such that $\tilde{J}J' = J'$ and $\text{supp } \tilde{J} \subset (-\infty, -\theta_2) \cup (-\theta_1, 0)$. Then, the first term in the right-hand-side of (III.5.32) can be written as

$$\frac{1}{t}\chi(H)\tilde{J}(\frac{r_*}{t})B(t)\tilde{J}(\frac{r_*}{t})\chi(H),$$

where $B(t)$ is uniformly bounded. By Propositions III.5.2 and III.5.5, this term is integrable along the evolution since $\theta_1 < 1$ and $\theta_2 > 1$.

Moreover, recall that $a \in T^{-1, -\infty}$ and $b \in T^{0, -\infty}$. Then, since $\text{supp } J \subset \mathbb{R}^-$ and $J = 0$ on a neighbourhood of 0, we have $a(r_*)J(\frac{r_*}{t}), b(r_*)J(\frac{r_*}{t}) \in O(t^{-\infty})$. Hence, the third term in (III.5.32) is integrable in norm.

Finally, we can rewrite (III.5.32) as follows

$$\mathbf{D}\phi(t) = \frac{1}{t}\chi(H)(\Gamma^1 - \frac{r_*}{t})J^2(\frac{r_*}{t})R''(\frac{r_*}{t})(\Gamma^1 - \frac{r_*}{t})\chi(H) + L^1(dt).$$

But observe that $J^2(\frac{r_*}{t})R''(\frac{r_*}{t}) - \mathbf{1}_{[-\theta_2, -\theta_1]}(\frac{r_*}{t}) \geq 0$. Thus,

$$\mathbf{D}\phi(t) \geq \frac{1}{t}\chi(H)(\Gamma^1 - \frac{r_*}{t})\mathbf{1}_{[-\theta_2, -\theta_1]}(\frac{r_*}{t})(\Gamma^1 - \frac{r_*}{t})\chi(H) + L^1(dt).$$

We conclude the proof of (III.5.30) applying Proposition III.5.1.

Let us now prove (III.5.31). Let $J \in C_0^\infty(\mathbb{R}^-)$ such that $J = 1$ on $[-\theta_2, -\theta_1]$. We define the propagation observable

$$\phi(t) = \chi(H)J(\frac{r_*}{t})(\Gamma^1 - \frac{r_*}{t})^2J(\frac{r_*}{t})\chi(H).$$

A straightforward computation leads to

$$\begin{aligned} -\mathbf{D}\phi(t) &= -\frac{1}{t}\chi(H)J'(\frac{r_*}{t})(\Gamma^1 - \frac{r_*}{t})^3J(\frac{r_*}{t})\chi(H) + hc \\ &\quad + 2\chi(H)J(\frac{r_*}{t})\{ia(r_*)[\Gamma^2, \Gamma^1] + ib(r_*)[\Gamma^0, \Gamma^1]\}\frac{r_*}{t}J(\frac{r_*}{t})\chi(H) \\ &\quad + \frac{2}{t}\phi(t). \end{aligned} \quad (\text{III.5.33})$$

As $\text{supp } J' \subset (-\infty, -\theta_2) \cup (-\theta_1, 0)$, the first term in (III.5.33) is integrable along the evolution thanks to Propositions III.5.2 and III.5.5.

The second term in (III.5.33) is integrable in norm by the same argument as previously.

The term $\frac{2}{t}\phi(t)$ in (III.5.33) is also integrable along the evolution according to (III.5.30). Therefore, using Lemma III.5.1, the following limit

$$\lim_{t \rightarrow +\infty} (e^{-itH}\psi, \phi(t)e^{-itH}\psi), \quad (\text{III.5.34})$$

exists. But, we also have

$$\int_1^\infty (e^{-itH}\psi, \phi(t)e^{-itH}\psi) \frac{dt}{t} < \infty,$$

by (III.5.30) again. We conclude that the limit (III.5.34) is zero. Hence, (III.5.31) is proved.

◇

Microlocal velocity estimates at spacelike infinity

At spacelike infinity, the mass m of the Dirac field plays an important role. In this case, the classical velocity observable is given by $\mathcal{V}_m = D_{r_*}(H_0^m)^{-1}$ when $m \neq 0$ and $\mathcal{V}_0 = \Gamma^1$ when $m = 0$. The microlocal velocity estimates take the following form.

Proposition III.5.7 *Let $0 < \theta_1 < \theta_2$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m, c_0\} = \emptyset$. Then, we have*

$$\int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{r_*}{t} \right) (\mathcal{V}_m - \frac{r_*}{t}) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{III.5.35})$$

and

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{r_*}{t} \right) (\mathcal{V}_m - \frac{r_*}{t}) \chi(H) e^{-itH} = 0. \quad (\text{III.5.36})$$

In particular, for massless Dirac fields, we have to replace \mathcal{V}_m by Γ^1 .

Proof : It is enough to assume that $\theta_1 < \epsilon_\chi$ and $\theta_2 > 1$ where ϵ_χ is defined by the minimal velocity estimate. We can also assume that $\text{supp } \chi$ is included on a closed interval in $\mathbb{R} \setminus \{-m, +m, c_0\}$. For this choice of χ , we have

$$H_0^m \chi(H_0^m) = \pm \lambda_m(D_{r_*}) \chi(H_0^m), \quad (\text{III.5.37})$$

$$\mathcal{V}_m \chi(H_0^m) = \pm \lambda'_m(D_{r_*}) \chi(H_0^m), \quad (\text{III.5.38})$$

depending on the interval in which $\text{supp } \chi$ is located. The only relevant values for $\lambda_m(p)$ being in the compact set $\Omega(\text{supp } \chi)$ and since $\text{supp } \chi$ avoids the thresholds values, we can assume that λ_m has been extended into a C_0^∞ function on \mathbb{R} . Finally, the proofs being identical, we only consider the case $H_0^m \chi(H_0^m) = \lambda_m(D_{r_*}) \chi(H_0^m)$. We shall prove

$$\int_1^\infty \left\| J \left(\frac{r_*}{t} \right) (\mathcal{V}_m - \frac{r_*}{t}) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{III.5.39})$$

where $J \in C_0^\infty(\mathbb{R}^+)$ and $J = 1$ on $[\theta_1, \theta_2]$.

In order to avoid technical problems due to the matrix-valued operator \mathcal{V}_m , we first show that we can replace \mathcal{V}_m by its scalar version $\lambda'_m(D_{r_*})$ in (III.5.39). Indeed, if we apply several times the following results $[\mathcal{V}_m, J(\frac{r_*}{t})] \in O(t^{-1})$, $[\lambda'_m(D_{r_*}), J(\frac{r_*}{t})] \in O(t^{-1})$ and $J(\frac{r_*}{t})(\chi(H) - \chi(H_0^m)) \in O(t^{-1})$ (the last one follows immediately from the proof of Lemma III.4.5), we have

$$\left\| J \left(\frac{r_*}{t} \right) (\mathcal{V}_m - \frac{r_*}{t}) \chi(H) e^{-itH} \psi \right\|^2 = \left\| J \left(\frac{r_*}{t} \right) (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) \chi(H) e^{-itH} \psi \right\|^2 + L^1(dt),$$

by (III.5.38). Therefore (III.5.39) is equivalent to

$$\int_1^\infty \left\| J \left(\frac{r_*}{t} \right) (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.5.40})$$

Let us define $M(t) = R(\frac{r_*}{t}) + (\lambda'_m(D_{r_*}) - \frac{r_*}{t})R'(\frac{r_*}{t})$, where $R \in C_0^\infty(\mathbb{R})$ satisfies $R(r_*) = \frac{r_*^2}{2}$ on $\text{supp } J$, and the uniformly bounded propagation observable

$$\phi(t) = \chi(H)J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})\chi(H).$$

We compute its Heisenberg derivative.

$$\begin{aligned} \mathbf{D}\phi(t) &= -\frac{1}{t}\chi(H)J'(\frac{r_*}{t})\frac{r_*}{t}M(t)J(\frac{r_*}{t})\chi(H) + hc \\ &\quad + \frac{1}{t}\chi(H)J(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t})R''(\frac{r_*}{t})(-\frac{r_*}{t})J(\frac{r_*}{t})\chi(H) \\ &\quad + \chi(H)i[H, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H). \end{aligned} \quad (\text{III.5.41})$$

Let us study the last term in (III.5.41). We can decompose the operator H into the sum $H_0^m + V_0(r_*)$. The term involving V_0 equals

$$\chi(H)i[V_0(r_*), J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H) = \chi(H)J(\frac{r_*}{t})i[V_0(r_*), \lambda'_m(D_{r_*})]R'(\frac{r_*}{t})J(\frac{r_*}{t})\chi(H).$$

Recall that $V_0 \in T^{0,-1}$. Hence, $V_0(r_*)$ is a pseudodifferential operator which belongs to $\psi^{0,0}$. As $\lambda'_m \in C_0^\infty(\mathbb{R})$, we also have $\lambda'_m(D_{r_*}) \in \Psi^{0,-\infty}$. Thus, by Theorem III.A.2, we have

$$i[V_0(r_*), \lambda'_m(D_{r_*})] = V_0'(r_*)\lambda''_m(D_{r_*}) + r^w(r_*, D_{r_*}),$$

with $r^w \in \Psi^{-2,-\infty}$. Now, since $J \in C_0^\infty(\mathbb{R}^+)$, we have $J(\frac{r_*}{t})V'(r_*) \in O(t^{-2})$. Thus,

$$\chi(H)J(\frac{r_*}{t})V_0'(r_*)\lambda''_m(D_{r_*})R'(\frac{r_*}{t})J(\frac{r_*}{t})\chi(H) \in O(t^{-2}).$$

Moreover, it is immediate that $J(\frac{r_*}{t})r^w(r_*, D_{r_*}) \in O(t^{-2})$ since $r^w \in \Psi^{-2,-\infty}$. This implies that

$$\chi(H)J(\frac{r_*}{t})r^w(r_*, D_{r_*})R'(\frac{r_*}{t})J(\frac{r_*}{t})\chi(H) \in O(t^{-2}).$$

Therefore, $\chi(H)i[V_0(r_*), J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H)$ is integrable in norm along the evolution. It remains to analyse the term involving H_0^m . We decompose it into

$$\begin{aligned} \chi(H)i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H) &= \\ &\chi(H_0^m)i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H_0^m) \\ &+ (\chi(H) - \chi(H_0^m))i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H_0^m) \\ &+ \chi(H_0^m)i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})](\chi(H) - \chi(H_0^m)) \\ &+ (\chi(H) - \chi(H_0^m))i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})](\chi(H) - \chi(H_0^m)). \end{aligned} \quad (\text{III.5.42})$$

We claim that the last three terms in (III.5.42) are integrable in norm. Indeed, we have

$$\begin{aligned} i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})] &= \frac{1}{t}\Gamma^1 J'(\frac{r_*}{t})M(t)J(\frac{r_*}{t}) + hc \\ &\quad + \frac{1}{t}J(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t})R''(\frac{r_*}{t})\Gamma^1 J(\frac{r_*}{t}), \\ &\in O(t^{-1}). \end{aligned}$$

Moreover, we already saw that $(\chi(H) - \chi(H_0^m))J(\frac{r_*}{t}) \in O(t^{-1})$ and the same result is true with $J(\frac{r_*}{t})$ replaced by $J'(\frac{r_*}{t})$. Hence, all terms involving $\chi(H) - \chi(H_0^m)$ in (III.5.42) belong to $O(t^{-2})$.

Eventually, since $\lambda'_m(D_{r_*}) \in \Psi^{0,-\infty}$ and $J(\frac{r_*}{t}), R(\frac{r_*}{t}) \in \Psi_t^{0,0}$, we conclude from (III.5.37) and Theorem III.A.2 that

$$\begin{aligned} \chi(H_0^m)i[H_0^m, J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H_0^m) &= \chi(H_0^m)i[\lambda'_m(D_{r_*}), J(\frac{r_*}{t})M(t)J(\frac{r_*}{t})]\chi(H_0^m), \\ &= \frac{1}{t}\chi(H_0^m)J'(\frac{r_*}{t})\lambda'_m(D_{r_*})M(t)J(\frac{r_*}{t})\chi(H_0^m) + hc \\ &\quad + \frac{1}{t}\chi(H_0^m)J(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t}R''(\frac{r_*}{t})\lambda'_m(D_{r_*}))J(\frac{r_*}{t})\chi(H_0^m) + O(t^{-2}). \end{aligned}$$

Putting all these results together, we obtain

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t}\chi(H)J'(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t}M(t)J(\frac{r_*}{t}))\chi(H) + hc \\ &\quad + \frac{1}{t}\chi(H)J(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t}R''(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t}J(\frac{r_*}{t})))\chi(H) \quad (\text{III.5.43}) \\ &\quad + L^1(dt). \end{aligned}$$

The first term in (III.5.43) can be written as $\frac{1}{t}\chi(H)\tilde{J}(\frac{r_*}{t})B(t)\tilde{J}(\frac{r_*}{t})\chi(H)$ where $\tilde{J} \in C_0^\infty(\mathbb{R}^+)$ and $\text{supp } \tilde{J} \subset (0, \epsilon_\chi) \cup (1, +\infty)$ and $B(t)$ is a uniformly bounded operator on \mathcal{H} . Then this term is integrable along the evolution by Propositions III.5.2 and III.5.5.

Commuting certain terms in (III.5.43), we get

$$\mathbf{D}\phi(t) = \frac{1}{t}\chi(H)(\lambda'_m(D_{r_*}) - \frac{r_*}{t}J^2(\frac{r_*}{t})(\lambda'_m(D_{r_*}) - \frac{r_*}{t}))\chi(H) + L^1(dt),$$

where we used $R'' = 1$ on $\text{supp } J$. Then (III.5.40) holds by Proposition III.5.1.

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III.6 Asymptotic velocity P^\pm

We now turn our attention to the construction of the asymptotic velocity operators P^\pm defined by

$$P^\pm = s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH} \frac{r_*}{t} e^{-itH}. \quad (\text{III.6.1})$$

The strong $-C_\infty$ -convergence means that for any $J \in C_\infty(\mathbb{R})$, we have

$$J(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH} J(\frac{r_*}{t}) e^{-itH},$$

and, in this case, the operators P^\pm are well defined (see [21], Proposition B.2.1). In dimension one, the strong $-C_\infty$ -convergence is equivalent to the usual strong resolvent convergence. The existence of (III.6.1) will follow from the weak propagation estimates obtained in Section III.5 and Lemma III.5.1. In the remaining of this paper, we only prove the results for $t \rightarrow +\infty$ since the proofs for $t \rightarrow -\infty$ are identical.

III.6.1 Existence of P^\pm

The main Theorem of this Section is.

Theorem III.6.1 *Let $J \in C_\infty(\mathbb{R})$. Then there exists the limit*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{r_*}{t}\right) e^{-itH}. \quad (\text{III.6.2})$$

Moreover, if $J(r_*) = 1$ on $[-1, 1]$ then

$$s - \lim_{R \rightarrow +\infty} \left(s - \lim_{t \rightarrow \pm\infty} e^{itH} J\left(\frac{r_*}{Rt}\right) e^{-itH} \right) = 1. \quad (\text{III.6.3})$$

Let us define P^\pm by (III.6.1). Then P^\pm is defined on a dense subspace of \mathcal{H} and is selfadjoint. Moreover, P^\pm commutes with H .

Proof : Since all the operators in (III.6.2) are spherically symmetric, we only establish the result on a given spin-weighted harmonics. For simplicity, we drop all indices in the expression of H^n . By a density argument, it is enough to prove the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J\left(\frac{r_*}{t}\right) \chi(H) e^{-itH},$$

where $J, \chi \in C_0^\infty(\mathbb{R})$ and such that $\text{supp } \chi$ is a closed interval of $\mathbb{R} \setminus \{-m, 0, +m\}$. Let us introduce three functions $j_-, j_0, j_+ \in C^\infty(\mathbb{R})$ satisfying $j_-^2(r_*) + j_0^2(r_*) + j_+^2(r_*) = 1$ and $j_+(r_*) = 1$ on $r_* > \epsilon_\chi - \delta$, $j_0(r_*) = 1$ on a neighbourhood of 0, $j_-(r_*) = 1$ on $r_* < -1 + \delta$, where ϵ_χ denotes the constant given in the minimal velocity estimates and $\delta \in]0, \epsilon_\chi[$. Then, we can separate the problems at the horizon and spatial infinity as follows

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J\left(\frac{r_*}{t}\right) \chi(H) e^{-itH} = s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) \left\{ J_-\left(\frac{r_*}{t}\right) + J_0\left(\frac{r_*}{t}\right) + J_+\left(\frac{r_*}{t}\right) \right\} \chi(H) e^{-itH},$$

where J_-, J_0, J_+ denote respectively $j_-^2 J, j_0^2 J$ and $j_+^2 J$. Since $\text{supp } J_0 \subset (-1 + \delta, \epsilon_\chi)$, it is immediate that

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J_0\left(\frac{r_*}{t}\right) \chi(H) e^{-itH} = 0,$$

by Proposition III.5.5. It remains to prove that the limits

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J_-\left(\frac{r_*}{t}\right) \chi(H) e^{-itH}, \quad (\text{III.6.4})$$

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J_+\left(\frac{r_*}{t}\right) \chi(H) e^{-itH}, \quad (\text{III.6.5})$$

exist. Let us first study what happens at the horizon. By Proposition III.5.6, the existence of (III.6.4) is equivalent to that of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) \left(J_-\left(\frac{r_*}{t}\right) + \left(\Gamma^1 - \frac{r_*}{t} \right) J'_-\left(\frac{r_*}{t}\right) \right) \chi(H) e^{-itH}. \quad (\text{III.6.6})$$

We denote $\phi_-(t) = \chi(H) \left(J_-\left(\frac{r_*}{t}\right) + \left(\Gamma^1 - \frac{r_*}{t} \right) J'_-\left(\frac{r_*}{t}\right) \right) \chi(H)$. We compute its Heisenberg derivative.

$$\begin{aligned} \mathbf{D}\phi_-(t) &= \frac{1}{t} \chi(H) \left(\Gamma^1 - \frac{r_*}{t} \right) J''_-\left(\frac{r_*}{t}\right) \left(\Gamma^1 - \frac{r_*}{t} \right) \chi(H) \\ &\quad + \chi(H) i \{ a(r_*) [\Gamma^2, \Gamma^1] + b(r_*) [\Gamma^0, \Gamma^1] \} J'_-\left(\frac{r_*}{t}\right) \chi(H). \end{aligned}$$

The first term is integrable along the evolution by Proposition III.5.6. The second term is integrable in norm since $a(r_*)J'(\frac{r_*}{t}), b(r_*)J'(\frac{r_*}{t}) \in O(t^{-\infty})$. Then the limit (III.6.6) exists by Lemma III.5.1.

At spatial infinity, the situation is almost identical. Using Proposition III.5.7, the existence of (III.6.15) is equivalent to that of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) \left(J_+(\frac{r_*}{t}) + (\mathcal{V}_m - \frac{r_*}{t}) J'_+(\frac{r_*}{t}) \right) \chi(H) e^{-itH}.$$

But, as we already saw in the proof of Proposition III.5.7, we can replace the velocity operator \mathcal{V}_m by its scalar expression $\pm \lambda'_m(D_{r_*})$ on $\chi(H_0^m)$. Here, we assume that $\mathcal{V}_m \chi(H_0^m) = \lambda'_m(D_{r_*}) \chi(H_0^m)$. Thus, the existence of (III.6.5) is equivalent to that of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) \left(J_+(\frac{r_*}{t}) + (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) J'_+(\frac{r_*}{t}) \right) \chi(H) e^{-itH}. \quad (\text{III.6.7})$$

We denote $\phi_+(t) = \chi(H) \left(J_+(\frac{r_*}{t}) + (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) J'_+(\frac{r_*}{t}) \right) \chi(H)$. In the same way as in the proof of the microlocal velocity estimates at spatial infinity, we obtain for the Heisenberg derivative

$$\begin{aligned} \mathbf{D}\phi_+(t) &= \frac{1}{t} \chi(H) (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) J''_+(\frac{r_*}{t}) (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) \chi(H) \\ &\quad + L^1(dt). \end{aligned}$$

Therefore, the limit (III.6.7) exists by Proposition III.5.7 and Lemma III.5.1.

The assertion (III.6.3) is equivalent to

$$s - \lim_{R \rightarrow +\infty} \left(s - \lim_{t \rightarrow +\infty} e^{itH} J\left(\frac{r_*}{Rt}\right) e^{-itH} \right) = 0,$$

for any function $J \in C^\infty(\mathbb{R})$ such that $\text{supp } J \cap [-1, +1] = \emptyset$ and $J' \in C_0^\infty(\mathbb{R})$. Thus, (III.6.3) follows directly from the proofs of the second assertions in Propositions III.5.6 and III.5.7.

We conclude the proof of the Theorem by applying Proposition B.2.1. in [21]. The asymptotic velocity operators defined by (III.6.2) are selfadjoint on \mathcal{H} with dense domain. Furthermore, they commute with H .

◇

Let us now give another expression for the asymptotic velocity operator P^\pm in terms of the classical velocity operators \mathcal{V}_m and Γ^1 .

Proposition III.6.1 *Let P^\pm defined in Theorem III.6.1. Then*

$$P^\pm \mathbf{1}_{\mathbb{R}^\pm}(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH} (\pm \mathcal{V}_m) e^{-itH} \mathbf{1}_{\mathbb{R}^\pm}(P^\pm), \quad (\text{III.6.8})$$

$$P^\pm \mathbf{1}_{\mathbb{R}^\mp}(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH} (\pm \Gamma^1) e^{-itH} \mathbf{1}_{\mathbb{R}^\mp}(P^\pm), \quad (\text{III.6.9})$$

$$\mathbf{1}_{\{0\}}(P^\pm) = \mathbf{1}^{pp}(H) = 0. \quad (\text{III.6.10})$$

In particular, when $m = 0$, we have

$$P^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} (\pm \Gamma^1) e^{-itH}.$$

Proof : Let us show (III.6.8). By a density argument, it suffices to prove

$$s - \lim_{t \rightarrow +\infty} e^{itH} \left(f(\mathcal{V}_m) - f\left(\frac{r^*}{t}\right) \right) J\left(\frac{r^*}{t}\right) \chi(H) e^{-itH} = 0,$$

for $f, \chi \in C_0^\infty(\mathbb{R})$, $J \in C_0^\infty(\mathbb{R}^+)$ and $\text{supp } \chi \cap \{-m, c_0, +m\} = \emptyset$. Now, using the Helffer-Sjöstrand formula (see for instance [14], Appendix A), we have

$$f(\mathcal{V}_m) - f\left(\frac{r^*}{t}\right) = B\left(\mathcal{V}_m - \frac{r^*}{t}\right) + O(t^{-1}),$$

where B is bounded on \mathcal{H} . Hence, we only have to prove

$$s - \lim_{t \rightarrow +\infty} e^{itH} B\left(\mathcal{V}_m - \frac{r^*}{t}\right) J\left(\frac{r^*}{t}\right) \chi(H) e^{-itH} = 0.$$

But, this follows immediately from Proposition III.5.7.

The proof of (III.6.9) is identical. We omit it.

It is an interesting fact that, in general, the states having 0 asymptotic velocity correspond to the bound states of H (see [14]). Since we already know that $\mathbb{P}^p(H) = 0$, it is enough to prove $\mathbf{1}_{\{0\}}(P^+) \leq \mathbb{P}^p(H)$. Let us point out that this assertion relies only on the minimal velocity estimates.

Let us consider a function $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, c_0, +m, \sigma_p(H)\} = \emptyset$. Let $J \in C_0^\infty(\mathbb{R})$ a positive function such that $J(0) = 1$ and $\text{supp } J \subset (-1 + \delta, \epsilon_\chi - \delta)$ where $\epsilon_\chi > 0$ is defined by the minimal velocity estimates and $\delta \in]0, \epsilon_\chi[$.

Then Theorem III.6.1 implies

$$\begin{aligned} s - \lim_{t \rightarrow +\infty} e^{itH} \chi(H) J\left(\frac{r^*}{t}\right) \chi(H) e^{-itH} &= \chi^2(H) J(P^+), \\ &\geq \chi^2(H) \mathbf{1}_{\{0\}}(P^+). \end{aligned} \quad (\text{III.6.11})$$

By Proposition III.5.5, the strong limit in (III.6.11) vanishes. Thus we have proved that

$$\mathbf{1}_{\{0\}}(P^+) \leq \mathbf{1}_{\{-m, c_0, +m\} \cup \overline{\sigma_p(H)}}(H).$$

This last operator is equal to 0 since H has no eigenvalue. This concludes the proof of (III.6.10).

◇

III.6.2 Spectrum of P^\pm

The spectrum of P^\pm is the physically relevant information offered by the asymptotic velocity operator. It corresponds to the admissible values (measured by an observer static at infinity) of the velocity of propagation for Dirac fields in the asymptotic regions.

Theorem III.6.2 *Let P^\pm be the asymptotic velocity defined in Theorem III.6.1. Then, if $m \neq 0$, we have*

$$\sigma(P^+) = \{-1\} \cup [0, 1], \quad \sigma(P^-) = [-1, 0] \cup \{1\}. \quad (\text{III.6.12})$$

In the particular case $m = 0$, we have

$$\sigma(P^\pm) = \{-1\} \cup \{1\}. \quad (\text{III.6.13})$$

Proof : We only prove (III.6.12) since the proof for (III.6.13) is identical. Let us first prove the inclusion

$$\sigma(P^+) \subset \{-1\} \cup [0, 1]. \quad (\text{III.6.14})$$

Let $\xi_0 \in \mathbb{R} \setminus (\{-1\} \cup [0, 1])$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighbourhood of ξ_0 and $\text{supp } \chi \cap (\{-1\} \cup [0, 1]) = \emptyset$. We have to show that

$$\chi(P^+) = 0.$$

We separate the cases corresponding to the positive and negative values of the asymptotic velocity. We have

$$\begin{aligned} \chi(P^+) &= \chi(P^+)(\mathbf{1}_{\mathbb{R}^-}(P^+) + \mathbf{1}_{\mathbb{R}^+}(P^+)), \\ &= \chi_-(P^+) + \chi_+(P^+), \end{aligned}$$

where $\text{supp } \chi_- \subset \mathbb{R}^- \setminus \{-1\}$ and $\text{supp } \chi_+ \subset (1, +\infty)$. Now, using the characterizations (III.6.8) and (III.6.9) of the asymptotic velocity, we obtain

$$\chi(P^+) = s - \lim_{t \rightarrow +\infty} e^{itH} \chi_-(\Gamma^1) e^{-itH} + s - \lim_{t \rightarrow +\infty} e^{itH} \chi_+(\mathcal{V}_m) e^{-itH}. \quad (\text{III.6.15})$$

Since $\sigma(\Gamma^1) = \{-1, +1\}$ and $\sigma(\mathcal{V}_m) = [-1, 1]$, both terms $\chi_-(\Gamma^1)$ and $\chi_+(\mathcal{V}_m)$ vanish. Thus the limit (III.6.15) is also zero and (III.6.14) holds. *Note that, when $m = 0$, we have $\mathcal{V}_0 = \Gamma^1$. Thus, $\sigma(\mathcal{V}_0) = \{-1, +1\}$ implies that $\sigma(P^+) \subset \{-1\} \cup \{+1\}$.*

Let us prove the converse inclusion. We divide the proof in two steps : we consider either the event horizon or spatial infinity since the techniques will differ slightly as for Propositions III.5.6 and III.5.7. We first prove

$$\sigma(P^+) \supset (0, 1]. \quad (\text{III.6.16})$$

Since $\sigma(P^+)$ is closed, it will follow that $\{0\}$ also belongs to the spectrum. Let $\xi_0 \in (0, 1]$ and $\xi_0 = r_2 > r_1 > r_0 > 0$. Let $g \in C_0^\infty(\mathbb{R})$ positive such that $g(\xi_0) \neq 0$ and $\text{supp } g \subset (\xi_0 - r_0, \xi_0 + r_0)$. To prove (III.6.16), we have to show $g(P^+) \neq 0$. Actually, it suffices to prove

$$\chi_\pm(H) g^2(P^+) \chi_\pm(H) \neq 0, \quad (\text{III.6.17})$$

for any $\chi_\pm \in C_0^\infty(\mathbb{R}^\pm)$, $\chi_\pm = 0$ on a neighbourhood of 0. Since the proofs are the same, we only show (III.6.17) for χ_+ .

Let $J \in C^\infty(\mathbb{R})$ such that $J = 1$ on $(-\infty, r_1]$, $J = 0$ on $[r_2 - \frac{r_2 - r_1}{2}, +\infty)$ and $j = -J' \geq 0$. Clearly J and g satisfy the following relations

$$\bullet \quad J(|\xi - \xi_0|) g(\xi) = g(\xi). \quad (\text{III.6.18})$$

$$\bullet \quad j(|\xi - \xi_0|) |\xi - \xi_0| \geq r_1 j(|\xi - \xi_0|). \quad (\text{III.6.19})$$

$$\bullet \quad g(\xi) |\xi - \xi_0| \leq r_0 g(\xi). \quad (\text{III.6.20})$$

We define the propagation observable

$$\phi_0(t) = \chi_+(H) J\left(\left|\frac{r_*}{t} - \xi_0\right|\right) g^2(\mathcal{V}_m) J\left(\left|\frac{r_*}{t} - \xi_0\right|\right) \chi_+(H).$$

Using (III.6.8) and (III.6.18), we have

$$\chi_+(H) g^2(P^+) \chi_+(H) = s - \lim_{t \rightarrow +\infty} e^{itH} \phi_0(t) e^{-itH}. \quad (\text{III.6.21})$$

As usual, we would like to replace \mathcal{V}_m by a scalar differential operator. We proceed as follows. Since $\text{supp } \chi_+ \subset \mathbb{R}^+$, we have

$$\chi_+(H_0^m)g^2(\mathcal{V}_m) = \chi_+(H_0^m)g^2(\lambda'_m(D_{r_*})), \quad (\text{III.6.22})$$

where $\lambda_m(p) = \sqrt{p^2 + m^2}$ on $\Omega_0(\text{supp } \chi_+)$ and has been extended into a $C_0^\infty(\mathbb{R})$ function outside $\Omega_0(\text{supp } \chi_+)$. Therefore, if we define the propagation observable

$$\phi(t) = \chi_+(H)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H),$$

then, using $(\chi_+(H) - \chi_+(H_0^m))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right) \in O(t^{-1})$ and $[\chi_+(H_0^m), J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)] \in O(t^{-1})$, it follows from (III.6.22) that

$$\phi(t) = \phi_0(t) + O(t^{-1}).$$

Thus, we also have

$$\chi_+(H)g^2(P^+)\chi_+(H) = s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t)e^{-itH}. \quad (\text{III.6.23})$$

Remark that the above argument remains true when $m = 0$. In particular, (III.6.22) holds with $\lambda_0(p) = |p|$ on $\Omega_0(\text{supp } \chi_+)$. The fact that $\chi_+ = 0$ on a neighbourhood of 0 implies that $0 \notin \Omega_0(\text{supp } \chi_+)$ and thus, λ_0 can be extended into a $C_0^\infty(\mathbb{R})$ function outside this set.

Let us compute the Heisenberg derivative of $\phi(t)$. We claim that

$$\mathbf{D}\phi \geq R(t) \text{ where } R \in L^1(dt). \quad (\text{III.6.24})$$

Indeed, we have

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t}\chi_+(H)j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\frac{\frac{r_*}{t} - \xi_0}{\left|\frac{r_*}{t} - \xi_0\right|}\frac{r_*}{t}g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H) + hc \\ &\quad + \chi_+(H)i\left[H_0^m, J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\right]\chi_+(H) \\ &\quad + \chi_+(H)i\left[V_0(r_*), J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\right]\chi_+(H). \end{aligned} \quad (\text{III.6.25})$$

From the proof of Proposition III.5.7 and Theorem III.A.2, we have

$$\chi_+(H)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)i[V_0(r_*), g^2(\lambda'_m(D_{r_*}))]J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H) \in O(t^{-2}).$$

Thus, the third term in (III.6.25) is integrable in norm.

