Localization of a polymer interacting with an interface.

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Localisation d'un polymère en interaction avec une interface.

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Abstract. We study different models of polymers (discrete and continuous) in the neighborhood of an interface between two solvents (oil-water). These models give rise to a transition between a localized phase and a delocalized phase. We prove first several convergence results of discrete models towards their associated continuous counterparts. These convergence hold when the coupling tends to 0 (for high temperatures) and concerns the free energy and the slope of the critical curve at the origin. To that aim, we develop a method of coarse graining, introduced by Bolthausen and den Hollander, which we generalize to the case of a copolymer under the influence of a random pinning potential along the oil-water interface. We prove also a pathwise result in the case of a copolymer, which is pulled up and away from the interface. We show in particular that inside the localized phase, the polymer comes back to the interface only a finite number of times. Finally, we study the case of an hydrophobic homopolymer in the neighborhood of an oil-water interface, and also under the influence of a random potential when touching the interface. Through a method consisting of adapting the law of each excursion to its local random environment, we take into account the fact that the polymer can target the sites in which it comes back to the interface. This allows us to improve in a quantitative way the lower bound of the quenched critical curve.
Keywords: Polymers, localization-delocalization transition, pinning, random walk, random media, coarse graining, wetting.

Résumé. Nous étudions différents modèles (discrets ou continus) de polymères au voisinage d'une interface entre 2 solvants (huile-eau). Ces modèles donnent tous lieu à une transition entre une phase localisée et une phase délocalisée. Nous prouvons tout d'abord plusieurs résultats de convergence de modèles discrets vers leurs modèles continus associés. Ces convergences ont lieu dans le cas d'un couplage faible (haute température) et concernent l'énergie libre d'une part, et la pente de la courbe critique à l'origine d'autre part. Pour cela, nous développons une méthode de coarse graining introduite par Bolthausen et den Hollander que nous généralisons au cas d'un copolymère soumis à un potentiel d'accrochage aléatoire le long de l'interface huile-eau. Nous prouvons ensuite un résultat trajectoriel, dans le cas d'un copolymère soumis, en l'une de ses extrémités, à une force qui le tire loin de l'interface. Nous montrons, en particulier qu'à l'intérieur de la phase localisée, le polymère ne touche l'interface qu'un nombre fini de fois. Enfin, nous étudions le cas d'un homopolymère hydrophobe au voisinage d'une interface (huile-eau) et soumis également a un potentiel aléatoire lorsqu'il touche cette interface. Par une méthode consistant à adapter la loi de chacune des excusions en dehors de l'interface à son environnement aléatoire local, nous prenons en compte le fait que le polymère peut viser les sites où il vient toucher l'interface. Ceci permet d'améliorer de façon quantitative la borne inférieure de la courbe critique du modèle quenched donnée jusqu'alors par la courbe critique du modèle à potentiel constant.

Mots clés: Polymères, transition de localisation-délocalisation, accrochage, marche aléatoire, milieu aléatoire, renormalisation, mouillage.

Contents

Chapter 1

Introduction

1.1 Polymères

Durant toute cette étude, nous considérons des modèles probabilistes susceptibles de décrire le comportement de polymères dans différents milieux aqueux, et notamment au voisinage d'une interface solide-liquide ou liquide-liquide. Il convient donc tout d'abord de nous remémorer ce qu'est un polymère, et de passer en revue certains des domaines (industriels, médicaux) dans lesquels on les retrouve. Un polymère est une longue chaîne (macromolécule), formée d'une suite de molécules élémentaires d'un ou plusieurs types appelées monomères. On les trouve à l'état liquide (certains polymères entrent dans la composition des shampooings), mais aussi à l'état solide: certains sont élastiques (élastomères), d'autres sont mous (adhésifs) ou bien encore extrèmement durs (fibres de kevlar). Ces macromolécules peuvent être séparées en deux classes. Les homopolymères d'une part, composés d'un seul type de monomère, les hétéropolymères d'autre part, qui eux admettent plusieurs sortes de monomères le long de leur chaîne.

Les homopolymères interviennent dans de nombreux champs industriels; l'alimentaire par exemple avec le sucre (cellulose, fructane) ou encore l'industrie automobile (caoutchouc), le batiment (polypropylène), etc... Il est surprenant de constater que les propriétés mécaniques, physiques et chimiques de ces homopolymères ne leur sont pas conférées uniquement par le monomère qui les constitue, mais aussi par leur structure spatiale. Ainsi la cellulose et le dextrane, lorsqu'ils sont intégrés dans des produits alimentaires, ont des propriétés différentes (concernant leur capacité épaississante par exemple) alors qu'ils sont tous deux composés de D-glucose. Viennent ensuite les hétéropolymères, dont la chaîne est constituée de plusieurs types de monomères. Les protéines en font partie, dont les constituants (acides aminés) sont au nombre de vingt. Mais on les trouve, eux aussi dans l'industrie, où par exemple, le propylène (mentionné avant) peut être associé à de l'éthlyène au sein d'un même polymère pour améliorer sa résistance à l'impact. Enfin, citons le cas de l'A.D.N., dont les séquences de monomères forment en réalité un code pour synthétiser des protéines.

1.2 Problèmes de localisation

Notons tout d'abord que nous envisageons dans cette étude des systèmes à l'équilibre. En effet, nous ne considérons pas la dynamique, l'évolution d'un polymère que l'on place près d'une interface, mais plutôt la manière dont il se positionne par rapport à l'interface une fois l'équilibre atteint. L'objectif est donc de déterminer (à paramètres fixés) si notre polymère a tendance à rester près de l'interface et à la traverser souvent (typiquement en une densité positive de sites), on parlera alors de localisation, ou bien au contraire si il s'éloigne fortement de cette interface et dans ce cas on dira qu'il est délocalisé. Notre objectif dans cette étude est d'envisager ces questions de localisation d'un polymère sous l'angle de la mécanique statistique. En effet, nous quantifions les interactions entre chaque monomère et son environnement direct, qui peut être l'interface elle même ou l'un des solvants. De cette manière, nous voulons améliorer notre compréhension du comportement du polymère, c'est-à-dire identifier (à paramètres fixés) des sous-familles de trajectoires, empruntées préférentiellement par le polymère.

Cette problématique a généré beaucoup de travaux ces dernières années, tant dans les domaines de la chimie, de la physique, que des probabilités. Ceci s'explique en partie par les progrès très rapides qui ont été réalisés récemment en ce qui concerne la connaissance de la structure de la chaine d'A.D.N. et de la façon dont elle code les protéines. En effet, les macromolécules qui la composent, appelées brins, sont liées entre elle par des liaisons de type AT (adénosine-thymine) ou CG (cytosine-guanine) d'intensités différentes. C'est cette structure particulière qui permet de la modéliser à l'aide, soit de deux marches aléatoires en interaction (cf [32]), soit, si l'on ne considère que la position relative de l'une par rapport à l'autre,d'une marche aléatoire en interaction avec une interface.

L'une des innovations les plus récentes dans le domaine de la biologie cellulaire, met en lumière de façon concrète, l'intérêt d'étudier mathématiquement ces phénomènes de localisation au voisinage d'une interface. En effet, il est possible à présent de manipuler à l'échelle microscopique les chaînes de polymères. Pour cela, il est nécessaire de pouvoir appliquer de très petites forces en certains points de la chaîne, ce qui est possible grâce à des technologies de type pinces optiques (cf [36]). De cette façon, on peut éloigner les deux brins d'une portion d'A.D.N., les délocaliser l'un par rapport à l'autre. Ceci permet de détecter certaines séquences d'acides aminés, en fonction de la force nécessaire à leur ouverture. Une méthode de recherche des mutations, qui interviennent lors de la réplication de la chaîne d'A.D.N., en a été tirée (cf [2]). Le modèle que nous étudions au chapitre 3 permet, par exemple, de modéliser cette situation.

1.3 Modèles discrets

1.3.1 Milieu

A présent, décrivons plus précisément les modèles de polymères auxquels nous nous intéressons. Tout au long de cette étude, le milieu dans lequel le polymère sera plongé est constitué de deux phases, l'une aqueuse, l'autre huileuse, séparées par une interface horizontale. Nous modélisons ceci dans le demi- plan droit de

$\mathbb{Z}^2$, c'est-à-dire $\{(x, y) \in \mathbb{N} \times \mathbb{Z}\}$. L'interface sera représentée par la demi- droite $\{(x, y) \in \mathbb{N} \times \{0\}\}$, la phase huileuse correspond à $\{(x, y) \in \mathbb{N} \times (\mathbb{N} - \{0\})\}$, tandis que la phase aqueuse occupe la zone $\{(x, y) \in \mathbb{N} \times -(\mathbb{N} - \{0\})\}$ (cf Fig. 0.1). Notons aussi que, dans le premier chapitre, nous prenons en compte, soit l'ajout de fines gouttelettes (microémulsions) d'un troisième solvant le long d'une bande $\{(x, y) \in \mathbb{N} \times \{-K, .., K\}\}$, soit l'épaisseur de l'interface, qui voit les deux solvants se mélanger le long d'une bande d'épaisseur finie (cf Fig. 0.2).

Fig. 0.1:

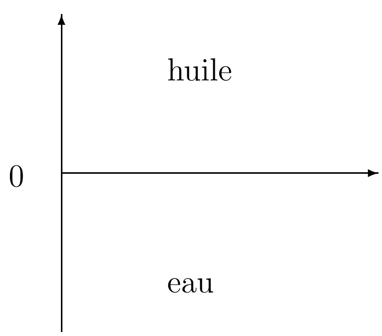
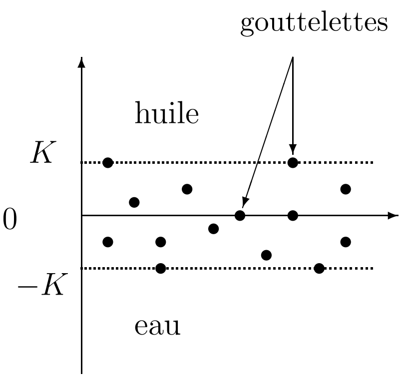


Fig. 0.2:



1.3.2 Trajectoires

Il nous faut choisir ensuite, les configurations (trajectoires) que le polymère peut emprunter. Pour ce faire, nous décidons de travailler avec une chaîne comportant un nombre fixé de monomères (N), quitte à laisser N tendre vers l'infini ensuite pour approximer le comportement de grands polymères. Il est naturel dans ce cas de considérer l'ensemble des trajectoires que peut emprunter une marche aléatoire. Si le choix d'une marche aléatoire auto-évitante en dimension 3 paraît être le plus réaliste, le plus adapté à la réalité physique, c'est un objet qui reste difficile à manier dans les calculs. Nous lui préférerons donc une marche aléatoire dirigée en dimension $1 + 1$, c'est-à-dire que chaque pas y est représenté par un vecteur de coordonées

$(1, +1)$ ou $(1, -1)$. Plus précisément, nous considérerons les trajectoires suivantes

$$\Big\{(i, S_i)_{i\geq 0}\Big\} \quad \text{avec } S_i = \sum_{j=1}^{i} X_j$$

et $(X_i)_{i\geq 1}$ une suite de variables aléatoires indépendantes, indentiquement distribuées, suivant une loi de Bernouilli de paramètre $1/2$, et prenant les valeurs -1 et $+1$. Il nous faut à présent perturber leur mesure de probabilités pour approcher au mieux le comportement du polymère dans le milieu physique que nous considérons. Pour ce faire, nous devons prendre en compte les contraintes à la fois chimiques et mécaniques auxquelles la macromolécule doit faire face. Nous associons donc à chaque trajectoire un hamiltonien (noté H_N), et une nouvelle mesure de probabilités, appelée mesure du polymère à taille N.

1.3.3 Hamiltonien

Désordre d'hydrophobicité.

Dans les deux premiers chapitres nous considérons un copolymère, doté de monomères hydrophiles et hydrophobes. Chaque monomère numéroté par $i \in \{1, .., N\}$, se voit associer une variable aléatoire w_i qui détermine son affinité pour l'un ou l'autre des solvants. Ainsi, si $w_i > 0$, le monomère i préfère l'huile, tandis qu'il préfère l'eau si $w_i < 0$. Les variables $(w_i)_{i\geq 1}$, sont indépendantes, symériques et identiquement distribuées, et l'on note Λ_i le signe de la marche aléatoire après i pas. Un premier terme intervient donc dans la construction de notre hamiltonien, il est donné par

$$\lambda \sum_{i=1}^{N} w_i \Lambda_i, \tag{1.3.1}$$

où λ est un réel stictement positif (c'est en réalité l'inverse de la température). On remarque que ce terme favorise les trajectoires qui traversent l'interface à de nombreuses reprises, de façon à placer un grand nombre de monomères dans leur solvant préféré.

Accrochage sur l'interface.

Dans les chapitres 1 et 3, nous considérons aussi la situation dans laquelle les monomères interagissent directement avec l'interface, c'est-à-dire que le polymère reçoit un prix γ_i^j si il atteint la hauteur $j \in \{-K, .., K\}$ après i pas $(S_i = j)$. Pour chaque j, $\left(\gamma_i^j\right)_{i \geq 1}$ est une suite de variables aléatoires indépendantes et identiquement distribuées. La contribution de cette interaction à l'hamiltonien est donnée par

$$\beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \, \mathbb{1}_{\{S_i=j\}}, \quad \text{où} \quad \beta \geq 0. \tag{1.3.2}$$

Notons que dans le chapitre 3, on se restreint au cas où les prix à l'origine sont de moyenne 1 et $K = 0$, c'est-à-dire que l'interface est réduite à une ligne.

Désordre d'hydrophobicité biaisée.

Un autre phénomène, que nous prendrons en compte dans le premier chapitre vient du fait que, dans le cas d'un copolymère, la loi des variables aléatoires w_i peut ne plus être symétrique. Deux phénomènes physiques peuvent être modélisés ainsi. Le premier correspond au cas d'un copolymère comportant exactement deux types de monomères, l'un d'eux réagissant plus fortement que l'autre avec chaque solvant. Dans ce cas, on choisit pour w_i une loi de Bernouilli de paramètre 1/2 et de valeurs -1 et $+1$, et on introduit un paramètre $h \in [0, 1]$ que l'on ajoute à w_i. La contribution à l'hamiltonien (1.3.1) devient

$$\lambda \sum_{i=1}^{N} (w_i + h)\Lambda_i. \tag{1.3.3}$$

Vient ensuite le cas d'un hétéropolymère dont la proportion de monomères hydrophobes est plus grande que celle de monomères hydrophiles. Dans ce cas $\mathbb{P}(w_i + h > 0) > 1/2$. Ce terme h favorise les trajectoires qui mettent un nombre important de monomères dans l'huile (le demi plan $y > 0$), et pousse le polymère à se délocaliser.

Force.

Enfin dans le second chapitre nous considérons un autre facteur délocalisant, il s'agit d'une force d'intensité F, qui s'applique à l'extrémité droite du polymère et qui tire ce dernier vers le haut. Elle intervient dans l'hamiltonien sous la forme

$$FS_N, \tag{1.3.4}$$

ce qui correspond au travail que doit fournir cette force pour amener l'extrémité de la chaîne de l'interface à la hauteur S_N. Ceci favorise donc les trajectoires dont l'extrémité est très éloignée de l'interface dans le demi plan $y > 0$.

Suivant les contraintes physiques et chimiques que nous prenons en compte pour bâtir un modèle, nous utiliserons une ou plusieurs de ces quatre contributions énergétiques. Elles nous permettent d'attribuer un hamiltonien $(H_N(S))$ à chaque trajectoire de la marche aléatoire simple (notée S). Nous pouvons définir à présent la mesure du polymère.

1.3.4 Mesures du polymère

Cas quenched (gelé)

Comme nous allons le voir maintenant, il y a deux façons de prendre en compte le désordre (w ou γ dans notre cas) pour définir la mesure du polymère. Pour des raisons que nous allons expliquer maintenant, ces deux approches sont véritablement distinctes et mènent, la plupart du temps à des résultats différents.

Le cas quenched consiste à considérer le modèle pour une réalisation donnée du désordre. Ceci signifie qu'une fois w ou γ tirés suivant la loi de probabilités qui les régit, ceux-ci sont considérés comme des paramètres du système au même titre que λ, β ou h. Ainsi pour chaque réalistion du désordre w ou γ, et chaque famille de paramètres (λ, h, β), on note P_N^{qch} la mesure du polymère à taille N et à désordre fixé. Par densité de Radon Nikodym, nous posons

$$\frac{dP_N^{\text{qch}}}{dP}(S) = \frac{\exp(H_N(S))}{Z_N}, \tag{1.3.5}$$

où Z_N est un facteur de renormalisation, également appelé fonction de partition du système, et de valeur

$$Z_N = E\big(\exp(H_N(S))\big).$$

Cas annealed (recuit)

Au contraire du cas quenched, dans le cas annealed, on ne définit plus la mesure du polymère à désordre fixé. En effet, on intègre sur le désordre toutes les quantités ou w et γ apparaissent. La mesure P_N^{ann} du polymère en taille N s'écrit donc

$$\frac{dP_N^{\mathrm{ann}}}{dP}(S) = \frac{\mathbb{E}(\exp(H_N(S)))}{\mathbb{E}(Z_N)}. \tag{1.3.6}$$

Dans tout ce travail nous nous intéresserons principalement au cas quenched, cependant, le cas annealed sera parfois utilisé comme un outil pour déterminer des bornes supérieures d'énergie libre et de courbes critiques (voir 1.6.2 et chapitre 4). L'idée sous-jacente à cette technique vient du fait que, dans le cas quenched, les trajectoires doivent en quelque sorte s'adapter au désordre pour optimiser leur hamiltonien, alors que dans le cas annealed, on peut adapter conjointement la trajectoire et le désordre pour augmenter la valeur de l'hamiltonien. C'est l'inégalité de Jensen qui traduit ceci mathématiquement.

1.4 Modèles continus

L'étude de ces modèles dans le cas d'un couplage faible (température élevée) nous conduit à envisager ce que l'on appellera par la suite des modèles continus, c'est-à-dire construits à partir de trajectoires de mouvements browniens. En effet, pour un polymère de taille N fixée, lorsque les différents paramètres de couplages (β, λ, h) tendent vers 0, l'hamiltonien de chaque trajectoire tend lui aussi vers 0. La mesure du polymère converge donc en loi vers celle de la marche aléatoire simple à N pas. Ceci implique que les grandes excursions loin de l'interface sont favorisées, et on peut éspérer alors, par un changement d'échelle approprié, prouver la convergence du modèle discret vers un modèle continu associé. Cette convergence aura lieu au sens de l'energie libre, et parfois aussi au sens de quantités telles que la pente de la courbe critique de l'espace des phases, etc...

1.4.1 Milieu et trajectoires

Dans le cas d'un modèle continu, on ne tient pas compte d'une éventuelle épaisseur de l'interface. Celle-ci est représentée par la demi- droite $\{(x, y) \in (0, \infty) \times \{0\}\}$, tandis que les quarts de plan $\{(x, y) \in (0, \infty) \times (0, \infty)\}$ et $\{(x, y) \in (0, \infty) \times (-\infty, 0)\}$ correspondent respectivement à l'huile et à l'eau.

Les configurations possibles du polymère sont alors données par les trajectoires d'un mouvement brownien B, c'est-à-dire

$$\Big\{(s, B_s)_{s \geq 0}\Big\}.$$

L'hamiltonien associé à taille t est noté H_t, et les différentes contributions qui le forment sont détaillées dans le paragraphe suivant.

1.4.2 Hamiltonien et mesures du polymère

L'accrochage à l'origine ne sera plus défini sur une bande autour de l'interface, comme c'est le cas dans le modèle discret. En effet, le modèle continu étant une limite d'échelle du cas discret, l'épaisseur de l'interface disparaît dans sa définition. On va donc considérer le temps local L_t, passé en 0 par le mouvement Brownien B entre les instants 0 et t. Et la contribution énergétique de cet accrochage sera donnée par

$$\beta L_t. \tag{1.4.1}$$

Pour prendre en compte les caractéristiques (hydro-phobie, -philie) de chaque monomère, on définit un second mouvement brownien $(R_s)_{s \geq 0}$ indépendant de B. Ce nouveau processus va jouer le rôle des variables w_i dans le modèle discret. Ainsi, en position s, l'interaction entre le polymère et les solvants prend la forme $\Lambda_s(dR_s + hds)$, où Λ_s est le signe de B en position s, et h l'assymétrie de l'interaction des monomères avec chaque solvant. Cette contribution énergétique s'écrira donc, pour l'ensemble de la chaîne

$$\lambda \int_0^t \Lambda_s (dR_s + h ds). \tag{1.4.2}$$

Enfin, et comme dans le cas du modèle discret, nous considérons au chapitre 2 une force qui tire le polymère à son extrémité droite vers le haut. Son intensité étant notée F, elle donnera lieu dans l'hamiltonien à la contribution

$$F S_t. \tag{1.4.3}$$

Pour chaque trajectoire de B, la mesure quenched du polymère sera définie par densité de Radon Nikodym vis à vis de la loi $(\widetilde{P})$ de B. Pour un polymère de taille t, elle sera notée $\widetilde{P}_t^{\text{qch}}$ et prendra la valeur

$$\frac{d\widetilde{P}_t^{\text{qch}}}{dP}(B) = \frac{\exp(H_t(B))}{\widetilde{Z}_t}, \tag{1.4.4}$$

où, comme dans le cas discret, $\widetilde{Z}_t$ est la fonction de partition du système.

La mesure annealed est définie comme dans le cas discret en intégrant sur le désordre, c'est-à-dire

$$\frac{d\widetilde{P}_t^{\text{ann}}}{d\widetilde{P}}(S) = \frac{\widetilde{\mathbb{E}}(\exp(\widetilde{H}_t(S)))}{\widetilde{\mathbb{E}}(\widetilde{Z}_t)}. \tag{1.4.5}$$

1.5 Energie libre, courbe critique

L'énergie libre est une quantité très étudiée par les physiciens, parce qu'elle permet souvent de caractériser l'état d'un système physique. Comme nous allons le voir dans ce travail, ceci est particulièrement vrai dans le cas des modèles de polymères que nous étudions. Plus précisément, elle nous donne un outil pour déterminer si le polymère est localisé près de l'interface ou pas. Dans les cas discrets, l'énergie libre notée Φ apparaît de la manière suivante

$$\Phi = \lim_{N \to \infty} \frac{1}{N} \log Z_N. \tag{1.5.1}$$

Nous traitons dans les chapitres 1 et 2 le problème de l'existence de l'énergie libre. Cependant, nous remarquons que, dans chacun des modèles présentés jusqu'ici, on peut extraire de l'ensemble des configurations, une sous-famille notée D_N (pour un polymère de taille N), regroupant des trajectoires qui restent dans le demi-plan positif et ont toutes le même hamiltonien H_N^D. Par exemple dans le cas du modèle de copolymère étudié au chapitre 1,

$$D_N = \{S \ : \ S_i > K, \forall i \in \{K+1, N\}\}, \quad \text{et} \quad H_N^D = \lambda h N + \sum_{i=1}^{N} w_i + \beta \sum_{j=1}^{K} \gamma_j^j.$$

Ceci nous permet d'obtenir une borne inférieure sur l'énergie libre, en calculant sa restriction a D_N, c'est-à-dire

$$\Phi \geq \lim_{N\to\infty} \frac{1}{N} \log E\Big(\exp(H_N^D)\, \mathbb{1}_{\{D_N\}}\Big) = \Phi_{\text{deloc}}. \tag{1.5.2}$$

En général, le terme de droite de (1.5.2) se calcule facilement. Ainsi, pour le copolymère mentionné avant, et puisque $P(\{D_N\})$ se comporte comme $c/\sqrt{N}$ lorsque N tend vers l'infini on obtient

$$\begin{aligned}\Phi_{\text{deloc}} = &\lim_{N\to\infty} \frac{H_N^D}{N} + \lim_{N\to\infty} \frac{1}{N} \log(P(D_N)) \\ =&\lambda h.\end{aligned}$$

Le critère de localisation se formule à partir de cette remarque. Suivant le modèle considéré, les paramètres changent, on note donc en toute généralité, X la famille des paramètres. On sépare alors l'espace des phases en une zone localisée $\mathcal{L}$ et une autre délocalisée $\mathcal{D}$, définies de la manière suivante

$$\begin{aligned} X \in \mathcal{L} \ \ &\text{si} \ \ \Phi > \Phi_{\text{deloc}} \\ X \in \mathcal{D} \ \ &\text{si} \ \ \Phi = \Phi_{\text{deloc}}. \end{aligned}$$

Ainsi, le polymère est délocalisé quand les trajectoires de D_N suffisent pour obtenir toute l'énergie libre. En revanche, lorsque d'autres groupes de trajectoire sont nécessaires (notamment des trajectoires qui visitent le demi-plan négatif) pour calculer Φ, on dit que le modèle est localisé. Cette distinction peut paraître grossière au premier abord, mais nous verrons, aux chapitres 1 et 2 notamment, qu'elle est en réalité très profonde.

Dans ces différents modèles, l'espace des phases est inclus dans $\mathbb{R}^2$ ou $\mathbb{R}^3$ (suivant le nombre de paramètres nécessaires pour définir l'hamiltonien). La courbe, où surface, séparant les zones localisée et délocalisée est appelée courbe (surface) critique. Dans le cas du modèle de copolymère du chapitre 1, à β fixé, cette courbe sera notée $h_c^\beta(\lambda)$, où encore $h_c(\beta)$ dans le chapitre 3, lorsque seuls h et β sont en jeu. L'existence et différentes propriétés de ces courbes sont étudiées en détail dans ce travail.

L'utilisation de l'énergie libre dans le modèle continu est similaire. Les trajectoires délocalisées D_t sont par exemple $\{B : B_s > 0 \, \forall s \in \; [1,t]\}$, et le critère de localisation reste le même. De plus, par les propriétés de scaling du mouvement Brownien, la courbe critique du modèle continu vérifie $\widetilde{h}_c^{\lambda\beta}(\lambda) = \lambda \widetilde{h}_c^\beta(1)$ pour tout $\lambda \geq 0$. Par la suite, on notera $h_c^\beta(1) = K_c^\beta$ et $K_c^0 = K_c$.

1.6 Résultats antérieurs

Ce type de modèle de polymère fut introduit pour la première fois de façon rigoureuse par Garel, Huse, Leibler et Orland en 1989 (cf [15]). Depuis, de nombreux travaux ont été publiés sur le sujet, et l'intérêt porté à ce domaine par les probabilistes n'a cessé de croître. Nous exposons ici les principaux résultats obtenus jusqu'à présent, et nous les séparons en quatre grandes catégories pour faciliter leur présentation. Nous considérons tout d'abord les travaux concernant les limites d'échelle des polymères, soit quand les paramètres de couplage tendent vers 0 à des vitesses adéquates, soit, à paramètres fixés, en renormalisant la chaîne par un facteur de croissance adapté. Dans les deux cas, le modèle limite, vers lequel le modèle discret converge, est construit à partir de processus continus du type mouvement Brownien, méandre Brownien etc... Ensuite, nous nous intéresserons aux résultats concernant les courbes critiques des différents modèles étudiés. Nous porterons une attention particulière aux travaux concernant les modèles de polymères au voisinage d'interfaces sélectives, et aux modèles d'accrochages. Dans un troisième paragraphe, nous exposerons différents résultats relatifs aux trajectoires que le polymère emprunte préférentiellement, suivant qu'il est localisé ou pas. Enfin, dans un quatrième paragraphe nous parlerons de l'extension très récente de ces modèles à des milieux multi-interfaces.

1.6.1 Limite d'échelle

Bolthausen et den Hollander ont étudié en 1996 (cf [6]) deux modèles de copolymères hydro-phile (-phobe) au voisinage d'une interface huile-eau. Ils ont considéré un modèle discret d'une part, d'hamiltonien (1.3.3) avec des variables w_i qui suivent une loi de Bernouilli, et un modèle continu d'autre part, d'hamiltonien (1.4.2). Par une technique de renormalisation que nous développerons dans ce travail, ils ont pu prouver une convergence du modèle discret vers le modèle continu lorsque les paramètres λ et h tendent vers 0 à la même vitesse. Ils obtiennent ainsi

$$\lim_{a\to 0} \frac{1}{a^2}\,\Phi(a\lambda, ah) = \widetilde{\Phi}(\lambda, h) \qquad \text{et} \qquad \lim_{\lambda\to 0} \frac{h_c(\lambda)}{\lambda} = K_c.$$

Rappelons que K_c est la pente de la courbe critique continue dont ils montrent par propriété de scaling du mouvement brownien que c'est une droite. Cette pente à l'origine ne dépend pas en réalité de la loi des variables aléatoires $(w_i)_{i\geq 1}$, pourvu que celles-ci soient indépendantes, identiquement distribuées, symétriques, bornées et de variance 1 (cf [18]).

La seconde famille de résultats concernant les limites d'échelle de polymères, a été obtenue par Isozaki et Yoshida d'une part (cf [19]), et par Giacomin Deuschel et Zambotti d'autre part (cf [10]). Dans [10] les auteurs ont généralisé les résultats de [19] à une marche aléatoire $(S_n)_{n\geq 1}$ en dimension $1+1$, dont les pas sont d'amplitude continue, centrée, et de carré intégrable. Ils conditionnent cette marche à rester positive, puis la perturbent en pondérant chaque trajectoire d'un facteur ε^{l_N}, avec $\varepsilon > 0$ et $l_N = \sum_{i=1}^{N} \mathbb{1}_{\{S_i=0\}}$. Ils obtiennent ainsi une mesure de polymère en taille N qu'il notent P_ε^N. En fonction de la valeur de ε, la chaîne se trouve localisée ou délocalisée, et ceci donne lieu à une transition de phase en $\varepsilon = \varepsilon_c$. Les auteurs ont alors considéré le polymère de taille N, renormalisé par un facteur proportionnel à $\sqrt{N}$, et ont noté Q_ε^N sa loi. Ils prouvent que, selon que ε est strictement inférieur, égal ou strictement supérieur à ε_c, Q_ε^N converge en loi, respectivement vers un méandre Brownien, un mouvement brownien réfléchi, ou vers 0 lorsque N tend vers ∞.

1.6.2 Courbe critique

Dans leur article de 1996, Bolthausen et Den Hollander prouvent (avec w_i prenant les valeurs 1 et -1 avec probabilité 1/2) l'existence et un certain nombre de propriétés de la courbe critique $h_c(\lambda)$ du modèle (1.3.3). Ils montrent par exemple que h_c est croissante, continue et que sa pente à l'origine verifie $K_c > 0$. Ils utilisent également une variation du modèle annealed, pour obtenir une borne supérieure de h_c, notée $\overline{h}$, et de valeur

$$\overline{h}(\lambda) = 1/(2\lambda) \log \cosh(2\lambda). \tag{1.6.1}$$

Ceci donne en outre la majoration $K_c \leq 1$. Il est à noter que si, comme nous l'expliquons ensuite, la borne inférieure de K_c a pu être améliorée par la suite, ce n'est pas le cas de la borne supérieure. En effet, Caravenna et Giacomin [7] ont montré qu'un raffinement de cette stratégie annealed (appelé méthode de Morita [28]), consistant à ajouter à l'hamiltonien un terme dépendant uniquement du désordre w (par exemple la moyenne des $\{w_i, i \in \{1,..,N\}\}$ ne peut pas améliorer la borne (1.6.1).

Par la suite, Bodineau et Giacomin [3] ont démontré que la courbe critique $\underline{h}(\lambda)$ conjecturée par Cécile Monthus [27], est en réalité une borne inférieure de $h_c(\lambda)$. Ils ont en effet, partitionné l'axe des abscisses en blocs consécutifs de taille $l < \infty$, et restreint le calcul de l'énergie libre aux trajectoires ne séjournant dans le demi-plan $\{y < 0\}$ que le long de blocs "atypiques", au sens qu'ils vérifient $\sum_{i \in \text{bloc}} w_i < -cl$ (pour un $c > 0$). Ils ont alors estimé, par un principe de grandes déviations, la probabilité de l'événement $\left\{\sum_{i=1}^{l} w_i < -cl\right\}$, pour comparer le gain d'énergie qu'induit cette stratégie à la perte d'entropie que nécessitent ces rares visites dans le demi plan $\{y < 0\}$. Ceci leur a permis d'optimiser le choix de c, et d'obtenir la borne inférieure $\underline{h}(\lambda) = 3/(4\lambda) \log \mathbb{E}(\exp(4\lambda w_1/3))$. Dans le cas où $\mathbb{P}(w_1 = \pm 1) = 1/2$, un développement limité de $\underline{h}$ au voisinage de zéro donne l'inégalité $K_c \geq 2/3$, qui n'a pas été améliorée jusqu'à présent.

Cependant, dans un travail plus récent, Caravenna Giacomin et Gubinelli (cf [8]) ont mené une étude numérique de ce modèle. Ils ont batti un test statistique qui montre, avec un niveau d'erreur très faible, que h_c ne coincide pas avec $\underline{h}$. D'autre part, une estimation numérique semble situer la valeur de K_c au dela de 0.8, soit strictement au dessus de la meilleure borne inférieure acctuelle (2/3).

1.6.3 Résultats trajectoriels

Dans son article de 1993 [37], Sinai s'est intéressé au modèle (1.3.3) dans sa phase localisée, pour donner une traduction en terme de trajectoire de la condition $\Phi > \Phi_{\text{deloc}}$. Biskup et Den Hollander ont poursuivi cette étude en 1999 [5] et mis en place un formalisme Gibbsien pour étudier les mesures limites possibles du polymère en volume infini. Ils ont prouvé que, dans la phase localisée, le polymère touche l'interface en une densité positive de sites, et que ses excursions hors de l'origine sont exponentiellement tendues.

Les résultats trajectoriels concernant la phase délocalisée sont plus récents dans la littérature. Le premier a été publié dans [5], et concerne l'intérieur de la phase localisée du modèle (1.3.3). Les auteurs ont montré que, la densité des pas réalisés par le polymère sous un niveau arbitraire k est nulle. Plus précisément, ils ont obtenu que $\mathbb{P}$ presque sûrement en w, pour tous $k \geq 1$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} P_N^{\text{Pol}}\big(S_i \geq k\big) = 1. \tag{1.6.2}$$

Giacomin et Tonninelli [18] ont également travaillé sur le sujet. Ils ont appliqué au modèle (1.3.3) certaines inégalités de concentration de la mesure pour contrôler l'écart entre les logarithmes des fonctions de partition quenched ($\log Z_N$) et leur moyenne $\mathbb{E}(\log Z_N)$. Ils ont pu ainsi estimer certaines quantités du type

$$P_N^{\mathrm{qch}}(\max \mathcal{A} \geq l), \quad \text{où} \quad \mathcal{A} = \{i \leq N \ : \ \text{such that } \Lambda_i = -1\},$$

et prouver que, pour $h > \overline{h}_c(\lambda)$, le nombre de sites que le polymère place dans le demi-plan $\{y < 0\}$ est borné indépendamment de N, et que ce nombre se comporte au pire comme $\log(N)$ lorsque $h \in (h_c(\lambda), \overline{h}(\lambda))$.

1.6.4 Milieux multi-interfaces

Den Hollander et Wuttricht [9] ont travaillé en 2004 sur des modèles de copolymères, plongés dans des milieux qui ne sont plus restreints à une seule interface. Ainsi, ils ont étudié un modèle de copolymère plongé dans un milieu en couches composées alternativement d'huile et d'eau. Ces couches ont toutes la même épaisseur, et celle-ci croît avec la taille N du polymère, à une vitesse comprise entre $\log(N)$ et $\log(\log(N))$. Ils prouvent alors que l'énergie libre de ce modèle est la même que celle du modèle à une seule interface, puis, ils donnent, sous la mesure annealed du polymère, une asymptotique de la vitesse à laquelle le polymère passe d'une couche d'huile à une couche d'eau d'une part, et d'une couche d'eau à une couche d'huile d'autre part.

1.7 Résultats de la thèse

Les résultats de cette étude s'organisent autour de trois modèles de polymères distincts. Nous donnons ici une brève présentation de chacun d'entre eux et des résultats obtenus les concernant.

Tout d'abord nous considérons un modèle de copolymère au voisinage d'une interface entre deux solvants, et en interaction directe avec une bande autour de cette interface. L'hamiltonien considéré sera obtenu à l'aide de (1.3.3), et (1.3.2). Il prend la forme suivante

$$\lambda \sum_{i=1}^{N} (w_i + h)\Lambda_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \, \mathbb{1}_{\{S_i = j\}}. \tag{1.7.1}$$

Comme nous l'avons vu auparavant, Bolthausen et Den Hollander se sont intéressés à ce même modèle sans le terme d'accrochage à l'origine. Ils ont mis en lumière une convergence, en terme d'énergie libre et de pente de la courbe critique à l'origine de ce modèle vers un modèle continu associé. Notre première série de résultats généralise cette convergence au cas où le copolymère interagit directement avec l'interface, par le biais d'un potentiel aléatoire, présent en chaque site de l'interface. Nous obtenons ainsi un modèle continu associé donné par l'hamiltonien

$$\lambda \int_0^t \Lambda_s(dR_s + hds) + \beta\Big(\sum_{j=-K}^{K} \mathbb{E}\big(\gamma_1^j\big)\Big) L_t, \tag{1.7.2}$$

et la convergence a lieu lorsque β, λ et h tendent vers 0 à la même vitesse, c'est-à-dire sous la forme $(a\beta, a\lambda, ah)$ avec $a \to 0$. On obtient alors

$$\lim_{a\to 0} \frac{1}{a^2}\Phi(a\beta, a\lambda, ah) = \widetilde{\Phi}(\beta, \lambda, h) \quad \text{et,}$$
$$\lim_{\lambda\to 0} \frac{h_c^{\lambda\beta}(\lambda)}{\lambda} = K_c^{\beta\Sigma}, \quad \text{avec} \quad \Sigma = \sum_{j=-k}^{k} \mathbb{E}\big(\gamma_1^j\big).$$

Enfin, nous terminons le chapitre 1 en appliquant ces résultats à un modèle d'homopolymère hydrophobe en interaction avec l'interface. Il s'agit en réalité du cas $\lambda = 1$ et $w \equiv 0$, l'hamiltonien s'écrivant

$$h\sum_{i=1}^{N} \Lambda_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \, \mathbb{1}_{\{S_i=j\}}. \tag{1.7.3}$$

Nous constatons alors que le modèle continu associé peut être résolu explicitement. On peut calculer, entre autres choses, sa courbe critique et son énergie libre. Ceci nous donne le comportement limite précis de certaines quantités liées au modèle discret (allure de la courbe critique à l'origine, proportion de pas effectués dans le demi-plan inférieur etc...) quand les paramètres β et h tendent vers 0 à des vitesses bien choisies.

Dans le second chapitre, nous étudions encore un copolymère (hydro-phile,-phobe). Notons que contrairement au modèle développé dans le premier chapitre, aucun des deux types de monomère ne réagit plus vivement que l'autre avec les solvants. C'est pourquoi nous fixons $h = 0$. En revanche, le polymère est soumis à une force verticale dirigée vers le haut ($y > 0$), et qui s'applique en son extrémité droite. L'hamiltonien associé s'écrit à l'aide des contributions énergétiques (1.3.1) et (1.3.4)

$$\lambda \sum_{i=1}^{N} w_i \Lambda_i + F S_N. \tag{1.7.4}$$

On s'intéresse également ici à la convergence de l'énergie libre du modèle discret lorsque le couplage devient faible. Un modèle continu limite apparaît alors, ayant pour hamiltonien

$$\lambda \int_0^t \Delta_s dR_s + F S_t. \tag{1.7.5}$$

Pour obtenir cette convergence, nous menons un calcul partiel des deux énergies libres (discrète et continue), qui s'expriment en réalité comme le maximum de deux fonctions, l'une de λ, l'autre de F. Pour cela, nous prenons en compte séparément le comportement de la chaîne avant son dernier retour à l'origine d'une part, et entre ce dernier retour et son extrémité droite d'autre part. Ceci, nous permet de faire apparaître deux contributions distinctes à l'énergie libre, l'une fonction de λ, l'autre de F. De cette façon , et en utilisant les résultats du premier chapitre, on obtient

$$\lim_{a \to 0} \frac{\Phi(a\lambda, aF)}{a^2} = \widetilde{\Phi}(\lambda, F).$$

Dans la deuxième partie de chapitre, nous prouvons un résultat trajectoriel fort, à l'intérieur de la zone délocalisée de l'espace des phases. Effectivement, en utilisant le calcul de l'énergie libre discrète évoqué précédemment, nous prouvons que dans ces conditions, le polymère ne touche l'interface qu'un nombre fini de fois, avant de se délocaliser définitivement dans l'huile.

