Quelques problemes de Reflexion-Transmission en optiquedispersive faiblement non linaire.

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QUELQUES PROBLÈMES DE RÉFLEXION-TRANSMISSION EN OPTIQUE DISPERSIVE FAIBLEMENT NON LINÉAIRE

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Chapitre 1

Introduction

Nous proposons dans cette thèse l'étude mathématique détaillée de la transmission d'ondes électromagnétiques propagées dans des milieux dispersifs à réponse non linéaire.

1.0.1 Contexte

Ce travail s'inscrit à la fois dans le cadre de l'optique géométrique et celui de l'optique ondulatoire.

L'optique géométrique datant de l'Antiquité puis établie au XVII$^{\text{ième}}$ siècle par Fermat, Snell et Descartes, propose une analyse mathématique de la lumière. Elle est fondée sur quelques principes simples : propagation rectiligne, réflexion, réfraction. Les lois de Snell Descarte permettent de décrire les phénomènes de dispersion tels que l'arc-en-ciel.

L'optique ondulatoire date du XIX$^{\text{ième}}$ siècle avec Fresnel, Maxwell. Elle postule le caractère vibratoire de la lumière et permet d'expliquer les phénomènes de polarisation, d'interférence et de diffraction.

Ces deux optiques se concilient à travers la représentation des rayons sous la forme phase-amplitude que l'on doit à [31]

$$u^\varepsilon(t,x) = \varepsilon^\alpha a^\varepsilon(t,x) e^{i\varphi(t,x)/\varepsilon}, \tag{1.0.1}$$

où ε correspond à la longueur d'onde de l'onde, α est un paramètre qui caractérise la taille de l'amplitude et a^ε est l'amplitude adimensionnée que l'on cherche sous la forme d'un développement Wentzel-Kramers-Brillouin (WKB)

$$a^\varepsilon(t,x) = \sum_{n\geq 0} \varepsilon^n a_n(t,x). \tag{1.0.2}$$

Cette représentation ondulatoire rend compte aussi de l'aspect corpusculaire de la lumière si l'on suppose a^ε à décroissance rapide (typiquement gaussien).

En cherchant une solution de cette forme de faible amplitude pour les équations de Maxwell on obtient tout d'abord une équation pour la phase $\varphi(t,x)$: l'équation éikonale, de type Hamilton-Jacobi, qui donne le trajet du rayon lumineux. Celui-ci est rectiligne pour les ondes planes. Une telle phase est dite caractéristique pour le problème linéaire. L'équation éikonale cesse d'être résoluble lorsque les caracteristiques se croisent. On dit alors qu'il y a focalisation.

Ensuite on obtient une cascade d'équations de type transport, pour les amplitudes a_n, le long des rayons dont le trajet est dicté par la phase.

L'analyse asymptotique cesse lorsque les rayons focalisent.

Cette description a donné lieu à un grand nombre de résultats pour des équations hyperboliques faiblement non linéaires dans l'espace entier. Entre autres, dans le cas d'une seule phase, [22] justifie le développement (1.0.2) pour des systèmes semi-linéaires et [18] traite le cas quasi-linéaire. La justification du développement dans un cadre dispersif est réalisée dans [13, 15] et [13, 14] justifient dans ce cadre le développement de l'optique diffractive avec l'adjonction d'une nouvelle échelle "lente", $\mathsf{x} = \varepsilon x$, dans l'amplitude.

Dans le cas de plusieurs phases [21, 23] introduisent les notions de *cohérence* et de *petit diviseur* qui sont requises pour justifier la convergence d'un développement multiphasique indépendemment de ε. Sans cette condition [24] montre qu'on peut focaliser en temps ε.

La liste des travaux est impressionante et nous nous bornerons à renvoyer le lecteur à [41] pour une bonne introduction sur le sujet et [16] qui présente une vue d'ensemble très complète et synthétique.

L'utilisation du développement phase-amplitude de [31] dans le cas de problèmes mixtes (problème de Cauchy dans un domaine borné ou semi-borné) hyperboliques est plus délicate. En effet, la théorie linéaire sur les problèmes mixtes hyperboliques montre (cf.[30, 42]) qu'il faut adjoindre certaines conditions dites de Lopatinski ou maximales dissipatives sur les équations supplémentaires au bord pour prétendre à un problème bien posé (existence et stabilité). Les conditions de Lopatinski permettent une résolution explicite des ondes réfléchies et/ou transmises qualifiées d'*entrantes*.

L'analyse du problème de transmission de l'optique géométrique non dispersive est faite dans [37] dans le cadre Lopatinski avec néanmoins l'hypothèse restrictive que les ondes entrantes sont purement oscillantes. [45] lève cette hypothèse et montre l'apparition de *couches limites* formées de modes évanescents. Dans [47] (§3.F) l'auteur donne une classification complète des ondes engendrées par le bord et traite le cas spécifique d'onde rasantes (*glancing*) pour lesquelles l'ansatz sur l'amplitude requiert une échelle en $\sqrt{\varepsilon}x$.

Le cadre dispersif est justifié par [13] dans le cas des équations de Maxwell quasilinéaires mais restreint à des faisceaux dit *paraxiaux* pour lesquels les conditions de transmission au bord peuvent être approchées par une condition *d'onde entrante* qui ne sélectionne que les modes oscillants parmi les modes entrants. La justification repose ici sur la maximale dissipativité de la condition *d'onde entrante* et fait appel au résultat de [19] sur l'existence de solutions pour des problèmes hyperboliques mixtes caractéristiques.

Le travail que nous proposons consiste à reprendre l'analyse de P.Donnat sans l'hypothèse *d'onde entrante* sur le problème de transmission à coefficients constants et à bord droit. Plus généralement il étend les travaux de M.Williams au cadre dispersif. Se faisant il met à jour le phénomène de génération infini de phases non liées, conséquence de l'interaction au bord d'ondes dispersives. De nouveaux effets secondaires de génération d'onde sont également recontrés (cf. chap.1).

1.0.2 Le modèle de transmission issu de la physique

La modélisation d'un tel problème commence dans [13] avec les équations de Maxwell écrites dans $\mathbb{R}^3$ pour des milieux diélectriques libres de charges et de courants. En notant

$\partial_t := \partial/\partial t$ et $\mathrm{rot}E = {}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \wedge E$ les équations s'écrivent

$$\left\{ \begin{array}{c} \partial_t B + \mathrm{rot}E = 0, \\ \frac{1}{c^2}\partial_t E - \mathrm{rot}B = -\mu_0 \partial_t P \\ div(\varepsilon_0 E + P) = 0, \quad divB = 0. \end{array} \right. \tag{1.0.3}$$

où $E, B \in \mathbb{R}^3$ sont les champs électrique et magnétique, $P \in \mathbb{R}^3$ la polarisation qui décrit l'interaction champ-matière et ε_0, μ_0 les perméabilités linéaires diélectrique et magnétique. Il existe plusieurs modèles de couplage ; l'un d'eux, le modèle de l'oscillateur harmonique non linéaire (cf.[13]), s'écrit

$$\partial_t^2 P + \frac{1}{T_a}\partial_t P + \omega_a^2 P - d|P|P = \gamma_a E. \tag{1.0.4}$$

Le problème de transmission nécessite des conditions de transmission à l'interface (également libre de charges et de courants)

$$[E] \wedge n = 0, \quad [B] \wedge n = 0, \tag{1.0.5}$$

$$[\varepsilon_0 E + P] \cdot n = 0, \quad [B] \cdot n = 0. \tag{1.0.6}$$

$[E]$ est le saut du champ électrique à travers l'interface de normale n.

Enfin, pour une transmission effective on se donne une onde incidente. Celle-ci est propagée par les équations de Maxwell (1.0.3) qui sont des équations d'évolution en temps et nécessitent seulement une donnée initiale. Dans le cadre d'ondes électromagnétiques les solutions recherchées sont fortement oscillantes et on peut prendre une donnée initiale $(E_{|_{t=0}}, B_{|_{t=0}})$ de la forme

$$E(t=0,x) = e(x)e^{ikx}, \qquad B(t=0,x) = b(x)e^{ikx}, \tag{1.0.7}$$

où le paramètre k est très grand de sorte que $|\partial_x e| << |k|e$. D'ores et déjà on suppose que cette donnée initiale vérifie les équations de divergence de sorte que la solution satisfait également ces équations (qui sont transportées par les deux premières équations d'évolution) que l'on peut "oublier".

Comme chacune des équations précédentes fait intervenir des paramètres physiques dont les tailles varient de plusieurs ordres, un adimentionnement préalable est nécéssaire. Nous renvoyons le lecteur à [13] qui a mis en évidence un petit paramètre ε dont la définition et la taille dépendent de la forme (étalement spatio-temporel) de la source (cf.[13] où est fait une classification géométrique des sources : de type boulet, cigare ou galette de lumière). Pour les boulets $\varepsilon = \frac{1}{T_{ref}\omega_{ref}}$ où T_{ref} est la largeur temporelle de l'amplitude et ω_{ref} la pulsation de l'onde). De manière approchée, $1/\varepsilon$ donne le nombre d'oscillations de la phase contenues dans le support de l'enveloppe.

La description la plus complète est réalisée pour le boulet de lumière pour lequel la source prend la forme

$$E(t=0,x) = \varepsilon^\alpha e(x)e^{ikx/\varepsilon}. \tag{1.0.8}$$

Pour le problème de Cauchy on peut citer [4] qui fait une analyse des constantes présentes dans les estimations de convergence asymptotique vis-à-vis de la forme de la source.

[13] montre que le système complet (1.0.3) (sans les équations de divergence),(1.0.4) forme un système hyperbolique. Il s'écrit sous la forme adimensionnée

$$L^{\varepsilon}(\varepsilon\partial_t, \varepsilon\partial_x) = \varepsilon L^1(\partial_t, \partial_x) + L^0 + \varepsilon^2 L^2,$$

où pour $u = (E, B, P, Q = \varepsilon\partial_t P)$

$$L^1(\partial_t, \partial_x)u = \begin{pmatrix} \partial_t B + rotE \\ \partial_t E - rotB \\ D_t P \\ D_t Q \end{pmatrix}, \quad L^0 u = \begin{pmatrix} 0 \\ Q \\ -Q \\ \omega_a^2 P - \gamma E \end{pmatrix}, \quad L^2 u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\tau_a} Q \end{pmatrix}.$$

L'opérateur constant $\varepsilon^2 L^2$ n'ayant d'action qu'à l'ordre supérieur, sa présence est assimilée à celle d'un terme source dans le cas de l'optique géométrique et l'analyse linéaire est celle de l'opérateur

$$L(\varepsilon\partial_t, \varepsilon\partial_x) = \varepsilon L^1(\partial_t, \partial_x) + L^0.$$

Dans le cas de l'optique diffractive L^2 s'ajoute à l'opérateur $L(\varepsilon\partial_{\mathsf{t}}, \varepsilon\partial_{\mathsf{x}})$ (sur l'échelle lente) et n'apporte qu'une dérive dans la propagation. Nous l'omettrons par la suite.

Le terme non linéaire est

$$\phi(u) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d|P|^2 P \end{pmatrix}.$$

Enfin en notant $\Gamma \in \mathbb{R}^3$ un bord régulier de dimension 2, les équations de transmission s'écrivent

$$T(x)u(x) = \mathrm{diag}([E] \wedge n(x), [B] \wedge n(x), 0, 0) = 0, \ x \in \Gamma. \tag{1.0.9}$$

Soit Ω_g, Ω_d une partition de $\mathbb{R}^3$ de frontière connexe Γ régulière, le problème de transmission pour le système de Maxwell s'écrit finalement

$$\begin{cases} L_g(\varepsilon\partial_t, \varepsilon\partial_x)u = \phi_g(u), & \text{dans } [0,T] \times \Omega_g \\ L_d(\varepsilon\partial_t, \varepsilon\partial_x)u = \phi_d(u), & \text{dans } [0,T] \times \Omega_d \\ T(x)u = 0, & \text{sur } \Gamma \\ u(t=0,x) = \varepsilon^{\alpha} u(x) e^{i\varphi(x)/\varepsilon}. \end{cases} \tag{1.0.10}$$

Sous certaines hypothèses, l'étude (existence, et régularité de solutions fortes) de ce système rentre dans le cadre hyperbolique linéaire de J.Rauch et quasi-linéaire d'O.Guès. Ces cadres généraux dépassent le cadre des équations de Maxwell avec par exemple les équations d'Euler chez [12] et s'étendent aux équations de la magnéto-hydrodynamique chez [39].

Cependant il faut noter que le cadre des équations de Maxwell se distingue très nettement sur deux points qui sont liés. Tout d'abord comme les solutions sont à divergence nulle on doit vérifier cette propriété à l'instant initial : si $u(t=0) = (e, b, p, q)$

$$div(e+p) + \frac{i}{\varepsilon}\nabla\varphi.(e+p) = 0 \tag{1.0.11}$$

$$div(b) + \frac{i}{\varepsilon}\nabla\varphi.b = 0\ . \tag{1.0.12}$$

Ensuite cette même condition de divergence nulle permet d'obtenir une régularité globale alors que celle du système (1.0.10) n'admet qu'une régularité inhomogène (cf. [19]).

Le but que nous poursuivons à présent est l'étude de solutions de type (1.0.1) où l'amplitude se développe ou non sous la forme (1.0.2).

Dans les paragraphes suivants nous présentons le cadre précis et les hypothèses fondamentales que nous faisons pour l'étude de (1.0.10) et nous formulons un premier Ansatz pour les solutions de type optique géométrique commun aux deux premiers chapitres.

1.0.3 Solutions pour le problème mixte hyperbolique (1.0.10)

Les hypothèses fondamentales

Nous présentons en premier lieu les hypothèses générales faites tout au long de ce travail concernant (1.0.10). Ces hypothèses sont celles de [13] et un cas particulier de [18]. Comme elles s'énoncent dans le cas de dimension quelconque nous notons $x = (x_0, \dots, x_d) \in \mathbb{R}^{d+1}$ la variable spatio-temporelle où x_0 est la variable temporelle et $x'' = (x_1, \dots, x_d)$ la variable d'espace. Nous notons également $x' = (x_0, \dots, x_{d-1})$.

Equations à coefficients constant et bord droit.

L'opérateur $L^1(\partial_x) = \partial_{x_0} + \sum_{j=1}^{d} A_j \partial_{x_j}$ a ses coefficients matriciels A_j indépendants de t, x. L'opérateur L^0 est constant.

Le bord Γ est supposé droit. On le choisit comme étant le plan $x_d = 0$.

EDP symétriques hyperboliques semi-linéaires.

Les matrices A_j sont symétriques et L^0 est anti-symétrique.

Système caractéristique maximal dissipatif.

L'opérateur L est caractéristique : A_d n'est pas inversible ($\ker A_d$ de codimension égale à 4 pour Maxwell).

L'étude du problème mixte nécessite alors des conditions sur les équations de bord (cf.[42]). D'abord l'obtention d'estimations L^2 requiert de la dissipativité : $A_{d|_{\ker T}} \geq 0,\ x \in \Gamma$. Ensuite l'existence de solutions pour le problème de transmission nécessite des conditions de maximalité : $\mathrm{rg} T = \dim E^+_{A_d}$ où $E^+_{A_d}$ est l'espace propre maximal où $A_d \geq 0$.

Sous ces hypothèses on peut appliquer le théorème d'existence de [19] qui donne l'existence d'une unique solution sur un intervalle de temps $[0, t^\varepsilon]$ où t^ε dépend de ε.

Notons que les résultats de [19] permettent d'élargir l'étude de (1.0.10) à des problèmes quasi-linéaires, par exemple le modèle de l'oscillateur harmonique pour lequel la polarisation s'écrit $P = P_L + P_{NL}, \quad P_{NL} = |E^2|E$ (cf.[13]).

On peut également noter que le cadre énoncé concerne les équations d'Euler (cf. [12] où les conditions de Rankine-Hugoniot sont assimilées à des équations de transmission).

Temps d'existence

On souhaite étudier la dépendance de t^ε en fonction de ε et notamment trouver des conditions pour que $t^\varepsilon \geq 1$.

[15, 14] montrent que deux paramètres interviennent : la taille de l'amplitude p et l'ordre d'annulation J de la nonlinéarité : $\phi(u) = \mathcal{O}(|u|^J)$. Le temps d'existence est alors en $t^\varepsilon \sim \varepsilon^{1-p(J-1)}$.

Dans le cas de (1.0.10) avec une nonlinéarité cubique $J = 3$ et le régime d'optique géométrique $t^\varepsilon \sim 1$ est réalisé avec $p = 1/2$. Le cas de l'optique diffractive $t^\varepsilon \sim 1/\varepsilon$ requiert $p = 1$.

L'étude de sources d'amplitude plus forte nécessite des hypothèses structurelles supplémentaires telles que la *transparence* (cf. [28]) ou le cas de champ linéairement dégénérés (cf. [10]) que nous n'aborderons pas dans ce travail.

1.0.4 Transmission de phases ; Ansatz

On se place dans le cadre de [13] où les impulsions possèdent un grand nombre d'oscillations $\sim 1/\varepsilon$ ce qui permet de considérer des solutions à deux échelles (1.0.1). Tout d'abord même si la donnée initiale ne contient qu'une seule phase, la nonlinéarité va engendrer des harmoniques et ce, en nombre infini. Une première approche consiste alors à chercher des solutions monophasées du type

$$u(x) \sim \sum_{j\geq 0} \varepsilon^j U^j(x, \varphi(x)/\varepsilon) \tag{1.0.13}$$

$$U^j(x,\theta) = \sum_{k\in\mathbb{Z}} U_k^j(x) e^{ik\theta}. \tag{1.0.14}$$

Cependant des mécanismes de génération d'ondes caractéristiques décrits ci-dessous révèlent l'insuffisance de cet Ansatz (1.0.14). Nous rappelons d'abord quelques définitions.

Variété caractéristique, régularité et harmoniques

Lorsqu'on utilise la forme (1.0.13),(1.0.14) de U dans (1.0.10) on obtient la cascade d'équations

$$L(\partial_X)U^0 = 0 \tag{1.0.15}$$

$$L^1(\partial_x)U^j + L(\partial_X)U^{j+1} = F^j,\ j \geq 0 \tag{1.0.16}$$

dans le cadre de l'optique faiblement non linéaire. Pour chaque mode de Fourier on obtient l'équation sur l'amplitude $L(k\nabla_{t,x}\varphi(x))U_k^0 = 0$. Le fondamental : U_1^0 est non nul si et seulement si la phase vérifie l'équation eïkonale

$$\det(L(\nabla\varphi) = 0.$$

On dit alors que φ est *caractéristique*. Comme le polynôme est à coefficients constants il existe des solutions phase plane $\varphi = i\xi.x$. On appelle *variété caractéristique* de l'opérateur L notée

$$\mathrm{char}L := \{\xi = (\xi_0, \xi'') \in \mathbb{R}^{d+1} \mid \det(L(i\xi)) = 0\}.$$

Comme le système (1.0.10) est hyperbolique, le polynôme caractéristique $p(\xi) := \det(L(i\xi))$ est scindé sur $\mathbb{R}_{\xi_0}$.

On dit que $\xi \in \mathrm{char}L$ est *régulier* s'il existe un voisinage de (ξ_0, ξ'') dans lequel $\mathrm{char}L$ est décrit par une unique équation $\xi_0 + \lambda(\xi'') = 0$ avec $\lambda\ \mathcal{C}^\infty$.

Milieux dispersifs : finitude des harmoniques caractéristiques

La présence de la matrice constante L^0 est responsable du caractère dispersif de l'opérateur $L(\partial_x)$.

Sans ce terme char$L$ est une réunion de cônes vectoriels $\mathcal{C}$ et de plans affines $\mathcal{P}$. Ainsi, si $\xi \in \mathcal{C} \subset \text{char}L$ alors toutes ses harmoniques sont caractéristiques : $k\xi \in \text{char}L,\ k \in \mathbb{Z}$.

Lorsque $L^0 \neq 0,\ \text{char}L$ contient en plus des nappes courbes qui ne sont ni des cônes ni des plans. Ces nappes sont paramétrées par $\xi_0 = \lambda(\xi'')$ où λ est une fonction non linéaire. Si ξ, point régulier de char$L$, se trouve sur une telle nappe courbe, il existe au plus un nombre fini d'harmoniques $k\varphi,\ k \in \mathbb{Z}$ caractéristiques. Ainsi la somme (1.0.14) est en général finie.

Génération d'harmoniques caractéristiques au bord

A.*Mécanismes linéaires.*

Soit $\varphi(x) = ix.\xi$ la phase plane présente dans (1.0.13). Cette phase et ses harmoniques sont donc présentes au bord. L'équation au bord pour le profil U^j se décompose alors en $TU^j_k = 0$ selon les modes de Fouriers $k\xi' = k(\xi_0, \ldots, \xi_{d-1})$ appelés harmoniques tangentielles. Cependant, si $k\varphi$ n'est pas caractéristique, U^j_k est déjà déterminé et $TU^j_k = 0$ n'est en général pas satisfaite.

Ceci montre qu'il faut considérer pour chaque phase $k\varphi$ dans (1.0.14) **toutes les phases caractéristiques dont la restriction au bord est** $k\varphi_{|_{x_d=0}} = ik\xi'.x'$.

Ces nouvelles phases sont $(ik\xi', i\alpha^l_k)$ et $(ik\xi', i\beta^l_k)$ solutions de

$$\begin{array}{l} \det(L_d(ik\xi', i\alpha^l_k) = 0, l \leq deg_{\xi_3} \det L_d, \\ \det(L_g(ik\xi', i\beta^l_k) = 0, l \leq deg_{\xi_3} \det L_g. \end{array} \tag{1.0.17}$$

Dans le cas où $L_g \neq L_d$ ces nouvelles phases caractéristiques ne sont en général pas des harmoniques de phases existantes. Elles peuvent être complexes puisque les équations précédentes ne sont pas nécessairement hyperbolique en x_3. De manière complète, on a chez [47] la classification des solutions de (1.0.17) suivante : Une phase réelle caractéristique $\xi_0 = \lambda(\xi'')$ régulière est *entrante*, *sortante* ou *rasante* selon que $\nabla_{\xi''}\lambda > 0,\ < 0$ ou $= 0$. Une phase complexe est *évanescente* ou *explosive* selon que $\text{Im}\,\xi_d > 0,\ < 0$.

Notons qu'une onde incidente arrivant à l'interface avec un angle supérieur à l'angle critique (s'il existe) ne transmet qu'une couche limite à droite au premier ordre. Notre analyse précise cette description en montrant qu'*en général des termes correcteurs oscillant et résonant sont transmis*. Sous certaines conditions (cf.chap.1,Introduction, 4.) de tels termes d'amplitude ε sont générés. De manière analogue si les ondes transmises relatives aux profils principaux sont purement oscillantes, il existe en général des termes correcteurs évanescents (cf.chap.1,Introduction, 3.).

B.*Mécanisme non linéaires*

Lorsqu'on considère les équations de la cascade (1.0.15) on voit que les $F^j,\ j \geq 0$ étendent le réseau de phases constitué par le spectre des profils $U^k,\ k \leq j$.

Tout d'abord le problème de génération infinie se pose pour le profil principal U^0 qui vérifie une équation de transport non linéaire. Comme la non linéarité F^0 est le développement de Taylor de ϕ en 0 elle est polynômiale. Nous avons alors fait l'hypothèse de l'existence d'un module Λ^0 stable (vis-à-vis des résonances) F^0 (voir chap.1, hypothèse 2.2.9, 1.).

Ensuite pour $j \geq 1$ la génération de nouvelles phases caractéristiques est en général finie puisque la variété caractéristique est courbe et la non linéarité F^j est polynômiale. Nous avons néanmoins fait l'hypothèse (cf. chap.1., hypothèse 2.2.9, 2.) assurant l'existence d'un ensemble Λ^j, fini, contenant le spectre de U^j.

Néanmoins si à chaque équation de la cascade seul un nombre fini de nouvelles phases caractéristiques est généré le processus créé de proche en proche un réseau de phases de dimension infinie.

La description habituelle du profil U par rapport à la variable rapide X (ex. [23]) selon un module généré par un nombre fini de phases est alors inconsistante.

Ansatz

Nous avons opté pour une description du profil $U = U_{os} + U_{ev}$ où U_{os} est un profil purement oscillant et U_{ev} un profil évanescent qui, contrairement à [47], regroupe les phases complexes correspondant aux ondes d'amplitudes exponentiellement décroissantes. De fait, $\mathcal{U}_{ev}$ est solution d'un système dissipatif. L'Ansatz utilisé s'écrit alors

$$\begin{array}{c} U(x,X) = \sum_{\xi'} U_{\xi'}(x,X_d) e^{iX'.\xi'} \\ U_{\xi'}(x,X_d) = \sum_{\xi_d} U_{\xi}(x) e^{iX_d\xi_d} + U_{ev,\xi'}(x,X_d) \end{array} \tag{1.0.18}$$

où les sommes sont finies et $U_{ev,\xi'}(x,X_d) \sim e^{-\delta X_d}$ pour un $\delta > 0$.

1.1 Présentation des chapitres

A présent nous décrivons les trois chapitres de cette thèse. Le premier chapitre concerne l'analyse de l'Optique Géométrique sur le problème de transmission pour des développements à tous ordres et en donne la convergence. Le second traite le même problème dans le cas diffractif c'est-à-dire pour des amplitudes plus petites et des temps plus longs. Enfin le troisième propose une étude sans développement WKB de l'amplitude devant satisfaire de manière la plus exacte possible l'équation linéaire. Dans ce cadre nous simplifions le problème de bord en ne choisissant qu'un seul mode entrant sans harmonique résonante et en négligeant toutes les autres polarisations ainsi que les couches limites.

1.1.1 Transmission de l'Optique Géométrique.

Dans cette partie nous construisons un développement WKB asymptotique multidimensionnel à tous ordres pour le problème (1.0.10) avec des coefficients constants et un bord droit infini. Chaque terme du développement satisfait à l'Ansatz (1.0.18).

La solution approchée WKB

Soit $x = (x_0, \ldots, x_d)$ où $x_0 = t$ et $x_d = 0$ est une paramétrisation du bord. Les équations considérées s'écrivent pour le premier profil

$$\begin{array}{c} L(\partial_X) U^0(x,X) = 0 \\ TU^0(x', x_d = 0) = 0 \\ U^0(x_0 = 0, x'') = u^0(x''), \end{array} \tag{1.1.1}$$

et pour les autres

$$\begin{array}{c} L(\partial_X) U^j(x,X) + L(\partial_x) U^{j-1}(x,X) = F^{j-1}(x,X) \\ TU^j(x', x_d = 0) = 0 \\ U^j(x_0 = 0, x'') = u^j(x''). \end{array} \tag{1.1.2}$$

Ces équations posent plusieurs problèmes. Tout d'abord il faut résoudre les EDP pour les profils du type (1.0.18). Ensuite il faut résoudre les problèmes mixtes avec deux échelles différentes. Enfin il s'agit de contrôler la génération de phases et tout particulièrement celles qui ne sont pas régulières.

La première équation et plus généralement l'équation sur les variables rapides s'écrit

$$L(\partial_X)U = F, \tag{1.1.3}$$

et décrit la polarisation des ondes associées aux phases caractéristiques. Comme nous souhaitons résoudre le problème de génération d'onde au bord notons $L(\partial_x) = A_d\partial_{x_d} + L'(\partial_{x'})$. Nous considérons alors l'équation générique pour chaque fréquence tangentielle ξ'

$$\big(A_d\partial_{X_d} + L'(i\xi')\big)U_{\xi'} = F_{\xi'} \tag{1.1.4}$$

Comme A_d n'est pas inversible on découple l'équation en deux équations, l'une sur l'image de A_d l'autre sur le noyau. On peut supposer que $U = (U^1, U^2)$ où $U^1 \in \operatorname{Im} A_d$ et $U^2 \in \ker A_d$. Si la seconde équation sur $\ker A_d$ permet d'exprimer U^2 en fonction de U^1 on obtient la nouvelle équation sur $\operatorname{Im} A_d$

$$(\partial_{X_d} + G(i\xi'))U^1_{\xi'} = \tilde{F}_{\xi'}. \tag{1.1.5}$$

Cette équation se découple en une équation sur les profils purement oscillants et une autre sur la partie évanescente. La première redonne la polarisation habituelle suivant chaque phase imaginaire pure et résonante associée à ξ'. La seconde est une équation d'évolution en X_d, elliptique car définie sur l'espace propre de $G(i\xi')$ associé aux valeurs propres de partie réelle non nulle. Cette équation nécessite donc une donnée en $X_d = 0$ qui permet de déterminer le profil évanescent. On obtient alors le résultat

Résultat 1.1.1. *Il existe des projecteurs $\mathbb{P}$ et $\mathbb{P}^i$ ainsi qu'un inverse partiel $\mathbb{Q}$ agissant dans l'espace des profils du type* (1.0.18) *tels que pour F, profil de type* (1.0.18)*, l'équation* (1.1.3) *a une solution U du type* (1.0.18) *si et seulement si $\mathbb{P}^i F = 0$. La solution générale s'écrit*

$$U = \mathbb{Q}F + \mathbb{P}U.$$

Nous renvoyons au chapitre 2, théorème 2.2.14 pour un énoncé exact.

La seconde équation concerne la variable lente x pour la partie oscillante du profil U^0. C'est en général une équation de transport non linéaire pour les profils associés aux phases régulières (cf. [15]) et une équation des ondes (problème hyperbolique tangent) pour les phases non régulières (ex : $\xi = 0$ cf.[32, 44]). Elle nécessite donc des données sur le bord $x_d = 0$ et/ou initiales selon que l'onde est entrante ou sortante. Plus précisemment la donnée initiale, polarisée, détermine les profils sortants et l'équation de bord fournit les données relatives aux profils entrants et évanescents pour chaque ξ' grâce à une condition de type Lopatinski sur T.

Cependant cette condition n'est pas utilisée pour les profils associés à des phases non régulières pour lesquelles on se sert de la maximale dissipativité héritée par le problème tangent. Chaque profil U^j de type (1.0.18) est donc solution d'un problème hyperbolique maximal dissipatif. Il faut enfin signaler le problème de compatibilité des données initiales et de bord pour la régularité en $x_0 = x_d = 0$. On obtient le résultat

Theorem 1.1.1. *Soit* $e(x'') = \sum_{0\le j\le n} \varepsilon^j E^j(x", x''/\varepsilon)$, *une donnée initiale WKB telle que* $E^j = \mathbb{P}E^j$, $j \le n$ *infiniment plate en* $x_d = 0$ *de spectre oscillant fini et spec*$E^0 = \Lambda^0$ *(cf.* 1.0.4*B).*

Il existe $t^\star > 0$, *une suite de profils* $U^n \in L^\infty_x([0, t*] \times \mathbb{R}^d_+; L^\infty_X(\mathbb{R}^{d+1}_+))$ *de type* (1.0.18), *solution des équations de la cascade* (1.1.1),(1.1.2).

Nous renvoyons au chapitre 1, Théorème 2.16 pour la version exacte.

Convergence

Ensuite on montre la convergence/stabilité du développement asymptotique vers la solution exacte du problème 1.0.10.

Le problème que nous obtenons sur les restes est un problème mixte hyperbolique singulier en ε et caractéristique. L'obtention d'inégalités d'énergie donc pose plusieurs problèmes.

Tout d'abord l'aspect caractéristique du bord fait que l'on doit dinstinguer la régularité normale (∂_{x_d}) et conormale ($\partial_{x'}$). Plus précisément grâce à l'hyperbolicité du système on obtient une certaine régularité conormale et les dérivées normales regagnées en utilisant les équations coûtent deux dérivées conormales. Nous utilisons donc des espaces d'analyse inhomogènes $E^{m,\varepsilon}([0, t^*])$ (cf. chap.1,§2.2) proches de ceux d'O.Guès [19] mais bornés en temps.

Ensuite le caractère singulier en ε oblige à considérer des ε-dérivées. En outre, là encore il faut distinguer comme dans [13] les dérivées normales qui nécessitent un facteur ε^2 (2ε-dérivées conormales).

Enfin comme dans [14] nous appliquons la méthode de [18] qui consiste à considérer $\approx d/2$ termes dans la solution approchée afin d'utiliser l'inégalité de plongement de L^∞ dans les Sobolev. En résumé on obtient le résultat

Theorem 1.1.2. *Sous les hypothèse du théorème 1.1.1 et* $n \ge M \ge m > \frac{d+1}{2}$, m *pair, il existe* $\varepsilon_0 > 0$ *tel que pour tout* $\varepsilon \in]0, \varepsilon_0]$ *le problème de Cauchy* (1.0.10) *avec donnée initiale*

$$u^\varepsilon(t = 0) = e + \varepsilon^M h$$

où h *est suffisamment plate au bord, a une unique solution* $u^\varepsilon = \sum_{0\le j\le n} U^j + \varepsilon^M v^\varepsilon$ *dans* $E^{m,\varepsilon}([0, t_*])$ *avec* $v^\varepsilon \in E^{m,\varepsilon}([0, t^*])$.

1.1.2 Transmission de l'Optique Diffractive.

Cette partie poursuit l'étude réalisée au premier chapitre et complète le second chapitre de [13]. Elle concerne des ondes de plus petite amplitude $\mathcal{O}(\varepsilon^2)$ et des temps/distances de propagation de l'ordre de $\mathcal{O}(1/\varepsilon)$. L'utilisation de l'ansatz à deux échelles montre qu'alors les termes correcteurs (du profil principal U^0) ne restent pas bornés sur cet intervalle. D'autre part on s'attend à voir sur ces distances, outre la propagation, la diffraction. L'analyse consiste (cf. [13]) à introduire une nouvelle échelle εx dans l'ansatz (1.0.18). Comme la nonlinéarité n'intervient qu'à la troisième équation le profil principal est défini globalement en la variable moyenne x.

Le but poursuivi est alors d'obtenir des correcteurs ayant également une croissance sous-linéaire (voire bornés) par rapport à cette variable moyenne. La différence essentielle avec les travaux [13, 14, 32] vient encore de la présence de phases linéairement indépendantes.

Plus précisemment les équations de transport dans la variable moyenne contiennent des termes sources non linéaires de plusieurs type : des termes

-transportés à la vitesse du champ de transport,

-transportés à une vitesse différente,

-non transportés.

Les travaux [13, 14] ne faisaient apparaître que le premier type de terme source soit à cause du peu de termes considérés soit parce qu'étant non dipersif. Ce terme est reponsable de croissance linéaire et doit donc être compensé par les termes linéaires en une équation sur les variables lentes.

Dans le cas dispersif de [32], le troisième type d'interaction apparaît. Ce terme n'induit pas de croissance et est laissé comme terme source dans l'équation de transport.

Enfin pour mieux décrire l'action du terme du second type, on a décomposé chaque profil U^j_α relatif à une phase $i\alpha X$, résonante

$$U^j_\alpha(\mathsf{x},x,X)=\mathcal{U}^j_{\alpha,\alpha}(\mathsf{x},x'-v_\alpha z,z,X)+\sum_{v_\beta\neq v_\alpha}\mathcal{U}^j_{\alpha,\beta}(\mathsf{x},x'-v_\beta z,z,X)+U^{j,\sharp}_\alpha(\mathsf{x},x,X),$$

où les β sont dans Λ, un réseau fini de modes résonants et v_β les vitesses de groupe associées. Le terme $U^{j,\sharp}_\alpha$ prend en compte les interactions du troisième type. C'est typiquement le produit d'ondes transportées à des vitesses différentes.

Les équations se découplent et pour chaque β tel que $v_\beta\neq v_\alpha$ on a une condition de résolubilité sur le terme source.

Nous n'avons pas cherché à expliciter plus avant cette condition de type intégrale (exprimée sur la caractéristique liée à v_α) qui imposait à première vue des profils impairs, non acceptables pour des impulsions laser. Nous avons alors fait l'hypothèse qu'aucun terme source du second type n'était généré. L'Ansatz approprié pour les profils est alors

$$U^j_\alpha(\mathsf{x},x,X)=\mathcal{U}^j_\alpha(\mathsf{x},x'-v_\alpha z,z,X)+U^{j,\sharp}_\alpha(\mathsf{x},x,X), \tag{1.1.6}$$

où la partie transportée à v_α est solution d'un problème de transport diffractif mixte qui découple les ondes sortantes de celles entrantes.

La convergence de ce développement vers la solution exacte pour des temps diffractifs en $\mathcal{O}(1/\varepsilon)$ est calquée sur celle du premier chapitre. Néanmoins l'équation sur les restes est modifiée par la présence dans la donnée initiale de $U^{j,\sharp}_\alpha|_{\mathsf{t}=-\mathsf{t}^0,t=-\mathsf{t}^0/\varepsilon,T=-\mathsf{t}^0/\varepsilon^2}$.

La remarque faite par [13] est que pour des impulsions à profil dans l'espace de Schwartz ce terme est à décroissance rapide. L'analyse appropriée est alors réalisée avec des espaces du type

$$\Gamma(\mathbb{R}^d)=\{(x''-vt)u(t,x'')\in H^\infty_{x''}(\mathbb{R}^d)\},$$

qui permettent de montrer que les $U^{j,\sharp}_{|_{\mathsf{t}=-\mathsf{t}^0,t=-\mathsf{t}^0/\varepsilon,T=-\mathsf{t}^0/\varepsilon^2}}$ sont en $\mathcal{O}(\varepsilon^\infty)$.

Résultat 1.1.2. *Soit $e(x'')=\sum_{0\leq j\leq n}\varepsilon^j E^j(\varepsilon x'',x'',x''/\varepsilon)$ une donnée initiale polarisée, plate au bord et de spectre fini et spec$E^0=\Lambda^0$.*

1. *Il existe $(U^j)_{0\leq j\leq n}$ une suite de profiles solutions des équations de la cascade* (3.2.1)*, s'écrivant sous la forme* (1.1.6) *et tels que $\mathcal{U}^j\in\Gamma(\mathbb{R}^d_{x'})$.*
2. *Il existe $\mathsf{t}_*>-\mathsf{t}^0$ et $^\varepsilon=\sum_{0\leq j\leq n}U^j+\varepsilon^M v^\varepsilon$ solution exacte de* (1.0.10) *définie pour $t\leq\mathsf{t}_*/\varepsilon$.*

Nous renvoyons le lecteur au chapitre 3, Théorèmes 3.3.1 et 3.3.2.

1.1.3 Modèle de transmission intermédiaire

Ce troisième chapitre est consacré à la dérivation d'un modèle d'optique géométrique dans l'esprit de [6], adapté au cadre de la transmission.

Le modèle dérivé dans [6]

L'obtention du modèle dans [6] débute par la recherche d'une solution exacte au problème

$$\begin{array}{c} L(\partial_x)U = F(U), \quad \text{in} \quad [0,t]\times\mathbb{R}^d \\ U(t=0) = \varepsilon^p U_0(x'')e^{i\xi_*''.x''/\varepsilon} \end{array} \tag{1.1.7}$$

sous forme de profil $U(x) = \varepsilon^p \mathcal{U}(x,\theta)$, 2π-périodique en $\theta = \xi_*.x/\varepsilon$ pour un ξ_* donné, résonant.

La décomposition spectrale de $L(\partial_x)$

$$L(\partial_x) = \partial_t + \sum_j \pi_j(\partial_x'')\lambda_j(\partial_x'')$$

conduit à chercher un modèle scalaire construit sur $\lambda_* = \lambda_1$ (associé à ξ_*), proche de (1.1.7) pour des données initiales oscillantes dont le profile vérifie :

$$U_0(x'') = \pi_*(\varepsilon\partial_x'')U_0(x'') \tag{1.1.8}$$

Le modèle s'écrit :

$$\left\{ \begin{array}{c} (\partial_t + \frac{\lambda_*(\varepsilon\partial_x''+\xi_*''\partial_\theta)-\xi_{*,0}\partial_\theta}{\varepsilon})\mathcal{U}_{app}(x,\theta) = \pi_*(\varepsilon\partial_x'' + \xi_*''\partial_\theta)F(\mathcal{U}_{app}(x,\theta)) \\ \mathcal{U}_{app}(t=0,x'',\theta) = \varepsilon^p U_0(x'')e^{i\theta}. \end{array} \right. \tag{1.1.9}$$

Il néglige donc toutes les autres polarisations créées par le terme non linéaire F. Elles sont en effet d'autant plus faibles que

-initialement nulles,

-non résonantes avec ξ_* et supposées non résonantes avec les harmoniques de ξ_*.

-$F(x) = \mathcal{O}(|x|^J)$, $x \sim 0$ avec J grand.

En introduisant le paramètre $\sigma = \varepsilon^{p(J-1)-1}$ le théorème obtenu dans [6] dans ce contexte s'énnonce :

Theorem 1.1.3. *Soit $U_0 \in H^s(\mathbb{R}^d\times\mathbb{T})$, $s > d/2$ une donnée initiale satisfaisant* (1.1.8)*. Il existe $t_0 > 0$ et $U = \varepsilon^p\mathcal{U}$,$\mathcal{U}_{app}$ solutions de* (1.1.7) *et de* (1.1.9) *dans $X_{t_0} = L^\infty(0,t_0;H^s(\mathbb{R}^d\times\mathbb{T}))$ tels que*

$$\varepsilon^{-p}\|\mathcal{U} - \mathcal{U}_{app}\|_{X_{t_0}} = \mathcal{O}(\varepsilon\sigma).$$

Le modèle que nous dérivons

Pour le problème de transmission on peut alors s'attendre à une équation de transport pseudo-différentielle en la variable x'. Le problème auquel on s'intéresse est l'analogue de (1.1.7) avec une donnée au bord au lieu de la donnée de Cauchy

$$\begin{array}{c} L(\partial_x)U = F(U), \quad \text{dans} \quad \mathbb{R}^d\times[0,z] \\ U(z=0) = \varepsilon^p U_0(x')e^{i\xi_*'.x'/\varepsilon} \end{array} \tag{1.1.10}$$

Mais comme $L(\partial_x)$ n'est pas hyperbolique dans la direction dz,(1.1.10) est en général mal posé. Il faut en effet adjoindre des conditions d'onde entrante au bord et comme le problème est non linéaire il faut également des conditions similaires sur les dérivées normales au bord.

Néanmoins dans le cas d'impulsions à spectre étroit, contenu dans une nappe réelle de char$L$ on peut dériver un modèle qui vérifie de manière approchée l'équation semi-linéaire dans (1.1.10). Ceci est réalisé en introduisant un opérateur de troncature χ_*, fonction caractéristique de la nappe considérée. Le modèle s'écrit alors

$$\mathcal{E}\begin{cases} (\partial_z + \frac{\zeta(\varepsilon\partial'_x+\xi'_*\partial_\theta)-\xi_{*,d}\partial_\theta}{\varepsilon})\mathcal{U}(x,\theta) = \tilde{\pi}_*(\varepsilon\partial'_x + \xi'_*\partial_\theta)\chi_*(\partial'_x)F(x,\theta) \\ \mathcal{U}(x', z=0,\theta) = \varepsilon^p\tilde{\pi}_*(\varepsilon\partial'_x + \xi'_*\partial_\theta)\chi_*(\partial'_x)U_0(x')e^{i\theta}, \end{cases} \tag{1.1.11}$$

et le profil $\mathcal{U}$ est cherché dans l'espace $\mathcal{C}(0,z;H^s(\mathbb{R}^d))$. On a alors le lemme d'existence suivant

Lemma 1.1.4. *Soit $\xi_\star$ associé à $\xi_{\star,n} = \zeta_1(\xi'_\star)$. Alors il existe $z_0 > 0$, et une unique solution $\mathcal{U} = \tilde{\pi}_*\chi_*\mathcal{U}$ solution de $(\mathcal{E})$ dans $X^s(z_0/\sigma)$.*

Ensuite l'obtention d'un théorème analogue à 1.1.3 consiste à chercher une solution exacte de (1.0.10) de type profil, $U(x,X)$, dans une algèbre de Wiener. Ce problème est l'analogue de [26] pour le problème de Cauchy, dans le cadre de problèmes mixtes hyperboliques. Ce problème paraît difficile car le spectre d'une telle solution n'est pas uniquement oscillant et le spectre oscillant est contenu dans un module de dimension infinie. On s'attend donc à des osillations denses semblables au cas des équations d'Euler dans [25]. L'obtention d'estimations indépendantes de ε pour chaque mode U_ξ solution d'un problème mixte hyperbolique couplé semble alors compromise.

Nous avons choisi alors de montrer un théorème analogue à 1.1.3 non plus avec (1.0.10) mais avec (1.1.10) que l'on a considéré comme la restriction à $z \geq 0$ de (1.0.10) avec une condition initiale sortante bien choisie de sorte que $U(z=0)$ soit la trace au bord de l'onde transmise.

Cependant, même dans ce cadre, la recherche d'une solution exacte de (1.1.10) sous la forme d'un profil $\mathcal{U}(x,X)$ ne semble pas réalisable. Nous avons évité cette analyse en cherchant une solution de (1.1.10) sous la forme

$$U \approx \mathcal{U} + \varepsilon\sigma r_1^\varepsilon$$

où r^ε est un correcteur WKB est purement oscillant.

Enfin l'obtention d'un théorème d'approximation nécessite d'introduire des troncatures $\psi_1(t), \psi_2(z)$ car contrairement à la solution exacte dont le support se propage à vitesse finie, la solution approchée résout des équations pour lesquelles la vitesse de propagation est infinie.

On obtient finalement le théorème :

Theorem 1.1.5. *Soient s un entier superieur strictement à $d/2$. Il existe $z_0 > 0$ et un correcteur $r^\varepsilon \in X^s(z_0/\sigma)$ tel que le problème mixte*

$$\begin{gathered} L(\partial_x)u = F(u), \quad \textit{dans} \quad \mathbb{R}^d \times [0,z] \\ u(z=0) = \varepsilon^p\Big(U_0(x')e^{i\xi'_*.x'/\varepsilon} + \varepsilon\sigma r_1^\varepsilon\Big) \\ u(t \leq 0) = 0 \end{gathered} \tag{1.1.12}$$

possède une unique solution

$$\varepsilon^{-p}u^\varepsilon = \psi_1(t)\psi_2(z)\Big(\mathcal{U}(x)e^{i\xi_*.x/\varepsilon} + \varepsilon\sigma r_1^\varepsilon(x,x/\varepsilon)\Big) + \varepsilon^M r^\varepsilon(x),$$

où $r^\varepsilon \in E^{s-n,\varepsilon}([0,t^\varepsilon])$ *pour* $t^\varepsilon := \tilde{z}_0/(v_{*,n}\sigma)$ *et* $\tilde{z}_0 < z_0$.

L'espace $E^{s,\varepsilon}([0,t^\varepsilon])$ est le même que pour le théorème (1.1.2).

Résultats numériques

Nous avons réalisés des calculs sur le modèle (1.1.11) et les avons comparés à ceux obtenus pour l'équation de Schrödinger associée.

Les simulations sont réalisées pour des impulsions à spectre large (impulsions courtes ou chirpées) de fréquence ζ_* située en divers points de forte courbure de char$L$.

On observe de grandes différences dans le cas linéaire où la solution du modèle (1.1.11) est exacte. Notamment, le support spatial de la solution de (1.1.11) est plus grand. En effet il tient mieux compte des grandes vitesses de groupe temporelles.

Des différences apparaissent également dans le cas non linéaire mais elles sont plus rares. Elles requièrent un équilibre entre la non-linéarité et les effets dispersifs.

1.2 Problèmes ouverts : équations à coefficients variables, bords et phases courbes

Ces thèmes qui ne sont pas abordés dans la thèse sont néanmoins sous-jacents au problème (1.0.10). Nous en décrivons quelques aspects comme perspectives possibles.

Tout d'abord, le cas de coefficients variables intervient dès que l'on considère des milieux inhomogènes : l'opérateur de dispersion devient $L^0(x)$. Les bords courbes introduisent des coefficients non constants dans L^1 tout en préservant la structure hyperbolique de L.

Dans le cas de systèmes hyperboliques à deux vitesses, [9] donne une analyse WKB à tous ordres. Cette analyse est locale dans un ouvert spatio-temporel qui intersecte le bord en un ouvert $\mathcal{O}$ de type "timelike" pour L.

Une telle approche géométrique locale pour le problème (1.0.10) semble appropriée pour une description détaillée des ondes réfléchies et transmises. Ensuite l'hypothèse sous-jacente aux problèmes à deux vitesses est une hypothèse de *cohérence* (cf. [23]) qui assure l'existence globale de deux phases imaginaires pures caractéristiques. Sans cette hypothèse [24] donne des exemples de phases qui focalisent avec perte de régularité et explosion en norme L^∞. Cependant [5] montre que pour des nonlinéarités "assez faibles" (sous critiques) on peut décrire la propagation des ondes au-delà des caustiques à l'aide d'opérateurs Fourier intégraux. Pour finir, les données au bord dans [9] sont sortantes et ce indépendemment de $x \in \Gamma$. Or dans le cas de bords courbes des rayons peuvent être rasants. L'étude de la transition au niveau d'un rayon rasant est faite chez [8]. Néanmoins une asymptotique à tous ordres reste ouverte.

Chapitre 2

Wave transmission in dispersive media

2.1 Introduction

The aim of this chapter is to make a detailed analysis of the reflected and transmitted high frequency waves at an interface for nonlinear dispersive equations. For simplicity, we consider a planar interface, constant coefficient equations and classical incoming wave packets

$$v^\varepsilon(t,x) = \varepsilon^p \mathrm{Re}\left\{A(t,x)e^{(k\cdot x-\omega t)/\varepsilon}\right\} + O(\varepsilon) \tag{2.1.1}$$

with real planar phases $\varphi(t,x) = k\cdot x - \omega t$. The main example we have in mind are Maxwell-Lorentz equations

$$\mathbf{rot}\,\mathbf{H} - \partial_t \mathbf{D} = 0, \qquad \mathbf{rot}\,\mathbf{E} + \partial_t \mathbf{B} = 0, \tag{2.1.2}$$

$$\nabla\cdot\mathbf{D} = 0, \qquad \nabla\cdot\mathbf{B} = 0, \tag{2.1.3}$$

where $\mathbf{D} = \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \mu\mathbf{B}$ with μ constant in each medium. These equations are satisfied on both side of the interface $\{x_3 = 0\}$ which separates two different media where the polarization $\mathbf{P}$ satisfies for instance anharmonic equations

$$\varepsilon^2\partial_t^2\mathbf{P} + \omega_m^2\mathbf{P} - h_m(\mathbf{P}) = \gamma_m\mathbf{E} \quad m\in\{l,r\}, \tag{2.1.4}$$

with $m = l$ [resp. $m = r$] on the left [resp. right] hand side $x_3 < 0$ [resp. $x_3 > 0$]. The nonlinear terms h_m are smooth functions of their argument vanishing at least at second order at $P = 0$ (typical examples are cubic interactions $h_m(P) = \delta_m|P|^2P$). The physical transmission conditions read

$$(\mathbf{E}_r - \mathbf{E}_\mathbf{l})\wedge\mathbf{n} = \mathbf{0}, \qquad (\mu_r\mathbf{B}_r - \mu_l\mathbf{B}_\mathbf{l})\wedge\mathbf{n} = \mathbf{0}, \tag{2.1.5}$$

$$(\mathbf{D}_r - \mathbf{D}_\mathbf{l})\cdot\mathbf{n} = \mathbf{0}, \qquad (\mathbf{B}_r - \mathbf{B}_\mathbf{l})\cdot\mathbf{n} = \mathbf{0}, \tag{2.1.6}$$

where $\mathbf{n} = (0,0,1)$ is the normal to the interface. We refer to P.Donnat [13] for a justification of this model and for a detailed discussion of different models arising in nonlinear optics.

This Maxwell Lorentz system can be embedded in a more general setting. Changing x_3 to $-x_3$ and forgetting the divergence equations Eq. (2.1.3) which are propagated together

with their transmission conditions Eq. (2.1.6) from the initial data to the solutions, leads to consider boundary value problems for first order semi-linear dispersive hyperbolic systems in $\mathbb{R}^{1+d}_+ = \{x_d > 0\}$:

$$\begin{aligned} L(\varepsilon\partial_x)v = \Phi(v), \qquad & \text{on} \quad x_d > 0, \\ Tv = 0, \qquad & \text{on} \quad x_d = 0 \end{aligned} \tag{2.1.7}$$

where $v = (\mathbf{E}_l, \mathbf{B}_l, \mathbf{P}_l, \varepsilon\partial_t\mathbf{P}_l, \mathbf{E}_r, \mathbf{B}_r\mathbf{P}_r, \varepsilon\partial_t\mathbf{P}_r)$ and

$$L(\varepsilon\partial_x) := \varepsilon L^1(\partial_x) + L^0 = \sum_{j=0}^{d} \varepsilon A_j \partial_{x_j} . + L^0. \tag{2.1.8}$$

Here $x = (x_0, x_1, \ldots, x_d)$ denote the space time variables and $x_0 = t$ is time. The nonlinear interaction Φ vanishes at order $J \geq 2$ at the origin, meaning that $\nabla_v^\alpha \Phi(0) = 0$ for all $|\alpha| \leq J - 1$. In this setting, the question is to study the reflection at the boundary of an incoming wave (2.1.1), in the regime of geometric optics.

The weakly nonlinear geometric optics regime concerns solutions of amplitude $O(\varepsilon^p)$ with $p = 1/(J-1)$ (see [13, 15, 14, 32]) : setting $w = \varepsilon^p u$ yields

$$L(\varepsilon\partial_x)u = \varepsilon F(u, \varepsilon^{1/p}) \tag{2.1.9}$$

where $F(u, \varepsilon^{1/p})$ is a smooth function of its arguments. Recall that these amplitudes are computed so that the nonlinear effects appear in time $t = O(1)$. Note that if f is an homogeneous polynomial of degree J, then $F(u, \varepsilon^{1/p}) = f(u)$. This holds in particular for the cubic anharmonic Maxwell-Lorentz equations ; in this case the choices of p are $p = 1/2$ in the weakly non linear geometric optics ($p = 1$ for diffractive optics). Motivated by this example we will consider in this paper equations

$$\begin{aligned} L(\varepsilon\partial_x)u = \varepsilon f(u), \qquad & \text{on} \quad x_d > 0, \\ Tu = 0 \qquad , \qquad & \text{on} \quad x_d = 0 \end{aligned} \tag{2.1.10}$$

with f polynomial.

In the non dispersive case $L^0 = 0$, the geometric optics regime has been studied by M.Williams [45, 46, 47]. We stress that the dispersive case involves a much more complicated pattern of phases. Not only harmonics are presents, but in general there are much more phases present in the correctors for a transmission or boundary value problem than for an interior propagation. These *extra phases* are created by the reflection-transmission process because the dispersion relation is not homogeneous when $L^0 \neq 0$. In general they span a space of infinite dimension over $\mathbb{Q}$. Other interesting phenomena can occur : for instance harmonics of interior propagating waves can produce boundary layers, see the discussion below. Moreover, we can relax several technical conditions that are present in [47] and motivated by the Maxwell system Eq. (2.1.2)-(2.1.6) we also consider the case of characteristic boundaries.

The question of transmission of dispersive nonlinear waves was first raised by P.Donnat. In [13] he assumes that the left (incident) wave is completely given (typically the medium is linear and the wave is purely monochromatic) and he computes an approximate transmitted wave solving a boundary value problem on the right. The point is that he does not solve the exact transmission conditions Eq. (2.1.5), Eq. (2.1.6). His solution satisfies them only at

the leading order, that is up to a $O(\varepsilon)$ error. The boundary conditions he considers are not explicit, they are precisely chosen to eliminate the generation of waves associated to the extra phases mentioned above. Our goal in this paper is to solve the exact transmission problem and give a detailed account of correctors including the extra waves.

First, such an analysis enables a global study in strong norms like L^∞. In fact as the leading profile satisfies a non linear equation the whole WKB approximate solution naturally lies in L^∞. It is then natural to estimate the difference between the exact solution and the WKB approximation in this space. This is made possible by taking enough term in the WKB expansion (when looking for the last mentioned error as a function which is not a profile, one loses $\varepsilon^{(d+1)/2}$ in the L^∞ norm (see lemma 2.5.14)).

A second reason for computing the correctors is when ε is not that small, say $\varepsilon > 10^{-1}$. This parameter roughly counts the number of fast oscillations in the wave train. So $\varepsilon \sim 1$ is rejected here since the ansatz 2.1.1 doesn't make sense (see [1] for a radically different analysis involving continuous spectrum). The case $\varepsilon \sim 10^{-1}$ is reached for femto second pulse which are delivered by lasers such as [Ti :sapphire] Lasers.

$\varepsilon \sim 10^{-1}$ might also be found in chirped lasers which show up under-structures such as "speckles" enhancing a new parameter $1 << \eta << \varepsilon$. This new scale may cause a dramatic change in the nature of the profile equations. However, supposing those structures remains "separated" while evolving dependently, the wave train could be modelized by $u^{\eta,\varepsilon}(x) = \sum_{j,finite} u_j(x/\eta)e^{ikx/\varepsilon}$. Taking this new scale as reference would enhance the rescaled parameter $\varepsilon/\eta >> \varepsilon$ which might be "big".

In the remaining part of this introduction we sketch the main features of our analysis, pointing out the interesting new phenomena which are described and studied in the paper.

1.*Profiles and the fast scale equation.* We consider the boundary value problem (2.1.10), assuming that L is symmetric hyperbolic, that the A_j are real symmetric and L^0 is real skew symmetric. We further assume that the boundary conditions (or transmission conditions) are maximal dissipative. In the geometric optics regime we look for solutions

$$u^\varepsilon(x) \sim \sum \varepsilon^n U^n(x, \frac{x}{\varepsilon}) \tag{2.1.11}$$

Plugging (2.1.11) in Eq. (2.1.10) gives a cascade of equations to solve

$$\begin{cases} L(\partial_X)U^0 = 0 \\ L(\partial_X)U^{j+1} + L(\partial_x)U^j = F^j, \quad j \geq 0 \end{cases} \tag{2.1.12}$$

and solving the equation in X leads to consider the general equation

$$L(\partial_X)U = F, \quad \text{on } X_d > 0. \tag{2.1.13}$$

Plane wave solutions are

$$U(x, X) = A(x)e^{i\beta\cdot X} \tag{2.1.14}$$

where β solves the dispersion relation

$$p(\beta) := \det\big(L(i\beta)\big) = 0, \tag{2.1.15}$$

and A satisfies the polarization condition

$$A(x) \in \ker L(i\beta). \tag{2.1.16}$$

Given an outgoing wave U^0_{out} of this form, the reflected term U^0_{ref} must satisfy Eq. (2.1.12) and the boundary condition $T(U_{out} + U_{ref}) = 0$. For higher order terms, this leads to solve Eq. (2.1.13) together with

$$TU_{|X_d=0} = Be^{i\beta'\cdot X'}, \tag{2.1.17}$$

where we use the notations $X = (X', X_d)$, $\beta = (\beta', \beta_d)$. As our analysis naturally involves time description and z description we will use for any $\mathbb{R}^{d+1}$ variable

Notations 2.1.1. *$x = (t, x")$ for a time description, $x = (x', z)$ for a z-description.*

Assuming that the wave is 2π-periodic in X' and seeing X_d as an evolution variable, the solution of (2.1.13) can be looked as $U(X) = U_{\beta'}(X_d)e^{i\beta'\cdot X'}$ where $U_{\beta'}(X_d)$ satisfies

$$L(i\beta', \partial_{X_d})U_{\beta'}(X_d) = 0, \quad TU_{\beta'|X_d=0} = B. \tag{2.1.18}$$

This linear constant coefficient ordinary differential equation leads to solve in ξ_d the dispersion equation $p(\beta', \xi_d) = 0$. The real roots correspond to reflected plane wave solutions of the form (2.1.14). Complex roots of the dispersion relation with negative imaginary part are not physical since give exponentially growing solutions. On the other hand, complex roots with positive imaginary part yield exponentially decaying solutions, which correspond to *evanescent waves* or boundary layer. This leads to consider *profiles* of the following form

$$\begin{aligned} U(x, X) &= \sum_{\beta'} U_{\beta'}(x, X_d)e^{i\beta'\cdot X'}, \\ U_{\beta'}(x, X_d) &= \sum_{\beta_d} U_{\beta',\beta_d}(x)e^{i\beta_d\cdot X_d} + U_{\beta',ev}(x, X_d) \end{aligned} \tag{2.1.19}$$

with $U_{\beta',ev}$ exponentially decaying in X_d. Note that this class of profiles is stable by nonlinear composition. This is formal if the sums (series) are infinite, but this is rigorous for finite sums and polynomial nonlinearities.

In [47] the evanescent term $U_{\beta',ev}$ is split into complex exponentials, assuming that the complex roots are semi-simple. Here, we make a global treatment of the decaying part and therefore we don't need this technical assumption on the complex roots.

2. *Generation of phases.* Given an outgoing phase with wave number $\underline{\beta}$, the reflected waves are associated to the roots $\beta^i = (\underline{\beta}', \beta^i_d)$ of the dispersion equation $p(\underline{\beta}', \beta_d) = 0$. Nonlinear interaction will produce oscillations associated to phases $\beta^\nu = \sum \nu_i\beta^i$ with $\nu_i \in \mathbb{Z}$. This is the classical discussion of nonlinear geometric optics in the interior, and for dispersive equations, harmonics β^ν are very rarely solutions of the dispersion relations (see [13, 15]) and thus very rarely propagated.

In the construction of correctors for boundary value problems, nonlinear interactions forces to consider equations Eq. (2.1.13), Eq. (2.1.17) where the phases $\beta^\nu \cdot X$ and $(\beta^\nu)' \cdot X'$

are present in the source term and in the boundary term respectively. Therefore for all the harmonics $n\underline{\beta}'$ of the tangential wave number, oscillations with space-time wave numbers

$$\beta^{n,i} = (n\underline{\beta}', \beta_d^{n,i}). \tag{2.1.20}$$

are expected, where the $\beta_d^{n,i}$ satisfy the dispersion equations

$$p(n\underline{\beta}', \beta_d) = 0. \tag{2.1.21}$$

For non dispersive equations, p is homogeneous and the roots are $\beta_d^{n,i} = n\beta_d^i$. Thus all the phases remain in the finitely generated group $\sum \mathbb{Z}\beta^i$. On the other hand, for dispersive equations the roots $\beta_d^{n,i}$ of (2.1.21) are different of $n\beta_d^i$, and in general they span a group which is not finitely generated. These *extra phases* carry oscillations that are propagated at their own group velocity. Next, the nonlinear term produce phases $\beta \in \sum \mathbb{Z}\beta^{n,i}$, but their tangential component $\beta' \in \mathbb{Z}\underline{\beta}'$ and the phases given by solving $p(\beta', \beta_d) = 0$ are already taken into account. This shows that for the profiles in (2.1.19) the natural *tangential spectrum* for the indices β' is

$$\Lambda' = \mathbb{Z}\underline{\beta}', \tag{2.1.22}$$

while the natural *spectrum* for the index $\beta = (\beta', \beta_d)$ is

$$\Lambda = \sum \mathbb{Z}\beta^{n,i}. \tag{2.1.23}$$

This discussion extends to the case where there are several outgoing phases, so that the tangential frequencies β' are restricted to belong to a $\mathbb{Z}$-module Λ' and Λ is the group generated by all the real solutions $\beta = (\beta', \beta_d)$ of $p(\beta) = 0$ with $\beta' \in \Lambda'$.