Similarly, we already saw that the second term in (III.6.25) can be written as

$$\chi_+(H_0^m)i\left[\lambda_m(D_{r_*}), J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\right]\chi_+(H_0^m) + O(t^{-2}),$$

Using Theorem III.A.2, this term is equal to

$$-\frac{1}{t}\chi_+(H_0^m)j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\frac{\frac{r_*}{t} - \xi_0}{\left|\frac{r_*}{t} - \xi_0\right|}\lambda'_m(D_{r_*})g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H_0^m) + hc + L^1(dt).$$

Putting all these results together, we obtain

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t}\chi_+(H)j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\left|\frac{r_*}{t} - \xi_0\right|g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H) + hc \\ &\quad + \frac{1}{t}\chi_+(H)j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\frac{\frac{r_*}{t} - \xi_0}{\left|\frac{r_*}{t} - \xi_0\right|}(\xi_0 - \lambda'_m(D_{r_*}))g^2(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H) + hc \\ &\quad + L^1(dt). \end{aligned}$$

Commuting certain terms and using Theorem III.A.2, we have

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t}\chi_+(H)g(\lambda'_m(D_{r_*}))J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\left|\frac{r_*}{t} - \xi_0\right|g(\lambda'_m(D_{r_*}))\chi_+(H) \\ &\quad - \frac{1}{t}\chi_+(H)(Jj)^{\frac{1}{2}}\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\frac{\frac{r_*}{t} - \xi_0}{\left|\frac{r_*}{t} - \xi_0\right|}(\lambda'_m(D_{r_*}) - \xi_0)g^2(\lambda'_m(D_{r_*}))(Jj)^{\frac{1}{2}}\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\chi_+(H) \\ &\quad + L^1(dt). \end{aligned}$$

Now, using (III.6.19) and (III.6.20), we have

$$\mathbf{D}\phi(t) \geq \frac{1}{t}(r_1 - r_0)\chi_+(H)g(\lambda'_m(D_{r_*}))(Jj)\left(\left|\frac{r_*}{t} - \xi_0\right|\right)g(\lambda'_m(D_{r_*}))\chi_+(H) + L^1(dt),$$

and since $r_1 - r_0 \geq 0$, we get

$$\mathbf{D}\phi(t) \geq R(t) \text{ where } R(t) \in L^1(dt).$$

Let us complete the proof. Using (III.6.23), for any $t_0 \in \mathbb{R}^+$, we have

$$\begin{aligned} \chi_+(H)g^2(P^+)\chi_+(H) &= e^{it_0H}\phi(t_0)e^{-it_0H} + \int_{t_0}^{+\infty} e^{itH}\mathbf{D}\phi(t)e^{-itH}dt, \\ &\geq e^{it_0H}\phi(t_0)e^{-it_0H} - \int_{t_0}^{+\infty} \|R(t)\|dt. \end{aligned} \quad (\text{III.6.26})$$

Since $R(t) \in L^1(dt)$, if we choose t_0 large enough, we can make the integral on the right-hand-side of (III.6.26) as small as we want. Now, we claim that

$$\lim_{t \rightarrow +\infty} \|e^{it_0H}\phi(t_0)e^{-it_0H}\|, \quad (\text{III.6.27})$$

exists and is non-zero. Indeed, we have

$$\begin{aligned} e^{it_0\xi_0 D_{r_*}}\phi(t_0)e^{-it_0\xi_0 D_{r_*}} &= \\ &\chi_+(H_0^m + V_0(r_* + t_0\xi_0))J\left(\frac{|r_*|}{t_0}\right)g^2(\lambda'_m(D_{r_*}))J\left(\frac{|r_*|}{t_0}\right)\chi_+(H_0^m + V_0(r_* + t_0\xi_0)) \\ &\quad + \chi_-(H_0^m + V_0(r_* + t_0\xi_0))J\left(\frac{|r_*|}{t_0}\right)g^2(-\lambda'_m(D_{r_*}))J\left(\frac{|r_*|}{t_0}\right)\chi_-(H_0^m + V_0(r_* + t_0\xi_0)). \end{aligned}$$

By (III.6.22), this tends strongly to $\chi_+(H_0^m)g^2(\mathcal{V}_m)\chi_+(H_0^m)$ which is a non-zero operator. Finally, since

$$\|e^{it_0H}\phi(t_0)e^{-it_0H}\| = \|e^{it_0\xi_0 D_{r_*}}\phi(t_0)e^{-it_0\xi_0 D_{r_*}}\|,$$

the limit (III.6.27) is non-zero and thus $g^2(P^+) \neq 0$.

Let us now prove that $\sigma(P^+) \supset \{-1\}$. Let $\xi_0 = -1$ and we define g, J as previously. We introduce the propagation observable $\phi(t) = J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)g^2(\Gamma^1)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)$. By (III.6.9) and (III.6.18), it follows that

$$g^2(P^+) = s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) e^{-itH}.$$

Then it suffices to show that $\mathbf{D}\phi(t) \geq R(t)$ with $R(t) \in L^1(dt)$ and the result will hold by the same argument. But we have

$$\begin{aligned} \mathbf{D}\phi(t) &= \frac{1}{t}j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\left|\frac{r_*}{t} - \xi_0\right|g^2(\Gamma^1)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right) + hc \\ &\quad - \frac{1}{t}j\left(\left|\frac{r_*}{t} - \xi_0\right|\right)\frac{\frac{r_*}{t} - \xi_0}{\left|\frac{r_*}{t} - \xi_0\right|}(\xi_0 - \Gamma^1)g^2(\Gamma^1)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right) + hc \quad (\text{III.6.28}) \\ &\quad + J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)i(a(r_*)[\Gamma^2, g^2(\Gamma^1)] + b(r_*)[\Gamma^0, g^2(\Gamma^1)])J\left(\left|\frac{r_*}{t} - \xi_0\right|\right). \end{aligned}$$

Since $a \in T^{-1, -\infty}$ and $\text{supp} J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)$ is strictly contained in \mathbb{R}^- , we have $a(r_*)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right) \in O(t^{-\infty})$. The same is true for $b(r_*)J\left(\left|\frac{r_*}{t} - \xi_0\right|\right)$. Therefore, the last term in (III.6.28) is integrable in norm. We conclude using (III.6.19) and (III.6.20) that

$$\mathbf{D}\phi(t) \geq R(t),$$

with $R(t) \in L^1(dt)$. This completes the proof of the Theorem.

◇

III.7 Wave operators

This Section is devoted to the construction of wave operators for the Dirac Hamiltonian H . We introduce different wave operators according to which asymptotic region we consider. An interesting application of the asymptotic velocity operator in this case, is the possibility to separate the two different ends without introducing supplementary cut-off functions. More precisely, P^\pm can serve to distinguish between the *incoming* Dirac fields (i.e. the part of the solution which goes towards the horizon) and *outgoing* Dirac fields (i.e. the part of the solution which goes towards spacelike infinity). We have the natural decomposition

$$\mathcal{H} = \mathcal{H}_{in}^\pm \oplus \mathcal{H}_{out}^\pm,$$

where $\mathcal{H}_{in}^\pm = \mathbf{1}_{\mathbb{R}^-}(P^\pm)\mathcal{H}$ and $\mathcal{H}_{out}^\pm = \mathbf{1}_{\mathbb{R}^+}(P^\pm)\mathcal{H}$ are the incoming (resp. outgoing) solutions when $t \rightarrow \pm\infty$. Therefore, if we consider $t \rightarrow +\infty$, it suffices to construct the wave operators at the horizon (resp. at infinity) only for the incoming states \mathcal{H}_{in}^+ (resp. outgoing states \mathcal{H}_{out}^+) and similarly for $t \rightarrow -\infty$.

III.7.1 Wave operators at the event horizon

We decompose H into $H_1 + V_1(r_*)$ where V_1 is exponentially decreasing when $r_* \rightarrow -\infty$. Since the potential V_1 is short-range at the horizon, the natural comparison operator is e^{-itH_1} . If we consider only

incoming Dirac fields, the standard Cook method can be applied to prove the existence and asymptotic completeness of classical wave operators. We have

Theorem III.7.1 *The following limits exist*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_1} \mathbf{1}_{\mathbb{R}^-} (\pm\Gamma^1). \quad (\text{III.7.1})$$

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_1} e^{-itH} \mathbf{1}_{\mathbb{R}^-} (P^\pm). \quad (\text{III.7.2})$$

Moreover, if Ω_H^\pm denotes the limit (III.7.1) then Ω_H^\pm is a partial isometry and

$$(\text{III.7.2}) = (\Omega_H^\pm)^*, \quad (\Omega_H^\pm)^* \Omega_H^\pm = \mathbf{1}_{\mathbb{R}^-} (\pm\Gamma^1), \quad \Omega_H^\pm (\Omega_H^\pm)^* = \mathbf{1}_{\mathbb{R}^-} (P^\pm). \quad (\text{III.7.3})$$

We also have the intertwining relations

$$P^\pm \mathbf{1}_{\mathbb{R}^-} (P^\pm) = \Omega_H^\pm (\pm\Gamma^1) (\Omega_H^\pm)^*. \quad (\text{III.7.4})$$

$$H \mathbf{1}_{\mathbb{R}^-} (P^\pm) = \Omega_H^\pm H_1 (\Omega_H^\pm)^*. \quad (\text{III.7.5})$$

Proof : First note that $\pm\Gamma^1$ are the asymptotic velocity operators for the Dirac Hamiltonian H_1 , i.e.

$$\pm\Gamma^1 = s - C_\infty - \lim_{t \rightarrow \pm\infty} e^{itH_1} \frac{r_*}{t} e^{-itH_1}.$$

Since the existence of (III.7.1) immediately follows from Cook's method and Huygens's principle, we only prove (III.7.2). Note however that the proof of (III.7.2) applies without any change to the proof of (III.7.1). This is another advantage of the present method that the proof of existence and the proof of completeness of wave operators are identical.

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, c_0, +m\} = \emptyset$. Let $J \in C^\infty(\mathbb{R}^-)$, $J = 0$ on neighbourhood of 0 and $J = 1$ on $(-\infty, -1 + \delta]$ with $\delta \in]0, 1[$. Since $\sigma(P^+) = \{-1\} \cup [0, 1]$, we have $\mathbf{1}_{\mathbb{R}^-} (P^+) = J(P^+)$. Then, by a density argument, the existence of (III.7.2) is equivalent to that of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH_1} e^{-itH} J(P^+) \chi^2(H). \quad (\text{III.7.6})$$

Let us define the propagation observable

$$\phi(t) = \chi(H_1) J\left(\frac{r_*}{t}\right) \chi(H).$$

It follows from Theorem III.6.1 and the usual result $(\chi(H) - \chi(H_1)) J\left(\frac{r_*}{t}\right) \in O(t^{-1})$ that

$$(\text{III.7.6}) = s - \lim_{t \rightarrow +\infty} e^{itH_1} \phi(t) e^{-itH}.$$

Let us compute the generalized Heisenberg derivative of $\phi(t)$.

$$\begin{aligned} \mathbf{D}_2 \phi(t) &= \frac{d}{dt} \phi(t) + iH_1 \phi(t) - i\phi(t)H, \\ &= -\frac{1}{t} \chi(H_1) \left(J' \left(\frac{r_*}{t} \right) \frac{r_*}{t} - J' \left(\frac{r_*}{t} \right) \Gamma^1 \right) \chi(H) \\ &\quad - i\chi(H_1) J\left(\frac{r_*}{t}\right) V_1(r_*) \chi(H). \end{aligned} \quad (\text{III.7.7})$$

Since $\text{supp } J' \subset (-1, 0)$, the first term in (III.7.7) is integrable along the evolution by Proposition III.5.5. Moreover, since $V_1 \in T^{0, -\infty}$, we have $J(\frac{r_*}{t})V_1(r_*) \in O(t^{-\infty})$. Hence, the second term in (III.7.7) is integrable in norm. This implies the existence of (III.7.6) by Lemma III.5.1.

By Proposition B.5 in [21], Ω_H^\pm is a partial isometry satisfying (III.7.3).

Let us prove (III.7.4). First note that

$$(\Omega_H^\pm)^* = \mathbf{1}_{\mathbb{R}^-}(P^\pm)(\Omega_H^\pm)^*, \quad (\text{III.7.8})$$

by (III.7.3). Using Proposition III.6.1, we have

$$\begin{aligned} P^+ \mathbf{1}_{\mathbb{R}^-}(P^+) &= s - C_\infty - \lim_{t \rightarrow +\infty} e^{itH} \Gamma^1 e^{-itH} \mathbf{1}_{\mathbb{R}^-}(P^+), \\ &= s - \lim_{t \rightarrow +\infty} e^{itH} e^{-itH_1} \Gamma^1 e^{itH_1} e^{-itH} \mathbf{1}_{\mathbb{R}^-}(P^+). \end{aligned}$$

Thus, we deduce from (III.7.1), (III.7.2) and (III.7.8) that (III.7.4) holds.

Finally, we prove (III.7.5). We have

$$\begin{aligned} H \mathbf{1}_{\mathbb{R}^-}(P^+) &= s - \lim_{t \rightarrow +\infty} e^{itH} H e^{-itH} \mathbf{1}_{\mathbb{R}^-}(P^+), \\ &= s - \lim_{t \rightarrow +\infty} e^{itH} e^{-itH_1} H_1 e^{itH_1} e^{-itH} \mathbf{1}_{\mathbb{R}^-}(P^+) \\ &\quad + s - \lim_{t \rightarrow +\infty} e^{itH} V_1(r_*) e^{-itH} \mathbf{1}_{\mathbb{R}^-}(P^+). \end{aligned} \quad (\text{III.7.9})$$

By (III.7.1), (III.7.2) and (III.7.8), the first term in (III.7.9) is exactly the term in the right-hand-side of (III.7.5). Moreover, observe that $\mathbf{1}_{\mathbb{R}^-}(P^+) = g(P^+)$ with $g \in C_0^\infty(\mathbb{R}^-)$ and $g = 1$ on a neighbourhood of -1 . Then, by Theorem III.6.1, the second term in (III.7.9) can be rewritten as follows

$$s - \lim_{t \rightarrow +\infty} e^{itH} V_1(r_*) e^{-itH} \mathbf{1}_{\mathbb{R}^-}(P^+) = s - \lim_{t \rightarrow +\infty} e^{itH} V_1(r_*) g\left(\frac{r_*}{t}\right) e^{-itH},$$

and this term vanishes since $V_1(r_*)g(\frac{r_*}{t}) \in O(t^{-\infty})$. Hence, the limit is zero when t tends to $+\infty$. This concludes the proof of the Theorem.

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III.7.2 Wave operators at spacelike infinity, I

The Dirac Hamiltonian H can be decomposed into $H_0^m + V_0(r_*)$ where $V_0(r_*) \simeq C\langle r_* \rangle^{-1}$ when $r_* \rightarrow +\infty$. Therefore, the potential V_0 is long-range at spacelike infinity. It is well known, in this case, that the dynamics $e^{-itH_0^m}$ cannot be used as the comparison operator and must be modified. There exist several possibilities for this. We shall use the modification introduced by Dollard and Velo [26] which has the double advantage of being very intuitive and adapted to matrix-valued equations.

The idea of Dollard modification is to add a phase $e^{-iX(t)}$ to the natural dynamics $e^{-itH_0^m}$ in such a way that this phase compensates the effect of the long-range potential V_0 . Assume a moment that $H = H_0^m + f(r_*)$ where f is a scalar long-range potential. Then, the usual Dollard modification is given in this case by

$$U_D(t) = e^{-itH_0^m} T\left(e^{-i \int_0^t f(s\nu_m) ds}\right),$$

where T denotes time-ordering. To prove the existence of the Dollard modified wave operator

$$s - \lim_{t \rightarrow +\infty} e^{itH} U_D(t),$$

it suffices to apply Cook's method. We have to show

$$\frac{d}{dt} \left(e^{itH} U_D(t) \right) = e^{itH} i(f(r_*) - f(t\mathcal{V}_m)) U_D(t) \in L^1(dt).$$

But, intuitively, it follows from the microlocal velocity estimate, Proposition III.5.7, that this will be true since the operator $t\mathcal{V}_m$ approaches r_* when $t \rightarrow +\infty$. We will use this strategy to construct the wave operators at infinity.

Nevertheless, there is an important difference with our example. Indeed, the potential V_0 is the sum of scalar and matrix-valued long-range perturbations. How to add the matrix-valued terms which do not commute with H_0^m in the Dollard modification? Roughly speaking, we will see that we can write V_0 as follows

$$V_0(r_*) = f(r_*) + g(r_*)M + h(r_*)N,$$

where f, g, h are scalar long-range functions and M, N two operators such that M commutes with H_0^m and N anticommutes with H_0^m . We shall see in the proof of Theorem III.7.3 that all these terms are not true long-range perturbations. Because of its anticommutation property, the term $h(r_*)N$ will not enter in the definition of the modification. Consequently, we will define the Dollard modification by

$$U_D(t) = e^{-itH_0^m} T \left(e^{-i \int_0^t (f(s\mathcal{V}_m) + g(s\mathcal{V}_m)M) ds} \right).$$

Note that the phase is well defined since M commutes with H_0^m and \mathcal{V}_m .

Let us apply these arguments to our case. We first treat the case $m \neq 0$. We define the operator $G_m = \Gamma^0 - m(H_0^m)^{-1}$ introduced in [29]. A straightforward calculation shows that the operator G_m anticommutes with H_0^m . Now, we rewrite V_0 as follows

$$V_0(r_*) = a_0(r_*)\Gamma^2 + b_0(r_*)G_m + b_0(r_*)m(H_0^m)^{-1} + c_0(r_*).$$

Since Γ^2 and G_m anticommute with H_0^m , they do not enter in the Dollard modification. Thus we define

$$U_D^m(t) = e^{-itH_0^m} T \left(e^{-i \int_0^t (b_0(s\mathcal{V}_m)m(H_0^m)^{-1} + c_0(s\mathcal{V}_m)) ds} \right).$$

Similarly, when $m = 0$, we define

$$U_D^0(t) = e^{-itH_0^0} T \left(e^{-i \int_0^t c_0(s\mathcal{V}_0) ds} \right),$$

where $\mathcal{V}_0 = \Gamma^1$. Now the main Theorem of this Section is

Theorem III.7.2 *The following limits exist*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} U_D^m(t) \mathbf{1}_{\mathbb{R}^+}(\pm\mathcal{V}_m). \quad (\text{III.7.10})$$

$$s - \lim_{t \rightarrow \pm\infty} (U_D^m)^*(t) e^{-itH} \mathbf{1}_{\mathbb{R}^+}(P^\pm). \quad (\text{III.7.11})$$

Moreover, if Ω_∞^\pm denotes the limit (III.7.10) then Ω_∞^\pm is a partial isometry and

$$(\text{III.7.11}) = (\Omega_\infty^\pm)^*, \quad (\Omega_\infty^\pm)^* \Omega_\infty^\pm = \mathbf{1}_{\mathbb{R}^+}(\pm\mathcal{V}_m), \quad \Omega_\infty^\pm (\Omega_\infty^\pm)^* = \mathbf{1}_{\mathbb{R}^+}(P^\pm). \quad (\text{III.7.12})$$

We also have the intertwining relations

$$P^\pm \mathbf{1}_{\mathbb{R}^+}(P^\pm) = \Omega_\infty^\pm (\pm\mathcal{V}_m) (\Omega_\infty^\pm)^*, \quad (\text{III.7.13})$$

$$H \mathbf{1}_{\mathbb{R}^+}(P^\pm) = \Omega_\infty^\pm H_0^m (\Omega_\infty^\pm)^*. \quad (\text{III.7.14})$$

For the proof of this theorem, we follow the presentation given in [21]. The first step is to obtain similar results for time-dependent long-range potentials. The second step is to use the properties of the asymptotic velocity operator P^+ to make the link between time-dependent and time-independent potentials.

Let us study time-dependent Dirac operators of the form

$$H(t) = H_0^m + V(t, r_*),$$

where $V(t, r_*) = V_1(t, r_*) + V_2(t, r_*)\Gamma^0 + V_3(t, r_*)\Gamma^2$. We assume that

$$|\partial_t V_j(t, r_*)| \leq C\langle t \rangle^{-2}, \forall j = 1, 2, 3, \quad (\text{III.7.15})$$

$$|\partial_{r_*}^\alpha V_j(t, r_*)| \leq C\langle t \rangle^{-1-|\alpha|}, \forall \alpha \in \mathbb{N}, \forall j = 1, 2, 3. \quad (\text{III.7.16})$$

For such Hamiltonians, it is possible to define an associated unitary dynamics (see [21], appendix B.3, Proposition B.3.6) which we will denote by $U(t, s)$ and which satisfies

- The map $(t, s) \rightarrow U(t, s)$ is strongly continuous with values in unitary operators in \mathcal{H} such that

$$U(t, t) = 1, \quad U(t, u)U(u, s) = U(t, s).$$

- If we denote $B = ((H_0^m)^2 + 1)^{\frac{1}{2}}$, we have

$$\begin{aligned} \partial_s U(t, s)B^{-1} &= U(t, s)iH(s)B^{-1}, \\ \partial_t B^{-1}U(t, s) &= -iB^{-1}H(t)U(t, s). \end{aligned}$$

We want to define the asymptotic velocity for such time-dependent Hamiltonians and to describe some of its properties. This has been done in [14], [15] for the case of three dimensional Dirac operators. Since the proofs are identical, we only recall the main results.

Proposition III.7.1 *For any $g \in C_\infty(\mathbb{R})$ and $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0, the limits*

$$s - \lim_{t \rightarrow \pm\infty} U(0, t)g(\mathcal{V}_m)\chi(H_0^m)U(t, 0) =: \mathcal{P}_{m, g, \chi}^\pm, \quad (\text{III.7.17})$$

exist. The operators $\mathcal{P}_{m, g, \chi}^\pm$ correspond to the asymptotic velocity operators for high energies associated to $U(t, s)$.

We also need propagation estimates for such time-dependent operators. The next proposition summarizes the large and minimal velocity estimates in this case.

Proposition III.7.2 *Suppose that $j, g \in C_0^\infty(\mathbb{R})$ and $\text{supp } j \cap \text{supp } g = \emptyset$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Then*

$$\int_1^\infty \left\| j\left(\frac{r_*}{t}\right)g(\mathcal{V}_m)\chi(H_0^m)U(t, 0)\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.7.18})$$

Furthermore, if $J \in C_0^\infty(\mathbb{R})$ such that $J = 1$ on a neighbourhood of $\text{supp } g$, then

$$s - \lim_{t \rightarrow \pm\infty} U(0, t)J\left(\frac{r_*}{t}\right)g(\mathcal{V}_m)\chi(H_0^m)U(t, 0) = \mathcal{P}_{m, g, \chi}^\pm. \quad (\text{III.7.19})$$

Similarly, the microlocal velocity estimates hold.

Proposition III.7.3 *Suppose that $g, J \in C_0^\infty(\mathbb{R})$ and $\text{supp } g \cap \text{supp } J' = \emptyset$. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0. Then*

$$\int_1^\infty \left\| \left(\frac{r_*}{t} - \mathcal{V}_m \right) J \left(\frac{r_*}{t} \right) g(\mathcal{V}_m) \chi(H_0^m) U(t, 0) \psi \right\|^2 \frac{dt}{t} \leq \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{III.7.20})$$

Moreover,

$$s - \lim_{t \rightarrow \pm\infty} \left(\frac{r_*}{t} - \mathcal{V}_m \right) J \left(\frac{r_*}{t} \right) g(\mathcal{V}_m) \chi(H_0^m) U(t, 0) = 0. \quad (\text{III.7.21})$$

Remark III.7.1 *In the previous propositions, we need to introduce cut-off for the low energies since we consider both massive and massless cases at the same time. Note however that, in the case of strictly massive Dirac fields, we can remove the energy cut-off $\chi(H_0^m)$ in Propositions III.7.1, III.7.2 and III.7.3 and define the full asymptotic velocity (see [14]).*

Let us define the Dollard comparison operator

$$U_{0,D}^m(t) = e^{-itH_0^m} T \left(e^{-i \int_0^t (V(s, s\mathcal{V}_m) ds)} \right),$$

where $V(s, s\mathcal{V}_m) = V_1(s, s\mathcal{V}_m) + V_2(s, s\mathcal{V}_m)m(H_0^m)^{-1}$ when $m \neq 0$ and $V(s, s\mathcal{V}_0) = V_1(s, s\mathcal{V}_0)$ when $m = 0$. In the next theorem, we prove the existence and asymptotic completeness of the time-dependent wave operators with energy cut-off.

Theorem III.7.3 *For any $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 0$ on a neighbourhood of 0, the limits*

$$s - \lim_{t \rightarrow \pm\infty} U(0, t) \chi^2(H_0^m) U_{0,D}^m(t), \quad (\text{III.7.22})$$

$$s - \lim_{t \rightarrow \pm\infty} (U_{0,D}^m)^*(t) \chi^2(H_0^m) U(t, 0), \quad (\text{III.7.23})$$

exist.

Proof : Since the proofs are identical, we only prove (III.7.23). We may assume without loss of generality that $\text{supp } \chi \subset (0, +\infty)$. By a density argument, it is enough to prove the existence of

$$s - \lim_{t \rightarrow +\infty} (U_{0,D}^m)^*(t) \chi^2(H_0^m) U(t, 0) \langle r_* \rangle^{-1}.$$

For technical reasons, we need to replace $\chi(H_0^m)$ by $\chi(H(t))$. Using the Helffer-Sjöstrand formula and (III.7.16), we have

$$\chi(H_0^m) - \chi(H(t)) \in O(t^{-1}). \quad (\text{III.7.24})$$

Hence, it suffices to prove the existence of

$$s - \lim_{t \rightarrow +\infty} (U_{0,D}^m)^*(t) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1}.$$

Let us compute the time derivative of this expression. We obtain

$$\begin{aligned} \frac{d}{dt} (U_{0,D}^m)^*(t) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \psi &= (U_{0,D}^m)^*(t) i(V(t, t\mathcal{V}_m) - V(t, r_*)) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \psi \\ &\quad + (U_{0,D}^m)^*(t) \left(\frac{d}{dt} \chi^2(H(t)) \right) U(t, 0) \langle r_* \rangle^{-1} \psi. \end{aligned}$$

Using the Helffer-Sjöstrand formula, (III.7.15) imply that

$$\frac{d}{dt}\chi(H(t)) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H(t))^{-1} \left(\frac{d}{dt} V(t, r_*) \right) (z - H(t))^{-1} dz \wedge d\bar{z} \in O(t^{-2}),$$

and thus the second term is integrable in norm. Now we can write the time derivative as

$$\begin{aligned} & \frac{d}{dt} (U_{0,D}^m)^*(t) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \psi = \\ & (U_{0,D}^m)^*(t) i \left\{ (V_1(t, t\mathcal{V}_m) - V_1(t, r_*)) + \Gamma^0 (V_2(t, t\mathcal{V}_m) - V_2(t, r_*)) + \Gamma^2 (V_3(t, t\mathcal{V}_m) - V_3(t, r_*)) \right. \\ & \quad \left. - G_m V_2(t, t\mathcal{V}_m) - \Gamma^2 V_3(t, t\mathcal{V}_m) \right\} \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \psi + O(t^{-2}), \\ & = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + O(t^{-2}). \end{aligned}$$

(Recall that $G_m = \Gamma^0 - m(H_0^m)^{-1}$ and $G_0 := \Gamma^0$). Let us prove that $I_1(t) = (U_{0,D}^m)^*(t) i (V_1(t, t\mathcal{V}_m) - V_1(t, r_*)) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \psi$ belongs to $L^1(dt)$. By the Helffer-Sjöstrand formula ([14], Appendix A), we have

$$V_1(t, t\mathcal{V}_m) - V_1(t, r_*) = C_1(t) (r_* - t\mathcal{V}_m) + O(t^{-2}),$$

where $C_1(t)$ is a bounded operator satisfying $C_1(t) \leq Ct^{-2}$. Therefore, we can rewrite $I_1(t)$ as follows

$$I_1(t) = (U_{0,D}^m)^*(t) C_1(t) (r_* - t\mathcal{V}_m) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \psi + L^1(dt).$$

Now, it suffices to show that

$$(r_* - t\mathcal{V}_m) \chi^2(H(t)) U(t, 0) \langle r_* \rangle^{-1} \in O(t^\mu), \quad 0 < \mu < 1, \quad (\text{III.7.25})$$

in order to prove $I_1(t) \in L^1(dt)$. Using (III.7.24) and the fact that $[\chi(H_0^m), r_*]$ is bounded on \mathcal{H} , (III.7.25) is equivalent to

$$\chi(H_0^m) (r_* - t\lambda'_m(D_{r_*})) \chi(H_0^m) U(t, 0) \langle r_* \rangle^{-1} \in O(t^\mu), \quad 0 < \mu < 1,$$

since $\mathcal{V}_m \chi(H_0^m) = \lambda'_m(D_{r_*}) \chi(H_0^m)$. Using once again (III.7.24), it is enough to show

$$\chi(H(t)) (r_* - t\lambda'_m(D_{r_*})) \chi(H(t)) U(t, 0) \langle r_* \rangle^{-1} \in O(t^\mu), \quad 0 < \mu < 1. \quad (\text{III.7.26})$$

Since $0 < \mu$, (III.7.26) will follow from

$$L(t) := \frac{1}{t^\mu} \frac{d}{dt} \left(U(0, t) \chi(H(t)) (r_* - t\lambda'_m(D_{r_*})) \chi(H(t)) U(t, 0) \langle r_* \rangle^{-1} \right) \in L^1((1, +\infty), dt). \quad (\text{III.7.27})$$

But we have

$$\begin{aligned} L(t) &= \frac{1}{t^\mu} U(0, t) \chi(H(t)) \left(i[H(t), r_* - t\lambda'_m(D_{r_*})] - \lambda'_m(D_{r_*}) \right) \chi(H(t)) U(t, 0) \langle r_* \rangle^{-1}, \\ &+ \frac{1}{t^\mu} U(0, t) \left(\frac{d}{dt} \chi(H(t)) \right) (r_* - t\lambda'_m(D_{r_*})) \chi(H(t)) U(t, 0) \langle r_* \rangle^{-1} + hc. \end{aligned} \quad (\text{III.7.28})$$

Since $\frac{d}{dt} \chi(H(t)) \in O(t^{-2})$, the second term in (III.7.28) is integrable in norm. Now, we have

$$i[H(t), r_* - t\lambda'_m(D_{r_*})] = i[H_0^m, r_*] - t[V(t, r_*), \lambda'_m(D_{r_*})].$$

Since $V(t, r_*) \in \Psi_t^{-1,0}$ and $\lambda'_m(D_{r_*}) \in \Psi^{0,0}$, it follows from Theorem III.A.2 that $[V(t, r_*), \lambda'_m(D_{r_*})]$ belongs to $O(t^{-2})$. Therefore, this term also gives a contribution that is integrable in norm. It remains to see that

$$\frac{1}{t^\mu} U(0, t) \chi(H(t)) \left(i[H_0^m, r_*] - \lambda'_m(D_{r_*}) \right) \chi(H(t)) U(t, 0) \langle r_* \rangle^{-1}, \quad (\text{III.7.29})$$

is integrable. But, using (III.7.24) and

$$\chi(H_0^m) i[H_0^m, r_*] \chi(H_0^m) = \chi(H_0^m) i[\lambda_m(D_{r_*}), r_*] \chi(H_0^m) = \chi(H_0^m) \lambda'_m(D_{r_*}) \chi(H_0^m),$$

it is immediate that

$$(\text{III.7.29}) = \frac{1}{t^\mu} U(0, t) \chi(H(t)) (\lambda'_m(D_{r_*}) - \lambda_m'(D_{r_*})) \chi(H(t)) U(t, 0) \langle r_* \rangle^{-1} + O(t^{-1-\mu}),$$

Since the first term vanishes, this term is also integrable in norm. Thus, we have proved that $I_1(t) \in L^1(dt)$. In the same manner, we can prove that $I_2(t)$ and $I_3(t)$ belong to $L^1(dt)$.

It remains to analyse the terms $I_4(t), I_5(t)$. We shall need the following lemma

Lemma III.7.1 *Let us denote*

$$E(t) = \left(\int_0^t e^{isH_0^m} G_m e^{-isH_0^m} ds \right) \chi^2(H_0^m), \quad F(t) = \left(\int_0^t e^{isH_0^m} \Gamma^2 e^{-isH_0^m} ds \right) \chi^2(H_0^m).$$

Then $E(t), F(t)$ are bounded operators, uniformly in t . As a consequence, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \|E(t)\| = 0, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \|F(t)\| = 0.$$

Proof : Since G_m anticommutes with H_0^m , we have

$$e^{isH_0^m} G_m e^{-isH_0^m} = G_m e^{-i2sH_0^m}, \quad \forall s \in \mathbb{R},$$

and

$$E(t) = G_m \left(\int_0^t e^{i2sH_0^m} ds \right) \chi^2(H_0^m).$$

Since $\chi = 0$ on a neighbourhood of 0, there exists $\delta > 0$ such that

$$\begin{aligned} \|E(t)\| &\leq C \sup_{\lambda \in \mathbb{R}} \left| \chi^2(\lambda) \int_0^t e^{-2is\lambda} ds \right|, \\ &\leq C \sup_{\lambda \in \mathbb{R} \setminus [-\delta, \delta]} \left| \frac{1}{2i\lambda} (1 - e^{-2it\lambda}) \right|, \\ &\leq C_\delta. \end{aligned}$$

Thus, $E(t)$ is uniformly bounded, in operator norm, with respect to t . The proof is identical for $F(t)$ since Γ^2 anticommutes with H_0^m .