Enfin, dans le troisième chapitre, nous envisageons le cas d'un homopolymère hydrophobe accroché le long d'une interface par des prix de type $1+s\zeta_i$ où $(\zeta_i)_{i\geq 1}$est une suite de variables aléatoires i.i.d., centrées, et de variance non nulle. L'hamiltonien considéré est de la forme

$$h \sum_{i=1}^{N} \Lambda_i + \beta \sum_{i=1}^{N} (1 + s\zeta_i)\, \mathbb{1}_{\{S_i = 0\}}. \tag{1.7.6}$$

L'objectif est de comparer la courbe critique $h_c^0(\beta)$ du modèle non désordonné avec celle $h_c^s(\beta)$ d'un cas où les prix à l'origine sont vraiment aléatoires ($s > 0$). Dans un article récent, Alexander et Sidoravicius ont prouvé, pour une large classe de marche aléatoire, qu'introduire un accrochage aléatoire à l'origine permet de localiser la chaîne strictement plus qu'avec un accrochage constant de même moyenne. Dans notre cas cela se traduit par $h_c^o(\beta) < h_c^s(\beta)$ dès lors que $s > 0$. Nous sommes allés au dela de ce résultat dans ce chapitre, en mettant en place une nouvelle stratégie qui permet de viser les sites en lesquels le polymère va revenir à l'origine. Ceci nous permet d'exploiter le caractère aléatoire des prix à l'origine pour améliorer la borne inférieure de $h_c^s(\beta)$.

Finalement en laissant h tendre vers l'infini et en gardant β proche de $\log(2)$, on prouve un corollaire de ce résultat concernant le modèle du wetting. En effet dans le cas d'une marche aléatoire simple accrochée par un hamiltonien du type $\sum_{i=1}^N(-u + s\zeta_i)\,\mathbb{1}_{\{S_i=0\}}$, on prouve qu'il existe $c > 0$ tel que pour s assez petit, et pour tout $u \geq -cs^2$, le polymère est localisé.

1.8 Techniques utilisées

1.8.1 Coarse graining

Plusieurs outils et techniques probabilistes nous ont permis de mener à bien ce travail. Les théorèmes 3 et 4 du premier chapitre ont été obtenus en mettant en oeuvre une méthode de renormalisation, également appelée "coarse graining". Cet outil avait été utilisé par Bolthausen et Den Hollander (cf. [6]), et nous le développons ici, pour le généraliser à un modèle de copolymère, accroché sur une bande de taille finie autour de l'interface par des prix aléatoires. Notre objectif principal est de prouver la convergence de l'énergie libre du modèle discret vers celle du modèle continu associé, lorsque les paramètres tendent vers 0 à des vitesses bien choisies. Cependant, l'énergie libre est une quantité obtenue après avoir laissé la taille du polymère tendre vers l'infini. Le "coarse graining" consiste alors à découper les N

pas du polymère en blocs consécutifs (éventuellement très grands) de taille Δ fixée. Sur chacun de ces blocs nous modifions, étape par étape, l'hamiltonien initial pour le transformer en celui du modèle continu associé. Lorsque la taille du polymère (N) tend vers l'infini, le nombre de ces blocs tend lui aussi vers l'infini. Mais, en choisissant les différents paramètres et Δ de façon adéquate, nous pouvons, bloc par bloc, c'est-à-dire à taille finie, approximer l'énergie libre du modèle continu par celle du modèle discret. Ensuite, quand N tend vers l'infini, nous utilisons des propriétés d'ergodicité sur ces blocs (notamment par des techniques de martingales), pour prouver que cette convergence obtenue bloc par bloc est également vraie en taille infinie.

1.8.2 Stratégie de localisation en milieu désordonné

Dans le troisième chapitre, nous développons une méthode de transformation locale de la loi des retours en zéro d'une marche aléatoire. Cette méthode avait été introduite elle aussi dans [6] pour trouver une borne inférieure de la courbe critique du copolymère. Elle consistait à remplacer, dans la fonction de partition, la marche aléatoire simple par une marche aléatoire dotée d'un drift la poussant vers l'origine. L'objectif était de considérer une marche aléatoire dont le comportement décrive mieux que la marche aléatoire simple, la façon dont le polymère revient à l'origine lorsqu'il est dans une configuration localisée. Dans notre étude, nous allons plus loin, en transformant la loi des retours à l'origine, excursion par excursion, pour prendre en compte le caractère inhomogène du désordre. Ainsi, par densité de Radon Nikodym, nous faisons en sorte que la loi de la taille de chaque excursion le long de la chaîne dépende du désordre environnant.

1.8.3 Couplage

Parallèlement à ces deux premiers outils, nous avons également recours à des techniques de couplage. En effet, pour obtenir le lemme 5 et plusieurs inégalités du chapitre 3, nous comparons le nombre de retours en 0 de différentes marches aléatoires, définies sur un même espace de probabilités. Ainsi dans la preuve du lemme 5, après avoir utilisé un résultat de couplage entre le nombre de retours à l'origine d'une marche aléatoire simple et le temps local d'un mouvement Brownien, nous sommes amenés à prouver l'uniforme intégrabilité d'une suite de variables aléatoires. Celles-ci s'expriment en réalité comme fonction du nombre de pas effectués par une marche dans une bande autour de l'origine. Pour ce faire, nous construisons une marche aléatoire réfléchie à l'aide de différentes quantités discrètes et indépendantes (excursion de marches aléatoires en dehors de l'origine, temps d'atteinte d'un niveau fixé etc...). Ceci nous permet de borner ce nombre de pas dans une bande par des quantités dont les lois sont bien connues, et qui nous permettent de prouver cette intégrabilité uniforme.

Chapter 2

Copolymer Pinned at an Interface

In this chapter we consider a copolymer, composed by hydrophobic and hydrophilic monomers.

2.1 Introduction

2.1.1 Discrete model

We consider a polymer of N monomers, and an interface separating two solvents (for example oil and water). This interface is given by the x axis.

- Configurations. The possible configurations of the polymer are given by the 2^N different trajectories of a simple random walk (S) of length N. Let $\{X_i\}_{i=1,2,..}$ be i.i.d. bernoulli trials satisfying $P(X_1 = \pm 1) = 1/2$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Let $\Lambda_i = \text{sign}(S_i)$ if $S_i \neq 0$, $\Lambda_i = \Lambda_{i-1}$ otherwise.

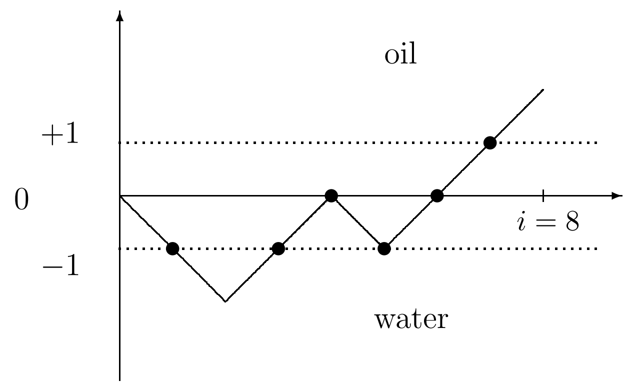
- Pinning potential. We define a pinning potential in a layer of finite width around the interface. For every $j \in \{-K, -K+1, .., K-1, K\}$, we let $\{\gamma_i^j\}_{i=1,2,..}$ be i.i.d. random variables, satisfying $\mathbb{E}\big(\exp\big(\beta|\gamma_1^j|\big)\big) < \infty$ for every $\beta \geq 0$.

- Copolymer. Let $\lambda \geq 0$, $h \geq 0$, and let $\{w_i\}_{i=1,2,..}$ be i.i.d. random variables, independent of γ, bounded, and satisfying $\mathbb{E}(w_1) = 0$ and $\mathbb{E}(w_1^2) = 1$. These variables define a rate of hydrophobicity at each monomer. Indeed, the higher w_i is, the more hydrophobic monomer number i is. We remark that the disorders γ and w are defined under the law $\mathbb{P}$.

- Hamiltonian. For each trajectory of the random walk, we define the following hamiltonian (see the example on Fig. 1)

$$H_{N,\beta,h}^{w,\gamma,\lambda}(S) = \lambda \sum_{i=1}^{N}(w_i + h)\Lambda_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \; \mathbb{1}_{\{S_i=j\}}. \tag{2.1.1}$$

Fig. 1:



On this picture $K = 1$, $N = 14$ and for the drawn trajectory, the hamiltonian takes the value

$$H_{14,\beta,h}^{w,\gamma,\lambda} = \lambda\Big(- \sum_{i=1}^{6} w_i + w_7 + w_8\Big) - 4\lambda h + \beta\Big(\sum_{i\in\{1,3,5\}} \gamma_i^{-1} + \gamma_0^4 + \gamma_0^6 + \gamma_1^7\Big)$$

To avoid heavy notations, the hamiltonian will be denoted by $H_N(S)$. Then, we perturb the law of the random walk as follow

$$\frac{dP_N^{w,\gamma}}{dP}(S) = \frac{\exp\left(H_N(S)\right)}{Z_N^{w,\gamma}}. \tag{2.1.2}$$

This new measure $P_N^{w,\gamma}$ is called polymer measure of size N, and seems to favor two particular subsets of trajectories. First, the trajectories that remain in the neighborhood of the interface and enter in the layer of width $2K$ around it to gain positive prices. These trajectories can also cross the interface often, to put as many monomers as possible in their preferred solvent, i.e. the water if $w_i < 0$, the oil otherwise. These trajectories, called "localized", are favored from the energetic point of view. In the same time, the energy term h favors another class of trajectories called "delocalized". These trajectories spend most of the time in the half upper plane and are much more numerous than the "localized ones". Thus, an energy-entropy competition arises and gives birth to a delocalization transition.

In the next section, we will see how the free energy of this system can be used as a tool to decide if the polymer is localized or not.

2.1.2 Free energy (proposition 1)

We define the free energy of the discrete model with the help of $Z_N^{w,\gamma}$.

Proposition 1 *For* $(\beta, \lambda, h) \in (\mathbb{R}^+)^3$*, there exists a real number, denoted by* $\Phi(\beta, \lambda, h)$*, which satisfy* $\mathbb{P}$ *a.s. the following convergence,*

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^{w,\gamma} = \Phi(\beta, \lambda, h).$$

This limit is called free energy of the model, and is constant $\mathbb{P}$ *a.s.. It means that its value does not depend on the realization of* (w, γ)*.*

This proposition has been proved in different papers (see [17] for example) for some quantities similar to $Z_N^{w,\gamma}$. In our case, the difference comes from the fact that the disorder is spread on a layer of finite width around the interface, but the proof remains essentially the same and is left to the reader. We notice that the convergence occurs also in $\mathbb{L}^1$, and that $\Phi(\beta, \lambda, h)$ is continuous, convex in each variable, and non decreasing in β.

This free energy gives us a tool to decide, for every (β, λ, h), if the system is localized or not. To that aim, we denote by D_N the subset $\{S : S_i > K \ \forall\ i \ \in$

$\{K+1,..,N\}\}$, that is to say trajectories that leave the attracting layer as soon as possible (at site K), and stay above the level K until N. These trajectories are called utterly delocalized and satisfy $\Lambda_i = 1$ for every $i \in \{1,\dots,N\}$. Thus, if we restrict the computation of the free energy Φ to D_N we obtain

$$\begin{aligned}\Phi(\beta,\lambda,h) &\geq \liminf_{N\to\infty} \frac{1}{N} \log E\Big(\exp\Big(\lambda \sum_{i=1}^{N}(w_i+h) + \beta \sum_{i=1}^{K} \gamma_i^i\Big) \mathbb{1}_{\{D_N\}}\Big) \\ &\geq \lambda h + \liminf_{N\to\infty} \frac{\lambda \sum_{i=1}^{N} w_i}{N} + \liminf_{N\to\infty} \frac{\beta \sum_{i=1}^{K} \gamma_i^i}{N} + \liminf_{N\to\infty} \frac{\log\left(P\left(D_N\right)\right)}{N} \geq \lambda h.\end{aligned} \tag{2.1.3}$$

The first inferior limit of (2.1.3) tends to 0 when N tends to ∞ because the law of large number can be applied to $(w_i)_{i\geq 1}$, the second inferior limit tends to 0 because it is a constant ($\sum_{i=1}^{K} \gamma_i^i$) divided by N and the last one tends to 0 because $P(D_N) = (1+o(1))c/\sqrt{N}$ as N tends to ∞. Therefore, the system is said to be delocalized when $\Phi(\beta,\lambda,h) = \lambda h$, because it suffices to consider the utterly delocalized trajectories to obtain the whole free energy, whereas the system is localized when $\Phi(\beta,\lambda,h) > \lambda h$. The (β,λ,h)-space is divided into a localized phase, denoted by $\mathcal{L}$, and a delocalized one denoted by $\mathcal{D}$.

This separation between the localized and delocalized phases has an interpretation in terms of trajectories of the polymer. This issue has been closely studied recently and we refer to [37] or [18] to find precise estimations about it. We mention here a result of [5] concerning the delocalized phase. It shows that the proportion of steps done by the polymer under an arbitrary level $L > 0$ is equal to 0, namely

$$\lim_{N\to\infty} \frac{1}{N} E_N^{w,\gamma}\left(\sharp\{i \in \{1,..,N\} : S_i \leq L\}\right) = 0, \quad \mathbb{P}\text{-a.s. in } w.$$

In the localized phase, the convexity of Φ in β gives after a simple derivation that the polymer comes back in the layer around the origin in a positive density of sites.

2.2 Motivations and objectives

2.2.1 A more realistic model of interface

Models of polymer pinned at an interface have attracted a lot of interest in the last years (see [21], [1], [31]). One of the physical situations that can be modelled by such systems is a polymer put in the neighborhood of an interface between two solvents (see [6]). It gives opportunities to study the localization of the polymer with respect to the interface. Nevertheless, these models do not take into account that such an interface has a width, that is to say a small layer in which the two solvents are more or less mixed together. The model that we develop in this chapter gives a more realistic image of an interface. It allows us also to consider a case, in which microemulsions of a third solvent are spread in a thin layer around the interface.

2.2.2 Continuous limit at weak coupling

One of the main issues of this chapter consists in proving the convergence (in a sense that will be specified) of this discrete model toward a continuous one, when the parameters (λ, h and β) go to zero at a certain speed. Such a convergence has been proved in [6], when there is no pinning term (i.e. $\beta = 0$). But, when $\beta \neq 0$, we know that some zones, in the interacting layer around the origin, concentrate a large number of high rewards and play a particular role from the localization point of view. Indeed, the chain can target the sites where it goes back close to the origin to get prices as high as possible. Consequently, some zones favor the localization of the polymer more than others (see [31]). But, if this model converges to a continuous one, does this continuous model still attract more the polymer in certain parts of the interface? Do the prices that the chain gets when it comes back to the origin remain random, or does the passage to very weak coupling lead to a complete averaging of the disorder?

We answer this question in this chapter, by constructing a continuous model which is in fact the limit of our discrete model. To perform this proof, we use an argument of coarse graining, previously introduced in [6]. We will show a convergence, in terms of free energy, of the discrete model towards the continuous one, as the parameters go to 0 at appropriate speeds.

This associated continuous model has a pinning term at the interface. It is given by the local time in zero of a Brownian motion, weighted by the expectation of the prices (Σ). Hence, we show that the randomness of the pinning term vanishes at weak coupling.

2.3 Pinning term and critical curve

2.3.1 Random copolymer

In this section, we consider the case $K = 0$ and a constant disorder $\gamma \equiv 1$. We want to insist on the fact that, for a fixed $\beta \in \mathbb{R}$, this model generates a critical curve $h_c^\beta(\lambda)$, separating the localized and delocalized phases. In the case $\beta = 0$, this curve has been studied by Bolthausen and Den Hollander in [6], and more recently, by Giacomin and Bodineau. They have found some upper and lower bounds of $h_c^0(\lambda)$ and tried to compute its exact slope at the origin. As proved in [18], this slope, denoted by K_c, has a certain universality because it takes the same value for many types of i.i.d. disorders. In [3], it is proved that $2/3 \leq K_c \leq 1$, and a numerical study seems to show that the exact value of K_c is close to 0.8 (see [8]). However, another important issue, that has not been investigated yet, consists in understanding the influence of a constant (de)pinning term on the critical curve.

At this moment, we know that such a pinning term can transform any delocalized configuration (with parameters $(0, \lambda, h) \in \mathcal{D}$) in a localized one ($(\beta, \lambda, h) \in \mathcal{L}$), as soon as β is large enough. Effectively, by restricting the expectation over S to the subset of localized trajectories $V_{2N} = \{S : S_{2i} = 0,\ \forall\ i \in \{1, \ldots, N\}\}$ and by noticing that $(w_i + h)\Lambda_i \geq -(|w_i| + h)$ for every $i \in \{1, \ldots, 2N\}$, we can bound from below the free energy as follow

$$\Phi(\beta,\lambda,h) \geq \liminf_{N\to\infty} \frac{1}{2N} \log E\Big(\exp\Big(\lambda \sum_{i=1}^{2N} (w_i+h)\Lambda_i + \beta N\Big) \mathbb{1}_{\{V_{2N}\}}\Big)$$
$$\geq -\lambda h + \liminf_{N\to\infty} \frac{-\lambda \sum_{i=1}^{2N} |w_i|}{2N} + \frac{\beta}{2} + \liminf_{N\to\infty} \frac{1}{2N} \log P\left(V_{2N}\right).$$

At this stage, we can apply the law of large numbers to $(|w_i|)_{i\geq 1}$. It implies that $\liminf_{N\to\infty}(-\lambda/2N)\sum_{i=1}^{2N}|w_i| = -\mathbb{E}(|w_1|)$. Then, since $1 = \mathbb{E}(w_1^2) \geq \mathbb{E}(|w_1|)$ we obtain

$$\Phi(\beta,\lambda,h) \geq -\lambda(1+h) + \frac{\beta}{2} + \liminf_{N\to\infty} \frac{1}{2N} \log P\left(V_{2N}\right)$$

and the equality $P\left(V_{2N}\right) = (1/2)^N$ implies $\Phi(\beta,\lambda,h) \geq -\lambda(1+h) + \beta/2 - 1/2 \log 2$. Therefore, for β large enough the free energy is strictly larger than λh.

In [3], Bodineau and Giacomin have enlightened a lower bound of $h_c^0(\lambda)$, denoted by $\underline{h_c^0}(\lambda)$ and equal to $3/(4\lambda)\log\mathbb{E}(\exp(4\lambda w_1/3))$. We can show that, if a configuration of parameters $(0,\lambda,h)$ satisfies $h \leq \underline{h_c^0}(\lambda)$, then $(-\beta,\lambda,h)$ remains localized even if β becomes very large. This point arises directly from the computation of their lower bound. Indeed, to build this curve, they consider some particular trajectories that make long excursions in the half upper plane, and come back sometimes to the interface to get the energy reward corresponding to untypical stretches of -1 of the disorder. But for (λ,h) between the lower bound of [3] and the real critical curve, we do not know at this moment if a negative pinning term can delocalize a localized situation.

2.3.2 Periodic copolymer

The model we have studied since the beginning of this chapter, can also be defined for a periodic copolymer. Effectively, instead of considering the disorder w as a sequence of i.i.d. random variables, we can choose a centered and T periodic sequence of $+1$ and -1 (centered means that $\sum_{i=1}^{T} w_i = 0$). As proved in [4], its free energy is well defined, the localization condition remains the same, and it gives birth to a continuous critical curve. One of the main interest of studying this model comes from the fact that it is expected to converge to the random model, when the period T tends to ∞.

To understand which sense we give to this limit, we must define a few notations:

- Let $\{w_i\}_{i=1,2,..}$ be a sequence of i.i.d. Bernouilli trials of law $\mathbb{P}\left(w_1 = \pm 1\right) = 1/2$. Then for every $k \in \mathbb{N} - \{0\}$, we note $p_k(w) = \sum_{l=1}^{k} w_l/k$ and we consider the mapping $T_k : \{-1,1\}^{\mathbb{N}-\{0\}} \to \{-1,1\}^{\mathbb{N}-\{0\}}$ defined by

$$(T_k(w)_i)_{i\geq 1} = \left(w_{i[\mod k]} - p_k(w)\right)_{i\geq 1}.$$

  In other words $(T_k(w)_i)_{i\geq 1}$ is a k periodic sequence equal to $w_i - p_k(w)$ for $i \in \{1,..k\}$. We subtract $p_k(w)$ to every term, to make sure that $(T_k(w))_{i\geq 1}$ is centered.

- Here again $K = 0$, $\gamma \equiv 1$ and, using the notation of (2.1.2), we note $\Phi_k^{\text{per}}(\beta,\lambda,h)$ $= \lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\log Z_N^{T_k(w),\gamma}\right)$.

- For fixed $k \geq 1$ and $\beta \geq 0$, we let $h_c^{k,\beta}(\lambda)$ be the critical curve of the $2k$ periodic copolymer, pinned by β.

Up to now, it has been proved in [17] that $\lim_{k\to\infty}\Phi_{\text{k}}^{\text{per}}(\beta,\lambda,h) = \Phi(\beta,\lambda,h)$ and that $\limsup_{k\to\infty} h_c^{k,\beta}(\lambda) \geq h_c^{\beta}(\lambda)$. But, an interesting point comes from the fact that, for every k and every couple (λ,h) satisfying $\Phi_k^{\text{per}}(0,\lambda,h) > \lambda h$, we can find β large enough, such that, $\Phi_k^{\text{per}}(-\beta,\lambda,h) = \lambda h$. It means that, contrary to what happens for the random copolymer, a k periodic copolymer in a localized configuration can always be delocalized by a large enough depinning term. Indeed, the contribution to the hamiltonian of an excursion out of the origin is bounded from above by $V_k = \lambda\left|\sum_{i=i_k+1}^{i_{k+1}} w_i + h\right|$ (with i_{k-1} the beginning of the excursion number k and i_k its end). Thus, since the disorder w is T periodic and since $\sum_{i=1}^{T} w_i = 0$ we have immediately that $V_k \leq T + \lambda h(i_k - i_{k-1})$. Therefore, if we choose $\beta > T$, the quantity $V_k - \beta$ is bounded from above by $\lambda h(i_k - i_{k-1})$, and necessarily $\Phi_k^{\text{per}}(-\beta,\lambda,h) = \lambda h$.

2.4 Continuous model (proposition 2)

In this section, we consider a polymer of length $t \in \mathbb{R}$. The parameters λ, h and β are still non negative.

- Configurations. In this continuous case, the configurations of the polymer will be given by the set of trajectories of the Brownian motion $(B_s)_{s\in[0,t]}$. The law of B will be denoted by $\widetilde{P}$, and we note $\Lambda_s = \text{sign}\big(B_s\big)$.

- Pinning potential. The pinning potential of this model will be given by the local time spent in 0 by B between 0 and t. It will be denoted by L_t.

- Copolymer. Independently of B, we let $(R_s)_{s\geq 0}$ be a standard Brownian motion of law $\widetilde{\mathbb{P}}$. We consider dR_s an elementary variation of R at position s. This quantity gives the hydrophobicity of the polymer around the position s, and plays the role of w_i in the discrete model.

- Hamiltonian: for a fixed trajectory of R we can define, for every trajectory of B, the following hamiltonian

$$\widetilde{H}^{R,t}_{\lambda,h,\beta}(B) = \lambda \int_0^t \Lambda(s)(dR_s + hds) + \beta L_t. \tag{2.4.1}$$

For simplicity, the hamiltonian will be denoted by $\widetilde{H}^R_t$. As in the discrete case, we define a new probability law of the B.M. trajectories, called polymer measure

$$\frac{d\widetilde{P}^R_t}{d\widetilde{P}}(B) = \frac{\exp\left(\widetilde{H}^R_t(B)\right)}{\widetilde{Z}^R_t}.$$

Now, we can define the free energy associated with this model.

Proposition 2 *For every $(\lambda, h, \beta) \in (\mathbb{R}^+)^3$, there exists a real number, denoted by $\widetilde{\Phi}(\lambda, h, \beta)$, which satisfies $\widetilde{\mathbb{P}}$ a.s. the following convergence,*

$$\lim_{t\to\infty} \frac{1}{t} \log \widetilde{Z}^R_t = \widetilde{\Phi}(\lambda, h, \beta).$$

This limit is called free energy of the model, and is constant $\widetilde{\mathbb{P}}$ a.s..

Remark 1 As in the discrete case, this convergence occurs in $\widetilde{\mathbb{L}}^1$ and $\widetilde{\Phi}$ is continuous and convex in each variable.

In [17], the proposition 2 has been proved in the case $\beta = 0$, that is to say without pinning term at the interface. In the Annex A, we give a detailed proof of the case $\beta \geq 0$. To that aim, we follow the scheme of the proof exposed in [17], and we modify some steps to take into account the presence of the β term.

Remark 2 We can define, for the continuous model also, a subset of delocalized trajectories, i.e. $\widetilde{D}_N = \{B : B_s > 0 \; \forall \; s \in [1,t]\}$. The computation of $\widetilde{\Phi}$ restricted to $\widetilde{D}_N$ gives a contribution of λh. Thus, the conditions of localization and delocalization of this continuous model are the same as in the discrete one.

It has been proved in [6] that, when $\beta = 0$, the critical curve $h_c^0(\lambda)$ of the continuous model is a straight line of slope K_c^0. It is still true when we add a pinning term. Indeed, the critical curve satisfies $\widetilde{h}_c^{\lambda\beta}(\lambda) = \lambda K_c^\beta$. Since $\widetilde{\Phi}$ is non decreasing in β, K_c^β is non decreasing in β, and we give here a short proof of the convexity of K_c^β. For that, we consider β_1 and β_2 such that $K_c^{\beta_1}$ and $K_c^{\beta_2}$ are finite. Then, we note $\alpha \in [0,1]$, and obtain

$$\widetilde{H}^{t,R}_{1,\alpha K_c^{\beta_1}+(1-\alpha)K_c^{\beta_2},\alpha\beta_1+(1-\alpha)\beta_2} = \alpha \widetilde{H}^{t,R}_{1,K_c^{\beta_1},\beta_1} + (1-\alpha)\widetilde{H}^{t,R}_{1,K_c^{\beta_2},\beta_2}.$$

We apply the Hölder's inequality (with $p = 1/\alpha$ and $q = 1/(1-\alpha)$), and we let t go to ∞, it gives

$$\begin{aligned}\widetilde{\Phi}\big(1, \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2}, \alpha\beta_1 + (1-\alpha)\beta_2\big) &\leq \alpha\widetilde{\Phi}\big(1, K_c^{\beta_1}, \beta_1\big) + (1-\alpha)\widetilde{\Phi}\big(1, K_c^{\beta_2}, \beta_2\big) \\ &\leq \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2}.\end{aligned}$$

Since $\widetilde{\Phi}(1,h,\beta) \geq h$ (see remark 2), we have

$$\widetilde{\Phi}\big(1, \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2}, \alpha\beta_1 + (1-\alpha)\beta_2\big) = \alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2},$$

which implies that

$$\alpha K_c^{\beta_1} + (1-\alpha)K_c^{\beta_2} \geq K_c^{\alpha\beta_1+(1-\alpha)\beta_2},$$

and then the convexity is proved. As a consequence, K_c^β is continuous in β as long as it is finite.

Finally we notice that the free energy of this model is also continuous convex and non decreasing in each variable.

2.5 Limit of weak coupling

2.5.1 Theorem 3

We began above to explain that the continuous model can be seen as the limit of the discrete model when λ, β and h go to 0, that is to say in the limit of weak coupling. As stated in the next theorem, the convergence occurs between the discrete and continuous free energies.

Theorem 3 *Let β, λ and h be non negative constants, and let $\Sigma = \sum_{j=-K}^{K} E\big(\gamma_1^j\big)$. We have the following convergence*

$$\lim_{a\to 0} \frac{1}{a^2}\ \Phi(a\beta, a\lambda, ah) = \widetilde{\Phi}\left(\beta\Sigma, \lambda, h\right). \tag{2.5.1}$$

Remark 3 This theorem will in fact be deduced from the next one but before, for simplicity, we define the quantities $\Psi_N(\beta,\lambda,h) = \Phi_N(\beta,\lambda,h) - \lambda h$ and $\widetilde{\Psi}_t(\beta,\lambda,h) = \widetilde{\Phi}_t(\beta,\lambda,h) - \lambda h$. Then, it is easy to notice that $\Psi_N(\beta,\lambda,h)$ converges $\mathbb{P}$ a.s. and in $\mathbb{L}^1$ to $\Psi(\beta,\lambda,h) = \Phi(\beta,\lambda,h) - \lambda h$ and similarly that $\widetilde{\Psi}_t(\beta,\lambda,h)$ converges $\widetilde{\mathbb{P}}$ a.s. and in $\widetilde{\mathbb{L}}^1$ to $\widetilde{\Psi}(\beta,\lambda,h) = \widetilde{\Phi}(\beta,\lambda,h) - \lambda h$. Therefore, to decide whether the polymer is localized or not, it suffices to compare Ψ or $\widetilde{\Psi}$ to zero.

We associate with Ψ_N the hamiltonian

$$\begin{aligned} H_N^{w,\gamma} &= \lambda \sum_{i=1}^{N} (w_i + h)\Lambda_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \ \mathbb{1}_{\{S_i = j\}} - \lambda h N \\ &= -2\lambda \sum_{i=1}^{N} (w_i + h)\Delta_i + \lambda \sum_{i=1}^{N} w_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \ \mathbb{1}_{\{S_i = j\}} \end{aligned}$$

with $\Delta_i = 1$ if $\Lambda_i = -1$ and $\Delta_i = 0$ otherwise. Moreover, $\mathbb{P}$ a.s. $\sum_{i=1}^N w_i = o(N)$ when N tends to ∞. Hence, the term $\lambda \sum_{i=1}^N w_i$ has no influence on the value of

$\Psi(\beta, \lambda, h)$, and we can delete it in the definition of $H_N^{w,\gamma}$, namely

$$H_N^{w,\gamma} = -2\lambda \sum_{i=1}^{N} (w_i + h)\Delta_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \; \mathbb{1}_{\{S_i = j\}}.$$

Similarly, $\widetilde{\Psi}_t(\beta, \lambda, h)$ is associated with

$$\widetilde{H}_t^R = -2\lambda \int_0^t \mathbb{1}_{\{B_s < 0\}} (dR_s + h ds) + \beta L_t^0,$$

and Ψ and $\widetilde{\Psi}$ are convex and continuous in each of the three variables, non decreasing in β, non increasing in h.

We insist on the fact that, proving the Theorem 3 with Φ and $\widetilde{\Phi}$ or Ψ and $\widetilde{\Psi}$ is absolutely equivalent. Now, we can introduce our next theorem.

2.5.2 Theorem 4 (and corollary 5)

We define a slightly modified hamiltonian. Let β_1 and β_2 be two non negative numbers and

$$I_1 = \{j \in \{-K, .., K\} \text{ such that } \mathbb{E}(\gamma_1^j) > 0\},$$
$$I_2 = \{j \in \{-K, .., K\} \text{ such that } \mathbb{E}(\gamma_1^j) < 0\}.$$

Then, if $\mathbb{E}(\gamma_1^j) \neq 0$, we define

$$H_N^{w,\gamma}(\beta_1, \beta_2, \lambda, h) = \beta_1 \sum_{j \in I_1} \sum_{i=1}^{N} \gamma_i^j \; \mathbb{1}_{\{S_i = j\}} + \beta_2 \sum_{j \in I_2} \sum_{i=1}^{N} \gamma_i^j \; \mathbb{1}_{\{S_i = j\}} + \lambda \sum_{i=1}^{N} (w_i + h)\Lambda_i. \tag{2.5.2}$$

The associated free energy $\Psi(\beta_1, \beta_2, \lambda, h)$ is defined as in proposition 1, and satisfies $\Psi(\beta, \lambda, h) = \Psi(\beta, \beta, \lambda, h)$. Thus, in the following, we will use the notation $\Psi(\beta_1, \beta_2, \lambda, h)$, only if $\beta_1 \neq \beta_2$. Otherwise we will use $\Psi(\beta, \lambda, h)$. Finally, let $\Sigma = \Sigma_1 + \Sigma_2$, with $\Sigma_1 = \sum_{j \in I_1} \mathbb{E}(\gamma_1^j)$ and $\Sigma_2 = \sum_{j \in I_2} \mathbb{E}(\gamma_1^j)$. Then, we can give the theorem.

Theorem 4 *If $\mathbb{E}(\gamma_1^j) \neq 0$ for every $j \in \{-K, .., K\}$, if $\beta_1 > 0$, $\beta_2 > 0$, and $(\mu_1, \mu_2) \in \mathbb{R}^2$ such that*

$$\mu_1 > \beta_1 \Sigma_1 + \beta_2 \Sigma_2 > \mu_2$$

and $\rho > 0$, $h > 0$, $h^{'} \geq 0$, $\lambda > 0$ such that $(1+\rho)h^{'} < h$, there exists $a_0 > 0$ such that for every $a < a_0$

$$\frac{1}{a^2}\Psi\left(a\beta_1, a\beta_2, a\lambda, ah\right) \leq (1+\rho)\widetilde{\Psi}\big(\mu_1, \lambda, h^{'}\big) \qquad (2.5.3)$$
$$\widetilde{\Psi}(\mu_2, \lambda, h) \leq (1+\rho)\frac{1}{a^2}\Psi\big(a\beta_1, a\beta_2, a\lambda, ah^{'}\big).$$

This result allows us to prove the convergence of the slope in 0 of the discrete critical curve towards the continuous one. We detail it in the next corollary.

Corollary 5 *For every $\beta \geq 0$, and even if for some $j \in \{-K, .., K\}$ $\mathbb{E}(\gamma_i^j) = 0$, we obtain the convergence*

$$\lim_{\lambda \to 0} \frac{h_c^{\lambda\beta}(\lambda)}{\lambda} = K_c^{\beta\Sigma}.$$

2.6 Proof of theorem 3 and corollary 5

In this section, we assume that the theorem 4 is satisfied. Its proof will be exposed in the next section.

2.6.1 Proof of corollary 5

We prove this corollary by applying the theorem 4 with particular parameters. We note $\rho = 1/n$, $\mu_1 = \beta\Sigma + 1/n$, $h = (1+2/n)K_c^{\mu_1}$, $h^{'} = K_c^{\mu_1}$, $\beta_1 = \beta_2 = \beta$, and $\lambda = 1$. For a small enough, the first inequality of theorem 4 gives

$$\frac{1}{a^2}\Psi\left(a\beta, a, a\Big(1+\frac{2}{n}\Big)K_c^{\beta\Sigma+\frac{1}{n}}\right) \leq \Big(1+\frac{1}{n}\Big)\widetilde{\Psi}\left(\beta\Sigma + \frac{1}{n}, 1, K_c^{\beta\Sigma+\frac{1}{n}}\right). \qquad (2.6.1)$$

By definition of $K_c^{(\cdot)}$, the right hand side of (2.6.1) is equal to zero. Therefore, we have the inequality

$$\liminf_{a \to \infty} \frac{h_c^{a\beta}(a)}{a} \leq \Big(1+\frac{2}{n}\Big)K_c^{\beta\Sigma+\frac{1}{n}}. \qquad (2.6.2)$$

Then, we let n go to ∞ and since K_c is continuous in β, the inequality (2.6.2) becomes $\liminf_{a\to\infty} h_c^{a\beta}(a)/a \leq K_c^{\beta\Sigma}$.

It remains to prove the opposite inequality. To that aim, we apply the second inequality of theorem 4 with the parameters $\rho = 1/n$, $\mu_2 = \beta\Sigma - 1/n$, $h = (1 + 2/n)K_c^{\mu_2}$, $h' = K_c^{\mu_2}$, $\beta_1 = \beta_2 = \beta$, and $\lambda = 1$. For a small enough we obtain

$$\widetilde{\Psi}\left(\beta\Sigma - \frac{1}{n}, 1, K_c^{\beta\Sigma - \frac{1}{n}} - \frac{1}{n}\right) \leq \frac{1 + 1/n}{a^2}\widetilde{\Psi}\left(a\beta, a, \frac{a}{1 + 1/n}\left(K_c^{\beta\Sigma - \frac{1}{n}} - \frac{2}{n}\right)\right). \quad (2.6.3)$$

Since the left hand side of (2.6.3) is strictly positive, we obtain

$$\limsup_{a \to \infty} \frac{h_c^{a\beta}(a)}{a} \geq \frac{K_c^{\beta\Sigma - \frac{1}{n}} - 2/n}{1 + 1/n}. \quad (2.6.4)$$

Finally, since $K_c^{(.)}$ is continuous in β, we let n go to ∞ and it completes the proof of the corollary.

2.6.2 Proof of theorem 3

Now, we prove that theorem 3 is a consequence of theorem 4. This proof is divided into 3 steps. In the first one, we show that theorem 3 is satisfied when $\lambda > 0$, $h > 0$, and every pinning price γ_1^j has a non zero average. In the second step, we prove that the result can be extended to the case in which some γ_1^j have a zero average. Finally, in the last step, we will consider the case $h = 0$.

Step I

First, we consider the case: $\lambda > 0$, $h > 0$ and $\mathbb{E}(\gamma_1^j) \neq 0$ for every $j \in \{-K, .., K\}$. We can apply the first inequality of Theorem 4 with the choices $\rho = 1/n$, $h' = h/(1 + 1/n)^2$ and $\beta_1 = \beta_2 = \beta$, $\mu_1(v) = \beta(1 + 1/v)\Sigma_1 + \beta(1 - 1/v)\Sigma_2$ (n and $v \in \mathbb{N} - \{0\}$). It gives, for every integers n and v strictly positive, that

$$\limsup_{a \to 0} \frac{1}{a^2}\Psi\left(a\beta, a\lambda, ah\right) \leq \left(1 + \frac{1}{n}\right)\widetilde{\Psi}\left(\mu_1(v), \lambda, \frac{h}{(1 + 1/n)^2}\right). \quad (2.6.5)$$

At this stage, we let successively n and v go to ∞, and, by continuity of $\widetilde{\Psi}$ in h and β we obtain $\limsup_{a \to 0} 1/a^2\Psi\left(a\beta, a\lambda, ah\right) \leq \widetilde{\Psi}(\beta\Sigma, \lambda, h)$. The lower bound is proved with the second inequality of theorem 4. Indeed, if we choose $\mu_2 = \beta(1 - 1/v)\Sigma_1 +$

$\beta(1+1/v)\Sigma_2$ and keep the other notations, we obtain

$$\widetilde{\Psi}\left(\mu_2, \lambda, h\left(1+\frac{1}{n}\right)^2\right) \leq \left(1+\frac{1}{n}\right) \liminf_{a\to 0} \frac{1}{a^2}\Psi\left(a\beta, a\lambda, ah\right). \tag{2.6.6}$$

We let n go to ∞, and after, we let v go to ∞. In that way, we can conclude that $\lim_{a\to 0} 1/a^2\Psi\left(a\beta, a\lambda, ah\right) = \widetilde{\Psi}(\beta\Sigma, \lambda, h)$ which implies the theorem 3.

Step II

Now, we prove the theorem 3 when there exists $j \in \{-K, .., K\}$ such that $\mathbb{E}\big(\gamma_1^j\big) = 0$. For that, we choose $\mu > 0$ and small enough, such that, $\mathbb{E}(\gamma_i^j + \mu) \neq 0$ for every $j \in \{-K, .., K\}$. With these new variables we can use the previous case with a new Σ, i.e. $\Sigma_\mu = \Sigma + (2K+1)\mu$. Therefore, we can apply the theorem 3 and since the free energy Φ_μ associated with the new variables γ_i^j is larger than Φ, we obtain

$$\limsup_{a\to 0} \frac{1}{a^2}\Phi(a\beta, a\lambda, ah) \leq \lim_{a\to 0} \frac{1}{a^2}\Phi_\mu(a\beta, a\lambda, ah) = \widetilde{\Phi}(\beta(\Sigma + (2K+1)\mu), \lambda, h).$$

As $\widetilde{\Phi}$ is continuous in β, we let μ go to 0 and write $\limsup_{a\to 0} 1/a^2\Phi(a\beta, a\lambda, ah) \leq \widetilde{\Phi}(\beta\Sigma, \lambda, h)$. Now, it suffices to do the same thing with $-\mu < 0$, and we obtain the other inequality, namely

$$\liminf_{a\to 0} \frac{1}{a^2}\Phi(a\beta, a\lambda, ah) \leq \lim_{\mu\to 0} \widetilde{\Phi}(\beta(\Sigma - (2K+1)\mu), \lambda, h) = \widetilde{\Phi}(\beta\Sigma, \lambda, h).$$

We can say that $\liminf_{a\to 0} \frac{1}{a^2}\Phi(a\beta, a\lambda, ah) = \widetilde{\Phi}(\beta\Sigma, \lambda, h)$

Step III

It remains to show the theorem 3 when $h = 0$. Since Ψ and $\widetilde{\Psi}$ are non increasing in h, the theorem 3, with strictly positive parameters, implies

$$\liminf_{a\to 0} \frac{1}{a^2}\Psi(a\beta, a\lambda, 0) \geq \widetilde{\Psi}(\beta\Sigma, \lambda, 0).$$

To prove the opposite inequality, we just notice that Φ is non decreasing in h. Effectively

$$\left.\frac{\partial\Phi}{\partial h}\right|_{(\beta,\lambda,0)} = \lim_{N\to\infty} \mathbb{E}\left(\sum_{\Lambda_1,..,\Lambda_N} P(\Lambda_1, .., \Lambda_N)\ \frac{\lambda\sum_{i=1}^N \Lambda_i}{N}\ \exp\left(\lambda\sum_{i=1}^N w_i\Lambda_i + \beta..\right)\right), \tag{2.6.7}$$

the $\{w_i\}_{i=1,2,..}$ are symmetric and the random walk also. Thus, we can transform w_i in $-w_i$, and $(\Lambda_1,..,\Lambda_N)$ in $(-\Lambda_1,..,-\Lambda_N)$, without changing (2.6.7). It gives

$$\frac{\partial\Phi}{\partial h}\Big|_{\beta,..} = \lim_{N\to\infty} \mathbb{E}\Bigg(\sum_{\Lambda_1,..,\Lambda_N} P(-\Lambda_1,..,-\Lambda_N)\ \frac{-\lambda\sum_{i=1}^N \Lambda_i}{N}\ \exp\Big(-\lambda\sum_{i=1}^N -w_i\Lambda_i+\beta..\Big)\Bigg)$$

$$= \mathbb{E}\Bigg(\sum_{\Lambda_1,..,\Lambda_N} P(\Lambda_1,..,\Lambda_N)\ \frac{-\lambda\sum_{i=1}^N \Lambda_i}{N}\ \exp\Big(\lambda\sum_{i=1}^N w_i\Lambda_i+\beta..\Big)\Bigg) = -\frac{\partial\Phi}{\partial h}\Big|_{(\beta,\lambda,0)}.$$

This derivative is equal to 0 and Φ is convex in h, hence, Φ is non-decreasing in h. Now, in (2.6.5), we let n and v go to ∞ and we add λh on both sides. We obtain, for $h>0$

$$\limsup_{a\to 0} 1/a^2\Phi\,(a\beta, a\lambda, ah) \le \widetilde{\Phi}(\beta\Sigma,\lambda,h). \tag{2.6.8}$$

Since Φ is non-decreasing in h, the inequality (2.6.8) implies,

$$\limsup_{a\to 0} 1/a^2\Phi\,(a\beta, a\lambda, 0) \le \widetilde{\Phi}(\beta\Sigma,\lambda,h).$$

Then, we let h go to zero, and the proof of theorem 3 is completed.

