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**THESE DE DOCTORAT
DE L'ÉCOLE NORMALE SUPERIEURE DE CACHAN**

présentée par

Julien Mathiaud

pour obtenir le grade de

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Etude de systèmes de type gaz-particules

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A mes parents et Gilliane

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INTRODUCTION

L'objet de ce travail est la modélisation, l'étude mathématique et numérique de sprays, notamment dans le cadre d'applications rencontrées au CEA/DAM à Bruyères-le-Châtel. Les sprays sont composés de particules (ou gouttelettes) en suspension dans un fluide environnant; ils permettent l'étude de phénomènes physiques complexes et très divers. Ainsi, l'étendue des applications va de l'automobile (les moteurs diesels : cf. [AOB89]), en passant par l'aéronautique (moteurs de fusée Ariane : cf. [Hy199] et [Duf05]), l'étude de flammes ([MV01]) jusqu'à la biologie avec l'étude de l'impact de sprays pharmaceutiques ou de dépôts nicotiniqueux sur les poumons ([Mau04]).

Dans cette thèse, les sprays étudiés à partir des données du CEA sont constitués de gouttelettes d'étain se mouvant dans l'air. La particularité de ce type de sprays est d'être constitué de gouttes très denses par rapport au fluide environnant: les gouttelettes vont donc jouer un rôle essentiel dans l'étude du système de par leur inertie.

Les sprays sont généralement décrits par un jeu d'équations aux dérivées partielles. Deux types d'approches essentiellement existent pour leur modélisation.

La première consiste à utiliser des équations issues de la mécanique des milieux continus (Euler ou Navier-Stokes) pour les deux fluides: c'est l'étude des systèmes semi-cinétiques et en particulier des systèmes diphasiques (appelés aussi bifluides et eulérien-eulériens): on peut trouver les équations de ces systèmes dans [Sai95], [Bou98], [Rov06], et [GHS04], ainsi que dans le **premier chapitre**. De nombreux modèles découlent de cette approche selon les applications rencontrées ([Duf05], [Lau02]) et selon les hypothèses que l'on fait sur la répartition en rayon, vitesse et énergie des gouttelettes: ainsi les modèles bifluides consistent à considérer que le brouillard de gouttelettes est constitué de gouttelettes aux caractéristiques identiques en un même point de l'espace et du temps alors que les modèles multi-fluides sont seulement basés sur le fait que les gouttelettes ont une même vitesse et une même énergie. pour une plage de rayons donnée (appelée section: [Tam85]), chaque section agissant comme un fluide à part entière, et du coup permettent de continuer à modéliser des phénomènes comme la fragmentation et la coalescence ([LMV04]), contrairement aux modèles bifluides. L'intérêt des modèles semi-cinétique provient du fait que

La deuxième approche, que nous adoptons dans la suite, consiste à décrire le mouvement des particules par une équation cinétique de type Vlasov-Boltzmann et celui du gaz par des équations issues de la mécanique des milieux continus ([Wil85], [CP83], [O'R81], [Ree91]). Nous ferons d'ailleurs le lien entre les deux approches dans le premier chapitre d'un point de vue purement formel.

Nous présentons maintenant précisément les modèles Eulériens-Lagrangiens qui seront les seuls étudiés dans cette thèse.

Commençons par décrire l'équation régissant le mouvement des particules. Dans le cadre de la théorie cinétique des gaz proposée par Maxwell et Boltzmann (cf. [CC70], [CIP94], [Cer88], [Car57] et [GPV04]), un système constitué de nombreuses particules n'est pas décrit par un système discret d'équations différentielles pour chaque particule, mais par une fonction de distribution de probabilité (f.d.p. ou p.d.f.) $f(t, x, u_p, e_p)$ dans l'espace des phases (x, u_p, e_p) appartenant à $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$, où x est la position des particules, u_p la vitesse des particules et e_p est l'énergie interne des particules: cette approche privilégie une étude au niveau mésoscopique (milieu continu) plutôt qu'au niveau microscopique (milieu discret). D'autres paramètres peuvent être rajoutés dans la description des particules comme le rayon (ou de façon équivalente la surface ou le volume: cf. [Duf05]), la masse... L'ajout de paramètres apporte une description plus précise malgré le surcoût numérique engendré et permet la modélisation de phénomènes complexes telle que l'abrasion, la fragmentation et la coalescence de gouttes ([Duf05]). Ce type d'approche a été formalisé par Williams et O'Rourke ([Wil85, O'R81]).

L'évolution de la fonction f est régie par l'équation cinétique de Vlasov-Boltzmann:

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p}(f\phi) = Q(f, f).$$

Le membre de gauche représente le transport des particules dans l'espace des phases, tandis que le membre de droite s'appelle noyau de collision et modélise l'impact des collisions entre particules.

Γ représente une force agissant sur les particules (traînée, masse ajoutée, gravité...). ϕ représente des échanges thermiques (conduction...). Nous renvoyons à [Duf05] pour une étude exhaustive de Γ et ϕ , selon les applications rencontrées. Ces deux quantités régissent l'évolution des particules dans l'espace des phases.

L'aspect polydispersé (rayons différents) n'est pas pris en compte ici dans un souci de simplification du modèle, étant donné que nous n'étudierons ni la fragmentation, ni la coalescence -qui devraient être placés dans le membre de droite en tant que termes sources - . Pour ces phénomènes complexes à étudier nous renvoyons aux travaux suivants, tant d'un point de vue numérique que d'un point de vue théorique et physique : [AO89], [O'R81], [MV01], [Hyl99], [VH97], [Lau02], [Bar04a], [Duf05], [HF92] et [HF95].

On suppose que les collisions sont binaires et localisées au même point de l'espace (on ne tient pas compte d'éventuelles collisions ternaires ou avec encore plus de particules, qui sont supposées très peu nombreuses). Dans les modèles que l'on va étudier, les masses des particules sont identiques (sprays **monodispersés**). De plus l'impulsion totale et l'énergie totale (énergie cinétique + énergie interne) sont conservées durant les collisions. On considère à la fois des collisions élastiques pour lesquelles l'énergie cinétique est conservée (et a fortiori l'énergie interne) et des collisions inélastiques pour lesquelles il y a des pertes d'énergie cinétique liées à des frottements visqueux ([AO89], [JUL92], [WO03]), cette approche étant très proche de l'étude des milieux granulaires ([Vil02], [GPV04], [Cer03]). Dans le cas inélastique, l'énergie cinétique perdue est transférée en énergie interne. Même dans le cas élastique, il peut y avoir des échanges d'énergie interne durant la collision car les deux gouttelettes restent collées l'une à l'autre pendant une durée courte mais non nulle. Par ailleurs, afin d'écrire le noyau de collision, on va faire l'hypothèse classique de chaos moléculaire : les vitesses de particules qui collisionnent sont totalement décorréliées avant collision ([Vil02], [Cer88]). Il reste à décrire la section efficace de collision. Dans les applications rencontrées au cours de cette thèse, les particules interagissent seulement lorsqu'elles sont en contact. Ceci amène à considérer la section efficace \mathcal{B} des sphères dures: nous renvoyons à [Vil02]

pour les autres modèles de section efficace. Pour cette section efficace, la probabilité de collisionner est proportionnelle à la différence de vitesse entre deux particules, mais ne dépend pas de l'angle entre les deux vitesses ([Cer88]) lorsque l'on décrit les collisions en "paramétrage σ " ($\mathcal{B} = 4\pi r^2 |u_p - u_{p*}|$, avec r rayon des particules, u_p et u_{p*} vitesses des deux particules). Le noyau de collision exact sera donné lors de la présentation du premier chapitre p.xvii.

Enfin, il faut noter que le calcul des moments de la p.d.f. permet de décrire le comportement moyen des particules: ainsi $\iint f(t, x, u_p, e_p) du_p de_p$ représente le nombre de particules se trouvant en x au temps t , $\frac{\iint f(t, x, u_p, e_p) m_p u_p du_p de_p}{\iint f(t, x, u_p, e_p) du_p de_p}$ représente la quantité de mouvement moyenne des particules dans ce volume, $\frac{\iint f(t, x, u_p, e_p) \frac{1}{2} u_p^2 du_p de_p}{\iint f(t, x, u_p, e_p) du_p de_p}$ représente l'énergie cinétique massique moyenne des particules et $\frac{\iint f(t, x, u_p, e_p) e_p du_p de_p}{\iint f(t, x, u_p, e_p) du_p de_p}$ représente l'énergie interne massique moyenne des particules.

Passons à l'étude du gaz. Dans le cadre de la mécanique des milieux continus (équations d'Euler ou de Navier-Stokes selon l'importance de la viscosité), le gaz est décrit par trois grandeurs macroscopiques : la densité $\rho_g := \rho_g(t, x)$, la vitesse $u_g := u_g(t, x)$ et l'énergie interne par unité de masse $e_g := e_g(t, x)$ (ou de façon équivalente par ρ_g , u_g et l'énergie totale par unité de masse $E_g = e_g + \frac{1}{2} |u_g|^2 := E_g(t, x)$). Les équations décrivant l'évolution de ces quantités sont obtenues en écrivant les lois de conservation des grandeurs physiques que sont la masse, l'impulsion et l'énergie ([LL59], [Lio96], [Lio98]). En ne tenant pas compte pour le moment de l'interaction avec les particules et d'éventuelles forces extérieures au système, on obtient les équations suivantes qui traduisent les conservations de ces trois quantités :

$$\begin{aligned} \partial_t(\rho_g) + \nabla_x \cdot (\rho_g u_g) &= 0, \\ \partial_t(\rho_g u_g) + \nabla_x \cdot (\rho_g u_g \otimes u_g) + \nabla_x p &= 0, \\ \partial_t(\rho_g E_g) + \nabla_x \cdot \left(\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) &= 0, \end{aligned}$$

où la pression p dépend de e_g et ρ_g via une loi d'état ($p = \rho_g e_g$ pour un gaz parfait par exemple, où e_g est définie par $e_g = E_g - \frac{1}{2} u_g^2$).

Ces équations sont les équations d'Euler compressibles. Elles ne tiennent pas compte des phénomènes de viscosité - contrairement aux équations de Navier-Stokes-, qui sont négligeables dans les applications rencontrées par la suite (à l'exception du chapitre 4, où l'on tient compte de la **viscosité turbulente**) : les nombres de Reynolds sont de l'ordre de 10^5 . Ces équations sont parfois moyennées lorsque la turbulence entre en jeu afin de traiter les différentes échelles de turbulence comme on le verra aux **troisième** et **quatrième** chapitres.

Il faut noter que les équations d'Euler compressibles peuvent s'obtenir en tant que limite de l'équation de Boltzmann dans le cadre de la limite hydrodynamique selon le développement de Hilbert ([Cer88], [CC70] et [Gra58]); de même les équations de Navier-Stokes compressible peuvent s'obtenir à partir du développement de Chapman-Enskog (cf. [CC70] et [Cer88] p.244). Nous montrerons ici que l'on retrouve les équations du diphasique à partir d'un développement de Hilbert dans le **premier chapitre**. Par ailleurs il faut noter que d'un point de vue mathématique, le problème des équations d'Euler est bien posé en temps petit pour des données régulières ([Maj84]), mais on ne sait pas montrer l'existence et l'unicité en temps grand pour des données initiales quelconques, du fait de l'apparition de singularités (cf. [Lax73], [Lio96] p.272 et [Ser96b]).

Nous pouvons maintenant décrire les sprays à l'aide de la représentation cinétique pour les particules et des équations d'Euler pour le fluide environnant. Cette approche a été initiée par Williams ([Wil85]) et Papanicolaou ([CP83]) et utilisée dans de nombreux codes industriels tel que le code Kiva ([AOB89]) développé à Los Alamos. On suppose ici que les gouttelettes sont **incompressibles**. Les sprays sont parfois classifiés en fonction du volume occupé par les particules (selon la caractérisation de O'Rourke [O'R81]) :

- les sprays très fins monodispersés, pour lesquels le gaz agit sur les particules (généralement via la force de traînée de Stokes notée Γ) mais les particules n'agissent pas sur le gaz. La fraction de volume occupée par les particules ainsi que la fraction massique des particules sont négligées. On ne tient pas compte non plus des collisions. Un système d'équations régissant l'évolution du spray est le suivant :

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0, \quad (0.1)$$

$$\partial_t (\rho_g u_g) + \nabla_x \cdot (\rho_g u_g \otimes u_g) + \nabla_x p = 0, \quad (0.2)$$

$$\partial_t (\rho_g E_g) + \nabla_x \cdot \left(\rho_g \left(E_g + \frac{p}{\rho_g} \right) \right) = 0, \quad (0.3)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \Gamma) + \partial_{e_p} (f \phi) = 0, \quad (0.4)$$

$$p = P_1(\rho_g, e_g), \quad (0.5)$$

$$m_p \Gamma = C_1 (u_g - u_p), \quad (0.6)$$

$$m_p \phi = C_2 (T_g - T_p), \quad (0.7)$$

$$T_g = T_1(e_g, \rho_g), \quad (0.8)$$

$$T_p = T_2(e_p), \quad (0.9)$$

où m_p est la masse des particules, T_g et T_p sont les températures du gaz et des particules, C_1 est le coefficient de traînée ([BDM03], [Sai95]) et C_2 est le coefficient de transfert thermique. Les expressions de P_1 , T_1 et T_2 dépendent des applications rencontrées ([BDM03]). On rappelle que e_g s'obtient via la relation $e_g = E_g - \frac{1}{2} u_g^2$. Pour ce type de sprays, le gaz agit comme un réservoir de quantité de mouvement et d'énergie pour les particules ([DMV03, Duf05]).

- les sprays fins monodispersés, pour lesquels la force de traînée et les échanges thermiques agissent entre les deux phases. La fraction de volume occupée par les particules est négligée ainsi que les collisions (ou autres phénomènes entre gouttelettes tels que la coalescence), mais pas la fraction massique des particules : typiquement, les particules ont une densité plus de mille fois supérieure à celle du gaz porteur. Un système d'équations régissant l'évolution du spray est le suivant :

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0, \quad (0.10)$$

$$\partial_t (\rho_g u_g) + \nabla_x \cdot (\rho_g u_g \otimes u_g) + \nabla_x p = \iint_{u_p, e_p} -m_p f \Gamma, \quad (0.11)$$

$$\partial_t (\rho_g E_g) + \nabla_x \cdot \left(\rho_g \left(E_g + \frac{p}{\rho_g} \right) \right) = \iint_{u_p, e_p} -m_p f \Gamma \cdot u_p - m_p f \phi, \quad (0.12)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \Gamma) + \partial_{e_p} (f \phi) = 0, \quad (0.13)$$

$$p = P_1(\rho_g, e_g), \quad (0.14)$$

$$m_p \Gamma = C_1 (u_g - u_p), \quad (0.15)$$

$$m_p \phi = C_2 (T_g - T_p), \quad (0.16)$$

$$T_g = T_1(e_g, \rho_g), \quad (0.17)$$

$$T_p = T_2(e_p). \quad (0.18)$$

Ce type de modèle est notamment utilisé par l'équipe de l'ONERA-CERT travaillant sur les moteurs d'Ariane ([Hyl99, Ces97]). D'un point de vue mathématique, Baranger et Devillettes ont récemment prouvé le caractère bien posé de ce système d'équations en temps petit pour des solutions régulières ([BD06]).

- les sprays modérément épais monodispersés, pour lesquels on tient compte en plus des phénomènes de collision. La fraction de volume occupée par les particules à un instant donné est négligée, mais pas la fraction de volume occupée par les particules au cours de leur "vol" pendant un temps caractéristique de l'expérience. Un système d'équations régissant l'évolution du spray est le suivant :

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0, \quad (0.19)$$

$$\partial_t (\rho_g u_g) + \nabla_x \cdot (\rho_g u_g \otimes u_g) + \nabla_x p = \iint_{u_p, e_p} -m_p f \Gamma, \quad (0.20)$$

$$\partial_t (\rho_g E_g) + \nabla_x \cdot \left(\rho_g \left(E_g + \frac{p}{\rho_g} \right) \right) = \iint_{u_p, e_p} -m_p f \Gamma \cdot u_p - m_p f \phi, \quad (0.21)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \Gamma) + \partial_{e_p} (f \phi) = Q(f, f), \quad (0.22)$$

$$p = P_1(\rho_g, e_g), \quad (0.23)$$

$$m_p \Gamma = C_1(u_g - u_p), \quad (0.24)$$

$$m_p \phi = C_2(T_g - T_p), \quad (0.25)$$

$$T_g = T_1(e_g, \rho_g), \quad (0.26)$$

$$T_p = T_2(e_p). \quad (0.27)$$

Ici, $Q(f, f)$ est l'opérateur de collisions. On pourrait aussi tenir compte des phénomènes de fragmentation et de coalescence dans le cadre de l'étude de sprays polydispersés ([Bar04b, FM05, LM03, AOB89, DMV03, Duf05, O'R81, LMV04]). Ce type de spray est notamment utilisé pour l'étude des moteurs diesels dans le code Kiva II ([AOB89]). L'objet du **deuxième chapitre** est l'étude mathématique de ce type de spray (recherche de solutions régulières locales en temps petit).

- les sprays très épais monodispersés, pour lesquels on tient compte des interactions entre les deux phases et des collisions entre gouttelettes ainsi que du volume occupé par les particules : le comportement du système est alors proche de celui des régimes eulériens-eulériens. On introduit la fraction de volume occupée par les particules $(1 - \alpha) := (1 - \alpha)(t, x)$ que l'on avait négligée jusqu'alors. Le jeu d'équations décrivant le spray devient le suivant :

$$\partial_t (\alpha \rho_g) + \nabla_x \cdot (\alpha \rho_g u_g) = 0, \quad (0.28)$$

$$\partial_t (\alpha \rho_g u_g) + \nabla_x \cdot (\alpha \rho_g u_g \otimes u_g) + \nabla_x p = \iint_{u_p, e_p} -m_p f \Gamma, \quad (0.29)$$

$$\begin{aligned} & \partial_t (\alpha \rho_g E_g) + \nabla_x \cdot \left(\alpha \rho_g \left(E_g + \frac{p}{\rho_g} \right) \right) + p \partial_t \alpha \\ &= \iint_{u_p, e_p} -m_p f \left(\Gamma + \frac{p}{\rho_p} \right) \cdot u_p - m_p f \phi, \end{aligned} \quad (0.30)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \Gamma) + \partial_{e_p} (f \phi) = Q(f, f), \quad (0.31)$$

$$1 - \alpha = \iint_{u_p, e_p} \frac{4}{3} \pi r^3 f, \quad (0.32)$$

$$p = P_1(\rho_g, e_g), \quad (0.33)$$

$$m_p \Gamma = -\frac{m_p}{\rho_p} \nabla_x p + C_1 (u_g - u_p), \quad (0.34)$$

$$m_p \phi = C_2 (T_g - T_p), \quad (0.35)$$

$$T_g = T_1(e_g, \rho_g), \quad (0.36)$$

$$T_p = T_2(e_p). \quad (0.37)$$

On a noté r le rayon d'une particule et ρ_p la densité des particules (elle est supposée constante). La pression n'agit plus uniquement sur le gaz : elle agit également sur les gouttelettes car leur volume n'est plus négligeable. De plus, on suppose que la pression vue par les particules et par le gaz est la même (voir [Sai95] pour des modèles à plusieurs pressions).

Le coefficient de traînée noté C_1 jusqu'à présent n'est pas forcément une constante. Nous verrons que dans certains modèles il peut dépendre de la différence de vitesse entre les deux phases, de la densité et de la viscosité du gaz ([BDM03, Ces97, Duf05, AOB89, LM01, Lau02]). De même, l'expression de C_2 dépend des modèles rencontrés ([BDM03, Duf05]).

Pour le modèle décrit par les équations (0.1)-(0.9), la conservation de la quantité de mouvement et de l'énergie n'est pas vérifiée, car on a considéré que la fraction massique des particules était négligeable pour construire ce modèle. Dans les modèles définis par les équations (0.10)-(0.18), (0.19)-(0.27) et (0.28)-(0.37), il y a conservation des masses, des quantités de mouvement et de l'énergie totale si le système physique est clos (pas d'apport ou de pertes aux frontières du domaine). Pour obtenir ces conservations, il suffit d'intégrer l'équation de Vlasov-Boltzmann contre la masse, l'impulsion et l'énergie ([Bar01]). Ainsi pour l'impulsion, on obtient :

$$\frac{d}{dt} \iiint_{x, u_p, e_p} f m_p u_p du_p de_p dx = \iiint_{x, u_p, e_p} m_p \Gamma f du_p de_p dx,$$

en faisant des intégrations par partie et en se servant du fait que durant les collisions l'impulsion est conservée. On voit immédiatement qu'en intégrant en espace l'équation portant sur $\rho_g u_g$ (ou $\alpha_g \rho_g u_g$ pour le système (0.28)-(0.37)), on obtient la conservation de la quantité de mouvement pour le système ([Bar04a]). Ces calculs sont détaillés dans le cadre du premier chapitre.

La classification proposée pour les sprays n'illustre qu'imparfaitement leur richesse et leur complexité. Selon les applications rencontrées, on peut rajouter différents paramètres à la p.d.f. : si on considère des phénomènes de fragmentation et de coalescence ou d'abrasion, on rajoute le rayon comme variable pour les particules ([AOB89, Bar04a, Lau02, Duf05, LMV04])... La viscosité moléculaire et la viscosité turbulente peuvent être intégrées aux équations du gaz. Les effets de compressibilité peuvent aussi être pris en compte pour les particules comme nous le verrons.

Cette thèse s'articule autour de 4 chapitres. Le **premier chapitre** est consacré à la transition d'un système lagrangien-eulérien vers un système eulérien-eulérien dans un cadre fortement collisionnel : il fait le lien entre les deux modélisations au niveau formel à partir d'un développement de Hilbert. Le **second chapitre** porte sur l'étude mathématique des sprays modérément épais : l'existence et l'unicité de solutions régulières locales en temps sont prouvées, le travail portant essentiellement sur l'équation cinétique. Le **troisième chapitre** a pour objet la construction d'un modèle turbulent de spray adapté aux situations rencontrées au CEA - à partir d'un modèle de spray non turbulent- en utilisant des méthodes usuelles telle qu'une fermeture $k - \varepsilon$ pour le gaz. Enfin le **quatrième** et dernier chapitre porte sur la phase gazeuse en mode turbulent : on prouve l'existence et l'unicité de solutions régulières locales en temps pour les équations incompressibles de Navier-Stokes couplées au modèle $k - \varepsilon$ de turbulence et on s'attache à faire un développement asymptotique en régime faiblement turbulent de ces équations, en se servant des propriétés paraboliques du système.

Limite hydrodynamique dans le contexte des sprays

L'objet de ce chapitre est l'étude de la transition des modèles de **sprays épais** vers les modèles eulériens-eulériens. Ici, les sprays considérés sont **monodispersés**. On va regarder ce qu'il se passe lorsque le nombre de collisions devient très grand. Ce processus appelé limite hydrodynamique (ou développement de Hilbert) permet d'obtenir les équations d'Euler (ou de Navier-Stokes) à partir de l'équation de Boltzmann dans le cas où le système est constitué uniquement de particules ([Cer88]). Il a été initialement proposé par Hilbert en 1912 ([CC70] et [Cer75]) et est basé sur la recherche d'équilibres en situations fortement collisionnelles tels que les Maxwelliennes dans le cas d'une section efficace de collision des sphères dures. Ce raisonnement a été généralisé par la recherche de fonctions de distributions proches de l'équilibre, dans le cadre de développement de type Chapman-Enskog ([Cer88, CC70]).

Jusqu'à récemment, cette limite était formelle ou pour des solutions locales en temps ([KMN79]) mais récemment, Golse et Saint Raymond ont prouvé le résultat d'un point de vue mathématique pour des solutions globales en temps ([GSR04]), dans le cadre d'un adimensionnement menant aux équations de Navier-Stokes incompressibles.

Ici, la limite hydrodynamique est formelle et a pour but de prouver que les systèmes diphasiques peuvent être obtenus à partir des sprays épais. On vérifie que cet adimensionnement correspond aux ordres de grandeur pour un spray utilisé dans les codes du CEA (le spray considéré est constitué de particules d'étain se mouvant dans l'air). Le modèle de spray étudié dans le cadre du CEA est le suivant ([BDM03]) :

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (0.38)$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = - \iint_{u_p, e_p} m_p \Gamma f du_p de_p, \quad (0.39)$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t \alpha = \\ \iint_{u_p, e_p} - \left(m_p \Gamma + \frac{m_p}{\rho_p} \nabla_x p \right) \cdot u_p f du_p de_p - \iint_{u_p, e_p} m_p \phi f du_p de_p, \end{aligned} \quad (0.40)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p} (f\phi) = Q(f, f), \quad (0.41)$$

$$\alpha = 1 - \frac{4}{3}\pi r^3 \iint_{u_p, e_p} f du_p de_p, \quad (0.42)$$

$$m_p \Gamma = - \frac{m_p}{\rho_p} \nabla_x p + C_1(u_g - u_p), \quad (0.43)$$

$$m_p \phi = C_2(T_g - T_p), \quad (0.44)$$

$$e_g = E_g - \frac{1}{2}u_g^2, \quad (0.45)$$

$$T_g = T_1(\rho_g, e_g), \quad (0.46)$$

$$p = P_1(\rho_g, e_g), \quad (0.47)$$

$$T_p = T_2(e_p). \quad (0.48)$$

La fonction de densité de particules f dépend de t , x , u_p et e_p . Les deux phases effectuent des échanges thermiques (équations (0.40) et (0.44)). Une hypothèse importante est que les pressions vues par les deux phases sont les mêmes (équilibre isobare). De plus on suppose que les particules sont **incompressibles**. Le coefficient de traînée C_1 est égal à $\frac{1}{2}\rho_g\pi r^2 C_d |u_p - u_g|$ dans ce chapitre, où C_d dépend du nombre de Reynolds des particules Re ($C_d = \frac{24}{Re}$ dans les cas les plus simples : cf. [BDM03, Bar04a, Hyl99, Ces97, Duf05] pour des lois plus réalistes). La quantité C_2 vaut $4\pi r\lambda Nu$, où λ est la conductivité thermique des particules (elle est supposée constante) et Nu représente le nombre de Nusselt. Le nombre de Nusselt est calculé à partir du nombre de Reynolds et du nombre de Prandtl Pr (cf. [BDM03]) : $Nu = 1 + 0.3Re^{\frac{1}{2}}Pr^{1/3}$ où le nombre de Reynolds est défini par $Re = \frac{2r\rho_g|u_p - u_g|}{\eta}$ (η viscosité du gaz, supposée constante) et le nombre de Prandtl, défini par $Pr = \frac{\eta C}{\lambda}$ (C chaleur spécifique du gaz, supposée constante), est supposé constant.

Nous écrivons maintenant le noyau de collision pour totalement décrire le système. Durant les collisions, on va considérer que les particules peuvent échanger de la chaleur - via de la conduction thermique - et qu'il peut y avoir des pertes d'énergie cinétique à cause de frottements visqueux. Si on note p (resp. p_*) une particule avant collision ayant pour vitesse u_p (resp. u_{p*}) et pour énergie interne e_p (resp. e_{p*}) et p'

(resp. p'_*) la même particule ayant pour vitesse u'_p (resp. $u'_{p'_*}$) et pour énergie interne e'_p (resp. $e'_{p'_*}$) après collision, on obtient les formules suivantes :

- $u'_p = \frac{u_p + u_{p_*}}{2} + \beta \frac{|u_p - u_{p_*}|}{2} \sigma,$
- $u'_{p_*} = \frac{u_p + u_{p_*}}{2} - \beta \frac{|u_p - u_{p_*}|}{2} \sigma,$
- $e'_p = \frac{2-a}{2} e_p + \frac{a}{2} e_{p_*} + b \Delta E'_c,$
- $e'_{p_*} = \frac{a}{2} e_p + \frac{2-a}{2} e_{p_*} + (1-b) \Delta E'_c,$
- $\Delta E'_c = \frac{1}{2} (u_{p_*}^2 + u_p^2 - u_p'^2 - u_{p_*}'^2) = \left(\frac{1-\beta^2}{4} \right) (u_p - u_{p_*})^2 > 0$: perte d'énergie cinétique durant la collision,
- β : paramètre d'inélasticité,
- a : paramètre d'échange d'énergie ($0 \leq a \leq 1$),
- b : paramètre de distribution de l'énergie cinétique perdue ($0 \leq b \leq 1$),
- σ : vecteur unitaire de la sphère \mathbb{S}^2 (on rappelle que l'intégrale de σ sur la sphère unité vaut 4π),

avec $\beta = 1$ pour des collisions élastiques et $0 \leq \beta < 1$ pour le cas inélastique. Le vecteur σ est quelconque et appartient à la sphère unité car les équations de conservation de l'impulsion et de l'énergie ne fournissent pas assez d'équations scalaires par rapport au nombre d'inconnues. Dans ce chapitre, on estime l'ordre de grandeur des paramètres a , b et β à l'aide de modèles très simples basés sur des phénomènes de relaxation en température pour a et de calcul des efforts de viscosité pour β .

Avec toutes ces hypothèses, le noyau de collision s'écrit :

$$Q(f, f)(t, x, u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p_*} \in \mathbb{R}^3 \\ e_{p_*} \in \mathbb{R}^+}} \left(\frac{1}{|1-a|} \frac{1}{\beta^4} 'f_*' f - f_* f \right) r^2 |u_p - u_{p_*}| d\sigma du_{p_*} de_{p_*}$$

en notations précollisionnelles (cf. [GPV04] et [Des93]). Le terme $\frac{1}{|1-a|} \frac{1}{\beta^4}$ est un pré-facteur lié aux notations précollisionnelles.

Nous rappelons que les notations précollisionnelles sont les suivantes :

- σ : vecteur de \mathbb{S}^2
- $'u_p = \frac{u_p + u_{p_*}}{2} + \frac{1}{\beta} \frac{|u_p - u_{p_*}|}{2} \sigma$: vitesse précollisionnelle,
- $'u_{p_*} = \frac{u_p + u_{p_*}}{2} - \frac{1}{\beta} \frac{|u_p - u_{p_*}|}{2} \sigma$: vitesse précollisionnelle,

- $'e_p = \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p*} - \frac{a-2b}{2-2a} \Delta' E_c$: énergie interne précollisionnelle,
- $'e_{p*} = -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p*} - \frac{a-2+2b}{2-2a} \Delta' E_c$: énergie interne précollisionnelle,
- $\Delta' E_c = \frac{1}{2} ('u_p^2 + 'u_{p*}^2 - u_{p*}^2 - u_p^2) = \left(\frac{1-\beta^2}{4\beta^2} \right) (u_p - u_{p*})^2$: perte d'énergie cinétique,
- $'f_* = f(t, x, 'u_{p*}, 'e_{p*})$,
- $'f = f(t, x, 'u_p, 'e_p)$,
- $f_* = f(t, x, u_{p*}, e_{p*})$,
- $f = f(t, x, u_p, e_p)$.

Avant de chercher la limite hydrodynamique, on définit des quantités macroscopiques associées aux particules en calculant les différents moments :

- $\rho = \frac{1}{1-\alpha} \iint_{u_p, e_p} f m_p du_p de_p$: densité moyenne des particules,
- $v = \frac{1}{(1-\alpha)\rho} \iint_{u_p, e_p} f m_p u_p du_p de_p$: vitesse moyenne des particules,
- $e = \frac{1}{(1-\alpha)\rho} \iint_{u_p, e_p} f \{m_p e_p\} du_p de_p$: énergie interne moyenne des particules,
- $E_p = \frac{1}{(1-\alpha)\rho} \iint_{u_p, e_p} f \left\{ \frac{1}{2} m_p |u_p|^2 + m_p e_p \right\} du_p de_p$: énergie totale moyenne des particules,
- $T = \frac{1}{(1-\alpha)\rho} \iint_{u_p, e_p} f \left\{ \frac{1}{3} m_p |u_p - v|^2 \right\} du_p de_p$: température d'agitation des particules,
- $P' = \frac{1}{(1-\alpha)} \iint_{u_p, e_p} f m_p (v - u_p) \otimes (v - u_p) du_p de_p$,
- $q = \frac{1}{(1-\alpha)} \iint_{u_p, e_p} f m_p (v - u_p)^2 (u_p - v) du_p de_p$.

Ces moments sont des valeurs moyennes pour les particules. Il faut noter que l'énergie totale des particules E_p **n'est pas égale à** $e + \frac{1}{2}v^2$ mais en fait à $e + \frac{1}{2}v^2 + \frac{3}{2}T$ car toutes les particules n'ont pas la même vitesse.

Nous montrons alors que ces moments vérifient des équations qui ne forment pas un système clos :

Conservation de la masse :

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (0.49)$$

$$\partial_t((1-\alpha)\rho) + \nabla_x \cdot ((1-\alpha)\rho v) = 0, \quad (0.50)$$

$$(\rho = \rho_p = cte). \quad (0.51)$$

Conservation de l'impulsion :

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha\nabla_x p = \iint_{u_p, e_p} D_p(u_p - u_g) f du_p de_p, \quad (0.52)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p + \nabla_x ((1-\alpha)P') = \\ - \iint_{u_p, e_p} D_p(u_p - u_g) f du_p de_p. \end{aligned} \quad (0.53)$$

Conservation de l'énergie :

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha \\ = \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p - \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p, \end{aligned} \quad (0.54)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho E_p) + \nabla_x \cdot \left((1-\alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)(P'v + q)) \\ = - \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p. \end{aligned} \quad (0.55)$$

Afin de clore le système d'équations, il est nécessaire de connaître la forme de la fonction de densité des particules dans la limite que l'on considère. On va rechercher les distributions qui annulent le noyau de collision afin de procéder à la limite hydrodynamique. Au préalable on aura établi des conditions sur les différentes variables dans lesquelles cette limite est valide, à partir d'un adimensionnement adapté de l'équation de Vlasov-Boltzman (0.40). Ces conditions traduisent le fait que les collisions sont prépondérantes. L'équilibre obtenu pour des collisions inélastiques ($0 \leq \beta < 1$) avec des échanges thermiques ($0 < a \leq 1$) est le suivant :

$$f(t, x, u_p, e_p) = G(t, x) \delta_{e_p=e(t,x)}(e_p) \otimes \delta_{u_p=v(t,x)}(u_p), \quad (0.56)$$

où $G(t, x)$ est lié au nombre de particules, $v(t, x)$ est la vitesse moyenne des particules et $e(t, x)$ est l'énergie interne moyenne des particules au point x , au temps t .

Il est clair que cette distribution annule le noyau de collision. Nous montrons formellement que cet équilibre est le seul qui soit stable d'un point de vue dynamique.

Une fois que l'on connaît les équilibres (0.56), on intègre l'équation de Vlasov-Boltzmann contre la masse m_p , l'impulsion $m_p u_p$ et l'énergie totale $m_p \left\{ \frac{1}{2} u_p^2 + e_p \right\}$ en utilisant la forme explicite de f .

On obtient alors les équations suivantes (toujours dans le cas où les collisions sont inélastiques et avec des échanges thermiques) :

$$\begin{aligned} \partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) &= 0, \\ \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p &= \frac{1}{2m_p}(1 - \alpha)\rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v), \\ \partial_t((1 - \alpha)\rho E_p) + \nabla_x \cdot \left((1 - \alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p \partial_t(1 - \alpha) \\ &= -\frac{1}{2m_p}(1 - \alpha)\rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v) \cdot v + 4\pi r \lambda N u (T_g - T_p) \frac{(1 - \alpha)\rho}{m_p}. \end{aligned}$$

Ces équations sont symétriques avec celles du gaz.

Si on les regroupe avec celles du gaz on retrouve alors les équations typiques des modélisations eulériennes-eulériennes des **fluides diphasiques** ([Bou98, Duf05, Rov06, Sai95, GHS04]) si bien que le lien entre les deux modélisations est établi :

Conservation de la masse :

$$\partial_t(\alpha \rho_g) + \nabla_x \cdot (\alpha \rho_g u_g) = 0, \quad (0.57)$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0, \quad (0.58)$$

$$\rho = \rho_p = cte. \quad (0.59)$$

Conservation de l'impulsion :

$$\partial_t(\alpha \rho_g u_g) + \nabla_x \cdot (\alpha \rho_g u_g \otimes u_g) + \alpha \nabla_x p = -\frac{1}{2m_p}(1 - \alpha)\rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v), \quad (0.60)$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha)\nabla_x p = \\ \frac{1}{2m_p}(1 - \alpha)\rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v). \end{aligned} \quad (0.61)$$

Conservation de l'énergie :

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \\ - \frac{1}{2m_p}(1-\alpha)\rho_g\rho\pi r^2 C_d |v - u_g| (u_g - v) \cdot v - 4\pi r \lambda Nu (T_g - T_p) \frac{(1-\alpha)\rho}{m_p}, \end{aligned} \quad (0.62)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho E_p) + \nabla_x \cdot \left((1-\alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) = \\ \frac{1}{2m_p}(1-\alpha)\rho_g\rho\pi r^2 C_d |v - u_g| (u_g - v) \cdot v + 4\pi r \lambda Nu (T_g - T_p) \frac{(1-\alpha)\rho}{m_p}. \end{aligned} \quad (0.63)$$

Equations d'état

$$p(t, x) = P_1(\rho_g, e_g), \quad (0.64)$$

$$T_g(t, x) = T_1(\rho_g, e_g), \quad (0.65)$$

$$T_p = T_2(e), \quad (0.66)$$

$$e_g = E_g - \frac{1}{2}u_g^2, \quad (0.67)$$

$$e = E_p - \frac{1}{2}v^2. \quad (0.68)$$

Les termes sources diffèrent selon les applications rencontrées : l'expression donnée ici n'est qu'un exemple parmi d'autres. Le modèle Eulérien-Eulérien obtenu reste isobare comme l'était le modèle de spray.

On a testé numériquement ensuite ces résultats formels. Dans le cadre d'un code homogène (en espace) où il n'y a que des particules, on a vérifié la loi de Haff ([Haf83, MM06, MMR06]) qui prédit que la vitesse de convergence vers la masse de Dirac est proportionnelle à $1/t^2$, où t est le temps. On a aussi observé la concentration des énergies internes.

Les collisions inélastiques avec échanges thermiques ont été implémentées et testées numériquement dans des cas-tests au CEA : la convergence de la p.d.f. vers des distributions de type (0.56) y a été observée pour différentes géométries et différents maillages.

D'autres limites hydrodynamiques seront aussi établies (cas où les collisions sont élastiques, cas où les particules sont compressibles). Il est à noter que l'on obtient sept équations au lieu de six dans le cas où les collisions sont élastiques car la température d'agitation n'est pas nulle (les équilibres en vitesse sont des Maxwelliennes et non des masses de Dirac) : ceci est lié au fait que l'énergie cinétique **et** l'énergie interne sont conservées pour des collisions élastiques.

Etude mathématique des sprays modérément épais

L'objet de ce chapitre est la recherche de solutions régulières \mathcal{C}^1 pour les sprays modérément épais (il s'agit d'un couplage (Euler)-(Vlasov-Boltzmann)). La difficulté de cette étude provient du couplage. Les connaissances sur l'équation de Boltzmann sont liées à l'existence de solutions renormalisées en temps long à large amplitude (à la Di Perna-Lions: [DL89]), ou perturbatives près des Gaussiennes ([UA82], [MP97b], [Guo03]) ou du vide ([IS84], [BT85]), alors que l'existence et l'unicité de solutions est connue en temps petit pour la partie eulérienne ([Maj84]), ainsi que dans le cadre 1D pour des solutions de Glim à variation totale petite ([Ser96b]). Ceci nous oblige à considérer des solutions en temps petit par la suite, le travail restant à faire devant essentiellement porter sur la partie Vlasov-Boltzmann du couplage.

Jusqu'à présent, les résultats mathématiques connus sur les sprays ne concernent pas les sprays collisionnels. Domelevo et Roquejoffre ont prouvé l'existence de solutions régulières pour un couplage Burgers visqueux-Vlasov ([DR99]) via une force de traînée en 1D. En multi-D, Hamdache a prouvé l'existence globale en temps grand de solutions pour un couplage Stokes-Vlasov ([Ham98]). Récemment, Baranger et Desvillettes ont considéré un couplage Euler isentropique-Vlasov et prouvé l'existence de solutions régulières \mathcal{C}^1 en temps petit (et à support compact pour f) en se servant de la théorie hyperbolique ([Maj84, Ser96a, AG91, Ser96b]) et de la méthode des caractéristiques ([Sch91]).

Nous poursuivons ici ce dernier travail en introduisant les collisions. De plus, nous tenons compte de l'énergie du gaz et des particules : elles n'avaient pas été prises en compte lors des travaux évoqués plus haut. La méthode des caractéristiques employée par Baranger ([Bar04b]) pour résoudre la partie cinétique ne convient plus car on ne peut assurer un contrôle du support de $f(t, x, v, e)$ à cause de l'apparition de grandes vitesses, du fait des collisions. On va à la place utiliser des fonctions à décroissance rapide dans les espaces de Sobolev H^s ($f \exp(v^2 + e)(t, \cdot) \in H^s(\mathbb{R}^7)$) avec v vitesse et e énergie interne, suivant la méthode initiée par Guo ([Guo03], [Guo02]). Les u_p deviennent des v et les e_p deviennent des e , afin de simplifier les écritures et de rester cohérent avec les travaux de Baranger et Desvillettes ([BD06]). La principale difficulté provient du contrôle du noyau de collision : la section des sphères dures en $|v - v_*|$ fait perdre un moment en vitesse comme nous le verrons. L'utilisation de la traînée va permettre de compenser cette perte de moment.

Le système de spray étudié mathématiquement est le suivant :

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0, \quad (0.69)$$

$$\partial_t u_g + (u_g \cdot \nabla_x) u_g + \frac{\nabla_x p}{\rho_g} = -\frac{1}{\rho_g} \iint_{v,e} F f dv de, \quad (0.70)$$

$$\partial_t e_g + u_g \cdot \nabla_x e_g + \frac{p}{\rho_g} \nabla_x \cdot u_g = \frac{1}{\rho_g} \iint_{v,e} (F \cdot (u_g - v) - \phi) f dv de \quad (0.71)$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (fF) + \partial_e (f\phi) = Q(f, f), \quad (0.72)$$

$$F = -(v - u_g), \quad (0.73)$$

$$\phi = T_g - T_p, \quad (0.74)$$

$$p = (\gamma - 1) \rho_g e_g, \quad (0.75)$$

$$T_g = e_g, \quad (0.76)$$

$$T_p = e. \quad (0.77)$$

C'est le modèle défini par les équations (0.19)-(0.27) pour lequel on préfère utiliser l'équation en énergie interne pour le gaz à celle en énergie totale : on écrit le système sous forme non conservative car les inconnues du gaz sont ρ_g , u_g et e_g dans ce chapitre (afin de simplifier l'écriture des matrices intervenant dans la partie hyperbolique). Nous rappelons que pour des sprays modérément épais, la fraction de volume occupée par le gaz est égale à 1. Par ailleurs le gaz est supposé parfait. On suppose aussi que la dépendance en température des énergies internes des deux phases est linéaire. Pour la traînée, nous prenons le modèle le plus simple où le coefficient est constant. Enfin on prend toutes les constantes physiques égales à 1 (notamment la constante de traînée C_1 qui est strictement positive et la constante d'échange thermique C_2 , les chaleurs spécifiques, le rayon, la masse...), mise à part la constante des gaz parfaits ($\gamma > 1$), dans un souci de simplification des estimations a priori.

Par ailleurs, on ne regarde que des collisions élastiques sans échanges thermiques. Le noyau de collision $Q(f, f)$ que l'on considère s'écrit :

$$Q(f, f)(t, x, v, e) = \iiint_{\sigma \in \mathbb{S}^2, v_* \in \mathbb{R}^3, e_* \in \mathbb{R}^+} (f'_* f' - f_* f) |v - v_*| d\sigma dv_* de_*, \quad (0.78)$$

où

- σ : vecteur de \mathbb{S}^2 ,
- $v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma$: vitesse post-collisionnelle,
- $v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma$: vitesse post-collisionnelle,
- $e' = e$: énergie interne post-collisionnelle,
- $e'_* = e_*$: énergie interne post-collisionnelle,

- $f_* = f(t, x, v'_*, e'_*),$
- $f' = f(t, x, v', e'),$
- $f_* = f(t, x, v_*, e_*),$
- $f = f(t, x, v, e).$

Le noyau est noté en forme post-collisionnelle ici, car les collisions élastiques sont des phénomènes réversibles.

On cherche des solutions $(\rho_g - 1)(t, \cdot), u_g(t, \cdot), (e_g - 1)(t, \cdot)$ et $(f \exp(v^2 + e))(t, \cdot)$ continues à valeurs dans H^s et dérivables à valeur dans H^{s-1} : ρ_g et e_g ne doivent pas s'annuler si on veut pouvoir utiliser la théorie hyperbolique pour le gaz. C'est pour cela que ces fonctions sont à valeurs dans H^s modulo une constante ([Maj84, AG91, Ser96a, Ser96b]).

On note $\partial_\alpha^{\beta, \gamma} := \partial_x^\alpha \partial_v^\beta \partial_e^\gamma$, $\|g\|$ la norme L^2 et $\|g\|_w$ la norme à poids L^2 ($w = (1 + |v|^2)$):

$$\|g\| = \left(\iiint_{x,v,e} g^2 dedvdx \right)^{\frac{1}{2}},$$

$$\|g\|_w = \left(\iiint_{x,v,e} g^2 w dedvdx \right)^{\frac{1}{2}}.$$

Tout d'abord on regarde le problème à "gaz fixé". On va supposer ρ_g, u_g et e_g réguliers et on va considérer l'équation (0.72). On commence par réécrire l'équation en utilisant la fonction $g := f \exp(v^2 + e)$. Celle-ci vérifie l'équation suivante :

$$\partial_t g + v \cdot \nabla_x g + \nabla_v \cdot (gF) + \partial_e (g\phi) - 2v \cdot gF - g\phi = \Gamma(g, g), \quad (0.79)$$

où

$$\begin{aligned} & \Gamma[g_1, g_2](t, x, v, e) \\ &= \Gamma^+[g_1, g_2](t, x, v, e) - \Gamma^-[g_1, g_2](t, x, v, e) \\ &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v'_*, e'_*) g_2(t, x, v', e') dv_* de_* d\sigma \\ &- g_2(t, x, v, e) \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v_*, e_*) dv_* de_* d\sigma. \end{aligned}$$

On commence par prouver une estimation a priori, puis on prouve l'existence et l'unicité de solutions régulières en temps petit pour l'équation (0.79), en utilisant la méthode des caractéristiques et le théorème de point fixe de Picard. Plus précisément on démontre le théorème suivant :

Théorème 0.1

On suppose que ρ_g et e_g sont des fonctions strictement positives. De plus, on prend $\tilde{\rho}_g = \rho_g - 1, u_g$ et $\tilde{e}_g = e_g - 1$ dans $\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))$ pour un temps $T > 0$ et s un entier qui vérifie $s \geq 5$. On suppose que g_0 vérifie

$$\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \frac{1}{2} \left\| \partial_\alpha^{\beta, \gamma} (g_0) \right\|^2 < +\infty.$$

Alors :

1. On peut trouver $T' > 0$ tel qu'il y ait une solution g à l'équation (0.79) avec g_0 pour donnée initiale. De plus, g reste positive et

$$\mathcal{E}_{s, \frac{1}{2}}(g(t)) = \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left[\frac{1}{2} \left\| \partial_{\alpha}^{\beta, \gamma} g \right\|_{H^s}^2(t) + \int_0^t \left\| \partial_{\alpha}^{\beta, \gamma} g \right\|_w^2(u) du \right]$$

reste bornée sur $[0, T']$. Finalement, T' est contrôlé par

$$T' \geq \frac{C}{\left[1 + \sup_{0 \leq t \leq T} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g\|_{H^s} + \frac{3}{2} \|g_0\|_{H^s}^2 \right]} \quad (0.80)$$

où C est strictement positive et ne dépend que de s .

Enfin, g appartient à $\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.

2. Si g_1 et g_2 sont deux solutions positives de l'équation (0.79) dans $\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ avec même donnée initiale, mêmes données ρ_g, u_g et e_g , et vérifiant $\sup_{0 \leq t \leq T} \mathcal{E}_{s, \frac{1}{2}}(g_1(t, \cdot)) < +\infty$ et $\sup_{0 \leq t \leq T} \mathcal{E}_{s, \frac{1}{2}}(g_2(t, \cdot)) < +\infty$, alors $g_1 = g_2$.