◇

Now let us treat the term $I_4(t)$. First, observe that using (III.7.24) and (III.7.16), we have

$$I_4(t) = (U_{0,D}^m)^*(t) G_m \chi^2(H_0^m) V_2(t, t\mathcal{V}_m) U(t, 0) \langle r_* \rangle^{-1} \psi + O(t^{-2}).$$

If we denote concisely $T\left(e^{-i\int_0^t V(s, s\mathcal{V}_m)ds}\right)$ by $e^{-iX(t)}$, we can rewrite I_4 as

$$I_4(t) = e^{iX(t)}(\partial_t E(t))e^{itH_0^m}V_2(t, t\mathcal{V}_m)U(t, 0)\langle r_* \rangle^{-1}\psi + O(t^{-2}).$$

Therefore, in order to apply Cook's Lemma, it is enough to show that

$$I_{s,u} := \left\| \int_s^u e^{iX(t)}(\partial_t E(t))e^{itH_0^m}V_2(t, t\mathcal{V}_m)U(t, 0)\langle r_* \rangle^{-1}\psi dt \right\|, \quad (\text{III.7.30})$$

tends to 0 when $s, u \rightarrow \infty$. From the definition of $E(t)$, it is clear that it anticommutes with H_0^m and with \mathcal{V}_m . Thus we can write $e^{iX(t)}(\partial_t E(t)) = (\partial_t E(t))e^{i\tilde{X}(t)}$ where

$$e^{i\tilde{X}(t)} = T\left(e^{i\int_0^t (V_1(s, -s\mathcal{V}_m) - m(H_0^m)^{-1}V_2(s, -s\mathcal{V}_m))ds}\right).$$

By an integration by part in (III.7.30), we obtain

$$\begin{aligned} I_{s,u} &\leq \left\| E(u)e^{i\tilde{X}(u)}e^{iuH_0^m}V_2(u, u\mathcal{V}_m)U(u, 0)\langle r_* \rangle^{-1}\psi \right\| \\ &\quad + \left\| E(s)e^{i\tilde{X}(s)}e^{isH_0^m}V_2(s, s\mathcal{V}_m)U(s, 0)\langle r_* \rangle^{-1}\psi \right\| \\ &\quad + \left\| \int_s^u E(t)\partial_t (e^{i\tilde{X}(t)}e^{itH_0^m}V_2(t, t\mathcal{V}_m)U(t, 0)\langle r_* \rangle^{-1}\psi) dt \right\|. \end{aligned}$$

Using $V_2(t, t\mathcal{V}_m) \in O(t^{-1})$ and Lemma III.7.1, the first two terms tend to 0 as $s, u \rightarrow \infty$. If we compute the derivative in the last term, we find

$$\begin{aligned} \partial_t (e^{i\tilde{X}(t)}e^{itH_0^m}V_2(t, t\mathcal{V}_m)U(t, 0)\langle r_* \rangle^{-1}\psi) &= \\ & i(V_1(t, -t\mathcal{V}_m) - m(H_0^m)^{-1}V_2(t, -t\mathcal{V}_m))e^{i\tilde{X}(t)}e^{itH_0^m}V_2(t, t\mathcal{V}_m)U(t, 0)\langle r_* \rangle^{-1}\psi \\ & + e^{i\tilde{X}(t)}e^{itH_0^m}(\partial_t V_2(t, t\mathcal{V}_m) + \mathcal{V}_m\partial_{r_*}V_2(t, t\mathcal{V}_m))U(t, 0)\langle r_* \rangle^{-1}\psi \\ & - ie^{i\tilde{X}(t)}e^{itH_0^m}V_2(t, t\mathcal{V}_m)V(t, r_*)U(t, 0)\langle r_* \rangle^{-1}\psi, \end{aligned}$$

All these terms belong to $O(t^{-2})$ by (III.7.15) and (III.7.16). Since $E(t)$ is bounded, the last integral also tends to 0 when $s, u \rightarrow \infty$. We treat the term $I_5(t)$ in a similar way which concludes the proof of the Theorem.

◇

Proof (of Theorem III.7.2) : So far, we have established complete scattering results concerning Dirac Hamiltonians with time-dependent long-range potentials. This concludes the first step of the proof of Theorem III.7.2. For the last step, we have to make the link between time-dependent and time-independent Hamiltonians. This is done in the following way. First, remark that by a density argument, the existence of (III.7.10), (III.7.11) are equivalent to that of the limits

$$s - \lim_{t \rightarrow +\infty} e^{itH}U_D^m(t)\chi^4(H_0^m)g^2(\mathcal{V}_m), \quad (\text{III.7.31})$$

$$s - \lim_{t \rightarrow +\infty} (U_D^m)^*(t)e^{-itH}\chi^4(H)g^2(P^+), \quad (\text{III.7.32})$$

where $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, c_0, +m\} = \emptyset$, $\chi = 0$ on a neighbourhood of 0 and $g \in C_0^\infty(\mathbb{R}^+)$ such that $g = 0$ on a neighbourhood of 0 and $g = 1$ on $(\epsilon_\chi, 1)$ (ϵ_χ is defined by the

minimal velocity estimates). Now, consider a function $J \in C_0^\infty(\mathbb{R}^+)$ such that $J = 1$ on a neighbourhood of $\text{supp } g$ and $J = 0$ on a neighbourhood of 0. Let us associate to any function $f \in S^{-1}(\mathbb{R})$ the time-dependent function

$$f_J(t, r_*) = f(r_*)J\left(\frac{r_*}{t}\right),$$

defined for $t \geq 1$. Such a function obviously satisfies the properties

- For any y in a neighbourhood of $\text{supp } g$,

$$f(ty) = f_J(t, ty) \quad (\text{III.7.33})$$

- For any $t \geq 1$ fixed, $r_* \rightarrow f_J(t, r_*) \in C_0^\infty(\mathbb{R})$ and there is a constant M such that

$$\text{supp } f_J(t, \cdot) \subset (0, Mt).$$

- The following estimates hold

$$|\partial_{r_*}^\alpha f_J(t, r_*)| \leq C\langle t \rangle^{-1-\alpha}, \quad \forall \alpha \in \mathbb{N}, \quad (\text{III.7.34})$$

$$|\partial_t^k f_J(t, x)| \leq C\langle t \rangle^{-1-k}, \quad \forall k \in \mathbb{N}. \quad (\text{III.7.35})$$

We introduce now some notations. We call *effective time-dependent potential* the potential $V_J(t, r_*) = a_{0,J}(t, r_*)\Gamma^2 + b_{0,J}(t, r_*)\Gamma^0 + c_{0,J}(t, r_*)$. We denote $H_J(t)$ the time-dependent Hamiltonian $H_0^m + V_J(t, r_*)$ and $U_J(t, s)$ the associated dynamics. We also denote $U_{D,J}^m(t)$ the following time-dependent Dollard modification

$$U_{D,J}^m(t) = e^{-itH_0^m} T \left(e^{-i \int_0^t (c_{0,J}(s, s\mathcal{V}_m) + m(H_0^m)^{-1} b_{0,J}(s, s\mathcal{V}_m)) ds} \right),$$

and

$$U_{D,J}^0(t) = e^{-itH_0^0} T \left(e^{-i \int_0^t c_{0,J}(s, s\mathcal{V}_0) ds} \right).$$

We rewrite (III.7.31) as follows

$$\begin{aligned} s - \lim_{t \rightarrow +\infty} e^{itH} U_D^m(t) \chi^4(H_0^m) g^2(\mathcal{V}_m) = \\ s - \lim_{t \rightarrow +\infty} e^{itH} U_J(t, 0) U_J(0, t) g(\mathcal{V}_m) \chi^2(H_0^m) U_J(t, 0) U_J(0, t) \chi^2(H_0^m) U_{D,J}^m(t) g(\mathcal{V}_m), \end{aligned}$$

where we used $U_D^m(t)g(\mathcal{V}_m) = U_{D,J}^m(t)g(\mathcal{V}_m) = g(\mathcal{V}_m)U_{D,J}^m(t)$ by (III.7.33). Now assume the existence of the limits

$$s - \lim_{t \rightarrow +\infty} U_J(0, t) \chi^2(H_0^m) U_{D,J}^m(t), \quad (\text{III.7.36})$$

$$s - \lim_{t \rightarrow +\infty} U_J(0, t) g(\mathcal{V}_m) \chi^2(H_0^m) U_J(t, 0) =: \mathcal{P}_{m,g,\chi^2}^+, \quad (\text{III.7.37})$$

$$s - \lim_{t \rightarrow +\infty} e^{itH} U_J(t, 0) \mathcal{P}_{m,g,\chi^2}^+, \quad (\text{III.7.38})$$

then the limit (III.7.31) will exist by the chain rule. Similarly, using Proposition III.6.1, we can rewrite (III.7.32) as

$$\begin{aligned} s - \lim_{t \rightarrow +\infty} (U_D^m)^*(t) e^{-itH} \chi^4(H) g^2(P^+) = \\ s - \lim_{t \rightarrow +\infty} g(\mathcal{V}_m) (U_{D,J}^m)^*(t) \chi^2(H_0^m) U_J(t, 0) U_J(0, t) e^{-itH} g(P^+) \chi^2(H). \end{aligned}$$

Therefore, if we prove the existence of the limits

$$s - \lim_{t \rightarrow +\infty} (U_{D,J}^m)^*(t) \chi^2(H_0^m) U_J(t, 0), \quad (\text{III.7.39})$$

$$s - \lim_{t \rightarrow +\infty} U_J(0, t) e^{-itH} g(P^+) \chi^2(H), \quad (\text{III.7.40})$$

then the limit (III.7.32) will exist by the chain rule.

Now, observe that we already proved (III.7.36), (III.7.37) and (III.7.39) in the previous results concerning time-dependent Dirac Hamiltonians (see Proposition III.7.1 and Theorem III.7.3). Hence, it remains to show

Lemma III.7.2 *The following limits exist*

$$s - \lim_{t \rightarrow +\infty} e^{itH} U_J(t, 0) \mathcal{P}_{m,g,\chi^2}^+, \quad (\text{III.7.41})$$

$$s - \lim_{t \rightarrow +\infty} U_J(0, t) e^{-itH} g(P^+) \chi^2(H). \quad (\text{III.7.42})$$

Proof : Since the proofs are identical, we only show (III.7.41). Moreover, since the other cases can be treated similarly, we may assume that $\text{supp } \chi \subset (+m, +\infty)$. Let us define two functions g_1 and g_2 belonging to $C_0^\infty(\text{supp } J)$ such that $g_1 g = g$, $g_2 g_1 = g_1$ and $J g_2 = g_2$. Then using Propositions III.7.2 and III.7.3, we have

$$\mathcal{P}_{m,g,\chi^2}^+ = s - \lim_{t \rightarrow +\infty} U_J(0, t) g_2\left(\frac{r_*}{t}\right) M_0(t) g_2\left(\frac{r_*}{t}\right) g(\mathcal{V}_m) \chi^2(H_0^m) U_J(t, 0),$$

where $M_0(t) = g_1\left(\frac{r_*}{t}\right) + (\mathcal{V}_m - \frac{r_*}{t}) g_1'\left(\frac{r_*}{t}\right)$. Let us define the propagation observable

$$\phi_0(t) = g_2\left(\frac{r_*}{t}\right) M_0(t) g_2\left(\frac{r_*}{t}\right) g(\mathcal{V}_m) \chi^2(H_0^m),$$

then the existence of (III.7.41) is equivalent to that of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi_0(t) U_J(t, 0).$$

To avoid technical problems, we must use a scalar version of $\phi_0(t)$. Let us define

$$\phi(t) = \chi(H) g_2\left(\frac{r_*}{t}\right) M(t) g_2\left(\frac{r_*}{t}\right) g(\mathcal{V}_m) \chi(H_J(t)),$$

with $M(t) = g_1\left(\frac{r_*}{t}\right) + (\lambda'_m(D_{r_*}) - \frac{r_*}{t}) g_1'\left(\frac{r_*}{t}\right)$. Noting that $\chi(H_0^m) \mathcal{V}_m = \chi(H_0^m) \lambda'_m(D_{r_*})$ and using

$$f\left(\frac{r_*}{t}\right) (\chi(H) - \chi(H_0^m)) \in O(t^{-1}), \quad f \in C_0^\infty(\mathbb{R}^+), \quad (\text{III.7.43})$$

$$\chi(H_J(t)) - \chi(H_0^m) \in O(t^{-1}), \quad (\text{III.7.44})$$

we have

$$\phi(t) = \phi_0(t) + O(t^{-1}).$$

Therefore, it is enough to prove the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH} \phi(t) U_J(t, 0). \quad (\text{III.7.45})$$

We compute the generalized Heisenberg derivative of $\phi(t)$. We obtain

$$\mathbf{D}_2\phi(t) = -\frac{1}{t}\chi(H)\left(g_2'\left(\frac{r_*}{t}\right)\frac{r_*}{t}M(t)g_2\left(\frac{r_*}{t}\right) + hc\right)g(\lambda'_m(D_{r_*}))\chi(H_J(t)) \quad (\text{III.7.46})$$

$$-\frac{1}{t}\chi(H)g_2\left(\frac{r_*}{t}\right)\left(\lambda'_m(D_{r_*}) - \frac{r_*}{t}g_1''\left(\frac{r_*}{t}\right)\frac{r_*}{t}g_2\left(\frac{r_*}{t}\right)g(\lambda'_m(D_{r_*}))\right)\chi(H_J(t)) \quad (\text{III.7.47})$$

$$+\chi(H)g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right)g(\lambda'_m(D_{r_*}))\left(\frac{d}{dt}\chi(H_J(t))\right) \quad (\text{III.7.48})$$

$$+\chi(H)i[H_0^m, g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right)g(\lambda'_m(D_{r_*}))]\chi(H_J(t)) \quad (\text{III.7.49})$$

$$+i\chi(H)\left(V(r_*)g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right) - g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right)V_J(t, r_*)\right)g(\lambda'_m(D_{r_*}))\chi(H_J(t)) \quad (\text{III.7.50})$$

$$+\chi(H)g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right)i[V_J(t, r_*), g(\lambda'_m(D_{r_*}))]\chi(H_J(t)). \quad (\text{III.7.51})$$

We already saw that $\frac{d}{dt}\chi(H_J(t)) \in O(t^{-2})$. Thus, the term (III.7.48) is integrable in norm. Similarly, by standard pseudodifferential calculus, we know that $[V_J(t, r_*), g(\lambda'_m(D_{r_*}))] \in O(t^{-2})$. Hence, the term (III.7.51) is also integrable in norm. Moreover, since $J_{\mathcal{Q}} = g_2$, we have $g_2\left(\frac{r_*}{t}\right)V_J(t, r_*) = V_{g_2}(t, r_*)$. Consequently, we can write the term (III.7.50) as

$$\begin{aligned} V(r_*)g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right) - g_2\left(\frac{r_*}{t}\right)M(t)g_2\left(\frac{r_*}{t}\right)V_J(t, r_*) \\ = \left(V_{g_2}(t, r_*)\lambda'_m(D_{r_*})g_2\left(\frac{r_*}{t}\right) - g_2\left(\frac{r_*}{t}\right)\lambda'_m(D_{r_*})V_{g_2}(t, r_*)\right)g_1'\left(\frac{r_*}{t}\right), \\ = \left([V_{g_2}(t, r_*), \lambda'_m(D_{r_*})]g_2\left(\frac{r_*}{t}\right) + [\lambda'_m(D_{r_*}), g_2\left(\frac{r_*}{t}\right)]V_{g_2}(t, r_*)\right)g_1'\left(\frac{r_*}{t}\right) \end{aligned}$$

Hence, by Theorem III.A.2, the term (III.7.50) belongs to $O(t^{-2})$ and is integrable in norm.

The term (III.7.46) can be written after some commutations as

$$\frac{1}{t}\chi(H)\eta\left(\frac{r_*}{t}\right)B(t)\eta\left(\frac{r_*}{t}\right)g(\lambda'_m(D_{r_*}))\chi(H_J(t)) + O(t^{-2}),$$

where $B(t)$ is a uniformly bounded operator in t and $\eta \in C_0^\infty(\mathbb{R}^+)$ such that $\text{supp } \eta \cap (\epsilon_\chi, 1) = \emptyset$ and $\text{supp } \eta \cap \text{supp } g = \emptyset$. Thus, the term (III.7.46) is integrable along the evolution using Propositions III.5.2, III.5.5 and III.7.2.

Now, the term (III.7.49) is equal to

$$\frac{1}{t}\chi(H)g_2\left(\frac{r_*}{t}\right)\left(\lambda'_m(D_{r_*}) - \frac{r_*}{t}g_1''\left(\frac{r_*}{t}\right)\lambda'_m(D_{r_*})\right)g_2\left(\frac{r_*}{t}\right)g(\lambda'_m(D_{r_*}))\chi(H_J(t)) + L^1(dt).$$

Therefore, we obtain for the generalized Heisenberg derivative.

$$\mathbf{D}_2\phi(t) = \frac{1}{t}\chi(H)g_2\left(\frac{r_*}{t}\right)\left(\lambda'_m(D_{r_*}) - \frac{r_*}{t}g_1''\left(\frac{r_*}{t}\right)\lambda'_m(D_{r_*})\right)g_2\left(\frac{r_*}{t}\right)g(\lambda'_m(D_{r_*}))\chi(H_J(t)) + L^1(dt).$$

The remaining term is integrable along the evolution by Propositions III.5.7 and III.7.3. We conclude that the limit (III.7.32) exists using Lemma III.5.1.

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Therefore, we have constructed the Dollard-modified wave operators Ω^+ and $\tilde{\Omega}^+$. The fact that $(\Omega^+)^* = \tilde{\Omega}^+$ follows from [21], Lemma B.5.1. Eventually, the intertwining relations (III.7.13) and (III.7.14) are proved in the same way as in Theorem III.7.1.

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III.7.3 Wave operators at spacelike infinity, II

We finish this Section by an alternative definition of the Dollard-modified wave operators at infinity. This second definition relies on the following observation : Reissner-Nordström spacetimes are *asymptotically flat* i.e. the metric g tends to the Minkowski metric $\eta = dt^2 - dr_*^2 - r_*^2 d\omega^2$ when $r_* \rightarrow +\infty$. Therefore, we expect the behaviour of Dirac fields at infinity to be the same as in Minkowski spacetime. Let us introduce the following Dirac Hamiltonian corresponding to Minkowski spacetime.

$$H_\infty = \Gamma^1 D_\rho + \frac{1}{\rho} H_{S^2} + b_m(\rho) \Gamma^0 + c_q(\rho), \quad D_\rho = -i\partial_\rho.$$

This is a selfadjoint operator acting on $\mathcal{H}_\infty = L^2(\mathbb{R}_\rho^+ \times S^2, \rho^2 d\rho d\omega; \mathbb{C}^4)$. Indeed, using polar coordinates, the Hamiltonian H_∞ is unitarily equivalent to the Hamiltonian

$$\hat{H}_\infty = \Gamma \cdot p + b_m(|x|) \Gamma^0 + c_q(|x|), \quad p = -i\nabla,$$

where $\Gamma = (\Gamma^1, \Gamma^2, \Gamma^3)$ stands for the Dirac matrices. Now, it is immediate from the Kato-Rellich Theorem that \hat{H}_∞ is selfadjoint on $L^2(\mathbb{R}^3; \mathbb{C}^4)$, see [75].

If we look at the region $r_* \rightarrow +\infty$ and if we make the natural identification $\rho = r_*$, we formally have

$$H = H_\infty + V_\infty(r_*),$$

where $V_\infty(r_*) = O(\frac{1}{r_*^2})$ when $r_* \rightarrow +\infty$. Therefore, the potential is short-range and e^{-itH_∞} is the natural comparison operator. Then, we will take advantage of our previous works [14], [15] to define complete Dollard-modified wave operators for H_∞ .

Since the Dirac operator H acts on \mathcal{H} whereas H_∞ acts on \mathcal{H}_∞ , we first need to introduce an operator of identification between these spaces. We proceed as follows. Let $j_\infty \in C^\infty(\mathbb{R})$ a function satisfying $j_\infty(r_*) = 0$ for $r_* \leq \frac{1}{2}$ and $j_\infty(r_*) = 1$ for $r_* \geq 1$. We define the operator $\mathcal{I} : \mathcal{H}_\infty \rightarrow \mathcal{H}$ which sends any element $\psi \in \mathcal{H}_\infty$ to an element $\mathcal{I}\psi$ in \mathcal{H} defined by

$$\mathcal{I}\psi(r_*) = \begin{cases} j_\infty(r_*)\psi(r_*), & r_* \geq 0, \\ 0, & r_* \leq 0, \end{cases}$$

where we made the identification $\rho = r_*$. We will also denote $\mathcal{I}^* : \mathcal{H} \rightarrow \mathcal{H}_\infty$ the adjoint of \mathcal{I} defined, for any $\psi \in \mathcal{H}$, by

$$\mathcal{I}^*\psi(\rho) = j_\infty(\rho)\psi(\rho).$$

We denote P_∞^\pm the asymptotic velocity operator related to H_∞ constructed in [14] (massive case) and in [15] (massless case). We recall that $\mathbf{1}^c(H_\infty) = \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P_\infty^+)$ where $\mathbf{1}^c(H_\infty)$ denotes the projection onto the continuous subspace of H_∞ .

Proposition III.7.4 *The following limits exist*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \mathcal{I} e^{-itH_\infty} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P_\infty^\pm), \quad (\text{III.7.52})$$

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_\infty} \mathcal{I}^* e^{-itH} \mathbf{1}_{\mathbb{R}^+}(P^\pm), \quad (\text{III.7.53})$$

Proof : Since the proofs are identical, we only treat (III.7.53). We also work on a given spin-weighted spherical harmonics. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, e_0, +m\} = \emptyset$. Let $J \in C_0^\infty(\mathbb{R}^+)$ such

that $J = 0$ on a neighbourhood of 0 and $J = 1$ on a neighbourhood of $[\epsilon_\chi, 1]$. By a density argument, it suffices to prove the existence of the following limit.

$$s - \lim_{t \rightarrow +\infty} e^{itH_\infty} \mathcal{I}^* e^{-itH} J(P^+) \chi^2(H).$$

Using Theorem III.6.1, this is equivalent to the existence of

$$s - \lim_{t \rightarrow +\infty} e^{itH_\infty} \mathcal{I}^* J\left(\frac{r_*}{t}\right) \chi^2(H) e^{-itH}. \quad (\text{III.7.54})$$

Let us note that, for t large enough, we have $\mathcal{I}^* J\left(\frac{r_*}{t}\right) = j_\infty(r_*) J\left(\frac{r_*}{t}\right) = J\left(\frac{r_*}{t}\right)$. Thus, $J\left(\frac{r_*}{t}\right)$ clearly can be considered as an operator from \mathcal{H} to \mathcal{H}_∞ . Moreover, we have

$$\chi(H_\infty) J\left(\frac{r_*}{t}\right) - J\left(\frac{r_*}{t}\right) \chi(H) \in O(t^{-1}). \quad (\text{III.7.55})$$

Indeed, using the Helffer-Sjöstrand formula, we get

$$\begin{aligned} (\text{III.7.55}) &= \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H_\infty)^{-1} (H_\infty J\left(\frac{r_*}{t}\right) - J\left(\frac{r_*}{t}\right) H) (z - H)^{-1} dz \wedge d\bar{z}, \\ &= \frac{1}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H_\infty)^{-1} \left(\frac{1}{t} J'\left(\frac{r_*}{t}\right) \Gamma^1 - i \left(\frac{l + \frac{1}{2}}{r_*} + a_l(r_*) \right) J\left(\frac{r_*}{t}\right) \Gamma^2 \right) (z - H)^{-1} dz \wedge d\bar{z}. \end{aligned}$$

Since $\left(-\frac{l + \frac{1}{2}}{r_*} - a_l(r_*)\right) J\left(\frac{r_*}{t}\right) \in O(t^{-2})$, the term under the integral is bounded by $O\left(\frac{1}{t|Im z|^2}\right)$. Hence, the integral converges in norm and (III.7.55) holds.

Let us define the propagation observable $\phi(t) = \chi(H_\infty) J\left(\frac{r_*}{t}\right) \chi(H)$. From (III.7.54) and (III.7.55), it is enough to prove the existence of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH_\infty} \phi(t) e^{-itH}. \quad (\text{III.7.56})$$

We compute its generalized Heisenberg derivative. We obtain

$$\begin{aligned} \mathbf{D}_2 \phi(t) &= -\frac{1}{t} \chi(H_\infty) J'\left(\frac{r_*}{t}\right) \frac{r_*}{t} \chi(H) \\ &\quad + \frac{1}{t} \chi(H_\infty) J'\left(\frac{r_*}{t}\right) \Gamma^1 \chi(H) \\ &\quad - \chi(H_\infty) J\left(\frac{r_*}{t}\right) i \left(a_l(r_*) + \frac{l + \frac{1}{2}}{r_*} \right) \chi(H). \end{aligned}$$

Therefore, we have

$$\begin{aligned} |(\psi, \mathbf{D}_2 \phi(t) \varphi)| &\leq C \left\| \frac{1}{\sqrt{t}} \tilde{J}\left(\frac{r_*}{t}\right) \chi(H_\infty) \psi \right\| \left\| \frac{1}{\sqrt{t}} \tilde{J}\left(\frac{r_*}{t}\right) \chi(H) \varphi \right\| \\ &\quad + \left\| \frac{1}{t} \chi(H_\infty) \psi \right\| \left\| t J\left(\frac{r_*}{t}\right) i \left(a_l(r_*) + \frac{l + \frac{1}{2}}{r_*} \right) \chi(H) \varphi \right\|, \quad (\text{III.7.57}) \end{aligned}$$

with $\tilde{J} \in C_0^\infty(\mathbb{R}^+)$ chosen such that $\tilde{J} J' = J'$ and $\text{supp } \tilde{J} \subset (0, \epsilon_\chi) \cup (1, +\infty)$. Propositions III.5.2 and III.5.5 and Propositions 4.2 and 4.4 in [14] valid for the Dirac operator H_∞ in Minkowski spacetime, imply that the first term in (III.7.57) is integrable along the evolution. Moreover, since $J\left(\frac{r_*}{t}\right) i \left(a_l(r_*) + \frac{l + \frac{1}{2}}{r_*} \right) \in O(t^{-2})$, the second term in (III.7.57) is also integrable along the evolution. We conclude that the

limit (III.7.56) exists applying Lemma III.5.1.

◇

We now finish the construction of the wave operators at spatial infinity introducing the Dollard modified dynamics for H_∞ . We decompose H_∞ as follows

$$H_\infty = H_\infty^m + b_\infty(|x|)\Gamma^0 + c_\infty(|x|),$$

where $H_\infty^m = \Gamma^1 D_\rho + \frac{1}{\rho} H_{S^2} + m\Gamma^0$ corresponds to the free Dirac operator in flat spacetime.

We define

$$U_D^m(t) = e^{-itH_\infty^m} T \left(e^{-i \int_0^t (b_\infty(|\mathcal{V}_m|) m (H_\infty^0)^{-1} + c_\infty(|\mathcal{V}_m|)) ds} \right),$$

$$U_D^0(t) = e^{-itH_\infty^0} T \left(e^{-i \int_0^t c_\infty(|\mathcal{V}_0|) ds} \right),$$

where \mathcal{V}_m denotes the classical velocity operator and T the time ordering. In the case $m \neq 0$, we have $\mathcal{V}_m = p(H_\infty^m)^{-1}$. This operator has a limit when $m \rightarrow 0$. We denote it \mathcal{V}_0 . From [14], [15], we have

Proposition III.7.5 *The following limits exist*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_\infty} U_D^m(t) \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(\pm\mathcal{V}_m), \quad (\text{III.7.58})$$

$$s - \lim_{t \rightarrow \pm\infty} (U_D^m)^*(t) e^{-itH_\infty} \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(P_\infty^\pm). \quad (\text{III.7.59})$$

As a consequence of Propositions III.7.4 and III.7.5, we have

Corollary III.7.1 *The Dollard modified wave operators at spacelike infinity, defined by the limits*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} \mathcal{I} U_D^m(t) \mathbf{1}_{\mathbb{R}^3 \setminus \{0\}}(\pm\mathcal{V}_m), \quad (\text{III.7.60})$$

$$s - \lim_{t \rightarrow \pm\infty} (U_D^m)^*(t) \mathcal{I}^* e^{-itH} \mathbf{1}_{\mathbb{R}^+}(P^\pm), \quad (\text{III.7.61})$$

exist and are complete.

APPENDIX

III.A Pseudodifferential operators

In this Appendix, we give some elementary results concerning pseudodifferential operators that we use throughout this paper. Let us consider the metric

$$\gamma(t) = \frac{dx^2}{\langle x \rangle_t^2} + \frac{d\xi^2}{\langle \xi \rangle^2} \quad \text{with} \quad \langle x \rangle_t^2 = x^2 + t^2 + 1.$$

We introduce the following class of symbols as subsets of $C^\infty(\mathbb{R}_x \times \mathbb{R}_\xi)$.

$$a \in S(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t)) \iff \forall \alpha, \beta \in \mathbb{N}, |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} f(t) \langle \xi \rangle^{k-\alpha} \langle x \rangle_t^{m-\beta}.$$

To any symbol $a \in S(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t))$, we can associate its Weyl pseudodifferential operator. We write

$$a^w \in \Psi(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t)) \iff a \in S(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t)).$$

For shortness, we will often write $a \in S_t^{k,m}$ (resp. $a^w \in \Psi_t^{k,m}$) instead of $a \in S(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t))$ (resp. $a^w \in \Psi(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t))$). Moreover, when $t = 0$, we will write γ instead of $\gamma(0)$ and $a \in S^{k,m}$ (resp. $a^w \in \Psi^{k,m}$) instead of $a \in S_0^{k,m}$ (resp. $a^w \in \Psi_0^{k,m}$).

The following results are the analogues of the ones given in the Appendix D.6-D.8 in [21].

We first state the Beal criterion. For any $m, k \in \mathbb{R}$, let us introduce the function space

$$H_t^{m,k} = \{u \in \mathcal{D}'(\mathbb{R}) \mid \langle x \rangle_t^m \langle D \rangle^k u \in L^2(\mathbb{R})\},$$

or equivalently,

$$H_t^{m,k} = \{u \in \mathcal{D}'(\mathbb{R}) \mid \langle D \rangle^k \langle x \rangle_t^m u \in L^2(\mathbb{R})\}.$$

We also will write $H^{m,k}$ when $t = 0$. We have the fundamental Theorem

Theorem III.A.1 *The following assertions are equivalent :*

(i) A_t is an operator on $L^2(\mathbb{R})$ such that $ad_D^\alpha ad_x^\beta A_t$ is bounded from $H_t^{m-\alpha, k-\beta}$ into $H_t^{0,0}$ for any $\alpha, \beta \in \mathbb{N}$ and

$$\|ad_D^\alpha ad_x^\beta A_t\|_{H_t^{m-\alpha, k-\beta} \rightarrow H_t^{0,0}} \leq C_{\alpha, \beta} f(t).$$

(ii) A_t is an operator on $L^2(\mathbb{R})$ such that $ad_D^\alpha ad_x^\beta A_t$ is bounded from $H_t^{m_0+m-\alpha, k_0+k-\beta}$ into $H_t^{m_0, k_0}$ for any $\alpha, \beta \in \mathbb{N}$ and

$$\|ad_D^\alpha ad_x^\beta A_t\|_{H_t^{m_0+m-\alpha, k_0+k-\beta} \rightarrow H_t^{m_0, k_0}} \leq C_{\alpha, \beta} f(t).$$

(iii) $A_t = a_t^w(x, D)$ with the symbol a_t belonging to $S_t^{k,m}$.

The composition and the commutator between two such pseudodifferential operators are well defined and have nice properties. We first introduce the notion of principal symbol. Set

$$S_{pr}(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t)) = S_t^{k,m} / S_t^{k-1, m-1}.$$

For $a_t \in S_t^{k,m}$, we denote $\sigma_{pr}(a_t)$ its canonic projection onto $S_{pr}(f(t)\langle \xi \rangle^k \langle x \rangle_t^m, \gamma(t))$. If $A_t \in \Psi_t^{k,m}$ and $A_t = a_t^w(x, D)$ then we define the principal symbol of A_t by

$$\sigma_{pr}(A_t) = \sigma_{pr}(a_t).$$

We have the fundamental result.

Theorem III.A.2 *Let $A_i(t) \in \Psi_t^{k_i, m_i}$, $i = 1, 2$. Then*

$$A_1(t)A_2(t) \in \Psi(f_1(t)f_2(t)\langle x \rangle_t^{m_1+m_2}\langle \xi \rangle^{k_1+k_2}, \gamma(t)),$$

$$[A_1(t), A_2(t)] \in \Psi(f_1(t)f_2(t)\langle x \rangle_t^{m_1+m_2-1}\langle \xi \rangle^{k_1+k_2-1}, \gamma(t)).$$

Moreover, we have

$$\sigma_{pr}(A_1(t)A_2(t)) = \sigma_{pr}(A_1(t))\sigma_{pr}(A_2(t)),$$

$$\sigma_{pr}([A_1(t), A_2(t)]) = \frac{1}{i} \{ \sigma_{pr}(A_1(t)), \sigma_{pr}(A_2(t)) \},$$

where $\{.,.\}$ is the Poisson bracket.

We also need the following compactness criterion

Theorem III.A.3 *Let $a \in S^{-\delta, -\epsilon}$ with $\delta, \epsilon > 0$. Then the operator $A = a^w(x, D)$ is compact on $L^2(\mathbb{R})$.*

We end this Appendix with a simple and useful Proposition.

Proposition III.A.1 *Let $j, F \in C^\infty(\mathbb{R})$ such that $j', F' \in C_0^\infty(\mathbb{R})$. Then*

$$[j(r_*), F(D_{r_*})] \in \Psi^{-\infty, -\infty}.$$

Chapitre IV

Scattering of charged Dirac fields by a Kerr-Newman black hole

IV.1 Introduction

This paper is concerned with the study of the scattering properties of charged massive Dirac fields in the outer region of Kerr-Newman black holes. These spacetimes form a three parameter family of axisymmetric exact solutions of the Einstein-Maxwell equations. They were discovered by Newman *et al.* [60] in 1965, shortly after the discovery of the family of solutions corresponding to the uncharged case, by R.P. Kerr [52] in 1963. They describe asymptotically flat spacetimes containing nothing but an eternal, charged, rotating black hole and generalize the previously known spherically symmetric solutions : the Schwarzschild (1916) and Reissner-Nordström (1918) black holes. Since all cosmological objects are in rotation, Kerr-Newman (or Kerr) black holes are the most realistic models of the exterior of a black hole we can study theoretically.