2.7 Proof of theorem 4

We first prove a lemma which will be very useful in the proof of the theorem.

2.7.1 Technical Lemma

Lemma 6 *For every $K\in\mathbb{N}$ and every $(f_{-K}, f_{-K+1}, ..., f_K)$ in $(\mathbb{R}^+)^{2K+1}$ the following convergence occurs:*

$$\lim_{N\to\infty} E\Bigg(\exp\Bigg(\frac{1}{\sqrt{N}}\sum_{j=-K}^{K} f_j \sum_{i=1}^{N} \mathbb{1}_{\{S_i=j\}}\Bigg)\Bigg) = E\Bigg(\exp\Bigg(\Big(\sum_{j=-K}^{K} f_j\Big)\ l_1^0\Bigg)\Bigg), \tag{2.7.1}$$

where l_1^0 is the local time in 0 of a Brownian motion $(B_s)_{s\ge 0}$ between 0 and 1.

Proof of the lemma

First, we prove the following intermediate result. For every $K\in\mathbb{N}$

$$\frac{1}{\sqrt{N}}\sum_{j=-K}^{K} f(j)\sum_{i=1}^{N}\mathbb{1}_{\{S_i=j\}} \to^{\text{Law}}_{N\to\infty} \Big(\sum_{j=-K}^{K} f_j\Big)\ l_1^0. \tag{2.7.2}$$

For simplicity, we only prove that $1/\sqrt{N}\left(\sum_{i=1}^N \mathbb{1}_{\{S_i=0\}}, \sum_{i=1}^N \mathbb{1}_{\{S_i=1\}}\right)$ converges in law to (l_1^0, l_1^0) as N tends to ∞. The proof for $2K+1$ levels is exactly the same. For this convergence in law, we use a result of [34], saying that we can build, on the same probability space $(\Omega, \mathcal{A}, P)$, a simple random walk $(S_i)_{i\geq 0}$ and a Brownian motion $(B_t)_{t\geq 0}$ such that a.s. in $w \in \Omega$

$$\lim_{n\to\infty} \sup_{j\in\{0,1\}} \frac{1}{\sqrt{n}} \left| U(j,n) - L(j,n) \right| = 0 \tag{2.7.3}$$

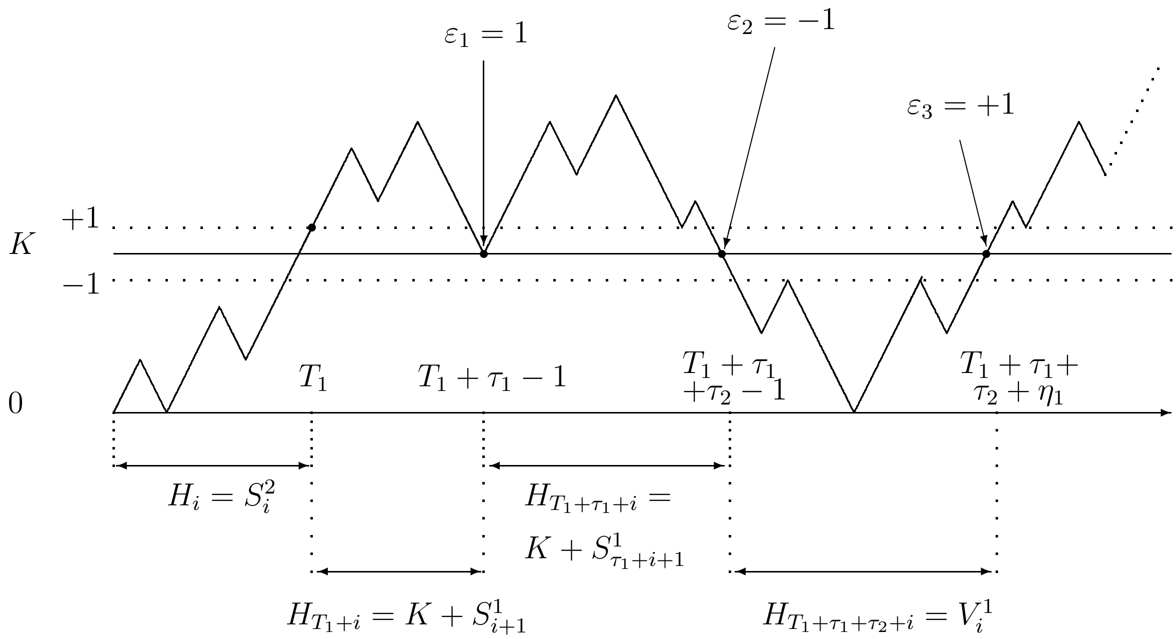
with $U(j,n) = \sum_{i=1}^n \mathbb{1}_{\{S_i=j\}}$ and $L(x,n)$ the local time in x of B between 0 and n. The equation (2.7.3) implies that $1/\sqrt{n}\ (U(0,n) - L(0,n))$ and $1/\sqrt{n}\ (U(1,n) - L(1,n))$ go a.s. to 0 as n tends to ∞. Therefore, the proof of (2.7.2) will be completed if we show that $1/\sqrt{n}\ (L(0,n), L(1,n))$ converges in law to $(l(0,1), l(0,1))$. By the scaling property of the Brownian motion, we obtain that, for every $n \geq 1$, $1/\sqrt{n}\ (L(0,n), L(1,n))$ has the same law as $(l(0,1), l(1/\sqrt{n}, 1))$. Thus, since $l_1(x)$ is a.s. continuous in 0, we obtain immediately the a.s. convergence of $(l_1(0), l_1(1/\sqrt{n}))$ towards $(l_1(0), l_1(0))$. This a.s. convergence implies the convergence in law and (2.7.2) is proved.

Now, since the function $\exp(x)$ is continuous, (2.7.2) gives us the convergence in law of $K_N = \exp\left(1/\sqrt{N} \sum_{j=-K}^K f_j \sum_{i=1}^N \mathbb{1}_{\{S_i=j\}}\right)$ towards $\exp\left(\left(\sum_{j=-K}^K f_j\right) l_1^0\right)$ as N tends to ∞. The uniform integrability of the family $(K_N)_{N\geq 1}$ will therefore be sufficient to complete the proof of lemma 6. To that aim we will use the following construction.

Let $(S_n^1)_{n\geq 0}$ be a reflected simple random walk, and denote by k_N the number of return to the origin before time N and $\tau_1, \tau_2, ..., \tau_{k_N}, N - \tau_1 - ... - \tau_{k_N}$ the length of the corresponding excursions out of the origin until time N. Independently, we let $(S_n^2)_{n\geq 0}$ be a reflected S.R.W. starting at $S_0 = 0$ and we denote by T_1 her first passage time in $K+1$. Next, for every $i \geq 1$, we let $(V_n^i)_{n\geq 0}$ be a reflected simple random walk, independent of all the other ones, and satisfying $V_0^i = K - 1$. We denote by η_i the first passage time in $K+1$ of V_n^i. Finally, we define a sequence $(\epsilon_i)_{i\geq 1}$ of independent Bernoulli trials satisfying $P(\epsilon_1 = \pm 1) = 1/2$.

Now, we build a new process (see Fig 2), denoted by $(H_i)_{i\geq 1}$, such that $H_i = S_i^2$ for every $i = 0, 1, .., T_1$. Thus, $H_{T_1} = K + 1$, and we note $H_{T_1+i} = K + S_{i+1}^1$ for every $i = 0, .., \tau_1 - 1$, so that $H_{T_1+\tau_1-1} = K$. At this stage, either $\epsilon_1 = 1$ and $H_{T_1+\tau_1+i} = K + S^1_{\tau_1+i+1}$ for every $i = 0, .., \tau_2 - 1$ and $H_{T_1+\tau_1+\tau_2-1} = K$, or $\epsilon_1 = -1$ and $H_{T_1+\tau_1+i} = V_1^i$ for every $i = 0, .., \eta_1$ and $H_{T_1+\tau_1+\eta_1} = K + 1$. We go on like this, that is to say, after the j^{th} excursion of H above K, if $\varepsilon_j = 1$, H describes above K the next excursion of $(S_n^1)_{n\geq 0}$, otherwise H describes an excursion between 0 and K until it reaches $K + 1$. At this moment, H describes above K the next excursion of $(S_n^1)_{n\geq 0}$ and so on...

Fig. 2:



We denote by k_N^1 the number of excursions between 0 and K done by H before time N, and by j_N the number of steps that H does between 0 and K before N. It comes easily that $k_N^1 \leq k_N$, and that

$$j_N \leq k_N + \sum_{j=1}^{k_N^1} \eta_j + T_1 \leq k_N + \sum_{j=1}^{k_N} \eta_j + T_1. \tag{2.7.4}$$

We note $F = \max\{f_{-K}, f_{-K+1}, ..., f_K\}$, and to prove the uniform integrability of K_N, it suffices to show that $V_N = \exp\left(Fj_N/\sqrt{N}\right)$ is bounded from above in L^2 norm, independently of N. By definition $(\zeta_i)_{i\geq 1}, T_1$ and k_N are independent, and, using the Jensen's inequality we obtain

$$\begin{aligned} E\left(V_N^2\right) &= E\left(\exp\left(\frac{2Fk_N}{\sqrt{N}} + \frac{2F}{\sqrt{N}}\sum_{j=1}^{k_N}\eta_j + \frac{2FT_1}{\sqrt{N}}\right)\right) \\ &= E\Bigg(E\left(\exp\left(\frac{2F\eta_1}{\sqrt{N}}\right)\right)^{k_N} E\left(\exp\left(\frac{2FT_1}{\sqrt{N}}\right)\right)\ \exp\left(\frac{2Fk_N}{\sqrt{N}}\right)\Bigg) \\ &\leq E\left(\exp\left(\frac{k_N}{\sqrt{N}}\Big(\log E\left(\exp\left(2F\eta_1\right)\right) + 2F\Big)\right)\right) E\left(\exp\left(2FT_1\right)\right). \end{aligned} \tag{2.7.5}$$

To complete our proof, it just remains to prove that for every $b > 0$ the sequence $\left(E\left(\exp(bK_N/\sqrt{N})\right)\right)_{N\geq 0}$ is bounded from above independently of N. To that aim, we notice that $K_N \leq K_{2N} \leq N$ and write the obvious inequality

$$E\left(\exp(bK_{2N}/\sqrt{N})\right) \leq \sum_{k=0}^{\left[\sqrt{N/2}\right]} e^{\sqrt{2}\,b(k+1)} P\left(K_{2N} \in \left[k\sqrt{2N}, (k+1)\sqrt{2N}\right[\right). \tag{2.7.6}$$

With [13] we can find an upper bound of $P\left(K_{2N} \in \left[k\sqrt{2N}, (k+1)\sqrt{2N}\right[\right)$. Indeed, for every $k \leq \left[\sqrt{N/2}\right]$ we obtain

$$\begin{aligned} &P\left(K_{2N} \in \left[k\sqrt{2N}, (k+1)\sqrt{2N}\right[\right) \\ &\qquad \leq \sum_{j=\left[k\sqrt{2N}\right]}^{\max\left(\left[(k+1)\sqrt{2N}\right], N\right)} P\left(S_{2N} = 0\right) \frac{\left(1-\frac{1}{N}\right)...\left(1-\frac{j-1}{N}\right)}{\left(1-\frac{1}{2N}\right)...\left(1-\frac{j-1}{2N}\right)}. \end{aligned} \tag{2.7.7}$$

We know that the function $\log(1-x)+x$ is decreasing on $[0,1)$. Hence, for every j in $\left\{\left[k\sqrt{2N}\right], .., \max\left(\left[(k+1)\sqrt{2N}\right], N\right)\right\}$, we have, $\log\left(1-j/N\right) - \log\left(1-j/2N\right) \leq -j/2N$. Therefore,

$$\frac{\left(1-\frac{1}{N}\right)...\left(1-\frac{j-1}{N}\right)}{\left(1-\frac{1}{2N}\right)...\left(1-\frac{j-1}{2N}\right)} \leq \exp\left(\sum_{i=1}^{j-1} -\frac{i}{2N}\right) = \exp\left(-\frac{j(j-1)}{4N}\right) \leq \exp\left(-\frac{(k-1)^2}{2}\right).$$

Moreover $\left[(k+1)\sqrt{2N}\right] - \left[k\sqrt{2N}\right] \leq \sqrt{2N}+1$ and there exists a constant $c > 0$ such that, $P\left(S_{2N} = 0\right) \leq c/\sqrt{2N}$ for every $N \geq 1$. That is why, the equation (2.7.7) becomes

$$P\left(K_{2N} \in \left[k\sqrt{2N}, (k+1)\sqrt{2N}\right]\right) \leq 2c\exp\left(-(k-1)^2/2\right).$$

This results allows us to rewrite (2.7.6) as

$$E\left(\exp(bK_{2N}/\sqrt{N})\right) \leq \sum_{k=0}^{\infty} 2ce^{b(k+1)}e^{-\frac{(k-1)^2}{2}},$$

and the r.h.s. of this inequality is the sum of a convergent series. Therefore, the proof of lemma 4 is completed.

2.7.2 Coarse graining

We define a relation (previously introduced in [6]), which is very useful to carry out the proof.

Definition 7 *let $f_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2)$ and $g_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2)$ be real-valued functions. The relation $f << g$ occurs if for every $\beta_3 > \beta_1$, $\beta_2 > \beta_4$, $\rho > 0$, and $h > h^{'} \geq 0$ satisfying $(1+\rho)h^{'} < h$, there exists δ_0 such that for $0 < \delta < \delta_0$ there exists $\varepsilon_0(\delta)$ such that for $0 < \varepsilon < \varepsilon_0$ there exists $a_0(\varepsilon,\delta)$ satisfying*

$$\limsup_{t\to\infty} f_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) - (1+\rho)g_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho),h^{'},\beta_3,\beta_4) \leq 0 \quad \text{for} \quad 0 < a < a_0 \tag{2.7.8}$$

In this proof we consider some functions of the form

$$F_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) = \mathbb{E}\Big(\frac{1}{t}\log E\big(\exp(aH_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2))\big)\Big),$$

and we denote

- $F^1_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) = \frac{1}{a^2}\Psi_{[t/a^2]}(a\beta_1,a\beta_2,a,ah)$

- $F^7_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) = \widetilde{\Psi}_t(\beta_1\Sigma_1+\beta_2\Sigma_2,1,h)$.

The proof of (2.5.3) will consist in showing that $F^1 << F^7$ and $F^7 << F^1$ (denoted by $F^1 \sim F^7$). To that aim, we will create the intermediate functions $F_2, ..., F_6$ associated with slight modifications of the hamiltonian to transform, step by step, the discrete hamiltonian into the continuous one. As the relation $\sim$ is transitive, we will prove at every step that $F^i \sim F^{i+1}$, to conclude finally that $F^1 \sim F^7$.

Scheme of the proof

To show that $F^i << F^{i+1}$ we notice that, by the Hölder's inequality, if $H^i = H^I + H^{II}$, F^i is bounded from above,

$$F^i_{t,\varepsilon,\delta}(a,h,\beta) \leq \frac{1}{t(1+\rho)}\mathbb{E}\left(\log E\left(\exp(a(1+\rho)H^I)\right)\right) \\ + \frac{1}{t\,(1+\rho^{-1})}\mathbb{E}\left(\log E\left(\exp(a(1+\rho^{-1})H^{II})\right)\right).$$

Thus, if we choose

$$H^I = H^{i+1}_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho), h', \beta_3, \beta_4),$$

we obtain

$$F^i_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2)) - (1+\rho)F^{i+1}_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho), h', \beta_3, \beta_4)) \\ \leq \frac{1}{t(1+\rho^{-1})}\mathbb{E}\left(\log E\left(\exp(a(1+\rho^{-1})H^{II})\right)\right).$$

Then, it suffices to prove that $\limsup_{t\to\infty} 1/t \log \mathbb{E}E\left(\exp\left(a(1+\rho^{-1})H^{II}\right)\right) \leq 0$ for a, ϵ and δ small enough.

STEP I

The first hamiltonian that we consider in this proof is given by

$$H^{(1)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) = -2\sum_{i=1}^{t/a^2}\Delta_i(w_i + ah) + \beta_1\sum_{j=I_1}\sum_{i=1}^{t/a^2}\gamma_i^j\ \mathbb{1}_{\{S_i=j\}} + \beta_2\sum_{j=I_2}\sum_{i=1}^{t/a^2}\gamma_i^j\ \mathbb{1}_{\{S_i=j\}},$$

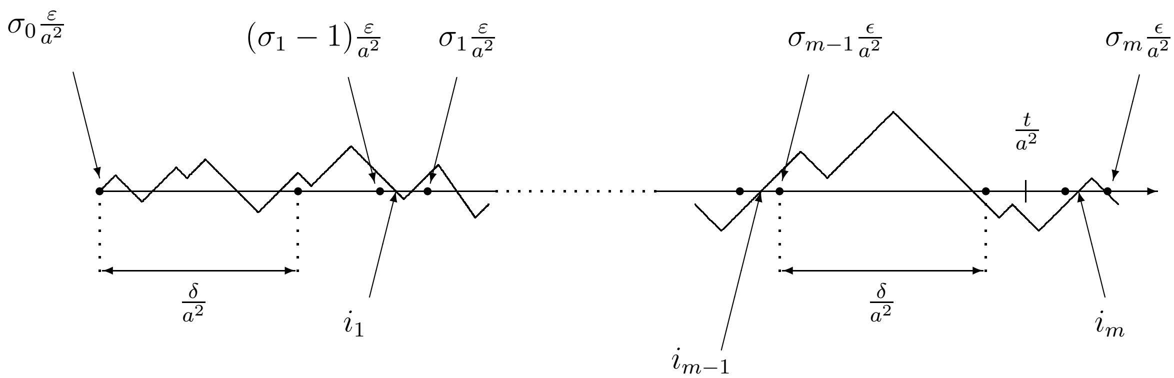
with $\Delta_i = 1$ if $\Lambda_i = -1$ and $\Delta_i = 0$ if $\Lambda_i = 1$.

Let us define some notations to build the following hamiltonians (see Fig. 3).

- $\sigma_0 = 0, \quad i_0^v = 0$ and $i_{k+1}^v = \inf\{\, n > \sigma_k \varepsilon/a^2 + \delta/a^2 : S_n = 0\}$
- $m \;=\; \inf\{k \geq 1 : i_m > t/a^2\}$
- $i_k = i_k^v$ for $k < m$ and $i_m = t/a^2$
- $\sigma_{k+1} = \inf\{\, n \geq 0 : i_{k+1} \in \,](n-1)\varepsilon/a^2, n\varepsilon/a^2]\}$,
- $\overline{I}_k \;= \,](\sigma_{k-1}+1)\,\varepsilon/a^2, \sigma_k\,\varepsilon/a^2\,] \cap \,]0, t/a^2], \;\; s_{k+1} = \operatorname{sign}(\,S_{i_{k+1}-1}\,)$

We give an example of this construction with the Fig.3.

Fig. 3:



Now, we define the first transformation of the Hamiltonian

$$H^{(2)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) \;=\; -\,2\sum_{k=1}^{m} s_k\left(\sum_{i\in\overline{I}_k} w_i + ah|\overline{I}_k|\right) \;+\; \beta_1 \sum_{j\in I_1}\sum_{i=1}^{t/a^2} \gamma_i^j \;\mathbb{1}_{\{S_i=j\}}$$
$$+\,\beta_2 \sum_{j\in I_2}\sum_{i=1}^{t/a^2} \gamma_i^j \;\mathbb{1}_{\{S_i=j\}},$$

we want to show that $F_1 << F_2$. For that, we denote

$$H^{(II)} \;=\; -2\sum_{i=1}^{t/a^2} \Delta_i(w_i+ah) \;+2\sum_{k=1}^{m} s_k\left(\sum_{i\in\overline{I}_k} w_i + a(1+\rho)h^{'}|\overline{I}_k|\right)$$
$$+\,(\beta_1-\beta_3)\sum_{j\in I_1}\sum_{i=1}^{t/a^2} \gamma_i^j \;\mathbb{1}_{\{S_i=j\}} + (\beta_2-\beta_4)\sum_{j\in I_2}\sum_{i=1}^{t/a^2} \gamma_i^j \;\mathbb{1}_{\{S_i=j\}}, \qquad (2.7.9)$$

and it remains to prove that

$$\limsup_{t\to\infty} \frac{1}{t} \log E\mathbb{E}(\exp(a(1+\rho^{-1})H^{(II)})) \le 0. \tag{2.7.10}$$

We integrate over the disorder γ and the second and third terms of the right hand side of (2.7.9) becomes in (2.7.10)

$$\exp\Big(\sum_{j\in I_1}\sum_{i=1}^{t/a^2} \log\mathbb{E}\big((\beta_1-\beta_3)a(1+\rho^{-1})\gamma_i^j\big)\ \mathbb{1}_{\{S_i=j\}}\Big) \\ \times\exp\Big(\sum_{j\in I_2}\sum_{i=1}^{t/a^2} \log\mathbb{E}\big((\beta_2-\beta_4)a(1+\rho^{-1})\gamma_i^j\big)\ \mathbb{1}_{\{S_i=j\}}\Big). \tag{2.7.11}$$

Since $\mathbb{E}\big(\exp(\lambda|\gamma_1^j|)\big) < \infty$ for every $j \in \{-K,..,K\}$ and $\lambda > 0$, we can write a first order development of $\log\mathbb{E}\big(\exp\big(Aa\gamma_1^j\big)\big)$ when a tends to 0. It gives

$$\log\mathbb{E}\big(\exp\big(Aa\gamma_1^j\big)\big) = Aa\mathbb{E}\big(\gamma_i^j\big) + o(a). \tag{2.7.12}$$

We assume in this proof that $\mathbb{E}\big(\gamma_1^j\big) \neq 0$ for every $j \in \{-K,..,K\}$ (see the assumptions of theorem 4), so that $\{-K,..,K\} = I_1 \cup I_2$. For every $i \in I_1$, $\mathbb{E}\big(\gamma_1^j\big) > 0$, and $\beta_1-\beta_3 < 0$. Thus, by (2.7.12), we obtain, for a small enough, that

$$\sum_{j\in I_1}\sum_{i=1}^{t/a^2} \log\mathbb{E}\big((\beta_1-\beta_3)a(1+\rho^{-1})\gamma_i^j\big)\ \mathbb{1}_{\{S_i=j\}} \le 0. \tag{2.7.13}$$

The sum over I_2 satisfies the same inequality for a small enough because $\beta_2-\beta_4 > 0$ and $\mathbb{E}\big(\gamma_i^j\big) < 0$ when $j \in I_2$. Therefore, we can delete (2.7.11) in (2.7.10), and it remains to prove that

$$\limsup_{t\to\infty} \frac{1}{t} \log E\mathbb{E}(\exp(a(1+\rho^{-1})H^{(II)})) \le 0$$

with

$$
\begin{aligned}
H^{(II)} =& -2\sum_{i=1}^{t/a^2} \Delta_i(w_i+ah) \;+2\sum_{k=1}^{m} s_k\left(\sum_{i\in\overline{I}_k} w_i + a(1+\rho)h^{'}|\overline{I}_k|\right)\\
H^{(II)} =& -2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\overline{I}_k} w_i\,(\Delta_i - s_k) - 2a(1+\rho)h^{'}\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\overline{I}_k}(\Delta_i - s_k)\\
&-2a(h-(1+\rho)h^{'})\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\overline{I}_k}\Delta_i.
\end{aligned}
$$

Thus, we integrate over the disorder w which is independent of the random walk. But, since $\mathbb{E}(w_i) = 0$ and $\mathbb{E}(\exp(\lambda|w_1|)) < \infty$ for every $\lambda > 0$, a second order expansion gives that for every $c \in \mathbb{R}$ there exists $A > 0$ such that for a small enough

$$
\log \mathbb{E}\left(\exp(c\,a\,w_i\;(\Delta_i - s_k))\right) \le Aa^2|\Delta_i - s_k|. \tag{2.7.14}
$$

Finally, we have to prove, for $A > 0$ and $B > 0$, that

$$
\limsup_{t\to\infty}\frac{1}{t}\log E\left(\exp\left(Aa^2\sum_{k=1}^{m_{t/a^2}}\sum_{i\in\overline{I}_k}\mid s_k-\Delta_i\mid -Ba^2\sum_{i=1}^{t/a^2}\Delta_i\right)\right)\le 0. \tag{2.7.15}
$$

This is explicitly proved in [6] (page 1355), and completes the step 1 because the proof of $F_2 << F_1$ is very similar and consists essentially in showing (2.7.15).

STEP II

In this step we aim at transforming the disorder w into a sequence $(\hat{w}_i)_{i\ge 1}$ of independent random variables of law $\mathcal{N}_{0,1}$. To that aim, we use a coupling method developed in [33] to redefine for every $j \in \mathbb{N}\setminus\{0\}$ the variables $(w_i)_{i\in\{(j-1)\varepsilon/a^2+1,..,j\varepsilon/a^2\}}$ and to define on the same probability space, some independent variables of law $\mathcal{N}_{0,1}$, denoted by $(\hat{w}_i)_{i\in\{(j-1)\varepsilon/a^2+1,..,j\varepsilon/a^2\}}$, such that for every $p > 2$ and $x > 0$

$$
\mathbb{P}\left(\left|\sum_{i=(j-1)\varepsilon/a^2+1}^{j\varepsilon/a^2} w_i - \hat{w}_i\right| \ge x\right) \le \frac{(Ap)^p\;\varepsilon}{x^p\;a^2}\;\mathbb{E}\left(w_1^p\right). \tag{2.7.16}
$$

These constructions are made independently on every blocs $\{(j-1)\varepsilon/a^2+1,..,j\varepsilon/a^2\}$. Thus, we can form the third hamiltonian as follow

$$\begin{aligned}H^{(3)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) \;=\; &-2\sum_{k=1}^{m} s_k\left(\sum_{i\in\overline{I}_k}\hat{w}_i + ah|\overline{I}_k|\right) \;+\; \beta_1\sum_{j\in I_1}\sum_{i=1}^{t/a^2}\gamma_i^j\ \mathbb{1}_{\{S_i=j\}}\\ &+\beta_2\sum_{j\in I_2}\sum_{i=1}^{t/a^2}\gamma_i^j\ \mathbb{1}_{\{S_i=j\}}.\end{aligned}$$

To prove that $F^2 << F^3$, we need the H^{II} hamiltonian. It takes the value

$$H^{II} = H^{(2)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) - H^{(3)}_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho),h',\beta_3,\beta_4). \qquad (2.7.17)$$

As in step I we delete the pinning term (see (2.7.13))

$$(\beta_1-\beta_3)\sum_{j\in I_1}\sum_{i=1}^{t/a^2}\gamma_i^j\ \mathbb{1}_{\{S_i=j\}} + (\beta_2-\beta_4)\sum_{j\in I_2}\sum_{i=1}^{t/a^2}\gamma_i^j\ \mathbb{1}_{\{S_i=j\}},$$

and it suffices to consider

$$\begin{aligned}H^{II} &= -2\sum_{k=1}^{m} s_k\sum_{i\in\overline{I}_k}(w_i-\hat{w}_i) + 2a\sum_{k=1}^{m} s_k(h-(1+\rho)h')|\overline{I}_k|\\ &\le 2\sum_{k=1}^{m} s_k\left(\sum_{j=\sigma_{k-1}+1}^{\sigma_k}\left|\sum_{i=j\varepsilon/a^2+1}^{(j+1)\varepsilon/a^2} w_i-\hat{w}_i\right| - (h-(1+\rho)h')\frac{\varepsilon}{a}\right).\end{aligned}$$

Now, we want to prove that $\limsup_{t\to\infty} 1/t\log\mathbb{E}E\left(\exp\left(a(1+\rho^{-1})H^{II}\right)\right)\le 0$. By independence of $(w,\hat{w})$ on each blocs $\{(j-1)\varepsilon/a^2+1,..,j\varepsilon/a^2\}$, it suffices to show that for every $C>0$ and $B>0$

$$\mathbb{E}\left(\exp(Ca\left|\sum_{i=1}^{\varepsilon/a^2} w_i-\hat{w}_i\right| - B\varepsilon\right) \le 1 \text{ for } \varepsilon, \text{ and } a \text{ small enough..} \qquad (2.7.18)$$

We prove this point as follow,

$$\begin{aligned}\mathbb{E}\left(\exp(Ca\left|\sum_{i=1}^{\varepsilon/a^2} w_i-\hat{w}_i\right|\right) \le \sum_{k=N}^{+\infty}\exp\left(Ca(k+1)\frac{\varepsilon}{\sqrt{a}}\right)\mathbb{P}\left(\left|\sum_{i=1}^{\varepsilon/a^2} w_i-\hat{w}_i\right|\ge k\frac{\varepsilon}{\sqrt{a}}\right)\\ (2.7.19)\\ +\exp\left(CN\sqrt{a}\varepsilon\right).\end{aligned}$$

By using (2.7.16) and the fact that $\mathbb{E}(w_1^k) \leq R^k$, we obtain that for every j and $k \geq 1$

$$\mathbb{P}\left(\left|\sum_{i=(j-1)\varepsilon/a^2+1}^{j\varepsilon/a^2} w_i - \hat{w}_i\right| \geq \frac{k\varepsilon}{\sqrt{a}}\right) \leq \frac{(AR\sqrt{a})^k}{\varepsilon^{k-1}\, a^2}. \tag{2.7.20}$$

We consider (2.7.19) with $N = 5$, and we use (2.7.20) to obtain

$$\begin{aligned}\mathbb{E}\left(\exp(Ca\left|\sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i\right|\right) \leq & \exp(5C\sqrt{a}\varepsilon) \\ & + \varepsilon\, \frac{\exp\left(C\sqrt{a}\varepsilon\right)}{a^2} \sum_{k=5}^{+\infty} \left(\exp\left(C\sqrt{a}\varepsilon\right) \frac{AR\sqrt{a}}{\varepsilon}\right)^k.\end{aligned} \tag{2.7.21}$$

Therefore, for $\varepsilon > 0$ fixed, there exists $K(\varepsilon, a) > 0$ that tends to zero when a tends to zero, and satisfy

$$\mathbb{E}\left(\exp(Ca\left|\sum_{i=1}^{\varepsilon/a^2} w_i - \hat{w}_i\right|\right) \leq (1 + K(\varepsilon, a)) \quad \exp(5C\varepsilon\sqrt{a}).$$

This implies (2.7.18), and completes the step 2 because the proof of $F^3 << F^2$ is exactly the same.

STEP III

In this step, we aim at making a link between the discrete and the continuous model. For that, we take into account the number of return to the origin of the R.W., and the local time of the Brownian motion. We define, independently of the random walk, a sequence $\left(l_1^k\right)_{k\geq 0}$ of independent local times of Brownian motion between 0 and 1. The law of this sequence is denoted by M. Then, we build the new hamiltonian

$$H^{(4)}_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2) = -2\sum_{k=1}^{m} s_k \left(\sum_{i\in\overline{I}_k} \hat{w}_i + ah|\overline{I}_k|\right) + \frac{(\beta_1\Sigma_1 + \beta_1\Sigma_1)\sqrt{\delta}}{a} \sum_{k=1}^{m} l_1^k. \tag{2.7.22}$$

To prove that $F_2 << F_3$, we consider

$$\begin{aligned}H^{(II)} &= -2a(h-(1+\rho)h')\sum_{k=1}^{m} s_k \mid \overline{I}_k \mid +\beta_1 \sum_{k=1}^{m}\sum_{j\in I_1}\sum_{i=i_{k-1}+1}^{i_k} \gamma_i^j \, 1\!\!1_{\{S_i=j\}} \\ &\quad + \beta_2 \sum_{k=1}^{m}\sum_{j\in I_2}\sum_{i=i_{k-1}+1}^{i_k} \gamma_i^j \, 1\!\!1_{\{S_i=j\}} - \frac{(\beta_4\Sigma_2+\beta_3\Sigma_1)\sqrt{\delta}\,\Sigma}{a}\sum_{k=1}^{m} l_1^k \\ &\le \beta_1 \sum_{k=1}^{m}\sum_{j\in I_1}\sum_{i=i_{k-1}+1}^{i_k} \gamma_i^j \, 1\!\!1_{\{S_i=j\}} - \frac{\beta_3\Sigma_1\sqrt{\delta}\,\Sigma}{a}\sum_{k=1}^{m} l_1^k \\ &\quad + \beta_2 \sum_{k=1}^{m}\sum_{j\in I_2}\sum_{i=i_{k-1}+1}^{i_k} \gamma_i^j \, 1\!\!1_{\{S_i=j\}} - \frac{\beta_4\Sigma_2\sqrt{\delta}\,\Sigma}{a}\sum_{k=1}^{m} l_1^k,\end{aligned}$$

and we have to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log E_{P\otimes M}\mathbb{E}(\exp(a(1+\rho^{-1})H^{(II)})) \le 0.$$

With the Hölder's inequality (applied with $p=q=2$), it suffices to prove that for $x=1$ and 2

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log E_{P\otimes M}\mathbb{E}\Bigg[\exp\Bigg(2\sum_{k=1}^{m}\sum_{j=I_x}\sum_{i=i_{k-1}+1}^{i_k} \\ a\beta_x\left(1+\rho^{-1}\right)\gamma_i^j \, 1\!\!1_{\{S_i=j\}} - 2\beta_{x+2}\sqrt{\delta}\Sigma_x(1+\rho^{-1})l_1^k\Bigg)\Bigg] \le 0.\end{aligned}$$

We integrate over the disorder γ and it remains to prove that for $x=1$ and 2

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log E_{P\otimes M}\Bigg[\exp\Bigg(\sum_{k=1}^{m}\sum_{j=I_x}\sum_{i=i_{k-1}+1}^{i_k} \\ \log\mathbb{E}\left(\exp\left(2a\beta_x\left(1+\rho^{-1}\right)\gamma_i^j\right)\right) 1\!\!1_{\{S_i=j\}} - 2\beta_{x+2}\sqrt{\delta}\Sigma_x(1+\rho^{-1})l_1^k\Bigg)\Bigg] \le 0.\end{aligned} \tag{2.7.23}$$

For simplicity, in the following we will use E instead of $E_{P\otimes M}$. We begin with the proof of (2.7.23) in the case $x=1$. To that aim, we recall (2.7.12), that gives

$$\log\mathbb{E}\left(\exp\left(2a\beta_1\left(1+\rho^{-1}\right)\gamma_i^j\right)\right) = 2\mathbb{E}(\gamma_1^j)a\beta_1\left(1+\rho^{-1}\right)+o(a). \tag{2.7.24}$$

If we choose β'' such that $\beta_1 < \beta'' < \beta_3$ and a small enough, we obtain for every $j\in I_1$ the inequality $\log\mathbb{E}\left(\exp\left(2a\beta_1\left(1+\rho^{-1}\right)\gamma_i^j\right)\right) \le 2a\beta''\left(1+\rho^{-1}\right)\mathbb{E}(\gamma_1^j)$. Finally, since $\mathbb{E}\left(\gamma_1^j\right) > 0$ for every j, we can replace $(i_k)_{k\in\{1,..,m\}}$ by $(i_k^v)_{k\in\{1,..,m\}}$ (see notations at the beginning of Step I), and it remains to prove that for $B > A > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \log E\Bigg(\exp\left(\sum_{k=1}^{m} \left(\sum_{j\in I_1} Aa\mathbb{E}\left(\gamma_1^j\right) \sum_{i=i_{k-1}^v+1}^{i_k^v} \mathbb{1}_{\{S_i=j\}} - B\sqrt{\delta}\ \Sigma_1\ l_1^k \right)\right)\Bigg) \leq 0. \tag{2.7.25}$$

For simplicity, we will use in the following the notation $\mathbb{E}\big(\gamma_1^j\big) = f(j)$, and consequently $\Sigma_1 = \sum_{j\in I_1} f(j)$. For every N, we build a new filtration, i.e., $F_N = \sigma\left(A_{i_N^v} \cup \sigma\big(l_1^1..., l_1^N\big)\right)$ with $A_k = \sigma(X_1, ..., X_k)$ and the random variable

$$M_N = \frac{\exp\left(\sum_{k=1}^{N} Aa \sum_{j\in I_1} f(j)\ \sharp\{v \in \{i_{k-1}^v+1, i_k^v\} : S_v = j\} - B\sqrt{\delta}\,\Sigma_1 \sum_{k=1}^{N} l_1^k\right)}{\mu^N E\left(\exp\left(Aa\sum_{j\in I_1} \sharp\{i \in \{0, \frac{\delta+\epsilon}{a^2}\} : S_i = j\} - B\sqrt{\delta}\,\Sigma_1\, l_1^1\right)\right)^N}$$

where μ is a constant > 1. We will precise the value of μ later, so that M_N is a positive surmartingale with respect to $(F_N)_{N\geq 0}$. To that aim, we define

$$P_N^j = \sharp\{u \in \{i_{N-1}^v+1, i_N^v\} : S_u = j\},$$

and a new filtration

$$G_{N-1} = \sigma\left(F_{N-1} \cup \sigma\left(X_{i_{N-1}^v+1}, ..., X_{i_{N-1}^v+(\delta+\varepsilon)/a^2}, l_1^N\right)\right).$$

Then, we consider the quantity $E\left(M_N|F_{N-1}\right)$, and, by independence of the random walk excursions out of the origin we obtain

$$E\left(M_N|F_{N-1}\right) = M_{N-1}\frac{\mu^{-1}E\left(\exp\left(Aa\sum_{j=I_1} f(j)\ P_N^j\ - B\sqrt{\delta}\,\Sigma_1\, l_1^N\right)\big|F_{N-1}\right)}{E\left(\exp\left(Aa\sum_{j=I_1} f(j)\ \sharp\{i \in \{0, \frac{\delta+\epsilon}{a^2}\} : S_i = j\} - B\sqrt{\delta}\Sigma_1 l_1^1\right)\right)}. \tag{2.7.26}$$

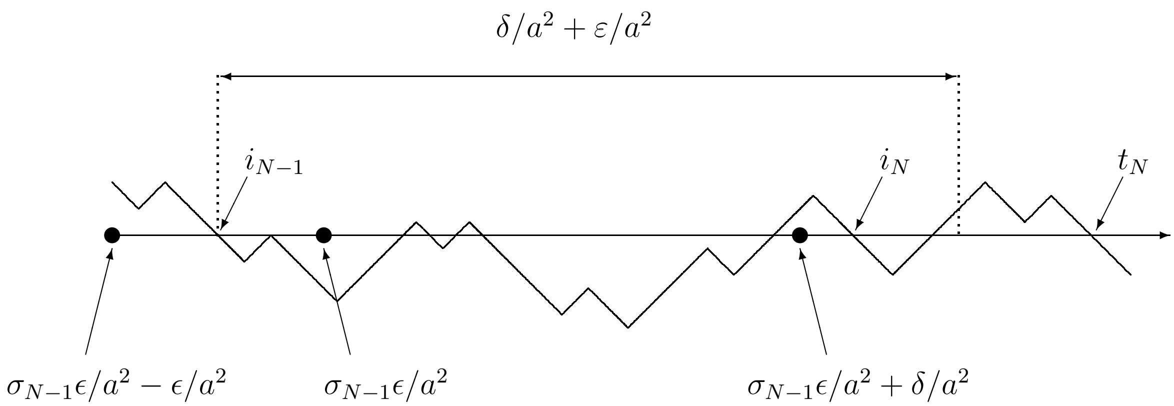
We define $t_N = \inf\{i > i_{N-1}^v + (\delta+\varepsilon)/a^2 : S_i = 0\}$ and notice that $t_N \geq i_N^v$ (see Fig 4 for an example in which $t_N > i_N^v$). Therefore, we can write $P_N^j \leq B_{1,N}^j + B_{2,N}^j$ with

$$\begin{aligned} B_{1,N}^j =& \sharp\{v \in \{i_{N-1}^v+1, .., i_{N-1}^v + (\delta+\varepsilon)/a^2\} : S_v = j\} \\ B_{2,N}^j =& \sharp\{v \in \{i_{N-1}^v + (\delta+\varepsilon)/a^2 + 1, .., t_N\} : S_v = j\}. \end{aligned}$$