Il faut noter qu'on obtient un gain de moment sur g d'après l'expression de l'énergie $\mathcal{E}_{s, \frac{1}{2}}(g(t, \cdot))$. Celui-ci provient du terme de traînée et permet le contrôle du noyau de collision dans les estimations a priori de la preuve du théorème 0.1. Plus précisément, on a l'estimation suivante sur le terme de traînée non linéaire de l'équation 0.79:

Lemme 0.2 (terme $gF \cdot v$)

Pour tout ε vérifiant $1 > \varepsilon > 0$,

$$\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \iiint \partial_{\alpha}^{\beta, \gamma} g \partial_{\alpha}^{\beta, \gamma} (gF \cdot v) dv dx \leq -(1 - \varepsilon) \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left\| \partial_{\alpha}^{\beta, \gamma} g \right\|_w^2 + C_1 \|g\|_{H^s}^2 + \frac{C_2}{\varepsilon} \|g\|_{H^s}^2 \|u_g\|_{H^s}^2. \quad (0.81)$$

Une fois résolu le problème à gaz fixé, on s'intéresse au système couplé (0.69)-(0.77). En se servant d'un schéma itératif et du résultat à "gaz fixé" établi au théorème 0.1, on démontre le résultat suivant :

Théorème 0.3

On note $I =]0, +\infty[\times \mathbb{R}^3 \times]0, +\infty[$, $s \in \mathbb{N}$ un entier vérifiant $s \geq 5$, et I_1, I_2 des ouverts de I tel que $\overline{I_1} \subset I_2$ et $\overline{I_1}, \overline{I_2}$ soient compacts dans I . Soit $(\rho_{g_0}, u_{g_0}, e_{g_0}) : \mathbb{R}^3 \rightarrow I_1$ des fonctions vérifiant $\tilde{\rho}_{g_0} = \rho_{g_0} - 1 \in H^s(\mathbb{R}^3)$, $u_{g_0} \in H^s(\mathbb{R}^3)$ et $\tilde{e}_{g_0} = e_{g_0} - 1 \in H^s(\mathbb{R}^3)$. Soit $\tilde{f}_0 = f_0 \exp(v^2 + e) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ une fonction de $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$.

- Alors, on peut trouver $T > 0$ tel qu'il existe une solution $(\rho_g, u_g, e_g; f)$ au système (0.69) - (0.77) qui appartient à $C^1([0, T] \times \mathbb{R}^3, \overline{I_2}) \times C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$. De plus, $\tilde{\rho}_g (= \rho_g - 1)$, u_g , $\tilde{e}_g (= e_g - 1) \in \mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))$, $\tilde{f} = f \exp(v^2 + e) \in \mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.
- Finalement, si $(\tilde{\rho}_{g_1}, u_{g_1}, \tilde{e}_{g_1}; \tilde{f}_1)$ et $(\tilde{\rho}_{g_2}, u_{g_2}, \tilde{e}_{g_2}; \tilde{f}_2)$ appartiennent à $(\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))) \times (\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)))$, si $(\rho_{g_1}, u_{g_1}, e_{g_1}; f_1)$ et $(\rho_{g_2}, u_{g_2}, e_{g_2}; f_2)$ appartiennent à $C^1([0, T] \times \mathbb{R}^3, \overline{I_2}) \times C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$ et si elles vérifient (0.69) - (0.77), alors $\rho_{g_1} = \rho_{g_2}$, $u_{g_1} = u_{g_2}$, $e_{g_1} = e_{g_2}$ et $f_1 = f_2$.

Les ensembles I_1 et I_2 sont introduits pour contrôler l'hyperbolicité ([Maj84]), afin d'être sûr que les valeurs propres des matrices intervenant dans les estimations a priori sur la partie hyperbolique du système et celles de leurs inverses restent bornées ([Maj84, Ser96a, BD06]).

Modèle turbulent de spray

L'objectif de ce chapitre est la construction d'un **modèle turbulent de spray épais** à partir du modèle étudié dans le premier chapitre pour les besoins du CEA, afin d'initialiser un code Eulérien-Eulérien qui tient compte de la turbulence : il faut donc que la turbulence soit déjà prise en compte dans le spray. On va utiliser un modèle $k - \varepsilon$ pour représenter cette turbulence.

L'apport de ce chapitre consiste surtout à adapter aux situations rencontrées au CEA des méthodes connues pour décrire l'impact de la turbulence sur les sprays.

Les équations d'Euler sont moyennées via le processus de Favre ([MP94]) et on obtient alors les équation $k - \varepsilon$ compressibles. Ce processus consiste à séparer les variables du gaz ρ_g , u_g et e_g en deux composantes ([Ver96, Van83]) : une composante moyenne (notée $\tilde{\cdot}$) appelée moyenne de Favre de la variable (elle-même issue de la moyenne de Reynolds - notée $\bar{\cdot}$ - comme nous le verrons, notamment pour la pression, dans ce chapitre) et une composante fluctuante ou turbulente (notée $\tilde{\cdot}$). Les variables moyennes sont obtenues sur un volume de contrôle V (on note g la fonction caractéristique du volume de contrôle dont l'intégrale vaut 1 sur ce volume) et on note V_g la partie du volume occupée par le gaz. Les composantes moyennes vérifient alors des équations similaires aux équations d'Euler à l'exception de l'apparition de termes liés aux valeurs turbulentes :

$$\begin{aligned} \partial_t(\alpha \tilde{\rho}_g) + \nabla_x \cdot (\alpha \tilde{\rho}_g \tilde{u}_g) &= 0, \\ \partial_t(\alpha \tilde{\rho}_g \tilde{u}_g) + \nabla_x \cdot (\alpha \tilde{\rho}_g \tilde{u}_g \otimes \tilde{u}_g) + \nabla_x \tilde{p} &= F_p + \nabla_x \cdot R, \\ \partial_t(\alpha \tilde{\rho}_g \tilde{E}_g) + \nabla_x \cdot \left(\alpha \tilde{\rho}_g \left(\tilde{E}_g + \frac{\tilde{p}}{\tilde{\rho}_g} \right) \tilde{u}_g \right) + \tilde{p} \partial_t \alpha &= Q_p + Q_t. \end{aligned}$$

- $R = - \int_{y \in V_g} \rho_g u_g'' \otimes u_g'' g(y) dy$: tenseur de Reynolds,

- F_p : terme d'interaction avec les particules (traînée),
- $Q_t = -\nabla_x \cdot \int_{y \in V_g} \rho_g E_g'' u_g'' g(y) dy$,
- Q_p : terme d'interaction avec les particules (efforts liés à la traînée et échanges thermiques modifiés par la turbulence).

Pour trouver l'expression du tenseur de Reynolds R , de F_p , de Q_t et de Q_p , on introduit deux quantités notées k et ε qui sont respectivement l'énergie cinétique de turbulence et la dissipation d'énergie turbulente. Le terme $\frac{k^2}{\varepsilon}$ a la dimension d'une viscosité et $\frac{k}{\varepsilon}$ représente le temps caractéristique de la turbulence. En se servant de l'analyse dimensionnelle et des propriétés d'invariance de R (cf. [MP94]), on établit l'expression du tenseur de Reynolds :

$$R = -\frac{2}{3}(\alpha \tilde{\rho}_g k) \mathbf{I} + \alpha \tilde{\rho}_g \nu_t \left[(\nabla_x \tilde{u}_g + \nabla_x \tilde{u}_g^T) - \frac{2}{3}(\nabla_x \cdot \tilde{u}_g) \mathbf{I} \right].$$

La quantité $\nu_t = c_\mu \frac{k^2}{\varepsilon}$ est la viscosité turbulente (c_μ est une constante) et \mathbf{I} désigne la matrice identité.

Il faut noter que \tilde{E}_g (l'énergie totale moyennée du gaz) est reliée à l'énergie interne moyennée \tilde{e}_g , l'énergie turbulente k et l'énergie cinétique moyennée $\frac{1}{2} \tilde{u}_g^2$ par

$$\tilde{E}_g = \tilde{e}_g + k + \frac{1}{2} \tilde{u}_g^2.$$

Le flux d'énergie turbulente Q_t est modélisé par :

$$Q_t = \nabla_x \cdot \left(\alpha \tilde{\rho}_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} \left(\nabla_x \tilde{e}_g + \bar{p} \nabla_x \left(\frac{1}{\tilde{\rho}_g} \right) \right) \right),$$

où Pr_t est le nombre de Prandtl turbulent (il est constant et vaut 0.7 dans les applications rencontrées : [Llo01, Llo05]).

Les quantités k et ε vérifient les équations suivantes (cf. [MP94]) :

$$\begin{aligned} \frac{\partial (\alpha \tilde{\rho}_g k)}{\partial t} + \nabla_x (\alpha \tilde{\rho}_g k \tilde{u}_g) - \frac{1}{2} R : (\nabla_x \tilde{u}_g + \nabla_x \tilde{u}_g^T) - \nabla_x \cdot \left(c_\mu \frac{\alpha \tilde{\rho}_g k^2}{\varepsilon} \nabla_x k \right) + \alpha \tilde{\rho}_g \varepsilon &= 0, \\ \frac{\partial (\alpha \tilde{\rho}_g \varepsilon)}{\partial t} + \nabla_x (\alpha \tilde{\rho}_g \varepsilon \tilde{u}_g) - \frac{c_{1\varepsilon}}{2} \frac{\varepsilon}{k} R : (\nabla_x \tilde{u}_g + \nabla_x \tilde{u}_g^T) - \nabla_x \cdot \left(c_\varepsilon \frac{\alpha \tilde{\rho}_g k^2}{\varepsilon} \nabla_x \varepsilon \right) + c_2 \frac{\alpha \tilde{\rho}_g \varepsilon^2}{k} &= 0, \end{aligned}$$

avec $c_1, c_2, c_\mu, c_\varepsilon$ des constantes calibrées à l'aide d'expériences physiques ([Llo01, Llo05, Bra05]).

Tout ce processus de moyenne pour le gaz est classique ([MP94],[O'R81]) - y compris pour α ([SN00])- et l'essentiel de cette étude porte en fait sur l'impact de la turbulence sur les particules.

Il reste maintenant à étudier l'impact de la turbulence sur les particules et les collisions. On montrera à l'aide de calculs sur les données du Cea que pour les applications rencontrées, les particules sont tellement lourdes que les tourbillons de turbulence ne modifient pas leur trajectoire et donc que les particules restent régies par la même équation, le noyau de collision n'étant pas modifié : lorsqu'on calcule le coefficient d'efficacité des collisions en présence de turbulence introduit par Langmuir ([Lan48]), on s'aperçoit qu'il est très proche de 1 et donc que la turbulence n'a pas d'impact sur les collisions. L'influence de la turbulence est donc nulle sur le mouvement des particules. Pour plus de détails sur l'impact de la turbulence pour des particules plus légères nous renvoyons le lecteur vers les travaux suivants : [VS04],[Sim00],[SF00],[MP97a],[MP01], [Pop94], [O'R81], [DV05].

En utilisant toutes ces hypothèses, on finit par établir le système d'équations pour les sprays turbulents épais (on a supprimé toutes les notations en $\bar{\cdot}$ et $\tilde{\cdot}$ liées au processus de moyenne dans un souci de simplification) :

Système gaz-particules turbulent :

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (0.82)$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = - \iint_{u_p, e_p} m_p \Gamma f du_p de_p + \nabla_x \cdot R, \quad (0.83)$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \nabla_x \cdot \left(\alpha\rho_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} \left(\nabla_x e_g + p\nabla_x \left(\frac{1}{\rho_g} \right) \right) \right) \\ - \iint_{u_p, e_p} (m_p \Gamma + \frac{m_p}{\rho_p} \nabla_x p) \cdot u_p f du_p de_p - \iint_{u_p, e_p} m_p \phi f du_p de_p, \end{aligned} \quad (0.84)$$

$$\frac{\partial(\alpha\rho_g k)}{\partial t} + \nabla_x (\alpha\rho_g k u_g) - R : \frac{\nabla_x u_g + \nabla_x u_g^T}{2} - \nabla_x \cdot \left(c_\mu \frac{\alpha\rho_g k^2}{\varepsilon} \nabla_x k \right) + \alpha\rho_g \varepsilon = 0, \quad (0.85)$$

$$\frac{\partial(\alpha\rho_g \varepsilon)}{\partial t} + \nabla_x (\alpha\rho_g \varepsilon u_g) - c_{1\varepsilon} \frac{\varepsilon}{k} R : \frac{\nabla_x u_g + \nabla_x u_g^T}{2} - \nabla_x \cdot \left(c_\varepsilon \frac{\alpha\rho_g k^2}{\varepsilon} \nabla_x \varepsilon \right) + c_2 \frac{\alpha\rho_g \varepsilon^2}{k} = 0, \quad (0.86)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p}(f\phi) = Q(f, f), \quad (0.87)$$

$$\alpha = 1 - \frac{4}{3} \pi r^3 \iint_{u_p, e_p} f du_p de_p, \quad (0.88)$$

$$m_p \Gamma = - \frac{m_p}{\rho_p} \nabla_x p - D_p(u_p - u_g), \quad (0.89)$$

$$m_p \phi = 4\pi r \lambda N u (T_g - T_p), \quad (0.90)$$

$$R = - \frac{2}{3} (\alpha\rho_g k) \mathbf{I} + c_\mu \frac{\alpha\rho_g k^2}{\varepsilon} \left[(\nabla_x u_g + \nabla_x u_g^T) - \frac{2}{3} (\nabla_x \cdot u_g) \mathbf{I} \right], \quad (0.91)$$

$$e_g(t, x) = E_g(t, x) - k(t, x) - \frac{1}{2}u_g^2(t, x), \quad (0.92)$$

$$T_g(t, x) = T_1(\rho_g(t, x), e_g(t, x)), \quad (0.93)$$

$$p(t, x) = P_1(\rho_g(t, x), e_g(t, x)), \quad (0.94)$$

$$T_p = T_2(e_p). \quad (0.95)$$

On prend dans les exemples air-étain du CEA les valeurs suivantes pour les constantes :

- $c_{1\varepsilon} = 1.45$,
- $c_2 = 1.92$,
- $c_\mu = 0.09$,
- $c_\varepsilon = 0.07$,
- C_0 entre 5.0 et 6.1,
- $Pr_t = 0.9$.

Solutions régulières en temps petit du modèle $k - \varepsilon$ incompressible

L'objet de ce dernier chapitre est l'étude du modèle $k - \varepsilon$ incompressible d'un point de vue mathématique. Il porte uniquement sur une phase gazeuse et visqueuse régie par des équations de Navier-Stokes de type $k - \varepsilon$. Ce modèle sert à étudier notamment les instabilités de Rayleigh-Taylor, Kelvin-Helmholtz et Richtmyer-Meshkov ([DG02, DLY99, BG90, LB03, HG03]). Ainsi on modélise l'explosion de supernovae telle que celle découverte en 1987 ([ABKW89],[KMC90]) et des expériences de fusion par confinement inertiel ([Mat03]).

On note $U := U(t, x)$ la vitesse du fluide, $k := k(t, x)$ l'énergie cinétique de turbulence, $\varepsilon := \varepsilon(t, x)$ le taux de dissipation d'énergie turbulente et R le tenseur de Reynolds. Le modèle étudié est le suivant :

$$\frac{\partial U}{\partial t} + U \cdot \nabla U + \nabla P - \nu \Delta U - \nabla \cdot R = 0, \quad (0.96)$$

$$\nabla \cdot U = 0, \quad (0.97)$$

$$\frac{\partial k}{\partial t} + U \cdot \nabla k - \frac{c_\mu}{2} \frac{k^2}{\varepsilon} |\nabla U + \nabla U^T|^2 - \nabla \cdot \left(c_\mu \frac{k^2}{\varepsilon} \nabla k \right) + \varepsilon = 0, \quad (0.98)$$

$$\frac{\partial \varepsilon}{\partial t} + U \cdot \nabla \varepsilon - \frac{c_1}{2} k |\nabla U + \nabla U^T|^2 - \nabla \cdot \left(c_\varepsilon \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + c_2 \frac{\varepsilon^2}{k} = 0, \quad (0.99)$$

$$U(0, x) = U^0(x), \quad k(0, x) = k^0(x), \quad \varepsilon(0, x) = \varepsilon^0(x), \quad (0.100)$$

$$R = -\frac{2}{3}kI + c_\mu \frac{k^2}{\varepsilon} (\nabla U + \nabla U^T), \quad (0.101)$$

avec

- $c_1 = 0.126$,
- $c_2 = 1.92$,
- $c_\mu = 0.09$,
- $c_\varepsilon = 0.07$.

Les deux premières équations sont celles de Navier-Stokes modifiées par l'apport du tenseur de Reynolds R . Les équations sur k et ε sont obtenues via des hypothèses statistiques sur les moyennes temporelles et spatiales des fluctuations turbulentes ([MP94, Ver96, Van83]). Les différentes constantes sont obtenues via des expériences canoniques ([MP94]). Le tenseur de Reynolds incompressible et celui compressible présenté au **troisième chapitre** sont reliés par le fait que $\nabla \cdot U = 0$. Ainsi, le terme $R : \frac{\nabla U + \nabla U^T}{2}$ est égal à $c_\mu \frac{k^2}{\varepsilon} |\nabla U + \nabla U^T|^2$ du fait de la relation d'incompressibilité ($I : \frac{\nabla U + \nabla U^T}{2} = \nabla \cdot U = 0$). Tout ceci permet de retrouver les équations incompressibles à partir des équations compressibles (0.85)-(0.86) (même si d'un point de vue historique le modèle incompressible est le premier à être apparu : cf. [Van83]).

L'étude porte d'abord sur l'obtention d'estimations a priori pour le système (0.96)-(0.101) dans des espaces de Sobolev sur le tore \mathbb{T}^3 ($= [0, 2\pi]^3$ avec conditions aux limites périodiques), puis sur l'existence et l'unicité de solutions régulières en temps petit. Enfin on étudie le régime faiblement turbulent lorsqu'on néglige U en effectuant un développement asymptotique. La principale difficulté provient du fait qu'il faut garder k et ε strictement positifs : c'est l'une des raisons qui nous pousse à faire l'étude sur le tore \mathbb{T}^3 . Le seul résultat connu sur ce type de système a été prouvé sur un système modifié appelé $\theta - \phi$ ($\phi = \frac{\varepsilon^3}{k^2}$ et $\theta = \frac{k}{\varepsilon}$) par Lewandowski et Mohammadi, système dont les propriétés de diffusion sont meilleures que celles du système $k - \varepsilon$ ([LM93, MP94]).

On établit d'abord le résultat suivant pour le système $k - \varepsilon$ incompressible (0.96)-(0.101) :

Théorème 0.4

- (a) Soient des données initiales U^0, k^0, ε^0 dans $H^s(\mathbb{T}^3)$ pour $s > 4 + 3/2$ ($s \in \mathbb{N}$) avec k^0 et ε^0 plus grands qu'une constante strictement positive. Alors il existe un temps $T > 0$ et une solution (U, k, ε) au système (0.96)-(0.101) sur $[0, T]$ appartenant à $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$, et telle que k et ε restent strictement positives sur $[0, T]$.
- (b) De plus si $(U_1, k_1, \varepsilon_1)$ et $(U_2, k_2, \varepsilon_2)$ sont deux solutions au sens des distributions du système (0.96)-(0.101) avec même donnée initiale, appartenant à $\mathcal{C}([0, T]; H^2(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; L^2(\mathbb{T}^3))$ et si k_1, ε_1, k_2 et ε_2 sont strictement positives du problème, alors $U_1 = U_2, k_1 = k_2, \varepsilon_1 = \varepsilon_2$ sur $[0, T]$.

La preuve est basée sur l'utilisation du principe du maximum pour s'assurer que k et ε restent strictement positifs et contrôler cette positivité, ainsi que l'établissement d'estimations a priori utilisant la viscosité du fluide.

On s'intéresse ensuite uniquement à k et ε via un adimensionnement lié à des expériences sur des instabilités Richtmyer-Meshkov ([Mat03]). On considère une situation initiale où la vitesse U du fluide est nulle, ce qui arrive lors d'expériences de Rayleigh-Taylor ([DLY99]) et on la négligera par la suite.

Le scaling effectué est le suivant :

- $x \mapsto \tilde{x} = \frac{x}{L}$ où L est la longueur caractéristique,
- $t \mapsto \tilde{t} = \frac{t}{T}$ où T est le temps caractéristique,
- $\varepsilon \mapsto \tilde{\varepsilon} = \frac{\varepsilon}{\varepsilon^0}$ où ε^0 est le taux de dissipation d'énergie cinétique turbulente caractéristique de ces expériences,
- $k \mapsto \tilde{k} = \frac{k}{k^0}$ où k^0 est l'énergie cinétique turbulente caractéristique de ces expériences.

Après scaling, on obtient le système suivant :

$$\frac{\partial k}{\partial t} - \eta \nabla \cdot \left(\frac{k^2}{\varepsilon} \nabla k \right) + A\varepsilon = 0, \quad (0.102)$$

$$\frac{\partial \varepsilon}{\partial t} - \eta \nabla \cdot \left(\frac{c_\varepsilon k^2}{c_\mu \varepsilon} \nabla \varepsilon \right) + c_2 A \frac{\varepsilon^2}{k} = 0, \quad (0.103)$$

avec

- $A = \varepsilon^0 T / k^0$,
- $\eta = \frac{c_\mu (k^0)^2 T}{\varepsilon^0 L^2}$.

Ce système est de type parabolique. En se servant d'estimations a priori et d'un schéma itératif, on peut prouver l'existence de solutions en temps petit régulières sur le tore \mathbb{T}^3 ([Lio96]). Par ailleurs, on prouvera que le temps d'existence de k et ε est plus grand qu'une constante strictement positive indépendante de η , à A fixé.

Dans des expériences de fusion par confinement inertiel ([Mat03]), η est très petit tandis que A est d'ordre 1. On établit un développement asymptotique en fonction de η pour k et ε , notamment en étudiant les 2 premiers termes du développement ($k_0 + \eta k_1$ et $\varepsilon_0 + \eta \varepsilon_1$). Ceux-ci obéissent aux équations suivantes :

Système à l'ordre 0

$$\frac{\partial k_0}{\partial t} + A\varepsilon_0 = 0, \quad k_0(0, \cdot) = k^0(\cdot), \quad (0.104)$$

$$\frac{\partial \varepsilon_0}{\partial t} + c_2 A \frac{\varepsilon_0^2}{k_0} = 0, \quad \varepsilon_0(0, \cdot) = \varepsilon^0(\cdot). \quad (0.105)$$

Système à l'ordre 1

$$\frac{\partial k_1}{\partial t} - \nabla \cdot \left(\frac{k_0^2}{\varepsilon_0} \nabla k_0 \right) + A \varepsilon_1 = 0, \quad k_1(0, \cdot) = 0, \quad (0.106)$$

$$\frac{\partial \varepsilon_1}{\partial t} - \nabla \cdot \left(\frac{c_\varepsilon k_0^2}{c_\mu \varepsilon_0} \nabla \varepsilon_0 \right) + c_2 A \left(\frac{2\varepsilon_0 \varepsilon_1}{k_0} - \frac{\varepsilon_0^2 k_1}{k_0^2} \right) = 0, \quad \varepsilon_1(0, \cdot) = 0. \quad (0.107)$$

Ces systèmes se résolvent de façon explicite pour (0.104)-(0.105) et via une formule de Duhamel pour (0.106)-(0.107). Il faut noter que k_0 et ε_0 sont bien définis et strictement positifs pour tout temps. On établit alors le théorème suivant pour comparer k et ε à leur développement asymptotique :

Théorème 0.5

On suppose que A est fixé. Soient k^0 et ε^0 appartenant à $H^7(\mathbb{T}^3)$, plus grands qu'une constante strictement positive. Soit k_0, ε_0 des solutions de (0.104)-(0.105). Soit k_1, ε_1 des solutions de (0.106)-(0.107). Soient k et ε des solutions du problème (0.102)-(0.103) plus grands qu'une constante strictement positive et $T(\eta)$ leur temps d'existence dans $C^1([0, T(\eta)]; H^5(\mathbb{T}^3))$. Alors il existe $T' \leq T$ strictement positif tel que $T(\eta) \geq T'$ et indépendant de η et une constante C tels que pour tout $t < T'$:

$$\|k - k_0 - \eta k_1\|_{L^\infty}(t) \leq C \eta^{\frac{3}{2}}, \quad \|\varepsilon - \varepsilon_0 - \eta \varepsilon_1\|_{L^\infty}(t) \leq C \eta^{\frac{3}{2}}.$$

La preuve s'effectue à partir d'estimations a priori et du principe du maximum en se servant de la viscosité.

BIBLIOGRAPHY

- [ABKW89] D. Arnett, J. Bahcall, R. Kirschner, and S. Woosley. *Ann. Rev. Astron. Astrophys.*, 27:629, 1989.
- [AG91] S. Alinhac and P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Interéditions/Éditions du CNRS, 1991.
- [AO89] A.A. Amsden and P.J. O'Rourke. The T.A.B. method for numerical calculation of spray droplet breakup. Technical report, Los Alamos National Laboratory, 1989.
- [AOB89] A.A. Amsden, P.J. O'Rourke, and T.D. Butler. Kiva II, a computer program for chemical reactive flows with spray. Technical report, Los Alamos National Laboratory, 1989.
- [Bar01] C. Baranger. Collisions, coalescences et fragmentations des goutelettes dans un spray: écriture précise des équations relatives au modèle T.A.B. Master's thesis, CMLA, Ecole normale supérieure de Cachan, 2001.
- [Bar04a] C. Baranger. Modelling of oscillations, breakup and collisions for droplets: the establishment of kernels for the T.A.B. model. *Math. Models Methods Appl. Sci.*, 14(5):775–794, 2004.
- [Bar04b] C. Baranger. *Modélisation, étude mathématique et simulation des collisions dans les fluides complexes*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2004.
- [BD06] C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays : local smooth solutions. *J. Hyperbolic Differ. Equ.*, 3:1–26, 2006.
- [BDM03] L. Boudin, L. Desvillettes, and R. Motte. A modeling of compressible droplets in a fluid. *Commun. Math. Sci.*, 1(4):657–669, 2003.
- [BG90] M. Bonnet and S. Gauthier. A $k - \varepsilon$ model for turbulent mixing in shock-tube flows induced by Rayleigh-Taylor instability. *Phys. Fluid A*, 10:2698–2700, 1990.
- [Bou98] M. Boucker. *Modélisation numérique multidimensionnelle d'écoulements diphasiques liquide-gaz en régimes transitoire et permanent: méthodes et applications*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 1998.
- [Bra05] J.-P. Braeunig. Une méthode de résolution de modèle de turbulence $k - \varepsilon$ préservant les supports compacts. Rapport CEA, 2005.
- [BT85] N. Bellomo and G. Toscani. On the Cauchy problem for the nonlinear Boltzmann equation: global existence, uniqueness and asymptotic stability. *J. Math. Phys.*, 26(2):334–338, 1985.

- [Car57] T. Carleman. *Problèmes mathématiques dans la théorie cinétique des gaz*. Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.
- [CC70] S. Chapman and T.G. Cowling. *The mathematical theory of non uniform gases*. Cambridge Mathematical Library, 1970.
- [Cer75] C. Cercignani. *Theory and application of the Boltzmann equation*. Elsevier, New York, 1975.
- [Cer88] C. Cercignani. *The Boltzmann Equation and its Applications*. Springer-Verlag, 1988.
- [Cer03] C. Cercignani. Recent results in the kinetic theory of granular materials. In *Perspectives and problems in nonlinear science*, pages 217–228. Springer, New York, 2003.
- [Ces97] N. Cesco. *Etude et modélisation de l'écoulement diphasique à l'intérieur des propulseurs à poudre*. PhD thesis, Ecole nationale supérieure d'aéronautique et de l'espace (Toulouse), 1997.
- [CIP94] C. Cercignani, R. Illner, and M. Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.
- [CP83] R. Caflisch and G. C. Papanicolaou. Dynamic theory of suspensions with Brownian effects. *SIAM J. Appl. Math.*, 43(4):885–906, 1983.
- [Des93] L. Desvillettes. *Equations cinétiques*, 1993. Ecole nationale des ponts et chaussées.
- [DG02] B. Desjardins and E. Grenier. On Rayleigh Taylor and Richtmyer instabilities, 2002.
- [DL89] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.
- [DLY99] S. B. Dalziel, P. F. Linden, and D. L. Youngs. Self-similarity and internal structure of turbulence induced by Rayleigh-Taylor instability. *J. Fluid Mech.*, 399:1–48, 1999.
- [DMV03] G. Dufour, M. Massot, and P. Villedieu. Étude d'un modèle de fragmentation secondaire pour les brouillards de gouttelettes. *C. R. Math. Acad. Sci. Paris*, 336(5):447–452, 2003.
- [DR99] K. Domelevo and J.-M. Roquejoffre. Existence and stability of travelling wave solutions in a kinetic model of two-phase flows. *Comm. Partial Differential Equations*, 24(1-2):61–108, 1999.
- [Duf05] G. Dufour. *Modélisation multi-fluide eulérienne pour les écoulements diphasiques à inclusions dispersées*. PhD thesis, Université Paul Sabatier Toulouse III, 2005.
- [DV05] K. Domelevo and P. Villedieu. A hierarchy of models for turbulent dispersed two-phase flows derived from a kinetic equation for the joint particle-gas pdf. *Phys. Fluid*, 2005.
- [FM05] N. Fournier and S. Mischler. A spatially homogeneous Boltzmann equation for elastic, inelastic and coalescing collisions. *J. Math. Pures Appl. (9)*, 84(9):1173–1234, 2005.
- [GHS04] T. Gallouët, J.-M. Hérard, and N. Seguin. Numerical modeling of two-phase flows using the two-fluid two-pressure approach. *Math. Models Methods Appl. Sci.*, 14(5):663–700, 2004.

- [GPV04] I. M. Gamba, V. Panferov, and C. Villani. On the Boltzmann equation for diffusively excited granular media. *Comm. Math. Phys.*, 246(3):503–541, 2004.
- [Gra58] H. Grad. *Principles of the kinetic theory of gases*. Springer, 1958.
- [GSR04] F. Golse and L. Saint-Raymond. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.*, 155(1):81–161, 2004.
- [Guo02] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys.*, 231(3):391–434, 2002.
- [Guo03] Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.*, 169(4):305–353, 2003.
- [Haf83] P.K. Haff. Grain flow as a fluid mechanical phenomenon. *J. Fluid. Mech.*, 134, 1983.
- [Ham98] K. Hamdache. Global existence and large time behavior of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.* 15,1, 1998.
- [HF92] L.P. Hsiang and G.M. Faeth. Near-limit drop deformation and secondary breakup. *International Journal of Multiphase Flow*, 18(5):635–652, 1992.
- [HF95] L.P. Hsiang and G.M. Faeth. Drop deformation and breakup due to shock-wave and steady disturbances. *International Journal of Multiphase Flow*, 21(5):545–560, 1995.
- [HG03] Hyung Ju Hwang and Yan Guo. On the dynamical Rayleigh-Taylor instability. *Arch. Ration. Mech. Anal.*, 167(3):235–253, 2003.
- [Hyl99] J. Hylkema. *Modélisation cinétique et simulation numérique d’un brouillard dense de gouttelettes. Application aux propulseurs à poudre*. PhD thesis, Ecole nationale supérieure de l’aéronautique et de l’espace (Toulouse), 1999.
- [IS84] R. Illner and M. Shinbrot. The Boltzmann equation: global existence for a rare gas in an infinite vacuum. *Comm. Math. Phys.*, 95(2):217–226, 1984.
- [JUL92] Y.J. Jiang, A. Umemura, and C.K. Law. An experimental investigation on the collision behaviour of hydrocarbon droplets. *J. Fluid. Mech.*, 172, 1992.
- [KMC90] R. I. Klein, C. F. McKee, and P. Colella. In the evolution of interstellar medium. *L. Blitz, Ed., Astronomical Society of the Pacific Conference Series*, 12:117, 1990.
- [KMN79] S. Kawashima, A. Matsumura, and T. Nishida. On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation. *Comm. Math. Phys.*, 70(2):97–124, 1979.
- [Lan48] I. Langmuir. The production of rain by a chain reaction in cumulous clouds at temperatures above freezing. *J. Atm. Sci.*, 5:175–192, 1948.
- [Lau02] F. Laurent. *Modélisation mathématique et numérique de la combustion de brouillards de gouttes polydispersés*. PhD thesis, Université Claude Bernard (Lyon), 2002.

- [Lax73] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.
- [LB03] A. Llor and P. Bailly. A new turbulent two-field concept for modeling Rayleigh-Taylor, Richtmyer-Meshkov, and Kelvin-Helmholtz mixing layers. *Laser and particle beams*, 21-3:305–315, 2003.
- [Lio96] P.-L. Lions. *Mathematical topics in fluid mechanics . vol. 1 , Incompressible models*. Clarendon Press, 1996.
- [Lio98] P.-L. Lions. *Mathematical topics in fluid mechanics . vol. 2 , Compressible models*. Clarendon Press, 1998.
- [LL59] L.D. Landau and E.M. Lifschitz. *Fluid Mechanics*. Pergamon Press, 1959.
- [Llo01] A. Llor. *Modèles hydrodynamiques statistiques pour les écoulements d'instabilités de mélange en régime développé: critères théoriques d'évaluation "0D" et comparaison des approches mono et bifluides*. C.E.A./ DAM, Septembre 2001.
- [Llo05] A. Llor. *Statistical Hydrodynamic Models for developed Mixing Instability flows*. Springer, 2005.
- [LM93] R. Lewandowski and B. Mohammadi. Existence and positivity results for the ϕ - θ and a modified k - ϵ two-equation turbulence models. *Math. Models Methods Appl. Sci.*, 3(2):195–215, 1993.
- [LM01] F. Laurent and M. Massot. Multi-fluid modeling of laminar polydisperse spray flames: origin, assumptions and comparison of sectional and sampling methods. *Combustion theory and modelling*, 5(4):537–572, 2001.
- [LM03] P. Laurençot and S. Mischler. Convergence to equilibrium for the continuous coagulation-fragmentation equation. *Bull. Sci. Math.*, 127(3):179–190, 2003.
- [LMV04] F. Laurent, M. Massot, and P. Villedieu. Eulerian multi-fluid modeling for the numerical simulation of coalescence in polydisperse dense liquid sprays. *J. Comput. Phys.*, 194(2):505–543, 2004.
- [Maj84] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Springer-Verlag, 1984.
- [Mat03] J. Mathiaud. Différents aspects du modèle k-epsilon. Master's thesis, Ecole normale supérieure de Lyon, 2003.
- [Mau04] B. Mauroy. *Hydrodynamique dans les poumons, relations entre flux et géométries*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2004.
- [MM06] S. Mischler and C. Mouhot. Cooling process for inelastic Boltzmann equations for hard spheres, part II: Self-similar solutions and tail behavior. *J. Statis. Phys.*, To appear, 2006.

- [MMR06] S. Mischler, C. Mouhot, and M. Rodriguez Ricard. Cooling process for inelastic Boltzmann equations for hard spheres, part I: The Cauchy problem. *J. Statis. Phys.*, To appear, 2006.
- [MP94] B. Mohammadi and O. Pironneau. *Analysis of the k - ε Turbulence Model*. Masson, 1994.
- [MP97a] J-P. Minier and J. Pozorski. Derivation of a PDF model for turbulent flows based on principles from statistical physics. *Phys. Fluids*, 9(6):1748–1753, 1997.
- [MP97b] S. Mischler and B. Perthame. Boltzmann equation with infinite energy: renormalized solutions and distributional solutions for small initial data and initial data close to a Maxwellian. *SIAM J. Math. Anal.*, 28(5):1015–1027, 1997.
- [MP01] J-P. Minier and E. Peirano. The pdf approach to turbulent polydispersed two-phase flows. *Phys. Rep.*, 352(1-3):1–214, 2001.
- [MV01] M. Massot and P. Villedieu. Modélisation multi-fluide eulérienne pour la simulation de brouillards denses polydispersés. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):869–874, 2001.
- [O’R81] P. O’Rourke. *Collective drop effects on vaporizing liquid sprays*. PhD thesis, Princeton University, 1981.
- [Pop94] S.B Pope. Lagrangian pdf methods for turbulent flows. *Annual review of fluid mechanics*, 1994.
- [Ree91] M.W. Reeks. On a kinetic equation for the transport of particles in turbulent flows. *Phys. Fluid A*, 3, 1991.
- [Rov06] J.M. Rovarch. *Un solveur volume fini 3D pour les écoulements diphasiques*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2006.
- [Sai95] L. Sainsaulieu. *Contribution à la modélisation mathématique et numérique des écoulements diphasiques constitués d’un nuage de particules dans un écoulement de gaz*. Habilitation à diriger des recherches. Université Paris VI, 1995.
- [Sch91] Jack Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335, 1991.
- [Ser96a] D. Serre. *Systèmes de lois de conservation I*. Diderot, 1996.
- [Ser96b] D. Serre. *Systèmes de lois de conservation II*. Diderot, 1996.
- [SF00] O. Simonin and P. Fevrier. Statistical and continuum modelling of turbulent reactive particulate flows. part 2: Application of a two-phase second-moment transport model for prediction of turbulent gas-particle flows. *Theoretical and Experimental Modeling of Particulate Flows, Lecture Series 2000-06, Von Karman Institute for Fluid Dynamics, Rhode Saint Genèse (Belgium)*., 2000.
- [Sim00] O. Simonin. Statistical and continuum modelling of turbulent reactive particulate flows. part 1: Theoretical derivation of dispersed eulerian modelling from probability density function kinetic equation. *Theoretical and Experimental Modeling of Particulate Flows, Lecture Series 2000-06, Von Karman Institute for Fluid Dynamics, Rhode Saint Genèse (Belgium)*., 2000.

-
- [SN00] N.R. Smirnov and V.F. Nikitin. Ignition and combustion of turbulized dust-air mixtures. *Combustion and flame*, 123(3), October 2000.
- [Tam85] Y. Tambour. A lagrangian sectional approach for simulating droplet size distribution of vaporizing fuel in a turbulent jet. *Combustion and flame*, 60:15–28, 1985.
- [UA82] S. Ukai and K. Asano. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publ. Res. Inst. Math. Sci.*, 18(2):477–519 (57–99), 1982.
- [Van83] D. Vandromme. *Contribution à la modélisation et à la prédiction d'écoulements turbulents à masse volumique variable*. PhD thesis, University of Lille, 1983.
- [Ver96] O. Vermorel. *Etude numérique et modélisation de la modulation de la turbulence dans un écoulement de nappe chargée en particules*. PhD thesis, Ecole Polytechnique (Palaiseau), 1996.
- [VH97] P. Villedieu and J. Hylkema. Une méthode particulière aléatoire reposant sur une équation cinétique pour la simulation numérique des sprays denses de gouttelettes liquides. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(3):323–328, 1997.
- [Vil02] C. Villani. *A review of mathematical topics in collisional kinetic theory*. Elsevier Science, 2002.
- [VS04] P. Villedieu and O. Simonin. Modeling of coalescence in turbulent gas-droplet flows. *Commun. Math. Sci.*, 2(suppl. 1):13–33, 2004.
- [Wil85] F.A. Williams. *Combustion theory*. Benjamin Cummings, 1985.
- [WO03] K. Willis and M. Orme. Binary droplet collisions in vacuum environment: an experimental investigation of the role of viscosity. *Exp. Fluids*, 34:28–41, 2003.

1. HYDRODYNAMIC LIMIT IN SPRAYS' CONTEXT

Aim

The aim of this chapter is to study the limit of gas-particle models when collisions are significant (for thick sprays for instance) so that particles may be considered as a fluid.

We establish the equations for this fluid of particles and end up with an Eulerian-Eulerian model, typical of two-phase flows models, consisting in a system of six or seven coupled equations (mass, momentum and total energy conservation for each phase).

What is at stake in this chapter is a better understanding of the transition from gas-particle (Lagrangian-Eulerian) models to two-phase flow (Eulerian-Eulerian) models in order to implement a numerical method which would allow to use simultaneously gas-particle and two-phase flows models in different areas of the same numerical grid or at different times of the computation. Indeed, the volume fraction of particles in a mesh cell may become large enough at some place (or some time) to justify the use of the two-phase flows model, while elsewhere (or at some other time) it is not the case.

In this chapter, all computations are done at a formal level. Two-phase flows equations are obtained by taking the first moments of the p.d.f. (particle density function) but the system which is obtained is not closed. In the limit when the number of collisions goes to infinity, the p.d.f. becomes a Maxwellian or a Dirac mass in velocity and a Dirac mass in internal energy, so that it is possible to close the limit system explicitly. This limit is also studied from the numerical viewpoint.

We recall that we only take into account **monodispersed sprays** so that phenomena like coalescence, break-up or abrasion are not considered in this work, especially as it depends thoroughly on the applications which are studied. We refer to [Bar04a] and [Duf05] for more information on these subjects.

The interesting point of this work is that we find a posteriori the form of the p.d.f., using the hydrodynamic limit, whereas in many works it is given a priori to obtain semi-kinetic models such as multi-fluid models ([Duf05]) and two-phase flow models ([Sai95]). It is mainly due to the fact that we consider monodispersed sprays.

When one deals with polydispersed sprays, the difficulty lies in the fact that the distribution in radii must be known. Moreover equilibria seem out of reach in this situation, since the phenomena of coalescence and break-up do not counterbalance each other. Few results exist on this topic except in some specific models in which break-up produces only two droplets from one droplet ([LM03, FM05]), but these models are far away from the complex applications which are considered here.

Outline of the chapter

- We first write down the equations of a monodispersed spray of incompressible droplets.
- Then, we model the collision mechanism.
- We define the macroscopic physical quantities linked to the particles and write down the corresponding non closed system of equations.
- The different terms appearing in the equation for the p.d.f. are then studied from the point of view of dimensional analysis, so that we obtain constraints on the typical values of the parameters of the spray for which the Hilbert expansion that we propose is justified.
- The transition towards an Eulerian-Eulerian model is then performed.
- We check numerically that our hypotheses are in accordance with a "real" case on a tin-air mixture used in a code of the CEA-DAM.
- Finally, the compressible extension of our model is studied (that is, a case when the droplets are compressible).

Notations

Quantities associated with the gas

- $\rho_g := \rho_g(t, x)$: density of the gas,
- $u_g := u_g(t, x)$: velocity of the gas,
- $e_g := e_g(t, x)$: internal energy of the gas per mass unit,
- $T_g := T_g(t, x)$: temperature of the gas,
- $E_g := E_g(t, x) = e_g(t, x) + \frac{1}{2}u_g^2(t, x)$: total energy of the gas per mass unit,
- $p := p(t, x)$: pressure,
- $\alpha := \alpha(t, x)$: gas volume fraction,
- λ : heat conductivity coefficient of the gas,
- C : gas specific heat at constant pressure,
- η : gas dynamic viscosity,
- $Pr = \frac{\eta C}{\lambda}$: Prandtl Number (supposed constant here),
- Re_{gas} Reynolds number of the gas.

Quantities associated with the particles

- u_p : velocity of a particle,
- e_p : internal energy of a particle,
- $f := f(t, x, u_p, e_p)$ (p.d.f.): number density of particles of internal energy e_p and velocity u_p located in x at time t ,
- $Q(f, f) := Q(f, f)(t, x, u_p, e_p)$: Boltzmann's collision operator,
- T_p : temperature of a particle,
- m_p : mass of a particle (all the particles are assumed to have the same mass since we deal with monodispersed sprays),
- r : radius of a particle (a constant except in the section 1.7 on compressible droplets),
- ρ_p : density of a particle, which is a constant (except in the last section of the chapter, where it depends on pressure and internal energy, when we study compressible particles).
One has to note that in all cases $m_p = \frac{4}{3}\pi\rho_p r^3$.
- Γ : variation of u_p in the phase space,

- ϕ : variation of e_p in the phase space,
- μ_p : dynamic viscosity of the particles,
- Pr_p : Prandtl number of the particles,
- C_p : specific heat of the particles,
- λ_p : heat conductivity coefficient of the particles,
- σ_T : surface tension of the particles,
- $1 - \alpha := (1 - \alpha)(t, x) = \iint_{u_p, e_p} f \frac{4}{3} \pi r^3 du_p de_p$: particles volume fraction,
- $\rho := \rho(t, x) = \frac{1}{1 - \alpha} \iint_{u_p, e_p} f m_p du_p de_p$: density of the fluid of particles,
- $v := v(t, x) = \frac{1}{(1 - \alpha)\rho} \iint_{u_p, e_p} f m_p u_p du_p de_p$: velocity of the fluid of particles ,
- $P' := P'(t, x) = \frac{1}{1 - \alpha} \iint_{u_p, e_p} f m_p (v - u_p) \otimes (v - u_p) du_p de_p$: Reynolds stress tensor ,
- $T = \frac{1}{3} \frac{tr(P')}{\rho}$: kinetic temperature per mass unit,
- $q := q(t, x) = \frac{1}{1 - \alpha} \iint_{u_p, e_p} f m_p (v - u_p)^2 (u_p - v) du_p de_p$: heat flux ,
- $e_c := e_c(t, x) = \frac{1}{(1 - \alpha)\rho} \iint_{u_p, e_p} \frac{1}{2} f m_p u_p^2 du_p de_p$: microscopic kinetic energy of the fluid of particles per mass unit,
- $e := e(t, x) = \frac{1}{(1 - \alpha)\rho} \iint_{u_p, e_p} f m_p e_p du_p de_p$: microscopic internal energy of the fluid of particles per mass unit,
- $E_p := E_p(t, x) = e(t, x) + e_c(t, x)$: total energy of the particles fluid per mass unit.

Physical constants and numbers associated with the gas and particles

- $Re = \frac{2r\rho_g |u_p - u_g|}{\eta}$: Reynolds number associated with the difference of velocity between the two fluids ,

- Nu : Nusselt number (see [Wil85]) which depends on Re and Pr ,

$$Nu = 1 + 0.3Re^{\frac{1}{2}}Pr^{\frac{1}{3}},$$

We recall that Pr is the Prandtl number of the gas which was defined above.

- $D_p = \frac{1}{2}\rho_g\pi r^2 C_d |u_p - u_g|$,

C_d : drag coefficient, the drag force expression is $D_p(u_p - u_g)$,

$C_d = \frac{24}{Re}$ (see [CGW78, O'R81, MR83, Ces97] for a more realistic law for C_d).

1.1 Equations of the spray

We write down a system of typical equations for thick sprays (cf. [O'R81], [AOB89] and the presentation of equations (0.28)-(0.37) in the introduction) of incompressible droplets having the same radius r (monodispersed sprays). This model is fully described in [BDM03]. Note that (as explained in the introduction) the main difference with thin sprays is that one cannot neglect the volume fraction $1 - \alpha$ of the particles (see [Bar04b]). The set of equations is the following:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.1)$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = - \iint_{u_p, e_p} m_p \Gamma f du_p de_p, \quad (1.2)$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t \alpha = \\ \iint_{u_p, e_p} - \left(m_p \Gamma + \frac{m_p}{\rho_p} \nabla_x p \right) \cdot u_p f du_p de_p - \iint_{u_p, e_p} m_p \phi f du_p de_p, \end{aligned} \quad (1.3)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p} (f\phi) = Q(f, f), \quad (1.4)$$

$$\alpha = 1 - \frac{4}{3} \pi r^3 \iint_{u_p, e_p} f du_p de_p, \quad (1.5)$$

$$m_p \Gamma = - \frac{m_p}{\rho_p} \nabla_x p - D_p(u_p - u_g), \quad (1.6)$$

$$m_p \phi = 4\pi r \lambda N u(T_g - T_p), \quad (1.7)$$

$$e_g = E_g - \frac{1}{2} u_g^2, \quad (1.8)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.9)$$

$$p = P_1(\rho_g, e_g), \quad (1.10)$$

$$T_p = T_2(e_p). \quad (1.11)$$

Equations (1.1)-(1.3) are balance laws for the density $\rho_g := \rho_g(t, x)$, the velocity $u_g := u_g(t, x)$ and the total energy (per mass unit) $E_g := E_g(t, x)$ of the gas, while equation (1.4) is a Vlasov-Boltzmann-type equation for the particle density function $f := f(t, x, u_p, e_p)$ of the particles.

The coupling between (1.1)-(1.3) on one hand and (1.4) on the other hand is made through the volume fraction of the gas ($\alpha \in [0, 1]$), which satisfies eq.(1.5), and through the source terms in (1.2) and (1.3). In these source terms appear the force $m_p \Gamma$ whose expression is given in (1.6), and the term of energy exchange $m_p \phi$ given by (1.7). The whole system is closed by the thermodynamic laws (1.9)-(1.11). One can find a thorough study of the interaction terms in [Duf05].

In our context, the viscosity is negligible in the gas equations (high Reynolds numbers are considered) but not in the drag force term $D_p(u_p - u_g)$ (D_p depends on η as it is explained in the notations).

Such a set of equations is valid only when the characteristic length L of the gas is larger than the typical mean free path σ of the droplets, which in turn is larger than the radius r of a droplet.

Typically, in the applications of the CEA-DAM that we consider, the volume fraction occupied by the particles can be 10%. As a consequence, one must consider the effect of particles collisions. The description of the remaining collision term $Q(f, f)$ is given in next section.

Remark 1.1

There is another way of writing down eq.(1.2) and eq.(1.3) which leads to classic formulas of two-phase flows (cf. [Sai95, Ram00, Bou98, Rov06]),

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha\nabla_x p = \iint_{u_p, e_p} D_p(u_p - u_g) f du_p de_p, \quad (1.12)$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha \\ = \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p - \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p. \end{aligned} \quad (1.13)$$

We also note that the equation for the total energy of the gas can be rewritten as an equation for the internal energy per mass unit e_g (see [BDM03]), namely

$$\begin{aligned} \partial_t(\alpha\rho_g e_g) + \nabla_x \cdot (\alpha\rho_g u_g e_g) + p(\partial_t\alpha + \nabla_x \cdot (\alpha u_g)) = \\ \iint_{u_p, e_p} D_p(u_p - u_g)^2 f du_p de_p - \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p. \end{aligned} \quad (1.14)$$

1.2 Monodispersed kernel of collision $Q(f, f)$

In this part, we describe the collision kernel $Q(f, f)$ appearing in (1.4). We consider situations in which kinetic energy is conserved (elastic collisions) or transferred into internal energy (inelastic collisions).

For more information on elastic collision kernels, we refer to [Bar01] , [Vil02] and while for inelastic collision kernels, we refer to [GPV04], [Vil02] or [SR96] and subsection 1.5.2.