The object of scattering theory on black hole spacetimes is to analyse the influence of the curvature, i.e. the influence of the black hole, on the propagation of fields (here Dirac fields) and more precisely, to provide a detailed study of the asymptotic behavior in time of these fields. For instance, we prove in this paper that the energy of Dirac fields does not remain trapped in compact sets of the exterior region of a Kerr-Newman black hole but instead, scatters towards the asymptotic regions where fields obey simpler equations. Note here that we have to deal with two asymptotic regions –the horizon and spacelike infinity– with *very different geometrical structures* : a peculiarity of scattering theory in black hole spacetimes. In consequence, the simpler equations we are looking for will be different according to the asymptotic region we consider and we will show how they are naturally related to the local geometry of the spacetime. This program is achieved by introducing (Dollard-modified at infinity) wave operators and proving their existence and asymptotic completeness (see [69], Vol 3, for a detailed presentation of the scattering formalism).

The motivation for the analysis of such problems are manifold. First, it provides a deeper insight into the physics of black holes as well as the theoretical study of the propagation of fields in curved spacetimes. Second, it is the first step towards the study of quantum effects in general relativity such as the *Hawking effect* (see [5], [6], [7] and [56] for a detailed analysis in Schwarzschild and Reissner-Nordström spacetimes). Another interesting subject is the study of the *resonances* of the equation. These can be understood as the (complex) frequencies which are the poles of the analytic continuation of the scattering operator (note also that the notion of resonances is closely related to the quasinormal modes of a black hole). We refer to [8] and [73] for such studies in Schwarzschild black holes. We also mention some closely related works by Finster, Kamran, Smoller and Yau [30], [31] and [32] where, among other results, the decay rates of Dirac fields in Kerr-Newman black hole geometries is studied and the work by Nicolas [64] analysing the solutions of a non linear Klein-Gordon equation on a Kerr background.

Compared to earlier works treating analogous but simpler geometries such as Schwarzschild and Reissner-Nordström black holes (see [3], [4], [16], [22], [23], [24], [25], [55], [62]) the analysis of the scattering properties of Dirac fields outside a Kerr-Newman black hole is faced with one additional fundamental difficulty : the *lack of spherical symmetry*. Indeed, the Kerr-Newman solutions possess only two commuting Killing vector fields given in the Boyer-Lindquist coordinate system by ∂_t and ∂_ϕ . Kerr-Newman spacetimes therefore have cylindrical but not spherical spatial symmetry. This lack of symmetry leads to several distinct problems.

- It prevents a straightforward decomposition into spin-weighted spherical harmonics, that reduces the problem to the study of a $1 + 1$ dimensional evolution system with potential. Thus, even in the case of short-range potential, we cannot use the classical trace-class methods (see [69], Vol 3) to construct the wave operators. In such a situation, the unique tool we have in our possession is the Mourre theory (see [58] for the original paper and [1], [36] for later versions).

- As already mentioned, the horizon and infinity have different geometrical structures. More precisely, an observer static at infinity (the point of view we adopt) perceives the propagation of fields outside the black hole as an evolution on a riemannian manifold (exactly the cylindrical manifold $\Sigma = \mathbb{R} \times S^2$) with one *asymptotically flat end* corresponding to infinity and one *exponentially large end* representing the horizon. It is well known (see [19], [33] and [46]) that, in the absence of spherical symmetry, the exponentially large end is awkward for scattering theory, more particularly for the choice of a conjugate operator in the framework of Mourre theory.
- The presence of rotation requires to have a good understanding of the geometry (better than what is required in the Schwarzschild case for similar purposes) in order to express the Dirac equation under a form that does not involve artificial long-range terms.

To solve these problems, we follow the ideas of Häfner [44] and Häfner and Nicolas [46]. In the latter, the authors treated the case of massless Dirac fields outside a Kerr black hole. The difference between [46] and the present work is the presence of long-range terms in the equation. In our case, these long-range terms are contributions due to the mass of the field and due to the interaction between the charge of the field and that of the black hole. To define properly the modification in the wave operators involved by these long-range terms, we use the time-dependent theory summarized by Dereźński and Gérard in [21] in the setting of non-relativistic quantum mechanics and already used for the study of Dirac fields in Reissner-Nordström spacetime in [16]. This formalism provides an elegant construction of the wave operators by a systematic use of natural and physically relevant observables such as the asymptotic velocity operators. These operators allow to considerably simplify the proof of existence and asymptotic completeness of the wave operators. Indeed, in order to deal with the analytical difficulties entailed by the existence of two distinct asymptotic regions, it is convenient to have a natural means of separating outgoing and incoming fields. This is precisely what the asymptotic velocity operators allow us to do. Another interesting feature of this formalism is that it helps us to choose a modification (in the definition of the wave operators) which is particularly convenient. Precisely, we will define a certain Dollard modification such that the modified wave operators satisfy classical intertwining relations between the asymptotic velocity operators and their simpler analogues : the classical velocity operators. Such intertwining relations are a basic claim in the definition of the wave operators.

Let us now explain the strategy of this paper. We first define an adapted null tetrad to write down the Dirac equation (using the Newman-Penrose formalism) in a very convenient form. This null tetrad, first introduced in [46], closely follows the local rotation of the spacetime and avoids the presence of artificial long-range terms in the equation. Next, we compare the physical dynamics with a simpler dynamics corresponding to a Dirac equation outside a Reissner-Nordström black hole (this is possible on each angular mode of a solution). The difference between these two dynamics is short-range. These points are the objects of section IV.2. Some fundamental properties of the generators of these dynamics such as selfadjointness, domain invariance, resolvent estimates and absence of eigenvalues, are then described in section IV.3. As a first approximation, we prove the existence and asymptotic completeness of the classical wave operators between these two dynamics. The crucial step is given in section IV.4 where we establish a Mourre theory for the physical Hamiltonian. The conjugate operator we propose here is slightly different from the one used in [46] : it is particularly adapted to the Dirac equation. We also prove in this section the related *minimal and maximal velocity estimates* which allow us to use the standard Cook's method to obtain the intermediate wave operators. Note that our analysis differs here (and in the following) from the analysis in [46] since no propagation estimate is established there and a limiting absorption principle is systematically used in order to obtain similar results. In section IV.5, we take advantage of these intermediate wave operators and of our previous work [16] concerning the scattering theory of massive charged Dirac fields outside a Reissner-Nordström black hole, to construct

the asymptotic velocity operators and study their spectra : the physically relevant information offered by these operators. Note that the minimal and maximal velocity estimates together with the asymptotic velocity operators, are almost more essential than the wave operators themselves : they provide in a natural and visual way all the scattering information we require. We conclude this paper defining the complete (Dollard-modified at infinity) wave operators using the corresponding results in [16] again.

Abstract index formalism : In this paper, we use the notations of [68]. The abstract index formalism is a notational device for keeping track of the nature of the objects in the course of calculations. *Abstract indices* are denoted by light face Latin letters, small for tensor indices and capital for spinor indices. They do not imply any reference to a coordinate basis. All expressions and calculations involving them are perfectly intrinsic. On the other hand, *concrete indices* defining components in reference to a basis are represented by bold face Latin letters. Concrete tensor indices, denoted by bold face small Latin letters, take their values in $\{0, 1, 2, 3\}$ while concrete spinor indices, denoted by bold face capital Latin letters, take their values in $\{0, 1\}$.

Einstein summation convention : Throughout this paper, we adopt Einstein's convention for the same index appearing twice, once up, once down, in the same term. For abstract indices, this means contraction. For concrete indices, the sum is taken over all the values of the index.

IV.2 Kerr-Newman black holes and Dirac's equation

IV.2.1 The Kerr-Newman metric

In the Boyer-Lindquist coordinates, a Kerr-Newman black hole is described by a smooth four dimensional manifold $\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r \times S_{\theta, \varphi}^2$ whose spacetime Lorentzian metric g_{ab} , simply denoted g , and electromagnetic vector potential Φ_a are given by

$$g = \left(1 + \frac{Q^2 - 2Mr}{\rho^2}\right) dt^2 + \frac{2a \sin^2 \theta (2Mr - Q^2)}{\rho^2} dt d\varphi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\sigma^2}{\rho^2} \sin^2 \theta d\varphi^2, \quad (\text{IV.2.1})$$

with

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + Q^2 - 2Mr + a^2, \quad \sigma^2 = (r^2 + a^2)\rho^2 + (2Mr - Q^2)a^2 \sin^2 \theta,$$

and

$$\Phi_a = -\frac{Qr}{\rho^2} (dt - a \sin^2 \theta d\varphi), \quad (\text{IV.2.2})$$

where M, a, Q denote respectively the mass, the angular momentum per unit mass and the charge of the black hole. When $Q = 0$, the electromagnetic vector potential vanishes and we recover the usual Kerr metric. When $a = 0$, the metric (IV.2.1) reduces to the Reissner-Nordström metric. If in addition $Q = 0$, (IV.2.1) reduces to the Schwarzschild metric. Therefore all eternal isolated black hole solutions are encompassed by spacetimes endowed with the metric (IV.2.1).

The metric (IV.2.1) has two types of singularities. First, the ring $\{r = 0, \theta = \frac{\pi}{2}\}$ which is a true *curvature* singularity and second, the zeros of the function Δ . The number of these zeros depends on the respective values of the constants M, a, Q .

- If $Q^2 + a^2 < M^2$ then Δ has two roots $r_{\pm} = M \pm \sqrt{M^2 - Q^2 - a^2}$ and the spheres $\{r = r_{\pm}\}$ are called *event horizons*.

- If $Q^2 + a^2 = M^2$ then Δ has a unique double root $r_0 = M$ and the sphere $\{r = r_0\}$ is the unique horizon. This case is called *extreme* Kerr-Newman spacetime.
- If $Q^2 + a^2 > M^2$ then Δ has no root. The ring singularity is called a naked singularity.

The horizons, where Δ vanishes, are in fact coordinate singularities. Using appropriate coordinate systems, they are understood as regular null hypersurfaces that can be crossed one way but would require speeds greater than that of light to be crossed the other way, hence their name : event horizons.

We shall here restrict attention to the case $M^2 > a^2 + Q^2$ which is usually considered as the generic description of an asymptotically flat universe containing simply a rotating charged black hole (the extreme case is believed to be unstable). The two horizons separate the spacetime into three connected components called Boyer-Lindquist blocks : block I, denoted here \mathcal{B}_I , is the exterior of the black hole $\{r > r_+\}$; block II, $\{r_- < r < r_+\}$, is a dynamic region where inertial frames are dragged from the outer horizon towards the inner horizon ; block III, $\{r < r_-\}$, is the part of the spacetime containing the ring singularity.

In this study, we shall take the point of view of observers static at infinity. The perception of time of such observers is well described by the time function t of the Boyer-Lindquist coordinates. Thus even though the metric is singular at the horizon in this coordinate system, it is the natural choice for our study. From the point of view of observers static at infinity, black hole spacetimes have the following remarkable property : the light rays emitted towards the black hole take an infinite time to reach the horizon $\{r = r_+\}$ and consequently, the horizon is perceived by them as an *asymptotic region*. Hence, we only work on block I equipped with the metric (IV.2.1). We denote $\mathcal{B}_I = \mathbb{R}_t \times \Sigma$ where Σ is the generic spacelike slice $]r_+, +\infty[\times S^2$. Moreover, since the quantities ρ^2 and σ^2 are positive on \mathcal{B}_I , we denote $\rho = \sqrt{\rho^2}$ and $\sigma = \sqrt{\sigma^2}$. Let us now briefly describe a few properties of block I.

Superradiance : If we look at the causal character of the vector field $\frac{\partial}{\partial t}$, we observe that there exists a region of the spacetime, called *ergosphere*, surrounding the horizon and given by

$$\mathcal{E} = \left\{ (t, r, \theta, \varphi); r_+ < r < M + \sqrt{M^2 - Q^2 - a^2 \cos^2 \theta} \right\},$$

where the vector $\frac{\partial}{\partial t}$ is spacelike. More generally, we cannot find a globally defined timelike Killing vector field on \mathcal{B}_I , that is to say that \mathcal{B}_I is not *stationary*. This gives rise to the superradiance phenomenon. For field equations of integral spin, such as the wave equation, Klein-Gordon or Maxwell, no positive definite conserved energy exists : this allows fields to extract energy from the ergosphere. For field equations of half integral spin such as the Dirac equation, there exists a conserved \bar{L} norm and therefore, there is no superradiance.

Global hyperbolicity A function $f \in C^1(\mathcal{B}_I)$ is a *time function* if its gradient is everywhere timelike (in particular, it does not vanish). This is clearly the case for the function t of the Boyer-Lindquist coordinates since its gradient $\nabla^a t$ is a smooth, non vanishing timelike vector field on block I. The foliation $\Sigma_t = \{t\} \times \Sigma$ by the level hypersurfaces of the function t , is a foliation of \mathcal{B}_I by Cauchy hypersurfaces. Block I is therefore globally hyperbolic (see for example Geroch [42] or Wald [77]). In consequence, we shall consider the Dirac equation as an evolution equation in t with initial data given on the spacelike hypersurface Σ_0 . Note that we need not impose some boundary conditions on the horizon $\{r = r_+\}$ since it is an asymptotic region of the spacetime.

Spin structure : In dimension 4, the global hyperbolicity of \mathcal{B}_I implies the existence of a spin structure (see Geroch [40, 41]). We denote \mathbb{S} (or \mathbb{S}^A in the abstract index formalism) the spin bundle over \mathcal{B}_I

and $\bar{\mathbb{S}}$ (or $\mathbb{S}^{A'}$) the same bundle with the conjugate complex structure. The dual bundles \mathbb{S}^* and $\bar{\mathbb{S}}^*$ will be denoted respectively \mathbb{S}_A and $\mathbb{S}_{A'}$. The complexified tangent bundle to \mathcal{B}_I is recovered as the tensor product of \mathbb{S} and $\bar{\mathbb{S}}$, i.e.

$$T\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S} \otimes \bar{\mathbb{S}} \text{ or } T^a \mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}^A \otimes \mathbb{S}^{A'},$$

and similarly

$$T^* \mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}^* \otimes \bar{\mathbb{S}}^* \text{ or } T_a \mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}_A \otimes \mathbb{S}_{A'}.$$

An abstract tensor index a is thus understood as an unprimed spinor index A and a primed spinor index A' clumped together : $a = AA'$. The spin bundle \mathbb{S} is equipped with a canonical symplectic form ϵ_{AB} , referred to as the Levi-Civita symbol. It is used to raise and lower spinor indices :

$$\epsilon_{AB} \kappa^A = \kappa_B, \quad \epsilon^{AB} \kappa_B = \kappa^A.$$

The complex conjugate $\overline{\epsilon_{AB}} = \bar{\epsilon}_{A'B'}$, simply denoted $\epsilon_{A'B'}$, plays a similar role on $\bar{\mathbb{S}}$. These symplectic structures are compatible with the metric, more precisely

$$g_{ab} = \epsilon_{AB} \epsilon_{A'B'}.$$

IV.2.2 Dirac's equation in the Newman-Penrose formalism

We can represent the four components of the wave function which satisfies Dirac's equation by two spinors $(\phi^A, \chi^{A'})$ and then the charged massive Dirac equation takes the form ([68], p. 419)

$$\begin{cases} (\nabla_{A'}^A - iq\Phi_{A'}^A)\phi_A = \mu\chi_{A'}, \\ (\nabla_A^{A'} - iq\Phi_A^{A'})\chi_{A'} = \mu\phi_A, \quad \mu = \frac{m}{\sqrt{2}}, \end{cases} \quad (\text{IV.2.3})$$

where $m \geq 0$ and q denote respectively the mass and the electric charge of the field. As mentioned previously, an important property of the Dirac equation is that it possesses a quantity which is conserved along the evolution. Precisely, the total charge outside the black hole given by

$$C(t) = \int_{\Sigma_t} (\phi_A \bar{\phi}_{A'} + \bar{\chi}_A \chi_{A'}) T^{AA'} d\text{Vol}, \quad (\text{IV.2.4})$$

is constant throughout time (see for example [65]). Here, T^a denotes the future oriented normal vector field to Σ_t , normalized so that $T_a T^a = 2$ and $d\text{Vol}$ is the volume form induced on Σ_t by g . The charge $C(t)$ defines a norm for $(\phi^A, \chi^{A'})$ on Σ_t .

We use the Newman-Penrose formalism to express the equation (IV.2.3) as a system of partial differential equations with respect to a coordinate basis. This formalism is based on the choice of a null tetrad, i.e. a set of four vectors fields l^a, n^a, m^a, \bar{m}^a , the first two being real and future oriented, \bar{m}^a being the complex conjugated of m^a , satisfying

$$l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a \bar{m}^a = 0, \quad (\text{IV.2.5})$$

and the normalization condition

$$l_a n^a = 1, \quad m_a \bar{m}^a = -1. \quad (\text{IV.2.6})$$

The tetrad $(l^a, n^a, m^a, \bar{m}^a)$ defines at each point a basis of the complexified tangent space to our manifold. To any such tetrad, we can associate a unitary spin-frame (o^A, ι^A) , defined uniquely up to an overall sign factor by the requirements that

$$o^A \bar{o}^{A'} = l^a, \quad \iota^A \bar{\iota}^{A'} = n^a, \quad o^A \bar{\iota}^{A'} = m^a, \quad \iota^A \bar{o}^{A'} = \bar{m}^a, \quad o_A \iota^A = 1. \quad (\text{IV.2.7})$$

The principle of the Newman-Penrose formalism is to decompose the covariant derivative into directional covariant derivatives along the vectors of the tetrad plus connections terms. The directional covariant derivatives are denoted

$$D = l^a \nabla_a, \quad D' = n^a \nabla_a, \quad \delta = m^a \nabla_a, \quad \delta' = \bar{m}^a \nabla_a.$$

The connections terms can be organised into combinations involving only derivatives of frame vectors along frame vectors, namely the twelve spin coefficients defined by (see [68])

$$\begin{aligned} \kappa &= m^a D l_a, \quad \tilde{\rho} = m^a \delta' l_a, \quad \tilde{\sigma} = m^a \delta l_a, \quad \tau = m^a D' l_a, \\ \epsilon &= \frac{1}{2}(n^a D l_a + m^a D \bar{m}_a), \quad \alpha = \frac{1}{2}(n^a \delta' l_a + m^a \delta' \bar{m}_a), \\ \beta &= \frac{1}{2}(n^a \delta l_a + m^a \delta \bar{m}_a), \quad \gamma = \frac{1}{2}(n^a D' l_a + m^a D' \bar{m}_a), \\ \pi &= -\bar{m}^a D n_a, \quad \lambda = -\bar{m}^a \delta' n_a, \quad \mu = -\bar{m}^a \delta n_a, \quad \nu = -\bar{m}^a D' n_a. \end{aligned} \quad (\text{IV.2.8})$$

The main interest of this procedure is the freedom to choose the Newman-Penrose tetrad to be the closest possible to the particular geometric structure of the spacetime. For example, the vectors l^a and n^a should be chosen such that they correspond to the “dynamic” (or scattering) directions, i.e. directions along which light rays may escape towards infinity, while the vectors m^a and \bar{m}^a should be typically chosen such that they generate rotations or more generally, such that they have spatially bounded integral curves.

We denote $\phi_0, \phi_1, \chi_{0'}, \chi_{1'}$ the components of ϕ_A and $\chi_{A'}$ in the spin-frames (o^A, l^A) and $(\bar{o}^{A'}, \bar{l}^{A'})$ associated to the null tetrad. Noting that the dual dyad for (o^A, l^A) is given by $(-l_A, o_A)$, we have

$$\phi_A = -\phi_0 l_A + \phi_1 o_A, \quad \chi_{A'} = -\chi_{0'} \bar{l}_{A'} + \chi_{1'} \bar{o}_{A'}.$$

We reexpress the charged massive Dirac equation as a system of partial differential equations acting on these components. We obtain (see [68])

$$\begin{cases} n^a (\partial_a - iq\Phi_a) \phi_0 - m^a (\partial_a - iq\Phi_a) \phi_1 + (\mu - \gamma) \phi_0 + (\tau - \beta) \phi_1 = \frac{m}{\sqrt{2}} \chi_{1'}, \\ l^a (\partial_a - iq\Phi_a) \phi_1 - \bar{m}^a (\partial_a - iq\Phi_a) \phi_0 + (\alpha - \pi) \phi_0 + (\epsilon - \tilde{\rho}) \phi_1 = -\frac{m}{\sqrt{2}} \chi_{0'}, \\ n^a (\partial_a - iq\Phi_a) \chi_{0'} - \bar{m}^a (\partial_a - iq\Phi_a) \chi_{1'} + (\bar{\mu} - \bar{\gamma}) \chi_{0'} + (\bar{\tau} - \bar{\beta}) \chi_{1'} = \frac{m}{\sqrt{2}} \phi_1, \\ l^a (\partial_a - iq\Phi_a) \chi_{1'} - m^a (\partial_a - iq\Phi_a) \chi_{0'} + (\bar{\alpha} - \bar{\pi}) \chi_{0'} + (\bar{\epsilon} - \bar{\rho}) \chi_{1'} = -\frac{m}{\sqrt{2}} \phi_0. \end{cases} \quad (\text{IV.2.9})$$

The Dirac equation in Kerr geometry was calculated and separated for the first time by Teukolski [74], Unruh [76] (these authors studied in fact neutrino fields) and Chandrasekhar [12] in the framework of the Newman-Penrose formalism. The case of a charged Dirac equation in a Kerr-Newman geometry has been treated by Page [67]. All these references make use of a tetrad introduced by Kinnersley [53]. This tetrad is naturally inherited from the Petrov type D structure (see for example [66], chap. 5). That is to say that the two real null vectors are chosen along the principal null directions

$$V^\pm = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} \pm \frac{\partial}{\partial r} + \frac{a}{\Delta} \frac{\partial}{\partial \varphi}.$$

Unfortunately, it has been shown in [46] that such a choice gives rise to artificial long-range terms in the equation. These terms are due to the fact that the vectors V^\pm are angularly accelerated : this entails apparent *long-range effects of the rotation at infinity*. In order to avoid these artificial long-range terms, we use another tetrad introduced in [46] and chosen in such a way that the apparent effect of the rotation be minimized.

In order to motivate the choice of this new tetrad, observe first that we can associate a preferred timelike vector field to any Newman-Penrose tetrad. It suffices to consider the vector field given by the

sum of the two real frame vectors i.e. $l^a + n^a$. Note that the norm of such a vector field must always be $\sqrt{2}$ since $(l_a + n_a)(l^a + n^a) = 2$. The idea is to choose l^a, n^a such that their sum describes *locally non-rotating* observers. Such observers are described by the future-oriented normal vector field T^a to the hypersurfaces Σ_t and normalized so that $T^a T_a = 2$. This vector field is given by

$$T^a = \sqrt{\frac{2\sigma^2}{\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{a(2Mr - Q^2)}{\sigma^2} \frac{\partial}{\partial \varphi} \right). \quad (\text{IV.2.10})$$

Since there are many possible choices for l^a, n^a satisfying $l^a + n^a = T^a$, we single out a pair of null vectors that are not accelerated in the angular directions, i.e. we choose l^a and n^a in the plane spanned by T^a and ∂_r . Requiring, in addition, that l^a be outgoing and n^a incoming and the two vectors have similar behavior at the horizon, we define

$$l^a = \frac{1}{2}T^a + \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r} = \sqrt{\frac{\sigma^2}{2\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{a(2Mr - Q^2)}{\sigma^2} \frac{\partial}{\partial \varphi} \right) + \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r}, \quad (\text{IV.2.11})$$

$$n^a = \frac{1}{2}T^a - \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r} = \sqrt{\frac{\sigma^2}{2\Delta\rho^2}} \left(\frac{\partial}{\partial t} + \frac{a(2Mr - Q^2)}{\sigma^2} \frac{\partial}{\partial \varphi} \right) - \sqrt{\frac{\Delta}{2\rho^2}} \frac{\partial}{\partial r}. \quad (\text{IV.2.12})$$

The choice of m^a and \bar{m}^a is now imposed except for the freedom of a complex factor of modulus 1. We obtain

$$m^a = \frac{1}{\sqrt{2\rho^2}} \left(\frac{\partial}{\partial \theta} + \frac{\rho^2}{\sqrt{\sigma^2}} \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right), \quad \bar{m}^a = \frac{1}{\sqrt{2\rho^2}} \left(\frac{\partial}{\partial \theta} - \frac{\rho^2}{\sqrt{\sigma^2}} \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right). \quad (\text{IV.2.13})$$

Remark IV.2.1 (Expression of the positive conserved quantity) *A good test to determine if this new Newman-Penrose tetrad is well chosen is to compute the exact expression of the charge current (IV.2.4). From [46], we have*

$$T^{\mathbf{A}\mathbf{A}'} = \begin{pmatrix} n_a T^a & -\bar{m}_a T^a \\ -m_a T^a & l_a T^a \end{pmatrix},$$

which is clearly equal to the identity Id . Hence the conserved positive quantity simply becomes

$$C(t) = \int_{\Sigma_t} (|\phi_0|^2 + |\phi_1|^2 + |\chi_{0'}|^2 + |\chi_{1'}|^2) dVol.$$

We are now ready to give a more explicit form to Dirac's equation (IV.2.9). Once the Newman-Penrose tetrad is chosen, the first task is to compute the spin coefficients (IV.2.8). This can be easily done with the help of Ricci rotation coefficients $\gamma_{(a)(b)(c)}$ (see for instance Chandrasekhar [13]). For this definition, let us denote the frame vectors by

$$l^a = e_{(1)}^a, \quad n^a = e_{(2)}^a, \quad m^a = e_{(3)}^a, \quad \bar{m}^a = e_{(4)}^a.$$

The Ricci rotation coefficients are defined by

$$\begin{aligned} \gamma_{(a)(b)(c)} &= \frac{1}{2} [\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}], \\ \lambda_{(a)(b)(c)} &= \left[\frac{\partial}{\partial x^j} e_{(b)i} - \frac{\partial}{\partial x^i} e_{(b)j} \right] e_{(a)}^i e_{(c)}^j, \end{aligned} \quad (\text{IV.2.14})$$

and the expression of the spin coefficients in terms of the $\gamma_{(a)(b)(c)}$ is

$$\begin{aligned}\kappa &= \gamma_{(3)(1)(1)}, \quad \tilde{\rho} = \gamma_{(3)(1)(4)}, \quad \epsilon = \frac{1}{2}(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}), \\ \tilde{\sigma} &= \gamma_{(3)(1)(3)}, \quad \mu = \gamma_{(2)(4)(3)}, \quad \gamma = \frac{1}{2}(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}), \\ \lambda &= \gamma_{(2)(4)(4)}, \quad \tau = \gamma_{(3)(1)(2)}, \quad \alpha = \frac{1}{2}(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}), \\ \nu &= \gamma_{(2)(4)(2)}, \quad \pi = \gamma_{(2)(4)(1)}, \quad \beta = \frac{1}{2}(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}).\end{aligned}\tag{IV.2.15}$$

Note that the Ricci rotation coefficients involve only usual partial derivatives which makes the following calculations easy. Using (IV.2.11), (IV.2.12), (IV.2.13) and the expressions for the associated 1-forms

$$\begin{aligned}l_a dx^a &= \sqrt{\frac{\Delta \rho^2}{2\sigma^2}} dt - \sqrt{\frac{\rho^2}{2\Delta}} dr, & n_a dx^a &= \sqrt{\frac{\Delta \rho^2}{2\sigma^2}} dt + \sqrt{\frac{\rho^2}{2\Delta}} dr, \\ m_a dx^a &= ia \sin \theta (2Mr - Q^2) \sqrt{\frac{1}{2\rho^2 \sigma^2}} dt - \sqrt{\frac{\rho^2}{2}} d\theta - i \sin \theta \sqrt{\frac{\sigma^2}{2\rho^2}} d\varphi,\end{aligned}$$

the spin coefficients are given by

$$\begin{aligned}\kappa &= \frac{a^2 \Delta \sin \theta \cos \theta}{2\sqrt{2}\rho}, \\ \tilde{\sigma} &= -\frac{a^2 \sin^2 \theta \sqrt{\Delta}}{\sqrt{2}\rho^3 \sigma^2} (r(2Mr - Q^2) + 2\rho^2(r - M)) - i \frac{a^3 \sqrt{\Delta} (2Mr - Q^2) \sin^2 \theta \cos \theta}{\sqrt{2}\rho^3 \sigma^2}, \\ \mu &= -\frac{\sqrt{\Delta} (2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta)}{2\sqrt{2}\rho \sigma^2}, \\ \gamma &= \frac{1}{2\sqrt{2}} \left(\frac{r - M}{\sqrt{\Delta} \rho} + \frac{\sqrt{\Delta} r}{\rho^3} - \frac{\sqrt{\Delta} (2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta)}{\rho \sigma^2} \right) \\ &\quad + \frac{i}{2\sqrt{2}} \frac{a^3 (2Mr - Q^2) \sqrt{\Delta} \sin^2 \theta \cos \theta}{\rho^3 \sigma^2}, \\ \tau &= \frac{1}{2\sqrt{2}} \left(\frac{a^2 \sin \theta \cos \theta}{\rho^3} + \frac{a^2 (r^2 + a^2) (2Mr - Q^2) \sin \theta \cos \theta}{\rho^3 \sigma^2} \right) \\ &\quad + \frac{i}{\sqrt{2}} \left(\frac{aM \sin \theta}{\rho^3} - \frac{a \sin \theta (2Mr - Q^2) (2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta)}{\rho^3 \sigma^2} \right), \\ \beta &= \frac{1}{2\sqrt{2}} \left(\frac{\cot \theta}{\rho} + \frac{a^2 \sin \theta \cos \theta (r^2 + a^2) (2Mr - Q^2)}{\rho^3 \sigma^2} \right) \\ &\quad + \frac{i}{2\sqrt{2}} \left(\frac{aM \sin \theta}{\rho^3} - \frac{a \sin \theta (2Mr - Q^2) (2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta)}{\rho^3 \sigma^2} \right),\end{aligned}\tag{IV.2.16}$$

and

$$\kappa = -\nu, \quad \tilde{\sigma} = \lambda, \quad \mu = \tilde{\rho}, \quad \gamma = \epsilon, \quad \tau = -\pi, \quad \beta = -\alpha.\tag{IV.2.17}$$

In order to simplify the notations, we also introduce the functions

$$C = -iql^a \Phi_a = -iqn^a \Phi_a = i \frac{qQr(r^2 + a^2)}{\sqrt{2\Delta} \rho \sigma}, \quad D = -iqm^a \Phi_a = iq\bar{m}^a \Phi_a = \frac{qQra \sin \theta}{\sqrt{2}\rho \sigma}.\tag{IV.2.18}$$

To the tetrad (IV.2.11)-(IV.2.13), we associate a spin-frame (σ^A, ι^A) satisfying (IV.2.7). In order to write Dirac's equation in the usual formalism involving 4×4 Dirac matrices and use our previous paper [16],

we consider the equation satisfied by the bi-spinor $\psi = {}^t(\phi_0, \phi_1, \chi_{1'}, -\chi_{0'})$. We refer to [65] for this point. Therefore, Dirac's equation takes the form

$$M_t \partial_t \psi + M_r \partial_r \psi + M_\theta \partial_\theta \psi + M_\varphi \partial_\varphi \psi + C \psi + P \psi = M_m \psi, \quad (\text{IV.2.19})$$

where

$$M_t = \frac{\sigma}{\sqrt{2\Delta}\rho} \text{Id}, \quad M_r = \frac{\sqrt{\Delta}}{\sqrt{2}\rho} \Gamma^1, \quad M_\theta = \frac{1}{\sqrt{2}\rho} \Gamma^2, \quad M_\varphi = \frac{a(2Mr - Q^2)}{\sqrt{2\Delta}\rho\sigma} \text{Id} + \frac{\rho}{\sqrt{2}\sigma \sin \theta} \Gamma^3,$$

$$P = \begin{pmatrix} \mu - \gamma & \tau - \beta - D & 0 & 0 \\ \tau - \beta + D & -(\mu - \gamma) & 0 & 0 \\ 0 & 0 & -(\bar{\mu} - \bar{\gamma}) & -(\bar{\tau} - \bar{\beta} - D) \\ 0 & 0 & -(\bar{\tau} - \bar{\beta} + D) & (\bar{\mu} - \bar{\gamma}) \end{pmatrix}, \quad M_m = -i \frac{m}{\sqrt{2}} \Gamma^0,$$

with

$$\Gamma^1 = \text{diag}(-1, 1, 1, -1),$$

$$\Gamma^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \Gamma^0 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

The Γ 's are 4×4 -Dirac matrices and satisfy the anticommutation relations $\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\text{Id}$. An explicit calculation of the coefficients in the term P gives

$$\begin{aligned} \mu - \gamma &= -\frac{1}{2\sqrt{2}} \left(\frac{r - M}{\sqrt{\Delta}\rho} + \frac{\sqrt{\Delta}r}{\rho^3} \right) - i \frac{a^3(2Mr - Q^2)\sqrt{\Delta} \sin^2 \theta \cos \theta}{2\sqrt{2}\rho^3\sigma^2}, \quad (\text{IV.2.20}) \\ \tau - \beta \pm D &= \left(\frac{a^2 \sin \theta \cos \theta}{2\sqrt{2}\rho^3} - \frac{\cot \theta}{2\sqrt{2}\rho} \pm \frac{qQra \sin \theta}{\sqrt{2}\rho\sigma} \right) \\ &\quad + i \left(\frac{aM \sin \theta}{2\sqrt{2}\rho^3} - \frac{a \sin \theta (2Mr - Q^2)(2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta)}{2\sqrt{2}\rho^3\sigma^2} \right). \quad (\text{IV.2.21}) \end{aligned}$$

We first write Dirac's equation (IV.2.19) as an evolution system with respect to t multiplying (IV.2.19) by M_t^{-1} . Hence, noting that the term P can be written as the sum of $M_\theta \frac{\cot \theta}{2}$ plus a remainder term, we obtain

$$\partial_t \psi + \tilde{M}_r \partial_r \psi + \tilde{M}_\theta \left(\partial_\theta + \frac{\cot \theta}{2} \right) \psi + \tilde{M}_\varphi \partial_\varphi \psi + \tilde{C} \psi + \tilde{P} \psi = \tilde{M}_m \psi, \quad (\text{IV.2.22})$$

where

$$\tilde{M}_r = \frac{\Delta}{\sigma} \Gamma^1, \quad \tilde{M}_\theta = \frac{\sqrt{\Delta}}{\sigma} \Gamma^2, \quad \tilde{M}_\varphi = \frac{a(2Mr - Q^2)}{\sigma^2} \text{Id} + \frac{\rho^2 \sqrt{\Delta}}{\sigma^2 \sin \theta} \Gamma^3, \quad \tilde{M}_m = -i \frac{m\rho\sqrt{\Delta}}{\sigma} \Gamma^0,$$

$$\tilde{S} = i \frac{qQr(r^2 + a^2)}{\sigma^2}, \quad \tilde{P} = \begin{pmatrix} p & p_- & 0 & 0 \\ p_+ & -p & 0 & 0 \\ 0 & 0 & -\bar{p} & -\bar{p}_- \\ 0 & 0 & -\bar{p}_+ & \bar{p} \end{pmatrix},$$

with

$$p = -\frac{1}{2} \left(\frac{r-M}{\sigma} + \frac{\Delta r}{\sigma \rho^2} \right) - i \frac{a^3 (2Mr - Q^2) \Delta \sin^2 \theta \cos \theta}{2\rho^2 \sigma^3}, \quad (\text{IV.2.23})$$

$$p_{\pm} = \left(\frac{a^2 \sqrt{\Delta} \sin \theta \cos \theta}{2\rho^2 \sigma} \pm \frac{qQ \sqrt{\Delta} r a \sin \theta}{\sigma^2} \right) + i \left(\frac{aM \sqrt{\Delta} \sin \theta}{2\rho^2 \sigma} - \frac{a \sqrt{\Delta} \sin \theta (2Mr - Q^2) (2r(r^2 + a^2) - (r-M)a^2 \sin^2 \theta)}{2\rho^2 \sigma^3} \right). \quad (\text{IV.2.24})$$

IV.2.3 First simplifications of the equation

In the form (IV.2.22), we seem to face a standard evolution problem with boundary conditions on the horizon $r = r_+$. But two facts indicate that we do not work with a “good” system of coordinates. We already said that the horizon is an asymptotic region of the spacetime. Therefore, no boundary conditions should be imposed on it. Moreover, the (t, r) principal part $\partial_t + \frac{\Delta}{\sigma} \Gamma^1 \partial_r$ of equation (IV.2.22) presents a default of strict hyperbolicity when $r \rightarrow r_+$. This follows from the vanishing of the coefficient $\frac{\Delta}{\sigma}$ when $r \rightarrow r_+$. Therefore, we introduce a new radial coordinate which turns out to be useful for the study of the scattering properties of the fields. We define the “Regge-Wheeler type” variable r_* (see [13] or more recently [46]) by

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad (\text{IV.2.25})$$

i.e.

$$r_* = r + M \log \Delta + \frac{2M^2 - Q^2}{2\sqrt{M^2 - a^2 - Q^2}} \log \frac{r - r_+}{r - r_-} + R_0, \quad (\text{IV.2.26})$$

where R_0 is any constant of integration. The variable r_* is chosen so that the principal null directions are expressed as

$$V^{\pm} = \frac{r^2 + a^2}{\Delta} \left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r_*} + \frac{a}{r^2 + a^2} \frac{\partial}{\partial \varphi} \right), \quad (\text{IV.2.27})$$

which implies that the principal null geodesics have radial speed ± 1 with respect to this coordinate. This property will entail the simplest expression of the asymptotic dynamics satisfied by the fields at the horizon. Note that, in the coordinate system $(t, r_*, \theta, \varphi)$, the horizon $\{r = r_+\}$ is pushed away to $\{r_* = -\infty\}$ which gives sense to the fact that this is an “asymptotic” region of the spacetime. The term $\tilde{M}_r \partial_r$ becomes $\tilde{M}_{r_*} \partial_{r_*}$ with $\tilde{M}_{r_*} = \frac{r^2 + a^2}{\sigma} \Gamma^1$. Since \tilde{M}_{r_*} remains invertible when $r_* \rightarrow -\infty$, the (t, r) part of the equation is now strictly hyperbolic.