The analysis above indicates what are the expected phases. For the corresponding oscillations to be actually present, one has to check that some interaction coefficients linking the polarizations (2.1.16) and the boundary conditions do not vanish. The approximate boundary conditions imposed in [13] are precisely chosen to cancel out these interactions so that the extra phases (2.1.20) with $n \neq \pm 1$ can be ignored.

3. *Generation of boundary layers.* It is very common that, depending on $\underline{\beta}$, the dispersion equation $p(\underline{\beta}, \beta_d) = 0$ may have non real roots. For instance, this is typical of total reflection. The new phenomenon due to the dispersion is that the roots of $p(\underline{\beta}, \beta_d) = 0$ can be real, while the roots for the second harmonic $p(2\underline{\beta}, \beta_d) = 0$ are non real.

Consider for instance the transmission problem for the Maxwell-Lorentz model Eq. (2.1.2) and Eq. (2.1.4). Denoting by $\beta = (\tau, k_1, \dots, k_d) = (\tau, k', k_d)$ the space-time wave numbers, the dispersion equations read ($m = l$ or r)

$$p_m(\tau, k) = \tau^2(\tau^2 - \omega_m^2 - \gamma_m)\Big(\tau^2(\tau^2 - \omega_m^2 - \gamma_m) - (\tau^2 - \omega_m^2)(|k|^2)\Big)^2 = 0. \tag{2.1.24}$$

For $\tau \neq 0, \pm\omega_m, \pm\sqrt{\omega_m^2 + \gamma_m}$, the roots in $|k|$ are given by

$$|k|^2 = \tau^2(1 + \chi_m(\tau)), \quad \chi_m(\tau) = \frac{\gamma_m}{\omega_m^2 - \tau^2}. \tag{2.1.25}$$

The (left) incident phase $\underline{\beta} = (\underline{\tau}, \underline{k})$ being real, there holds

$$|\underline{k}'|^2 < \underline{\tau}^2(1 + \chi_l(\underline{\tau})). \tag{2.1.26}$$

There is no real transmitted wave (total reflection) when

$$\chi_l(\underline{\tau}) > \chi_r(\underline{\tau}) \quad \text{and} \quad |\underline{k}'|^2 > \underline{\tau}^2(1 + \chi_r(\underline{\tau})). \tag{2.1.27}$$

The additional phenomenon due to dispersion is that for $\omega_l < |\tau| < \sqrt{\omega_l^2 + \gamma_l}$ there are no real roots in k_d of the equation $p(2\underline{\tau}, 2\underline{k}', k_d) = 0$. Thus if the real incident wave number $\underline{\beta} = (\underline{\tau}, \underline{k})$ satisfies $|\underline{\tau}| < \omega_l < 2|\underline{\tau}| < \sqrt{\omega_l^2 + \gamma_l}$, the second harmonic $2\underline{\beta}' = 2(\underline{\tau}, \underline{k}')$ will necessarily produce a boundary layer in the correctors, on the left side. There is a similar phenomenon for the transmitted wave : $\underline{\beta}'$ can produce a real transmitted wave (present in the main term) and $2\underline{\beta}'$ produces a transmitted boundary layer (in the first corrector).

4. *Transmission of harmonics of totally reflected waves.* This is the converse phenomenon. The main wave can be evanescent and the harmonics (correctors) can propagate in the interior. Consider again the transmission problem for Maxwell-Lorentz equation. Denote by $\underline{\beta} = (\underline{\tau}, \underline{k})$ the incident wave number. If (2.1.27) holds (total reflection) the main transmitted wave is a boundary layer. But if in addition

$$\chi_r(2\underline{\tau}) > \chi_r(\underline{\tau}) \quad \text{and} \quad |\underline{k}'|^2 < \underline{\tau}^2(1 + \chi_r(2\underline{\tau})) \tag{2.1.28}$$

the second harmonic transmitted wave is real the second harmonic present in the corrector term U^1 in the expansion (2.1.11) is propagated inside the medium on the right.

Note that the first condition in (2.1.28) holds if $2\underline{\tau} < \omega_r$.

5. *Propagation equations.* In the expansion (2.1.19), the propagating modes $U_\beta(x)e^{i\beta\cdot X}$ are associated to real roots $\beta = (\beta', \beta_d) \in \Lambda$ of the dispersion relation. For the leading order term U^0 in the expansion (2.1.11), the amplitudes satisfy the polarization principle $U^0_\beta = \pi(\beta)U^0_\beta$ where $\pi(\beta)$ is the spectral projection on $\ker L(i\beta)$. For the correctors U^n, $n \geq 1$, the non polarized part $(\mathrm{Id} - \pi(\beta))U^n_\beta$ are determined by the previous terms $(U^0, \dots, U^{n-1})$ of the expansion. The evanescent parts $U^n_{\beta',ev}$ are determined by the previous terms when $n \geq 1$ and the values on the boundary of $\pi_{ev}(\beta')U^n_{\beta',ev|X_d=0}$ where $\pi_{ev}(\beta')$ is a spectral projector which corresponds to modes ξ_d with $\operatorname{Im}\xi_d > 0$.

We will assume that all $\beta \in \Lambda \setminus \{0\}$ with $p(\beta) = 0$ is a regular point of the characteristic manifold (see below) so that, in the regime of geometric optics, the polarized part $\pi(\beta)U_\beta$ is determined by a transport equation

$$\Big(\partial_t + \sum_{j=1}^{d} \mathbf{v}_j(\beta)\partial_{x_j}\Big)\pi(\beta)U_\beta = f_\beta \tag{2.1.29}$$

where $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_d)$ is the group velocity of the β-wave. The position of $\mathbf{v}(\beta)$ with respect to the boundary plays a crucial role : the mode is incoming/outgoing/glancing when $\mathbf{v}_d(\beta)$ is positive/negative/zero . The treatment of glancing modes is explained in [47], they involve a third scale in $x/\sqrt{\varepsilon}$. In this paper, we will assume for simplicity that glancing modes are never launched that is that $\mathbf{v}_d(\beta) \neq 0$ when $\beta \in \Lambda \setminus \{0\}$ with $p(\beta) = 0$. For Maxwell-Lorentz

equations, this assumption is satisfied for almost all choice of $\underline{\beta}$, the wave number of the source.

On the other hand, still for the Maxwell-Lorentz system, the frequency $\beta = 0$ is not regular, implying that the propagation equation for $\pi(0)U_0$ is a non diagonal system [32, 44]

$$\pi(0)L^1(\partial_x)\pi(0)U_0 = f_0. \tag{2.1.30}$$

The equations Eq. (2.1.29) and Eq. (2.1.30) are coupled through the source terms f_β, which depend (non-linearly for the principal term) on the amplitudes U_β. Moreover, these equations are supplemented with boundary conditions which couple the traces of the oscillatory amplitudes U_{β',β_d} and of the evanescent parts $U_{\beta',ev}$:

$$T\Big(\sum_{\beta_d} \pi(\beta',\beta_d)U_{\beta',\beta_d|x_d=0} + \pi_{ev}(\beta')U_{\beta',ev|x_d=X_d=0}\Big) = g_{\beta'}. \tag{2.1.31}$$

The well posedness of the boundary problem (2.1.29) (2.1.31) is expressed by two conditions :

1. for $\beta' \neq 0$ we impose a Lopatinski type condition which says that the evanescent part $\pi_{ev}(\beta')U_{\beta',ev|x_d=X_d=0}$ and the oscillatory incoming $\pi(\beta',\beta_d)U_{\beta',\beta_d|x_d=0}$ can be uniquely determined from the outgoing $\pi(\beta',\beta_d)U_{\beta',\beta_d|x_d=0}$ and $g_{\beta'}$.
2. for $\beta' = 0$, we assume that only the equation Eq. (2.1.30) is present and the maximal dissipativity of the initial problem implies that the boundary value problem for Eq. (2.1.30) is maximal dissipative.

Note that the maximal dissipativity of the initial problem also implies that for $\beta' \neq 0$ the problem is maximal dissipative. The condition (i) is slightly stronger and would be automatic if the initial problem were strictly dissipative. Note also that the Lopatinski condition in (i) is not exactly the usual hyperbolic Lopatinski condition for L which involves only the principal part of L ; it is a semi-classical Lopatinski condition.

We also mention that there is an additional difficulty for $\beta' = 0$. The boundary is characteristic for Maxwell-Lorentz equations and the hyperplane $\tau = 0$ is entirely contained in the characteristic set, implying by nonlinear resonances that all frequencies $(0, \beta_d)$ could be present. Indeed, a detailed analysis of the interaction coefficients shows that these modes are not created by interaction and are not present if they are not present in the initial data. This is linked to the propagation of the vanishing of div$B$ and div$D$.

6. *Reflected waves and their harmonics propagate in different directions.* Consider an outgoing wave number $\underline{\beta} = (\underline{\tau}, \underline{k}_1, \dots, \underline{k}_d) = (\underline{\tau}, \underline{k}', \underline{k}_d)$ for the Maxwell-Lorentz system. The dispersion relation Eq. (2.1.25) implies that

$$|\underline{k}|^2 = \underline{\tau}^2(1 + \chi(\underline{\tau})) := \Psi(\underline{\tau}). \tag{2.1.32}$$

The group velocity is

$$\underline{\mathbf{v}} = \mathbf{v}(\underline{\beta}) = \frac{2}{\Psi'(\underline{\tau})}\,\underline{k}. \tag{2.1.33}$$

The wave is outgoing exactly when $\underline{k}_d < 0$.

The main reflected incoming phase is associated to $\underline{\beta}^1 = (\underline{\tau}, \underline{k}_1, \dots, \underline{k}_{d-1}, -\underline{k}_d)$. Its group velocity is $\underline{\mathbf{v}}^1 = (\underline{\mathbf{v}}_1, \dots, \underline{\mathbf{v}}_{d-1}, -\underline{\mathbf{v}}_d) = (\underline{\mathbf{v}}', -\underline{\mathbf{v}}_d)$.

The second harmonic reflected wave is $\underline{\beta}^2 = (2\underline{\tau}, 2\underline{k}', -\tilde{k}_d)$ where $\tilde{k}_d$ is the positive root (when it exists) of

$$4|\underline{k}'|^2 + \tilde{k}_d^2 =:= \Psi(2\underline{\tau}). \tag{2.1.34}$$

Its group velocity is

$$\underline{\mathbf{v}}^2 = \frac{2}{\Psi'(2\underline{\tau})}\,(2\underline{k}', -\tilde{k}_d). \tag{2.1.35}$$

As usual for dispersive equations, we see that the first and second harmonic time-oscillations $e^{i\underline{\tau}t/\varepsilon}$ and $e^{2i\underline{\tau}t/\varepsilon}$ propagate at different speeds $|\underline{\mathbf{v}}^1|$ and $|\underline{\mathbf{v}}^2|$ respectively. But because $\tilde{k}_d \neq 2\underline{k}_d$, we see that they also propagate in different spatial directions if $\underline{k}' \neq 0$.

We end this introduction with several additional remarks.

Remarks 2.1.2. *1.* In this paper we consider the equation Eq. (2.1.10) with polynomial source term $f(u)$. The case of general equation Eq. (2.1.9) with source term $F(u, \varepsilon^{1/p})$ is quite similar (see [27, 32]) : one replaces the expansion (2.1.11) by

$$u^\varepsilon(x) \sim \sum \varepsilon^{n/p} U^n(x, \frac{x}{\varepsilon}) \tag{2.1.36}$$

and one uses the expansion

$$F(u, \varepsilon^{1/p}) \sim \sum \varepsilon^{np} f_p(u) \tag{2.1.37}$$

where the polynomials f_p are given by a Taylor expansion of the original source term $\Phi(v)$ in Eq. (2.1.7).

2. The dispersive character of the equation implies that in general the principal term U^0 is expected to have a finite oscillating spectrum (see [42, 15]). The polynomial character of f is crucial in the analysis below : it implies that nonlinear interactions create at each step only finitely many new phases, so that each term in the expansion U^n is expected to have a finite spectrum, increasing with n but finite for each corrector. This allows to work with profiles (2.1.19) which are finite sums and eliminate the questions about the convergence of these series.

For non dispersive equations, the spectra are not finite in general, but they are contained in finitely generated groups so that it is possible to represent the profiles U^n as functions $\mathcal{U}^n(x, \varphi_1/\varepsilon, \dots, \varphi_m/\varepsilon)$ with a *finite* number of phases $\varphi_j(x)$ and profiles $\mathcal{U}^n(x, \theta_1, \dots, \theta_m)$ *periodic* in the fast variables θ_j (see e.g. [23, 24, 45, 47]). In this case, the analysis can be carried out in classical function spaces for the $\mathcal{U}^n$, typically Sobolev spaces. In the dispersive case, in general the expected spectrum is not contained in a finitely generated group, as explained above. Thus the consideration of non polynomial interactions f in Eq. (2.1.10) immediately raises the difficult question of convergence of the series (2.1.19) for correctors and of the choice of good functional spaces. A possibility, for real analytic f, would be to work within the Wiener algebra of *almost periodic* profiles U as in [26, 33].

3. Finally, we mention the important question of curved boundary and/or non constant coefficients equations and/or non planar phases. In this case, the dispersion relation equations are replaced by eikonal equations. Using the polynomial character of f, to find a finite expansion (2.1.11), we only have to solve a finite number of eikonal equations. This can be done if one assumes that there is one incident phase $\underline{\varphi}$ and that $(x', n\partial'_x\underline{\varphi})$ is never a glancing point for all x' on the boundary or the interface and all harmonic $n \in \mathbb{Z}\backslash 0$. In this context, we refer to [9] for a complete analysis of geometric optics for non dispersive wave equations. We also refer to [7] for the analysis of a nonlinear glancing case. In the dispersive case, assuming that all the necessary phases are found, the analysis below can probably be repeated in the geometric optics regime, using the appropriate non constant coefficient transport equations to find the profiles and the appropriate estimates for non constant coefficients hyperbolic boundary value problems to prove the convergence.

2.2 Dispersive geometric optics at boundaries

2.2.1 Notations and Assumptions

We consider a $D \times D$ constant coefficient system (2.1.8) $L(\varepsilon\partial_x) = \sum \varepsilon A_j \partial_j + L^0$ on the half space $\mathbb{R}^{1+d}_+ = \{x_d > 0\}$ with boundary conditions T.

We next give many assumptions often tricky and not obvious at first sight. However all are duly satisfied by the maxwell system (2.1.2)-(2.1.4). The next section is devoted to check them.

We assume that this system is hyperbolic symmetric, that the boundary is characteristic and T is maximal dissipative (see [38, 42] for example) :

Assumption 2.2.1. (H1) *there is a matrix S such that the matrices SA_j are symmetric with SA_0 definite positive, and SL^0 is skew symmetric.*

(H2) $\dim(\ker A_d) = D_2 \geq 1$.

(H3) *SA_d is non negative on the space $\ker T$ and the rank of T is equal to the number of negative eigenvalues of SA_d counted with their multiplicity.*

Multiplying the system by on the left by $(SA_0)^{-1/2}S$ and on the right by $(SA_0)^{-1/2}$ reduces to the case where $S = A_0 = \mathrm{Id}$ and the A_j are symmetric. and L^0 is skew symmetric :

$$A_0 = \mathrm{Id}, \quad A_j = {}^tA_j, \quad L_0 = -{}^tL_0. \tag{2.2.1}$$

Changing bases, we can further assume that

$$A_d = \begin{pmatrix} A_d^{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.2.2}$$

with A_d^{11} invertible. The diagonal blocks have dimension D_1 and D_2 respectively, with $D_1 + D_2 = D$. Since (H3) implies that $\ker A_d \subset \ker T$, there holds

$$T = \begin{pmatrix} T^1 & 0 \end{pmatrix} \tag{2.2.3}$$

that is $Tu = T^1u^1$ for $u = {}^t(u^1, u^2)$.

The *symbol* of L is

$$L(i\xi) = \sum_{j=0}^{d} i\xi_j A_j + L^0. \tag{2.2.4}$$

It is a skew adjoint matrix. Recall the following definitions (see e.g. [15] for the dispersive framework)

Definition 2.2.2. *1. The real [resp. complex] characteristic set of L is the set of $\xi \in \mathbb{R}^{1+d}$ [resp. $\xi \in \mathbb{C}^{1+d}$ such that* $\det(L(i\xi)) = 0$.

2. A real *mode $\underline{\xi} \in \mathrm{char}L$ is regular if there is a $\mathcal{C}^\infty$ function $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that near $\underline{\xi} \in \mathrm{char}L$ is locally given by the equation $\xi_0 + \lambda(\xi_1, \dots, \xi_d) = 0$.*

Remark 2.2.3. *A real root may be non regular. This happens for Maxwell-Lorentz system at $\xi = 0$ where three characteristic sheets meet and at $\xi = (\pm\omega_l, 0, 0, 0)$ where two sheets meet tangentially.*

As mentioned in the introduction, the "microscopic" equation reads $L(\partial_X)U = F$. By Fourier expansion in $\xi' = (\xi_0, \dots, \xi_{d-1}) \in \mathbb{R}^d$ this equation is transformed into

$$\big(A_d\partial_{X_d} + L'(i\xi')\big)U := L(i\xi', \partial_{X_d})U = F, \tag{2.2.5}$$

where, using the same block decomposition as in (2.2.2) :

$$L'(i\xi') = \sum_{j=0}^{d-1} i\xi_j A_j + L^0 := \begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix} \tag{2.2.6}$$

This is a system of ordinary differential equation in X_d, which is singular since A_d is not invertible. Plane wave solutions like $e^{i\xi_d X_d}$ are found by solving the equation in ξ_d

$$\det L(i\xi', i\xi_d) = 0. \tag{2.2.7}$$

From hypothesis (H2), the characteristic polynomial expands

$$\det L(i\xi) = (i\xi_d)^{d_1} \det A_d^{11} \det(L^{22}(i\xi')) + \ \mathcal{O}(\xi_d^{d_1-1}).$$

In particular, the number of roots in ξ_d depends on the invertibility of $L^{22}(i\xi')$. (see the analysis in [38]). Referring to [47] we use the following classification :

Definition 2.2.4. *1. A real root in ξ_d of* (2.2.7) *is*

(a) hyperbolic incoming if $\xi = (\xi', \xi_d)$ is regular and $\partial_{\xi_d}\lambda(\xi) > 0$.

(b) hyperbolic outgoing if $\xi = (\xi', \xi_d)$ is regular and $\partial_{\xi_d}\lambda(\xi) < 0$.

(c) glancing if it is regular and $\partial_{\xi_d}\lambda(\xi) = 0$.

2. A complex root in ξ_d of (2.2.7) *is*

(a) elliptic incoming (evanescent) if $\mathrm{Im}\,\xi_d > 0$,

(b) elliptic outgoing (explosive) if $\mathrm{Im}\,\xi_d < 0$,

We denote by $\mathcal{I}$, $\mathcal{O}$, $\mathcal{G}$, $\mathcal{E}_{ev}$ and $\mathcal{E}_{ex}$ the associated sets.

In (i), λ denotes the local equation of $\mathrm{char}(L)$ as in Definition 2.2.2. Complex phases appear in cases of total reflection or transmission.

Writing $U = (U_1, U_2)$, $F = (F_1, F_2)$, equation Eq. (2.2.5) reads :

$$\left\{ \begin{array}{c} A_d^{11}\partial_{X_d} U_1 + L^{11}(i\xi')U_1 + L^{12}(i\xi')U_2 = F_1 \\ L^{21}(i\xi')U_1 + L^{22}(i\xi')U_2 = F_2 \end{array} \right. \tag{2.2.8}$$

Its solvability depends on the invertibility of $L^{22}(i\xi')$. As it is skew adjoint let $\pi^{22}(\xi')$ be the orthogonal projector on $\mathrm{Ker} L^{22}(\xi')$ and $Q^{22}(\xi')$ the partial inverse such that $Q^{22}(\xi')\pi^{22}(\xi') = 0$. Thus Eq. (2.2.8) is equivalent to

$$\left\{ \begin{array}{rl} M(\xi')U_1 = \pi^{22}(\xi')F_2 & (1) \\ (\partial_{X_d} + G(\xi'))U_1 = \tilde{F}_1 + (A_d^{11})^{-1}(M(\xi'))^* U_2 & (2) \\ (1 - \pi^{22}(\zeta'))U_2 = Q^{22}(\zeta')(F_2 - L^{21}U_1) & (3) \end{array} \right. \tag{2.2.9}$$

where

$$\left\{ \begin{array}{l} M(\xi') = \pi^{22}(\xi')L^{21}(i\xi'), \\ G(\xi') = (A_d^{11})^{-1}(L^{11}(i\xi') - L^{12}(i\xi')Q^{22}(\xi')L^{21}(i\xi')), \\ \tilde{F}_1 = (A_d^{11})^{-1}(F_1 - L^{12}(i\xi')Q^{22}(\xi')F_2). \end{array} \right. \tag{2.2.10}$$

We have used the notation for the hermitian transpose $M^* = \overline{{}^t M}$. Note that G is square but not symmetric.

To solve the equations Eq. (2.2.5), we need assumptions on the matrices $L^{ab}(i\xi')$. At this stage, we do not pretend to full generality but consider cases which are realistic for the Maxwell's equations. First, because the profiles we are looking for are periodic or quasi-periodic functions of the fast variables X, the tangential frequencies ξ' are restricted to belong to a discrete subset Λ' of $\mathbb{R}^d$. Because we solve nonlinear equations, we have to consider linear combinations of oscillations and we assume that Λ' is an additive subgroup of $\mathbb{R}^d$. Typically, for one incoming oscillation of space-time frequency $\underline{\xi} = (\underline{\xi}', \underline{\xi}_d)$, the choice is

$$\Lambda' = \mathbb{Z}\underline{\xi}'. \tag{2.2.11}$$

Motivated by the example of Maxwell's equations, we make the following hypothesis :

Assumption 2.2.5. *We are given an additive subgroup Λ' of $\mathbb{R}^d$ and for all $\xi' \in \Lambda' \setminus 0$*

1. *$L^{22}(i\xi')$ is invertible,*
2. *the real roots in ξ_d of the characteristic equation Eq. (2.2.7) are* regular *and* non glancing.

Under this Assumption, the equations Eq. (2.2.9) simplify and

$$L(i\xi', \partial_{X_d})U = 0 \;\Leftrightarrow\; \left\{ \begin{array}{l} \partial_{X_d} U_1 = -G(\xi')U_1, \\ U_2 = -(L^{22}(\xi'))^{-1}L^{21}(\xi')U_1 \end{array} \right. \tag{2.2.12}$$

Moreover, ξ_d is a root of the characteristic equation Eq. (2.2.7) if and only if $-i\xi_d$ is an eigenvalue of $G(\xi')$. They split into real roots, which are by Assumption 2.2.5 either incoming or outgoing, and non real roots, which are also either incoming or outgoing, according to Definition 2.2.4. Denote by $\mathbb{E}^-_{H,G}(\xi')$ [resp. $\mathbb{E}^-_{E,G}(\xi')$] the spectral space of $G(\xi')$ associated to

the incoming hyperbolic [resp. elliptic] roots. Define similarly $\mathbb{E}^+_{H,G}$ and $\mathbb{E}^+_{E,G}$ the spectral spaces associated to outgoing modes.

The non-glancing assumption implies for all $\xi' \in \Lambda' \setminus \{0\}$

$$\mathbb{C}^{D_1} = \mathbb{E}^-_{H,G}(\xi') \oplus \mathbb{E}^-_{E,G}(\xi') \oplus \mathbb{E}^+_{H,G}(\xi') \oplus \mathbb{E}^+_{E,G}(\xi'). \tag{2.2.13}$$

By (2.2.12), the spaces $\mathbb{E}^\pm_{H,G} \subset \mathbb{C}^{D_1}$ and $\mathbb{E}^\pm_{E,G}$ can be lifted to $\mathbb{C}^D$:

$$\mathbb{E}^-_{H,L}(\xi') = \begin{pmatrix} Id \\ -(L^{22})^{-1}(i\xi')L^{21}(i\xi') \end{pmatrix} \mathbb{E}^-_{H,G}(\xi')$$

with a similar formula for the $\mathbb{E}^+_{H,L}$ and the elliptic spaces $\mathbb{E}^\pm_{E,L}$. Moreover,

$$\mathbb{E}^-_{H,L}(\xi') = \bigoplus_{\xi_d |\ (\xi',\xi_d)\in\mathcal{I}} \ker L(i\xi', i\xi_d), \tag{2.2.14}$$

with similar formula for $\mathbb{E}^+_{H,L}$ and $\mathbb{E}^+_{E,L}$ summing over the roots ξ_d such that $(\xi', \xi_d) \in \mathcal{I}$ and $(\xi', \xi_d) \in \mathcal{E}_{ev}$ respectively.

So far, the boundary conditions were ignored in the discussion. They come in when one solves the propagation equations for profiles. Assumption 2.2.5 implies that when $\xi' \neq 0$, they are scalar transport equations concerning the hyperbolic modes, see section 4 below. To solve them, one is lead to impose the following Lopatinski-type conditions (see [30, 38])

Assumption 2.2.6. Lopatinski type conditions *: For all $\xi' \in \Lambda' \setminus 0$*

$$\begin{aligned} \dim(\mathbb{E}^-_{H,L}(\xi')) + \dim(\mathbb{E}^-_{E,L}(\xi')) &= rg(T), \\ \ker T \cap \left(\mathbb{E}^-_{H,L}(\xi') \oplus \mathbb{E}^-_{E,L}(\xi')\right) &= \{0\}. \end{aligned}$$

Equivalently, the conditions read

$$\begin{aligned} \dim(\mathbb{E}^-_{H,G}(\xi')) + \dim(\mathbb{E}^-_{E,G}(\xi')) &= rg(T^1), \\ \ker T^1 \cap \left(\mathbb{E}^-_{H,G}(\xi') \oplus \mathbb{E}^-_{E,G}(\xi')\right) &= \{0\}. \end{aligned}$$

The case of the frequency $\xi' = 0$ is quite different. Inspired by Maxwell's equation, we make the following assumption

Assumption 2.2.7. *For $\xi' = 0$, the following conditions are satisfied :*

1. $\{0\} \neq \ker\ L^{22}(0) \subset \ker\ L^{12}(0)$,
2. $G(0) = 0$,
3. *the real roots in ξ_d of the characteristic equation $L(0, i\xi_d) = 0$ are* regular *except for $\xi_d = 0$.*

Condition (i) is equivalent to $L^{12}(0)\pi^{22}(0) = 0$, hence by symmetry to $M(0) := \pi^{22}(0)L^{21}(0) = 0$. It implies that for all ξ_d,

$$\begin{pmatrix} 0 \\ \ker L^{22}(0) \end{pmatrix} \subset \ker L(0, i\xi_d) \tag{2.2.15}$$

with equality exactly when $i\xi_d$ is not an eigenvalue of $G(0)$. Moreover, (2.2.15) implies that

$$\{0\} \times \mathbb{C} \subset \mathrm{char}L. \tag{2.2.16}$$

If in addition the regularity condition iii) holds, the dimension of $\ker L(0, i\xi_d)$ is constant for ξ_d real positive and ξ_d real negative. Thus, $G(0)$ has no purely imaginary nonzero eigenvalues. In this context, condition ii) is natural.

Under Assumption 2.2.7, equation Eq. (2.2.8) is equivalent to

$$\begin{cases} \pi^{22}(0)F_2 = 0, \\ \partial_{X_d} U_1 = \tilde{F}_1, \\ (1 - \pi^{22}(0))U_2 = Q^{22}(0)(F_2 - L^{21}(0)U_1). \end{cases} \tag{2.2.17}$$

We consider now the nonlinear equation Eq. (2.1.10) in the context of geometric optics :

$$\begin{cases} L(\varepsilon\partial_x)u = \varepsilon f(u), & \text{on } \ x_d > 0, \\ \quad Tu = 0 \qquad , & \text{on } \ x_d = 0 \end{cases} \tag{2.2.18}$$

where

$$f(u) = \sum_{j=0}^{J} f^j(u) \tag{2.2.19}$$

is a *polynomial* of degree J, sum of homogeneous terms f^j of degree j. According to the decomposition (2.2.2), we use the notations

$$f(u) = \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix}.$$

As in [15], the dispersive character of the equation leads to consider oscillations with *finite spectrum*. However, property (2.2.16) and the generation of new characteristic modes from oscillations of boundary terms make the situation more delicate. If $(0, \xi_d)$ belongs to the spectrum of U, then all the harmonics are created by nonlinearity and are characteristic. To prevent the spontaneous formation of such modes we make the following hypothesis :

Assumption 2.2.8. *For all u, $\pi^{22}(0)f_2(u) = 0$.*

Having taken apart the component $\{0\} \times \mathbb{R}^*$ of $\mathrm{char}L$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, let introduce the notation

$$(\mathrm{char}L)^\sharp = (\mathrm{char}L) \setminus (\{0\} \times \mathbb{R}^*). \tag{2.2.20}$$

Next, we assume (see Assumption 2.2.9) that we can find finite spectra Λ for which any resonant harmonic ξ, $\xi \neq 0$ lies in $\Lambda \cap (\mathrm{char}L)^\sharp$.

In practice, such resonances are discrete phenomena, and spectra without resonances are "generic". We will explain this on the example of Maxwell's equations.

We denote by Λ_0 a finite set which will contain the spectrum of the principal profile U^0. We assume that Λ_0 is symmetric, that is $\Lambda_0 = -\Lambda_0$, so that we can consider real valued profiles. If U^0 has its spectrum contained in Λ_0, then the spectrum of $f(U^0)$ is contained in

$$\mathcal{N}(\Lambda_0) := \bigcup_{\{l: f^l \neq 0\}} l * \Lambda_0 \tag{2.2.21}$$

where for all $k \in \mathbb{N}$ and all subset Γ of some space $\mathbb{R}^n$

$$k * \Gamma := \underbrace{\Gamma + \ldots + \Gamma}_{k \text{ terms}}. \tag{2.2.22}$$

with the usual convention that for $k = 0$ the set in the right hand side is $\{0\}$. Similarly, the spectrum of $\nabla_u f(U^0)$ is contained in

$$\mathcal{N}^\flat(\Lambda_0) := \bigcup_{\{j: f^j \neq 0\}} (j-1) * \Lambda_0. \tag{2.2.23}$$

We make the following remarks : using the trivial relation $\xi - \xi + \xi = \xi$, we see that if Γ is symmetric, that is if $-\Gamma = \Gamma$, then $k * \Gamma \subset (k+2) * \Gamma$. If $0 \in \Gamma$, then $k * \Gamma \subset (k+1) * \Gamma$.

Assumption 2.2.9. *1. We are given symmetric finite sets $\Lambda_0' \subset \Lambda'$ and $\Lambda_0 \subset \Lambda' \times \mathbb{R}$ such that*

$$(\Lambda_0' \times \mathbb{R}) \cap (\mathrm{char} L)^\sharp \subset \Lambda_0 \subset (\Lambda_0' \times \mathbb{R}) \cap \mathrm{char} L \tag{2.2.24}$$

and

$$\mathcal{N}(\Lambda_0) \cap (\mathrm{char} L)^\sharp \subset \Lambda_0. \tag{2.2.25}$$

2. For all $k \geq 1$ in general and for all k odd if f is odd, there is a symmetric finite set $\Lambda_1' \subset \Lambda'$ such that $\Lambda_1' \supset k * \Lambda_0'$ and $\Lambda_1 = (\Lambda_1' \times \mathbb{R}) \cap (\mathrm{char} L)^\sharp$ satisfies

$$\big(\mathcal{N}^\flat(\Lambda_0) + \Lambda_1\big) \cap (\mathrm{char} L)^\sharp \subset \Lambda_1. \tag{2.2.26}$$

Remark 2.2.10. Condition 2.2.24 is required in order to include all the possible reflected and transmitted oscillations. Condition (2.2.25) is a natural extension of Donnat-Rauch condition. The existence of wave numbers $(0, \xi_d) \neq 0$ in $\mathcal{N}(\Lambda_0)$ cannot be completely ruled out. These frequencies are necessarily in $\mathrm{char} L$ by (2.2.16) and cannot belong to Λ_0 if this set is finite. In this case Assumption 2.2.8, a transparency-like condition, implies that the bad extra oscillations will not be created, if not already present in the initial data.

Remark 2.2.11. Let $\underline{\xi}' \in \mathbb{R}^d \setminus \{0\}$. For all $s \in \mathbb{R}^*$ denote by $\xi_d = \mu^j(s)$ the real roots of $\det L(is\underline{\xi}', i\xi_d)$. The number of roots may depend on s, as in the example of Maxwell anharmonic model. Take $\Lambda' = \mathbb{Z}s\underline{\xi}'$, $\Lambda_0' = \{\pm s\underline{\xi}', 0\}$ and $\Lambda_0 = \{(\pm s\underline{\xi}', \mu^j(\pm s))\} \cup \{(0,0)\}$. Condition (2.2.25) is violated if there are signs $\varepsilon_k \in \{\pm 1\}$ and choices j_k and j such that

$$\mu^j(ns) = \sum \mu^{j_k}(\varepsilon_k s), \quad n = \sum \varepsilon_k, \quad |n| \geq 2.$$

In general, the μ^j are nonlinear (and non homogeneous) functions of s, and for a given choice of signs and indices, the condition is satisfied at most for a discrete set of values of s. This shows that the condition (2.2.25) is satisfied for almost all choice of s. The same reasoning applies

for all finite number of harmonics $\Lambda_1' = \{ns\underline{\xi}, |n| \leq k\}$, and (2.2.26) is generically satisfied. This will be made precise for the Maxwell anharmonic model below. This reasoning indicates that in general, there are no resonances at all. However, our setting allows a finite number of resonances, to be included in Λ_0, and this is important for applications for instance to the harmonic generation.

Remark 2.2.12. When f is odd, one can choose Λ_0' and Λ_1' which do not contain 0. This means that there are no interaction of oscillations with the mean field (no rectification). In this case, the analysis is much simpler. However, we want to include in our analysis the possibility of rectification, which in particular occurs if f contains quadratic terms. Thus, the possibility that $0 \in \Lambda_0$ is studied in detail.

2.2.2 Main results

From now on, we always assume that the above assumptions are satisfied.

Profiles.

With $\Lambda' \subset \mathbb{R}^d$ the set of tangential modes, one defines $\mathcal{P}$ as the set of profile U (2.1.19) with tangential spectrum contained in Λ'. For the reader's convenience we recall the definition and introduce new notations :

Definition 2.2.13. *1. $\mathcal{P}^z_{os}$ is the set of functions $U(x,Z)$ which are finite sums $U(x,Z) = \sum e^{i\xi_d Z} U_{\xi_d}(x)$ with coefficients $U_{\xi_d} \in H^\infty(\mathbb{R}^{d+1}_+)$.*
$\mathcal{P}^z_{os}$ is the subspace of sums with frequencies $\xi_d \neq 0$.
2. $\mathcal{P}^z_{ev}$ is the set of functions $U(x,Z)$ such that for some $\delta > 0$, $Ue^{\delta Z} \in H^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^+)\}$.
3. Define $\mathcal{P}^z := \mathcal{P}^z_{os} \oplus \mathcal{P}^z_{ev}$.
4. The space of profiles $\mathcal{P}$ is the space of finite sums

$$U(x,X) = \sum_{\xi' \in \Lambda'} U_{\xi'}(x, X_d) e^{i\xi' X'}, \quad \text{with} \quad U_{\xi'} \in \mathcal{P}^z. \tag{2.2.27}$$

For $U \in \mathcal{P}$ given by (2.2.27), the tangential spectrum of U is the (finite) set of ξ' such that $U_{\xi'} \neq 0$. Each $U_{\xi'}$ is split into

$$U_{\xi'} = U_{\xi',os} + U_{\xi',ev} \quad \text{with} \quad U_{\xi',os} \in \mathcal{P}^z_{os},\ U_{\xi',ev} \in \mathcal{P}^z_{ev}. \tag{2.2.28}$$

Moreover,

$$U_{\xi',os}(x,Z) = \sum_{\xi_d} U_{(\xi',\xi_d)}(x) e^{i\xi_d Z}. \tag{2.2.29}$$

The spectrum of U is the set of (ξ', ξ_d) such that $U_{(\xi',\xi_d)} \neq 0$. As seen in the previous section, the mean mode $\xi = 0$ plays a very particular role and for $\xi' = 0$ we also use the following decomposition

$$U_{0,os}(x,Z) = U^*_{0,os}(x,Z) + \underline{U}_0(x) \tag{2.2.30}$$

where $\underline{U}_0 = U_{(0,0)}$ corresponds to the frequency $\xi_d = 0$ in (2.2.29).

The definition above is given for functions of $x \in \mathbb{R}^{d+1}_+ = \{x_d \geq 0\}$. It immediately extends to functions of (x,X) with $x_0 = t \in [0, t_*]$, a finite time interval. When it is necessary to make this explicit, we denote by $\mathcal{P}([0,t_*])$ the corresponding space.

The microscopic equation.

Our first result concerns the general solution of equation Eq. (2.2.5) for source term expanding as a trigonometric sum of tangential modes.

Theorem 2.2.14. *There are projectors $\mathbb{P}$ and $\mathbb{P}^i$ and partial inverse $\mathbb{Q}$ acting in $\mathcal{P}$ such that such that for all $F \in \mathcal{P}$ the equation*

$$L(\partial_X)V = F$$

has a solution $V \in \mathcal{P}$ if and only if $\mathbb{P}^i F = 0$. The general solution is

$$V = \mathbb{Q}F + \mathbb{P}V.$$

The precise definition of the projectors is given in section 3. Note that in this theorem, the slow variable x is a parameter and that the operators act tangential frequency by tangential frequency : for U given by Eq. (2.2.27), there holds

$$\mathbb{P}U = \sum_{\xi'} e^{i\xi' X'} \mathbb{P}(\xi') U_{\xi'} \tag{2.2.31}$$

with similar formulas for $\mathbb{P}^i$ and $\mathbb{Q}$. Moreover, for each ξ', the operators decompose according to $U_{\xi'} = U_{\xi',os} + U_{\xi',ev}$, into

$$\mathbb{P}(\xi')U_{\xi'} = \mathbb{P}_{os}(\xi')U_{\xi',os} + \mathbb{P}_{ev}(\xi')U_{\xi',ev} \tag{2.2.32}$$

the first term belonging to $\mathcal{P}^z_{os}$ and the second one to $\mathcal{P}^z_{ev}$. In addition, when $\xi' = 0$,

$$\mathbb{P}_{os}(0)U_0 = \mathbb{P}^*_{os}U^*_{\xi',os} + \underline{\mathbb{P}}_0\underline{U}_0 \tag{2.2.33}$$

with $\mathbb{P}^*_{os}U^*_{\xi',os} \in \mathcal{P}^{z,*}_{os}$ and $\underline{\mathbb{P}}_0\underline{U}_0(x)$ constant in Z.

Lastly note that the pojectors $\mathbb{P}$ and $\mathbb{P}^i$ differ in two ways : For $\xi' \neq 0, \mathbb{P}^i_{ev} = 0$ while $\mathbb{P}_{ev} \neq 0$. Then for $\xi = 0$, $\underline{\mathbb{P}}^i_0 = {}^t\underline{\mathbb{P}}_0$.

WKB solutions.

We look for asymptotic solutions (2.1.11) of the equation Eq. (2.2.18).

We look for a two scales expansion $u^\varepsilon = U(\varepsilon, x, x/\varepsilon)$ with

$$U(\varepsilon, x, X) \sim \sum_{j\geq 0} \varepsilon^j U^j(x, X), \tag{2.2.34}$$

with profiles $U^j \in \mathcal{P}$.

Then plugging the WKB expansion (2.2.34) in (2.2.18) yields

$$L(\varepsilon\partial_x)u^\varepsilon - \varepsilon f(u^\varepsilon) \sim \sum_{j\geq 0} \varepsilon^j \big(L(\partial_X)U^{j+1} + L_1(\partial_x)U^j - F^j\big) \tag{2.2.35}$$

Canceling successively all the ε^j coefficients gives the equations :

$$L(\partial_X)U^0 = 0, \tag{2.2.36}$$

$$L(\partial_X)U^j = F^{j-1} - L_1(\partial_x)U^{j-1}, \quad j \geq 1. \tag{2.2.37}$$

Similarly, the boundary condition $Tu^\varepsilon{}_{|x_d=0} = 0$ yields the equations

$$TU^j{}_{|x_d=X_d=0} = 0. \tag{2.2.38}$$

In particular the equations for the leading profile U^0 read

$$U^0 = \mathbb{P}U^0, \tag{2.2.39}$$

$$\mathbb{P}^i L_1(\partial_x)\mathbb{P}U^0 = \mathbb{P}^i f(U^0). \tag{2.2.40}$$

$$TU^0_{|x_d=X_d=0} = 0. \tag{2.2.41}$$

Initial conditions for U^0 read

$$U^0_{|x_0=X_0=0} = H^0. \tag{2.2.42}$$

For solving these equations, the data must satisfy *polarization* conditions for the oscillations, and *compatibility* conditions at the corner $\{x_0 = x_d = 0\}$. As usual in the analysis of WKB solutions, we satisfy the polarization condition by choosing

$$H^0 = \tilde{H}^0{}_{|X_0=0} \quad \text{with} \quad \tilde{H}^0 \in \mathbb{P}\mathcal{P}_{|x_0=0}. \tag{2.2.43}$$

Moreover, we choose $\tilde{H}^0$ with spectrum contained in Λ_0 and instead of Eq. (2.2.42) we solve

$$U^0(x_0 = 0) = \tilde{H}^0. \tag{2.2.44}$$

We do not make explicit the compatibility condition at the corner. We just note that they are automatically satisfied if $\tilde{H}^0$ is infinitely flat at $x_d = 0$.

Theorem 2.2.15. *Suppose that $\tilde{H}^0 \in \mathbb{P}\mathcal{P}_{|x_0=0}$ has its spectrum contained in Λ_0 and is infinitely flat at $x_d = 0$. Then, there is $t_* > 0$ such that the profile equations Eq.* (2.2.39)*, Eq.* (2.2.40) *with boundary condition* (2.2.41) *and Cauchy data* (2.2.44) *has a solution $U^0 \in \mathcal{P}([0, t_*])$ with spectrum contained in Λ_0.*

This theorem is proved in section 4, where we also make explicit the profile equations. We just point out here several remarks. The polarization condition (2.2.39) implies that

$$\begin{aligned} U^0(x, X) &= U^0_{os}(x, X) + U^0_{ev}(x, X) \\ &= \sum_{(\xi', \xi_d) \in \Lambda_0} U^0_{(\xi', \xi_d)}(x) e^{i(\xi' X' + \xi_d X_d)} \\ &\qquad + \sum_{\xi' \in \Lambda'_0} U^0_{\xi', ev}(x, X_d) e^{i\xi' X'} \end{aligned}$$

with the $U^0_{\xi', ev}$ exponentially decaying in X_d. The $U^0_{(\xi', \xi_d)}$ satisfy the usual polarization condition

$$U^0_{(\xi', \xi_d)} \in \ker L(i\xi', i\xi_d).$$

The evanescent terms $U^0_{\xi', ev}$ satisfy

$$U^0_{(\xi', ev)} \in \mathbb{E}^-_{E,L}(\xi')$$

Using the properties (2.2.32) and (2.2.33) and the assumptions of section 2.1, we show in section 4 that the equations Eq. (2.2.40) read

$$\begin{cases} X_{(\xi',\xi_d)} U^0_{(\xi',\xi_d)} = F^0_{(\xi',\xi_d)}, & \xi' \neq 0 \\ \underline{X}_0 U^0_{(0,0)} = F^0_{(0,0)}, \\ Y U^{0,*}_{0,os} = Y U^0_{0,ev} = 0, \end{cases} \tag{2.2.45}$$

where the $F^0_{(\xi',\xi_d)}$ are the Fourier coefficients of $F^0 = f(U^0_{os})$. Moreover, the $X_{(\xi',\xi_d)}$ are the usual transport at the group velocity corresponding to the regular modes $(\xi',\xi_d) \neq 0$ (see [15]) and $\underline{X}_0$ is the hyperbolic systems associated to the singular mode 0 (see [32]). The new operator Y is a transport field *parallel* to the boundary.

The boundary conditions read

$$\begin{cases} T\big(\sum_{\xi_d} U^0_{(\xi',\xi_d)|x_d=0}\big) + T U^0_{\xi',ev|x_d=X_d=0} = 0, & \xi' \neq 0 \\ T U_{(0,0)} = 0. \end{cases} \tag{2.2.46}$$

Thus, we see that we have a system of hyperbolic equations for the U^0_{os}, coupled through the nonlinearity f, and coupled to the evanescent part U^0_{ev} only through the boundary conditions. That the right hand side of the third equation in Eq. (2.2.45) vanishes, follows from Assumption 2.2.8. In particular, if the initial data for $U^{0,*}_{0,os}$ and $U^0_{0,ev}$ vanish, these terms remain equal to zero for all time.

The key point for proving Theorem 2.2.15 is that systems (2.2.45) (2.2.46) look like an hyperbolic maximal dissipative initial-boundary value problem, so that the methods of [42, 19] can be adapted.

There are similar but linear equations for the other profiles U^n. They have the general form

$$(1-\mathbb{P})U^n = \mathbb{Q}R^{n-1}, \tag{2.2.47}$$

$$\mathbb{P}^i L_1(\partial_x)\mathbb{P}U^n = \mathbb{P}^i\big(f'(U^0)\mathbb{P}U^n + R^{n-1}\big), \tag{2.2.48}$$

$$T\mathbb{P}U^n_{|x_d=X_d=0} = R^{n-1}, \tag{2.2.49}$$

$$\mathbb{P}U^n_{|x_0=0} = \tilde{H}^n, \tag{2.2.50}$$

where the R^{n-1} denote various terms in $\mathcal{P}$ which depend only on $U^0,\dots,U^{n-1}$.

The next result gives the existence of reflected-transmitted WKB geometric optics solutions of infinite order to the problem (2.2.18), under the assumptions of Section 2.1. Given Λ_0, we denote by $\mathcal{P}_{fin}$ the set of profiles with tangential spectrum contained in a multiple $k * \Lambda'_0$, with k odd if f is odd.

Theorem 2.2.16. *Let $(\tilde{H}^n)_{n\geq 0} \in \mathbb{P}\mathcal{P}_{fin}|_{t=0}$ be a sequence of initial data, infinitely flat at $x_d = 0$, with the spectrum of $\tilde{H}^0$ contained in Λ_0. There is a $t_* > 0$, and a unique sequence of solutions $U^n \in \mathcal{P}_{fin}([0,t_*])$ solution of the profile equations.*

Convergence.

The WKB solutions provide asymptotic solutions u^ε (2.1.11), to infinite order. Keeping a finite number of terms in the series we obtain *approximate* solutions

$$u^\varepsilon_{app}(x) = \sum_{k=0}^{n} \varepsilon^k U^k(x, \frac{x}{\varepsilon}) \tag{2.2.51}$$

which satisfy the equations up to order ε^n. We look for exact solutions

$$u^\varepsilon(x) = u^\varepsilon_{app}(x) + \varepsilon^M v^\varepsilon(x). \tag{2.2.52}$$

We follow the method developed by O.Guès [20], constructing and estimating v^ε in functional spaces where the derivatives are weighted by ε, so that functions $U(x, x/\varepsilon)$ are naturally uniformly bounded in such families of spaces. However, we need to adapt the analysis to the case of boundary value problem, with possibly characteristic boundaries (see also [13]).

Before stating the results, we introduce the necessary functional spaces, which are also used in the proof of Theorem 2.2.15.

We use the notations $x = (x_0, x_1, \dots, x_d) = (x_0, x'') = (t, x'')$, and work in the half space $\mathbb{R}^d_+ = \{x'' : x_d \geq 0\}$. Following [42] the tangential spatial Sobolev spaces are :

$$H^m_{tan}(\mathbb{R}^d_+) = \{u \in L^2(\mathbb{R}^d_+) \ : \ \partial_1^{m_1} \dots \partial_{d-1}^{m_{d-1}} u \in L^2 \quad m_1 + \dots + m_{d-1} \leq m\}.$$

Then the C^m-tangential spaces are

$$T^m([0,t_*] \times \mathbb{R}^d_+) = \bigcap_{r \leq m} C^r([0,t_*], H^{m-r}_{tan}(\mathbb{R}^d_+)).$$

Then, as it is known for maximal dissipative characteristic boundary problems with constant coefficients one can easily get estimates in T^m. But since the boundary is characteristic, in general, one can't fully express the normal derivative through the equation. Following [19], the natural estimates require two tangential derivatives for one normal derivative. This leads to introduce the spaces

$$E^m_{sp}(\mathbb{R}^d_+) = \{u \ : \ \partial^k_{x_d} u \in H^r_{tan}(\mathbb{R}^d_+) \quad \text{for } 2k + r \leq m\}.$$

Then the C^m-space

$$E^m([0,t_*] \times \mathbb{R}^d_+) = \bigcap_{r \leq m} C^r([0,t_*], E^{m-r}_{sp}).$$

These spaces are equipped with the obvious norms.

To prove the stability of the WKB expansion we take into account the fast oscillations by taking weighted norms : in the definitions above, we replace the tangential derivatives $\partial_{x_0}, \dots, D_{x_{d-1}}$ by $\varepsilon\partial_{x_0}, \dots, \varepsilon D_{x_{d-1}}$ and the normal derivative ∂_{x_d} by $\varepsilon^2\partial_{x_d}$. This does not change the space but changes the norms. We denote them by using the superscript ε. For instance

$$\|u(t)\|_{E^{m,\varepsilon}([0,t_*])} = \sum_{2\alpha_d + |\alpha'| \leq m} \varepsilon^{2\alpha_d + |\alpha'|} \sup_{t \in [0,t_*]} \|\partial^\alpha_x u(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}.$$

Similar explicit definitions are given in Section 5. As usual, we say that a family v^ε is bounded in such a family, say $E^{m,\varepsilon}$, if the family $\{\|v^\varepsilon\|_{E^{m,\varepsilon}}\}$ is bounded.

We can now state the stability theorem. We suppose that we are given a WKB solution on $[0,t_*] \times \mathbb{R}^d$ as in Theorem 2.2.16, and we consider and approximate solution u^ε_{app} as in (2.2.51). We consider the Cauchy problem for (2.2.18) with initial data

$$u^\varepsilon{}_{|t=0} = u^\varepsilon_{app|t=0} + \varepsilon^M h^\varepsilon. \tag{2.2.53}$$

For the initial data, we use the following weighted norms :

$$\|h^\varepsilon\|_{H^m_\varepsilon(\mathbb{R}^d_+)} := \sum_{|\alpha''|\le m} \varepsilon^{|\alpha''|}\|\partial^{\alpha''}_{x''} h\|_{L^2(\mathbb{R}^d_+)}.$$

Theorem 2.2.17. *Suppose that $n \ge M \ge m$ are integers, with $m > \frac{d+1}{2}$ even. Suppose that the family h^ε is bounded in H^m_ε and satisfies*

$$\partial^k_{x_d} h_{|x_d=0} = 0, \quad \text{for } k \in \{0,\dots,m-1\}.$$

$x_d = 0$. Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0,\varepsilon_0]$ the Cauchy problem (2.2.18) (2.2.53) has a unique solution u^ε in $E^{m,\varepsilon}([0,t_*])$. Moreover, $u^\varepsilon = u^\varepsilon_{app} + \varepsilon^M v^\varepsilon$. with v^ε bounded in $E^{m,\varepsilon}([0,t_*])$.

Note that the bounds imply that $\varepsilon^{M-(d+1)/2} v^\varepsilon$ is bounded in L^∞ (see Section 5) together with similar estimates for the derivatives. Thus, choosing M and n large enough, we see that $u^\varepsilon - u^\varepsilon_{app}$ can be made small in any Sobolev space.

2.2.3 Verification of Assumptions for the Maxwell anharmonic oscillator model.

Assumption 2.2.1 (H1) : Symmetrisable system.

Reducing the problem to a the half space $\{x_d \ge 0\}$ leads to $L(\partial_x)U = F, \quad x_d \ge 0$ with

$$L(\partial_x) = \begin{pmatrix} \tilde{L}_l(\partial_x) & 0 \\ 0 & L_r(\partial_x) \end{pmatrix}; \quad F = \begin{pmatrix} F_l \\ F_r \end{pmatrix},$$

where $\tilde{L}_l = L_l(\partial_{x'}, -\partial_z)$ and L_l [resp. L_r] is the maxwell operator in the region $x_d \le 0$ [resp. $x_d \ge 0$].

$$L_l(i(\xi_0,\xi'')) = \begin{pmatrix} i\xi_0 I_3 & i\xi''\wedge\cdot & 0_3 & I_3 \\ -i\xi''\wedge\cdot & i\xi_0 I_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & i\xi_0 I_3 & -I_3 \\ -\gamma_l I_3 & 0_3 & \omega_l^2 I_3 & i\xi_0 I_3 \end{pmatrix}$$

As L is not symmetric we multiply L on the left by $S = \begin{pmatrix} S_l & 0 \\ 0 & S_r \end{pmatrix}$ where S_l is a symmetrizer for L_l.

$$S_l = \mu_l \,\mathrm{diag}(I_3, I_3, \omega_l^2/\gamma_l I_3, 1/\gamma_l I_3).$$

Interverting the indexes l, r one gets a symmetrizer for L_r. The factor μ_l allows not to symmetrise the boundary conditions.

Then, the decomposition of the operator by blocks is achieved by a reordering of the basis in $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ is the base related to E_x, E_y, B_x, B_y and $\mathcal{B}_2$ is related to $E_z, B_z, \mathbf{P}, \mathbf{Q}$. The boundary matrix then writes

$$A_3 = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} & 0_{4,8} \\ 0_{8,4} & 0_{8,8} \end{pmatrix}$$

thus $dim(ker(A_3)) = 8$.

Assumption 2.2.1 (H3) : Maximal dissipative boundary condition

.

The boundary operator in (2.1.10) takes the form $T = (-A_3, A_3)$ (12×24 matrix) and the normal matrix of the symmetric operator SL is

$$\mathcal{A}_3 = \begin{pmatrix} -S_l\, A_3 & 0 \\ 0 & S_r\, A_3 \end{pmatrix}.$$

One has

$$(\mathcal{A}_3 U, U) = \mathbf{E}_l \wedge \mathbf{e}_3 \cdot \mu_l \mathbf{B}_l - \mathbf{E}_r \wedge \mathbf{e}_3 \cdot \mu_r \mathbf{B}_r = (\mathbf{E}_l \wedge \mathbf{e}_3) \cdot (\mu_r \mathbf{B}_r - \mu_l \mathbf{B}_l) = 0$$

Thus T is dissipative. Then $rg(T) = 4$ and the number of positive eigenvalue of $\mathcal{A}_3$ is 4 (2 $\mu_l\gamma_l$ and 2 $\mu_r\gamma_r$). This proves the maximality.

Assumption 2.2.5.

Here, we compute the characteristic values of $L^{22}(i\xi')$ and the values which are non regular and glancing for L to deduce a suitable set Λ' for which Assumption 2.2.5 is valid.

However using the diagonal shape of L we just need computing the zeros of the characteristic polynomial of L_l. Indeed remark that under a simple basis change one has

$$L(\partial_x) \sim \begin{pmatrix} L^{11}(\partial_x) & L^{12}(\partial_x) \\ L^{21}(\partial_x) & L^{22}(\partial_x) \end{pmatrix}; \quad L^{ij}(\partial_x) = \begin{pmatrix} \tilde{L}^{ij}(\partial_x) & 0 \\ 0 & L^{ij}(\partial_x) \end{pmatrix}.$$

So the bad values for L are the union of the bad values of L_l and L_r.

$\det L_l^{22}(i\xi') = \xi_0^2(\xi_0^2 - \omega_l^2 - \gamma_l)(\xi_0^2 - \omega_l^2)^2$.

Note $\Lambda_l^{22\prime} = \{\xi' = (\xi_0, \xi_1, \xi_2)$ such that $\xi_0 = 0, \pm\omega_l, \pm\sqrt{\omega_l^2 + \gamma_l}\}$. Choosing a tangential resonance set ${\Lambda_0}' \subset \mathbb{R}^3 \setminus \Lambda_l^{22\prime}$ a root in $\xi_0 \in {\Lambda_0}'$ of $\det L_l(i(\xi_0, \eta, \xi_d)) = 0$ is a root of

$$\xi_0^2(\xi_0^2 - \omega_l^2 - \gamma_l) - (\xi_0^2 - \omega_l^2)(|\eta|^2 + |\xi_d|^2) = 0 \tag{2.2.54}$$

Setting $\sigma_l = \omega_l^2 + \gamma_l + |\xi''|^2$, one has $\xi_0 = \pm\sqrt{\sigma_l/2 \pm \sqrt{\sigma_l^2 - 4\omega_l^2|\xi''|^2}/2}$ which is C^∞ for all $(\eta, \xi_d) \neq 0$ so it is regular. And $\frac{d\xi_0}{d\xi_d} = \frac{2\xi_d}{\xi_0(\xi_0\chi' + 2(1+\chi))}$. Thus to be non glancing one requires $\xi_d \neq 0$ that is

(P_l) $\quad \xi_0^2(\xi_0^2 - \omega_m^2 - \gamma_m) \neq (\xi_0^2 - \omega_m^2)|\eta|^2$.

Thus to satisfy Assumption 2.2.5, one must take ${\Lambda_0}' \subset \mathbb{R}^3 \setminus (\Lambda_l^{22\prime} \cup \Lambda_r^{22\prime} \cup {\Lambda_g}')$ where Λ_g' is the set of tangential resonances satisfying (P_l) or (P_r).

Assumption 2.2.6 : The Lopatinski conditions.

First we look for the eigenspaces of G. For each ξ', on each side, there holds

$$\mathbb{C}^4 = \mathbb{E}_H^-(\xi') \oplus \mathbb{E}_H^+(\xi') \;\text{ or }\; \mathbb{C}^4 = \mathbb{E}_E^-(\xi') \oplus \mathbb{E}_E^+(\xi').$$

Indeed let ξ an eigenvalue of G then $(\xi', \xi_d) \in \text{char}(L)$ and it is solution to Eq. (2.2.54). Thus $(\xi', -\xi_d)$ is also in $\text{char}(L)$ and $-\xi_d$ an eigenvalue of G. Now, from [13] the dimension of the eigenspace is two. It looks (in the initial basis)

$$\pi(\xi)U_\xi = (\mathbf{E},\ \frac{\xi''}{\xi_0} \wedge \mathbf{E},\ \chi(\xi)\mathbf{E},\ i\xi_0\chi(\xi)\mathbf{E}), \quad \xi' \neq 0 \text{ with } \xi'' \cdot \mathbf{E} = 0.$$

Next, to show the second part of Assumption 2.2.6 let solve $T(\pi_l U_l, \pi_r U_r) = 0$ where T is given in the introduction (Eq. (2.1.5), Eq. (2.1.6)) and $\pi_l U_l \in \mathbb{E}^-_{H,l}(\xi')$ or $\mathbb{E}^-_{E,l}(\xi')$ and the same with $\pi_r U_r$ (π_m is a projector on $\mathbb{E}^-_{H,m}(\xi')$ or $\mathbb{E}^-_{E,m}(\xi')$ depending on ξ is real or complex).
Note $\xi = (\xi_0, \eta, \xi_3)$ and $\mathbf{E} = \binom{\mathbf{E}_{//}}{E_3}$.
From the jump equation, $\mathbf{E}_{r//} = \mathbf{E}_{l//} = \mathbf{E}_{//}$. From the jump equation on $\mathbf{B}$, $\mu_r(\xi_3^r \mathbf{E}_{r//} - \eta^r E_{r3}) - \mu_l(\xi_3^l \mathbf{E}_{l//} - \eta^l E_{l3}) = 0$.
As $\eta^r = \eta^l = \eta, \quad (\mu_r \xi_3^r - \mu_l \xi_3^l)\mathbf{E}_{//} = (\mu_r E_{r3} - \mu_l E_{l3})\eta$.
Using the condition $\mathbf{E}_m \cdot \xi_m = 0,\ m \in \{l, r\} \Rightarrow E_{r3}\xi_3^r = E_{l3}\xi_3^l$. Thus scalar-multiplying by $\eta\xi_3^r$
$\xi_3^r(\mu_r\xi_3^r - \mu_l\xi_3^l)(-\xi_3^l E_{l3}) = E_{l3}(\mu_r\xi_3^l - \mu_l\xi_3^r)|\eta|^2$. And

$$\mathbf{E} = 0 \Longleftrightarrow (\mu_r\xi_3^r - \mu_l\xi_3^l)\xi_3^r\xi_3^l \neq -(\mu_r\xi_3^l - \mu_l\xi_3^r)|\eta|^2 \quad (Q)$$

with $|\xi_3^r|^2 = \xi_0^2(\chi_r(\xi_0) - \chi_l(\xi_0)) + |\xi_3^l|^2$.
Thus one chooses $\Lambda_0 \subset (\Lambda_0' \times \mathbb{R}) \cap \{\xi,\ (Q) \text{ is true}\}$

Verifying Assumption 2.2.7.

- $det(L^{22}(0)) = 0$.
- $L^{12}(0)\pi^{22}(0) : \pi^{22}(0)W = (E_3, B_3, 0, 0, \chi(0)E_3, 0, 0, 0)$, and $L^{21}(0)V = (0, 0, 0, 0, 0, E_1, E_2, 0)$ so $\pi^{22}(0)L^{21}(0) = 0$.
- $\xi = (0, \xi_d),\ \xi_d \neq 0$ is regular. However $\xi = 0$ is not regular since three distinct sheets cross in this point : $\xi_0 = 0$ and $\xi_0 = \pm\sqrt{\sigma_l/2 - \sqrt{\sigma_l^2 - 4\omega_l^2|\xi''|^2}/2}$. The first exhibits a glancing situation but as all the derivatives of $\xi_0 = 0$ are null one can't expect the apparition of a new intermediate scale as described in [47].
- For $\xi_d = 0$ we make the change of basis : $\mathcal{B}_2 \to \mathcal{B}_2^\star = (e_7, e_8, e_9 - \chi(0)e_5, e_{10}, e_{11}, (\omega_l^2 + \chi(0)\gamma_l)e_{12}, e_5 + \chi(0)e_9, e_6)$ and call P the associated matrix. Thus setting $L_l^\star = P^{-1}L_l P$,

$$L_l^\star = \begin{pmatrix} 0_4 & L_\star^{12} & 0_{3,2} \\ -{}^tL_\star^{12} & L_\star^{22} & 0_{6,2} \\ 0_{2,3} & 0_{2,6} & 0_{2,2} \end{pmatrix}$$

Where

$$L_\star^{22} = \begin{pmatrix} 0_3 & -D \\ D & 0_3 \end{pmatrix}, \quad L_\star^{12} = \begin{pmatrix} 0_{4,3} & M^{12} \end{pmatrix}$$

with D diagonal invertible. Remark in this basis $G(0)$ is equivalent to $\binom{G^\star}{0}$ with $G^\star = L_\star^{12}(L_\star^{22})^{-1}{}^tL_\star^{12}$. But

$$G^\star = (0, M^{12}) \begin{pmatrix} 0_3 & -D^{-1} \\ D^{-1} & 0_3 \end{pmatrix} \begin{pmatrix} 0 \\ M^{12} \end{pmatrix} = 0.$$

Thus $G(0) = 0$.