The quantity $B^j_{1,N}$ is measurable with respect to G_{N-1} and $F_{N-1} \subset G_{N-1}$. Therefore, we write

$$
\begin{aligned}
C &= E\left(\exp\left(Aa \sum_{j\in I_1} f(j)\ P^j_N\ - B\sqrt{\delta}\,\Sigma_1\, l_1^N \right) \middle| F_{N-1} \right) \\
&\leq E\left(\exp\left(Aa \sum_{j\in I_1} f(j)\ B^j_{1,N}\ - B\sqrt{\delta}\,\Sigma_1\, l_1^N \right) \right. \\
&\qquad\qquad\qquad\qquad \left. \times\, E\left(\exp\left(Aa \sum_{j\in I_1} f(j)\ B^j_{2,N} \right) \middle| G_{N-1} \right) \middle| F_{N-1} \right).
\end{aligned}
$$

Fig. 4:



If we denote by H the quantity $E\Big(\exp\Big(Aa \sum_{j\in I_1} f(j)\ B^j_{2,N} \Big)\ \big| G_{N-1} \Big)$, the fact that the local times $(l_1^1, .., l_1^N)$ are independent of the random walk gives the equality $H = E\Big(\exp\Big(Aa \sum_{j\in I_1} f(j)\ B^j_{2,N} \Big)\ \big| A_{i_{N-1}+(\delta+\varepsilon)/a^2} \Big)$. The strong Markov property can be applied here. Indeed, if $(V_n)_{n\geq 0}$ is a simple random walk with $V_0 = S_{i^v_{N-1}+(\delta+\varepsilon)/a^2}$, and if $s = \inf\{n > 1 : V_n = 0\}$, we can write

$$
H = E_V\left(\exp\left(Aa \sum_{j\in I_1} f(j)\sharp\ \{i \in \{1, ., s\} : V_i = j\} \right) \right).
$$

Thus, if we denote $f = \max_{j\in I_1}\{f_j\}$, we can bound H from above

$$
H \leq E_V\big(\exp\big(Aaf\sharp\ \{i \in \{1, ., s\} : V_i \in \{-K, .., K\}\}\big)\big). \tag{2.7.27}
$$

We want to find an upper bound of H independent of the starting point $S_{i_{N-1}+(\delta+\varepsilon)/a^2}$. The r.h.s. of (2.7.27) is even with respect to the starting point, and we do not transform it if we consider that V is a reflected random walk. That is why it suffices to bound from above the quantities $L(x,a) = E_x\big(\exp\big(Aaf\sharp\,\{i\in\{1,.,s\} : |V_i| \in \{0,..,K\}\}\big)\big)$ with $x\in\mathbb{N}$. Moreover, the Markov property implies that $L(x,a) = L(K,a)$ for every $x \geq K$, and $L(x,a) < L(K,a)$ if $x < K$ because the random walk starting in K touches necessarily in x before reaching 0. So we can write an upper bound of C

$$C \leq E\Bigg(\exp\Bigg(Aa\sum_{j\in I_1} f(j)\ B_{1,N}^j - B\sqrt{\delta}\,\Sigma_1\, l_1^N\Bigg)\Bigg|F_{N-1}\Bigg)L(K,a),$$

and, by independence of the excursions of a random walk, $B_{1,N}^j$ is independent of F_{N-1}. Hence,

$$E\Bigg(\exp\Bigg(Aa\sum_{j=I_1} f(j)\ B_{1,N}^j - B\sqrt{\delta}\,\Sigma_1\, l_1^N\Bigg)\Bigg|F_{N-1}\Bigg) =$$
$$E\Bigg(\exp\Bigg(Aa\sum_{j\in I_1} f(j)\ \sharp\Big\{i\in\Big\{0,\frac{\delta+\epsilon}{a^2}\Big\} : S_i = j\Big\} - B\sqrt{\delta}\,\Sigma_1\, l_1^1\Bigg)\Bigg),$$

and (2.7.26) becomes $E\left(M_N|F_{N-1}\right) \leq M_{N-1}\ L(K,a)/\mu$. But $L(K,a)$ tends to 1 as a tends to 0 and becomes smaller than μ for a small enough. That is why for a small enough $(M_N)_{N\geq 0}$ is a surmartingale. Since the stopping time m_{t/a^2} is bounded from above by t/a^2, we can apply a stopping time theorem and say that $E\left(M_m\right) \leq E\left(M_1\right) \leq 1$. Then, to complete the proof of (2.7.25), it suffices to show that, for δ,ϵ,a small enough the quantity $V_{\delta,\epsilon,a}$, defined in (2.7.28), is smaller than 1.

$$V_{\delta,\epsilon,a} = \mu E\Bigg(\exp\Bigg(Aa\sum_{j\in I_1} f(j)\ \sharp\Big\{i\in\Big\{0,\frac{\delta+\epsilon}{a^2}\Big\} : S_i = j\Big\} - B\sqrt{\delta}\,\Sigma_1\, l_1^1\Bigg)\Bigg). \tag{2.7.28}$$

To that aim, recall that the random walk and the local time l_1^1 are independent. Therefore, we can write

$$V_{\delta,\epsilon,a} = \mu E\Bigg(\exp\Bigg(Aa\sum_{j\in I_1} f(j)\,\sharp\Big\{i\in\Big\{0,\frac{\delta+\epsilon}{a^2}\Big\} : S_i = j\Big\}\Bigg)\Bigg)E\Big(\exp\big(-B\sqrt{\delta}\Sigma_1\, l_1^1\big)\Big).$$

By lemma 4, we know that

$$\lim_{a\to 0} V_{\delta,\epsilon,a} = \mu E\left(\exp(A\sqrt{\delta+\epsilon}\Sigma_1\, l_1^1\right) E\left(\exp(-B\sqrt{\delta}\Sigma_1\, l_1^1\right).$$

Since Σ_1 is fixed, it enters in the constant A and B without changing the fact that $B > A$. We denote, for every x in $\mathbb{R}$, $f(x) = E\left(\exp(x l_1^1)\right)$. The law of l_1^1 is known (see [35]), and the derivative of f in 0 satisfies $f^{'}(0) = E\left(l_1^1\right) > 0$. Therefore, a first order development of f gives $f\left(A\sqrt{\delta+\epsilon}\right) = 1 + f^{'}(0)A\sqrt{\delta+\epsilon} + o(\sqrt{\delta+\epsilon})$ and $f\left(-B\sqrt{\delta}\right) = 1 - f^{'}(0)B\sqrt{\delta} + o(\sqrt{\delta})$. If we take $\epsilon \leq \delta^2$, we obtain

$$f\left(A\sqrt{\delta+\epsilon}\right) f\left(-B\sqrt{\delta}\right) \leq 1 + f^{'}(0)\sqrt{\delta}\left(A\sqrt{1+\delta} - B\right) + o(\sqrt{\delta}). \qquad (2.7.29)$$

Since $B > A$, the right hand side of (2.7.29) is strictly smaller than 1 for δ small enough . For such a δ, for $\epsilon \leq \delta^2$ and for $\mu > 1$ but small enough we obtain $\lim_{a\to 0} V_{\delta,\epsilon,a} < 1$. As a consequence, for a small enough, $V_{\delta,\epsilon,a} < 1$. This completes the proof of (2.7.25), and therefore, the proof of (2.7.23) for $x = 1$.

The proof of (2.7.23) for $x = 2$, is easier than the previous one. Indeed, $\mathbb{E}\left(\gamma_i^j\right) < 0$ for every $j \in I_2$, and therefore, if we choose $\beta^{''}$ such that $\beta_2 > \beta^{''} > \beta_4$, the first order development of (2.7.12) gives, for a small enough, that

$$\log \mathbb{E}\left(\exp\left(2a\beta_2\left(1+\rho^{-1}\right)\gamma_i^j\right)\right) \leq 2a\beta''\left(1+\rho^{-1}\right)\mathbb{E}(\gamma_1^j).$$

By following the scheme of the previous proof (for x=1), we notice that it suffices to replace $\{u \in \{i_{k-1}^v + 1, i_k^v\} : S_u = j\}$ by $\{u \in \{i_{k-1}^v + 1, i_{k-1}^v + \delta + \varepsilon/a^2\} : S_u = j\}$ in the definition of M_N. Moreover, there is no need to introduce $\mu > 1$ in the definition of M_N, which is in this case a positive martingale. The rest of the proof is totally similar to the case $x = 1$.

The proof of $F_4 << F_3$ is almost the same, we just exchange the role of β_1, β_2 and β_3, β_4 in the definition of H^{II}. Consequently, the role of A and $-B$ in (2.7.25) are also exchanged, and, as in the previous proof, the Lemma 1 implies the result.

STEP IV

We notice that the quantities $m, \sigma_1, \sigma_2, ..\sigma_m, s_1, s_2, .., s_m$ can also be defined for a Brownian motion on the interval $[0, t]$. Indeed, we denote $\sigma_0 = 0, p_0 = 0$, and recursively $p_{k+1} = \inf\{s > \sigma_k\epsilon + \delta : B_s = 0\}$ and σ_{k+1} the only integer satisfying

$p_{k+1} \in \big((\sigma_{k+1}-1)\epsilon, \sigma_{k+1}\epsilon\big]$. Finally, we let $m_t = \inf\{k \geq 1 : p_k > t\}$ and $p_m = t$. Now, we want to transform the random walk that gives the possible trajectories of the polymer into a Brownian motion. For that (as in [6]), we denote by Q the measure of $(m_{t/a^2}, \sigma_1, \sigma_2, ...\sigma_m, s_1, s_2, ..., s_m)$ associated with the random walk on $[0, t/a^2]$ and by $\widetilde{Q}$ the measure of $(m_t, \sigma_1, \sigma_2, ..., \sigma_m, s_1, s_2, ..., s_m)$ associated with the Brownian motion on $[0, t]$.

As proved in [6] (page 1362) Q and $\widetilde{Q}$ are absolutely continuous and their Radon-Nikodým derivative satisfies that there exists a constant $K_{a,\epsilon,\delta} > 0$ such that for every $\delta > 0$

$$\lim_{\epsilon\to 0}\limsup_{a\to 0} K_{a,\epsilon,\delta} = 0 \quad \text{and} \quad (1-K)^m \leq \frac{d\widetilde{Q}}{dQ} \leq (1+K)^m . \tag{2.7.30}$$

We denote by R the law of the local times $(l_1^1, l_1^2, ..., l_1^m)$, which are independent of the random walk and consequently of Q. Moreover, $|\overline{I}_k| = (\sigma_k - \sigma_{k-1})\epsilon/a^2$. Hence, the equation (2.7.22) gives that $H^{(4)}_{t,\varepsilon,\delta}(a, h, \beta)$ depends only on $(m_{t/a^2}, \sigma_1, \sigma_2, ...\sigma_m, s_1, s_2, ..., s_m)$ and $(l_1^1, l_1^2, ..., l_1^m)$. That is why, we can write

$$F^4_{t,\epsilon,\delta}(a, h, \beta_1, \beta_2) = 1/t \log E_{R\otimes Q}\left(\exp\left(H^{(4)}_{t,\epsilon,\delta}(a, h, \beta)\right)\right).$$

At this stage, we define F_5 by replacing the random walk by a Brownian motion, namely, by integrating over $R \otimes \widetilde{Q}$ instead of $R \otimes Q$. We define

$$H^{(5)}_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2) = H^{(4)}_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2) + \log\left(d\widetilde{Q}/dQ\right),$$

and

$$\begin{aligned} F^5_{t,\epsilon,\delta}(a, h, \beta_1, \beta_2) &= \frac{1}{t}\log E_{R\otimes\widetilde{Q}}\left(\exp\left(H^{(4)}_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2)\right)\right) \\ &= \frac{1}{t}\log E_{R\otimes Q}\left(\exp\left(H^{(5)}_{t,\varepsilon,\delta}(a, h, \beta_1, \beta_2)\right)\right). \end{aligned}$$

Therefore we aim at proving that $F^4 << F^5$ and we calculate H^{II},

$$\begin{aligned}
H^{II} &= H^{(4)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) - H^{(5)}_{t(1+\rho)^5,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(a(1+\rho),h',\beta_3,\beta_4)\\
&= -\frac{2}{a}(h-(1+\rho)h')\sum_{k=1}^{m} s_k(\sigma_k-\sigma_{k-1})\epsilon\\
&\qquad + \big((\beta_1-\beta_3)\Sigma_1+(\beta_2-\beta_4)\Sigma_2\big)\frac{\sqrt{\delta}}{a}\Sigma\sum_{k=1}^{m} l_1^k - \frac{1}{a(1+\rho)}\log\frac{d\widetilde{Q}}{dQ}\\
&\le -\frac{2}{a}(h-(1+\rho)h')\sum_{k=1}^{m} s_k(\sigma_k-\sigma_{k-1})\epsilon - \frac{1}{a(1+\rho)}\log\frac{d\widetilde{Q}}{dQ}.
\end{aligned}$$

We do not detail the end of this step, because it is done in detail in [6] (page $1361-1362$). To prove that $F_5 << F_4$, we consider the density $dQ/d\widetilde{Q}$ in H^{II}, and (2.7.30) can also be applied. It completes the step IV.

STEP V

Now, we integrate over $R\otimes\widetilde{Q}$. That is why the term $\log\big(d\widetilde{Q}/dQ\big)$ does not appear in H^5 any more. In this step, we aim at transforming the local times $(l_1^1,...,l_1^k,...)$ into the local times of the Brownian motion that determines $\widetilde{Q}$. We will denote by L_t the local time spent in zero by $(B_s)_{s\ge 0}$ between the times 0 and t.

But before, we define $(R_s)_{s\ge 0}$ a Brownian motion, independent of B, and we shed light on the fact that, for every $k\in\{1,..,m\}$,

$$a\sum_{i\in\overline{I}_k}\hat{w}_i \overset{D}{=} R_{\sigma_k\varepsilon}-R_{\sigma_{k-1}\varepsilon} \quad\text{and}\quad a^2|\overline{I}_k| = (\sigma_k-\sigma_{k-1})\varepsilon. \tag{2.7.31}$$

Then, we can rewrite the fifth hamiltonian as

$$\begin{aligned}
H^{(5)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) &= -\frac{2}{a}\sum_{k=1}^{m_t} s_k\left(R_{\sigma_k\varepsilon}-R_{\sigma_{k-1}\varepsilon}+h(\sigma_k-\sigma_{k-1})\epsilon\right)\\
&\qquad + \frac{(\beta_1\Sigma_1+\beta_2\Sigma_2)\sqrt{\delta}}{a}\sum_{k=1}^{m} l_1^k.
\end{aligned}\tag{2.7.32}$$

We define the sixth hamiltonian as follow,

$$H^{(6)}_{t,\varepsilon,\delta}(a,h,\beta_1,\beta_2) = -\frac{2}{a}\sum_{k=1}^{m_t} s_k\left(R_{\sigma_k\varepsilon}-R_{\sigma_{k-1}\varepsilon}+h(\sigma_k-\sigma_{k-1})\epsilon\right) + \frac{\beta_1\Sigma_1+\beta_2\Sigma_2}{a}L_t.$$

At this stage, we notice that F^5 and F^6 do not depend on a anymore. Hence, to simplify the following steps, we transform a bit the general scheme of the proof. Indeed, from now on, we will denote, for $i = 5, 6$ or 7,

$$F^i_{t,\varepsilon,\delta}(h, \beta_1, \beta_2) = \frac{1}{t} \log E_{\widetilde{Q}} \left(\exp(\overline{H}^i_{t,\varepsilon,\delta}(h, \beta_1, \beta_2)) \right) \tag{2.7.33}$$

with $\overline{H}^i_{t,\varepsilon,\delta}(h, \beta_1, \beta_2) = aH^i_{t,\varepsilon,\delta}(h, \beta_1, \beta_2)$. Therefore, to prove that $F^i << F^j$ we use

$$H^{II} = \overline{H}^i_{t,\varepsilon,\delta}(h, \beta_1, \beta_2) - \frac{1}{1+\rho} \overline{H}^j_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}(h', \beta_3, \beta_4), \tag{2.7.34}$$

and we show that

$$\limsup_{t \to \infty} 1/t \log \widetilde{\mathbb{E}} E \left(\exp((1 + \rho^{-1}) H^{II}) \right) \leq 0. \tag{2.7.35}$$

We want to prove that $F^5 << F^6$. By the scaling property of the Brownian motion, it is not difficult to show that for $i = 5$ or 6

$$\overline{H}^i_{t(1+\rho)^2, \varepsilon(1+\rho)^2, \delta(1+\rho)^2}(h, \beta_1, \beta_2) = (1 + \rho) \overline{H}^i_{t,\varepsilon,\delta}((1 + \rho)h, \beta_1, \beta_2). \tag{2.7.36}$$

That is why, by (2.7.34), we can write $H^{II} = \overline{H}^4_{t,\varepsilon,\delta}(h, \beta_1, \beta_2) - \overline{H}^5_{t,\varepsilon,\delta}((1+\rho)h', \beta_3, \beta_4)$, and, since $(1 + \rho)h' < h$, we obtain

$$\begin{aligned}
H^{II} = &-2 \left(h - (1 + \rho)h' \right) \sum_{k=1}^{m} s_k \left(\sigma_k - \sigma_{k-1} \right) \epsilon + \beta_1 \Sigma_1 \sqrt{\delta} \sum_{k=1}^{m} l_1^k \\
&- \beta_3 \Sigma_1 \sum_{k=1}^{m} L_{p_k} - L_{p_{k-1}} + \beta_2 \Sigma_2 \sqrt{\delta} \sum_{k=1}^{m} l_1^k - \beta_4 \Sigma_2 \sum_{k=1}^{m} L_{p_k} - L_{p_{k-1}} \\
H^{II} \leq &\ \beta_1 \sqrt{\delta} \Sigma_1 \sum_{k=1}^{m} l_1^k - \beta_3 \Sigma_1 \sum_{k=1}^{m} L_{p_k^v} - L_{p_{k-1}^v} \\
&+ \beta_3 \Sigma_1 \left(L_{t+\delta} - L_t \right) + \beta_2 \Sigma_2 \sqrt{\delta} \sum_{k=1}^{m} l_1^k - \beta_4 \Sigma_2 \sum_{k=1}^{m} L_{p_k} - L_{p_{k-1}}
\end{aligned}$$

with $p_j^v = p_j$ for every $j < m$ and $p_m^v = \inf\{t > \sigma_{m-1}\epsilon + \delta : B_t = 0\}$. Finally, by the Hölder's inequality, it suffices to prove, for $B > A$, that

$$\limsup_{t \to \infty} \frac{1}{t} \log E \left(\exp \left(A \sum_{k=1}^{m} \sqrt{\delta} l_1^k - B \sum_{k=1}^{m} L_{p_k^v} - L_{p_{k-1}^v} \right) \right) \leq 0 \ , \tag{2.7.37}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log E\left(\exp\left(A\sum_{k=1}^{m}L_{p_k^v}-L_{p_{k-1}^v}-B\sum_{k=1}^{m}\sqrt{\delta}l_1^k\right)\right)\leq 0\ , \tag{2.7.38}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log E\left(\exp\left(B\left(L_{t+\delta}-L_t\right)\right)\right)=0. \tag{2.7.39}$$

We denote by C_t the first return to the origin after time t. Proving (2.7.39) is immediate because for every $t\geq 0$ we can write

$$E\Big(\exp\left(B\left(L_{t+\delta}-L_t\right)\right)\Big)=\int_t^{t+\delta}E\left(\exp\left(B\left(L_{t+\delta}-L_u\right)\right)\big|C_t=u\right)dC_t(u).$$

since C_t is a stopping time with respect to the natural filtration of B, we can apply the strong Markov property, and we obtain

$$E\Big(\exp\left(B\left(L_{t+\delta}-L_t\right)\right)\Big)=\int_t^{t+\delta}E\Big(\exp\left(B\left(L_{t+\delta-u}-L_u\right)\right)\Big)dC_t(u)\leq E\Big(\exp\left(BL_\delta\right)\Big). \tag{2.7.40}$$

This implies (2.7.39), and it remains to prove (2.7.37), and (2.7.38). We define a new filtration, $F_N=\sigma\left(\sigma\left(\left(B_s\right)_{s\leq p_N^v}\right)\bigcup\sigma\left(l_1^1,...,l_1^N\right)\right)$. We notice that $(p_N^v)_{N\geq 0}$ is a sequence of increasing stoping times, and consequently, F_N is an increasing filtration. We denote by M_N the quantity

$$M_N=\frac{\exp\left(A\sum_{k=1}^N\sqrt{\delta}l_1^k-B\sum_{k=1}^N L_{p_k^v}-L_{p_{k-1}^v}\right)}{E\left(\exp\left(-BL_\delta+A\sqrt{\delta}l_1^1\right)\right)^N}, \tag{2.7.41}$$

which is a surmartingale M_N with respect to F_N. Indeed, L and $(l_1^k)_{k\geq 1}$ are independent, $(L_s)_{s\geq p_N^v}$ is independent of F_N (because $B_{p_N^v}=0$) and $L_{p_{N+1}^v}-L_{p_N^v}\geq L_{p_N^v+\delta}-L_{p_N^v}$. Thus, since $E\Big(\exp\Big(-B(L_{p_N^v+\delta}-L_{p_N^v})\Big)\Big)=E\left(\exp\left(-B(L_\delta)\right)\right)$, we obtain $E\left(M_{N+1}|F_N\right)\leq M_N$. Moreover, m_t is a stoping time with respect to F_N and is bounded from above by t/δ. Therefore, to prove (2.7.37), it suffices to show (as in step III) that for $B>A$ and δ small enough, $V=E\left(\exp\left(A\sqrt{\delta}l_1^1-BL_\delta\right)\right)\leq 1$. Moreover, L_δ and $\sqrt{\delta}l_1^1$ have the same law and are independent. That is why we can write $V=E\left(\exp\left(A\sqrt{\delta}l_1^1\right)\right)E\left(\exp\left(-B\sqrt{\delta}l_1^1\right)\right)$, which is strictly smaller than 1 for δ small enough (as proved in step III).

We prove (2.7.38) in a very similar way. Effectively, since $L_{p^v_{N+1}} - L_{p^v_N} \leq L_{p^v_N+\delta+\varepsilon} - L_{p^v_N}$, we prove that the inequality (2.7.37) is still satisfied when A and $-B$ are exchanged. Therefore, the proof of $F^5 << F^6$ is completed. To end this step, we notice that (2.7.38) and (2.7.37) imply directly that $F^6 << F^5$. Thus, the proof of step V is completed.

STEP VI

Let $\mu_1 = \beta_1\Sigma_1 + \beta_2\Sigma_2$ and $\mu_3 = \beta_3\Sigma_1 + \beta_4\Sigma_2$. This step is the last one, therefore, the following hamiltonian is the one of the continuous model, namely,

$$\overline{H}^{(7)}_{t,\varepsilon,\delta}(h,\beta_1,\beta_2) = -2\int_0^t \mathbb{1}_{\{B_s<0\}}(dR_s + hds) + \mu_1 L_t.$$

For simplicity, we define $(\phi_s)_{s\in[0,t]}$ by $\phi_s = s_k$ for every $s \in (\sigma_{k-1}\epsilon, \sigma_k\varepsilon]$. In that way, $\sum_{k=1}^m s_k(R_{\sigma_k\varepsilon} - R_{\sigma_{(k-1)\varepsilon}} + h(\sigma_k - \sigma_{k-1})\epsilon) = \int_0^t \phi_s(dR_s + hds)$. Moreover, the scaling property of the Brownian motion gives, for $i = 6$ or 7,

$$\overline{H}^{(i)}_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(h,\beta) \overset{D}{=} \overline{H}^{(i)}_{t,\varepsilon,\delta}((1+\rho)^2h,(1+\rho)\beta_1,(1+\rho)\beta_2).$$

Hence, to show that $F^6 << F^7$, we must consider (as in step V) the difference

$$H^{II} = \overline{H}^{(6)}_{t,\varepsilon,\delta}(h,\beta_1,\beta_2) - 1/\left(1+\rho\right)\overline{H}^{(7)}_{t(1+\rho)^2,\varepsilon(1+\rho)^2,\delta(1+\rho)^2}(h',(1+\rho)\beta_3,(1+\rho)\beta_4),$$

which is equal to $\overline{H}^{(6)}_{t,\varepsilon,\delta}(h,\beta_1,\beta_2) - \overline{H}^{(7)}_{t,\varepsilon,\delta}((1+\rho)h',(1+\rho)\beta_3,(1+\rho)\beta_4)$. Thus, we can bound H^{II} from above as follow

$$\begin{aligned}
H^{II} &= -2\int_0^t \left(\phi_s - \mathbb{1}_{\{B_s<0\}}\right)dR_s - 2\int_0^t \left(h\phi_s - (1+\rho)h'\,\mathbb{1}_{\{B_s<0\}}\right)ds + (\mu_1-\mu_3)L_t \\
H^{II} &\leq -2\int_0^t \left(\phi_s - \mathbb{1}_{\{B_s<0\}}\right)dR_s - 2h\int_0^t \left(\phi_s - \mathbb{1}_{\{B_s<0\}}\right)ds + (\mu_1-\mu_3)\Sigma L_t.
\end{aligned}$$

We want to prove that

$$\limsup_{t\to\infty} 1/t\log\widetilde{\mathbb{E}}E\left(\exp((1+\rho^{-1})H^{II})\right) \leq 0,$$

and, after the integration over $\widetilde{\mathbb{E}}$, it remains to prove that for $A > 0$ and $B > 0$

$$\limsup_{t\to\infty} 1/t\log\widetilde{\mathbb{E}}E\left(\exp\left(A\int_0^t \left|\phi_s - \mathbb{1}_{\{B_s<0\}}\right|ds - BL_t\right)\right) \leq 0.$$

But in fact, between p_{k-1} and p_k, if we find an excursion of length larger than $\delta+\epsilon$, it is necessarily the one which ends at p_k and gives the value of s_k (see figure 2). It means that, apart eventually from the very beginning of such an excursion (between p_{k-1} and $\sigma_{k-1}\epsilon$), s_k and ϕ_s have the same value along the excursion. Finally, we obtain

$$\int_0^t |\,\mathbb{1}_{\{B_s<0\}}-\phi_s|ds \leq P_{0,t,\delta,\varepsilon}+m\varepsilon,$$

where $P_{u,v,\delta,\varepsilon}$ is the sum between u and v of the excursion lengths, which is smaller than $\delta+\varepsilon$. The term $m\varepsilon$ allows us to take into account the previously mentioned situation between p_{k-1} and $\sigma_{k-1}\varepsilon$.

With this upper bound, we can write $H^{II}\leq AP_{0,t,\delta,\varepsilon}+Am\varepsilon-BL_t$ with $A>0$ and $B>0$. Therefore, to complete the proof we must show that the inequality $\limsup_{t\to\infty}1/t\log E(\exp(1/(1+\rho)H^{II}))\leq 0$ occurs for δ,ε small enough. Thus, by applying the Hölder's inequality, it suffices to prove that, for two strictly positive constants A and B, we have

$$\limsup_{t\to\infty}\frac{1}{t}\log E(\exp(A\varepsilon m-BL_t))\leq 0, \tag{2.7.42}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log E(\exp(AP_{0,t,\delta,\epsilon}-BL_t))\leq 0. \tag{2.7.43}$$

We begin with the proof of (2.7.42). To that aim, we recall that, for every $k<m$, we have $p_k>p_{k-1}+\delta$. Therefore, we can write

$$A\varepsilon m-BL_t\leq A\varepsilon m-B\sum_{k=1}^{m}L_{p_{k-1}+\delta}-L_{p_{k-1}}+B(L_{t+\delta}-L_t).$$

With the equation (2.7.39), and the Hölder's inequality we deduce that the term $B(L_{t+\delta}-L_t)$ does not change the result. That is why we only consider the quantity $1/t\log E\left(\exp\left(\sum_{k=1}^m A\varepsilon-B(l_{p_{k-1}+\delta}-L_{p_k})\right)\right)$, when t tends to ∞. As in (2.7.41), we define the martingale

$$M_N=\frac{1}{(V_{\varepsilon,\delta})^N}\exp\left(\sum_{k=1}^{N}A\varepsilon-B(L_{p_{k-1}+\delta}-L_{p_k})\right)\ \text{with } V_{\varepsilon,\delta}=E(\exp(A\varepsilon-BL_\delta)), \tag{2.7.44}$$

and as m is a stopping time bounded from above by t/δ. It suffices again to show that, $V_{\varepsilon,\delta} < 1$, for δ, ϵ small enough. It is the case because, $E(\exp(-BL_\delta)) < 1$ for every $B > 0$. Therefore, we take ε small enough and it completes the proof.

Now, it remains to prove (2.7.43). But, we notice that $P_{0,t,\delta,\varepsilon} = \sum_{k=1}^m P_{p_{k-1},p_k,\delta,\varepsilon}$ and that $P_{p_{k-1},p_k,\delta,\epsilon} \leq 2(\delta + \varepsilon)$ (see fig 3). Therefore, we obtain the following upper bound

$$AP_{0,t,\delta,\epsilon} - BL_t \leq 2A(\delta+\varepsilon)m - B\sum_{k=1}^{m} L_{p_{k-1}+\delta} - L_{p_{k-1}} + B(L_{t+\delta} - L_t).$$

The term $B(L_{t+\delta} - L_t)$ is removed, as before (in (2.7.39)), and it remains to consider $1/t \log E\left(\sum_{k=1}^m A(\varepsilon+\delta) - B(l_{p_{k-1}+\delta} - L_{p_k})\right)$ when t tends to ∞. To that aim, we build again the martingale

$$M_N = \frac{1}{(D_{\epsilon,\delta})^N} \exp\left(\sum_{k=1}^{N} A(\varepsilon+\delta) - B(l_{p_{k-1}+\delta} - L_{p_k})\right) \tag{2.7.45}$$

with $D_{\epsilon,\delta} = E(\exp(A(\delta+\varepsilon) - BL_\delta))$. The term m is a stopping time, therefore, it suffices to show, for δ, ε small enough, that $D_{\epsilon,\delta} < 1$. To that aim, we choose $\varepsilon \leq \delta$, and it remains to consider the quantity $E\left(\exp\left(2A\delta - BL_\delta\right)\right)$. Moreover, $L_\delta \overset{D}{=} \sqrt{\delta} L_1$, and if we note $f(x) = E(\exp(xL_1))$, we can use a first order development of f in 0. It gives $f(-B\sqrt{\delta}) = 1 - f'(0)B\sqrt{\delta} + \varepsilon_1(\delta)\sqrt{\delta}$ with $f'(0) > 0$ and $\lim_{x\to 0} \varepsilon_1(x) = 0$. We also know that, $\exp(2A\delta) = 1 + 2A\delta + \varepsilon_2(\delta)\delta$ with $\lim_{x\to 0}\varepsilon_2(x) = 0$. Hence, for $\varepsilon \leq \delta$ and δ small enough, we obtain $E\left(\exp\left(2A\delta - BL_\delta\right)\right) = \exp(2A\delta) f(-B\sqrt{\delta}) < 1$. The proof of $F_6 << F_5$ is exactly the same and the Step VI is completed.

2.8 The homopolymer case (proposition 8)

In this part, we want to consider another model, more simple, and called the h-model in the following. We look at a polymer, constituted only by hydrophobic monomers. To that aim, we fix $w_i \equiv 0$ for every $i \geq 1$ and $\lambda = 1$ in the hamiltonian defined in (2.1.1). Therefore, the hamiltonian becomes

$$h\sum_{i=1}^{N} \Delta_i + \beta \sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \, \mathbb{1}_{\{S_i = j\}}.$$

Apart from the fact that this model corresponds to a different physical situation (homopolymer instead of copolymer), there are two principal reasons to study it. First, this type of model with a pinning term at the interface in competition with a repulsion effect (here given by $h\sum_{i=1}^{N}\Delta_i$) is often investigated in the literature (see [20], or [11]). In the wetting model for instance, the repulsion is given by the fact that the involved random walk is conditioned to stay positive. We will prove in chapter 4 that this model can be seen as the limit of the h-model when h tends to infinity. In that way, we will translate some results concerning the h-model to the wetting one. We show also in Chapter 4 that this h-model has a critical curve, denoted by $h_c(\beta)$, that separates the (h,β)-plane into a localized and a delocalized phase. This curve is increasing, convex, $h_c(0)=0$ and for every $h \geq h_c(\beta)$ the couple (h,β) belongs to $\mathcal{D}$, whereas (h,β) belongs to $\mathcal{L}$ if $h < h_c(\beta)$. The other particularity of this system comes from the simplicity of its continuous limit. Indeed, applied to this case, the Theorem 3 implies that the continuous hamiltonian is given by

$$h\int_0^t \Delta_s ds + \beta\Sigma L_t. \tag{2.8.1}$$

Thus, the disorder disappears and we can compute some quantities related to this limit. If we denote by $\widetilde{\phi}(\beta\Sigma, h)$ the continuous free energy, which is associated with (2.8.1), we obtain the following proposition.

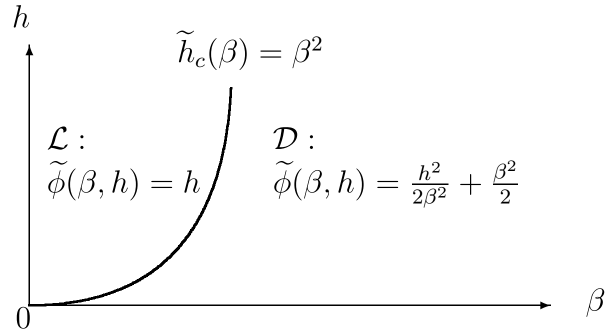
For simplicity, we state this proposition in the case $\Sigma = 1$.

Proposition 8 *Let $\beta \geq 0$ and $h \geq 0$, then*

$$\widetilde{\phi}(\beta,h) = h \quad \textit{if} \quad h \geq \beta^2 \qquad \textit{and} \qquad \widetilde{\phi}(\beta,h) = \frac{h^2}{2\beta^2} + \frac{\beta^2}{2} \quad \textit{if} \quad h < \beta^2$$

Moreover, the localization condition remains the same for the h-model, i.e. $(\beta,h) \in \mathcal{L}$ when $\phi(\beta,h) > h$ or $\widetilde{\phi}(\beta,h) > h$ in the continuous case. Therefore, since $h^2/(2\beta^2) + \beta^2/2 > h$ when $h < \beta^2$, we obtain the whole continuous critical curve, namely, in the case $\Sigma = 1$, $\widetilde{h}_c(\beta) = \beta^2$ (see Fig 5).

Fig. 5:



We come back to the general h-model, i.e. with a general Σ. We can also deduce from the Theorem 3, the behavior of some quantities linked to the discrete model as β tends to zero. Indeed, we can compute the slope at the origin of the discrete critical curve

$$\lim_{\beta \to 0} \frac{h_c(\beta)}{\beta^2} = \Sigma^2. \tag{2.8.2}$$

This limit is conform to the intuition, to the extend that a stronger pinning along the interface enlarges the localized area, and consequently, increases the slope of the critical curve at the origin. This result is also confirmed by the bounds of the critical curve found in the Chapter 4 (in that particular case K is equal to 1, γ satisfies $E(\gamma_1) = 1$ and $h_c(\beta) = (1 + o(\beta))\beta^2$ as β tends to 0).

With the Proposition 8, we derive $\widetilde{\Phi}(h, \beta)$ with respect to β and we find the asymptotic behavior of the price average that the polymer gains along the interface in the localized area. Indeed, when $h < \beta^2$, by convexity of Φ_N in β, we can write that, a.s. in w,

$$\lim_{a \to 0} \lim_{N \to \infty} \frac{1}{aN} E_{N,a\beta}^{a^2 h, w} \left(\sum_{j=-K}^{K} \sum_{i=1}^{N} \gamma_i^j \, \mathbb{1}_{\{S_i = j\}} \right) = \beta - \frac{h^2}{\beta^3}.$$

The same derivation with respect to h gives an approximation, for a small, of the time proportion that the polymer spends under the interface, i.e.

$$\lim_{a \to 0} \lim_{N \to \infty} E_{N,a\beta}^{a^2 h, w} \left(\frac{\Gamma_-(N)}{N} \right) = \frac{\beta^2 - h}{2\beta^2}.$$

2.9 Computation of $\widetilde{\phi}$

We must compute $\lim_{t\to\infty}\widetilde{\phi}_t(h,\beta) = h + \lim_{t\to\infty} 1/t \log E\left(\exp\left(-2h\Gamma^-(t) + \beta L_t^0\right)\right)$. To that aim, we will use the joint density of the random variables $\Gamma^-(t)$ and L_t^0, which is explicitly given in [23], i.e.

$$dP_{\left(\Gamma^-(t),L_t^0\right)}(\tau,b) = \mathbb{1}_{\{0<\tau<t\}}\,\mathbb{1}_{\{b>0\}}\frac{b\,t\exp\left(-\frac{t\,b^2}{8\,\tau\,(t-\tau)}\right)}{4\,\pi\,\tau^{\frac{3}{2}}\,(t-\tau)^{\frac{3}{2}}}\,db\;d\tau. \tag{2.9.1}$$

From now on, we will denote $R_t = E\left(\exp\left(-2h\Gamma^-(t) + \beta L_t^0\right)\right)$, and, applying (2.9.1) and the new variables $s = \tau/t$ and $v = b/\sqrt{t}$, we obtain

$$R_t = \int_0^\infty \frac{v\exp\left(\beta v\sqrt{t}\right)}{4\pi}\int_0^1 \exp(-2hst)\frac{\exp\left(-\frac{v^2}{8s(1-s)}\right)}{s^{\frac{3}{2}}(1-s)^{\frac{3}{2}}}dsdv. \tag{2.9.2}$$

At this stage, we can delete the constant term 4π, that does not change the limit, and write $\int_0^1$ of (2.9.2) as the sum of $A_1(t) = \int_0^{1/2}$ and $A_2(t) = \int_{1/2}^1$. Then, we introduce the new variable $u = s(1-s)$ in $A_1(t)$ and $A_2(t)$, and we obtain

$$A_1(t) = \int_0^{\frac{1}{4}} \frac{\exp\left(h(\sqrt{1-4u}-1)t - \frac{v^2}{8u}\right)}{u^{\frac{3}{2}}\,\sqrt{1-4u}}\,du,$$
$$A_2(t) = \int_0^{\frac{1}{4}} \frac{\exp\left(-h(\sqrt{1-4u}+1)t - \frac{v^2}{8u}\right)}{u^{\frac{3}{2}}\,\sqrt{1-4u}}\,du. \tag{2.9.3}$$

It gives immediately the inequalities $A_1(t) \le A_1(t) + A_2(t) \le 2A_1(t)$. Therefore, instead of studying the convergence of $1/t\,\log R(t)$, it suffices to consider $1/t\log S(t)$ with $S(t) = \int_0^\infty v\exp(\beta v\sqrt{t})A_1(t)dv$. We apply the Fubini Tonnelli theorem which gives

$$S(t) = \int_0^{\frac{1}{4}} \frac{\exp(ht\sqrt{1-4u})}{u^{\frac{3}{2}}\,\sqrt{1-4u}}\int_0^\infty v\exp\left(\beta v\sqrt{t} - \frac{v^2}{8u}\right)dvdu\;\;\exp(-ht). \tag{2.9.4}$$

Thus, for every $u \in [0,1/4]$, we change the variables of the second integral of (2.9.4). To that aim, we denote $r = v^2/u$. After that, we transform the variable u into $x = 4u$, and we obtain

$$S(t) = \frac{1}{4}\int_0^1 \frac{\exp(ht\sqrt{1-x})}{\sqrt{1-x}}\;\frac{\int_0^\infty \exp\left(\frac{\beta\sqrt{rxt}}{2} - \frac{r}{8}\right)dr}{\sqrt{x}}dx\;\;\exp(-ht). \tag{2.9.5}$$

The constant factor 1/4 can be deleted and the laplace method allows us to find the asymptotic behavior of $Y(x) = \int_0^\infty \exp\big(\beta\sqrt{rxt}/2 - r/8\big)\,dr$ when x tends to ∞. It gives $Y(x) \sim_{x\to\infty} c\sqrt{xt}\exp\left(\beta^2 xt/2\right)$ with $c > 0$ that depends on β. Thus, by considering (2.9.5), for every $\varepsilon > 0$, we can write the following lower bound,

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log S(t) + h &\geq \liminf_{t\to\infty}\frac{1}{t}\left(\log\int_0^\varepsilon \frac{\exp(ht\sqrt{1-u})}{\sqrt{1-u}\,\sqrt{u}}du + \log\int_0^\infty e^{-\frac{r}{8}}dr\right)\\ &\geq h\sqrt{1-\varepsilon}. \end{aligned} \tag{2.9.6}$$

Thus, since (2.9.6) is true for every $\varepsilon > 0$, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log S(t) + h \geq h. \tag{2.9.7}$$

But we can also bound $\liminf_{t\to\infty}\frac{1}{t}\log S(t)+h$ as follow. By the asymptotic behavior of $Y(x)$, if we choose $c^{'} < c$ and t large enough, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log S(t) + h \geq \liminf_{t\to\infty}\frac{1}{t}\log\int_\varepsilon^1 \frac{\exp(ht\sqrt{1-x})}{\sqrt{1-x}\,\sqrt{x}}c^{'}\sqrt{xt}\exp\left(\frac{\beta^2 xt}{2}\right)dx,$$

so that, after simplifications,

$$\liminf_{t\to\infty}\frac{1}{t}\log S(t) + h \geq \liminf_{t\to\infty}\frac{1}{t}\log\int_\varepsilon^1 \frac{\exp(ht\sqrt{1-x}+\frac{t\beta^2 x}{2})}{\sqrt{1-x}}dx. \tag{2.9.8}$$

With the formerly mentioned laplace method, we can find the asymptotic behavior of the integral of the r.h.s. of (2.9.8).As t tends to ∞, it behaves as $d\exp\big(t\big(\frac{h^2}{2\beta^2}+\frac{\beta^2}{2}\big)\big)/\sqrt{t}$ with $d > 0$. Therefore, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log S(t) + h \geq \frac{h^2}{2\beta^2} + \frac{\beta^2}{2}. \tag{2.9.9}$$

Finally, (2.9.7) and (2.9.9) give

$$\liminf_{t\to\infty}\frac{1}{t}\log S(t) + h \geq \max\big\{\frac{h^2}{2\beta^2} + \frac{\beta^2}{2}, h\big\}. \tag{2.9.10}$$

Now, we want to show that the r.h.s. of (2.9.9) is also an upper bound of the quantity $\limsup_{t\to\infty} 1/t\log S(t)+h$. To that aim, we use the fact that $\limsup_{t\to\infty} 1/t\log S(t)+h$ is equal to the maximum of $\limsup_{t\to\infty} 1/t\log\int_0^\varepsilon$ and $\limsup_{t\to\infty} 1/t\log\int_\varepsilon^1$. The same kind of estimates allows us to perform the computation. Hence, we have

$$\lim_{t\to\infty}\frac{1}{t}\log S(t) + h = \max\left(\frac{h^2}{2\beta^2} + \frac{\beta^2}{2}, h\right).$$

Finally, $\widetilde{\phi}(\beta,h) = h + \lim_{t\to\infty} 1/t \log S(t)$, and therefore,

$$\widetilde{\phi}(h,\beta) = h \quad \text{if} \quad h > \beta^2 \quad \text{and} \quad \widetilde{\phi}(h,\beta) = \frac{h^2}{2\beta^2} + \frac{\beta^2}{2} \quad \text{if} \quad h \leq \beta^2.$$

2.10 Appendix

2.10.1 A: proof of proposition 2

2.10.2 Step I

First, we define some important notations. Let

$$H_{s,t}(\lambda,h,\beta) = \lambda \int_s^t \Lambda_u (dR_u + hdu) + \beta(L_t - L_s),$$

and

$$V_x^l(s,t) = E_x \left(\exp\left(H_{s,t}(l\lambda, h, l\beta) \right) \mathbb{1}_{\{B_t^x \in [1,2]\}} \right).$$

In this first step, we aim at proving that $\widetilde{\mathbb{P}}$ almost surely, as t tends to ∞, the quantity $G(t) = 1/t \log \inf_{x\in[1,2]} V_x^1(0,t)$ converges toward a constant denoted by $\widetilde{\Phi}(\lambda,h,\beta)$. To that aim, we denote $S_{s,t}(R) = \log\left(\inf_{x\in[1,2]} V_x^1(s,t)\right)$, and we show that the four hypothesis of the Kingman's super additive theorem (see [24]), are satisfied by the process $f_{s,t}(R)$. This process is defined $\widetilde{\mathbb{P}}$ a.s. by

$$f_{s,t}(R) = \frac{S_{s,t}(R)}{t-s}. \tag{2.10.1}$$

For $s \geq 0$, we define the operator θ_s by, $\theta_s(R)(.) = R(s+.) - R(s)$, and we stress the fact that the convergence of $f_{0,t}(R)$ implies immediately the same convergence for $G(t)$.