Some long-range terms remain in the expression of \tilde{P} . Multiplying the spinor ψ by a well chosen measure density, we can get rid of them. Note that $d\text{Vol}$ has the following expression in the coordinate system (r_*, θ, φ)

$$d\text{Vol} = \frac{\sqrt{\Delta} \sigma \rho}{r^2 + a^2} dr_* d\omega, \quad d\omega = \sin \theta d\theta d\varphi.$$

In order to work in a standard (i.e. not weighted) L^2 space, we introduce the spinor density

$$\Psi = \left(\frac{\sqrt{\Delta} \sigma \rho}{r^2 + a^2} \right)^{\frac{1}{2}} \psi.$$

The equation satisfied by Ψ is

$$\partial_t \Psi + \mathbb{M}_{r_*} \partial_{r_*} \Psi + \mathbb{M}_{\theta} \left(\partial_{\theta} + \frac{\cot \theta}{2} \right) \Psi + \mathbb{M}_{\varphi} \partial_{\varphi} \Psi + \mathbb{C} \Psi + \mathbb{P} \Psi = \mathbb{M}_m \Psi, \quad (\text{IV.2.28})$$

where $\mathbb{M}_{r_*} = \tilde{M}_{r_*}$, $\mathbb{M}_\theta = \tilde{M}_\theta$, $\mathbb{M}_\varphi = \tilde{M}_\varphi$, $\mathbb{M}_m = \tilde{M}_m$, $\mathbb{C} = \tilde{C}$ and $\mathbb{P} = \tilde{P} + A_r + A_\theta$ with

$$\begin{aligned} A_r &= M_{r_*}^1 \left(\frac{\sqrt{\Delta}\sigma\rho}{r^2 + a^2} \right)^{\frac{1}{2}} \left[\partial_{r_*}, \left(\frac{\sqrt{\Delta}\sigma\rho}{r^2 + a^2} \right)^{-\frac{1}{2}} \right] \\ &= \left(-\frac{r-M}{2\sigma} - \frac{r\Delta}{2\rho^2\sigma} + \frac{(r-M)\Delta a^2 \sin^2 \theta}{2\sigma^3} - \frac{r\Delta^2 a^2 \sin^2 \theta}{\sigma^3(r^2 + a^2)} \right) \Gamma^1, \\ A_\theta &= M_\theta^1 \left(\frac{\sqrt{\Delta}\sigma\rho}{r^2 + a^2} \right)^{\frac{1}{2}} \left[\partial_\theta, \left(\frac{\sqrt{\Delta}\sigma\rho}{r^2 + a^2} \right)^{-\frac{1}{2}} \right] = \left(\frac{a^2 \Delta^{\frac{3}{2}} \sin \theta \cos \theta}{2\sigma^3} + \frac{a^2 \sqrt{\Delta} \sin \theta \cos \theta}{2\rho^2 \sigma} \right) \Gamma^2. \end{aligned}$$

Certain of these terms cancel with certain terms in (IV.2.23) and (IV.2.24). We obtain

$$\mathbb{P} = \begin{pmatrix} \mathbf{p} & \mathbf{p}_- & 0 & 0 \\ \mathbf{p}_+ & -\mathbf{p} & 0 & 0 \\ 0 & 0 & -\bar{\mathbf{p}} & -\bar{\mathbf{p}}_- \\ 0 & 0 & -\bar{\mathbf{p}}_+ & \bar{\mathbf{p}} \end{pmatrix},$$

with

$$\begin{aligned} \mathbf{p} &= -\frac{(r-M)\Delta a^2 \sin^2 \theta}{2\sigma^3} + \frac{r\Delta^2 a^2 \sin^2 \theta}{\sigma^3(r^2 + a^2)} - i \frac{a^3(2Mr - Q^2)\Delta \sin^2 \theta \cos \theta}{2\rho^2 \sigma^3}, \\ \mathbf{p}_\pm &= -\frac{a^2 \Delta^{\frac{3}{2}} \sin \theta \cos \theta}{2\sigma^3} \pm \frac{qQ\sqrt{\Delta}ra \sin \theta}{\sigma^2} \\ &\quad + i \left(\frac{aM\sqrt{\Delta} \sin \theta}{2\rho^2 \sigma} - \frac{a\sqrt{\Delta} \sin \theta (2Mr - Q^2)(2r(r^2 + a^2) - (r-M)a^2 \sin^2 \theta)}{2\rho^2 \sigma^3} \right). \end{aligned}$$

The matrix-valued potential \mathbb{P} is short-range at infinity, i.e. its fall-off when $r_* \rightarrow +\infty$ is of the order of $(r_*)^{-1-\epsilon}$, $\epsilon > 0$. Thus, it will not give a contribution in the definition of the wave operators.

IV.2.4 Comparison with a spherically symmetric dynamics

Our goal in this section is to reexpress equation (IV.2.28) as a spherically symmetric equation plus short-range terms. The reduction to a spherically symmetric setting is almost forced by the analytical difficulties arising from the full equation.

The first step is to isolate the Dirac operator on the 2-sphere, denoted \mathbb{D}_{S^2} , in (IV.2.28). We define

$$\mathbb{D}_{S^2} = \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}, \quad \text{with } S = i \begin{pmatrix} 0 & \partial_\theta + \frac{\cot \theta}{2} + \frac{i}{\sin \theta} \partial_\varphi \\ \partial_\theta + \frac{\cot \theta}{2} - \frac{i}{\sin \theta} \partial_\varphi & 0 \end{pmatrix}. \quad (\text{IV.2.29})$$

We obtain the following Hamiltonian form for the Dirac equation (IV.2.28)

$$i\partial_t \Psi = \mathbb{D}\Psi, \quad (\text{IV.2.30})$$

with

$$\mathbb{D} = \frac{r^2 + a^2}{\sigma} \Gamma^1 D_{r_*} + \frac{\sqrt{\Delta}}{\sigma} \mathbb{D}_{S^2} + A_\varphi D_\varphi + \frac{m\rho\sqrt{\Delta}}{\sigma} \Gamma^0 + \frac{qQr(r^2 + a^2)}{\sigma^2} + \mathbb{V}, \quad (\text{IV.2.31})$$

where

$$A_\varphi = \frac{a(2Mr - Q^2)}{\sigma^2} \text{Id} + \frac{\sqrt{\Delta}}{\sigma \sin \theta} \left(\frac{\rho^2}{\sigma} - 1 \right) \Gamma^3, \quad \mathbb{V} = -i\mathbb{P}, \quad D_{r_*} = -i\partial_{r_*}, \quad D_\varphi = -i\partial_\varphi.$$

An advantage of expression (IV.2.31) is that it involves only the operators \mathbb{D}_{S^2} and D_φ which are regular on the whole 2-sphere. The singularities appearing in $\cot \theta$ and $\frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}$ are thus understood as coordinate singularities. Furthermore, the lack of spherical symmetry of the equation is measured by the D_φ term and by the θ -dependence of the coefficients. In order to simplify the later analysis, we now replace these terms by spherically symmetric terms plus short-range potentials.

We use the cylindrical symmetry of \mathbb{D} to treat the long-range term $A_\varphi D_\varphi$ as a potential. We decompose the elements of \mathcal{H} on the basis $e^{in\varphi}$, $n \in \mathbb{Z}$. We denote $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$ this decomposition and for any $\Psi \in \mathcal{H}$, we have

$$\Psi(r_*, \theta, \varphi) = \sum_{n \in \mathbb{Z}} \Psi_n(r_*, \theta) e^{in\varphi}.$$

Each component Ψ_n satisfies the equation

$$i\partial_t \Psi_n = \mathbb{D}^n \Psi_n,$$

where

$$\mathbb{D}^n = h^2(r_*, \theta) \Gamma^1 D_{r_*} + a(r_*, \theta) \mathbb{D}_{S^2} + b(r_*, \theta) \Gamma^0 + c(r_*, \theta) + V, \quad (\text{IV.2.32})$$

and

$$\begin{aligned} h(r_*, \theta) &= \sqrt{\frac{r^2 + a^2}{\sigma}}, \quad a(r_*, \theta) = \frac{\sqrt{\Delta}}{\sigma}, \quad b(r_*, \theta) = \frac{m\rho\sqrt{\Delta}}{\sigma}, \\ c(r_*, \theta) &= \frac{na(2Mr - Q^2)}{\sigma^2} + \frac{qQr(r^2 + a^2)}{\sigma^2}, \quad V = \mathbb{V} + \frac{n\sqrt{\Delta}}{\sigma \sin \theta} \left(\frac{\rho^2}{\sigma} - 1 \right) \Gamma^3. \end{aligned}$$

The term V only involves the terms out of the diagonal coming from $A_\varphi D_\varphi$. These terms are short-range and thus, V remains short-range. On the other hand, the diagonal part of $A_\varphi D_\varphi$ gives rise to a long-range scalar contribution which appears in $c(r_*, \theta)$. Note that we conserved the full Dirac operator \mathbb{D}_{S^2} in the expression of \mathbb{D}^n since we aim to compare it with a spherically symmetric operator plus short-range term.

The second step is to replace the functions a, b, c by functions which only depend on the radial variable. Roughly speaking, this is done by approaching ρ by $\sqrt{r^2 + a^2}$ and σ by $r^2 + a^2$. Then, we introduce the spherically symmetric Dirac operator (of Reissner-Nordström type, see [16])

$$\mathbb{D}_0^n = \Gamma^1 D_{r_*} + a_0(r_*) \mathbb{D}_{S^2} + b_0(r_*) \Gamma^0 + c_0(r_*),$$

where

$$a_0(r_*) = \frac{\sqrt{\Delta}}{r^2 + a^2}, \quad b_0(r_*) = \frac{m\sqrt{\Delta}}{\sqrt{r^2 + a^2}}, \quad c_0(r_*) = \frac{na(2Mr - Q^2)}{(r^2 + a^2)^2} + \frac{qQr}{r^2 + a^2}.$$

Now, a straightforward computation shows that the operator \mathbb{D}^n can be written as

$$\mathbb{D}^n = h \mathbb{D}_0^n h + V_0,$$

where

$$V_0 = \mathbb{V}_0 + \frac{n\sqrt{\Delta}}{\sigma \sin \theta} \left(\frac{\rho^2}{\sigma} - 1 \right) \Gamma^3 + \frac{m\sqrt{\Delta}}{\sigma} (\rho - \sqrt{r^2 + a^2}) \Gamma^0 + \left(\frac{na(2Mr - Q^2)}{(r^2 + a^2)\sigma^2} + \frac{qQr}{\sigma^2} \right) (r^2 + a^2 - \sigma),$$

and

$$\mathbb{V}_0 = \begin{pmatrix} v_0 & v_1 & 0 & 0 \\ \bar{v}_1 & -v_0 & 0 & 0 \\ 0 & 0 & v_0 & v_1 \\ 0 & 0 & \bar{v}_1 & -v_0 \end{pmatrix}$$

$$\begin{aligned}
v_0 &= \frac{a^3(2Mr - Q^2)\Delta \sin^2 \theta \cos \theta}{2\rho^2 \sigma^3}, \\
v_1 &= \left(\frac{aM\sqrt{\Delta} \sin \theta}{2\rho^2 \sigma} - \frac{a\sqrt{\Delta} \sin \theta(2Mr - Q^2)(2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta)}{2\rho^2 \sigma^3} \right) \\
&\quad + i \frac{qQ\sqrt{\Delta}ra \sin \theta}{\sigma^2}.
\end{aligned}$$

We will show rigorously in the next section that V_0 is short-range and therefore will not play a role in the definition of the wave operators.

The operators \mathbb{D}^n and \mathbb{D}_0^n , acting on \mathcal{H}^n , will be the operators at the heart of our analysis. From now on, we will assume that these operators act on the total Hilbert space \mathcal{H} as follows : we make the parts of \mathbb{D}^n and \mathbb{D}_0^n involving \mathbb{D}_{S^2} act on \mathcal{H} in a natural way whereas we still consider the parts of \mathbb{D}^n and \mathbb{D}_0^n involving the angular mode n as a potential. To avoid confusion, we denote H and H_0 the extensions of \mathbb{D}^n and \mathbb{D}_0^n to \mathcal{H} for a given $n \in \mathbb{N}$. It is important to keep in mind that H_0 (resp. H) only coincide with \mathbb{D}_0^n (resp. \mathbb{D}^n) on the particular mode $e^{in\varphi}$.

The operator H_0 is a spherically symmetric Dirac operator of Reissner-Nordström type (see [16]) whereas H remains an axisymmetric Dirac operator. The next sections are devoted to the study of the basic properties of H_0 and H such as selfadjointness, domain invariance, absence of eigenvalues and in particular, we will turn our attention on finding a conjugate operator (in the sense of Mourre's theory) for both of them. This will provide important informations on their spectra and allow us to establish the so-called minimal velocity estimates. As a consequence, since V_0 is short-range, we will be able in a first approximation, to compare asymptotically the dynamics e^{-itH} and e^{-itH_0} and construct the classical wave operators between these by a standard Cook's argument. The important point is that the results proved for H_0 and H will remain true for \mathbb{D}_0^n and \mathbb{D}^n *simply by restriction to the angular mode* $e^{in\varphi}$. Since we can do that for any $n \in \mathbb{N}$, we will then recombine these results to obtain a global result involving the full dynamics $e^{-it\mathbb{D}}$.

IV.3 Abstract analytic framework and fundamentals properties of Dirac Hamiltonians

IV.3.1 Symbol classes

We introduce in this section some classes of functions which measure the r_* -decay of the potentials in the asymptotic regions. Let us first analyse the asymptotic behavior of the function $\Delta(r(r_*))$ which enters in the definition of the potentials a_0 and b_0 . Recall that $\Delta(r) = (r - r_+)(r - r_-)$. From (IV.2.26), it is immediate that $r_* \simeq r(r_*)$ as $r_* \rightarrow +\infty$ and thus, the function $\Delta(r)$ behaves as r_*^2 at spacelike infinity. On the other hand, when considering the horizon, we have $\Delta(r) \rightarrow 0$ when $r \rightarrow r_+$. More precisely, we obtain from (IV.2.26) the expression

$$r_* - r = \frac{r_+^2 + a^2}{r_+ - r_-} \log(r - r_+) - \frac{r_-^2 + a^2}{r_+ - r_-} \log(r - r_-) + R_0.$$

Denoting $\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$ the surface gravity at the black hole horizon and $\delta = \frac{r_-^2 + a^2}{r_+^2 + a^2}$, we can express $r - r_+$ and Δ as

$$r - r_+ = e^{2\kappa_+(r_* - r - R_0)}(r - r_-)^\delta, \quad \Delta = e^{2\kappa_+(r_* - r - R_0)}(r - r_-)^{\delta+1}. \quad (\text{IV.3.1})$$

Since $r(r_*) \rightarrow r_+$ as $r_* \rightarrow -\infty$, the functions $r(r_*) - r_+$ and $\Delta(r_*)$ are equivalent to $e^{2\kappa+r_*}$ when $r_* \rightarrow -\infty$ and thus are *exponentially decreasing at the horizon*.

This leads us to define the following symbol classes as subsets of $C^\infty(\Sigma)$. Let $\eta > 0$.

$$\bullet f \in \mathbf{S}^{m,n} \text{ if and only if } \forall \alpha \in \mathbb{N}, \beta \in \mathbb{N}^{n-1}, \partial_{r_*}^\alpha \partial_\omega^\beta f \in \begin{cases} O(\langle r_* \rangle^{m-\alpha}), r_* \rightarrow +\infty, \\ O(e^{\eta|r_*|}), r_* \rightarrow -\infty. \end{cases}$$

$$\bullet f \in \mathbf{S}^m \text{ if and only if } \forall \alpha \in \mathbb{N}, \beta \in \mathbb{N}^{n-1}, \partial_{r_*}^\alpha \partial_\omega^\beta f \in O(\langle r_* \rangle^{m-\alpha}).$$

We denote $S^{m,n}$ and S^m the same spaces for spherically symmetric functions i.e. for $f \in C^\infty(\mathbb{R}_{r_*})$. We list some properties of these spaces.

$$(i) \quad \mathbf{S}^{m,n} \times \mathbf{S}^{p,q} \subset \mathbf{S}^{m+p,n+q}, \quad S^{m,n} \times S^{p,q} \subset S^{m+p,n+q}, \quad (\text{IV.3.2})$$

$$(ii) \quad \partial_{r_*}^\alpha : \mathbf{S}^{m,n} \rightarrow \mathbf{S}^{m-\alpha,n}, \quad \partial_{r_*}^\alpha : S^{m,n} \rightarrow S^{m-\alpha,n}, \quad \forall \alpha \in \mathbb{N}, \quad (\text{IV.3.3})$$

$$(iii) \quad \partial_\omega^\beta : \mathbf{S}^{m,n} \rightarrow \mathbf{S}^{m,n}, \quad \forall \beta \in \mathbb{N}. \quad (\text{IV.3.4})$$

Let us set $\eta = \kappa_+$ in the definition of the above spaces. The potentials a_0, b_0, c_0 satisfy the following assumptions.

Proposition IV.3.1 *There exist two constants $c_1 \neq 0, c_2$ and $\epsilon > 0$ such that the potentials a_0, b_0, c_0 satisfy*

$$a_0 \in S^{-1,-1}, \text{ and } \begin{cases} \left(a_0(r_*) - \frac{1}{r_*}\right)^{(i)} \in O(\langle r_* \rangle^{-1-\epsilon-i}), r_* \rightarrow +\infty, \\ a_0(r_*) - c_1 e^{\kappa+r_*} \in O(e^{(\kappa_++\epsilon)r_*}), r_* \rightarrow -\infty, \end{cases} \quad (\text{IV.3.5})$$

$$b_0 \in S^{0,-1}, b'_0 \in S^{-2,-1}, \text{ and } b_0 - m \in O(\langle r_* \rangle^{-1}), r_* \rightarrow +\infty, \quad (\text{IV.3.6})$$

$$c_0 \in S^{-1,0}, c'_0 \in S^{-2,-2}, \text{ and } c_0(r_*) - c_2 \in O(e^{2\kappa+r_*}), r_* \rightarrow -\infty. \quad (\text{IV.3.7})$$

Proof : To prove these assertions, we need to make the link between the respective asymptotic behaviors of functions $f(r)$ and $f(r(r_*))$. This is done in the following Lemma. Let us first introduce the class of functions

$$f(r, \theta) \in \Pi^m \text{ if and only if } \forall \alpha, \beta \in \mathbb{N}, \partial_r^\alpha \partial_\theta^\beta f \in \begin{cases} O(\langle r \rangle^{m-\alpha}), r \rightarrow +\infty, \\ O(1), r \rightarrow r_+. \end{cases}$$

Then, from [44], Lemme 9.7.1, we have

Lemma IV.3.1 (i) $f(r) \in \Pi^m \implies f(r(r_*)) \in \mathbf{S}^{m,0}$ and $\forall \alpha \in \mathbb{N}^*, \partial_{r_*}^\alpha f(r(r_*)) \in \mathbf{S}^{m-\alpha,-2}$,
(ii) $f(r_*) \in \mathbf{S}^{m,n}, g(r) \in \Pi^k \implies f(r_*)g(r(r_*)) \in \mathbf{S}^{m+k,n}$.

From (IV.2.26), (IV.3.1) and Lemma IV.3.1, it is immediate that

$$\Delta \in S^{2,-2}, \quad \sqrt{\Delta} \in S^{1,-1}, \quad (\text{IV.3.8})$$

Recall that $a_0(r_*) = \frac{\sqrt{\Delta}}{r_+^2 + a^2}$. Since $\frac{1}{r_+^2 + a^2} \in \Pi^{-2}$, Lemma IV.3.1 and (IV.3.8) imply $a_0 \in S^{-1,-1}$.

Let us set $c_1 = \frac{e^{-\kappa+(r_++R_0)}(r_+-r_-)^{\frac{\delta+1}{2}}}{r_+^2+a^2}$. Then using (IV.3.1), we have

$$a_0(r_*) - c_1 e^{\kappa+r_*} = f(r_*) e^{\kappa+r_*},$$

with $f(r_*) = \frac{e^{-\kappa+(r-R_0)(r-r_-)} \frac{\delta+1}{2}}{r^2+a^2} - c_1 \in O(r-r_+)$ when $r_* \rightarrow -\infty$. Hence, (IV.3.1) implies $a_0(r_*) - c_1 e^{\kappa+r_*} \in O(e^{(\kappa+\epsilon)r_*})$ when $r_* \rightarrow -\infty$. Now we write

$$a_0(r_*) - \frac{1}{r_*} = \frac{\Delta}{r^2+a^2} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r_*}.$$

The term $\frac{\Delta}{r^2+a^2} - \frac{1}{r}$ belongs to $O(\langle r_* \rangle^{-2})$ and the term $\frac{1}{r} - \frac{1}{r_*}$ belongs to $O(\langle r_* \rangle^{-2+\epsilon})$ for any $\epsilon > 0$ by (IV.2.25). Therefore, $a_0(r_*) - \frac{1}{r_*} \in O(\langle r_* \rangle^{-1-\epsilon})$. Similarly, an explicit calculation shows that $a'_0(r_*) + \frac{1}{r_*^2} \in O(\langle r_* \rangle^{-2-\epsilon})$.

Since $\frac{1}{\sqrt{r^2+a^2}} \in \Pi^{-1}$, $b_0(r_*) = \frac{m\sqrt{\Delta}}{\sqrt{r^2+a^2}} \in S^{0,-1}$ by Lemma IV.3.1 and (IV.3.8). Similarly, we have $b'_0(r_*) = m\sqrt{\Delta} \frac{((r-M)(r^2+a^2)-\Delta r)}{(r^2+a^2)^{\frac{5}{2}}}$. Since $\frac{(r-M)(r^2+a^2)-\Delta r}{(r^2+a^2)^{\frac{5}{2}}} \in \Pi^{-2}$, we obtain $b'_0(r_*) \in S^{-2,-1}$ by Lemma IV.3.1 and (IV.3.8). Finally, it is straightforward to see that $b_0 - m \in O(\langle r_* \rangle^{-1})$ when $r_* \rightarrow +\infty$ by (IV.2.25).

Since $c'_0(r) = \frac{na(2Mr-Q^2)}{(r^2+a^2)^2} + \frac{qQr}{r^2+a^2} \in \Pi^{-1}$, Lemma IV.3.1 implies $c_0 \in S^{-1,0}$ and $c'_0 \in S^{-2,-2}$. Let us set $c_2 = \frac{na(2Mr_+-Q^2)}{(r_+^2+a^2)^2} + \frac{qQr_+}{r_+^2+a^2}$. Then using (IV.2.26), we have $r - r_+ \in O(e^{2\kappa+r_*})$ and $\frac{1}{r^2+a^2} - \frac{1}{r_+^2+a^2} \in O(e^{2\kappa+r_*})$. Therefore, $c_0(r_*) - c_2 \in O(e^{2\kappa+r_*})$.
◇

We shall also need the following result

Lemma IV.3.2 *The function h and the potential V_0 satisfy*

$$|h^2 - 1| < 1, \quad h - 1 \in \mathbf{S}^{-2,-2}, \quad \partial_\theta h \in \mathbf{S}^{-2,-2}, \quad V_0^{ij} \in \mathbf{S}^{-2,-1}.$$

In particular, $V_0 \in \mathbf{S}^{-2}$ is a short-range matrix-valued potential.

Proof : We have

$$0 \leq h^2 - 1 \leq \frac{h^4 - 1}{h^2 + 1} \leq h^4 - 1 = \frac{(r^2 + a^2)^2 - \sigma^2}{\sigma^2} = \frac{\Delta a^2 \sin^2 \theta}{\sigma^2}$$

Since $\sigma^2 \geq (r^2 + a^2)^2$ and $\Delta \leq r^2 + a^2$ for $r \geq r_+$, we obtain

$$0 \leq h^2 - 1 \leq \frac{a^2}{r^2 + a^2} < 1.$$

Similarly, we have

$$h - 1 = \frac{h^2 - 1}{h + 1} = \frac{h^4 - 1}{(h + 1)(h^2 + 1)} = \frac{\Delta a^2 \sin^2 \theta}{\sigma^2 (h + 1)(h^2 + 1)} \in \mathbf{S}^{-2,-2},$$

and

$$\partial_\theta h = \frac{\sqrt{r^2 + a^2} \Delta a^2 \sin \theta \cos \theta}{2\sigma^{\frac{5}{2}}} \in \mathbf{S}^{-2,-2}.$$

Now, let us study the term V_0 . We deduce from its explicit expression that V_0 belongs to $\mathbf{S}^{-2,-1}$. Eventually, using that

$$\begin{aligned}\rho - \sqrt{r^2 + a^2} &= -\frac{a^2 \sin^2 \theta}{\rho + \sqrt{r^2 + a^2}} \in \mathbf{S}^{-1,0}, \\ \rho^2 - (r^2 + a^2) &= -a^2 \sin^2 \theta \in \mathbf{S}^{0,0}, \\ \sigma - (r^2 + a^2) &= -\frac{\Delta a^2 \sin^2 \theta}{\sigma + (r^2 + a^2)} \in \mathbf{S}^{-2,-2},\end{aligned}$$

the remaining terms in V_0 belong to $\mathbf{S}^{-2,-1}$. Thus the assumption on V_0 holds.

◇

IV.3.2 Spin-weighted spherical harmonics

The operator \mathbb{D}_{S^2} has compact resolvent and hence can be diagonalized into an infinite sum of matrix-valued multiplication operators. For this, we introduce a generalization of usual spherical harmonics called *spin-weighted harmonics*. We refer to I.M. Gel'Fand and Z.Y. Sapiro [35] for a detailed presentation.

For each spinorial weight m such that $2m \in \mathbb{Z}$, for each $l - |m| \in \mathbb{N}$ and $l - |n| \in \mathbb{N}$, we define the functions W_{mn}^l by

$$W_{mn}^l(\theta, \varphi) = e^{in\varphi} u_{mn}^l(\theta) \in L^2(S^2), \quad (\text{IV.3.9})$$

with

$$\frac{d^2 u_{mn}^l}{d\theta^2} + \cot \theta \frac{du_{mn}^l}{d\theta} + \left[l(l+1) - \frac{n^2 - 2mn \cos \theta + m^2}{\sin^2 \theta} \right] u_{mn}^l = 0, \quad (\text{IV.3.10})$$

and

$$\frac{du_{mn}^l}{d\theta} - \frac{m - n \cos \theta}{\sin \theta} u_{mn}^l = \begin{cases} -i[(l+n)(l+n-1)]^{\frac{1}{2}} u_{m,n-1}^l = 0, & n > -l, \\ 0, & n = l. \end{cases} \quad (\text{IV.3.11})$$

$$\frac{du_{mn}^l}{d\theta} + \frac{m - n \cos \theta}{\sin \theta} u_{mn}^l = \begin{cases} -i[(l+n)(l+n-1)]^{\frac{1}{2}} u_{m,n+1}^l = 0, & n < l, \\ 0, & n = l. \end{cases} \quad (\text{IV.3.12})$$

The functions u_{mn}^l are normalized by

$$\int_0^\pi |u_{mn}^l(\theta)|^2 \sin \theta d\theta = \frac{1}{2\pi},$$

and they satisfy the relations of induction

$$\frac{du_{mn}^l}{d\theta} \mp \frac{n - m \cos \theta}{\sin \theta} u_{mn}^l = -i[(l \pm m)(l \mp m + 1)]^{\frac{1}{2}} u_{m \mp 1, n}^l.$$

We set $\mathcal{I}_m = \{(l, n), l - |m| \in \mathbb{N}, l - |n| \in \mathbb{N}\}$. The family $\{W_{mn}^l(\theta, \varphi)\}_{(l,n) \in \mathcal{I}_m}$ forms a Hilbert basis of $L^2(S^2)$. Thus, any function in $L^2(\mathbb{R}_{r_*} \times S^2)$ can be decomposed into an infinite sum

$$u(r_*, \theta, \varphi) = \sum_{(l,n) \in \mathcal{I}_m} u_{ln}(r_*) W_{mn}^l(\theta, \varphi),$$

where $u_{ln}(r_*) \in L^2(\mathbb{R})$.