Verifying Assumption 2.2.8

$F_l = (0_3, 0_3, 0_3, h_l(P))$. In the basis $(\mathcal{B}_1, \mathcal{B}_2)$ it writes $F_l = (F_l^1, F_l^2) = (0, F_l^2)$ with $F_l^2 = (0, 0, 0_3, h_l(P))$ and $\ker L_l^{22}(0) = Vect((0, 1, 0_3, 0_3), (\omega_l^2, 0, 0, 0, \gamma_l, 0_3))$. Thus $\pi_l^{22} F_l^2 = 0$. The same analysis runs for the subscript r and lastly in the product space.

An example of Λ_0 which fulfills Assumption 2.2.9.

We consider again the example of Remark 2.2.11. Choose $\underline{\xi}' \in \mathrm{char} L_l \cap \mathbb{Q}$ and take Λ_0 as a net of basis $\{\pm\underline{\xi}, \pm\check{\underline{\xi}}\} = \{(\pm\underline{\xi}', \pm\xi_d(\pm\underline{\xi}'))\}$.

In general except the basis itself, no mode in Λ_0 is resonant. Indeed let $\xi = \alpha_1\underline{\xi} + \alpha_2\check{\underline{\xi}} = (\sigma\underline{\xi}', \delta\xi_d(\underline{\xi}'))$ with $\sigma = \alpha_1 + \alpha_2,\ \delta = \alpha_1 - \alpha_2$.

$$\xi \in \mathrm{char} L_r \Longleftrightarrow \delta^2 = \sigma^2 \frac{1 + \chi_r(\sigma\underline{\xi}_0)}{1 + \chi_l(\underline{\xi}_0)} \ .$$

The fraction must be rational : this is very seldom. In particular, taking all the coefficients of χ_r, χ_l rational except γ_r , irrational, leads to irrational values of the fraction whatever σ.

2.3 The microscopic equation. Proof of Theorem 2.2.14

We consider the fast scale equation $L(\partial_X)U = F$ in the space of profiles $\mathcal{P}$, with the goal of making explicit the kernel and the range of $L(\partial_X)$. In this analysis, the slow variable x is a parameter and is omitted in the notations for simplicity. Expanding the equation in Fourier independent tangential modes reduces to solve the equation Eq. (2.2.5), $(A_d\partial_{X_d} + L'(i\xi'))U_{\xi'} = F_{\xi'}$ for every $\xi' \in \Lambda'$.

– $\boxed{\text{First case : } \xi' \neq 0}$. The spaces $\mathcal{P}_{os}^z$ and $\mathcal{P}_{ev}^z$ are clearly invariant by $A_d\partial_{X_d} + L'(i\xi')$, thus is sufficient to study the equation in each space separately.

1. *Oscillating terms.*
   For $\xi' \in \Lambda' \setminus \{0\}$ and $\xi_d \in \mathbb{R}$, let $\pi(\xi', \xi_d)$ denote the orthogonal projector on the kernel of $L(i\xi', i\xi_d)$. It satisfies

   $$L(i\xi', i\xi_d)\pi(\xi', \xi_d) = \pi(\xi', \xi_d)L(i\xi', i\xi_d) = 0$$

   and $\pi(\xi', \xi_d) \neq 0$ if and only if (ξ', ξ_d) belongs to $\mathrm{char} L$. Let $Q(\xi', \xi_d)$ denote the partial inverse of $L(i\xi', i\xi_d)$ such that

   $$\begin{aligned} &L(i\xi', i\xi_d)Q(\xi', \xi_d) = Q(\xi', \xi_d)L(i\xi', i\xi_d) = (1 - \pi(\xi', \xi_d)), \\ &Q(\xi', \xi_d)\pi(\xi', \xi_d) = 0. \end{aligned}$$

   For $U(Z) = \sum U_{\xi_d} e^{i\xi_d Z} \in \mathcal{P}_{os}^z$ and $F = \sum F_{\xi_d} e^{i\xi_d Z} \in \mathcal{P}_{os}^z \in \mathcal{P}_{os}^z$ let

   $$\mathbb{P}_{os}(\xi')U(Z) = \mathbb{P}_{os}^i(\xi')U(Z) := \sum_{\xi_d} \pi(\xi', \xi_d)U_{\xi_d} e^{i\xi_d Z}, \tag{2.3.1}$$

   $$\mathbb{Q}_{os}(\xi')F(Z) := \sum_{\xi_d} Q(\xi', \xi_d)F_{\xi_d} e^{i\xi_d Z}. \tag{2.3.2}$$

   With these notations, there holds

Lemma 2.3.1. *For all $\xi' \in \Lambda' \setminus \{0\}$ and $F \in \mathcal{P}^z_{os}$ the equation*

$$(A_d\partial_Z + L'(i\xi'))U = F \tag{2.3.3}$$

has a solution $U \in \mathcal{P}^z_{os}$ if and only if $\mathbb{P}^i_{os}(\xi')F = 0$. In this case, U is a solution if and only if

$$U = \mathbb{Q}_{os}(\xi')F + \mathbb{P}_{os}(\xi')U.$$

Démonstration. The equation reads $L(i\xi', i\xi_d)U_{\xi_d} = F_{\xi_d}$ for all ξ_d in the (finite) normal spectra of U and F. This has a solution, if and only if $\pi(\xi', \xi_d)F_{\xi_d} = 0$ and U_{ξ_d} is a solution if and only if $U_{\xi_d} = Q(\xi', \xi_d)F_{\xi_d} + \pi(\xi', \xi_d)U_{\xi_d}$.
Adding up in ξ_d gives the result. □

2. *Evanescent terms.*
We consider next the equation Eq. (2.3.3) in spaces of evanescent profiles. Using the block decomposition $U = (U_1, U_2)$, Assumption 2.2.5 implies that the equation reads

$$\partial_Z U_1 + G(\xi')U_1 = A_d^{-1}\big(F_1 - L^{12}(i\xi')\{L^{22}(i\xi')\}^{-1}F_2\big), \tag{2.3.4}$$

$$U_2 = \{L^{22}(i\xi')\}^{-1}\big(F_2 - L^{21}(i\xi')U_1\big), \tag{2.3.5}$$

Recall that $\mathbb{E}^-_{E,G}(\xi')$ denotes the invariant subspace of $G(\xi')$ associated to eigenvalues in $\{\mathrm{Re} > 0\}$. Denote by $\Pi^-_{E,G}(\xi')$ the corresponding spectral projector. The evanescent solutions of $(\partial_Z + G(\xi'))U = 0$ are $e^{-ZG(\xi')}U(0)$ with $U(0) \in \mathbb{E}^-_{E,G}(\xi')$. This leads to define the projector

$$\mathbb{P}_{ev}(\xi')U(Z) = \begin{pmatrix} \mathrm{Id} \\ -\{L^{22}(\xi')\}^{-1}\,L^{21}(\xi') \end{pmatrix} e^{-G\,Z}\mathbb{P}^-_{E,G}U_1(0) \tag{2.3.6}$$

It maps $\mathcal{P}^z_{ev}$ onto the kernel $\ker(L(i\xi', \partial_Z)) \cap \mathcal{P}_{ev}$.

Lemma 2.3.2. *For all $\mathcal{F} \in \mathcal{P}^z_{ev}$, the problem*

$$\partial_Z\mathcal{U} + G(\xi')\mathcal{U} = \mathcal{F}, \quad \Pi^-_{E,G}(\xi')\mathcal{U}(0) = 0, \tag{2.3.7}$$

has a unique solution $\mathcal{U} \in \mathcal{P}^z_{ev}$. It is given by

$$\begin{aligned}\mathcal{U}(Z) = \mathbb{I}(\mathcal{F})(Z) := &\int_0^Z e^{(Z-y)G(\xi')}\Pi^-_{E,G}(\xi')\mathcal{F}_{ev}(y)dy \\ &- \int_Z^\infty e^{(Z-y)G(\xi')}(1 - \Pi^-_{E,G}(\xi'))\mathcal{F}_{ev}(y)dy.\end{aligned} \tag{2.3.8}$$

Démonstration. The uniqueness follows from the remark preceding the lemma. The convergence of the integrals in Eq. (2.3.8) follows from the estimates

$$|e^{(Z-y)G}\Pi^-_{E,G}| \le e^{-\delta(Z-y)}, \quad \text{for } Z - y \ge 0,$$

$$|e^{(Z-y)G}(1 - \Pi^-_{E,G})| \le |Z - y|^N, \quad \text{for } Z - y \le 0,$$

for some $\delta > 0$ and N. Indeed, the Assumption 2.2.5 implies that the real eigenvalues of G are semi-simple so that one can take $N = 0$ in the estimate above. Thus, for $\mathcal{F}$ exponentially decaying, Eq. (2.3.8) defines $\mathbb{I}(\mathcal{F})$ which is also exponentially decaying, and solution to Eq. (2.3.7). □

For $\xi' \in \Lambda' \setminus \{0\}$ and $F \in \mathcal{P}^z_{ev}$ define

$$\mathbb{Q}_{ev}(\xi')F = \begin{pmatrix} \mathrm{Id} \\ -\{L^{22}(\xi')\}^{-1} L^{21}(\xi') \end{pmatrix} \mathbb{I}(\mathcal{F}) + \begin{pmatrix} 0 \\ \{L^{22}(\xi')\}^{-1} F_2 \end{pmatrix}$$

with $\mathcal{F} = A_d^{-1}\big(F_1 - L^{12}(\xi')\{L^{22}(i\xi')\}^{-1}$. With Eq. (2.3.4) Eq. (2.3.5), we see that $U = \mathbb{Q}_{ev}(\xi')F$ is a solution in $\mathcal{P}^z_{ev}$ of Eq. (2.3.3), and is the unique solution such that $\Pi^-_{E,G}(\xi')U_1(0) = 0$, or equivalently, the unique solution such that $\mathbb{P}_{ev}(\xi')U = 0$. Collecting the results above, we have proved

Lemma 2.3.3. *For all $\xi' \in \Lambda' \setminus \{0\}$ and all $F \in \mathcal{P}^z_{ev}$ the equation Eq.* (2.3.3) *has a solution $U \in \mathcal{P}^z_{ev}$. Moreover, $U \in \mathcal{P}^z_{ev}$ is a solution if and only if*

$$U = \mathbb{Q}_{ev}(\xi')F + \mathbb{P}_{ev}(\xi')U.$$

Because, the equation has a solution for all F, we set

$$\mathbb{P}^i_{ev}(\xi') \equiv 0. \tag{2.3.9}$$

– Second case : $\xi' = 0$.

From Assumption 2.2.7, the equation Eq. (2.3.3) for $\xi' = 0$ reads (see Eq. (2.2.17)) :

$$\begin{aligned} &\partial_Z U_1 = \mathcal{F} := (A_d^{11})^{-1}(F_1 - L^{12}(0)Q^{22}(0)F_2), \\ &(1 - \pi^{22}(0))U_2 = -Q^{22}(0)(L^{21}(0)U_2 - F_2), \\ &\pi^{22}(0)F_2 = 0. \end{aligned}$$

1. *Evanescent terms.* For $\mathcal{F} \in \mathcal{P}^z_{ev}$, the equation $\partial_Z U_1 = \mathcal{F}$ has a unique solution in $\mathcal{P}^z_{ev}$, namely

$$U_1(Z) = \mathbb{I}(\mathcal{F})(Z) := -\int_Z^\infty \mathcal{F}(y)dy.$$

Introducing

$$\mathbb{P}_{ev}(0) = \mathbb{P}^i_{ev}(0) = \begin{pmatrix} 0 & 0 \\ 0 & \pi^{22}(0) \end{pmatrix} := \Pi^{22}. \tag{2.3.10}$$

and

$$\mathbb{Q}_{ev}(0)F = \begin{pmatrix} \mathrm{Id} \\ -Q^{22}(0)\, L^{21}(0) \end{pmatrix} \mathbb{I}(\mathcal{F}) + \begin{pmatrix} 0 \\ Q^{22}(0)F_2 \end{pmatrix}$$

with $\mathcal{F} := (A_d^{11})^{-1}(F_1 - L^{12}(0)Q^{22}(0)F_2)$ as above, there holds :

Lemma 2.3.4. *For $\xi' = 0$ and $F \in \mathcal{P}^z_{ev}$ the equation Eq.* (2.3.3) *has a solution $U \in \mathcal{P}^z_{ev}$ if and only if $\mathbb{P}^i_{ev}F = 0$. In this case, $U \in \mathcal{P}^z_{ev}$ is a solution if and only if*

$$U = \mathbb{Q}_{ev}(0)F + \mathbb{P}_{ev}(0)U.$$

2. *Oscillating terms.* Consider $U = \sum U_{\xi_d} e^{i\xi_d Z} \in \mathcal{P}^z_{os}$ and $F = \sum F_{\xi_d} e^{i\xi_d Z} \in \mathcal{P}^z_{os}$. The equation $L(0, \partial_Z)U = F$ amounts to solve for all ξ_d $L(0, i\xi_d)U_{\xi_d} = F_{\xi_d}$, that is

$$\begin{aligned} &i\xi_d U_{\xi_d,1} = \mathcal{F}_{\xi_d} := (A_d^{11})^{-1}(F_{\xi_d,1} - L^{12}(0)Q^{22}(0)F_{\xi_d,2}),\\ &(1-\pi^{22}(0))U_{\xi_d,2} = -Q^{22}(0)(L^{21}(0)U_{\xi_d,1} - F_{\xi_d,2}),\\ &\pi^{22}(0)F_{\xi_d,2} = 0. \end{aligned}$$

The analysis depends on whether ξ_d vanishes or not. This leads to split apart the frequency zero and use the notations (2.2.30) $U = \underline{U} + U^*$ and $F = \underline{F} + F^*$. Consider first the nonzero frequencies. Introduce the projectors

$$\mathbb{P}^*_{os} = \mathbb{P}^{i,*}_{os} = \Pi^{22} \begin{pmatrix} 0 & 0 \\ 0 & \pi^{22}(0) \end{pmatrix}. \tag{2.3.11}$$

which act on $\mathcal{P}^{z,*}_{ev}$. Introduce next

$$\mathbb{Q}^*_{os}(0)F^* = \begin{pmatrix} \mathrm{Id} \\ -Q^{22}(0)\, L^{21}(0) \end{pmatrix} \sum_{\xi_d \neq 0} \frac{1}{i\xi_d} \mathcal{F}_{\xi_d} e^{i\xi_d Z} + \begin{pmatrix} 0 \\ Q^{22}(0)F_2^* \end{pmatrix}$$

with $\mathcal{F} := (A_d^{11})^{-1}(F_1 - L^{12}(0)Q^{22}(0)F_2)$ as above,

3. *non oscillating terms*
Finally, for the frequency $\xi_d = 0$, consider the projectors and partial inverse

$$\underline{\mathbb{P}}_0 = \begin{pmatrix} Id & 0 \\ -Q^{22}(0)L^{21}(0)) & \pi^{22}(0) \end{pmatrix}, \quad \underline{\mathbb{P}}^i_0 = {}^t\underline{\mathbb{P}}_0, \quad \underline{\mathbb{Q}}_0 = \begin{pmatrix} 0 & 0 \\ 0 & Q^{22}(0) \end{pmatrix}.$$

Adding up we define $\mathbb{P}_{os}(0)U = \underline{\mathbb{P}}_0\underline{U} + \mathbb{P}^*_{os}U^*$ and similarly $\mathbb{P}^i_{os}(0)F$ and $\mathbb{Q}_{os}(0)F$. With these notations, there holds

Lemma 2.3.5. *For all $F \in \mathcal{P}^z_{os}$ the equation*

$$(A_d\partial_Z + L'(0))U = F$$

has a solution $U \in \mathcal{P}^z_{os}$ if and only if $\mathbb{P}^i_{os}(0)F = 0$. Moreover, $U \in \mathcal{P}^z_{os}$ is a solution, if and only if

$$U = \mathbb{Q}_{os}(0)F + \mathbb{P}_{os}(0)U.$$

Remark 2.3.6. For $\xi_d = 0$ one might have chosen the usual Hermitian projector $\pi(0)$ on $\ker L(0)$ but to take into account the characteristic block decomposition above, our choice of $\underline{\mathbb{P}}_0$ and $\underline{\mathbb{P}}^i_0$ which are not symmetric is more natural.

Collecting the Lemmas above, the Theorem 2.2.14 follows.

2.4 Geometric optics, WKB solutions

In this section we solve the mixed boundary value problem (2.2.18) for times t of order $\mathcal{O}(1)$ uniformly with respect to ε, proving Theorems 2.2.16. We use a WKB asymptotic expansion coupled with a discrete mode by mode analysis. We first expose the cascade of equation and then determine the profiles. We end with the proof of Theorem 2.2.15 and 2.2.16.

2.4.1 The cascade of profile equations

Here we refer to the equations (2.2.36)-(2.2.38) and we expand the non-linear term when plugging the infinite WKB expansion in it.

Lemma 2.4.1. *1. In the sense of formal Taylor expansions, if U is given by Eq. (2.2.34), then*

$$f(U) \sim \sum \varepsilon^j F^j(x, X) \qquad F^j \in \mathcal{P}. \tag{2.4.1}$$

Moreover,

$$F^0 = f(U^0), \quad F^0_{os} = F(U^0_{os}), \tag{2.4.2}$$

and for $j > 1$,

$$F^j = \nabla_u f(U^0)U^j + \tilde{F}^j, \quad F^j_{os} = \nabla_u f(U^0_{os})U^j_{os} + \tilde{F}^j_{os} \tag{2.4.3}$$

where $\tilde{F}^j$ depends only on $(U^0, \dots, U^{j-1})$.
Moreover, if f is odd and the tangential spectrum of $(U^0, \dots, U^{j-1})$ contained in an odd multiple $(2k+1) * \Lambda'_0$, then the tangential spectrum of $\tilde{F}^j$ is contained in a odd multiple $(2l+1) * \Lambda'_0$.

2. If the spectrum of $U^0 \in \mathcal{P}$ is contained in Λ_0, then the spectrum of $\mathbb{P}F^0$ is also contained in $\Lambda_0 \cap (\mathrm{char}L)^\sharp$. Moreover, if Λ_1 satisfies (2.2.26) and the spectrum of $V \in \mathcal{P}$ is contained in Λ_1, then the spectrum of $\mathbb{P}(f'(U^0)V)$ is contained in $\Lambda_1 \cap (\mathrm{char}L)^\sharp$.

Démonstration. Let us note $U = U^0 + \varepsilon R^1$. Then writing the finite Taylor expansion for the polynomial f, one has

$$f(U) = f(U^0) + \sum_{j=1}^{J} \varepsilon^j \frac{1}{j!} \phi^j(\underbrace{R^1, \dots, R^1}_{j}),$$

where J is the degree of f and $\phi^j = \nabla^j f(U^0)$ is j-linear with coefficients that are polynomial in U^0. Expanding R^1, there holds in the sense of formal Taylor expansions :

$$\phi^j(R^1, \dots, R^1) = \sum_{k_1, \dots, k_j \geq 1} \varepsilon^{k_1 - 1} \cdot \dots \cdot \varepsilon^{k_j - 1} \phi^j(U^{k_1}, \dots, U^{k_j}).$$

Thus Eq. (2.4.1) follows with $F^0 = f(U^0)$ and for $j \geq 1$, F^j is a polynomial in $U_0, \dots, U_j$:

$$F^j = \sum_{k_1 + \dots + k_l = j} \phi^j_{k_1, \dots, k_l} U^{k_1} \dots U^{k_l} \tag{2.4.4}$$

Moreover, if f is odd, then in the sum above only terms with l odd are present. The tangential Fourier expansion reads

$$F^j_{\xi'} = \sum_{k_1+\ldots+k_l=j} \sum_{\xi'_1+\ldots+\xi'_l=\xi'} \phi^j_{k_1,\ldots,k_l} U^{k_1}_{\xi'_1} \ldots U^{k_l}_{\xi'_l}. \tag{2.4.5}$$

This shows (i).

Next, we note that f is a polynomial and that the multiplication of any profile with an exponentially decaying profile in $\mathcal{P}_{ev}$ is exponentially decaying and belongs to $\mathcal{P}_{ev}$. Thus $f(U^0) - f(U^0_{os}) \in \mathcal{P}_{ev}$, implying the second part of (2.4.2). The proof of (2.4.3) is similar.

Suppose now that the spectrum of U^0 is contained in Λ_0. Then, expanding U^0_{os} as a linear combination of exponential $e^{i\xi X}$ with $\xi \in \Lambda_0$, we see that the spectrum of $f(U^0_{os})$ is contained in the set $\mathcal{N}(\Lambda_0)$, union of the sets $\mathcal{N}_j(\Lambda_0) = \{\xi^1 + \ldots + \xi^j \,:\, \xi^k \in \Lambda_0\}$, for j the degrees of nonvanishing monomials in f, see (2.2.21). Thus

$$F^0_{os} = \sum_{\xi\in\mathcal{N}(\Lambda_0)} F^0_\xi e^{i\xi X} \quad \text{and} \quad \mathbb{P}_{os} F^0_{os} = \sum_{\xi\in\mathcal{N}(\Lambda_0)} G^0_\xi e^{i\xi X}.$$

If $\xi \in \{0\} \times \mathbb{R}^*$, $G^0_\xi = \mathbb{P}^*_{os} F^0_\xi$ vanishes by (2.3.11) and Assumption 2.2.8. If $\xi \notin \mathrm{char}L$, then $G^0_\xi = 0$ by definition of the projectors $\mathbb{P}_{os}(\xi')$. Thus $G^0_\xi = 0$ if $\xi \notin \mathcal{N}(\Lambda_0) \cap (\mathrm{char}L)$ which by condition (2.2.25) of Assumption 2.2.9 implies that $G^0_\xi = 0$ if $\xi \notin \Lambda_0$.

Similarly, the spectrum of $\big(f'(U^0)\big)_{os} = f'(U^0_{os})$ is contained in $\mathcal{N}^\flat(\Lambda_0)$ and if the spectrum of V is contained in Λ_1, the spectrum of $G = \mathbb{P}(f'(U^0)V)$ is contained in $\mathcal{N}^\flat(\Lambda_0) + \Lambda_1$. With notations as above, the Fourier components G_ξ of G_{os} vanish when $\xi \in \{0\} \times \mathbb{R}^*$ by Assumption 2.2.8 and when $\xi \notin \mathrm{char}L$. Thus (2.2.26) implies that that $G_\xi = 0$ if $\xi \notin \Lambda_1$. The proof of the lemma is complete. □

Applying Theorem 2.2.14, we see that the cascade Eq. (2.2.36) Eq. (2.2.37) is equivalent to

$$\begin{cases} U^0 = \mathbb{P}U^0, \\ \mathbb{P}^i L_1(\partial_x)\mathbb{P}U^0 = \mathbb{P}^i F^0, \end{cases} \tag{2.4.6}$$

and for $j \geq 1$:

$$\begin{cases} (1-\mathbb{P})U^j = \mathbb{Q}\big(F^{j-1} - L_1(\partial_x)U^{j-1}\big) \\ \mathbb{P}^i L_1(\partial_x)\mathbb{P}U^j = \mathbb{P}^i\big(F^j - L_1(\partial_x)(1-\mathbb{P})U^j\big). \end{cases} \tag{2.4.7}$$

From [14, 32] one identifies the first order operator $\mathbb{P}^i L_1(\partial_x)\mathbb{P}$. If $U(x,X) = \sum e^{i\xi' X'} U_{\xi'}(x, X_d)$, then

$$\mathbb{P}^i L_1(\partial_x)\mathbb{P}U = \sum e^{i\xi' X'} \big(\mathbb{P}^i(\xi') L_1(\partial_x)\mathbb{P}(\xi') U_{\xi'}\big) \tag{2.4.8}$$

and

$$\begin{aligned} \mathbb{P}^i(\xi') L_1(\partial_x)\mathbb{P}(\xi') U_{\xi'} = {} & \mathbb{P}^i_{os}(\xi') L_1(\partial_x)\mathbb{P}_{os}(\xi') U_{\xi',os} \\ & + \mathbb{P}^i_{ev}(\xi') L_1(\partial_x)\mathbb{P}_{ev}(\xi') U_{\xi',ev}. \end{aligned} \tag{2.4.9}$$

Moreover, when $\xi' = 0$ we separate off the 0 frequency and write

$$\begin{aligned}\mathbb{P}^i_{os}(0)L_1(\partial_x)\mathbb{P}_{os}(0)U_{0,os} = \mathbb{P}^{i,*}_{os}L_1(\partial_x)\mathbb{P}^*_{os}U^*_{0,os}\\ + \underline{\mathbb{P}}^i_0 L_1(\partial_x)\underline{\mathbb{P}}_0\underline{U}_0.\end{aligned} \tag{2.4.10}$$

Proposition 2.4.2. *1. If $\xi' \in \Lambda' \setminus \{0\}$ and $U(x,Z) = \sum e^{i\xi_d Z}U_{\xi_d}(x) \in \mathcal{P}^z_{os}$, then*

$$\begin{aligned}\mathbb{X}_{os}(\xi')U &:= \mathbb{P}^i_{os}(\xi')L_1(\partial_x)\mathbb{P}_{os}(\xi')U\\ &= \sum_{\{\xi_d:(\xi',\xi_d)\in \mathrm{char}L\}} e^{i\xi_d Z}\Big(\big(\partial_t + \mathbf{v}_{(\xi',\xi_d)}\cdot\partial_{x''}\big)\pi_{\xi',\xi_d}U_{\xi_d}\Big)\end{aligned} \tag{2.4.11}$$

where, for a regular real mode $\xi \in \mathrm{char}L$, associated to the eigenvalue $\lambda(\xi'')$ (see Definition 2.2.2), $\mathbf{v}_\xi = \partial_{\xi''}\lambda$ is the associated group velocity.

2. If $\xi' \in \Lambda' \setminus \{0\}$, then $\mathbb{P}^i_{ev}(\xi')L_1(\partial_x)\mathbb{P}_{ev}(\xi') = 0$.

3. For $\xi' = 0$, there holds

$$\mathbb{P}^{i,*}_{os}L_1(\partial_x)\mathbb{P}^*_{os} = \mathbb{P}^i_{ev}(0)L_1(\partial_x)\mathbb{P}_{ev}(0) = \begin{pmatrix}0 & 0\\ 0 & Y\pi^{22}(0)\end{pmatrix} \tag{2.4.12}$$

where $Y = \partial_t + \mathbf{v}_0\cdot\partial_{x''}$ is a scalar vector field, tangent to the boundary $\{x_d = 0\}$.

4. For $\xi = 0$, $\underline{X}_0(\partial_x) := \underline{\mathbb{P}}^i_0 L^1(\partial_x)\underline{\mathbb{P}}_0$ is a matricial hyperbolic operator on the range of $\underline{\mathbb{P}}_0$, whose characteristic variety is the cone tangent to charL at $\xi = 0$.

Démonstration. Property (i) follows from Assumption 2.2.5 and the well known fact that $\pi(\xi)L_1(\partial_x)\pi(\xi) = (\partial_t + \mathbf{v}_\xi\partial_{x''})\pi(\xi)$ if ξ is a regular mode (see [15]). Similarly, $iv)$ follows from [32]. Point $ii)$ is trivial since $\mathbb{P}^i(\xi') = 0$ when $\xi' \neq 0$. By (2.3.10) and (2.3.11), the operators $\mathbb{P}^{i,*}_{os}L_1(\partial_x)\mathbb{P}^*_{os}$ and $\mathbb{P}^i_{ev}(0)L_1(\partial_x)\mathbb{P}_{ev}(0)$ are both equal to

$$\begin{pmatrix}0 & 0\\ 0 & \pi^{22}(0)L^{22}(\partial'_x)\pi^{22}(0)\end{pmatrix}.$$

Recall from Assumption 2.2.7 that $(0,\xi_d)$ is characteristic and regular for all $\xi_d \in \mathbb{R}\setminus 0$. Moreover, $\mathbb{P}_{os}(0)$ is the orthogonal projector on $\mathrm{Ker}L(0,i\xi_d)$ for all $\xi_d \neq 0$, for instance at $\xi_d = 1$. Thus $\mathbb{P}^{i,*}_{os}L_1(\partial_x)\mathbb{P}^*_{os} = Y\mathbb{P}^*_{os}$ where $Y = \partial_t + \mathbf{v}_0\partial_{x''}$ is the scalar transport field with group velocity associated to the eigenvalue λ such that $0 = \lambda(0,1)$. Since $(0,\xi_d)$ is characteristic for all ξ_d there holds $\lambda(0,\xi_d) \equiv 0$ implying that $\partial_{\xi_d}\lambda(0,1) = 0$, hence that $\mathbf{v}_0$ and Y are tangent to the boundary $\{x_d = 0\}$. For (iv) we refer to [32].

□

Example 2.4.3. For Maxwell Y is the tangential transport associated to the eigenvalue $\lambda = 0$ so that the group velocity is zero : $Y = \partial_t$. In fact this equation concerns the divergence and if the initial condition is divergence free this extends to the WKB expansion of the solution up to an error of truncation.

Next we consider the boundary conditions (2.2.38). For a profile $U(x,X) = \sum U_{\xi'}(x,X_d)e^{i\xi',X'} \in \mathcal{P}$ define

$$\mathbb{T}U(x',X') = TU_{|x_d=X_d=0} = \sum e^{i\xi' X'}TU_{\xi'|x_d=X_d=0}$$

Then $\mathbb{T}$ acts from $\mathcal{P}$ to $\mathcal{P}_b$, the space of profiles $V(x',X')$ that are finite sums $\sum V_{\xi'}(x')e^{i\xi' X'}$.

2.4.2 The linearized profile equations

In this section, we solve for $U = \mathbb{P}U$ the linear system

$$\mathbb{P}^i L_1(\partial_x)\mathbb{P}U = \mathbb{P}^i F, \quad F \in \mathcal{P} \tag{2.4.13}$$

together with boundary condition

$$TU_{|x_d=X_d=0} = TK, \quad K \in \mathcal{P}_b \tag{2.4.14}$$

and initial conditions

$$U_{|x_0=0} = \mathbb{P}H, \quad H \in \mathcal{P}_{|x_0=0}. \tag{2.4.15}$$

We decompose $\mathbb{P}U$ into $\sum U_{\xi'}(x, X_d)e^{i\xi' X'}$ and U_0 into $\underline{U}_0 + \tilde{U}_0$ with $\tilde{U}_0 = U^*_{0,os} + U_{0,ev}$. We use similar notations for F and H and decompose K into $K(x', X') = \sum K_{\xi'}(x')e^{i\xi' X'}$. Proposition 2.4.2 implies that the system above splits into

$$\begin{cases} \tilde{U}_0 = \Pi^{22}\tilde{U}_0, \quad Y\tilde{U}_0 = \Pi^{22}\tilde{F}_0, \\ \tilde{U}_{0|x_0=0} = \Pi^{22}\tilde{H}_0, \end{cases} \tag{2.4.16}$$

where Π^{22} is the projector defined in Eq. (2.3.10),

$$\begin{cases} \underline{U}_0 = \underline{P}_0\underline{U}_0, \quad \underline{X}_0\underline{U}_0 = \underline{P}^i_0\underline{F}_0, \\ \underline{U}_{0|x_0=0} = \underline{P}_0\underline{H}_0, \quad T\underline{U}_{0|x_d=0} = TK_0, \end{cases} \tag{2.4.17}$$

and for $\xi' \in \Lambda' \setminus \{0\}$

$$\begin{cases} U_{\xi',os} = \mathbb{P}_{os}(\xi')U_{\xi',os}, \quad \mathbb{X}_{os}(\xi')U_{\xi',os} = \mathbb{P}_{os}(\xi')F_{\xi',os}, \\ U_{\xi',os|x_0=0} = \mathbb{P}_{os}(\xi')H_{\xi',os}, \\ TU_{\xi',os|x_d=X_d=0} + TU_{\xi',ev|x_d=X_d=0} = TK_{\xi'}, \\ U_{\xi',ev} = \mathbb{P}_{ev}(\xi')U_{\xi',ev}. \end{cases} \tag{2.4.18}$$

Note that there is no boundary condition in (2.4.16) since the boundary conditions only involve the first block of components (see (2.2.3)) so that $T\Pi^{22} = 0$.

We study the solvability of these equations in spaces of smooth functions.

- Because Y is tangent to the boundary, the oscillating and the evanescent modes of $\tilde{U}_0$ are determined from Eq. (2.4.16) by integration along the integral curves of Y.
- By Proposition 2.4.2, the equation in (2.4.17) is hyperbolic symmetric on the space $\ker L(0)$.

Lemma 2.4.4. *The boundary condition T is maximal dissipative for $\underline{X}_0$ on the space* $\ker L(0)$.

Démonstration. The boundary matrix of $\underline{X}_0$ is ${}^t\underline{P}_0 A_d \underline{P}_0 = A_d$. Since T and A_d act only on the first components, Assumption 2.2.1 implies that T is also maximal dissipative w.r.t A_d on $\ker L(0)$. □

Therefore, the initial boundary value problem for Eq. (2.4.17) has a unique solution in $C^0(L^2)$ for data in L^2 (see [17, 42]). When the data are smooth, for the solution to be smooth compatibility conditions at the edge $\{x_0 = x_d = 0\}$ are necessary and sufficient (see [43, 19]). We recall briefly how these conditions are determined. Write the equation as :

$$\partial_{x_0}\mathcal{U} + \mathcal{A}\mathcal{U} = \mathcal{F}, \quad \mathcal{U}_{|x_0=0} = \mathcal{H}, \quad T\mathcal{U}_{|x_d=0} = \mathcal{K}. \tag{2.4.19}$$

From the equation and the initial condition, one determines the Taylor expansion of the solution at $\{x_0 = 0\}$: for $k = 0$ set $\Phi_0(\mathcal{H}) = \mathcal{H}$ and for $k \geq 1$, let

$$\partial_t^k \mathcal{U}_{|x_0=0} = \sum_{j=0}^{k-1} (-\mathcal{A})^{k-j} \partial_t^j \mathcal{F}_{|x_0=0} + (-\mathcal{A})^k \mathcal{H} := \Phi_k(\mathcal{H}, \mathcal{F}). \tag{2.4.20}$$

The compatibility conditions read

$$T\Phi_k(\mathcal{H}, \mathcal{F})_{|x_d=0} = \partial_{x_0}^k \mathcal{K}_{|x_0=0}. \tag{2.4.21}$$

They are automatically satisfied if the data vanish on a neighborhood of the corner $\{x_d = x_0 = 0\}$. They are satisfied up to order k if $\mathcal{H}$ and $\mathcal{K}$ [resp. $\mathcal{F}$] vanish at order k [resp. $k-1$] at the corner.

Consider next the system Eq. (2.4.18) when $\xi' \in \Lambda' \setminus \{0\}$. Let $\Xi_d(\xi')$ denote the finite set of real ξ_d such that $(\xi', \xi_d) \in \mathrm{char} L$. The polarization condition implies that

$$U_{\xi'}(x, X_d) = \sum_{\Xi_d(\xi')} U_{(\xi',\xi_d)}(x) e^{i\xi_d X_d} + U_{\xi',ev}(x, X_d). \tag{2.4.22}$$

The propagation equations decouple into a system of transport equations for the U_{ξ',ξ_d} :

$$\begin{cases} U_{(\xi',\xi_d)} = \pi(\xi', \xi_d) U_{(\xi',\xi_d)} \\ \partial_t + \mathbf{v}_{(\xi',\xi_d)} U_{(\xi',\xi_d)} = \pi(\xi', \xi_d) F_{(\xi',\xi_d)} \ , \qquad \xi_d \in \Xi_d(\xi'). \\ U_{(\xi',\xi_d)|x_0=0} = \pi(\xi', \xi_d) H_{(\xi',\xi_d)} \end{cases} \tag{2.4.23}$$

The evanescent term $U_{\xi',ev}$ appears only in the boundary condition and the polarization condition $U_{\xi',ev} = \mathbb{P}_{ev}(\xi') U_{\xi',ev}$. This constraint is equivalent to

$$U_{\xi',ev}(x, X_d) = \begin{pmatrix} \mathrm{Id} \\ -\{L^{22}(\xi')\}^{-1} L^{21}(\xi') \end{pmatrix} e^{-X_d G(\xi')} V(x),$$

with $V(x)$ taking its values in $\mathbb{E}^-_{E,G}(\xi')$ (see (2.3.6)). Equivalently, this shows that $U_{\xi',ev}(x, X_d)$ is uniquely determined by an arbitrary data

$$U_{\xi',ev}(x, 0) \in \mathbb{E}^-_{E,L}(\xi'). \tag{2.4.24}$$

Therefore, the boundary conditions for Eq. (2.4.23) read

$$TU_{\xi',os|x_d=X_d=0} \in TK_{\xi'} + T\mathbb{E}^-_{E,L}(\xi'). \tag{2.4.25}$$

By Assumption 2.2.5, the transport field $\partial_t + \mathbf{v}_\xi \partial_{x''}$ are never tangent to the boundary : the normal speed $\mathbf{v}_{\xi,d} \neq 0$. They are incoming or outgoing according to the classification of (ξ', ξ_d). Accordingly, we split $\Xi_d(\xi')$ into $\Xi_{in}(\xi') \cup \Xi_{out}(\xi')$ and

$$U_{\xi',os}(x, X_d) = U_{\xi',in}(x, X_d) + U_{\xi',out}(x, X_d).$$

The polarization condition and (2.2.14) imply that

$$U_{\xi',in} \in \bigoplus_{\xi_d \in \Xi_{in}(\xi')} \ker L(i\xi', i\xi_d) = \mathbb{E}^-_{H,L}(\xi'). \tag{2.4.26}$$

Assumption 2.2.6 implies the following :

Lemma 2.4.5. *For $\xi' \in \Lambda' \setminus \{0\}$, there are matrices $S_{(\xi',\xi_d)}$ for $\xi_d \in \Xi_{in}(\xi')$ and $S_{ev}(\xi')$ such that the equations*

$$TU_{in} + TU_{ev} = TK, \quad U_{in} \in \mathbb{E}^-_{H,L},\ U_{ev} \in \mathbb{E}^-_{E,L} \tag{2.4.27}$$

are equivalent to

$$U_{(\xi',\xi_d)} = S_{(\xi',\xi_d)}K \ \textit{for}\ \xi_d \in \Xi_{in}(\xi'), \quad \textit{and}\ \ U_{ev} = S_{ev}K. \tag{2.4.28}$$

The strategy to solve Eq. (2.4.18) is now clear.

– First one solves the transport equations Eq. (2.4.23) for the outgoing modes $\xi = (\xi', \xi_d) \in \mathcal{O}$, :

$$U_\xi(x) = \pi(\xi) H_\xi(x'' - \mathbf{v}_\xi x_0) + \int_0^{x_0} \pi(\xi) F_\xi(s, x'' - \mathbf{v}_\xi(s_0 - s))ds. \tag{2.4.29}$$

No boundary condition is needed since the vector field is outgoing. Moreover, if the data are smooth, the solutions are smooth. This determines completely $U_{\xi',out}$.

– Knowing the boundary value of $U_{\xi',out|x_d=X_d=0}$, the boundary condition and Lemma 2.4.5 uniquely determine

$$\begin{aligned} &U_{\xi',ev|x_d=X_d=0} \in \mathbb{E}^-_{E,L}, \\ &U_{(\xi',\xi_d)|x_d=0} \in \ker L(i\xi', i\xi_d) \quad \textit{for}\ \xi_d \in \Xi_{in}(\xi'). \end{aligned} \tag{2.4.30}$$

– For $\xi = (\xi', \xi_d) \in \mathcal{I}$, U_ξ satisfies a mixed initial boundary value scalar problem. When $x_d > \mathrm{v}_{\xi,d} x_0$ the solution is given by (2.4.29) and when $x_d < \mathrm{v}_{\xi,d} x_0$ by

$$U_\xi(x' - \mathbf{w}_\xi x_d, 0) + \frac{1}{\mathbf{v}_{\xi,d}} \int_0^{x_d} \pi(\xi) F_\xi(x' - \mathbf{w}_\xi(x_d - s, s)ds \quad z \leq v_\zeta t \tag{2.4.31}$$

where $\mathbf{w}_\xi = \frac{1}{\mathbf{v}_{\xi,d}}(1, \mathbf{v}_{\xi,1}, \ldots, , \mathbf{v}_{\xi,d-1})$. To be smooth, the initial datum must satisfy the compatibility conditions, which can be written down in the spirit of (2.4.20). They are satisfied if the data are flat at the corner.

– Knowing $U_{\xi',ev|x_d=X_d=0}$ from step ii), we extend it arbitrarily to $x_d > 0$, for instance one can choose

$$U_{\xi',ev}(x', x_d, X', 0) = \varphi(x_d) U_{\xi',ev}(x', 0, X', 0) \tag{2.4.32}$$

with $\varphi \in C^\infty_0(\mathbb{R})$ such that $\varphi(0) = 1$. This determines $U_{\xi',ev}$.

Remark 2.4.6. There is a lack of uniqueness in the fourth step. This corresponds to a lack of uniqueness in the representation of boundary layers as $\sum \varepsilon^n U^n(x', x_d, x_d/\varepsilon)$ since Taylor expansions of the profiles in x_d yields other expansion. Uniqueness would be restored if one would choose the description $\sum \varepsilon^n V^n(x', x_d/\varepsilon)$ for the layers (see [40]).

Summing up, we have proved the following

Proposition 2.4.7. *Given profiles $F \in \mathcal{P}([0,t_*]\times\mathbb{R}^d_+)$, $H \in \mathcal{P}_{|x_0=0}(\mathbb{R}^d_+)$ and $K \in \mathcal{P}_b([0,t_*]\times\mathbb{R}^{d-1})$, compatible at the corner, for instance infinitely flat at $\{x_0 = x_d = 0\}$, the problem* (2.4.13), (2.4.14), (2.4.14) *has solutions $U = \mathbb{P}U \in \mathcal{P}([0,t_*]\times\mathbb{R}^d_+)$.*

Moreover, the spectrum of $\tilde{U}_0$ is contained in the union of spectra of $\tilde{F}_0$ and $\tilde{H}_0$; if the spectrum of $\mathcal{K}$ is contained in Λ_1' and the spectra of F and H are contained in $\Lambda_1'\times\mathbb{R}$, then the spectrum of $U - \tilde{U}_0$ is contained in $\Lambda_1 = (\Lambda_1'\times\mathbb{R}) \cap (\mathrm{char}L)^\sharp$.

To solve the profile equations, we use iterative schemes and need estimates for the solutions. We use the Sobolev spaces H^m, the spaces with tangential regularity H^m_{tan}, the anisotropic spaces E^m_{sp} and their time dependent versions T^m and E^m introduced in Section 2.2.2. We denote by $T^m([0,t_*])$ and $E^m([0,t_*])$, the spaces defined on $[0,t_*]$. We use the following notations

$$\begin{aligned} \|V(t)\|_{T^m} &:= \sum_{j\le m} \|\partial_{x_0}^j V(t,\cdot)\|_{H^{m-j}_{tan}}, \\ \|V(t)\|_{E^m} &:= \sum_{j\le m} \|\partial_{x_0}^j V(t,\cdot)\|_{E^{m-j}_{sp}} = \sum_{2k+|\alpha|\le m} \|\partial_{x'}^\alpha \partial_{x_d}^k V(t,\cdot)\|_{L^2}. \end{aligned} \tag{2.4.33}$$

Proposition 2.4.8. *For all even $m \in \mathbb{N}$, there is a constant C such that for all smooth $\underline{U}_0$ satisfying* (2.4.17) *with $K = 0$, there holds for $t \in [0,1]$:*

$$\|\underline{U}_0(t)\|_{E^m} \le C\|\underline{U}_0(0)\|_{E^m} + C\int_0^t \|\underline{F}_0(s)\|_{E^m} ds \tag{2.4.34}$$

Démonstration. We follow [19]. Since the problem is maximal dissipative, one has the usual L^2 estimates

$$\|\underline{U}_0(t)\|_{L^2} \le C\|\underline{U}_0(0)\|_{L^2} + C\int_0^t \|\underline{F}_0(s)\|_{L^2} ds.$$

Differentiating tangentially the equation yields the estimates :

$$\|\underline{U}_0(t)\|_{T^m} \le C\|\underline{U}_0(0)\|_{T^m} + C\int_0^t \|\underline{F}_0(s)\|_{T^m} ds.$$

To get the normal derivatives, one uses the equation. We have already noticed that the boundary matrix in $\underline{X}_0$ is A_d. Next we compute the lower right block. Using the explicit form of $\underline{\mathbb{P}}_0$ we get

$$\underline{X}_0^{22}(\partial_{x'}) = \pi^{22}(0) L^{22}(\partial_{x'}) \pi^{22}(0)$$

which already appears in $\Pi^{22} L(\partial_x)\Pi^{22}$. Thus $\underline{X}_0$ has the following structure :

$$\underline{X}_0 = \begin{pmatrix} A_d^{11}\partial_{x_d} + L^{11}(\partial_x') & \underline{X}_0^{12}(\partial_x') \\ \underline{X}_0^{21}(\partial_x') & Y(\partial_x')\pi^{22}(0) \end{pmatrix}.$$

The first equation immediately implies

$$\|\partial_{x_d}(\underline{U}_0)_1(t)\|_{T^{m-1}} \leq \|\underline{U}_0(t)\|_{T^m} + \|\underline{F}_0(t)\|_{T^{m-1}}.$$

Differentiating the second equation yields

$$Y\partial_{x_d}(\underline{U}_0)_2 = \underline{X}_0^{21}(\partial'_x)\partial_{x_d}(\underline{U}_0)_1 + \partial_{x_d}(\underline{F}_0)_2.$$

This is a transport equation for $\partial_{x_d}(\underline{U}_0)_2$, therefore

$$\|\partial_{x_d}(\underline{U}_0)_2(t)\|_{T^{m-2}} \leq C\Big(\|\partial_{x_d}(\underline{U}_0)_2(0)\|_{T^{m-2}} + \int_0^t \big(\|\partial_{x_d}(\underline{U}_0)_1(s)\|_{T^{m-1}} + \|\partial_{x_d}(\underline{F}_0)_2(s)\|_{T^{m-2}}\big)ds\Big).$$

these estimates imply

$$\|\partial_{x_d}\underline{U}_0(t)\|_{T^{m-2}} \leq C\Big(\|\partial_{x_d}\underline{U}_0(0)\|_{T^{m-2}} + \int_0^t \|\underline{U}_0(s)\|_{T^m}ds \int_0^t \big(\|\underline{F}_0(s)\|_{T^{m-1}} + \|\partial_{x_d}\underline{F}_0(s)\|_{T^{m-2}}\big)ds\Big).$$

Differentiating the equation with respect to x_d the estimate (2.4.34) follows by induction. □

Next we consider the system (2.4.23) for $U_{\xi',os} = \sum U_{\xi',\xi_d} e^{i\xi_d X_d}$. Given a space X of functions of slow variables we use here the notations

$$\|U_{\xi',os}\|_X = \sum_{\xi_d \in \Xi_d(\xi')} \|U_{\xi',\xi_d}\|_X. \tag{2.4.35}$$

Proposition 2.4.9. *For $\xi' \neq 0$ in Λ' and all even $m \in \mathbb{N}$, there is a constant C such that for all smooth $\underline{U}_{\xi',os}$ satisfying Eq. (2.4.23) and Eq. (2.4.25) with $K_{\xi'} = 0$, there holds for $t \leq 1$:*

$$\|\underline{U}_{\xi'}(t)\|_{E^m} \leq C\sum_{j\leq m} \|\partial_{x_0}^j \underline{U}_{\xi'}(0)\|_{H^{m-j}} + C\int_0^t \|\underline{F}_{\xi'}(s)\|_{E^m}ds. \tag{2.4.36}$$

The proof is similar and is indeed simpler since the system is non characteristic. However, since we will couple the two kinds of systems, we give a common estimate in anisotropic spaces.

2.4.3 Construction of the leading profile

We now proceed to the proof of Theorem 2.2.15. We suppose that the finite sets Λ'_0 and Λ_0 satisfy Assumption 2.2.9. The initial data H^0 is given in $\mathcal{P}_{|x_0=0}$ with spectrum contained in Λ_0 and vanishes at infinite order at $x_d = 0$. With notations as above, we first note that the profile equations Eq. (2.4.6) for U^0 read

$$\tilde{U}_0^0 = \Pi^{22}\tilde{U}_0^0, \quad Y\tilde{U}_0^0 = 0, \quad \tilde{U}_{0\,|x_0=0}^0 = \Pi^{22}\tilde{H}_0^0, \tag{2.4.37}$$

$$\begin{cases} \underline{U}^0_0 = \underline{P}_0 \underline{U}^0_0, \quad \underline{X}_0 \underline{U}^0_0 = \underline{P}^i_0 \underline{F}^0_0, \\ \underline{U}^0_{0|x_0=0} = \underline{P}_0 \underline{H}^0_0, \quad T\underline{U}_{0|x_d=0} = 0, \end{cases} \tag{2.4.38}$$

and for $\xi' \in \Lambda' \setminus \{0\}$

$$\begin{cases} U^0_{\xi',os} = \mathbb{P}_{os}(\xi') U^0_{\xi',os}, \quad \mathbb{X}_{os}(\xi') U^0_{\xi',os} = \mathbb{P}_{os}(\xi') F^0_{\xi',os}, \\ U^0_{\xi',os|x_0=0} = \mathbb{P}_{os}(\xi') H^0_{\xi',os}, \\ TU^0_{\xi',os|x_d=X_d=0} \in \mathbb{E}^-_{E,L}(\xi'), \end{cases} \tag{2.4.39}$$

and

$$U^0_{\xi',ev} = \mathbb{P}_{ev}(\xi') U^0_{\xi',ev}, \quad TU^0_{\xi',ev|x_d=X_d=0} = -TU^0_{\xi',os|x_d=X_d=0}, \tag{2.4.40}$$

with $F^0 = f(U^0)$.

- The right hand side in Eq. (2.4.37) vanishes thanks to Assumption 2.2.8. In particular, this equation uniquely determines the oscillating part $\tilde{U}^0_{0,os}$ and evanescent part $U^0_{0,ev}$ of $\tilde{U}^0_0$ and for all time x_0. The spectrum of $\tilde{U}^0_{0,os}$ is contained in the spectrum of $\tilde{H}^0_0$ and $\tilde{U}^0_0$ vanishes at infinite order at the corner $\{x_0 = x_d = 0\}$.
- Next we solve the nonlinear systems (2.4.38) and (2.4.39) for all ξ', for the unknowns $V = U^0_{os} - \tilde{U}^0_{0,os}$. These systems are coupled through their right hand sides

$$\mathbb{P}_{os} F^0 = \mathbb{P}_{os} f(U^0) = \mathbb{P}_{os} f(U^0_{os}) = \mathbb{P}_{os} f(V + \tilde{U}^0_{0,os}).$$

  The spectrum of V is contained in $\Lambda_0^\sharp := \Lambda_0 \cap (\mathrm{char} L)^\sharp$:

$$V = \sum_{\xi \in \Lambda_0^\sharp} V_\xi(x) e^{i\xi X}.$$

  By Lemma 2.4.1, for such V, the spectrum of $f(V + \tilde{U}_{0,os})$ is contained in $\Lambda_0^\sharp$. Therefore, (2.4.38) (2.4.39) form a finite dimensional system for V of the form

$$\mathcal{L}V = F(V + W), \quad V_{|x_0=0} = V_0, \quad \mathcal{T} V_{|x_d=0} = 0, \tag{2.4.41}$$

  where $W \in H^\infty([0,1] \times \mathbb{R}^d_+)$ and $V_0 \in H^\infty(\mathbb{R}^d_+)$ are given and flat at the corner, and F is polynomial with $F(0) = 0$.

Lemma 2.4.10. *$\mathcal{L}$ is hyperbolic symmetric and the boundary conditions $\mathcal{T}$ are maximal dissipative.*

Démonstration. The system $(\mathcal{L}, \mathcal{T})$ is diagonal for $\underline{V}_0$ and $V_{\xi',os}$. The system for $\underline{V}_0$ is symmetric and maximal dissipative by Lemma 2.4.4. The system for $V_{\xi',os}$ is a diagonal system of non tangential transport equations for the components V_{ξ',ξ_d}, which are split into two groups, the incoming ones and the outgoing ones. The boundary matrix is diagonal, with positive [resp. negative] entries for the incoming [resp. outgoing] components. By Lemma 2.4.5, the boundary conditions read $V_{\xi',in} = SV_{\xi',out}$. Thus there are diagonal symmetrizers which make the boundary condition strictly maximal dissipative. □

Proposition 2.4.11. *For all data $H^0 \in \mathcal{P}_{|x_0=0}$ with spectrum contained in Λ_0 and vanishing at infinite order at $x_d = 0$, there is $t_* > 0$ such that the initial boundary value problem for V has a unique solution $V \in \mathcal{P}_{os}([0,t_*])$ with spectrum contained in $\Lambda_0^\sharp$.*

Démonstration. This theorem follows from the results of [19], adapted to the semi-linear case. For the convenience of the reader, we recall the main steps of the proof.

1. *Reduction to vanishing initial data.* We fix $m > \frac{d+1}{2}$. We determine the Taylor expansion of the solution V at $t = x_0 = 0$ using the equation. This determines $V_j = \partial_t^j V_{|t=0}$ as a polynomial function of V_0 and W and their derivatives. Thus $W_j \in H^\infty(\mathbb{R}^d_+)$ and V_j is flat at the corner. Thus $V^a = \sum_{j=1}^m \frac{t^j}{j!} V_j \in H^\infty([0,1]\times\mathbb{R}^d_+)$ and is flat at the boundary. In particular $\mathcal{T} V^a{}_{|x_d=0} = 0$. We look for the solution as $V = V^a + V^r$. The equation for V^r is similar to Eq. (2.4.41) with vanishing initial data and W replaced by $W + V^a$. Thus, it is sufficient to solve this equation when $W \in H^\infty([0,1]\times\mathbb{R}^d_+)$ is flat at the corner and

$$V_0 = 0 \quad \text{and} \quad \partial_t^j f(W)_{|t=0} = 0 \quad \textit{for } j < m. \tag{2.4.42}$$

2. *Local existence in spaces E^m.* We assume that (2.4.42) is satisfied.

One uses an iterative scheme to determine V :

$$\mathcal{L} V^{n+1} = f(V^n + W), \quad V^{n+1}{}_{|x_0=0} = 0, \quad \mathcal{T} V^{n+1}{}_{|x_d=0} = 0. \tag{2.4.43}$$

One initializes the scheme by $V^0 = 0$.

By Proposition 2.4.7 there is a sequence V^n with (coefficients of) $V^n \in H^\infty([0,1]\times\mathbb{R}^d_+$ and flat at the corner. Moreover, by induction, one shows that $\partial_t^j V^n{}_{|t=0} = 0$ for $j \le m$. By Propositions 2.4.8 and 2.4.9 they satisfy the following estimates :

$$\|V^{n+1}(t)\|_{E^m} \le C \int_0^t \|f(V^n + W)(s)\|_{E^m} ds. \tag{2.4.44}$$

Next we use nonlinear estimates. For $t > 0$ and m an even integer, denote by $\tilde{F}^m(t)$ the space of functions on $]-\infty, t]\times\mathbb{R}^d_+$ such that $\partial_{x'}^{\alpha'}\partial_{x_d}^{\alpha_d} u \in L^2(]-\infty,t]\times\mathbb{R}^d_+)$ for all $|\alpha'| + 2\alpha_d \le m$. Introduce next the space $F^m(t)$ of functions on $[0,t]\times\mathbb{R}^d_+$ whose extension by 0 for $t < 0$ belongs to $\tilde{F}^m(t)$. This space is equipped with the obvious norm, which reads

$$\|u\|_{F^m(t)} = \Big(\int_0^t \|u(s)\|^2_{E^m} ds \Big)^{\frac{1}{2}}. \tag{2.4.45}$$

Gagliardo-Nirenberg estimates are valid in the space $\tilde{F}(0)$, thus in $\tilde{F}(t)$ by translation in time, with constants independent of t (see [19]). In particular, writing $f(W+V) = f(W) + g(W,V)V$ and noticing that $f(W) \in F(1)$ by (2.4.42), we see that there is a constant C_0 such that for all $t \in]0,1]$ and all $V \in F^m(t)$:

$$\|f(W+V)\|_{F^m(t)} \le C_0 + C\big(\|V\|_{L^\infty}\big)\|V\|_{F^m(t)} \tag{2.4.46}$$

where $C(\cdot)$ is some function on $\mathbb{R}$. Moreover, the Sobolev embedding $E^m_{sp}(\mathbb{R}^d_+) \subset L^\infty(\mathbb{R}^d_+)$ is valid for m even, $m > \frac{d+1}{2}$ (see [19] and note that the quasi-homogeneous dimension of $\mathbb{R}^d$ associated to E^m_{sp} is $d+1$). Therefore

$$\|V(t)\|_{L^\infty(\mathbb{R}^d_+)} \le C_1 \|V(t)\|_{E^m_{sp}(\mathbb{R}^d_+)} \le C_1 \|V(t)\|_{E^m}. \tag{2.4.47}$$

Since the V^n vanish at order m on $x_0 = 0$, they belong to $F^m(1)$. Moreover, by (2.4.45), its norm in $F^m(t)$ is bounded by $\sqrt{t}$ its norm in E^m. Adding up, we see that for all $t \in]0,1]$ and all n

$$\|V^{n+1}\|_{E^m(t)} \leq C_2 + t\tilde{C}\big(\|V^n\|_{E^m(t)}\big).$$

Thus, for $t_* > 0$ small enough, the sequence V^n is bounded by $2C_2$ in $E^m(t_*)$, thus in $L^\infty([0,t_*] \times \mathbb{R}^d_+)$.

Similarly, writing the equation for the difference $V^{n+1} - V^n$ and using the L^∞ bounds, yield for all $t \in]0,t_*]$ and all n

$$\|V^{n+1} - V^n\|_{E^m(t)} \leq C_3 t\big(\|V^n - V^{n-1}\|_{E^m(t)}\big).$$

Thus, for t small enough, the scheme is a contraction and V^n converges in $E^m(t)$. The limit is solution to the equation.

Uniqueness of solutions in E^m follows from the L^2 energy estimate applied to the difference of two solutions.

3. *Propagation of smoothness.* Fix $m_0 > \frac{d+1}{2}$ even. There is $t_* > 0$ and a solution $V \in E^{m_0}(t_*)$. We prove that $V \in E^m(t_*)$ for all even $m > m_0$. As in step 2, we look for V as $V = V^a + V^r$ and it is sufficient to prove that $V^r \in E^m(t_*)$. By step 2, and local uniqueness, we know that $V^r \in E^m(t)$ for t small, possibly depending on m. To extend the smoothness, it is sufficient to prove that the norm of V^r in $E^m(t)$ is bounded independently of $t \leq t_*$. But, from step 2 and (2.4.47) we have an L^∞ estimate of V^r up to time t_*. Therefore, using the energy estimate for $\mathcal{L}$, the Gagliardo-Nirenberg estimate (2.4.46) and squaring , yields

$$\begin{aligned}
\|V^r(t)\|^2_{E^m} &\leq C + C\Big(\int_0^t \|f(W + V^a + V^r)(s)\|_{E^m} ds\Big)^2 \\
&\leq C + Ct \int_0^t \|f(W + V^a + V^r)(s)\|^2_{E^m} ds \\
&\leq \tilde{C}(t) + \tilde{C}'(t) \int_0^t \|V^r(s)\|^2_{E^m} ds.
\end{aligned}$$

Gronwall's lemma implies the desired uniform estimate of $\|V^r(t)\|_{E^m}$ and the proof of Proposition 2.4.11 is complete. □

• To finish the proof of Theorem 2.2.15, it remains to determine the components $U^0_{\xi',ev}$ such that (2.4.40) is satisfied. When $\xi' \notin \Lambda'_0$, $U^0_{\xi',ev} = 0$, since the source term vanishes. When $\xi' \in \Lambda'_0$, the source term $-TU^0_{\xi',os} x_d = X_d = 0$ takes its values in the space $\mathbb{E}^-_{H,L}(\xi')$ by (2.4.39). As in the linear case, together with the polarization condition, this determines uniquely $U^0_{\xi',ev}(x', 0, X_d) \in \mathcal{P}^z_{ev}$, and we can extend its definition arbitrarily to $x_d > 0$, using (2.4.32) for instance.

2.4.4 Higher order profiles

In this paragraph, we prove Theorem 2.2.16, solving the sequence of equations Eq. (2.4.7). We proceed by induction, showing that $U^j \in \mathcal{P}_{fin}([0,t_*])$, the set of profiles in $\mathcal{P}([0,t_*])$ with tangential spectrum contained in a multiple $k * \Lambda'_0$ of Λ'_0 with k odd if f is odd (we use here the notations (2.2.22)). Moreover, U^j is flat at the corner.

In this case, F^{j-1} is a polynomial function of $(U^0, \dots, U^{j-1})$ and $L_1(\partial_x)U^{j-1}$ belongs to $\mathcal{P}_{fin}$. Thus, the first equation determines explicitly $(1-\mathbb{P})U^j \in \mathcal{P}_{fin}([0,t_*])$. Substituting in the second equation and using (2.4.3), we see that the equation for $\mathbb{P}U^j$ reads

$$\mathbb{P}^i L_1(\partial_x)\mathbb{P}U^j = \mathbb{P}^i(f'(U^0)\mathbb{P}U^j) + \mathbb{P}^j F \tag{2.4.48}$$

where $F \in \mathcal{P}_{fin}([0,t_*])$ is known. Similarly, the boundary condition reads

$$T\mathbb{P}U^j{}_{|x_d=X_d=0} = TK \tag{2.4.49}$$

where $-K \in \mathcal{P}_b$ is the trace of $(1-\mathbb{P})U^j$ which is known. Its spectrum is contained in a multiple $k * \Lambda_0'$. By assumption, we are also given the initial values

$$\mathbb{P}U^j{}_{|x_0=0} = \mathbb{P}H^j \tag{2.4.50}$$

with $H^j \in \mathcal{P}_{fin}|_{x_0=0}$ given. For simplicity, we now drop the superscript j. This is a linear system for $U = \mathbb{P}U^j$ and all the data vanish at infinite order at the corner.

The equation Eq. (2.4.48) is similar to Eq. (2.4.13) except for the coupling term $\mathbb{P}^i f'(U_0)U$. However, by Assumption 2.2.8 this term does not contribute to the frequencies $\xi' = 0$ except for the mean term. Therefore, with notations as in (2.4.16), the equation for $\tilde{U}_0$ reads

$$\tilde{U}_0 = \Pi^{22}\tilde{U}_0, \quad Y(\partial_x')\tilde{U}_0 = \Pi^{22}\tilde{F}_0, \quad \tilde{U}_{0|x_0=0} = \Pi^{22}\tilde{H}_0.$$

This uniquely determines $\tilde{U}_0$.