We suppose that during a collision, there is an exchange of internal energy modeled through a probability law ϕ , and that the repartition of the loss of kinetic energy (for inelastic collisions) is modeled through a probability law ψ .

After writing the general form of the kernel, these probability laws will be modeled. Finally, classical mathematical properties of the kernel, which are linked to the conservations of moments, are given.

We write $Q(f, f)$ as:

$$Q(f, f)(t, x, u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p_*} \in \mathbb{R}^3 \\ e_{p_*} \in \mathbb{R}^+, a \in [0,1], b \in [0,1]}} \left(\frac{1}{|1-a|} \frac{1}{\beta^4} 'f_*'f - f_*f \right) \times \\ r^2 |u_p - u_{p_*}^*| \phi(a) da \psi(b) db d\sigma du_{p_*} de_{p_*} \quad (1.15)$$

with (see [Bar01]),

- σ : vector in \mathbb{S}^2 ,
- $'u_p = \frac{u_p + u_{p_*}}{2} + \frac{1}{\beta} \frac{|u_p - u_{p_*}|}{2} \sigma$: pre-collisional velocity,
- $'u_{p_*} = \frac{u_p + u_{p_*}}{2} - \frac{1}{\beta} \frac{|u_p - u_{p_*}|}{2} \sigma$: pre-collisional velocity,
- $'e_p = \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p_*} - \frac{a-2b}{2-2a} \Delta' E_c$: pre-collisional internal energy,
- $'e_{p_*} = -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p_*} - \frac{a-2+2b}{2-2a} \Delta' E_c$: pre-collisional internal energy,
- $\Delta' E_c = \frac{1}{2} ('u_p^2 + 'u_{p_*}^2 - u_{p_*}^2 - u_p^2) = \left(\frac{1-\beta^2}{4\beta^2} \right) (u_p - u_{p_*})^2$: loss of kinetic energy (gain of internal energy),
- β : parameter of inelasticity ,
- a : parameter of exchange of internal energy ($0 \leq a < 1$) between the two particles ,
- b : parameter of transfer from kinetic energy to internal energy for each particle ($0 \leq b \leq 1$) during the collision ,

- ϕ and ψ are two probability laws,
- $'f_* = f(t, x, 'u_{p_*}, 'e_{p_*})$,
- $'f = f(t, x, 'u_p, 'e_p)$,
- $f_* = f(t, x, u_{p_*}, e_{p_*})$,
- $f = f(t, x, u_p, e_p)$.

Further explanations on the exchanges between internal energy and kinetic energy during the collisions can be found in [Des97]. The model presented here is similar to the one of Borgnakke-Larsen except that the process of exchange of internal energy is not reversible. As a consequence no H -theorem is expected (like the one of [Des97] and [DMS05]).

We have chosen a model of inelasticity in which kinetic energy is lost in all directions. It is also possible to consider a loss of kinetic energy in the direction of the impact only (see [BCG00]) but it does not lead to a significant change of behavior of equations.

Remark 1.2

As far as we know, the parameter β is really hard to assess since very few experimental results are available. Furthermore, these experiments are rather based on wall particle interactions: somehow a wall particle interaction can be thought of as a collision with a droplet of infinite radius (some models of wall-particle interaction were developed for the Kiva code in [SR96] and [SR98]).

Note that $\frac{1}{|1-a|} \frac{1}{\beta^3}$ is the Jacobian of the pre-collisional transformation

$$(u_p, e_p, u_{p_*}, e_{p_*}) \leftrightarrow ('u_p, 'e_p, 'u_{p_*}, 'e_{p_*})$$

and that $1/\beta$ comes from the hard sphere cross section, so that we obtain $\frac{1}{|1-a|} \frac{1}{\beta^4}$ as a prefactor in (1.15).

The case $\beta = 1$ corresponds to elastic collisions, that is collisions in which kinetic and internal energies are separately conserved.

The characteristic function χ is used to ensure the positivity of pre-collisional internal energies.

1.2.1 Probability laws of exchange of energy

We now explain the process of transfer of energy during the collisions. We obtain a hint of the expression of ϕ and ψ . However we keep general notations (ϕ and ψ) in the sequel.

A collision of droplets may be considered as a coalescence of two particles which finally part away, so that the two droplets remain stuck together for a certain amount of time. While the particles are stuck,

they exchange energy through thermal conduction (convection effects are negligible in our applications) and there is also a loss of kinetic energy due to viscosity.

Since typical collision time is small in our experiments (compared with the characteristic time of the gas), inelasticity and thermal exchange are supposed to be independent, so that ϕ and ψ are two independent probability laws and so that we consider only the elastic case in order to assess a . We shall be looking for typical times of relaxation for the two phenomena taken into account here.

Law of probability of a

The exchange of internal energy is introduced in our model because during collisions, the particles remain stuck together during a certain time. So all along the collision they can exchange some heat; the parameter a assesses the amount of exchanged internal energy. This thermalization of the particles can be thought as a relaxation phenomena.

The time of collision between two particles p (whose velocity is u_p , internal energy e_p and temperature T_p) and p_* (whose velocity is u_{p_*} , internal energy e_{p_*} and temperature T_{p_*}) is generally assessed as $\Delta\tau_{coll} = \frac{2r}{|u_p - u_{p_*}|}$ (see [Cer88, Hy199]).

Because of heat transfers, we get that the amount $4\pi r\lambda_p(T_p - T_{p_*})$ of internal energy is exchanged by unit of time (with λ_p : thermal conductivity) from particle p towards particle p_* . The 4π factor might not be the good geometric factor because one particle does not surround the other during the collision as the gas surrounds a particle, but this is the right order of magnitude. Hence, we get

$$\frac{d}{dt}(e_{p_*} - e_p) = \frac{4\pi r\lambda_p}{m_p}(T_p - T_{p_*}).$$

The variation of internal energy during the collision can be assessed through

$$\frac{d}{dt}(e_{p_*} - e_p) = C_p \frac{d}{dt}(T_{p_*} - T_p),$$

where C_p is the specific heat of the particles. We suppose that C_p is constant.

Finally, we obtain that $T_{p_*} - T_p$ obeys to the following ordinary differential equation:

$$\frac{d}{dt}(T_{p_*} - T_p) = -\frac{4\pi r\lambda_p}{C_p m_p}(T_{p_*} - T_p).$$

Hence,

$$(T_{p_*} - T_p)(t) = \exp\left(-\frac{4\pi r\lambda_p}{C_p m_p}t\right) \times (T_{p_*} - T_p)(0),$$

and

$$(e_{p_*} - e_p)(t) = \exp\left(-\frac{4\pi r\lambda_p}{C_p m_p}t\right) \times (e_{p_*} - e_p)(0).$$

Using the last equality for $t = \Delta\tau_{coll}$, one gets

$$a = 1 - \exp\left(-\frac{4\pi r \lambda_p}{C_p m_p} \frac{2r}{|u_p - u_{p*}|}\right),$$

since $(e_p - e_{p*})(\Delta\tau_{coll}) = (1 - a)(e_p - e_{p*})(0)$ by definition of a .

Therefore, in our computations, we shall take the probability law for a as the Dirac mass of parameter

$$a = 1 - \exp\left(-\frac{4\pi r \lambda_p}{C_p m_p} \frac{2r}{|u_p - u_{p*}|}\right). \quad (1.16)$$

Note that a cannot be equal to zero (the radii of the particles and their thermal conductivity are strictly positive...), so that for each collision there is an exchange of energy. If there were no gas, the particles would adopt after some time a unique internal energy since the process of exchange is made through convex linear combinations. Note also that a is a function of $|u_p - u_{p*}|$.

Finally, for small parameters in the exponential function, a takes a simpler form,

$$\begin{aligned} a &= \frac{4\pi r \lambda_p}{C_p m_p} \frac{2r}{|u_p - u_{p*}|} \\ &= \frac{6\lambda_p}{C_p \rho_p r |u_p - u_{p*}|}. \end{aligned}$$

Nonetheless, formula (1.16) is used in our numerical experiments because it guarantees that $0 \leq a \leq 1$. In fact, the linearized formula could be used when one considers the phenomena of coalescence: great times of collision no longer exist since this large time collision are in fact a coalescences (see [Duf05] and [Bar04a]).

Law of probability of b

It is difficult to really assess b since this parameter is related to the distribution of the gain of energy for both particles due to inelastic collisions. The most reasonable guess is that b must be proportional to the mass of the particles, that is to say,

$$b = \frac{1}{2},$$

for monodispersed sprays.

Hence, we choose for ψ a Dirac mass centered in $b = \frac{1}{2}$.

1.2.2 Estimate of the inelasticity

We assess the effect of inelasticity through some computations using the T.A.B. model (Taylor Analogy Break-up model) used in the Kiva code (see [Bar04a] and [AO89]). The basic idea is to assess viscosity effects through this model because viscosity has been established as the main factor of loss of kinetic energy during collisions between liquid droplets ([JUL92] and [WO03]): the more viscosity increases, the more energy dissipation increases during collisions.

Let us first recall that the T.A.B. model is based on the oscillations of the droplets. If oscillations are important enough, the droplets break up. The main point of the model is the analogy made between a droplet and an oscillator: the energy which is lost during collisions is transferred into oscillation energy through the work of surface tension and viscous efforts. We suppose here that the effect of surface tension is negligible with respect to the viscosity effects (in the applications of the CEA-DAM, this approximation is relevant).

We recall that the distortion of sphericity y obeys to the following ordinary differential equation (see [Bar04a]),

$$\ddot{y} = \frac{2}{3} \frac{\rho_g}{\rho_p} \frac{|u_p - u_g|^2}{r^2} - \frac{8\sigma_T}{\rho_p r^3} y - \frac{10\mu_p}{\rho_p r^2} \dot{y},$$

with

- σ_T : surface tension,
- μ_p : dynamic viscosity of the particles.

When inelasticity effects occur during collisions, the loss of kinetic energy is due to the viscous term since we neglect surface tension effects (see [GBWQ04]). So if we do not take into account the gas, the characteristic time τ_c (viscous time) of evolution of the derivative function of \dot{y} is

$$\tau_c = 1 / \left(\frac{10\mu_p}{\rho_p r^2} \right),$$

since we solve the equation $\ddot{y} = -\frac{10\mu_p}{\rho_p r^2} \dot{y}$.

Remark 1.3

One can note that the characteristic time of surface tension is

$$\tau_{\sigma_T} = \sqrt{\frac{\rho_p r^3}{8\sigma_T}}.$$

Therefore, in order to neglect tension surface work, we must have the following relation:

$$\frac{\Delta\tau_{coll}}{\tau_{\sigma_T}} \ll \frac{\tau_{\sigma_T}}{\tau_c}, \tag{1.17}$$

or equivalently $\frac{\sigma_T}{\mu_p |u_p - u_{p*}|} \ll 1$.

Since oscillatory energy is proportional to \dot{y}^2 , we get that the kinetic energy (in the center of mass) of the two particles $E(t) = \frac{1}{4}(u_p(t) - u_{p^*}(t))^2$ is controlled through an exponential evolution of parameter $\frac{\tau_c}{2}$. Hence,

$$E(t) = E(0) \exp\left(-\frac{2t}{\tau_c}\right).$$

Besides, because of the definition of the parameter β , we know that the value of the loss of kinetic energy during collisions (which last $\Delta\tau_{coll} = \frac{2r}{|u_p - u_{p^*}|}$) is

$$\begin{aligned} \Delta E_c &= \frac{1}{4}(u_{p^*}(\Delta\tau_{coll}) - u_p(\Delta\tau_{coll}))^2 - \frac{1}{4}(u_{p^*}(0) - u_p(0))^2 \\ &= \left(\frac{1 - \beta^2}{4}\right)(u_p(0) - u_{p^*}(0))^2. \end{aligned}$$

Using that ΔE_c is also equal to $E(0) \left(1 - \exp\left(-\frac{2\Delta\tau_{coll}}{\tau_c}\right)\right)$, we get

$$\left(\frac{1 - \beta^2}{4}\right)(u_p(0) - u_{p^*}(0))^2 = \frac{1}{4}(u_p(0) - u_{p^*}(0))^2 \left(1 - \exp\left(-\frac{2\Delta\tau_{coll}}{\tau_c}\right)\right)$$

and

$$\beta = \exp\left(-\frac{\Delta\tau_{coll}}{\tau_c}\right). \quad (1.18)$$

Eventually, since in our context the time τ_c is much larger than the time of collision $\Delta\tau_{coll}$, we are able to assess β :

$$\begin{aligned} \beta &= \exp\left(-\frac{\Delta\tau_{coll}}{\tau_c}\right) \\ &\sim 1 - \frac{\Delta\tau_{coll}}{\tau_c} \\ &\sim 1 - \frac{20\mu_p}{\rho_p r |u_p - u_{p^*}|}. \end{aligned}$$

Of course, we have just found an order of magnitude for β : we do not claim to obtain the exact value. We just want to know whether our regime of collision is elastic or not because the behavior of the cloud of droplets is sensibly different in the two cases. As it was done for a , when it comes to numerical experiments, we use formula (1.18) in order to ensure that $0 \leq \beta \leq 1$, even if as explained before for the a parameter, when coalescences are taken into account the linearized formula should work.

Remark 1.4 (Polydispersed sprays: modifications on a , b and β)

For polydispersed sprays, the formula for a and β are slightly modified because one has to take into account the different masses of the particles. After some computations, we obtain the following expressions for a , b and β ,

$$a = \frac{2m_{p^*}}{m_p + m_{p^*}} \left(1 - \exp \left(-\frac{4\pi\lambda_p \min^2(r_p, r_{p^*})}{C_p |u_p - u_{p^*}|} \frac{m_p + m_{p^*}}{m_p m_{p^*}} \right) \right),$$

$$b = \frac{m_p}{m_p + m_{p^*}},$$

$$\beta = \exp \left(-\frac{20\mu_p}{\rho_p \min(r_p, r_{p^*}) |u_p - u_{p^*}|} \right).$$

We recover the expressions for monodispersed sprays by using that $m_p = m_{p^*}$ (and therefore $r_p = r_{p^*}$) for these sprays.

1.2.3 Properties of the collision kernel

We have (at the formal level) the following mathematical properties for both elastic and inelastic kernels of collision in the sense of distributions (see [Bar04b] and [Des93]):

For all test function Ψ ,

$$\begin{aligned} & \iint_{u_p, e_p} Q(f, f) \Psi(u_p, e_p) du_p de_p & (1.19) \\ &= \iiint \iiint \iiint \iiint_{u_p, e_p, u_{p^*}, e_{p^*}, a, b} f f_* [\Psi' - \Psi] 4\pi r^2 |u_p - u_p^*| \phi(a) da \psi(b) db du_p de_p du_{p^*} de_{p^*} \\ &= \frac{1}{2} \iiint \iiint \iiint \iiint_{u_p, e_p, u_{p^*}, e_{p^*}, a, b} f f_* [\Psi'_* + \Psi' - \Psi_* - \Psi] 4\pi r_p^2 |u_p - u_p^*| \phi(a) da \psi(b) db du_p de_p du_{p^*} de_{p^*}, & (1.20) \end{aligned}$$

with:

- $u'_p = \frac{u_p + u_{p^*}}{2} + \beta \frac{|u_p - u_{p^*}|}{2} \sigma$: post-collisional velocity,
- $u'_{p^*} = \frac{u_p + u_{p^*}}{2} - \beta \frac{|u_p - u_{p^*}|}{2} \sigma$: post-collisional velocity,
- $e'_p = \frac{2-a}{2} e_p + \frac{a}{2} e_{p^*} + b \Delta E'_c$: post-collisional internal energy,
- $e'_{p^*} = \frac{a}{2} e_p + \frac{2-a}{2} e_{p^*} + (1-b) \Delta E'_c$: post-collisional internal energy,

- $\Delta E'_c = \frac{1}{2}(u_{p_*}^2 + u_p^2 - u_p'^2 - u_{p_*}'^2) = \left(\frac{1-\beta^2}{4}\right)(u_p - u_{p_*})^2 > 0$: loss of kinetic energy (gain of internal energy),
- $\Psi = \Psi(t, x, u_p, e_p)$,
- $\Psi_* = \Psi(t, x, u_{p_*}, e_{p_*})$,
- $\Psi' = \Psi(t, x, u_p', e_p')$,
- $\Psi'_* = \Psi(t, x, u_{p_*}', e_{p_*}')$.

Using now the functions $\Psi(u_p, e_p) = m_p$, $\Psi(u_p, e_p) = m_p u_p$ and $\Psi(u_p, e_p) = \frac{1}{2}m_p u_p^2 + m_p e_p$, we get the conservations of mass, momentum and total energy :

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p du_p de_p = 0, \quad (1.21)$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p u_p du_p de_p = 0, \quad (1.22)$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \left[\frac{1}{2}m_p u_p^2 + m_p e_p \right] du_p de_p = 0. \quad (1.23)$$

Moreover, for elastic collisions ($\beta = 1$) -in which both kinetic and internal energy are conserved-, we have

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \left[\frac{1}{2}m_p u_p^2 \right] du_p de_p = 0, \quad (1.24)$$

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) [m_p e_p] du_p de_p = 0. \quad (1.25)$$

The quantities m_p , $m_p u_p$ and $\frac{1}{2}m_p u_p^2 + m_p e_p$, which cancel the collision kernel, are generally called invariants.

1.3 Fluid of particles

We now consider the particles as a fluid which can be characterized through macroscopic quantities. We establish a system of equations for these quantities. This system is not closed.

1.3.1 Notations

We define the following quantities associated with the moments of order zero (mass), one (momentum), two (energy) and three (flux of energy) of the fluid of particles:

Definition 1.1 (Moments of the particle density function)

- $(1 - \alpha)\rho = \iint_{u_p, e_p} f m_p du_p de_p,$
- $(1 - \alpha)\rho_p v = \iint_{u_p, e_p} f m_p u_p du_p de_p,$
- $(1 - \alpha)\rho_p e_c = \iint_{u_p, e_p} \frac{1}{2} f m_p |u_p|^2 du_p de_p,$
- $(1 - \alpha)\rho_p e = \iint_{u_p, e_p} f m_p e_p du_p de_p,$
- $(1 - \alpha)\rho_p E_p = \iint_{u_p, e_p} f \left\{ \frac{1}{2} m_p |u_p|^2 + m_p e_p \right\} du_p de_p,$
- $(1 - \alpha)P' = \iint_{u_p, e_p} f m_p (v - u_p) \otimes (v - u_p) du_p de_p,$
- $T = \frac{1}{3} \frac{tr(P')}{\rho},$
- $(1 - \alpha)q = \iint_{u_p, e_p} f m_p (v - u_p)^2 (u_p - v) du_p de_p,$

where:

- $1 - \alpha$: proportion of volume occupied by the particles
- ρ : density of the fluid of particles. The density ρ is equal to ρ_p since ρ_p is supposed to be constant (the droplets are incompressible).
- v : mean velocity of the fluid of particles

- e_c : its microscopic kinetic energy per mass unit
- e : its microscopic internal energy per mass unit
- $E_p = e + e_c$: total energy of the particles fluid per mass unit
- P' : Reynolds tensor of pressure
- T is an averaged kinetic temperature of the particles fluid
- q : heat flux

The pressure P' will appear in our set of equations of the particles fluid because the fluid of particles does not "see" the same pressure as the gas. This extra term of pressure, sometimes called interfacial pressures, appears in many works concerned with the modeling of two-phase flows -see [Sai95] and [GHS04] for the modeling of this pressure-. This pressure tensor vanishes if all the particles have the same velocity (for a Dirac mass in velocities); q also vanishes in such a case.

1.3.2 Mass and momentum equation for the fluid of particles

We integrate the Boltzmann equation against the invariants $m_p du_p de_p$ (mass conservation) and $m_p u_p du_p de_p$ (momentum conservation). We use the properties (1.21) and (1.22) of the collision kernel. By doing this, we easily obtain the following equations for both quantities of the fluid of particles.

Mass equation:

$$\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v) = 0.$$

Momentum equation:

$$\begin{aligned} \partial_t((1 - \alpha)\rho_p v) + \nabla_x \cdot ((1 - \alpha)\rho_p v \otimes v) + (1 - \alpha)\nabla_x p + \nabla_x \cdot ((1 - \alpha)P') = \\ - \iint_{u_p, e_p} D_p(u_p - u_g) f du_p de_p. \end{aligned}$$

1.3.3 Energy equations for the fluid of particles

We integrate over $\left(m_p e_p + \frac{1}{2} m_p u_p^2\right) du_p de_p$ in order to obtain the energy equation, using the property (1.23).

Microscopic internal energy equation

We begin with the expression of the internal energy equation of the fluid of particles. If we integrate the Boltzmann equation against $m_p e_p du_p de_p$, we get:

$$\partial_t ((1 - \alpha)\rho_p e) + \nabla_x \cdot ((1 - \alpha)\rho_p e v) = \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p + \iint_{u_p, e_p} Q(f, f) m_p e_p du_p de_p$$

thanks to the identity:

$$\begin{aligned} \iint_{u_p, e_p} \partial_{e_p} (f \phi) e_p du_p de_p &= - \iint_{u_p, e_p} f \phi du_p de_p \\ &= - \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p. \end{aligned}$$

Note that there is no pressure term (that is, involving p) in the equation of internal energy: this is due to the hypothesis of incompressibility for the droplets.

Microscopic kinetic energy equation

We now integrate against $\frac{1}{2} m_p u_p^2 du_p de_p$ and get:

$$\begin{aligned} \partial_t ((1 - \alpha)\rho_p e_c) + \nabla_x \cdot ((1 - \alpha)(\rho_p e_c + P')v) + (1 - \alpha)v \cdot \nabla_x p + \nabla_x \cdot ((1 - \alpha)q) = \\ - \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p. \end{aligned} \quad (1.26)$$

Total energy equation

We add the two energy contributions, defining the total energy of the fluid of particles E_p as $e_c + e$ and using (1.23). We get:

$$\begin{aligned} \partial_t ((1 - \alpha)\rho_p E_p) + \nabla_x \cdot ((1 - \alpha)\rho_p E_p v) + (1 - \alpha)v \cdot \nabla_x p + \nabla_x \cdot ((1 - \alpha)(P'v + q)) = \\ - \iint_{u_p, e_p} D_p(u_p - u_g) \cdot f u_p du_p de_p + \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p. \end{aligned} \quad (1.27)$$

Using the mass conservation, one gets

$$\begin{aligned} 0 &= p(\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)v)) \\ &= p\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)pv) - (1-\alpha)v \cdot \nabla_x p. \end{aligned} \quad (1.28)$$

Therefore, we end up with the following total energy equation

$$\begin{aligned} \partial_t((1-\alpha)\rho E_p) + \nabla_x \cdot \left((1-\alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)(P'v + q)) \\ = - \iint_{u_p, \epsilon_p} D_p(u_p - u_g) \cdot u_p f du_p d\epsilon_p + \iint_{u_p, \epsilon_p} 4\pi r \lambda N u (T_g - T_p) f du_p d\epsilon_p. \end{aligned} \quad (1.29)$$

Apart from the P' and q terms and provided that some terms be computed, this equation is exactly the energy equation of the second phase in two-phase flows systems (see [GHS04, Rov06]) - for an incompressible fluid -.

1.3.4 Diphasic system with microscopic terms

We recall there the complete system of formulas obtained up to now.

Conservation of mass:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.30)$$

$$\partial_t((1-\alpha)\rho) + \nabla_x \cdot ((1-\alpha)\rho v) = 0, \quad (1.31)$$

$$(\rho = \rho_p = cte). \quad (1.32)$$

Conservation of momentum:

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha\nabla_x p = - \iint_{u_p, \epsilon_p} D_p(u_g - u_p) f du_p d\epsilon_p, \quad (1.33)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p + \nabla_x((1-\alpha)P') = \\ - \iint_{u_p, \epsilon_p} D_p(u_p - u_g) f du_p d\epsilon_p. \end{aligned} \quad (1.34)$$

Conservation of total energy:

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha \\ = \iint_{u_p, \epsilon_p} D_p(u_p - u_g) \cdot u_p f du_p d\epsilon_p - \iint_{u_p, \epsilon_p} 4\pi r \lambda N u (T_g - T_p) f du_p d\epsilon_p, \end{aligned} \quad (1.35)$$

$$\begin{aligned}
& \partial_t((1-\alpha)\rho E_p) + \nabla_x \cdot \left((1-\alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)(P'v + q)) \\
& = - \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p. \quad (1.36)
\end{aligned}$$

Equations of state

$$p = P_1(\rho_g, e_g), \quad (1.37)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.38)$$

$$E_g = e_g + \frac{1}{2}u_g^2 \quad (1.39)$$

$$T_p = T_2(e_p). \quad (1.40)$$

Finally, in order to get a closed system of equations, it remains to use the predominance of the collision term (through an Hilbert expansion). In the framework of this scaling, the limit f has a given form in (u_p, e_p) , so that P' , q and the source terms can be computed explicitly.

1.4 Dimensional analysis and validity of the Hilbert expansion

We now establish conditions under which an Hilbert expansion is valid for the gas-particle system, that is when the leading term in the Boltzmann equation is the collision term (see [Cer88] and [CC70]). Note that since a quadratic term and a linear term coexist in the equation on the p.d.f., we are sure to obtain a set of parameters for which the Hilbert expansion makes sense.

1.4.1 Parameters of the experiment

We introduce some new parameters in order to be able to compare the different terms of the p.d.f.'s equation:

Parameters:

- t_g : typical time of the experiment,
- L : typical length of the experiment,
- P : typical pressure of the experiment,
- r_p : typical radius of the particles,
- $\rho_p = \frac{m_p}{\frac{4}{3}\pi r_p^3}$: typical density of the particles,
- ρ_g : typical density of the gas,
- N : typical number of particles of the experiment,
- σ : mean free path (see [Cer88] and [CC70] for more details).

We have the following relation between the different characteristic lengths and N (see [Cer88]) in order to define σ :

$$\sigma = \frac{L^3}{r_p^2 N}.$$

- V : typical mean velocity of the particles,
- v_T : typical thermal velocity of the particles (variance of the velocity distribution),
- $P' = \rho_p v_T^2$,
- I_p : characteristic internal energy of the particles per mass unit,
- I_g : characteristic internal energy of the gas per mass unit,
- T_g : typical temperature of the gas,

- T_p : typical temperature of the particles,
- Re : Reynolds number of the particles ($Re = \frac{2\rho_g r_p |u_g - u_p|}{\eta}$)
- $Re_{gas} = \frac{\rho_g L^2}{\eta t_g}$: Reynolds number associated with the gas (we recall that in our context, η can be neglected in the equations of the spray, except in the drag force term: see numerical values of the parameters in section 1.6),
- U_g : typical velocity of the gas for the experiment. Note that U_g satisfies $U_g = \frac{L}{t_g}$.

In order to keep the computations tractable, we shall assume that the typical velocities V (mean velocity) , v_T (thermal velocity) and U_g are of the same order. For this scaling it leads also to the following relation:

$$Re \sim \frac{r}{L} Re_{gas}.$$

In the experiments that are presented later, this condition is satisfied. We also assume the temperature of the gas T_g , the temperature of the particles T_p and the difference of temperature between the gas and the particles $T_g - T_p$ to be of the same order.

We denote by ε the quantity $\varepsilon = \frac{\sigma}{L}$.

This quantity is at the basis of the Hilbert expansion since it is supposed to tend to zero in the hydrodynamic limit (see [Cer88] and [CC70]). This is the Knudsen number (see [Cer88]).

1.4.2 Dimensional analysis

We analyze the five different terms of the Boltzmann equation in order to obtain their characteristic values in our experiments.

First, using the definition of these parameters and using that f has the same order as $\frac{N}{I_p V^3 L^3}$, we get the following orders of magnitude (it is easy to obtain them for the time derivative term, the transport term, the pressure term and the collisional term).

Orders of magnitude:

1. Time derivative term:

$$\partial_t f \sim \frac{N}{I_p V^3 L^3} \frac{1}{t_g}. \quad (1.41a)$$

2. Transport term:

$$\begin{aligned} \nabla_x \cdot (f u_p) &\sim \frac{N}{I_p V^3 L^3} \frac{V}{L} \\ &\sim \partial_t f. \end{aligned} \quad (1.41b)$$

3. Pressure term:

$$\begin{aligned}\nabla_{u_p} \cdot \left(\frac{\nabla_x p}{\rho_p} f \right) &\sim \frac{N}{I_p} \frac{V}{v^3} \frac{P}{L^3} \frac{P}{L P'} \\ &\sim \frac{P}{P'} \partial_t f.\end{aligned}\tag{1.41c}$$

4. Drag force term:

$$\begin{aligned}\nabla_{u_p} \cdot \left(\frac{D_p}{m_p} (u_p - u_g) f \right) &\sim \frac{N}{I_p} \frac{1}{V^3} \frac{1}{L^3} \frac{1}{t} C_d \left(\frac{\rho_g L}{\rho_p r_p} \right) \\ &\sim C_d \left(\frac{\rho_g L}{\rho_p r_p} \right) \partial_t f.\end{aligned}\tag{1.41d}$$

5. Thermal term:

$$\begin{aligned}\partial_{e_p} (f \Phi) &\sim \frac{N}{I_p} \frac{1}{V^3} \frac{1}{L^3} \frac{1}{t} \frac{1}{Pr} \frac{\rho_g}{Re_{gas}} \frac{L}{\rho_p} \frac{I_g}{r_p I_p} \\ &\sim \frac{Nu}{Pr} \frac{\rho_g}{Re_{gas}} \frac{L^2}{\rho_p} \frac{I_g}{r_p^2 I_p} \partial_t f.\end{aligned}\tag{1.41e}$$

6. Collisional term:

$$\begin{aligned}Q(f, f) &\sim r_p^2 v_T I_p V^3 \left(\frac{N}{I_p V^3 L^3} \right)^2 \\ &\quad \text{(we recall that we use a hard sphere cross section)} \\ &\sim \frac{1}{\varepsilon} \partial_t f.\end{aligned}\tag{1.41f}$$

We now explain in detail how the orders of magnitudes for the drag force term and the thermal term are obtained:

Dimensional analysis of drag force and thermal terms:

1. First, in order to get the expression of the drag force, we recall the expression of D_p :

$$D_p = C_d \rho_g r_p^2 |u_p - u_g|.$$

We get that $D_p \sim C_d \rho_g r_p^2 v_T$ and using the simplest expression of C_d ($C_d = \frac{24}{Re}$: cf. [CGW78])

$$\begin{aligned}\nabla_{u_p} \cdot \left(\frac{D_p}{m_p} (u_p - u_g) f \right) &\sim C_d \frac{1}{V} \frac{\rho_g r_p^2 v_T}{m_p} \times \frac{N}{I_p V^3 L^3} v_T, \\ &\sim \frac{N}{I_p} \frac{1}{V^3} \frac{1}{L^3} \frac{1}{t_g} \frac{1}{Re_{gas}} \left(\frac{\rho_g L^2}{\rho_p r_p^2} \right).\end{aligned}$$

2. Concerning the thermal terms, we recall that the expression of the Nusselt number directly depends on Reynolds and Prandtl numbers. The Prandtl number is defined by

$$Pr = \frac{C\eta}{\lambda}.$$

Moreover, since $\phi = \frac{4\pi r}{m_p} \lambda Nu (T_g - T_p)$, we get:

$$\partial_{e_p}(f\phi) \sim \frac{1}{I_p} \frac{N}{I_p V^3 L^3} \frac{r_p}{m_p} Nu \lambda |T_g - T_p|.$$

Using that $Re_{gas} = \frac{\rho_g L^2}{\eta t_g}$, that $m_p = \frac{4}{3} \pi \rho_p r_p^3$ that $I_g \sim CT_g$ and that $(T_g - T_p) \sim T_g$, we eventually obtain:

$$\partial_{e_p}(f\Phi) \sim \frac{N}{I_p V^3 L^3} \frac{1}{t} \frac{Nu}{Pr Re_{gas}} \frac{\rho_g}{\rho_p} \frac{L^2}{r_p^2} \frac{I_g}{I_p}.$$

1.4.3 Dimensionless Boltzmann equation

We now make a change of variables in order to focus on the preponderant terms of the Boltzmann equation.

- $x \hookrightarrow \tilde{x}$ with $\tilde{x} = \frac{x}{L}$,
- $t \hookrightarrow \tilde{t}$ with $\tilde{t} = \frac{t}{t_g}$,
- $u_p \hookrightarrow \tilde{u}_p$ with $\tilde{u}_p = \frac{u_p}{U_g}$,
- $u_g \hookrightarrow \tilde{u}_g$ with $\tilde{u}_g = \frac{u_g}{U_g}$,
- $e_p \hookrightarrow \tilde{e}_p$ with $\tilde{e}_p = \frac{e_p}{I_p}$,
- $r \hookrightarrow \tilde{r}$ with $\tilde{r} = \frac{r}{r_p}$,
- $p \hookrightarrow \tilde{p}$ with $\tilde{p} = \frac{p}{P}$,
- $T_p \hookrightarrow \tilde{T}_p$ with $\tilde{T}_p = \frac{T_p}{T_{gas}}$,
- $T_g \hookrightarrow \tilde{T}_g$ with $\tilde{T}_g = \frac{T_g}{T_{gas}}$,
- $f \hookrightarrow \tilde{f}$ with $\tilde{f}(\tilde{t}, \tilde{x}, \tilde{u}_p, \tilde{e}_p) = \frac{I_p L^3 v_T^3}{N} f(t_g \tilde{t}, L \tilde{x}, v_T \tilde{u}_p, I_p \tilde{e}_p)$.

Non dimensional equation:

The equation satisfied by \tilde{f} is

$$\partial_{\tilde{t}} \tilde{f} + \tilde{u}_p \cdot \nabla_{\tilde{x}} \tilde{f} + \nabla_{\tilde{u}_p} \cdot (\tilde{f} \tilde{\Gamma}) + \partial_{\tilde{e}_p} (\tilde{f} \tilde{\Phi}) = \frac{1}{\varepsilon} Q(\tilde{f}, \tilde{f}), \quad (1.42)$$

where:

- $\tilde{\Gamma} = C_1 \nabla_{\tilde{x}} \tilde{p} + C_2 \frac{\tilde{\rho}_g}{\tilde{\rho}_p \tilde{r}} |\tilde{u}_p - \tilde{u}_g| (\tilde{u}_p - \tilde{u}_g),$
- $\tilde{\Phi} = C_3 \frac{1}{\tilde{\rho} \tilde{r}^2} (\tilde{T}_g - \tilde{T}_p),$
- $C_1 = \frac{P}{P'},$
- $C_2 = \frac{24}{Re_{gas}} \frac{\rho_g L^2}{\rho_p r_p^2},$
- $C_3 = 4\pi \frac{Nu}{Pr Re_{gas}} \frac{L^2 I_g}{r_p^2 I_p}.$

1.4.4 Hilbert approach and meaning of the different approximations

The Hilbert approach consists in studying the limit of the Boltzmann equation when the dominant term in the Boltzmann equation is the collision term. We see that it corresponds to a situation in which:

$$\text{from (1.41a) and (1.41b), } 1 \gg \varepsilon, \quad (1.43a)$$

$$\text{from (1.41c), } \frac{P'}{P} \gg \varepsilon, \quad (1.43b)$$

$$\text{from (1.41d), } Re_{gas} \frac{\rho_p r_p^2}{\rho_g L^2} \gg \varepsilon, \quad (1.43c)$$

$$\text{from (1.41e), } \frac{Pr Re_{gas}}{Nu} \frac{\rho_p I_p}{\rho_g I_g} \left(\frac{r_p}{L} \right)^2 \gg \varepsilon. \quad (1.43d)$$

Inequality (1.43a) means that we are in the regime of small Knudsen number, so that the hydrodynamic limit (that is $\varepsilon \rightarrow 0$) is valid.

1.5 Closure of the system

In order to close the system (1.30)-(1.40), we assume that we are in a situation when the number of collisions is large (this hypothesis is validated with the numerical data we give in section 1.6). Using equation (1.43d), we wish to obtain the limit f of the sequence f^ε satisfying

$$\partial_t f^\varepsilon + u_p \cdot \nabla_x f^\varepsilon + \nabla_{u_p} \cdot (f^\varepsilon \Gamma) + \partial_{e_p} (f^\varepsilon \phi) = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon). \quad (1.44)$$

According to the type of collisions that we consider (elastic or not), we obtain two different hydrodynamic limits.

Before determining the limit note that:

1. we only consider the homogeneous Boltzmann equation $\frac{d}{dt} f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)$ in order to determine the form of the p.d.f. in the limit,
2. we shall give conditions under which such a limit remains relevant when we take the gas into account (that is, terms $u_p \cdot \nabla_x f^\varepsilon$, $\nabla_{u_p} \cdot (f^\varepsilon \Gamma)$ and $\partial_{e_p} (f^\varepsilon \phi)$: see remark 1.7 p.40).

1.5.1 Elastic collisions and Maxwellian distribution

We now suppose that all collisions are elastic (this does not correspond to the realistic data of section 1.6), so that $\beta = 1$ and $Q(f, f)$ is written down as:

$$Q(f, f)(t, x, u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p*} \in \mathbb{R}^3 \\ e_{p*} \in \mathbb{R}^+}} \left(\frac{1}{|1-a|} {}'f_* {}'f - f_* f \right) \chi_{e_p, e_p^* \geq 0}(e_p, e_p^*) r^2 |u_p - u_{p*}| d\sigma du_{p*} de_{p*}$$

with,

- σ : vector in \mathbb{S}^2 ,
- ${}'u_p = \frac{u_p + u_{p*}}{2} + \frac{|u_p - u_{p*}|}{2} \sigma$: pre-collisional velocity,
- ${}'u_{p*} = \frac{u_p + u_{p*}}{2} - \frac{|u_p - u_{p*}|}{2} \sigma$: pre-collisional velocity,
- ${}'e_p = \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p*}$: pre-collisional internal energy,
- ${}'e_{p*} = -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p*}$: pre-collisional internal energy,

- a : parameter of exchange of internal energy ($0 \leq a < 1$) between the two particles,
- $'f_* = f(t, x, 'u_{p_*}, 'e_{p_*})$,
- $'f = f(t, x, 'u_p, 'e_p)$,
- $f_* = f(t, x, u_{p_*}, e_{p_*})$,
- $f = f(t, x, u_p, e_p)$.

We recall that both kinetic and internal energy are conserved during the collisions.

Formal limit of the Boltzmann equation

For the moment, we do not take into account the gas in the Boltzmann equation. We let the Knudsen number ε tend to zero in equation (1.44): in this limit $Q(f^\varepsilon, f^\varepsilon) = O(\varepsilon)$, so that the limit f of f^ε must cancel the collision kernel. Formally f takes the form of a Maxwellian distribution in velocities thanks to Boltzmann's H-theorem (see [Cer88] and [CC70]).

For all convex function Ψ , one can note that using the post-collisional form of the kernel, one gets

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \Psi(e_p) du_p de_p \leq 0, \quad (1.45)$$

since post-collisional internal energies are convex linear combinations of internal energies.

More precisely we have the following property in the homogeneous case (in space):

Lemma 1.1 (elastic H-theorem)

If $f(u_p, e_p)$ is a positive measure with compact support in e_p and the thermal exchange parameter a satisfies $0 < a(|u_p - u_{p_*}|) \leq 1$, then we have the following equivalent properties:

1. There exists a strictly convex function Ψ in $C^\infty(\mathbb{R}^+)$ such that

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \Psi(e_p) du_p de_p = 0, \quad (1.46)$$

and the following inequality holds:

$$\iint_{u_p, e_p} Q(f, f)(u_p, e_p) \log(f) du_p de_p = 0; \quad (1.47)$$

2. $Q(f, f) = 0$;

3. There exists Z, T, v and e such that

$$f(u_p, e_p) = \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p), \quad (1.48)$$

where

- Z represents the number of particles
- v is the mean velocity of the particles
- e is the mean internal energy of the particles
- T is an averaged kinetic temperature of the particles.

Proof of Lemma 1.1:

We clearly have that (3) \Rightarrow (2) \Rightarrow (1). There remains to prove that (1) \Rightarrow (3).

Let f be a positive measure with compact support in e_p which satisfies relations (1.46) and (1.47). $f \otimes f$ is also a positive measure with compact support in e_p and e_{p^*} . Furthermore, the support of $f \otimes f$ is linked to the support of f through (see [Hor90] or [Sch66])

$$\text{supp}(f \otimes f) = \text{supp}(f) \otimes \text{supp}(f). \quad (1.49)$$

Let Ψ be the strictly convex function in $\mathcal{C}^\infty(\mathbb{R}^+)$ which satisfies (1.46), that is,

$$\langle Q(f, f)(u_p, e_p), \Psi(e_p) \rangle = 0.$$

Using the properties of the collision kernel (see subsection 1.2.3), one gets that in the sense of distributions,

$$\langle 4\pi r^2 |u_p - u_{p^*}| f \otimes f, \Psi + \Psi_* - \Psi' - \Psi'_* \rangle = 0.$$

First, one sees that the test function $\Psi + \Psi_* - \Psi' - \Psi'_*$ - that we note $\phi := \phi(e_p, e_{p^*})$ - is strictly positive except when $e_p = e_{p^*}$, since

$$\phi(e_p, e_{p^*}) = \Psi(e_p) + \Psi(e_{p^*}) - \Psi\left(\left(1 - \frac{a}{2}\right)e_p + \frac{a}{2}e_{p^*}\right) + \Psi\left(\left(1 - \frac{a}{2}\right)e_{p^*} + \frac{a}{2}e_p\right).$$

The function Ψ is strictly convex and $\frac{a}{2}$ satisfies $0 < \frac{a}{2} < 1$, so that the following relations are satisfied for $e_p \neq e_{p^*}$:

$$\begin{aligned} \Psi\left(\left(1 - \frac{a}{2}\right)e_p + \frac{a}{2}e_{p^*}\right) &< \left(1 - \frac{a}{2}\right)\Psi(e_p) + \frac{a}{2}\Psi(e_{p^*}), \\ \Psi\left(\left(1 - \frac{a}{2}\right)e_{p^*} + \frac{a}{2}e_p\right) &< \left(1 - \frac{a}{2}\right)\Psi(e_{p^*}) + \frac{a}{2}\Psi(e_p). \end{aligned}$$

We now prove that the support in (e_p, e_{p^*}) is restricted to a single point or empty.

First of all, if the support is empty, then the measure $4\pi r^2 |u_p - u_{p_*}| f \otimes f$ is equal to zero and (1.48) is satisfied using 0 for Z .

We now suppose that the support of the measure is not empty. Let $\chi := \chi(u_p, e_p, u_{p_*}, e_{p_*})$ be a test function in $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$ which vanishes in a neighborhood of the set A :

$$A = \{(u_1, e_1, u_2, e_2) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+ / e_1 = e_2\}.$$

We want to prove that χ cancels the collision kernel.

Since χ vanishes on a neighborhood of A , we can define the function $\varphi := \frac{\chi}{\phi}$. This function is well defined since χ vanishes in a neighborhood of A , whereas ϕ vanishes only in A (we extend φ on A by zero) and belongs to $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$. Furthermore, φ is equal to zero as soon as u_p, e_p, u_{p_*} and e_{p_*} are large enough since χ is compactly supported. Finally, we get by continuity that φ is bounded and that there exists a constant C such that

$$|\chi| \leq C|\phi| = C\phi. \quad (1.50)$$

Using the fact that $4\pi r^2 |u_p - u_{p_*}| f \otimes f$ is a positive measure and inequality (1.50), one gets that χ cancels the collision kernel since ϕ cancels it.

Finally, since this is true for every χ which vanishes in a neighborhood of A , the support of the measure is contained in A .

In order to conclude, we use the property (1.49) of the support of a product measure: if (u_1, e_1) and (u_2, e_2) are in the support of f (which is not empty by hypothesis) then (u_1, e_1, u_2, e_2) is in the support of $f \otimes f$. Moreover, (u_1, e_1, u_2, e_2) must belong to A since the support of $f \otimes f$ is in A so that necessarily $e_1 = e_2$. Eventually, we get that the support in energy is restricted to one point for f since its support is not empty.

As a consequence, since f is a positive measure in e_p whose support in e_p is restricted to a point, we get that f must have the following shape (see [Hor90]):

$$f(u_p, e_p) = \mu(u_p) \otimes \delta_{e_p=e}(e_p),$$

with μ positive measure.

We now find the expression of μ . Since the measure f is a Dirac mass in e_p , the expression of the collision kernel is exactly similar to the formula for the Boltzmann collision kernel of monoatomic rarefied gas in the case of hard spheres. It is well-known that only Maxwellian functions of u_p cancel this kernel (see [Cer88]) since it is equivalent to

$$\langle Q(f, f), \log(f) \rangle = 0.$$

So finally, a measure f which cancels the kernel takes the following shape:

$$f(u_p, e_p) = \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p).$$

This shape also takes into account the case of the degenerate Maxwellian - that is to say a Dirac mass in velocity (when $T = 0$)-, which can occur since μ is a measure.

It concludes the proof of lemma 1.1. ■

In the non homogeneous case, all the quantities Z , T , v and e depend on t and x so that if the collision kernel $Q(f, f)$ vanishes in the whole space and for all time, f takes the following shape:

$$f(t, x, u_p, e_p) = \frac{Z(t, x)}{(2\pi T(t, x))^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v(t, x))^2}{2T(t, x)}\right) \delta_{e_p=e(t, x)}(e_p). \quad (1.51)$$

Remark 1.5

- Note that the condition on $a := a(|u_p - u_{p*}|)$ is satisfied by the model which is proposed in section (1.2).
- For f satisfying (1.51), the tensor P' can be identified to a scalar quantity p' since f is "isotropic" ($f(t, x, v + u_p, e_p) = f(t, x, v + |u_p|\sigma, e_p)$ for all σ in \mathbb{S}^2) with respect to velocity and T depends on p' through the relation $p' = \rho T$. Moreover the third moment q is null.

Moments of the Maxwellian

Considering back the coupling with the gas, the hydrodynamic limit remains valid when the collision term is much larger than the drag force term and the term of exchange of energy in the Boltzmann equation, that is to say conditions (1.43a) to (1.43d) are satisfied. Since it is the case in the experiments of the CEA-DAM, we consider that locally in time and space, f is at equilibrium and consequently takes the following shape:

$$f(t, x, u_p, e_p) = \frac{Z(t, x)}{((2\pi T(t, x)))^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v(t, x))^2}{2T(t, x)}\right) \delta_{e_p=e(t, x)}(e_p).$$

We can now compute the moments and close system (1.30)-(1.40); one can note that the set of independent variables for the fluid of particles is now $(1 - \alpha)\rho$, v , E_p and T . Therefore, we need a set of seven equations in order to close the system.

Moments:

- $(1 - \alpha)\rho = \iint_{u_p, e_p} m_p f du_p de_p = m_p Z,$
- $(1 - \alpha)\rho v = \iint_{u_p, e_p} f m_p u_p du_p de_p,$

- $(1 - \alpha)P' = \iint_{u_p, e_p} f m_p (u_p - v) \otimes (u_p - v) du_p de_p$
 $= (1 - \alpha)\rho T \mathbf{I}$,
- $(1 - \alpha)\rho e_c = \iint_{u_p, e_p} \frac{1}{2} f m_p u_p^2 du_p de_p = \frac{1}{2}(1 - \alpha)\rho v^2 + \frac{3}{2}(1 - \alpha)\rho T$,
- $(1 - \alpha)\rho e = \iint_{u_p, e_p} f m_p e_p du_p de_p$,
- $T_p = T_2(e)$,
- $(1 - \alpha)q = 0$.

As it can be noticed, P' can be identified to a scalar and is directly linked to the quantities e_c and v through

$$\text{Trace}(P') = 2\rho(e_c - \frac{1}{2}v^2),$$

so that we can choose e_c rather than P' as an unknown.

We now write down the expressions of the transfers between the two phases. We obtain integrals with parameters which depend on u_g , v , ρ , $(1 - \alpha)$ and T .

- transfer of momentum:

$$\begin{aligned} & \mathcal{M}(\rho, 1 - \alpha, v, T, u_g) \\ &= \iint_{u_p, e_p} D_p(u_p - u_g) \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p) du_p de_p \\ &= \frac{1}{2}\rho g \pi r^2 \frac{(1 - \alpha)\rho}{m_p (2\pi T)^{\frac{3}{2}}} \int_{u_p} C_d |u_p - u_g| (u_p - u_g) \exp\left(-\frac{(u_p - v)^2}{2T}\right) du_p. \end{aligned} \tag{1.52}$$

- transfer of energy due to drag force :

$$\begin{aligned} & \mathcal{I}(\rho, 1 - \alpha, v, T, u_g) \\ &= \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p \frac{Z}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v)^2}{2T}\right) \delta_{e_p=e}(e_p) du_p de_p \\ &= \frac{1}{2}\rho g \pi r^2 \frac{(1 - \alpha)\rho}{m_p (2\pi T)^{\frac{3}{2}}} \int_{u_p} C_d |u_p - u_g| (u_p - u_g) \cdot u_p \exp\left(-\frac{(u_p - v)^2}{2T}\right) du_p. \end{aligned} \tag{1.53}$$

We recall that C_d is a coefficient which depends on the applications ([CGW78, MR83, Duf05, Ces97]). The most simple law is $C_d = \frac{24}{Re}$ but for high Reynolds number, one has to use a more realistic law depending on the Reynolds number but also on α and some other data ([BDM03]).

System associated with elastic collisions

Thanks to the computations above, and system (1.30)-(1.40), we eventually obtain a closed set of seven equations for seven unknowns.