Now, we consider the Hilbert basis of $L^2(S^2; \mathbb{C}^4)$ given by $F_n^l = {}^t \left\{ W_{-\frac{1}{2}, n}^l, W_{\frac{1}{2}, n}^l, W_{-\frac{1}{2}, n}^l, W_{\frac{1}{2}, n}^l \right\}$, for any index $(l, n) \in \mathcal{I}_{\frac{1}{2}}$. The elements F_n^l diagonalize the Dirac operator on S^2 . Precisely, we have

$$\mathbb{D}_{S^2} F_n^l = -\left(l + \frac{1}{2}\right) \Gamma^2 F_n^l. \quad (\text{IV.3.13})$$

IV.3.3 Selfadjointness of H_0 and H

The operator H_0 is spherically symmetric and thus can be decomposed onto the spin-weighted spherical harmonics F_n^l . For any $\psi \in \mathcal{H}$, we write

$$H_0 \psi = \sum_{(l,n) \in \mathcal{I}_{\frac{1}{2}}} H_0^{ln} \psi_{ln} F_n^l = \sum_{(l,n) \in \mathcal{I}_{\frac{1}{2}}} (\Gamma^1 D_{r_*} + k a_0(r_*) \Gamma^2 + b_0(r_*) \Gamma^0 + c_0(r_*)) \psi_{ln}(r_*) F_n^l,$$

where we used (IV.3.13) and denoted $k = -(l + \frac{1}{2})$. Hence, we identify the operator H_0^{ln} with the operator $\Gamma^1 D_{r_*} + k a_0(r_*) \Gamma^2 + b_0(r_*) \Gamma^0 + c_0(r_*)$ acting on $\mathcal{H}_{ln} = [L^2(\mathbb{R}_{r_*})]^4$. From [16], we have

Lemma IV.3.3 *For any $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, the Dirac operators H_0^{ln} are selfadjoint on \mathcal{H}_{ln} with domain*

$$D(H_0^{ln}) = [H^1(\mathbb{R}_{r_*})]^4.$$

In consequence, the Dirac operator H_0 is selfadjoint on \mathcal{H} and its domain $D(H_0)$ is given by

$$D(H_0) = \left\{ \psi = \sum_{(l,n) \in \mathcal{I}_{\frac{1}{2}}} \psi_{ln}; \psi_{ln} \in D(H_0^{ln}), \sum_{(l,n) \in \mathcal{I}_{\frac{1}{2}}} \|H_0^{ln} \psi_{ln}\|^2 < \infty \right\}.$$

Corollary IV.3.1 *The Dirac operator H is selfadjoint on \mathcal{H} and its domain is given by $D(H) = D(H_0)$.*

Proof : Let us write H as

$$H_0 + (h^2 - 1)H_0 + \tilde{V}_0, \quad (\text{IV.3.14})$$

where $\tilde{V}_0 = h[H_0, h] + V_0$. Using lemma IV.3.2, we have $\tilde{V}_0 \in \mathbf{S}^{-2}$ and for any $\psi \in \mathcal{H}$,

$$\|(H - H_0)\psi\| \leq a \|H_0 \psi\| + b \|\psi\|,$$

with $a < 1$. This implies the result using the Kato-Rellich Theorem.

◇

IV.3.4 Compactness criteria and resolvent estimates

In this subsection, we collect some standard results useful for the later analysis, especially for Mouré's theory. Let us first begin with compactness criteria. For this, we need a better description of the domains $D(H_0) = D(H)$. By the same argument as in [46], we know that

$$D(H) \subset [H_{loc}^1(\Sigma)]^4. \quad (\text{IV.3.15})$$

Then we have the standard compactness criterion

Lemma IV.3.4 *If $f = f_{ij}, g \in C_\infty(\mathbb{R})$ then $f(r_*)g(H)$ and $f(r_*)g(H_0)$ are compact on \mathcal{H} .*

As a consequence, we have

Corollary IV.3.2 *Let $\chi \in C_0^\infty(\mathbb{R})$. Then the operator $\chi(H) - \chi(H_0)$ is compact.*

Proof : Using the helffer-Sjöstrand formula, we obtain

$$\chi(H) - \chi(H_0) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) (z - H)^{-1} (H - H_0) (z - H_0)^{-1} dz \wedge d\bar{z}.$$

The integral converges in operator norm since $\|(z - H)^{-1} (H - H_0) (z - H_0)^{-1}\| \leq C \frac{\langle z \rangle}{|Imz|^2}$. Hence, it is enough to prove that the operator under the integral is compact. Using (IV.3.14), we have

$$(z - H)^{-1} (H - H_0) (z - H_0)^{-1} = (z - H)^{-1} ((h^2 - 1)H_0 + \tilde{V}_0) (z - H_0)^{-1},$$

where $(h^2 - 1), \tilde{V}_0 \in \mathbf{S}^{-2}$. Hence the result follows from Lemma IV.3.4.

◇

We will also need the following resolvent estimates (see [46]). For any $\psi \in D(H_0)$,

$$\|a_0(r_*)\mathbb{D}_{S^2}\psi\| \leq C(\|H_0\psi\| + \|\psi\|), \quad (\text{IV.3.16})$$

$$\|\Gamma^1 D_{r_*}\psi\| \leq C(\|H_0\psi\| + \|\psi\|). \quad (\text{IV.3.17})$$

The same estimates are true if we replace H_0 by H .

IV.3.5 Domain invariance

The domains $D(\langle r_* \rangle^n)$, $n \in \mathbb{N}$, are stable under the action of the resolvents of H_0, H . Precisely, we have

Proposition IV.3.2 *Let $n \in \mathbb{N}$, $z \in \mathbb{C} \setminus \sigma(H)$ and $\chi \in C_0^\infty(\mathbb{R})$. Then*

$$(H - z)^{-1} D(\langle r_* \rangle^n) \subset D(\langle r_* \rangle^n), \quad (\text{IV.3.18})$$

$$\chi(H) D(\langle r_* \rangle^n) \subset D(\langle r_* \rangle^n). \quad (\text{IV.3.19})$$

Moreover, (IV.3.18) and (IV.3.19) remain true if we replace H by H_0 .

Proof : We prove (IV.3.18) by induction. The result is obvious for $n = 0$. Now let us suppose that $(H - z)^{-1} D(\langle r_* \rangle^{n-1}) \subset D(\langle r_* \rangle^{n-1})$. By a basic criterion, it suffices to prove

$$\sup_{|s| \leq 1} \left\| \frac{e^{isr_*^n} - 1}{s} (H - z)^{-1} u \right\| < \infty, \quad \forall u \in D(\langle r_* \rangle^n).$$

We have

$$\frac{e^{isr_*^n} - 1}{s} (H - z)^{-1} = \frac{e^{isr_*^n} (H - z)^{-1} e^{-isr_*^n} - (H - z)^{-1}}{s} + \frac{e^{isr_*^n} (H - z)^{-1} (1 - e^{-isr_*^n})}{s}.$$

Since $u \in D(\langle r_* \rangle^n)$, we have

$$\sup_{|s| \leq 1} \left\| \frac{e^{isr_*} (H - z)^{-1} (1 - e^{-isr_*})}{s} u \right\| < \infty.$$

Now, denoting $H_s = e^{isr_*} H e^{-isr_*}$, we obtain

$$\frac{e^{isr_*} (H - z)^{-1} e^{-isr_*} - (H - z)^{-1}}{s} = (H_s - z)^{-1} \frac{H - H_s}{s} (H - z)^{-1}.$$

Since $H - H_s = snh^2 \Gamma^1 r_*^{n-1}$, (IV.3.18) follows from the induction hypothesis.

The proof of (IV.3.19) is an easy consequence of the Helffer-Sjöstrand formula and (IV.3.18).

◇

IV.3.6 Absence of eigenvalues for H_0, \mathbb{D}

In this section, we begin our study of the spectral properties of H_0, H, \mathbb{D} . Let us first recall some notations. Given a selfadjoint operator A , we denote $\sigma(A)$ its spectrum. We also denote $\sigma_{disc}(A)$, $\sigma_{ess}(A)$, $\sigma_{pp}(A)$, $\sigma_{ac}(A)$ and $\sigma_{sing}(A)$ the discrete, essential, pure point, absolutely continuous and singular continuous spectra respectively.

Our first result concerns the absence of eigenvalues for H_0 .

Proposition IV.3.3 $\sigma_{pp}(H_0) = \emptyset$.

Proof : Thanks to the spherical symmetry of H_0 , it is sufficient to prove that $H_0 - c_2$ has no eigenvalue on any spin-weighted harmonics. Assume that $u \in L^2(\mathbb{R}; \mathbb{C}^4)$ is a non-zero eigenvector of $H_0^{ln} - c_2$ with eigenvalue $\lambda \in \mathbb{R}$. Clearly, $u \in H^1(\mathbb{R}; \mathbb{C}^4)$. We define $w(r_*) = e^{-i\lambda \Gamma^1 r_*} u(r_*)$. This function also belongs to $H^1(\mathbb{R}; \mathbb{C}^4)$ and thus, $\lim_{r_* \rightarrow -\infty} w(r_*) = 0$. Moreover, $w(r_*)$ satisfies the following equation

$$\frac{dw(r_*)}{dr_*} = e^{-i\lambda \Gamma^1 r_*} (-i\Gamma^1) f(r_*) e^{i\lambda \Gamma^1 r_*} w(r_*),$$

where $f(r_*) = -(l + \frac{1}{2})a_0(r_*)\Gamma^2 + b_0(r_*)\Gamma^0 + (c_0(r_*) - c_2)$. Integrating this expression between $-\infty$ and r_* , we obtain

$$|w(r_*)| \leq \int_{-\infty}^{r_*} |f(s)| |w(s)| ds.$$

Since $\int_{-\infty}^{r_*} |f(s)| ds < \infty$ by Proposition IV.3.1, we conclude by Gronwall's Lemma that $w = 0$.

◇

If we want to apply the same technique in order to prove the absence of eigenvalues for \mathbb{D} , we must use the separability of the equations. This was the technique used in [30] where the class of solutions is specified by means of matching conditions across the horizon. In fact, it is enough to impose the physical requirement that solutions should have finite total charge (see [46]) to obtain this result. Precisely, we show

Proposition IV.3.4 *There are no stationary finite charge Dirac fields outside a Kerr-Newman black hole, i.e. they are no non-zero solutions*

$$\Psi = (\phi_A, \chi_{A'}) \in C(\mathbb{R}_t; L^2((\Sigma; dVol); \mathbb{S}_A \otimes \mathbb{S}_{A'})), \quad (\text{IV.3.20})$$

of (IV.2.3) of the form

$$\Psi(t, r, \theta, \varphi) = e^{-i\omega t} \psi(t, r, \theta, \varphi).$$

This is equivalent to the absence of pure point spectrum for \mathbb{D} , i.e.

$$\sigma_{pp}(\mathbb{D}) = \emptyset.$$

Proof : We use the separable form of Dirac's equation obtained in [30]. First, we choose the symmetric frame $(l^a, n^a, m^a, \bar{m}^a)$

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2\rho^2\Delta}} \left((r^2 + a^2) \frac{\partial}{\partial t} + \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right), & n^a &= \frac{1}{\sqrt{2\rho^2\Delta}} \left((r^2 + a^2) \frac{\partial}{\partial t} - \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right), \\ m^a &= \frac{1}{\sqrt{2\rho^2}} \left(ia \sin \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right), & \bar{m}^a &= \frac{1}{\sqrt{2\rho^2}} \left(-ia \sin \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right), \end{aligned}$$

to compute the equation. After the regular and time-dependent transformation $\Psi \rightarrow \hat{\Psi} = S\Psi$ where

$$S = \Delta^{\frac{1}{4}} \text{diag} \left((r - ia \cos \theta)^{\frac{1}{2}}, (r - ia \cos \theta)^{\frac{1}{2}}, (r + ia \cos \theta)^{\frac{1}{2}}, (r + ia \cos \theta)^{\frac{1}{2}} \right), \quad (\text{IV.3.21})$$

and using the ansatz

$$\hat{\Psi}(t, r, \theta, \varphi) = e^{-i\omega t} e^{-i(k+\frac{1}{2})\varphi} \begin{pmatrix} X_-(r)Y_-(\theta) \\ X_+(r)Y_+(\theta) \\ X_+(r)Y_-(\theta) \\ X_-(r)Y_+(\theta) \end{pmatrix}, \quad (\text{IV.3.22})$$

the Dirac equation (IV.2.3) can be written as the decoupled system of ODEs :

$$\begin{pmatrix} \sqrt{\Delta} \mathcal{D}_+ & imr - \lambda \\ -imr - \lambda & \sqrt{\Delta} \mathcal{D}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} = 0, \quad (\text{IV.3.23})$$

$$\begin{pmatrix} \mathcal{L}_+ & -am \cos \theta + \lambda \\ am \cos \theta + \lambda & -\mathcal{L}_- \end{pmatrix} \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix} = 0,$$

with

$$\mathcal{D}_{\pm} = \frac{\partial}{\partial r} \pm \frac{i}{\Delta} \left(\omega(r^2 + a^2) + (k + \frac{1}{2})a + qQr \right), \quad \mathcal{L}_{\pm} = \frac{\partial}{\partial \theta} + \frac{\cot \theta}{2} \mp \left(a\omega \sin \theta + \frac{k + \frac{1}{2}}{\sin \theta} \right).$$

We only need the ODEs concerning the radial unknowns $X_{\pm}(r)$ in the following. Using the radial variable r_* , we reexpress (IV.3.23) as

$$\begin{aligned} \left(\frac{\partial}{\partial r_*} + \frac{iK}{r^2 + a^2} \right) X_+ &= (\lambda - imr) \frac{\sqrt{\Delta}}{r^2 + a^2} X_-, \\ \left(\frac{\partial}{\partial r_*} - \frac{iK}{r^2 + a^2} \right) X_- &= (\lambda + imr) \frac{\sqrt{\Delta}}{r^2 + a^2} X_+, \end{aligned}$$

where $K = \omega(r^2 + a^2) + (k + \frac{1}{2})a + qQr$ is a real number. We can multiply X_{\pm} by phase factors in order to get rid of the terms involving iK . We define

$$U_+ = \exp \left(\int_0^{r_*} \frac{iK}{(r(s)^2 + a^2)} ds \right) X_+ =: \beta X_+,$$

$$U_- = \exp\left(-\int_0^{r_*} \frac{iK}{(r(s)^2 + a^2)} ds\right) X_- =: \bar{\beta} X_-,$$

where $s \rightarrow r(s)$ is the reciprocal function of $r \rightarrow r_*$. Now the pair (U_+, U_-) satisfy the differential system

$$\frac{\partial}{\partial r_*} \begin{pmatrix} U_+ \\ U_- \end{pmatrix} = \begin{pmatrix} 0 & (\lambda - imr) \frac{\sqrt{\Delta}}{r^2 + a^2} \beta^2 \\ (\lambda + imr) \frac{\sqrt{\Delta}}{r^2 + a^2} \bar{\beta}^2 & 0 \end{pmatrix} \begin{pmatrix} U_+ \\ U_- \end{pmatrix}. \quad (\text{IV.3.24})$$

Note that the condition (IV.3.20) is equivalent (after the transformations (IV.3.21 and IV.3.22)) to

$$X_{\pm} \in L^2(\mathbb{R}_{r_*}; dr_*), \text{ or, } U_{\pm} \in L^2(\mathbb{R}_{r_*}; dr_*).$$

Since the factors $(\lambda - imr) \frac{\sqrt{\Delta}}{r^2 + a^2} \beta^2$ and $(\lambda + imr) \frac{\sqrt{\Delta}}{r^2 + a^2} \bar{\beta}^2$ are bounded on \mathbb{R} , if the pair (U_+, U_-) belongs to $L^2(\mathbb{R}_{r_*}; dr_*)$ and satisfies (IV.3.24) then $(U_+, U_-) \in H^1(\mathbb{R}_{r_*}; dr_*)$ and in consequence, $\lim_{r_* \rightarrow \infty} U_{\pm}(r_*) = 0$. Moreover, $(\lambda - imr) \frac{\sqrt{\Delta}}{r^2 + a^2} \beta^2$ and $(\lambda + imr) \frac{\sqrt{\Delta}}{r^2 + a^2} \bar{\beta}^2$ fall off exponentially fast as $r_* \rightarrow -\infty$ and therefore, are integrable at $-\infty$. Hence, we can conclude the proof by the same argument as in Proposition IV.3.3.

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IV.4 Mourre theory and minimal velocity estimate

IV.4.1 Abstract theory

We go on now with our inspection of the spectral properties of H_0, H . Mourre's theory will be our basic tool throughout this section. The principle of Mourre theory is to find a selfadjoint operator A on \mathcal{H} so that the pair (H, A) satisfies the following assumptions (see [58]). Let $I \subset \mathbb{R}$ an open interval.

(M1) $e^{-itA} D(H) \subset D(H)$.

(M2) $i[H, A]$ defined as a quadratic form on $D(H) \cap D(A)$ extends to an element of $\mathcal{B}(D(H), \mathcal{H})$.

(M3) $[A, [A, H]]$ well defined as a quadratic form on $D(H) \cap D(A)$ by **(ii)**, extends to an element of $\mathcal{B}(D(H), D(H)^*)$.

(M4) There exists a strictly positive constant μ and a compact operator K such that

$$\mathbf{1}_I(H) i[H, A] \mathbf{1}_I(H) \geq \mu \mathbf{1}_I(H) + K. \quad (\text{IV.4.1})$$

The fundamental assumption here is the Mourre estimate (M4). Its meaning is that we must find an observable A which essentially increases along the evolution e^{-itH} . The other conditions are more technical and turn out to be difficult to check directly in the case where A and H are unbounded selfadjoint operators having no explicitly known domains. We give below some useful criteria to verify them. If the pair (H, A) satisfy these assumptions then we say that A is a *conjugate operator* for H on I . The existence of a conjugate operator provides important informations on the spectrum of H . Precisely, we have (see for example [1])

Theorem IV.4.1 *Let H, A two selfadjoint operators on \mathcal{H} . Assume that A is a conjugate operator for H on the interval I . Then H has no singular continuous spectrum in I . Furthermore, the number of eigenvalues of H in I is finite (counting multiplicity).*

We will discuss in section IV.4.3 another important consequence of the existence of the conjugate operator : the minimal velocity estimate.

How can we choose a conjugate operator ? For Schrödinger or Dirac operators in flat spacetime, the usual generator of dilations is a good choice. Let us mention however that in the particular case of Dirac operators, there exist other choices (see [11], [14], [15], [16], [37]) which could be used. For the spherically symmetric operator H_0 , the situation is similar to that of Dirac operators in flat spacetime thanks to the possibility of decomposing the problem into spin-weighted spherical harmonics. The existence of conjugate operators for H_0 has been established first in [55] and more recently, in [16], (the conjugate operator used in [16] is particularly adapted to Dirac operators). On the other hand, although we can consider H as a short-range perturbation of H_0 , this perturbation is not spherically symmetric and thus, we cannot use a decomposition into spin-weighted spherical harmonics. The full operator \mathbb{D}_{S^2} must be conserved in the course of the calculations.

Recall that the evolution described by H can be understood as an evolution on a Riemannian manifold (given here by $\Sigma = \mathbb{R}_{r_*} \times S^2$) having two different ends. At infinity, the metric on Σ tends to the flat metric. Therefore, we can still use the generator of dilations there. At the horizon, this metric is exponentially large and the choice of a conjugate operator turns out to be much more complicated. Analogous situations have been studied before, first by Froese and Hislop [33] in the case of a second-order-elliptic Hamiltonian, then by De Bièvre, Hislop and Sigal [19] for the wave equation and more recently, by Häfner and Nicolas [46] for a massless Dirac equation in a Kerr background. We follow here the presentation given in [46]. Let us study a toy model of the situation at the black hole horizon and consider the operator

$$H = \Gamma^1 D_{r_*} + e^{\kappa+r_*} \mathbb{D}_{S^2}, \text{ acting on } \mathbb{R}_- \times S^2.$$

Let us try to use the operator $A = \Gamma^1 r_*$ introduced in [15] as a conjugate operator for H . This very simple operator is well adapted to the case of massless Dirac operator. We have

$$i[H, A] = 1 + 2r_* e^{\kappa+r_*} \mathbb{D}_{S^2} \Gamma^1.$$

The second term has no sign and is not controlled by H . The origin of the problem comes from the fact that the term $r_* e^{\kappa+r_*}$ does not decay faster than $e^{\kappa+r_*}$ when $r_* \rightarrow -\infty$. The same problem happens if we use the generator of dilations instead of A . Therefore, the Mourre estimate has no chance to hold if we proceed in this manner. Instead, we introduce the unitary transformation

$$U = e^{-i\kappa_+^{-1} D_{r_*} \ln |\mathbb{D}_{S^2}|},$$

and we try to find a conjugate operator for

$$\hat{H} = U^* H U = \Gamma^1 D_{r_*} + e^{\kappa+r_*} \frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|}.$$

This is equivalent to the initial problem since U is unitary. Now, if we compute the commutator between \hat{H} and A , we obtain

$$i[\hat{H}, A] = 1 + 2r_* e^{\kappa+r_*} \frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|} \Gamma^1.$$

Since $\frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|}$ is bounded, the term $\chi(\hat{H}) r_* e^{\kappa+r_*} \frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|} \Gamma^1 \chi(\hat{H})$ is now compact for any $\chi \in C_0^\infty(\mathbb{R})$. Therefore, we can use $U A U^*$ as conjugate operator for the toy Hamiltonian H . The conjugate operator for the true Hamiltonian H we propose below, closely follows this procedure.

Before we do this, let us give some precisions concerning the conditions (M1), (M2) and (M3) of Mourre's theory. One of the difficulties in Mourre theory consists in working with commutators $i[H, A]$ (see (M2)) between unbounded selfadjoint operators. We have to be careful to define correctly such quantities since $D(H)$ and $D(A)$ can be unknown or have an intersection which is not even dense in \mathcal{H} . Similarly, the assumption (M1) is not easy to prove since the action of e^{isA} may also be unknown. Therefore, it is useful to have different criterion. Let us first define the class of operator $C^k(A)$:

Definition IV.4.1 *For a selfadjoint operator A , we say that another selfadjoint operator H belongs to $C^k(A)$, $k \in \mathbb{N}$, if and only if*

$$(ABG) \quad \exists z \in \mathbb{C} \setminus \sigma(H), s \longrightarrow e^{isA}(H - z)^{-1}e^{-isA} \in C^k(\mathbb{R}_s; \mathcal{B}(\mathcal{H})),$$

for the strong topology of $\mathcal{B}(\mathcal{H})$.

It has been shown in [1] that we can replace the assumptions (M1) and (M2) by the unique assumption $H \in C^1(A)$ without changing the conclusions of Theorem IV.4.1. Roughly speaking, this condition allows that the following equality

$$[A, (z - H)^{-1}] = (z - H)^{-1}[A, H](z - H)^{-1},$$

makes sense on \mathcal{H} . From [1], $H \in C^1(A)$ is equivalent to

$$\begin{aligned} (i)' \quad & \exists z \in \mathbb{C} \setminus \sigma(H), (H - z)^{-1}D(A) \subset D(A), (H - \bar{z})^{-1}D(A) \subset D(A), \\ (ii)' \quad & |(Hu, Au) - (Au, Hu)| \leq C(\|Hu\|^2 + \|u\|^2), \quad \forall u \in D(H) \cap D(A), \end{aligned}$$

which is close to both conditions (M1) and (M2). Nevertheless, they remain complicated to check when the domains of H and A are not explicite known. One way to remedy this problem consists in finding first a common core for H and A . This procedure is described in [38]. We only recall the two results we shall need.

Lemma IV.4.1 (Nelson) *Let $N \geq 1$ a self-adjoint operator on \mathcal{H} . Let A a symmetric operator on \mathcal{H} such that $D(N) \subset D(A)$. Assume that*

$$\begin{aligned} \|Au\| &\leq C\|Nu\|, \quad \forall u \in D(N), \\ |(Au, Nu) - (Nu, Au)| &\leq C\|N^{\frac{1}{2}}u\|^2, \quad \forall u \in D(N). \end{aligned} \tag{IV.4.2}$$

Then A is essentially self-adjoint on $D(N)$. Furthermore every core of N is also a core for A .

Lemma IV.4.2 (Gérard, Laba) *Let H, H_0 and N three self-adjoint operators on \mathcal{H} satisfying $N \geq 1$, $D(H) = D(H_0)$ and $(H - z)^{-1}D(N) \subset D(N)$. Let A a symmetric operator on $D(N)$. Assume that H_0 and A satisfy the assumptions of Lemma IV.4.1 and*

$$|(Au, Hu) - (Hu, Au)| \leq C(\|Hu\|^2 + \|u\|^2), \quad \forall u \in D(N). \tag{IV.4.3}$$

Then we have

- $D(N)$ is dense in $D(A) \cap D(H)$ with the norm $\|Hu\| + \|Au\| + \|u\|$,
- the quadratic form $i[H, A]$ defined on $D(A) \cap D(H)$ is the unique extension of $i[H, A]$ on $D(N)$,
- $H \in C^1(A)$.

The operator N above is called a *comparison operator*. When the assumptions of Lemma IV.4.2 are satisfied, it is enough to compute the commutator $i[H, A]$ as a quadratic form on the common core $D(N)$ or on any core of N . We finish this subsection by two observations. First, the condition $H \in C^1(A)$ together with the condition (M2) imply the condition (M1) thanks to a result due to Gérard and Georgescu [36]

Lemma IV.4.3 (Georgescu, Gérard) *Let H and A two self-adjoint operators such that $H \in C^1(A)$ and $i[H, A] \in \mathcal{B}(D(H), \mathcal{H})$ then $e^{isA}D(H) \subset D(H)$ for all $s \in \mathbb{R}$*

Second, in order to prove the minimal velocity estimate below, we will need to prove that $H \in C^2(A)$. But this follows automatically from conditions (M1) to (M3) thanks to Theorem 6.3.4 in [1].

IV.4.2 Conjugate operators for H_0 and H

Preliminaries

In order to separate the problems at the horizon and infinity, we define two cut-off functions $j_{\pm} \in C^\infty(\mathbb{R})$ satisfying

$$\begin{aligned} j_-(r_*) &= 1, \text{ for } r_* \leq \frac{1}{2}, & j_-(r_*) &= 0, \text{ for } r_* \geq 1, \\ j_+(r_*) &= 1, \text{ for } r_* \geq 2, & j_+(r_*) &= 0, \text{ for } r_* \leq \frac{3}{2}. \end{aligned}$$

In this definition, the support of j_- must contain $(-\infty, 0]$ whereas the support of j_+ only has to contain a neighbourhood of $+\infty$. Now, for $S \geq 1$, we set

$$R_-(r_*, \mathbb{D}_{S^2}) = (r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|) j_-^2 \left(\frac{r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|}{S} \right), \quad R_+(r_*) = r_* j_+^2 \left(\frac{r_*}{S} \right).$$

We define the local conjugate operators A_-, A_+ by

$$A_- = R_-(r_*, \mathbb{D}_{S^2}) \Gamma^1, \quad A_+ = \frac{1}{2} (D_{r_*} R_+(r_*) + R_+(r_*) D_{r_*}).$$

At infinity, the conjugate operator A_+ is sort of generator of dilations whereas at the horizon, A_- is a combination of the conjugate operators used in [14], [15] and [46]. The true conjugate operator A will be the sum or the difference between A_- and A_+ depending on the energy interval we consider. Precisely, let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi$ is included in one of the intervals $(-\infty, -m)$, $(-m, +m)$, $(+m, +\infty)$. Then we define the conjugate operator A_χ on $\text{supp } \chi$ by

$$\begin{aligned} A_\chi &= A_- + A_+ \text{ when } \text{supp } \chi \subset (+m, +\infty), \\ A_\chi &= A_- - A_+ \text{ when } \text{supp } \chi \subset (-\infty, -m), \\ A_\chi &= A_- \text{ when } \text{supp } \chi \subset (-m, +m). \end{aligned}$$

We will prove the assumptions (M1) to (M3) for $A = A_- + A_+$. The other cases are analogous.

Let us first study the operator $R_-(r_*, \mathbb{D}_{S^2})$. On each spin-weighted spherical harmonics, it reduces to the operator of multiplication $R_-(r_*, k)$ on \mathcal{H}_{ln} , where $k = -(l + \frac{1}{2})$ denotes the eigenvalue associated to the given harmonics. Hence, $-k \in \mathbb{N}^*$. We have

Lemma IV.4.4 For all $S \geq 1$,

$$\begin{aligned} |R_-(r_*, k)| &\leq C\langle r_* \rangle, \text{ uniformly in } k, \\ |R_-^{(i)}(r_*, k)| &\leq C, \text{ uniformly in } k, \quad i = 1, 2. \end{aligned}$$

As a consequence, the domains of the operators $R_-(r_*, \mathbb{D}_{S^2})$ and A_- contain $D(\langle r_* \rangle)$. Moreover, if $j_1 \in C^\infty(\mathbb{R})$ satisfies $j_1(r_*) = 1$, for $r_* \leq 1$ and $j_1(r_*) = 0$, for $r_* \geq \frac{3}{2}$, then

$$R_-^{(i)}(r_*, \mathbb{D}_{S^2}) = R_-^{(i)}(r_*, \mathbb{D}_{S^2}) j_1^2\left(\frac{r_*}{S}\right), \quad i = 0, 1, 2.$$

Proof : The operator $R_-(r_*, k)$ is a translation by $\kappa_+^{-1} \ln |k|$ of the operator of multiplication $r_* j_-^2\left(\frac{r_*}{S}\right)$. Since $-k \in \mathbb{N}^*$, we have

$$r_* \leq r_* + \kappa_+^{-1} \ln |k|, \quad \forall k. \quad (\text{IV.4.4})$$

Therefore, on $\text{supp } j_-$ we have

$$r_* \leq r_* + \kappa_+^{-1} \ln |k| \leq S,$$

and thus

$$|r_* + \kappa_+^{-1} \ln |k|| \leq \max(S, |r_*|) \leq C\langle r_* \rangle, \quad \forall k.$$

This proves the first assertion.

Now we have

$$R_-^{(1)}(r_*, k) = j_-^2\left(\frac{r_* + \kappa_+^{-1} \ln |k|}{S}\right) + \left(\frac{r_* + \kappa_+^{-1} \ln |k|}{S}\right) (j_-^2)' \left(\frac{r_* + \kappa_+^{-1} \ln |k|}{S}\right).$$

Since $\text{supp } (j_-^2)'$ is compact, the second assertion holds.

Finally, it is immediate from (IV.4.4) that

$$(j_-^2)^{(i)}\left(\frac{r_* + \kappa_+^{-1} \ln |k|}{S}\right) j_1^2\left(\frac{r_*}{S}\right) = (j_-^2)^{(i)}\left(\frac{r_* + \kappa_+^{-1} \ln |k|}{S}\right), \quad \forall k, \quad i = 0, 1, 2.$$

This concludes the proof of the lemma.

◇

We shall now prove the Mourre assumptions (M1) to (M3) for (H, A) . We first define the comparison operator

$$N = D_{r_*}^2 + a_0^2(r_*) \mathbb{D}_{S^2}^2 + \langle r_* \rangle^2.$$

We recall from [46] that $D(N) = D(H^2) \cap D(\langle r_* \rangle^2) = D(H_0^2) \cap D(\langle r_* \rangle^2)$. From this and Proposition IV.3.2, we have for any $z \in \mathbb{C} \setminus \sigma(H)$,

$$(H - z)^{-1} D(N) \subset D(N).$$

We also recall some useful estimates (immediate from the definition of N). For $u \in D(N)$,

$$\|D_{r_*} u\| \leq C \|N^{\frac{1}{2}} u\|, \quad \|a_0(r_*) \mathbb{D}_{S^2} u\| \leq C \|N^{\frac{1}{2}} u\|, \quad \|r_* u\| \leq C \|N^{\frac{1}{2}} u\|. \quad (\text{IV.4.5})$$

Lemma IV.4.5 (A_\pm, N) satisfy the hypotheses of Nelson's Lemma IV.4.1 and thus A_\pm are essentially selfadjoint on $D(N)$.

Proof : By Lemma IV.4.4, $D(A_-) \supset D(\langle r_* \rangle) \supset D(N)$ and for any $u \in D(N)$, we have

$$\|A_- u\| \leq C \|\langle r_* \rangle u\| \leq C \|Nu\|.$$

Hence, it remains to show that $|(u, i[A_\pm, N]u)| \leq C \|N^{\frac{1}{2}}u\|^2$. But, using Lemma IV.4.4 and (IV.4.5), we have

$$|(u, [A_-, N]u)| \leq 2|(D_{r_*} u, R_-^{(1)}(r_*, \mathbb{D}_{S^2})\Gamma^1 u)| \leq C \|u\| \|D_{r_*} u\| \leq C \|N^{\frac{1}{2}}u\|^2.$$

The proof for A_+ is identical to that given in [46]. We omit it.