2.10.3 Hypothesis 1

$$\widetilde{\mathbb{P}}(\theta_s(R) \in \ .\ ) = \widetilde{\mathbb{P}}(R \in \ .\ ) \quad \text{for every } s \geq 0.$$

Effectively, the Kingman theorem gives a limit, which is "a priori" a function of the trajectories $(R_s)_{s\geq 0}$. But, in this case, R is invariant by θ_s for every $s \geq 0$. It implies that the limit is measurable with respect to the σ algebra $\cap_{t>0}\sigma((R_s)_{s\geq t})$. Hence, by the Blumenthal $0-1$ law, we obtain that the limit $\widetilde{\Phi}$ is $\widetilde{\mathbb{P}}$ a.s. constant.

2.10.4 Hypothesis 2

Let $s<t$. We want to prove that

$$tf_{0,t}(R) \geq sf_{0,s}(R) + (t-s)f_{0,t-s}(\theta_s(R)). \tag{2.10.2}$$

We consider the quantity $S_{0,t}(R)$ and we restrict, for every $x \in [1,2]$, the quantity $V_x^1(0,t)$ to the event $\{B_x(s) \in [1,2]\}$. Then, by applying the Markov property, we obtain $S_{0,t}(R) \geq S_{0,s}(R) + S_{0,t-s}(\theta_s(R))$, which is equivalent to (2.10.2).

2.10.5 Hypothesis 3

We have to show that

$$\sup_{t\geq 1} \widetilde{\mathbb{E}}\Big(f_{0,t}(R)\Big) < \infty.$$

For that, we use the Jensen's inequality as follow,

$$\sup_{t\geq 1} \widetilde{\mathbb{E}}\big(f_{0,t}(R)\big) \leq \sup_{t\geq 1} \frac{1}{t} \log E_0 \left(\widetilde{\mathbb{E}}\Big(\exp\Big(\lambda \int_0^t \Lambda_s dR_s + \lambda h t\Big)\Big) \exp(\beta L_t)\right).$$

Moreover,

$$\widetilde{\mathbb{E}}\left(\exp\left(\lambda \int_0^t \Lambda_s dR_s\right)\right) = \exp(\lambda^2 t/2)$$

because the process $\big(\int_0^t \Lambda_u dR_u\big)_{t\geq 0}$ is a Brownian motion under $\widetilde{\mathbb{P}}$. Then, we can write

$$\sup_{t\geq 1} \widetilde{\mathbb{E}}\big(f_{0,t}(R)\big) \leq \lambda h + \frac{\lambda^2}{2} + \sup_{t\geq 1} \frac{1}{t} \log E_0\left(\exp(\beta L_t)\right).$$

Finally, since the density of L_t with respect to the Lebesgue measure is given by (see [23])

$$dP_{L_t}(b) = \sqrt{2/\pi t}\, \exp(-b^2/2t)\, \mathbb{1}_{\{b>0\}} db,$$

a short computation shows that, $E_0\left(\exp(\beta L_t)\right) \leq C \exp(\beta^2 t/2)$ with $C>1$. Therefore, for every $t \geq 1$, we have

$$\frac{1}{t} \log\left(E_0\left(\exp(\beta L_t)\right)\right) \leq \log(C) + \frac{\beta^2}{2},$$

and the hypothesis 3 is satisfied.

2.10.6 Hypothesis 4

It remains to prove that, for every $T > 0$, the following inequality occurs

$$G_T = \widetilde{\mathbb{E}}\left(\sup_{0\le s<t\le T} |(t-s) f_{s,t}(R)|\right) < \infty.$$

A first computation gives

$$\begin{aligned} G_T &\le \widetilde{\mathbb{E}}\left(E_1\left(\exp\left(\lambda \sup_{0\le s<t\le T}\left|\int_s^t \Lambda_u dR_u\right| + \lambda h T + \beta L_T\right)\right)\right) \\ &\le \lambda h T + \log E_1 \otimes \widetilde{\mathbb{E}}\left(\exp\left(\lambda \sup_{0\le s<t\le T}\left|\int_s^t \Lambda_u dR_u\right|\right) + \beta L_T\right) \end{aligned}$$

We use again the fact that, under $\widetilde{\mathbb{P}}$, the process $L_{t\ge 0} = \left(\int_0^t \Lambda_u dR_u\right)_{t\ge 0}$ is a Brownian motion. Hence, $\widetilde{\mathbb{P}}$ a.s., we can write that $\sup_{s\in[0,t]}\left(\int_0^s \Lambda_u dR_u\right) = o(t)$, and therefore,

$$\widetilde{\mathbb{E}}\left(\sup_{0\le s<t\le T} (t-s) f_{s,t}(R)\right) \le \lambda h T + \frac{\lambda^2 T}{2} + \log E_2\left(\exp(\beta L_T)\right).$$

It remains to bound from above $\sup_{0\le s<t\le T} -(t-s) f_{s,t}(R)$. To that aim, we constrain the brownian motion (B) to stay positive between 0 and $t-s$, namely

$$\begin{aligned} (t-s) f_{s,t}(R) &\ge \log \inf_{x\in[1,2]} E_x\left(\exp\left(H_{s,t}(\lambda,h,\beta)\right) \mathbb{1}_{\{B_t^x\in[1,2]\}} \mathbb{1}_{\{B_u^x>0,\, \forall u\in[s,t]\}}\right) \\ &\ge \lambda(R_t - R_s) + \inf_{x\in[1,2]} \log P\left(B_u^x > 0\ \forall u\in[s,t],\ \text{and} B_{t-s}^x \in [1,2]\right). \end{aligned} \tag{2.10.3}$$

The second term of the right hand side of (2.10.3) is easily bounded by below, uniformly in $0\le s<t\le T$, by a negative constant $-c_T$. Hence,

$$\widetilde{\mathbb{E}}\left(\sup_{0\le s<t\le T} -S_{t-s}(\theta_s(R))\right) \le 2\widetilde{\mathbb{E}}\left(\max_{s\in[0,T]}\{|R_s|\}\right) + c_T < \infty.$$

The proof of the first step is therefore completed.

2.10.7 Step II

We proved in the former step that $\lim_{t\to\infty} 1/t \log \inf_{x\in[1,2]} V_x^1(t) = \widetilde{\Phi}(\lambda,h,\beta)$. Now, we aim at showing that the same convergence occurs if we replace the quantity $\inf_{x\in[1,2]} V_x^1(t)$ by $V_2^1(t)$. One of the two required inequalities is obvious, i.e.,

$\liminf_{t\to\infty} 1/t \log V_2^1(t) \geq \widetilde{\Phi}(\lambda, h, \beta)$. Therefore, it remains to prove the opposite one, with lim sup instead of lim inf.

We denote $\tau_x = \inf\{s \geq 0 \ : \ B_s^x = 0\}$, and the density h_x of τ_x is given by (see [23])

$$h_x(u) = \frac{|x|}{\sqrt{2\pi u^3}} \exp\left(-\frac{x^2}{2u}\right) \mathbb{1}_{\{u>0\}} du. \tag{2.10.4}$$

This implies that for every $x \in [1,2]$ and $u > 0$, $h_2(u) \leq 2h_x(u)$. Then we can write

$$\begin{aligned} V_2^1(t) = & \int_0^t E_2\left(\exp\left(H_{0,u}(\lambda,h,\beta)\right) | \tau_2 = u\right) E_0\Big(\exp\left(H_{u,t-u}(\lambda,h,\beta)\right) \mathbb{1}_{\{B^0_{t-u}\in[1,2]\}}\Big) h_2(u) du \\ & + E_2\left(\exp\left(H_{0,t}(\lambda,h,\beta)\right) \mathbb{1}_{\{B^2_t\in[1,2]\}} \mathbb{1}_{\{\tau_2>t\}}\right). \end{aligned}$$

Moreover, if $\tau_2 \geq u$, then $H_{0,u}(\lambda,h,\beta) = \lambda R_u + \lambda h u$. Thus, for every $x \in [1,2]$, we obtain

$$\begin{aligned} V_2^1(t) \leq 2 & \int_0^t \exp\left(\lambda R_u + \lambda h u\right) E_0\left(\exp\left(H_{u,t-u}(\lambda,h,\beta)\right) \mathbb{1}_{\{B^0_{t-u}\in[1,2]\}}\right) h_x(u) du \\ & + \exp(\lambda R_t + \lambda h t) P_2\left(\{B_t^2 \in [1,2]\} \cup \{\tau_2 > t\}\right). \end{aligned}$$

For every $t \geq 1$, we can show that there exists two strictly positive constants C_1 and C_2, such that, for every x in $[1,2]$

$$\frac{C_1}{t^{\frac{3}{2}}} \leq P_x\left(\{B_t^x \in [1,2]\} \cup \{\tau_2 > t\}\right) \leq \frac{C_2}{t^{\frac{3}{2}}}.$$

Then, we can write

$$V_2^1(t) \leq \max\left\{2, \frac{C_2}{C_1}\right\} \inf_{x\in[1,2]} \Big\{V_x(t)\Big\},$$

and the proof of $\lim_{t\to\infty} 1/t \log V_2(t) = \widetilde{\Phi}(\lambda,h,\beta)$ is completed. This convergence gives the convexity of $\widetilde{\Phi}$, as limit of convex functions.

2.10.8 Step III

In this step we want to show that $\lim_{t\to\infty} 1/t \log V_0^1(t) = \widetilde{\Phi}(\lambda,h,\beta)$. We begin with the proof of $\limsup_{t\to\infty} 1/t \log V_0^1(t) \leq \widetilde{\Phi}(\lambda,h,\beta)$. To that aim, we denote $\kappa_x = \inf\{t \geq 1 \ : \ B_x(t) = 0\}$, and we notice that, for $x = 0$ or 2, the quantity

$1/t \log E_x\left(\exp\left(H_{0,1}(\lambda,h,\beta)\right)\right)$ vanishes when t tends to ∞. Indeed, for any fixed trajectory of B, the random process $(\int_0^s \Lambda_u d\mathbb{R}_u)_{s\geq 0}$ is a Brownian motion under $\widetilde{\mathbb{P}}$. Then, by Jensen's inequality,

$$\begin{aligned}
\widetilde{\mathbb{E}}\Big(\log\Big(E_x\Big(\exp\Big(\lambda \int_0^1 \Lambda_s(dR_s + hds) + \beta L_1\Big)\Big)\Big)\Big) \\
&\leq \log\left(E_x\left(\widetilde{\mathbb{E}}\Big(\exp\Big(\lambda \int_0^1 \Lambda_s(dR_s+hds)\Big)\Big) \exp(\beta L_1)\right)\right) \\
&\leq \frac{\lambda^2}{2} + \lambda h + \log\big(E_x\left(\exp(\beta L_1)\right)\big) \ < \infty.
\end{aligned}$$

Thus, the random variable $\log E_x\left(\exp\left(H_{0,1}(\lambda,h,\beta)\right)\right)$ is integrable, with respect to $\widetilde{\mathbb{P}}$, and is consequently $a.s.$ finite.

Now, if we apply the Hölder's inequality with $1/L + 1/l = 1$, we obtain

$$\begin{aligned}
\frac{1}{t}\log V_0^1(t) \leq &\frac{1}{Lt}\log E_0\left(\exp\left(H_{0,1}(L\lambda,h,L\beta)\right)\right) \\
&+ \frac{1}{lt}\log E_0\left(\exp\left(l\lambda\int_1^t \Lambda_s(dR_s+hds) + l\beta(L_t - L_1)\right) \mathbb{1}_{\{B_0(t)\in[1,2]\}}\right).
\end{aligned} \tag{2.10.5}$$

Then, the superior limit of $1/t \log V_0^1(t)$ is smaller than the one of the second term of the right hand side of (2.10.5). For simplicity, in the following, we denote

$$D_x^l(t) = E_x\left(\exp\left(l\lambda\int_1^t \Lambda_s(dR_s+hds) + l\beta(L_t-L_1)\right)\mathbb{1}_{\{B_x(t)\in[1,2]\}}\right)$$

and we rewrite $D_x^l(t)$ in dependence of the position of κ_x with respect to t. We recall that B keeps a constant sign (i.e. Λ_1) between 1 and κ_x. Therefore, we can write $D_x^l(t) = A_x^l(t) + C_x^l(t)$ with

$$C_x^l(t) = E_x\Big(\exp\left(l\lambda\Lambda_1(R_t - R_1 + h(t-1))\right)\mathbb{1}_{\{B_x(t)\in[1,2]\}}\,\mathbb{1}_{\{\kappa_x>t\}}\Big),$$

and

$$\begin{aligned}
A_x^l(t) = \int_1^t E_x\Big(\exp\left(\lambda l\Lambda_1(R_u - R_1 + h(u-1))\right)\Big|\kappa_x = u\Big) \\
\times E_0\Big(\exp\left(H_{u,t-u}(l\lambda,h,l\beta)\right)\mathbb{1}_{\{B_0(t-u)\in[1,2]\}}\Big)\, dP_{\kappa_x}(u).
\end{aligned} \tag{2.10.6}$$

Now, since $\widetilde{\mathbb{P}}$ a.s., $\max_{s\in[0,t]} |R_s| = o(t)$ when t tends to ∞, we observe that the term $\exp(\lambda l \Lambda_1(R_u - R_1))$ has no influence on the value of $\limsup_{t\to\infty} 1/t \log A_x^l(t)$. We notice also that, for $x = 0$ or 2, $P_x(B_x(1) > 0|\kappa_x = u) \geq P_x(B_x(1) < 0|\kappa_x = u)$. Consequently, for $x = 0$ or 2, in $A_x^l(t)$, we can take the restriction of $E_x\big(\exp(\lambda l \Lambda_1(R_u - R_1 + h(u-1)))|\kappa_x = u\big)$ to the event $\{B_x(1) > 0\}$, without changing the value of $\limsup_{t\to\infty} 1/t \log A_x^l(t)$.

At this stage we must prove the two following inequalities, i.e.

$$\limsup_{t\to\infty} 1/lt \log A_0^l(t) \leq \widetilde{\Phi}(l\lambda, h, l\beta)/l \tag{2.10.7}$$

and

$$\limsup_{t\to\infty} 1/lt \log C_0^l(t) \leq \widetilde{\Phi}(l\lambda, h, l\beta)/l. \tag{2.10.8}$$

Thus, the equation (2.10.5) will give, for every $l > 1$, $\limsup_{t\to\infty} 1/t \log V_0^1(t) \leq \widetilde{\Phi}(l\lambda, h, l\beta)/l$. But, as mentioned in the previous step, $\widetilde{\Phi}(l\lambda, h, l\beta)$ is convex and continuous in l. Then, we will let l tend to 1, and the proof of $\limsup_{t\to\infty} 1/t \log V_0^1(t) \leq \widetilde{\Phi}(\lambda, h, \beta)$ will be completed.

For the first inequality (2.10.7), we compute the density $l_x(s)$ of P_{κ_x} with respect to the Lebesgue's measure. It gives, for $s > 1$,

$$2l_x(s)ds = \int_{\mathbb{R}} \widetilde{P}_v\big(\tau_v \in [s-1-ds, s-1+ds]\big) dP_{B_x(1)}(v).$$

With the help of (2.10.4), we can compute these densities $(l_x(s))$. Indeed, for $s > 1$,

$$l_x(s) = \frac{\exp\left(-\frac{x^2}{2s}\right)}{2\pi(s-1)^{\frac{3}{2}}} \int_{\mathbb{R}} \left| u + \frac{x(s-1)}{s} \right| \exp\left(-\frac{su^2}{2(s-1)}\right) du. \tag{2.10.9}$$

Thus, we notice that there exists $c_1 > 0$, such that for every $s > 1$, $l_0(s) \leq 2c_1 l_2(s)$ (for instance $c_1 = \exp(2)$). We use this inequality in the expression of $A_0^l(t)$ (see (2.10.6)). It implies that, $A_0^l(t) \leq c_1 A_2^l(t)$. Then, we consider the equality

$$\exp\left(l\lambda \int_1^t \Lambda_s(dR_s + hds) + l\beta(L_t - L_1)\right) = \exp\big(H_{0,t}(l\lambda, h, l\beta)\big)$$
$$\times \exp\big(-H_{0,1}(l\lambda, h, l\beta)\big),$$

and we apply the Hölder's inequality (similarly to what we did in (2.10.5)). Thus, for $l' > l$, we obtain the inequality

$$\limsup_{t\to\infty} 1/lt \log D_2^l(t) \le \limsup_{t\to\infty} 1/l't \log V_2^{l'}(t),$$

whose right hand side is equal to $\widetilde{\Phi}(l'\lambda, h, l'\beta)/l'$ (see step 2). This is true for every $l' > l > 1$. Consequently, if we let l' go to l, it comes

$$\limsup_{t\to\infty} \frac{1}{lt} \log A_0^1(t) \le \widetilde{\Phi}(l\lambda, h, l\beta)/l. \tag{2.10.10}$$

Now, it remains to prove the second inequality (2.10.8). But, we noticed in remark 2 that $\widetilde{\Phi}(\lambda, h, \beta) \ge \lambda h$. Clearly,

$$C_0^l(t) \le \exp\Big(l\lambda(\Lambda_1 \max_{s\in[0,t]}\{|R_s|\} + ht)\Big),$$

therefore, since a.s. in R $\max_{s\in[0,t]}\{|R_s|\} = o(t)$, we obtain

$$\limsup_{t\to\infty} 1/lt \log C_0^l(t) \le \ \lambda h \le \ \widetilde{\Phi}(l\lambda, h, l\beta)/l. \tag{2.10.11}$$

The proof of $\limsup_{t\to\infty} 1/t \log V_0^1(t) \le \widetilde{\Phi}(\lambda, h, \beta)$ is completed.

Now, it remains to prove that $\liminf_{t\to\infty} 1/t \log V_0^1(t) \ge \widetilde{\Phi}(\lambda, h, \beta)$. To that aim, we use the Hölder's inequality (as in (2.10.5)), to bound $\widetilde{\Phi}(\lambda, h, \beta)$ from above. For $l > 1$, we obtain

$$\begin{aligned}\widetilde{\Phi}(\lambda, h, \beta) &= \lim_{t\to\infty} \frac{1}{t} \log V_2^1(t)\\ &\le \limsup_{t\to\infty} \frac{1}{lt} \log E_2\Big(\exp\Big(l\lambda \int_1^t \lambda_s(dR_s + hds) + l\beta(L_t - L_1)\Big) 1\!\!1_{\{B_0(t)\in[1,2]\}}\Big)\end{aligned} \tag{2.10.12}$$

Here again, we use the formula $D_2^l(t) = A_2^l(t) + C_2^l(t)$. Therefore, if we can prove that the two quantities $\limsup_{t\to\infty} 1/t \log A_2^l(t)$ and $\limsup_{t\to\infty} 1/t \log C_2^l(t)$ are smaller than $\liminf_{t\to\infty} 1/t \log V_0^l(t)$, it will imply for every $l > 1$ that

$$\widetilde{\Phi}(\lambda, h, \beta) \le \liminf_{t\to\infty} \frac{1}{lt} \log V_0^l(t).$$

Thus, by considering $(\lambda/l, h, \beta/l)$ instead of (λ, h, β), we will obtain the inequality $l\widetilde{\Phi}(\lambda/l, h, \beta/l) \leq \liminf_{t\to\infty} 1/t \log V_0^1(t)$, and, by letting l go to 1, we will complete the proof.

We begin with the computation of $\liminf_{t\to\infty} 1/t \log V_0^l(t)$, restricted to the subset of trajectories $\{B : \Lambda_s = 1$ for every $s \in [1,t]$ and $B_0(t) \in [1,2]\}$. We obtain, as explained in remark 2, that $\liminf_{t\to\infty} 1/t \log V_0^l(t) \geq \lambda hl$. But, in (2.10.11), we have seen that $\limsup_{t\to\infty} \frac{1}{t} \log C_2^l(t) \leq l\lambda h$. Therefore,

$$\limsup_{t\to\infty} \frac{1}{t} \log C_2^l(t) \leq l\lambda h \leq \liminf_{t\to\infty} \frac{1}{t} \log V_0^l(t)$$

It remains to prove that $\limsup_{t\to\infty} 1/t \log A_2^l(t) \leq \limsup_{t\to\infty} 1/t \log A_0^l(t)$. Since $A_0^l(t) \leq V_0^l(t)$, the proof of step 3 will be completed. To that aim, using (2.10.9) we notice that there exists $K > 0$ such that, for every $s > 1$, $l_2(s) \leq K l_0(s)$. Therefore, we bound from above $A_2^l(t)$ by $K A_0^l(t)$, by replacing $l_0(s)$ in (2.10.6) by $K l_0(s)$. We obtain finally, $\limsup_{t\to\infty} 1/t \log A_2^l(t) \leq \limsup_{t\to\infty} 1/t \log A_0^l(t)$ and the proof is completed.

Thus, we have the convergence $\lim_{t\to\infty} 1/t \log V_0^1(t) = \widetilde{\Phi}(\lambda, h, \beta)$.

2.10.9 Step IV

This last step is dedicated to the relaxation of the condition $B_0(t) \in [1,2]$. If we denote

$$K_t^l = E_0\left(\exp\left(H_{0,t}(l\lambda, h, l\beta)\right)\right),$$

it appears obviously that $\liminf_{t\to\infty} 1/t \log K_t^1 \geq \widetilde{\Phi}(\lambda, h, \beta)$. Therefore, it remains to prove the opposite inequality with lim sup instead of lim inf. To that aim, we let $K_t^l(+\text{ or }-)$ be the restriction of K_t^l to the event $\{B_0(t) > 0\}$ or $\{B_0(t) < 0\}$. It comes

$$\limsup_{t\to\infty} \frac{1}{t} \log K_t^1 = \max\left\{\limsup_{t\to\infty} \frac{1}{t} \log K_t^1(-), \limsup_{t\to\infty} \frac{1}{t} \log K_t^1(+)\right\}.$$

Now, we can rewrite

$$K_t^1(-) = \int_0^t E_0\Big(\exp\big(H_{0,u}(\lambda,h,\beta)\big)\,|g_t = u\Big)$$
$$\times \exp\Big(-\lambda(R_t - R_0) - \lambda h(t-u)\Big) P_0\left(B^0_{t-u} < 0 | g_{t-u} = 0\right) dP_{g_t}(u).$$

Moreover, $\widetilde{\mathbb{P}}$ a.s., we have that $\sup_{s\in[0,t]}\{|R_s|\} = o(t)$ when t tends to ∞. Therefore, $\exp(-\lambda(R_t - R_0))$ does not influence the value of $\limsup_{t\to\infty} 1/t \log K_t^1(-)$. Thus, since $P_0(B^0_{t-u} < 0 | g_{t-u} = 0) = P_0(B^0_{t-u} > 0 | g_{t-u} = 0)$, it comes

$$\limsup_{t\to\infty} \frac{1}{t} \log K_t^1(-) \leq \limsup_{t\to\infty} 1/t \log K_t^1(+).$$

Hence, it suffices to prove that $\limsup_{t\to\infty} 1/t \log K_t^1(+) \leq \widetilde{\Phi}(\lambda,h,\beta)$. We do it in three steps. The first and the last one are dedicated to subtract and re-add the contribution of the Hamiltonian between $t-1$ and t. For that, we let $l > 1$ and $L > 1$, such that, $1/l + 1/L = 1$, and we obtain

$$\begin{aligned}\frac{1}{t}\log K_t^1(+) \leq\ & \frac{1}{lt}\log E_0\left(\exp\left(H_{0,t-1}(l\lambda,h,l\beta)\right)\ \mathbb{1}_{\{B_0(t)>0\}}\right)\\ &+ \frac{1}{Lt}\log E_0\Big(\exp\Big(L\lambda\int_{t-1}^t \Lambda_s(dR_s + hds) + L\beta(L_t - L_{t-1})\Big)\ \mathbb{1}_{\{B_0(t)>0\}}\Big).\end{aligned} \tag{2.10.13}$$

Then, we denote $P_{s,t}(L) = E_0\Big(\exp\Big(L\lambda\int_s^t \Lambda_u(dR_u + hdu) + L\beta(L_t - L_{t-1})\Big)\Big)$, and, since every $t > 0$ belongs to an interval of type $[n, n+1]$ with $n \in \mathbb{N}$, the Cauchy Schwarz's inequality gives

$$\log P_{t-1,t}(L) \leq \frac{1}{2}\log\left(\sup_{\zeta\in[0,1]} P_{n-\zeta,n}(2L)\right) + \frac{1}{2}\log\left(\sup_{\zeta\in[0,1]} P_{n,n+\zeta}(2L)\right). \tag{2.10.14}$$

By using the fact that, on the one hand, there exists $C > 0$ such that for every $n \in \mathbb{N}$, $\widetilde{\mathbb{E}}\Big(\exp\big(\sup_{\zeta\in[0,1]}\int_n^{n+\zeta}\Lambda_u dR_u\big)\Big) < C$, and that, on the other hand, for every $t \geq 0$, $E_0\Big(\exp(2\beta(L_t - L_{t-1}))\Big) \leq E_0(\exp(2\beta L_1))$, we obtain a constant $Z > 0$, which satisfies

$$\forall n \in \mathbb{N}, \qquad \widetilde{\mathbb{E}}\Big(\sup_{\zeta\in[0,1]} P_{n,n+\zeta}(2L)\Big) < Z.$$

Hence, by Markov's inequality, and for $\varepsilon > 0$, we obtain

$$\forall n \in \mathbb{N}, \qquad \widetilde{\mathbb{P}}\Big(\frac{1}{n}\log\left(\sup_{\zeta\in[0,1]} P_{n,n+\zeta}(2L)\right) > \varepsilon\Big) < Z\exp(-\varepsilon n).$$

Thus, with the help of the Borel Cantelli Lemma, we obtain, $\widetilde{\mathbb{P}}$ almost surely, the inequality $\limsup_{n\to\infty} 1/n \log\big(\sup_{\zeta\in[0,1]} P_{n,n+\zeta}(2L)\big) \leq 0$. We can prove the same result with $P_{n-\zeta,n}(2L)$ instead of $P_{n,n+\zeta}(2L)$. Consequently, with (2.10.14) we can say that the second term of the right hand side of (2.10.13) vanishes as t tends to ∞.

Now, let $I = \{B_0(t-1-u) > 0,\ B_0(t-u) > 0\}$ and also $II = \{B_0(t-1-u) < 0,\ B_0(t-u) > 0\}$. We rewrite the first term of the right hand side of (2.10.13) in dependence of g_{t-1}. It gives

$$\begin{aligned}
&\frac{1}{lt}\log E_0\Big(\exp\Big(H_{0,t-1}(l\lambda,h,l\beta)\Big)\,\mathbb{1}_{\{B_0(t)>0\}}\Big)\\
&\qquad=\frac{1}{lt}\log\int_0^{t-1} E_0\Big(\exp\left(H_{0,u}(l\lambda,h,l\beta)\right)\,\big|\,g_{t-1}=u\Big)\\
&\qquad\qquad\times\Big[e^{\lambda L(R_{t-1}-R_u)+\lambda Lh(t-1-u)}P(I\mid g_{t-1-u}=0)\\
&\qquad\qquad\quad+e^{-\lambda L(R_{t-1}-R_u)-\lambda Lh(t-1-u)}\ P(II\mid g_{t-1-u}=0)\Big]dP_{g_{t-1}}(u).
\end{aligned} \tag{2.10.15}$$

At this stage, we notice that

$$\begin{aligned}
&P\big(B_0(s)\in[1,2],B_0(s-1)>0\mid g_{s-1}=0\big)\\
&\qquad\qquad\geq P\big(B_0(s)\in[1,2],B_0(s-1)\in[0,1]\mid g_{s-1}=0\big),
\end{aligned}$$

and we can use the explicit formulas of the three-dimensional Bessel process (see [35]). It gives

$$P\big(B_0(s-1)\in[0,1]\mid g_{s-1}=0\big)=c\int_0^{1/\sqrt{s-1}} b^2\exp(-b^2/2)db,$$

which behaves like $1/s^{3/2}$ when s tends to ∞. Therefore, there exists a constant $C>0$, such that

$$P\big(B_0(s)\in[1,2],\ B_0(s-1)>0\mid g_{s-1}=0\big)\geq\frac{C}{s^{3/2}}. \tag{2.10.16}$$

The same inequality as (2.10.16) is satisfied for $B_0(s-1)<0$ instead of $B_0(s-1)>0$. In that way, we can write

$$P(I\mid g_{t-1-u}=0)\leq 1\leq P\big(B_0(s)\in[1,2],\ B_0(s-1)>0\mid g_{s-1}=0\big)\frac{T^{3/2}}{C}.$$

The same is true for II instead of I, and $B_0(s-1)<0$ instead of $B_0(s-1)>0$. Therefore, we can bound from above the left hand side of (2.10.15) by

$$\frac{1}{lt}\log E_0\Big(\exp\Big(H_{0,t-1}(l\lambda,h,l\beta)\Big)\,1\!\!1_{\{B_0(t)>0\}}\Big)\le\frac{2}{lt}\log\big(\frac{t^{\frac{3}{2}}}{C}\big)+J,$$

where J takes the value

$$\begin{aligned}J=\frac{1}{lt}\log\int_0^{t-1}&E_0\Big(\exp\left(H_{0,u}(l\lambda,h,l\beta)\right)\Big|\,g_{t-1}=u\Big)\\&\times\Big[e^{\lambda L(R_{t-1}-R_u)+\lambda Lh(t-1-u)}P\big(B_0(s)\in[1,2],\ B_0(s-1)>0\mid g_{s-1}=0\big)\\&+e^{-\lambda L(R_{t-1}-R_u)-\lambda Lh(t-1-u)}P\big(B_0(s)\in[1,2],\ B_0(s-1)<0\mid g_{s-1}=0\big)\Big]\\&\hfill dP_{g_{t-1}}(u).\end{aligned}$$

Then, $J=E_0\big(\exp\big(H_{0,t-1}(l\lambda,h,l\beta)\big)\,1\!\!1_{\{B_0(t)\in[1,2]\}}\big)$, and we obtain the inequality

$$\limsup_{t\to\infty}\frac{1}{t}\log K_t^1\le\limsup_{t\to\infty}\frac{1}{lt}\log E_0\big(\exp\big(H_{0,t-1}(l\lambda,h,l\beta)\big)\,1\!\!1_{\{B_0(t)\in[1,2]\}}\big).$$

Now, it suffices to add the contribution of the hamiltonian between $t-1$ and t. To that aim, (as in step 3), we use the Hölder's inequality with $l'>l$, and the equality between $H_{0,t-1}(l\lambda,h,l\beta)$ and $H_{0,t}(l\lambda,h,l\beta)+H_{t-1,t}(l\lambda,h,l\beta)$. Finally, we obtain that for every $l'>1$, $\limsup_{t\to\infty}(1/t)\log K_t^1\le\widetilde{\Phi}(l'\lambda,h,l'\beta)/l'$. We let l' go to 1 and, by continuity of $\widetilde{\Phi}(l'\lambda,h,l'\beta)$ in l', the proof is completed.

Chapter 3

Copolymer pulled by a force

3.1 Introduction

3.1.1 Discrete model

In this chapter, as in the former one, we consider an (hydrophilic-hydrophobic) copolymer. The type of monomer number i is given by the random variable w_i defined in section 1.1.1. But in this model, contrary to what happens in the former one, the hydrophobic monomers do not interact stronger with both solvents than the hydrophilic monomers (see h factor in 1.1.1). This time, we apply a vertical force of intensity F at the right extremity of the polymer. It is a way to pull the polymer up and away from the interface in the half upper plane. Finally, the possible configurations of the polymer are given, once again, by the trajectories of a simple random walk.

Now for each trajectory of the random walk we define the following hamiltonian

$$H_{N,F}^{w,\lambda}(S) = \lambda \sum_{i=1}^{N} w_i \Delta_i + F S_N.$$

With this hamiltonian we perturb the law of the random walk as follow

$$\frac{dP_{N,F}^{w,\lambda}}{dP}(S) = \frac{\exp\left(H_{N,F}^{w,\lambda}(S)\right)}{Z_{N,F}^{w,\lambda}}.$$

This new measure $P^{w,\lambda}_{N,F}$ is called polymer measure of size N. In this model, the two parts of the hamiltonian have opposite effects on the polymer. The force pulls the chain up and away from the interface in the half upper plane, whereas, similarly to what happens in the former model, the two types of monomers (given by the w_i variables) constrain the chain to remain close to the interface to put as many monomers as possible in their preferred solvent. Thus, a competition between these two possible behaviors arises.

Finally we denote also by i_N the last hitting time of the origin before time N.

3.1.2 Physical motivation

One of the major reasons explaining the recent interest around the question of polymer localization close to an interface is probably related to the huge progresses that have been achieved concerning the structure of the DNA strand. Indeed, a fruitful way to model the two strands of a DNA chain consists in studying either the relative position of two interacting random walks (see [32], [29]), or a single random walk interacting with a flat interface (see [30]).

Recently, a technological innovation called optical tweezer (see [36]) has allowed the micromanipulation of polymer molecules. In particular, it can be applied to denature the DNA chain (separating its two strands) by applying a very small force in a certain point of the chain (see [12]). This gives important new possibilities to manipulate locally a DNA strand, which is a crucial aim of the current research.

Similar models have been studied by physicists (see [29]), but we investigate very precisely in this paragraph the case of an hydro(philic)-(phobic) copolymer in the neighborhood of an oil-water interface and pulled by a force away from the interface at one of its extremities. We give in particular precise conditions of delocalisation with respect to the parameters (λ and F).

3.1.3 Free energy

For this model the free energy is defined as in the previous chapter, i.e.,

$$\Phi^w(\lambda, F) = \lim_{N\to\infty} \frac{1}{N} \log Z^w_{N,F}.$$

Here, denoting the free energy by Φ is an abuse of notation with respect to the first chapter. Indeed, we must keep in mind that, what we call $\Phi(\lambda, 0)$ in this chapter, would have been written $\Phi(0, \lambda, 0)$ with the notations of the previous chapter.

The proof of [17] can not be directly adapted to this model because of the term FS_N, but this free energy will still be a good tool to decide, for fixed parameters (λ, F), if the polymer is localized or not. Therefore, we begin with a different proof of its existence. To that aim we will distinguish what happens before and after i_N (the last hitting time of the origin). Indeed, we will take into account the fact that the force influence depends only on the random walk behavior between i_N and N. This idea allows us to go further in the free energy computation than what we did in the previous chapter. This implies the following theorem.

3.1.4 Theorem 9

Theorem 9 *If λ and F are two non negative parameters, we obtain a.s. in w*

$$\Phi^w(\lambda, F) = \max\left(\Phi(\lambda, 0),\, \frac{1}{2}\log\left(\frac{\cosh(2F)+1}{2}\right)\right).$$

Since this free energy does not depend on w, from now on, it will be denoted by $\Phi(\lambda, F)$. The proof of this theorem is detailed in appendix A.

Remark 4 With theorem 9, we notice that the polymer is delocalized as soon as the quantity $1/2\log((\cosh(2F)+1)/2)$ becomes larger than $\Phi(\lambda, 0)$. Indeed, in this case, the localization effect (given by the term $\lambda \sum_{i=1}^N w_i \Delta_i$) does not influence the value of the free energy. Moreover, a partial differentiation of the free energy with respect to F gives that there exists $\mu(F) > 0$ such that a.s. in w

$$\lim_{N\to\infty} E^{w,F}_{N,\lambda}\left(\frac{S_N}{N}\right) = \mu(F).$$

We will go further in section 2.4 by showing that in this case, the polymer comes back to the origin only a finite number of times. On the contrary, in the localized area, the force F does not give any contribution to the free energy.

Thus, the former remark gives directly the existence of a critical force, above which the polymer is delocalized. This force will be denoted by $F_c(\lambda)$. Moreover $\Phi(\lambda, 0)$ and $1/2 \log((\cosh(2F)+1)/2)$ are continuous and increasing, respectively in λ and F. This implies that the critical curve, described by $F_c(\lambda)$, is continuous and increasing in λ.

3.2 Continuous model

As we did in the second chapter we can define a continuous version of the present model. The possible configurations of a polymer of length t are given by the set of trajectories of a Brownian motion $(B_s)_{s\in[0,t]}$. The law of B will be denoted by $\widetilde{P}$, and we set $\Lambda_s = \text{sign}\left(B_s\right)$. Independently of B, we define $(R_s)_{s\geq 0}$ a standard Brownian motion of law $\widetilde{\mathbb{P}}$. The polymer hydrophobicity around position s is given by dR_s, the elementary variation of R at position s. Finally, the force influence appears (as in the discrete model) with the term FB_t.