Conservation of mass:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.54)$$

$$\partial_t((1-\alpha)\rho) + \nabla_x \cdot ((1-\alpha)\rho v) = 0, \quad (1.55)$$

$$(\rho = \rho_p). \quad (1.56)$$

Conservation of momentum:

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = \mathcal{M}(\rho, 1-\alpha, v, T, u_g), \quad (1.57)$$

$$\partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p + \nabla_x((1-\alpha)\rho T) = -\mathcal{M}(\rho, 1-\alpha, v, T, u_g). \quad (1.58)$$

Energy equations:

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \\ \mathcal{I}(\rho, 1-\alpha, v, T, u_g) - 4\pi r \lambda N u (T_g - T_p) \frac{(1-\alpha)\rho}{m_p}, \end{aligned} \quad (1.59)$$

$$\partial_t((1-\alpha)\rho e_c) + \nabla_x \cdot ((1-\alpha)\rho(e_c + T)v) + (1-\alpha)v \cdot \nabla_x p = -\mathcal{I}(\rho, 1-\alpha, v, T, u_g), \quad (1.60)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho e) + \nabla_x \cdot ((1-\alpha)\rho e v) + p(\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)v)) = \\ 4\pi r \lambda N u (T_g - T_p) \frac{(1-\alpha)\rho}{m_p}. \end{aligned} \quad (1.61)$$

Equations of state

$$e_g = E_g - \frac{1}{2}u_g^2, \quad (1.62)$$

$$p = P_1(\rho_g, e_g), \quad (1.63)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.64)$$

$$T_p = T_2(e), \quad (1.65)$$

$$T = \frac{2}{3}(e_c - \frac{1}{2}v^2). \quad (1.66)$$

The system of energy equations can be put under another form: as it was done to obtain the unclosed two-phase flows system, one can obtain the total energy equation of the fluid of particles by summing the two energy equations of the particles. Furthermore one can obtain an equation on the quantity $(1 - \alpha)\rho v^2$ using linear combinations of the momentum and mass equations for the particles, so that, using the equation of e_c , one gets an equation on the thermal temperature since $T = \frac{2}{3}(e_c - v^2)$.

1.5.2 Inelastic collisions

We now study the case of inelastic collisions. From now on, we suppose that $\beta < 1$ (β is fixed) and a is also fixed. We shall explain in remark 1.7 p.40 how we can come back to the realistic case, using the expression found in subsections 1.2.1 (for a) and 1.2.2 (for β).

Inelastic collision kernel

We recall the form of the inelastic collision kernel (see subsection 1.2.3):

$$Q(f, f)(t, x, u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2, u_{p_*} \in \mathbb{R}^3 \\ e_{p_*} \in \mathbb{R}^+}} \left(\frac{1}{|1 - a|} \frac{1}{\beta^4} f_*' f' - f_* f \right) \chi_{e_p, e_p^* \geq 0}(e_p, e_p^*) r^2 |u_p - u_{p_*}| d\sigma du_{p_*} de_{p_*}. \quad (1.67)$$

For inelastic collisions, we have $0 \leq \beta < 1$ and the microscopic kinetic energy e_c decreases when there is no gas; this can be seen by integrating the kinetic equation against $\frac{1}{2}m_p u_p^2$ and by computing the collision kernel:

$$\begin{aligned} \partial_t ((1 - \alpha)\rho e_c) + \nabla_x \cdot ((1 - \alpha)(\rho e_c + \rho T)v) + (1 - \alpha)v \cdot \nabla_x p \\ = - \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p. \end{aligned}$$

In the inelastic case, $\frac{1}{2}m_p u_p^2$ does not cancel the collision kernel (see [GPV04]) as it can be seen below. Thanks to the results of section 1.2.3,

$$\iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p = - \iiint \iiint \frac{1 - \beta^2}{8} f f_* 4\pi r^2 m_p |u_p - u_{p_*}|^3 du_{p_*} de_{p_*} du_p de_p d\sigma. \quad (1.68)$$

The effect of inelastic collisions is to concentrate the velocities of the particles according to (1.68), and except when f is a Dirac mass (in the velocity variable),

$$\iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p < 0.$$

Hence the p.d.f. is due to tend to a Dirac mass when there is no gas (this convergence towards a Dirac mass is proven rigorously in [Vil02], [BGP04] or [FM05]).

In the hydrodynamic limit, ε tends to zero so that the kernel of collision vanishes at the limit. Lemma 1.2 below leads to claim that the distribution of velocities follows a Dirac mass of parameter v (velocity of the fluid of particles). Concerning the distribution of energy, it is more complicated: when f is a Dirac mass in velocity, every distribution of internal energy μ satisfies

$$Q(\delta_{u_p=v}(u_p) \otimes \mu(e_p), \delta_{u_p=v}(u_p) \otimes \mu(e_p)) = 0,$$

since the cross section for hard spheres is null.

In fact, these equilibriums are not stable from a dynamical point of view, except if μ is also a Dirac mass: we are going to "formally" prove that if the initial data is not a Dirac mass in velocity, then there are collisions for all times, and these collisions are numerous enough to entail convergence towards a Dirac mass in internal energy.

More precisely, we have the following lemma for the homogeneous case (in time and space) which is valid for β **fixed** as supposed at the beginning of this section, but also more generally for β **variable** as soon as $0 \leq \beta(|u_p - u_{p*}|) < 1$:

Lemma 1.2 (inelastic static H-theorem)

If f is a positive tempered distribution and the inelasticity parameter β satisfies $0 \leq \beta(|u_p - u_{p*}|) < 1$, then we have the following equivalent properties:

1. $\langle Q(f, f), u_p \hookrightarrow u_p^2 \rangle = 0$ in the sense of distributions;
2. $Q(f, f) = 0$;
3. there exists some positive measure μ and velocity v such that,

$$f(u_p, e_p) = \delta_{u_p=v}(u_p) \otimes \mu(e_p). \quad (1.69)$$

Proof of Lemma 1.2:

We clearly have (3) \Rightarrow (2) \Rightarrow (1). We now prove (1) \Rightarrow (3).

Let f be a positive tempered distribution. Then $4\pi r^2 |u_p - u_{p*}| f \otimes f$ is also a positive tempered distribution.

By hypothesis, $u_p \hookrightarrow u_p^2$ is in $C^\infty(\mathbb{R}^3, \mathbb{R})$ and satisfies (1), that is

$$\langle Q(f, f)(u_p, e_p), u_p \hookrightarrow u_p^2 \rangle = 0,$$

so that

$$0 = \iint_{u_p, e_p} Q(f, f) \frac{1}{2} m_p u_p^2 du_p de_p \quad (1.70)$$

$$= - \iiint \iiint \frac{1 - \beta^2}{8} f f_* 4\pi r^2 m_p |u_p - u_{p_*}|^3 du_p de_p d\sigma du_{p_*} de_{p_*}. \quad (1.71)$$

We prove that the support of f is restricted to one point in the space of velocities.

First of all, when the support is empty, the measure $4\pi r^2 |u_p - u_{p_*}| f \otimes f$ is equal to zero and (1.69) is satisfied using 0 for μ .

We now suppose that the support of the measure is not empty. Let $\chi := \chi(u_p, e_p, u_{p_*}, e_{p_*})$ be a test function in $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$ which vanishes in a neighborhood of the set $A := \{(u_1, e_1, u_2, e_2) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+ / u_1 = u_2\}$.

We want to prove that χ cancels the collision kernel.

Since χ vanishes in a neighborhood of A , we can define the function φ :

$$\varphi := \frac{\chi}{(1 - \beta^2) |u_p - u_{p_*}|^3}.$$

This function is well defined because χ vanishes in a neighborhood of A , whereas ϕ vanishes only in A , and because $0 \leq \beta < 1$ (we extend φ on A by zero) and belongs to $\mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$. Furthermore, φ is equal to zero as soon as u_p, e_p, u_{p_*} and e_{p_*} are large enough since χ is compactly supported. Finally, we get by continuity that φ is bounded and that there exists a constant C such that

$$|\chi| \leq C(1 - \beta^2) |u_p - u_{p_*}|^3. \quad (1.72)$$

Using the fact that $4\pi r^2 |u_p - u_{p_*}| f \otimes f$ is a positive measure (since it is a positive distribution) and inequality (1.72), one gets that χ cancels the collision kernel because of equality (1.70).

Finally, since this is true for every χ which vanishes in a neighborhood of A , the support of the distribution is contained in A .

In order to conclude, we use the property (1.49) of the support of a product measure: if (u_1, e_1) and (u_2, e_2) are in the support of f (which is not empty by hypothesis) then (u_1, e_1, u_2, e_2) is in the support of $f \otimes f$. Moreover (u_1, e_1, u_2, e_2) must belong to A since the support of $f \otimes f$ is in A so that necessarily $u_1 = u_2$. Eventually, we get that the support in velocity is restricted to one point for f since it is not empty.

As a consequence, since f is a positive distribution in u_p whose support in u_p is restricted to a point, we get that f must have the following expression (see [Hor90]):

$$f(u_p, e_p) = \delta_{u_p=v}(u_p) \otimes \mu(e_p),$$

with μ positive tempered distribution.

It ends the proof of lemma 1.2. ■

We now establish a formal result in order to obtain the instability of these equilibria when the distribution in internal energy is not a Dirac mass. We prove that the stable equilibria take the following shape:

$$f(t, u_p, e_p) = G\delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e), \quad (1.73)$$

for v mean velocity of the particles and e mean internal energy of the particles:

$$v(t) = \frac{\iint_{u_p, e_p} f(t, u_p, e_p) m_p u_p du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p},$$

$$e(t) = \frac{\iint_{u_p, e_p} f(t, u_p, e_p) m_p e_p du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p},$$

We first recall Haff's law ([Haf83]): if one considers inelastic collisions ($\beta < 1$) in a spatially homogeneous case, then the distribution f tends to a Dirac mass in velocity of parameter v (mean velocity of the particles), and we have

$$\iint_{u_p, e_p} f(t, u_p, e_p) \frac{1}{2} m_p (u_p - v)^2 du_p de_p \underset{+}{\sim} \frac{1}{t^2}. \quad (1.74)$$

Until recently, only the upper bound was mathematically proved. The complete result has been proved by Mischler and Mouhot using renormalized solutions (in [MM06] and [MMR06]), in order to control the concentration of the measure:

$$\frac{m}{1+t^2} \leq T(t) \leq \frac{M}{1+t^2}, \quad (1.75)$$

for $T(t) = \frac{\iint_{u_p, e_p} f(t, u_p, e_p) \frac{1}{3} m_p (u_p - v)^2 du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p}$, and some m and M depending on initial data.

Remark 1.6

For instance, m and M can easily be computed and are equal when the initial datum is a Maxwellian since the solution of the homogeneous problem remains a Maxwellian in this case (whose parameters evolve in time). The temperature T satisfies the following equality:

$$\frac{d}{dt}T = -\frac{3}{r}(1-\beta^2)\frac{8(1-\alpha)}{3\sqrt{\pi}}T^{\frac{3}{2}}, \quad (1.76)$$

where we recall that $(1-\alpha)$ is the fraction of volume occupied by the particles (see appendix A.2 p.152).

It leads to

$$T(t) = \frac{T(0)}{\left(1 + \frac{3}{r}(1-\beta^2)\frac{4(1-\alpha)}{3\sqrt{\pi}}\sqrt{T(0)}t\right)^2}. \quad (1.77)$$

We now show the convergence in energy when the time goes to infinity in the spatially homogenous Boltzmann equation

$$\partial_t f(t, u_p, e_p) = Q(f, f)(t, u_p, e_p) \quad (1.78)$$

where Q is given by (1.67) with $\beta < 1$.

We define the function g as

$$g(t) = \frac{\iint_{u_p, e_p} f(t, u_p, e_p) m_p (e_p - e(t))^2 du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p}.$$

Some immediate computations lead to the following ODE for g :

$$g'(t) = \frac{\iint_{u_p, e_p} Q(f, f)(t, u_p, e_p) m_p (e_p - e(t))^2 du_p de_p}{\iint_{u_p, e_p} f(t, u_p, e_p) m_p du_p de_p}. \quad (1.79)$$

Using the weak form of the collision kernel, one gets that

$$\begin{aligned} & \iint_{u_p, e_p} Q(f, f)(u_p, e_p) m_p (e_p - e)^2 du_p de_p \\ &= \frac{1}{2} \iiint \iint f f^* [\Psi'_* + \Psi' - \Psi_* - \Psi] 4\pi r^2 |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \end{aligned} \quad (1.80)$$

with $\Psi = m_p(e_p - e)^2$ using the post-collisional notations and the properties of the collision kernel defined in subsection 1.2.3 p.14.

Hence

$$\Psi'_* + \Psi' - \Psi_* - \Psi = m_p \left(-a\left(1 - \frac{a}{2}\right)(e_p - e_{p_*})^2 + \frac{1}{2}\Delta E_c'^2 + \Delta E_c'(e_p + e_{p_*} - 2e)\right), \quad (1.81)$$

so that

$$\begin{aligned}
g'(t) &= \left(-a\left(1 - \frac{a}{2}\right) \iiint \frac{1}{2} f f^* 4\pi r^2 (e_p - e_{p_*})^2 |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \right. \\
&\quad + \frac{1}{2} \iiint 4\pi r^2 f f^* \left(\frac{1}{2} \Delta E_c'^2 + \Delta E_c'(e_p + e_{p_*} - 2e) \right) |u_p - u_{p_*}| du_p du_{p_*} de_p de_{p_*} \left. \right) \\
&\quad \Big/ \iint_{u_p, e_p} f du_p de_p .
\end{aligned} \tag{1.82}$$

From now on, we make the following approximation based on Haff's law: in all computations we replace $|u_p - u_{p_*}|$ by $\sqrt{3T(t)}$ (the 3 comes from the fact that we are in 3D). We still keep equalities in our computations even if it is an approximation. It leads to the following "equality":

$$\begin{aligned}
g'(t) &= \left(-a\left(1 - \frac{a}{2}\right) \sqrt{3T(t)} \iiint \frac{1}{2} f f^* 4\pi r^2 (e_p - e_{p_*})^2 du_p du_{p_*} de_p de_{p_*} \right. \\
&\quad + \iiint 2\pi r^2 f f^* \left(\frac{1}{2} \left(\frac{1 - \beta^2}{4} 3T \right)^2 + \frac{1 - \beta^2}{4} 3T(t) (e_p + e_{p_*} - 2e) \right) \sqrt{3T(t)} du_p du_{p_*} de_p de_{p_*} \left. \right) \\
&\quad \Big/ \iint_{u_p, e_p} f du_p de_p .
\end{aligned} \tag{1.83}$$

Using the definition of $e(t)$, one gets that

$$\begin{aligned}
&\iiint \frac{1}{2} f f^* (e_p - e_{p_*})^2 du_p du_{p_*} de_p de_{p_*} \\
&= \iiint f f^* [(e_p - e(t))^2 + (e(t) - e_{p_*})^2] du_p du_{p_*} de_p de_{p_*} \\
&= 2g(t) \left(\iint_{u_p, e_p} f du_p de_p \right)^2 ,
\end{aligned} \tag{1.84}$$

and that

$$\iiint f f^* (e_p + e_{p_*} - 2e(t)) du_p du_{p_*} de_p de_{p_*} = 0. \tag{1.85}$$

So we end up with

$$g'(t) = (1 - \alpha) \left(-a\left(1 - \frac{a}{2}\right) \sqrt{3} \frac{3}{r} T(t)^{1/2} g(t) + \frac{1}{4} \left(\frac{1 - \beta^2}{4} \right)^2 \frac{3}{r} (3T(t))^{5/2} \right), \tag{1.86}$$

using that $1 - \alpha = \iint_{u_p, e_p} f \frac{4}{3} \pi r^3 du_p de_p$.

We now use the form of the expression found for $T(t)$ when the initial datum is a Maxwellian. This other approximation is valid in the sense that Haff's law tells us that the behavior of $T(t)$ is the same as the one of a Maxwellian when the time goes to infinity. Therefore we write down $T(t)$ under the following

form: $T(t) = \frac{c_1^2}{(1+c_2t)^2}$ where c_1 and c_2 can be fitted with the data of the Maxwellian for instance. We end up with two cases:

- first case: $\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2) \neq 4$

$$g(t) = \frac{g(0)}{(1+c_2t)^{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2)}} + \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{3}c_1)^5/c_2}{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2) - 4} \left[(1+c_2t)^{-4} - (1+c_2t)^{-\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2)} \right]; \quad (1.87)$$

- second case: $\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2) = 4$

$$g(t) = \frac{g(0)}{(1+c_2t)^{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2)}} + \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{3}c_1)^5}{c_2} \frac{\ln(1+c_2t)}{(1+c_2t)^{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2)}}. \quad (1.88)$$

Using the different expressions found for g and supposing that initially $T(0) > 0$, four cases arise for the asymptotic behavior according to the values of a and β (we now use the expression coming from the Maxwellian to replace $T(t)$: $c_1 = \sqrt{T(0)}$ and $c_2 = \frac{3}{r}(1-\beta^2)\frac{4(1-\alpha)}{3\sqrt{\pi}}\sqrt{T(0)}$):

- $4 < \frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2) = \frac{3\sqrt{\pi}\sqrt{3a}(1-a/2)}{4(1-\beta^2)}$:

We get

$$g(t) \underset{+\infty}{\sim} \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{3}c_1)^5/c_2}{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3a}(1-a/2) - 4} \frac{1}{(1+c_2t)^4}.$$

Thermal exchanges are predominant so that $\sqrt{g(t)}$ converges to zero as rapidly as the temperature $T(t)$ (\sqrt{g} has the same dimension as an energy). This case does not exist if collisions are too inelastic because since a is bounded by 1, we need β to satisfy

$$1 - \beta^2 \leq \frac{3\sqrt{3\pi}}{32},$$

in order to obtain a parameter a corresponding to this case.

$$\bullet \quad 4 = \frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3}a(1-a/2) = \frac{3\sqrt{\pi}\sqrt{3}a(1-a/2)}{4(1-\beta^2)}.$$

We get

$$g(t) \underset{+\infty}{\sim} \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{3}c_1)^5}{c_2} \frac{\ln(1+c_2t)}{(1+c_2t)^4}.$$

It is the limit case. The quantity $\sqrt{g(t)}$ converges to zero almost as rapidly as the temperature $T(t)$ (that is, up to a logarithm).

$$\bullet \quad 0 < \frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3}a(1-a/2) = \frac{3\sqrt{\pi}\sqrt{3}a(1-a/2)}{4(1-\beta^2)} < 4:$$

We get

$$g(t) \underset{+\infty}{\sim} \left(g(0) + \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{3}c_1)^5/c_2}{4 - \frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3}a(1-a/2)} \right) \frac{1}{(1+c_2t)^{\frac{3}{r}(1-\alpha)\frac{c_1}{c_2}\sqrt{3}a(1-a/2)}}.$$

In this case, we still have a convergence towards zero of $\sqrt{g(t)}$, but the speed of convergence is slower than the one of the temperature $T(t)$: the more a decreases, the more the speed of convergence towards zero decreases.

$$\bullet \quad a = 0:$$

We get

$$g \underset{+\infty}{\sim} g(0) + \frac{1}{4} \left(\frac{1-\beta^2}{4} \right)^2 \frac{(\sqrt{3}c_1)^5/c_2}{4} > 0.$$

In this case, there is no thermal exchanges, and g does not tend to zero.

Finally in the spatially homogeneous case, as long as $\beta < 1$ and $a > 0$, we have convergence of the p.d.f. towards a Dirac mass with respect to internal energy except in the case $T(0) = 0$ (that is to say, when initially there are no collisions - this equilibrium being unstable as we have seen-). In the next subsection, we shall see that the previous computations are validated at the numerical level.

Remark 1.7 ("Real" case, characteristic time of equilibrium at fixed a and β)

- We no longer consider that a and β are constant.

– First note that the condition on β for static equilibrium ($\beta := \beta(|u_p - u_{p*}|) < 1$) is satisfied by the model which is proposed in subsection 1.2.

- In equality (1.68), one must note that using the expression found for β in subsection 1.2.2 ($1 - \beta^2 = 1 - \exp(-2\Delta\tau_{\text{coll}}/\tau_c)$: we recall that t_c is the typical viscous time and $\Delta\tau_{\text{coll}}$ the typical time of collision) and replacing $|u_p - u_{p*}|$ by \sqrt{T} , one gets the following "equality" for T :

$$\begin{aligned} \frac{d}{dt}T &\sim -\frac{C}{r} \left(1 - \exp\left(-2\frac{\Delta\tau_{\text{coll}}}{\tau_c}\right)\right) (1 - \alpha)T^{\frac{3}{2}} \\ &\sim -\frac{C}{r} \left(1 - \exp\left(-\frac{4r}{\sqrt{3T}\tau_c}\right)\right) (1 - \alpha)T^{\frac{3}{2}}, \end{aligned} \quad (1.89)$$

for some C strictly positive that we do not compute.

When T is large, $\frac{d}{dt}T \sim -\frac{4C}{\tau_c}(1 - \alpha)T$ so that T decays exponentially. When T becomes small enough, T satisfies $\frac{d}{dt}T \sim -\frac{C}{r}(1 - \alpha)T^{\frac{3}{2}}$ and we are back to the situation we have studied with β constant and close to 0 (eq.(1.76) p.37).

Using the same type of computations for a , we get that a tends to 1 as T tends to zero.

Finally, in the "real" case, the decay of g should be of order $\frac{1}{t^{\frac{3\sqrt{\pi}\sqrt{3}a(1-a/2)}{4(1-\beta^2)}}} \sim \frac{1}{t^{\frac{3\sqrt{3}\pi}{8}}}$ for large time.

- Using the physical linearized expressions of a and β obtained in subsections 1.2.1 and 1.2.2, one gets

$$\frac{3\sqrt{\pi}\sqrt{3}a(1-a/2)}{4(1-\beta^2)} \sim \frac{\lambda_p}{C_p\mu_p} = Pr_p.$$

The Prandtl number of the particles Pr_p is a good diagnostic of the situation: high Prandtl numbers lead to a rapid convergence in internal energy towards the Dirac mass, whereas low Prandtl numbers lead to a slow convergence. This is expected since Prandtl number compares thermal effects with viscous effects.

- We now assess the different characteristic times at a and β fixed in the homogeneous case.

- The characteristic time t_v of convergence toward the Dirac mass with respect to velocity is

$$t_v = \frac{r}{(1 - \alpha)(1 - \beta^2)\sqrt{T(0)}},$$

according to our computations (at time t_v , $T(t_v) = \frac{T(0)}{4}$).

Approximating $\sqrt{T(0)}$ by $\frac{2r}{\Delta\tau_{\text{coll}}}$, one gets that the characteristic time is of order

$$\begin{aligned} t_v &\sim \frac{2\Delta\tau_{\text{coll}}}{(1 - \alpha)\left(1 - \exp\left(-2\frac{\Delta\tau_{\text{coll}}}{\tau_c}\right)\right)} \\ &\sim \frac{\tau_c}{1 - \alpha}. \end{aligned} \quad (1.90)$$

The quantity t_v is the viscous time up to the factor $\frac{1}{1-\alpha}$ which takes into account whether there is a lot of collisions or not, since it is the volume occupied by the particles.

- The characteristic time t_e of convergence towards the Dirac mass with respect to internal energy must be computed according to the different cases we have found for the behavior of g . t_e is defined as $g(t_e) = \frac{g(0)}{4}$.

We know that $g \sim \left(\frac{1}{c_2 t}\right)^p$ for some p in all cases, with $0 \leq p \leq 4$.

Some immediate computations lead to $t_e = \frac{4^{1/p} - 1}{4^{1/4} - 1} t_v$.

When g converges as rapidly as T , $p = 4$ so that $t_e \sim t_v$, whereas $t_e \gg t_v$ when p tends to zero (for small thermal exchanges).

- We finally consider the inhomogeneous case (in space)
 - All the quantities G , v and e depend on t and x when we no longer consider the homogenous case so that if the collision kernel vanishes in the whole space and for all time, f takes the following shape,

$$f(t, x, u_p, e_p) = G(t, x) \delta_{u_p=v(t,x)}(u_p) \otimes \delta_{e_p=e(t,x)}(e_p).$$
 - If we take into account the gas, we can perform the hydrodynamic limit in the whole space if the characteristic time of convergence of thermal effects t_e and the one of inelasticity t_v are much smaller than the one of the gas t_g everywhere.

Numerical results on the convergence towards the Dirac mass in internal energy and velocity

We now present some numerical results about the convergence towards a Dirac mass with respect to internal energy and velocity. Here, we consider a spatially homogeneous situation where there is no gas. We use a particle method in order to get the results (cf. [Bar04a, PR04]); the p.d.f. is described at each time by a discrete measure made of numerical particles, which represent a certain number of real particles: here, we suppose that all the numerical particles have the same numerical weight. The collisions are computed using Bird's method ([Bir94]): we assess the number of collisions of the numerical particles during a given time step -by computing the total cross-section- and then choose randomly as many couples of particles as necessary and make them collide as if they were real particles: we will precisely describe later Bird's method when we give the whole algorithm for the spray used in the French Atomic Agency p.52. The code we use was first developed by Baranger ([Bar01]) in \mathbf{C} , in order to compute the T.A.B. model. Here, it is modified in order to take into account internal energies so that inelastic collisions with thermal exchanges can be performed.

The spray is monodispersed ($r = 10^{-4}m$) and initially its distribution is uniform in velocity and internal energy: the three components of the velocity take uniformly their values in $[-10000, 10000]$ whereas the

internal energies are uniformly distributed between $[5 \cdot 10^5, 5 \cdot 10^6]$. We use 10000 numerical particles and the fraction of volume occupied by the particles is equal to 0.1, so that the Knudsen number is equal to 10^{-3} . The initial time step is chosen such that the CFL of the spurious collision method is checked.

Convergence towards the Dirac mass in velocity with β fixed: Haff's law

We now verify Haff's law for $a = 1$ and $\beta = 0.99, 0.95, 0.8$. In order to check the convergence towards the Dirac mass with respect to velocity, we plot the results in logarithm scale: since $T(t) \sim_{\infty} \frac{1}{t^2}$, we must get a line whose main coefficient is equal to -2 . As it can be seen below, Haff's law is well satisfied since $\ln T(t)$ decreases linearly when t goes to infinity. Moreover, the more β decreases, the more rapidly $T(t)$ converges to zero, as expected.

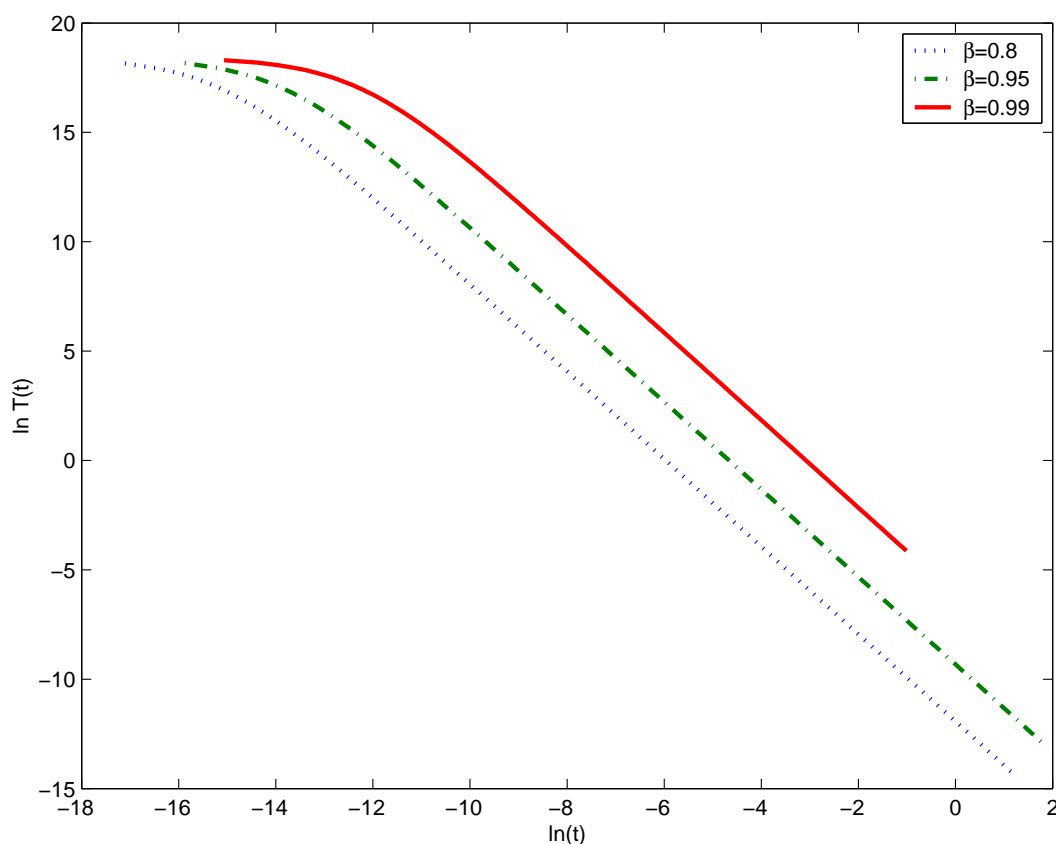


Fig. 1.1: Behavior of kinetic temperature: $\ln T$ as a function of $\ln t$ for different β

Convergence towards the Dirac mass in internal energy in the inelastic case with a and β fixed

We now check the convergence towards the Dirac mass in internal energy. We fix $\beta = 0.99$. Since $1 - \beta^2 \leq \frac{3\sqrt{\pi}}{32}$, we can observe all the types of convergence for g according to the values of a . We plot $\ln\left(\frac{\iint f(t, u_p, e_p)|e_p - e(t)|de_p du_p}{\iint f(t, u_p, e_p)de_p du_p}\right)$. Its behavior as a function of $\ln(t)$ is due to be more or less identical to that of

$$\ln(\sqrt{g(t)}) = \ln\left(\sqrt{\frac{\iint f(t, u_p, e_p)|e_p - e(t)|^2 de_p du_p}{\iint f(t, u_p, e_p)de_p du_p}}\right),$$

since $\iint f(t, u_p, e_p)de_p du_p$ is a constant.

The critical a satisfies: $\frac{3\sqrt{3\pi}\sqrt{3a}(1-a/2)}{4(1-\beta^2)} = 4$, according to our previous computations.

Since $a \leq 1$, the critical a is equal to $1 - \sqrt{1 - \frac{32(1-\beta^2)}{3\sqrt{3\pi}}}$, so that $a = 0.035$ (for $\beta = 0.99$).

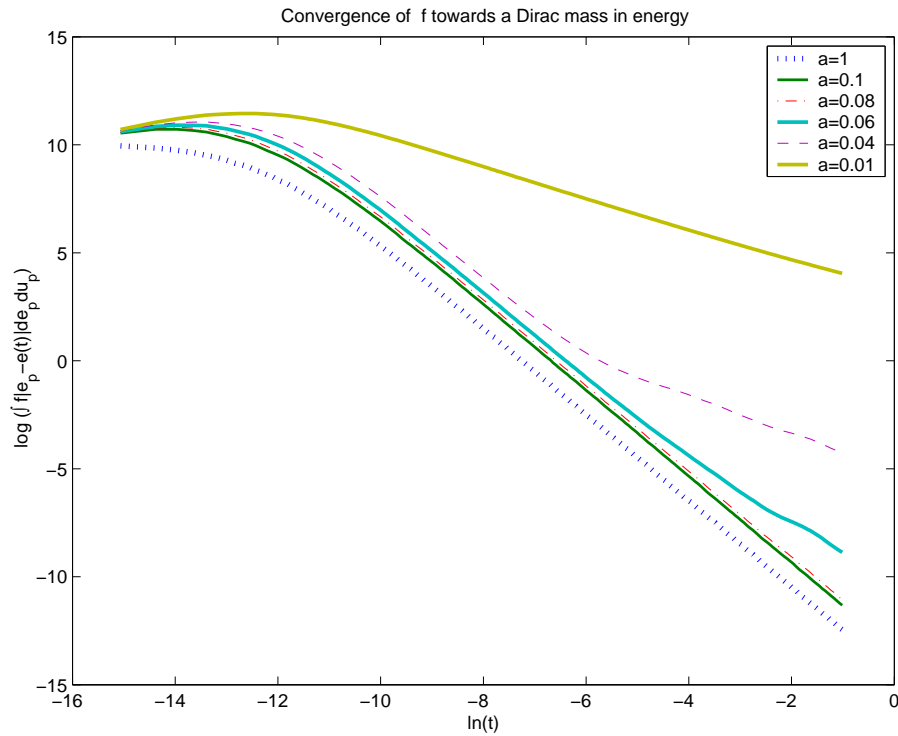


Fig. 1.2: Convergence towards the Dirac mass in internal energy: $\ln\left(\frac{\iint f(t, u_p, e_p)|e_p - e(t)|de_p du_p}{\iint f(t, u_p, e_p)de_p du_p}\right)$ as a function of $\ln t$ for different a

As it can be seen in figure 1.2, the more a increases, the more the coefficients of the lines tend to -2 . The numerical critical value seems to be near 0.06, which corresponds to the order of magnitude of the

theoretical value.

Moreover it can be seen that for small a , the function increases before decreasing and after that the coefficient of the line is smaller than -2 : thermal exchanges are not important enough to completely counterbalance the positive term in equation (1.86) at all times.

One last remark is that near the critical value ($a = 0.04$ and $a = 0.06$), there is a break of the coefficient of the line. At the beginning it is equal to -2 and it increases later: it takes the predicted value only for long times because there are enough collisions before (we recall that in order to obtain the behavior of g we have neglected one term in equation 1.86 -because of equation 1.85 - using Haff's law and that this term cannot be neglected for medium times).

Convergence towards the Dirac masses in internal energy and velocity in the "real" case

We now look at the "real" case: we use the expression of a and β found in subsections 1.2.1 and 1.2.2. All the physical constants of the particles are taken equal to one ($\rho_p, \mu_p, \lambda_p, C_p$).

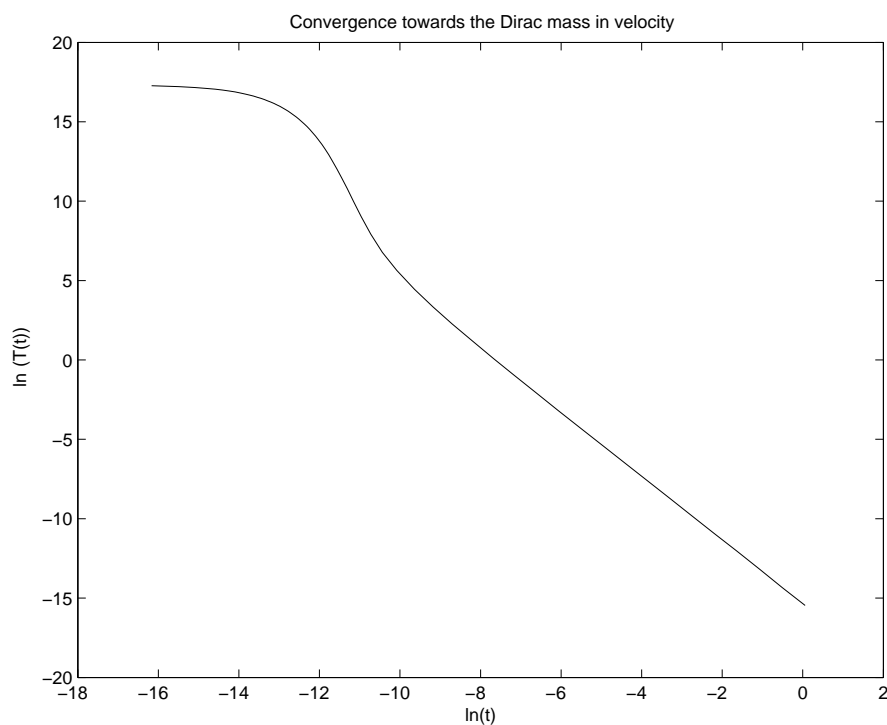


Fig. 1.3: Convergence towards the Dirac mass in velocity: $\ln T$ as a function of $\ln t$

As one can see in figure 1.3, $T(t)$ converges to zero. At the beginning, the convergence is exponential as it was asserted for large T , whereas as soon as T is small enough, the rate of convergence is equivalent

to $\frac{1}{t^2}$.

We now observe the convergence in internal energy for $\sqrt{g(t)}$ in figure 1.4. According to our computations, the main coefficient of the line should be $\frac{3\sqrt{3\pi}}{16} = 0.58$. On the graph we find 0.77. So we get the good order of magnitude. The different probably comes from the fact that we replace $|u_p - u_{p*}|$ by $\sqrt{3T}$: the constant $\sqrt{3}$ is only a rough approximation.

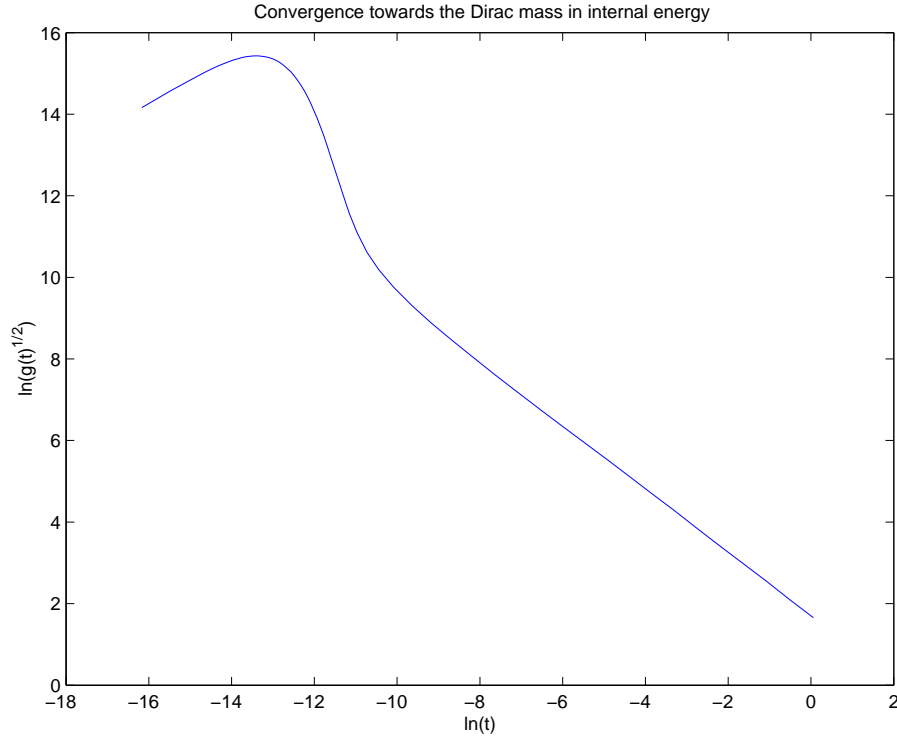


Fig. 1.4: Convergence towards the Dirac mass in internal energy: $\ln \left(\frac{\iint f(t, u_p, e_p) |e_p - e(t)| de_p du_p}{\iint f(t, u_p, e_p) de_p du_p} \right)$ as a function of $\ln t$

Finally in the "real" case, we also observe the convergence towards the Dirac masses in velocity and energy.

Moments of the Dirac distribution

In the inelastic hydrodynamic limit, when ε tends to zero in (1.44), we have formally proved that f^ε

tends to f , where f can be written down as

$$f(t, x, u_p, e_p) = G(t, x) \delta_{u_p=v(t,x)}(u_p) \otimes \delta_{e_p=e(t,x)}(e_p).$$

We now compute the moments of the p.d.f. :

Moments of the p.d.f.

- $(1 - \alpha)\rho = \iint_{u_p, e_p} m_p G \delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p) du_p de_p,$
- $(1 - \alpha)\rho v = \iint_{u_p, e_p} G \delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p) m_p u_p du_p de_p,$
- $(1 - \alpha)P' = \iint_{u_p, e_p} G (\delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p)) m_p (v - u_p) \otimes (v - u_p) du_p de_p = 0,$
- $(1 - \alpha)\rho e_c = \iint_{u_p, e_p} \frac{1}{2} G \delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p) m_p u_p^2 du_p de_p = \frac{1}{2}(1 - \alpha)\rho v^2,$
- $(1 - \alpha)\rho e = \iint_{u_p, e_p} G \delta_{u_p=v}(u_p) \otimes \delta_{e_p=e}(e_p) m_p e_p du_p de_p,$
- $T_p = T_2(e),$
- $(1 - \alpha)q = 0.$

Diphasic system

Using the computations of the moments above, and the system (1.30)-(1.40), we obtain the following set of equations:

Conservation of mass:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.91)$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0, \quad (1.92)$$

$$\rho = \rho_p = cte. \quad (1.93)$$

Conservation of momentum:

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = -\frac{1}{2m_p}(1 - \alpha)\rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v), \quad (1.94)$$

$$\partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p = \frac{1}{2m_p}(1-\alpha)\rho_g\rho\pi r^2 C_d |v - u_g|(u_g - v). \quad (1.95)$$

Conservation of total energy:

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \\ - \frac{1}{2m_p}(1-\alpha)\rho_g\rho\pi r^2 C_d |v - u_g|(u_g - v) \cdot v - 4\pi r \lambda Nu (T_g - T_p) \frac{(1-\alpha)\rho}{m_p}, \end{aligned} \quad (1.96)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho E_p) + \nabla_x \cdot \left((1-\alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) = \\ \frac{1}{2m_p}(1-\alpha)\rho_g\rho\pi r^2 C_d |v - u_g|(u_g - v) \cdot v + 4\pi r \lambda Nu (T_g - T_p) \frac{(1-\alpha)\rho}{m_p}. \end{aligned} \quad (1.97)$$

Equations of state

$$p = P_1(\rho_g, e_g), \quad (1.98)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.99)$$

$$T_p = T_2(e), \quad (1.100)$$

$$e_g = E_g - \frac{1}{2}u_g^2, \quad (1.101)$$

$$e = E_p - \frac{1}{2}v^2. \quad (1.102)$$

Finally the assumption of inelastic collisions leads to a standard one pressure two-phase flow model (see [GHS04, Bou98, Rov06] for more details about two-phase flow models and [Duf05, Lau02] for other ways to obtain this system using the multi-fluid approach). Note that the system is totally closed since we have 6 equations for 6 unknowns.

1.6 Numerical data and results on the gas particle system used at the CEA-DAM

We now give some realistic data used in numerical experiments at the CEA. We show that these data are compatible with our different hypothesis on the kernel of collision and that the Hilbert expansion is also valid for these data. We consider a gas-particle system based on air and droplets of tin, whose characteristics are given below (see [KG05] for tin droplets):

1.6.1 Numerical data

We have the following physical properties at the beginning of the experiment:

- Physical properties of the air :
 - density: $0,616 \text{ kg.m}^{-3}$
 - characteristic length: 5.10^{-3} m
 - characteristic time: 25.10^{-6} s
 - characteristic velocity: 200 m.s^{-1}
 - mean pressure: 10^5 Pa
 - mean temperature: 600 K
 - mean internal energy: 10^4 J.kg^{-1}
 - specific heat: $1000 \text{ J.kg}^{-1}.\text{K}^{-1}$
 - dynamic viscosity: 25.10^{-6} Pa.s
 - thermal conductivity: $45.10^{-3} \text{ W.m}^{-1}.\text{K}^{-1}$

- Physical properties of the tin droplets:
 - number of particles: 10^{10}
 - density: 7310 kg.m^{-3}
 - mean radius: 10^{-6} m
 - mean velocity: 200 m.s^{-1}
 - mean internal energy: 10^6 J.kg^{-1}
 - mean temperature: 600 K
 - specific heat: $200 \text{ J.kg}^{-1}.\text{K}^{-1}$
 - dynamic viscosity: 8.10^{-3} Pa.s
 - surface tension: 0.56 N.m^{-1}
 - thermal conductivity: $60 \text{ W.m}^{-1}.\text{K}^{-1}$
 - fraction of volume occupied by the droplets: 10%

- Non dimensional numbers in our experiments:

- Reynolds number of the gas: 10^5
- Reynolds number of the particles: 10
- Prandtl Number of the gas: 68
- Prandtl Number of the particles: 35
- Nusselt Number: 1.3

1.6.2 Validity of the Hilbert expansion

We now check the validity of inequalities (1.43a)-(1.43d) with our numerical data.

- first, the Knudsen number is equal to 0.006, so that inequality (1.43a) is satisfied.
- since $\frac{P}{P'}$ is equal to 2900, inequality (1.43b) is also satisfied.
- we have that $\frac{24}{Re} \frac{\rho_p}{\rho_g} \frac{r_p}{L} = 2.3$, so inequality (1.43c) is also satisfied.
- finally, as $\frac{Pr Re_{gas}}{Nu} \frac{\rho_p I_p}{\rho_g I_g} \left(\frac{r_p}{L}\right)^2 \sim 10^5$, inequality (1.43d) is also valid.

1.6.3 Study of the collisions

In the next pages, we shall present images made with numerical experiments which were done thanks to a numerical code of the CEA. We give below an image representing the initial situation:

Initially, the gas is at rest and some particles are injected on a thin layer; the geometry which is used is cylindric (in (r, z) coordinates: r is represented in X -coordinate and z in Y -coordinate). Moreover the z -components of the velocities are negative (that is to say, oriented towards the bottom of the figure). All particles have the same temperature.

A splitting is used to perform the computations: a Lagrange-remap method is used for the gas (see [Bar04a], [Lag00] and [Bou00]), whereas a particle method of Bird's type ([Bir94] is used for the droplets, which is close to the one used in the homogeneous code of subsection 1.5.2 (see [PR04] and [Bar04a]). We have implemented inelasticity and thermal effects in the numerical code of the CEA. Only elastic collisions without thermal exchanges were implemented before this work.

More precisely, the numerical method is a finite volume scheme for the equations of the gas (1.1)-(1.3), written in the Eulerian form. This is made in two steps (which come from a splitting of equations (1.1)-(1.3)): the first step consists in solving the equations without the advection terms. During the second step, one solves the convection part of the equations.

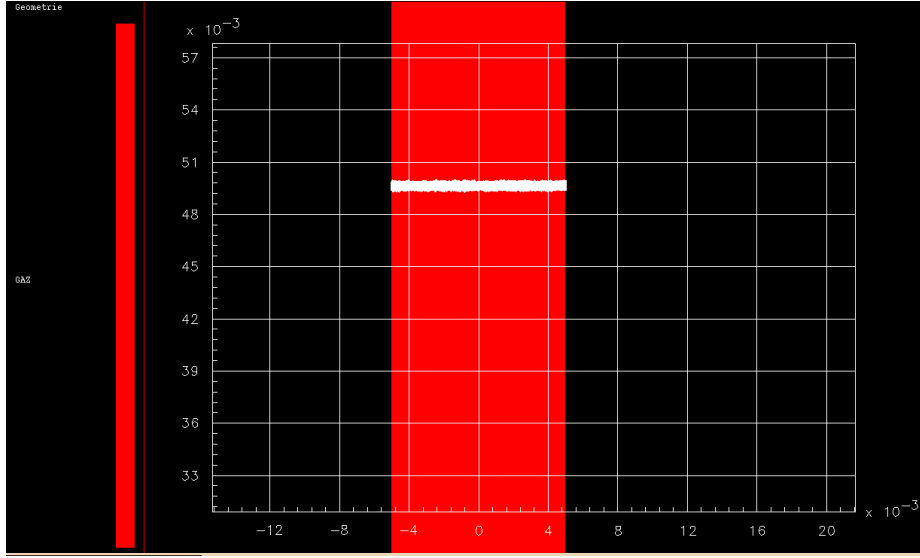


Fig. 1.5: Initial data: cloud of particles at the beginning

First step:

$$\begin{cases} \partial_t(\alpha \rho_g) + \alpha \rho_g \nabla_x \cdot u_g = 0, \\ \alpha \rho_g \partial_t u_g + \alpha \nabla_x p = \iint D_p(u_p - u_g) f du_p de_p, \\ \alpha \rho_g \partial_t e_g + p (\partial_t \alpha + \nabla_x \cdot (\alpha u_g)) = \iint (D_p(u_p - u_g)^2 + 4\pi r \lambda N u (T_p - T_g)) f du_p de_p. \end{cases} \quad (1.103)$$

The computations are made thanks to an explicit prediction-correction type scheme, which is of second order in both time and space. The velocities are computed at the mesh edges of the grid, and the other quantities at the mesh centers. This step is itself divided in 3 phases, that are described in [Mot00].

Second step:

$$\begin{cases} \partial_t(\alpha \rho_g) + u_g \cdot \nabla_x(\alpha \rho_g) = 0, \\ \partial_t u_g + u_g \cdot \nabla_x u_g = 0, \\ \partial_t e_g + u_g \cdot \nabla_x e_g = 0, \\ \alpha = 1 - \iint \frac{4\pi}{3} r^3 f du_p de_p. \end{cases} \quad (1.104)$$

In this step, the gas quantities are given by projection of their previous values on the fixed initial Eulerian grid. This procedure is also of second order in both time and space.

Finally, the Vlasov-Boltzmann equation is computed in two steps. The droplets are represented by numerical particles with same numerical weight, using Bird's method ([Bar04a]). The collisions can occur only between numerical particles in the same cell of the grid.

First step:

The Vlasov equation is solved, using the characteristics:

$$\left\{ \begin{array}{l} \frac{d}{dt}x = u_p, \\ \frac{d}{dt}u_p = \Gamma, \\ \frac{d}{dt}e_p = \phi, \end{array} \right. \quad (1.105)$$

where x is the position of a numerical particle, u_p its velocity and e_p its internal energy.