◇

Lemma IV.4.6 *The pair (H_0, N) satisfies the hypotheses of Nelson's Lemma IV.4.1.*

Proof : Using (IV.4.5), we have for any $u \in D(N)$

$$\|H_0 u\| \leq \|D_{r_*} u\| + \|a_0(r_*)\mathbb{D}_{S^2} u\| + C \|u\| \leq C \|Nu\|.$$

Moreover,

$$\begin{aligned} |(u, [H_0, N]u)| &\leq 2 \left(|(u, \Gamma^1 a'_0(r_*) a_0(r_*) \mathbb{D}_{S^2}^2 u)| + |(u, \Gamma^1 r_* u)| + |(D_{r_*} u, a'_0(r_*) \mathbb{D}_{S^2} u)| \right. \\ &\quad \left. + |(D_{r_*} u, b'_0(r_*) u)| + |(D_{r_*} u, c'_0(r_*) u)| \right). \end{aligned}$$

Now remark that there exists a constant C such that $|a'_0(r_*)| \leq C |a_0(r_*)|$. Hence we have

$$\begin{aligned} |(u, [H_0, N]u)| &\leq C \left(\|a_0(r_*)\mathbb{D}_{S^2} u\|^2 + \|u\| \|r_* u\| + \|D_{r_*} u\| (\|a_0(r_*)\mathbb{D}_{S^2} u\| + \|u\|) \right), \\ &\leq C \|N^{\frac{1}{2}}u\|^2. \end{aligned}$$

◇

Lemma IV.4.7 $H \in C^1(A)$. *Moreover, the commutator $i[H, A]$ belongs to $\mathcal{B}(D(H), \mathcal{H})$. Consequently, assumptions (M1) and (M2) of Mourre's theory are satisfied.*

Proof : Thanks to Lemmata IV.4.5 and IV.4.6, it suffices to show that $|(u, i[H, A_\pm]u)| \leq C(\|Hu\|^2 + \|u\|^2)$ in order to apply Lemma IV.4.2. We only prove it for A_- since the proof for A_+ is identical to that given in [46]. We first calculate $i[H_0, A_-]$. We have

$$i[H_0, A_-] = R_-^{(1)}(r_*, \mathbb{D}_{S^2}) + 2ia_0(r_*)R_-(r_*, \mathbb{D}_{S^2})\mathbb{D}_{S^2}\Gamma^1 + 2ib_0(r_*)R_-(r_*, \mathbb{D}_{S^2})\Gamma^0\Gamma^1.$$

Let $u \in D(N)$. By Lemma IV.4.4, we estimate the first term by

$$\|R_-^{(1)}(r_*, \mathbb{D}_{S^2})u\| \leq C \|u\|.$$

We can estimate the second term by

$$\begin{aligned} \|a_0(r_*)R_-(r_*, \mathbb{D}_{S^2})\mathbb{D}_{S^2}\Gamma^1 u\| &\leq \\ &\left\| \frac{a_0(r_*) - c_1 e^{\kappa+r_*}}{a_0(r_*)} j_1^2\left(\frac{r_*}{S}\right) R_-(r_*, \mathbb{D}_{S^2}) \right\| \|a_0(r_*)\mathbb{D}_{S^2} (H_0 + i)^{-1}\| \|(H_0 + i)u\| \\ &+ \|c_1 R_-(r_*, \mathbb{D}_{S^2}) e^{\kappa+(r_*+\kappa_+^{-1} \ln |\mathbb{D}_{S^2}|)} \frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|} \Gamma^1 u\|. \end{aligned}$$

Observe that $\|\frac{a_0(r_*)-c_1 e^{\kappa+r_*}}{a_0(r_*)} j_1^2(\frac{r_*}{S}) R_-(r_*, \mathbb{D}_{S^2})\|$ is bounded thanks to Proposition IV.3.1 and Lemma IV.4.4. Furthermore, $\|a_0(r_*) \mathbb{D}_{S^2} (H_0 + i)^{-1}\|$ and $\|R_-(r_*, \mathbb{D}_{S^2}) e^{\kappa+(r_*+\kappa_+^{-1} \ln |\mathbb{D}_{S^2}|)}\|$ are bounded by (IV.3.16) and by definition of $R_-(r_*, \mathbb{D}_{S^2})$. Hence, we get

$$\|a_0(r_*) R_-(r_*, \mathbb{D}_{S^2}) \mathbb{D}_{S^2} \Gamma^1 u\| \leq C(\|H_0 u\| + \|u\|).$$

Finally, the last term is estimated as follows

$$\|b_0(r_*) R_-(r_*, \mathbb{D}_{S^2}) \Gamma^0 \Gamma^1 u\| \leq C \|b_0(r_*) j_1^2(\frac{r_*}{S}) R_-(r_*, \mathbb{D}_{S^2})\| \|u\| \leq C \|u\|,$$

by Proposition IV.3.1 again.

We now estimate $i[H, A_-]$. We have

$$i[H, A_-] = h i[H_0, A_-] h + i[V_0, A_-].$$

Since h is bounded as an operator from $D(H)$ into itself, the first term belongs to $\mathcal{B}(D(H), \mathcal{H})$ by the previous estimate. Moreover, since $V_0 \in \mathbf{S}^{-2}(\mathbb{R})$, the remaining term $i[V_0, A_-]$ is clearly bounded by Lemma IV.4.4. This concludes the proof of the Lemma.

◇

Lemma IV.4.8 *The double commutator $[i[H, A], A]$ extends to a bounded operator in $\mathcal{B}(D(H), \mathcal{H})$.*

Proof : We first estimate $[i[H, A_-], A_-]$. We have

$$[i[H, A_-], A_-] = h [i[H_0, A_-], A_-] h + [i[V_0, A_-], A_-].$$

The second term is clearly bounded since $V_0 \in \mathbf{S}^{-2}$ and A_- is bounded from $D((r_*))$ to \mathcal{H} . Moreover, the first term

$$[i[H_0, A_-], A_-] = 4i a_0(r_*) R_-^2(r_*, \mathbb{D}_{S^2}) \mathbb{D}_{S^2} + 4i b_0(r_*) R_-^2(r_*, \mathbb{D}_{S^2}) \Gamma^0,$$

is bounded from $D(H)$ to \mathcal{H} by the same argument as in the proof of Lemma IV.4.7.

Now we estimate $[i[H, A_-], A_+]$. We have

$$[i[H, A_-], A_+] = \left(i[h, A_+][H_0, A_-] h + h c \right) + h [i[H_0, A_-], A_+] h + [i[V_0, A_-], A_+]. \quad (\text{IV.4.6})$$

Since $i[h, A_+] = -R_+(r_*)(\partial_{r_*} h) \in \mathbf{S}^{-1}$ by Lemma IV.3.2, the first term in (IV.4.6) is bounded from $D(H)$ to \mathcal{H} by Lemma IV.4.7. From the exact expression of $i[H_0, A_-]$, the second term in (IV.4.6) is written as

$$\begin{aligned} [i[H_0, A_-], A_+] &= i R_+(r_*) R_-^{(2)}(r_*, \mathbb{D}_{S^2}) - 2(a_0(r_*) R_-(r_*, \mathbb{D}_{S^2}))^{(1)} R_+(r_*) \mathbb{D}_{S^2} \Gamma^1 \\ &\quad - 2(b_0(r_*) R_-(r_*, \mathbb{D}_{S^2}))^{(1)} R_+(r_*) \Gamma^0. \end{aligned} \quad (\text{IV.4.7})$$

Recall that $R_-^{(i)}(r_*, \mathbb{D}_{S^2}) = R_-^{(i)}(r_*, \mathbb{D}_{S^2}) j_1^2(\frac{r_*}{S})$, $\forall i = 0, 1, 2$. We can assume that the function j_+ has been chosen such that $\text{supp } j_1 \cap \text{supp } j_+ = \emptyset$. Hence, (IV.4.7) vanishes. Eventually, since $[V_0, A_-] \in \mathbf{S}^{-1}$, the last term in (IV.4.6) is also bounded from $D(H)$ to \mathcal{H} .

Similarly, $[i[H, A_+], A_-]$ is bounded from $D(H)$ to \mathcal{H} . The same result holds for $[i[H, A_+], A_+]$ (see [46]).

◇

The Mourre estimate for (H, A)

The strategy in this section is the following. We first establish Mourre estimates between H_0 and A_- (resp. H_0 and A_+). The main difficulties arise with the part of the proof concerned with A_- . We shall use here several technical results from [46]. Since they are instructive, short proofs will be sketched in the course of the calculations. The rigorous results can be found in section 5.5 of [46]. Finally, we will show that the remaining term involving $H - H_0$ is compact on \mathcal{H} .

Mourre estimate for (H_0, A_-) : When working at the horizon of the black hole, it is convenient to introduce the operator

$$H_e = \Gamma^1 D_{r_*} + c_1 e^{\kappa+r_*} \mathbb{D}_{S^2} + c_2,$$

which corresponds to the formal limit of the operator H_0 when $r_* \rightarrow -\infty$. Let us recall some basic properties of H_e (see sections 3.2 and 5.5 in [46]). On each spin-weighted spherical harmonics, we denote this operator H_e^{ln} and $D(H_e^{ln}) = \{u \in \mathcal{H}_{ln}, H_e^{ln}u \in \mathcal{H}_{ln}\}$ its natural domain. Then we have

$$\text{Resolvent estimates : } \forall u \in D(H_e^{ln}), \max(\|\Gamma^1 D_{r_*} u\|, \|e^{\kappa+r_*} \Gamma^2 u\|) \leq C(\|H_e^{ln} u\| + \|u\|) \quad \text{(IV.4.8)}$$

$$\text{Selfadjointness : } (H_e^{ln}, D(H_e^{ln})) \text{ is selfadjoint on } \mathcal{H}_{ln}, \quad \text{(IV.4.9)}$$

$$\text{Domain : } D(H_e^{ln}) \subset [H^1(\mathbb{R})]^4, \quad \text{(IV.4.10)}$$

$$\text{Compactness : } \forall f, g \in C_\infty(\mathbb{R}), f(r_*)g(H_e^{ln}) \text{ is compact}, \quad \text{(IV.4.11)}$$

$$\text{Spectrum : } H_e^{ln} \text{ has no eigenvalue i.e. } \sigma_{pp}(H_e^{ln}) = \emptyset. \quad \text{(IV.4.12)}$$

We also denote $D(H_e) = \{u = \sum u_{ln}, u_{ln} \in D(H_e^{ln}), \sum(\|H_e^{ln} u\|^2 + \|u\|^2) < \infty\}$ the domain of H_e . We have

$$\text{Resolvent estimates : } \forall u \in D(H_e), \max(\|\Gamma^1 D_{r_*} u\|, \|e^{\kappa+r_*} \mathbb{D}_{S^2} u\|) \leq C(\|H_e u\| + \|u\|) \quad \text{(IV.4.13)}$$

$$\text{Selfadjointness : } (H_e, D(H_e)) \text{ is selfadjoint on } \mathcal{H}. \quad \text{(IV.4.14)}$$

Let us state three results which will be useful later.

Lemma IV.4.9 *Let $\chi \in C_0^\infty(\mathbb{R})$ and $j \in C^\infty(\mathbb{R})$ a cut-off function at the horizon, i.e. $j = 1$ on a neighbourhood of $-\infty$ and $j = 0$ on a neighbourhood of $+\infty$. Then $j(r_*)(\chi(H_0) - \chi(H_e))$ is compact on \mathcal{H} .*

Proof : We use the Helffer-Sjöstrand formula. We obtain

$$j(r_*)(\chi(H_0) - \chi(H_e)) = \frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\chi}(z) j(r_*)(z - H_0)^{-1} (H_0 - H_e)(z - H_e)^{-1} dz \wedge d\bar{z}.$$

The integral converges in operator norm since $\|j(r_*)(z - H_0)^{-1} (H_0 - H_e)(z - H_e)^{-1}\| \leq C \frac{\langle z \rangle}{|Im z|^2}$. Hence, it is enough to prove that the operator under the integral is compact. But

$$\begin{aligned} j(r_*)(z - H_0)^{-1} (H_0 - H_e)(z - H_e)^{-1} &= (z - H_0)^{-1} j(r_*)(H_0 - H_e)(z - H_e)^{-1} \\ &\quad + i(z - H_0)^{-1} j'(r_*) \Gamma^1 (z - H_0)^{-1} (H_0 - H_e)(z - H_e)^{-1}, \end{aligned}$$

Using that $j(r_*)(H_0 - H_e) = j(r_*)((a_0(r_*) - c_1 e^{\kappa+r_*}) \mathbb{D}_{S^2} + b_0(r_*) \Gamma^0 + (c_0(r_*) - c_2))$, both terms are compact by Proposition IV.3.1 and the standard compactness criterion IV.3.4.

◇

Lemma IV.4.10 *Let $f, \chi \in C_\infty(\mathbb{R})$. Then*

$$f(r_* + \kappa_+^{-1} \ln |k|) \chi(H_e^{l_n}) \text{ is compact on } \mathcal{H}_{l_n}. \quad (\text{IV.4.15})$$

Moreover, for any $\lambda_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(r_* + \kappa_+^{-1} \ln |k|) \mathbf{1}_{[\lambda_0 - \delta, \lambda_0 + \delta]}(H_e^{l_n})\| < \epsilon, \text{ uniformly in } k. \quad (\text{IV.4.16})$$

Proof : For any k , the function $g(r_*) := f(r_* + \kappa_+^{-1} \ln |k|)$ belongs to $C_\infty(\mathbb{R})$. Hence the compactness of $f(r_* + \kappa_+^{-1} \ln |k|) \chi(H_e^{l_n})$ follows from (IV.4.11).

In order to prove (IV.4.16), let us introduce the unitary operator $U^{l_n} = e^{-i\kappa_+^{-1} \ln |k| D_{r_*}}$. Conjugating the operator in (IV.4.16) by U^{l_n} , we have to show that for δ small enough

$$\|f(r_*) \mathbf{1}_{[\lambda_0 - \delta, \lambda_0 + \delta]}(\Gamma^1 D_{r_*} - c_1 e^{\kappa_+ r_*} \Gamma^2 + c_2)\| < \epsilon, \text{ uniformly in } k. \quad (\text{IV.4.17})$$

Clearly, the operator in (IV.4.17) is independent of k and is compact by (IV.4.15). Hence the result follows from (IV.4.12).

◇

This lemma immediately yields the ‘‘smallness-result’’

Corollary IV.4.1 *Let $f \in C_\infty(\mathbb{R})$ and $\lambda_0 \in \mathbb{R}$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|) \mathbf{1}_{[\lambda_0 - \delta, \lambda_0 + \delta]}(H_e)\| < \epsilon.$$

Having recalled these technical results, we turn now to the Mourre estimate for (H_0, A_-) .

Lemma IV.4.11 *Let $\lambda_0 \in \mathbb{R}$. Then there exist a function $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi$ containing λ_0 , a strictly positive constant μ and a compact operator K such that*

$$\chi(H_0) i[H_0, A_-] \chi(H_0) \geq \mu \chi(H_0) j_1^2\left(\frac{r_*}{S}\right) \chi(H_0) + K,$$

for S large enough.

Proof : Recall that

$$i[H_0, A_-] = R_-^{(1)}(r_*, \mathbb{D}_{S^2}) + 2ia_0(r_*) R_-(r_*, \mathbb{D}_{S^2}) \mathbb{D}_{S^2} \Gamma^1 + 2ib_0(r_*) R_-(r_*, \mathbb{D}_{S^2}) \Gamma^0 \Gamma^1.$$

Let us choose $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi$ contains λ_0 . We decompose $\chi(H_0) i[H_0, A_-] \chi(H_0)$ into four terms

$$\chi(H_0) i[H_0, A_-] \chi(H_0) = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \chi(H_0) j_1^2\left(\frac{r_*}{S}\right) \chi(H_0)$$

$$I_2 = \chi(H_0) j_1^2\left(\frac{r_*}{S}\right) \left(j_1^2\left(\frac{r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|}{S}\right) - 1 \right) \chi(H_0)$$

$$I_3 = \chi(H_0) \left\{ \frac{r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|}{S} (j_1^2)' \left(\frac{r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|}{S} \right) + 2ie^{(\kappa_+ r_* + \ln |\mathbb{D}_{S^2}|)} R_- \frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|} \Gamma^1 \right\} \chi(H_0)$$

$$I_4 = \chi(H_0) \left\{ 2i \frac{a_0(r_*) - c_1 e^{\kappa_+ r_*}}{a_0(r_*)} R_- a_0(r_*) \mathbb{D}_{S^2} \Gamma^1 + 2ib_0(r_*) R_- \Gamma^0 \Gamma^1 \right\} \chi(H_0).$$

The term I_4 is compact on \mathcal{H} . For instance, let us treat the first term in I_4 . We can write it as

$$-2i \left(\chi(H_0) \frac{a_0(r_*) - c_1 e^{\kappa+r_*}}{a_0(r_*)} j_1^2 \left(\frac{r_*}{S} \right) \langle r_* \rangle \right) \left(\langle r_* \rangle^{-1} R_-(r_*, \mathbb{D}_{S^2}) \Gamma^1 \right) (a_0(r_*) \mathbb{D}_{S^2} \chi(H_0)).$$

Using Proposition IV.3.1, the function $\frac{a_0(r_*) - c_1 e^{\kappa+r_*}}{a_0(r_*)} j_1^2 \left(\frac{r_*}{S} \right) \langle r_* \rangle$ vanishes at infinity and thus

$$\chi(H_0) \frac{a_0(r_*) - c_1 e^{\kappa+r_*}}{a_0(r_*)} j_1^2 \left(\frac{r_*}{S} \right) \langle r_* \rangle$$

is compact by Lemma IV.3.4. This implies the compactness of the full term since $\langle r_* \rangle^{-1} R_-(r_*, \mathbb{D}_{S^2}) \Gamma^1$ and $a_0(r_*) \mathbb{D}_{S^2} \chi(H_0)$ are bounded by Lemma IV.4.4 and (IV.3.16). The other term in I_4 is treated similarly.

Thanks to Lemmata IV.4.4 and IV.4.9, the term I_3 can be written as

$$\chi(H_e) j_1 \left(\frac{r_*}{S} \right) \left\{ f_S(r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|) + g_S(r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|) \frac{\mathbb{D}_{S^2}}{|\mathbb{D}_{S^2}|} \Gamma^1 \right\} j_1 \left(\frac{r_*}{S} \right) \chi(H_e) + K,$$

where K compact and $f_S, g_S \in C_\infty(\mathbb{R})$. By Corollary IV.4.1, the part involving f_S and g_S tends to 0 in operator norm when $\text{supp } \chi$ is small enough. Precisely, for all S and $\epsilon > 0$, we can choose χ with $\text{supp } \chi$ small enough such that

$$I_3 \geq -\epsilon \chi(H_0) j_1^2 \left(\frac{r_*}{S} \right) \chi(H_0) + K.$$

We now prove that the term I_2 is the sum of compact operator plus a term which tends to 0 in operator norm when S tends to infinity. We first introduce the bounded operator

$$W = j_1 \left(\frac{r_*}{S} \right) \eta \left(\frac{r_* + \kappa_+^{-1} \ln |\mathbb{D}_{S^2}|}{S} \right),$$

where $\eta \in C^\infty(\mathbb{R})$ and $\eta^2 = 1 - j_-^2$. Using Lemma IV.4.9, we have

$$I_2 = \chi(H_0) W^2 \chi(H_0) = \chi(H_e) W^2 \chi(H_e) + K, \quad K \text{ compact.}$$

We claim that

$$\lim_{S \rightarrow \infty} \|\chi(H_e) W^2 \chi(H_e)\| = 0. \quad (\text{IV.4.18})$$

To see this, we study the operator $B = \chi(H_e) e^{\kappa+r_*} \mathbb{D}_{S^2} W^2 e^{\kappa+r_*} \mathbb{D}_{S^2} \chi(H_e)$. Since W preserves $D(H_e)$ ([46], Lemma 5.17), B is well defined on \mathcal{H} . Moreover, using (IV.4.13), B is in fact bounded. Now, since W commutes with $e^{\kappa+r_*} \mathbb{D}_{S^2}$, B can be written as $\chi(H_e) W e^{2\kappa+r_*} \mathbb{D}_{S^2}^2 W \chi(H_e)$. But, note that on $\text{supp } \eta$, we have

$$e^{\kappa+r_*} |\mathbb{D}_{S^2}| \geq e^{\kappa+S}.$$

Hence, $B \geq e^{2\kappa+S} \chi(H_e) W^2 \chi(H_e)$ and since B is bounded, there exists a constant C such that

$$\chi(H_e) W^2 \chi(H_e) \leq C e^{-2\kappa+S} \rightarrow 0, \quad S \rightarrow \infty,$$

which implies (IV.4.18). Finally, using that $[\chi(H_e), j_1 \left(\frac{r_*}{S} \right)] \in O(S^{-1})$ and the previous estimate, we obtain : for any $\epsilon > 0$, we can find S_0 large enough such that for any $S > S_0$,

$$\chi(H_0) W^2 \chi(H_0) \geq -\epsilon \chi(H_0) j_1^2 \left(\frac{r_*}{S} \right) \chi(H_0) + K.$$

This concludes the proof of the lemma.

◇

The Mourre estimate for (H_0, A_+) : The situation at infinity is more standard. The unique subtlety comes from the choice of the conjugate operator since this choice depends on the energy interval we consider. In particular, two threshold values appear for which we are not able to establish a Mourre estimate. Precisely, we have

Lemma IV.4.12 (a) For any $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi \subset (+m, +\infty)$, there exist $\mu > 0$ and a compact operator K such that

$$\chi(H_0)i[H_0, A_+]\chi(H_0) \geq \mu \chi(H_0)j_+^2\left(\frac{r_*}{S}\right)\chi(H_0) + K. \quad (\text{IV.4.19})$$

(b) For any $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi \subset (-\infty, -m)$, (IV.4.19) is true if we replace A_+ by $-A_+$.

(c) For any $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi \subset (-m, +m)$, then $j_+^2\left(\frac{r_*}{S}\right)\chi(H_0)$ is compact on \mathcal{H} and (IV.4.19) is valid for any $\mu > 0$.

Proof : Let us show (a). Using (IV.3.17) and the fact that $j_+^2\left(\frac{r_*}{S}\right)\chi(H_0)$ is compact, we obtain

$$\begin{aligned} \chi(H_0)i[H_0, A_+]\chi(H_0) &= \chi(H_0)j_+\left(\frac{r_*}{S}\right)\Gamma^1 D_{r_*} j_+\left(\frac{r_*}{S}\right)\chi(H_0) \\ &\quad - \chi(H_0)R_+(r_*)\left\{a'_0(r_*)\mathbb{D}_{S^2} + b'_0(r_*)\Gamma^0 + c'_0(r_*)\right\}\chi(H_0) + K, \end{aligned}$$

where K compact. We now make H_0 appear in the first term and using that the three operators

$$\begin{aligned} &\chi(H_0)j_+^2\left(\frac{r_*}{S}\right)\left\{r_*a'_0(r_*) + a_0(r_*)\right\}\mathbb{D}_{S^2}\chi(H_0), \\ &\chi(H_0)\left\{R_+(r_*)b'_0(r_*) + (b_0(r_*) - m)\right\}\Gamma^0\chi(H_0), \\ &\chi(H_0)\left\{R_+(r_*)c'_0(r_*) + c_0(r_*)\right\}\chi(H_0), \end{aligned}$$

are compact by Proposition IV.3.1 and Lemma IV.3.4 (compactness criterion), we get

$$\chi(H_0)i[H_0, A_+]\chi(H_0) = \chi(H_0)j_+\left(\frac{r_*}{S}\right)H_0j_+\left(\frac{r_*}{S}\right)\chi(H_0) - \chi(H_0)j_+^2\left(\frac{r_*}{S}\right)m\Gamma^0\chi(H_0) + K.$$

Finally, using again that $j_+^2\left(\frac{r_*}{S}\right)\chi(H_0)$ is compact and the fact that $\text{supp } \chi \subset (+m, +\infty)$, there exists a strictly positive constant μ such that

$$\chi(H_0)j_+\left(\frac{r_*}{S}\right)H_0j_+\left(\frac{r_*}{S}\right)\chi(H_0) \geq (m + \mu)\chi(H_0)j_+^2\left(\frac{r_*}{S}\right)\chi(H_0) + K,$$

while the second term is obviously estimated by

$$\chi(H_0)j_+^2\left(\frac{r_*}{S}\right)m\Gamma^0\chi(H_0) \geq -m\chi(H_0)j_+^2\left(\frac{r_*}{S}\right)\chi(H_0).$$

This entails (IV.4.19).

The proof of (b) is identical to the previous one with A_+ replaced by $-A_+$.

In order to prove (c), we introduce the selfadjoint operator $H_m = \Gamma^1 D_\rho + \frac{1}{\rho}\mathbb{D}_{S^2} + m\Gamma^0$ acting on $\mathcal{H}_\infty = [L^2(\mathbb{R}_\rho^+ \times S^2; \rho d\rho d\omega)]^4$. H_m is exactly the free Dirac Hamiltonian in flat spacetime written in

polar coordinates. Observe that H_m corresponds to the formal limit of H_0 when $r_* \rightarrow +\infty$. It is well known that (see for instance [75])

$$\sigma(H_m) = \sigma_{ac}(H_m) = (-\infty, -m] \cup [+m, +\infty),$$

which implies

$$\chi(H_m) = 0, \tag{IV.4.20}$$

if $\text{supp } \chi \subset (-m, +m)$. Using (IV.4.20), we can express $\chi(H_0)$ as the difference $\chi(H_0) - \chi(H_m)$ and we want to use the Helffer-Sjöstrand formula and the standard compactness criterion to prove (c). Since H_0 and H_m do not act on the same Hilbert space, we have to be cautious in this procedure. We proceed as follows. Since $[j_+(\frac{r_*}{S}), \chi(H_0)]$ is compact, it is enough to prove that $j_+(\frac{r_*}{S})\chi(H_0)j_+(\frac{r_*}{S})$ is compact. Remark that the cut-off function $j_+(\frac{r_*}{S})$ obviously plays the role of (bounded) identifying operator between \mathcal{H} and \mathcal{H}_∞ (and conversely). Indeed, we have

$$\begin{aligned} \mathcal{I} : \quad \mathcal{H} &\longrightarrow \mathcal{H}_\infty, & \mathcal{I}^* : \quad \mathcal{H}_\infty &\longrightarrow \mathcal{H}, \\ \psi(r_*) &\longrightarrow j_+(\frac{\rho}{S})\psi(\rho), & \psi(\rho) &\longrightarrow j_+(\frac{r_*}{S})\psi(r_*), \quad r_* \geq 0 \text{ and } 0 \text{ otherwise.} \end{aligned}$$

Hence, using these identifying operators and (IV.4.20), the following identity makes sense on \mathcal{H}

$$j_+(\frac{r_*}{S})\chi(H_0)j_+(\frac{r_*}{S}) = j_+(\frac{r_*}{S})(\chi(H_0) - \chi(H_m))j_+(\frac{r_*}{S}).$$

Now using the Helffer-Sjöstrand formula, it suffices to show that

$$L = j_+(\frac{r_*}{S})((z - H_0)^{-1} - (z - H_m)^{-1})j_+(\frac{r_*}{S}),$$

is compact on \mathcal{H} . We introduce $\tilde{j}_+ \in C^\infty(\mathbb{R})$ satisfying $\tilde{j}_+j_+ = j_+$ and $\tilde{j}_+ = 0$ on $(-\infty, 0]$. Then we have

$$\begin{aligned} L &= j_+(\frac{r_*}{S})((z - H_0)^{-1}\tilde{j}_+(\frac{r_*}{S}) - \tilde{j}_+(\frac{r_*}{S})(z - H_m)^{-1})j_+(\frac{r_*}{S}), \\ &= j_+(\frac{r_*}{S})(z - H_0)^{-1}(H_0\tilde{j}_+(\frac{r_*}{S}) - \tilde{j}_+(\frac{r_*}{S})H_m)(z - H_m)^{-1}j_+(\frac{r_*}{S}), \\ &= j_+(\frac{r_*}{S})(z - H_0)^{-1}\left\{ -\frac{i}{S}\Gamma^1(\tilde{j}_+)'(\frac{r_*}{S}) + \tilde{j}_+(\frac{r_*}{S})(a_0(r_*) - \frac{1}{r_*})\mathbb{D}_{S^2} \right. \\ &\quad \left. + \tilde{j}_+(\frac{r_*}{S})(b_0(r_*) - m)\Gamma^0 + \tilde{j}_+(\frac{r_*}{S})c_0(r_*) \right\}(z - H_m)^{-1}j_+(\frac{r_*}{S}). \end{aligned}$$

Thus L is compact using Proposition IV.3.1 and Lemma IV.3.4. This concludes the proof of the lemma.

◇

Summarizing all the previous results, we have proved

Proposition IV.4.1 *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi$ is included in one of the intervals $(-\infty, -m)$, $(-m, +m)$, $(+m, +\infty)$ and $\text{supp } \chi$ small enough. Then for S sufficiently large, there exist $\mu > 0$ and a compact operator K such that*

$$\chi(H_0)i[H_0, A_\chi]\chi(H_0) \geq \mu \chi^2(H_0) + K,$$

where $A_\chi = A_- + A_+$ when $\text{supp } \chi \subset (+m, +\infty)$, $A_\chi = A_- - A_+$ when $\text{supp } \chi \subset (-\infty, -m)$ and $A_\chi = A_-$ when $\text{supp } \chi \subset (-m, +m)$.

The Mourre estimate for (H, A_χ) : Let $\chi \in C_0^\infty(\mathbb{R})$ satisfying the assumptions of Proposition IV.4.1. We only treat the case $\text{supp } \chi \subset (+m, +\infty)$ since the other cases are analogous. Hence $A_\chi = A = A_- + A_+$. Using Lemma IV.4.7 and Corollary IV.3.2, we have

$$\chi(H)i[H, A]\chi(H) = \chi(H_0)i[H, A]\chi(H_0) + K,$$

where K compact. We write $H = H_0 + (h-1)H_0 + H_0(h-1) + (h-1)H_0(h-1) + V_0 = H_0 + H_1$ and we show that

$$\chi(H_0)i[H_1, A]\chi(H_0) \text{ is compact.}$$

For instance, we have

$$\chi(H_0)i[(h-1)H_0, A]\chi(H_0) = \chi(H_0)i[(h-1), A]H_0\chi(H_0) + \chi(H_0)(h-1)i[H_0, A]\chi(H_0).$$

But, $i[(h-1), A] = -R_+(r_*)\partial_{r_*}h$ and $h-1$ belong to \mathbf{S}^{-2} by Lemma IV.3.2. Hence this term is compact. The other terms involving $h-1$ are treated similarly. Eventually, since $V_0 \in \mathbf{S}^{-2}$, it is immediate that $[V_0, A] \in \mathbf{S}^{-1}$ and thus, the last term $\chi(H_0)i[V_0, A]\chi(H_0)$ is also compact. Hence, we have proved

Proposition IV.4.2 *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi$ is included in one of the intervals $(-\infty, -m)$, $(-m, +m)$, $(+m, +\infty)$ and $\text{supp } \chi$ small enough. Then for S sufficiently large, there exist $\mu > 0$ and a compact operator K such that*

$$\chi(H)i[H, A_\chi]\chi(H) \geq \mu \chi^2(H) + K,$$

where A_χ defined as in Proposition IV.4.1.

The first application of Theorem IV.4.1 concerns the spectrum of H_0 . Using Proposition IV.3.3, we have

Theorem IV.4.2 $\sigma(H_0) = \sigma_{ess}(H_0) = \sigma_{ac}(H_0) = \mathbb{R}$.

Since $(H - i)^{-1} - (H_0 - i)^{-1}$ is compact (see the proof of Corollary IV.3.2), it follows from the Weyl Theorem (see [69], Vol IV) and Proposition IV.4.2 that $\sigma_{ess}(H) = \mathbb{R}$. Using this and Theorem IV.4.1 again, we obtain

Theorem IV.4.3 *The operator H has no singular continuous spectrum ($\sigma_{sing}(H) = \emptyset$) and the only eigenvalues for H can be $\{-m, +m\}$ ($\sigma_{pp}(H) \subset \{-m, +m\}$). If either m or $-m$ is an eigenvalue, it has infinite multiplicity.*

IV.4.3 Propagation estimates

In this section, we establish the minimal and maximal velocity estimates. Similar estimates appeared for the first time in [72]. They turned out useful for the study of time-dependent scattering problems (in particular for the study of the n-body problem in quantum mechanics). We follow here the formalism given by Dereziński and Gérard in [21] who placed these estimates at the centre of their work.

Minimal velocity estimate

The meaning of the minimal velocity estimate is the following : the energy of the field must escape from a narrow cone $\mathcal{C} = \{ \frac{|r_*|}{t} \leq c \}$ at late times (in a classical picture, the constant c represents the minimal velocity of the fields). Such an estimate corresponds to a weak form of the *Huygens principle*. Precisely, we show

Proposition IV.4.3 *Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \subset \mathbb{R} \setminus \{-m, +m\}$. Then there exists a constant $\epsilon_\chi > 0$ such that*

$$\int_1^\infty \left\| \mathbf{1}_{[0, \epsilon_\chi]} \left(\frac{|r_*|}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{IV.4.21})$$

Furthermore,

$$s - \lim_{t \rightarrow +\infty} e^{itH} \mathbf{1}_{[0, \epsilon_\chi]} \left(\frac{r_*}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{IV.4.22})$$

The same is true if we replace the operator H by H_0 .

Proof : We prove this proposition in two steps. We first establish an estimate very close to (IV.4.21) involving the conjugate operator A_χ . This estimate is originally due to Sigal and Soffer [72]. We also refer to [39], appendix A. We first choose $\chi \in C_0^\infty(\mathbb{R})$ with small enough support and S large such that

$$\chi(H) i[H, A_\chi] \chi(H) \geq \mu \chi^2(H) + K, \quad K \text{ compact.}$$

Now using the compactness of K , we can shrink the support of χ such that there exists $\mu_0 \in (0, \mu)$ satisfying

$$\chi(H) i[H, A_\chi] \chi(H) \geq \mu_0 \chi^2(H). \quad (\text{IV.4.23})$$

Since $H \in C^2(A_\chi)$ (see the previous section), this implies that for any $g \in C_0^\infty(\mathbb{R})$ satisfying $\text{supp } g \subset (-\infty, \mu_0)$, we have

$$\int_1^{+\infty} \left\| g \left(\frac{A_\chi}{t} \right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad (\text{IV.4.24})$$

and

$$s - \lim_{t \rightarrow +\infty} g \left(\frac{A_\chi}{t} \right) \chi(H) e^{-itH} = 0. \quad (\text{IV.4.25})$$

Now, we want to replace the operator A_χ by a more meaningful observable such as the position operator $|r_*|$ in (IV.4.24) and (IV.4.25). This is done using a slight extension of a result due to Gérard and Nier [39]. (The proof of the following lemma is analogous to that of [39]).