Then, for a fixed trajectory of R we can define, for every trajectory of B, the following hamiltonian

$$\widetilde{H}^R_{t,\lambda,F}(B) = \lambda \int_0^t \Lambda(s) dR_s + FB_t. \tag{3.2.1}$$

For simplicity, the hamiltonian will be denoted by $\widetilde{H}^R_t$, and we define the polymer measure

$$\frac{d\widetilde{P}^R_t}{d\widetilde{P}}(B) = \frac{\exp\left(\widetilde{H}^R_t(B)\right)}{\widetilde{Z}^R_{t,,h}},$$

and the associated free energy

$$\widetilde{\Phi}^R_t(\lambda, F) = \lim_{t\to\infty} \frac{1}{t} \log \widetilde{E}\left(\widetilde{Z}^R_t(\lambda, F)\right).$$

But for the continuous model also, the existence of the free energy is far from being easy to obtain. Therefore, as in the discrete case, we are going to prove its existence by using a partial computation of this free energy. It leads us to the following theorem.

3.2.1 Theorem 10

Theorem 10 *Let F and λ be two non negative parameters, then*

$$\widetilde{\Phi}(\lambda, F) = \max\left(\widetilde{\Phi}(\lambda, 0), \frac{F^2}{2}\right).$$

The computations of the discrete and continuous free energies allow us to state the analogue of theorem 3 for this model. Effectively, the theorem 3 applied in the case $(\beta = 0, \lambda, h = 0)$, and an easy computation gives

$$\lim_{a \to 0} \frac{1}{a^2} \Phi(a\lambda, 0) = \widetilde{\Phi}(\lambda, 0) \quad \text{and} \quad \lim_{a \to 0} \frac{1}{a^2} \log\left(\frac{1 + \cosh(2aF)}{2}\right) = \frac{F^2}{2}.$$

Therefore,

$$\lim_{a \to 0} \frac{1}{a^2} \Phi(a\lambda, aF) = \widetilde{\Phi}(\lambda, F),$$

which means that, in the limit of weak coupling, we have the convergence, in terms of free energy, of the discrete model toward the continuous one.

3.3 The delocalized phase (proposition 11)

In this section we aim at understanding deeper the behavior of the polymer in the delocalized phase. Indeed, the localized phase has been closely studied for different models like copolymer at a selective interface and sharp results are available about it (see [37], [5]). But on the contrary the delocalized phase is more complicated to analyze. This is essentially because the copolymer does not come back to the interface with a positive density of times. Therefore, the differentiation of the free energy in h only tells us that the density of steps done by the polymer in the half lower plane tends to 0 as N tends to ∞. But we would like to go further and show that after a finite number of steps the polymer never comes back to the interface. This is detailed in the following proposition, but for simplicity we set first $F_l^N = \{S : \exists k \in \{l, .., N\}$ such that $S_{2k} = 0\}$, and $Z_F = (1 + \cosh(2F))/2$.

Proposition 11 *We consider a couple (λ, F) inside the delocalized area, i.e. satisfying $\phi(\lambda, 0) < \log(Z_F)/2$. Then, $\mathbb{P}$ a.s. in w, there exists $\mu > 0$ and $D > 0$ (depending on w), such that, for every $N \geq 1$ and $l \geq 1$ we have*

$$P_{2N,\lambda}^{w,F}\big(F_l^N\big) \leq D \exp\left(-\mu 2l\right). \tag{3.3.1}$$

Then, if we multiply the right hand term of (3.3.1) by l, we obtain the general term of a convergent serie. Therefore, a.s. in w we can bound from above, independently of N, the quantity $E_{2N,\lambda}^{w,F}\big(\tau_{2N}\big)$, with $\tau_{2N} = \max\{i \in \{0, .., 2N\}$ such that $S_i = 0\}$. This really means that the polymer comes back only a finite number of times to the interface.

3.4 Proof of theorem and proposition

3.4.1 Proof of theorem 10

In this proof we consider the quantity $\widetilde{Z}_t^R(\lambda, F)$, and denote by g_t the last hitting time of zero by $(B_s)_{s\in[0,t]}$ before time t. Then we can write

$$\widetilde{Z}_t^R(\lambda, F) = \int_0^t \widetilde{E}\left(\exp(FB_t)\ \exp\left(\lambda \int_0^u \Lambda_s dR_s + \lambda\Lambda_t(R_t - R_u)\right)\Big| g_t = u\right) d\widetilde{P}_{g_t}(u).$$

But (as proved in [35], chapter XII, p. 492), under the event $\{g_t = u\}$ the processes $(B_s)_{s\in[0,u]}$ and $(B_s)_{s\in[u,t]}$ are independent. This implies

$$\begin{aligned}\widetilde{Z}_t^R(\lambda, F) = \int_0^t \widetilde{E}&\left(\exp\left(\lambda \int_0^u \Lambda_s dR_s\right)\Big| g_t = u\right)\\ &\times \widetilde{E}\left(\exp\left(\lambda\Lambda_t(R_t - R_u) + FB_t\right)\Big| g_t = u\right) d\widetilde{P}_{g_t}(u).\end{aligned}$$

At this stage, we notice that $|R_t - R_u|$ is bounded by $2\max\{|R_s|; s \in [0, t]\}$ uniformly in $u \in [0, t]$. But a.s. in R, we have that $\max\{|R_s|; s \in [0, t]\} = o(t)$ as t tends to ∞.

Therefore, the term $\lambda\Lambda_t(R_t - R_u)$ does not influence the existence and the value of the limit and it suffices to prove

$$\lim_{t\to\infty} \frac{1}{t} \log K_t = \max\left(\widetilde{\Phi}(\lambda, 0), \frac{F^2}{2}\right), \tag{3.4.1}$$

with the quantities $K_t = \int_0^t H_u V_t^u dP_{g_t}(u)$, $H_u = \widetilde{E}\big(\exp\big(\lambda \int_0^u \Lambda_s dR_s\big)\big|g_t = u\big)$ and $V_t^u = \widetilde{E}\big(\exp\big(FB_t\big)\big|g_t = u\big)$.

The proof of (3.4.1) will be performed in three steps. The first one is dedicated to the convergence of $1/t \log V_t^0$ towards $F^2/2$. In the second step, we show that $1/t \log H_u$ tends to $\Phi(\lambda, 0)$, and we complete the proof in a third step.

3.4.2 Step I

First, we notice that for every $u \le t$, $V_t^u = V_{t-u}^0$. That is why it suffices to consider the quantity $1/t \log V_t^0$. The measure of B_t under the event $\{g_t = 0\}$ is explicitely known (see [35], chapter XII, p. 493). It is defined with respect to the Lebesgue measure by the Radon-Nikodým density: $h(x) = |x| \exp(-x^2/2t)/2t$. Hence, we set $v = x/\sqrt{t}$, and an easy computation gives us

$$V_t^0 = \int_{\mathbb{R}} \frac{|x| \exp\left(Fx - \frac{x^2}{2t}\right)}{2t} dx = \frac{\exp(\frac{F^2 t}{2})}{2} \int_{\mathbb{R}} |v + \sqrt{t}F| \exp\left(-\frac{v^2}{2}\right) dv.$$

Then we can bound the quantity $A_t^F = (1/2) \int_{\mathbb{R}} |v + \sqrt{t}F| \exp\big(-\frac{v^2}{2}\big) dv$ as follow

$$\int_0^\infty \frac{v}{2} \exp\left(-\frac{v^2}{2}\right) dv \le A_t^F \le \int_{\mathbb{R}} \frac{|v|}{2} \exp\left(-\frac{v^2}{2}\right) dv + \frac{\sqrt{t}F}{2} \int_{\mathbb{R}} \exp\left(-\frac{v^2}{2}\right) dv. \tag{3.4.2}$$

Therefore, the term A_t^F has no contribution to the limit of $1/t \log V_t^0$ when t tends to ∞. We obtain

$$\lim_{t\to\infty} \frac{1}{t} \log V_t^0 = \frac{F^2}{2}. \tag{3.4.3}$$

3.4.3 Step II

From now on we let $\widetilde{P}_{b.b.}^{u,b}$ be the law of a Brownian Bridge, starting in 0 and arriving in b at time u. We consider the quantity $H_u = \widetilde{E}(\exp(\lambda \int_0^u \Lambda_s dR_s))|g_t = u)$, and we notice that $\widetilde{P}(.|g_t = u) = \widetilde{P}^{u,0}_{b.b.}$ (see [35]), so that we can rewrite $H_u = \widetilde{E}^{u,0}_{b.b.}(\exp(\lambda \int_0^u \Lambda_s dR_s))$.

To go further in this step, we must prove first the following result

$$\lim_{t\to\infty} \frac{1}{t} \log \widetilde{E}^{t,0}_{b.b.} \left(\exp\left(\lambda \int_0^{t-1} \Lambda_s dR_s \right) \mathbb{1}_{\{B_{t-1}\in[-1,1]\}} \right) = \widetilde{\Phi}(\lambda, 0). \tag{3.4.4}$$

In the previous chapter we obtained

$$\lim_{t\to\infty} \frac{1}{t} \log \widetilde{E} \left(\exp\left(\lambda \int_0^{t} \Lambda_s dR_s \right)\right) = \widetilde{\Phi}(\lambda, 0), \tag{3.4.5}$$

but with the help of an adequate Hölder inequality, we obtain (as we did in the step 4 of the proof of proposition 2) that the limit (3.4.5) remains the same when we replace $\int_0^t \Lambda_s dR_s$ by $\int_0^{t-1} \Lambda_s dR_s$. Then, we notice that for every $s \in [0,t]$, we have the inequality

$$\widetilde{P}(B_0(s) \in [-1,1]|g_s = 0) \;\geq\; \widetilde{P}(B_0(t) \in [-1,1]|g_t = 0) = 1 - e^{-\frac{1}{2t}}. \tag{3.4.6}$$

Finally, by rewriting $M_t = \widetilde{E}\big(\exp(\lambda \int_0^{t-1} \Lambda_s dR_s) \, \mathbb{1}_{\{B_{t-1}\in[-1,1]\}}\big)$ in dependence of g_{t-1}, we obtain

$$\begin{aligned} M_t = \int_0^{t-1} & \widetilde{E}^{u,0}_{b.b.}\Big(\exp\left(\lambda \int_0^u \Lambda_s dR_s \right)\Big) \\ & \times \widetilde{E}\Big(\exp\big(\lambda \Lambda_{t-1-u}(R_{t-1} - R_u)\big) | g_{t-1-u} = 0\Big) d\widetilde{P}_{g_{t-1}}(u). \end{aligned}$$

Thus, since $\max\{|R_s|; s \in [0,t]\} = o(t)$ when t tends to ∞, we do not need to take into account the term $\exp\big(\lambda \Lambda_{t-1-u}(R_{t-1} - R_u)\big)$. Then, using (3.4.6), and since $1/t \log(1 - \exp(-1/2t))$ tends to zero as t tends to ∞, we obtain $\lim_{t\to\infty} 1/t \log M_t = \widetilde{\Phi}(\lambda, 0)$. Now, we notice that

$$\begin{aligned} N_t &= \widetilde{E}^{t,0}_{b.b.}\Big(\exp\left(\lambda \int_0^{t-1} \Lambda_s dR_s \right) \mathbb{1}_{\{B_{t-1}\in[-1,1]\}} \Big) \\ &= \int_{-1}^{1} \widetilde{E}^{t-1,u}_{b.b.}\Big(\exp\left(\lambda \int_0^{t-1} \Lambda_s dR_s \right)\Big) d\widetilde{P}^{t,0}_{b.b.}(B_{t-1} = u), \end{aligned} \tag{3.4.7}$$

and a simple computation gives us

$$\begin{aligned} d\widetilde{P}^{t,0}_{b.b.}(B_{t-1} = u) &= \sqrt{\frac{t}{2\pi(t-1)}} \, \exp\Big(- \frac{u^2 t}{2(t-1)} \Big) \\ &= \sqrt{t} \exp\Big(- \frac{u^2}{2} \Big) \, d\widetilde{P}(B_{t-1} = u). \end{aligned} \tag{3.4.8}$$

Therefore, for every $u \in [-1,1]$, we have

$$e^{-1}\sqrt{t}\,d\widetilde{P}(B_{t-1}=u)\ \le d\widetilde{P}^{t,0}_{b.b.}(B_{t-1}=u)\le\ \sqrt{t}\,d\widetilde{P}(B_{t-1}=u).$$

Consequently, replacing $d\widetilde{P}(B_{t-1}=u)$ by $d\widetilde{P}^{t,0}_{b.b.}(B_{t-1}=u)$ in (3.4.7) does not change the value of $\lim_{t\to\infty} 1/t\log N_t$. It is enough to complete the proof of (3.4.4).

Now, it remains to show the same convergence, without restricting the computation to the event $\{B_{t-1}\in[-1,1]\}$. Then, rewriting N_t with respect to g_{t-1} we obtain

$$\begin{aligned} N_t=\frac{1}{t}\log\int_0^{t-1}&\widetilde{E}^{t,0}_{b.b.}\left(\exp\left(\int_0^u\lambda\Lambda_s dR_s\right)\Big|g_{t-1}=u\right)\\ &\times E^{t,0}_{b.b.}\left(\exp\left(\lambda\Lambda_{t-1}(R_{t-1}-R_u)\right)\mathbb{1}_{\{B_{t-1}\in[-1,1]\}}\Big|g_{t-1}=u\right)d\widetilde{P}_{g_{t-1}}(u). \end{aligned}$$

We notice that

$$\widetilde{E}^{t,0}_{b.b.}\left(\exp\left(\int_0^u\lambda\Lambda_s dR_s\right)\Big|g_{t-1}=u\right)=\widetilde{E}^{u,0}_{b.b.}\left(\exp\left(\int_0^u\lambda\Lambda_s dR_s\right)\right)=H_u,$$

and, using the property $\max\{|R_s|;s\in[0,t]\}=o(t)$, we get that N_t has the same limit as

$$V_t=\frac{1}{t}\log\int_0^{t-1}H_u\ \widetilde{P}^{t,0}_{b.b.}\big(B_{t-1}\in[-1,1]|g_{t-1}=u\big)\ d\widetilde{P}_{g_{t-1}}(u).$$

We proved in appendix B that independently of t and u, there exists a constant $c>0$ satisfying

$$\widetilde{P}^{t,0}_{b.b.}\big(B_{t-1}\in[-1,1]|g_{t-1}=u\big)\ge c.$$

Therefore, we have the following convergence

$$\lim_{t\to\infty}\frac{1}{t}\log\int_0^{t-1}H_u\ d\widetilde{P}_{g_{t-1}}(u)=\widetilde{\Phi}(\lambda,0),$$

and since $\sup_{s\in[0,t]}|R_s|=o(t)$, we obtain easily

$$\lim_{t\to\infty}\frac{1}{t}\log\int_0^{t-1}H_u\widetilde{E}^{t,0}_{b.b.}\left(\exp\left(\lambda\Lambda_{t-1}(R_{t-1}-R_u)\right)\Big|g_{t-1}=u\right)d\widetilde{P}_{g_{t-1}}(u)=\widetilde{\Phi}(\lambda,0). \tag{3.4.9}$$

But we have also

$$H_u\ \widetilde{E}^{t,0}_{b.b.}\left(\exp\left(\lambda\Lambda_{t-1}(R_{t-1}-R_u)\right)\Big|g_{t-1}=u\right)=\widetilde{E}^{t,0}_{b.b.}\left(\exp\left(\lambda\int_0^{t-1}\Lambda_s dR_s\right)\Big|g_{t-1}=u\right),$$

then (3.4.9) can also be writen as

$$\lim_{t\to\infty}\frac{1}{t}\log\widetilde{E}^{t,0}_{b.b.}\left(\exp\left(\lambda\int_0^{t-1}\Lambda_s dR_s\right)\right)=\widetilde{\Phi}(\lambda,0).$$

Finally, an adapted Hölder inequality allows us to prove (3.4.9) with $\int_0^t$ instead of $\int_0^{t-1}$, and the proof of step 2 is completed.

3.4.4 Step III

In this step, we mimic the step 2 of theorem 9. We begin with proving the first of the two required inequalities, i.e., $\liminf_{t\to\infty} 1/t\log K(t)\geq\max\{\widetilde{\Phi}(\lambda,0),F^2/2\}$. To that aim, for every $\varepsilon>0$, we bound from below the term $K(t)$ by $\int_0^\varepsilon H_u V^0_{t-u}d\widetilde{P}_{g_t}(u)$.

As proved in step 2, there exists s_0 such that for every $s\geq s_0$ the inequality $V^0_s\geq\exp((\frac{F^2}{2}-\varepsilon)s)$ occurs. Then if we choose $t\geq s_0+\varepsilon$ we can write

$$\frac{1}{t}\log\int_0^\varepsilon H_u V^0_{t-u}d\widetilde{P}_{g_t}(u)\geq(\frac{F^2}{2}-\varepsilon)(1-\varepsilon)+\frac{1}{t}\log\int_0^\varepsilon H_u d\widetilde{P}_{g_t}(u). \tag{3.4.10}$$

At this stage we notice that $d\widetilde{P}_{g_t}(u)/du=1/(\pi\sqrt{u}\sqrt{t-u})$ and we can write

$$\frac{1}{t}\log\int_0^\varepsilon H_u d\widetilde{P}_{g_t}(u)\geq\frac{1}{t}\log\int_0^\varepsilon H_u\frac{1}{\sqrt{\varepsilon t}}du\geq-\frac{1}{2t}\log(\varepsilon t)+\frac{1}{t}\log\int_0^\varepsilon H_u du.$$

Therefore, the term $\int_0^\varepsilon H_u du$ does not depend on t any more and consequently the r.h.s. of (3.4.10) tends to $(\frac{F^2}{2}-\varepsilon)(1-\varepsilon)$ as t tends to ∞. Then, we let ε tend to zero and we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log K(t)\geq\frac{F^2}{2}.$$

Then, it remains to prove the same inequality with $\widetilde{\Phi}(\lambda,0)$ instead of $F^2/2$. To that aim, we use the same method, i.e., we bound K_t from below by $\int_{t-\varepsilon}^t H_u V^0_{t-u}d\widetilde{P}_{g_t}(u)$ and we perform the same type of computation. I do not give the details here because the proof is too similar to the one we just did.

At this stage, we must show that $\liminf_{t\to\infty} 1/t \log K(t) \leq \max\{\widetilde{\Phi}(\lambda,0), F^2/2\}$, and this will complete the proof of theorem 9. To that aim we separate $K(t) = \int_0^t H_u V^0_{t-u} du$ in three parts. Thus, we set

$$A_1(t) = \int_0^{\sqrt{t}} H_u \; V^0_{t-u} \; d\widetilde{P}_{g_t}(u), \qquad A_2(t) = \int_{\sqrt{t}}^{t-\sqrt{t}} H_u \; V^0_{t-u} \; d\widetilde{P}_{g_t}(u),$$
$$A_3(t) = \int_{t-\sqrt{t}}^{t} H_u \; V^0_{t-u} \; d\widetilde{P}_{g_t}(u),$$

so that $K(t) = A_1(t) + A_2(t) + A_3(t)$. A well known formula gives

$$\limsup_{t\to\infty} \frac{1}{t} \log K(t) = \max_{i\in\{1,2,3\}} \left\{ \limsup_{t\to\infty} \frac{1}{t} \log A_i(t) \right\}.$$

We recall that (see (3.4.2)), $V^0_t = \exp(F^2 t/2) A^F_t$ with

$$\max_{u\in[0,t]} A^F_u \leq \int_{\mathbb{R}} \frac{|v|}{2} \exp\Big(-\frac{v^2}{2}\Big) dv + \frac{\sqrt{t}F}{2} \int_{\mathbb{R}} \exp\Big(-\frac{v^2}{2}\Big) dv. \tag{3.4.11}$$

Therefore, we obtain

$$\lim_{t\to\infty} \frac{1}{t} \log \max_{u\in[0,t]} A^F_u = 0,$$

and instead of $A_i(t)$ it suffices to consider $A'_i(t) = \int H_u \exp\left(F^2(t-u)/2\right) dP_{g_t}(u)$.

To begin, we consider the case $i = 1$ and we notice immediately that

$$\frac{1}{t} \log A'_1(t) \leq \frac{1}{t} \log \int_0^{\sqrt{t}} H_u dP_{g_t}(u) + \frac{F^2}{2}.$$

As proved in step 2, there exists $M > 0$ and $t_0 > 0$ such that for every $t \geq t_0$, $H_u \leq \exp(Mu)$. Therefore, we obtain the following upper bound

$$\frac{1}{t} \log A'_1(t) \leq \frac{1}{t} \log \int_0^{t_0} H_u dP_{g_t}(u) + \frac{M(\sqrt{t}-t_0)}{t} + \frac{F^2}{2} \tag{3.4.12}$$

and the r.h.s. of (3.4.12) tends to $F^2/2$ as t tends to ∞. It completes the case $i = 1$.

The case $i = 3$ is very similar to the former one. We write first that

$$\frac{1}{t} \log A'_3(t) \leq \frac{F^2\sqrt{t}}{2t} + \frac{1}{t} \log \int_{t-\sqrt{t}}^{t} H_u dP_{g_t}(u),$$

and as proved in step 2, we have the convergence of $1/t \log H_u$ towards $\widetilde{\Phi}(\lambda,0)$. Then, for every $\varepsilon > 0$ there exists $t_0 > 0$ such that for every $u \geq t_0$ $H_u \leq \exp\Big(u(\widetilde{\Phi}(\lambda,0)+\varepsilon)\Big)$. Hence, for t satisfying $t - \sqrt{t} \geq t_0$, we obtain

$$\frac{1}{t} \log \int_{t-\sqrt{t}}^{t} H_u dP_{g_t}(u) \leq (\Phi(\lambda,0)+\varepsilon)\Big(1 - \frac{1}{\sqrt{t}}\Big).$$

Therefore, for every $\varepsilon > 0$, we can write $\limsup_{t\to\infty} \frac{1}{t} \log A_3'(t) \leq \Phi(\lambda, 0) + \varepsilon$, and this completes the proof of case $i = 3$.

It remains to conclude with the case $i = 2$. For simplicity we assume that $F^2/2 > \widetilde{\Phi}(\lambda, 0)$, and the case $\widetilde{\Phi}(\lambda, 0) \geq F^2/2$ will not be detailed here because it is absolutely similar to this one. Then, we know by step 2 that for every $\varepsilon > 0$ satisfying $\widetilde{\Phi}(\lambda, 0) + \varepsilon < F^2/2$, there exists t_0 large enough such that $t \geq t_0$ implies $H_u \leq \exp\big(u(\widetilde{\Phi}(\lambda, 0) + \varepsilon)\big)$. This gives

$$\begin{aligned} \frac{1}{t} \log A_2'(t) &\leq \frac{1}{t} \log \int_{\sqrt{t}}^{t-\sqrt{t}} \exp\big(u(\widetilde{\Phi}(\lambda, 0) + \varepsilon)\big) \exp\Big(\frac{F^2(t-u)}{2}\Big) dP_{g_t}(u) \\ &\leq \frac{F^2}{2} + \frac{1}{t} \log \int_{\sqrt{t}}^{t-\sqrt{t}} \exp\Big(\Big(\widetilde{\Phi}(\lambda, 0) + \varepsilon - \frac{F^2}{2}\Big) u\Big) dP_{g_t}(u). \end{aligned} \tag{3.4.13}$$

But, since $\widetilde{\Phi}(\lambda, 0) + \varepsilon - F^2/2 < 0$, the superior limit of the r.h.s. of (3.4.13) is equal to $F^2/2$. It completes the proof of theorem 10.

3.4.5 Proof of proposition 11

First we set a few notations,

- $F_l^N = \{S : \exists k \in \{l, .., N\}$ such that $S_{2k} = 0\}$,

- $H_{2k} = \{S : S_i \neq 0, \forall\, i \in < 1, 2k >\}$,

- $P_k^N = \{S : S_{2k} = 0$ and $S_i \neq 0, \forall\, i \in < 2(k+1), 2N >\}$,

- $H_{2k}^+ = \{S : S_i > 0, \forall\, i \in < 1, 2k >\}, \quad H_{2k}^- = \{S : S_i < 0, \forall\, i \in < 1, 2k >\}$.

Then, we consider the quantity

$$P_{2N,\lambda}^{w,F}(F_l^N) = \frac{1}{Z_{2N,\lambda}^{w,F}} E\Big(\exp\Big(-2\lambda \sum_{i=1}^{2N} w_i \Delta i + F S_{2N}\Big) 1_{\{F_l^N\}}\Big),$$

and by using the position of the last return to origin and the Markov property, we can write

$$P^{w,F}_{2N,\lambda}\big(F^N_l\big) = \sum_{k=l}^{N} P^{w,F}_{2N,\lambda}\big(P^N_k\big)$$
$$= \frac{1}{Z^{w,F}_{2N,\lambda}} \sum_{k=l}^{N} E\Big(\exp\Big(-2\lambda \sum_{i=1}^{2k} w_i \Delta i\Big) 1_{\{S_{2k}=0\}}\Big)$$
$$\times E\Big(\exp(-2\lambda\Delta_1(W_{2N}-W_{2k}) + FS_{2(N-k)}) 1_{\{H_{2(N-k)}\}}\Big). \tag{3.4.14}$$

We denote by $R^N_k = E\big(\exp(-2\lambda\Delta_1(W_{2N}-W_{2k}) + FS_{2(N-k)}) 1_{\{H_{2(N-k)}\}}\big)$ and we can take into account the fact that the last excursion can be either above or under the interface. It gives

$$R^N_k = E\Big(\exp(FS_{2N-k}) 1_{\{H^+_{2(N-k)}\}}\Big) + E\Big(\exp(-2\lambda(W_{2N}-W_{2k}) + FS_{2(N-k)}) 1_{\{H^-_{2(N-k)}\}}\Big).$$

Now, if we let $M_{2N} = \max\{\mid W_i \mid, i \in < 1, 2N >\}$ we can bound R^N_K from above as

$$R^N_K \le E\Big(\exp(FS_{2(N-k)}) 1_{\{H^+_{2(N-k)}\}}\Big) + \exp(4\lambda M_{2N}).$$

Therefore, with (3.4.14) we obtain an upper bound of $P^{w,F}_{2N,\lambda}\big(P^N_l\big)$, i.e., $P^{w,F}_{2N,\lambda}\big(P_l\big) \le V_{l,N} + K_{l,N}$ with

$$V_{l,N} = \frac{1}{Z^{w,F}_{2N,\lambda}} \sum_{k=l}^{N} E\left(\exp\left(-2\lambda \sum_{i=1}^{2k} w_i \Delta i\right) 1_{\{S_{2k}=0\}}\right) E\left(\exp(FS_{2(N-k)}) 1_{\{H^+_{2(N-k)}\}}\right) \tag{3.4.15}$$

and

$$K_{l,N} = \frac{\exp(4\lambda M_{2N})}{Z^{w,F}_{2N,\lambda}} \sum_{k=l}^{N} E\left(\exp\left(-2\lambda \sum_{i=1}^{2k} w_i \Delta i\right) 1_{\{S_{2k}=0\}}\right). \tag{3.4.16}$$

Thus, we consider the term $V_{l,N}$ and as proved before

$$E\Big(\exp(FS_{2N}) 1_{\{H^+_{2N}\}}\Big) = E\Big(\exp(FS_{2N}) 1_{\{H_{2N}\}}\Big) + O(1)$$
$$= Z^{N-1}_F \sinh(2F) + O(1). \tag{3.4.17}$$

But almost surely in w we know that

$$\lim_{N\to\infty} \frac{1}{2N} \log E\left(\exp\left(\lambda \sum_{i=1}^{2N} w_i \Delta i\right) 1_{\{S_{2N}=0\}}\right) = \phi(\lambda, 0),$$

and if we set $c > 1$ such that $c\phi(\lambda, 0) < \log(Z_F)/2$ we obtain a $N_0 > 0$ such that for every $N \geq N_0$

$$E\left(\exp\left(\lambda \sum_{i=1}^{2N} w_i \Delta_i\right) 1_{\{S_{2N}=0\}}\right) \leq \exp\left(c\ \phi\left(\lambda, 0\right)\ 2N\right). \tag{3.4.18}$$

Moreover, $Z_{2N,\lambda}^{w,F} \geq E\Big(\exp(FS_{2N})1_{\{H_{2N}^{+}\}}\Big) \geq \sinh\left(2F\right) Z_F^{N-1} + O\left(1\right)$, so using (3.4.15), (3.4.17) and (3.4.18) we can write for $B > 0$ large enough,

$$\begin{aligned} V_{l,N} &\leq \sum_{k=l}^{N} \frac{\exp\left(c\ \phi\left(\lambda, 0\right)\ 2k\right)\left(Z_F^{N-k-1}\sinh\left(2F\right) + O\left(1\right)\right)}{\sinh\left(2F\right) Z_F^{N-1} + O\left(1\right)} \\ &\leq B \sum_{k=l}^{N} \exp\left(\left(c\ \phi\left(\lambda, 0\right) - \frac{1}{2}\log\left(Z_F\right)\right) 2k\right) \\ &\leq B \sum_{k=l}^{\infty} \exp\left(\left(c\ \phi\left(\lambda, 0\right) - \frac{1}{2}\log\left(Z_F\right)\right) 2k\right). \end{aligned}$$

Thus, the right hand side tends to 0 as l tends to ∞ and if we denote by $\mu = -c\ \phi\left(\lambda, 0\right) + \log\left(Z_F\right)/2$ we obtain for $B' > 0$ large enough,

$$V_{l,N} \leq B' \exp\left(-\mu 2l\right). \tag{3.4.19}$$

Then, we consider the term $K_{l,N}$, and we aim at proving again that it tends to 0 as N tends to ∞. That is to say for $l > N_0$,

$$K_{l,N} = \frac{\exp\left(4\lambda M_{2N}\right)}{Z_{2N,\lambda}^{w,F}} \sum_{k=l}^{N} \exp\left(c\ \phi\left(\lambda, 0\right)\ 2k\right).$$

Therefore, as above

$$K_{l,N} \leq \frac{\exp\left(4\lambda M_{2N}\right)}{Z_F^{N-1}\sinh\left(2F\right) + O\left(1\right)} \sum_{k=l}^{N} \exp\left(c\ \phi\left(\lambda, 0\right)\ 2k\right),$$

hence, for $A > 0$ large enough,

$$K_{l,N} \leq A \frac{\exp\left(4\lambda M_{2N}\right)}{Z_F^{N}} \sum_{k=l}^{N} \exp\left(c\ \phi\left(\lambda, 0\right)\ 2k\right).$$

But the law of large number implies that a.s. in w, $M_{2N}/2N \overset{N\to\infty}{\to} 0$. Now, we set $\epsilon > 0$ such that: $c\,\phi(\lambda,0)+\epsilon < \log(Z_F)/2$ and for this ϵ, there exists N_1 such that, for all $N > N_1$, $M_{2N} < 2N\epsilon$. Hence, for $l > \max(N_0, N_1)$,

$$K_{l,N} \leq A \sum_{k=l}^{\infty} \exp\left(\left(\phi(\lambda,0)+\epsilon - \frac{1}{2}\log(Z_F)\right) 2k\right). \tag{3.4.20}$$

The right hand side of (3.4.20) tends to 0 as l tends to ∞ and if we denote by $\mu' = -\phi(\lambda,0) - \epsilon + \log(Z_F)/2$ we obtain, for A' large enough,

$$K_{l,N} \leq A' \exp(-\mu' 2k). \tag{3.4.21}$$

Combining (3.4.18) and (3.4.21), we put $\mu'' = \min\{\mu, \mu'\}$ and for D large enough we obtain

$$P^{w,F}_{2N,\lambda}\left(F^N_l\right) \leq D \exp(-\mu'' 2l).$$

The proof of proposition 11 is therefore completed.

3.5 Copolymer under an asymmetric random walk

In this part we draw a link between this model (copolymer under the influence of a force) and a model of copolymer defined under an asymmetric random walk.

We set $\varepsilon \in [0,1]$ and we define and i.i.d. sequence of random variables $(X_i)_{i\geq 0}$ as follow

$$P_\varepsilon\big(X_1 = 1\big) = \frac{1-\varepsilon}{2}, \quad P_\varepsilon\big(X_1 = 0\big) = \varepsilon \quad \text{and} \quad P_\varepsilon\big(X_1 = -1\big) = \frac{1-\varepsilon}{2}.$$

Now we build $P^{F,w}_{N,\lambda}$, a copolymer measure under the influence of the force F. By Radon Nikodým density, we define this measure with respect to P_ε as

$$\frac{dP^{F,w}_{N,\lambda}}{dP_\varepsilon}(S) = \frac{\exp\left(\lambda \sum_{i=1}^N w_i \Delta_i + F S_N\right)}{E_\varepsilon\left(\exp\left(\lambda \sum_{i=1}^N w_i \Delta_i + F S_N\right)\right)}.$$

We define also p, q, r, three non negative parameters and $(X_i)_{i\geq 0}$ an i.i.d. sequence of asymmetric random variables given by

$$P_{\text{asy}}\big(X_1 = 1\big) = p, \quad P_{\text{asy}}\big(X_1 = 0\big) = q \quad \text{and} \quad P_{\text{asy}}\big(X_1 = -1\big) = r.$$

Now we define $P^w_{N,\lambda}$, a copolymer measure, built by Radon Nikodým density with respect to P_{asy}

$$\frac{dP^w_{N,\lambda}}{dP_{\text{asy}}}(S) = \frac{\exp\left(\lambda \sum_{i=1}^N w_i \Delta_i\right)}{E_{\text{asy}}\left(\exp\left(\lambda \sum_{i=1}^N w_i \Delta_i\right)\right)}.$$

At this stage, we notice that with an adapted choice of ε and F, this two systems are equivalent. Namely, if p, q, r are fixed (and therefore satisfy $p + q + r = 1$), we can denote by

$$\begin{aligned} &U_1 = \sharp\{i \in \{1,..,N\} : X_i = 1\}, \quad U_2 = \sharp\{i \in \{1,..,N\} : X_i = 0\}, \\ &\text{and} \quad U_3 = \sharp\{i \in \{1,..,N\} : X_i = -1\}, \end{aligned}$$

and if we choose

$$\varepsilon = \frac{q}{q + 2\sqrt{rp}}, \quad c = q + 2\sqrt{rp}, \quad F = \frac{1}{2}\log\left(\frac{p}{r}\right),$$

we obtain the following equalities

$$p = c\, e^F \frac{1-\varepsilon}{2}, \quad q = c\varepsilon \quad \text{and} \quad r = c\, e^{-F}\frac{1-\varepsilon}{2}.$$

Hence, for every trajectory of the random walk we obtain

$$P^w_{N,\lambda}(S) = P^{F,w}_{N,\lambda}(S) = \frac{\exp\left(\lambda \sum_{i=1}^N w_i \Delta_i\right)}{Z}\, p^{U_1} q^{U_2} r^{U_3},$$

and consequently $P^{F,w}_{N,\lambda} = P^w_{N,\lambda}$.

3.6 Appendix

3.6.1 A: proof of theorem 9

We recall that the variables $(w_i)_{i\geq 1}$ are bounded (see the definition of w in the chapter 2). In this proof, for simplicity, we will assume that they are bounded by 1.

First of all we use the Markov property to rewrite $Z^{F,w}_{N,\lambda}$ in dependence of i_N. We obtain

$$Z^{F,w}_{2N,\lambda} = \sum_{i=0}^{N} U_{2i}\; V^{2i}_{2(N-i)},$$

with the notations

$$U_{2i} = E\left(\exp\left(\lambda\sum_{j=1}^{2i} w_j\,\Delta_j\right)\,\mathbb{1}_{\{S_{2i}=0\}}\right)$$

$$\text{and}\quad V_{2(N-i)}^{2i} = E\Bigg(\exp\left(\lambda\sum_{j=1}^{2(N-i)} w_{j+2i}\,\Delta_j + FS_{2(N-i)}\right)\mathbb{1}_{\{S_j\neq 0,\forall j\in\{1,2(N-i)\}\}}\Bigg).$$

Chapter 1 gives us, a.s. in w, the convergence of $1/2N\log U_{2N}$ toward $\Phi(\lambda,0)$, and in a first step we will show that, for every $i\in\mathbb{N}$

$$\lim_{N\to\infty}\frac{1}{2N}\log V_{2N}^{2i} = \frac{1}{2}\log\Big(\frac{\cosh(2F)+1}{2}\Big). \tag{3.6.1}$$

With this two limits, and keeping in mind the convergence, when N tends to ∞, of the term $1/N\log\sum_{i=0}^{N}\exp(ai)\exp(b(N-i))$ toward $\max(a,b)$, we will show in a second step that a.s. in w

$$\lim_{N\to\infty}\frac{1}{2N}\log\sum_{k=0}^{N}U_{2k}V_{2(N-k)}^{2k} = \max\left(\Phi(\lambda,0),\frac{1}{2}\log\Big(\frac{\cosh(2F)+1}{2}\Big)\right). \tag{3.6.2}$$

This will complete the proof.

3.6.2 Step I

We prove (3.6.1) in the case $i=0$, and the proof is exactly the same for the other $i\in\mathbb{N}$. We denote by

$$p_{2N}(x) = P\big(S_j\neq 0\ \forall j\in\{1,..,2N\},\text{and } S_{2N}=x\big),$$

and we can write

$$V_{2N}^{0} = \sum_{x\in\mathbb{Z}-\{0\}}\exp\left(\lambda\left(\sum_{j=1}^{2N}w_j\right)\text{sign}(x)\right)\exp(2Fx)p_{2N}(x).$$

But the law of large number can be applied to $(w_i)_{i\geq 1}$. It implies $\sum_{j=1}^{2N}w_j = o(2N)$. That is why it suffices to obtain the limit of $1/2N\log\big(\sum_{x\in\mathbb{Z}-\{0\}}\exp(2Fx)p_{2N}(x)\big)$. Moreover, we notice that $\varepsilon_1(N)=\sum_{x<0}\exp(2Fx)p_{2N}(x)$ is a bounded function.

To go further in this computation, we consider $(S_n^*)_{n\geq 0}$ the random walk satisfying $S_n^* = S_{2n}/2$ for every $n \geq 0$. Then for every $x \in \mathbb{N} - \{0\}$, the Markov property and the reflection principle allow us to write

$$p_{2N}(x) = \frac{1}{4}\big(P(S_{N-1}^* = x-1) - P(S_{N-1}^* = -x-1)\big). \tag{3.6.3}$$

By using (3.6.3), we can write

$$\sum_{x\in\mathbb{Z}-\{0\}} \exp(2Fx)p_{2N}(x)\big)$$
$$= \sum_{x\geq 1} \frac{\exp(2Fx)}{4}\left(P(S_{N-1}^* = x-1) - P(S_{N-1}^* = -x-1)\right) + \epsilon_1(N).$$

But here again, if we denote by $\epsilon_2(N)$ the sum over $\{x \leq 0\}$ of the quantities $\exp(2Fx)(P(S_{N-1}^* = x-1) - P(S_{N-1}^* = -x-1))/4$, we notice that ϵ_2 is a bounded function. Finally we denote by $\varepsilon = \varepsilon_1 - \varepsilon_2$ and we obtain

$$\begin{aligned}\sum_{x\in\mathbb{Z}-\{0\}} \exp(2Fx)p_{2N}(x) &= \sum_{x\in\mathbb{Z}} \frac{\exp(2Fx)}{4} P(S_{N-1}^* = x-1)\\ &\quad - \sum_{x\in\mathbb{Z}} \frac{\exp(2Fx)}{4} P(S_{N-1}^* = -x-1) + \varepsilon(N)\\ &= \frac{E\big(\exp(2F(1+S_{N-1}^*))\big)}{4} - \frac{E\big(\exp(-2F(1+S_{N-1}^*))\big)}{4} + \varepsilon(N).\end{aligned}$$

Moreover, an easy computation gives us that $E(\exp(2FS_N^*)) = (1/2 + \cosh(2F)/2)^N$, and $\varepsilon(N)$ is a bounded function. Consequently, we obtain that

$$\lim_{N\to\infty} \frac{1}{2N} \log\left(\sum_{x\in\mathbb{Z}-\{0\}} \exp(2Fx)p_{2N}(x)\right) = \frac{1}{2}\log\left(\frac{1+\cosh(2F)}{2}\right)$$

and the proof is completed.

3.6.3 Step II

For simplicity we set $a = \Phi(\lambda, 0)$ and $b = \frac{1}{2}\log\big(1/2 + \cosh(2F)/2\big)$. Then since U_0 and V_0^{2N} are equal to 1, (3.6.1) gives us

$$\liminf_{N\to\infty} \frac{1}{2N} \log \sum_{k=0}^{N} U_{2k} V_{2(N-k)}^{2k} \geq \max(a,b). \tag{3.6.4}$$

Thus, it remains to prove the opposite inequality. To obtain this result we divide the sum in three parts, i.e.,

$$A_1(N)=\sum_{k=0}^{[\sqrt{N}]} U_{2k}V_{2(N-k)}^{2k},\ A_2(N)=\sum_{[\sqrt{N}]}^{N-[\sqrt{N}]} U_{2k}V_{2(N-k)}^{2k},\ A_3(N)=\sum_{N-[\sqrt{N}]}^{N} U_{2k}V_{2(N-k)}^{2k}$$

and the l.h.s. of (3.6.4) is equal to $\max_{i\in\{1,2,3\}}\{\limsup 1/2N\log A_i(N)\}$. Therefore, for every $i\in\{1,2,3\}$, we must prove that $\limsup_{N\to\infty} 1/2N\log A_i(N)\le \max(a,b)$.