Second step:

The collision step is solved using the spurious collision method. First, an upper bound \mathcal{B}_{max} is found for the numerical cross section (it is the maximum of all the cross-sections of the particles). Then, $wN^2\mathcal{B}_{max}\frac{\Delta t}{2}$ couples of particles are randomly picked (w is the weight of the numerical particle, N is the number of particles in the cell and Δt is the time step which is chosen such as $\Delta t \ll \frac{1}{wN\mathcal{B}_{max}}$, in order to entail the convergence of the method). One has to note that a particle can be picked several times. These couple of particles collide with probability $\frac{\mathcal{B}}{\mathcal{B}_{max}}$, where \mathcal{B} is their exact cross section ([Bir94, Bar04a]). When a numerical couple (p, p_*) collides, we modify the velocities and internal energies as if they were physical particles, using the following formula:

- $u'_p = \frac{u_p + u_{p_*}}{2} + \beta \frac{|u_p - u_{p_*}|}{2} \sigma,$
- $u'_{p_*} = \frac{u_p + u_{p_*}}{2} - \beta \frac{|u_p - u_{p_*}|}{2} \sigma,$
- $e'_p = \frac{2-a}{2} e_p + \frac{a}{2} e_{p_*} + b \Delta E'_c,$
- $e'_{p_*} = \frac{a}{2} e_p + \frac{2-a}{2} e_{p_*} + (1-b) \Delta E'_c,$
- $\Delta E'_c = \left(\frac{1-\beta^2}{4} \right) (u_p - u_{p_*})^2,$

where σ is picked randomly in the sphere \mathbb{S}^2 , and a , b and β are computed using the formula found in subsections 1.2.1 and 1.2.2.

The time step of the splitting scheme is chosen such that the CFL for solving the gas equations and the CFL for the convergence of the spurious method for collisions is ensured. Here, the time step is fixed

and does not vary all along the experiment and is equal to $2.10^{-7}s$. The CFL is equal to 0.33 for the gas and 0.2 for the spurious collision method.

It must be noted that these experiments were also done for variable time steps (for which the CFL conditions were checked) and that the time step imposed by the CFLs was much larger than the one imposed here, except at the beginning of these experiments, so that the time of computation was divided by four.

The grid is made of squares of length $4.10^{-5}m$ and contains approximately 10^5 cells.

Thermal exchanges and inelasticity: numerical data

We now show that the hypotheses on thermal exchanges and inelasticity are completely fulfilled by our data in our numerical experiments.

Measure of thermal exchange parameter a :

We now assess a whose expression is,

$$a = 1 - \exp\left(-\frac{4\pi r \lambda_p}{C_p m_p} \frac{2r}{|u_p - u_p^*|}\right).$$

Computing this expression, we obtain,

$$a \sim 0.71.$$

So thermal exchanges are important in our experiments.

Measure of inelasticity parameter β :

First, we have to check that we can neglect surface tension effect (see relation (1.17)), that is to say,

$$1.6 \frac{\sigma_T}{\mu_p |u_p - u_{p^*}|} \ll 1.$$

Here the ratio, $1.6 \frac{\sigma_T}{\mu_p |u_p - u_{p^*}|}$ is equal to 0.56 so that the approximation is not completely valid.

We recall that β is equal to

$$\beta = \exp\left(-\frac{\Delta\tau_{coll}}{\tau_c}\right) = \exp\left(-\frac{20\mu_p}{\rho_p r |u_p - u_{p^*}|}\right).$$

Computing the expression above and using that $|u_p - u_{p^*}| \sim 200 m.s^{-1}$ we get

$$\beta \sim 0.90.$$

So β is far from one: the regime of collision of the droplets is very inelastic.

The values of a and β show that the phenomena we have taken into account to model the collision kernel must have an impact on our numerical experiments.

Convergence towards the Dirac mass with respect to velocity and internal energy

We now study the convergence towards the Dirac mass in velocity and internal energy. We have

$$\frac{3\sqrt{\pi} \sqrt{3}a(1 - a/2)}{4(1 - \beta^2)} = 5.35 > 4.$$

Consequently, the speed of convergence towards the Dirac mass in velocity is due to be in $1/t^2$ and the convergence towards the Dirac mass in energy is also in $1/t^2$, according to the computations done in subsection 1.5.2.

Finally we compute the characteristic times of convergence towards the Dirac masses:

- characteristic time of convergence towards the Dirac mass with respect to velocity:

$$t_v = 9.10^{-7} s.$$

- characteristic time of convergence towards the Dirac mass with respect to energy:

$$t_e = 1.2 \cdot 10^{-6} s.$$

These two times are much smaller than the characteristic time of the gas ($t_g = 25 \cdot 10^{-6} s$).

Effect of thermal exchanges on collisions

We recall that initially, the particles have the same temperature. We have taken two pictures of the gas-particle system (figures 1.6 and 1.8) slightly after the beginning of the experiment ($t = 7, 5 \cdot 10^{-7} s$), because the thermalization between the particles occurs on very short times compared with the characteristic time of the gas. Moreover, for long times, there is also a thermalization with the gas which could interfere with the result.

Figure 1.6 was made for elastic collisions without exchanges of internal energy by conduction whereas figure 1.8 was made for elastic collisions with exchanges of internal energy that is,

- $u'_p = \frac{u_p + u_{p_*}}{2} + \frac{|u_p - u_{p_*}|}{2} \sigma,$
- $u'_{p_*} = \frac{u_p + u_{p_*}}{2} - \frac{|u_p - u_{p_*}|}{2} \sigma,$
- $e'_p = \frac{2 - a}{2} e_p + \frac{a}{2} e_{p_*},$

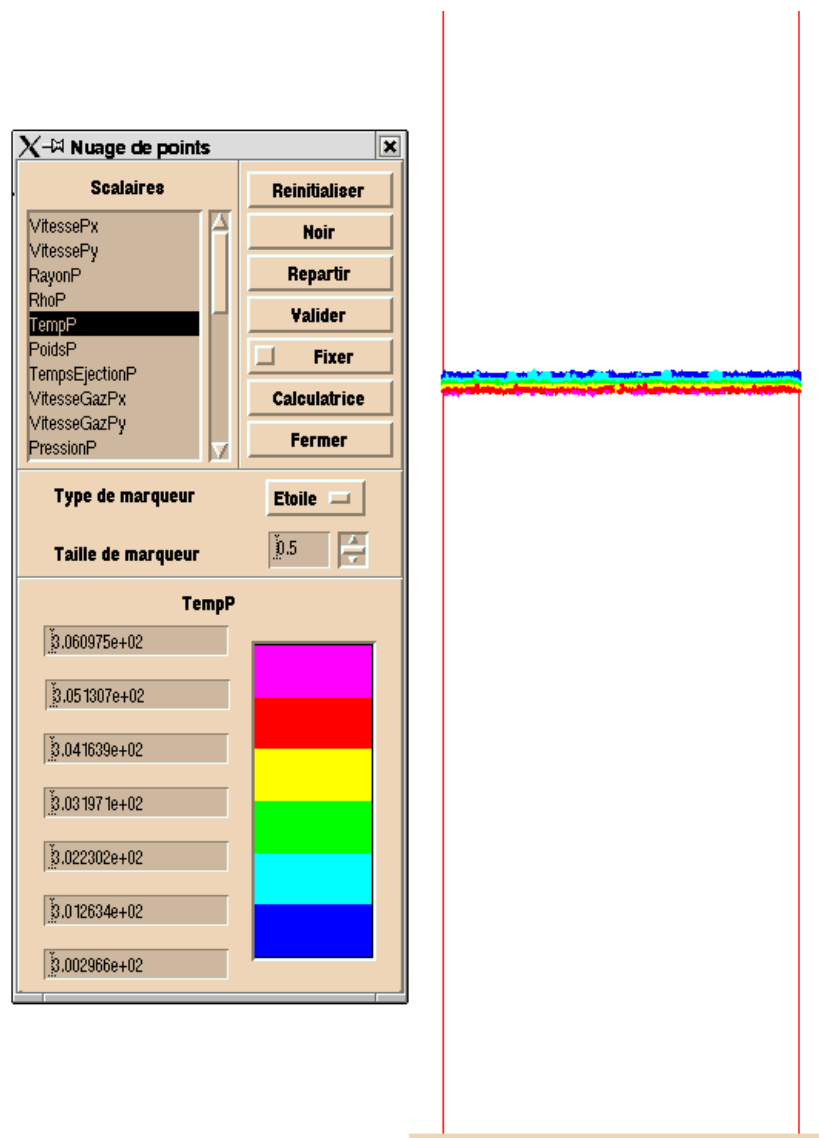


Fig. 1.6: Elastic case without thermal exchange: temperature of the droplets in Celsius

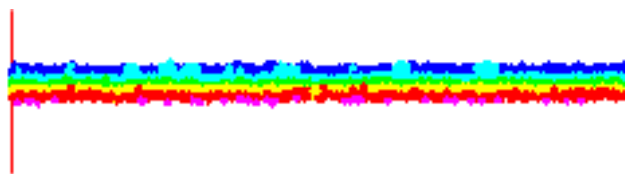


Fig. 1.7: Elastic case without thermal exchange: zoom

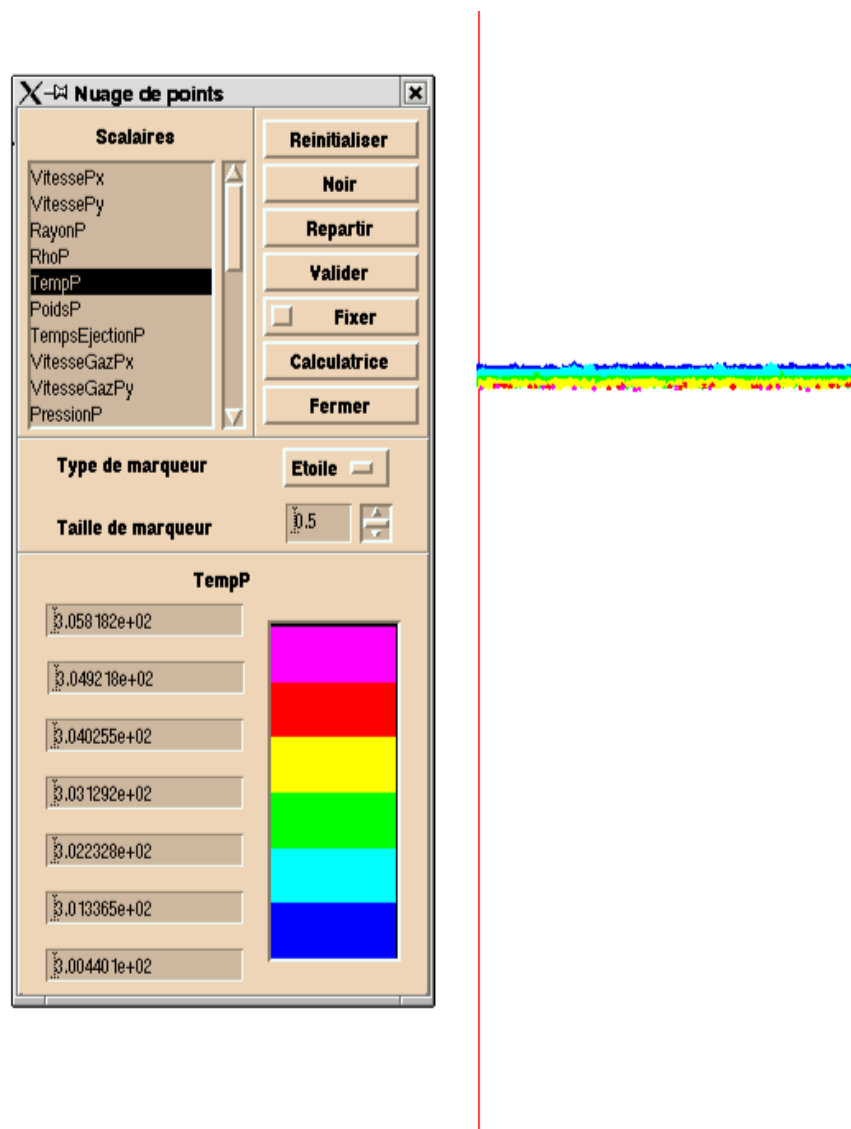


Fig. 1.8: Elastic case with thermal exchange: temperature of the droplets in Celsius

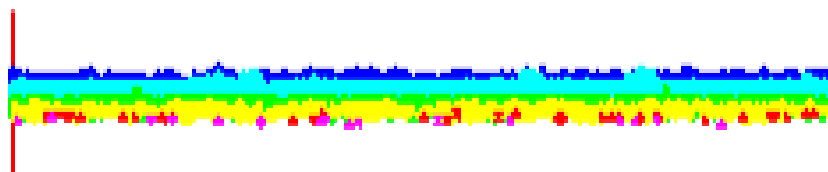


Fig. 1.9: Elastic case with thermal exchange: zoom

- $e_{p_*}' = \frac{a}{2} e_p + \frac{2-a}{2} e_{p_*}'.$

As can be seen, the distribution of temperature is less dispersed in figures 1.8 and 1.9: most of the particles have temperatures in $[300, 304]$ whereas most of the particles have temperatures in $[300, 305]$ in figures 1.6 and 1.7. Besides, the central layers (in cyan, green and yellow) are larger. As can also be seen, the convergence towards the Dirac mass in energy is also well-observed when the coupling with the gas is taken into account.

Effect of inelasticity on collisions

We consider here elastic and inelastic collisions (using the formula found for β in subsection 1.2.2). We have taken two pictures of the gas-particle system (figures 1.10 and 1.11) at the end of the experiment. As can be seen, the behavior of the cloud of particles is totally different. In figure 1.11, the particles remain together, whereas the cloud is very dispersed in the elastic case: at the beginning of the experiment, there are so many collisions that particles in a given cell take the same velocity and after that there is almost no collisions: they stick together all along the experiment.

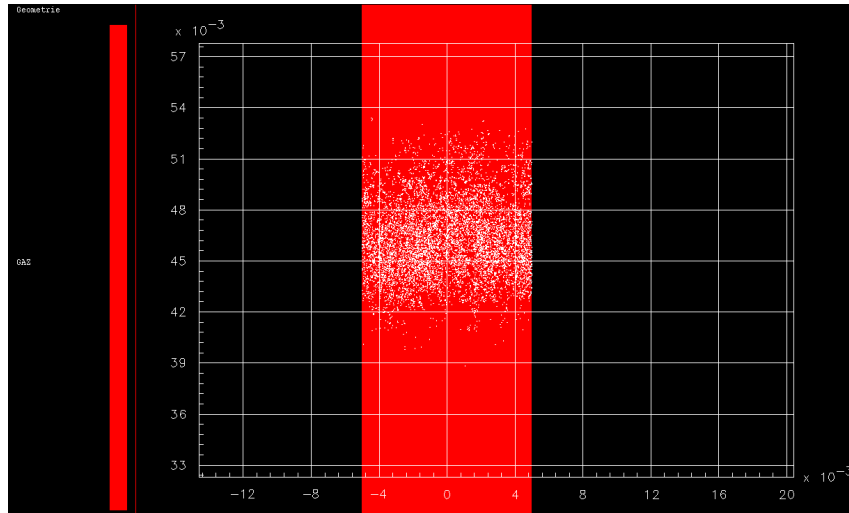


Fig. 1.10: Elastic case: form of the cloud of particles

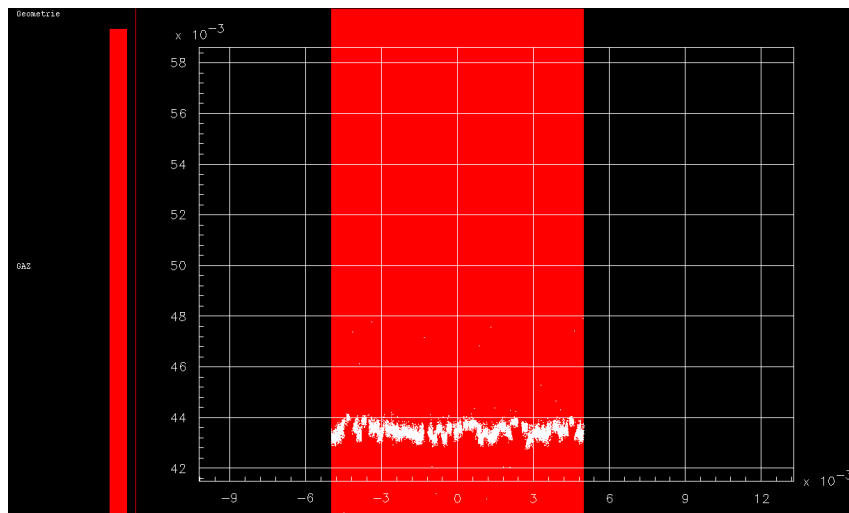


Fig. 1.11: Inelastic case: form of the cloud of particles

As can also be seen in figure 1.12, the convergence towards a Dirac mass in velocity in the inelastic

case is observed when the coupling with the gas is taken into account.

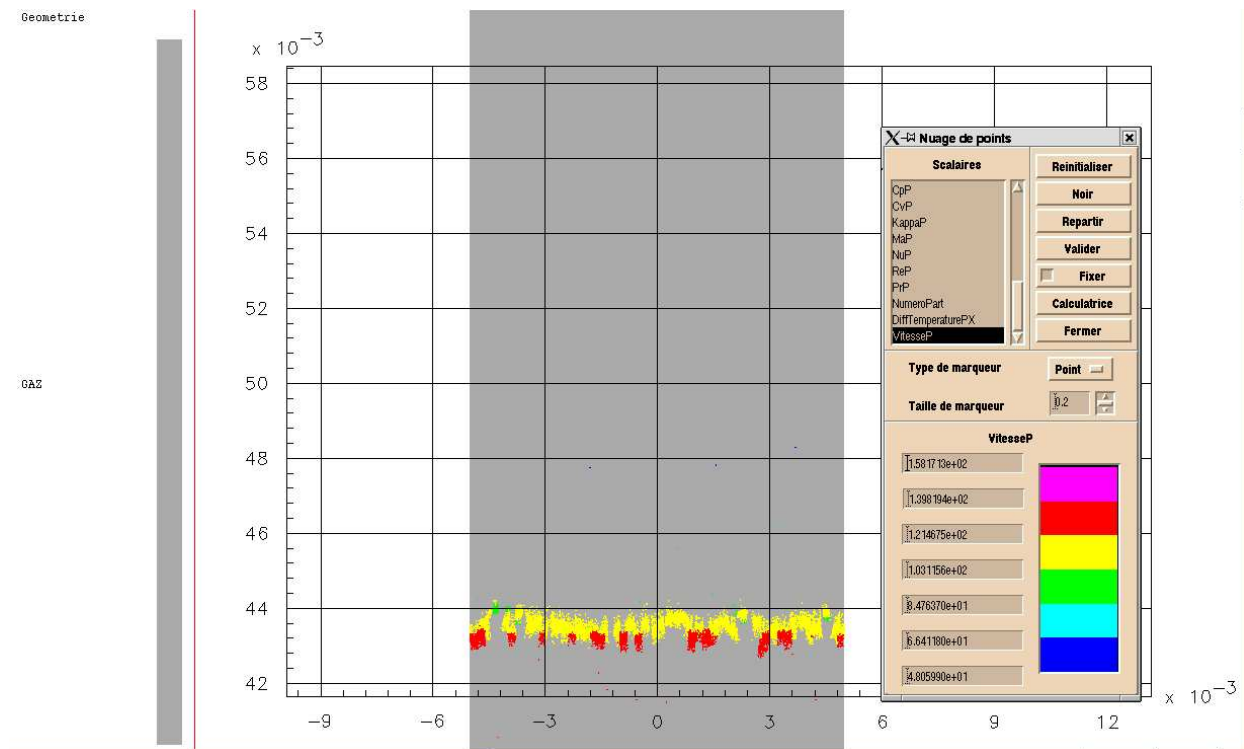


Fig. 1.12: Inelastic case: norm of the velocity of the particles

Convergence and stability in various cases

First, the behavior of the code does not change with the size of the grid: the typical length used here is $4 \cdot 10^{-5} m$ but the same experiment for $4 \cdot 10^{-6} m$ and $4 \cdot 10^{-3} m$ without any significant change. We recall that the mesh cannot be too refined since we need to have enough particles in the cells in order that the particles numerical scheme be effective.

The code is not affected by a change of the time step as soon as the CFL conditions are checked.

Although we do not provide other images for confidentiality purposes, the convergence towards Dirac masses in both internal energy and velocity was observed for different geometries (cylindric and spherical), grids and gas-particles systems.

Moreover, some tests were performed for different a and β fixed in order to observe the convergence, even if it has no physical relevance to fix them.

1.7 Extension in the case of compressible droplets

We now suppose that droplets are compressible. The aim is to obtain the elastic and inelastic hydrodynamic limits.

Let's first recall the equations governing the compressible gas-particle system (see [BDM03] and [O'R81] for more modelling) for which the collision kernel has been added (we do not take into account the added mass term which is in [BDM03]):

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.106)$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = - \iint_{u_p, e_p} m_p \Gamma f du_p de_p, \quad (1.107)$$

$$\partial_t(\alpha\rho_g e_g) + \nabla_x \cdot (\alpha\rho_g e_g u_g) + p[\partial_t \alpha + \nabla_x \cdot (\alpha u_g)] =$$

$$\iint_{u_p, e_p} [(m_p \Gamma + \frac{m_p}{\rho_p} \nabla_x p) \cdot (u_g - u_p) - 4\pi r \lambda Nu(T_g - T_p)] f du_p de_p, \quad (1.108)$$

$$\alpha = 1 - \iint_{u_p, e_p} \frac{m_p}{\rho_p} f du_p de_p, \quad (1.109)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \Gamma) + \partial_{e_p} (f \phi) = Q(f, f), \quad (1.110)$$

$$m_p \Gamma = -\frac{m_p}{\rho_p} \nabla_x p - D_p(u_p - u_g), \quad (1.111)$$

$$m_p \phi = 4\pi r \lambda Nu(T_g - T_p) + \frac{p m_p}{\rho_p^2} \frac{d\rho_p}{dt}, \quad (1.112)$$

$$\frac{d\rho_p}{dt} = \partial_t \rho_p + u_p \cdot \nabla_x \rho_p + \Phi \partial_{e_p} \rho_p, \quad (1.113)$$

$$E_g = e_g + \frac{1}{2} u_g^2 \quad (1.114)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.115)$$

$$p = P_1(\rho_g, e_g) = P_2(\rho_p, e_p), \quad (1.116)$$

$$T_p = T_2(\rho_p, e_p). \quad (1.117)$$

One can notice that one more relation is added to replace the incompressibility of the droplets (equation (1.116)). Since m_p is supposed to be the same for all particles, r is now a function of ρ_p and depends on

t , x and e_p . The collision kernel Q is slightly changed (its cross section depends on ρ_p and ρ_{p^*}) because the radii of the particles depend on their thermodynamic state. Finally compressibility effects are taken into account for the droplets in equation (1.112).

1.7.1 Preliminary results

The main changes occur in the internal energy equation, for which some extra computations must be done (especially for the elastic limit) because the expression of ϕ has changed.

We now prove that there is a work of the pressure p for the internal energy equation of the particles (since particles are compressible). In order to do so, let us recall that we would like to see a term of the form

$$p \left(\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v) \right)$$

appearing in the equations.

The two following results are proved in lemma A.1 in appendix A.1:

- $p(\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v)) = \iint_{u_p, e_p} p m_p f \frac{d}{dt} \left(\frac{1}{\rho_p} \right) de_p du_p,$
- $\iint_{u_p, e_p} \partial_{e_p} (f m_p \Phi) e_p du_p de_p = p(\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v)) - \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p.$

One can notice that thanks to the mass equation, the term of compressibility vanishes if ρ_p is constant. One gets

$$\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v) = (1 - \alpha)\rho \left(\partial_t \frac{1}{\rho} + v \cdot \nabla_x \frac{1}{\rho} \right)$$

so that the link with the incompressible case is clear.

1.7.2 Diphasic system with microscopic terms

We are now able to establish the equations in the compressible case. They are similar to those of the incompressible case apart from the compressibility terms:

Conservation of mass:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \tag{1.118}$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0. \tag{1.119}$$

Conservation of momentum:

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha\nabla_x p = \iint_{u_p, e_p} D_p(u_p - u_g) f du_p de_p, \quad (1.120)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p + \nabla_x((1-\alpha)P') = \\ - \iint_{u_p, e_p} D_p(u_p - u_g) f du_p de_p. \end{aligned} \quad (1.121)$$

Total energy equations:

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha \\ = \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p - \iint_{u_p, e_p} 4\pi r \lambda N u(T_g - T_p) f du_p de_p, \end{aligned} \quad (1.122)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho E_p) + \nabla_x \cdot \left((1-\alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)(P'v + q)) \\ = - \iint_{u_p, e_p} D_p(u_p - u_g) \cdot u_p f du_p de_p + \iint_{u_p, e_p} 4\pi r \lambda N u(T_g - T_p) f du_p de_p. \end{aligned} \quad (1.123)$$

Remark 1.8 (radius in the hydrodynamic limits)

One consequence of the convergence towards a Dirac mass in internal energy in both elastic and inelastic cases - that is of small importance in the incompressible case- is that the particles have the same radii and density at a given point x , because they have the same mass and internal energy. Nonetheless, the density of the particles ρ_p is not uniform, since e is also not uniform.

1.7.3 Compressible elastic model

As it was done for the model with incompressible droplets, we suppose that the velocities distribution is a Maxwellian and that the internal energy distribution is a Dirac mass when the collisions between particles are elastic and predominant: the limit is the same because, since all the particles have the same mass, the invariants of the collision kernel do not change (mass, momentum, internal energy and kinetic energy).

We recall the expression of f :

$$f(t, x, u_p, e_p) = \frac{Z(t, x)}{(2\pi T(t, x))^{\frac{3}{2}}} \exp\left(-\frac{(u_p - v(t, x))^2}{2T(t, x)}\right) \delta_{e_p = \epsilon(t, x)}(e_p).$$

By using the conservation of both kinetic and internal energy during collisions, we obtain two energy equations for the fluid of particles, so that we end up with a set of 7 equations for 7 unknowns and the system is closed. Finally we obtain a compressible two-phase flows system (see [GHS04, Bou98, Rov06]):

Conservation of mass:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.124)$$

$$\partial_t((1-\alpha)\rho) + \nabla_x \cdot ((1-\alpha)\rho v) = 0. \quad (1.125)$$

Conservation of momentum:

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = \mathcal{M}(\rho, 1-\alpha, v, T, u_g), \quad (1.126)$$

$$\partial_t((1-\alpha)\rho v) + \nabla_x \cdot ((1-\alpha)\rho v \otimes v) + (1-\alpha)\nabla_x p + \nabla_x((1-\alpha)\rho T) = -\mathcal{M}(\rho, 1-\alpha, v, T, u_g). \quad (1.127)$$

Total Energy equations:

$$\partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \mathcal{I}(\rho, 1-\alpha, v, T, u_g) - 4\pi r\lambda Nu(T_g - T_p), \quad (1.128)$$

$$\partial_t((1-\alpha)\rho e_c) + \nabla_x \cdot ((1-\alpha)\rho(e_c + T)v) + (1-\alpha)v \cdot \nabla_x p = -\mathcal{I}(\rho, 1-\alpha, v, T, u_g), \quad (1.129)$$

$$\begin{aligned} \partial_t((1-\alpha)\rho e) + \nabla_x \cdot ((1-\alpha)\rho e v) + p(\partial_t(1-\alpha) + \nabla_x \cdot ((1-\alpha)v)) = \\ 4\pi r\lambda Nu(T_g - T_p). \end{aligned} \quad (1.130)$$

Equations of state

$$p = P_1(\rho_g, e_g) = P_2(\rho, e), \quad (1.131)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.132)$$

$$T_p = T_2(\rho, e), \quad (1.133)$$

$$m_p = \frac{4}{3}\pi\rho r^3, \quad (1.134)$$

$$T = \rho(e_c - \frac{1}{2}v^2). \quad (1.135)$$

The quantities \mathcal{M} and \mathcal{I} are defined by equations (1.52) and (1.53), p.31, and $E_g = e_g + \frac{1}{2}u_g^2$. The equation on α is replaced by the relation 1.131 between the pressures: the quantity α becomes a Lagrangian multiplier for the system (see [Ram00]).

1.7.4 Compressible inelastic model

As it was done for the model with incompressible droplets, we suppose that the velocities distribution is a Dirac mass and that the internal energy distribution is a Dirac mass when the collisions between

particles are inelastic and predominant, since the invariants of the collision kernel are the same as in the incompressible model. We recall that we can write down f as:

$$f(t, x, u_p, e_p) = \frac{(1 - \alpha(t, x))\rho(t, x)}{m_p} \delta_{e_p=e(t, x)}(e_p) \otimes \delta_{u_p=v(t, x)}(u_p).$$

We finally obtain the following set of 6 equations for 6 unknowns (we recall that α is a Lagrangian multiplier),

Conservation of mass:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (1.136)$$

$$\partial_t((1 - \alpha)\rho) + \nabla_x \cdot ((1 - \alpha)\rho v) = 0. \quad (1.137)$$

Conservation of momentum:

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \alpha \nabla_x p = \frac{1}{2m_p} (1 - \alpha) \rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v), \quad (1.138)$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho v) + \nabla_x \cdot ((1 - \alpha)\rho v \otimes v) + (1 - \alpha) \nabla_x p = \\ - \frac{1}{2m_p} (1 - \alpha) \rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v). \end{aligned} \quad (1.139)$$

Total energy equations:

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p \partial_t \alpha = \\ - \frac{1}{2m_p} (1 - \alpha) \rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v) \cdot v - 4\pi r \lambda Nu (T_g - T_p) \frac{(1 - \alpha)\rho}{m_p}, \end{aligned} \quad (1.140)$$

$$\begin{aligned} \partial_t((1 - \alpha)\rho E_p) + \nabla_x \cdot \left((1 - \alpha)\rho \left(E_p + \frac{p}{\rho} \right) v \right) + p \partial_t(1 - \alpha) = \\ \frac{1}{2m_p} (1 - \alpha) \rho_g \rho \pi r^2 C_d |v - u_g| (u_g - v) \cdot v + 4\pi r \lambda Nu (T_g - T_p) \frac{(1 - \alpha)\rho}{m_p}. \end{aligned} \quad (1.141)$$

Equations of state

$$p = P_1(\rho_g, e_g) = P_2(\rho, e), \quad (1.142)$$

$$T_g = T_1(\rho_g, e_g), \quad (1.143)$$

$$T_p = T_2(\rho, e), \quad (1.144)$$

$$m_p = \frac{4}{3}\pi\rho r^3, \quad (1.145)$$

$$e_g = E_g - \frac{1}{2}u_g^2, \quad (1.146)$$

$$e = E_p - \frac{1}{2}v^2. \quad (1.147)$$

BIBLIOGRAPHY

- [AO89] A.A. Amsden and P.J. O'Rourke. The T.A.B. method for numerical calculation of spray droplet breakup. Technical report, Los Alamos National Laboratory, 1989.
- [AOB89] A.A. Amsden, P.J. O'Rourke, and T.D. Butler. Kiva II, a computer program for chemical reactive flows with spray. Technical report, Los Alamos National Laboratory, 1989.
- [Bar01] C. Baranger. Collisions, coalescences et fragmentations des gouttelettes dans un spray: écriture précise des équations relatives au modèle T.A.B. Master's thesis, CMLA, Ecole normale supérieure de Cachan, 2001.
- [Bar04a] C. Baranger. Modelling of oscillations, breakup and collisions for droplets: the establishment of kernels for the T.A.B. model. *Math. Models Methods Appl. Sci.*, 14(5):775–794, 2004.
- [Bar04b] C. Baranger. *Modélisation, étude mathématique et simulation des collisions dans les fluides complexes*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2004.
- [BCG00] A. V. Bobylev, J. A. Carrillo, and I. M. Gamba. On some properties of kinetic and hydrodynamic equations for inelastic interactions. *J. Statist. Phys.*, 98(3-4):743–773, 2000.
- [BDM03] L. Boudin, L. Desvillettes, and R. Motte. A modeling of compressible droplets in a fluid. *Commun. Math. Sci.*, 1(4):657–669, 2003.
- [BGP04] A. V. Bobylev, I. M. Gamba, and V. Panferov. Moment inequalities and high-energy tails for the boltzmann equations with inelastic interactions. *J. Statist. Phys.*, 116(5-6):1651–1682, 2004.
- [Bir94] G.A. Bird. *Molecular gas dynamics and the direct simulation of gas flows*. Oxford Science publications, 1994.
- [Bou98] M. Boucker. *Modélisation numérique multidimensionnelle d'écoulements diphasiques liquide-gaz en régimes transitoire et permanent: méthodes et applications*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 1998.
- [Bou00] L. Boudin. *Etude mathématique des équations aux dérivées partielles cinétiques et hyperboliques de la physique*. PhD thesis, Université d'Orléans, 2000.
- [CC70] S. Chapman and T.G. Cowling. *The mathematical theory of non uniform gases*. Cambridge Mathematical Library, 1970.
- [Cer88] C. Cercignani. *The Boltzmann Equation and its Applications*. Springer-Verlag, 1988.

- [Ces97] N. Cesco. *Etude et modélisation de l'écoulement diphasique à l'intérieur des propulseurs à poudre*. PhD thesis, Ecole nationale supérieure d'aéronautique et de l'espace (Toulouse), 1997.
- [CGW78] R. Clift, J.R. Grace, and M.E. Weber. *Bubbles, drops and particles*. Academic press, 1978.
- [Des93] L. Desvillettes. *Equations cinétiques*, 1993. Ecole nationale des ponts et chaussées.
- [Des97] L. Desvillettes. Sur un modèle de type Borgnakke-Larsen conduisant à des lois d'énergie non linéaires en température pour les gaz parfaits polyatomiques. *Ann. Fac. Sci. Toulouse Math. (6)*, 6(2):257–262, 1997.
- [DMS05] L. Desvillettes, R. Monaco, and F. Salvarani. A kinetic model allowing to obtain the energy law of polytropic gases in the presence of chemical reactions. *Eur. J. Mech. B Fluids*, 24(2):219–236, 2005.
- [Duf05] G. Dufour. *Modélisation multi-fluide eulérienne pour les écoulements diphasiques à inclusions dispersées*. PhD thesis, Université Paul Sabatier Toulouse III, 2005.
- [FM05] N. Fournier and S. Mischler. A spatially homogeneous Boltzmann equation for elastic, inelastic and coalescing collisions. *J. Math. Pures Appl. (9)*, 84(9):1173–1234, 2005.
- [GBWQ04] P-G. De Gennes, F. Brochard-Wyart, and D. Quéré. *Capillarity and Wetting phenomena*. Springer, 2004.
- [GHS04] T. Gallouët, J-M. Hérard, and N. Seguin. Numerical modeling of two-phase flows using the two-fluid two-pressure approach. *Math. Models Methods Appl. Sci.*, 14(5):663–700, 2004.
- [GPV04] I. M. Gamba, V. Panferov, and C. Villani. On the Boltzmann equation for diffusively excited granular media. *Comm. Math. Phys.*, 246(3):503–541, 2004.
- [Haf83] P.K. Haff. Grain flow as a fluid mechanical phenomenon. *J. Fluid. Mech.*, 134, 1983.
- [Hor90] L. Hormander. *The analysis of Linear Partial Differential Operators I*. Springer-Verlag, 1990. 2nd Edition.
- [Hyl99] J. Hylkema. *Modélisation cinétique et simulation numérique d'un brouillard dense de gouttelettes. Application aux propulseurs à poudre*. PhD thesis, Ecole nationale supérieure de l'aéronautique et de l'espace (Toulouse), 1999.
- [JUL92] Y.J. Jiang, A. Umemura, and C.K. Law. An experimental investigation on the collision behaviour of hydrocarbon droplets. *J. Fluid. Mech.*, 172, 1992.
- [KG05] S. Kamnis and S. Gu. Numerical modelling of droplet impingement. *J. Phys. D.*, 38:3664–3673, 2005.
- [Lag00] F. Lagoutière. *Modélisation mathématique et résolution numérique de problèmes de fluides compressibles à plusieurs constituants*. PhD thesis, PARIS VI, 2000.
- [Lau02] F. Laurent. *Modélisation mathématique et numérique de la combustion de brouillards de gouttes polydispersés*. PhD thesis, Université Claude Bernard (Lyon), 2002.

- [LM03] P. Laurençot and S. Mischler. Convergence to equilibrium for the continuous coagulation-fragmentation equation. *Bull. Sci. Math.*, 127(3):179–190, 2003.
- [MM06] S. Mischler and C. Mouhot. Cooling process for inelastic Boltzmann equations for hard spheres, part II: Self-similar solutions and tail behavior. *J. Statis. Phys.*, To appear, 2006.
- [MMR06] S. Mischler, C. Mouhot, and M. Rodriguez Ricard. Cooling process for inelastic Boltzmann equations for hard spheres, part I: The Cauchy problem. *J. Statis. Phys.*, To appear, 2006.
- [Mot00] R. Motte. A numerical method for solving particle-fluid equations. In *Trends in Numerical and Physical Modeling for Industrial Multiphase Flows*, 2000.
- [MR83] R. Maxey and J. Riley. Equation of motion of a small rigid sphere in a non uniform flow. *Physics of Fluids*, 26(4):883–889, 1983.
- [O’R81] P. O’Rourke. *Collective drop effects on vaporizing liquid sprays*. PhD thesis, Princeton University, 1981.
- [PR04] L. Pareschi and G. Russo. *An introduction to the numerical analysis of the Boltzmann equation in Summer School on methods and models of kinetic theory*. Rivista di Matematica della Università di Parma, 2004.
- [Ram00] D. Ramos. *Quelques résultats mathématiques et simulations numériques d’écoulements régis par des modèles bifluïdes*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2000.
- [Rov06] J.M. Rovarch. *Un solveur volume fini 3D pour les écoulements diphasiques*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2006.
- [Sai95] L. Sainsaulieu. *Contribution à la modélisation mathématique et numérique des écoulements diphasiques constitués d’un nuage de particules dans un écoulement de gaz*. Habilitation à diriger des recherches. Université Paris VI, 1995.
- [Sch66] L. Schwartz. *Théorie des distributions*. Hermann, 1966.
- [SR96] D. Stanton and C. Rutland. Modeling fuel film interaction and wall interaction in diesel engines. *SAE 960628*, 1996.
- [SR98] D. Stanton and C. Rutland. Multi-dimensional modeling of heat and mass transfer of fuel films resulting from impinging sprays. *SAE 980132*, 1998.
- [Vil02] C. Villani. *A review of mathematical topics in collisional kinetic theory*. Elsevier Science, 2002.
- [Wil85] F.A. Williams. *Combustion theory*. Benjamin Cummings, 1985.
- [WO03] K. Willis and M. Orme. Binary droplet collisions in vacuum environment: an experimental investigation of the role of viscosity. *Exp. Fluids*, 34:28–41, 2003.

2. MATHEMATICAL STUDY OF A THIN SPRAY MODEL WITH COLLISIONS

Aim

Sprays are made of particles (droplets) in an underlying gas. A coupling of the droplets and the gas was first proposed by Williams ([Wil85]), Papanicolaou and Caffisch ([CP83]), etc... The aim of this chapter is a mathematical study of thin sprays with collisions. This means that we consider a gas in which the volume fraction of droplets is such that collisions must be taken into account but not such that the theory of thick sprays has to be used (cf. the classification of sprays page xi in the introduction). Our study is based on a gas-particle system used at the CEA (see [BDM03]).

The gas is generally described through Navier-Stokes or Euler equations. Here, the Euler representation is chosen because the gas evolves at high Reynolds number in the model proposed in [BDM03] ($Re = 10^5$). The density ρ_g , the velocity u_g and the internal energy e_g characterize the behavior of the gas; the gas is considered to be perfect for mathematical purposes.

The particles are described via a particle density function f which obeys to a Vlasov-Boltzmann equation (see [Cer88] and [Vil02]). Here the parameters of the p.d.f. are the time t , the position x , the velocity v and the internal energy e . The collision kernel corresponds to hard spheres.

The coupling between the two phases is made through the drag force (in the momentum equation) and thermal exchanges through thermal conduction (in the energy equation). Although viscosity is neglected because of the value of the Reynolds number, the drag force (which is proportional to viscosity) has to be taken into account because particles are very small in the applications ($r = 10^{-6}m$).

From a mathematical viewpoint, some results exist for both Boltzmann equation and Euler equations. Concerning the Boltzmann equation, there exists global renormalized solutions (a la Di Perna-Lions: [DL89]), perturbative solutions near gaussians ([UA82], [MP97], [Guo03]) or vacuum ([IS84], [BT85]). On the Euler equations, only local smooth solutions are known ([Maj84]) in all dimensions whereas for long time, only the 1D Glim solutions exist up to now ([Ser96b]). Therefore, in order to be able to consider the coupling between the two equations, we will deal with local in time solutions

Mathematical results have already been proved for sprays. For instance, Domelevo and Roquejoffre have shown the existence and uniqueness of regular solutions for thin sprays without collisions (see [DR99]), considering the viscous Burgers equation for the gas. Hamdache proved the existence and uniqueness of solutions for large times for a Stokes-Vlasov system ([Ham98]). Recently Baranger and Desvillettes proved

the existence and uniqueness of \mathcal{C}^1 solutions for an Euler-Vlasov system ([BD06]), locally in time. The present work is the next step towards thick sprays, where the volume fraction occupied by the particles is explicitly appearing in the equations. As far as we know, it is also the first work which deals with the energies of the two phases.

Model

We now write down the model:

Gas-particle system

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0, \quad (2.1)$$

$$\partial_t u_g + (u_g \cdot \nabla_x) u_g + \frac{\nabla_x p}{\rho_g} = -\frac{1}{\rho_g} \iint_{v,e} m_p F f dv de, \quad (2.2)$$

$$\partial_t e_g + u_g \cdot \nabla_x e_g + \frac{p}{\rho_g} \nabla_x \cdot u_g = \frac{1}{\rho_g} \iint_{v,e} (m_p F \cdot (u_g - v) - m_p \phi) f dv de \quad (2.3)$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f F) + \partial_e (f \phi) = Q(f, f), \quad (2.4)$$

$$F = D_p (u_g - v), \quad (2.5)$$

$$\phi = \frac{4\pi r}{m_p} \lambda Nu (T_g - T_p), \quad (2.6)$$

$$p = (\gamma - 1) \rho_g e_g, \quad (2.7)$$

$$T_g = \frac{e_g}{C_{vg}}, \quad (2.8)$$

$$T_p = \frac{e}{C_{vp}}. \quad (2.9)$$

Here, D_p is the drag coefficient. One can note that, in some models, D_p is proportional to the density of the gas ([BD06]): it would not modify the problem which is studied here, from a mathematical viewpoint. The quantity C_{vg} (resp. C_{vp}) is the specific heat at constant volume of the gas (resp. particles). λ is the thermal conductivity of the gas and Nu is the Nusselt number. γ is the specific heat ratio of the gas. The particles are supposed to have the same mass m_p and radius r . All these data are supposed to be constant.

Equations (2.1)-(2.3) are balance laws for the density $\rho_g := \rho_g(t, x)$, the velocity $u_g := u_g(t, x)$ and the internal energy (per mass unit) $e_g := e_g(t, x)$ of the gas, while equation (2.4) is a Vlasov-Boltzmann equation for the particle density function $f := f(t, x, v, e)$ of the particles. The gas is supposed to be perfect (equation (2.7) and (2.8)).

The coupling between (2.1)-(2.3) on one hand, and (2.4) on the other hand is made through the source terms in (2.2) and (2.3). In these source terms appear the force $m_p \Gamma$ and the term of energy exchange $m_p \phi$. The whole system is closed by the thermodynamic laws (2.7), (2.8) and (2.9).

Eventually, $Q(f, f)$ is a suitable hard-sphere type collision kernel taking into account internal energies:

$$Q(f, f)(t, x, v, e) = \iiint_{\sigma \in \mathbb{S}^2, v_* \in \mathbb{R}^3, e_* \in \mathbb{R}^+} (f'_* f' - f_* f) r^2 |v - v_*| d\sigma dv_* de_*, \quad (2.10)$$

with (see [Bar01]),

- σ : vector in \mathbb{S}^2 ,
- $v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma$: post-collisional velocity,
- $v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma$: post-collisional velocity,
- $e' = e$: post-collisional internal energy,
- $e'_* = e_*$: post-collisional internal energy.
- $f'_* = f(t, x, v'_*, e'_*)$
- $f' = f(t, x, v', e')$
- $f_* = f(t, x, v_*, e_*)$
- $f = f(t, x, v, e)$

For mathematical purposes, we do not introduce a complex collision kernel like in chapter one. We rather take the simplest reasonable model: collisions are elastic and no internal energy is exchanged during the contact between droplets.

The constants $\lambda, C_{vg}, C_{vp}, D_p, m_p, r$ are taken equal to one for mathematical purposes, and γ is such that $\gamma > 1$. Nu is chosen so that $\frac{4\pi r}{m_p} \lambda Nu = 1$ in order to simplify computations.

We can now write down the system that we shall study. The unknowns are $\rho_g := \rho_g(t, x) \geq 0$, $u_g := u_g(t, x)$, $e_g := e_g(t, x) \geq 0$, and $f := f(t, x, v, e) \geq 0$, and they satisfy:

$$\partial_t \rho_g + \nabla_x \cdot (\rho_g u_g) = 0, \quad (2.11)$$

$$\partial_t u_g + (u_g \cdot \nabla_x) u_g + \frac{\nabla_x p}{\rho_g} = -\frac{1}{\rho_g} \iint_{v, e} F f dv de, \quad (2.12)$$

$$\partial_t e_g + u_g \cdot \nabla_x e_g + \frac{p}{\rho_g} \nabla_x \cdot u_g = \frac{1}{\rho_g} \iint_{v, e} (F \cdot (u_g - v) - \phi) f dv de, \quad (2.13)$$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f F) + \partial_e (f \phi) = Q(f, f), \quad (2.14)$$

$$F = u_g - v, \quad (2.15)$$

$$\phi = e_g - e, \quad (2.16)$$

$$p = (\gamma - 1) \rho_g e_g, \quad (2.17)$$

where $Q(f, f)$ is defined by (2.10) (with $r = 1$).

Functional space and norms

We give here some notations and define the space of functions of this work. We denote by $\|g\|$ the L^2 -norm and by $\|g\|_w$ the weighted- L^2 norm ($w = (1 + |v|^2)$),

$$\|g\| = \left(\iiint_{x,v,e} g^2 dedvdx \right)^{\frac{1}{2}},$$

$$\|g\|_w = \left(\iiint_{x,v,e} g^2 w dedvdx \right)^{\frac{1}{2}}.$$

We note $|g|$ and $|g|_w$ the local L^2 -norms,

$$|g| = \left(\iint_{v,e} g^2 dedv \right)^{\frac{1}{2}}, \quad |g|_w = \left(\iint_{v,e} g^2 w dedv \right)^{\frac{1}{2}}.$$

Sobolev spaces $W^{s,2}$ are denoted by H^s .

We note $\partial_\alpha^{\beta,\gamma} := \partial_x^\alpha \partial_v^\beta \partial_e^\gamma$. Let s be an integer such that $s > 4$. Using Guo's method (see [Guo03] and [Guo02]), we define the energy of a function as

$$\mathcal{E}_{s,\varepsilon}(g(t, \dots)) = \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left[\frac{1}{2} \left\| \partial_\alpha^{\beta,\gamma} g \right\|^2(t) + \int_0^t 2(1-\varepsilon) \left\| \partial_\alpha^{\beta,\gamma} g \right\|_w^2(u) du \right], \quad (2.18)$$

where ε is a constant which satisfies $0 < \varepsilon < 1$.

We refer here to Guo's papers because even if their main results are for large time and small initial data with respect to this energy $\mathcal{E}_{s,\varepsilon}$, in fact the result for all initial data is also obtained locally in time.

We note $E_{s,\varepsilon,T}$ the intersection of the nonnegative functions and the functional space made of the functions f in $\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ such that

$$\sup_{0 \leq t \leq T} \mathcal{E}_{s,\varepsilon}(f)(t)^{\frac{1}{2}} < +\infty.$$

The norm associated with $E_{s,\varepsilon,T}$ is denoted $\|\cdot\|_{E_{s,\varepsilon,T}}$ and defined through

$$\|f\|_{E_{s,\varepsilon,T}} = \sup_{0 \leq t \leq T} \mathcal{E}_{s,\varepsilon}(f)(t)^{\frac{1}{2}}.$$

The space $E_{s,\varepsilon,T}$ is complete since it is the intersection of a complete functional space and the cone of positive functions.

Outline of the chapter

- We first establish an a priori estimate for the Boltzmann equation, thanks to the use of a simple change of unknown in the equation. We define $g := f \times \exp(v^2 + e)$ using Guo's method ([Guo03]). The function g satisfies

$$\partial_t g + v \cdot \nabla_x g + \nabla_v \cdot (gF) + \partial_e(g\phi) - 2v \cdot gF - g\phi = \Gamma(g, g), \quad (2.19)$$

where Γ is defined through

$$\begin{aligned} & \Gamma[g_1, g_2](t, x, v, e) \\ &= \Gamma^+[g_1, g_2](t, x, v, e) - \Gamma^-[g_1, g_2](t, x, v, e) \\ &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v'_*, e'_*) g_2(t, x, v', e') dv_* de_* d\sigma \\ &- g_2(t, x, v, e) \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v_*, e_*) dv_* de_* d\sigma. \end{aligned}$$

We shall show that the norm of g ($\|g\|_{E_{s,\varepsilon,T}}$) remains bounded for small T . The proof is based on the fact that the "new" collision kernel Γ can be controlled by the term coming from the drag force $-2v \cdot gF$.