Lemma IV.4.13 *Let A, H two selfadjoint operators on \mathcal{H} and $\chi \in C_0^\infty(\mathbb{R})$ such that $H \in C^2(A)$ and (IV.4.23) is satisfied. Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\chi \tilde{\chi} = \chi$. We denote A_1 the selfadjoint extension of the closed symmetric operator $\tilde{\chi}(H) A \tilde{\chi}(H)$ originally defined on $D(A)$. Let $B \geq 1$ another selfadjoint operator on \mathcal{H} satisfying*

$$\begin{aligned} (i) \quad & D(B) \subset D(A_1), \\ (ii) \quad & A_1 \leq cB, \\ (iii) \quad & [A_1, B]B^{-1} \in \mathcal{B}(\mathcal{H}). \end{aligned} \quad (\text{IV.4.26})$$

then there exists a small enough constant ϵ such that

$$\left\| \mathbf{1} \left(\frac{B}{t} \leq \epsilon \right) \mathbf{1} \left(\frac{A}{t} \geq \mu_0 \right) \chi(H) \right\| \in O(t^{-1}). \quad (\text{IV.4.27})$$

We apply this lemma with $B = \langle r_* \rangle$. We can assume without loss of generality that $\text{supp } \chi$ is included in $(+m, +\infty)$. Thus $A_\chi = A_- + A_+$. We define $A_1 = \tilde{\chi}(H)A_- \tilde{\chi}(H) + \tilde{\chi}(H)A_+ \tilde{\chi}(H)$. Thanks to (IV.3.17) and Proposition IV.3.2, the hypotheses in (IV.4.26) are clearly satisfied. Hence, for a small enough constant ϵ , we get

$$\left\| \mathbf{1}\left(\frac{\langle r_* \rangle}{t} \leq \epsilon\right) \mathbf{1}\left(\frac{A_\chi}{t} \geq \mu_0\right) \chi(H) \right\| \in O(t^{-1}). \quad (\text{IV.4.28})$$

Combining (IV.4.28) with (IV.4.24) and (IV.4.25), we obtain

$$\int_1^{+\infty} \left\| \mathbf{1}_{[0, \epsilon]}\left(\frac{\langle r_* \rangle}{t}\right) \chi(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad (\text{IV.4.29})$$

and

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[0, \epsilon]}\left(\frac{\langle r_* \rangle}{t}\right) \chi(H) e^{-itH} = 0. \quad (\text{IV.4.30})$$

Eventually, since $\mathbf{1}_{[0, \epsilon]}\left(\frac{\langle r_* \rangle}{t}\right) - \mathbf{1}_{[0, \epsilon]}\left(\frac{\langle r_* \rangle}{t}\right) \in L^1(dt)$, (IV.4.29) and (IV.4.30) imply (IV.4.21) and (IV.4.22) for any $\chi \in C_0^\infty(\mathbb{R})$ with a small enough support included in one of the intervals $(-\infty, -m)$, $(-m, +m)$ or $(+m, +\infty)$. Hence (IV.4.21) and (IV.4.22) for $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi \subset \mathbb{R} \setminus \{-m, +m\}$.
 \diamond

Maximal velocity estimate

This section uses the commutator methods explained in appendix IV.A. The maximal velocity estimate states, in a weak sense, that the field cannot travel faster than the speed of light (equal to 1 in our convention). Precisely, we have

Proposition IV.4.4 *Let $\delta \in]0, 1[$ and $f \in C_0^\infty(\mathbb{R})$ such that $\text{supp } f \cap [-1 - \delta, +1 + \delta] = \emptyset$. Then*

$$\int_1^{+\infty} \left\| f\left(\frac{r_*}{t}\right) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{IV.4.31})$$

Let $g \in C^\infty(\mathbb{R})$ such that $g = 0$ on $[-1 - \delta, +1 + \delta]$ and $g(r_) = 1$ for $|r_*|$ large. Then*

$$s - \lim_{t \rightarrow +\infty} g\left(\frac{r_*}{t}\right) e^{-itH} = 0. \quad (\text{IV.4.32})$$

Proof : We assume that $\text{supp } f \subset [1 + \delta, +\infty)$. We define $F(s) = \int_{-\infty}^s f^2(\sigma) d\sigma$ and $\phi(t) = F\left(\frac{r_*}{t}\right)$. The time-dependent observable $\phi(t)$ is a uniformly bounded function of t with values in selfadjoint operators. We compute its Heisenberg derivative

$$\begin{aligned} -\mathbf{D}\phi(t) &:= -\frac{d}{dt}\phi(t) - i[H, \phi(t)], \\ &= \frac{1}{t} f^2\left(\frac{r_*}{t}\right) \frac{r_*}{t} - \frac{1}{t} f^2\left(\frac{r_*}{t}\right) h^2(r_*, \theta) \Gamma^1. \end{aligned}$$

From the proof of Lemma IV.3.2, we have $h^2 \leq 1 + \frac{a^2}{r^2 + a^2}$. Therefore, on $\text{supp } f$, we can find T large enough such that for any $t \geq T$, we have $f^2\left(\frac{r_*}{t}\right) h^2 \leq (1 + \frac{\delta}{2}) f^2\left(\frac{r_*}{t}\right)$. On the other hand, we have $f^2\left(\frac{r_*}{t}\right) \frac{r_*}{t} \geq (1 + \delta) f^2\left(\frac{r_*}{t}\right)$. Hence, we obtain for $t \geq T$

$$-\mathbf{D}\phi(t) \geq \frac{\delta}{2t} f^2\left(\frac{r_*}{t}\right).$$

We conclude the proof of (IV.4.31) applying Lemma IV.A.1. (The proof for $\text{supp } f \subset (-\infty, -1 - \delta]$ is identical. It suffices to compute $\mathbf{D}\phi(t)$ instead of $-\mathbf{D}\phi(t)$).

In order to prove (IV.4.32), we compute the Heisenberg derivative of $g(\frac{r_*}{t})$. We get

$$\mathbf{D}g(\frac{r_*}{t}) = -\frac{1}{t}g'(\frac{r_*}{t})\frac{r_*}{t} + \frac{1}{t}g'(\frac{r_*}{t})h^2(r_*, \theta)\Gamma^1.$$

Since $\text{supp } g'$ is compact and h bounded, we have

$$\mathbf{D}g(\frac{r_*}{t}) = \frac{1}{t}g'(\frac{r_*}{t})B(t)\tilde{g}(\frac{r_*}{t}),$$

where $B(t)$ is a uniformly bounded operator and $\tilde{g} \in C^\infty(\mathbb{R})$ satisfies $g\tilde{g} = g$ and $\tilde{g}(r_*) = 0$ on $[-1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}]$. Hence the existence of the limit

$$s - \lim_{t \rightarrow +\infty} e^{itH} g(\frac{r_*}{t}) e^{-itH}, \quad (\text{IV.4.33})$$

follows from Lemma IV.A.1 and (IV.4.31). Assume moreover that g has compact support. Then the limit (IV.4.33) vanishes using (IV.4.31) again. The general case (i.e. g has no compact support) is proved by a standard limit procedure (see for example [14]). We omit the details.

◇

We can be a little more precise on what happens for fields having energy in $(-m, +m)$. Exactly,

Proposition IV.4.5 *Let $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi \subset (-m, +m)$ and $j_+ \in C^\infty(\mathbb{R})$ satisfying $j_+ = 1$ on $(\delta, +\infty)$ and $j_+ = 0$ on $(-\infty, \frac{\delta}{2})$ for some $\delta > 0$. Then, there exists $\epsilon > 0$ such that*

$$j_+^2(\frac{r_*}{t})\chi(H) \in O(t^{-\epsilon}). \quad (\text{IV.4.34})$$

As a consequence,

$$\int_1^{+\infty} \left\| j_+^2(\frac{r_*}{t})\chi(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad (\text{IV.4.35})$$

and

$$s - \lim_{t \rightarrow +\infty} j_+^2(\frac{r_*}{t})\chi(H)e^{-itH} = 0. \quad (\text{IV.4.36})$$

Proof : We first decompose $j_+^2(\frac{r_*}{t})\chi(H)$ as

$$j_+^2(\frac{r_*}{t})\chi(H) = j_+^2(\frac{r_*}{t})(\chi(H) - \chi(H_0)) + j_+^2(\frac{r_*}{t})\chi(H_0).$$

Using the Helffer-Sjöstrand formula and Proposition IV.3.2, it is immediate that

$$j_+^2(\frac{r_*}{t})(\chi(H) - \chi(H_0)) \in O(t^{-2}).$$

Furthermore, since $\text{supp } \chi \subset (-m, +m)$, we prove that there exists $\epsilon > 0$ such that

$$j_+^2(\frac{r_*}{t})\chi(H_0) \in O(t^{-\epsilon}),$$

mimicking the proof of Lemma IV.4.12, (c) and using Proposition IV.3.2 once again.

◇

Combining Propositions IV.4.3 and IV.4.5, we see that fields having not enough energy (i.e. energy included in $(-m, +m)$) cannot escape at infinity but must fall into the black hole with probability 1. We refer to [32] for another proof of this assertion.

IV.5 Wave operators

IV.5.1 Intermediate wave operators (between H and H_0)

We begin with an easy application of the previous minimal velocity estimate : we prove that the standard wave operators between H_0 and H exist and are complete. Precisely, we show

Theorem IV.5.1 *The wave operators defined by the strong limits*

$$\Omega^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}, \quad (\text{IV.5.1})$$

$$\tilde{\Omega}^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH} \mathbf{1}^c(H), \quad (\text{IV.5.2})$$

exist on \mathcal{H} . Moreover, $(\Omega^\pm)^* = \tilde{\Omega}^\pm$, $(\Omega^\pm)^* \Omega^\pm = \mathbf{1}_{\mathcal{H}}$ and $\Omega^\pm (\Omega^\pm)^* = \mathbf{1}_{\mathcal{H}_c}$. Eventually, we have the intertwining relations

$$H = \Omega^\pm H_0 (\Omega^\pm)^*.$$

Proof : Since the proofs are identical, we only prove (IV.5.2). Since $\sigma_{pp}(H) \subset \{-m, +m\}$ (see Theorem IV.4.3), by a density argument, it is enough to show the existence of the limit

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_0} \chi(H) e^{-itH},$$

for any $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp } \chi \cap \{-m, +m\} = \emptyset$. We introduce two functions j_0, j satisfying $j_0 + j = 1$, $j_0 = 1$ on a neighbourhood of 0 and $\text{supp } j_0 \subset (-\epsilon_\chi, \epsilon_\chi)$ where ϵ_χ is defined by the minimal velocity estimate. Using Proposition IV.4.3, we have

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_0} j_0\left(\frac{r^*}{t}\right) \chi(H) e^{-itH} = 0,$$

and thus, it suffices to prove the existence of the remaining limit

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_0} j\left(\frac{r^*}{t}\right) \chi(H) e^{-itH}. \quad (\text{IV.5.3})$$

In order to prove (IV.5.3), we apply Lemma IV.A.1. Let us compute the Heisenberg derivative of $\phi(t) = j\left(\frac{r^*}{t}\right) \chi(H)$. We have

$$\mathbf{D}\phi(t) = -\frac{1}{t} j' \left(\frac{r^*}{t}\right) \frac{r^*}{t} \chi(H) + i(H_0 j\left(\frac{r^*}{t}\right) - j\left(\frac{r^*}{t}\right) H) \chi(H). \quad (\text{IV.5.4})$$

Using that $H = H_0 + (h^2 - 1)H_0 + \tilde{V}_0$ with $\tilde{V}_0 \in \mathbf{S}^{-2}$, we rewrite (IV.5.4) as

$$\mathbf{D}j\left(\frac{r^*}{t}\right) \chi(H) = \frac{1}{t} \tilde{\chi}(H) \tilde{j}\left(\frac{r^*}{t}\right) B(t) j'\left(\frac{r^*}{t}\right) \chi(H) - j\left(\frac{r^*}{t}\right) i((h^2 - 1)H_0 + \tilde{V}_0) \chi(H), \quad (\text{IV.5.5})$$

where $B(t)$ is a uniformly bounded operator and $\tilde{\chi}, \tilde{j} \in C_0^\infty(\mathbb{R})$ satisfy $\tilde{\chi}\chi = \chi$, $\text{supp } \tilde{\chi} \cap \{-m, +m\} = \emptyset$, $\tilde{j}j' = j'$ and $\tilde{j} = 0$ on a neighbourhood of 0. Since $h^2 - 1, \tilde{V}_0 \in \mathbf{S}^{-2}$, the second term in (IV.5.5) belongs to $O(t^{-2})$ and is integrable in norm. Using that $(\tilde{\chi}(H) - \tilde{\chi}(H_0))\tilde{j}\left(\frac{r^*}{t}\right) \in O(t^{-1})$ and Proposition IV.4.3, the first term in (IV.5.5) is also integrable along the evolution in the sense of Lemma IV.A.1. Thus, the limit (IV.5.3) exists which concludes the proof of the theorem.

◇

Comments : Since \mathbb{D} and H coincide on the subspace \mathcal{H}_n , Theorem IV.5.1 implies that on any angular mode, the dynamics generated by \mathbb{D}^n can be compared (and replaced) at late times with the dynamics generated by \mathbb{D}_0^n i.e. a dynamics generated by a Dirac operator outside a Reissner-Nordström black hole type. For any $n \in \mathbb{N}$, the limits

$$\Omega_n^\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH^n} e^{-itH_0^n} = s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}^n} e^{-it\mathbb{D}_0^n}, \quad (\text{IV.5.6})$$

$$(\Omega_n^\pm)^* = s - \lim_{t \rightarrow \pm\infty} e^{itH_0^n} e^{-itH^n} = s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}_0^n} e^{-it\mathbb{D}^n}, \quad (\text{IV.5.7})$$

exist. Note here that we need not add a projection onto the continuous spectral subspace of $\mathbb{D} = H^n$ in (IV.5.7) since \mathbb{D}^n has no eigenvalue (Proposition IV.3.4). The scattering for spherically symmetric operators such as H_0 has been thoroughly studied in [55] and [16]. We now use the results of [16] to construct the so-called asymptotic velocity operators and define the full (Dollard-modified at infinity) wave operators associated to the dynamics $e^{-it\mathbb{D}}$.

IV.5.2 Asymptotic velocity operators

Let us first state a result of “existence”

Theorem IV.5.2 *For any $g \in C_\infty(\mathbb{R})$, the limits*

$$g(P^\pm) := s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}} g\left(\frac{r_*}{t}\right) e^{-it\mathbb{D}}, \quad (\text{IV.5.8})$$

exist on \mathcal{H} . This defines P^\pm as selfadjoint operators on \mathcal{H} with dense domain. Moreover, the operators P^\pm commute with \mathbb{D} .

Proof : It suffices to prove (IV.5.8) on each angular mode $e^{in\varphi}$. Since $\mathbb{D}^n = H^n$, we have

$$g(P_n^\pm) := s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}^n} g\left(\frac{r_*}{t}\right) e^{-it\mathbb{D}^n} = s - \lim_{t \rightarrow \pm\infty} e^{itH^n} e^{-itH_0^n} e^{itH_0^n} g\left(\frac{r_*}{t}\right) e^{-itH_0^n} e^{itH_0^n} e^{-itH^n},$$

and using the chain rule, (IV.5.6) and (IV.5.7), we only have to prove the existence of

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_0^n} g\left(\frac{r_*}{t}\right) e^{-itH_0^n}. \quad (\text{IV.5.9})$$

But this has been done in [16] using the spherical symmetry of H_0 . Let us denote $g(P_{0,n}^\pm)$ the limit (IV.5.9). Hence we have

$$g(P_n^\pm) = \Omega_n^\pm g(P_{0,n}^\pm) (\Omega_n^\pm)^*. \quad (\text{IV.5.10})$$

◇

We now propose another characterization of the asymptotic velocity operators P^\pm useful for studying their spectra. For this, we introduce the “classical velocity operator” $\mathcal{V}_m = D_{r_*} H_\infty^{-1}$ where $H_\infty = \Gamma^1 D_{r_*} + m\Gamma^0$ when $m \neq 0$ and $\mathcal{V}_0 = \Gamma^1$ when $m = 0$ (we refer to [16] or [75] for the definition of these operators). We have

Proposition IV.5.1 *For any $g \in C_\infty(\mathbb{R})$,*

$$g(P^\pm) \mathbf{1}_{\mathbb{R}^+}(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}} g(\pm\mathcal{V}_m) e^{-it\mathbb{D}} \mathbf{1}_{\mathbb{R}^+}(P^\pm), \quad (\text{IV.5.11})$$

$$g(P^\pm) \mathbf{1}_{\mathbb{R}^-}(P^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}} g(\pm\Gamma^1) e^{-it\mathbb{D}} \mathbf{1}_{\mathbb{R}^-}(P^\pm). \quad (\text{IV.5.12})$$

Proof : We prove (IV.5.11). Again, it is enough to work on a given angular mode $e^{jn\varphi}$. Using (IV.5.10), we have

$$g(P_n^\pm) \mathbf{1}_{\mathbb{R}^+}(P_n^\pm) = \Omega_n^\pm g(P_{0,n}^\pm) \mathbf{1}_{\mathbb{R}^+}(P_{0,n}^\pm) (\Omega_n^\pm)^*.$$

From [16], we know that

$$g(P_{0,n}^\pm) \mathbf{1}_{\mathbb{R}^+}(P_{0,n}^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH_0^n} g(\pm\mathcal{V}_m) e^{-itH_0^n} \mathbf{1}_{\mathbb{R}^+}(P_{0,n}^\pm).$$

Hence, by the chain rule, (IV.5.6) and (IV.5.7), this yields (IV.5.11).

The proof of (IV.5.12) is identical if we use (from [16])

$$g(P_{0,n}^\pm) \mathbf{1}_{\mathbb{R}^-}(P_{0,n}^\pm) = s - \lim_{t \rightarrow \pm\infty} e^{itH_0^n} g(\pm\Gamma^1) e^{-itH_0^n} \mathbf{1}_{\mathbb{R}^-}(P_{0,n}^\pm).$$

◇

This characterization of P^\pm allows us to study the physically relevant information given by the asymptotic velocities : their spectra.

Theorem IV.5.3

$$\begin{aligned} \sigma(P^+) &= \{-1\} \cup [0, 1], \quad \text{when } m \neq 0, \\ \sigma(P^-) &= [-1, 0] \cup \{1\}, \quad \text{when } m \neq 0, \\ \sigma(P^\pm) &= \{-1\} \cup \{+1\}, \quad \text{when } m = 0. \end{aligned} \tag{IV.5.13}$$

Moreover, the states having zero asymptotic velocity are the bound states of \mathbb{D} , i.e.

$$\mathbf{1}^{pp}(\mathbb{D}) = \mathbf{1}_{\{0\}}(P^\pm) = 0. \tag{IV.5.14}$$

Proof : From [16], the assertions in (IV.5.13) and (IV.5.14) are true with \mathbb{D} , P^\pm replaced by H_0^n , $P_{0,n}^\pm$. By the usual argument, this remains true for \mathbb{D} , P^\pm .

◇

Comments : At late times, Dirac fields emerge from or fall into the black hole horizon with radial velocity equal to 1 whereas at spacelike infinity, this velocity can be smaller than 1 because of the mass m (and is equal to 1 if $m = 0$). We now make more precise these distinct behaviors in the asymptotic regions by introducing the complete (Dollard-modified at infinity) wave operators. A useful application of the asymptotic velocity operators P^\pm is that they help us to separate outgoing and incoming Dirac fields. Precisely, we define

$$\begin{aligned} \mathcal{H}_{in}^\pm(\mathbb{D}) &= \mathbf{1}_{\mathbb{R}^-}(P^\pm) \mathcal{H}, \\ \mathcal{H}_{out}^\pm(\mathbb{D}) &= \mathbf{1}_{\mathbb{R}^+}(P^\pm) \mathcal{H}. \end{aligned}$$

IV.5.3 Wave operators at the horizon

Let us introduce the selfadjoint operator acting on \mathcal{H}

$$\mathbb{D}_H = \Gamma^1 D_{r_*} + \frac{a}{r_+^2 + a^2} D_\varphi + \frac{qQr_+}{r_+^2 + a^2},$$

which is formally the limit of \mathbb{D} when $r_* \rightarrow -\infty$. The dynamics generated by \mathbb{D}_H is a transport equation in r_* with a rotation around the axis of symmetry (note that the constant $\frac{a}{r_*^2+a^2}$ is the angular rotation speed of the black hole horizon as perceived by an observer at infinity) plus a phase term due to the interaction between the charge of the black hole and that of the field. We can construct the asymptotic velocity operators P_H^\pm in a similar way as for \mathbb{D} and it turns out that

$$P_H^\pm = \pm \Gamma^1.$$

We also define the outgoing and incoming spaces by

$$\begin{aligned}\mathcal{H}_{in}^\pm(\mathbb{D}_H) &= \mathbf{1}_{\mathbb{R}^-}(\pm\Gamma^1)\mathcal{H}, \\ \mathcal{H}_{out}^\pm(\mathbb{D}_H) &= \mathbf{1}_{\mathbb{R}^+}(\pm\Gamma^1)\mathcal{H}.\end{aligned}$$

Since the difference between \mathbb{D} and \mathbb{D}_H is short-range, the dynamics $e^{-it\mathbb{D}_H}$ is a good comparison dynamics for $e^{-it\mathbb{D}}$, as showed by the following theorem

Theorem IV.5.4 *The wave operators defined by the strong limits*

$$W_H^\pm = s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}} e^{-it\mathbb{D}_H}, \quad (\text{IV.5.15})$$

$$\tilde{W}_H^\pm = s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}_H} e^{-it\mathbb{D}}, \quad (\text{IV.5.16})$$

exist on $\mathcal{H}_{in}^\pm(\mathbb{D}_H)$, (resp. $\mathcal{H}_{in}^\pm(\mathbb{D})$). Moreover, they satisfy $(W_H^\pm)^* = \tilde{W}_H^\pm$, $(W_H^\pm)^* W_H^\pm = \mathbf{1}_{\mathbb{R}^-}(\pm\Gamma^1)$, $W_H^\pm (W_H^\pm)^* = \mathbf{1}_{\mathbb{R}^-}(P^\pm)$ and the intertwining relations

$$\mathbb{D}\mathbf{1}_{\mathbb{R}^-}(P^\pm) = W_H^\pm \mathbb{D}_H (W_H^\pm)^*, \quad (\text{IV.5.17})$$

$$P^\pm \mathbf{1}_{\mathbb{R}^-}(P^\pm) = W_H^\pm (\pm\Gamma^1) (W_H^\pm)^*. \quad (\text{IV.5.18})$$

Proof : Let us prove (IV.5.15). It suffices to show that, on any angular mode $e^{jn\varphi}$, the limits

$$s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}^n} e^{-it\mathbb{D}_H^n} \mathbf{1}_{\mathbb{R}^-}(\pm\Gamma^1), \quad (\text{IV.5.19})$$

exist. Here we denote $\mathbb{D}_H^n = \Gamma^1 D_{r_*} + \frac{na+qQr_+}{r_*^2+a^2}$. Let us write

$$s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}^n} e^{-it\mathbb{D}_H^n} \mathbf{1}_{\mathbb{R}^-}(\pm\Gamma^1) = s - \lim_{t \rightarrow \pm\infty} e^{itH^n} e^{-itH_0^n} e^{itH_0^n} e^{-it\mathbb{D}_H^n} \mathbf{1}_{\mathbb{R}^-}(\pm\Gamma^1).$$

From [16], we know that the limits

$$s - \lim_{t \rightarrow \pm\infty} e^{itH_0^n} e^{-it\mathbb{D}_H^n} \mathbf{1}_{\mathbb{R}^-}(\pm\Gamma^1), \quad (\text{IV.5.20})$$

exist. Hence the existence of (IV.5.19) follows from (IV.5.6), (IV.5.20) and the chain rule.

The proofs of the other assertions are analogous and use the corresponding results in [16]. We omit them.

◇

IV.5.4 Dollard-modified wave operators at infinity

At infinity, the operator \mathbb{D} tends formally to the operator

$$\mathbb{D}_\infty = H_\infty = \Gamma^1 D_{r_*} + m\Gamma^0.$$

This operator is an one dimensional Dirac operator in flat spacetime. Intuitively, we expected such a comparison operator since a Kerr-Newman black hole is an asymptotically flat spacetime. Once again, we can construct the asymptotic velocity operators P_∞^\pm associated to \mathbb{D}_∞ and it turns out that

$$P_\infty^\pm = \pm \mathcal{V}_m,$$

where $\mathcal{V}_m = D_{r_*} \mathbb{D}_\infty^{-1}$ when $m \neq 0$ and $\mathcal{V}_0 = \Gamma^1$ when $m = 0$. The outgoing and incoming spaces are given by

$$\begin{aligned} \mathcal{H}_{in}^\pm(\mathbb{D}_\infty) &= \mathbf{1}_{\mathbb{R}^-}(P_\infty^\pm)\mathcal{H}, \\ \mathcal{H}_{out}^\pm(\mathbb{D}_\infty) &= \mathbf{1}_{\mathbb{R}^+}(P_\infty^\pm)\mathcal{H}. \end{aligned}$$

However, the difference between \mathbb{D} and \mathbb{D}_∞ at infinity is not short-range. Indeed, we have

$$\mathbb{D} = \mathbb{D}_\infty + b_m(r_*)\Gamma^0 + c_q(r_*) + \text{short-range terms},$$

where $b_m(r_*) = b_0(r_*) - m$ and $c_q(r_*) = \frac{qQr}{r^2+a^2}$ are potentials of Coulombian type. We thus need to modify the comparison dynamics $e^{-it\mathbb{D}_\infty}$ by a phase term. From [16], we introduce the Dollard modified wave operators

$$\begin{aligned} U_D^m(t) &= e^{-it\mathbb{D}_\infty} T\left(e^{-i\int_0^t (b_m(s\mathcal{V}_m)m\mathbb{D}_\infty^{-1} + c_q(s\mathcal{V}_m))ds}\right), \quad m \neq 0, \\ U_D^0(t) &= e^{-it\mathbb{D}_\infty} T\left(e^{-i\int_0^t c_q(s\mathcal{V}_0)ds}\right), \quad m = 0, \end{aligned}$$

where T denotes time ordering. Now, using (IV.5.6), (IV.5.7) and the results in [16], we have

Theorem IV.5.5 *The wave operators defined by the strong limits*

$$\begin{aligned} W_\infty^\pm &= s - \lim_{t \rightarrow \pm\infty} e^{it\mathbb{D}} U_D^m(t), \\ \tilde{W}_\infty^\pm &= s - \lim_{t \rightarrow \pm\infty} (U_D^m)^*(t) e^{-it\mathbb{D}}, \end{aligned}$$

exist on $\mathcal{H}_{out}^\pm(\mathbb{D}_\infty)$, (resp. on $\mathcal{H}_{out}^\pm(\mathbb{D})$). Moreover, $(W_\infty^\pm)^* = \tilde{W}_\infty^\pm$, $(W_\infty^\pm)^* W_\infty^\pm = \mathbf{1}_{\mathbb{R}^+}(\pm\mathcal{V}_m)$ and $W_\infty^\pm (W_\infty^\pm)^* = \mathbf{1}_{\mathbb{R}^+}(P^\pm)$. Eventually, we have the intertwining relations

$$\begin{aligned} \mathbb{D}\mathbf{1}_{\mathbb{R}^+}(P^\pm) &= W_\infty^\pm \mathbb{D}_\infty (W_\infty^\pm)^*, \\ P^\pm \mathbf{1}_{\mathbb{R}^+}(P^\pm) &= W_\infty^\pm (\pm\mathcal{V}_m) (W_\infty^\pm)^*. \end{aligned}$$

IV.5.5 Global Wave operators

We finish this paper by describing in a more synthetic manner the preceding scattering results. We introduce the space of scattering data

$$\mathcal{H}_{scat}^\pm = \mathcal{H}_{in}^\pm(\mathbb{D}_H) \oplus \mathcal{H}_{out}^\pm(\mathbb{D}_\infty),$$

and the global wave operators

$$\begin{aligned} W^\pm : \quad \mathcal{H}_{scat}^\pm &\longrightarrow \mathcal{H}, \\ (\psi_1, \psi_2) &\longrightarrow W_H^\pm \psi_1 + W_\infty^\pm \psi_2, \\ \tilde{W}^\pm : \quad \mathcal{H} &\longrightarrow \mathcal{H}_{scat}^\pm, \\ \psi &\longrightarrow (\tilde{W}_H^\pm \psi, \tilde{W}_\infty^\pm \psi). \end{aligned}$$

Using Theorems IV.5.4 and IV.5.5, the operators W^\pm are isometries and satisfy

$$\tilde{W}^\pm W^\pm = \mathbf{1}_{\mathcal{H}_{scat}^\pm}, \quad W^\pm \tilde{W}^\pm = \mathbf{1}_{\mathcal{H}}.$$

APPENDIX

IV.A Commutator methods

Let H a selfadjoint operator on \mathcal{H} . We denote $\mathbf{D} = \frac{d}{dt} + i[H, \cdot]$ the Heisenberg derivative which acts on time-dependent selfadjoint observables $B(t)$ and satisfies the following property

$$\frac{d}{dt} \left[e^{itH} B(t) e^{-itH} \right] = e^{itH} \mathbf{D} B(t) e^{-itH}.$$

Propagation estimates are obtained using the fundamental criterion.

Proposition IV.A.1 *Let $\Phi(t)$ be a family of selfadjoint operators belonging to $W_{loc}^{1,1}(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ i.e. there exists $B(t) \in L_{loc}^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ such that*

$$\Phi(t_1) - \Phi(t_2) = \int_{t_1}^{t_2} B(s) ds.$$

(i) *Assume that $\mathbf{D}\Phi(t) \in L_{loc}^1(\mathbb{R}^+, \mathcal{B}(\mathcal{H}))$. Then*

$$\|\Phi(t) e^{-itH} \psi\| \leq \|\Phi(0) \psi\| + \int_0^t \|\mathbf{D}\Phi(s) e^{-isH} \psi\| ds.$$

(ii) *Assume that $\Phi(t)$ is uniformly bounded and that there exists $C_0 > 0$ and some operator valued functions $B(t)$ and $B_i(t)$, $i = 1, \dots, n$ such that*

$$\mathbf{D}\Phi(t) \geq C_0 B^*(t) B(t) - \sum_{i=1}^n B_i^*(t) B_i(t),$$

with

$$\int_1^\infty \|B_i(t) e^{-itH} \psi\|^2 dt \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}, \quad i = 1, \dots, n.$$

Then there exists a constant C such that

$$\int_1^\infty \|B(t) e^{-itH} \psi\|^2 dt \leq C \|\psi\|^2, \quad \forall \psi \in \mathcal{H}. \quad (\text{IV.A.1})$$

Estimates taking the form of (IV.A.1) are called *weak propagation estimates* since the only information we have concerning the time-decay of $\frac{1}{t}\|B(t)e^{-itH}\psi\|$ is that this quantity belongs to $L^1(dt)$. In particular, we cannot conclude that $\|B(t)e^{-itH}\psi\| \in O(t^{-\rho})$ for any $\rho > 0$. Nevertheless, this weak information is enough to prove the existence of *asymptotic observables*. By this, we mean the following : if $B(t)$ is a time-dependent uniformly bounded observable, the asymptotic observable related to $B(t)$ along the evolution e^{-itH} is given by the limit

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} B(t) e^{-itH}. \quad (\text{IV.A.2})$$

We state here two criteria, due to Cook and Kato, to prove the existence of (IV.A.2). We refer to [21], Lemma B.4.2, for the proof of this result.

Lemma IV.A.1 (Cook, Kato) *Let \mathcal{H} a Hilbert space and H a selfadjoint operator on \mathcal{H} . Let $\phi(t)$ be an uniformly bounded function belonging to $W_{loc}^{1,1}(\mathbb{R}; \mathcal{B}(\mathcal{H}))$. We suppose that $[H, \phi(t)]$, originally defined as a quadratic form on $D(H)$ extends to an element of $L_{loc}^1(\mathbb{R}; \mathcal{B}(\mathcal{H}))$. Let \mathcal{D} a dense subspace of \mathcal{H} .*

(i) (Cook) Assume that for $\psi \in \mathcal{D}$,

$$\int_1^\infty \|\mathbf{D}\Phi(t)e^{-itH}\psi\| dt < \infty,$$

then there exists

$$s - \lim_{t \rightarrow \infty} e^{itH} \Phi(t) e^{-itH}. \quad (\text{IV.A.3})$$

(ii) (Kato) Assume that

$$|(\psi_2, \mathbf{D}\Phi(t)\psi_1)| \leq \sum_{i=1}^n \|B_{2i}(t)\psi_2\| \|B_{1i}(t)\psi_1\|,$$

with

$$\begin{aligned} \int_1^\infty \|B_{2i}(t)e^{-itH}\psi\|^2 dt &\leq C\|\psi\|^2, & \psi \in \mathcal{H}, i = 1, \dots, n, \\ \int_1^\infty \|B_{1i}(t)e^{-itH}\psi\|^2 dt &\leq C\|\psi\|^2, & \psi \in \mathcal{D}, i = 1, \dots, n, \end{aligned}$$

then the limit (IV.A.3) exists.

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