Next, we solve the system for $V = U - \tilde{U}_0$ that is for $\underline{U}_0$ and the $U_{\xi',os}$:

$$\begin{cases} \underline{U}_0 = \underline{P}_0\underline{U}_0, \quad \underline{X}_0\underline{U}_0 = \underline{P}_0^i(\underline{\Phi}_0 + \underline{\hat{F}}_0), \\ \underline{U}_{0|x_0=0} = \underline{P}_0\underline{H}_0, \quad T\underline{U}_{0|x_d=0} = TK_0, \end{cases} \tag{2.4.51}$$

and for $\xi' \neq 0$

$$\begin{cases} U_{\xi',os} = \mathbb{P}_{os}(\xi')U_{\xi',os}, \quad \mathbb{X}_{os}(\xi')U_{\xi',os} = \mathbb{P}_{os}(\xi')(\Phi_{\xi',os} + \hat{F}^0_{\xi',os}), \\ U_{\xi',os|x_0=0} = \mathbb{P}_{os}(\xi')H_{\xi',os}, \\ TU_{\xi',os|x_d=X_d=0} - TK_{\xi'} \in \mathbb{E}^-_{E,L}(\xi'), \end{cases} \tag{2.4.52}$$

with $\Phi = f'(U_0)V$ and $\hat{F} = F + f'(U_0)\tilde{U}_0$. This system is easily solved by usual iterations, as in the previous section, as soon as it is noticed that it involves only a finite number of frequencies ξ'. The tangential spectra of the source terms and data, K, H, $\hat{F}$ are finite and contained in a multiple $k * \Lambda_0'$, with k odd if f is odd. By Assumption 2.2.9, these spectra are contained in a set Λ_1' which satisfies (2.2.26). Therefore, if the spectrum of V is contained in Λ_1, the spectrum of $\mathbb{P}(f'(U_0)V)$ is also contained in Λ_1. Therefore, the Eq. (2.4.51) Eq. (2.4.52) form a finite dimensional system for $\{\underline{V}_0,\ V_{\xi'}, \xi' \in \Lambda_1'\}$, which admits a unique solution in $\mathcal{P}([0,t_*])$ with spectrum contained in Λ_1'.

It remains to solve

$$U_{\xi',ev} = \mathbb{P}_{ev}(\xi')U_{\xi',ev}, \quad TU_{\xi',ev|x_d=X_d=0} = TK_{\xi'} - TU_{\xi',os|x_d=X_d=0}, \tag{2.4.53}$$

for ξ' in a finite set. Solutions are constructed as for Proposition 2.4.7. This finishes the construction of $\mathbb{P}U^j \in \mathcal{P}_{fin}([0,t_*])$ and the proof of Theorem 2.2.16.

2.5 Convergence

In this section, we assume that the profiles $U_j \in \mathcal{P}([0,t_*])$ are given, flat at the corner, and solve the profile equations. Let u^ε_{app} denote the approximate solution (2.2.51) at order n. We show that it is an approximation at order ε^M of an exact solution of $u^\varepsilon = u^\varepsilon_{app} + \varepsilon^M v^\varepsilon$ of (2.1.10) for times in $[0,t_*]$, proving Theorem 2.2.17.

The truncated WKB solution $\sum_{j\le n} \varepsilon^j U^j$ satisfies the equations Eq. (2.2.37) up to order n. Therefore, plugging u^ε_{app} in the equation, we see that all the terms in the right hand side of (2.2.35) of order $\le n$ vanish. Therefore

$$L(\varepsilon\partial_x)u^\varepsilon_{app} - f(u^\varepsilon_{app}) = \varepsilon^{n+1} r^\varepsilon, \quad r^\varepsilon(x) = R(x, \frac{x}{\varepsilon}). \tag{2.5.1}$$

with $R \in \mathcal{P}([0,t_*])$. Moreover, by construction there holds

$$(Tu^\varepsilon_{app})_{|x_d=0} = 0. \tag{2.5.2}$$

The equation for the remainder v^ε reads

$$\begin{cases} L(\varepsilon\partial_x)v^\varepsilon = \varepsilon\Big(\varepsilon^{n+1-M} r^\varepsilon + F(u^\varepsilon_{app}, \varepsilon^M v^\varepsilon)v^\varepsilon\Big), \\ Tv^\varepsilon{}_{|x_d=0} = 0, \end{cases} \tag{2.5.3}$$

with

$$F(u,v) = \int_0^1 f'(u+sv)ds.$$

The equation Eq. (2.5.3) is solved by iterations. We follow closely both [19] and [20]. Because the derivative of the coefficients are not bounded, we are led to introduce the weight ε in front of the derivatives as in [20]. However, because we are dealing with a dispersive equation, we carefully check that the analysis of characteristic boundary value problem made in [19] extends to our case (see also [13]).

2.5.1 Linear estimates

We study first the linearized equation :

$$L(\varepsilon\partial_x)u = \varepsilon f, \quad Tu_{|x_d=0} = 0, \quad u_{|x_0=0} = h. \tag{2.5.4}$$

Following [19, 42] this equation has a unique solution, which is smooth if f and h are smooth and satisfy the compatibility conditions at the corner, for instance if they are flat at the corner.

Since the system is symmetric and maximal dissipative, multiplying the equation by u and integrating by parts, yields the usual L^2 estimates

$$\|u(t)\|_{L^2} \le C\|h\|_{L^2} + C\int_0^t \|f(s)\|_{L^2} ds.$$

Next, one differentiates the equation in the tangential directions $x' = (x_0, \dots, x_{d-1})$, yielding immediately the estimates :

$$\|u(t)\|_{T^{m,\varepsilon}} \le C\|u(0)\|_{T^{m,\varepsilon}} + C\int_0^t \|f(s)\|_{T^{m,\varepsilon}} ds. \tag{2.5.5}$$

where we use the notations

$$\|u(t)\|_{T^{m,\varepsilon}} := \sum_{|\alpha'|\leq m} \varepsilon^{|\alpha'|}\ \|\partial_{x'}^{\alpha'} u(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}, \tag{2.5.6}$$

To get the normal derivatives, one uses the equation. We use the block decomposition of A_d and L. From the first equation we deduce that

$$\varepsilon A_d^{11}\partial_{x_d} u^1 = \varepsilon f^1 - L^{11}(\varepsilon\partial_{x'})u^1 - L^{12}(\varepsilon\partial_{x'})u^2. \tag{2.5.7}$$

Therefore,

$$\|\varepsilon\partial_{x_d} u^1(t)\|_{T^{m-1,\varepsilon}} \leq C\big(\|\varepsilon f^1(t)\|_{T^{m-1,\varepsilon}} + \|u(t)\|_{T^{m,\varepsilon}}\big). \tag{2.5.8}$$

The second equation reads

$$L^{22}(\varepsilon\partial_{x'})u^2 = \varepsilon f^2 - L^{21}(\varepsilon\partial_{x'})u^1, \tag{2.5.9}$$

implying that

$$L^{22}(\varepsilon\partial_{x'})(\varepsilon^2\partial_{x_d} u^2) = \varepsilon\Big(\varepsilon^2\partial_{x_d} f^2 - L^{21}(\varepsilon\partial_{x'})(\varepsilon\partial_{x_d} u^1)\Big). \tag{2.5.10}$$

The system $L^{22}(\varepsilon\partial_{x'})$ is symmetric hyperbolic, as a diagonal block of the original symmetric hyperbolic system. Moreover, it only involves the tangential directions. Therefore, an estimate similar to (2.5.5) holds for this system. It implies that

$$\begin{aligned}\varepsilon^2\|\partial_{x_d} u^2(t)\|_{T^{m-2,\varepsilon}} \leq C\Big(&\varepsilon^2\|\partial_{x_d} u^2(0)\|_{T^{m-2,\varepsilon}} \\ &+ \int_0^t \big(\varepsilon\|\partial_{x_d} u^1(s)\|_{T^{m-1,\varepsilon}} + \varepsilon^2\|\partial_{x_d} f^2(s)\|_{T^{m-2,\varepsilon}}\big)ds\Big).\end{aligned}$$

Substituting (2.5.8) in the estimate above yields

$$\begin{aligned}\varepsilon^2\|\partial_{x_d} u(t)\|_{T^{m-2,\varepsilon}} \leq C\Big(&\varepsilon^2\|\partial_{x_d} u(0)\|_{T^{m-2,\varepsilon}} + \int_0^t \|u(s)\|_{T^{m,\varepsilon}} ds \\ &\int_0^t \big(\varepsilon\|f(s)\|_{T^{m-1,\varepsilon}} + \varepsilon^2\|\partial_{x_d} f(s)\|_{T^{m-2,\varepsilon}}\big)ds\Big).\end{aligned}$$

Next we proceed by induction to estimate the higher order derivatives in x_d, differentiating the equations Eq. (2.5.7) and Eq. (2.5.10). Using the notations

$$\begin{aligned}\|u(t)\|_{E^{m,\varepsilon}} &:= \sum_{2k\leq m} \varepsilon^{2k}\|\partial_{x_d}^k u(t)\|_{T^{m-j,\varepsilon}} \\ &= \sum_{2\alpha_d+|\alpha'|\leq m} \varepsilon^{2\alpha_d+|\alpha'|}\|\partial_x^\alpha u(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}.\end{aligned} \tag{2.5.11}$$

we obtain the next proposition.

Proposition 2.5.1. *For all even $m \in \mathbb{N}$, there is a constant C such that for all smooth u satisfying* (2.5.4) *there holds for $t \in [0,1]$:*

$$\|u(t)\|_{E^{m,\varepsilon}} \leq C\|u(0)\|_{E^{m,\varepsilon}} + C\int_0^t \|f(s)\|_{E^{m,\varepsilon}} ds \tag{2.5.12}$$

Remark 2.5.2. The assumption that the boundary $\{x_d = 0\}$ is characteristic means that $\nu = (0, \dots, 0, 1)$ belongs to the characteristic variety of the principal term $L_1(\partial_x)$. If ν is a regular point of this variety, then the principal part in the block $L^{22}(\partial'_x)$ is a transport field. By restriction to $\ker L^{22}(0)$, it must be equal to the field $Y(\partial_{x'})$ introduced in Proposition 2.4.2. For Maxwell's equation, this is ∂_t. However, note that in general, the zero'th order part in $L^{22}(\varepsilon\partial_{x'})$, which is $L^{22}(0)$, is not diagonal.

2.5.2 Nonlinear estimates

The proof of Theorem 2.2.17 is based on the the estimates of the last proposition together with estimates for the nonlinear term $F(u^\varepsilon_{app}, \varepsilon^M v^\varepsilon)v^\varepsilon$. We need a Sobolev embedding estimate and Gagliardo-Nirenberg inequalities.

Consider the weighted norms on $E^m_{sp}(\mathbb{R}^d_+)$:

$$\|u\|_{E^{m,\varepsilon}_{sp}} = \sum_{2\alpha_d+|\alpha''|\le m} \varepsilon^{2\alpha_d+|\alpha''|} \|\partial^{\alpha''}_{(x_1,\dots,x_{d-1})} \partial^{\alpha_d}_{x_d} u\|_{L^2(\mathbb{R}^d_+)}. \tag{2.5.13}$$

The next lemma is a refinement of (2.4.47).

Lemma 2.5.3. *For $m > \frac{d+1}{2}$ even, there is a constant C such that for all $\varepsilon \in]0, 1]$ and all $u \in E^m_{sp}$*

$$\varepsilon^{\frac{d+1}{2}} \|u\|_{L^\infty} \le C \|u\|_{E^{m,\varepsilon}_{sp}}. \tag{2.5.14}$$

Démonstration. Using an extension operator, it is sufficient to prove a similar estimate on $\mathbb{R}^d$. For m even, the weighted norm is equivalent to

$$\|u\|_{\tilde{E}^{m,\varepsilon}_{sp}} = \|(1 + \varepsilon|\xi''| + \varepsilon\sqrt{|\xi_d|})^m \hat{u}\|_{L^2}$$

where $\hat{u}$ denotes the Fourier transform of u and $\xi'' = (\xi_1, \dots, \xi_{d-1})$. The L^∞ norm of u is dominated by the L^1 norm of $\hat{u}$ and the estimate follows from Cauchy-Schwartz inequality and the identity

$$\|(1 + \varepsilon|\xi''| + \varepsilon\sqrt{|\xi_d|})^{-m}\|_{L^2} = \varepsilon^{-\frac{d+1}{2}} \|(1 + |\xi''| + \sqrt{|\xi_d|})^m\|_{L^2}$$

the last norm being finite when $m > (d + 1)/2$. □

As in Section 2.4, we denote by $\tilde{F}^m(t)$ the space of functions on $]-\infty, t] \times \mathbb{R}^d_+$ such that $\partial^{\alpha'}_{x'} \partial^{\alpha_d}_{x_d} u \in L^2(]-\infty, t] \times \mathbb{R}^d_+)$ for $|\alpha'| + 2\alpha_d \le m$ together with the weighted norms

$$\|u\|_{\tilde{F}^{m,\varepsilon}(t)} = \sum_{2\alpha_d+|\alpha''|\le m} \varepsilon^{2\alpha_d+|\alpha''|} \|\partial^\alpha_x u\|_{L^2(]-\infty,t]\times\mathbb{R}^d_+)}. \tag{2.5.15}$$

Quasi-homogeneous Gagliardo Niremberg estimates are valid on $\mathbb{R} \times \mathbb{R}^d$ (see [19]) : for m even, there holds

$$\|\partial^\beta_x u\|_{L^p} \le C \|u\|^{1-\frac{2}{p}}_{L^\infty} \sum_{|\alpha'|+2\alpha_d=m} \|\partial^\alpha_x u\|^{\frac{2}{p}}_{L^2}, \quad \text{for } \frac{2}{p} = \frac{|\beta'| + 2\beta_d}{m} \le \frac{1}{2}. \tag{2.5.16}$$

Using classical extension operators, these estimates are also valid, on $]-\infty,t]\times\mathbb{R}^d_+$, with constants independent of t. Together with Hölder inequality for exponents p between 2 and the maximal value given in (2.5.16), this implies that

$$\varepsilon^{2\beta_d+|\beta''|}\|\partial_x^\beta u\|_{L^p}\le C\|u\|_{L^\infty}^{1-\frac{2}{p}}\ \|u\|_{F^{m,\varepsilon}}^{\frac{2}{p}},\quad \text{for } 1\le\frac{2}{p}\le\frac{|\beta'|+2\beta_d}{m}. \tag{2.5.17}$$

Lemma 2.5.4. *Suppose that G is a smooth function of its argument. For $m>\frac{d+1}{2}$ even, there is a function $C(\cdot)$ from $[0,+\infty[$ to $\mathbb{R}_+$ such that for all $\varepsilon\in]0,1]$ and all $t\in[0,t_*]$, there holds for all v and w in $\tilde F^m(t)$*

$$\|F(u^\varepsilon_{app},\varepsilon^M v)w\|_{\tilde F^{m,\varepsilon}(t)}\le C(\varepsilon^M\|v\|_{L^\infty})\\ \big(\|w\|_{\tilde F^{m,\varepsilon}(t)}+\varepsilon^M\|w\|_{L^\infty}(1+\|v\|_{\tilde F^{m,\varepsilon}(t)})\big)$$

Démonstration. Noticing that the derivatives $\varepsilon^{|\alpha|}\partial_x^\alpha u^\varepsilon_{app}$ are uniformly bounded and using the chain rule to compute derivatives, we see that it is sufficient to estimate the L^2 norms of terms

$$\varepsilon^{2\alpha_d+|\alpha'|}\varepsilon^{M(k-1)}\ \partial_x^{\alpha^1}v.\dots.\partial_x^{\alpha^{k-1}}v\ \partial_x^{\alpha^k}w$$

with $2\alpha_d+|\alpha'|=2\alpha_d^1+|\alpha^{1\prime}|+\ldots+2\alpha_d^k+|\alpha^{k\prime}|$, $k\ge1$ and $\alpha^j\neq0$ if $j<k$. Choosing the p^j such that $\frac{2}{p^j}\le\frac{|\alpha^{j\prime}|+2\alpha_d^j}{m}$ and $\sum\frac{2}{p^j}=1$, we bound the L^2 norm of this term by

$$C\varepsilon^{M(k-1)}\|v\|_{L^\infty}^{k-2+\frac{2}{p^k}}\|v\|_{F^{m,\varepsilon}}^{1-\frac{2}{p^k}}\|w\|_{L^\infty}^{1-\frac{2}{p^k}}\|w\|_{F^{m,\varepsilon}}^{\frac{2}{p^k}}.$$

In addition, if $k=1$, there are no v. Using Hölder inequality, the lemma follows. □

We also need products estimates on $H^m(\mathbb{R}^d_+)$, using the weighted norms :

$$\|h^\varepsilon\|_{H^m_\varepsilon(\mathbb{R}^d_+)}:=\sum_{|\alpha''|\le m}\varepsilon^{|\alpha''|}\|\partial_{x''}^{\alpha''}h\|_{L^2(\mathbb{R}^d_+)}.$$

Recall that $x''=(x_1,\dots,x_d)$ denote the spatial variables.

Lemma 2.5.5. *For $m>\frac{d}{2}$, the product $(h_1,h_1)\mapsto h_1h_2$ maps $H^{m-m_1}\times H^{m-m_2}$ to $H^{m-m_1-m_2}$ and there is C such that for all $\varepsilon\in]0,1]$*

$$\|h_1h_2\|_{H^{m-m_1-m_2}_\varepsilon(\mathbb{R}^d_+)}\ \le C\ \varepsilon^{-\frac{d}{2}}\ \|h_1\|_{H^{m-m_1}_\varepsilon(\mathbb{R}^d_+)}\ \|h_2\|_{H^{m-m_2}_\varepsilon(\mathbb{R}^d_+)}. \tag{2.5.18}$$

Démonstration. The result is classical when $\varepsilon=1$. Using the dilations $h_j(x'')=\varepsilon^{-\frac{d}{2}}\tilde h_j(x''/\varepsilon)$ reduces to this case. □

2.5.3 Proof of Theorem 2.2.17

We suppose that $n\ge M\ge m$ are given with $m>\frac{d+1}{2}$ even. We fix a C_0 and we consider the Cauchy problem for (2.5.3) with initial data $h\in H^m(\mathbb{R}^d_+)$, such that

$$\|h\|_{H^m_\varepsilon(\mathbb{R}^d_+)}\le C_0. \tag{2.5.19}$$

$$\partial_{x_d}^k h_{|x_d=0}=0,\quad \text{for } k\in\{0,\dots,m-1\}. \tag{2.5.20}$$

In a first step, we assume that

$$h \in C_0^\infty(\{x_d > 0\}). \tag{2.5.21}$$

We look for the solution v as the limit of a sequence v^p satisfying

$$\begin{cases} L(\varepsilon\partial_x)v^{p+1} = \varepsilon\Big(\varepsilon^{n+1-M} r^\varepsilon + F(u_{app}^\varepsilon, \varepsilon^M v^p)v^p\Big), \\ Tv^{p+1}{}_{|x_d=0} = 0 \\ v^{p+1}{}_{|t=0} = h. \end{cases} \tag{2.5.22}$$

Following [43], we construct the first term as follows. From the nonlinear equation, one computes the Taylor expansion of solutions of (2.5.3) in terms of the spatial derivatives of the initial data. Writing $L(\varepsilon\partial) = \varepsilon\partial_{x_0} - A(\varepsilon\partial_{x''})$ and denoting by $\Phi^\varepsilon(x,v)$ the right hand side of the equation, the $h_j = \varepsilon^j \partial_{x_0}^j v_{|x_0=0}$ are computed by induction

$$h_j = A^j(\varepsilon\partial_{x''})h + \sum_{k<j} A^{j-k-1}(\varepsilon\partial_{x''})\varepsilon^k \big(\partial_{x_0}^k \Phi(x,v)\big)_{|x_0=0} \tag{2.5.23}$$

and $\varepsilon^k \big(\partial_{x_0}^k \Phi(x,v)\big)_{|x_0=0}$ is a linear combination of terms

$$\varepsilon G(x'', \frac{x''}{\varepsilon}) h_{j_1} \dots h_{j_l}, \quad \text{with } j_1 + \dots + j_l \le k$$

where the coefficients $G(x'', X'')$ are profiles, flat at $x_d = 0$.

Lemma 2.5.6. *For $h = h_0$ satisfying* (2.5.21)*, the induction relation* (2.5.23) *defines $h_j \in H^\infty(\mathbb{R}^d_+)$, flat on the boundary $\{x_d = 0\}$ then there is $v^0 \in H^\infty(\mathbb{R}\times\mathbb{R}^d_+)$, flat on the boundary, supported in the strip $\{|x_0| \le 1\}$ and such that*

$$\varepsilon^j \partial_{x_0}^j v^0{}_{|x_0=0} = h_j, \quad j \le m, \tag{2.5.24}$$

Moreover, there is a constant C_1 such that if h satisfies (2.5.19) *then h_j satisfy for $j \le m$:*

$$\|h_j\|_{H^{m-j}_\varepsilon} \le C_1, \tag{2.5.25}$$

$$\sup_{t\ge 0} \|\varepsilon^j \partial_{x_0}^j v^0(t,\cdot)\|_{H^{m-j}_\varepsilon(\mathbb{R}^d_+)} \le C_1. \tag{2.5.26}$$

Démonstration. The first statement is clear and (2.5.25) follows from Lemma 2.5.5.

Next we lift the traces h_j to $v^0 \in H^\infty(\mathbb{R}\times\mathbb{R}^d_+)$. Using dilations, it is sufficient to do it and prove the estimates when $\varepsilon = 1$. Because the h_j are flat on the boundary, their extension by 0 for $x_d < 0$ is smooth. Using adapted lifting operators, these extended traces can be lifted to $x_0 \neq 0$ and there is $v^0 \in H^\infty(\mathbb{R}\times\mathbb{R}^d)$ satisfying (2.5.24) and vanishing for $x_d < 0$. Moreover, v^0 is bounded in $\cap C^j(\mathbb{R}, H^{m-j})$ if the h_j are bounded in H^{m-j}.

At last, we can multiply v^0 by a cut-off function in time to make it compactly supported in x_0, increasing the constant C_1 if necessary. □

Following [43], we initialize the scheme by taking v^0 given by the previous lemma. The existence and regularity results of [19, 42] recalled at the beginning of Section 2.5.1, imply by induction on p, that the scheme (2.5.22) defines for $p \ge 1$, smooth solutions $v^p \in H^\infty([0,t_*]\times\mathbb{R}^d_+)$, which are flat at the corner. We show that if ε is small enough, the sequence v^p is bounded and converges in the norm $E^m_\varepsilon([0,t_*])$. This will be a consequence of the energy estimates.

Following [43], comparing (2.5.23) to the similar relation valid for (2.5.22), we first check by induction that for all p the Taylor expansion of v^p at $x_0 = 0$ is that of v^0 :

$$\varepsilon^j \partial_{x_0}^j v^p{}_{|x_0=0} = h_j, \quad j \le m. \tag{2.5.27}$$

As a consequence, we see that there is a constant C_2 such that

$$\|v^p(0)\|_{E^{m,\varepsilon}} = \sum \|h_j\|_{E_{sp}^{m-j,\varepsilon}} \le \sum \|h_j\|_{H^{m-j,\varepsilon}} \le C_2. \tag{2.5.28}$$

Next, we look at the nonlinear source term in the right hand side of (2.5.22). In order to apply the nonlinear estimates of Lemma 2.5.4 on the intervals $[0,t]$, we use the following trick : because v^p and v^0 have the same Taylor expansion, the extension of $v^p - v^0$ by 0 for $x_0 < 0$, denoted by $\widetilde{v^p - v^0}$, is of class C^m ; thus, on $[0,t_*]$, v^p is the restriction of $v^0 + (\widetilde{v^p - v^0}) \in \tilde{F}^m(t_*)$. Moreover,

$$F(u_{app}^\varepsilon, \varepsilon^M v)v = F(u_{app}^\varepsilon, \varepsilon^M v^0)v^0 + G(u_{app}^\varepsilon, \varepsilon^M v^0, \varepsilon^M(v - v^0))(v - v^0) \tag{2.5.29}$$

where G is a polynomial function of its arguments. Introduce the notations

$$\|f\|_{F^{m,\varepsilon}([0,t])} = \sum_{2\alpha_d + |\alpha''| \le m} \varepsilon^{2\alpha_d + |\alpha''|} \|\partial_x^\alpha f\|_{L^2([0,t]\times\mathbb{R}^d_+)}. \tag{2.5.30}$$

Recall that

$$\|f\|^2_{F^{m,\varepsilon}([0,t])} \approx \int_0^t \|f(s)\|_{E^{m,\varepsilon}} ds.$$

Lemma 2.5.7. *Let*

$$f^p = \varepsilon^{n+1-M} r^\varepsilon + F(u_{app}^\varepsilon, \varepsilon^M v^p)v^p.$$

There are constants C_3 and C_4, such that for all $\varepsilon \in]0,1]$ and h satisfying (2.5.19) (2.5.21), *if*

$$\varepsilon^M \|v^0\|_{L^\infty([-1,t_*]\times\mathbb{R}^d_+)} \le 1, \quad \varepsilon^M \|v^p\|_{L^\infty([0,t_*]\times\mathbb{R}^d_+)} \le 1,$$

then for all $t \in [0,t_*]$:

$$\|f^p\|_{F^{m,\varepsilon}([0,t])} \le C_3 + C_4 \|v^p\|_{F^{m,\varepsilon}([0,t])}. \tag{2.5.31}$$

Démonstration. Use (2.5.29) and replace $v^p - v^0$ by its extension $\widetilde{v^p - v^0}$ in G. Then Lemma 2.5.4 implies that if v^p and v^0 satisfy the L^∞ bounds, then

$$\|f^p\|_{F^{m,\varepsilon}([0,t])} \le C_3 + C_4\big(\|v^p - v^0\|_{F^{m,\varepsilon}([0,t])} + \|v^0\|_{\tilde{F}^{m,\varepsilon}(t)}\big).$$

Next, note that by Lemma 2.5.6, the norm $\|v^0\|_{\tilde{F}^{m,\varepsilon}(t)}$ is uniformly bounded. □

Proposition 2.5.8. *There are $\varepsilon_0 > 0$ and C such that for all $\varepsilon \in]0, \varepsilon_0]$, h satisfying* (2.5.19) (2.5.21)*, and all $p \geq 0$:*

$$\varepsilon^M \|v^p\|_{L^\infty([0,t_*]\times\mathbb{R}^d_+)} \leq 1, \tag{2.5.32}$$

$$\sup_{t\in[0,t_*]} \|v^p(t)\|_{E^{m,\varepsilon}} \leq C. \tag{2.5.33}$$

Démonstration. We show that one can choose C, $K > 0$ and ε_0 such that the estimates (2.5.32) and

$$\|v^p(t)\|_{E^{m,\varepsilon}} \leq Ce^{Kt}. \tag{2.5.34}$$

can be proved by induction on p. For $p = 0$, we know from (2.5.26) and Lemma 2.5.3 that

$$\varepsilon^{\frac{d+1}{2}} \|v^0\|_{L^\infty([-1,t_*]\times\mathbb{R}^d_+)} \leq C_5 \sup_{t\in[-1,t_*]} \|v^0(t)\|_{E^{m,\varepsilon}_{sp}} \leq C_5 C_1.$$

Thus, the estimates are satisfied if $C \geq C_5 C_1$ and $\varepsilon^{M-\frac{d+1}{2}} \leq \frac{1}{C_5 C_1}$. In addition,

$$\varepsilon^{\frac{d+1}{2}} \|v^0\|_{L^\infty([-1,t_*]\times\mathbb{R}^d_+)} \leq 1$$

as needed in Lemma 2.5.7

For $p > 0$, we use Proposition 2.5.1 applied to the linear equation Eq. (2.5.22). Squaring Eq. (2.5.13) and using (2.5.28) and (2.5.31), we see that there is C_6 such that if the induction hypothesis (2.5.32) is satisfied at the order p, there holds for all $t \in [0, t_*]$

$$\begin{aligned}\|v^{p+1}(t)\|^2_{E^{m,\varepsilon}} &\leq C_6 + C_7\, t\, \|f^p\|^2_{F^{m,\varepsilon}([0,t])} \\ &\leq C_6 + C_8 \int_0^t \|v^p(s)\|^2_{E^{m,\varepsilon}} ds.\end{aligned}$$

Thus, if (2.5.34) is satisfied at the order p,

$$\|v^{p+1}(t)\|^2 \leq C_6 + \frac{C_8 C^2}{2K}(e^{2Kt} - 1) \leq C^2 e^{2Kt},$$

proving (2.5.34) at the order $p + 1$, if $C^2 \geq C_6$ and $2K \geq C_8$.

Finally, we note that

$$\varepsilon^M \|v^{p+1}\|_{L^\infty([0,t_*]\times\mathbb{R}^d_+)} \leq C_5 \varepsilon^{M-\frac{d+1}{2}} \sup_{t\in[0,t_*]} \|v^{p+1}(t)\|_{E^{m,\varepsilon}_{sp}} \leq C_5 \varepsilon^{M-\frac{d+1}{2}} Ce^{Kt_*} \leq 1$$

if $\varepsilon \leq \varepsilon_0$ and ε_0 is small enough. □

Proposition 2.5.9. *For all $\varepsilon \in]0, \varepsilon_0]$, and h satisfying* (2.5.19) (2.5.21)*, the sequence v^p converges in $E^m(t_*)$. The limit is the unique solution v of the Cauchy problem for* (2.5.4) *in the space $E^m(t_*)$. It satisfies*

$$\sup_{t\in[0,t_*]} \|v(t)\|_{E^{m,\varepsilon}} \leq C \tag{2.5.35}$$

where C depends only on the initial bound C_0.

Démonstration. The energy estimate for $v^{p+1} - v^p$ reads

$$\|(v^{p+1} - v^p)(t)\|^2_{E^{m,\varepsilon}} \leq C_7\, t \int_0^t \|(f^p - f^{p-1})(s)\|^2_{E^{m,\varepsilon}} ds$$

since v^{p+1} and v^p have the same Taylor expansion at $x_0 = 0$. We use again (2.5.29) to compute the difference $f^p - f^{p-1}$ which can be written as

$$\widetilde{G}\big(u^\varepsilon_{app}, \varepsilon^M v^0, \varepsilon^M (v^p - v^0), \varepsilon^M (v^{p-1} - v^0)\big)(v^p - v^{p-1}).$$

Arguing as in the proof of Lemma 2.5.7, using the bounds given by Proposition 2.5.8 Lemma 2.5.4 and applying Lemma 2.5.3 to bound the L^∞ norm of $v^p - v^{p-1}$, we obtain that

$$\|(f^p - f^{p-1})(s)\|^2_{F^{m,\varepsilon}([0,t])} \leq C_9 \|(v^p - v^{p-1})(s)\|^2_{F^{m,\varepsilon}([0,t])}.$$

Therefore

$$\|(v^{p+1} - v^p)(t)\|^2_{E^{m,\varepsilon}} \leq C_{10}\, t \int_0^t \|(v^p - v^{p-1})(s)\|^2_{E^{m,\varepsilon}} ds$$

which implies that the series $\sum (v^{p+1} - v^p)$ converges in the norm $E^{m,\varepsilon}([0, t_*])$. The limit is solution to the Cauchy problem.

Uniqueness easily follows from the basic L^2 estimate for the difference of two solutions in $L^2 \cap L^\infty$. □

To finish the proof of Theorem 2.2.17 it remains to drop the assumption (2.5.21). If h satisfies (2.5.19) and (2.5.20), there is a sequence h^ν which satisfies (2.5.19) and (2.5.21) and converges to h in $H^m(\mathbb{R}^d_+)$.

By Proposition 2.5.9, for $\varepsilon \in]0, \varepsilon_0]$, the Cauchy problem with initial data h^ν has a unique solution v^ν in $E^m([0, t_*])$, and the sequence is uniformly bounded in the norm $E^{m,\varepsilon}$. Thus, by Lemma 2.5.3 the $\varepsilon^M v^\nu$ are also uniformly bounded in L^∞. Moreover, $v^\nu = v^{\nu,0} + w^\nu$ where $v^{\nu,0}$ depends only on the initial data h^ν and $\partial^k_{x_0} w^\nu{}_{|x_0=0} = 0$ for $k \leq m$. We prove that, for all fixed $\varepsilon \in]0, \varepsilon_0]$, the sequence v^ν is a Cauchy sequence in $E^{m,\varepsilon}$. We use the energy estimate for the difference $v^\nu - v^\mu$. With obvious notations we obtain

$$\begin{aligned} \|(v^\nu - v^\mu)(t)\|^2_{E^{m,\varepsilon}} \leq C(&\|(v^{\nu,0} - v^{\mu,0})(0)\|^2_{E^{m,\varepsilon}} \\ &+ C \int_0^t \|(f^\nu - f^\mu)(s)\|^2_{E^{m,\varepsilon}} ds. \end{aligned} \tag{2.5.36}$$

The difference $f^\nu - f^\mu$ reads

$$\Phi(u^\varepsilon_{app}, \varepsilon^M v^\nu, \varepsilon^M v^\mu)\ (v^\nu - v^\mu).$$

Arguing as in Lemma 2.5.7 and using the known bounds yields the estimate

$$\|f^\nu - f^\mu\|_{F^{m,\varepsilon}([0,t])} \leq C\|v^{\nu,0} - v^{\mu,0}\|_{F^{m,\varepsilon}([-1,t])} + C\|v^\nu - v^\mu\|_{F^{m,\varepsilon}([0,t])}.$$

Substituting in (2.5.36) and using Gronwall's lemma implies that

$$\|(v^\nu - v^\mu)(t)\|^2_{E^{m,\varepsilon}} \leq C\|(v^{\nu,0} - v^{\mu,0})(0)\|^2_{E^{m,\varepsilon}} + \|v^{\nu,0} - v^{\mu,0}\|_{F^{m,\varepsilon}([-1,t])}$$

Because the initial data h^ν form a Cauchy sequence in H^m, the right hand side tends to 0 as ν and μ tend to $+\infty$. Therefore, the sequence v^ν converges, the limit is solution of the Cauchy problem for (2.5.4) with initial data h, and satisfies the bound (2.5.35). This finishes the proof of Theorem 2.2.17.

Chapitre 3

Diffractive wave transmission in dispersive media

3.1 Introduction, Definitions and Assumptions

The aim of this chapter is to make a detailed analysis of the reflected and transmitted high frequency waves in the frame of weakly nonlinear diffractive geometric optics for dispersive hyperbolic equations.

We start by looking for phase-amplitude like approximate solutions U_{app} with infinite order WKB expansion of the amplitude. For planar waves this enhances at least two scales x (for the amplitude), $X = x/\varepsilon$ (for the phase) where ε is the small adimentioned wavelength (see [13], $\varepsilon \sim 10^{-3}$).

Then to investigate the regime of diffractive optics [14, 27, 32] one must consider time and distances of order $\mathcal{O}(1/\varepsilon)$. On such a long scale, in the vacuum, the wave undergoes modifications which are not due to propagation so the corresponding profile should take it into account by the mean of a new slow variable $\mathsf{x} = \varepsilon x$.

In [13], chap.2, the author shows that the regime of diffractive optics in this context is essentially similar to that for the Cauchy problem set in the whole space, without boundary. Nevertheless the transmission equations at the boundary are not solved exactly and the WKB expansion stops at the second corrector. One knows from [35] that solving the boundary condition increases the set of resonant waves so that the analysis of the nonlinear interaction becomes trickier. Here we aim at constructing a WKB expansion up to any order. This requires the profile to have a sub-linear growth with respect to the middle scale (see [32]). In [13] the first corrector is taken constant on the characteristics. We show that for the next corrector this cannot be achieved. Moreover we show that the nonlinear interactions can contribute to not integrable profiles (see lemma 3.4.3). We thus choose bounded profiles as in [14] and we make an assumption to eliminate any non resonant nonlinear interaction traveling at the characteristic speed (see Assumption 3.2.20).

We also make the assumption of cubic non-linearity to avoid the mean mode (cf. [14], oddness hypothesis). The generation of this mode corresponding to a non oscillating phase is called rectification. This could be considered but in a different context with the use of a Wiener algebra (see (3.2.3)).

3.1.1 The equations and main assumptions

Consider a classical planar incoming wave packets

$$v^\varepsilon(t,x) = \varepsilon^p \mathrm{Re}\left\{ A(\varepsilon t, \varepsilon x, t, x) e^{i(k\cdot x - \omega t)/\varepsilon} \right\} + O(\varepsilon) \tag{3.1.1}$$

with real planar phases $\varphi(t,x) = k\cdot x - \omega t$. We have in mind the Maxwell-Lorentz equations with laser-like solution.

Taking planar phases requires considering planar interface and constant coefficient semi-linear equations. The reflection transmission problem in $\mathbb{R}^{d+1}_+ = \{x_d > 0\}$ then writes

$$\begin{aligned} L(\varepsilon\partial_x)v = \Phi(v), \qquad & \text{on} \quad x_d > 0, \\ Tv = 0, \qquad & \text{on} \quad x_d = 0 \\ v(t=0) = v_0 & \end{aligned} \tag{3.1.2}$$

with T a constant matrix and

$$L(\varepsilon\partial_x) := \varepsilon L^1(\partial_x) + L^0 = \sum_{j=0}^{d} \varepsilon A_j \partial_{x_j} . + L^0. \tag{3.1.3}$$

Here $x = (x_0, x_1, \dots, x_d)$ denote the space time variables and $x_0 = t$ is time. The nonlinear interaction Φ vanishes at order $J \geq 2$ at the origin, meaning that $\nabla_v^\alpha \Phi(0) = 0$ for all $|\alpha| \leq J-1$. The weakly nonlinear diffractive regime concerns solutions of amplitude $O(\varepsilon^p)$ with $p = 2/(J-1)$ (see [13, 15, 14, 32]) : setting $v = \varepsilon^p u$ yields

$$L(\varepsilon\partial_x)u = \varepsilon^2 F(u, \varepsilon^{1/p}) \tag{3.1.4}$$

where $F(u, \varepsilon^{1/p})$ is a smooth function of its arguments. Recall that these amplitudes are computed so that the nonlinear effects appear in time $t = O(1/\varepsilon)$. Note that if f is an homogeneous polynomial of degree J, then $F(u, \varepsilon^{1/p}) = f(u)$. This holds in particular for the cubic anharmonic Maxwell-Lorentz equations ; in this case the choices of p are $p = 1/2$ in the weakly non linear geometric optics and $p = 1$ for diffractive optics. Motivated by this example we will consider in this paper the mixed problem

$$\begin{aligned} L(\varepsilon\partial_x)u = \varepsilon^2 f(u), \qquad & \text{on} \quad x_d > 0, \\ Tu = 0 \qquad , \qquad & \text{on} \quad x_d = 0 \\ u(t=0) = u_0 & \end{aligned} \tag{3.1.5}$$

with f a polynomial.

We make several assumptions on this system as in [35]. We first give the fundamental assumptions under which the initial boundary value problem is well posed (cf. [19]).

Assumption 3.1.1. (H1) *The matrices A_j are symmetric with $A_0 = Id$ and L^0 is skew symmetric.*

(H2) $\dim(\ker A_d) = D_2 \geq 1$.

(H3) *A_d is non negative on the space* $\ker T$ *and the rank of T is equal to the number of positive eigenvalues of A_d counted with their multiplicity.*

Changing bases, we can further assume that

$$A_d = \begin{pmatrix} A_d^{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.1.6}$$

with A_d^{11} invertible.

3.1.2 The profiles

As our analysis naturally involves propagation we will use for any $\mathbb{R}^{d+1}$ variable

Notations 3.1.2. *$x = (x_0, \dots, x_d) = (x_0, x'') = (x', x_d)$ and as x_0 is the time variable we use instead t. One will also use z instead of x_d.*

We use the same splitting for the dual variable $\xi = (\xi_0, \xi'') = (\xi', \xi_d)$.

We consider WKB solution of (3.1.5) with profile description :

$$u^\varepsilon(x) = U(\varepsilon, \varepsilon x, x, x/\varepsilon) \tag{3.1.7}$$

$$U(\varepsilon, \mathsf{x}, x, X) = \sum_{j \geq 0} \varepsilon^j U^j(\mathsf{x}, x, X). \tag{3.1.8}$$

Each profile U^j is expected to solve an equation on the fast scale, a boundary equation and other interior equations with respect to the other scales. Because of the boundary equation, one knows that both oscillating and evanescent waves are to be generated (cf. [45]). We thus assume each $U^j(\mathsf{x}, .)$ has the following *finite* decomposition (cf. chapter 2) :

$$\begin{aligned} U(\mathsf{x}, x, X) &= \sum_{\xi'} U_{\xi'}(\mathsf{x}, x, X_d) e^{i\xi' \cdot X'}, \\ U_{\xi'}(\mathsf{x}, x, X_d) &= U_{\xi', os}(\mathsf{x}, x, X_d) + U_{\xi', ev}(\mathsf{x}, x, X_d) \\ U_{\xi', os}(\mathsf{x}, x, X_d) &= \sum_{\xi_d} U_{\xi', \xi_d}(\mathsf{x}, x) e^{i\xi_d \cdot X_d} \end{aligned} \tag{3.1.9}$$

where $U_{\xi', ev}$ is exponentially decaying in X_d. Note that this class of profiles is stable by nonlinear composition. This is formal if the sums (series) are infinite, but this makes sense for finite sums and polynomial nonlinearities.

Then extending the notation 3.1.2 on subsets of $\mathbb{R}^{d+1}$ let introduce $\Lambda' \subset \mathbb{R}^d$.

Definition 3.1.3. *Let Λ' a $\mathbb{Z}$-module on $\mathbb{R}^d$. Then define $\mathcal{P}$ the formal space of profiles decomposing finitely according to* (3.1.9) *on $\Lambda' \times \mathbb{R}$.*

The set of ξ' such that $U_{\xi'} \neq 0$ is called the tangential spectrum of U and noted $spec_t U$. Similarly, the set of $\xi \in \mathbb{R}^d$ such that $U_\xi \neq 0$ is called the oscillating spectrum of U and noted $spec_{os} U$.

3.1.3 The initial data

Since the wave is expected to propagate on long distances the geometric scale of reflexion-transmission is actually the slow scale.

This suggests taking a Gaussian shaped initial data placed away from the boundary on this scale.

Then we know from the standard linear analysis on the hyperbolic systems that the data decomposes into different polarizations which generate waves propagating in different directions. Some of which (outgoing) strike the boundary at different time with respect to the slow scale and for clearness (but without loss of generality) we choose to describe the reflexion-transmission of a single such outgoing wave.

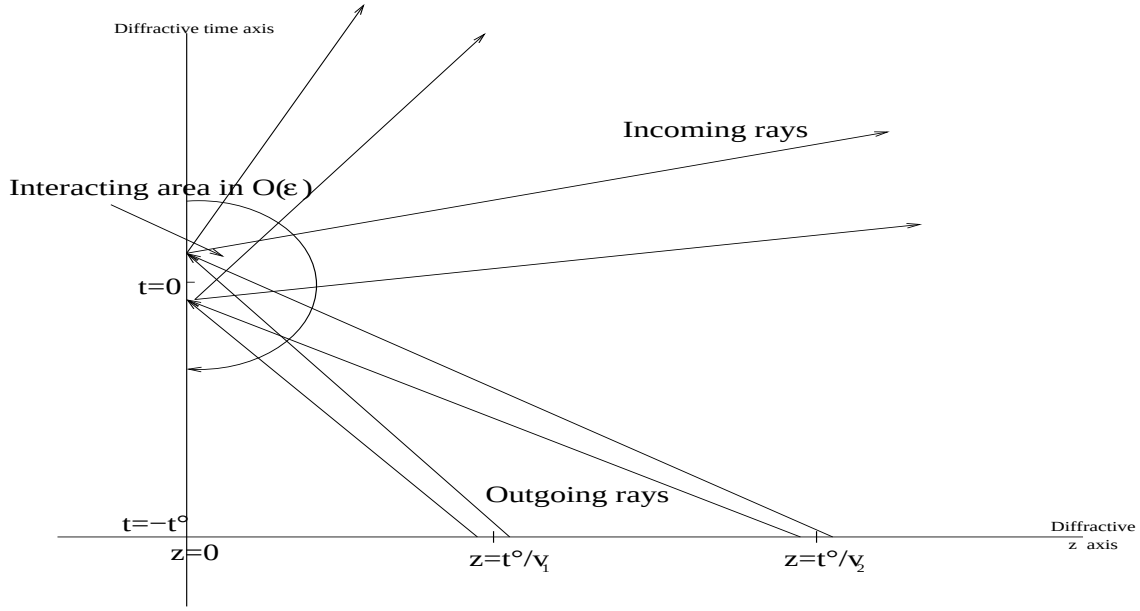

FIG. 3.1 – Diffractive transmission.

Then we rescale the time and choose an initial negative time on the slow scale $-\mathsf{t}^0$ so that the outgoing wave strikes the boundary at time $\mathsf{t} = 0$. So one must place it initially according to its group velocity (see fig.1) on the z axis (on the slow scale). Remark that our study would be unchanged if other waves would strike the boundary at the same time (see fig.1).

Note that according to the ε-expansion (3.1.7) the initial center of the wave is $\mathsf{z} = -v\mathsf{t}^0$ if it moves with normal speed v.

Remark that to "see" the transmission one needs to solve (3.1.5) for time duration greater than t^0. One could do as in [13] where the author chooses an initial time $t = -1/\varepsilon^\delta$, $0 < \delta < 1$ so that the transmission always happens.

3.2 The cascade of equations.

Plugging (3.1.7) in (3.1.5) gives a cascade of equations to solve

$$\text{(3.2.1)} \qquad \begin{cases} L(\partial_X)U^0 = 0 \\ L(\partial_X)U^1 + L^1(\partial_x)U^0 = 0 \\ L(\partial_X)U^{j+1} + L^1(\partial_x)U^j + L^1(\partial_\mathsf{x})U^{j-1} = F^{j-1}, \quad j \geq 1 \end{cases}$$

Each profile U^j appear in three consecutive equations. We next analyse them singly. For sake of easiness we define and use many projectors to break the equations up in elementary pieces. We also make assumptions, already given in the previous chapter, which fit the Maxwell equations. A summary of the profile construction is then given in section §3.3.

3.2.1 Equation on the fast scale : the microscopic equation

It writes $L(\partial_X)U^j = \mathcal{F}^j$ where $\mathcal{F}^j$ depends on the profiles $(U^l)_{l\leq j-1}$ and on their derivatives. Before recalling the main facts we would like to refer the reader to the previous chapter where an extensive study is performed. Let consider the first homogeneous equation for U^0. Using the tangential Fourier decomposition (3.1.9) of $L(\partial_X)U^0 = 0$ with U^0 decomposing according to (3.1.9) leads to solve

$$L(i\xi', \partial_Z)U_{\xi'} = (A_d\partial_Z + L'(i\xi'))U^0_{\xi'} = 0. \tag{3.2.2}$$

Decomposing again $U^0_{\xi'}$ in normal modes leads to solve $L(i\xi)U^0_\xi = 0$. Looking for $U^0_\xi \neq 0$ thus requires $\det L(i\xi) = 0$. Such a ξ is said to be characteristic.

Definition 3.2.1. *i) The real [resp. complex] characteristic set of L is the set of $\xi \in \mathbb{R}^{1+d}$ [resp. $\xi \in \mathbb{C}^{1+d}$] such that $\det(L(i\xi)) = 0$. We note $p(\xi) := \det(L(i\xi))$ the characteristic polynomial of L.*

ii) A real *mode $\underline{\xi} \in \mathrm{char}L$ is regular if there is a $\mathcal{C}^\infty$ function $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that near $\underline{\xi}$ $\mathrm{char}L$ is locally given by the equation $\xi_0 + \lambda(\xi_1, \ldots, \xi_d) = 0$.*

The analysis of equation (3.2.2) thus rests on finding the roots of $p(\xi', \xi_d)$ in ξ_d for a given $\xi' \in \mathbb{R}^d$.

We make the following classification of the modes (see [47, 35]) :

Definition 3.2.2. *Let given $\xi' \in \mathbb{R}^d$.*

i) A real root in ξ_d of $p(\xi', \xi_d) = 0$ is

1) hyperbolic incoming if $\xi = (\xi', \xi_d)$ is regular and $\partial_{\xi_d}\lambda(\xi) > 0$.

2) hyperbolic outgoing if $\xi = (\xi', \xi_d)$ is regular and $\partial_{\xi_d}\lambda(\xi) < 0$.

3) glancing if it is regular and $\partial_{\xi_d}\lambda(\xi) = 0$.

ii) A complex root in ξ_d of $p(\xi', \xi_d) = 0$ is

4) elliptic incoming (evanescent) if $\mathrm{Im}\,\xi_d > 0$,

5) elliptic outgoing (explosive) if $\mathrm{Im}\,\xi_d < 0$,

We denote by $\mathcal{I}$, $\mathcal{O}$, $\mathcal{G}$, $\mathcal{E}_{ev}$ and $\mathcal{E}_{ex}$ the associated sets.

For $\xi \in \mathbb{R}^{d+1}$, $\partial_{\xi_d}\lambda(\xi'')$ is the last component of the group velocity.

From now on we note $v_\xi = \nabla_{\xi''}\lambda(\xi'') \in \mathbb{R}^d$.

To deal with the non invertibility of A_d in equation (3.2.2) we place our analysis in the frame of [35, 38] where one makes a reduction of equation (3.2.2) on the range of A_d. This leads to decompose L' according to (3.1.6).

$$L'(i\xi) = \begin{pmatrix} L^{11} & L^{12} \\ L^{21} & L^{22} \end{pmatrix}.$$

Writing $U = (U^1, U^2)$ the equation (3.2.2) then writes

$$\begin{aligned} &A^{11}\partial_z U^1 + L^{11}U^1 + L^{12}U^2 = 0 \\ &L^{21}U^1 + L^{22}U^2 = 0. \end{aligned}$$

One solves this equation expressing U^2 in term of U^1 to get an evolution equation in z on U^1. This process requires inverting L^{22}. One thus must insure this operation is valid for the

Maxwell's case. In [35] one makes two differents assumptions according on that $\xi' \neq 0$ or $\xi' = 0$. But in [35] and [13] it is shown that the modes $\xi \neq 0$ such that $\xi' = 0$ are ruled out because of the Maxwell divergence equations. We thus don't consider those modes. Lastly one does not consider the case of rectification : $\xi = 0$ for technical reasons explained in the next remark.

Remark 3.2.3. *As one expects solving a non-linear equation for the leading profile (see §2.3) the mean mode is bound to be generated. Contrary to [13] and because of the boundary one cannot make a pseudo-differential analysis. Moreover, taking many correctors increases the nonlinear interactions and one should adapt to the mixted case the results of [34]. An interesting problem would be working with only two correctors in a Wiener algebra.*

Finally one makes the assumption :

Assumption 3.2.4. *For all $\xi' \in \Lambda'$ we assume that*
1) $L^{22}(i\xi')$ is invertible,
2) The real roots in ξ_d of p are regular and non glancing.

So for Maxwell one does not consider the ξ' such that $\xi_0 = 0, \pm\omega_a$ or $\xi_0 = \pm\sqrt{\gamma + \omega_a^2}$ (see [35]). As we are in a dispersive context, the finiteness of nonlinear generation shows that all those special modes can be avoided (with a suitable assumption on the initial set of resonances) except $\xi_0 = 0$ and more particularly $\xi = 0$.

As we do not want to consider this rectification we must give an example of radical assumption preventing for its generation. A way to avoid this situation is to take F as a polynomial with odd powers and Λ' odd which means

Definition 3.2.5. *for all $k \in \mathbb{N}$ and all subset Γ of some space $\mathbb{R}^n$ we note*

$$k * \Gamma := \underbrace{\Gamma + \ldots + \Gamma}_{k \text{ terms}}. \tag{3.2.3}$$

with the usual convention that for $k = 0$ the set in the right hand side is $\{0\}$.

So by Λ' odd we mean that there is a $\Xi' \subset \mathbb{R}^d \setminus 0$ and an odd integer $k = 2p + 1 \in \mathbb{Z}$ so that $\Lambda' = k * \Xi'$.

Assumption 3.2.6. *F and Λ' are odd.*

Remark 3.2.7. *There are some cases where $\mathrm{Im}\,(F) \subset \ker(\mathbb{P}_0)$ ($\mathbb{P}_0$ is a projector on $ker L(0)$) for example nonlinear Maxwell with anharmonic oscillator model. In this case Assumption 3.2.6 is not needed. The second point of Assumption 3.2.6 is thus not met (ex. $\tau = 0$ for Maxwell) but every mode is still described through a scalar transport equation.*

Lastly we quote the result obtained on equation (3.2.2) in [35].

Theorem 3.2.8. *There are projectors $\mathbb{P}$ and $\mathbb{P}^i$ and partial inverse $\mathbb{Q}$ acting in $\mathcal{P}$ such that for all $F \in \mathcal{P}$ the equation*

$$L(\partial_X)V = F$$

has a solution $V \in \mathcal{P}$ if and only if $\mathbb{P}^i F = 0$. The general solution is

$$V = \mathbb{Q}F + \mathbb{P}V.$$

The precise definition of the projectors will be made explicit when needed. We just need to notice they are defined for a given tangential mode ξ' and they decompose naturally into an oscillating and evanescent part. For example $\mathbb{P}^i = \mathbb{P}^i_{ev} + \mathbb{P}^i_{os}$ with $\mathbb{P}^i_{ev} = 0$; so there is no condition on the evanescent part of F.

This theorem thus determines generically $(1-\mathbb{P})U^j$. Then $\mathbb{P}_{os}U^j_{os}$ and $\mathbb{P}_{ev}U^j_{ev}$ are determined from their initial value at $X_d = 0$. Those values are determined by solving the boundary equation.

For a profile $U \in \mathcal{P}$ we define

$$\mathbb{T}U(\mathsf{x}', x', X') = TU_{|_{\mathsf{x}_d = x_d = X_d = 0}} = \sum e^{i\xi' X'} TU_{\xi'}(., \mathsf{x}_d = 0, ., x_d = 0, ., X_d = 0)$$

and $\mathbb{T}$ acts from $\mathcal{P}$ to $\mathcal{P}_b$, the space of profiles $V(\mathsf{x}', x', X')$ that are finite sums $\sum V_{\xi'}(\mathsf{x}', x')e^{i\xi' X'}$.

As in [35] we impose Lopatinski condition on T which are satisfied by the Maxwell equations. But to this goal introduce $\mathbb{E}^-_{H,L}, \mathbb{E}^+_{H,L}, \mathbb{E}^-_{E,L}$ the spectral spaces associated to the eigenvalue which are in $\mathcal{I}, \mathcal{O}, \mathcal{E}_{ev}$. In particular

$$\mathbb{E}^-_{H,L}(\xi') = \bigoplus_{\xi_d |\, (\xi', \xi_d) \in \mathcal{I}} \ker L(i\xi', i\xi_d).$$

Assumption 3.2.9. Lopatinski type conditions *: For all $\xi' \in \Lambda'$*

$$\dim(\mathbb{E}^-_{H,L}(\xi')) + \dim(\mathbb{E}^-_{E,L}(\xi')) = rg(T),$$
$$\ker T \cap \left(\mathbb{E}^-_{H,L}(\xi') \oplus \mathbb{E}^-_{E,L}(\xi')\right) = \{0\}.$$

This allows to find the boundary value for the incoming modes and the evanescent modes in term of the outgoing ones. Moreover since $\mathbb{P}^i_{ev} = 0$ there is no PDE for $\mathbb{P}_{ev}U_{ev}$ which must be chosen as a smooth extension for all $\mathsf{z} > 0, z > 0$ of $\mathbb{P}_{ev}U_{ev|_{\mathsf{z}=z=0}}$. This indetermination could be removed looking for elliptic profiles U_{ev} such that $\mathbb{P}_{ev}U_{ev}$ is a function of (x', x', X).

3.2.2 Equation on the middle scale : transport equation

Using the compatibility condition of theorem 3.2.8 on the second equation gives

$$\mathbb{P}^i L^1(\partial_x)\mathbb{P}U^0 = 0 \tag{3.2.4}$$

This equation concerns only the oscillating modes since $\mathbb{P}^i_{ev} = 0$. Equation (3.2.4) decomposes according to a set $\Lambda \subset (\Lambda' \times \mathbb{R}) \cap \mathrm{char}L$. Let us recall $\mathbb{P}^i_{os}U_{os} = \mathbb{P}_{os}U_{os}$ and by definition

$$\mathbb{P}_{os}U_{os} := \sum_{\xi \in \Lambda \cap (\mathcal{I} \cup \mathcal{O})} \pi(\xi)U_\xi e^{i\xi X},$$

with $\mathcal{I}, \mathcal{O}$ introduced in definition 3.2.2 and $\pi(\xi)$ is the orthogonal projector onto $\ker L(i\xi)$. Fourier decomposing (3.2.4) gives

$$\pi_\xi L^1(\partial_x)\pi_\xi U^0_\xi = 0, \quad \xi \in \Lambda \cap (\mathcal{I} \cup \mathcal{O}).$$

We thus recover the usual transport equation on the middle scale (see [14]). In the sequel we will note

$X_\xi(\partial_x)\pi_\xi := \pi_\xi L^1(\partial_x)\pi_\xi = (\partial_t + v_\xi.\nabla_{x''})\pi_\xi$ and

$$\mathbb{X}(\partial_x)\mathbb{P}_{os}U_{os} = \sum_{\xi \in \Lambda \cap (\mathcal{I} \cup \mathcal{O})} X_\xi(\partial_x)\pi_\xi U_\xi e^{i\xi X} = \mathbb{P}_{os}L^1(\partial_x)\mathbb{P}_{os}U_{os}.$$

Now let consider the third equation. Using again the compatibility condition of theorem 3.2.8 gives

$$\mathbb{P}_{os}L^1(\partial_x)\mathbb{P}_{os}U^1 = \mathbb{P}_{os}\left[-L^1(\partial_{\mathsf{x}}) + L^1(\partial_x)\mathbb{Q}L^1(\partial_x)\right]\mathbb{P}_{os}U^0 + \mathbb{P}_{os}F^0, \tag{3.2.5}$$

We are thus led to solve the non homogeneous analogue of (3.2.4) :

$$\mathbb{P}_{os}L^1(\partial_x)\mathbb{P}_{os}U^1 = \mathbb{P}_{os}G \tag{3.2.6}$$

It is well known (see [14, 32]) that $\mathbb{P}_{os}G$ must satisfy conditions for $\mathbb{P}_{os}U$ to have a controlled growth in time or equivalently in z. Indeed εU^{j+1} is a corrector of U^j only if it satisfies this sub-linear growth

$$\lim_{z\to\infty}\frac{1}{z}\sup_{\mathsf{x}}\|U^j_\xi(\mathsf{x},.,z)\|_{L^2(\mathbb{R}^d_+)} = 0, \quad \forall j. \tag{3.2.7}$$

One can check that if equation (3.2.6) reduces to

$$X_\xi(\partial)U^1_\xi = \pi(\xi)G_\xi, \quad \text{with} \quad X_\xi(\partial)G_\xi = 0, \tag{3.2.8}$$

then U_ξ doesn't satisfy condition 3.2.7 and $\pi(\xi)G_\xi$ should cancel.

However in [14] there is just one characteristic mode and thus one speed so equation (3.2.6) reduces to (3.2.8). Here the multiple speed interaction may lead to intermediate interactions (see equation (3.2.16)) and locally supported interaction. For consistency we search the profile in the space of profile decomposing into a sum indexed by $\Lambda \subset (\Lambda' \times \mathbb{R}) \cap \mathrm{char}L$ of pure transports and a remainder with global decrease at infinity. First define

Definition 3.2.10. *For $p \in \mathbb{R}^d$*

$$\Gamma^s_p(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid (1+|x'-p|^2)^{l/2}(1+\partial^2_{x'})^{k/2}u \in L^2, \quad l+k \le s\}.$$

When $p = 0$ we forget the subscript.

Then let

Definition 3.2.11. *$\mathcal{E}^s(\Lambda)$ the set of profiles decomposing into $U = \tilde{U} + \tilde{\tilde{U}}$ with*

1. *$\tilde{U}(x) = \sum_\alpha \mathcal{U}_\alpha(x' - v'_\alpha z)$ for finite families $(v'_\alpha)_{\alpha\in\Lambda} \in \mathbb{R}^d$ and $(\mathcal{U}_\alpha)_{\alpha\in\Lambda} \in \Gamma^s(\mathbb{R}^d)$*
2. *$\tilde{\tilde{U}} \in \Gamma^s(\mathbb{R}^{d+1}_+)$.*

For $U \in \mathcal{E}$ the decomposition $U = \tilde{U} + \tilde{\tilde{U}}$ is unique for $s > (d+1)/2$ (see next section lemma 3.4.1).

Remark 3.2.12. *When looking for an outgoing $U^0_\xi \in \mathcal{E}^\infty(\Lambda)$ one is lead to solve the Cauchy problem for $X_\xi(\partial)U_\xi = 0$. One finds $U^0_\xi = \tilde{\mathcal{U}}^0(x'' - v_\xi t)$. Expressing the latter in term of $x' - v'_\xi z$ shows that $\mathcal{U}^0$ must lie in a $\Gamma^\infty_p(\mathbb{R}^d)$ with $p = -v_\xi/(\varepsilon \mathsf{t}_0)$ (see lemma 3.5.5).*

So we look for a solution of equation (3.2.6) in $\mathcal{P}$ and such that $U_\xi \in \mathcal{E}^\infty(\Lambda)$. We note

$$U_\xi = \tilde{U}_\xi + \tilde{\tilde{U}}_\xi, \tag{3.2.9}$$

$$\tilde{U}_\xi(x) = \sum_{\alpha\in\Lambda} \mathcal{U}_{\xi,\alpha}(x' - v'_\alpha z). \tag{3.2.10}$$

Now we can define precisely the space for the profiles. Let $\Omega_{\mathsf{t}} := [-\mathsf{t}^0, \mathsf{t}] \times \mathbb{R}^d_+$. Define the space for the profiles : $\mathcal{P}^s$

Definition 3.2.13. *i) Let $\mathcal{P}^{z,s}_{os}$, the set of functions $U(\mathsf{x}, x, X_d)$ which are finite sums*

$$U(\mathsf{x}, x, X_d) = \sum e^{i\xi_d X_d} U_{\xi_d}(\mathsf{x}, x)$$

with coefficients $U_{\xi_d} \in H^m(\Omega_{\mathsf{t}}; \mathcal{E}^\infty(\Lambda))$.

ii) Let $\mathcal{P}^{z,s}_{ev}$, the set of functions $U(\mathsf{x}, x, X_d)$ such that for some $\delta > 0$, $Ue^{\delta X_d} \in H^m(\Omega_{\mathsf{t}}; \Gamma^\infty(\mathbb{R}^{d+1}))\}$.

iii) Let $\mathcal{P}^{z,s} := \mathcal{P}^{z,s}_{os} \oplus \mathcal{P}^{z,s}_{ev}$.

iv) The space of profiles $\mathcal{P}^s$ (also noted $\mathcal{P}^s(\Omega_{\mathsf{t}}, \Lambda)$) is the space of finite sums

$$U(\mathsf{x}, x, X) = \sum_{\xi'\in\Lambda'} U_{\xi'}(\mathsf{x}, x, X_d) e^{i\xi' X'}, \quad \text{with} \quad U_{\xi'} \in \mathcal{P}^{z,s},\ spec_{os} U \in \Lambda. \tag{3.2.11}$$

It splits into $\mathcal{P}^s = \mathcal{P}^s_{os} \oplus \mathcal{P}^s_{ev}$ where $\mathcal{P}^s_{os}$ [resp. $\mathcal{P}^s_{ev}$] is the space of finite sums like (3.2.11) *with $U_{\xi'} \in \mathcal{P}^{z,s}_{os}$ [resp. $U_{\xi'} \in \mathcal{P}^{z,s}_{ev}$.*

For $\xi \in \mathrm{char}L$ we want to distinguish $\mathcal{U}_{\xi,\xi}$ from $\mathcal{U}_{\xi,\alpha}$, $\alpha \neq \xi$ in notation (3.2.10). This is suggested by the paragraph following (3.2.8). So let us define the projectors p, p^i defined for the profiles writing as (3.2.9),(3.2.10) :

Definition 3.2.14. *Let $\xi \in \Lambda \cap charL$. First define $\mathsf{p}(\xi), \mathsf{p}^i(\xi) \ : \ \mathcal{E}^\infty(\Lambda) \to \mathcal{E}^\infty(\Lambda)$ by*

$$\mathsf{p}(\xi) U_\xi := \mathcal{U}_{\xi,\xi}(x' - v_\xi z),$$
$$\mathsf{p}^i(\xi) U_\xi := \tilde{U}_\xi$$

Then for $U \in \mathcal{P}^s_{os}$ define

$$\mathsf{p}U := \sum_\xi \mathsf{p}(\xi) U_\xi e^{i\xi X},$$
$$\mathsf{p}^i U := \tilde{U}.$$

Remark 3.2.15. *Remark that $\mathbb{P}_{os} L^1(\partial_x) \mathsf{p} \mathbb{P}_{os} = \mathbb{X}(\partial) \mathsf{p} \mathbb{P}_{os} = 0$. Moreover p^i and $\mathbb{X}(\partial)$ commute on $\mathcal{E}^\infty(\Lambda)$.*

Making use of these projectors on equation (3.2.6) we get

$$\mathsf{p}^i \mathbb{P}_{os} L^1(\partial_x) \mathbb{P}_{os} U = \mathsf{p}^i \mathbb{P}_{os} G, \tag{3.2.12}$$

$$(1 - \mathsf{p}^i) \mathbb{P}_{os} L^1(\partial_x) \mathbb{P}_{os} U = (1 - \mathsf{p}^i) \mathbb{P}_{os} G\ . \tag{3.2.13}$$

From remark 3.2.15 the second equation is an equation for $\tilde{\tilde{U}}$. It writes

$$\mathbb{X}(\partial) \tilde{\tilde{U}} = \widetilde{\widetilde{\mathbb{P}_{os} G}}.$$

In lemma 3.4.4 we show this equation has a unique solution in $\Gamma^\infty(\mathbb{R}^{d+1}_+)$ and we note q the corresponding inverse defined on $\Gamma^\infty(\mathbb{R}^{d+1}_+)$.