- Then, we solve the modified Boltzmann equation (2.19) at "fixed gas" with respect to the new unknown function g : we suppose that the gas is known and establish a result of existence and uniqueness for g . In order to do so, we use Picard's fixed point theorem and estimates based on the characteristics method (as in [BD06]). More precisely, we prove the following result:

Theorem 2.1 (Existence and uniqueness for equation (2.19))

We suppose that ρ_g and e_g are strictly positive functions. Furthermore, we assume that $\tilde{\rho}_g = \rho_g - 1, u_g$ and $\tilde{e}_g = e_g - 1$ belong to $\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))$ for some $T > 0$ and s integer such that $s \geq 5$. We suppose that $g_0 := f_0 \exp(v^2 + e)$ satisfies

$$\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \frac{1}{2} \left\| \partial_{\alpha}^{\beta, \gamma} (g_0) \right\|^2 < +\infty.$$

Then:

1. One can find $T' > 0$ such that there exists a solution g to equation (2.19) with g_0 for initial data (and F and ϕ given by (2.15) and (2.16)). Besides, g remains positive and

$$\mathcal{E}_{s, \frac{1}{2}}(g(t, \cdot)) = \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left[\frac{1}{2} \left\| \partial_{\alpha}^{\beta, \gamma} g \right\|^2(t) + \int_0^t \left\| \partial_{\alpha}^{\beta, \gamma} g \right\|_w^2(u) du \right]$$

remains bounded on $[0, T']$. Finally, T' is controlled through

$$T' \geq \frac{3}{8C \left[1 + \sup_{0 \leq t \leq T} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g\|_{H^s} + \frac{3}{2} \|g_0\|_{H^s}^2 \right]} \quad (2.20)$$

with C strictly positive constant depending only on s .

Moreover, g belongs to $\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.

2. If g_1 and g_2 are two positive solutions of eq. (2.19) in $\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ with same initial data - u_g and \tilde{e}_g being fixed- and with finite energy on $[0, T]$ (i.e. $\sup_{0 \leq t \leq T} \mathcal{E}_{s, \frac{1}{2}}(g_1(t, \cdot)) < +\infty$, $\sup_{0 \leq t \leq T} \mathcal{E}_{s, \frac{1}{2}}(g_2(t, \cdot)) < +\infty$), then $g_1 = g_2$.

- Finally, we tackle the global system. Using an iterative scheme based on hyperbolic theory for the gas and the result at "fixed gas" for the particles (theorem 2.1), we prove the following result:

Theorem 2.2

We consider $I =]0, +\infty[\times \mathbb{R}^3 \times]0, +\infty[$, $s \in \mathbb{N}$ such that $s \geq 5$, and I_1, I_2 open sets of I such that $\overline{I_1} \subset I_2$, and such that $\overline{I_1}, \overline{I_2}$ are compact in I . Let $(\rho_{g_0}, u_{g_0}, e_{g_0}) : \mathbb{R}^3 \rightarrow I_1$ be functions satisfying $\tilde{\rho}_{g_0} = \rho_{g_0} - 1 \in H^s(\mathbb{R}^3)$, $u_{g_0} \in H^s(\mathbb{R}^3)$ and $\tilde{e}_{g_0} = e_{g_0} - 1 \in H^s(\mathbb{R}^3)$. Let also $g_0 = f_0 \exp(v^2 + e) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a function of $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$.

Then, one can find $T > 0$ such that there exists a solution $(\rho_g, u_g, e_g; f)$ to system (2.11) - (2.17) belonging to $C^1([0, T] \times \mathbb{R}^3, \overline{I_2}) \times C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$. Moreover, $\tilde{\rho}_g (= \rho_g - 1)$, u_g , $\tilde{e}_g (= e_g - 1) \in \mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))$, $\tilde{f} = f \exp(v^2 + e) \in \mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.

Finally, if $(\tilde{\rho}_{g_1}, u_{g_1}, \tilde{e}_{g_1}; f_1)$ and $(\tilde{\rho}_{g_2}, u_{g_2}, \tilde{e}_{g_2}; f_2)$ belong to $(\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))) \times (\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)))$, if $(\rho_{g_1}, u_{g_1}, e_{g_1}; f_1)$ and $(\rho_{g_2}, u_{g_2}, e_{g_2}; f_2)$ belong to $C^1([0, T] \times \mathbb{R}^3, \overline{I_2}) \times C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$ and they satisfy (2.11) - (2.17), then $\rho_{g_1} = \rho_{g_2}$, $u_{g_1} = u_{g_2}$, $e_{g_1} = e_{g_2}$ et $f_1 = f_2$.

2.1 A priori estimates for Boltzmann equation with fixed gas

We want to study the following equation,

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (fF) + \partial_e(f\phi) = Q(f, f), \quad (2.21)$$

with F, ϕ, Q given by (2.14) – (2.16)

We suppose that $\tilde{\rho}_g = \rho_g - 1, u_g$ and $\tilde{e}_g = e_g - 1$ belong to the following space: $\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))$ for some $T > 0$ and s integer such that $s \geq 5$ (typical space for hyperbolic problems -cf [Maj84]-): the gas is at rest at infinity. Moreover we suppose that $f_0 \exp(v^2 + e)$ is in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$.

2.1.1 Rewriting Boltzmann equation

We note g the following function: $g : (t, x, v, e) \mapsto f(t, x, v, e) \exp(v^2 + e)$. Putting g in the Vlasov-Boltzmann equation, one obtains the equation for g :

$$\partial_t g + v \cdot \nabla_x g + \nabla_v \cdot (gF) + \partial_e(g\phi) - 2v \cdot gF - g\phi = \Gamma(g, g), \quad (2.22)$$

where Γ is defined through

$$\begin{aligned} \Gamma[g_1, g_2](t, x, v, e) &= \Gamma^+[g_1, g_2](t, x, v, e) - \Gamma^-[g_1, g_2](t, x, v, e) \\ &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v_*, e_*) g_2(t, x, v', e') dv_* de_* d\sigma \\ &\quad - g_2(t, x, v, e) \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v_*, e_*) dv_* de_* d\sigma. \end{aligned} \quad (2.23)$$

The form of Γ is obtained by noticing that both kinetic and internal energy are conserved during collisions.

2.1.2 Some functional results

We now give and prove some results which are useful for the different estimates. We first recall the following Sobolev embedding (see [Bre83] and [Tay96]):

Lemma 2.3

1. Let h_1 be in $W^{3,1}(\mathbb{R}^3)$. The following embedding holds:

$$\|h_1\|_\infty \leq C \|h_1\|_{W^{3,1}}. \quad (2.24)$$

2. Let h_2 be in $H^p(\mathbb{R}^n)$ with p such that $p \geq \frac{n}{2}$. The following embedding holds:

$$\|h_2\|_\infty \leq C\|h_2\|_{H^p}. \quad (2.25)$$

Thanks to these embeddings we prove the following inequality:

Lemma 2.4

Let $s \in \mathbb{N}$ such that $s \geq 4$, h_1 be in $H^s(\mathbb{R}^3)$ and h_2 be in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$. Then for all $\alpha = \alpha_1 + \alpha_2$, β and γ multi-indices such that $|\alpha| + |\beta| + |\gamma| \leq s$,

$$\|\partial_{\alpha_1} h_1 \partial_{\alpha_2}^{\beta, \gamma} h_2\| \leq C\|h_1\|_{H^s(\mathbb{R}^3)}\|h_2\|_{H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)}.$$

As a consequence, $h_1 h_2$ is in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$.

Proof of Lemma 2.4:

Let α , β and γ be some multi-indices such that $|\alpha| + |\beta| + |\gamma| \leq s$. $\partial_{\alpha}^{\beta, \gamma}(h_1 h_2)$ is a sum of terms like this (with $\alpha_1 + \alpha_2 = \alpha$):

$$\partial_{\alpha_1} h_1 \partial_{\alpha_2}^{\beta, \gamma} h_2.$$

If $|\alpha_1| \leq s - 2$, then using the second embedding, we get that $\partial_{\alpha_1} h_1$ is L^∞ -bounded by $\|h_1\|_{H^s}$ and that

$$\begin{aligned} \|\partial_{\alpha_1} h_1 \partial_{\alpha_2}^{\beta, \gamma} h_2\| &\leq C\|\partial_{\alpha_1} h_1\|_\infty \|\partial_{\alpha_2}^{\beta, \gamma} h_2\| \\ &\leq C\|h_1\|_{H^s} \|h_2\|_{H^s}. \end{aligned} \quad (2.26)$$

Else, we see that $|\alpha_2| + |\beta| + |\gamma| \leq 1$ and consequently (using the first embedding),

$$\begin{aligned} \sup_{x \in \mathbb{R}^3} \iint \left(\partial_{\alpha_2}^{\beta, \gamma} h_2 \right)^2 dv de &\leq C \sum_{|\delta| \leq 3} \left\| \partial_{\alpha_2 + \delta}^{\beta, \gamma} h_2 \right\|^2 \\ &\leq C\|h_2\|_{H^{3+|\alpha_2|+|\beta|+|\gamma|}} \\ &\leq C\|h_2\|_{H^{3+1}} \\ &\leq C\|h_2\|_{H^s} \quad \text{since } s \geq 4. \end{aligned}$$

Then,

$$\begin{aligned} \iiint |\partial_{\alpha_1} h_1 \partial_{\alpha_2}^{\beta, \gamma} h_2|^2 dv dedx &\leq \sup_{x \in \mathbb{R}^3} \iint \left(\partial_{\alpha_1}^{\beta, \gamma} h_2 \right)^2 dv de \times \int |\partial_{\alpha_1} h_1|^2 dx \\ &\leq C\|h_2\|_{H^s}^2 \|h_1\|_{H^s}^2. \end{aligned} \quad (2.27)$$

Eventually, we get in both cases

$$\|\partial_{\alpha_1} h_1 \partial_{\alpha_2}^{\beta, \gamma} h_2\| \leq C\|h_2\|_{H^s} \|h_1\|_{H^s}.$$

■

2.1.3 Collision kernel

We now consider the modified collision operator introduced in (2.23),

$$\begin{aligned}
\Gamma[g_1, g_2](t, x, v, e) &= \Gamma^+[g_1, g_2] - \Gamma^-[g_1, g_2] \\
&= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v'_*, e'_*) g_2(t, x, v', e') dv_* de_* d\sigma \\
&\quad - g_2(t, x, v, e) \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_1(t, x, v_*, e_*) dv_* de_* d\sigma,
\end{aligned} \tag{2.28}$$

and obtain its control.

Using the change of variables $v_* \mapsto v_* + v$ in the integral, and the fact that collisions do not change internal energy, one gets:

$$\begin{aligned}
&\Gamma[g_1, g_2](t, x, v, e) \\
&= \Gamma^+[g_1, g_2](t, x, v, e) - \Gamma^-[g_1, g_2](t, x, v, e) \\
&= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v_*| \exp(-((v_* + v)^2 + e_*)) g_1(t, x, v + \frac{1}{2}(v_* - |v_*|\sigma), e_*) \\
&\quad \times g_2(t, x, v + \frac{1}{2}(v_* + |v_*|\sigma), e) dv_* de_* d\sigma \\
&\quad - g_2(t, x, v, e) \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v_*| \exp(-((v_* + v)^2 + e_*)) g_1(t, x, v_* + v, e_*) dv_* de_* d\sigma.
\end{aligned} \tag{2.29}$$

Using this form of the kernel, one can differentiate this expression (see [Guo03]). Then, using the change of variables backwards, one obtains

$$\partial_\alpha^{\beta, \gamma} \Gamma(g_1, g_2) = \sum C(|\alpha|, |\alpha_1|, |\alpha_2|, |\beta|, |\beta_0|, |\beta_1|, |\beta_2|, |\gamma|, |\gamma_0|, |\gamma_1|, |\gamma_2|) \Gamma_0 \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right], \tag{2.30}$$

where the summation is over $\beta = \beta_0 + \beta_1 + \beta_2$, $\alpha = \alpha_1 + \alpha_2$ and $\gamma = \gamma_0 + \gamma_1 + \gamma_2$. The coefficients in the expression are integers that we do not compute. We define Γ_0 as

$$\begin{aligned}
\Gamma_0 \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right] &= \Gamma_0^+ \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right](t, x, v, e) - \Gamma_0^- \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right](t, x, v, e) \\
&= \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \partial^{\beta_0, \gamma_0} [\exp(-(v_*^2 + e_*))] (\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1)'_* (\partial_{\alpha_2}^{\beta_2, \gamma_2} g_2)' \\
&\quad - \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \partial^{\beta_0, \gamma_0} [\exp(-(v_*^2 + e_*))] (\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1)_* \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2.
\end{aligned} \tag{2.31}$$

One has to note that in fact $\gamma_1 = \gamma_0 = 0$, and $\gamma_2 = \gamma$ for all γ since the derivatives in e are only the derivatives of g_2 .

According to our future needs, we prove two different properties on the gain and loss terms of the collision kernel:

Lemma 2.5

Let $\beta = \beta_0 + \beta_1 + \beta_2$, $\alpha = \alpha_1 + \alpha_2$ and $\gamma = \gamma_0 + \gamma_1 + \gamma_2$.

1. The following inequalities hold:

$$\begin{aligned} \left| \iiint \Gamma_0^+ \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right] \partial_{\alpha}^{\beta, \gamma} g_3 dv dedx \right| \\ \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right] \left\| \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right\| \left\| \partial_{\alpha}^{\beta, \gamma} g_3 \right\|_w, \end{aligned} \quad (2.32)$$

$$\begin{aligned} \left| \iiint \Gamma_0^- \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right] \partial_{\alpha}^{\beta, \gamma} g_3 dv dedx \right| \\ \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right] \left\| \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right\| \left\| \partial_{\alpha}^{\beta, \gamma} g_3 \right\|_w. \end{aligned} \quad (2.33)$$

2. The following inequalities hold:

$$\begin{aligned} \left| \iiint \Gamma_0^+ \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right] \partial_{\alpha}^{\beta, \gamma} g_3 dv dedx \right| \\ \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_2 + \delta}^{\beta_2, \gamma_2} g_2 \right\| \right] \left\| \partial_{\alpha_1}^{\beta_1, \gamma_1} g_1 \right\| \left\| \partial_{\alpha}^{\beta, \gamma} g_3 \right\|_w, \end{aligned} \quad (2.34)$$

$$\begin{aligned} \left| \iiint \Gamma_0^- \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right] \partial_{\alpha}^{\beta, \gamma} g_3 dv dedx \right| \\ \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_2 + \delta}^{\beta_2, \gamma_2} g_2 \right\| \right] \left\| \partial_{\alpha_1}^{\beta_1, \gamma_1} g_1 \right\| \left\| \partial_{\alpha}^{\beta, \gamma} g_3 \right\|_w. \end{aligned} \quad (2.35)$$

Proof of Lemma 2.5:

We first consider inequalities (2.33) and (2.35).

Inequalities (2.33) and (2.35):

Using the Sobolev injection $W^{3,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we get that $x \mapsto \iint \left(\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1 \right)^2 dv de$ is in $L^\infty(\mathbb{R}^3)$ (see [Bre83]) and that

$$\sup_{x \in \mathbb{R}^3} \iint \left(\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1 \right)^2 dv de \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right]^2. \quad (2.36)$$

Moreover, we have the following inequalities,

$$\iint |v - v_*|^2 \exp(-(v_*^2 + e_*)) dv_* de_* \leq Cw, \quad (2.37)$$

$$\left| \partial^{\beta_0, \gamma_0} [\exp(-(v_*^2 + e_*))] \right| \leq C \exp\left(-\frac{v_*^2 + e_*}{2}\right). \quad (2.38)$$

For the sake of simplicity, we shall note the integral over v, v_*, e, e_*, σ and x only \int .

We see that

$$\begin{aligned} & \left| \iiint \Gamma_0^- \left[\partial_{\alpha_1}^{\beta_1, \gamma_1} g_1, \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right] \partial_{\alpha}^{\beta, \gamma} g_3 dv de dx \right| \\ & \leq \int |v - v_*| \left| \partial^{\beta_0, \gamma_0} [\exp(-(v_*^2 + e_*))] \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*} \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*} \partial_{\alpha}^{\beta, \gamma} g_3 \right| \\ & \leq C \int \left| \exp\left(-\frac{v_*^2 + e_*}{2}\right) \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*} \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*} \partial_{\alpha}^{\beta, \gamma} g_3 \right| |v - v_*| \\ & \leq C \left(\int \left| \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*} \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*} \right|^2 \right)^{1/2} \times \left(\int \left| \partial_{\alpha}^{\beta, \gamma} g_3 \right|^2 |v - v_*|^2 \exp(-(v_*^2 + e_*)) \right)^{1/2} \end{aligned}$$

using Cauchy-Schwarz inequality. From now on, g_1 and g_2 have symmetric roles, so that if inequality (2.33) holds, inequality (2.35) also holds.

Because of inequality (2.36), we have

$$\sup_{x \in \mathbb{R}^3} \iint \left(\partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*} \right)^2 dv_* de_* \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right]^2. \quad (2.39)$$

Eventually, one gets

$$\begin{aligned} \left(\int \left| \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*} \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*} \right|^2 \right)^{1/2} & \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right] \left(\int \left| \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right|^2 \right)^{1/2} \\ & \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right] \left\| \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right\|. \end{aligned} \quad (2.40)$$

Integrating in v_*, e_* and after that, in the other variables, we get using (2.37):

$$\begin{aligned} \left(\int \left| \partial_{\alpha}^{\beta, \gamma} g_3 \right|^2 |v - v_*|^2 \exp(-(v_*^2 + e_*)) \right)^{1/2} & \leq C \left(\int \left| \partial_{\alpha}^{\beta, \gamma} g_3 \right|^2 w \right)^{1/2} \\ & \leq C \left\| \partial_{\alpha}^{\beta, \gamma} g_3 \right\|_w. \end{aligned} \quad (2.41)$$

Using (2.40) and (2.41), one gets the result.

We finally consider inequalities (2.32) and (2.34).

Inequalities (2.32) and (2.34):

$$\begin{aligned}
& \left| \iiint \Gamma_0^+ \left[\partial_\alpha^{\beta_1, \gamma_1} g_1, \partial_\alpha^{\beta_2, \gamma_2} g_2 \right] \partial_\alpha^{\beta, \gamma} g_3 dv dedx \right| \\
& \leq \int |v - v_*| \left| \partial^{\beta_0, \gamma_0} [\exp(-(v_*^2 + e_*))] \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*}' \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*}' \partial_\alpha^{\beta, \gamma} g_3 \right| \\
& \leq C \left[\int \left| \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*}' \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*}' \right|^2 dv_* dv dede_* dx d\sigma \right]^{1/2} \\
& \quad \times \left[\int \left| \partial_\alpha^{\beta, \gamma} g_3 \right|^2 |v - v_*|^2 \exp(-(v_*^2 + e_*)) dv_* dv dede_* dx d\sigma \right]^{1/2}
\end{aligned} \tag{2.42}$$

still using Cauchy-Schwarz inequality.

The roles of v_*' and v' are now symmetric in the last expression so that proving inequality (2.32) is the same as proving inequality (2.34) once one switches g_1 and g_2 . We only prove (2.32).

Using Sobolev embeddings, one gets

$$\sup_{x \in \mathbb{R}^3} \iint \left(\partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*} \right)^2 \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right]^2.$$

Therefore,

$$\left(\int \left| \partial_{\alpha_1}^{\beta_1, \gamma_1} g_{1*}' \partial_{\alpha_2}^{\beta_2, \gamma_2} g_{2*}' \right|^2 \right)^{1/2} \leq C \left[\sum_{|\delta| \leq 3} \left\| \partial_{\alpha_1 + \delta}^{\beta_1, \gamma_1} g_1 \right\| \right] \left\| \partial_{\alpha_2}^{\beta_2, \gamma_2} g_2 \right\|$$

using the change of variables $(v, v_*, \sigma) \mapsto (v', v_*', \sigma')$ at fixed internal energies (we can use this change of variables since we consider reversible collisions: see [Des93]).

Combining the last inequality and (2.41) gives inequality (2.32) and symmetrically inequality (2.34).

It concludes the proof of lemma 2.5. ■

2.1.4 Derivative terms

We have the following result for the transport term of equation (2.19):

Lemma 2.6 (Transport term)

The following inequality holds:

$$\left| \iiint \partial_\alpha^{\beta, \gamma} g \partial_\alpha^{\beta, \gamma} (\nabla_x \cdot (gv)) dv dedx \right| \leq C \|g\|_{H^{|\alpha| + |\beta| + |\gamma|}}^2, \tag{2.43}$$

so that for all integer s ,

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (\nabla_x \cdot (gv)) dv dx \right| \leq C \|g\|_{H^s}^2. \quad (2.44)$$

Proof of Lemma 2.6:

From $\partial_\alpha^{\beta,\gamma} (\nabla_x \cdot (gv))$, only two kinds of terms appear: $\nabla_x \cdot \left(\left(\partial_\alpha^{\beta,\gamma} g \right) v \right)$ and $\partial_{\alpha+\delta}^{\beta-\delta,\gamma} g$ (only when $\beta \geq 1$ and with $|\delta| = 1$). Multiplying by $\partial_\alpha^{\beta,\gamma} g$, one sees that the first kind of term vanishes (integrating by parts), and that the second kind of term is controlled through

$$\begin{aligned} \left| \iiint \partial_\alpha^{\beta,\gamma} g \left(\partial_{\alpha+\delta}^{\beta-\delta,\gamma} g \right) dv dx \right| &\leq \|g\|_{H^{|\alpha|+|\beta|+|\gamma|}} \|g\|_{H^{|\alpha|+|\delta|+|\beta|-|\delta|+|\gamma|}} \\ &\leq \|g\|_{H^{|\alpha|+|\beta|+|\gamma|}}^2, \end{aligned} \quad (2.45)$$

so that the first result is proved. By summing over all the multi-indices, we get ineq. (2.44). \blacksquare

We have the following estimate for the drag force term:

Lemma 2.7 (Drag force term)

For $s \geq 5$, the following inequality holds:

$$\begin{aligned} \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} \nabla_v \cdot (gF) dv dx \right| &\leq C_1 (\|g\|_{H^s}^2 \sum_{1 \leq |\alpha| \leq s} \|\partial_\alpha u_g\| + \|g\|_{H^s}^2) \\ &\leq C_2 (\|g\|_{H^s}^2 \|u_g\|_{H^s} + \|g\|_{H^s}^2). \end{aligned} \quad (2.46)$$

Proof of Lemma 2.7:

We split gF in its two parts in order to get the result.

We first deal with the gu_g term. The quantity $\partial_\alpha^{\beta,\gamma} (\nabla_v \cdot gu_g)$ is made of terms like $\partial_{\alpha_1}^{\beta,\gamma} (\nabla_v g) \cdot \partial_{\alpha_2} (u_g)$ with $\alpha = \alpha_1 + \alpha_2$. Two different cases appear:

- $|\alpha_2| > 0$: using the same arguments as in lemma 2.4, one gets that $\partial_{\alpha_1}^{\beta,\gamma} (\nabla_v g) \cdot \partial_{\alpha_2} (u_g)$ is in $L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ and its norm is controlled by $C \|g\|_{H^s} \|\partial_{\alpha_2} u_g\|_{H^{s-|\alpha_2|}}$. One has to note that the condition $s \geq 4$ becomes $s \geq 5$ because in the case $|\alpha_2| > s - 2$ of lemma 2.4 (in that case $|\alpha_1| + |\beta| + |\gamma| \leq 1$), the L^∞ -norm of $\iint |\partial_{\alpha_1}^{\beta,\gamma} (\nabla_v g)|^2 dv dx$ is only controlled by $\|g\|_{H^{|\alpha_1|+|\beta|+1+|\gamma|+3}}$ and consequently by $\|g\|_{H^5}$.

- $\alpha_2 = 0$: in this case, we have

$$\begin{aligned} \iiint \partial_\alpha^{\beta,\gamma} g \left(\partial_\alpha^{\beta,\gamma} (\nabla_v g) \cdot (u_g) \right) dv dx &= \iiint \nabla_v \cdot \left(\frac{1}{2} \left(\partial_\alpha^{\beta,\gamma} g \right)^2 \right) \cdot u_g dv dx \\ &= 0 \text{ integrating by parts.} \end{aligned}$$

Eventually, for the gas part of the force we have

$$\left| \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \iiint \partial_\alpha^{\beta,\gamma} g \left(\partial_\alpha^{\beta,\gamma} (\nabla \cdot (gu_g)) \right) v dv dx \right| \leq C \|g\|_{H^s}^2 \sum_{1 \leq |\alpha| \leq s} \|\partial_\alpha u_g\|. \quad (2.47)$$

We now deal with the $-gv$ term in F . We do not cope with its negative sign since we obtain results in absolute values. First, we note that $\nabla_v \cdot (gv) = 3g + v \cdot \nabla_v g$, so that $\partial_\alpha^{\beta,\gamma} \nabla_v \cdot (gv)$ is made of terms like this:

- $3\partial_\alpha^{\beta,\gamma} g$: integrating, we get

$$\left| \iiint \partial_\alpha^{\beta,\gamma} g 3\partial_\alpha^{\beta,\gamma} g dv dx \right| \leq 3 \|\partial_\alpha^{\beta,\gamma} g\|^2.$$

- $v \cdot (\nabla_v (\partial_\alpha^{\beta,\gamma} g))$: integrating, we get

$$\begin{aligned} \left| \iiint (\partial_\alpha^{\beta,\gamma} g) v \cdot (\nabla_v (\partial_\alpha^{\beta,\gamma} g)) dv dx \right| &= \left| \iiint v \cdot \left(\nabla_v \cdot \left(\frac{1}{2} (\partial_\alpha^{\beta,\gamma} g)^2 \right) \right) dv dx \right| \\ &\leq \frac{3}{2} \|\partial_\alpha^{\beta,\gamma} g\|^2 \text{ integrating by parts.} \end{aligned}$$

- $\partial_0^{\beta_i,0} v \cdot \partial_\alpha^{\beta-\beta_i,\gamma} (\nabla_v g)$ with $|\beta_i| = 1$ (if $|\beta_i| \geq 2$, the first part $\partial_0^{\beta_i,0} v$ is null): immediate computations lead to

$$\partial_0^{\beta_i,0} v \cdot \partial_\alpha^{\beta-\beta_i,\gamma} (\nabla_v g) = \partial_0^{\beta_i,0} v_i \partial_\alpha^{\beta-\beta_i,\gamma} (\partial_0^{i,0} g) = \partial_\alpha^{\beta,\gamma} g.$$

We are back to the first type of terms which we were able to control.

Finally, for the gv part of gF , we have

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \left(\partial_\alpha^{\beta,\gamma} (\nabla_v (gv)) \right) dv dx \right| \leq C \|g\|_{H^s}^2. \quad (2.48)$$

Combining the result with the one above for the $-gu_g$ term, we get lemma 2.7. ■

We finally obtain this inequality for the last derivative term:

Lemma 2.8 (Thermic term)

For $s \geq 5$, the following inequality holds,

$$\begin{aligned} \left| \sum_{1 \leq |\alpha| \leq s} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (\partial_e (g\phi)) dv dx \right| &\leq C_1 (\|g\|_{H^s}^2 \sum_{1 \leq |\alpha| \leq s} \|\partial_\alpha e_g\| + \|g\|_{H^s}^2) \\ &\leq C_2 (\|g\|_{H^s}^2 \|\tilde{e}_g\|_{H^s} + \|g\|_{H^s}^2). \end{aligned} \quad (2.49)$$

We do not prove the result since the proof is exactly the same as the one for the drag force derivative term which was done just before.

2.1.5 Non linear terms

The most important result of this subsection is the following, which allows to recover the loss of moments due to the collision kernel:

Lemma 2.9 (*$gF \cdot v$ -term*)

For all ε such that $1 > \varepsilon > 0$,

$$\begin{aligned} \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (gF \cdot v) dv dx \\ \leq -(1-\varepsilon) \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 + C_1 \|g\|_{H^s}^2 + \frac{C_2}{\varepsilon} \|g\|_{H^s}^2 \|u_g\|_{H^s}^2. \end{aligned} \quad (2.50)$$

Proof of Lemma 2.9:

Let's note that $gF \cdot v = -g(v^2 - u_g \cdot v)$. We study the contributions of the two terms.

From $-\partial_\alpha^{\beta,\gamma}(gv^2)$, three types of terms appear (at worst):

- $-(\partial_\alpha^{\beta,\gamma}g)v^2$, so that integrating,

$$\iiint \partial_\alpha^{\beta,\gamma} g \left(-(\partial_\alpha^{\beta,\gamma}g)v^2 \right) dx dv = -\|\partial_\alpha^{\beta,\gamma}g\|_w^2 + \|\partial_\alpha^{\beta,\gamma}g\|^2,$$

- $-(\partial_\alpha^{\beta-\beta_i,\gamma}g)2v_i$ (if $|\beta| \geq 1$, with $|\beta_i| = 1$ and ∂^{β_i} the derivative in the v_i direction) so that integrating by parts,

$$\iiint \partial_\alpha^{\beta,\gamma} g \left(-(\partial_\alpha^{\beta-\beta_i,\gamma}g)2v_i \right) dx dv = \iiint \left(\partial_\alpha^{\beta-\beta_i,\gamma}g \right)^2 dv dx = \|\partial_\alpha^{\beta-\beta_i,\gamma}g\|^2,$$

- $-(\partial_\alpha^{\beta-\beta_i,\gamma}g)2$ (if $|\beta| \geq 2$ and with $|\beta_i| = 2$), so that

$$\iiint \partial_\alpha^{\beta,\gamma} g \left(-(\partial_\alpha^{\beta-\beta_i,\gamma}g)2 \right) \leq \|\partial_\alpha^{\beta-\beta_i,\gamma}g\|^2 + \|\partial_\alpha^{\beta,\gamma}g\|^2.$$

Eventually we sum the results, and get

$$\begin{aligned} & \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (gF \cdot v) dv dx \\ & \leq \sum_{|\alpha|+|\beta|+|\gamma|\leq s} -\|\partial_\alpha^{\beta,\gamma}g\|_w^2 + \|\partial_\alpha^{\beta,\gamma}g\|^2 + \sum_{\substack{|\alpha|+|\beta|+|\gamma|\leq s \\ |\beta|\geq 1, |\beta_i|=1}} \|\partial_\alpha^{\beta-\beta_i,\gamma}g\|^2 \\ & \quad + \sum_{\substack{|\alpha|+|\beta|+|\gamma|\leq s \\ |\beta|\geq 2, |\beta_i|=2}} \|\partial_\alpha^{\beta-\beta_i,\gamma}g\|^2 + \|\partial_\alpha^{\beta,\gamma}g\|^2 \\ & \leq 4\|g\|_{H^s}^2 - \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma}g\|_w^2. \end{aligned} \quad (2.51)$$

There remains to study the terms depending on the gas. Since $w = 1 + v^2$, we clearly have

$$\|gv\|_{H^s} \leq C \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 \right)^{1/2},$$

so that gv is in H^s .

Using lemma 2.4 for gv and u_g gives that $\partial_\alpha^{\beta,\gamma}(gv \cdot u_g)$ is in L^2 and that

$$\|\partial_\alpha^{\beta,\gamma}(gu_g \cdot v)\| \leq C \|u_g\|_{H^s} \|gv\|_{H^s} \quad (2.52)$$

$$\leq C \|u_g\|_{H^s} \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 \right)^{1/2}. \quad (2.53)$$

Finally, for the terms depending on the gas, we get

$$\iiint \left| \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (gu_g \cdot v) \right| \leq C \|g\|_{H^s} \|u_g\|_{H^s} \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 \right)^{1/2}.$$

Summing the results for all terms, we get

$$\begin{aligned} \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (gF \cdot v) dv dx &\leq - \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 + 4 \|g\|_{H^s}^2 \\ &\quad - C \|g\|_{H^s} \|u_g\|_{H^s} \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 \right)^{1/2}. \end{aligned} \quad (2.54)$$

Using the inequality $2ab \leq a^2 + b^2$, we get

$$C \|g\|_{H^s} \|u_g\|_{H^s} \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2 \right)^{1/2} \leq \frac{1}{2\varepsilon} C^2 \|g\|_{H^s}^2 \|u_g\|_{H^s}^2 + \frac{1}{2}\varepsilon \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \|\partial_\alpha^{\beta,\gamma} g\|_w^2,$$

and putting the result in the next-to-last inequality ends the proof of lemma 2.9. ■

Lemma 2.10 (*gφ-term*)

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left| \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma} (g\phi) dv dx \right| \leq C_1 \|g\|_{H^s}^2 (1 + \|\tilde{e}_g\|_{H^s}). \quad (2.55)$$

Proof of Lemma 2.10:

We split $g\phi$ into its two parts $-ge$ and ge_g .

Only two types of terms come from the $\partial_\alpha^{\beta,\gamma}(-ge)$ term: $(\partial_\alpha^{\beta,\gamma-\gamma_i}(-g))$ and $(\partial_\alpha^{\beta,\gamma}(-g))e$ (with $\gamma_i \leq \gamma$ and $|\gamma_i| = 1$). Finally, multiplying by $(\partial_\alpha^{\beta,\gamma}g)$, we get

$$\begin{aligned} & \sum_{|\alpha|+|\beta|+|\gamma|\leq s, \gamma_i \leq \gamma, |\gamma_i|=1} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma-\gamma_i}(-g) dv dx \\ &= - \sum_{|\alpha|+|\beta|+|\gamma|\leq s, \gamma_i \leq \gamma, |\gamma_i|=1} \iiint \partial_0^{\beta,\gamma_i} \left(\partial_\alpha^{\beta,\gamma-\gamma_i} \left(\frac{g^2}{2} \right) \right) dv dx \\ &= 0, \end{aligned} \tag{2.56}$$

and

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma}(-g) e dv dx \leq 0, \tag{2.57}$$

since internal energies of the particles are positive.

For the $\partial_\alpha^{\beta,\gamma}(ge_g)$ terms, since $\tilde{e}_g = e_g - 1$ is in $H^s(\mathbb{R}^3)$ and g is in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ and thanks to lemma 2.4, we get that $ge_g = g + g\tilde{e}_g$ is in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ so that for all $|\alpha| + |\beta| + |\gamma| \leq s$, $\partial_\alpha^{\beta,\gamma}(ge_g)$ is in $L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ and

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \iiint \partial_\alpha^{\beta,\gamma} g \partial_\alpha^{\beta,\gamma}(ge_g) dv dx \leq C \|g\|_{H^s}^2 (1 + \|\tilde{e}_g\|_{H^s}). \tag{2.58}$$

It ends the proof of lemma 2.10. ■

2.1.6 A priori estimate

Thanks to all the previous results, we are able to get the following a priori estimate for solutions of the modified Boltzmann equation (2.19). We now sum up all the results through the following proposition:

Proposition 2.11

Let g be a nonnegative solution of (2.19) in $\mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$, with ρ_g and e_g strictly positive functions and $\tilde{\rho}_g = \rho_g - 1$ and $\tilde{e}_g = e_g - 1$ belonging to $\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3))$ for some $T > 0$. For all $s \geq 5$, $1 > \varepsilon > 0$, we have the following a priori estimate for the energy defined $\mathcal{E}_{s,\varepsilon}$ by (2.18),

$$\frac{d}{dt} \mathcal{E}_{s,\varepsilon} \leq \frac{C}{\varepsilon} (1 + \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} + \mathcal{E}_{s,\varepsilon}) \mathcal{E}_{s,\varepsilon}. \tag{2.59}$$

where C depends only on the physical constants and s .

Proof of Proposition 2.11:

First, let's recall that

$$\mathcal{E}_{s,\varepsilon}(g(t, x, v, e)) = \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left[\frac{1}{2} \left\| \partial_\alpha^{\beta,\gamma} g \right\|^2 + \int_0^t 2(1-\varepsilon) \left\| \partial_\alpha^{\beta,\gamma} g(u, \dots, \dots) \right\|_w^2 \right] du.$$

For the collision term, we use the lemma 2.5 putting $g_1 = g_2 = g_3 = g$ so that using the two different estimates for both Γ_0^+ and Γ_0^- ,

$$\iiint \partial_\alpha^{\beta,\gamma} \Gamma(g, g) \partial_\alpha^{\beta,\gamma} g \leq C_1 \|g\|_{H^s}^2 \left\| \partial_\alpha^{\beta,\gamma} g \right\|_w.$$

The condition $s \geq 5$ is used in the last inequality since we need $\partial_{\alpha_1}^{\beta_1,\gamma_1} g_1$ or $\partial_{\alpha_2}^{\beta_2,\gamma_2} g_2$ to be in H^3 (with $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$ and $\gamma = \gamma_1 + \gamma_2$). It is the case as soon as the integer part of $\frac{s}{2}$ is less than $s - 3$ or equivalently $s \geq 5$.

We sum all the contributions of the different terms of equation (2.19), using lemmas 2.5 to 2.10. Eventually, we obtain the following inequality:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|g\|_{H^s}^2 &\leq C \|g\|_{H^s}^2 \left[\left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left\| \partial_\alpha^{\beta,\gamma} g \right\|_w^2 \right)^{1/2} + 1 + \|u_g\|_{H^s} + \frac{2}{\varepsilon} \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} \right] \\ &\quad - 2 \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta,\gamma} g(s) \right\|_w^2, \end{aligned} \quad (2.60)$$

using the lemma 2.9 on the non-linear drag term for $\frac{1}{2}\varepsilon$ instead of ε (the 2 factor comes from equation (2.19)). The condition $s \geq 5$ also comes from the control of the derivative terms in lemmas 2.7 and 2.8.

Using once again $2ab \leq \frac{1}{2\varepsilon} a^2 + 2\varepsilon b^2$ on $C \|g\|_{H^s}^2 \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left\| \partial_\alpha^{\beta,\gamma} g \right\|_w^2 \right)^{1/2}$, we obtain the following inequality:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|g\|_{H^s}^2 + \sum_{|\alpha|+|\beta|+|\gamma|\leq s} 2\left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta,\gamma} g(s) \right\|_w^2 &\leq \\ \frac{1}{2\varepsilon} C \|g\|_{H^s}^2 (1 + \|u_g\|_{H^s} + \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} + \|g\|_{H^s}^2) + \varepsilon \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left\| \partial_\alpha^{\beta,\gamma} g(s) \right\|_w^2, \end{aligned} \quad (2.61)$$

so that

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|g\|_{H^s}^2 + \sum_{|\alpha|+|\beta|+|\gamma|\leq s} 2(1-\varepsilon) \left\| \partial_\alpha^{\beta,\gamma} g(s) \right\|_w^2 &\leq \\ \frac{1}{\varepsilon} C_1 \|g\|_{H^s}^2 (1 + \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} + \|g\|_{H^s}^2). \end{aligned} \quad (2.62)$$

Using that $\frac{1}{2}\|g\|_{H^s}^2 \leq \mathcal{E}_{s,\varepsilon}$ ends the proof of proposition 2.11. ■

2.2 Solving the Boltzmann equation with "fixed gas"

We now want use proposition 2.11 in order to prove theorem 2.1. We first prove the existence of solutions thanks to an iterative scheme as it is done in [Guo03] and [BD06]. Then we prove uniqueness.

Proof of theorem 2.1:

For the sake of simplicity, we shall suppose that the initial data g_0 is C^∞ and compactly supported (to be able to do integrations by parts), and ρ_g, u_g, e_g are also smooth C^∞ functions.

2.2.1 Existence: iterative scheme and characteristics

We adopt the following iterative scheme in order to prove the existence of a solution to equation (2.19) in theorem 2.1:

$$\partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \nabla_v \cdot (g^{n+1} F) + \partial_e (g^{n+1} \phi) = \Gamma^+(g^n, g^n) - \Gamma^-(g^n, g^{n+1}) + 2v \cdot g^{n+1} F + g^{n+1} \phi, \quad (2.63)$$

$$g(0, \cdot, \cdot, \cdot) = g_0(\cdot, \cdot, \cdot). \quad (2.64)$$

The first iterate g^0 of the sequel $(g^n)_{n \in \mathbb{N}}$ is equal to the constant function (in time) g_0 .

There are two main reasons for using this iterative scheme. First, we ensure positiveness of all iterates if g^0 is positive by taking $\Gamma^-(g^n, g^{n+1})$. Secondly, we can use the characteristics method in order to prove the existence of all iterates (see [BD06]).

We note $X(s; x, v, e, t), V(s; x, v, e, t), E(s; x, v, e, t)$ the characteristics linked to the flow of the modified Boltzmann equation (2.19). They satisfy the following equations:

$$\begin{aligned} \frac{d}{dt} X &= V & , & \quad X(s; x, v, e, s) = x, \\ \frac{d}{dt} V &= -(V - u_g(t, X)) & , & \quad V(s; x, v, e, s) = v, \\ \frac{d}{dt} E &= -(E - e_g(t, X)) & , & \quad E(s; x, v, e, s) = e. \end{aligned}$$

The characteristics are well defined since ρ_g, u_g and e_g are smooth functions. We note Y the point in the phase space (X, V, E) .

We notice that since $\frac{d}{dt} E = -(E - e_g(X))$, E is due to remain positive and the internal energy of the particles remains positive. Then, g takes its values in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$.

2.2.2 Solution

We now give the expression of the solution of the linear Vlasov equation appearing in (2.63) using the characteristics. Equation (2.63) can be rewritten

$$\partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + (u_g - v) \cdot \nabla_v g^{n+1} + (e_g - e) \partial_e g^{n+1} = \Gamma^+(g^n, g^n) + h(g_n) g^{n+1}, \quad (2.65)$$

where $h(g_n)$ is defined through

$$\begin{aligned} h(g_n)(t, x, v, e) &= 4 + 2v \cdot (u_g - v) + (e_g - e) \\ &- \iiint_{\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{S}^2} |v - v_*| \exp(-(v_*^2 + e_*)) g_n(t, x, v_*, e_*) dv_* de_* d\sigma. \end{aligned} \quad (2.66)$$

The solution of the iterative scheme at step n can be written explicitly thanks to the formula

$$\begin{aligned} g^{n+1}(t, x, v, e) &= \exp\left(\int_0^t h(g_n)(\tau, Y(\tau; x, v, e, t)) d\tau\right) g_0(Y(0; x, v, e, t)) \\ &+ \int_0^t \exp\left(\int_\tau^t h(g_n)(\tau, Y(\tau; x, v, e, t)) d\tau\right) \Gamma^+(g_n, g_n)(Y(\tau; x, v, e, t)) d\tau. \end{aligned} \quad (2.67)$$

Note that all iterates are compactly supported and \mathcal{C}^∞ by induction since $g^0 = g_0$ has these properties. Contrary to [BD06], we do not need to control the support of all iterates: because of collisions, we already know that the limit of the sequence g_n is not compactly supported ... Moreover, the formula shows that all iterates remain positive.

2.2.3 Convergence of the iterative scheme, uniqueness

We prove the convergence of the scheme using Picard's fixed point theorem. We define the following application G :

$$\begin{aligned} G &: E_{s,1/2,T'} \cap \mathcal{C}^\infty \mapsto \mathcal{C}^\infty, \\ &k \mapsto \ell \end{aligned} \quad (2.68)$$

where ℓ is the unique solution given by formula (2.67) of the equation:

$$\partial_t \ell + v \cdot \nabla_x \ell + (u_g - v) \cdot \nabla_v \ell + (e_g - e) \partial_e \ell = \Gamma^+(k, k) + h(k) \ell, \quad (2.69)$$

$$\ell(0, \cdot) = g(0, \cdot) \in \mathcal{C}^\infty. \quad (2.70)$$

The operator G is well-defined: ℓ exists, is nonnegative and smooth (k is nonnegative). We recall that $E_{s,1/2,T'}$ is complete: in order to use Picard's theorem, it remains to prove that for each g_0 there exists a radius R and T' small enough, such that the application maps the ball of radius R of $E_{s,1/2,T'}$ into itself and is a contraction on this ball. Eventually, once we know that the fixed point exists, it will lie in $E_{s,1/2,T'}$, and therefore to $\mathcal{C}([0, T'], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$. Using equation (2.19), one gets that the

partial derivative in time of the fixed point will lie in $\mathcal{C}([0, T'], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ so that the fixed point belongs to $\mathcal{C}^1([0, T'], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.

We choose the radius R of the ball in the following way:

$$R = \|g_0\|_{H^s} = 2\mathcal{E}_{s,1/2}(g_0)(0)^{\frac{1}{2}}.$$

Then, we choose T' :

$$T' = \frac{3}{4C \left[1 + \sup_{0 \leq t \leq T} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g\|_{H^s} + 2R^2 \right]}, \quad (2.71)$$

for some C that is given later.

We prove that the ball of radius R of $E_{s,1/2,T'}$ is stable. We suppose k to be in this ball and we show that $\ell = G(k)$ is in this ball. Using all the lemmas of the a priori estimate (lemma 2.5 to lemma 2.10), one gets for all $1 > \varepsilon > 0$,

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\ell\|_{H^s}^2 &\leq C \|\ell\|_{H^s}^2 \left[\left(\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left\| \partial_\alpha^{\beta,\gamma} \ell \right\|_w^2 \right)^{1/2} + 1 + \|u_g\|_{H^s} + \frac{2}{\varepsilon} \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} \right] \\ &\quad - 2 \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta,\gamma} \ell \right\|_w^2 \\ &\quad + C \|k\|_{H^s} (\|k\|_{H^s} + \|\ell\|_{H^s}) \left(\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left\| \partial_\alpha^{\beta,\gamma} \ell \right\|_w^2 \right)^{1/2} \\ &\leq C \|\ell\|_{H^s}^2 \left[1 + \|u_g\|_{H^s} + \frac{2}{\varepsilon} \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} \right] \\ &\quad + \frac{C^2}{2\varepsilon} \|\ell\|_{H^s}^4 + \frac{\varepsilon}{2} \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left\| \partial_\alpha^{\beta,\gamma} \ell \right\|_w^2 \\ &\quad - 2 \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta,\gamma} \ell \right\|_w^2 \\ &\quad + \frac{C^2}{2\varepsilon} \|k\|_{H^s}^2 (2\|k\|_{H^s}^2 + 2\|\ell\|_{H^s}^2) + \frac{\varepsilon}{2} \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left\| \partial_\alpha^{\beta,\gamma} \ell \right\|_w^2. \end{aligned} \quad (2.72)$$

The only difference with the proof of proposition 2.11 is that the functions in the collision kernel are not the same: there is a "mixed term" in k and ℓ . Choosing $\varepsilon = \frac{1}{2}$, one gets

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{s,1/2}(\ell)(t) &\leq C \left[\mathcal{E}_{s,1/2}(\ell)(t) \left(1 + \mathcal{E}_{s,1/2}(\ell)(t) + \sup_{0 \leq t \leq T} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g\|_{H^s} \right) \right. \\ &\quad \left. + (\mathcal{E}_{s,1/2}(k)(t))^2 \right], \end{aligned} \quad (2.73)$$

for some C only depending on s (we note C for all constants).

Let's T_1 be the first time for which $\mathcal{E}_{s,1/2}(\ell)(t) = R^2$ on $[0, T']$ (it may not exist but in this case, the ball is stable since we have $\mathcal{E}_{s,1/2}(\ell)(0) < R^2$ on $[0, T']$). For all $t \leq T_1$, we have

$$\frac{d}{dt}\mathcal{E}_{s,1/2}(\ell)(t) \leq C \left[R^2 \left(1 + R^2 + \sup_{0 \leq t \leq T} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g\|_{H^s} \right) + R^4 \right], \quad (2.74)$$

so that

$$\begin{aligned} \frac{3R^2}{4} = R^2 - \frac{1}{4}R^2 &= \mathcal{E}_{s,1/2}(\ell)(T_1) - \mathcal{E}_{s,1/2}(\ell)(0) \\ &\leq CT_1 \left[R^2 \left(1 + 2R^2 + \sup_{0 \leq t \leq T} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g\|_{H^s} \right) \right], \end{aligned} \quad (2.75)$$

since k is in the ball of radius R of $E_{s,1/2,T'}$. Eventually, $T_1 = T'$, and we get that the ball of radius R of the space $\|\cdot\|_{E_{s,1/2,T'}}$ is stable through the application G .