First, we consider the case $i=1$. The case $i=3$, which is very similar to $i=1$ is let to the reader. We notice easily that for every $k\in\{0,..,[\sqrt{N}]\}$, $U_{2k}\le \exp(2\lambda\sqrt{N})$ and $|\sum_{j=1}^{2(N-k)} w_{j+2k}|\le|\sum_{j=1}^{2N} w_j|+2\sqrt{N}$. Therefore, we recall that a.s. in w, $\sum_{j=1}^{2N} w_j=o(N)$ and since all the trajectories involved in the computation of V_{2N}^{2k} keep a constant sign between $j=1$ and $j=2k$, it suffices to show that

$$\limsup_{N\to\infty}\frac{1}{2N}\log\sum_{k=0}^{[\sqrt{N}]} S_{2(N-k)}\le\max(a,b) \tag{3.6.5}$$

with $S_{2j}=E\big(\exp(FS_{2j})\,\mathbb{1}_{\{S_i\neq 0,\forall i\in\{1,2j\}\}}\big)$.

We proved in step 1 that $1/2N\log S_{2N}$ tends to b as N tends to ∞, and since $N-\sqrt{N}$ tends to ∞ with N, we obtain that for every $\varepsilon>0$, there exists N_0 such that for every $N\ge N_0$ and every $k\in\{0,..,[\sqrt{N}]\}$ we have

$$S_{2(N-k)}\le\exp(2(N-k)b+\varepsilon(N-k)). \tag{3.6.6}$$

By using (3.6.6), we obtain easily that $\limsup_{N\to\infty} 1/2N\log\sum_{k=0}^{[\sqrt{N}]} S_{2(N-k)}\le b+\varepsilon$. Thus, we let $\varepsilon\to 0$ and the case $i=1$ is completed.

In the same way we show that $\limsup_{N\to\infty}\frac{1}{2N}\log A_3(N)\le a$. Then it remains only the proof of the case $i=2$. To that aim, we stress the fact that

$$V_{2(N-k)}^{2k}\le\exp\left(\lambda\left|\sum_{j=2k+1}^{2N} w_j\right|\right)S_{2(N-k)}.$$

But, for $k\ge\sqrt{N}$ we can bound from above the former summation over w as follow

$$\left|\sum_{j=2k+1}^{2N} w_j\right|\le\left|\sum_{j=1}^{2N} w_j\right|+\max_{k\in\{[\sqrt{N}],N\}}\left|\sum_{j=1}^{2k} w_j\right|.$$

By the law of large numbers we can say that for every $\varepsilon > 0$, there exists N_0 such that for every $k \geq \sqrt{N_0}$ we have $\left|\sum_{j=1}^{2k} w_j\right| \leq 2\varepsilon k$. Then for every $N \geq N_0$ and $k \in \{\sqrt{N}, N\}$ we obtain $|\sum_{j=2k}^{2N} w_j| \leq |\sum_{j=1}^{2N} w_j| + 2\varepsilon N$. Therefore, if $N \geq N_0$ we can write

$$A_2(N) \leq \exp\left(\lambda \left|\sum_{j=1}^{2N} w_j\right| + 2\lambda\varepsilon N\right) \sum_{k=[\sqrt{N}]}^{N-[\sqrt{N}]} U_{2k} S_{2(N-k)} \tag{3.6.7}$$

and consequently the proof of case 2 will be completed if we show

$$\limsup_{N\to\infty} \frac{1}{2N} \sum_{k=[\sqrt{N}]}^{N-[\sqrt{N}]} U_{2k} S_{2(N-k)}^{2k} \leq \max(a, b).$$

But (3.6.1) gives us that for every $\varepsilon > 0$, there exists k_0 such that for every $k \geq k_0$ we have $U_{2k} \leq \exp(2ak + \varepsilon k)$ and $S_{2k} \leq \exp(2bk + \varepsilon k)$. Then, if $N \geq k_0^2$, for every $k \in \{\sqrt{N}, .., N - \sqrt{N}\}$ we obtain that $k \geq k_0$ and $N - k \geq k_0$. Therefore, if we assume that $b \geq a$, we have the inequality

$$\sum_{k=[\sqrt{N}]}^{N-[\sqrt{N}]} U_{2k} S_{2(N-k)}^{2k} \leq \exp(2bN) \sum_{k=[\sqrt{N}]}^{N-[\sqrt{N}]} \exp(2(a-b)k) \exp(2\varepsilon N) \leq N \exp(2bN+2\varepsilon N).$$

As a consequence, we obtain $\limsup_{N\to\infty} 1/2N \sum_{k=[\sqrt{N}]}^{N-[\sqrt{N}]} U_{2k} V_{2(N-k)}^{2k} \leq b+\varepsilon$, and we let ε tend to zero to obtain the result. Of course the case $a > 0$ is totaly similar to this one. Hence, the proof of theorem 9 is completed.

3.6.4 B

We denote by $V_{u,t} = \widetilde{P}_{b.b.}^t\big(B_{t-1} \in [-1, 1] | g_{t-1} = u\big)$, and we notice (by scaling property) that under $\widetilde{P}_{b.b.}^t$ the process $\Big(\big((s+1)/\sqrt{t}\big)\, B_{st/(s+1)}\Big)_{s\geq 0}$ is a brownian motion (see[35]). The function $h(s) = st/(s + 1)$ is a bijection of $[0, \infty)$ on $[0, t)$. Therefore, there exist a unique $s_1 \in [0, \infty)$ and a unique $s_u \in [0, \infty)$ satisfying $h(s_1) = t - 1$ and $h(s_u) = u$. More precisely $s_1 = t - 1$ and $s_u = u/(t - u)$. With these tools we can write

$$\begin{aligned}
V_{u,t} &= \widetilde{P}^{t}_{b.b.}\left(\frac{1+s_1}{\sqrt{t}}B_{\frac{s_1 t}{1+s_1}} \in [-\sqrt{t},\sqrt{t}] \;\Big|\; B_{\frac{s_u t}{1+s_u}} = 0 \text{ and } \forall s \in (s_u, s_1]\ B_{\frac{st}{s+1}} \neq 0\right)\\
&= \widetilde{P}_{\text{brow. motion}}\left(X_{s_1} \in [-\sqrt{t},\sqrt{t}] \mid X_{s_u} = 0 \text{ and } \forall s \in (s_u, s_1]\ X_s \neq 0\right)\\
&= \widetilde{P}_{\text{brow. motion}}\left(X_{t-1} \in [-\sqrt{t},\sqrt{t}] \mid g_{t-1} = \frac{u}{t-u}\right).
\end{aligned}$$

As mentionned before,we know that the density of $\widetilde{P}_{X_{t-1}}(.|g_{t-1} = u/(t-u))$ is given by the function

$$v(x) = \frac{|x| \exp\left(-\frac{x^2}{2(t-1-\frac{u}{t-u})}\right)}{2\left(t-1-\frac{u}{t-u}\right)}.$$

Hence, we can compute $V_{u,t}$ as follow

$$V_{u,t} = 2\int_0^{\sqrt{t}} \frac{x \exp\left(-\frac{x^2}{2(t-1-\frac{u}{t-u})}\right)}{2\left(t-1-\frac{u}{t-u}\right)} dx = 1 - \exp\left(-\frac{t}{2\left(t-1-\frac{u}{t-u}\right)}\right).$$

Moreover, $u \in [0, t-1]$ implies $2\left(t-1-\frac{u}{t-u}\right) \in [0, 2(t-1)/t]$, and since $1-\exp(-1/x)$ is deacreasing in x, we obtain for $t \geq 2$ the inequality

$$V_{u,t} \geq 1 - \exp\left(-\frac{t}{2(t-1)}\right) \geq \frac{1-\exp(-1)}{2}.$$

Chapter 4

Pinning

4.1 Introduction

4.1.1 The model

Let $S = (S_n)_{n \geq 0}$ be a simple symmetric random walk starting at 0, i.e., $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, where $\{X_i\}_{i \geq 1}$ are i.i.d. random variables such that $P(X_1 = \pm 1) = 1/2$. Let $\Lambda_i = \text{sign}(S_i)$ if $S_i \neq 0$, $\Lambda_i = \Lambda_{i-1}$ otherwise. Let $\{\zeta_i\}_{i \geq 1}$ be i.i.d. random variables, non a.s. equal to 0, such that $\mathbb{E}(\zeta_1) = 0$ and $\mathbb{E}\left(e^{\lambda|\zeta_1|}\right) < \infty$ for every $\lambda > 0$.

For $h \geq 0$, $s \geq 0$ and for every trajectory S of the random walk, we define the hamiltonian

$$H_{N,\beta,h}^{\zeta,s}(S) = \beta \sum_{i=1}^{N} (1 + s\zeta_i) \ \mathbb{1}_{\{S_i = 0\}} + h \sum_{i=1}^{N} \Lambda_i, \tag{4.1.1}$$

and the probability measure $P_{N,\beta,h}^{\zeta,s}$

$$\frac{dP_{N,\beta,h}^{\zeta,s}}{dP}(S) = \frac{\exp\left(H_{N,\beta,h}^{\zeta,s}(S)\right)}{Z_{N,\beta,h}^{\zeta,s}} \tag{4.1.2}$$

with the partition function

$$Z_{N,\beta,h}^{\zeta,s} = \mathbb{E}\Big(\exp\left(H_{N,\beta,h}^{\zeta,s}(S)\right)\Big). \tag{4.1.3}$$

The law $P_{N,\beta,h}^{\zeta,s}$ is called the polymer measure of size N. Under this measure, two types of trajectories seem to be favoured: the localized trajectories that come back often to the origin to receive a positive pinning reward along the x axis, on the other hand, the delocalized trajectories that spend almost all the time in the upper half plane. The latter are favoured at the same time by the second term of the hamiltonian and by the fact that they are much more numerous than the former. Thus, a competition between these two possible behaviors arises.

4.1.2 Free energy

To decide, at fixed parameters, if the system is localized or not, we introduce the free energy, denoted by $\Psi^s(\beta, h)$, and defined by

$$\Psi^s(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z^{\zeta,s}_{N,\beta,h}.$$

This limit is non-random and occurs $\mathbb{P}$ almost surely in ζ and $\mathbb{L}^1$. The proof of this convergence is similar to the one given in [17] or [6]. For this reason, we do not detail it in this article.

The free energy can be bounded from below by computing its restriction to the subset D_N defined by $D_N = \{S : S_i > 0 \ \forall\, i \in \{1, \dots, N\}\}$. For each trajectory of D_N, the hamiltonian is equal to hN, because the chain stays in the upper half plane and never comes back to the interface. Moreover, $P(D_N) \sim c/N^{1/2}$ as $N \to \infty$. Hence,

$$\Psi^s(\beta, h) \geq \liminf_{N\to\infty} \frac{1}{N} \log E\left(e^{hN}\, \mathbb{1}_{\{D_N\}}\right) \geq h + \liminf_{N\to\infty} \frac{\log\left(P\left(D_N\right)\right)}{N} \geq h,$$

and so the free energy is larger than or equal to h. We will say that the polymer is delocalized if $\Psi^s(\beta, h) = h$ (because then the trajectories of D_N give us the whole free energy) and delocalized if $\Psi^s(\beta, h) > h$.

This separation between the localized and delocalized regimes seems a bit crude. Indeed, many trajectories that come back only a few times to the origin, and spend almost all the time in the upper half plane, should also be called delocalized. Thus, taking only into account the trajectories of D_N could be insufficient. However, the convexity of the free energy ensures that throughout the localized phase the chain comes back to the interface in a positive density of sites. Another result helps us to understand the localization phenomenon. This result is due to Sinai [37], and we can adapt it to our pinning model to control the vertical displacement of the chain in the localized area. To that aim, we transform the hamiltonian to $\beta\sum_{i=1}^{N}(1+s\zeta_{N-i})\,\mathbb{1}_{\{S_i=0\}}+h\sum_{i=1}^{N}\Lambda_i$. Thus, the disorder is fixed in the neighborhood of S_N, while the free energy is not modified. Then, for $\Psi^s(\beta,h)>0$ and $\epsilon>0$, we can show that, $\mathbb{P}$ almost surely in ζ, there exists a finite constant $C_\zeta^\epsilon>0$ such that, for every $L\geq 0$ and $N\geq 0$,

$$P_{N,\beta,h}^{\zeta,s}\left(|S_N|>L\right)\leq C_\zeta^\epsilon\exp\left(-\left(\Psi^s(\beta,h)-\epsilon\right)L\right).$$

This result cannot occur if we keep the original hamiltonian, because the disorder is not fixed close to S_N. Therefore, $\mathbb{P}$ almost surely in ζ, we meet arbitrary long stretches of negative rewards, which push S_N far away from the interface.

Some pathwise results have been proved in the delocalized area. In our case, we can use the method developed in the last part of [5] to prove that $\mathbb{P}$ almost surely in ζ, and for every $K>0$,

$$\lim_{N\to\infty}E_{N,\beta,h}^{\zeta,s}\left(\sharp\{i\in\{1,\ldots,N\}:S_i>K\}/N\right)=1.$$

These results allow us to understand more deeply what localization and delocalization mean.

4.1.3 Simplification of the model

We transform the hamiltonian to simplify the localization condition. To that aim, we notice that

$$\Psi^s(\beta,h)-h=\lim_{N\to\infty}\frac{1}{N}\log\left(E\left(\exp\left(\beta\sum_{i=1}^{N}(1+s\zeta_i)\,\mathbb{1}_{\{S_i=0\}}+h\sum_{i=1}^{N}(\Lambda_i-1)\right)\right)\right)$$

and we let $\Phi^s(\beta,h)$ be $\Psi^s(\beta,h)-h$. The delocalization condition becomes $\Phi^s(\beta,h)=0$ and the localization condition $\Phi^s(\beta,h)>0$. Finally, we set $\Delta_i=1$ if $\Lambda_i=-1$ and

$\Delta_i = 0$ if $\Lambda_i = 1$. Then the hamiltonian becomes

$$H^{\zeta,s}_{N,\beta,h}(S) = \beta \sum_{i=1}^{N} (1 + s\zeta_i)\ \mathbb{1}_{\{S_i=0\}} - 2h \sum_{i=1}^{N} \Delta_i,$$

and we keep $Z^{\zeta,s}_{N,\beta,h} = E\big(e^{H^{\zeta,s}_{N,\beta,h}}\big)$. Thus, we obtain

$$\Phi^s(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \log Z^{\zeta,s}_{N,\beta,h}.$$

The function Φ^s is convex and continuous in both variables, non-decreasing in β and non-increasing in h.

4.2 Motivation and Preview

4.2.1 Physical motivation

Systems of random walk attracted by a potential at an interface are closely studied at this moment (see [17]). One of the major issue in the subject consists in understanding better the influence of a random potential compared to a constant one (with the same expectation). Indeed, while it seems intuitively clear that a random potential has a stronger power of attraction than a constant one, it is much less obvious how to quantify this difference.

In this article, we consider a potential at the interface together with the fact that the polymer prefers lying in the upper half plane than in the lower half plane. Such a type of system has been studied numerically in [20] and describes, for instance, a hydrophobic homopolymer at an interface between oil and water. Close to this interface, some very small droplets of a third solvent (microemulsions) are placed. These droplets have a strong capacity of attraction on the monomers composing our chain. Thus, the pinning rewards that the chain can receive when it comes back to the origin represent the attractive emulsions that the polymer touches close to the interface.

4.2.2 Preview

In this article, we investigate new strategies of localization for the polymer, consisting in targeting the sites where it comes back to the interface. We find an explicit lower bound on the critical curve that lies strictly above the non-random one.

Our result covers, as a limit case when h tends to infinity, the wetting transition model. Indeed, in the last ten years the wetting problem, i.e., the case of a polymer interacting with an (impenetrable) interface, has attracted a lot of interest, because it can be seen as a Polland-Sheraga model of the DNA strand (see [32], [16]). The localization transition with a constant disorder occurs for the pinning reward $\log 2$, and several open questions are linked with the effect of a small random perturbation added to the reward $\log 2$. Moreover, with the constant pinning reward $\log 2$, the simple random walk conditioned to stay positive has the same law as the reflected random walk (see [19]). That is why, to study the wetting model around the pinning reward $\log 2$, it suffices to consider the pure pinning model, i.e., a reflected random walk pinned at the origin by small random variables.

This pure pinning model has been closely studied. For example, in [21] a particular type of positive potential has been considered and a criterium has been given to decide for every disorder realization whether it localizes the polymer or not. But a very difficult question consists in estimating, for small s, the critical delocalization average $u_c(s)$ of a pinning potential $\{-u+s\zeta_i\}_{i\geq 1}$, where $\{\zeta_i\}_{i\geq 1}$ are i.i.d., centered and of variance 1 (i.e., $Var(-u+s\zeta_i)=s^2$). The annealed critical curve, denoted by $u_a(s)$, is an upper bound of $u_c(s)$ and satisfies

$$u_a(s) = \log E\left(\exp(s\zeta_i)\right) = (1+o(1))s^2/2 \quad \text{when } s \text{ tends to } 0.$$

Moreover, $u_a(s)$ is equal to $s^2/2$ when ζ_i follows an $N(0,1)$ law.

In the last 20 years, there has been a lot of activity on this question, mostly from the physics side, and it is now widely believed that $u_c(s)$ behaves as $s^2/2$. But it is still an open question whether $u_c(s) = s^2/2$ (see [14]) for s small or $u_c(s) < s^2/2$ for all s (see [11] or [22]).

However, on the mathematics side the only rigorous fact that has been proved is in [1], where Alexander and Sidoravicius have studied a general class of random walks pinned either by an interface between two solvents or by an impenetrable wall. If we apply their results in our case, we obtain that the quenched quantity $u_c(s)$ is strictly larger than the non-disordered one $u_c(0)$. In this paper, we develop a new localization strategy, which allows us to go further, by giving a lower bound of $u_c(s)$ which has the same scale as the annealed upper bound for s small (i.e. $-cs^2$ with $c > 0$).

4.3 Critical curve

In this article, we are particularly interested in the critical curve of the system, namely, the curve that divides the (h,β)-plane into a localized and a delocalized phase. Before defining this curve precisely, it is helpful to consider the non-disordered case ($s = 0$), which is easier to understand and gives a good intuition of what happens in the disordered case ($s \neq 0$).

4.3.1 Non-disordered case (proposition 12)

Above the critical curve the system is delocalized, and below localized. In appendix B, we compute the equation of this curve when $s = 0$. We obtain

$$\begin{aligned} h_c^0 :\ & [0, \log 2) \to \mathbb{R} \\ & \beta \longrightarrow h_c^0(\beta) = -\frac{1}{4} \log\left(1 - 4\left(1 - e^{-\beta}\right)^2\right). \end{aligned} \tag{4.3.1}$$

This curve is increasing, convex and tends to ∞ when β tends to $\log 2$ from the left. When $\beta \geq \log 2$ the system is always localized. In fact, when h is chosen large, the free energy is strictly positive. That is why this critical curve is only defined on $[0, \log 2)$ (see Fig 1).

Our first result concerns $s \neq 0$ and shows that the critical curve has a form that is qualitatively similar to (4.3.1).

Proposition 12 *For $s \geq 0$ and $\beta \geq 0$ the following properties are satisfied.*

i) There exists $h_c^s(\beta) \in [0, +\infty]$ such that

$$\Phi^s(\beta, h) > 0 \quad \textit{if} \quad h < h_c^s(\beta),$$
$$\Phi^s(\beta, h) = 0 \quad \textit{if} \quad h \geq h_c^s(\beta).$$

ii) The function $\beta \to h_c^s(\beta)$ is convex and increasing.

iii) For $s \geq 0$ there exists $\beta_0(s) \in (0, \infty]$ such that $h_c^s(\beta) < +\infty$ when $\beta < \beta_0(s)$ and $h_c^s(\beta) = +\infty$ when $\beta > \beta_0(s)$.

iv) The non-disordered critical curve $h_c^0(\beta)$ is a lower bound for $h_c^s(\beta)$.

v) $\beta_0(s) \leq \beta_0(0) = \log 2$.

Remark 5 The case $\beta = \beta_0(s)$ remains open. More precisely, two different behaviors of the curve may occur. Either $\lim_{\beta \to \beta_0^-(s)} h_c(\beta) = +\infty$, or there exists $h_0^s < \infty$ such that $\lim_{\beta \to \beta_0^-(s)} h_c(\beta) = h_0^s$. In the latter case, by continuity of Φ^s in β, we obtain $\Phi(\beta_0(s), h_0^s) = 0$ and $h_c(\beta_0(s)) = h_0^s$.

4.3.2 Annealed case

We obtain an upper bound of $h_c^s(\beta)$, as usual, by computing the annealed free energy. This is, by Jensen's inequality, an upper bound on the quenched free energy. The annealed system gives a critical curve $(\beta \to h_c^{an,s}(\beta))$, which is an upper bound on the quenched critical curve. The annealed free energy is given by

$$\Phi^s_{ann.}(h, \beta) = \lim_{N \to \infty} \frac{1}{N} \log E\mathbb{E}\left(\exp\left(\beta \sum_{i=1}^N (1 + s\zeta_i)\, \mathbf{1}_{\{S_i = 0\}} - 2h \sum_{i=1}^N \Delta_i\right)\right).$$

We integrate over $\mathbb{P}$ to obtain

$$\Phi^s_{ann.}(h,\beta) = \lim_{N\to\infty} \frac{1}{N} \log E\left(\exp\left(\left(\beta + \log \mathbb{E}(e^{\beta s \zeta_1})\right) \sum_{i=1}^N \mathbb{1}_{\{S_i=0\}} - 2h \sum_{i=1}^N \Delta_i\right)\right)$$
$$= \Phi^0\big(h, \beta + \log \mathbb{E}\big(e^{\beta s \zeta_1}\big)\big). \tag{4.3.2}$$

Finally, we denote by β^s_{an} the unique solution of $\beta + \log \mathbb{E}(e^{\beta s \zeta_1}) = \log 2$, and for $\beta \in [0, \beta^s_{an})$ we obtain $h_c^{an,s}(\beta) = h_c^0\left(\beta + \log \mathbb{E}\left(e^{\beta s \zeta_1}\right)\right)$ (see Fig 1).

Remark 6 We notice that $h_c^{an,s}(\beta)$ and $h_c^0(\beta)$ are both equal to $\beta^2(1+o(1))$ when β tends to 0.

4.3.3 Disordered model (theorem 13)

Up to now, two types of localization strategy have been used to find lower bounds on the quenched critical curve. The first one consists in computing the free energy on a particular subset of trajectories, i.e., trajectories that come back often to the interface ([3]). The other one consists in transforming (by using Radon-Nikodým derivatives) the law of the excursions out of the origin. Bolthausen and den Hollander have used this second method in [6], to constrain the chain to come back to the origin in a positive density of sites. We go further here, because we make the chain choose, at each excursion, a law adapted to the local disorder.

Proposition 12 tells that $h_c^s(\beta) = \infty$ when $s \geq 0$ and $\beta \geq \log 2$. Therefore, the critical curve is not defined after $\log 2$. For this reason, we will only consider the case $\beta \leq \log 2$.

Theorem 13 *If $Var(\zeta_1) \in (0,\infty)$, then there exist $c_1 > 0$, $c_2 > 0$ such that, for every $s \leq c_1$ and $\beta \in [0, \log 2 - c_2 s^2 \beta^2)$,*

$$h_c^s(\beta) \geq -\frac{1}{4} \log\left(1 - 4\left(1 - e^{-\beta - c_2 s^2 \beta^2}\right)^2\right) = m^s(\beta).$$

On Fig. 1 below, we draw the curves which we have mentioned up to now.

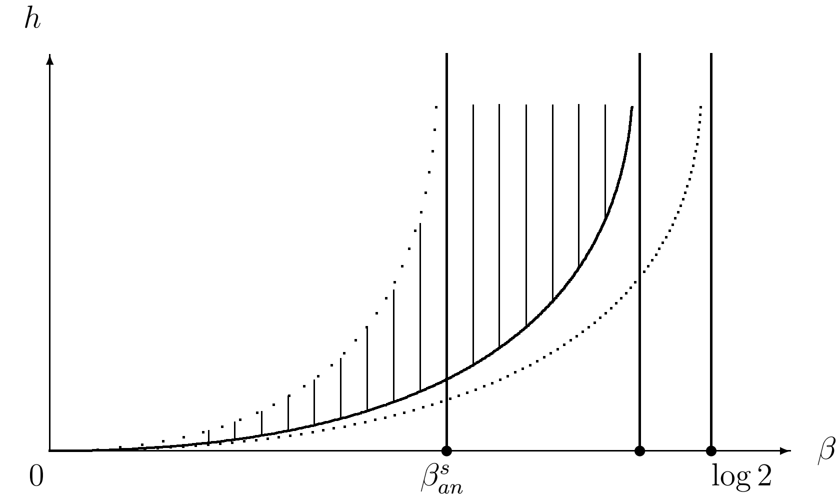


Fig. 1: .... $h_c^{an,s}(\beta)$

——— $m^s(\beta)$ possible location of $h_c^s(\beta)$

.... $h_c^0(\beta)$

Remark 7 In the proof of Theorem 13, we restrict to $\mathbb{P}(\zeta_1 > 0) = 1/2$ and to $\mathbb{E}\left(\zeta_1 \mathbb{1}_{\{\zeta_1 > 0\}}\right) = 1$. In this case, $c_1 = 1$ and $c_2 = 1/(5 \times 2^{14})$. With other conditions on $\mathbb{P}(\zeta_1 > 0)$ and $\mathbb{E}\left(\zeta_1 \mathbb{1}_{\{\zeta_1 > 0\}}\right)$, the constants c_1 and c_2 would have to be chosen differently, but the strategy to obtain the lower bound still works.

4.4 Pure pinning and wetting model (corollary 14)

The pure pinning model is different from the previous one. The $h-$term is removed, and the rewards at the interface take the form $-u + s\zeta_i$ with $u \geq 0$. The corresponding hamiltonian is

$$H_{N,s}^{\zeta,u}(S) = \sum_{i=1}^{N} (-u + s\zeta_i)\ \mathbb{1}_{\{S_i = 0\}}.$$

The localization and delocalization conditions associated with the free energy remain the same. We obtain a critical u denoted by $u_c(s)$, such that the system is localized when $u < u_c(s)$ and delocalized when $u \geq u_c(s)$.

For this model, the annealed model gives an upper bound on $u_c(s)$, denoted by $u_c^{an}(s)$. If $Var(\zeta_1) = 1$, then this annealed upper bound satisfies $u_c^{an}(s) = (1 + o(1))s^2/2$ when $s \to 0$. A corollary of Theorem 13 gives a lower bound on $u_c(s)$, which has the same scale (i.e., cs^2 as $s \to 0$).

Corollary 14 *If $Var(\zeta_1) \in (0, \infty)$, then there exist $c_3, c_4 > 0$ such that, for every $s \le c_3$,*

$$u_c(s) \ge c_4 s^2.$$

Remark 8 The values of c_3 and c_4 depend on the law of ζ_1. In the proof of Corollary 3, we will consider the conditions of Remark 7 concerning ζ_1. In this case, $c_3 = \log 2$ and $c_4 = 1/(5 \times 2^{16})$.

4.5 Proof of theorem and proposition

4.5.1 Proof of Proposition 12

The proof of parts i)-v) are given below.

i) For $\beta \ge 0$ and $s \ge 0$, let $J_\beta^s = \{h \ge 0 : \Phi^s(\beta, h) = 0\}$. Let $h_c^s(\beta)$ be the infimum of J_β^s. Recall that Φ is positive, continuous, and non-increasing in h. Hence, $J_\beta^s = [h_c^s(\beta), +\infty)$ and i) is proved.

iii) The function Φ is convex in β, positive, and $\Phi^s(0, h) = 0$ for every $h \ge 0$. Therefore, Φ is non-decreasing in β, and $h_c^s(\beta)$ is non-decreasing. If we define $\beta_0(s) = \sup\{\beta \ge 0 : J_\beta^s \ne \emptyset\}$, then the annealed computation gives $\beta_0(s) > 0$. Indeed, $J_{ann.\beta}^s \subset J_\beta^s$ because $\Phi^s(h, \beta) \le \Phi_{ann.}^s(h, \beta)$. Thus, $\beta_0(s) \ge \beta_{an}^s > 0$ and iii) is proved.

iv) We want to show that $h_c^s(\beta) \ge h_c^0(\beta)$ when $s \ge 0$. To that aim, we prove that $\Phi^s(\beta, h) > 0$ when $s \ge 0$, $\beta \ge 0$ and $h < h_c^0(\beta)$. For β and h fixed, $\Phi^s(\beta, h)$ is convex in s, because it is the limit as $N \to \infty$ of $\Phi^s_N(\beta, h) = \mathbb{E}\big(1/N \log E\big(\big(\exp\big(H^{\zeta,s}_{N,\beta,h}\big)\big)\big)$, which is convex in s. Moreover, for every $N > 0$, $\Phi^s_N(\beta, h)$ can be differentiated w.r.t. s. This gives

$$\frac{\partial \Phi^s_N(\beta, h)}{\partial s} = \frac{1}{N}\mathbb{E}\left(\frac{E\left(\beta \sum_{i=1}^N \zeta_i \, \mathbb{1}_{\{S_i=0\}} \exp\left(H^{\zeta,s}_{N,\beta,h}\right)\right)}{E\left(\exp\left(H^{\zeta,s}_{N,\beta,h}\right)\right)}\right).$$

But, when $s = 0$, the hamiltonian does not depend on the disorder ζ. Therefore, by the Fubini-Tonelli Theorem and the fact that $\mathbb{E}(\zeta_i) = 0$, we can write

$$\left.\frac{\partial \Phi^s_N(\beta, h)}{\partial s}\right|_{s=0} = \frac{1}{N}\frac{E\left(\beta \sum_{i=1}^N \mathbb{E}\left(\zeta_i\right) \, \mathbb{1}_{\{S_i=0\}} \exp\left(H^{\zeta,0}_{N,\beta,h}\right)\right)}{E\left(\exp\left(H^{\zeta,0}_{N,\beta,h}\right)\right)} = 0.$$

Hence, the convergence of Φ_N towards Φ and their convexity allows us to write

$$\left.\frac{\partial_{right} \Phi^s\left(\beta, h\right)}{\partial s}\right|_{s=0} \geq \lim_{N \to \infty} \left.\frac{\partial_{right} \Phi^0_N\left(\beta, h\right)}{\partial s}\right|_{s=0} = 0.$$

Thus, by convexity in s, we can assert that $\Phi^s\left(\beta, h\right)$ is non-decreasing in s. Hence, $s \geq 0$ implies $\Phi^s\left(\beta, h\right) \geq \Phi^0\left(\beta, h\right) > 0$. That is why $h^s_c\left(\beta\right) \geq h^0_c\left(\beta\right)$, and iv) is satisfied.

v) This is a direct consequence of iv).

ii) We want to prove that $h^s_c(\beta)$ is convex, and therefore continuous on $[0, \beta_0(s))$. To prove convexity, we let $0 < a < b$ and $\lambda \in [0, 1]$. Then, since

$$H^{\zeta,s}_{N,\ \lambda a + (1-\lambda) b,\ \lambda h^s_c(a) + (1-\lambda) h^s_c(b)} = H^{\zeta,s}_{N,\ \lambda a,\ \lambda h^s_c(a)} + H^{\zeta,s}_{N,\ (1-\lambda)b,\ (1-\lambda) h^s_c(b)},$$

the Hölder inequality gives

$$\begin{aligned}\frac{1}{N}\log E\left(\exp\left(Z^{\zeta,s}_{N,\ \lambda(a, h^s_c(a)) + (1-\lambda)(b, h^s_c(b))}\right)\right) \leq{}& \frac{\lambda}{N}\log E\left(\exp\left(Z^{\zeta,s}_{N,\ a,\ h^s_c(a)}\right)\right) \\ &+ \frac{1-\lambda}{N}\log E\left(\exp\left(Z^{\zeta,s}_{N,\ b,\ h^s_c(b)}\right)\right).\end{aligned} \tag{4.5.1}$$

Therefore, if $N \to \infty$, the r.h.s. of (4.5.1) tends to zero, because, by continuity of Φ in h, $\Phi(a, h_c^s(a)) = \Phi(b, h_c^s(b)) = 0$. Hence,

$$\Phi^s(\lambda a + (1-\lambda)b\,,\ \lambda h_c^s(a) + (1-\lambda)h_c^s(b)) = 0,$$

and

$$h_c^s(\lambda a + (1-\lambda)b) \leq \lambda h_c^s(a) + (1-\lambda)h_c^s(b).$$

This completes the proof of the first part of ii). To get the second part of ii), we show that $h_c^s(\beta)$ is increasing in β. Indeed, since $h_c^s(0) = 0$ and $h_c^s(\beta) \geq h_c^0(\beta) > 0$ for $\beta > 0$, the convexity of $h_c^s(\beta)$ gives us the result.

4.5.2 Proof of theorem 13

In the following we consider $h > 0$, $\beta \leq \log 2$, $\mathbb{P}(\zeta_1 > 0) = 1/2$, $\mathbb{E}\left(\zeta_1 \mathbb{1}_{\{\zeta_1 > 0\}}\right) = 1$ and $s \leq 1$.

4.5.3 Step I

Transformation of the excursion law.

Definition 15 *From now on, we denote by i_j the site of the j^{th} return to the origin. Thus, $i_0 = 0$ and $i_j = \inf\{i > i_{j-1} : S_i = 0\}$. Let τ_j be the length of the j^{th} excursion away of the origin, i.e., $\tau_j = i_j - i_{j-1}$. Also, let l_N be the number of returns to the origin before time N.*

By independence of the excursion signs, we can rewrite the partition function as

$$H_N = E\Bigg(\exp\left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j}\right) \exp\left(\beta l_N\right) \prod_{j=1}^{l_N} \left(\frac{1+\exp\left(-2h\tau_j\right)}{2}\right) \times \frac{1+\exp\left(-2h\left(N - i_{l_N}\right)\right)}{2} \Bigg). \quad (4.5.2)$$

We want to transform the law of the excursions away of the origin to constrain the chain to come back to zero in a positive density of sites. For that, we introduce $P_{\alpha,h}^{\beta}$, the law of a homogeneous positive recurrent Markov process. Its excursions law is given by

$$\forall\, n \in \mathbb{N}\backslash\{0\} \qquad P^{\beta}_{\alpha,h}\left(\tau_1 = 2n\right) = \left(\frac{1+\exp\left(-4hn\right)}{2}\right)\alpha^{2n}\frac{P\left(\tau = 2n\right)}{H^{\beta}_{\alpha,h}}\exp\left(\beta\right), \tag{4.5.3}$$

with

$$H^{\beta}_{\alpha,h} = \sum_{i=1}^{\infty}\frac{\exp\left(-4hi\right)+1}{2}e^{\beta}\alpha^{2i}P\left(\tau = 2i\right) = e^{\beta}\left(1 - \frac{\sqrt{1-\alpha^2}+\sqrt{1-e^{-4h}\alpha^2}}{2}\right). \tag{4.5.4}$$

We notice that the term inside the expectation of (4.5.2) only depends on l_N and on the positions of the returns to the origin, i.e., $i_1, \dots, i_{l_N}$. Therefore, we can rewrite H_N as an expectation under $P^{\beta}_{\alpha,h}$, because we know the Radon-Nikodým derivative $dP/dP^{\beta}_{\alpha,h}(\{i_1, \dots, i_{l_N}\})$. Hence, H_N becomes

$$H_N = E^{\beta}_{\alpha,h}\left(\exp\left(\beta s\sum_{j=1}^{l_N}\zeta_{i_j}\right)\prod_{j=1}^{l_N}\frac{H^{\beta}_{\alpha,h}}{\alpha^{\tau_j}}\left(\frac{1+e^{-2h\left(N-i_{l_N}\right)}}{2}\right)\frac{P\left(\tau \geq N - i_{l_N}\right)}{P^{\beta}_{\alpha,h}\left(\tau \geq N - i_{l_N}\right)}\right).$$

Next we aim at transforming the excursion law again, so that the chain comes back more often in sites where the pinning reward is large. Indeed, we want the chain to take into account its local environment. For that, we define $P^{\beta,\zeta,\alpha_1}_{\alpha,h}$ the law of a non-homogenous Markov process, that depends on the environment. Its excursion law is defined as follow. Let

$$\alpha_1 < \frac{1-P^{\beta}_{\alpha,h}(\tau=2)}{P^{\beta}_{\alpha,h}(\tau=2)} \ \text{ such that } \ \mu_1 = 1 - \alpha_1\frac{P^{\beta}_{\alpha,h}\left(\tau=2\right)}{\left(1-P^{\beta}_{\alpha,h}\left(\tau=2\right)\right)} > 0,$$

and let

$$\begin{aligned} P^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\tau = 2\right) &= P^{\beta}_{\alpha,h}\left(\tau=2\right)\left(1+\alpha_1\right)^{\mathbb{1}_{\{\zeta_2>0\}}} \\ P^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\tau = 2r\right) &= P^{\beta}_{\alpha,h}\left(\tau=2r\right)\mu_1^{\mathbb{1}_{\{\zeta_2>0\}}} \ \text{ for } r \geq 2. \end{aligned} \tag{4.5.5}$$

Under the law of this process, if the chain comes back to the origin at time i, then the law of the following excursion is $P^{\beta,\zeta_{i+\cdot},\alpha_1}_{\alpha,h}$. Thus, the chain checks whether the reward at time $i+2$ is positive or negative. If $\zeta_{i+2} \geq 0$, then the probability to come back to zero at time $i+2$ increases. Else it remains the same.

With this new process we can write

$$
\begin{aligned}
H_N = E_{\alpha,h}^{\beta,\zeta,\alpha_1} \Bigg(\exp\left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j} \right) \prod_{j=1}^{l_N} \left(\frac{H_{\alpha,h}^{\beta}}{\alpha^{\tau_j}} \right) \left(\frac{1}{2} + \frac{e^{-2h\left(N-i_{l_N}\right)}}{2} \right) \\
\times \prod_{j=1}^{l_N} \left(\frac{P_{\alpha,h}^{\beta}\left(\tau_j\right)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+.},\alpha_1}\left(\tau_j\right)} \right) \frac{P\left(\tau \geq N - i_{l_N}\right)}{P_{\alpha,h}^{\beta,\zeta_{i_{l_N}+.},\alpha_1}\left(\tau \geq N - i_{l_N}\right)} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\geq E_{\alpha,h}^{\beta,\zeta,\alpha_1} \Bigg(\exp\left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j} \right) \left(H_{\alpha,h}^{\beta} \right)^{l_N} \\
\times \frac{1}{2} \prod_{j=1}^{l_N} \left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+.},\alpha_1}\left(\tau_j\right)} \right) P\left(\tau \geq N - i_{l_N}\right) \Bigg).
\end{aligned}
$$

We apply Jensen's inequality to obtain

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{N} \log H_N \right) \geq & \frac{\beta s}{N}\, \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \zeta_{i_j} \right) + \log\left(H_{\alpha,h}^{\beta} \right) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) + \frac{1}{N} \log\left(\frac{1}{2} \right) \\
& + \frac{1}{N} \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \log\left(\frac{P_{\alpha,h}^{\beta}\left(\tau_j\right)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+.},\alpha_1}\left(\tau_j\right)} \right) \right) + \frac{\log\left(P\left(\tau \geq N\right)\right)}{N}.
\end{aligned}
\tag{4.5.6}
$$

At this stage, we can divide the lower bound of (4.5.6) in two parts. The first part (called $E_1(N)$) is a positive energetic term, corresponding to the additional reward that the chain can expect by coming back often in "high reward" sites, namely,

$$
E_1(N) = \frac{\beta s}{N}\, \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \zeta_{i_j} \right).
$$

The second part (called $E_2(N)$) is a negative entropic term, because the measure transformations we performed have an entropic cost, namely,

$$
\begin{aligned}
E_2(N) = & \log\left(H_{\alpha,h}^{\beta} \right) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) + \frac{1}{N} \log\left(\frac{1}{2} \right) \\
& + \frac{1}{N} \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \log\left(\frac{P_{\alpha,h}^{\beta}\left(\tau_j\right)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+.},\alpha_1}\left(\tau_j\right)} \right) \right) + \frac{1}{N} \log\left(P\left(\tau \geq N\right)\right).
\end{aligned}
$$

4.5.4 Step II

Energy term computation.

Notice that

$$\begin{aligned}\sum_{j=1}^{l_N} \zeta_{i_j} =& \sum_{i=0}^{N-2} \zeta_{i+2} \ \mathbb{1}_{\{S_i=0\}} \ \mathbb{1}_{\{S_{i+2}=0\}} \\ &+ \sum_{k=3}^{N} \sum_{s=0}^{N-k} \zeta_{s+k} \ \mathbb{1}_{\{S_s=0\}} \ \mathbb{1}_{\{S_i\neq 0 \ \forall \ i\in\{s+1,\dots,s+k-1\}\}} \ \mathbb{1}_{\{S_{s+k}=0\}}. \end{aligned} \tag{4.5.7}$$

Let $A = \sum_{i=0}^{N-2} \zeta_{i+2} \ \mathbb{1}_{\{S_i=0\}} \ \mathbb{1}_{\{S_{i+2}=0\}}$ and

$B = \sum_{k=3}^{N} \sum_{s=0}^{N-k} \zeta_{s+k} \ \mathbb{1}_{\{S_s=0\}} \ \mathbb{1}_{\{S_i\neq 0 \ \forall \ i\in\{s+1,\dots,s+k-1\}\}} \ \mathbb{1}_{\{S_{s+k}=0\}}$.