Again from remark 3.2.15 the first equation reads $X(\partial)\tilde{U} = \widetilde{\mathbb{P}_{os}G}$. It decomposes according to p :

$$0 = \mathsf{p}\mathbb{P}_{os}G, \tag{3.2.14}$$

$$\mathbb{X}(\partial)(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}U = (\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}G\ . \tag{3.2.15}$$

Equation (3.2.14) was expected since it corresponds to example 3.2.8.

Equation (3.2.15) again splits into distinct transport equations.

$$X_\xi(\partial)\mathcal{U}_{\xi,\alpha} = \mathcal{F}_{\xi,\alpha},\ \alpha \neq \xi. \tag{3.2.16}$$

This compels $\int_{-\infty}^{\infty} \mathcal{F}_{\xi,\alpha}(x' - (v'_\alpha - v'_\beta)z)dz = 0$ (see lemma 3.4.3). This enables to define a projector on the image of X_ξ :

Definition 3.2.16. *First define the projector* $j_{v,w} : \Gamma^\infty(\mathbb{R}^d) \to \mathcal{C}^\infty(\mathbb{R}^d)$ *by*

$$j_{v,w}\mathcal{F} = \int_{-\infty}^{\infty} \mathcal{F}(x' - (w' - v')z)dz.$$

Let $\xi \in \Lambda \cap charL$. *Define* $\mathsf{j}(\xi)\ :\ \mathcal{E}^\infty(\Lambda) \to \mathcal{E}^\infty(\Lambda)$ *by*

$$J(\xi)f_\xi = \sum_{\alpha \in \Lambda\setminus\{\xi\}} J_{v_\xi,v_\alpha}\mathcal{F}_{\xi,\alpha}.$$

Lastly define $\mathsf{j}\ :\ \mathcal{P}_{os} \to \mathcal{P}_{os}$ *as in definition 3.2.14.*

If $j_{v_\alpha,v_\xi}\mathcal{F}_{\xi,\alpha} = 0$ equation (3.2.16) has a unique solution in $\Gamma^\infty(\mathbb{R}^d)$ (see lemma 3.4.3) and one can define an inverse noted q_j.

Summarizing we have

Theorem 3.2.17. *Let* $G \in \mathcal{P}^s_{os}$. *Then the equation*

$$\mathbb{P}_{os}L^1(\partial_x)\mathbb{P}_{os}U = G \tag{3.2.17}$$

has a solution in $\mathcal{P}^s_{os}$ *iff* $\mathsf{j}G = 0$ *and* $\mathsf{p}G = 0$. *The general solution writes*

$$\mathbb{P}_{os}U = \mathsf{q}G + \mathsf{q}_jG + \mathsf{p}U.$$

Unfortunately it doesn't seem easy to describe $\ker J$. Some solutions re given by $f(x') = a(x'.\delta')b(x' - (x'.\delta')\delta')$ with a odd, where $\delta' = v'_\alpha - v'_\xi$. But such solutions are not physical. In [13] the profile are Gaussian shaped profiles thus more even than odd. We thus set a definitive assumption to avoid those situations : we restrict our analysis to characteristic profiles propagating at the characteristic speed.

Ansatz 3.2.18.

$$\mathsf{p}^i\mathbb{P}_{os}U = \mathsf{p}\mathbb{P}_{os}U. \tag{3.2.18}$$

As a consequence $\pi(\xi)U_\xi(x) = \mathcal{U}_{\xi,\xi}(x' - v'_\xi z) + (1 - \mathsf{p}^i)\pi(\xi)U_\xi(x)$. We use the notation

Notations 3.2.19. $\mathcal{U}_\xi := \mathcal{U}_{\xi,\xi}$.

Using this on the projection (3.2.15) of the third equation (3.2.5) gives $(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}F^0 = 0$.

As F^0 is chosen polynomial, F^0_ξ involves products of profiles propagating at the characteristic speed v_ξ. One thus could meet products of kind

$$\prod_{k\leq d} U^0_{\xi_k}, \quad \sum_{k\leq d}\xi_k = \xi, \tag{3.2.19}$$

$$v_{\xi_1} = \ldots = v_{\xi_d} = v^\sharp \neq v_\xi. \tag{3.2.20}$$

The ansatz is thus not self-consistent and one must give an assumption leading to a systematic cancellation of the extra speeds $(\mathsf{p}^i - \mathsf{p})G$ in equation (3.2.13). It reads

Assumption 3.2.20. *Let $\xi \in charL \cap \Lambda$.*

For all finite odd sequence $(\beta^j)_j \in charL \cap \Lambda$ such that $\sum_j \beta^j = \xi$ with $v^j = v^\sharp\ \forall j$, then $v^\sharp = v_\xi$.

From assumptions 3.2.22 and 3.2.23 the later is supposed to be valid for $\Lambda = \Lambda_n$ the finite union of spectrum of $F^j,\ j \leq n$ with n the order of the WKB expansion.

Lastly one expects to determine $\mathsf{p}\mathbb{P}_{os}U$ in (3.2.6) from the compatibility condition $\mathsf{p}G = 0$ of theorem 3.2.17. But this latter equation is a PDE on the slow scale x which requires data known for all x, X. One thus need to determine $\mathsf{p}(\xi)\pi(\xi)U_\xi(x) = \mathcal{U}_\xi(x' - v'_\xi z),\ \forall \xi \in \Lambda \cap \mathrm{char}L$. We thus need the initial data and solve the boundary equation.

First the outgoing profiles $\mathcal{U}_\xi(\mathsf{x}, x'),\ \xi \in \mathcal{I}$ is determined from its initial value at $-\mathsf{t}^0$ computed at $x' - v'_\xi z$.

Then the incoming waves are determined from the boundary conditions. Thanks to the Lopatinski condition 3.2.9 one finds $\mathcal{U}_{\xi_{|\mathsf{z}=z=0}}$. The boundary value on the slow scale is then given by $\mathcal{U}_{\xi_{|\mathsf{z}=z=0}}$ comuted at $x' - v'_\xi z$.

3.2.3 Equation on the slow scale : the Schrödinger equation

Using the remaining compatibility condition of theorem 3.2.17, $\mathsf{p}G = 0$ gives for the third equation

$$\mathbb{P}_{os}\left[L^1(\partial_\mathsf{x}) - L^1(\partial_x)\mathbb{Q}_{os}L^1(\partial_x)\right]\mathsf{p}\mathbb{P}_{os}U^0 = \mathsf{p}\mathbb{P}_{os}F^0\ . \tag{3.2.21}$$

More generally for the j^{ieth} profile

$$\begin{aligned}&\mathbb{P}_{os}\left[L^1(\partial_\mathsf{x}) - L^1(\partial_x)\mathbb{Q}_{os}L^1(\partial_x)\right]\mathsf{p}\mathbb{P}_{os}U^j = \\ &\quad \mathsf{p}\mathbb{P}_{os}\left\{F^j_{os} - L^1(\partial_\mathsf{x})(1-\mathbb{P}_{os})U^j - L^1(\partial_x)\mathbb{Q}_{os}[F^{j-1}_{os} - L^1(\partial_\mathsf{x})U^{j-1}]\right\}\ .\end{aligned}$$

Remark 3.2.21. *The equation* (3.2.21) *is a nonlinearly coupled system of equations indexed by $\xi \in charL$. For $\xi \in charL$ the corresponding nonlinear equation involves products $\Pi_k\mathcal{U}_{\alpha_k}$ with $\sum_k \alpha_k = \xi$ and $v'_{\alpha_k} = v'_\xi$. This shows that* (3.2.21) *splits into distinct sets of equations indexed on the set of characteristic speeds.*

The non-linear terms are $F^0 = f(U^0)$ and for $j > 0$

$$F^j = \nabla_u f(U^0)U^j + \tilde{F}^j, \quad F^j_{os} = \nabla_u f(U^0_{os})U^j_{os} + \tilde{F}^j_{os} \tag{3.2.22}$$

where $\tilde{F}^j$ depends only on $(U^0, \dots, U^{j-1})$ and $\tilde{F}^j_{os}$ on $(U^0_{os}, \dots, U^{j-1}_{os})$.

First note $j = 0$ is the only index for which F^j is non-linear in U^j. Let Λ_0 be the spectrum of U_0 that we suppose made of characteristic modes. Then equation (3.2.21) generates harmonics which are combinations of resonant modes of Λ_0. They are not resonant in general since the characteristic variety is curved however this generation could be infinite for some exceptional choice of Λ_0. As in [35] we do not want to consider those exceptional situations and we recall assumption 2.9 which is given.

Let us introduce the spectrum related to the polynomial non-linearity f and ∇f :

$$\mathcal{N}(\Lambda_0) := \bigcup_{\{l: f^l \neq 0\}} l * \Lambda_0, \qquad \mathcal{N}^\flat(\Lambda_0) := \bigcup_{\{l: f^l \neq 0\}} (l-1) * \Lambda_0. \tag{3.2.23}$$

Then as physical waves are real, they involve each mode and its opposite. It is thus physically meaningfull to suppose that $\Lambda = -\Lambda$. One will say that such a set is symmetric. In this case there holds $k * \Lambda \subset (k+2) * \Lambda,\ k \in \mathbb{Z}$. moreover from assumptiom 3.2.6 Λ and F are odd thus for Λ symmetric $\Lambda \subset \mathcal{N}(\Lambda)$. Then one can make the following assumption :

Assumption 3.2.22. *We are given symmetric finite sets $\Lambda'_0 \subset \Lambda'$ and $\Lambda_0 \subset (\Lambda'_0 \times \mathbb{R}) \cap \mathrm{char}L$ such that*

$$(\Lambda'_0 \times \mathbb{R}) \cap \mathcal{I} \subset \Lambda_0. \tag{3.2.24}$$

$$\mathcal{N}(\Lambda_0) \cap (\mathrm{char}L) \subset \Lambda_0. \tag{3.2.25}$$

The first condition (3.2.24) takes into account the generation at the boundary of all the incoming waves thanks to the Lopatinski condition. For Maxwell this means that if $\xi' \in \Lambda'_0$ then $(\xi', \xi_d) \in \Lambda_0 \cap \mathcal{I} \Rightarrow (\xi', -\xi_d) \in \Lambda_0$. Though this is a restriction to the choice of Λ'_0, Λ_0 in Assumption 3.2.22, one can still generically satisfy the second condition (3.2.25) (cf.[35], Remark (2.11) and §2.3.7).

Next, for $j > 1$ the fast scale equation implies $\mathrm{spec}U^j \supset \mathrm{spec}F^{j-2}$. Moreover $\mathrm{spec}U^j$ should be consistent with the non-linearity F^j and the boundary equation. We would like to emphasize that most resonant generation is synthesized at the boundary. The new resonant modes are thus constructed through the tangential spectrum $(\mathrm{spec}F^{j-2})'$ which is sum of kind $k * \Lambda'_0$ with k odd. We assume as in [35]

Assumption 3.2.23. *For all k odd there is a symmetric finite set $\Lambda'_1 \subset \Lambda'$ such that $\Lambda'_1 \supset k * \Lambda'_0$ and $\Lambda_1 = (\Lambda'_1 \times \mathbb{R}) \cap (\mathrm{char}L)$ satisfies*

$$\big(\mathcal{N}^\flat(\Lambda_0) + \Lambda_1\big) \cap (\mathrm{char}L) \subset \Lambda_1. \tag{3.2.26}$$

Expressing equation (3.2.21) through $\mathcal{U}^j_\xi(x' - v_{\xi'}z) = \mathsf{p}U^j_\xi(x', z)$ shows that it requires data known for all $x' - \mathbf{v}'_\xi z \in \mathbb{R}^d$. We thus need to extend $I^{j,\varepsilon}_\xi$ to all z. The initial condition for the outgoing modes is then $I^{j,\varepsilon}_\xi(\mathsf{x}'', x'' - v_\xi t)$ expressed in the variable $x' - v'_\xi z$. It is noted $\mathcal{I}^{j,\varepsilon}_\xi$.

As for the incoming modes one needs solving the boundary data for $\mathsf{t} \leq \mathsf{t}^*$:

$$\mathbb{T}\left(\mathsf{p}^- U_{os}(\mathsf{x}', x') + \mathbb{P}_{ev} U_{ev}(\mathsf{x}', x')\right) = -\mathbb{T}(1 - \mathbb{P}_{os}) U_{os}(\mathsf{x}', x') \\ - \mathbb{T}\mathsf{p}^+ U_{os}(\mathsf{x}', x') - \mathbb{T}(1 - \mathsf{p})\mathbb{P}_{os} U_{os}(\mathsf{x}', x'),$$

and from assumption 3.2.20 $(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os} U_{os} = 0$. Thus the last term is in fact $\mathbb{T}_{os}\mathsf{q}G_{os}(\mathsf{z} = 0, x', z = 0)$ supposing $\mathbb{P}U$ satisfies the ansatz 3.2.18. The remaining question is then how to satisfy the compatibility conditions at the corner $\mathsf{t} = \mathsf{z} = 0$.

But as usual (see [38, 19, 35]) one chooses initial outgoing data vanishing near $\mathsf{z} = 0$ which guaranties the outgoing waves restricted at the bound $\mathsf{z} = 0$ to vanish near $\mathsf{t} = 0$.

3.3 Main results

We first define the space for the initial data. Let

$$X_p^s(\mathbb{R}^d_+) = H^s(\mathbb{R}^d_+; \Gamma^\infty_p(\mathbb{R}^d)) \text{ and } Y_p^s(\mathbb{R}^d_+) = \bigcap_{j \leq s} C^j([-\mathsf{t}^0, \mathsf{t}]; X_p^{s-j}),$$

forgetting the indices when $p = 0$.

We choose the initial data so that the solution of the Schrödinger equation (3.5.4) strikes the boundary at time $\mathsf{t} = 0$.

$$U^\varepsilon_{|_{\mathsf{t}=t=T=0}} = \sum_{\xi \in \Lambda_0 \cap (\mathcal{I} \cup \mathcal{O})} \mathcal{I}^\varepsilon_\xi(\mathsf{x}'', x'') e^{i\xi \cdot X''} \tag{3.3.1}$$

where $\mathcal{I}^\varepsilon_\xi \in H^s(\mathbb{R}^d_+; \Gamma^\infty_{p_\xi}(\mathbb{R}^d))$ with $p_\xi = -v_\xi \mathsf{t}^0/\varepsilon \in \mathbb{R}^d$ when $\xi \in \mathcal{O}$. Moreover it satisfies $\mathcal{I}^\varepsilon_\xi = \pi(\xi)\mathcal{I}^\varepsilon_\xi$. Lastly we suppose there is no initial incoming wave : $\mathcal{I}^\varepsilon_\xi = 0,\ \xi \in \mathcal{I}$.

Let $I^{s,\varepsilon}(\Lambda)$ be the space of $\mathcal{I}^\varepsilon$ decomposing as finite sum of kind (3.3.1). We endow this space with the norm

$$\|\mathcal{I}^\varepsilon\|_{I^{s,\varepsilon}} := \sum_{\xi \in \Lambda_0 \cap (\mathcal{I} \cup \mathcal{O})} \|\mathcal{I}^\varepsilon_\xi\|_{H^s(\mathbb{R}^d_+; \Gamma^\infty_{p_\xi}(\mathbb{R}^d))}.$$

3.3.1 WKB approximate solution

Before we give the theorem stating the existence of a WKB approximate solution for (3.1.5) we summarize the set of equations for each profile.

First the leading profile is solution to

$$U^0 = \mathbb{P}U^0 = \mathbb{P}_{ev} U_{ev} + \mathsf{p}\mathbb{P}_{os} U_{os}, \\ \mathbb{P}^i L^1(\partial_x)\mathbb{P}U^0 = 0, \\ \mathbb{P}_{os}\left[L^1(\partial_\mathsf{x}) - L^1(\partial_x)\mathbb{Q}_{os} L^1(\partial_x)\right] \mathsf{p}\mathbb{P}_{os} U^0 = \mathsf{p}\mathbb{P}_{os} F^0, \\ \mathbb{T}U^0 = 0 \ .$$

The initial condition for U^0 reads

$$U^0_{|_{\mathsf{x}_0 = -\mathsf{t}^0, x_0 = -\mathsf{t}^0/\varepsilon, X_0 = -\mathsf{t}^0/\varepsilon^2}} = \mathcal{I}^{0,\varepsilon},$$

Since U^0 is polarized, its initial value must also be so : $\mathcal{I}^{0,\varepsilon} \in I^{s,\varepsilon}(\Lambda_0)$. This value just corresponds to the hyperbolic outgoing modes which reach the boundary at time $t = 0$; so it is an initial condition for $\mathsf{p}\mathbb{P}_{os}U_{os}$. Then this data must satisfy compatibility conditions at the corner $\mathsf{x}_0 = -\mathsf{t}^0,\ x_0 = -\mathsf{t}^0/\varepsilon,\ \mathsf{x}_d = x_d = 0$ (cf. [19, 35]). However our choice of space $(\mathcal{E})$ in the middle scale implies that the profiles U^0_ξ are determined either globally in $\mathbb{R}^{d+1}$ or in $\mathbb{R}^d_{x'}$. So they don't solve mixted boundary value problem on this scale and compatibility conditions shouldn't appear on this scale.

In fact the geometry (reflexion and refraction law) of the problem is described on the slow scale so one must solve the compatibility conditions on this scale in a way which makes them also true on the middle scale. Finally note that choosing an initial data vanishing at infinity at $\mathsf{x}_d = 0$ implies that the whole compatibility conditions are satisfied.

Next, the profiles $U^j,\ j \geq 1$ are determined through the equations :

$$
\begin{aligned}
&(1-\mathbb{P})U^j = \mathbb{Q}(F^{j-2} - L^1(\partial_x)U^{j-1} - L^1(\partial_{\mathsf{x}})U^{j-2}),\\
&\mathbb{P}^i L^1(\partial_x)\mathbb{P}U^j = \mathbb{P}^i(F^{j-1} - L^1(\partial_{\mathsf{x}})U^{j-1} - L^1(\partial_x)(1-\mathbb{P})U^j),\\
&\mathbb{P}_{os}\left[L^1(\partial_{\mathsf{x}}) - L^1(\partial_x)\mathbb{Q}_{os}L^1(\partial_x)\right]\mathsf{p}\mathbb{P}_{os}U^j =\\
&\mathsf{p}\mathbb{P}_{os}\left\{F^j_{os} - L^1(\partial_{\mathsf{x}})(1-\mathbb{P}_{os})U^j - L^1(\partial_x)\mathbb{Q}_{os}[F^{j-1}_{os} - L^1(\partial_{\mathsf{x}})U^{j-1}]\right\}\ .,\\
&\mathbb{T}U^j = 0\ .
\end{aligned}
$$

The initial condition for U^j reads

$$
\mathsf{p}\mathbb{P}_{os}U^j_{|_{\mathsf{x}_0=-\mathsf{t}^0,x_0=-\mathsf{t}^0/\varepsilon,X_0=-\mathsf{t}^0/\varepsilon^2}} = \mathcal{I}^{j,\varepsilon},
$$

with $\mathcal{I}^{j,\varepsilon} \in I^{s-j,\varepsilon}(\Lambda_0)$.

Theorem 3.3.1. *Let $s - n > d/2$, suppose given a finite module Λ_0 satisfying Assumption 3.2.20 and a sequence of initial data $(\mathcal{I}^{j,\varepsilon})_{j\leq n} \in I^{s-j,\varepsilon}(\Lambda_0)$ vanishing at $\mathsf{z} = 0$ at infinite order, then there are $c, \mathsf{t}_\star > -\mathsf{t}_0$, a unique leading term $U^0 \in \mathcal{P}^s(\Omega_{\mathsf{t}}, \Lambda_0),\ \mathsf{t} < \mathsf{t}_\star$ and for all $j \in \mathbb{N}$ a unique $U^j \in \mathcal{P}^{s-j}(\Omega_{\mathsf{t}}, \Lambda_j)$ satisfying the cascade of problem $(P^j)_{j\leq n}$ (see section 5) with $\Lambda_j \supset \mathcal{N}^\flat(\Lambda_{j-1})$ finite. If $\mathsf{t}_\star < \infty$ then there is a ξ such that*

$$
\lim_{\mathsf{t}\to\mathsf{t}_\star} \|\mathcal{U}^0_\xi\|_{L^\infty(\mathbb{R}^d_+\times\mathbb{R}^d)} = +\infty.
$$

This theorem is proved by recurence in section 5. Let us summarize the main steps :

Thanks to the Ansatz 3.2.18 one can write $U^j = (1-\mathbb{P})U^j + \mathbb{P}_{ev}U^j + \mathsf{p}^i\mathbb{P}_{os}U^j + \mathsf{p}\mathbb{P}_{os}U^j$. First one determines $(1-\mathbb{P})U^j$ from the microscopic equation and then $\mathsf{p}^i\mathbb{P}_{os}U^j$ from the $L^2(\mathbb{R}^{d+1})$ part of the middle scale equation. Next, one expresses the outgoing part of $\mathsf{p}\mathbb{P}_{os}U^j(\mathsf{t} = 0)$ with the initial data and then one gets $\mathsf{p}\mathbb{P}_{os}U^j(\mathsf{t}),\ \mathsf{t} < \mathsf{t}_*$ by solving a problem on the slow scale. This problem is in fact a set of independant systems each of which involving nonlinearly coupled Schrödinger equations related to one characteristic speed (see section §5.2.1). We point that thanks to the Ansatz the nonlinearity involves only the outgoing part of $\mathsf{p}\mathbb{P}_{os}U^j$.

One then gets a boundary value which enables to solve the boundary equation in term of the incoming waves : $\mathbb{P}_{ev}U^j(\mathsf{x}_d = 0, x_d = 0, X_d = 0)$ and the incoming hyperbolic part of $\mathsf{p}\mathbb{P}_{os}U^j(\mathsf{x}_d = 0, x_d = 0, X_d = 0)$. One finally gets $\mathbb{P}_{ev}U^j$ thanks to the microscopic equation and lastly the incoming hyperbolic part of $\mathsf{p}\mathbb{P}_{os}$ exactly as the outgoing part of $\mathsf{p}\mathbb{P}_{os}U^j$.

Finally one must check that the Ansatz 3.2.18 is stable by non-linearity. This is acheived by using the Assumption 3.2.20 (see lemma 3.5.9).

3.3.2 Convergence

As the initial system (3.1.5) is characteristic one must consider spaces with non-homogeneous regularity as in [19]. So as in [42] introduce the tangential spatial Sobolev spaces :

$$H^m_{tan}(\mathbb{R}^d_+) = \{u \in L^2(\mathbb{R}^d_+) \,:\, \partial_1^{m_1} \dots \partial_{d-1}^{m_{d-1}} u \in L^2 \quad m_1 + \dots + m_{d-1} \leq m\}.$$

Then the C^m-tangential spaces are

$$T^m([0,t_*] \times \mathbb{R}^d_+) = \bigcap_{r \leq m} C^r([0,t_*], H^{m-r}_{tan}(\mathbb{R}^d_+)).$$

Then, as it is known for maximal dissipative characteristic boundary problems with constant coefficients one can easily get estimates in T^m. But since the boundary is characteristic, in general, one can't fully express the normal derivative through the equation. Following [19], the natural estimates require two tangential derivatives for one normal derivative. This leads to introduce the spaces for m even

$$E^m_{sp}(\mathbb{R}^d_+) = \{u \,:\, \partial^k_{x_d} u \in H^r_{tan}(\mathbb{R}^d_+) \quad \text{for } 2k + r \leq m\}.$$

Then the C^m-space

$$E^m([0,t_*] \times \mathbb{R}^d_+) = \bigcap_{r \leq m} C^r([0,t_*], E^{m-r}_{sp}).$$

These spaces are equipped with the obvious norms.

To prove the stability of the WKB expansion we take into account the fast oscillations by taking weighted norms : in the definitions above, we replace the tangential derivatives $\partial_{x_0}, \dots, \partial_{x_{d-1}}$ by $\varepsilon\partial_{x_0}, \dots, \varepsilon\partial_{x_{d-1}}$ and the normal derivative ∂_{x_d} by $\varepsilon^2\partial_{x_d}$. This does not change the space but changes the norms. We denote them by using the superscript ε. For instance

$$\|u(t)\|_{E^{m,\varepsilon}([0,t_*])} = \sum_{2\alpha_d + |\alpha'| \leq m} \varepsilon^{2\alpha_d + |\alpha'|} \sup_{t \in [0,t_*]} \|\partial^\alpha_x u(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}. \tag{3.3.2}$$

We get the following theorem for the convergence.

Theorem 3.3.2. *Let n, $m = s - n$ even and M such that $m > (d+1)/2$, $M \leq n$ and $s \geq m + M + d + 2$.*

Let $e^\varepsilon(x'') = \sum_{l \leq n} \varepsilon^l \mathcal{I}^{l,\varepsilon}(\varepsilon x'', x'', x''/\varepsilon) + \varepsilon^M g^\varepsilon(x'')$ with $\mathcal{I}^{l,\varepsilon} \in I^{s-l}(\Lambda_0)$ as in theorem 3.3.1 and $g^\varepsilon \in H^m(\mathbb{R}^d)$, vanishing up to order m at $z = 0$.

Then there is a $\mathtt{t}^\sharp \leq \mathtt{t}^\star$, independent from ε such that for all $t < \mathtt{t}^\sharp/\varepsilon$ there is a $r \in E^{m,\varepsilon}$ such that $u^\varepsilon(\mathsf{x}) = \sum_{l \leq n} \varepsilon^j U^j(\varepsilon x, x, x/\varepsilon) + \varepsilon^M r^\varepsilon(x)$ solves (3.1.5).

Remarks 3.3.3. *From assumption 3.2.6, the profiles solve non characteristic scalar equations so that the natural spaces are Sobolevs'.*

The time of existence is, as expected, in $\mathcal{O}(1/\varepsilon)$ with respect to the middle variables t but may not be positive. This would mean no "physical" reflection.

3.4 Equation on the middle scale

3.4.1 Properties of $\mathcal{E}^s$

We first give the essential properties of $\mathcal{E}^s$.

Lemma 3.4.1. *For $u \in \mathcal{E}^s,\ s > (d+1)/2$ the decomposition $u = \tilde{u} + \tilde{\tilde{u}}$ is unique.*

Démonstration. Indeed suppose $u = \tilde{u}^1 + \tilde{\tilde{u}}^1 = \tilde{u}^2 + \tilde{\tilde{u}}^2$.

Suppose $u^1 \neq 0$ so consider a speed v'_p and set $x' = x'_0 + v'_p z$. Then express $u(x'_0 + v'_p z, z) = \sum_j \mathcal{U}^1_j(x'_0 + (v'_p - v'_j)z) + \tilde{\tilde{u}}^1(x'_0 + v'_p z, z) \underset{z\to\infty}{\to} \mathcal{U}^1_p(x'_0)$. And using the second expression

$$u(x'_0 + v'_p z, z) \underset{z\to\infty}{\to} \begin{cases} 0 & \text{if } v'_k \neq v'_p,\ \forall k \\ \mathcal{U}^2_q(x'_0) & \text{if } \exists\ q \text{ such that } v'_q = v'_p \end{cases}$$

This entails $\mathcal{U}^j = 0, \mathcal{U}^k = 0,\ \forall j,k$ except for the (p,q) such that $v'_p = v'_q$. Thus $\tilde{\tilde{u}}^1 = \tilde{\tilde{u}}^2$. □

Then we give a lemma for non-linear estimates.

Lemma 3.4.2. *Let $s > (d+1)/2$ and $u_1, u_2 \in \mathcal{E}^s$ and note $w = u_1 u_2$. Define $\tilde{w}(x) = \sum_{(\alpha,\beta)\in F} \mathcal{U}_{1,\alpha}(x' - v'_\alpha z)\mathcal{U}_{2,\beta}(x' - v'_\beta z)$ for a finite set F where $v'_\alpha = v'_\beta$. Set $\tilde{\tilde{w}} = w - \tilde{w}$ Then*

$$\|\tilde{\tilde{w}}\|_{\Gamma^\sigma} \leq C(\|u_1\|_{L^\infty}, \|u_2\|_{L^\infty})\|u_1\|_{\mathcal{E}^s}\|u_2\|_{\mathcal{E}^s}, \quad \forall \sigma < s - 1/2.$$

Démonstration. We review the different terms involved in $\tilde{\tilde{w}}$:

1) From [14] as $s > (d+1)/2,\ \Gamma^s(\mathbb{R}^{d+1}_+)$ is an algebra and $\tilde{\tilde{u}}_1\tilde{\tilde{u}}_2 \in \Gamma^s(\mathbb{R}^{d+1}_+)$.

2) Let show that if $v'_\alpha \neq v'_\beta$ then $\mathcal{U}_{1,\alpha}\mathcal{U}_{2,\beta} \in \Gamma^\sigma(\mathbb{R}^{d+1}_+)$. For sake of readability we note $a = \mathcal{U}_{1,\alpha}$ and $b = \mathcal{U}_{2,\beta}$. It is sufficient to prove

$$< x >^\gamma \partial^p_{x'} a \partial^q_{x'} b \in L^2(\mathbb{R}^d),\ |p| + |q| + \gamma \leq \sigma.$$

Then note $\mathcal{D}^p a = < x' - v'_\alpha z >^{s-|p|} \partial^p_{x'} a$.

$$\Big\| \frac{< x >^\gamma}{< x' - v^\alpha z >^{s-|p|} < x' - v^\beta z >^{s-|q|}} \mathcal{D}^p a \mathcal{D}^q b \Big\|_{L^2(\mathbb{R}^{d+1}_+)} \leq$$
$$\|H^\theta_{p,q}(x) Q(x)^{\gamma\theta} \mathcal{D}^p a\|_{L^{2s/j}(\mathbb{R}^{d+1}_+)}\ \|H^{1-\theta}_{p,q}(x) Q(x)^{\gamma(1-\theta)} \mathcal{D}^q b\|_{L^{2s/m}(\mathbb{R}^{d+1}_+)},$$

where $0 \leq \theta \leq 1,\ j + m = s,\quad H_{p,q}(x) = \frac{1}{< x' - v^\alpha z >^{s-|p|-\gamma} < x' - v^\beta z >^{s-|q|-\gamma}}$ and $Q(x) = \frac{< x >}{< x' - v^\alpha z >< x' - v^\beta z >}$.

If $s - |p| \leq s - |q|$ then $H_{p,q}(x) \leq \frac{1}{< x >^{s-|p|-\gamma}}$. It entails that Q is bounded.

Thus $\|H^\theta_{p,q}(x) Q(x)^{\gamma\theta} \mathcal{D}^p a\|_{L^{2s/j}(\mathbb{R}^{d+1}_+)} \leq \|\frac{1}{< x >^{\theta(s-|p|-\gamma)}} \mathcal{D}^p a\|_{L^{2s/j}(\mathbb{R}^{d+1}_+)}$

Make the change of variable $\tilde{x}' = x' - v^\alpha z$ in the first integral.

$$\|\frac{1}{< x >^{\theta(s-|p|-\gamma)}} \mathcal{D}^p a\|^{2s/j}_{L^{2s/j}} = \int_{\tilde{x}'} |\mathcal{D}^p a(\tilde{x}')|^{2s/j} \int_z \frac{1}{< (\tilde{x}' + v^\alpha z, z) >^{2s\theta(s-|p|-\gamma)/j}}. \tag{3.4.1}$$

The last integral is bounded iff $2s\theta(s-|p|-\gamma)/j > 1$ and this uniformly in $\tilde{x}'$. The integral on $\tilde{x}'$ is bounded for $j \leq |p|$ thanks to the Gagliardo-Nirenberg inequality in [14].

Applying the same for the other integral it leads $2s(1-\theta)(s-|p|-\gamma)/m > 1$. Thus $s-|p|-\gamma > \frac{1}{2s}\max\{j/\theta, m/(1-\theta)\} \geq 1/2$. Hence $\gamma + |p| < s - 1/2$ which means that one loses $1/2$ derivative.

3) The same analysis shows that $\tilde{u}^1\tilde{\tilde{u}}^2 \in \Gamma^\sigma(\mathbb{R}^{d+1}_+),\ \sigma < s-1/2$ (with the difference that there is no condition coming from $\tilde{\tilde{u}}^2$). □

3.4.2 Proof of Theorem 3.2.17

To show Theorem 3.2.17 one first make a Fourier mode by mode analysis. One has to solve

$$X_\xi(\partial)\pi(\xi)u_\xi = \pi(\xi)G_\xi, \quad \text{in } \mathbb{R}^{d+1}_+\,, \qquad\qquad u_\xi, G_\xi \in \mathcal{E}^\infty(\Lambda)\,. \tag{3.4.2}$$

As in the introduction we separate the transports from the global decreasing part. This is done through the operator $\mathsf{p}^i(\xi)$:

$$\begin{aligned}
&X_\xi(\partial)\mathsf{p}(\xi)\pi(\xi)u_\xi = 0 = \mathsf{p}(\xi)\pi(\xi)G_\xi\\
&X_\xi(\partial)(\mathsf{p}^i - \mathsf{p}(\xi))\pi(\xi)u_\xi = (\mathsf{p}^i(\xi) - \mathsf{p}(\xi))\pi(\xi)G_\xi\\
&X_\xi(\partial)\widetilde{\widetilde{\pi(\xi)u_\xi}} = \widetilde{\widetilde{\pi(\xi)G_\xi}}
\end{aligned}$$

The first equation gives the first compatibility condition of theorem 3.2.8.

The second equation requires a compatibility condition given in the lemma :

Lemma 3.4.3. *Let $f \in \Gamma^s(\mathbb{R}^d)$. The equation*

$$(\partial_z + v'\nabla_{x'})u = f(x' - w'z), \quad v' \neq w',$$

has a unique solution $u(x) = \mathcal{U}(x'-w'z), \quad \mathcal{U} \in \Gamma^s(\mathbb{R}^d)$ iff $j_{v,w}f = \int_{-\infty}^{\infty} f(x'-(w'-v')z)dz = 0$.

Démonstration. 1. Suppose $u(x) = \mathcal{U}(x' - w'z)$ with $\mathcal{U} \in \Gamma^s(\mathbb{R}^d)$. Then $u(x', z) \xrightarrow[z\to\infty]{} 0$. It reads $j_{v,w}f = 0$.

2. Conversely we show that if $j_{v,w}f = 0$ then there is a unique $\mathcal{U} \in \Gamma(\mathbb{R}^d)$ such that $u(x) = \mathcal{U}(x' - w'z)$.

Set $a(x) = u(x' + v'z, z)$. Then $\partial_z a(x', z) = f(x' - (w' - v')z)$. Since $f \in \Gamma^s(\mathbb{R}^d)$ and $v' \neq w', \quad U(x) = \int_{-\infty}^{z} f(x' - (w' - v')s)ds$ converges for all x.

Thus one can look for $a(x) = a_0(x') + \int_{-\infty}^{z} f(x' - (w' - v')s)ds$ where $\lim_{z\to-\infty} a(x', z) = a_0(x')$.

But from the assumption $j_{v,w}f = 0$, $\lim_{z\to+\infty} a(x', z) = 0,\ \forall x' \in \mathbb{R}^d$. Thus $a_0 = 0$ and $u(x) = U(x' - v'z, z)$.

Then note that $U(x - v'z, z) = \mathcal{U}(x' - w'z)$ with $\mathcal{U}(x') = \int_{-\infty}^{0} f(x' - (w' - v')s)ds$.

To complete the proof it remains to show that $\mathcal{U} \in \Gamma^s(\mathbb{R}^d)$.

Set $\mathcal{D}_x^p = (\partial_x, < x >)^p$ and remark $\mathcal{D}_{x'}^s \mathcal{U} = \sum_{\alpha \le s} \int_{-\infty}^0 < x' >^\alpha \partial_{x'}^{s-\alpha} f(x' - \delta' s)$. Thus one needs only showing the convergence of $\| \int_{-\infty}^0 < x' >^\alpha \partial_{x'}^{s-\alpha} f(x' - \delta' s) ds \|_{L^2(\mathbb{R}^d_{x'})}$.

Then set $\delta' = (\delta_1, \tilde{\delta}) = w' - v'$, and suppose $\delta_1 \neq 0$. One has

$$\| \int_{-\infty}^0 < x' >^\alpha \partial_{x'}^{s-\alpha} f(x' - \delta' s) ds \|_{L^2(\mathbb{R}^d_{x'})} = \frac{1}{\delta_1} \| \int_{-\infty}^{x_1} < x_1, \tilde{y} - x_1 \tilde{\delta}/\delta_1 >^\alpha g(u, \tilde{y}) du \|_{L^2(\mathbb{R}^d_{x_1, \tilde{y}})}$$

where $u = x_1 - \delta_1 s, \quad x' = (x_1, \tilde{x}), \ \tilde{y} = \tilde{x} + x_1 \tilde{\delta}/\delta_1$ and $g(u, \tilde{y}) = \partial_{x'}^{s-\alpha} f(u, \tilde{y} - \tilde{\delta}/\delta_1 u)$. One can check that $g \in \Gamma^\alpha(\mathbb{R}^d)$.

The crucial observation is that from assumption $j_{v,w} f = 0, \quad \int_{-\infty}^{x_1} g(u, \tilde{y}) du = - \int_{x_1}^{\infty} g(u, \tilde{y}) du$. Thus it is sufficient to show the convergence of the previous integral in $L^2(\mathbb{R}^+_{x_1} \times \mathbb{R}^{d-1}_{\tilde{y}})$.

Next using that $|u| \ge |x_1|$ allows to replace the weight by $< u, \tilde{y} - x_1 \tilde{\delta}/\delta_1 >$. Then using the Cauchy-Schwartz inequality gives

$$\int_{-\infty}^{x_1} < u, \tilde{y} - x_1 \tilde{\delta}/\delta_1 >^\alpha g(u, \tilde{y}) du \le \int_{-\infty}^{x_1} \frac{1}{u^{2p}} du \int_{-\infty}^0 | u^p < u, \tilde{y} - x_1 \tilde{\delta}/\delta_1 >^\alpha g(u, \tilde{y})|^2 du.$$

Replacing g by its definition shows the second integral depends on $\tilde{y} - x_1 \tilde{\delta}/\delta_1$ which by integration over $\mathbb{R}^{d-1}_{\tilde{y}}$ doesn't depend on x_1. Lastly

$$\| \int_{-\infty}^{x_1} < x_1, \tilde{y} - x_1 \tilde{\delta}/\delta_1 >^\alpha g(u, \tilde{y}) du \|_{L^2(\mathbb{R}^d_{x_1, \tilde{y}})} \le \int_{x_1 \le 0} \int_{-\infty}^{x_1} \frac{1}{u^{2p}} du \| f \|_{\Gamma^{s+p}(\mathbb{R}^- \times \mathbb{R}^{d-1})}.$$

Take $p > 1$ for the first double integral to be convergent. □

Lastly for the third equation we have the lemma

Lemma 3.4.4. *The equation $X(\partial) u = f$ with $f \in \Gamma^s(\mathbb{R}^{d+1}_+)$ has a unique solution which belongs to $\Gamma^\sigma(\mathbb{R}^{d+1}_+), \ \sigma < s - 1$. It reads*

$$u = - \int_z^\infty f(x' - v_\alpha(z - s), s) ds. \tag{3.4.3}$$

Démonstration. One makes the change of variable $\check{u} = u(x' + v' z, z)$ so the equation writes

$$\partial_z \check{u}(x', z) = f(x' + v' z, z).$$

Next one shows that $\int_z^\infty f(x' - v'(z - s), s) ds \in \Gamma^s(\mathbb{R}^{d+1}_+)$.

$$< x >^\gamma \int_z^\infty \partial_x^q f(x' - v'(z - s), s) ds = \int_z^\infty < u + v' z, z >^\gamma \partial_x^q f(u + v' s, s) ds$$

Where $u = x' - v' z$. Then note for $s \ge z$ there is a constant C such that $\frac{<u + v' z, z>}{<u + v' s, s>} \le C$. Then using the Cauchy-Schwartz inequality

$$\int_z^\infty < u+v'z, z >^\gamma \partial_x^q f(u+v's,s)ds \leq \| \frac{1}{< (u+v'.,.) >^\delta} \|_{L^2([z,\infty[)} \tag{3.4.4}$$

$$\| f^l_{\gamma+\delta,\xi}(u+v'.,.) \|_{L^2([z,\infty[)} \tag{3.4.5}$$

With the notation $f^l_{\gamma+\delta,q}(x) =< x >^{\gamma+\delta} \partial_x^q f(x)$.
At last note that $\frac{1}{<u+v's,s>} \leq \frac{c}{<s>}$. Gathering the results

$$\| < x >^\gamma \int_z^\infty \partial_x^\xi f(x'-v'(z-s),s)ds \|_{L^2_x} \leq \| f^l_{\gamma+\delta,\xi} \|_{L^2_{\tilde{u},s}} \int_{z\geq 0} \int_{s=z}^\infty \frac{c}{< s >^{2\delta}} ds \tag{3.4.6}$$

The first norm exists if $\gamma+\delta+|\xi| \leq s$ and the double integral exists if $\delta > 1$. □

We prove a lemma of restriction to $\mathbb{R}^d$ of function lying in $\Gamma^\infty(\mathbb{R}^{d+1}_+)$.

Lemma 3.4.5. *Let $u \in \Gamma^s(\mathbb{R}^{d+1}_+)$. Then $u_{|_{x_d=0}}$ belongs to $\Gamma^\sigma(\mathbb{R}^d)$, $\sigma < s-3/2$.*

Démonstration. We prove a little more general property : let κ be an hyperplane in $\mathbb{R}^{d+1}$ with normal ν and consider $v = u_{|_\kappa}$. We show that $v \in \Gamma^\sigma(\mathbb{R}^d), \sigma < s-1/2$. One has

$$\begin{aligned} \| < x' >^\alpha \partial^q_{x'} v \|_2 &\leq c \big\| < x' >^\alpha \int_{s\in\mathbb{R}} (\nu\cdot\nabla)\partial^q_{x'} u(s)ds \big\|_2 \\ &\leq \big\| \frac{1}{< x' >^\delta} \big\|_{L^2(\mathbb{R}^d)} \big\| < x' >^{\alpha+\delta} (\nu\cdot\nabla)\partial^q_{x'} u \big\|_{L^2(\mathbb{R}^{d+1}_+)} \end{aligned}$$

This is bounded iff $\delta > 1/2$ and $\alpha+\delta+|q| \leq s$. So $\alpha+|q| < s-1/2$. □

3.5 Construction of the profiles.

We expand the solution :

$$u^\varepsilon = \sum_{j=0}^n \varepsilon^n U^j(\varepsilon x, x, x/\varepsilon) + \varepsilon^M v(x) = U^n_{\text{app}}(\varepsilon x, x, x/\varepsilon) + \varepsilon^M v^\varepsilon, \tag{3.5.1}$$

$$u^\varepsilon_{|_{t=0}} = \sum_{l\leq n} \varepsilon^l \mathcal{I}^l(\varepsilon x'', x'', x''/\varepsilon) + \varepsilon^n r(x'') = \mathcal{I}^{n,\varepsilon}_{\text{app}} + \varepsilon^n r^\varepsilon. \tag{3.5.2}$$

We plug (3.5.1) in (3.1.5) and obtains

$$L^\varepsilon(\partial)u^\varepsilon - f(u^\varepsilon) = \Big[L^\varepsilon(\varepsilon\partial_{\mathsf{x}} + \partial_x + \frac{1}{\varepsilon}\partial_X) U^n_{\text{app}} + \varepsilon^M L^\varepsilon(\partial) v^\varepsilon \Big]_{\mathsf{x}=\varepsilon x,\, X=x/\varepsilon} - f(u^\varepsilon)$$

Equating to zero the coefficients of the expansion gives

$$(P^0) \begin{cases} L(\partial_X)U^0 = 0 \\ TU^0 = 0, \quad \mathsf{z} = z = 0 \\ (\mathsf{p}U^0)|_{\mathsf{t}=-\mathsf{t}^0, t=-\mathsf{t}^0/\varepsilon, T=-\mathsf{t}^0/\varepsilon^2} = \mathcal{I}^{0,\varepsilon}, \end{cases}$$

$$(P^1)\begin{cases} L(\partial_X)U^1 + L_1(\partial_x)U^0 = 0 \\ TU^1 = 0, \quad \mathsf{z} = z = 0 \\ (\mathsf{p}U^1)|_{\mathsf{t}=-\mathsf{t}^0, t=-\mathsf{t}^0/\varepsilon, T=-\mathsf{t}^0/\varepsilon^2} = \mathcal{I}^{1,\varepsilon}, \end{cases}$$

$$(P^{j+1})\begin{cases} L(\partial_X)U^{j+1} + L_1(\partial_x)U^j + L_1(\partial_\mathsf{x})U^{j-1} = F^{j-1} \\ TU^{j+1} = 0, \quad \mathsf{z} = z = 0 \\ (\mathsf{p}U^{j+1})|_{\mathsf{t}=-\mathsf{t}^0, t=-\mathsf{t}^0/\varepsilon, T=-\mathsf{t}^0/\varepsilon^2} = \mathcal{I}^{j+1,\varepsilon}. \end{cases}$$

3.5.1 Construction of the leading profile.

A) First we analyse the microscopic equations :

1. Consider the first equation in (P^0). From theorem 3.2.8 there holds $U^0 = \mathbb{P}U^0$.
2. Consider the first equation in (P^1). From theorem 3.2.8 this equation has a solution in $\mathcal{P} \supset \mathcal{P}^s$ if and only if $\mathbb{P}^i L^1(\partial_x)\mathbb{P}U^0 = 0$. The solution then reads $U^1 = \mathbb{P}U^1 - \mathbb{Q}L^1(\partial_x)\mathbb{P}U^0$. Fourier decomposing the compatibility condition on Λ_0 satisfying assumption 3.2.22 gives

$$X_\xi(\partial_x)U^0_\xi = 0 \tag{3.5.3}$$

   From Theorem 3.2.17 $\mathbb{P}_{os}U^0 = \mathsf{p}^i\mathbb{P}_{os}U^0$ and as we look for U^0 satisfying the ansatz (3.2.18) $\mathbb{P}_{os}U^0 = \mathsf{p}\mathbb{P}_{os}U^0$.
3. Consider the first equation in (P^2). Again from Theorem 3.2.8 this equation has a solution if and only if the equation (3.2.5) holds.

B) Analysys of the middle scale equation (3.2.5).

From Theorem 3.2.17 the equation (3.2.5) has a solution iff $\mathsf{j}G^0 = 0$ and $\mathsf{p}G^0 = 0$ with $G^0 = \mathbb{P}_{os}\left[-L^1(\partial_\mathsf{x}) + L^1(\partial_x)\mathbb{Q}L^1(\partial_x)\right]\mathbb{P}_{os}U^0 + \mathbb{P}_{os}F^0$.

1. Consider the first compatibility condition $\mathsf{j}G^0 = 0$. This equation only involves $(\mathsf{p}^i - \mathsf{p})G^0 = (\mathsf{p}^i - \mathsf{p})F^0$. We next show that under assumption 3.2.20 this quantity vanishes so that the compatibility condition is satisfied.

   Lemma 3.5.1. *Let $\Lambda \in \mathbb{R}^{d+1} \cap charL$ and $(U_\alpha)_{\alpha\in\Lambda}$ a (finite) family of profiles in $\mathcal{E}^\infty(\Lambda)$. Suppose that Assumption 3.2.20 is satisfied.*
   Then for all $\xi \in \Lambda$ and for all $k-$uplet $(U_{\alpha_1}, \dots, U_{\alpha_k})$ with k odd and $\sum_{1\le j\le k}\alpha_j = \xi$ there holds $(\mathsf{p}^i(\xi) - \mathsf{p}(\xi))\prod_{1\le j\le k}U_{\alpha_j} = 0$.

   Démonstration. $U_{\alpha_j} \in \mathcal{E}(\Lambda)$ writes $U_{\alpha_j}(x) = \sum_{\beta\in\Lambda}\mathcal{U}_{\alpha_j,\beta}(x' - v'_\beta z) + (1 - \mathsf{p}^i(\alpha_j))U_{\alpha_j}(x)$. Then from lemma 3.4.2 the product belongs to $\mathcal{E}(\Lambda)$. It writes :

$$P_\xi(x) := \prod_{1\le j\le k} U_{\alpha_j}(x) = \sum_{\gamma\in\Lambda} P_{\xi,\gamma}(x' - v'_\gamma z) + \tilde{\tilde{P}}_\xi(x),$$

   where $P_{\xi,\gamma}(x') = \sum_{\beta_1,\dots,\beta_j\in\Lambda}\prod_{1\le j\le k}\mathcal{U}_{\alpha_j,\beta_j}(x')$, $v'_{\beta_1} = \dots = v'_{\beta_k} = v'_\gamma$. From assumption 3.2.20 for $v'_\xi \neq v'_\gamma$ one has $P_{\xi,\gamma} = 0$. Thus $(\mathsf{p}^i(\xi) - \mathsf{p}(\xi))P_\xi(x) = \sum_{v'_\gamma \neq v'_\xi} P_{\xi,\gamma}(x' - v'_\gamma z) = 0$. □

2. Then consider the second compatibility condition $\mathsf{p}G^0 = 0$. This equation involves the second order operator $\mathbb{P}_{os}L_1(\partial_x)\mathbb{Q}L_1(\partial_x)\mathbb{P}_{os}$ which is described in [14], Proposition 3.2 that we recall :

Proposition 3.5.2.

$$\mathbb{P}_{os}L_1(\partial_x)\mathbb{Q}_{os}L_1(\partial_x)\mathbb{P}_{os} = iR_{os}(\partial_x)\mathbb{P}_{os}.$$

where $R_{os}(\partial_x)$ decomposes on $\mathcal{P}^s_{os}$ into $R_{os}(\partial_x)U = \sum_\xi R(\xi)(\partial_x)U_\xi$ where $R(\xi)(\partial_x)$ is a second order scalar operator.

Definition 3.5.3. *Let $\mathcal{R}(\xi)(\partial'_x)$ the operator defined on $\Gamma^\infty(\mathbb{R}^d)$ by $(\mathcal{R}_\xi(\partial_{x'})\mathcal{U}_\xi)\circ h = R_\xi(\partial_x)(\mathcal{U}_\xi \circ h)$, $\quad h : (x', z) \to x' - v'_\xi z$.*
Then define $\mathcal{R}_{os}$ similarly as $\mathbb{X}_{os}$ on $\mathsf{p}\mathbb{P}_{os}\mathcal{P}^s(\Omega_\mathsf{t}, \Lambda)$ by

$$\mathcal{R}_{os}\mathsf{p}\mathbb{P}_{os}U = \sum_{\xi\in\Lambda} \mathcal{R}_\xi(\partial_{x'})\pi(\xi)\mathcal{U}_\xi e^{i\xi X}.$$

So $\mathsf{p}(\xi)G^0_\xi = 0$ writes

$$\left[\mathbb{X}(\partial_\mathsf{x}) + \frac{i}{2}\mathcal{R}(\xi)(\partial_{x'})\right]\mathcal{U}^0_\xi(\mathsf{x}, x') = \mathsf{p}(\xi)\pi(\xi)F^0_\xi(\mathsf{x}, x'), \quad \xi \in \Lambda_0 \tag{3.5.4}$$

Remark 3.5.4. *The Schrödinger equation* (3.5.4) *expresses through a unique slow variable associated to the operator X_ξ. Remark that the non-linear term writes as a combination of resonant harmonic of ξ and one essentially expects a sum of odd products $U^0_\xi U^0_{-\xi}U^0_\xi$ such as in [13]. Remark this requires real systems for which $-\xi$ is resonant when ξ is resonant and $v_{-\xi} = v_\xi$.*

Every other resonant combination is nearly unexpected since all combination $U^0_{n\xi}U^0_{-n\xi}U^0_\xi$, would require $n\xi$ resonant with $v_{n\xi} = v_\xi$.

3.5.2 Solvability of the leading profile equation

Here we use the fact that the outgoing modes and the incoming modes are completely decoupled in the equations (3.5.4). Let introduce the projectors

$$\mathsf{p}^+ := \sum_{\xi\in\mathcal{O}} \mathsf{p}(\xi)\pi(\xi), \qquad \mathsf{p}^- := \sum_{\xi\in\mathcal{I}} \mathsf{p}(\xi)\pi(\xi)\ ,$$

on the outgoing and incomind modes.

Solvability of the outgoing modes

The outgoing modes are solution to Cauchy problems. So we construct the initial data $\mathcal{U}^0_\xi(\mathsf{t} = -\mathsf{t}^0),\ \xi \in \mathcal{O}\cap\Lambda_0$ for the equation (3.5.4) from the initial value $\mathcal{I}^{\varepsilon,0}_\xi \in I^{s,\varepsilon}$.

Lemma 3.5.5. *The Cauchy problem*

$$X_\xi(\partial_x)u_\xi = 0, \qquad u_\xi(t = 0) = \mathcal{I}^\varepsilon_\xi, \quad \xi \in \mathcal{O}$$

has a unique solution $u_\xi = \mathcal{U}_\xi(x' - v'_\xi z)$ with $\mathcal{U}_\xi \in \Gamma^\infty(\mathbb{R}^d)$.

Démonstration. For sake of readability we drop the subscript ξ.

Set $v' = (1, v_1 \ldots, v_d)/v_d$. Then, let $h_v^\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ the isomorphism defined by

$$h_v^\varepsilon(x') = (x'' - vt)|_{z=0} - v\mathsf{t}^0/\varepsilon, \quad (h_v^\varepsilon)^{-1}(x'') = (x' - v'z)|_{t=0} - (\mathsf{t}^0/\varepsilon, 0).$$

There holds $h_v^\varepsilon(x' - v'z) - h_v^\varepsilon(0) = x'' - vt$.

Let us set

$$\mathcal{U} = \mathcal{I}^\varepsilon \circ (h_v^\varepsilon - h_v^\varepsilon(0)).$$

Then $u(x) = \mathcal{I}^\varepsilon(x'' - vt) = \mathcal{U}(x' - v'z)$ and since $\mathcal{I}^\varepsilon \in \Gamma^s_{h_v^\varepsilon(0)}(\mathbb{R}^d_{x''})$ then $\mathcal{U} \in \Gamma^s(\mathbb{R}^d)$. □

Now one solves the Cauchy problem for the equation (3.5.4) for p^+U^0.

Lemma 3.5.6. *Let $s > d/2$ and suppose given an outgoing polarized initial data : $\mathcal{I}_+ \in I^{s,\varepsilon}(\Lambda_0)$, then there is a t^+ and a unique $\mathsf{p}^+U^0 \in \mathcal{P}^s(\Omega_{\mathsf{t}}, \Lambda_0)$, $\mathsf{t} \leq \mathsf{t}^-$ satisfying the Cauchy problem with* (3.5.4) *for the outgoing modes.*

Démonstration. (3.5.4) is a non-linearly coupled system and it decomposes into smaller problems $P_{\underline{\xi}}$ combining only the modes propagating at speed $v_{\underline{\xi}}$. From assumption 3.2.22 this set is finite.

Each profile involved in $P_{\underline{\xi}}$ is a function of the variables $\mathsf{x}, x' - \mathbf{v}_{\underline{\xi}} z$. Denoting by $O_\xi(w)$ the operator $e^{-\mathcal{R}_\xi(\partial_{x'})w}$, the solution reads

$$\begin{aligned}\mathcal{U}^0_\xi(\mathsf{t}, \mathsf{x}", x') &= O_\xi(\mathsf{t})\mathcal{I}^{0,\varepsilon}_\xi(\mathsf{x}'' - \mathbf{v}_\xi \mathsf{t}, x') \\ &\quad + \int_0^{\mathsf{t}} O_\xi(\mathsf{t} - s)\mathsf{p}(\xi)\pi(\xi)F^0_\xi(s, \mathsf{x}'' - \mathbf{v}_\xi(\mathsf{t} - s), x')ds \quad \mathsf{t} \leq \mathsf{t}_\xi.\end{aligned}$$

For an energy estimate one must first notice $O_\xi(\mathsf{t})$ maps $\Gamma^s(\mathbb{R}^d)$ into $\Gamma^s(\mathbb{R}^d)$ boundedly with a bound equal to $C(s)e^{c_\xi \mathsf{t}}$ where c_ξ is a constant (cf. [14], lemma 4.2). Then one takes the X^s norm of the previous equation

$$\|\mathcal{U}^0_\xi(\mathsf{t})\|_{X^s} \leq C(s)e^{c_\xi \mathsf{t}}\|\tilde{\mathcal{I}}^{0,\varepsilon}_\xi\|_{X^s} + c\sqrt{\mathsf{t}}\int_0^{\mathsf{t}} e^{c_\xi(\mathsf{t}-\sigma)}\|\mathsf{p}(\xi)\pi(\xi)F^0_\xi(\sigma)\|_{X^s}d\sigma.$$

And since in H^s, Γ^q the interpolation inequalities are valid one has the Gagliardo-Niremberg inequality

$$\|\partial^{l_1}_{\mathsf{x}''}(< x' >, \partial_{x'})^{l_2}U\|_{L^{2s/l}(\mathbb{R}^d \times \mathbb{R}^d)} \leq C\|U\|^{1-l/s}_{L^\infty(\mathbb{R}^d_{\mathsf{x}''} \times \mathbb{R}^d_{x'})}\|U\|^{l/s}_{X^s}$$

where $l_1 + l_2 \leq l$, $l_1 \leq s$. And since $F^0(0) = 0$,

$$\|F^0(U)\|_{X^s} \leq C(\|U\|_\infty)\|U\|_{X^s}.$$

Then setting $N^0(\mathsf{t}) = \sum_{\beta \in P_{\underline{\xi}}} \|U^0_\beta(\mathsf{t})\|_{X^s}$ we get

$$N^0(\mathsf{t}) \leq c_1 e^{c_2 \mathsf{t}}N^0(0) + c_3\sqrt{\mathsf{t}}\int_0^{\mathsf{t}} e^{c_4(t-s)}C(\|U\|^0_{L^\infty})N^0(s)ds.$$

So one can use Picard iterates to solve $P_{\underline{\xi}}$ in the Banach algebra X^s and find a time existence t_ξ.

Then by a Gronwall argument one gets a bound for $\|\mathcal{U}^0_\xi(\mathsf{t})\|_{\mathcal{C}^0([0,\mathsf{t}];X^s)}$ which depends on $\sup_{\mathsf{t}} \|\mathcal{U}^0_\xi(\mathsf{t})\|_{L^\infty(\mathbb{R}^d_+\times\mathbb{R}^d)}$. This suppremum does not depend on s hence neither does t_ξ. When the suppremum tends to infinity the solution stops existing. This gives an explosion criterion. Lastly use equation (3.5.4) repeatedly to see that $\|\mathcal{U}^0_\xi\|_{\tilde{E}^s(\Omega_\mathsf{t})} < \infty,\ \forall \mathsf{t} < \mathsf{t}_\xi$.

Then set $\mathsf{t}^+ = \min_{\alpha\in\mathcal{O}\cap\Lambda_0} \mathsf{t}_\alpha$. □

Next we show the well-posedness of the boundary value problem associated to the equation (3.5.4) for the incoming modes. We need to solve the boundary problem $\mathbb{TP}U^0 = 0$. Writing $\mathbb{P}U^0 = \mathsf{p}^+U^0 + \mathsf{p}^-U^0 + \mathbb{P}_{ev}U^0$ one gets the convenient form

$$\mathbb{T}\mathsf{p}^-U^0 + \mathbb{TP}_{ev}U^0 = -\mathbb{T}\mathsf{p}^+U^0.$$

Then from the Lopatinski conditions 3.2.9 one can define partial inverse for all ξ' satisfying assumption 3.2.4 :

$$\mathbb{T}^{-1}_{os}(\xi',\xi_d),\ \mathbb{T}^{-1}_{ev}(\xi')\ :\ \mathcal{P}_b \to \mathcal{P}_b, \qquad \xi_d \in \mathcal{I},$$

so that one gets $\mathbb{P}_{ev}U^0_{|\mathsf{z}=z=Z=0} = \mathbb{T}^{-1}_{ev}(\xi')(-\mathbb{T}\mathsf{p}^+U^0)$. This value is smoothly extended to all z,z. Next $\mathsf{p}^-U^0_{|\mathsf{z}=z=0} = \mathbb{T}^{-1}_{os}(\xi',\xi_d)(-\mathbb{T}\mathsf{p}^+U^0)$ is extended to all z through $x'-v'z$.

Thanks to the previous lemma $\mathsf{p}^+U^0_{|\mathsf{z}=0}$ is known for all $\mathsf{t} < \mathsf{t}^+$. We note $B = \mathsf{p}^-U^0_{|\mathsf{z}=0}$. Since B only involves terms of kind $\mathsf{p}^+U^0_{|\mathsf{z}=0}$ whose initial value vanish at the boundary , it vanishes at $\mathsf{t} = -\mathsf{t}^0$.

We get the following result for the boundary problem associated to equation (3.5.4).