There remains to prove that G is contracting. Let k, ℓ be two functions in $E_{s,1/2,T'}$ in the ball of radius R and let's define m as $m = G(k) - G(\ell)$. It satisfies the following equations:

$$\begin{aligned} \partial_t m + v \cdot \nabla_x m + \nabla_v \cdot (mF) + \partial_e(m\phi) = \\ \Gamma^+(k, k) - \Gamma^-(k, G(k)) - (\Gamma^+(\ell, \ell) - \Gamma^-(\ell, G(\ell))) + 2v \cdot mF + m\phi, \end{aligned} \quad (2.76)$$

$$m(0, \cdot, \cdot, \cdot) = 0. \quad (2.77)$$

All the linear terms in m can be controlled, using lemma 2.5 to lemma 2.10. There remains to use the bilinearity of the collision kernel in order to obtain the a priori estimate on m . Since we have the two following equalities:

$$\Gamma^+(k, k) - \Gamma^+(\ell, \ell) = \Gamma^+(k - \ell, k) + \Gamma^+(\ell, k - \ell), \quad (2.78)$$

$$\Gamma^-(k, G(k)) - \Gamma^-(\ell, G(\ell)) = \Gamma^-(k - \ell, G(k)) + \Gamma^-(\ell, m), \quad (2.79)$$

we get an a priori estimate using lemma 2.5 on the collision kernel:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|m\|_{H^s}^2 \leq C \|m\|_{H^s}^2 \left[1 + \|u_g\|_{H^s} + \frac{2}{\varepsilon} \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} \right] - 2 \sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta, \gamma} m \right\|_w^2 \\ + C \left(\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left\| \partial_\alpha^{\beta, \gamma} m \right\|_w^2 \right)^{1/2} [(\|k\|_{H^s} + \|\ell\|_{H^s} + \|G(k)\|_{H^s}) \|k - \ell\|_{H^s} + \|\ell\|_{H^s} \|m\|_{H^s}]. \end{aligned} \quad (2.80)$$

Using that k, ℓ and $G(k)$ are in the ball of radius R ,

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|m\|_{H^s}^2 &\leq C \|m\|_{H^s}^2 \left[1 + \|u_g\|_{H^s} + \frac{2}{\varepsilon} \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} \right] - 2 \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta,\gamma} m \right\|_w^2 \\ &\quad + C [3R\|k - \ell\|_{H^s} + R\|m\|_{H^s}] \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left\| \partial_\alpha^{\beta,\gamma} m \right\|_w^2 \right)^{1/2}, \end{aligned} \quad (2.81)$$

so that

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|m\|_{H^s}^2 &\leq C \|m\|_{H^s}^2 \left[1 + \|u_g\|_{H^s} + \frac{2}{\varepsilon} \|u_g\|_{H^s}^2 + \|\tilde{e}_g\|_{H^s} \right] - 2 \sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left(1 - \frac{\varepsilon}{2}\right) \left\| \partial_\alpha^{\beta,\gamma} m \right\|_w^2 \\ &\quad + \frac{1}{2} C^2 [3R\|k - \ell\|_{H^s} + R\|m\|_{H^s}]^2 + \frac{1}{2} \left(\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left\| \partial_\alpha^{\beta,\gamma} m \right\|_w^2 \right). \end{aligned} \quad (2.82)$$

Choosing $\varepsilon = \frac{1}{2}$ and still denoting by C a generic constant,

$$\frac{d}{dt} \mathcal{E}_{s,1/2}(m)(t) \leq C \|m\|_{E_{s,1/2,T'}}^2 \left[1 + \sup_{0 \leq t \leq T'} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T'} \|\tilde{e}_g\|_{H^s} + R^2 \right] + CR^2 \|k - \ell\|_{E_{s,1/2,T'}}^2, \quad (2.83)$$

so that for all $t \leq T'$,

$$\begin{aligned} \mathcal{E}_{s,1/2}(m)(t) - 0 &\leq \\ &\left(C \|m\|_{E_{s,1/2,T'}}^2 \left[1 + \sup_{0 \leq t \leq T'} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T'} \|\tilde{e}_g\|_{H^s} + R^2 \right] + CR^2 \|k - \ell\|_{E_{s,1/2,T'}}^2 \right) t. \end{aligned} \quad (2.84)$$

Taking the sup over $\{t/0 \leq t \leq T'\}$, one gets

$$\left(1 - CT' \left[1 + \sup_{0 \leq t \leq T'} \|u_g\|_{H^s}^2 + \sup_{0 \leq t \leq T'} \|\tilde{e}_g\|_{H^s} + R^2 \right] \right) \|m\|_{E_{s,1/2,T'}}^2 \leq T' CR^2 \|k - \ell\|_{E_{s,1/2,T'}}^2. \quad (2.85)$$

Finally, G is contracting for T' small enough. Using Picard's theorem, one gets that the modified Boltzmann equation (2.19) has a solution g in $E_{s,1/2,T'}$ (limit of the sequence g_n defined by eq.(2.63) and eq.(2.64)).

We now explain how we have obtained the time of existence given in theorem 2.1. For ordinary differential equations with locally Lipschitz coefficients, it is known that if the solution is bounded on $[0, T_1]$, then it can be extended for a time T_2 strictly greater than T_1 using Cauchy's theorem. The same result holds here as long as the energy of g is bounded, it can be extended and the a priori estimate given in proposition 2.11 controls the solution. Using this a priori estimate, we obtain the time given in theorem 2.1.

Obviously, the uniqueness of the solution comes from the proof of the constricting behavior of G . Theorem 2.1 is now proved.

We conclude this subsection with some remarks.

Remark 2.1 (Boltzmann equation)

A solution g of the modified Boltzmann equation (2.19) becomes a classical \mathcal{C}^1 solution f of the Boltzmann equation (2.14) where $f(t, x, v, e) = g(t, x, v, e) \exp(-(v^2 + e))$ and

$$\sum_{|\alpha|+|\beta|+|\gamma|\leq s} \left[\frac{1}{2} \left\| \partial_{\alpha}^{\beta, \gamma} (f \exp(v^2 + e)) \right\|^2(t) + \int_0^t \left\| \partial_{\alpha}^{\beta, \gamma} (f \exp(v^2 + e)) \right\|_w^2(u) du \right] < +\infty.$$

In order to obtain the full theorem for non smooth ¹ data, one has to make successive approximations of the initial data and regularize ρ_g, u_g, e_g (see [Ser96a, BD06])

¹ by smooth we mean \mathcal{C}^∞

2.3 Gas-particle system: existence and uniqueness of H^s -solutions

We now prove theorem 2.2. In order to obtain this theorem, we combine theorem 2.1 and results on hyperbolicity for perfect gases (see [Ser96a] and [Maj84]).

The proof of theorem 2.2 is close to what is done in [BD06], especially for the hyperbolic aspect of the problem. Therefore we only describe the main steps of the proof. We adopt the following plan:

- in subsection 2.3.1, we first recall some classical results for hyperbolic systems (especially in the linearized case),
- in subsection 2.3.2, we write down an iterative scheme based on the results obtained for the Boltzmann-Vlasov equation at "fixed gas" (theorem 2.1) and on linearized hyperbolic systems,
- in subsection 2.3.3, we finally pass to the limit in the scheme in order to obtain theorem 2.2.

2.3.1 Hyperbolic part of the system

We recall here some facts about hyperbolicity of perfect gases (see [Ser96a] for far more details). Let's define $I =]0, +\infty[\times \mathbb{R}^3 \times]0, +\infty[$ as the space in which (ρ_g, u_g, e_g) lies. The system of equations for perfect gases (2.11)-(2.12)-(2.13) can be put in a symmetrized form:

Proposition 2.12

We note $U_g = {}^t(\rho_g, u_g, e_g)$, $\delta_{j,i}$ the Kronecker's symbol, and c the sound speed of the gas defined by $c := \sqrt{\gamma(\gamma - 1)e_g}$. System (2.11)-(2.12)-(2.13) is symmetrized thanks to the following formula:

$$S(U_g)\partial_t U_g + \sum_i (SA_i)(U_g)\partial_{x_i} U_g = S(U_g) b(U_g, f), \quad (2.86)$$

where

$$S = \begin{pmatrix} \left(\frac{(\gamma-1)e_g}{\rho_g}\right)^2 & 0 & 0 \\ 0 & \frac{1}{\gamma}c^2 & 0 \\ 0 & 0 & (\gamma - 1) \end{pmatrix},$$

for $i = 1, 2, 3$,

$$A_i = u_{g_i} Id_5 + \begin{pmatrix} 0 & \rho_g \delta_{1,i} & \rho_g \delta_{2,i} & \rho_g \delta_{3,i} & 0 \\ \frac{(\gamma-1)e_g}{\rho_g} \delta_{1,i} & 0 & 0 & 0 & e_g \delta_{1,i} \\ \frac{(\gamma-1)e_g}{\rho_g} \delta_{2,i} & 0 & 0 & 0 & e_g \delta_{2,i} \\ \frac{(\gamma-1)e_g}{\rho_g} \delta_{3,i} & 0 & 0 & 0 & e_g \delta_{3,i} \\ 0 & (\gamma-1)e_g \delta_{1,i} & (\gamma-1)e_g \delta_{2,i} & (\gamma-1)e_g \delta_{3,i} & 0 \end{pmatrix},$$

and

$$b = \begin{pmatrix} 0 \\ - \int \frac{1}{\rho_g} f(u_{g_1} - v) dvde \\ - \int \frac{1}{\rho_g} f(u_{g_2} - v) dvde \\ - \int \frac{1}{\rho_g} f(u_{g_3} - v) dvde \\ - \int \frac{1}{\rho_g} f(e_g - e) dvde + \int \frac{1}{\rho_g} f(u_g - v)^2 dvde \end{pmatrix}.$$

For $s \geq 5$, if $f \exp(v^2 + e)(t, \cdot)$ is in $H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$, if moreover $(\rho_g - 1, u_g, e_g - 1)(t, \cdot)$ is in $(H^s(\mathbb{R}^3))^5$ and belongs to I_2 , then b is in $H^s(\mathbb{R}^3)$ and we have the following control:

$$\begin{aligned} \|b(t, \cdot)\|_{H^s(\mathbb{R}^3)} &\leq C(s, \overline{I_2}) \times (1 + \|\rho_g - 1\|_{H^s}) \\ &\quad \times \|f \exp(v^2 + e)(t, \cdot)\|_{H^s} \times (1 + \|e_g - 1\|_{H^s} + \|u_g\|_{H^s}^2), \end{aligned} \quad (2.87)$$

with C constant only depending on the compact set $\overline{I_2}$ and on the Sobolev index s .

Moreover, the symmetric definite positive matrix $S(U_g)$ is a smooth function of U_g satisfying

$$d Id_5 \leq S(U_g) \leq d^{-1} Id_5, \quad (2.88)$$

when $U_g \in I_1$ (or I_2), for some constant $d > 0$ (depending on I_1 (or I_2)). Finally, all the matrices $SA_i(U_g)$ are symmetric.

Proof of Proposition 2.12:

The computations on the matrices are trivial. Inequality (2.88) is a consequence of the diagonal form of the matrix S . The H^s regularity of b is a consequence of lemma 2.4. Contrary to [BD06], we do not need compact support for the estimate (2.87) on b : the exponential decay of f is sufficient. ■

In fact, we shall only consider a linearized hyperbolic problem of the type

$$S(U)\partial_t V + \sum_i (SA_i)(U)\partial_{x_i} V = S(U)b(U, f), \quad (2.89)$$

$$V(0, \cdot) = U(0, \cdot) \in I_1, \quad (2.90)$$

in the iterative scheme. Here, $U = (U_1, U_2, U_3)$ and $V = (V_1, V_2, V_3)$ are in $(\mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+)$. The matrices SA_i are the ones defined above.

This linear system is strictly hyperbolic as soon as $U_1 > 0$ and $U_3 > 0$. Then, if f is known and regular enough ($f \exp(v^2 + e) \in \mathcal{C}([0, T]; H^s(\mathbb{R}^7))$), there exists a (local in time) solution to (2.89) – (2.90) since $b(t, \cdot)$ belongs to $H^s(\mathbb{R}^3)$. This solution exists as long as f and U exist but may stay in I_2 for a shorter time. More precisely, we have the following a priori estimate for these hyperbolic systems with smooth coefficient (cf. [Maj84], [Ser96a] and [AG91]):

Proposition 2.13 (Control of linear hyperbolic systems)

Let s be an integer such that $s \geq 5$. If $f \exp(v^2 + e)$ is in $\mathcal{C}([0, T]; H^s(\mathbb{R}^7))$ and $U - U(0, \cdot)$ is in $\mathcal{C}([0, T]; (H^s(\mathbb{R}^3))^5)$, then the linearized problem (2.89)-(2.90) has a solution V such that $V(t, \cdot) - V(0, \cdot)$ belongs to $\mathcal{C}([0, T]; (H^s(\mathbb{R}^3))^5) \cap \mathcal{C}^1([0, T]; (H^{s-1}(\mathbb{R}^3))^5)$, and such that for a strictly positive time T_p , V remains in I_2 on $[0, T_p]$. Besides, the following inequality holds:

$$\forall t \leq T_p, \|V(t, \cdot) - V(0, \cdot)\|_{H^s(t)} \leq Ct, \quad (2.91)$$

where C is a constant depending on $\sup_{0 \leq u \leq T} \|f \exp(v^2 + e)(u, \cdot)\|_{H^s}$, $\sup_{0 \leq u \leq T} \|U(u, \cdot) - U(0, \cdot)\|_{H^s}$, the compact set $\overline{I_2}$, and the Sobolev index s .

2.3.2 Iterative scheme

We now start the proof of theorem 2.2.

We recall that $U_{g_0} - {}^t(1, 0, 0, 0, 1) \in H^s$ and that $f_0 \exp(v^2 + e) \in H^s$. Furthermore, we recall that $U_{g_0}(x)$ is in I_1 for all x . We define R by $R := \|f_0 \exp(v^2 + e)\|_{H^s}$.

We define by induction the following iterative scheme,

$$U_g^0(t, \cdot) = U_{g_0}(\cdot), \quad (2.92)$$

$$f^0(t, \cdot) = f_0(\cdot), \quad (2.93)$$

$$T_0 = +\infty. \quad (2.94)$$

Note that U_g^0 lies in I_2 since U_{g_0} is in I_1 .

Then,

$$S(U_g^k) \partial_t U_g^{k+1} + \sum_{i=1}^N (SA_i)(U_g^k) \partial_{x_i} U_g^{k+1} = S(U_g^k) b(U_g^k, f^k), \quad (2.95)$$

$$U_g^{k+1}(0, x) = U_0(x), \quad (2.96)$$

$$\partial_t f^{k+1} + \nabla_x \cdot (v f^{k+1}) + \nabla_v \cdot (f^{k+1}(u_g^k - v)) + \partial_e (f^{k+1}(e_g^k - e)) = Q(f^{k+1}, f^{k+1}), \quad (2.97)$$

$$f^{k+1}(0, x, v) = f_0(x, v). \quad (2.98)$$

We define T_{k+1} (for $k \geq 0$) as the minimum of:

- the maximal time of existence for U_g^{k+1} (which is T_k thanks to proposition 2.13),
- the maximal time such that U_g^{k+1} remains in $\overline{I_2}$,
- the maximal time such that $\|U_g^{k+1} - U_{g_0}\|_s \leq 2\|U_{g_0}^{-t}(1, 0, 0, 0, 1)\|_s$,
- the maximal time of existence for f^{k+1} ,
- the maximum time such that $\forall t/0 \leq t \leq T_{k+1}$,

$$\sum_{|\alpha|+|\beta|+|\gamma| \leq s} \left[\frac{1}{2} \left\| \partial_{\alpha}^{\beta, \gamma} \left(f^{k+1} \exp(v^2 + e) \right) \right\|^2(t) + \int_0^t \left\| \partial_{\alpha}^{\beta, \gamma} \left(f^{k+1} \exp(v^2 + e) \right) \right\|_w^2(u) du \right] \leq R^2. \quad (2.99)$$

The iterates are well defined since initial data and the first iterate (U_g^0, f_0) are in H^s . U_g^{k+1} and f^{k+1} are solved "separately". The hyperbolic part is solved using results on hyperbolic linear systems with smooth coefficients (proposition 2.13). The kinetic part is solved using theorem 2.1 for the Boltzmann equation with fixed gas: for T_{k+1} small enough strictly positive, the condition (2.99) is verified. Finally using the results of regularity in proposition 2.13 and theorem 2.1, $\rho_g^{k+1} - \rho_{g_0}$, u_g^{k+1} , $e_g^{k+1} - e_{g_0}$ $\in \mathcal{C}([0, T_{k+1}], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T_{k+1}], H^{s-1}(\mathbb{R}^3))$, and $g^{k+1} = f^{k+1} \exp(v^2 + e) \in \mathcal{C}([0, T_{k+1}], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T_{k+1}], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$.

The aim is now to prove that the sequence of existence times $(T_k)_{k \in \mathbb{N}}$ has a strictly positive lower bound T .

Using the conditions giving T_k , one gets that

$$\sup_{0 \leq t \leq T_k} \|U_g^k - U_{g_0}\|_s \leq 2\|U_{g_0}^{-t}(1, 0, 0, 0, 1)\|_s. \quad (2.100)$$

Using the control on the hyperbolic system (proposition 2.13), we get

$$\forall t \leq T_{k+1}, \|U_g^{k+1} - U_{g_0}\|_{H^s}(t) \leq C_1 t, \quad (2.101)$$

where C_1 depends on $2\|U_{g_0}^{-t}(1, 0, 0, 0, 1)\|_s$, I_2 and R .

Moreover, using Sobolev embeddings, there exists C_2 such that

$$\begin{aligned} \|U_g^{k+1} - U_{g_0}\|_{L^\infty} &\leq C_2 \|U_g^{k+1} - U_{g_0}\|_{H^s} \\ &\leq C_1 C_2 t. \end{aligned} \quad (2.102)$$

Since $\bar{I}_1 \subset I_2$, the distance $\text{dist}(fr(I_2), I_1)$ between the frontier of I_2 (called $fr(I_2)$) and I_1 is strictly positive. As long as $t \leq \frac{1}{C_1 C_2} \text{dist}(fr(I_2), I_1)$, U_g^{k+1} remains in I_2 .

There remains to obtain the time for which all iterates f_k satisfy $\|f^k\|_{E_{s,1/2,T_k}}^2 \leq \|f_0 \exp(v^2 + e)\|_{H^s}^2$. Using the theorem 2.1, one gets that f_k is at least defined for all t such that

$$\begin{aligned} t &\geq \frac{3}{8C \left[1 + \sup_{0 \leq t \leq T} \|u_g^{k-1}\|_{H^s}^2 + \sup_{0 \leq t \leq T} \|\tilde{e}_g^{k-1}\|_{H^s} + \frac{3}{2} \|f_0 \exp(v^2 + e)\|_{H^s}^2 \right]} \\ &\geq \frac{3}{8C \left[1 + \sup_{0 \leq t \leq T} 2\|U_{g_0}^{-t}(1, 0, 0, 0, 1)\|_s + \frac{3}{2} \|f_0 \exp(v^2 + e)\|_{H^s}^2 \right]}. \end{aligned} \quad (2.103)$$

Using this inequality, one gets a time T_* independent of k (since the constants no longer depend on k) for which all iterates f^k satisfy $\|f^k\|_{E_{s,1/2,T_*}}^2 \leq \|f_0 \exp(v^2 + e)\|_{H^s}^2$. Eventually, taking the minimum T of T_* and of $\frac{d}{C_1 C_2} \text{dist}(fr(I_2), I_1)$, one gets a time T strictly positive for which the scheme is stable: all T_k are larger than T .

2.3.3 Passing to the limit

We pass to the limit. To begin with, the following proposition holds:

Proposition 2.14

We consider the sequence T_k , U_g^k , f^k of the iterative scheme, and $T > 0$ the time of stability of the scheme. Then one can find $T_ \in]0, T[$ such that (for $k \geq 2$),*

$$\sup_{0 \leq t \leq T_*} \| (f^k - f^{k-1}) \exp(v^2 + e) \| \leq C_1(I_2, s, U_{g_0}, f_0) \sup_{0 \leq t \leq T_*} \| U_g^{k-1} - U_g^{k-2} \|, \quad (2.104)$$

$$\sup_{0 \leq t \leq T_*} \| U_g^{k+1} - U_g^k \| \leq \frac{1}{4} \sup_{0 \leq t \leq T_*} \| U_g^k - U_g^{k-1} \| + \frac{1}{4} \sup_{0 \leq t \leq T_*} \| U_g^{k-1} - U_g^{k-2} \|. \quad (2.105)$$

Proof of Proposition 2.14:

We define $m^k = (f^k - f^{k-1}) \exp(v^2 + e) = g^k - g^{k-1}$. Then, m^k satisfies the following equality:

$$\begin{aligned} \partial_t m^k + v \cdot \nabla_x(m^k) - v \cdot \nabla_v(m^k) - e \cdot \partial_e(m^k) &= h(U_g^k, U_g^{k-1}, g^k, g^{k-1}) \\ &+ 2v \cdot (g^k u_g^{k-1} - g^{k-1} u_g^{k-2}) + (4 - 2v^2)m^k - em^k + \Gamma(g^k, g^k) - \Gamma(g^{k-1}, g^{k-1}), \end{aligned} \quad (2.106)$$

where

$$\begin{aligned} h(U_g^k, U_g^{k-1}, g^k, g^{k-1}) &= -u_g^{k-1} \cdot \nabla_v(g^k) + u_g^{k-2} \cdot \nabla_v(g^{k-1}) - e_g^{k-1} \partial_e(g^k) + e_g^{k-2} \partial_e(g^{k-1}) \\ &+ e_g^{k-1} g^k - e_g^{k-2} g^{k-1}. \end{aligned} \quad (2.107)$$

The term in $-v^2$ in the equation for k comes from the drag force, and it is essential to keep it in order to counterbalance the collision term.

Multiplying by m^k and integrating over x, e, v , one gets

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|m^k\|^2 + 4 \|m^k\|^2 &= -2 \|m^k\|_w^2 + \iiint (-m^k) e + 2v \cdot (g^k u_g^{k-1} - g^{k-1} u_g^{k-2}) m^k dv dedx \\ &+ \iiint \left(h(U_g^k, U_g^{k-1}, g^k, g^{k-1}) + 4m^k + \Gamma(g^k, g^k) - \Gamma(g^{k-1}, g^{k-1}) \right) m^k dedvdx. \end{aligned} \quad (2.108)$$

Using the equality $a_1 b_1 - a_2 b_2 = (a_1 - a_2) b_1 + a_2 (b_1 - b_2)$, remembering that U_g^k is in $H^5(\mathbb{R}^3)$ (at least), and that the first derivatives of g^k are in $L^\infty(\mathbb{R}^7)$ (g^k is in $H^5(\mathbb{R}^7)$ at least), one gets that

$$\iiint \left(h(U_g^k, U_g^{k-1}, g^k, g^{k-1}) \right) m^k dedvdx \leq C \left(\|m^k\|^2 + \|U_g^k - U_g^{k-1}\|^2 \right), \quad (2.109)$$

where C depends on s , $\|f_0\|_{H^s}$ and $\|U_{g_0}\|_{L^\infty}$. The constant C does not depend on k because all the iterates are in the same bounded domain by construction on $[0, T]$.

Moreover, immediate computations lead to

$$\begin{aligned} &\iiint (-m^k) e + 2v \cdot (g^k u_g^{k-1} - g^{k-1} u_g^{k-2}) m^k dv de \\ &\leq 0 + 2 \iiint \left((m^k)^2 |v| \|u_g^{k-1}\|_{L^\infty} + g^{k-1} v \cdot (u_g^{k-1} - u_g^{k-2}) m^k \right) dv dedx \\ &\leq 2 \|u_g^{k-1}\|_{L^\infty} \|m^k\| \|m^k\|_w + C \|m^k\|_w \|g^{k-1}\|_s \|u_g^{k-1} - u_g^{k-2}\| \\ &\leq C (\|u_g^{k-1} - u_g^{k-2}\| \|m^k\|_w + \|m^k\| \|m^k\|_w), \end{aligned} \quad (2.110)$$

using Cauchy-Schwarz inequality, Sobolev embeddings and the fact that all iterates are bounded in H^s -norm.

For the collision part, we get the following inequality, using the bilinearity of the kernel and the estimate on the modified collision kernel in lemma 2.5,

$$\begin{aligned} \iiint \left(\Gamma(g^k, g^k) - \Gamma(g^{k-1}, g^{k-1}) \right) m^k dedvdx &= \iiint \left(\Gamma(m^k, g^k) + \Gamma(g^{k-1}, m^k) \right) m^k dedvdx \\ &\leq C \|m^k\|_w \times \|m^k\| \times (\|g^k\|_{H^s} + \|g^{k-1}\|_{H^s}) \\ &\leq 2C \|f_0 \exp(v^2 + e)\|_{H^s} \|m^k\|_w \times \|m^k\|. \end{aligned} \quad (2.111)$$

Finally, using Young's inequality, we see that on $[0, T]$,

$$\frac{1}{2} \frac{d}{dt} \|m^k\|^2 \leq C_1 \left(\|m^k\|^2 + \|m^k\| \|m^k\|_w + \|U_g^k - U_g^{k-1}\|^2 \right) - 2 \|m^k\|_w^2, \quad (2.112)$$

where C_1 is independent of k .

Using $ab \leq a^2 + b^2$, one gets

$$\frac{1}{2} \frac{d}{dt} \|m^k\|^2 \leq C_2 \left(\|m^k\|^2 + \|U_g^k - U_g^{k-1}\|^2 \right). \quad (2.113)$$

The last inequality and a Gronwall lemma gives inequality (2.104) since $m^k(0, \cdot) = 0$.

Concerning inequality (2.105), it is classically obtained for hyperbolic systems through a Gronwall lemma on the function $\|U_g^{k+1} - U_g^k\|$ in L^2 (as it is done in [BD06]): we refer to [Ser96a] p.112 for the precise inequality. The dependance in $\|U_g^{k-1} - U_g^{k-2}\|$ comes from the control of $\|b(U_g^k, f^k) - b(U_g^{k-1}, f^{k-1})\|$. More precisely, we recall that

$$b(U_g, f) = \begin{pmatrix} 0 \\ - \int \frac{1}{\rho_g} f(u_{g1} - v) dvde \\ - \int \frac{1}{\rho_g} f(u_{g2} - v) dvde \\ - \int \frac{1}{\rho_g} f(u_{g3} - v) dvde \\ - \int \frac{1}{\rho_g} f(e_g - e) dvde + \int \frac{1}{\rho_g} f(u_g - v)^2 dvde \end{pmatrix}.$$

Furthermore, we have the following inequalities (using that $U_g^k - U_g^0$ is L^∞ -bounded since it is H^s and that $\int f^k dvde$, $\int f^k e dvde$, $\int f^k v dvde$ and $\int f^k v^2 dvde$ are in $L^\infty(\mathbb{R}^3)$ since $f^k \exp(v^2 + e) dvde$ is H^s):

$$\|b(U_g^k, f^k) - b(U_g^{k-1}, f^{k-1})\| \leq C(s, I_2, \|f^k \exp(v^2 + e)\|_{H^s}, \|U_g^k\|_{L^\infty}, \|U_g^{k-1}\|_{L^\infty}) \|U_g^k - U_g^{k-1}\|, \quad (2.114)$$

$$\begin{aligned} & \|b(U_g^{k-1}, f^k) - b(U_g^{k-1}, f^{k-1})\| \\ & \leq C(s, I_2, \|f^k \exp(v^2 + e)\|_{H^s}, \|f^{k-1} \exp(v^2 + e)\|_{H^s}, \|U_g^{k-1}\|_{L^\infty}) \|(f^k - f^{k-1}) \exp(v^2 + e)\| \\ & \leq C(s, I_2, \|f^k \exp(v^2 + e)\|_{H^s}, \|f^{k-1} \exp(v^2 + e)\|_{H^s}, \|U_g^{k-1}\|_{L^\infty}) \\ & \quad \times C_1(I_2, s, U_{g0}, f_0) \sup_{0 \leq t \leq T_*} \|U_g^{k-1} - U_g^{k-2}\|, \end{aligned} \quad (2.115)$$

using inequality (2.104).

Since all the data are bounded in H^s independently of k , we obtain

$$\|b(U_g^k, f^k) - b(U_g^{k-1}, f^k)\| \leq C(s, I_2, f_0, U_{g_0}) \left(\|U_g^{k-1} - U_g^{k-2}\| + \|U_g^k - U_g^{k-1}\| \right).$$

Finally a Gronwall lemma classically gives inequality (2.105) even if we may have to take a smaller time T^* (see [BD06]) (T^* still strictly positive).

It ends the proof of inequality 2.105. ■

Conclusion of the proof of theorem 2.2:

Thanks to (2.105), we see that

$$\sum_k \sup_{0 \leq t \leq T_*} \|U_g^{k+1} - U_g^k\| < +\infty. \quad (2.116)$$

Then, the sequence U_g^k is a Cauchy sequence and converges in $L^\infty([0, T_*], (L^2(\mathbb{R}^3))^5)$ towards some limit U_g which satisfies $U_g - {}^t(1, 0, 1) \in L^\infty([0, T_*], (L^2(\mathbb{R}^3))^5)$.

It is clear that $U_g - {}^t(1, 0, 0, 0, 1) \in C([0, T_*], (L^2(\mathbb{R}^3))^5)$ (remember that for all $k \geq 0$, $U_g^k - {}^t(1, 0, 1)$ is $\in C([0, T_*], (L^2(\mathbb{R}^3))^5)$).

Moreover, thanks to (2.104) and (2.116),

$$\sum_k \sup_{0 \leq t \leq T_*} \|(f^{k+1} - f^k) \exp(v^2 + e)\|_{L^2} < +\infty. \quad (2.117)$$

Then, the sequence $f^k \exp(v^2 + e)$ is a Cauchy sequence and converges in the space $L^\infty([0, T_*], L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ towards some function $f \exp(v^2 + e)$ such that $f \exp(v^2 + e) \in C([0, T_*], L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ (remember that for all $k \geq 0$, f^k is $\in C([0, T_*], L^2(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$).

Since we know that $\sup_{0 \leq t \leq T_*} \|U_g^k(t, \cdot) - U_{g_0}\|_s$ is bounded, we also have that $U_g - U_{g_0}$ lies in $L^\infty([0, T_*], H^s(\mathbb{R}^3))$, and that $\sup_{0 \leq t \leq T_*} \|U_g(t, \cdot) - U_{g_0}\|_s$ is finite. In particular, $U_g = \begin{pmatrix} \rho_g \\ u_g \\ e_g \end{pmatrix}$ lies in $L^\infty([0, T_*], C^1(\mathbb{R}^3))$ and takes its values in $\overline{I_2}$ (since all iterates take their value in $\overline{I_2}$).

For the same reason, $f \exp\left(\frac{1}{2}(v^2 + e)\right)$ lies in $L^\infty([0, T_*], C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ since $H^5(\mathbb{R}^7) \hookrightarrow C^1(\mathbb{R}^7)$ (see [Tay96]).

Since $\sup_{0 \leq t \leq T_*} \|f^k \exp(v^2 + e)\|_s \leq \sqrt{2} \|f_0\|_s$, we see that

$$\sup_{0 \leq t \leq T_*} \|f \exp(v^2 + e)\|_s \leq \sqrt{2} \|f_0\|_s.$$

Moreover, the sequences $\int f^k dvde$, $\int v f^k dvde$ and $\int e f^k dvde$ are bounded in $L^\infty([0, T_*], H^s(\mathbb{R}^3))$ and converge in $L^\infty([0, T_*], L^2(\mathbb{R}^3))$ towards $\int f dv$, $\int v f dv$ and $\int e f dvde$. Then, $b(U_g^k, f^k)$ converges in $L^\infty([0, T_*], L^1(\mathbb{R}^3))$ towards $b(U_g, f)$.

By interpolation in Sobolev spaces, for all $s' \in]0, s[$

$$\begin{aligned} \sup_{0 \leq t \leq T_*} \|U_g^{k+1} - U_g^k\|_{s'} &\leq C(s) \sup_{0 \leq t \leq T_*} \|U_g^{k+1} - U_g^k\|_0^{1-\frac{s'}{s}} \\ &\quad \times \left(\sup_{0 \leq t \leq T_*} \|U_g^{k+1} - U_{g_0}\|_{\frac{s'}{s}} + \sup_{0 \leq t \leq T_*} \|U_g^k - U_{g_0}\|_{\frac{s'}{s}} \right) \\ &\leq C(s, s', I_2) \sup_{0 \leq t \leq T_*} \|U_g^{k+1} - U_g^k\|_0^{1-\frac{s'}{s}}. \end{aligned} \quad (2.118)$$

Using (2.116), the sequence U_g^k converges to U_g in $L^\infty([0, T_*], H^{s'}(\mathbb{R}^3))$ for $s' \in]0, s[$, and $U_g \in C([0, T_*[, H^{s'}(\mathbb{R}^3)) \subset C([0, T_*[, C^1(\mathbb{R}^3))$. Then, $\partial_{x_i} U_g^k$ converges to $\partial_{x_i} U_g$ in $C([0, T_*[\times \mathbb{R}^3)$, and $\sum_i A_i(U^k) \partial_{x_i} U_g^{k+1}$ converges to $\sum_i A_i(U_g) \partial_{x_i} U_g$ in $C([0, T_*[\times \mathbb{R}^3)$.

Still by interpolation, $\int f^k dvde$, $\int v f^k dvde$ and $\int e f^k dvde$ converge in $L^\infty([0, T_*], H^{s'}(\mathbb{R}^3))$ towards $\int f dvde$, $\int v f dvde$ and $\int e f dvde$, and therefore they also converge in $C([0, T_*[\times \mathbb{R}^3)$. Finally, $b(U_g^k, f^k)$ converges towards $b(U_g, f)$ in $C([0, T_*[\times \mathbb{R}^N)$.

Passing to the limit in the sense of distributions in

$$\partial_t U_g^{k+1} = - \sum_i A_i(U_g^k) \partial_{x_i} U_g^{k+1} + b(U_g^k, f^k),$$

we get the equation

$$\partial_t U_g = - \sum_i A_i(U_g) \partial_{x_i} U_g + b(U_g, f), \quad (2.119)$$

and $\partial_t U_g \in C([0, T_*] \times \mathbb{R}^3)$, so that (2.119) is satisfied in the classical sense.

We now pass to the limit in (2.97) in the sense of distributions. We recall that $\rho_g^k - 1$, u_g^k and $e_g^k - 1$ converge in $L^\infty([0, T_*[, H^{s'}(\mathbb{R}^3))$ towards ρ_g , u_g and e_g . Furthermore, using once again an interpolation, we get that $f^{k+1} \exp(v^2 + e)$ converges in $L^\infty([0, T_*[, H^{s'}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ towards $f \exp(v^2 + e)$, so that $Q(f^{k+1}, f^{k+1})$ converges toward $Q(f, f)$ in $L^\infty([0, T_*[, H^{s'}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ for all s' such that $\frac{7}{2} < s' < s$ since $H^{s'}(\mathbb{R}^7)$ is an algebra (one gets that $Q(f, f)$ is bounded in H^s using lemma 2.5 on the collision kernel). We get that eq. (2.14) is satisfied in the sense of distributions.

We see that $f \in C^1([0, T_*[\times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)$ using Sobolev embeddings (for s' close enough to s), so that (2.14) is satisfied in the classical sense. Then passing to the limit in (2.96), (2.98), it concludes the proof of existence of theorem 2.2.

Uniqueness of solutions is a consequence of the proof of proposition 2.14. Namely, we consider two solutions $(\rho_g^1, u_g^1, e_g^1, f^1)$ and $(\rho_g^2, u_g^2, e_g^2, f^2)$. We suppose that both $(\tilde{\rho}_g^1, u_g^1, \tilde{e}_g^1, \tilde{f}^1)$ and $(\tilde{\rho}_g^2, u_g^2, \tilde{g}^2, \tilde{f}^2)$ belong

to $\mathcal{C}([0, T], H^s(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3)) \times \mathcal{C}([0, T], H^s(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+))$ and that $(\rho_{g_1}, u_{g_1}, e_{g_1}; f_1)$ and $(\rho_{g_2}, u_{g_2}, e_{g_2}; f_2)$ belong to $C^1([0, T] \times \mathbb{R}^3, \bar{I}_2) \times C^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$, then we can prove that for some $T_{**} \in]0, T_*[$,

$$\sup_{0 \leq t \leq T_{**}} \|U_g^1 - U_g^2\| \leq \frac{1}{4} \sup_{0 \leq t \leq T_{**}} \|U_g^1 - U_g^2\| + \frac{1}{4} \sup_{0 \leq t \leq T^*} \|U_g^1 - U_g^2\|,$$

$$\sup_{0 \leq t \leq T_{**}} \|f^1 - f^2\| \leq C(I_2, s) \sup_{0 \leq t \leq T_{**}} \|U_g^1 - U_g^2\|.$$

As a consequence $U_g^1 = U_g^2$ and $f_1 = f_2$ on this time interval and (by considering the maximal interval where this identity holds) $U_1 = U_2$ and $f_1 = f_2$ on $[0, T_*]$.

This concludes the proof of theorem 2.2.

BIBLIOGRAPHY

- [AG91] S. Alinhac and P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Interéditions/Editions du CNRS, 1991.
- [Bar01] C. Baranger. Collisions, coalescences et fragmentations des gouttelettes dans un spray: écriture précise des équations relatives au modèle T.A.B. Master's thesis, CMLA, Ecole normale supérieure de Cachan, 2001.
- [BD06] C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays : local smooth solutions. *J. Hyperbolic Differ. Equ.*, 3:1–26, 2006.
- [BDM03] L. Boudin, L. Desvillettes, and R. Motte. A modeling of compressible droplets in a fluid. *Commun. Math. Sci.*, 1(4):657–669, 2003.
- [Bre83] H. Brezis. *Analyse fonctionnelle*. Masson, 1983.
- [BT85] N. Bellomo and G. Toscani. On the Cauchy problem for the nonlinear Boltzmann equation: global existence, uniqueness and asymptotic stability. *J. Math. Phys.*, 26(2):334–338, 1985.
- [Cer88] C. Cercignani. *The Boltzmann Equation and its Applications*. Springer-Verlag, 1988.
- [CP83] R. Caffisch and G. C. Papanicolaou. Dynamic theory of suspensions with Brownian effects. *SIAM J. Appl. Math.*, 43(4):885–906, 1983.
- [Des93] L. Desvillettes. Equations cinétiques, 1993. Ecole nationale des ponts et chaussées.
- [DL89] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.
- [DR99] K. Domelevo and J.-M. Roquejoffre. Existence and stability of travelling wave solutions in a kinetic model of two-phase flows. *Comm. Partial Differential Equations*, 24(1-2):61–108, 1999.
- [Guo02] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys.*, 231(3):391–434, 2002.
- [Guo03] Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.*, 169(4):305–353, 2003.
- [Ham98] K. Hamdache. Global existence and large time behavior of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.* 15,1, 1998.
- [IS84] R. Illner and M. Shinbrot. The Boltzmann equation: global existence for a rare gas in an infinite vacuum. *Comm. Math. Phys.*, 95(2):217–226, 1984.

-
- [Maj84] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Springer-Verlag, 1984.
- [MP97] S. Mischler and B. Perthame. Boltzmann equation with infinite energy: renormalized solutions and distributional solutions for small initial data and initial data close to a Maxwellian. *SIAM J. Math. Anal.*, 28(5):1015–1027, 1997.
- [Ser96a] D. Serre. *Systèmes de lois de conservation I*. Diderot, 1996.
- [Ser96b] D. Serre. *Systèmes de lois de conservation II*. Diderot, 1996.
- [Tay96] M. Taylor. *Partial differential equations, basic theory*. Springer, 1996.
- [UA82] S. Ukai and K. Asano. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publ. Res. Inst. Math. Sci.*, 18(2):477–519 (57–99), 1982.
- [Vil02] C. Villani. *A review of mathematical topics in collisional kinetic theory*. Elsevier Science, 2002.
- [Wil85] F.A. Williams. *Combustion theory*. Benjamin Cummings, 1985.

3. A TURBULENT MODEL

Aim

The aim of this chapter is to adapt the model presented in the first chapter (eq.(1.1)-(1.11)) to turbulent regimes for the gas. The derivation of such a turbulent model is obtained by adapting in our case the construction of the equations of the Kiva code ([O'R81]).

We obtain a system of equations similar to the gas-particles system (eq.(1.1)-(1.11)) except for turbulent terms (Reynolds stress tensor) which now appear. Two more equations are added to take into account these new terms: a classical compressible $k - \varepsilon$ model is used to compute the Reynolds stress tensor (see [MP94], [LS72] or [Van83] for more details about the compressible $k - \varepsilon$ model).

We must also study the effects of turbulence on the collision kernel (see [O'R81], [Hyl99] and [VH97]) and the Vlasov equation (see [VS04]).

What is new here is that contrary to most of works on the subject, we deal with a real (non perfect) gas, so that some terms need a special treatment in the closure of the gas equations (see [BG90, Llo01, Llo05]). Concerning the particles, what is also unusual is that because of their high density, the vortices of the gas have no impact on their trajectories as we shall see (the Stokes number is very high).

Outline of the chapter

- We first explain the Favre averaging process in order to derive a turbulent model for both the gas and the particles.
- Then, we give a non closed model of turbulent equations for our system,
- We close this model using a $k - \varepsilon$ closure,
- Finally, we write down the whole system of equations, and explain the limits of effectiveness of this system.

Notations

- x : characteristic space variable of the Favre average,
- y : local space variable,
- V : control volume around x ,
- $V_g(t, x)$: volume occupied by the gas in the control volume around the point x ,
- $\alpha(t, x) = \frac{V_g(t, x)}{V}$: volume fraction occupied by the gas around the point x ,
- η : dynamic gas viscosity,
- g : characteristic function of the control volume V . It satisfies $\int_{y \in V} g(y) dy = 1$.
- $\alpha \tilde{\rho}_g(x) = \int_{y \in V_g} g(y) \rho_g(y) dy$: mean flow density ,
- $\tilde{u}_g(x) = \frac{1}{\alpha \tilde{\rho}_g} \int_{y \in V_g} g(y) \rho_g u_g(y) dy$: mean flow momentum per volume unit ,
- $\tilde{e}_g(x) = \frac{1}{\alpha \tilde{\rho}_g} \int_{y \in V_g} g(y) \rho_g e_g(y) dy$: mean internal energy of the flow per unit of volume,
- \tilde{T}_g : averaged temperature obtained as a function of the mean internal energy \tilde{e}_g and $\tilde{\rho}_g$, thanks to an equation of state,
- $\bar{p} = \frac{1}{\alpha} \int_{y \in V_g} p g(y) dy$: Reynolds averaged pressure,
- u_g'' : local Favre velocity fluctuation which satisfies $u_g''(y) = u_g(y) - \tilde{u}_g(x)$,
- $k(x) = \frac{1}{\alpha \tilde{\rho}_g} \int_{y \in V_g} g(y) \rho_g u_g''(y)^2 dy$: kinetic turbulent energy of the flow per volume unit ,
- $\varepsilon(x) = \frac{1}{\alpha \tilde{\rho}_g} \eta \int_{y \in V_g} g(y) \rho_g \left(\frac{1}{2} |\nabla_x u_g''(y) + (\nabla_x u_g''(y))^T|^2 - \frac{2}{3} |\nabla_x \cdot u_g''(y)|^2 \right) dy$: rate of dissipation of turbulent kinetic energy per volume unit,
- $\tilde{E}_g = \frac{1}{2} \tilde{u}_g^2 + e_g + k$: total energy of the turbulent fluid per mass unit,
- $F_t = -\nabla_x \cdot \int_{y \in V_g} \rho_g u_g'' \otimes u_g'' g(y) dy$: turbulent force ,

-
- F_p : turbulent transfer of momentum between the particles and the gas ,
 - $Q_t = -\nabla_x \cdot \int_V \rho_g E_g'' u_g'' g(y) dy$: turbulent energy transfer,
 - Q_p : turbulent transfer of energy between the particles and the gas.

3.1 Derivation of the model

3.1.1 Main hypotheses

In the next sections, we adapt the following system of equations to the phenomena of turbulence and close the obtained system. The system is the same as the one of the **first chapter** (eq.(1.1)-(1.11)):

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (3.1)$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = - \iint_{u_p, e_p} m_p \Gamma f du_p de_p, \quad (3.2)$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \\ \iint_{u_p, e_p} - \left(m_p \Gamma + \frac{m_p}{\rho_p} \nabla_x p \right) \cdot u_p f du_p de_p - \iint_{u_p, e_p} m_p \phi f du_p de_p, \end{aligned} \quad (3.3)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p}(f\phi) = Q(f, f), \quad (3.4)$$

$$\alpha = 1 - \frac{4}{3}\pi r^3 \iint_{u_p, e_p} f du_p de_p, \quad (3.5)$$

$$m_p \Gamma = -\frac{m_p}{\rho_p} \nabla_x p - D_p(u_p - u_g), \quad (3.6)$$

$$m_p \phi = 4\pi r \lambda N u(T_g - T_p), \quad (3.7)$$

$$T_g = T_1(\rho_g, e_g), \quad (3.8)$$

$$p = P_1(\rho_g(t, x), e_g), \quad (3.9)$$

$$T_p = T_2(e_p). \quad (3.10)$$

We recall that some assumptions were necessary to obtain this gas-particle system. Particles are supposed to be spherical. We also suppose that the velocity is the same for every part of the particle and that density and temperature are uniform inside a particle (we shall see later the importance of this hypothesis). We make an isobaric hypothesis: the pressure at the surface of a droplet is the same as the one of the gas (see [BDM03]).

3.1.2 Averaged gas equations

Since we have different length scales, we must average the equations. In order to do so, we use a characteristic function g of a volume V centered around x (see [O'R81]): all the data are averaged over this "mesoscopic" volume, which is of the same order as L^3 (L being the characteristic length of the flow). The volume occupied by the gas in V at point x is denoted $V_g(t, x)$.

So g is defined as

$$g(y) = \begin{cases} \frac{1}{V} & \text{if } y \in V, \\ 0 & \text{else,} \end{cases}$$

and the gas volume fraction can be assessed as

$$\alpha(t, x) = \frac{V_g(t, x)}{V}.$$

This approach of the problem -which relies on capturing a spatially averaged flow at large scale- can also be considered as an "ensemble average" approach: numerical experiments are performed and an average of all the results yield the mean behavior of the fluid. Or equivalently turbulence is assumed to be ergodic (see [MP94, Llo05]): all spatially averaged quantities are equal to time averaged quantities or to statistical averaged quantities. This averaging process is known as Favre process (see [FKD⁺76]).

We are able to define averaged quantities and fluctuations of these quantities around the volume V .

Averaged quantities

Definition 3.1 (Extensive and intensive averaged variables of state for the gas)

- Let q be a physical variable of the problem. We define \bar{q} its Reynolds averaged quantity by:

$$\alpha\bar{q}(x) = \int_{V_g} q(y)g(y)dy.$$

- Let q be a physical variable of the problem. We define \tilde{q} its Favre averaged quantity by:

$$\alpha\bar{\rho}_g\tilde{q}(x) = \int_{V_g} \rho_g(y)q(y)g(y)dy.$$

Note that $\tilde{\rho}_g = \bar{\rho}_g$ (taking $q = 1$).

The new variables \bar{q} and \tilde{q} are less local: their typical length scale is L . All the new averaged variables are defined p.108.

Then, we define another variable q'' which measures the fluctuations of the variable q associated with turbulence effects.

Definition 3.2 (Fluctuation of a variable)

We define the Favre fluctuation of a variable q as:

$$q'' = q - \tilde{q}.$$

We have $\alpha\tilde{\rho}_g\tilde{q}(x) = \int_{V_g} \rho_g q(y)g(y)dy$, so that one has $\int_{V_g} \rho_g(y)q''(y)g(y)dy = 0$ ([MP94]).

Averaged mass equation

This equation is obtained (as in [O'R81] and [Ver96]) thanks to the local conservation of mass:

$$\partial_t \rho_g + \nabla_y \cdot (\rho_g u_g) = 0.$$

After some manipulations (we integrate over the control volume), we get an equation depending on x :

$$\partial_t(\alpha \tilde{\rho}_g) + \nabla_x \cdot (\alpha \tilde{\rho}_g \tilde{u}_g) = 0.$$

This averaged equation is exactly the same as the local equation. There is no fluctuation terms here: by construction they appear only if there are quadratic terms like $u_g \otimes u_g$ or $E_g u_g$.

Averaged momentum equation

We keep using O'Rourke method (see [O'R81]) to derive the averaged momentum equation:

$$\partial_t(\alpha \tilde{\rho}_g \tilde{u}_g) + \nabla_x \cdot (\alpha \tilde{\rho}_g \tilde{u}_g \otimes \tilde{u}_g) + \nabla_x \bar{p} = F_p + F_t$$

with:

- $F_t = -\nabla_x \cdot \int_{V_g} \rho_g u_g'' \otimes u_g'' g(y) dy$,
- F_p : turbulent transfer of momentum between the particles and the gas (it is described in subsection 3.1.3).

Averaged energy equation

Using the same procedure, one obtains the following total energy equation (by total we mean the sum of the averaged internal energy, the averaged kinematic energy and turbulent energy):

$$\partial_t(\alpha \tilde{\rho}_g \tilde{E}_g) + \nabla_x \cdot \left(\alpha \tilde{\rho}_g \left(\tilde{E}_g + \frac{\bar{p}}{\tilde{\rho}_g} \right) \tilde{u}_g \right) + \bar{p} \partial_t \alpha = Q_t + Q_p, \quad (3.11)$$

with

- $Q_t = -\nabla_x \cdot \int_{V_g} \rho_g E_g'' u_g'' g(y) dy$,
- Q_p : turbulent transfer of energy between the particles and the gas (it is described in subsection 3.1.3).

Averaged gas equations

So finally we obtain the following set of equations for the gas,

$$\begin{aligned}\partial_t(\alpha\tilde{\rho}_g) + \nabla_x \cdot (\alpha\tilde{\rho}_g\tilde{u}_g) &= 0, \\ \partial_t(\alpha\tilde{\rho}_g\tilde{u}_g) + \nabla_x \cdot (\alpha\tilde{\rho}_g\tilde{u}_g \otimes \tilde{u}_g) + \nabla_x \bar{p} &= F_p + F_t, \\ \partial_t(\alpha\tilde{\rho}_g\tilde{E}_g) + \nabla_x \cdot \left(\alpha\tilde{\rho}_g \left(\tilde{E}_g + \frac{\bar{p}}{\tilde{\rho}_g} \right) \tilde{u}_g \right) + \bar{p}\partial_t\alpha &= Q_t + Q_p.\end{aligned}$$

Up to now, no modelling of the different terms has been made.