We compute separately the contributions of A and B. We begin with

$$\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(B\right) = \sum_{k=3}^{N}\sum_{s=0}^{N-k} \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\zeta_{s+k} \ \mathbb{1}_{\{S_s=0\}} \ \mathbb{1}_{\{S_i\neq 0 \ \forall \ i\in\{s+1,\dots,s+k-1\}\}} \ \mathbb{1}_{\{S_{s+k}=0\}}\right).$$

By the Markov property,

$$\begin{aligned}\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(B\right) = \sum_{k=3}^{N}\sum_{s=0}^{N-k} & \mathbb{E}\left(\mathbb{1}_{\{\zeta_{s+2}>0\}} E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbb{1}_{\{S_s=0\}}\right) P_{\alpha,h}^{\beta}\left(k\right) \mu_1 \, \zeta_{s+k}\right) \\ &+ \mathbb{E}\left(\mathbb{1}_{\{\zeta_{s+2}\leq 0\}} E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbb{1}_{\{S_s=0\}}\right) P_{\alpha,h}^{\beta}\left(k\right) \, \zeta_{s+k}\right). \end{aligned}$$

But $E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbb{1}_{\{S_s=0\}}\right)$ only depends on $\{\zeta_1,\zeta_2,\dots,\zeta_s\}$, and the $\{\zeta_i\}_{i\geq 1}$ are independent and centered. For this reason, and since $k\geq 3$ we have $\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(B\right)=0$.

The contribution of part A in (4.5.7) is given by

$$\begin{aligned}\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(A\right) =& \sum_{i=0}^{N-2} \mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbb{1}_{\{S_i=0\}}\right) \ P_{\alpha,h}^{\beta}\left(2\right)\left(1+\alpha_1\right)\zeta_{i+2}\,\mathbb{1}_{\{\zeta_{i+2}>0\}}\right) \\ &+ \sum_{i=0}^{N-2} \mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\mathbb{1}_{\{S_i=0\}}\right) \ P_{\alpha,h}^{\beta}\left(2\right)\zeta_{i+2}\,\mathbb{1}_{\{\zeta_{i+2}\leq 0\}}\right) \\ =& \ \alpha_1 P_{\alpha,h}^{\beta}\left(2\right)\mathbb{E}\left(\zeta_1 \ \mathbb{1}_{\{\zeta_1\}>0}\right)\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\sharp\{i\in\{0,\dots,N-2\}:S_i=0\}\right). \end{aligned}$$

Therefore, the contribution of this energy term is

$$\begin{aligned}E_1(N) =& \beta s\alpha_1 P_{\alpha,h}^{\beta}\left(2\right)\frac{\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\sharp\{i\in\{0,\dots,N-2\}:S_i=0\}\right)}{N} \\ \geq& \beta s\alpha_1 P_{\alpha,h}^{\beta}\left(2\right)\frac{\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(l_N\right)}{N}. \end{aligned} \tag{4.5.8}$$

4.5.5 Step III

Computation of the entropic term.

Notice that the terms $1/N \log(P(\tau \geq N))$ and $1/N \log(1/2)$ tend to 0 as $N \to \infty$, independently of all the other parameters. Hence, if we denote by R_N the quantity $1/N \log\left(P\left(\tau \geq N\right)\right) + 1/N \log\left(1/2\right)$, then we can write

$$E_2(N) = \frac{S_N}{N} + \log\left(H^{\beta}_{\alpha,h}\right) \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right) + R_N,$$

with

$$S_N = \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\sum_{j=1}^{l_N} \log\left(\frac{P^{\beta}_{\alpha,h}\left(\tau_j\right)}{P^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}_{\alpha,h}\left(\tau_j\right)}\right)\right). \tag{4.5.9}$$

The definitions (4.5.3) and (4.5.5) of $P^{\beta,\zeta_{i_{j-1}+\cdot},\alpha_1}_{\alpha,h}$ and $P^{\beta}_{\alpha,h}$ immediately give

$$\begin{aligned}
S_N = &- \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\sum_{j=1}^{l_N} \mathbb{1}_{\{\zeta_{i_{j-1}+2}>0\}}\left(\mathbb{1}_{\{\tau_j=2\}} \log\left(1+\alpha_1\right) + \mathbb{1}_{\{\tau_j>2\}} \log\left(\mu_1\right)\right)\right) \\
= &- \sum_{i=0}^{N-2} \mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbb{1}_{\{S_i=0\}}\ \mathbb{1}_{\{S_{i+2}=0\}}\right) \mathbb{1}_{\{\zeta_{i+2}>0\}} \log\left(1+\alpha_1\right)\right) \\
&- \sum_{k=3}^{N} \sum_{s=0}^{N-k} \mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbb{1}_{\{S_s=0\}}\ \mathbb{1}_{\{S_i\neq 0\ \forall\, i\in\{s+1,\ldots,s+k-1\}\}}\ \mathbb{1}_{\{S_{s+k}=0\}}\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. \times\ \mathbb{1}_{\{\zeta_{s+2}>0\}} \log\left(\mu_1\right)\right).
\end{aligned}$$

By the Markov property, we can write

$$\mathbb{1}_{\{\zeta_{i+2}>0\}}\ E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbb{1}_{\{S_i=0\}} \mathbb{1}_{\{S_{i+2}=0\}}\right) = \mathbb{1}_{\{\zeta_{i+2}>0\}}\ E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbb{1}_{\{S_i=0\}}\right)\left(1+\alpha_1\right) P^{\beta}_{\alpha,h}\left(2\right),$$

and we notice that $E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\mathbb{1}_{\{S_i=0\}}\right)$ is independent of ζ_{i+2} and $\mathbb{P}\left(\zeta_{i+2}>0\right) = 1/2$. Hence,

$$\begin{aligned}
S_N = &- \frac{P^{\beta}_{\alpha,h}\left(2\right)}{2}\left(1+\alpha_1\right) \log\left(1+\alpha_1\right) \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(l_{N-2}\right) \\
&- \sum_{k=3}^{N} \frac{\mu_1 \log\left(\mu_1\right)}{2}\ P^{\beta}_{\alpha,h}\left(k\right)\ \mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(l_{N-k}\right).
\end{aligned}$$

Finally, the entropic contribution is

$$E_2(N) = \log\left(H^{\beta}_{\alpha,h}\right)\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right) - \frac{P^{\beta}_{\alpha,h}(2)}{2}(1+\alpha_1)\log(1+\alpha_1)\,\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-2}}{N}\right)$$
$$-\sum_{k=3}^{N}\frac{\mu_1\log(\mu_1)}{2}\,P^{\beta}_{\alpha,h}(k)\;\mathbb{E}E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-k}}{N}\right) + R_N, \tag{4.5.10}$$

and (4.5.8) and (4.5.10) give us a lower bound of formula (4.5.6) of the form

$$\mathbb{E}\Big(\frac{1}{N}\log(H_N)\Big) \geq E_1(N) + E_2(N). \tag{4.5.11}$$

4.5.6 Step IV

Estimation of $H^{\beta}_{\alpha,h}$ and choice of α and α_1.

Next we want to evaluate $H^{\beta}_{\alpha,h}$ with the expression of (4.5.4), namely,

$$H^{\beta}_{\alpha,h} = e^{\beta}\left(1 - \frac{\sqrt{1-\alpha^2}+\sqrt{1-e^{-4h}\alpha^2}}{2}\right).$$

To compare $\log\left(H^{\beta}_{\alpha,h}\right)$ with the other terms of (4.5.11), we denote $\alpha^2 = 1 - c\alpha_1^2$, with $c > 0$ and $\sqrt{c}\alpha_1 \leq 1$. In this way, we obtain

$$\begin{aligned} H^{\beta}_{\alpha,h} &= e^{\beta}\left(1 - \frac{\sqrt{1-e^{-4h}}}{2} + \frac{\sqrt{1-e^{-4h}} - \sqrt{1-e^{-4h}\left(1-c\alpha_1^2\right)} - \sqrt{c}\alpha_1}{2}\right) \\ &= e^{\beta}\left(1 - \frac{\sqrt{1-e^{-4h}}}{2}\right)\left[1 + \frac{\sqrt{1-e^{-4h}}\left(1-\sqrt{1+\frac{ce^{-4h}\alpha_1^2}{1-e^{-4h}}}\right) - \sqrt{c}\alpha_1}{2-\sqrt{1-e^{-4h}}}\right]. \end{aligned}$$

Since $\sqrt{1+x} \leq 1 + x/2$ for $x \in (-1, +\infty)$, and since $2 - \sqrt{1-e^{-4h}} \geq 1$, we obtain

$$\log\left(H^{\beta}_{\alpha,h}\right) \geq \log\left(e^{\beta}\left(1 - \frac{\sqrt{1-e^{-4h}}}{2}\right)\right) + \log\left(1 - \sqrt{c}\alpha_1 - \frac{c\alpha_1^2 e^{-4h}}{2\sqrt{1-e^{-4h}}}\right).$$

As $\sqrt{c}\alpha_1 \leq 1$, we can bound from above the term

$$\sqrt{c}\alpha_1 + \frac{c\alpha_1^2 e^{-4h}}{2\sqrt{1-e^{-4h}}} = \sqrt{c}\alpha_1\left(1 + \frac{\sqrt{c}\alpha_1 e^{-4h}}{2\sqrt{1-e^{-4h}}}\right) \leq \sqrt{c}\alpha_1\left(1 + \frac{1}{2\sqrt{1-e^{-4h}}}\right). \tag{4.5.12}$$

To continue this computation, we need to choose precise values for α_1 and c. That is why, recalling that $(\alpha^2 = 1 - c\alpha_1^2)$, we denote

$$\alpha_1 = \beta s / \left(5 \times 2^8\right) \qquad \sqrt{c} = \beta s / \left(3 \times 2^4 \left(1 + \frac{1}{2\sqrt{1 - e^{-4h}}}\right)\right). \tag{4.5.13}$$

Notice that $\log(1-x) \geq -3x/2$ if $x \in [0, 1/3]$, and since $\beta s \leq \log(2)$ the r.h.s. of (4.5.12) satisfies

$$\sqrt{c}\alpha_1 \left(1 + \frac{1}{2\sqrt{1 - e^{-4h}}}\right) \leq \frac{\beta^2 s^2}{15 \times 2^{12}} \leq \frac{1}{3}.$$

Hence $\log\left(H^\beta_{\alpha,h}\right)$ becomes

$$\begin{aligned}\log\left(H^\beta_{\alpha,h}\right) &\geq \log\left(e^\beta\left(1 - \frac{\sqrt{1 - e^{-4h}}}{2}\right)\right) - \frac{3}{2}\sqrt{c}\alpha_1\left(1 + \frac{1}{2\sqrt{1 - e^{-4h}}}\right) \\ &\geq \log\left(e^\beta\left(1 - \frac{\sqrt{1 - e^{-4h}}}{2}\right)\right) - \frac{\beta^2 s^2}{5 \times 2^{13}}.\end{aligned}$$

Then, since $\log(1 + \alpha_1) \leq \alpha_1$, we can rewrite (4.5.6) as

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq &\Bigg[\beta s \alpha_1 P^\beta_{\alpha,h}(2) - \frac{1}{2} P^\beta_{\alpha,h}(2)\left(1 + \alpha_1\right)\alpha_1 \\ &+ \log\left(e^\beta\left(1 - \frac{\sqrt{1 - e^{-4h}}}{2}\right)\right) - \frac{\beta^2 s^2}{5 \times 2^{13}}\Bigg]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \\ &- \sum_{k=3}^{N} P^\beta_{\alpha,h}(k)\frac{\mu_1 \log\left(\mu_1\right)}{2}\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-k}}{N}\right)\right) + R_N.\end{aligned} \tag{4.5.14}$$

4.5.7 Step V

Intermediate computations.

In the following steps, we need some inequalities on $P^\beta_{\alpha,h}$ and $H^\beta_{\alpha,h}$. As $\beta s \leq \log 2$, the equations in (4.5.13) show that $\alpha_1\sqrt{c} \in [0, 1/4]$. Therefore, $\alpha^2 = 1 - c\alpha_1^2 \geq 1 - 1/2^4 \geq 3/4$, and we can bound from above and below the quantity $H^\beta_{\alpha,h}$ (introduced in (4.5.4))

$$e^\beta \geq H^\beta_{\alpha,h} \geq e^\beta\left(1 - \frac{\sqrt{c}\alpha_1}{2} - \frac{1}{2}\right) \geq \frac{3e^\beta}{8}.$$

At this stage, we need to bound from above and below the quantity $P^{\beta}_{\alpha,h}(2)$, which has been defined in (4.5.3). With the previous inequalities, we have $e^{\beta}/H^{\beta}_{\alpha,h} \geq 1$ and $\sqrt{1-\alpha^2} \leq 1/4$. Thus,

$$\begin{aligned} P^{\beta}_{\alpha,h}(2) = 1 - \sum_{i=2}^{\infty} P^{\beta}_{\alpha,h}(2i) &\leq 1 - \sum_{i=2}^{\infty} \frac{1}{2}\alpha^{2i} P(\tau = 2i) \\ &= 1 - \frac{1}{2}\left(1 - \sqrt{1-\alpha^2} - \frac{\alpha^2}{2}\right) \leq \frac{7}{8}, \end{aligned} \tag{4.5.15}$$

and

$$\frac{1}{8} = \frac{1}{4} \times \frac{e^{\beta}}{2e^{\beta}} \leq P^{\beta}_{\alpha,h}(2). \tag{4.5.16}$$

Finally, with (4.5.15) and (4.5.16), we notice that

$$\frac{1}{8} \leq 1 - P^{\beta}_{\alpha,h}(2) \quad \text{and} \quad \frac{1}{7} \leq \frac{P^{\beta}_{\alpha,h}(2)}{1 - P^{\beta}_{\alpha,h}(2)} \leq 7. \tag{4.5.17}$$

Hence, the condition $\alpha_1 < P^{\beta}_{\alpha,h}(\tau = 2) / \big(1 - P^{\beta}_{\alpha,h}(\tau = 2)\big)$ is obviously satisfied.

4.5.8 Step VI

Conclusion

In (4.5.14), we still have to calculate the term

$$\sum_{k=3}^{N} P^{\beta}_{\alpha,h}(k)\, \mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-k}}{N}\right)\right).$$

If $N \geq N_0$, then

$$\begin{aligned} \sum_{k=3}^{N} P^{\beta}_{\alpha,h}(k)\, \mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-k}}{N}\right)\right) \geq & P^{\beta}_{\alpha,h}(\{3,\dots,N_0\}) \mathbb{E} E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_{N-N_0}}{N}\right) \\ \geq & \left(1 - P^{\beta}_{\alpha,h}(2)\right) \mathbb{E} E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right) - \frac{N_0}{N} \\ & - P^{\beta}_{\alpha,h}(\{N_0+1,\dots,\infty\})\, \mathbb{E} E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right), \end{aligned}$$

and equation (4.5.14) becomes

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq &\Bigg[\beta s\alpha_1 P^{\beta}_{\alpha,h}\left(2\right) - \frac{1}{2}P^{\beta}_{\alpha,h}\left(2\right)\left(1+\alpha_1\right)\alpha_1 - \frac{\beta^2 s^2}{5\times 2^{13}}\\ &+ \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right) - \left(1-P^{\beta}_{\alpha,h}\left(2\right)\right)\frac{\mu_1\log\left(\mu_1\right)}{2}\\ &+ P^{\beta}_{\alpha,h}\left(\{N_0+1,\ldots,\infty\}\right)\frac{\mu_1\log\left(\mu_1\right)}{2}\Bigg]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right)\\ &+ \frac{N_0}{N}\frac{\mu_1\log\left(\mu_1\right)}{2} + R_N. \end{aligned} \tag{4.5.18}$$

With (4.5.13) and (4.5.16), we can now bound from below

$$\beta s\alpha_1 P^{\beta}_{\alpha,h}\left(2\right) \geq \frac{\beta s}{2^3}\frac{\beta s}{5\times 2^8} = \frac{\beta^2 s^2}{5\times 2^{11}}.$$

Moreover, $\mu_1 = 1 - \left(\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)/\left(1-P^{\beta}_{\alpha,h}\left(2\right)\right)\right.$ and $-\log(1-x)\geq x$ for $x\in[0,1)$. Therefore, we obtain

$$-\frac{1-P^{\beta}_{\alpha,h}\left(2\right)}{2}\mu_1\log\left(\mu_1\right) \geq \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{2} - \frac{\alpha_1^2 P^{\beta}_{\alpha,h}\left(2\right)^2}{2\left(1-P^{\beta}_{\alpha,h}\left(2\right)\right)}.$$

In (4.5.16) and (4.5.17) we had $P^{\beta}_{\alpha,h}\left(2\right)\leq 7/8$ and $P^{\beta}_{\alpha,h}\left(2\right)/\left(2\left(1-P^{\beta}_{\alpha,h}\left(2\right)\right)\right)\leq 7/2$. Therefore,

$$-\frac{1-P^{\beta}_{\alpha,h}\left(2\right)}{2}\mu_1\,\log\mu_1 \;\geq\; \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{2} - \frac{7^2\alpha_1^2}{2^4} \geq \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{2} - 4\alpha_1^2.$$

In that way, the inequality in (4.5.18) can be written as

$$\begin{aligned}\mathbb{E}\left(\frac{1}{N}\log\left(H_N\right)\right) \geq &\Bigg[\frac{\beta^2 s^2}{5\times 2^{12}} - \frac{1}{2}P^{\beta}_{\alpha,h}\left(2\right)\left(1+\alpha_1\right)\alpha_1 + \frac{\alpha_1 P^{\beta}_{\alpha,h}\left(2\right)}{2} - 4\alpha_1^2\\ &+ \log\left(e^{\beta}\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\\ &+ P^{\beta}_{\alpha,h}\left(\{N_0+1,\ldots,\infty\}\right)\frac{\mu_1\log\left(\mu_1\right)}{2}\Bigg]\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right)\\ &+ \frac{N_0}{N}\mu_1\log\mu_1 + R_N. \end{aligned} \tag{4.5.19}$$

By (4.5.17) and (4.5.16), we know that $P^{\beta}_{\alpha,h}(2) \leq 7/8$ and $P^{\beta}_{\alpha,h}(2) / \left(1 - P^{\beta}_{\alpha,h}(2)\right) \leq 7$. Thus, we have the inequalities

$$-\frac{1}{2} P^{\beta}_{\alpha,h}(2)(1+\alpha_1)\alpha_1 + \frac{\alpha_1 P^{\beta}_{\alpha,h}(2)}{2} - 4\alpha_1^2 \geq -5\alpha_1^2 \geq -\frac{\beta^2 s^2}{5 \times 2^{16}}, \tag{4.5.20}$$

$$\text{and} \quad \frac{\alpha_1 P^{\beta}_{\alpha,h}(2)}{1 - P^{\beta}_{\alpha,h}(2)} \leq 7\alpha_1 = \frac{7\beta s}{5 \times 2^8} < \frac{1}{3}.$$

Since $\mu_1 \leq 1$ and $\log(1-x) \geq -3x/2$ for $x \in [0, 1/3]$, the second inequality of (4.5.20) allows us to bound from below

$$\mu_1 \log \mu_1 \geq -\frac{3}{2} \frac{P^{\beta}_{\alpha,h}(2)}{1 - P^{\beta}_{\alpha,h}(2)} \alpha_1 \geq -\frac{21\beta s}{5 \times 2^9} \geq -1.$$

Then, (4.5.19) becomes

$$\begin{aligned} \mathbb{E}\left(\frac{1}{N}\log(H_N)\right) \geq & \Bigg[\frac{\beta^2 s^2}{5 \times 2^{13}} + \log\left(e^{\beta}\left(1 - \frac{\sqrt{1 - e^{-4h}}}{2}\right)\right) \\ & - P^{\beta}_{\alpha,h}(\{N_0 + 1, \dots, \infty\})\Bigg] \mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) - \frac{N_0}{N} + R_N. \end{aligned} \tag{4.5.21}$$

As proved in Appendix A.1, $P^{\beta}_{\alpha,h}(\{N_0 + 1, \dots, \infty\})$ tends to zero as $N_0 \to \infty$, independently of $h \geq 0$. Therefore, for N_0 large enough and for all $h > 0$,

$$P^{\beta}_{\alpha,h}(\{N_0 + 1, \dots, \infty\}) \leq \frac{\beta^2 s^2}{5 \times 2^{14}}.$$

If we denote $q(s) = \frac{\beta^2 s^2}{5 \times 2^{14}}$, then, for $N \geq N_0$ and $h > 0$, (4.5.21) gives

$$\mathbb{E}\left(\frac{1}{N}\log(H_N)\right) \geq \left[q(s) + \log\left(e^{\beta}\left(1 - \frac{\sqrt{1 - e^{-4h}}}{2}\right)\right)\right] \mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) + R^{N_0}_N \tag{4.5.22}$$

with $R^{N_0}_N = R_N - N_0/N$. As proved in appendix A.2, for every $N \geq 1$ we have $\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}(l_N/N)\right) \geq \mathbb{E}\left(E^{\beta}_{\alpha,h}(l_N/N)\right)$. If we denote by $h_0(\beta)$ the only solution of

$$\log\left(e^{\beta}\left(1 - \sqrt{1 - e^{-4h_o(\beta)}}/2\right)\right) = -q(s),$$

then, for every $h < h_0(\beta)$ and $N \geq N_0$, we have

$$\mathbb{E}\left(\frac{1}{N}\log(H_N)\right) \geq \left[q(s) + \log\left(e^{\beta}\left(1 - \frac{\sqrt{1 - e^{-4h}}}{2}\right)\right)\right] \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) + R^{N_0}_N.$$

Consequently,

$$\Phi^s(\beta,h) \geq \left[q(s) + \log\left(e^\beta\left(1 - \frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right] \times \liminf_{N\to\infty} \mathbb{E}\left(E_{\alpha,h}^\beta\left(\frac{l_N}{N}\right)\right).$$

Notice also that $\liminf_{N\to\infty} \mathbb{E}\big(E_{\alpha,h}^\beta(l_N/N)\big) > 0$ (because $\alpha \in (0,1)$). Hence, for every β in $[0, \log(2) - q_s)$, $h_0(\beta)$ is a lower bound for $h_c(\beta)$, namely,

$$h_c(\beta) \geq h_0(\beta) = -\frac{1}{4}\log\left(1 - 4\left(1-e^{-\beta-q(s)}\right)^2\right).$$

This completes the proof of Theorem 13.

Remark 9 The precise value of $c_2 = 1/(5\times 2^{14})$ could certainly be improved, by using more complicated laws of return to the origin. For instance, some laws that depend more deeply on the environment (by taking into account ζ_{i+2}, ζ_{i+4}, etc.). However, the computations would be more complicated, and our aim here is not to optimize the value of c_1, c_2 but rather to expose a simple strategy that improves the non-disordered lower bound of a term $cs^2\beta^2$ with $c > 0$.

4.5.9 Proof of corollary 14

As shown just before in (4.5.22), there exists $N_0 \in \mathbb{N}\setminus\{0\}$ such that, for $h > 0$ and $N \geq N_0$,

$$\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\beta\sum_{i=1}^N \mathbb{1}_{\{S_i=0\}}(s\zeta_i+1) - 2h\sum_{i=1}^N \Delta_i\right)\right)\right)$$
$$\geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} + \log\left(e^\beta\left(1-\frac{\sqrt{1-e^{-4h}}}{2}\right)\right)\right]\mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right)\right) + R_N^{N_0}.$$

Moreover, in appendix A.2, we prove the following inequalities:

$$\mathbb{E}\left(E_{\alpha,h}^{\beta,\zeta,\alpha_1}\left(\frac{l_N}{N}\right)\right) \geq \mathbb{E}\left(E_{\alpha,h}^{\beta}\left(\frac{l_N}{N}\right)\right) \geq \mathbb{E}\left(E_{\alpha,\infty}^{0}\left(\frac{l_N}{N}\right)\right) > 0. \tag{4.5.23}$$

Thus, for β, s and N fixed, we let $h\to\infty$ and obtain

$$\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\beta\sum_{i=1}^N \mathbb{1}_{\{S_i=0\}}(s\zeta_i+1)\right)\mathbb{1}_{\{S_i\geq 0, \forall\, i\in\{1,\dots,N\}\}}\right)\right)$$
$$\geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} + \log\left(e^\beta\frac{1}{2}\right)\right]\mathbb{E}\left(E_{\alpha,\infty}^{0}\left(\frac{l_N}{N}\right)\right) + R_N^{N_0}.$$

Since $P(\{S_i \geq 0, \forall\, i \in \{1,\ldots,N\}\}) = (1+o(1))\, D/N^{1/2}$ when $N \to \infty$ (with $D > 0$), the lower bound becomes

$$\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\beta\sum_{i=1}^{N} \mathbb{1}_{\{S_i=0\}}\,(s\zeta_i+1)\right)\Big|\{S_i \geq 0, \forall\, i \in \{1,\ldots,N\}\}\right)\right) \\ \geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} + \log\left(e^{\beta}\frac{1}{2}\right)\right]\mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right) + K_N^{N_0}$$

with $K_N^{N_0} = R_N^{N_0} - 1/N\log\left(P(\{S_i \geq 0, \forall\, i \in \{1,\ldots,N\}\})\right)$, so that it tends to 0 as $N \to \infty$ independently of all the other parameters. By [19], we can apply the fact that, for an odd number of steps, the random walk conditioned to stay positive, and pinned by $\log 2$ along the x axis, becomes the reflected random walk. Indeed,

$$\frac{P_{\text{refl.RW}}}{P_{\text{RWcond.to be}\geq 0}}(S) = \frac{\exp\left((\log 2)\sum_{i=1}^{2N+1}\mathbb{1}_{\{S_i=0\}}\quad \mathbb{1}_{\{S_i\geq 0\ \forall\, i\in\{0,2N+1\}\}}\right)}{V_{2N+1}}.$$

The term $\frac{1}{N}\log V_N$ tends to 0 as $N \to \infty$. Hence, we denote $\beta = \log 2 - u$, and we obtain

$$\mathbb{E}\left(\frac{1}{2N+1}\log E\left(\exp\left(\log(2)\sum_{i=1}^{2N+1}\mathbb{1}_{\{S_i=0\}} + \sum_{i=1}^{2N+1}\mathbb{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\Bigg| \\ \{S_i \geq 0, \forall\, i \leq 2N+1\}\right)\right) \geq \left[\frac{\beta^2 s^2}{5\times 2^{14}} - u\right]\mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_{2N+1}}{2N+1}\right)\right) + K_{2N+1}^{N_0}$$

and

$$\mathbb{E}\left(\frac{1}{2N+1}\log E\left(\exp\left(\sum_{i=1}^{2N+1}\mathbb{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right)\right) \geq \\ \left[\frac{\beta^2 s^2}{5\times 2^{14}} - u\right]\mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_{2N+1}}{2N+1}\right)\right) + K_{2N+1}^{N_0} + \frac{1}{2N+1}\log V_{2N+1}.$$

Let $N \to \infty$, and recall that $\beta = \log(2) - u$. Then

$$\lim_{N\to\infty}\mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\sum_{i=1}^{N}\mathbb{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right)\right) \geq \\ \left[\frac{\beta^2 s^2}{5\times 2^{14}} - u\right]\lim_{N\to\infty} E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right),$$

and, for $u \le \log(2)/2$ (i.e., $\beta \ge (\log 2)/2$), we have

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\sum_{i=1}^{N} \mathbb{1}_{\{S_i=0\}}(-u+\beta s\zeta_i)\right)\right)\right) \ge \left[\frac{\log(2)^2 s^2}{5\times 2^{16}} - u\right] \lim_{N\to\infty} E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right).$$

By convexity, the free energy Φ, defined by

$$\Phi(u,v) = \lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\log E\left(\exp\left(\sum_{i=1}^{N} \mathbb{1}_{\{S_i=0\}}(-u+v\zeta_i)\right)\right)\right),$$

is not decreasing in v. Therefore,

$$\Phi(u,\log(2)s) \ge \left[\frac{\log(2)^2 s^2}{5\times 2^{16}} - u\right] \lim_{N\to\infty} E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right),$$

and, for $s \in [0,\log 2]$,

$$u_c(s) \ge \frac{s^2}{5\times 2^{16}}.$$

4.6 Appendix

4.6.1 A.1

We have to prove that $P^{\beta}_{\alpha,h}(\{N_0,\dots,+\infty\})$ tends to 0 as $N_0 \to \infty$ independently of $h \ge 0$. To that aim, we bound the quantity in (4.5.3) as follows:

$$\begin{aligned} P^{\beta}_{\alpha,h}(\tau_1 = 2n) &= \left(\frac{1+\exp(-4hn)}{2}\right)\alpha^{2n}\frac{P(\tau=2n)}{H^{\beta}_{\alpha,h}}\exp(\beta) \\ &\le \frac{\alpha^{2n}P(\tau=2n)}{\sum_{j=1}^{+\infty}\frac{1}{2}\alpha^{2j}P(\tau=2j)}. \end{aligned}$$

The r.h.s. of this inequality does not depend on h, and is the general term of a convergent series. Hence, we have uniform convergence in h.

4.6.2 A.2

We want to prove the inequalities of (4.5.23), i.e.,

$$\mathbb{E}\left(E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \ge \mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right) \ge \mathbb{E}\left(E^0_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right). \tag{4.6.1}$$

For that, we recall a coupling theorem (see [25] or [26]):

Theorem 16 *μ_1 and μ_2 are two probability measures on $2\mathbb{N}\setminus\{0\}$. If, for every bounded and non-decreasing function f defined on $2\mathbb{N}\setminus\{0\}$, $\mu_1(f)\le\mu_2(f)$, then we define on the same probability space (Ω,P) two random variables (T_1,T_2) of law (μ_1,μ_2) such that, P almost surely, $T_1\le T_2$.*

Remark 10 We notice that, to satisfy the hypothesis of the theorem, it is enough to show that there exists an integer i_0 such that, $\mu_1(2i)\ge\mu_2(2i)$ for every $i\ge i_0$ and $\mu_1(2i)\le\mu_2(2i)$ for every $i\ge i_0+1$. We can prove this easily by writing

$$\mu_2(f)-\mu_1(f)=\sum_{i=1}^{i_0}(\mu_2(2i)-\mu_1(2i))f(2i)+\sum_{i=i_0+1}^{\infty}(\mu_2(2i)-\mu_1(2i))f(2i).$$

As f is non-decreasing, $f(2i)\ge f(2i_0)$ for every $i\ge i_0+1$, and $f(2i)\le f(2i_0)$ for every $i\le i_0$. Moreover, since $\mu_2(2i)-\mu_1(2i)$ is positive when $i\ge i_0+1$ and negative otherwise, we have the inequality

$$\begin{aligned}\mu_2(f)-\mu_1(f)&\ge f(2i_0)\ \sum_{i=1}^{i_0}\mu_2(2i)-\mu_1(2i)\ +f(2i_0)\ \sum_{i=i_0+1}^{\infty}\mu_2(2i)-\mu_1(2i)\\&\ge -f(2i_0)\ (\mu_1-\mu_2)(\{2,\dots,2i_0\})\\&\qquad\qquad +f(2i_0)\ (\mu_2-\mu_1)(\{2(i_0+1),\dots,\infty\}).\end{aligned}$$

Since $(\mu_2-\mu_1)\left(\{2(i_0+1),\dots,\infty\}\right)=-(\mu_2-\mu_1)\left(\{2,\dots,2i_0\}\right)$, we obtain

$$\mu_2(f)-\mu_1(f)\ge -f(2i_0)(\mu_1-\mu_2)\left(\{2,\dots,2i_0\}\right)+f(2i_0)(\mu_1-\mu_2)\left(\{2,\dots,2i_0\}\right)\ge 0.$$

This is why we can use Theorem 16 in this situation.

We want to apply this remark to the following probability measures on $2\mathbb{N}\setminus\{0\}$: $P^0_{\alpha,\infty}$, $P^\beta_{\alpha,h}$ and $P^{\beta,+,\alpha_1}_{\alpha,h}$, which is the law defined in (4.5.5) when $\zeta_2\ge 0$. For that, we compare $P^\beta_{\alpha,h}$ and $P^{\beta,+,\alpha_1}_{\alpha,h}$, which is easy because

$$\begin{aligned}&P^{\beta,+,\alpha_1}_{\alpha,h}(\tau=2)=P^\beta_{\alpha,h}(\tau=2)\,(1+\alpha_1)\\&P^{\beta,+,\alpha_1}_{\alpha,h}(\tau=2r)=P^\beta_{\alpha,h}(\tau=2r)\,\mu_1\ \text{ for } r>2.\end{aligned}$$

Since $\alpha_1 > 0$ and $\mu_1 < 1$, we have the inequalities $P^{\beta,+,\alpha_1}_{\alpha,h}(\tau = 2) > P^{\beta}_{\alpha,h}(\tau = 2)$ and $P^{\beta,+,\alpha_1}_{\alpha,h}(\tau = 2r) < P^{\beta}_{\alpha,h}(\tau = 2r)$ for $r \geq 2$. Thus, Remark 10 tells us that we can use Theorem 16 and define on a probability space (Ω, P) a sequence of i.i.d. random variables $(T^1_i, T^2_i)_{i\geq 1}$ such that

- $P^{\beta,+,\alpha_1}_{\alpha,h}$ is the law of T^1_i for every $i \geq 1$,
- $P^{\beta}_{\alpha,h}$ the law of T^2_i for every $i \geq 1$,
- P almost surely $T^1_i \leq T^2_i$ for every $i \geq 1$.

At this stage, for every fixed disorder ζ, we define by recurrence another process $(T^3_i)_{i\geq 1}$ with

$$\begin{aligned} T^3_i = T^2_i \quad &\text{if} \quad \zeta_{T^3_1+\cdots+T^3_{i-1}+2} \geq 0 \\ = T^1_i \quad &\text{if} \quad \zeta_{T^3_1+\cdots+T^3_{i-1}+2} < 0. \end{aligned}$$

With these notations, $(T^2_i)_{i\geq 1}$ is the sequence of the excursion lengths of a random walk under the law $P^{\beta}_{\alpha,h}$, and $(T^3_i)_{i\geq 1}$ the one of a random walk under the law $P^{\beta,\zeta,\alpha_1}_{\alpha,h}$. By construction, $T^3_i \leq T^2_i$ for every $i \geq 1$. Thus, for $j =$ 2 or 3, we note $l^j_N = \max\{s \geq 1 : T^j_1 + \cdots + T^j_s \leq N\}$, and we have immediately that $l^3_N \geq l^2_N$ P almost surely. Therefore, for every ζ, we have

$$E^{\beta,\zeta,\alpha_1}_{\alpha,h}\left(\frac{l_N}{N}\right) = E_P\left(\frac{l^3_N}{N}\right) \geq E_P\left(\frac{l^2_N}{N}\right) = E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right),$$

and, by integration with respect to ζ, we obtain the l.h.s. of inequality (4.6.1).

To finish with these inequalities, we must show that the same argument allow us to compare $\mathbb{E}\left(E^{\beta}_{\alpha,h}\left(\frac{l_N}{N}\right)\right)$ and $\mathbb{E}\left(E^{0}_{\alpha,\infty}\left(\frac{l_N}{N}\right)\right)$. Indeed, we want to prove that Remark 10 also occurs. Recall that

$$\begin{aligned} P^{\beta}_{\alpha,h}(\tau_1 = 2n) &= \left(\frac{1+\exp(-4hn)}{2}\right)\alpha^{2n}\frac{P(\tau = 2n)}{H^{\beta}_{\alpha,h}}\exp(\beta) \\ P^{0}_{\alpha,\infty}(\tau_1 = 2n) &= \frac{\alpha^{2n}P(\tau = 2n)}{2H^{0}_{\alpha,\infty}}. \end{aligned}$$

If we note

$$L_n = \frac{P^{\beta}_{\alpha,h}(\tau_1 = 2n)}{P^{0}_{\alpha,\infty}(\tau_1 = 2n)} = (1 + \exp(-4hn)) \frac{H^{0}_{\alpha,\infty}}{H^{\beta}_{\alpha,h}} \exp(\beta),$$

then we immediately notice that L_n decreases with n, but we also have

$$\sum_{i=1}^{\infty} P^{\beta}_{\alpha,h}(\tau_1 = 2i) = \sum_{i=1}^{\infty} P^{0}_{\alpha,\infty}(\tau_1 = 2i) = 1.$$

Hence, there exists necessarily an i_0 in $\mathbb{N} \setminus \{0\}$ such that for $i \le i_0$ $P^{\beta}_{\alpha,h}(\tau_1 = 2i) \ge P^{0}_{\alpha,\infty}(\tau_1 = 2i)$ whereas for $i > i_0$ $P^{\beta}_{\alpha,h}(\tau_1 = 2i) \le P^{0}_{\alpha,\infty}(\tau_1 = 2i)$. This completes the proof.

4.6.3 B

First we recall a classical property, which tells us that we do not transform the free energy if we force the last monomer of the chain to touch the x axis. This is proved for a different case in [6], but the same technique can be applied to our hamiltonian. Therefore, we can write

$$\Phi^0(h,\beta) = \lim_{N\to\infty} \mathbb{E}\left(\frac{1}{2N} \log E\left(\exp\left(\beta \sum_{i=1}^{2N} \mathbb{1}_{\{S_i=0\}} - 2h \sum_{i=1}^{2N} \Delta_i\right) \mathbb{1}_{\{S_{2N}=0\}}\right)\right).$$

We note $Z_{2N,\beta,h} = E\left(\exp\left(\beta \sum_{i=1}^{2N} \mathbb{1}_{\{S_i=0\}} - 2h \sum_{i=1}^{2N} \Delta_i\right) \mathbb{1}_{\{S_{2N}=0\}}\right)$, and we remark that $Z_{2N,\beta,h}$ can be rewritten as

$$\begin{aligned} Z_{2N,\beta,h} &= \sum_{j=1}^{N} E\left(e^{\beta j} e^{-2h \sum_{i=1}^{2N} \Delta_i} \mathbb{1}_{\{l_{2N}=j\}} \mathbb{1}_{\{S_{2N}=0\}}\right) \\ &= \sum_{j=1}^{N} \sum_{\substack{\bar{l}\in\mathbb{N}^{*j} \\ |\bar{l}|=N}} \prod_{i=1}^{j} \left(e^{\beta j}\ V_{h,l_j}\right) \end{aligned}$$

with $V_{h,l} = P(\tau = 2l)\left(e^{-4hl}+1\right)/2$. We aim at computing the generating function of $Z_{2N,\beta,h}$, called $\theta_h(z)$. This gives

$$\begin{aligned}
\theta_h(z) = \sum_{N=1}^{\infty} Z_{2N,\beta,h} z^{2N} &= \sum_{N=1}^{\infty} z^{2N} \sum_{j=1}^{N} e^{\beta j} \sum_{\substack{\bar{l}\in\mathbb{N}^{*j}\\ |\bar{l}|=N}} \prod_{i=1}^{j} V_{h,l_j} \\
&= \sum_{j=1}^{\infty}\sum_{N=j}^{\infty} \sum_{\substack{\bar{l}\in\mathbb{N}^{*j}\\ |\bar{l}|=N}} \prod_{i=1}^{j} \left(e^{\beta} z^{2l_j}\, V_{h,l_j}\right) \\
&= \sum_{j=1}^{\infty}\left(\sum_{l=1}^{\infty} e^{\beta} z^{2l}\, V_{h,l}\right)^{j} = \sum_{j=1}^{\infty}\left(\sum_{l=1}^{\infty} \frac{P(\tau=2l)}{2}\left(1+e^{-4hl}\right) e^{\beta} z^{2l}\right)^{j}.
\end{aligned}$$

Finally, since

$$\sum_{l=1}^{\infty} P(\tau = 2l) z^{2l} = 1 - \sqrt{1-z^2},$$

we obtain

$$\theta_h(z) = \sum_{j=1}^{\infty}\left(\frac{e^{\beta}}{2}\left(2-\sqrt{1-z^2}-\sqrt{1-z^2 e^{-4h}}\right)\right)^{j}.$$

This series converges when $e^{\beta}\left(2-\sqrt{1-z^2}-\sqrt{1-z^2e^{-4h}}\right) < 2$, and if we denote by R its convergence radius, then we have $\Phi(\beta,h) = -\log(R)$. That is why $\Phi(\beta,h) > 0$ if and only if $R < 1$. So, we can exclude that (h,β) is on the critical curve if and only if, for $z = 1$, $e^{\beta}\left(2-\sqrt{1-z^2}-\sqrt{1-z^2e^{-4h}}\right) = 2$, i.e., $\sqrt{1-e^{-4h}} = 2\left(1-e^{-\beta}\right)$. This gives us the critical curve equation

$$h_c^0(\beta) = \frac{1}{4}\log\left(1-4\left(1-e^{-\beta}\right)^2\right).$$

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