Lemma 3.5.7. *Let $s > d/2$ an integer . Then there is a $\mathsf{t}^- \le \mathsf{t}^+$ and a unique $\mathsf{p}^-U^0 \in \mathbb{P}^-_{os} \in \mathcal{P}^s(\Omega_\mathsf{t},\Lambda_0),\ \mathsf{t}\le\mathsf{t}^-$ with boundary value B and whose profile $\mathcal{U}^0_\xi \in Y^s$ satisfy*

$$\begin{array}{lll} \left[X_\xi(\partial_\mathsf{x}) + \frac{i}{2}R_\xi(\partial_{x'})\right]\mathcal{U}^0_\xi = \mathsf{p}(\xi)\pi(\xi)F^0_\xi & & \\ \mathcal{U}^0_\xi(\mathsf{z}=0) = B_\xi & & \xi\in\mathcal{I},\ \mathsf{t}<\mathsf{t}^- \\ \mathcal{U}^0_\xi(\mathsf{t}=-\mathsf{t}^0) = 0 & & \end{array} \tag{3.5.5}$$

Démonstration. The proof is almost the same as for the previous lemma except that we get an energy estimate in z instead of t.

We thus needs to re-express the energy estimate so as to perform Picard iterates. We can't simply say there is an energy estimate in z exactly as in the previous lemma concerning t and recall $\mathsf{z} \le \mathsf{t}^+ v_z$. Indeed, the common space Y^s is designed for a time description.

We recall the notation $v_\xi = (v_y, v_z),\ v'_\xi = (1, v_y)/v_z$ with $v_y = (v_{y_1},\dots,v_{y_{d-1}})$.

Then for $\mathsf{z} \le \mathsf{t}^+ v_z$, $\mathcal{U}^0_\xi$ solves a Schrödinger equation in z. Let $J_1 = \|\mathcal{U}^0_\xi\|_{H^s(\mathbb{R}^{d-1}\times[0,v_{\xi,z}\mathsf{t}])\times\Gamma^s}$ and $J_2 = |\mathcal{U}^0_\xi\|_{H^s(\mathbb{R}^{d-1}\times]v_{\xi,z}\mathsf{t},\infty[)\times\Gamma^s}$. Because of the vanishing initial data $J_2 = 0$. Then

$$J_1 = \| \int_0^\mathsf{z} O_\xi(\mathsf{z}-s)\mathsf{p}(\xi)\pi(\xi)F^0_\xi(\mathsf{x}' - \mathbf{v}'_\xi(\mathsf{z}-s), s, x')ds\|_{L^2(\mathsf{y},\mathsf{z}\le v_{\xi,z}\mathsf{t},x')},$$

which can be estimated in term of t : set $u = (\mathsf{z}-s)$ and $b = \mathsf{y} - v'_{\xi,y}u$

$$J_1 \le c\int_{\mathsf{z}=0}^{v_{\xi,z}\mathsf{t}} \int_{u=0}^{\mathsf{z}} \sqrt{\mathsf{z}}e^u \|\mathsf{p}(\xi)\pi(\xi)F^0_\xi(\mathsf{t}-u/v_z, ., \mathsf{z}-u, .)\|_{L^2(\mathsf{b},x')}dud\mathsf{z},$$

using $\sqrt{\mathsf{z}} \le \sqrt{v_{\xi,z}\mathsf{t}}$ and switching the integrals it is majorized by

$$c\sqrt{v_{\xi,z}\mathsf{t}}\int_{u=0}^{v_{\xi,z}\mathsf{t}}\int_{\mathsf{z}=0}^{v_{\xi,z}\mathsf{t}} e^{u}\|\mathsf{p}(\xi)\pi(\xi)F_\xi^0(\mathsf{t}-u/v_z,.,\mathsf{z}-u,.)\|_{L^2(\mathsf{b},x')}d\mathsf{z}du.$$

We bound the integral on $\mathsf{z}\le v_{\xi,z}\mathsf{t}$ by an integral on $\mathsf{z}\in[0,\infty]$ then we make the translation $a=\mathsf{z}-u$ and we set $r=\mathsf{t}-u/v_z$. One has finally gets the bound

$$c\sqrt{v_{\xi,z}\mathsf{t}}\int_{r=0}^{\mathsf{t}} e^{v_{\xi,z}(\mathsf{t}-r)}\|\mathsf{p}(\xi)\pi(\xi)F_\xi^0(r,.)\|_{L^2(\mathsf{b},a,x')}dr.$$

Thus one gets an estimate for all $\mathsf{z}\ge 0$ (for $\mathsf{z}\ge v_{\xi,z}\mathsf{t}$, $\mathcal{U}^0=0$) and one can perform time Picard iterates as in lemma 3.5.6. Similarly the solution lies in Y^s, for all $\mathsf{t}<\mathsf{t}_\xi$.

Let $\mathsf{t}^-=\min_{\alpha\in\mathcal{I}\cap\Lambda_0}\mathsf{z}_\alpha$. □

Setting $\mathsf{t}^\star=\min(\mathsf{t}^-,\mathsf{t}^+)$, from the previous two lemma one has $U^0\in\mathcal{P}^s(\Omega_\mathsf{t},\Lambda_0),\quad\forall\mathsf{t}<\mathsf{t}^\star$.

3.5.3 Construction of the other profiles.

We explain how to construct U^1 and more generally the next profiles solving the cascade of problem P^j. Let us call E^j the first equation in P^j and write $\mathbb{P}_{os}(E^j)\ \mathbb{P}_{os}L_1(\partial_x)\mathbb{P}_{os}U^{j-1}=\mathcal{R}^j$ ((3.2.5) illustrates this with $\mathbb{P}_{os}(E^2)$).

Theorem 3.5.8. *Let U^0 determined as above and let $(\mathcal{I}^{j,\varepsilon})_{1\le j\le n}\in I^{s-j,\varepsilon}(\Lambda_0)$ a sequence of data. Then under assumption 3.2.20 there is a unique solution $U^\varepsilon=\sum_{j\le n}\varepsilon^jU^j$ to the cascade of problem $(P^j)_{j\le n+2}$ satisfying the Ansatz 3.2.18 where each profile belongs to $\mathcal{P}^{s-j}(\Omega_\mathsf{t},\Lambda_j),\quad\forall\mathsf{t}<\mathsf{t}^\star$ with $\Lambda_1=\Lambda_0$ and $\Lambda_j\supset\mathcal{N}^b(\Lambda_{j-1})$ finite.*

Démonstration. In the proof we first construct the profiles U^1 and $U^j,\ j>1$ solving $(1-\mathbb{P}^i)(E^j),(1-\mathsf{p}^i)\mathbb{P}_{os}(E^{j+1}),\mathsf{p}\mathbb{P}_{os}(E^{j+2})$ together with the initial and boundary condition of (P^j).

Next we give a lemma stating such a construction satisfies the remaining equations that is $(\mathsf{p}^i-\mathsf{p})\mathbb{P}_{os}(\mathcal{R}^j),\ \forall j\le n$.

Construction of U^1.

1. First solve the microscopic equation, $(1-\mathbb{P}^i)E^1:(1-\mathbb{P})U^1=-\mathbb{Q}L^1(\partial_x)U^0$. Note that $\mathbb{Q}_{os}L^1(\partial_x)U^0_{os}=0$. Also remark $(1-\mathbb{P})U^1\in\mathcal{P}^s$ and $\mathrm{spec}U^1=\Lambda_0$.

2. Next solve the global decreasing part of equation $\mathbb{P}^iE^2=\mathbb{P}_{os}E^2$ (see (3.2.5)). Taking $(1-\mathsf{p}^i)\mathbb{P}_{os}E^2$ one gets $\tilde{\tilde{U}}^1=\mathsf{q}\widetilde{\widetilde{\mathbb{P}_{os}F^0}}$.

3. Lastly determine $\mathsf{p}\mathbb{P}_{os}U^1$ and $\mathbb{P}_{ev}U^1$ from the initial and boundary conditions and using $\mathbb{P}_{os}E_3$. Indeed Theorem 3.2.17 gives the first solvability condition $\mathsf{p}\mathbb{P}_{os}\mathcal{R}_3=0$. It gives

$$\begin{aligned}(\mathbb{X}(\partial_\mathsf{x})+iR_{os}(\partial_x))\mathsf{p}\mathbb{P}_{os}U^1=&\mathsf{p}\mathbb{P}_{os}(F^1-L^1(\partial_x)F^0)+2K(\partial_x,\partial_\mathsf{x})U^0\\&-\mathbb{X}(\partial_x)\tilde{\tilde{U}}^1+K(\partial_x,\partial_x)(\tilde{\tilde{U}}^1-\mathbb{Q}L^1(\partial_x)U^0),\end{aligned}$$

where $K(\partial_x,\partial_x)$ is a second order linear operator defined on $\mathcal{P}^s$. Contrary to the nonlinear Schrödinger equation for the leading term this equation is linear in U^1. Since U^1 is constructed so as to satisfy the Ansatz 3.2.18, $\mathsf{p}\mathbb{P}_{os}F^1$ only depends on $\mathsf{p}U^1$. Thus, for each caracteristic velocity v, there is a linear system involving $U^1_{\xi_k},\ k<\infty$ where $v_{\xi_k}=v,\ \forall k$.

Thus one solves this equation as for the leading term (except that it is linear).

1. First determine the outgoing modes. The Schrödinger initial data is given solving the midle scale equation

$$\begin{aligned}\mathsf{p}(\xi)U^1_\xi(\mathsf{t}=-\mathsf{t}^0,\mathsf{x}'',x) =&\mathcal{I}^{1,\varepsilon}_\xi(\mathsf{t}=-\mathsf{t}^0,\mathsf{x}'',x''-v_\xi t)\\ =&\mathcal{U}^1_\xi(\mathsf{t}=-\mathsf{t}^0,\mathsf{x}'',x'-v'_\xi z),\quad \xi\in\Lambda_1\cap\mathcal{O},\end{aligned}$$

And a linear version of lemma 3.5.6 gives $\mathsf{p}(\xi)U^1_\xi$ for all $\mathsf{t}\le\mathsf{t}_+$.

2. Then determine the incoming modes from the boundary condition.
Next the boundary data at the slow scale for p^-U^1 is given through the middle variable $x'-v'_\xi z$ and don't involve $(\mathsf{p}^i-\mathsf{p})U^1$ so it reads

$$\begin{aligned}\mathsf{p}(\xi)U^1_\xi(\mathsf{x}',\mathsf{z}=0,x) =&-\mathbb{T}^{-1}_{os}(\xi)\mathbb{T}(\xi')\big((1-\mathbb{P}(\xi'))U^1(\mathsf{x}',\mathsf{z}=0,x'-v'_\xi z)\\ &-\mathbb{P}_{os+}(\xi'')U^1(\mathsf{x}',\mathsf{z}=0,x'-v'_\xi z)\\ &-\tilde{\tilde{\mathcal{U}}}^1(\mathsf{x}',\mathsf{z}=0,x'-v'_\xi z,x_d=0)\big),\quad \xi\in\Lambda_1\cap\mathcal{I}.\end{aligned}$$

It is defined for all $\mathsf{t}\le\mathsf{t}_+$. From lemma 3.5.1 $\tilde{\tilde{\mathcal{U}}}^1(x_d=0)\in\Gamma^\infty(\mathbb{R}^r)$ and vanishes infinitely at the corner $\mathsf{t}=\mathsf{z}=0$.

One thus solves the transport-Schrödinger equation with those boundary data which vanish up to order m at $\mathsf{t}=0$ thus compatible with the null initial data.

Because of $K(\partial_x,\partial_\mathsf{x})U^0$, $\mathsf{p}\mathbb{P}_{os}U^1\in Y^{s-1}$. And thanks to linearity (of F^1 w.r.t. U^1), the time existence is also $\mathsf{t}^\star$.

6. Then $\mathbb{P}_{ev}(\xi')U^1_{ev,\xi'}(\mathsf{x},x,X_d)$ is determined from the boundary value $U^1_{ev,\xi'}(\mathsf{x}',x',X_d=0)$ involving outgoing profile lying in $Y^s(\mathbb{R}^{d-1})$ as a Schwartz extension to all $\mathsf{z}\ge0,z$.

Construction of U^j.

The same approach applies though some modifications occurs :

1. First $(1-\mathbb{P}^i)E^j$ gives $(1-\mathbb{P})U^j=-\mathbb{Q}\left(F^{j-2}+L^1(\partial_x)U^{j-1}+L^1(\partial_\mathsf{x})U^{j-2}\right)$ Thus $(1-\mathbb{P})U^j\in\mathcal{P}^{s-j+1}$.

Then $\mathrm{spec}_tU^j=\mathrm{spec}_tF^{j-2}\neq\Lambda^{j-1\prime}$: there are new higher tangential harmonics created by F^{j-2}. Thanks to Assumption 3.2.23 there is a finite module Λ_j constructed on $\mathrm{spec}_{os}F^{j-2}$ and stable in the sense of (3.2.26). So $\mathrm{spec}_{os}U^j\subset\Lambda_j$.

2. Next $\mathbb{P}_{os}E^{j+1}$ reads

$$\mathbb{P}_{os}L^1(\partial_x)\mathbb{P}_{os}U^j=\mathbb{P}_{os}F^{j-1}-Sch\mathbb{P}_{os}U^{j-1}+\mathbb{P}_{os}H(U^k)_{k\le j-2},$$

where $H(U^k)_{k\le j-2}=L^1(\partial_\mathsf{x})(1-\mathbb{P}_{os})U^{j-1}+L^1(\partial_x)\mathbb{Q}_{os}(F^{j-2}+L^1(\partial_\mathsf{x})U^{j-2})$.

3. Then $(1-\mathsf{p}^i)\mathbb{P}_{os}E^{j+1}$ gives $\tilde{\tilde{U}}^j=\mathsf{q}\widetilde{\widetilde{\mathcal{R}^{j+1}}}$.

4. Next we determine $\mathsf{p}\mathbb{P}_{os}U^j$ from $\mathbb{P}_{os}E^{j+2}$ exactly as for U^1. We find $\mathcal{U}^j_\xi\in Y^{s-j}$. And thanks to linearity (of F^j w.r.t. U^j), the time existence is also $\mathsf{t}^\star$.

Checking the remaining equations.

Lemma 3.5.9. *Let U^ε constructed as previously. Then the aquations $(\mathsf{p}^i-\mathsf{p})\mathbb{P}_{os}(E^j)$, $\forall j\le n$ are true.*

Démonstration. Since U^j was constructed satisfying the ansatz, one just need to check $(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}(\mathcal{R}^j) = 0$.

First note this equation only involves the polarizations $\mathbb{P}_{os}U^k,\ k \leq j$. Then all linear expression $(L_1(\partial_x), R(\partial_x))$ preserve the ansatz so they are killed by the projector. Thus one needs only to check $(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}F^j = 0$ which is true from lemma 3.5.1. □

□

Remark 3.5.10. *Remark assumption 3.2.20 is fundamental in this construction. In fact it is sufficient to the construction : indeed if U^j satisfies the Ansatz 3.2.18 then one can check that $(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}\mathcal{R}^{j+1} = 0$ so $\mathbb{X}_{os}(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}U^{j+1} = 0$ and lemma 3.4.3 implies $(\mathsf{p}^i - \mathsf{p})\mathbb{P}_{os}U^{j+1} = 0$. This shows the propagation of the Ansatz.*

3.6 convergence

Here we prove theorem 3.3.2.

In the last section we have found a WKB approximation for a three scale solution $U^n_{app}(\varepsilon, \varepsilon x, x, x/\varepsilon) = \sum_{j\leq n} \varepsilon^j U^j$ of (3.1.5). We have shown the profiles satisfy a Schrödinger equation each involving a global middle variable $x' - \mathbf{v}'z$.

However the initial condition $e^{n,\varepsilon}_{app} = \mathcal{I}^{n,\varepsilon}_{app}(.\varepsilon, ., ./\varepsilon)$ is defined for $z \geq 0$ and one has to extend it for all z. Since it vanishes smoothly at $z = 0$, one keeps the same notation for the extension by 0 to $z \leq 0$.

Plugging $u^n_{app} = U^n_{app}(\varepsilon x, x, x/\varepsilon)$ in the equation (3.1.5), we see that all the terms of order $\leq n$ vanish. Therefore

$$L(\varepsilon\partial_x)u^n_{app} - \varepsilon^2 f(u^n_{app}) = \varepsilon^{n+1}g^\varepsilon, \quad g^\varepsilon(x) = G(\varepsilon x, x, \frac{x}{\varepsilon}). \tag{3.6.1}$$

with $G \in \mathcal{P}^{s-n-1}([0, t_*])$. Moreover, by construction there holds

$$(TU^n_{app})_{|_{x_d=0, x_d=0}} = 0. \tag{3.6.2}$$

Lastly one must remark the previous WKB construction modifies the initial data since the initial data are that of $\mathsf{p}\mathbb{P}u^n_{\rm app}$ and there are the extra term $e^i = (1-\mathsf{p}^i)\mathbb{P}_{os}U^n_{\rm app}(\mathsf{t} = 0, \varepsilon x'', t = 0, x'', T = 0, x''/\varepsilon)$.

Now to see this WKB approximation is really close to the initial problem (3.1.5) we look for the exact solution under the form $u^\varepsilon = u^n_{\rm app} + \varepsilon^M v^\varepsilon$ with initial condition $e^\varepsilon = e^n_{\rm app} + e^i + \varepsilon^M r^\varepsilon$.

The remainder v^ε is solution to :

$$(\mathcal{R}) \begin{cases} L^\varepsilon(\partial_x)v^\varepsilon = \varepsilon^2\left(\varepsilon^{n-M-1}g^\varepsilon + F(u^n_{app}, \varepsilon^M v^\varepsilon)v^\varepsilon\right) \\ Tv^\varepsilon = 0, \quad z = 0 \\ v^\varepsilon(t = -\mathsf{t}^0/\varepsilon) = r^\varepsilon + \varepsilon^{-M}e^i \end{cases}$$

where $F(u, v) = \int_0^1 \partial_u f(u + tv)dt$.

Since e^i vanish at $z = 0$ the compatibility for the whole problem is then insured by taking r^ε flat up to order m at $z = 0$.

Now we show that $\varepsilon^{-M}e^i$ is bounded.

Lemma 3.6.1. *There is a constant c_M independent from ε such that*

$$\varepsilon^{-M}\|e^i\|_{H^{m,\varepsilon}(\mathbb{R}^d_+)} < c_M \sup_{j\le n}\|(1-\mathsf{p}^i)\mathbb{P}_{os}U^j\|_{Y^{m+M+d+2}} .$$

Démonstration. $e^i = \sum_{\substack{j\le n\\ \xi\in\Lambda_n}} \varepsilon^j(1-\mathsf{p}^i(\xi))\pi(\xi)U^{j,\varepsilon}_\xi{}_{|_{\mathsf{t}=-\mathsf{t}^0,t=-\mathsf{t}^0/\varepsilon}}$. It is then sufficient to show the inequality for the leading profile. Note $a^0_\xi = (1-\mathsf{p}^i(\xi))\pi(\xi)U^{0,\varepsilon}_\xi$

$$\begin{aligned}
&\|a^0_\xi(\mathsf{t}=-\mathsf{t}^0,\varepsilon.,t=-\mathsf{t}^0/\varepsilon,.)\|_{H^{m,\varepsilon}(\mathbb{R}^d_+)} \le\\
&\|\int_{\mathsf{b}_0=0}^{\varepsilon t}\int_{\substack{\mathsf{b}_j\le\varepsilon x_j\\ 1\le j\le d-1}}\int_{\mathsf{b}_d=0}^{\varepsilon x_d}\int_{t=-\infty}^{-\mathsf{t}^0/\varepsilon}\frac{1}{|t|^M}|t|^M\partial_t\partial_{\mathsf{b}_0}\dots\partial_{\mathsf{b}_d}a^0_\xi(\mathsf{b},t,.)d\mathsf{b}\,dt\|_{H^{m,\varepsilon}(\mathbb{R}^d_+)}\\
&\le\|\int_{t=-\infty}^{-\mathsf{t}^0/\varepsilon}\frac{1}{|t|^M}|t|^M\int_{\mathsf{b}_0>0,\mathsf{b}_d>0}\int_{\substack{\mathsf{b}_j\in\mathbb{R}\\ 1\le j\le d-1}}\partial_t\partial_{\mathsf{b}_0}\dots\partial_{\mathsf{b}_d}a^0_\xi(\mathsf{b},t,.)d\mathsf{b}\,dt\|_{H^{m,\varepsilon}(\mathbb{R}^d_+)}\\
&\le c\sup_{t\le-\mathsf{t}^0/\varepsilon}\left(\frac{1}{|t|}\right)^M\|a^0_\xi\|_{Y^{m+M+d+2}} .
\end{aligned}$$

And $\sup_{\mathsf{t}\le-\mathsf{t}^0}\left(\frac{\varepsilon}{|\mathsf{t}|}\right)^M\le c\varepsilon^M$. □

Then for the convergence one needs to give an estimate of u^n_{app} in the inhomogeneous, ε-space $E^{m,\varepsilon}$ (cf. (3.3.2)).

Lemma 3.6.2. $\|u^n_{app}\|_{E^{m,\varepsilon}}\le C\|U^n_{app}\|_{Y^{m+d+1}}$.

For the convergence one can apply the results of [35]. However we recall the main steps.

1.*Initialization of the scheme*

We suppose that $n\ge M\ge m$ are given with $m>\frac{d+1}{2}$ even. We fix a C_0 and we consider the Cauchy problem for (3.6.1),(3.6.2) with initial data $h\in H^m(\mathbb{R}^d_+)$, such that

$$\|h\|_{H^m_\varepsilon(\mathbb{R}^d_+)}\le C_0. \tag{3.6.3}$$

$$\partial^k_{x_d}h_{|x_d=0}=0,\quad \text{for } k\in\{0,\dots,m-1\}. \tag{3.6.4}$$

In a first step, we assume that

$$h\in C^\infty_0(\{x_d>0\}). \tag{3.6.5}$$

To solve $(\mathcal{R})$ one uses a scheme giving a sequence v^p with limit v :

$$\begin{cases} L(\varepsilon\partial_x)v^{p+1}=\varepsilon\Big(\varepsilon^{n+1-M}r^\varepsilon+F(u^\varepsilon_{app},\varepsilon^M v^p)v^p\Big),\\ Tv^{p+1}{}_{|x_d=0}=0\\ v^{p+1}{}_{|t=0}=h. \end{cases} \tag{3.6.6}$$

As in [35] we initialize the scheme with v^0 such that $v^0(t=0)=h$ and $(\varepsilon\partial_t)^jv^0,\quad j\le m$ is computed by using the equation. The existence and regularity results of [19, 42], imply by induction on p, that the scheme (3.6.6) defines for $p\ge1$ smooth solutions $v^p\in H^\infty([0,t_*]\times\mathbb{R}^d_+)$, which are flat at the corner. Then for all p the Taylor expansion of v^p at $t=0$ is that of v^0.

2. *Global energy Bounds for the sequence.*

Proposition 3.6.3. *There are $\varepsilon_0 > 0$ and C such that for all $\varepsilon \in]0, \varepsilon_0]$, h satisfying* (3.6.3) (3.6.5)*, and all $p \geq 0$:*

$$\varepsilon^M \|v^p\|_{L^\infty([0,\mathtt{t}_*/\varepsilon]\times\mathbb{R}^d_+)} \leq 1, \tag{3.6.7}$$

$$\sup_{t\in[0,\mathtt{t}_*/\varepsilon]} \|v^p(t)\|_{E^{m,\varepsilon}} \leq C. \tag{3.6.8}$$

This is a consequence of the linear following estimate (see [35]) :

Lemma 3.6.4. *For all even $m \in \mathbb{N}$, there is a constant C such that for all smooth u satisfying $L(\varepsilon\partial_x)u = \varepsilon^2 f(u)$ there holds for $t \in [0, 1]$:*

$$\|u(t)\|_{E^{m,\varepsilon}} \leq C\|u(0)\|_{E^{m,\varepsilon}} + C\varepsilon \int_0^t \|f(s)\|_{E^{m,\varepsilon}} ds \tag{3.6.9}$$

and a non-linear estimates on the full system $(\mathcal{R})$ (cf. [35]). Note the added ε in (3.6.9) in front of the integral. Thanks to it one gets the supremum in (3.6.8) over $[0, \mathtt{t}_*/\varepsilon]$.

We next recall the main nonlinear estimate in [35]. It requires the definition of a new space : for $t > 0$ and m an even integer, denote by $\tilde{F}^m(t)$ the space of functions on $]-\infty, t] \times \mathbb{R}^d_+$ such that $\partial^{\alpha'}_{x'}\partial^{\alpha_d}_{x_d} u \in L^2(]-\infty, t] \times \mathbb{R}^d_+)$ for all $|\alpha'| + 2\alpha_d \leq m$.

Lemma 3.6.5. *Suppose that G is a smooth function of its argument. For $m > \frac{d+1}{2}$ even, there is a function $C(\cdot)$ from $[0, +\infty[$ to $\mathbb{R}_+$ such that for all $\varepsilon \in]0, 1]$ and all $t \in [0, t_*]$, there holds for all v and w in $\tilde{F}^m(t)$*

$$\|F(u^\varepsilon_{app}, \varepsilon^M v)w\|_{\tilde{F}^{m,\varepsilon}(t)} \leq C(\varepsilon^M\|v\|_{L^\infty}) \big(\|w\|_{\tilde{F}^{m,\varepsilon}(t)} + \varepsilon^M\|w\|_{L^\infty}(1 + \|v\|_{\tilde{F}^{m,\varepsilon}(t)})\big)$$

where the space $\tilde{F}^{m,\varepsilon}(t)$ is a subset of functions of $L^2(]-\infty, t] \times \mathbb{R}^d_+)$ and has associate weighted norm

$$\|u\|_{\tilde{F}^{m,\varepsilon}(t)} = \sum_{2\alpha_d+|\alpha''|\leq m} \varepsilon^{2\alpha_d+|\alpha''|}\|\partial^\alpha_x u\|_{L^2(]-\infty,t]\times\mathbb{R}^d_+)}. \tag{3.6.10}$$

3. *convergence*

Proposition 3.6.6. *For all $\varepsilon \in]0, \varepsilon_0]$, and h satisfying* (3.6.3) (3.6.5)*, the sequence v^p converges in $E^m(\mathtt{t}_*/\varepsilon)$. The limit is the unique solution v of the Cauchy problem for the limit system in the space $E^m(\mathtt{t}_*/\varepsilon)$. It satisfies*

$$\sup_{t\in[0,\mathtt{t}_*/\varepsilon]} \|v(t)\|_{E^{m,\varepsilon}} \leq C \tag{3.6.11}$$

where C depends only on the initial bound C_0.

4. *Relaxing hypothesis* (3.6.5)

This is done by taking a sequence $h^\nu \in C^\infty_0(z > 0)$ which converges to a h satisfying (3.6.4). Using the linear and non-linear estimate one gets a Cauchy sequence in $E^{m,\varepsilon}$ for the related sequence v^ν solution to $(\mathcal{R})$. Since $E^{m,\varepsilon}$ is a Banach space one gets a limit v solution to the Cauchy problem.

Chapitre 4

Intermediate models for laser propagation in nonlinear media

4.1 Introduction

4.1.1 Motivations

The aim of this paper is as in [6] to propose new models for the simulation of the propagation of a laser in nonlinear media. The classical framework [13] is to use the par-axial approximation. More precisely, one looks to the electric field under the form

$$E \sim \mathcal{E}(t,\mathbf{x})e^{i(\omega t-\mathbf{kx})/\varepsilon}, + \ c.c. \tag{4.1.1}$$

where ε is a small parameter. Then one finds equations satisfied by $\mathcal{E}$ thanks to a WKB expansion on $\mathcal{E}$. The amplitude $\mathcal{E}(t,\mathbf{x})$ is indeed searched as a formal (not necessarily convergent) Taylor series :

$$\mathcal{E}(t,\mathbf{x}) = \varepsilon^p \sum_j \varepsilon^j \mathcal{E}^j(t,\mathbf{x}), \tag{4.1.2}$$

with p a real which parametrizes the amplitude of the wave. It corresponds to that of the initial datum and its value determines the regime of optics : the smallest it is the farther the rays can propagate.

Here we consider the envelope of a plane wave (4.1.1) but as in [6] we do not perform the WKB expansion and look instead for an intermediate equation on the profile $\mathcal{E}$.

One will use this ansatz in the Maxwell system in order to provide an approximation of quasi-monochromatic rays propagating in dielectric media (see the next examples). In [6] the case of the Cauchy problem is addressed. Here we address the case of the boundary value problem.

4.1.2 Two model examples

We introduce two models studied in [13] and [11]. These models are 3-dimensional in space but we will mainly do computations on the one dimensional problem in order to avoid

technicalities. Both contains the Maxwell equations for a dielectric which are

$$\left\{\begin{array}{c} \partial_t H + \mathrm{rot} E = 0, \\ \partial_t D - \mathrm{rot} H = 0, \\ divD = 0, \quad divB = 0. \end{array}\right. \tag{4.1.3}$$

We consider a transmission problem through an interface. The transmission conditions for a dielectric read :

$$[E] \wedge n = 0, \quad [H] \wedge n = 0, \tag{4.1.4}$$

$$[D] \cdot n = 0, \quad [B] \cdot n = 0. \tag{4.1.5}$$

$[E]$ is the jump of the electric field through the interface whose normal is n. Note that choosing a divergence free initial condition, the solution of (4.1.3) is divergence free for all time.

(E, B) is the electromagnetic field and (D, H) the electromagnetic impulsion field. D is linked to the electric field through a constitutive relation :

$$D = \sum \chi^j (\underbrace{E, \ldots, E}_{j}).$$

(see physic text books : [3],[29], [13]).

In [13] $P = \chi^1(E) + \chi^3(E, E, E)$ where χ^1 is linear in E and gives the linear index of the medium. χ^3 is a trilinear function which describes the interaction of two populations of ions in a gas corresponding to two different levels of ionization of a same molecule.

$$\left\{\begin{array}{c} \partial_t H + \mathrm{rot} E = 0, \\ \partial_t E - \mathrm{rot} H + \frac{Q}{\varepsilon} = -\partial_t(|E|^2 E), \\ \partial_t P - \frac{Q}{\varepsilon} = 0 \\ \partial_t Q + \omega_a^2 \frac{P}{\varepsilon} = \gamma E. \end{array}\right. \tag{4.1.6}$$

Here, the small parameter is $\varepsilon = \frac{1}{T_{ref}\omega_{ref}}$ where T_{ref} is the characteristic time of the initial pulse and ω_{ref} its frequency. $\omega_a \sim 1$ is the dimensionless resonance pulsation of the medium and $\gamma \sim 1$ is the dimensionless nonlinear strength.

In [11], one considers a non-isotropic crystal. We work in the crystallographic coordinates via a two angles rotation. Then P reads $P = \chi^1(E) + \chi^2(E, E)$ with $\chi^1(E) = \chi_\infty^{(1)} E + \alpha_a \chi_a^{(1)} E + \alpha_b \chi_b^{(1)} E$ and χ^2 bilinear. The second order tensors $\chi_\infty^{(1)}, \alpha_a, \alpha_b$ are diagonal with constant coefficients and $\alpha_a + \alpha_b = Id$. $\chi_a^{(1)}, \chi_b^{(1)}$ are defined by :

$$\chi_a^{(1)} = \left(\frac{\chi_s^{(1)} - \chi_\infty^{(1)}}{1 - \frac{\omega^2}{\omega_a^2}}\right), \qquad \chi_b^{(1)} = \left(\frac{\chi_s^{(1)} - \chi_\infty^{(1)}}{1 - \frac{\omega^2}{\omega_b^2}}\right),$$

with $\chi_s^{(1)}$ a constant coefficient diagonal tensor. This polarization P describes the interaction between a fundamental wave and its first harmonic when it is resonant (this requires a phase matching condition between the pulsation ω and the orientation of the Cristal).

The corresponding system is written in the crystallographic coordinates and reads (in the dimensionless form)

$$\left\{\begin{array}{c} \partial_t\mathcal{H} + \mathbb{P}\mathrm{rot}\mathcal{E} = 0, \\ \chi_\infty^{(1)}\partial_t\mathcal{E} - \mathbb{P}\mathrm{rot}\mathcal{B} + \frac{\omega_a}{\varepsilon}\mathcal{U} + \frac{\omega_b}{\varepsilon}\mathcal{V} = -\gamma\partial_t\chi^{(2)}(\mathcal{E},\mathcal{E}), \\ \partial_t\mathcal{F} - \frac{\omega_a}{\varepsilon}\mathcal{U} = 0, \\ \partial_t\mathcal{G} - \frac{\omega_b}{\varepsilon}\mathcal{V} = 0, \\ \partial_t\mathcal{U} + \frac{\omega_a}{\varepsilon}\mathcal{F} - \alpha_a\frac{\omega_a}{\varepsilon}\mathcal{T}\mathcal{E} = 0, \\ \partial_t\mathcal{V} + \frac{\omega_b}{\varepsilon}\mathcal{G} - \alpha_b\frac{\omega_b}{\varepsilon}\mathcal{T}\mathcal{E} = 0, \end{array}\right. \tag{4.1.7}$$

The tensors $\omega_a, \omega_b \in \mathcal{M}_3(\mathbb{R})$ are associated to Lorentz resonances frequencies of the medium and

$$\mathbb{P} = \begin{pmatrix} \cos(\varphi)\cos(\alpha) & -\sin(\alpha) & \sin(\varphi)\cos(\alpha) \\ \cos(\varphi)\sin(\alpha) & \cos(\alpha) & \sin(\varphi)\sin(\alpha) \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}$$

is the rotation matrix. Finally $\mathcal{T} = \chi_s^{(1)} - \chi_\infty^{(1)}$.

Setting $x = (t, x_1, x_2, x_3)$ Those two systems can be written under the quasi-linear form :

$$\left(L(\partial_x) + \frac{L_0}{\varepsilon}\right)u = -\partial_t f(u). \tag{4.1.8}$$

In the first system $u = (H, E, Q, P)$ and $f(u) = (0, |E|^2E, 0, 0)$; in the second $u = (\mathcal{H}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{U}, \mathcal{V})$ and $f(u) = (0, \gamma\chi^{(2)}(\mathcal{E},\mathcal{E}), 0, 0, 0, 0)$.

Moreover the partial differential operator $L(\partial_x) = \partial_t + \sum_{j=1}^3 A_j\partial_{x_j}$ (with A_j symmetric and $A_0 = I$) is hyperbolic symmetric in the direction dt. The last matrix L_0 is skew-adjoint. For the sake of simplicity and as our analysis mainly deals with the linear operator $L(\partial_x) + L_0/\varepsilon$ one will replace $-\partial_t f(u)$ by a semi-linear term $f(u)$. We refer the interested reader to [23] for extending the theoretical arguments to the quasi-linear case. Nevertheless the numerical computation will be performed on (4.1.8). See also [11] for physical justification of using semi-linear systems instead of quasi-linear ones

4.1.3 General setting

Notations 4.1.1. *Let $\Omega = \mathbb{R}^n \times \mathbb{R}^+$ and $x = (x_0, \ldots, x_n)$. Sometimes one may use $x_0 = t$, $\tilde{x} = (x_1, \ldots, x_{n-1})$, $x_n = z$. We also use $x' = (t, \tilde{x})$ and $x'' = (\tilde{x}, z)$.*

For any subset $W \subset \mathbb{R}^{n+1}$ we will note $W' \subset \mathbb{R}^n$ the subset made of the x' for $x \in W$ and $W'' \subset \mathbb{R}^n$ that of the x''. W' will be often referred to as the tangential set of W.

we propose to solve the semilinear boundary-value problem related to (4.1.8) for an incoming wave :

$$\mathcal{L}(\varepsilon\partial_x)u^\varepsilon = f(u^\varepsilon), \qquad x' \in \mathbb{R}^n,\ z > 0 \tag{4.1.9}$$

$$u^\varepsilon = b^\varepsilon, \qquad \text{on } \ z = 0 \tag{4.1.10}$$

where $\mathcal{L}(\varepsilon\partial_x) = \varepsilon L(\partial_x) + L_0$, (with the notations from (4.1.8)) and $L(\partial_x) = \partial_t + \sum_{j=1}^n A_j\partial_{x_j}$. This problem is the natural representation of the transmission of an incident wave whose value at the boundary $z = 0$ is b^ε.

Nevertheless, as $\mathcal{L}$ is not hyperbolic in z and because of the nonlinearity, this problem is in general not well posed.

In fact one should consider the subclass of causal solutions for which one can get estimates using the hyperbolicity in the direction dt of $\mathcal{L}$. A way is to consider (4.1.9),(4.1.10) as the restriction to $z \geq 0$ of

$$\begin{array}{c} \mathcal{L}(\varepsilon\partial_x)u_l^\varepsilon = f_l(u_l), \quad x' \in \mathbb{R}^n,\ z < 0 \\ \mathcal{L}(\varepsilon\partial_x)u_r^\varepsilon = f_r(u_r), \quad x' \in \mathbb{R}^n,\ z > 0 \\ T(u_l^\varepsilon, u_r^\varepsilon) = 0, \quad z = 0 \\ u_l^\varepsilon(t = 0) = V_0^\varepsilon, \quad u_r^\varepsilon(t = 0) = 0\ , \end{array} \tag{4.1.11}$$

with $f_r = f$ and the linear boundary operator T is supposed to satisfy a Lopatinski condition (see [30]). This initial boundary value problem is well posed (cf. [19]). The problem is then to find V_0^ε so that $u_r^\varepsilon(z = 0) = b^\varepsilon$. But since the equation (4.1.9) is nonlinear one can just hope solving approximately this inverse problem (see section 2 for a discussion about this problem).

Moreover from [35] one expects the value $u_r^\varepsilon(z = 0)$ of (4.1.11) to show new propagating waves and evanescent waves. This gives rise to quite an intricate analysis. A way to simplify it is to consider (4.1.9),(4.1.10) with (4.1.10) chosen so that it cancels all the waves generated by the boundary except the considered incoming wave. With such a boundary data there is an energy estimate for approximate solutions of (4.1.11) involving only oscillating waves. This is explained in section 4.

If $b^\varepsilon \sim \varepsilon^p$ and f polynomial one can expect solutions propagating over long distances $z^\varepsilon \sim 1/\varepsilon^q$ with q depending on p and f (see Remark 4.1.2).

Next as we consider $x_n = z$ to be the evolution variable one must pay attention on the invertibility of A_n (recall $L(\partial_x) = \sum_{j\leq n} A_j\partial_j$). As it is not invertible in the two previous examples we will suppose, up to a change of basis that A_n is under the form

$A_n = \begin{pmatrix} A_n^{11} & 0 \\ 0 & 0 \end{pmatrix}$ with A_n^{11} invertible. Let us denote by d_1 the rank of A_n^{11}.

Now according to (4.1.1),(4.1.2) we look for a solution of (4.1.9) as a modulated plane wave :

$$u^\varepsilon(x) = \varepsilon^p\mathcal{U}(\varepsilon, x, \theta), \quad \theta = x.\xi_\star/\varepsilon,\ \xi_\star \in \mathbb{R}^{n+1}, \tag{4.1.12}$$

$$u^\varepsilon(x', z = 0) = b^\varepsilon = \varepsilon^p\mathcal{U}_0(x')e^{i\theta}{}_{|_{\theta = x'.\xi'_\star/\varepsilon}}\ , \tag{4.1.13}$$

where $\xi_\star$ is the wave frequency. From the 2π-periodicity in θ of the boundary data it is reasonable to look for oscillating profiles $\mathcal{U}(\varepsilon, x, \theta)$ that are also 2π-periodic in θ.

Then following [14], we write the singular equation satisfied by the profile $\mathcal{U}(\varepsilon, x, \theta)$ as follows

$$\mathcal{L}(\varepsilon\partial_x + \xi_\star\partial_\theta)\varepsilon^p\mathcal{U}(\varepsilon, x, \theta) = f(\varepsilon^p\mathcal{U}(\varepsilon, x, \theta)) \qquad \forall x = (x', z) \in \mathbb{R}^d \times \mathbb{R}^+,\ \theta = \frac{x.\xi_*}{\varepsilon} \tag{4.1.14}$$

and we impose that this equation is satisfied for all x', z and for all $\theta \in \mathbb{T}$ with $\mathbb{T} = \mathbb{Z}/2\pi\mathbb{Z}$ thus not only for $\theta = \frac{x.\xi_*}{\varepsilon}$.

Remark 4.1.2. *The size of the profile is $\mathcal{O}(\varepsilon^p)$ and the power p determines the distance of influence of the non-linearity. Indeed, take $\mathcal{L} = \partial_z$, $f(u) = u^\alpha$ and set $u^\varepsilon = \varepsilon^p U$. Then the equation reads*

$$(\varepsilon\partial_z + \xi_\star\partial_\theta)U = \varepsilon^{p(\alpha-1)}U^\alpha.$$

If ε is small enough and $p\alpha > 1$ the solution is defined at least on a z interval of size $\mathcal{O}(\frac{1}{\varepsilon^{p(\alpha-1)}})$ and this result is optimal as shown by the case $\xi_* = 0$ for which an exact solution can be found.

In the sequel, we suppose that $f(U) = \Phi(U;U,\dots,U)$ where $V \to \Phi(X;V,\dots,V)$ is J linear in V so that $f(\varepsilon^p U) = \varepsilon^{Jp}\Phi(\varepsilon^p U;U,\dots,U)$ and we note $\Phi(\varepsilon^p U;U,\dots,U) = F(x,\theta,\varepsilon)$.

Then writing $\mathcal{L} = A_n\partial_z + L'$ and setting $\sigma = \varepsilon^{p(J-1)-1}$ one gets

$$\left(A_n\partial_z + \frac{1}{\varepsilon}(L'(\varepsilon\partial'_x + \xi'_\star\partial_\theta) + \xi_{\star,n}A_n\partial_\theta)\right)\mathcal{U}(x,\theta) = \sigma F(x,\theta,\varepsilon^p). \tag{4.1.15}$$

Remark 4.1.3. *In the frame of geometric optics (for $z \in \mathcal{O}(1)$ thus $p = 1/J - 1$), one would use the expansion 4.1.2 for the profile so that one could expand the equation* (4.1.14) *:*

$$\sum_j \varepsilon^j \left(L^1(\partial_x)U^j(x,\theta) + \frac{1}{\varepsilon}\mathcal{L}(\xi_\star\partial_\theta)U^j(x,\theta)\right) = \sum_j \varepsilon^j F^j(x,\theta) \tag{4.1.16}$$

where $\mathcal{L}(\varepsilon u) = \varepsilon L^1 + L_0$. The solution in then found by cancelling the coefficients of the series, providing two equations for each profile :

$$\begin{aligned}\mathcal{L}(\xi_\star\partial_\theta)U^0 &= 0\\ L^1(\partial_x)U^j + \mathcal{L}(\xi_\star\partial_\theta)U^{j+1} &= F^j\end{aligned}$$

One solves this infinite system inductively and one can see it mainly depends on the invertibility (or not) of $\mathcal{L}(\xi_\star\partial_\theta)$.

Define $p(\xi) := \det(\mathcal{L}(i\xi))$ the characteristic polynomial of $\mathcal{L}$. Then the first equation has a nontrivial solution iff $p(\xi) = 0$. We say that ξ is characteristic for $\mathcal{L}$.

Definition 4.1.4. *Denote char$\mathcal{L} = \{\xi \in \mathbb{C}^{n+1} \mid p(\xi) = 0\}$.*

We say that $\underline{\xi} \in$ char$\mathcal{L}$ is regular if it is real and if there is a real C^∞ function λ defined on a neighbourhood of $\underline{\xi}''$ such that char$\mathcal{L}$ is locally parametrized by $\xi_0 - \lambda(\xi'') = 0$.

Next write $\mathcal{L} = \partial_t + L''(\partial'')$ with $L''(i\eta'')$ skew-symmetric. L'' has a semi simple decomposition with q distinct eigenvalues λ_j defined for all η''. By definition $p(\lambda_j(\eta''),\eta'') = 0$. This shows that char$\mathcal{L}$ is made of q distinct sheets separated with gaps in the time direction (see figure 4.1). Finally as the cascade of equation mainly gives transport equations one introduces the group velocity related to a characteristic mode $\underline{\eta} = (\lambda(\underline{\eta}''),\eta'') : v_{\underline{\eta}} = \nabla_{\eta''}\lambda(\underline{\eta}'')$.

Contrary to remark (4.1.3) we look for an exact solution of (4.1.15) which is not a WKB solution. Unfortunately as the roots of $p(\xi',\xi_n)$ in ξ_n can take real values one can't expect the existence of an exact solution of equation (4.1.15). However using cutoff functions one can get an approximate solution which is better as σ tends to zero. The derivation of the corresponding model is given in section 3 and the existence and stability are discussed in section 4 and proved in section 5. We next summarize the main results.

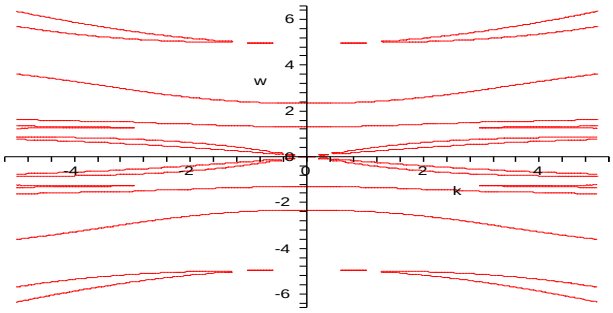

FIG. 4.1 – characteristic manifold $\omega = f(k)$ for maxwell-Bloch equation (4.1.7).

4.1.4 Main results

We present the results obtained through the example of the system (4.1.6) in the one dimensional case.

Let $\omega_*, k_* \in \mathbb{R}$ satisfying the Maxwell-Lorentz dispersion relation : $k_*^2 = \omega_*^2(1+\chi(\omega_*))$ with $\chi(\omega) = \frac{\gamma}{\omega_a^2 - \omega^2}$. Let us denote $k(\omega) = \omega\sqrt{1+\chi(\omega)}$ the root satisfying $k(\omega_*) = k_*$.

Then let $b^\varepsilon(t) = \varepsilon^p B(t) e^{i\omega_* t}$ a boundary data for (4.1.6) with $B(t)$ polarized and with Fourier compact support :

$$B(t) = \left(\frac{k_*}{\omega_* + \varepsilon\partial_t} e_2, e_1\right) A(t),\ e_1 = (1,0,0),\ e_2 = (0,1,0), \quad \chi_*(\varepsilon\partial_t) B(t) = B(t). \tag{4.1.17}$$

$A(t)$ is the scalar amplitude and χ_* is a cutoff function which vanishes outside a closed set containing k_* and on which k is well defined.

Theorem 4.1.5. *There is a unique solution $u^\varepsilon(t,z) = \varepsilon^p \mathcal{U}(t,z) e^{i\frac{\omega_* t + k_* z}{\varepsilon}} + \varepsilon^p \sigma \mathcal{O}(\varepsilon)$ to the system (4.1.6) with boundary value $\varepsilon^p B(t) e^{i\omega_* t} + \varepsilon^p \sigma \mathcal{O}(\varepsilon)$ and where the profile $\mathcal{U}(t,z)$ is solution given by*

$$U(t,z) = l_1(\omega_* + \varepsilon\partial_t)\alpha_1(t,z) + l_2(\omega_* + \varepsilon\partial_t)\alpha_2(t,z), \tag{4.1.18}$$

where α_1 and α_2 are solution to

$$\begin{aligned} \left(\partial_z + \tfrac{i}{\varepsilon}(\omega_* + \varepsilon\partial_t)\sqrt{1+\chi(\omega_* + \varepsilon\partial_t)} - k_*\right)\alpha_j(t,z) = \\ \sigma \frac{1}{2\sqrt{1+\chi(\omega_*+\varepsilon\partial_t)}} \chi_*(\varepsilon\partial_t) < l_j(\omega_* + \varepsilon\partial_t), F(U) >, \quad j \in \{1,2\},\ z \geq 0, \\ \alpha_j(t, z=0) =< l_j, B(t) >, \quad j \in \{1,2\} \end{aligned} \tag{4.1.19}$$

with l_j given in section §6.1. Moreover there is a $c > 0$, $z_0 > 0$ and $t_0 > 0$ such that the following error estimate holds :

$$\sup_{t\in[0,t_0/\sigma]} \sup_{z\in[0,z_0/\sigma]} \| u^\varepsilon - \varepsilon^p \mathcal{U}(t,z) e^{i\frac{\omega_* t + k_* z}{\varepsilon}} \|_\infty \leq c\sigma\varepsilon^{p+1}.$$

Remark 4.1.6. *Performing the limit $\varepsilon \to 0$ in* (4.1.19) *and expanding the square root $\sqrt{1+\chi(\omega_* + \varepsilon\partial_t)}$ one recovers the usual geometric optic model at first order and the Schrödinger equation at second order. Our model is therefore an intermediate model between the complete Maxwell-Lorentz system and the usual NLS. We expect this model to have an extended range of validity indeed.*

We have performed calculations on (4.1.6) (see section 6) and compared the solution of the equation (4.1.19) with that of the usual Schrödinger equation. We have checked the errors predicted by the previous inequality and the known estimate for the Schrödinger model. Then we have highlighted the difference between the two models. This difference is big mainly in the linear case for pulses with broad spectrum (that is short or spectrally chirped). In particular the shape of the profiles greatly differ. In the non-linear case the non-linear effects prevails on the condition that the dispersion is weak and then the two models are very close.

4.2 The boundary problem (4.1.9),(4.1.10) : a hidden transmission problem.

The aim of this section is to give a framework for (4.1.9),(4.1.10) to be well-posed and to discuss about a question related to the link between a Cauchy problem and boundary problem associated to a given hyperbolic equation.

(4.1.9),(4.1.10) being well-defined, one can then look for the intermediate model (4.1.19) (see next section for the derivation and sections $4, 5$ for a justification).

Here we do not perform any rigorous justification but just give some hints.

We propose to re-write (4.1.9),(4.1.10) as the restriction to $z \geq 0$ of the transmission problem

$$(P)\begin{cases} \mathcal{L}(\varepsilon\partial_x)u_l = \varepsilon\phi_l(u_l), \quad x_n < 0 \\ \mathcal{L}(\varepsilon\partial_x)u_r = \varepsilon\phi_r(u_r), \quad x_n > 0 \\ T(u_l, u_r) = 0, \quad x_n = 0 \\ u_l(t=0) = \varepsilon^p \pi_{out}(\varepsilon\partial_{x''})V_0(x'')e^{ix''.\xi''_{out}/\varepsilon}, \quad u_r(t=0) = 0\ . \end{cases} \tag{4.2.1}$$

where $\phi_r = \phi$ (as in (4.1.9)) and π_{out} is the orthogonal projector on $\ker \mathcal{L}(i\xi_{out})$ with $\xi_{out,0} = \lambda_{out}(\xi''_{out})$. Moreover ξ_{out} is an outgoing mode (so called "incident") for $\mathcal{L}$: it is real characteristic and $v_{\xi_{out},n} > 0$. The boundary condition is supposed to satisfy the Lopatinski conditions which enables to find every incoming mode (the "reflected" on the left side, the "transmitted" on the right side) in term of the outgoing ones (cf. Kreiss. See [47] for the analysis in the Geometric optics context).

One checks that u_r is indeed solution to (4.1.9),(4.1.10) with boundary value $u_{r|_{z=0}}$. Since (4.2.1) is well posed, the solution of (4.1.9),(4.1.10) obtained through this process is well defined. The main drawback of this construction is that one cannot take general boundary data b^ε for (4.1.9),(4.1.10). Moreover since the interior problems are non-linear one cannot compute explicitly $u_r(z=0)$ in term of the incident wave. However one can hope to get an approximate relation between $u_r(z=0)$ and V_0.

First, as it is more natural, let us find an approximation of $u_r(z=0)$ in term of V_0.

Using the fact that the searched solution is spatially almost compactly supported and assuming that the nonlinear interaction is small, a good approximation of u_l is given by

$u_l^{out} + u_l^{refl}$ where u_l^{out} solves the following problem that is posed on the whole space

$$\left\{ \begin{array}{c} \mathcal{L}(\varepsilon\partial_x)u = \varepsilon\phi(u), \quad x \in [0,T]\times\mathbb{R}^n \\ u(t=0) = \varepsilon^p \pi_{out}(\varepsilon\partial_{x''})V_0(x'')e^{ix''.\xi''_{out}/\varepsilon} \end{array} \right. \tag{4.2.2}$$

This approximation corresponds to a decoupling between this outgoing wave and all the incoming ones generated at the boundary. The smaller the nonlinear interaction is the better the approximation.

This Cauchy problem has a solution u_l^{out} defined for $t \leq T = c/\sigma$. This solution decomposes into k distinct waves $\pi_j(\varepsilon\partial_{x''})u_l^{out}$, $j \leq k$ among which $\pi_{out}(\varepsilon\partial_{x''})u_l^{out}$ is the main contribution. From [6] we can replace u_l^{out} by $We^{ix''.\xi''_{out}/\varepsilon} = \pi_{out}(\varepsilon\partial_{x''})u_l^{out}$ where W solves

$$\left\{ \begin{array}{c} \partial_t + \frac{1}{\varepsilon}(\lambda_{out}(\varepsilon\partial_{x''} + \xi''_{out}) - \xi_{out,0})W = \sigma\pi_{out}(\varepsilon\partial_{x''} + \xi''_{out})\phi(W), \quad x \in [0,T]\times\mathbb{R}^n \\ W(t=0) = \pi_{out}(\varepsilon\partial_{x''} + \xi''_{out})V_0(x'') \end{array} \right. \tag{4.2.3}$$

and the accuracy of the approximation increases when σ decreases.

Now choosing a rapidly decreasing (in z) initial profile V_0, vanishing infinitely at $z = 0$ and ε small enough, the Cauchy problem (4.2.2) provides a boundary data at $z = 0$ nearly known for all $t > 0$ and vanishing infinitely at $t = 0$. This value is close to $[\pi_{out}(\varepsilon\partial_{x''} + \xi''_{out})u_l^{out}]_{|z=0}$.

Finally decomposing $[\pi_{out}(\varepsilon\partial_{x''})u_l^{out}]_{|z=0}$ according to the transmission problem allows to use the Lopatinski condition and solve analytically $T([\pi_{out}(\varepsilon\partial_{x''})u_l^{out}]_{|z=0}, b^\varepsilon) = 0$.

Next one raises the question of the inverse problem which consists in determining V_0 in term of a given $u_{r|_{z=0}} = b^\varepsilon$. This requires to solve $T(u_{l|_{z=0}}, b^\varepsilon) = 0$ in $u_{l|_{z=0}}$ and then to find V_0 in term of $u_{l|_{z=0}}$.

The first step is however not natural : one usually solves the boundary equation by determining the incoming waves from the outgoing ones (using the Lopatinski condition).

So one supposes given instead, the trace of a polarized outging wave at the left of the boundary : $u_l^0 = \tilde{\pi}_{out}(\varepsilon\partial_{x'})u_l^0$ where $\tilde{\pi}_{out}(\eta') = \pi_{out}(\eta_1, \ldots, \eta_n, \zeta_{out}(\eta'))$ with $\xi_{out,n} = \zeta_{out}(\xi'_{out})$. This is the symmetric projector onto $\ker(\mathcal{L}(\varepsilon\partial_{x'}, \zeta_{out}(\varepsilon\partial_{x'})))$ defined for $\eta' \in \mathcal{F}_{\xi_{out}}$. Finally one supposes that u_l^0 has compact spectrum : $u_l^0 = \chi_{out}u_l^0$.

Then one supposes that the boundary data b^ε for the equation (4.3.8) is computed by solving $T(u_l^0, b^\varepsilon) = 0$.

Next we look for an approximate way to determine V_0 thanks to u_l^0. From the discussion before (4.2.3) one expects that u_l^0 can be written under the form $u_l^0 = [\pi_{out}(\varepsilon\partial_{x''})u_l^{out}]_{|z=0}$. Nevertheless u_l^0 and $\pi_{out}u_l^{out}{}_{|z=0}$ are not polarized on the same space. This is due to the nonlinearity in equation (4.2.3) and the fact that V_0 may have a large spectrum. However, if σ is small the two terms are all more the closer. More precisely one has

Lemma 4.2.1. *Consider the equation* (4.2.3)*. Let*

$$\Omega = \{(\xi_1, \ldots, \xi_d) \mid \exists\tau \text{ such that } (\tau, \xi_1, \ldots, \xi_{d-1}) \in \mathcal{F}_{\xi_{out}}\}.$$

Suppose $\sigma = 0$ and $\mathcal{F}_{x''}V_0 \in \mathcal{D}^0(\Omega)$ where $\mathcal{D}^0(\Omega)$ is the set of $\mathcal{C}^\infty(\Omega)$ functions compactly supported in Ω and $\mathcal{F}_{x''}$ is the Fourier transform with respect to x''. Then

$$(1 - \tilde{\pi}_{out}(\varepsilon\partial_{x'} + \xi'_{out}))W_{|z=0} = 0.$$

Démonstration. Indeed, if $\sigma = 0$ then W is solution to

$$\begin{aligned}
&\mathcal{L}(\partial_x + i\xi_{out}/\varepsilon)W = 0, \qquad t \in \mathbb{R}\\
&W(t=0) = V_0, \qquad t = 0\\
&V_0 = \pi_{out}(\varepsilon\partial_{x''})V_0, \quad \mathcal{F}_{x''}V_0 \in \mathcal{D}^0(\Omega).
\end{aligned}$$

Let $\mathcal{F}_{x''}u(t,\eta'')$ the Fourier transform of u in x'' then the solution reads $\mathcal{F}_{x''}u(t,\eta'') = e^{-it(\lambda_{out}-\xi_{out,0})/\varepsilon}\mathcal{F}_{x''}V_0$.

with the notations 4.3.1 we have for all $z \in \mathbb{R}$

$$\begin{aligned}
\mathcal{F}_{\tilde{x}}u(t,\tilde{\xi},z) = \int_{\eta_n} e^{iz\eta_n} e^{it(\lambda_{out}(\varepsilon\tilde{\xi}+\tilde{\xi}_{out},\varepsilon\eta_n+\xi_{out,n})-\xi_{out,0})/\varepsilon}\\
\pi_{out}(\varepsilon\tilde{\xi}+\tilde{\xi}_{out},\varepsilon\eta_n+\xi_{out,n})\mathcal{F}_{x''}V_0(\tilde{\xi},\eta_n)d\eta_n.
\end{aligned}$$

Thanks to the spectral compactness of V_0, one can make the change of variable $\eta_n \to \varepsilon\eta_0 + \xi_{out,0} = \lambda_{out}(\varepsilon\tilde{\xi}+\tilde{\xi}_{out},\varepsilon\eta_n+\xi_{out,n})$. Thus $\varepsilon\eta_n + \xi_{out,n} = \zeta_{out}(\varepsilon\eta_0+\xi_{out,0},\varepsilon\tilde{\xi}+\tilde{\xi}_{out})$ and the bounded Jacobien is $J^\varepsilon(\eta_0,\tilde{\xi}) = \partial\zeta_{out}/\partial_{\eta_0}$. The projector then becomes $\pi_{out}(\varepsilon\tilde{\xi}+\tilde{\xi}_{out},\zeta_{out}(\varepsilon\eta_0+\xi_{out,0},\varepsilon\tilde{\xi}+\tilde{\xi}_{out})) = \tilde{\pi}_{out}(\varepsilon\eta_0+\xi_{out,0},\varepsilon\tilde{\xi}+\tilde{\xi}_{out})$. As a result the formula reads

$$\begin{aligned}
\mathcal{F}_{\tilde{x}}W(t,\tilde{\xi},z) =&\mathcal{F}_{\eta_0}^{-1}(J^\varepsilon(.,\tilde{\xi})e^{iz(\zeta_{out}(\varepsilon.+\xi_{out,0},\varepsilon\tilde{\xi}+\tilde{\xi}_{out})-\xi_{out,n})/\varepsilon}\\
&\tilde{\pi}_{out}(\varepsilon.+\xi_{out,0},\varepsilon\tilde{\xi}+\tilde{\xi}_{out})\mathcal{F}_{x''}V_0(\tilde{\xi},(\zeta_{out}(\varepsilon.+\xi_{out,0},\varepsilon\tilde{\xi}+\tilde{\xi}_{out})-\xi_{out})/\varepsilon)
\end{aligned}$$

where $\mathcal{F}_{\eta_0}^{-1}$ is the inverse Fourier transform with respect to η_0. Applying $\mathcal{F}_{x_0}$ gives

$$\hat{W}(\xi',z) = J^\varepsilon(\xi')\tilde{\pi}_{out}(\varepsilon\xi'+\xi'_{out})V_0(\tilde{\xi},(\zeta_{out}(\varepsilon\xi'+\xi'_{out})-\xi_{out,n})/\varepsilon)e^{iz(\zeta_{out}(\varepsilon\xi'+\xi'_{out})-\xi_{out,n})/\varepsilon}$$

which means that $\hat{W}(\xi',z)$ is polarized according to $\tilde{\pi}_{out}$. This ends the proof. □

Then one expects that $V_0 \approx w(t=0)$ where w is solution to the boundary backward problem related to (4.2.3) for $z \geq -Z < 0$:

(4.2.4)

$$\begin{cases}
\partial_z + \frac{1}{\varepsilon}(\zeta_{out}(\varepsilon\partial_{x'}+\xi'_{out})-\xi_{out,n})w = \sigma\tilde{\pi}_{out}(\varepsilon\partial_{x'}+\xi'_{out})\chi_{out}(\varepsilon\partial_{x'})\phi(w), \quad x \in \mathbb{R}^n\times]-Z,0],\\
w(z=0) = u_l^0
\end{cases}$$

Note that even if $w(z) = \tilde{\pi}_{out}w(z)$ one has $w(t=0) \neq \pi_{out}w(t=0)$. Thus V_0 should be approximated by $\pi_{out}w(t=0)$. A lemma similar to lemma 4.2.1 would show that the approximation improves when σ decreases.

In the next section we derive a model from (4.1.9),(4.1.10) then in section 4 we show a precised convergence theorem. But for the sake of simplicity the result does not concern the convergence of the model to (4.2.1) but to (4.1.9),(4.1.10) with a boundary data artificially chosen to be well posed but close to that given in (4.3.8). This is meaningful as long as the approximation of the boundary data is of same order of that done in the previous discussion (see section 4, (4.4) and the remark that follows).

4.3 Formal derivation of the intermediate model

Notations 4.3.1. *Let $\mathcal{F}_{x'}$ the Fourier transform with respect to x', θ. Denote ξ', p the dual variables associated to x, θ.*

Since $\mathcal{U}$ is supposed to be 2π periodic in θ, $\mathcal{F}_{x'}\mathcal{U}(\xi', z, p)$ is in fact the p Fourier coefficient of the Fourier transform in x' of $\mathcal{U}$.

$$\mathcal{F}_{x'}(\mathcal{U})(\xi', z, p) = \frac{1}{2i\pi}\int_\theta \int_{x'} e^{i(\xi'.x'+p\theta)}\mathcal{U}(x', z, \theta)dx'd\theta.$$

For the sake of simplicity we omit the factor ε when it is just used as a parameter (we note $F(x, \theta)$ instead of $F(x, \theta, \varepsilon^p)$). Then we use the notation $\widehat{\mathcal{U}}_p(\xi') = \mathcal{F}_{x'}\mathcal{U}(\xi', p)$. Equation (4.1.15) then writes

$$\left(A_n\partial_z + \frac{1}{\varepsilon}(L'(i\varepsilon\xi' + i\xi'_\star\, p) + i\xi_{\star,n}A_n\, p)\right)\widehat{\mathcal{U}}_p(\xi', z) = \sigma\widehat{F}_p(\xi', z) \tag{4.3.1}$$

4.3.1 Reduction of equation (4.3.1)

First we display the skew-symmetric matrix $L'(i\eta') = \begin{pmatrix} L^{11}(i\eta') & L^{12}(i\eta') \\ L^{21}(i\eta') & L^{22}(i\eta') \end{pmatrix}$ in the same bloc decomposition as A_n. As $L'(i\eta')$ is skew-symmetric $L^{11}(i\eta')$ and $L^{22}(i\eta')$ are also skew-symmetric while $L^{12}(i\eta') = -(L^{21}(i\eta'))^*$ where M^* is the hermitian transpose of A.

For the sake of simplicity, η' will be used instead of $\xi' + \xi'_\star p/\varepsilon$ and ∂_z instead of $\partial_z + \xi_{\star,n}p/\varepsilon$ for algebraic calculation. Then using $\mathcal{U} = (\mathcal{U}^1, \mathcal{U}^2)$ equation (4.3.1) splits into

$$\begin{cases} A_n^{11}\partial_z\widehat{\mathcal{U}}^1 + L^{11}(i\eta')\widehat{\mathcal{U}}^1 + L^{12}(i\eta')\widehat{\mathcal{U}}^2 = \sigma\widehat{F}^1 \\ L^{21}(i\eta')\widehat{\mathcal{U}}^1 + L^{22}(i\eta')\widehat{\mathcal{U}}^2 = \sigma\widehat{F}^2 \end{cases} \tag{4.3.2}$$

The first equation is an evolution equation on the unknown $\widehat{\mathcal{U}}^1$ in the variable z so we wish to compute $\widehat{\mathcal{U}}^2$ in term of $\widehat{\mathcal{U}}^1$ from the second equation. We have to make an assumption on the invertibility of $L^{22}(\eta')$ (see next subsection, remark 2).

When L^{22} is invertible, system (4.3.1) reduces to

$$\begin{cases} (\partial_z + G)\,\widehat{\mathcal{U}}^1 = \sigma(A_n^{11})^{-1}\widehat{F}^3, \\ \widehat{\mathcal{U}}^2 = (L^{22})^{-1}\left(\sigma\widehat{F}^2 - L^{21}\widehat{\mathcal{U}}^1\right), \\ G = (A_n^{11})^{-1}\big(L^{11} - L^{12}(L^{22})^{-1}L^{21}\big), \\ \widehat{F}_3 = \widehat{F}^1 - L^{12}(L^{22})^{-1}\widehat{F}^2 \end{cases} \tag{4.3.3}$$

Solving the first equation is strongly linked with the study of the eigenvalues $i\zeta_j,\ \zeta_j \in \mathbb{C}$ of $G(i\eta')$. Note that necessarily $(i\eta', i\zeta_j) \in \mathrm{char}\mathcal{L}$.

If η_0 is located in a gap in $\mathrm{char}\mathcal{L}$ (cf. fig 1, $\eta_0 \in [1,6; 2, 3]$), the eigenvalue ζ_j is not real. This would change dramatically our profile description. So one must insure "artificially" the nonlinearity spectrum doesn't spread over the region where ζ_j is real (see subsection 2.3).

Now we define the good tangential set according to the last remark.

Definition 4.3.2. *Let $\underline{\eta} \in \mathrm{char}\mathcal{L}$, regular and such that $\underline{\eta}_d = \zeta_j(\underline{\eta}')$. As $\underline{\eta}$ is regular there is a unique λ_ι, a root in η_0 of p such that $\underline{\eta}_0 = \lambda_\iota(\underline{\eta}'')$. Suppose $\partial_n\lambda_\iota(\underline{\eta}'') \neq 0$.*

By the implicit function theorem there is a neighbourhood $\mathcal{O}_{\underline{\eta}} \subset \mathbb{R}^{n+1}$ of $\underline{\eta}$ such that $\eta_0 = \lambda_\iota(\eta'') \Leftrightarrow \eta_n = \zeta_j(\eta')$.

Define the maximal such neighbourhood : $\mathcal{C}_{\underline{\eta}} = \bigcup \mathcal{O}_{\underline{\eta}}$.

Then define $\mathcal{F}_{\underline{\eta}}$ to be the connected component of $\mathcal{C}'_{\underline{\eta}} \cap \{\eta',\ \det L^{22}(\eta') \neq 0\}$ containing $\underline{\eta}'$.

As η' stands for $\varepsilon\eta' + \xi'_*$, let ζ_1 be the root of p in η_n associated to ξ_*. We note $\mathcal{F}_1 := \mathcal{F}_{\xi_*}$. Then $i\zeta_1$ is an eigenvalue of G which is regular and real on $\mathcal{F}_1$.

Remark 4.3.3. *If the λ_ι doesn't cross any other sheet excepted tangentially at its extrema one has*

$$\mathcal{F}_{\underline{\eta}} = \mathrm{Im}\,(\lambda_\iota) \times \mathbb{R}^{n-1}.$$

Then define a cut-off function related to $\mathcal{F}_{\underline{\eta}}$

$$\chi_{\underline{\eta}}(\xi') = \begin{cases} 1 \text{ on } K^1_{\underline{\eta}} \subset \mathcal{F}_{\underline{\eta}} - \underline{\eta}' \text{ compact} \\ 0 \text{ on } {}^cK^2_{\underline{\eta}},\ \text{with } K^1_{\underline{\eta}} \subset K^2_{\underline{\eta}} \subset \mathcal{F}_{\underline{\eta}} - \underline{\eta}' \ dist(K^1_{\underline{\eta}}, K^2_{\underline{\eta}}) > 0. \end{cases}$$

For proving the convergence of the model toward the initial boundary value problem one may impose some restrictions on the size of $K^1_{\underline{\eta}}, K^2_{\underline{\eta}}$ (cf. lemma 4.5.2). We note $\chi_1 := \chi_{\xi_*}$.

After applying this cut-off on equation (4.3.3) one can reasonably solve $\widehat{\mathcal{U}}^1$. Indeed define π^1_j the symmetric projector on $\mathbb{E}^j_G$, the spectral space associated to ζ_j.

First note the spectral decomposition of G is semi-simple on $\mathcal{F}_{\underline{\eta}}$ for $\underline{\eta} \in \mathrm{char}\mathcal{L}$.

Lemma 4.3.4. *Let $\underline{\eta}' \in \mathcal{F}_{(\underline{\eta}', \zeta_j(\underline{\eta}'))}$. Then*

$$\dim \mathbb{E}^j_G = \dim \ker \mathcal{L}(i\underline{\eta}', i\zeta_j)\,.$$

Démonstration. Introduce the matrices

$$P(i\eta') = \begin{pmatrix} I & 0 \\ -(L^{22})^{-1}L^{21} & I \end{pmatrix}, \quad \widetilde{A}^{-1}_n = \begin{pmatrix} (A^{11}_n)^{-1} & 0 \\ 0 & I \end{pmatrix}\,.$$

Next make the product

$$\widetilde{A}^{-1}_n \mathcal{L}(i\eta)P(i\eta) = \begin{pmatrix} i\eta_n + G(i\eta') & (A^{11}_n)^{-1}L^{12}(i\eta') \\ 0 & L^{22}(i\eta') \end{pmatrix}\,.$$

Thus

$$p(\eta) = det(i\eta_n + G(i\eta'))det(L^{22}(i\eta'))det(A^{11}_n)\,. \tag{4.3.4}$$

let λ_ι the root related to $\mathcal{F}_{(\underline{\eta}', \zeta_j(\underline{\eta}'))}$. From the hyperbolicity of $\mathcal{L}$ the characteristic polynomial is broken down on $\mathbb{R}^{n+1}$ and writes

$p(\eta) = p_1(\eta)(\lambda_\iota(\eta'') - \eta_0)^m$ with $p_1(\underline{\eta}', \zeta_j(\underline{\eta}')) \neq 0$.

Then as ζ_j is a root of $det(i\eta_n + G(i\eta'))$ in η_n the latter writes $det(i\eta_n + G(i\eta')) = C(\eta)(\eta_n - \zeta_j(\eta'))^k$ with $C(\eta) \neq 0$ in a neighbourhood of $(\underline{\eta}', \zeta_j(\underline{\eta}')$.

From (4.3.4) and since $\det L^{22}(i\underline{\eta}') \neq 0$ one sees that $m = k$.

We conclude by the two facts : 1) $\dim\ker\mathcal{L}(i\underline{\eta}',i\zeta_j)=m$ since $\mathcal{L}$ is symmetric and 2) $\dim\mathbb{E}^j_G\geq\dim\ker\mathcal{L}(i\underline{\eta}',i\zeta_j)$ since $U\in\ker\mathcal{L}(i\underline{\eta}',i\zeta_j)\Rightarrow U^1\in\ker(i\zeta_j(\eta')+G(i\eta'))$. This ends the proof of the lemma.

□

Thus π^1_j is the symmetric projector on $\ker(i\zeta_jA^{11}_n+A^{11}_nG)$. As a consequence $\pi^1_jA^{11}_n\pi^1_k=c_k\delta_{jk}$ where δ_{jk} is the Kronecker symbol. Equation (4.3.3) thus splits for all $\eta'\in\mathcal{F}_\star$ into

$$\begin{aligned}&\big(\partial_z-i\zeta_j(\eta')\big)\,\pi^1_jA^{11}_n\pi^1_j\widehat{\mathcal{U}}^1=\sigma\pi^1_j\widehat{F}^3,\quad j\leq q\\&\widehat{\mathcal{U}}^2=(L^{22})^{-1}(\sigma\widehat{F}^2-L^{21}\widehat{\mathcal{U}}^1).\end{aligned}$$

Moreover doing as in Proposition 3.1 in [14] we get $M_j=\pi^1_jA^{11}_n\pi^1_j=\frac{1}{v_{j,n}}\pi^1_j(I+H^*H)\pi^1_j$ with $v_{j,n}=\partial_{\eta_0}\zeta_j(\eta')$ and $H=(L^{22})^{-1}L^{21}$. Remark since $I+H^*H$ is a definite positive matrix, M_j is definite on $\operatorname{Im}\pi^1_j$ and has a sign.

Let q^1_j the symmetric partial inverse of M_j such that $q^1_j(1-\pi^1_j)=0$. The equation on $\pi^1_j\widehat{\mathcal{U}}^1$ is thus

$$\big(\partial_z-i\zeta_j(\eta')\big)\,\pi^1_j\widehat{\mathcal{U}}^1=\sigma q^1_j\pi^1_j\widehat{F}^3.$$

Next since $(\pi^1_j\widehat{\mathcal{U}}^1,-H\pi^1_j\widehat{\mathcal{U}}^1)\in\ker(\mathcal{L}(i\eta',i\zeta_j))$ one has the following spectral decomposition

$$\widehat{\mathcal{U}}=\sum_{j=1}^q\Pi_j\widehat{\mathcal{U}}+\widehat{\mathcal{V}},\quad\widehat{\mathcal{V}}={}^t(0,\widehat{\mathcal{V}}^2)\in\ker A_n \tag{4.3.5}$$

where $\Pi_j=\begin{pmatrix}\pi^1_j&0\\-H\pi^1_j&0\end{pmatrix}$ is a projector on $\ker(\mathcal{L}(i\eta',i\zeta_j))$.

Plugging this decomposition in equation (4.3.3) and applying successively the projectors Π^*_j we get

$$\big(\partial_z-i\zeta_j(\eta')\big)\,\Pi^*_jA_n\Pi_j\widehat{\mathcal{U}}=\sigma\Pi^*_j\widehat{F},\quad j\leq q \tag{4.3.6}$$

$$\widehat{\mathcal{V}}^2=\sigma(L^{22})^{-1}\widehat{F}^2. \tag{4.3.7}$$

As previously and referring again to [14] $\Pi^*_jA_n\Pi_j=v_{j,n}\Pi^*_j\Pi_j$. Then note Q_j the symmetric partial inverse of $v_{j,n}\Pi^*_j\Pi_j$ such that $(1-\Pi_j)Q_j=0$.

Finally replace η' by $\xi'+\xi'_\star p/\varepsilon$ and ∂_z by $\partial_z+\xi_{\star,n}p/\varepsilon$. We make a few remarks :

Remark 4.3.5. *1. First as F is a polynomial of $\mathcal{U}$ in* (4.3.7) *the spectrum of $\mathcal{U}$ is $\mathbb{R}^n$ as soon as $z>0$ even if the initial data is spectrally compactly supported. Thus one needs to apply χ_1 on the right hand side of* (4.3.7) *for* (4.3.5) *to be valid.*

2. Remember we look for solutions of (4.1.9) *of kind* (4.1.13)*. From the previous equation we see that q distinct modes are to be generated, each propagating at different speed. Then, to select the mode related to ξ_* requires the boundary data to be polarized according to Π_1.*

3. *If there exists $p \in \mathbb{Z}$ such that $\xi_\star p \in \mathrm{char}\mathcal{L}$ then the solution of* (4.1.9) *will generate this resonant harmonic. As this phenomenon is not generic we assume as in [15] there is a finite number of resonant harmonics. A sufficient condition given by the authors is*

Assumption 4.3.6. $\det(L^1(\xi_\star)) \neq 0$.

Let R_* be the finite set of resonant harmonics $\xi_\star p \in \mathrm{char}\mathcal{L}$. We note $\mathcal{F}_p$, χ_p instead of $\mathcal{F}_{\xi_\star p}$, $\chi_{\xi_\star p}$ for $\xi_\star p \in \mathrm{char}\mathcal{L}$ and j_p the index of the root ζ_{j_p} related to $\mathcal{F}_p$ such that $p\xi_{*,n} = \zeta_{j_p}(p\xi'_*)$.

One expects $\mathcal{U} = \sum_{p\in R_*} \mathcal{U}_p(x', z)e^{ip\theta}$ with $\mathcal{U}_p$ solution of the coupled problem

$$\left\{\begin{array}{c} \left(\partial_z - \frac{i}{\varepsilon}(\zeta_{j_p}(\varepsilon\xi' + \xi'_\star p) - p\xi_{\star,n})\right)\widehat{\mathcal{U}}_p(\xi', z) = \\ \sigma Q_{j_p}(\varepsilon\xi' + \xi'_\star p)\chi_p(\varepsilon\xi')\widehat{F}_p(\mathcal{U})(\xi', z) \\ \widehat{\mathcal{U}}_{\pm 1}(\xi', z = 0) = \Pi_{\pm 1}(\varepsilon\xi' + \xi'_\star p)\chi_p(\varepsilon\xi')\widehat{\mathcal{U}}_0(\xi') \\ \widehat{\mathcal{U}}_p(\xi', z = 0) = 0 \quad p \neq \pm 1. \end{array}\right. \tag{4.3.8}$$

to provide a profile for an oscillating function of kind (4.1.13) which is a good approximation of (4.1.9),(4.1.10). As in [6] one expects this model to give a better approximation as the size of the rescaled initial data σ decreases (see section 4 for a precise approximation theorem).

Remark we neglect the part $\mathcal{V}_p$ of $\mathcal{U}_p$ since $L^{22}(\xi'_\star p)$ is invertible and $L^{22}(\xi' + \xi'_\star p/\varepsilon)\chi_p$ is of size $1/\varepsilon$.

4.3.2 Remarks on the model (4.3.8)

1. *New model vs Geometric and diffractive model.* Let us note that the new model is similar to the transport equation of Geometric optics or the Schrödinger equation of diffractive optics and the main change is that it is linearly exact while the other two are Taylor approximations respectively of first order and second order. Indeed the new model is based on the exact computation of $\zeta_1(\xi'_* + \varepsilon\xi')$ for all $\xi' \in \mathcal{F}_1 \subset \mathbb{R}$ while the other two models (Geometric and Diffractive optics) use a first order (and second order) approximation $\zeta_1 \approx \xi_{*,n} + i\varepsilon\xi'.\nabla_{\xi'}\zeta_1(\xi'_*)$ $(-\varepsilon^2\nabla^2_{\xi'}\zeta_1(\xi'_*; \xi', \xi'))$. One thus sees our model outperforms the other two in the linear regime. Next, as the computation of ζ_1 is easy for the Maxwell system (one only needs to solve a second order polynomial) the new model has an algorithmic complexity very close to the other two models.
2. *Typical solutions better ruled by the new model than the Geometric and diffractive ones.* The model is spectrally compact so it requires quite regular initial data. When studying the convergence toward the boundary problem (4.1.9),(4.1.10) one will also need spatial decay. Thus typical solutions are sharp Gaussian or spectrally chirped Gaussian. In both cases the wave shows a broad spectrum on which ζ_1 may not be well approximated by the Schrödinger model, typically where char$\mathcal{L}$ is very curved. In the case of a chirped wave one may enhance an intermediate small parameter (between 1 and ε) associated to intermediate under-structures that should be taken into account for a more accurate WKB development. On the contrary the new model doesn't need such a discussion. Nevertheless one must take care to the spectrum broadening of U since the validity of our approach stops when the spectrum reaches the elliptic zone of $\mathcal{L}$.

3. *Determining $\mathcal{F}_p$.* For the numerical implementation one needs to determine the $\mathcal{F}_p$ on which L^{22} is invertible. One thus gives a simple way to identify those roots on the graph of char$\mathcal{L}$. Those exceptional points $\underline{\eta'}$ are either in char$L$ and then all the sheet $\{\eta' = \underline{\eta'}\}$ are in char$L$ or they do not belong to char$L$ and then they correspond to an asymptotic value of a λ_j.

   To see this let write the characteristic polynomial p and $p^{22} := \det L^{22}$

$$p(\eta) = p_0(\eta') \sum_{j \leq k} c_j(\eta') \eta_n^j, \quad p^{22}(\eta') = p_0(\eta') c_k(\eta').$$

   The first case corresponds to a root of p_0. The second is a root of c_k. Indeed writing $\sum_{j \leq k} c_j(\eta') \eta_n^j = c_k(\eta') \eta_n^k + p^\sharp_{\eta'}(\eta_n)$ one sees there is at least one root of the polynomial which is not a root of $p^\sharp_{\underline{\eta'}}$ (its degree is strictly smaller). This root goes to infinity as $\eta' \to \underline{\eta'}$.

   By definition of $\mathcal{F}_1$ every $(\eta', \zeta_1(\eta'))$, $\eta' \in \mathcal{F}_1$ is a regular point of char$\mathcal{L}$ thus the projector Π_* is regular even if L^{22} is not invertible.

   Nevertheless $(L^{22})^{-1} \widehat{F}^2$ may not be definite and the model too. Even if it is finite one cannot neglect it anymore and the model should then take it into account.

   Yet, since L^{22} is almost always invertible it doesn't matter for numerical calculations. Indeed if a point of spectral discretization happens to be such that L^{22} is not invertible a slight change of the discretization will avoid it.

4. *L^2-like conservation.* Finally let us note that both systems given in the first section have nonlinearity deriving from a potential : $\varepsilon \partial_t F(U) \cdot U = \varepsilon \partial_t \Phi(U)$. The solutions of our model then satisfy a L^2-like conservation property. Indeed taking the scalar product of equation (4.3.8) (written as in equation (4.3.7)) by $\Pi_j \mathcal{U}_p$, taking the real part and summing over $p \in R_*$ gives

$$\partial_z \sum_{p \in R_*} < v_{j_p,n} \Pi_{j_p} \mathcal{U}_p, \Pi_{j_p} \mathcal{U}_p > = < \chi(\varepsilon \partial_t + \xi_* \partial_\theta) F(\mathcal{U}), \mathcal{U} >$$

   where $<,>$ is the $L^2_{x',\theta}$ scalar product and χ is the operator $\chi \mathcal{U} = \sum_{p \in R_*} \chi_{j_p} \mathcal{U}_p$. From equation (4.3.8) $\mathcal{U} = \chi \mathcal{U}$, thus one can forget the truncation in this identity.

   The extra non resonant harmonics generated by $F(\mathcal{U})$ have no influence. Therefore the r.h.s. of the identity is $\int_{x',\theta} (\varepsilon \partial_t + \xi_* \partial_\theta) \Phi(\mathcal{U}) dx' d\theta = 0$ if $\Phi(\mathcal{U})(z) \in L^2(\mathbb{R}^n \times \mathbb{T})$. Finally we get the conservation $N(z) = \sum_{p \in R_*} < v_{j_p,n} \Pi_{j_p} \mathcal{U}_p, \Pi_{j_p} \mathcal{U}_p > = N(0)$. If the $v_{j_p,n}$ have not the same sign, some harmonic could blow up. Otherwise as the $v_{j_p,n}$ do not vanish on $\mathcal{F}_p$ they keep the same sign and if all of them have the same sign one recovers a L^2-like conservation.

4.4 A partial justification of the derivation. Results

We first give a result for the existence of solution to (4.3.8) in a space of fast transverse decay. We will motivate this choice in the next discussion.

Then we give a convergence result between (4.3.8) and (4.1.9),(4.1.10) viewed as the restriction of (4.2.1) to $z > 0$ in the spirit of that obtained in [6] (Theorem 1.).

From section 2 and 4 one could reasonably expect a convergence result between (4.3.8) and (4.2.1) (with initial data found in the previous section). Nevertheless such a result requires a drastically different analysis compared to that in [6]. Indeed in [6] the authors look for an exact solution of the Cauchy problem as $\varepsilon^{-p}u^\varepsilon(x) = \mathcal{U}^\varepsilon(x,\theta) + \sigma\varepsilon r^\varepsilon(x,\theta)$ with $\theta = \xi_*.x/\varepsilon$ and r^ε 2π-periodic in θ.

In contrast it is known (cf.[35]) that the convergence for mixed problems can't just involve 2π-periodic remainder. The boundary indeed generates an infinite dimensional module of resonant harmonics where some of which are boundary layers. Moreover the well-known energy estimates using dissipation (cf.[42]) or Kreiss theory (cf.[30]) don't give obvious ε-independent estimates for a two scale remainder.

That is why one looks for the solution of (4.2.1) as a perturbation depending on the strength of the non-linearity (σ) :

$$u^\varepsilon(x) = \mathcal{U}^\varepsilon(x,\xi_*x/\varepsilon) + \sigma\varepsilon r_1^\varepsilon(x) + \varepsilon^M r^\varepsilon(x),$$

and the WKB corrector $r_1^\varepsilon(x) = \sum_{1\leq j\leq N}\varepsilon^j\mathcal{U}^{j,\varepsilon}(x,\xi_*x/\varepsilon)$ must be precise enough for r^ε to solve the remaining equation in strong norms (see [14] in a Cauchy context and [35] for a mixed problem).

However the study of mixed problems requires the use of oscillating profiles $\mathcal{U}^{j,\varepsilon}$ with boundary layer (that is exponentially decaying function in the direction transverse to the boundary).

For the sake of simplicity one chooses to prove the "convergence" of (4.3.8) toward (4.1.9),(4.1.10) in a sense close to [13]. We actually mean the succession of those two steps

1. First one constructs a WKB oscillating corrector r_1^ε with 2π-periodic profiles such that $\mathcal{U}^\varepsilon + \sigma\varepsilon r_1^\varepsilon$ solves accurately the equation $\mathcal{L}(\varepsilon\partial_x)U = \varepsilon\sigma F(U)$. In view of the convergence one sets U_{app} this approximate solution restricted to $t \geq 0$ and enlarged to all $z \geq 0$.
2. Then one takes the *artificial* boundary value for (4.1.9),(4.1.10) $U_{app}|_{z=0}$ and one shows the convergence between U_{app} and the solution of (4.1.9),(4.1.10) as an homogeneous mixed hyperbolic problem.

Remark 4.4.1. *This choice is similar to that of [13] who chooses to consider only one incoming wave and taking boundary data absorbing the other waves. What is important is that the difference between the* artificial *boundary value and the "physical" value* $\mathcal{U}^\varepsilon_{|_{z=0}}$ *given by the physical measurements is of same order* $\varepsilon\sigma$ *of that between the solution of the new model* (4.3.8) *and the physical solution of* (4.2.1)*.*

This "convergence" raises three difficulties. The first concerns the construction of r_1^ε *which requires a cut-off* χ_{j_p} *for any resonant mode* $p\xi_\star$ *(see the Ansatz and* (4.4.4)*). The other two come from the restriction of* $\mathcal{U}^\varepsilon + \sigma\varepsilon r_1^\varepsilon$ *to* $t \geq 0$ *and its extension to all* $z \geq 0$*. We refer to [13], chapter* 3 *and to fig.2.*

We next give a brief commentary on the construction of the corrector.

For the sake of simplicity we strengthen assumption 4.3.6 assuming there is no resonant harmonics.

Assumption 4.4.2. $p(k\xi_\star) = 0 \Longleftrightarrow k = \pm 1$.

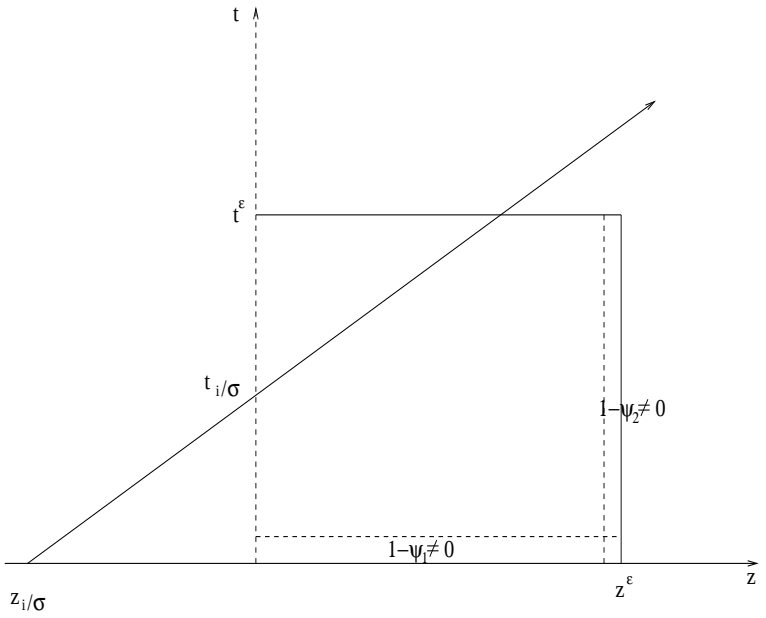

FIG. 4.2 – Convergence frame

Thanks to this assumption the higher harmonics can be found through an elliptic inversion. As for $\mathcal{U}^\varepsilon$ the fundamental mode must be searched with truncated spectrum (cf. section 2). Then define the truncations operator on profiles $\mathcal{V}(\xi', z, \theta)$, 2π-periodic in θ.

$$\chi_f(\varepsilon\xi')\mathcal{V}(\xi', z, \theta) = \chi_1(\varepsilon\xi')\mathcal{V}_1(\xi', z)e^{i\theta} + c.c. \tag{4.4.1}$$

$$\Pi_f(\varepsilon\xi)\mathcal{V}(\xi', z, \theta) = \Pi_1(\varepsilon\xi + \xi_*)\mathcal{V}_1(\xi', z)e^{i\theta} + c.c. \tag{4.4.2}$$

$$\chi(\varepsilon\xi')\mathcal{V}(\xi', z, \theta) = \chi_f(\varepsilon\xi')\mathcal{V}(\xi', z, \theta) + \sum_{p\neq\pm1} \mathcal{V}_p(\xi', z)e^{ip\theta}. \tag{4.4.3}$$

Ansatz 4.4.3. *$\mathcal{U}^{j,\varepsilon} \in \mathcal{C}([0, z]; L^2(\mathbb{R}^n \times \mathbb{T}))$ and*

$$\mathcal{U}^{j,\varepsilon} = \chi(\varepsilon\partial_{x'})\mathcal{U}^{j,\varepsilon}.$$

The approximate solution $U_{app} = \mathcal{U}^\varepsilon + \sigma\varepsilon r_1^\varepsilon$ will be designed to solve approximately

$$\mathcal{L}(\varepsilon\partial_x + \xi_*\partial_\theta)U_{app} \approx \varepsilon\sigma\chi F(U_{app}). \tag{4.4.4}$$

This approximation makes sense if $(1-\chi)F(U_{app})$ is smaller than any corrector. Clearly one needs a lot of regularity (see lemma 4.5.4).

Then one solves $(\chi - \chi_f)\mathcal{U}^{j,\varepsilon}$ and $(1-\Pi_f)\chi_f\mathcal{U}^{j,\varepsilon}$ through elliptic inversions. $\Pi_f\chi_f\mathcal{U}^{j,\varepsilon}$ satisfies a linear equivalent of (4.3.8).

Finally one gives the setting for the convergence. This is summarized in (fig.4.2).

If one refers to the lasers which are spatially localized, the truncation for $t \geq 0$ shouldn't change the remainder dramatically. We just need to take an initial value for (4.2.1) with

support far enough from the boundary so that the incident wave hits the boundary for large enough time in the future.

We first use this localization property taking profiles in $\Gamma^s_{y'}$ (see [13, 36]) :

$$\Gamma^s_{y'}(\mathbb{R}^n) = \{u(x') \mid \ < . - y' >^\alpha \partial^{s-\alpha}_{x'} u(.) \in L^2(\mathbb{R}^n),\ \alpha \leq s\}.$$

It is endowed with the related norm and is an algebra as soon as $s > n/2$.

Then we look for the boundary data in $\Gamma^s_{x'_i/\sigma}(\mathbb{R}^n)$ with s big. This means we assume the incident wave propagates up to t_i/σ. As shown in (figure 4.2) one considers that $x'_i = (t_i, \tilde{x}_i) \in \mathbb{R}^n$ comes from a Cauchy data initially placed at $x''_i = (0, z_i)$. As the ray propagates at speed v_* one has $(\tilde{d}_i, 0) - v_* t_i = (0, z_i)$.

Next showing $U_{app} - \mathcal{U}^\varepsilon + \sigma\varepsilon r^\varepsilon_1$ is as small as one wants, requires $\sigma < 1$ (cf. [13] in the frame of diffractive optics with $\sigma = \varepsilon$).

Assumption 4.4.4. $p(J-1) > 1$.

Since (4.3.8) is almost a transport equation we want to do the analysis in the frame that propagates with the group velocity v'_*.

Definition 4.4.5. *Let $\mathcal{P}^s(z)$ the space of profiles satisfying the ansatz and such that the profile expressed in the group velocity transport frame $\mathcal{V}^\varepsilon(x', z) := \mathcal{U}^\varepsilon(x' + v'_* z, z)$ lies in $\mathcal{V}^\varepsilon \in X^s_\sigma(z)$ with*

$$X^s_\sigma(z) = \bigcup_{a+b\leq s} \mathcal{C}^a([0,z]; \Gamma^b_{x'_i/\sigma}(\mathbb{R}^n) \times H^b(\mathbb{T})).$$

Lemma 4.4.6. *Let $\xi_\star$ with $\xi_{\star,n} = \zeta_1(\xi'_\star)$. Let $\mathcal{B}_1 \in \Gamma^s_{x'_i/\sigma}(\mathbb{R}^n)$ $\mathcal{B}_1 = \Pi_1\chi_1\mathcal{B}_1$ a polarized truncated boundary amplitude for*

$$\mathcal{U}^\varepsilon_{|_{z=0}} = \mathcal{B}_1 e^{i\xi'_* x'/\varepsilon} + c.c.$$

Then there is a $z_0 > 0$ and a unique $\mathcal{U}^\varepsilon = \Pi_f\chi_f\mathcal{U}^\varepsilon$ solution of the pseudo-differential system $\mathcal{F}^{-1}$(4.3.8) such that $\mathcal{V}^\varepsilon$ lies in $X^s(z_0/\sigma)$.

We have a similar existence result for the WKB corrector r^ε_1. With the previous notations

Lemma 4.4.7. *There is a $r^\varepsilon_1 = \sum_{1\leq j\leq N} \varepsilon^j \mathcal{U}^{j,\varepsilon}$ and r^ε_2 satisfying the ansatz such that*

$$\begin{aligned} &\mathcal{L}(\varepsilon\partial)(\mathcal{U}^\varepsilon + \varepsilon\sigma r^\varepsilon_1) = \varepsilon\sigma\chi F(\mathcal{U}^\varepsilon + \varepsilon\sigma r^\varepsilon_1) + \varepsilon^N r^\varepsilon_2 \\ &\Pi_f\chi_f r^\varepsilon_1(z=0) = 0\ . \end{aligned}$$

In addition $\mathcal{U}^{j,\varepsilon} \in \mathcal{P}^{s-j}(z_0/\sigma)$ and $r^\varepsilon_2 \in \mathcal{P}^{s-N}(z_0/\sigma)$.

In order to solve an homogeneous problem for the remainder between (4.3.8) and (4.1.9),(4.1.10) one cuts the WKB approximation as explained in 4.4. We thus introduce $\psi_1(t)$ [resp.$\psi_2(z)$] a smooth truncation around $t = 0$ [resp. around $z = z_0/\sigma$] (see step 2 in the convergence). Then one takes the boundary data

$$\varepsilon^{-p} u^\varepsilon(z=0) = \psi_1(t)(\mathcal{U}^\varepsilon(x',0)e^{i\xi'_*.x'/\varepsilon} + \varepsilon\sigma r^\varepsilon_1(x', 0, x'/\varepsilon, 0)). \tag{4.4.5}$$

Finally introduce the inhomogeneous space with twice as much co-normal derivatives than normal ones (see [35]) : $E^{m,\varepsilon}([0,t_*])$ endowed with the norm

$$\|u(t)\|_{E^{m,\varepsilon}([0,t_*])} = \sum_{2\alpha_n + |\alpha'| \leq m} \varepsilon^{2\alpha_n + |\alpha'|} \sup_{t\in[0,t_*]} \|\partial^\alpha_x u(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}.$$

Theorem 4.4.8. *Let s and N two integers chosen so that $s-N > n/2$ and $s-N$ even. Problem 4.1.9 with boundary value* (4.4.5) *has a unique solution vanishing for $t \leq 0$*

$$\varepsilon^{-p}u^\varepsilon = \psi_1(t)\psi_2(z)\big(\mathcal{U}^\varepsilon(x)e^{i\xi_*.x/\varepsilon} + \varepsilon\sigma r_1^\varepsilon(x,x/\varepsilon)\big) + \varepsilon^M r^\varepsilon(x),$$

where $\mathcal{U}^\varepsilon$ and r_1^ε are given by the previous lemma and $r^\varepsilon \in E^{s-n,\varepsilon}([0,t^\varepsilon])$, $t^\varepsilon := \tilde{z}_0/(v_{*,n}\sigma)$ with $\tilde{z}_0 < z_0$.

In particular one has the Corollary

Corollary 4.4.9. *Let $\mathcal{U}^\varepsilon$ the solution given by lemma 4.4.6. Then there is a constant c and a unique solution u^ε of* (4.1.9) *lying in $E^{s-n,\varepsilon}([0,t^\varepsilon])$ such that*

$$\sup_{t\in[0,t^\varepsilon]}\sup_{z\in[0,z_0/\sigma]} \|\varepsilon^{-p}u^\varepsilon - \mathcal{U}^\varepsilon\|_\infty \leq c\sigma\varepsilon.$$

4.5 Proof of the Results

4.5.1 Lemma 4.4.6

First write the equations for the amplitude $\mathcal{V}_1^\varepsilon(x',z) := \mathcal{U}_1^\varepsilon(x'+v_*'z,z)$

$$\left(\partial_z - \frac{i}{\varepsilon}(\zeta_1(\varepsilon\xi'+\xi_\star) - \xi_{\star,n} - \varepsilon\xi'.v_*')\right)\widehat{\mathcal{V}}_1^\varepsilon(\xi',z,\theta) = \tag{4.5.1}$$

$$\sigma\chi_1(\varepsilon\xi')Q_1(\varepsilon\xi'+\xi_\star')\widehat{F}_1(\mathcal{V}_1^\varepsilon)(\xi',z,\theta), \tag{4.5.2}$$

Let us use the Taylor formula for the differential operator :

$$\zeta_1(\varepsilon\xi'+\xi_\star) - \xi_{\star,n} - \varepsilon\xi'.v_*' = \varepsilon^2\int_0^1 \frac{(1-s)^2}{2}\nabla^2\zeta_1(s\varepsilon\xi'+\xi_\star,\xi',\xi')ds.$$

As usual in order to prove the existence of $\mathcal{V}^\varepsilon \in X^s_{z_0/\sigma}$ we just give an energy estimate. Apply $\partial^{s'} :=< x' >^\alpha \partial_{x'}^{s-\alpha}$ on (4.5.2). Then taking $\int_{\mathbb{R}^n}\Re(\partial^{s'}(4.5.2),\partial_z^j\Pi_1\mathcal{V}^\varepsilon)$ gives

$$\partial_z\|\partial^{s'}\Pi_1\chi_1\mathcal{V}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \sigma\|\partial^{s'}\chi_1 Q_1\widehat{F}(\mathcal{V}^\varepsilon)\|_{L^2(\mathbb{R}^n)}.$$

Next $Q_1\chi_1$ maps Γ^s onto Γ^s with a norm that is bounded independently of ε. Indeed it commutes with x'-derivatives and one has $\| < x' >^\alpha Q_1\chi_1 u\|_{L^2(\mathbb{R}^n)} \leq \sum_{j\leq\alpha} c_j\|\partial_{\xi'}^j Q_1(\varepsilon\xi'+\xi_\star')\chi_1\widehat{u}\|_{L^2(\mathbb{R}^n)}$. Since $Q_1\chi_1$ as a compactly supported symbol, its j^{ieth}-derivative is also compactly supported and smaller by a factor ε^j.

Finally, from [14], for $s > n/2$ there is a Gagliardo-Niremberg Moser like estimate in Γ^s. All together there is a smooth function C

$$\partial_z\|\mathcal{V}_1^\varepsilon\|_{\Gamma^s_{x_i'/\sigma}} \leq \sigma C(\|\mathcal{V}_1^\varepsilon\|_\infty)\|\mathcal{V}_1^\varepsilon\|_{\Gamma^s_{x_i'/\sigma}}.$$

By standard Picard iterates and using Gronwall inequality one gets the existence of a $z_0 > 0$ and a unique solution in $X^s_\sigma(z_0/\sigma)$.

4.5.2 Theorem 4.4.8

This proof divides into two parts :

1. Constructing the WKB corrector r_1^ε,

2. Proving the convergence of the approximate solution, $\mathcal{U}^\varepsilon + \varepsilon r_1^\varepsilon$ to the solution of problem (4.1.9),(4.1.10).

1) First consider (4.4.4) and expand the nonlinear term of degree d :

$$F(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon) = F(\mathcal{U}^\varepsilon) + \varepsilon\sigma F^\varepsilon(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon, r_1^\varepsilon)$$
$$F^\varepsilon = \sum_{1\le j\le d} \frac{(\varepsilon\sigma)^{k-1}}{k!} F^{(k)}(\mathcal{U}^\varepsilon; r_1^\varepsilon, \dots, r_1^\varepsilon)$$

and $F^{(k)}$ is the k^{ieth} derivative of F. Then one expands F^ε in power of ε :

$$F^\varepsilon = \sum_{j=0}^{(n+1)d} \varepsilon^j F_\sigma^j, \quad F_\sigma^j = \sum_{k=1}^{d} \frac{\sigma^{k-1}}{k!} \sum_{j_1+\dots+j_k=j-k+1} F^{(k)}(\mathcal{U}^\varepsilon; \mathcal{U}^{j_1,\varepsilon}, \dots, \mathcal{U}^{j_k,\varepsilon}).$$

Note that $F_\sigma^j = \nabla_u F(\mathcal{U}^\varepsilon; \mathcal{U}^{j,\varepsilon}) + \tilde{F}_\sigma^j((\mathcal{U}^{k,\varepsilon})_{k\le j-1})$. Computing the difference (4.4.4)- (4.3.8) and dividing by σ gives

$$\mathcal{L}(\varepsilon\partial_x + \xi_*\partial_\theta) r_1^\varepsilon = (\chi - \Pi_f^*\chi_f) F(\mathcal{U}^\varepsilon) + \varepsilon\sigma\chi F^\varepsilon.$$

Π_f^* is defined similarly as Π_f in (4.3.5) but using the hermitian transpose of Π_1 : Π_1^*. The first nonlinear term has two contributions : $\chi - \Pi_f^*\chi_f = (\chi - \chi_f) + (1 - \Pi_1^*)\chi_f$. Let split F

$$F = \underline{F} + F_*, \quad \underline{F} = F_h + F^\flat, \tag{4.5.3}$$
$$F_h = (\chi - \chi_f)F, \quad F^\flat = (1 - \Pi_f^*)\chi_f F, \quad F_* = \Pi_f^*\chi_f F. \tag{4.5.4}$$

Then decompose the first profile in a similar way :

$$\mathcal{U}^{j,\varepsilon} = \underline{\mathcal{U}}^j + \mathcal{U}_*^j, \qquad \underline{\mathcal{U}}^j = \mathcal{U}_h^j + \mathcal{U}^{j,\flat} \tag{4.5.5}$$
$$\mathcal{U}_h^j = (\chi - \chi_f)\mathcal{U}^{j,\varepsilon}, \quad \mathcal{U}^{j,\flat} = (1 - \Pi_f)\chi_f\mathcal{U}^{j,\varepsilon}, \quad \mathcal{U}_*^j = \Pi_f\chi_f\mathcal{U}^{j,\varepsilon}. \tag{4.5.6}$$

Note that $\mathcal{U}^\varepsilon = \mathcal{U}_*^\varepsilon$.

The key point is that $\underline{\mathcal{U}}^j$ is not resonant with the linear operator while $\mathcal{U}_*^j$ is. We next split $\mathcal{L}$ according to (4.5.5) and the last remark

$$\mathcal{L}(\varepsilon\partial_x + \xi_*\partial_\theta)\mathcal{U} = \left\{\underline{\mathcal{L}}(\varepsilon\partial_x' + \xi_*'\partial_\theta, \xi_{*,n}\partial_\theta) + \varepsilon l(\partial_x)\right\}\underline{\mathcal{U}} + \varepsilon\mathcal{L}_*(\varepsilon\partial_x' + \xi_*'\partial_\theta, \partial_z)\mathcal{U}_*$$
$$\underline{\mathcal{L}}(\varepsilon\partial_x + \xi_*\partial_\theta, \xi_{*,n}\partial_\theta)\underline{\mathcal{U}} = \mathcal{L}(\xi_*\partial_\theta)\mathcal{U}_h + \{\mathcal{L}'(\varepsilon\partial_{x'} + i\xi_*'\partial_\theta) + i\xi_{*,n}\partial_\theta\}\mathcal{U}^\flat$$
$$l(\partial_x)\underline{\mathcal{U}} = \partial_x\mathcal{U}_h + \partial_z\mathcal{U}^\flat$$
$$\mathcal{L}_*(\varepsilon\partial_x' + \xi_*'\partial_\theta, \partial_z)\mathcal{U}_* = \big(\partial_z - \frac{i}{\varepsilon}(\zeta_1(\varepsilon\partial_{x'} + i\xi_*'\partial_\theta) - \xi_{*,n}\partial_\theta)\big)\mathcal{U}_*$$

Note that $\mathcal{L}_*\mathcal{U}_* = \mathcal{L}\mathcal{U}_*$. The equation (4.3.8) thus reads $\mathcal{L}_*\mathcal{U}_*^\varepsilon = \sigma F_*$. Next rewrite (4.4.4) as :

$$\underline{\mathcal{L}}\mathcal{U}^{0,\varepsilon} + \varepsilon\sum_{j\geq 0}\varepsilon^j(\underline{\mathcal{L}\mathcal{U}}^{j+1,\varepsilon} + \mathcal{L}_*\mathcal{U}_*^{j,\varepsilon} + l(\partial_x)\underline{\mathcal{U}}^{j,\varepsilon}) = \underline{F}(\mathcal{U}^\varepsilon) + \varepsilon\chi\sum_{j\geq 0}\varepsilon^j(\sigma\underline{F}_\sigma^j + \sigma F_{\sigma,*}^j).$$

We solve this equation canceling the coefficient of the Taylor series in ε :

$$\begin{aligned}
&(0) \qquad \underline{\mathcal{L}\mathcal{U}}^{0,\varepsilon} = \underline{F}(\mathcal{U}^\varepsilon)\\
&(0_*) \qquad \mathcal{L}_*\mathcal{U}_*^{0,\varepsilon} = \sigma F_{\sigma,*}^0\\
&(j) \qquad \underline{\mathcal{L}\mathcal{U}}^{j,\varepsilon} = \sigma\underline{F}_\sigma^{j-1} - l(\partial_x)\underline{\mathcal{U}}^{j-1,\varepsilon}, \quad j\geq 1\\
&(j_*) \qquad \mathcal{L}_*\mathcal{U}_*^{j,\varepsilon} = \sigma F_{\sigma,*}^j.
\end{aligned}$$

The equations (j) are solved through elliptic inversions while (j_*) are propagation equations linear in $\mathcal{U}_*^{j,\varepsilon}$ with zero boundary data.

We first give a lemma to solve (j) :

Lemma 4.5.1. *Let $F \in \mathcal{P}^s(z_0/\sigma)$. Then there is a unique solution $U = \underline{U} \in \mathcal{P}^s(z_0/\sigma)$ solving $\underline{\mathcal{L}}U = \sigma\underline{F}$.*

Démonstration. One looks for $U = U_h + U^\flat$ with $U_h = \sum_{p\neq 1} U_p e^{ip\theta}$ and $U^\flat = \sum_{k\geq 2}\Pi_k\mathcal{U}^\flat + V^\flat$, $V^\flat = (0, V^{\flat,2}) \in \ker A_n$. Then

1. $U_p = -\sigma(\mathcal{L}(ip\xi_\star))^{-1}F_p, \quad p \neq \pm 1,$
2. $(\zeta_k(\varepsilon\xi' + \xi'_*) + \xi_{*,n})\Pi_k^*\Pi_k\mathcal{U}^\flat = \sigma\Pi_k^*\chi_f F^\flat, \quad k\geq 2$ and
   $V^{\flat,2} = \sigma(L^{22})^{-1}\chi_f F^{\flat,2}.$

Let $h_*^j(\varepsilon\partial'_x/i) := \zeta_j(\varepsilon\partial'_x/i + \xi'_\star) + \xi_{\star,n}$. Using the definition of χ_f in (4.3.5) one has to show that $h_*^j(\varepsilon\partial'_x/i)\chi_1(\varepsilon\partial'_x/i)$ and that L^{22} are invertible.

Lemma 4.5.2. *There is a choice of χ_1 for which there is an ε-independent constant a such that for all $j \geq 2$*

$$\|h_*^j(\varepsilon\partial'_x/i)\chi_1(\varepsilon\partial'_x/i)\|_{H^s\to H^s} > a, \quad \|L^{22}(\varepsilon\partial'_x + i\xi'_\star)\chi_1(\varepsilon\partial'_x/i)\|_{H^s\to H^s} > a.$$

Démonstration. By Parceval formula the operator applied to a test function u gives

$$\begin{aligned}
&\| <.>^{s/2} h_*^j(\varepsilon.)\chi_f(\varepsilon.)\hat{u}\|_{L^2} \geq\\
&c\inf_{\varepsilon\xi'\in K_1'} |\int_0^1 \nabla_{x'}\zeta_j(\xi'_\star + \varepsilon\xi' s)\varepsilon\xi' ds + \xi_{\star,n} + \zeta_j(\xi'_\star)| \ \| <.>^{s/2}\hat{u}(.)\|_{L^2}.
\end{aligned}$$

Here K_1' is the support of χ_1. By definition of j, $\xi_{\star,n} + \zeta_j(\xi'_\star) \neq 0$ and one just needs a control on the integral. Choosing K_1' small enough such that $\sup_j \sup_{\varepsilon\xi'\in K_1'} |\nabla_{x'}\zeta_j(\xi'_\star + \varepsilon\xi' s)\varepsilon\xi'| \leq c < \min_k(\xi_{\star,n} + \zeta_k(\xi'_\star))$ so that the term before the norm is bounded from below independently from ε and the result follows. □

□

Next we give the linear version of lemma 4.4.6 for the equation (j_*) .

Lemma 4.5.3. *Let Ψ an affine function defined on 2π-periodic profiles by $\Psi(V) = \nabla_u F(\mathcal{U}^\varepsilon; V) + G$ with $G \in \mathcal{P}^s(z_0/\sigma)$. Then there is a unique solution $V = V_* \in \mathcal{P}^s(z_0/\sigma)$ solving $\mathcal{L}_* V = \sigma \Psi_*(V)$ with $V_{|_{z=0}} = 0$.*

One then constructs the full corrector r_1^ε by recurrence using the two lemma on the cascade. Then $r_{1,p}^\varepsilon(x),\ p \neq \pm 1$ has a tangential Fourier spectrum as big as $\underbrace{K_1' + \ldots + K_1'}_{(N+1)J-N \text{ term}}$. But what is important is that it doesn't depend on ε so that $\chi_1(\varepsilon.)$ which is supported by K_1'/ε encircles it.

Finally there is $r_2^\varepsilon \in \mathcal{P}^{s-N}(z_0/\sigma)$ such that

$$\mathcal{L}(\varepsilon\partial)(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon) = \varepsilon\sigma\chi F(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon) + \varepsilon^{N+1} r_2^\varepsilon.$$

2) Convergence.

It begins with the second step in the enumeration 4.4. We refer to figure (fig.2). First because of the pseudo-differential form of the equation for $\mathcal{U}^\varepsilon$ the compatibility conditions are not satisfied at $z = t = 0$. However the energy at this time is expected to be very small.

Thus let $\psi^1(t)$ the smooth truncation function which is 0 for $t \leq 0$ and is 1 for $t \geq \eta > 0$ close to 0.

Next $\mathcal{U}^\varepsilon$ is defined for $z \leq z_0/\sigma$ and one needs uniform estimates in $E^{m,\varepsilon}([0, t^\varepsilon])$ which requires integrating the profile for all $z > 0$. Thus let $\psi^2(z)$ the smooth truncation which is 0 for $z \geq z_0/\sigma$ and is 1 for $z \leq z_0/\sigma - \eta$.

Then set $U_{app} = (\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon)\psi^1\psi^2$ and look for $\varepsilon^{-p}u^\varepsilon = U_{app} + \varepsilon^M r^\varepsilon$. The remainder satisfies

$$\left\{ \begin{array}{c} \mathcal{L}(\varepsilon\partial)\varepsilon^M r^\varepsilon = \varepsilon\sigma(F(\varepsilon^{-p}u^\varepsilon) - \chi\psi^1\psi^2 F(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon)) \\ +(\mathcal{U}^\varepsilon + \sigma\varepsilon r_1^\varepsilon)\mathcal{L}(\varepsilon\partial)(\psi^1\psi^2) + \varepsilon^{N+1}\psi^1\psi^2 r_2^\varepsilon \\ r^\varepsilon(z=0) = 0, \quad r^\varepsilon(t=0) = 0. \end{array} \right.$$

One thus need to show $\chi\psi^1\psi^2 F(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon) - F(U_{app})$ and
$(\mathcal{U}^\varepsilon + \sigma\varepsilon r_1^\varepsilon)\mathcal{L}(\varepsilon\partial)(\psi^1\psi^2)$ are as small as one wants in $E^{m,\varepsilon}([0, t^\varepsilon])$.

This mainly relies on the properties of the truncation operator $(1 - \chi\psi^1\psi^2)$. Writing $(1 - \chi\psi^1\psi^2) = (1 - \chi)\psi^1\psi^2 + (1 - \psi^1)\psi^2 + (1 - \psi^2)$ the smallness then relies both on a frequency $(1 - \chi)$ and time/space $(1 - \psi^1), (1 - \psi^2)$ localization.

As ψ^1, ψ^2 are bounded in H^s we just need

Lemma 4.5.4. *1. $(1 - \chi(\varepsilon\partial_{x'}))$ applies from $X_\sigma^m(z_0/\sigma)$ to $X_\sigma^{m-a}(z_0/\sigma)$ with a bound in $c\varepsilon^a$.*

2. $(1 - \psi^1)$ applies from $X_\sigma^m(z_0/\sigma) \cap \{t \geq 0\}$ to $X_\sigma^{m-a}(z_0/\sigma) \cap \{t \geq 0\}$ with a bound in $c\varepsilon^a$.

Let $Y_\sigma^m = \bigcup_{s \leq m} H^s([0, t^\varepsilon], \Gamma^{s-j}_{x_i''/\sigma}(\mathbb{R}^{n-1}_{\tilde{x}} \times [0, z_0/\sigma])$

3. $(1 - \psi^2)$ applies from Y_σ^m to Y_σ^{m-a} with a bound in $c\varepsilon^a$.

Démonstration. 1. Let $F \in X_\sigma^m(\mathbb{R}, [0, z_0/\sigma])$. As the spectral decay uses only regularity one applies only tangential derivatives. Then using the Parceval formula one only needs to estimate

$$\| < . >^{(m-a)/2} (1 - \chi_1(\varepsilon.))\hat{F}(., z)\|_{L^2(\mathbb{R}^n))} = \| < . >^{(m-a)/2} (1 - \chi_1(\varepsilon.))\hat{F}(., z)\|_{L^2}.$$

Since $(1-\chi_1(\varepsilon\xi'))$ is uniformly bounded in ξ' one has

$$\| <.>^{(m-a)/2} (1-\chi_1(\varepsilon.))\hat{F}(.,z)\|_{L^2} \leq c \sup_{\varepsilon\xi'\notin K_1'} \frac{1}{<\xi'>^{a/2}} \| <.>^{m/2} \hat{F}(.,z)\|_{L^2}$$

Remark by definition of $K_1' \subset \mathcal{F}_1$, $\varepsilon\xi_0 \geq c > 0$. Consequently there is a $\tilde{c} > 0$ such that $\sup_{\varepsilon\xi'\notin K_1'} \frac{1}{<\xi'>^{a/2}} \leq \tilde{c}\varepsilon^a$. Finally one gets the bound

$$c\varepsilon^a \| <.>^{m/2} \hat{F}(.,z)\|_{L^2} = c\varepsilon^a \|F(.,z)\|_{H^m} \leq c\varepsilon^a \|F(.,z)\|_{\Gamma^m_{x_i'/\sigma}}.$$

Integrating in $z \leq z_0/\sigma$ leaves the result unchanged.

2. For this second truncation one uses the spatial decay. Let us extend $(1-\psi^1)$ smoothly to $t < 0$ by 0 for $t \leq -\eta$.

$$\begin{aligned}\|(1-\psi^1)F(.,z)\|_{\Gamma^{m-a}_{x_i'/\sigma}(\mathbb{R}_+\times\mathbb{R}^{n-1})} &\leq \|(1-\psi^1)F(.,z)\|_{\Gamma^{m-a}_{x_i'/\sigma}(\mathbb{R}^n)} \\ &\leq \sup_{-\eta<t<\eta} \frac{1}{<t-z/v_{*,n}-t_i/\sigma>^{a/2}} \|F(.,z)\|_{\Gamma^m_{x_i'/\sigma}(\mathbb{R}^n)}.\end{aligned}$$

Then $v_{*,n} > 0$ implies that the fraction is bounded by $c\sigma^{a/2}$. Integrating in $z \leq z_0/\sigma$ leaves the result unchanged.

3. The third point is similar to 2 and uses that $t^\varepsilon = \tilde{z}_0/(\sigma v_{*,n})$ with $\tilde{z}_0 < z_0$. □

One applies this lemma on $\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon$ where each profile lies in a X^q_σ. However the third point requires that

Lemma 4.5.5. *$\mathcal{U}^\varepsilon(t,x'') = \mathcal{W}^\varepsilon(t,x''-v_*t)$ with $\mathcal{W}^\varepsilon(t,.) \in Y^m_\sigma$.*

Démonstration. This is just an other way of expressing $\mathcal{V}^\varepsilon(.,z) \in X^m_\sigma(z_0/\sigma)$ with $(\tilde{x}_i,0) - v_*t_i = (0,-z_i)$. As this result enhances the spatial decay we just estimate (for $t \leq t^\varepsilon = z_0/\sigma/v_{*,n}$)

$$\begin{aligned}I(t) &= \| <.+z_i/\sigma>^{m/2} \mathcal{W}^\varepsilon(t,.)\|_{L^2(\mathbb{R}^{n-1}_y\times[0,z_0/\sigma])} \\ &= \| <.-v_*t+(0,z_i/\sigma)>^{m/2} \mathcal{U}^\varepsilon(t,.)\|_{L^2(\mathbb{R}^{n-1}_y\times[v_{*,n}t,z_0/\sigma+v_{*,n}t])} \\ &= -v_{*,n}\| <(\tilde{u}-y_i/\sigma,0)-v_*(u_0-t_i/\sigma)>^{m/2} \mathcal{V}^\varepsilon(u',v_{*,n}(t-u_0))\|_{L^2([-t^\varepsilon,0]\times\mathbb{R}^{n-1})}.\end{aligned}$$

In the last step we have made the change of variable $u' = (u_0,\tilde{u}) = (t-z/v_{*,n}, \tilde{x}-\tilde{v}_*z)$. Then there is a constant c such that $<(\tilde{u}-\tilde{x}_i/\sigma,0)-v_*(u_0-t_i/\sigma)> \leq c<u'-x_i'>$.

Finally taking the $L^2([0,t^\varepsilon])$ norm in time gives the bound $\|I(t)\|_{L^2([0,t^\varepsilon])} \leq \|\mathcal{V}^\varepsilon\|_{X^m_\sigma(z_0/\sigma)}$. □

As the support of $\partial_t\psi^1$ is that of $1-\psi^1$ for $t > 0$ and the support of $\partial_z\psi^2$ is that of $1-\psi^2$ for $z \leq z_0/\sigma$ the previous lemma applies for $(\mathcal{U}^\varepsilon + \varepsilon r_1^\varepsilon)\mathcal{L}(\varepsilon\partial)\psi^1\psi^2$ showing it is as small as one wants in X^m_σ.

All together we have shown the equation for the remainder is

$$\mathcal{L}^\varepsilon(\partial) r^\varepsilon = \sigma \int_0^1 \nabla F(\mathcal{U}^\varepsilon + \varepsilon\sigma r_1^\varepsilon + s\varepsilon^M r^\varepsilon) r^\varepsilon ds \; + \varepsilon^{N-M} \psi^1 \psi_2 \tilde{r}_2^\varepsilon. \tag{4.5.7}$$

Finally we give the main features of the convergence using the estimates in [35].

Proposition 4.5.6. *For all even $m \in \mathbb{N}$, there is a constant C such that for all smooth u satisfying* (4.5.7) *with homogeneous initial and boundary data there holds for $t \in [0,1]$:*

$$\|u(t)\|_{E^{m,\varepsilon}} \leq C\|u(t=0)\|_{E^{m,\varepsilon}} + C\int_0^t \|f(s)\|_{E^{m,\varepsilon}} ds. \tag{4.5.8}$$

For non linear estimate one defines the space $\tilde{F}^m(t)$ as follows. For $t > 0$ and m an even integer, denote by $\tilde{F}^m(t)$ the space of functions on $]-\infty, t] \times \mathbb{R}^d_+$ such that $\partial_{x'}^{\alpha'} \partial_{x_d}^{\alpha_d} u \in L^2(]-\infty, t] \times \mathbb{R}^d_+)$ for all $|\alpha'| + 2\alpha_d \leq m$.

Lemma 4.5.7. *Suppose that G is a smooth function of its argument. For $m > \frac{d+1}{2}$ even, there is a function $C(\cdot)$ from $[0, +\infty[$ to $\mathbb{R}_+$ such that for all $\varepsilon \in]0,1]$ and all $t \in [0, t_*]$, there holds for all v and w in $\tilde{F}^m(t)$*

$$\|F(u_{app}^\varepsilon, \varepsilon^M v) w\|_{\tilde{F}^{m,\varepsilon}(t)} \leq C(\varepsilon^M \|v\|_{L^\infty}) \big(\|w\|_{\tilde{F}^{m,\varepsilon}(t)} + \varepsilon^M \|w\|_{L^\infty} (1 + \|v\|_{\tilde{F}^{m,\varepsilon}(t)}) \big)$$

Then recalling $\mathcal{U}^\varepsilon$ is defined in $\mathcal{P}^{s-N}(z_0/\sigma)$ one shall try to get uniform estimates on $[0, z_0/\sigma]$ for 4.5.7.

However from the energy estimate (4.5.8) one actually expects to get uniform bound on interval $[0, t^\varepsilon]$. This is fulfilled for small boundary data of order ε^p (see below). Then the theorem says the solution of problem (4.5.7) nearly propagates as the leading profile thus up to $\frac{z_0}{\sigma v_{*,n}}$.

Now setting $\varepsilon^{-p} u^\varepsilon = U_{app}^n + \varepsilon^M r^\varepsilon$ and using the linear estimate on (4.5.7) with initial and boundary data $r^\varepsilon(z=0) = 0, \quad r^\varepsilon(t=0) = 0$ with r^ε vanishing infinitely at the corner $z = t = 0$ we get

$$\|r^\varepsilon(t)\|_{E^{m,\varepsilon}} \leq \sigma \int_0^t \big(C(\|U_{app}^n(s)\|_\infty, \varepsilon^M \|r^\varepsilon(s)\|_\infty) \|r^\varepsilon(s)\|_{E^{m,\varepsilon}} + \varepsilon^{N-M} \|\tilde{r}_2^\varepsilon(s)\|_{E^{m,\varepsilon}} \big) ds.$$

By Gronwall lemma

$$\|r^\varepsilon(t)\|_{E^{m,\varepsilon}} \leq \varepsilon^{N-M} \int_0^t e^{\sigma \int_s^t C(\varepsilon^M \|r^\varepsilon(s)\|_\infty) d\tau} \|\tilde{r}_2^\varepsilon(s)\|_{E^{m,\varepsilon}} ds.$$

Using $\|r^\varepsilon(t)\|_\infty \leq \varepsilon^{n/2} \|r^\varepsilon(t)\|_{E^{m,\varepsilon}}$ we see $\|\varepsilon^M r^\varepsilon(t)\|_\infty, \|r^\varepsilon(t)\|_{E^{m,\varepsilon}}$ remain bounded independently from ε as long as $N - M \geq n/2$ and $t \leq \alpha/\sigma$ for $\alpha > 0$ small enough.

4.6 Numerical results in 1D

The aim of this part is to provide some numerical result to show that the new model (4.3.8) provides quite different solutions compared to the model of diffractive optics. We next perform the calculations of section 1 on the first model (4.1.6) with the third order non-linearity.

4.6.1 Calculations for the first model (4.1.6)

We compute the projectors of equation (4.3.8) for the first model. Similar computations applies for the second model (see [11]).

$$\left\{\begin{array}{c} i\omega H - ik\wedge E = 0,\\ i\omega E + ik\wedge H + Q = 0,\\ i\omega P - Q = 0\\ i\omega Q + \omega_a^2 P = \gamma E. \end{array}\right. \tag{4.6.1}$$

Here $n=3$. The case $n=1$ will be deduced by just removing the variable $\tilde{x}=(x_1,x_2)$. As the matrix A_3 is not of the block form of the section 2 we rearrange the vector $U=(H,E,P,Q)$ into $U^\sharp=(U^1,U^2)$ with $U^1=(H_{//},E_{//})$ and $U^2=(H_3,E_3,P,Q)$ where $E=(E_{//},E_3)$.

The system (4.6.1) re-writes similarly with A_3 with the desired block decomposition and $A_3^{11}=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}$.

Then

$$L^{22}(i\omega,ik_1,ik_2)=\begin{pmatrix} i\omega I_2 & 0_{2,3} & \begin{matrix}0&0&0\\0&0&1\end{matrix}\\ 0_{3,2} & i\omega I_3 & -I_3\\ \begin{matrix}0&0\\0&0\\0&-\gamma\end{matrix} & \begin{matrix}\omega_a^2 I_3\end{matrix} & i\omega I_3\end{pmatrix},\quad L^{21}(i\omega,ik_1,ik_2)=\begin{pmatrix}0&0&ik_2&-ik_1\\ik_2&-ik_1&0&0\\ \multicolumn{2}{c}{0_{3,2}} & \multicolumn{2}{c}{0_{3,2}}\\ -1&0&&\\0&-1&\multicolumn{2}{c}{0_{3,2}}\\0&0&&\end{pmatrix}$$

We have $\det(L^{22}(i\omega,ik_1,ik_2))=\omega^2(\omega^2-\omega_a^2-\gamma)(\omega^2-\omega_a^2)^2$
thus we have to avoid the frequency $0,\pm\omega_a,\pm\sqrt{\omega_a^2+\gamma}$.

Next we compute the roots in k_3 of the characteristic polynomial by looking for solutions of (4.6.1). The calculation have been done in [13]. Let $k=(k_1,k_2,k_3)$

- $(H,E,P,Q)=(\frac{k}{\omega}\wedge E, E, \gamma E/(\omega_a^2-\omega^2), i\omega P)$ together with $k\cdot E=0$. So there are two orthogonal vector polarization v_1,v_2 according to $E_1=k\wedge u$ (for u not collinear with k) and $E_2=k\wedge E_1$.
- For $\omega\neq 0,\pm\sqrt{\omega_a^2+\gamma}$

$$\omega^2(1+\chi(\omega))=|k|^2,\quad \chi(\omega)=\frac{\gamma}{\omega_a^2-\omega^2}.$$

Thus there are two roots. If k_* is an incoming wave the corresponding root is $\zeta_1=\sqrt{\omega^2(1+\chi(\omega))-k_1^2-k_2^2}$. Remark that for any Maxwell problem the characteristic equation is bi-square in k_3 so one can compute the roots analytically.

A. **The equation for the new model**

Here we assume we have just one space dimension. The variables are $(t,z)\in\mathbb{R}\times\mathbb{R}^+$ and the dual (ω,k) satisfy the dispersion relation $k^2=\omega^2\chi(\omega)$. let us give (ω_*,k_*) satisfying the dispersion relation.

We re-write (4.3.8) for this example supposing there is no characteristic harmonic.

Let e_1, e_2 the two first basis vector of $\mathbb{R}^3$ and let (b_1, b_2) be the orthogonal basis of $\ker(A^{11}(G(i\omega) - i\zeta_1))$:

$$b_1 = {}^t(\frac{(\omega_* + \varepsilon\omega)\sqrt{1+\chi(\omega_*+\varepsilon\omega)}}{\omega_* + \varepsilon\omega}e_2, e_1,), \quad b_2 = {}^t(-\frac{(\omega_* + \varepsilon\omega)\sqrt{1+\chi(\omega_*+\varepsilon\omega)}}{\omega_* + \varepsilon\omega}e_1, e_2)$$

Then set $f_1 = {}^t(b_1, -Hb_1)$, $f_2 = {}^t(b_2, -Hb_2)$ where $H = (L^{22}(i\omega_* + i\varepsilon\omega))^{-1}L^{21}$ (here, in 1d, L^{21} is a constant matrix since there is no k_1 nor k_2). U solves

$$\text{(4.6.2)} \qquad \begin{cases} \left(\partial_z - \frac{i}{\varepsilon}((\varepsilon\partial_t + \omega_*)\sqrt{1+\chi(\varepsilon\partial_t+\omega_*)} - k_*)\right) a_j(t,z) = \\ \sigma\frac{1}{2\sqrt{1+\chi(\omega_*+\varepsilon\partial_t)}}\chi_1(\varepsilon\partial_t) < f_1(\varepsilon\partial_t + \omega_*)), F_1(U)(t,z) >, \quad j \in \{1,2\} \\ U(t,z) = f_1(\varepsilon\partial_t + \omega_*)a_1(t,z) + f_2(\varepsilon\partial_t + \omega_*)a_2(t,z). \end{cases}$$

In practice (cf. §4.6.3) we choose initial data with spectrum contained in $\mathcal{F}_1$ so that as long as the spectrum of the solution of (4.6.2) remains contained in $\mathcal{F}_1$ one can forget the truncation χ_1.

B. **The equation for the diffractive model**

This model differs from the previous by an approximation of order 2 of ζ_1 in ω and by the Fourier coefficients which are expressed at ω_* :

$$\text{(4.6.3)} \qquad \begin{cases} \left(\partial_z + v_*\partial_t + ia_*\partial_t^2)\right) a_j = \sigma\frac{1}{2\sqrt{1+\chi(\omega_*)}} < f_j(\omega_*), F_1(U)(t,z) >, \quad j \in \{1,2\} \\ U(t,z) = f_1(\omega_*)a_1(t,z) + f_2(\omega_*)a_2(t,z). \end{cases}$$

We have noted $v_* = \partial_\omega\zeta_1(\omega_*) = \frac{\omega_*/k_*}{1+\chi(\omega_*)^2\omega_a^2/\gamma}$ and $a_* = \partial_\omega^2\zeta_1(\omega_*)/2 = v_*(1/\omega_* - v_*/k_*) + 4(\omega_a\omega_*)^2\chi(\omega_*)^3/(k_*\gamma^2)$. The equation for the geometric model would not involve the second order derivative term.

4.6.2 The numerical scheme

Now we present the numerical scheme used to solve the equation (4.6.2) (with additional possible resonances as in the example of (4.1.7) when there is phase matching). We give a numerical scheme adapted from the Duhamel formulation. We do it for the fundamental wave (it works similarly with the resonant harmonics with coupling). Let $S_1(z)$ the unitary group operator associated to ζ_1 in the linear equation.

First one computes numerically $u(z) = S_1(z)u_0$ with $\hat{u}_0 = \chi_1\pi(\varepsilon\omega+\omega_*)\hat{u}_0$ by a FFT (Fast Fourier Transform) :

$$\text{(4.6.4)} \qquad \begin{cases} \left(\partial_z + \frac{i}{\varepsilon}(\zeta_1(\varepsilon\omega + \omega_*) + k_*)\right)\hat{u}(\omega, z) = 0 \\ \hat{u}(\omega, z=0) = \hat{u}_0(\omega, z=0) \end{cases}$$

This "propagation" step requires a FFT and its inverse.

Next the solution to $\mathcal{L}u = F(u)$ is computed by Picard iterations with the Duhamel formula

$$U(\delta z) = S_1(\delta z)U_0 + \int_0^{\delta z} S_1(\delta z - s)F(U(s))ds \tag{4.6.5}$$

where one approximates the integral by a second order formula (a two step Runge Kutta scheme)

$$\int_0^{\delta z} S_1(\delta z - s)F(U(s))ds \approx \delta z\, S_1(\frac{\delta z}{2})F(U^{\delta z/2})$$

$$U^{\delta z/2} = S_1(\frac{\delta z}{2})(U_0 + \frac{\delta z}{2}F(U_0))$$

The scheme thus reads as follows. Let u_n given, representing $\mathcal{U}(n\delta z)$; first compute $v_n = u_n + \frac{\delta z}{2}F(u_n)$. Then compute $u_{n+1/2} = S_1(\frac{\delta z}{2})v_n$ and $l_n = S_1(\frac{\delta z}{2})u_n$ through (4.6.4). Finally compute similarly $u_{n+1} = S_1(\frac{\delta z}{2})(l_n + \delta z F(u_{n+1/2}))$.

Remark that the nonlinear terms are computed in the spatial domain while the linear equation (4.6.4) is solved in the Fourier domain. Unfortunately this prevents from any L^2 conservation property.

Finally note that the method requires two FFT, FFT^{-1} for each step.

4.6.3 Numerical results

We have performed simulations on the dimensionless equations (4.6.2) and (4.6.3) for some "extreme" examples considering broad spectrums and laser pulsations located on points where char$\mathcal{L}$ is very curved. For most examples we have compared the two models in the frame of theorem 4.4.8 and Corollary 4.4.9 and then performed computations out of this frame.

For physical signification we recall how to obtain a dimensionless model :

$t = T_{ref}\tilde{t},\ z = Z_{ref}\tilde{z},\ E = E_{ref}\tilde{E},\ \omega = \omega_{ref}\tilde{\omega}$ with $Z_{ref} = cT_{ref}$. Then $\varepsilon = \frac{1}{T_{ref}\omega_{ref}}$. For the applications we take the values :

$T_{ref} \sim 10^{-13}s,\ \omega_{ref} \sim 10^{16}s^{-1},\ E_{ref} \sim 10^9$ in s.i. units.

We choose the physical boundary data

$$E(z=0) = E_0 e^{-(t/t_c)^2} e^{i\alpha\cos(\alpha t)}, \quad \alpha \in [0, 10]. \tag{4.6.6}$$

where $E_0 \sim E_{ref},\ t_c \sim T_{ref}$. We note $\tilde{E}_0, \tilde{t}_c$ the dimensionless parameters. This data can describe short and/or spectrally chirped pulse. The chirp can be obtained for example with a diffractive element.

The energy(=power in 1D) of the wave is $E_n = \|E\|^2_{L^2(\mathbb{R}_t)} = E_{n,ref}\tilde{E}$ with $E_{n,ref} = \sqrt{\pi}/2E^2_{ref}T_{ref} \sim 10^5$ and $\tilde{E}_n = \tilde{E}_0^2\tilde{t}_c$.

In the next subsections we present a few numerical computations. The diffractive model is L^2 conservative and our new model obeys the conservation law

$$N^{\text{new}}(z) =< v_1(\varepsilon.+\omega_*)\Pi_1(\varepsilon.+\omega_*)U^{\text{new}}(.,z), \Pi_1(\varepsilon.+\omega_*)U^{\text{new}}(.,z) >_2 = C^{ste}.$$

We thus display the following quantity

$$\text{cons}^2(z) = \frac{|N^{\text{new}}(z)| + \|U^{\text{dif}}(.,z)\|_{L^2}}{|N^{\text{new}|}(z)}$$

which is expected to remain small. Then setting

$$\delta^{\infty}(z) = \frac{\|U^{\text{new}}(.,z) - U^{\text{dif}}(.,z)\|_{\infty}}{\|U^{\text{new}}(.,z)\|_{\infty}},$$

$$\delta^{2}(z) = \frac{\|U^{\text{new}}(.,z) - U^{\text{dif}}(.,z)\|_{2}}{\|U^{\text{new}}(.,z)\|_{2}},$$

we also draw

$$\text{err}^{\infty}(z) = |\delta^{\infty}(z) - \delta^{\infty}(0)| \quad \text{and} \quad \text{err}^{2}(z) = |\delta^{2}(z) - \delta^{2}(0)|.$$

4.6.4 Short pulse : linear case

We first investigate the cases with low energy, that is when non-linear effects are negligible. This corresponds roughly to energies $E_n << 100J$. For those energies the nonlinearity is very weak and one sees only the linear dispersive effects. Then as the transport is far more important with respect to diffraction and higher dispersive effects we set in the frame that travels at the group velocity v_* related to k_*. Then we wish to compare the diffractive model with the new model. We give a first example to check the Corollary (4.4.9) when $\varepsilon = 10^{-3}$ and $p = 1$ which is the typical diffractive case since it corresponds to an initial data of amplitude ε. One thus make computations up to $z_{max} = 1/\varepsilon$. One gets on figure (4.3) L^{∞} and L^{2} errors of order ε. Meanwhile the solutions are superposed.

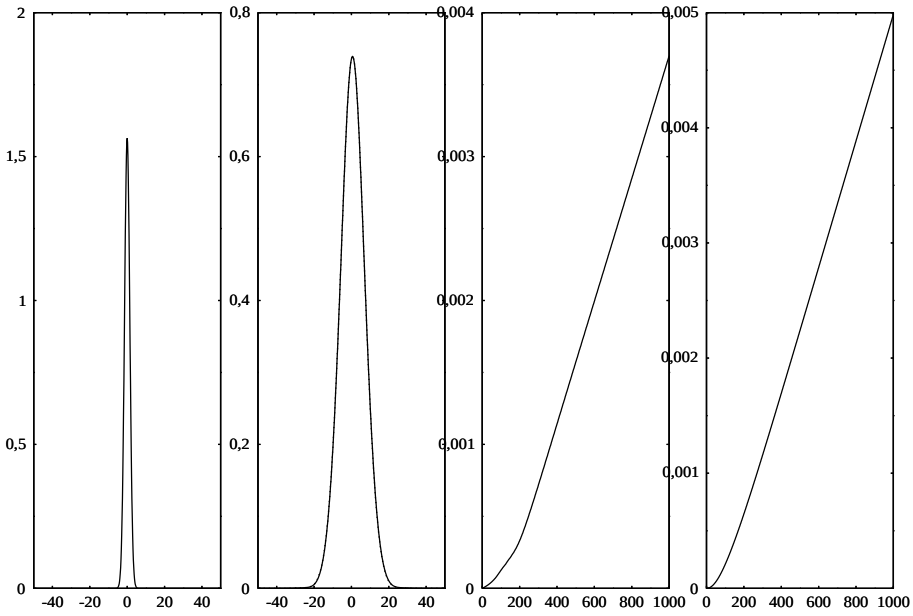

FIG. 4.3 – $E_n = 1J,\ \tilde{\omega} = 0,7,\ \tilde{t}_c = 2,\ z_{max} = 3cm$. Left-right : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), err^{∞}, err^{2}.

Then we want to investigate shorter pulses (thus with larger spectrum) to take into account the variation of the curvature of char$\mathcal{L}$. This is done in figure (4.4).

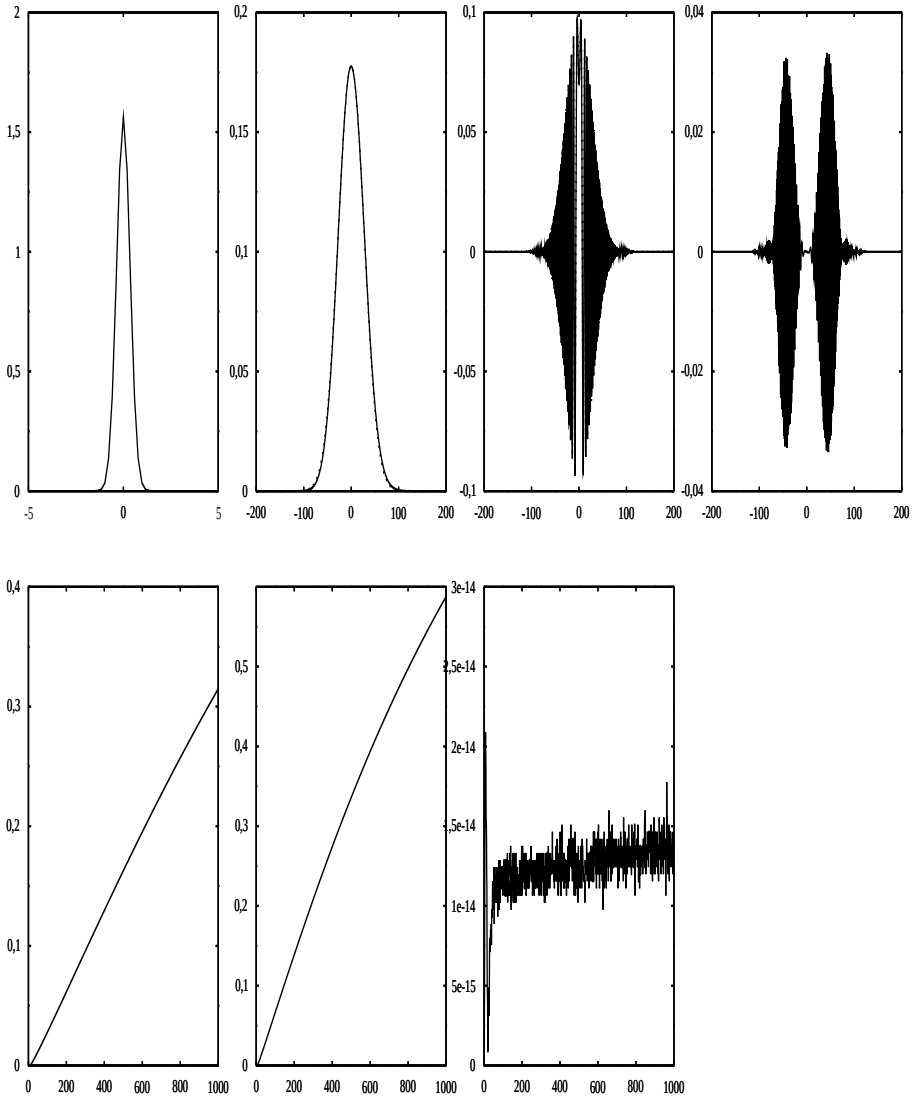

FIG. 4.4 – $E_n = 1J,\ \tilde{\omega} = 0,7,\ \tilde{t}_c = 0,5,\ z_{max} = 3cm$. From left-right and up-down : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), $Real(U^{\text{new}}(z_{max}))$, $Real(U^{\text{new}}(z_{max}) - U^{\text{dif}}(z_{max}))$,$\text{err}^{\infty}$, err^2, cons^2.

At first sight err^{∞} and err^2 seem too big compared with the visual closeness of the curves (almost superposed) in the second drawing. In fact one can remark for the evolution of the L^{∞} norm that there is a 10 factor between the initial data and the modulus of the solution at the final step. This is due to a greater dispersion of the short pulse. Next one can see that on figure (4.4) the real (and imaginary) part of $U^{\text{new}}(.,z) - U^{\text{dif}}(.,z)$ is big as it oscillates rapidly.

Next in the figure 4.5 we compare the solution of (4.6.2) and (4.6.3) for an energy $E_n = 1J$ for different values of $\varepsilon = 10^{-3},\ 10^{-2},\ 10^{-1}$. We do it by taking different values for T_{ref} at given ω_{ref} and $\tilde{\omega}_* = 0,7$.

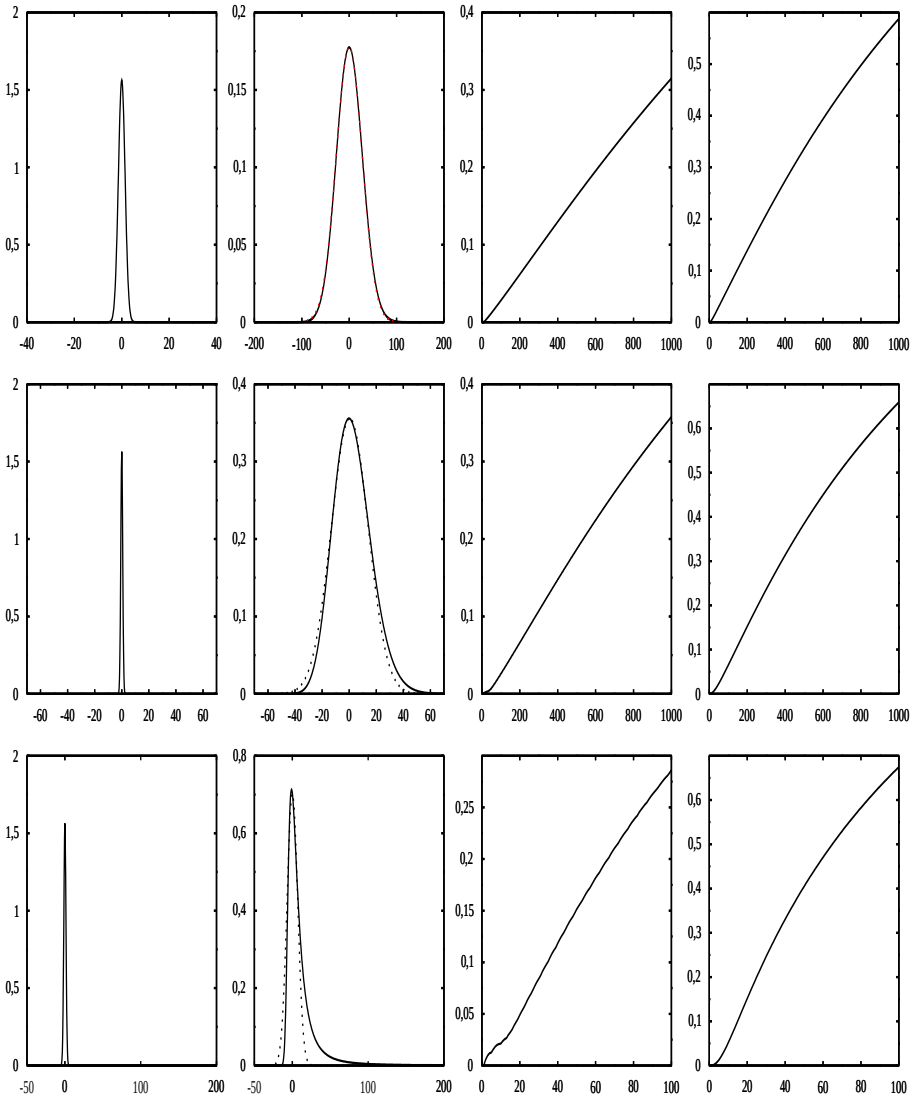

FIG. 4.5 – $E_n = 1J,\ \tilde{\omega}_* = 0,7$. Up-down : $\varepsilon = 10^{-3},\ t_c = 0,5,\ z_{max} = 3cm$; $\varepsilon = 10^{-2},\ t_c = 1,\ z_{max} = 0,3mm$; $\varepsilon = 10^{-1},\ t_c = 2,\ z_{max} = 3\mu m$. Left-right : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), err^∞, err^2.

In this example the wave length is $\lambda \sim 2,7\,10^{-7}$. In comparison the spatial steps are $\delta_z \sim 100\lambda, \lambda, \lambda$ for $\varepsilon = 10^{-3}, 10^{-2}, 10^{-1}$. The distances are computed so that $z_{max} = 1/\varepsilon$. As expected, the shorter (ε small) is the pulse the larger is the spectrum and the more sensitive is the variation of the dispersion. For $\varepsilon = 10^{-1}$ the result begins to be qualitatively bad.

Remark on the errors : why are they so close while the qualitative results deeply differ ? In fact when ε becomes bigger, the propagation distances are smaller so that the dispersive effects are less important : $\|U^{\text{new}}(z)\|_\infty$ becomes bigger thus err^∞ becomes smaller and the real and imaginary part of the solution oscillates less so that the error $\text{Re}\,(U^{new} - U^{dif})$ is also smaller.

Next we give a second example where $\varepsilon = 10^{-2}$ and a propagation over a longer distance (we use 4096 points) :

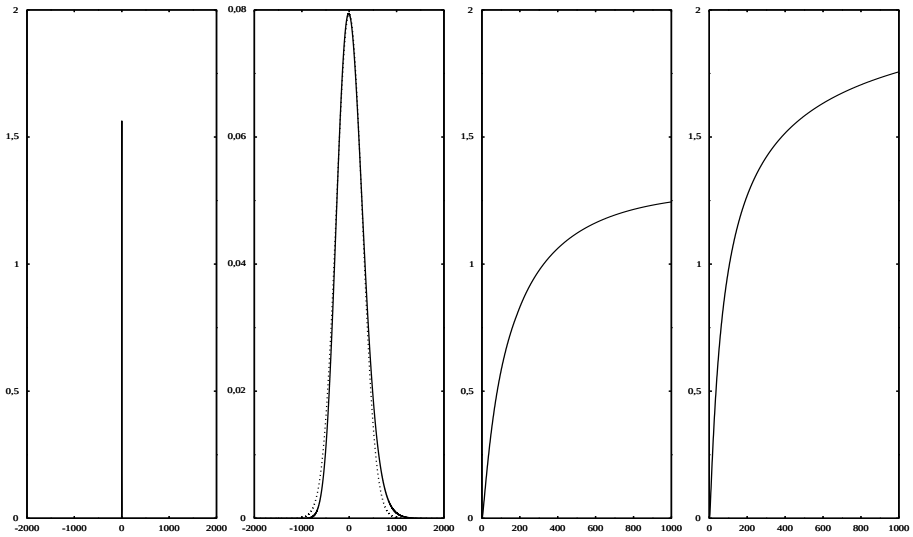

FIG. 4.6 – $E_n = 1J,\ \tilde{\omega} = 0,7,\ t_c = 1,\ z_{max} = 6mm$. Left-right : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), err^∞, err^2 .

Conclusion for low energies. the diffractive model gives a quite good approximation to the exact solution of the Maxwell equation in general except when the pulse is very short ($\varepsilon = 10^{-3}$ and $\tilde{t}_c = 0,5$ or $\varepsilon \leq 10^{-2}$) and the dispersion is strongly inhomogeneous (for the model (4.1.6) for example $\tilde{\omega} \in [0.8; 1[$). For $\varepsilon \geq 10^{-2}$ the Schrödinger doesn't account for the shape distortion of the profiles. For $\varepsilon \leq 10^{-2}$ it is to be noticed that the modulus of the solution is better approximated by the Schrödinger model than the real part and the imaginary part. Those results remain also true on long distances : $z > \sigma$.

4.6.5 Short pulse : nonlinear case

Now we investigate the nonlinear regime. It is well-known that for this cubic non-linearity autofocalization effects can occur if the wave if powerful enough. This is reached for energies of order $E_n = 100J$.

We next give an example to check the L^∞ estimate given in Corollary 4.4.9. We choose to propagate over distances of size $1/\sigma$ so that the expected L^∞ error between our model and the Schrödinger equation is of size ε. Nevertheless without dispersive effects the two models would give the same solution. One thus place in the case where the dispersion is either stronger or roughly balances the nonlinear effects.

Nevertheless it is to be noticed that the next examples are quite carefully chosen. In fact, in the general case either the non-linear effects or the dispersion prevail. This splitting becomes more clear-cut as the pulse is short. The limit is reached in [1] where the pulse is so short that all the non-linear effects can be neglected compared to the very strong dispersive effects.

In the next example one reaches a balance between the dispersion and the nonlinear effects with $\varepsilon = 10^{-2}$ and $\tilde{t}_c = 0,7$. The figure (4.7) gives the results for two different pulsations.

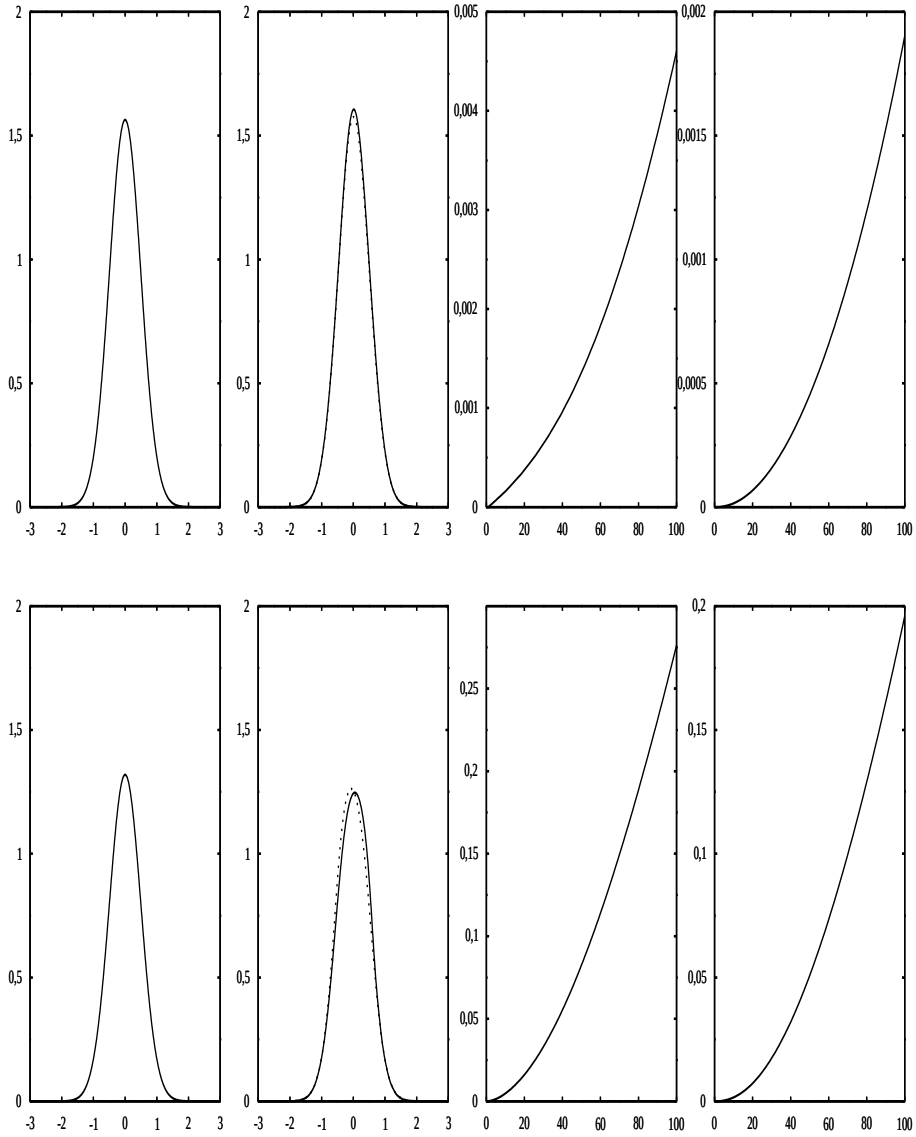

FIG. 4.7 – $E_n = 500J$, $\varepsilon = 10^{-2}, z_{max} = 0,6\mu m$. Up $\tilde{\omega} = .7$, down $\tilde{\omega} = 1.7$. Left-right : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), err^{∞}, err^{2}.

For $\tilde{\omega} = 0,7$ in figure 4.7 there is a beginning of focusing and it is quite well described by the diffractive model. On the contrary for $\tilde{\omega} = 1,7$ the error is bigger than the expected ε. In fact we observe a steepening of the pulse tail which seems to be due to higher order dispersive effects not taken into account by the Schrödinger model. The reason of the difference of approximation between the case $\tilde{\omega} = 0,7$ and $\tilde{\omega} = 1,7$ is probably that char$L$ is better approximated by a second order polynomial for $\omega < 1$ than for $\omega > \sqrt{2}$.

Let consider larger propagation distances :

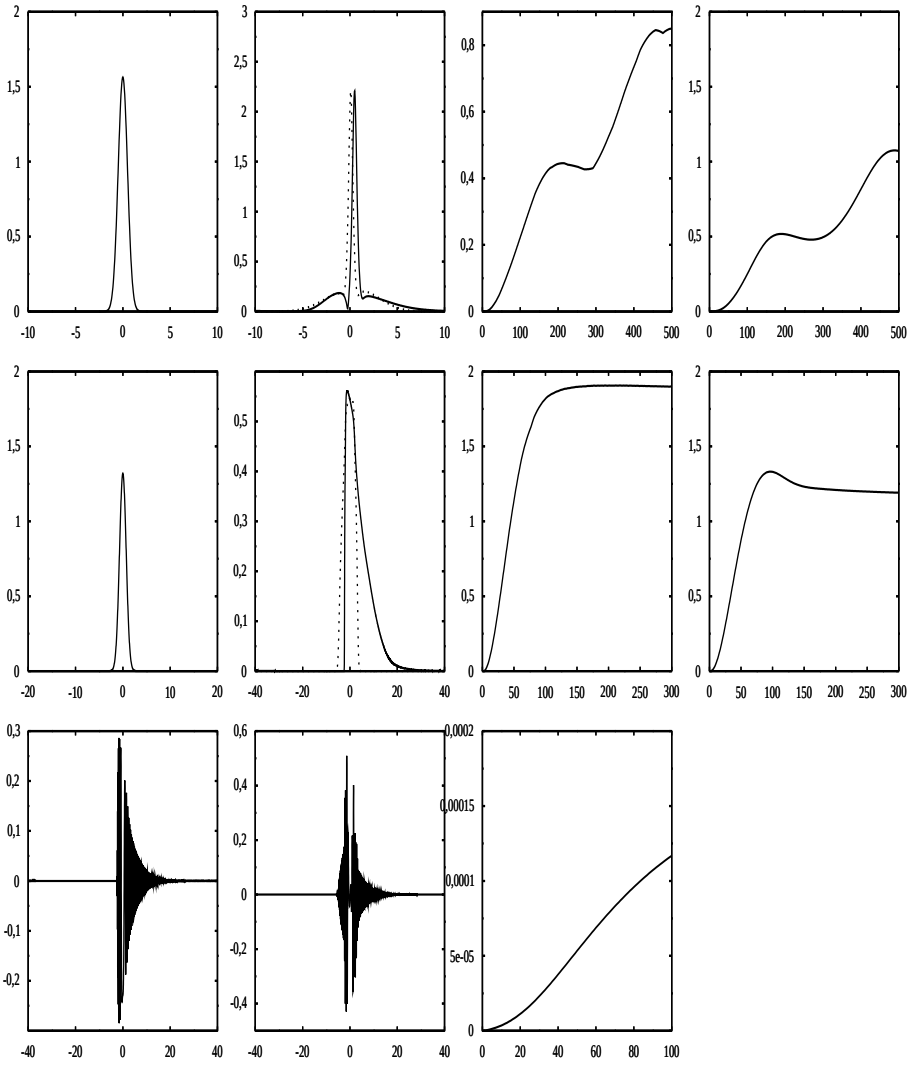

FIG. 4.8 – $E_n = 500J$, $\varepsilon = 10^{-2}$. Up $\tilde{\omega} = .7$ and $z_{max} = 30\mu m$, down $\tilde{\omega} = 1.7$ and $z_{max} = 3\mu m$. Left-right : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), $Real(U^{\text{new}}(z_{max}))$, $Real(U^{\text{new}}(z_{max}) - U^{\text{dif}}(z_{max}))$, cons^2.

We see that when $\tilde{\omega} = 0,7$ the two models differ radically : despite a similar shape, the predicted focused peak are almost separated out. then for $\tilde{\omega} = 1,7$ one sees that the steepening corresponds to fast oscillations not well taken into account by the Schrödinger model. Note that on longer distances the support of the new model solution grows larger too much fast for our computations.

Conclusion for high energies for the model (4.1.6) **of Maxwell 1D with kerr effect**. There is a big difference between the cases $\tilde{\omega} = .7$ and $\tilde{\omega} = 1.7$ which correspond to two distinct sheets of char$L$. We refer to [4] for similar remarks. For $\tilde{\omega} = .7$ there is focusing but there is an important time delay observed between the solutions of both models (4.6.2),(4.6.3). For $\tilde{\omega} = 1.7$ the two models give completely different profile solutions.

Now still in the nonlinear context we state an example highlighting the efficiency of solving the boundary problem instead of the Cauchy one. For this we refer to the result of [11] where computations are performed with respect to the Cauchy problem while the initial data is a boundary data. The example involves the generation of a second resonant harmonic which is described through the second example (4.1.7) given in the introduction. As the physical scale are shorter we compare the new model with the geometric optics model. Both models are considered in the one dimensional case and they are described by similar equations as (4.6.2),(4.6.3).

For this example we compute err^∞ through $\delta_2^\infty(z)$ (instead of $\delta^\infty(z)$) defined as follows :

$$\delta_2^\infty(z) = \sup_{j\in[1,2]} \frac{\|U_j^{\mathrm{new}}(.,z) - U_j^{\mathrm{dif}}(.,z)\|_\infty}{\|U_j^{\mathrm{new}}(.,z)\|_\infty},$$

The physical values are taken from [11] : $\varepsilon = 10^{-3}$, $T_{ref} = 10^{-13}$, $\lambda_* = 815nm$, $\tilde{E}_0/\varepsilon^p = 0,75$, $p = 1/2$, $t_c = 1,5$.

Those values are less critical compared to the previous calculations : the dispersion is weak so the nonlinear effects prevail. We thus compare our model with the geometric optics one.

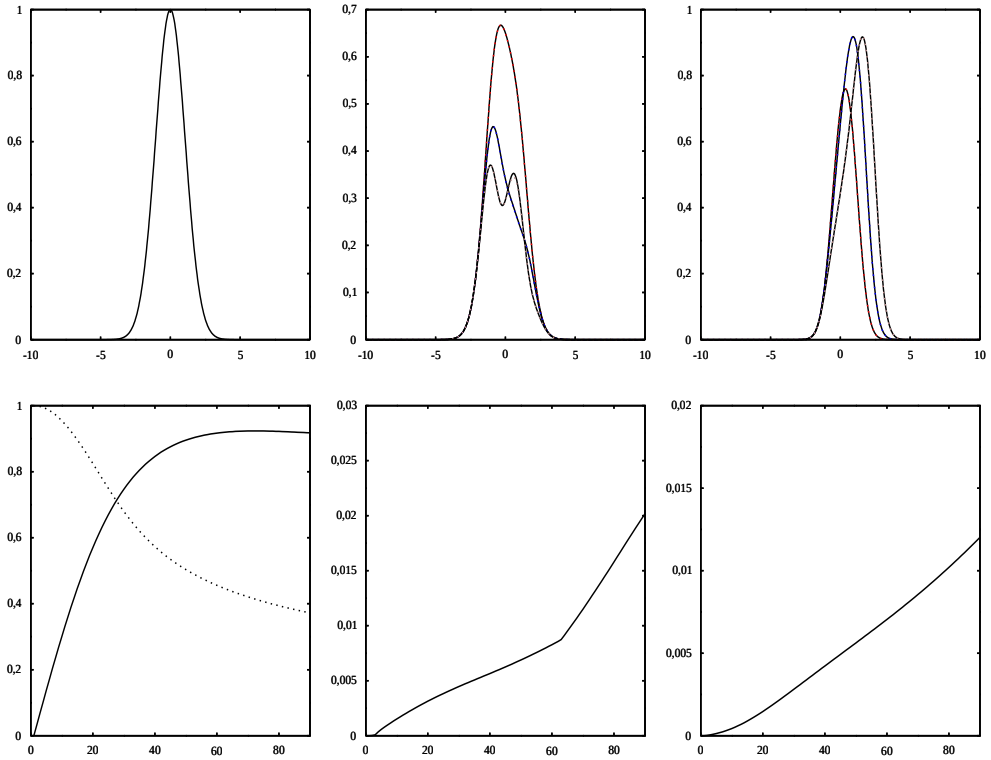

FIG. 4.9 – $z_{max} = 3mm$. Up : $|U^{\mathrm{new}}(z=0)|$, the fundamental $|U_1^{\mathrm{new}}|$ and $|U_1^{\mathrm{dif}}|$ (in dots) at $z = z_{max}/3, 2z_{max}/3, z_{max}$; the second harmonics $|U_2^{\mathrm{new}}|$ and $|U_2^{\mathrm{dif}}|$ (in dots) at $z = z_{max}/3, 2z_{max}/3, z_{max}$. Down : the evolution of the L^∞ norm of the fundamental (dots) and the second harmonic, err^∞, err^2.

We reach the numerical convergence with a z-step of order $\delta z = 10^{-5} \sim 12\lambda$ which is almost ten as much as the value 1.2λ in [11] for which there is convergence of the profiles.

We would point that the two upper right figures are made of 3 curves each of which are in fact two superposed curved : one from our model and the other from the geometric model.

On longer distances one gets :

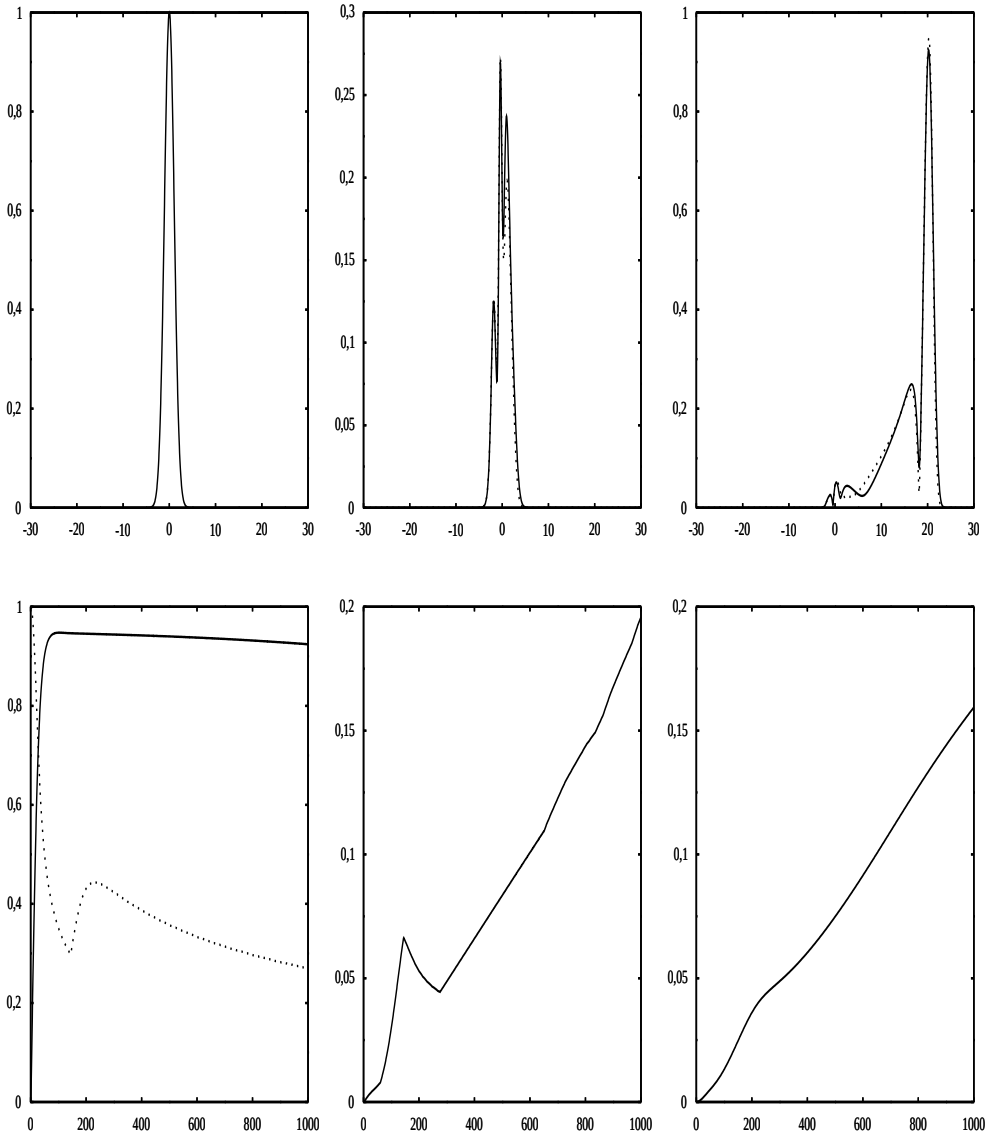

FIG. 4.10 – $z_{max} = 3cm$. Up : $|U^{\text{new}}(z=0)|$, the fundamental $|U_1^{\text{new}}(z_{max})|$ and $|U_1^{\text{dif}}(z_{max})|$ (in dots) ; the second harmonics $|U_2^{\text{new}}(z_{max})|$ and $|U_2^{\text{dif}}(z_{max})|$ (in dots). Down : the evolution of the L^∞ norm of the fundamental (dots) and the second harmonic, err^∞, err^2.

One can note that despite the errors, the two models do agree on the generation of the second harmonic. For shorter pulses ($\varepsilon = 10^{-2}$) the geometric optics model and the new model disagree :

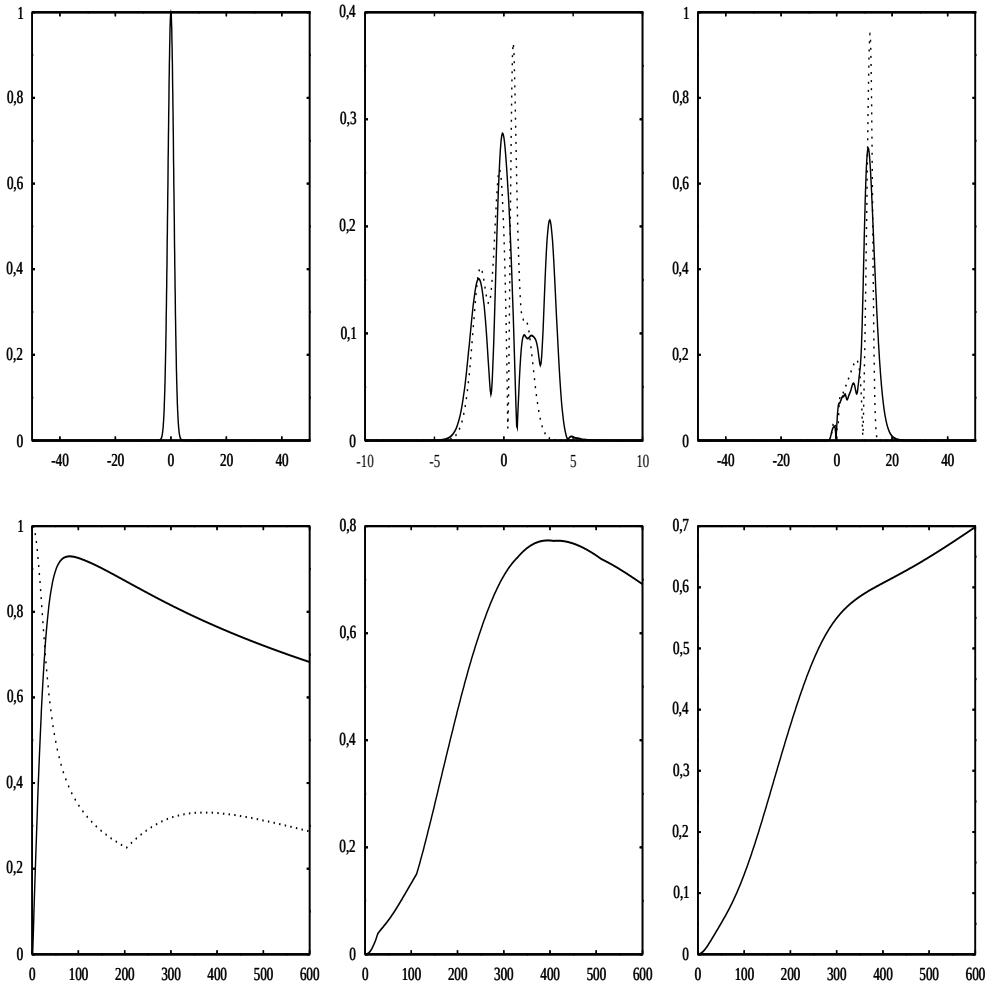

FIG. 4.11 – $z_{max} = 3cm$. Up : $|U^{\text{new}}(z=0)|$, the fundamental $|U_1^{\text{new}}(z_{max})|$ and $|U_1^{\text{dif}}(z_{max})|$ (in dots) ; the second harmonics $|U_2^{\text{new}}(z_{max})|$ and $|U_2^{\text{dif}}(z_{max})|$ (in dots). Down : the evolution of the L^∞ norm of the fundamental (dots) and the second harmonic, err^∞, err^2.

Conclusion for the generation of the second harmonic. Solving the problem in z seems to be well adapted for the generation of a second harmonic. We see that the geometric optics model and the new model provide very close results for small $\varepsilon < 10^{-3}$ even on long distance (up to $1000/\sigma,\ \sigma = 1$). When the pulse is shorter, that is when $\varepsilon \sim 10^{-2}$, the results greatly differ. This is due to the dispersive effects which play a much more important role and are not at all taken into account by the geometrical optic model. One should compare the new model with the diffractive model instead.

4.6.6 Spectrally Chirped pulse.

We present an example of spectrally chirped data according to (4.6.6) with $\alpha = 5$ and $\alpha = 10$. Such waves are typically obtained through a diffractive element. The interesting thing is that since the power is conserved the Fourier amplitude is smaller and this is safer

for the optics material. Contrary to §4.6.4 the spatial width of the pulse is not small so we do computations for $\varepsilon = 10^{-3}$.

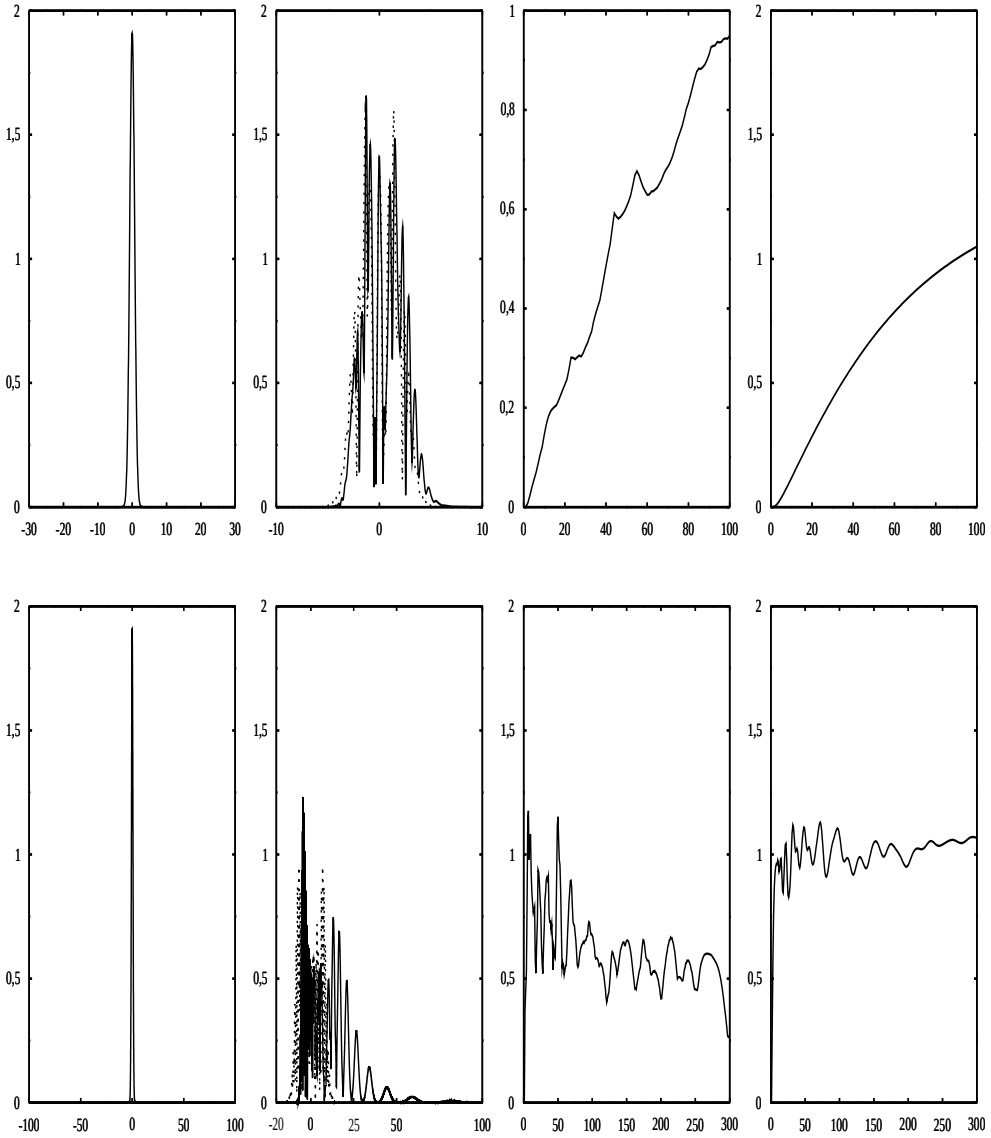

FIG. 4.12 – $E_n = 1J$, $\tilde{\omega} = .8$ and $z_{max} = 10^{-4}m$. Up : $\alpha = 5$, down : $\alpha = 10$. Left-right : $|U^{\text{new}}(z=0)|$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), err^{∞}, err^{2} .

One sees on figure 4.12 that as in [6] the diffractive model is very bad. One sees as in §4.6.4 that contrary to the diffractive effects the full dispersive effects imply a much more important asymmetry on the profile. This is of great importance for estimating the broadness of the pulse.

For longer waves, $\varepsilon = 10^{-4}$, all the frequential modes of the chirp are taken into account by the spatially longer profile but at the same time its spectrum is more punctual and finally the Schrödinger model gives a "closer" solution :

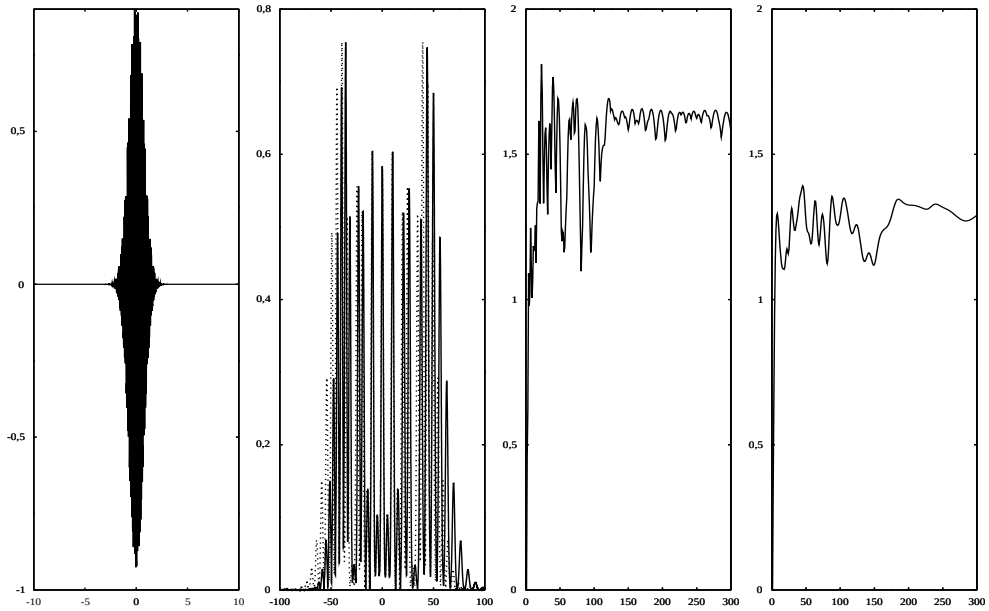

FIG. 4.13 – $E_n = 1J$, $\tilde{\omega} = .8$, $\alpha = 10$, $z_{max} = 9cm$. Left-right : $Real(U^{\text{new}}(z = 0))$, $|U^{\text{new}}(z_{max})|$ and $|U^{\text{dif}}(z_{max})|$ (in dots), err^∞, err^2 .

Conclusion for chirped pulses. For pulses such that $\varepsilon = 10^{-3}$ (and even bigger) with hard chirp the Schrödinger model is not good at all and the new model is recommended all the more as one propagates on long distances.

For $\varepsilon \sim 10^{-4}$ the Schrödinger model gives better qualitative results.

4.6.7 Conclusion on the numerical results ; prospects

From the computations one draws two main points :

1) In every (linear or weakly nonlinear) regime one should use the new model instead of the Schrödinger model as soon as the dispersive effects are non homogeneous : that is when the characteristic variety of L is curved over the spectrum of the data. The most important recommended application being addressed for spectrally chirped data.

2) Using the new model for simulating propagation of waves with large spectrum needs care. First one should check the spectrum broadening of the solution with respect to char$L$. Then, for a good interpretation of the L^∞ and L^2 errors between two solutions given by two models involving high order dispersive operators one needs to check the phase shift on the real part of the error.

Some perspectives would be :

1) To consider the crossing at an interface between two or more media of a spectrally chirped wave.

2) To perform computations on the 3D Maxwell model to study the additional transversal effects which are shown to be non negligible in [2].

Bibliographie

[1] K. Barrailh, D. Lannes A general framework for diffractive optics and its applications to lasers with large spectrum and short pulses, SIAM, Journal on Mathematical Analysis 34 (2003), no. 3, 636-674

[2] M. Billaud, rapport de stage de DEA à MATMECA, *Modèles de Schrödinger généralisés pour la simulation d'impulsions à spectre spatio-temporel large*, 2006.

[3] N.Bloembergen, *Nonlinear Optics* (Benjamin, NewYork, 1965).

[4] T. Bouchères, Thèse : *Hiérarchie de modèles asymptotiques pour les systèmes de Maxwell-Bloch et simulations numériques*, 2006.

[5] R. Carles, *Geometric optics with caustic crossing for some nonlinear Schrödinger equations*, Indiana Univ. Math. Journal 49 (2000), no. 2, 475-551.

[6] T. Colin, G. Gallice, K. Laurioux, *Intermediate models for laser beam propagation*, SIAM Journal on Mathematical Analysis, vol 36, No5, pp. 1664-1688.

[7] C. Cheverry, *The modulation equations of non linear geometric optics*, Comm. in Part. Diff. Eq, 21,(1996), 1119-1140.

[8] C. Cheverry, *Propagation d'oscillations près d'un point diffractif*, J. Math. Pures Appl, 75,(1996), 419-467.

[9] C. Jamel *On oscillation reflection for a two-speed system* C. R. Acad. Sci. Paris Sér. I Math. 313 (1991), no. 10, 675–678. (Reviewer : Michel Rascle) 35L50

[10] Cheverry, Christophe ; Guès, Olivier ; Métivier, Guy *Strong oscillations on a linearly degenerate field* Ann. Sci. École Norm. Sup. (4) 36 (2003), no. 5, 691–74

[11] T. Colin and B. Nkonga, *Multiscale numerical method for nonlinear Maxwell equations*, DCDS B, Volume 5, Number : 3 , August 2005, Pages 631 – 658.

[12] Corli, Andrea *Weakly non-linear geometric optics for hyperbolic systems of conservation laws with shock waves*. Asymptotic Anal. 10 (1995), no. 2, 117–172

[13] P.Donnat, thèse, *Quelques contributions mathématiques en optique non linéaire*. 1996 chapter 1 and 2.

[14] P.Donnat, J.L.Joly, G.Métivier, J.Rauch, *Diffractive nonlinear geometric optics.* , Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 1996, Exp. No. XVII, 25 pp.

[15] P.Donnat and J.Rauch, *Dispersive nonlinear geometric optics* J. Math. Phys., 38(1997), no. 3,1484-1523,

[16] E.Dumas, *About nonlinear geometric optics* (2005), http ://hal.ccsd.cnrs.fr/ccsd-00014461

[17] Friedrichs, K. O. ; Lax, P. D. *Boundary value problems for first order operators.* Comm. Pure Appl. Math. 18 1965 355–388.

[18] Guès, Olivier *Asymptotic expansion of exact solutions of quasilinear hyperbolic systems* Asymptotic Anal. 6 (1993), no. 3, 241–269

[19] O. Gues, *The characteristic quasilinear hyperbolic mixed problem* Comm. Partial Differential Equations 15 (1990), no. 5, 595–645

[20] O. Gues, *Multidimensional ϵ-stratified waves and oscillations* Duke Math. J. 68 (1992), no. 3, 401–446

[21] Hunter, J. K. ; Majda, A. ; Rosales, R. *Resonantly interacting, weakly nonlinear hyperbolic waves.* II. Several space variables. Stud. Appl. Math. 75 (1986), no. 3, 187–226

[22] Joly, Jean-Luc ; Rauch, Jeffrey *Justification of multidimensional single phase semilinear geometric optics.* Trans. Amer. Math. Soc. 330 (1992), no. 2, 599–623

[23] J.L.Joly, G.Métivier, J.Rauch, *Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves.* Duke Math. J. 70 (1993), no. 2, 373–404.

[24] J.L.Joly, G.Métivier, J.Rauch, *Coherent and focusing multidimensional nonlinear geometric optics.* Ann. Sci. École Norm. Sup. (4) 28 (1995), no. 1, 51–113

[25] J.L.Joly, G.Métivier, J.Rauch, *Dense oscillations for the Euler equations II.* pp. 425-430 in Hyperbolic Problems : Theory Numerics, Applications, eds. J. Glimm, M.J. Graham, J.W. Grove, B.J. Plohr., World Scientific, 1996.

[26] J.L.Joly, G.Métivier, J.Rauch, *Coherent nonlinear waves and the Wiener algebra.* Ann. Inst. Fourier (Grenoble) 44 (1994), no. 1, 167–196

[27] J.L.Joly, G.Métivier, J.Rauch, *Diffractive Nonlinear Geometric Optics with Rectification* Indiana U. Math. J. 47(1998), no. 4, 1167-1242,

[28] Joly, Jean-Luc ; Metivier, Guy ; Rauch, Jeffrey *Transparent nonlinear geometric optics and Maxwell-Bloch equations.* J. Differential Equations 166 (2000), no. 1, 175–250

[29] J.Y. Courtois, Optique Nonlinéaire (http ://www.bibsciences.org/bibsup/opt-coll/pub/page.php ?vol=4&art=courtois&cont=info).

[30] Kreiss, Heinz-Otto, *Initial boundary value problems for hyperbolic systems.* Comm. Pure Appl. Math. 23 (1970), 277–298

[31] Lax, Peter D. *Asymptotic solutions of oscillatory initial value problems.* Duke Math. J. 24 1957 627–646.

[32] D. Lannes, *Dispersion effects for nonlinear geometrical optics with rectification*, Asymptotic Analysis 18 (1998) 111-146.

[33] D. Lannes *Nonlinear geometrical optics for oscillatory wave trains with a continuous oscillatory spectrum*, Adv. Differential Equations 6 (2001), no. 6, 731–768.

[34] D. Lannes *secular growth estimates for hyperbolic systems.* J. Differential Equations 190 (2003), no. 2, 466–503

[35] V. Lescarret *Wave transmission in dispersive media.* M3AS 190 (2006).

[36] V. Lescarret *Diffractive Wave transmission in dispersive media.* To be submited.

[37] Majda, Andrew J. ; *Artola, Miguel Nonlinear geometric optics for hyperbolic mixed problems.* Analyse mathématique et applications, 319–356, Gauthier-Villars, Montrouge, 1988.

[38] A. Majda, S. Osher, *Initial Boundary value problems for hyperbolic equations with uniformly characteristic boundary* Comm. Pure Applied Math., Vol. XXVIII, 607-675 (1975)

[39] G. Métivier ; K. Zumbrun, *Hyperbolic boundary value problems for symmetric systems with variable multiplicities*. J. Differential Equations 211 (2005), no. 1, 61–134.

[40] G. Métivier, *Small viscosity and Boundary Layer Methods*. Birkhäuser, 2003

[41] J.Rauch *Lectures on Geometric Optics*, 152 pp, http ://www.math.lsa.umich.edu/ rauch/myresearch.html

[42] J.Rauch, *Symmetric Positive systems with boundary characteristic of constant multiplicity*. Trans. Amer. Math. Soc. 291 (1985), no. 1, 167–187

[43] J.Rauch, F.Massey, *Differentiability of solutions to hyperbolic initial-boundary value problems*. Trans. Amer. Math. Soc. 189 (1974), 303–318.

[44] B.Texier, *The short-wave limit for nonlinear, symmetric, hyperbolic systems*. Adv. Differential Equations 9 (2004), no. 1-2, 1–52.

[45] M. Williams, *Nonlinear geometric optics for hyperbolic boundary problems*, Comm. Partial Differential Equations 21 (1996), pp 1829–1895.

[46] M. Williams, *Nonlinear geometric optics for reflecting and glancing oscillations* in Singularities and oscillations (Minneapolis, MN, 1994/1995), IMA Vol. Math. Appl., 91, pp 137–151, Springer, New York, 1997.

[47] M. Williams, *Boundary layers and Glancing blow-up in nonlinear geometric optics*, Ann. Sci. École Norm. Sup., 33 (2000), pp 132–209