3.1.3 Vlasov-Boltzmann equation

We now establish the impact of turbulence on the Vlasov-Boltzmann equation. The approach presented here was first proposed by Reeks ([Ree91] and [Ree01]).

Because of turbulence, each particle in the control volume centered in x should not "see" the same velocity field of the gas (see [VS04], [Pop94], [O'R81] and [Sim00] for a theoretical approach, [SF00] and [Ver96] for a numerical approach). But in fact, for the kind of sprays used at the French Atomic Agency in which the particles density is much larger than the one of the gas, the same velocity field is "viewed" by the particles ([Ree91]).

Turbulence impact on the collision kernel

In order to show that the gas has no influence on the collisions, we compute the collision efficiency factor E_{coll} ($0 \leq E_{coll} \leq 1$; the new cross section is equal to $4r^2|u_p - u_{p^*}|E_{coll}$) as it is usually done in order to take into account the impact of the turbulence on the particles: it was first proposed by Langmuir in 1948 (see [Lan48] and [VS04]). For droplets having radii of the same order -which is satisfied in our applications-, Langmuir proposed (see [Hyl99]):

$$E_{coll} = \frac{1}{1 + Re/R_c} \frac{1}{\left(1 + \frac{3\ln(2K)}{4(K-1.214)}\right)^2} + \frac{Re/R_c}{1 + Re/R_c} \frac{K^2}{\left(\frac{1}{2} + K\right)^2},$$

with:

- $K = \frac{\rho_l}{\rho_g} Re$,
- $R_c = 60$,
- $Re = \frac{2r\rho_g|u_p - u_g|}{\eta}$: Reynolds number of the particles.

We give below the graphs of collision efficiency at different Reynolds number for the air-tin mixture used at the CEA (p.49). One can see that for large K , the efficiency is nearly equal to one ($K = 10^9$ and $Re = 5$ in our applications): particles are due to collide because the drag force which depends on ρ_g is not important enough to make the particles adopt the velocity of the gas (see [Hyl99]): turbulent eddies do not deviate particles from their trajectories.

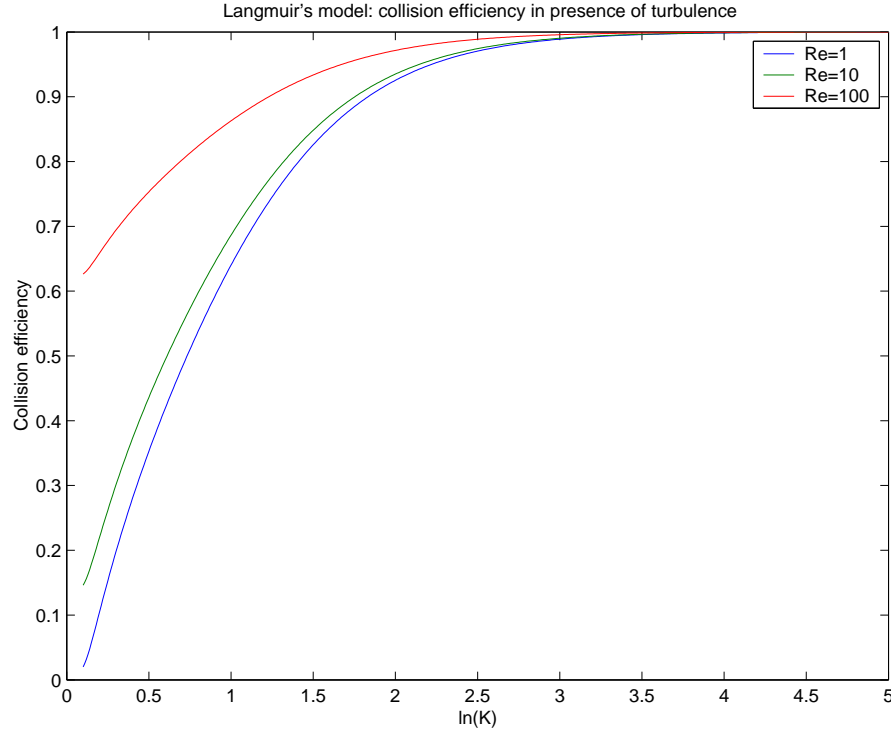


Fig. 3.1: Langmuir model: E_{coll} as a function of $\ln(K)$ for different Reynolds numbers

Finally, we keep the same collision kernel,

$$Q(f, f)(t, x, u_p, e_p) = \iiint_{\substack{\sigma \in \mathbb{S}^2 \\ u_{p*} \in \mathbb{R}^3, e_{p*} \in \mathbb{R}^+}} \left(\frac{1}{|1-a|} \frac{1}{\beta^4} 'f_*' f - f_* f \right) r^2 |u_p - u_p^*| d\sigma du_{p*} de_{p*} \quad (3.12)$$

with,

- σ : vector in \mathbb{S}^2 ,

- $'u_p = \frac{u_p + u_{p*}}{2} + \frac{1}{\beta} \frac{|u_p - u_{p*}|}{2} \sigma$: pre-collisional velocity,
- $'u_{p*} = \frac{u_p + u_{p*}}{2} - \frac{1}{\beta} \frac{|u_p - u_{p*}|}{2} \sigma$: pre-collisional velocity ,
- $'e_p = \frac{2-a}{2-2a} e_p - \frac{a}{2-2a} e_{p*} - \frac{a-2b}{2-2a} \Delta' E_c$: pre-collisional internal energy,
- $'e_{p*} = -\frac{a}{2-2a} e_p + \frac{2-a}{2-2a} e_{p*} - \frac{2b+a-2}{2-2a} \Delta' E_c$: pre-collisional internal energy,
- $\Delta' E_c = \frac{1}{2} ('u_p^2 + 'u_{p*}^2 - u_{p*}^2 - u_p^2) = \left(\frac{1-\beta^2}{4\beta^2} \right) (u_p - u_{p*})^2$: loss of kinetic energy (gain of internal energy),
- β : parameter of inelasticity ,
- a : parameter of exchange of internal energy ($0 \leq a \leq 1$) between the two particles,
- b : parameter of transfer of kinetic energy to internal energy of each particle ($0 \leq b \leq 1$) during the collision,
- $'f_* = f(t, x, 'u_{p*}, 'e_{p*})$,
- $'f = f(t, x, 'u_p, 'e_p)$,
- $f_* = f(t, x, u_{p*}, e_{p*})$,
- $f = f(t, x, u_p, e_p)$.

We finally obtain the following Vlasov-Fokker-Planck-Boltzmann equation for f :

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f \tilde{\Gamma}) + \partial_{e_p} (f \tilde{\phi}) = Q(f, f). \quad (3.13)$$

where,

- $m_p \tilde{\Gamma} = -\frac{m_p}{\rho_p} \nabla_x \bar{p} - \frac{1}{2} \rho_g \pi r^2 C_d |u_p - (\tilde{u}_g + u_g'')| ((u_p - (\tilde{u}_g + u_g'')))$,
- $m_p \tilde{\phi} = 4\pi r \lambda N u (\tilde{T}_g - T_p)$.

Remark 3.1

1. One can easily adapt this kernel to the case of elastic collisions: it just means taking $\beta = 1$.
2. Although there is one more variable in the space space, the numerical cost is not very large compared with the total cost of the implementation of the global model.
3. We have new invariants in the collision kernel: all the functions depending only on the turbulent velocity "seen" by the particles are invariants.
4. In the final model u_g'' is neglected in $\tilde{\Gamma}$ since all the particles see the same velocity as said before.

3.2 Closure of the model

Since the equations of the gas do not form a closed system, we now explain how to close the system. As it is generally done in industrial codes, we adopt a first-order isotropic closure: the compressible $k - \varepsilon$ model (see [Ces97], [Hyl99] and [AOB89] for industrial codes and [MP94] for the model). The two main differences with other codes is that we use dense sprays, so that volume fraction cannot be neglected in the inertia term of the momentum equation, and that we deal with non perfect gases.

Moreover, we also have to close the Vlasov-Boltzmann equation, since the characteristic time τ_g and the amplitude of turbulence σ are not yet defined.

3.2.1 Reynolds closure

As announced, we close the system by using a $k - \varepsilon$ model for the compressible gas.

There remains to model the F_t term in the averaged momentum equation:

$$\partial_t(\alpha\tilde{\rho}_g\tilde{u}_g) + \nabla_x \cdot (\alpha\tilde{\rho}_g\tilde{u}_g \otimes \tilde{u}_g) + \nabla_x \bar{p} = F_p + F_t.$$

The F_t term is generally known as the divergence of the Reynolds stress tensor and noted $\nabla_x \cdot R$. When the fluid is supposed to be compressible, its standard expression is:

$$R = -\frac{2}{3}(\alpha\tilde{\rho}_g k)\mathbf{I} + \alpha\tilde{\rho}_g\nu_t \left[(\nabla_x \tilde{u}_g + \nabla_x \tilde{u}_g^T) - \frac{2}{3}(\nabla_x \cdot \tilde{u}_g)\mathbf{I} \right],$$

with:

- k : kinetic turbulent energy defined by

$$\alpha\tilde{\rho}_g k(x) = \int_{y \in V_g} g(y)\rho_g \frac{1}{2} u_g''(y)^2 dy.$$

The quantity k is the kinetic turbulent energy linked to the fluctuations.

- ε : rate of dissipation of turbulent kinetic energy defined in such a way that

$$\alpha\tilde{\rho}_g \varepsilon(x) = \eta \int_{y \in V_g} g(y)\rho_g \left(\frac{1}{2} |\nabla_x u_g''(y) + (\nabla_x u_g''(y))^T|^2 - \frac{2}{3} |\nabla_x \cdot u_g''|^2 \right) dy$$

(we recall that η is the viscosity of the fluid: we take into account its effect for turbulence but not for the momentum equation since we deal with high Reynolds numbers).

The quantity ε measures the dissipation rate of turbulence through viscosity.

- ν_t : turbulent gas viscosity, defined by,

$$\nu_t = c_\mu \frac{k^2}{\varepsilon},$$

$c_\mu := 0.09$, physical constant .

- \mathbf{I} : identity matrix.

One can note that the closure of the Reynolds tensor is isotropic.

3.2.2 Equations for k and ε

We have the following classical set of equations for k and ε (see [MP94]) when the fluid is compressible:

$$\begin{aligned} \frac{\partial(\alpha\tilde{\rho}_g k)}{\partial t} + \nabla_x(\alpha\tilde{\rho}_g k\tilde{u}_g) - \frac{1}{2}R : (\nabla_x\tilde{u}_g + \nabla_x\tilde{u}_g^T) - \nabla_x \cdot \left(c_\mu \frac{\alpha\tilde{\rho}_g k^2}{\varepsilon} \nabla_x k \right) + \alpha\tilde{\rho}_g \varepsilon &= 0, \quad (3.14) \\ \frac{\partial(\alpha\tilde{\rho}_g \varepsilon)}{\partial t} + \nabla_x(\alpha\tilde{\rho}_g \varepsilon\tilde{u}_g) - \frac{c_{1\varepsilon}}{2} \frac{\varepsilon}{k} R : (\nabla_x\tilde{u}_g + \nabla_x\tilde{u}_g^T) - \nabla_x \cdot \left(c_\varepsilon \frac{\alpha\tilde{\rho}_g k^2}{\varepsilon} \nabla_x \varepsilon \right) + c_2 \frac{\alpha\tilde{\rho}_g \varepsilon^2}{k} &= 0, \quad (3.15) \end{aligned}$$

where, $c_{1\varepsilon}$, c_2 , c_μ and c_ε are strictly positive physical constants. Typical numerical values taken in practical computations are:

- $c_{1\varepsilon} = 1.45$,
- $c_2 = 1.92$,
- $c_\mu = 0.09$,
- $c_\varepsilon = 0.07$,

(see [MP94], [Llo01] and [Cou89] for more details).

3.2.3 Energy equations

We now give the turbulent energy equations of the gas. We first deal with the internal energy equation for convenience. Using equation (3.11) and the fact that $\tilde{E}_g = \frac{1}{2}\tilde{u}_g^2 + \tilde{e}_g + k$, one gets the following averaged internal energy equation,

$$\begin{aligned} \partial_t(\alpha\tilde{\rho}_g \tilde{e}_g) + \nabla_x \cdot (\alpha\tilde{\rho}_g \tilde{u}_g \tilde{e}_g) + \bar{p}(\partial_t\alpha + \nabla_x \cdot (\alpha\tilde{u}_g)) = \\ \iint_{u_p, e_p} \left((m_p \tilde{\Gamma} + \frac{m_p}{\rho_p} \nabla_x \bar{p}) \cdot (u_g - u_p) - m_p \tilde{\phi} \right) f du_p de_p + Q_t + \alpha\tilde{\rho}_g \varepsilon, \quad (3.16) \end{aligned}$$

where Q_t (still not modelled) represents a flux of enthalpy and $\alpha\rho_g\varepsilon$ represents the transfer of turbulent energy k to internal energy (we recall that ε is the dissipation rate of turbulent energy).

Generally, the expression of Q_t is written for perfect gases under the following form (see [SN00, MP94]),

$$\nabla_x \cdot \left(\alpha \tilde{\rho}_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} c_p \nabla_x \tilde{T}_g \right),$$

with:

- c_p : gas specific heat capacity at constant pressure,
- \tilde{T}_g : temperature of the gas,
- Pr_t : turbulent Prandtl Number .

The main idea in order to obtain Q_t in the case of non perfect gases is to come back to the definition of the enthalpy by mass unit \tilde{h}_g :

$$\tilde{h}_g = \tilde{e}_g + \frac{\bar{p}}{\tilde{\rho}_g}.$$

The turbulent flux of enthalpy at constant pressure is equal to

$$Q_t = \nabla_x \cdot \left(\alpha \tilde{\rho}_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} \left(\nabla_x \tilde{e}_g + \bar{p} \nabla_x \left(\frac{1}{\tilde{\rho}_g} \right) \right) \right),$$

using Favre averaging process: cf. [Bra05] and [LB03].

The constant Pr_t is usually equal to 0.7 for thermodynamic reasons (see [Llo01]).

Finally, the internal energy equation is

$$\begin{aligned} \partial_t(\alpha \tilde{\rho}_g \tilde{e}_g) + \nabla_x \cdot (\alpha \tilde{\rho}_g \tilde{u}_g \tilde{e}_g) + \bar{p}(\partial_t \alpha + \nabla_x \cdot (\alpha \tilde{u}_g)) = \\ \iint_{u_p, e_p} \left((m_p \tilde{\Gamma} + \frac{m_p}{\rho_p} \nabla_x \bar{p}) \cdot (\tilde{u}_g - u_p) - m_p \tilde{\phi} \right) f du_p de_p \\ + \nabla_x \cdot \left(\alpha \tilde{\rho}_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} \left(\nabla_x \tilde{e}_g + \bar{p} \nabla_x \left(\frac{1}{\tilde{\rho}_g} \right) \right) \right) + \alpha \tilde{\rho}_g \varepsilon. \end{aligned} \quad (3.17)$$

Since we have the internal energy and turbulent kinetic energy equations, we can get the equation on the total energy of the fluid \tilde{E}_g :

$$\begin{aligned} \partial_t(\alpha \tilde{\rho}_g \tilde{E}_g) + \nabla_x \cdot \left(\alpha \tilde{\rho}_g \left(\tilde{E}_g + \frac{\bar{p}}{\tilde{\rho}_g} \right) \tilde{u}_g \right) + \bar{p} \partial_t \alpha = \nabla_x \cdot \left(\alpha \tilde{\rho}_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} \left(\nabla_x \tilde{e}_g + \bar{p} \nabla_x \left(\frac{1}{\tilde{\rho}_g} \right) \right) \right) \\ - \iint_{u_p, e_p} (m_p \tilde{\Gamma} + \frac{m_p}{\rho_p} \nabla_x \bar{p}) \cdot u_p f du_p de_p - \iint_{u_p, e_p} m_p \tilde{\phi} f du_p de_p. \end{aligned} \quad (3.18)$$

3.3 Conclusion

3.3.1 System of equations

We are now able to write the full turbulent system of equations of the gas-particle model. We get the following model (from now on we omit all the $\bar{\cdot}$ and $\tilde{\cdot}$ on the physical averaged quantities in order to keep the notations tractable).

Turbulent gas-particle system:

$$\partial_t(\alpha\rho_g) + \nabla_x \cdot (\alpha\rho_g u_g) = 0, \quad (3.19)$$

$$\partial_t(\alpha\rho_g u_g) + \nabla_x \cdot (\alpha\rho_g u_g \otimes u_g) + \nabla_x p = - \iint_{u_p, e_p} m_p \Gamma f du_p de_p + \nabla_x \cdot R, \quad (3.20)$$

$$\begin{aligned} \partial_t(\alpha\rho_g E_g) + \nabla_x \cdot \left(\alpha\rho_g \left(E_g + \frac{p}{\rho_g} \right) u_g \right) + p\partial_t\alpha = \nabla_x \cdot \left(\alpha\rho_g \frac{1}{Pr_t} \frac{c_\mu k^2}{\varepsilon} \left(\nabla_x e_g + p\nabla_x \left(\frac{1}{\rho_g} \right) \right) \right), \\ - \iint_{u_p, e_p} \left(m_p \Gamma + \frac{m_p}{\rho_p} \nabla_x p \right) \cdot u_p f du_p de_p - \iint_{u_p, e_p} m_p \phi f du_p de_p, \end{aligned} \quad (3.21)$$

$$\frac{\partial(\alpha\rho_g k)}{\partial t} + \nabla_x(\alpha\rho_g k u_g) - R : \frac{\nabla_x u_g + \nabla_x u_g^T}{2} - \nabla_x \cdot \left(c_\mu \frac{\alpha\rho_g k^2}{\varepsilon} \nabla_x k \right) + \alpha\rho_g \varepsilon = 0, \quad (3.22)$$

$$\frac{\partial(\alpha\rho_g \varepsilon)}{\partial t} + \nabla_x(\alpha\rho_g \varepsilon u_g) - c_{1\varepsilon} \frac{\varepsilon}{k} R : \frac{\nabla_x u_g + \nabla_x u_g^T}{2} - \nabla_x \cdot \left(c_\varepsilon \frac{\alpha\rho_g k^2}{\varepsilon} \nabla_x \varepsilon \right) + c_2 \frac{\alpha\rho_g \varepsilon^2}{k} = 0, \quad (3.23)$$

$$\partial_t f + u_p \cdot \nabla_x f + \nabla_{u_p} \cdot (f\Gamma) + \partial_{e_p}(f\phi) = Q(f, f), \quad (3.24)$$

$$\alpha = 1 - \frac{4}{3}\pi r^3 \iint_{u_p, e_p} f du_p de_p, \quad (3.25)$$

$$m_p \Gamma = -\frac{m_p}{\rho_p} \nabla_x p - D_p(u_p - u_g), \quad (3.26)$$

$$m_p \phi = 4\pi r \lambda N u(T_g - T_p), \quad (3.27)$$

$$R = -\frac{2}{3}(\alpha\rho_g k)\mathbf{I} + c_\mu \frac{\alpha\rho_g k^2}{\varepsilon} \left[(\nabla_x u_g + \nabla_x u_g^T) - \frac{2}{3}(\nabla_x \cdot u_g)\mathbf{I} \right], \quad (3.28)$$

$$e_g = E_g - k - \frac{1}{2}u_g^2, \quad (3.29)$$

$$T_g = T_1(\rho_g, e_g), \quad (3.30)$$

$$p = P_1(\rho_g, e_g), \quad (3.31)$$

$$T_p = T_2(e_p), \quad (3.32)$$

with the following values for the constants (see [MP94] and [VS04]):

- $c_{1\varepsilon} = 1.45$,
- $c_2 = 1.92$,
- $c_\mu = 0.09$,
- $c_\varepsilon = 0.07$,
- $Pr_t = 0.9$.

3.3.2 Limits and effectiveness of the closure of the model

We now explain the limits and the effectiveness of the model.

The drawbacks of the $k - \varepsilon$ compressible model are well-known for the gas ([MP94, Bra05]):

- first of all, the model of turbulence is isotropic, which is not realistic.
- no modeling of the boundary conditions is introduced.
- the constants of the model slightly vary according to the fluids which are used ([Llo05, Bra05]).
- the model is not very effective in low Reynolds number areas (in the applications of the CEA, it is not important since we deal with high Reynolds number areas almost everywhere).

Concerning the particles, the main flaw is that the model presented here may be difficult to adapt, once we deal with poly-dispersed sprays, because very small particles could appear because of break-ups, so that turbulence could have an impact on them even if their density is large. We would be due to consider back the velocity of the gas seen by the particles as explained in many works ([O'R81, VS04, Pop94, RPMK04]).

BIBLIOGRAPHY

- [AOB89] A.A. Amsden, P.J. O'Rourke, and T.D. Butler. Kiva II, a computer program for chemical reactive flows with spray. Technical report, Los Alamos National Laboratory, 1989.
- [BDM03] L. Boudin, L. Desvillettes, and R. Motte. A modeling of compressible droplets in a fluid. *Commun. Math. Sci.*, 1(4):657–669, 2003.
- [BG90] M. Bonnet and S. Gauthier. A $k - \varepsilon$ model for turbulent mixing in shock-tube flows induced by Rayleigh-Taylor instability. *Phys. Fluid A*, 10:2698–2700, 1990.
- [Bra05] J.-P. Braeunig. Une méthode de résolution de modèle de turbulence $k - \varepsilon$ préservant les supports compacts. Rapport CEA, 2005.
- [Ces97] N. Cesco. *Etude et modélisation de l'écoulement diphasique à l'intérieur des propulseurs à poudre*. PhD thesis, Ecole nationale supérieure d'aéronautique et de l'espace (Toulouse), 1997.
- [Cou89] J. Cousteix. *Turbulence et couche limite*. Cepadues, Septembre 1989.
- [FKD⁺76] A. Favre, L.S.G. Kovasnay, R. Dumas, J. Gaviglio, and M. Coantic. *La turbulence en mécanique des fluides*. Gauthier Villars, 1976.
- [Hyl99] J. Hylkema. *Modélisation cinétique et simulation numérique d'un brouillard dense de gouttelettes. Application aux propulseurs à poudre*. PhD thesis, Ecole nationale supérieure de l'aéronautique et de l'espace (Toulouse), 1999.
- [Lan48] I. Langmuir. The production of rain by a chain reaction in cumulous clouds at temperatures above freezing. *J. Atm. Sci.*, 5:175–192, 1948.
- [LB03] A. Llor and P. Bailly. A new turbulent two-field concept for modeling Rayleigh-Taylor, Richtmyer-Meshkov, and Kelvin-Helmholtz mixing layers. *Laser and particle beams*, 21-3:305–315, 2003.
- [Llo01] A. Llor. *Modèles hydrodynamiques statistiques pour les écoulements d'instabilités de mélange en régime développé: critères théoriques d'évaluation "0D" et comparaison des approches mono et bifluïdes*. C.E.A./ DAM, Septembre 2001.
- [Llo05] A. Llor. *Statistical Hydrodynamic Models for developed Mixing Instability flows*. Springer, 2005.
- [LS72] B.E. Launder and D.B. Spalding. *Mathematical models of turbulence*. Academic press, 1972.
- [MP94] B. Mohammadi and O. Pironneau. *Analysis of the $k-\varepsilon$ Turbulence Model*. Masson, 1994.

- [O'R81] P. O'Rourke. *Collective drop effects on vaporizing liquid sprays*. PhD thesis, Princeton University, 1981.
- [Pop94] S.B Pope. Lagrangian pdf methods for turbulent flows. *Annual review of fluid mechanics*, 1994.
- [Ree91] M.W. Reeks. On a kinetic equation for the transport of particles in turbulent flows. *Phys. Fluid A*, 3, 1991.
- [Ree01] M.W. Reeks. An introduction to the pdf approach for modelling dispersed particle flows. In *Trends in Numerical and Physical Modeling for Industrial Multiphase Flows*, 2001.
- [RPMK04] J. Réveillon, C. Pera, M. Massot, and R. Knikker. Eulerian analysis of the dispersion of evaporating sprays in a statistically stationary turbulent flow. *Journal of Turbulence*, 1:1–27, 2004.
- [SF00] O. Simonin and P. Fevrier. Statistical and continuum modelling of turbulent reactive particulate flows. part 2: Application of a two-phase second-moment transport model for prediction of turbulent gas-particle flows. *Theoretical and Experimental Modeling of Particulate Flows, Lecture Series 2000-06, Von Karman Institute for Fluid Dynamics, Rhode Saint Genèse (Belgium).*, 2000.
- [Sim00] O. Simonin. Statistical and continuum modelling of turbulent reactive particulate flows. part 1: Theoretical derivation of dispersed eulerian modelling from probability density function kinetic equation. *Theoretical and Experimental Modeling of Particulate Flows, Lecture Series 2000-06, Von Karman Institute for Fluid Dynamics, Rhode Saint Genèse (Belgium).*, 2000.
- [SN00] N.R. Smirnov and V.F. Nikitin. Ignition and combustion of turbulized dust-air mixtures. *Combustion and flame*, 123(3), October 2000.
- [Van83] D. Vandromme. *Contribution à la modélisation et à la prédiction d'écoulements turbulents à masse volumique variable*. PhD thesis, University of Lille, 1983.
- [Ver96] O. Vermorel. *Etude numérique et modélisation de la modulation de la turbulence dans un écoulement de nappe chargée en particules*. PhD thesis, Ecole Polytechnique (Palaiseau), 1996.
- [VH97] P. Villedieu and J. Hylkema. Une méthode particulière aléatoire reposant sur une équation cinétique pour la simulation numérique des sprays denses de gouttelettes liquides. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(3):323–328, 1997.
- [VS04] P. Villedieu and O. Simonin. Modeling of coalescence in turbulent gas-droplet flows. *Commun. Math. Sci.*, 2(suppl. 1):13–33, 2004.

4. LOCAL SMOOTH SOLUTIONS OF THE INCOMPRESSIBLE K - ε MODEL AND THE LOW TURBULENT DIFFUSION LIMIT

The aim of this paper is to study the local in time well posedness of the incompressible k - ε model in a 3d periodic domain. In the case when turbulent diffusion is much smaller than dissipation, asymptotic expansions are also derived.

4.1 Introduction

The k - ε model is widely used in various physical models to assess isotropic turbulence effects (see [MP94]). It is based on two scalar quantities characterizing turbulence: the kinetic turbulent energy and the rate of dissipation of turbulent energy. This model, proposed by Launder and Spalding ([LS72]), was designed to model the evolution of large turbulent structures and their effect on the large scale mean flow. Its main applications can be found in Aerodynamics, for instance to study the influence of turbulence on airfoils boundary layers (cf. [Cou89] and [GHP91]). It is also considered for modelling turbulent mixing induced by Rayleigh–Taylor, Kelvin–Helmholtz and Richtmyer–Meshkov instabilities, for instance in Astrophysical framework.

As a matter of fact, observations of the famous supernova 1987A have indicated that radioactive cobalt is far more thoroughly distributed among the explosive debris in the envelope than was predicted by model calculations of thin-shell nucleosynthesis in the pre-supernova star. It suggests the occurrence of large-scale mixing in the ejecta during the explosion [KSN⁺89] [ABKW89]. The most promising mechanism for explaining mixing in the ejecta is a combination of the Rayleigh-Taylor and Kelvin-Helmholtz instabilities. The Rayleigh-Taylor instability can arise in the supernova envelope when the outwardly moving shock wave from the initial explosion propagates through layers of the star with radial stratification of the heavy elements. As the shock passes through the composition interfaces (i.e., oxygen/silicon, helium/carbon+oxygen and hydrogen/helium), a rarefaction front moves back into the star, resulting in an effective reversal of gravity as low-density composition is pressure-accelerated into the underlying high-density composition. Any perturbation at the interface (i.e., velocity perturbation or spatial perturbation) will get amplified by the Rayleigh-Taylor and Richtmyer-Meshkov instabilities and result in the overturning of light and heavy elements. This results in the mixing of heavy elements throughout the envelope of the supernova remnant, with associated observational consequences in the light curve. A further mixing will occur as the dense "tongues" of the heavy elements experience differential shear with the lighter elements, resulting in Kelvin-Helmholtz instabilities. Thus the fingers of heavy and light fluid that developed initially get far more distorted and the mixing layer increases its width. Eventually, these instabilities become so nonlinear that the mixing layer appears to become fully turbulent. The properties of turbulently mixed layers may be equally important for understanding how interstellar clouds get reprocessed back into the interstellar medium. Efficient mixing of cloud and inter-cloud matter has been shown to occur after

clouds get crushed by the interaction of strong shocks from supernova remnants [KMC90] [KMC94].

Here we study the mathematical properties of the incompressible model. Let us emphasize that this model is widely used in industrial codes because of its physical relevance and its simplicity. In order to introduce the mathematical setting and write the model equations, let us first introduce some notations. The domain will be a 3-dimensional box $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ with periodic boundary conditions in order to avoid additional difficulties (specific physical modelling and mathematical tools are needed to handle boundaries). The system of equations can be written as follows (cf.[MP94]):

$$\frac{\partial U}{\partial t} + U \cdot \nabla U + \nabla P - \nu \Delta U - \nabla \cdot R = 0, \quad (4.1)$$

$$\nabla \cdot U = 0, \quad (4.2)$$

$$\frac{\partial k}{\partial t} + U \cdot \nabla k - \frac{c_\mu}{2} \frac{k^2}{\varepsilon} |\nabla U + \nabla U^T|^2 - \nabla \cdot \left(c_\mu \frac{k^2}{\varepsilon} \nabla k \right) + \varepsilon = 0, \quad (4.3)$$

$$\frac{\partial \varepsilon}{\partial t} + U \cdot \nabla \varepsilon - \frac{c_1}{2} k |\nabla U + \nabla U^T|^2 - \nabla \cdot \left(c_\varepsilon \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + c_2 \frac{\varepsilon^2}{k} = 0, \quad (4.4)$$

$$U(0, x) = U^0(x), \quad k(0, x) = k^0(x), \quad \varepsilon(0, x) = \varepsilon^0(x), \quad (4.5)$$

where $U := U(t, x) \in \mathbb{R}^3$ denotes the large scale flow, $k := k(t, x)$ the kinetic turbulent energy, $\varepsilon := \varepsilon(t, x)$ its dissipation rate. $P = P(t, x)$ stands for the mean pressure of the fluid; as usual in incompressible fluid models, it may be interpreted as a lagrangian multiplier of the constraint (4.2). Moreover, $R := R(t, x)$ denotes the Reynolds stress tensor given by,

$$R = -\frac{2}{3} k I + c_\mu \frac{k^2}{\varepsilon} (\nabla U + \nabla U^T). \quad (4.6)$$

Finally, ν denotes the constant positive molecular viscosity of the fluid, while c_1 , c_2 , c_μ and c_ε are given positive constants that allow to capture the large scale features of turbulence (typical numerical values taken in realistic computations are: $c_1 = 0.126$, $c_2 = 1.92$, $c_\mu = 0.09$ and $c_\varepsilon = 0.07$).

For a survey about uniqueness and existence results concerning the Navier – Stokes equation without the k - ε extension, we refer to [Lio96]. There also exist some results on a modified k - ε model (the so-called ϕ - θ model) given by Mohammadi and Lewandowski ([LM93]) when U is supposed to be known, so that one has only to solve the equations on ϕ and θ (which are very close to (4.3) and (4.4)). As far as we know, no existence result has been mathematically proven for the coupled equations (4.1)-(4.5).

Remark 4.1

The incompressible Reynolds tensor can be obtained from the compressible Reynolds tensor presented in the third chapter, using the incompressibility relation (4.2), as explained in introduction p. xxx.

The aim of this article is to provide a first study of this problem:

1. First we prove the following result for short time:

Theorem 4.1 (existence and uniqueness of smooth solutions)

The following two results hold:

- (a) Let the initial data U^0, k^0, ε^0 be in H^s for $s > 4 + 3/2$ ($s \in \mathbb{N}$) with k^0 and ε^0 bounded away from zero by a positive constant. Then, there exists a positive T and a strong¹ solution (U, k, ε) to system (4.1)-(4.5) on $[0, T]$ which belongs to $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$, such that k and ε remain positive on $[0, T]$.
- (b) Moreover, let $(U_1, k_1, \varepsilon_1)$ and $(U_2, k_2, \varepsilon_2)$ be two solutions of system (4.1)-(4.5) in the sense of distributions. We suppose that they belong to $\mathcal{C}([0, T]; H^2(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; L^2(\mathbb{T}^3))$ and that k_1, ε_1, k_2 and ε_2 are positive functions. If $U_1^0 = U_2^0, k_1^0 = k_2^0, \varepsilon_1^0 = \varepsilon_2^0$, then $U_1 = U_2, k_1 = k_2, \varepsilon_1 = \varepsilon_2$ on $[0, T]$.

2. Then we study a particular regime when turbulent diffusion effects are small compared with dissipation and when the mean flow is supposed to be at rest (so that U is considered to be identically 0). We consider the following non dimensional system (see the beginning of Section 4.4):

$$\frac{\partial k}{\partial t} - \eta \nabla \cdot \left(\frac{k^2}{\varepsilon} \nabla k \right) + A \varepsilon = 0, \quad (4.7)$$

$$\frac{\partial \varepsilon}{\partial t} - \eta \nabla \cdot \left(\frac{c_\varepsilon k^2}{c_\mu \varepsilon} \nabla \varepsilon \right) + c_2 A \frac{\varepsilon^2}{k} = 0, \quad (4.8)$$

with $A = \varepsilon^0 T / k^0, \eta = c_\mu (k^0)^2 T / (\varepsilon^0 L^2)$, k^0 denoting the typical kinetic turbulent energy, ε^0 the rate of kinetic turbulent energy dissipation, T and L the typical time and length scales of the physical situation. The time of existence of k and ε a priori depends on η and A .

We make an asymptotic expansion with respect to η for this model since for some typical physical set of values, η is negligible while A 's value is of order 1: A is now fixed. The difference between the solution (k, ε) of equations (4.7)-(4.8) and the first terms of its η expansion $(k_0 + \eta k_1, \varepsilon_0 + \eta \varepsilon_1)$ can be controlled through the following result:

Theorem 4.2 (Asymptotic expansion)

Let A be fixed. Let k^0, ε^0 belong to $H^7(\mathbb{T}^3)$ and be bounded from zero by positive constants. Let k and ε be positive solutions of (4.7) and (4.8); let $T(\eta)$ be their time of existence. We suppose that k and ε are bounded away from zero and belong to $\mathcal{C}^1([0, T(\eta)]; H^5(\mathbb{T}^3))$. Then there exist a strictly positive time T' independent of η such that $T(\eta) \geq T'$ for all η and a constant C such that for all $t < T'$:

$$\|k - k_0 - \eta k_1\|_{L^\infty}(t) \leq C \eta^{\frac{3}{2}}, \quad \|\varepsilon - \varepsilon_0 - \eta \varepsilon_1\|_{L^\infty}(t) \leq C \eta^{\frac{3}{2}},$$

where $k_0, k_1, \varepsilon_0, \varepsilon_1$ are solutions of the following ordinary differential and partial differential equations systems:

1. Zero-th order system

$$\begin{aligned} \frac{\partial k_0}{\partial t} + A \varepsilon_0 &= 0, & k_0(0, \cdot) &= k^0(\cdot), \\ \frac{\partial \varepsilon_0}{\partial t} + c_2 A \frac{\varepsilon_0^2}{k_0} &= 0, & \varepsilon_0(0, \cdot) &= \varepsilon^0(\cdot). \end{aligned}$$

¹ By strong we mean a classical $\mathcal{C}_t^1(\mathcal{C}_x^2)$ solution

2. First order system

$$\begin{aligned} \frac{\partial k_1}{\partial t} - \nabla \cdot \left(\frac{k_0^2}{\varepsilon_0} \nabla k_0 \right) + A\varepsilon_1 &= 0, \quad k_1(0, \cdot) = 0, \\ \frac{\partial \varepsilon_1}{\partial t} - \nabla \cdot \left(\frac{c_\varepsilon k_0^2}{c_\mu \varepsilon_0} \nabla \varepsilon_0 \right) + c_2 A \left(\frac{2\varepsilon_0 \varepsilon_1}{k_0} - \frac{\varepsilon_0^2 k_1}{k_0^2} \right) &= 0, \quad \varepsilon_1(0, \cdot) = 0. \end{aligned}$$

From a mathematical viewpoint, the main difficulty in the proof of Theorems 1 and 2 is the control of the positivity of k and ε since the mathematical model degenerates when k or ε vanish: because of the ε (resp. k^2/ε) term in equation (4.3) (resp.(4.4)) we cannot ensure strict positivity of k (resp. ε), except for a small time.

Three classical mathematical tools are used all along the article to carry on the study. The first one is the maximum principle for parabolic PDE's. The second one is the use of energy methods for parabolic PDE's to obtain *a priori* estimates. The last one is the use of Sobolev embeddings (see [Bre83] and [Tay96a]) and Gagliardo-Nirenberg inequalities in order to control the different norms. Since we need smoothness of the solutions in order to be able to use these inequalities, our study is for data which belong to H^s with $s > 4 + 3/2$ so that the L^∞ -norms of the gradients of the data are bounded by the H^s -norms.

In Section 4.2, we give an *a priori* estimate and in section 4.3, we solve the problem of existence and uniqueness of solutions of (4.1)–(4.5), proving theorem 4.1; in Section 4.4 we perform the expansion with respect to η and prove theorem 4.2. All the results are given for small times since there is no hope in controlling the strict positivity of k and ε for long times.

4.2 Preliminary results

We establish *a priori* estimates on U , k and ε solutions of system (4.1)–(4.5). In all the computations, we consider solutions (U, k, ε) belonging to $C([0, T]; H^s(\mathbb{T}^3)) \cap C^1([0, T]; H^{s-2}(\mathbb{T}^3))$ for some $T > 0$ and for s integer such that $s > 4 + 3/2$ (and consequently belonging to $C^2([0, T]; H^{s-4}(\mathbb{T}^3))$ at least). All the integrals are computed on \mathbb{T}^3 ; k and ε are supposed to be strictly positive quantities for $t \in [0, T]$. We systemically use Sobolev embedding $H^s(\mathbb{T}^3) \hookrightarrow C^{s-2}(\mathbb{T}^3)$ so that the $(s-2)^{\text{th}}$ space derivatives of U , k and ε are L^∞ -bounded by the H^s norm of U , k and ε (see [Bre83]) in dimension 3.

4.2.1 A priori estimates on the Navier–Stokes equations

We obtain the following estimates for smooth solutions of the incompressible Navier–Stokes equation:

Proposition 4.3 (Estimates on U)

Let U , k and ε be solutions of (4.1)–(4.5) which belong to $C([0, T]; H^s(\mathbb{T}^3)) \cap C^1([0, T]; H^{s-2}(\mathbb{T}^3))$, with k and ε strictly positive and $s \geq 4 + 3/2$. We have:

$$\frac{d}{dt} \|U\|_{H^s}^2 \leq C Q_{4s+8}(U, k, \varepsilon)(t),$$

where C is a generic constant and Q_1, \dots, Q_n are functions defined by:

$$Q_1(U, k, \varepsilon)(t) = 1 + \|U(t)\|_{H^s} + \|k(t)\|_{H^s} + \|\varepsilon(t)\|_{H^s} + \frac{1}{k_{\min}(t)} + \frac{1}{\varepsilon_{\min}(t)},$$

$$Q_n = (Q_1)^n,$$

where we note

$$k_{\min}(t) = \min_{x \in \mathbb{T}^3} k(t, x), \quad k_{\max}(t) = \max_{x \in \mathbb{T}^3, 0 \leq s \leq t} k(s, x),$$

$$\varepsilon_{\min}(t) = \min_{x \in \mathbb{T}^3, 0 \leq s \leq t} \varepsilon(s, x), \quad \varepsilon_{\max}(t) = \max_{x \in \mathbb{T}^3, 0 \leq s \leq t} (\varepsilon(s, x)),$$

and $D^i h$ denote the i^{th} derivative of a function h .

Proof of Proposition 4.3:

First note that is enough to prove

$$\frac{d}{dt} \|D^s U\|_{L^2}^2 \leq C Q_{4s+8}(U, k, \varepsilon)(t),$$

since for all integer $s' < s$ the result on $D^{s'} U$ is easier to prove.

From now on, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ denotes a multi-index of differentiation such that

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad \text{with } |\alpha| = \sum_{i=1}^3 \alpha_i$$

denoting its length. Here, $\beta \leq \alpha$ means that $\forall i, \beta_i \leq \alpha_i$ while $\beta + \gamma = \alpha$ means $\forall i, \beta_i + \gamma_i = \alpha_i$. We first obtain estimates on $\partial^\alpha U$ ($|\alpha| = s$): we differentiate the Navier–Stokes equations and integrate the equation against $\partial^\alpha U$.

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha U\|_{L^2}^2 + \nu \|D^1(\partial^\alpha U)\|_{L^2}^2 = (A) + (B) + (C) + (D), \quad (4.9)$$

with:

$$\begin{aligned}
(A) &= -b(U, \partial^\alpha U, \partial^\alpha U), \\
(B) &= -b(\partial^\alpha U, U, \partial^\alpha U), \\
(C) &= - \sum_{\beta+\gamma=\alpha, s>|\beta|, |\gamma|\geq 1} b(\partial^\beta U, \partial^\gamma U, \partial^\alpha U), \\
(D) &= \int \partial^\alpha U \cdot \left(\nabla \cdot \partial^\alpha \left(\frac{k^2}{\varepsilon} (\nabla U + \nabla U^T) \right) \right) dx,
\end{aligned}$$

where $b(u, v, w)$ is the trilinear form linked to the incompressible Navier–Stokes equation and defined on $(H^1(\mathbb{T}^3))^3$ by

$$b(u, v, w) = \sum_{1 \leq i, j \leq 3} \int u_i \partial_i v_j w_j dx$$

(see [CF88] or [Tem84] for notations and properties). Using $\nabla \cdot U = 0$, one may note that the term in which pressure and $\nabla_x \cdot (kI)$ appear vanished after integrating by parts. Moreover, (A) vanishes as its last two arguments are equal and the first one is divergence free. (B) and (C) can be estimated by $C \|U\|_{H^s}^3$ thanks to Sobolev embeddings (C denotes a generic constant). For instance, for the (C) term, either β or δ is less than $s - 3$ ($s \geq 5$ and $|\beta| + |\delta| = s$), so that $\partial^\beta U$ or $\partial_i (\partial^\gamma U)$ are L^∞ -bounded by $\|U\|_{H^s}$ and since the two other terms of the trilinear form belong to L^2 at least, the trilinear form is well defined and bounded by $C \|U\|_{H^s}^3$.

There remains the last term (D): integrating by parts and using the symmetry of $\nabla U + \nabla U^T$, one gets (there is no boundary terms since we use periodic conditions),

$$\begin{aligned}
& \int \partial^\alpha U \cdot \left(\nabla \cdot \partial^\alpha \left(\frac{k^2}{\varepsilon} (\nabla U + \nabla U^T) \right) \right) dx \\
&= - \int \partial^\alpha \left(\frac{k^2}{\varepsilon} (\nabla U + \nabla U^T) \right) : \partial^\alpha \frac{\nabla U + \nabla U^T}{2} dx.
\end{aligned}$$

Then, studying the terms coming from $\partial^\alpha ((\nabla U + \nabla U^T)k^2/\varepsilon)$, we get three kinds of terms which satisfy the following inequalities (with $\beta + \gamma = \alpha$, $s \geq |\beta|$, $|\gamma| \geq 0$):

- when $\gamma = \alpha$ and $\beta = 0$:

$$-\frac{1}{2} \int \frac{k^2}{\varepsilon} \partial^\alpha (\nabla U + \nabla U^T) : \partial^\alpha (\nabla U + \nabla U^T) dx \leq 0,$$

- when $2 \leq |\gamma| \leq s - 1$:

$$\begin{aligned}
& -\frac{1}{2} \int \partial^\beta \left(\frac{k^2}{\varepsilon} \right) \partial^\gamma (\nabla U + \nabla U^T) : \partial^\alpha (\nabla U + \nabla U^T) dx \\
& \leq \frac{4}{\nu} \left(\left\| \partial^\beta \left(\frac{k^2}{\varepsilon} \right) \partial^\gamma (\nabla U + \nabla U^T) \right\|_{L^2} \right)^2 + \frac{\nu}{4} \|D^1(\partial^\alpha U)\|_{L^2}^2,
\end{aligned}$$

- when $|\gamma| \leq 1$:

$$\begin{aligned}
& -\frac{1}{2} \int \partial^\beta \left(\frac{k^2}{\varepsilon} \right) \partial^\gamma (\nabla U + \nabla U^T) : \partial^\alpha (\nabla U + \nabla U^T) dx \\
& \leq \frac{\nu}{4} \|D^1(\partial^\alpha U)\|_{L^2}^2 + \frac{4}{\nu} \left(\left\| \partial^\beta \left(\frac{k^2}{\varepsilon} \right) \partial^\gamma (\nabla U + \nabla U^T) \right\|_{L^2} \right)^2.
\end{aligned}$$

In order to complete the proof, one has to note that in the last two cases, either $\partial^\beta(k^2/\varepsilon)$ or $\partial^\gamma(\nabla U + \nabla U^T)$ belong to $L^\infty(\mathbb{T}^3)$ because of the embedding $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$. Therefore, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial^\alpha U\|_{L^2}^2 + \nu \|D^1(\partial^\alpha U)\|_{L^2}^2 \\ & \leq \frac{\nu}{2} \|D^1(\partial^\alpha U)\|_{L^2}^2 + C \left(\|U\|_{H^s}^3 + \left\| \frac{k^2}{\varepsilon} \right\|_{H^s}^2 \|U\|_{H^s}^2 \right). \end{aligned}$$

Differentiating k^2/ε , one gets after some computations that

$$\left\| \frac{k^2}{\varepsilon} \right\|_{H^s} \leq C \left(\|k(t)\|_{H^s} + \|\varepsilon(t)\|_{H^s} + \frac{1}{k_{\min}(t)} + \frac{1}{\varepsilon_{\min}(t)} \right)^{2s+3},$$

and, using the definition of Q_n , we get a bound for $\partial^\alpha U$:

$$\frac{d}{dt} \|\partial^\alpha U\|_{L^2}^2 \leq C Q_{4s+8}(U, k, \varepsilon)(t).$$

This result holds for all the third derivatives so this ends up the proof of proposition 4.3. ■

4.2.2 Positive lower bounds for k and ε

We are now able to control positive lower bounds k and ε through the following lemma.

Lemma 4.4 (Maximum principle)

Let U , k and ε be solutions of System (4.1)-(4.5) which belong to $C([0, T]; H^s(\mathbb{T}^3)) \cap C^1([0, T]; H^{s-2}(\mathbb{T}^3))$, with k and ε strictly positive. We have for all $t \geq 0$:

$$\begin{aligned} \frac{1}{k_{\min}(t)} - \frac{1}{k_{\min}(0)} & \leq C \frac{t}{k_{\min}^2(t)} \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s}, \\ \frac{1}{\varepsilon_{\min}(t)} - \frac{1}{\varepsilon_{\min}(0)} & \leq C \frac{t}{k_{\min}(t)\varepsilon_{\min}^2(t)} \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s}^2. \end{aligned}$$

Proof of Lemma 4.4:

We only prove the result for k_{\min} . One has to note that k belongs to $C^2([0, T]; H^2(\mathbb{T}^3))$ and consequently to $C^2([0, T]; C(\mathbb{T}^3))$ since k is a solution of the problem, so that k_{\min} is continuous. For all $t \leq T$, $\delta > 0$ and $u \leq t$, we define the following function,

$$r_\delta(u, x) = \delta(1 + u) + C \frac{u}{k_{\min}^2(t)} \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s} - \frac{1}{k(u, x)} + \frac{1}{k_{\min}(0)},$$

which belongs to $C^1([0, t]; H^{s-2}(\mathbb{T}^3))$, where C is the Sobolev constant of the embedding $H^s(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ ($s > 3/2$). One can note that thanks to the definition of k_{\min} ,

$$\forall x \in \mathbb{T}^3, \quad r_\delta(0, x) \geq \delta > 0.$$

Let us suppose that r_δ vanishes at some point in $(0, t) \times \mathbb{T}^3$ and let denote t_1 the first time such that there exists $x_1 \in \mathbb{T}^3$ satisfying $r_\delta(t_1, x_1) = 0$ (note that t_1 is uniquely defined whereas x_1 may not be unique).

Since $0 < t_1 < t$, $\partial_t k^{-1}(t_1, x_1)$ is well defined. Moreover, thanks to the definition of t_1 , we get that (t_1, x_1) is a local minimum in space of r_δ , and consequently a minimum of $x \mapsto -k^{-1}(t_1, x)$ and a minimum of k . As a result, $\nabla k(t_1, x_1) = 0$ and $\Delta k(t_1, x_1) \geq 0$ (\mathbb{T}^3 is an open set). Hence the k equation at point (t_1, x_1) gives:

$$\frac{\partial k}{\partial t}(t_1, x_1) \geq -\varepsilon(t_1, x_1).$$

Using Sobolev embeddings, one gets for all the minima at time t_1 :

$$\frac{\partial k}{\partial t}(t_1, x_1) \geq -C \|\varepsilon(t_1, \cdot)\|_{H^s}, \quad \text{so that} \quad \frac{\partial}{\partial t} \left(\frac{1}{k} \right) (t_1, x_1) \leq C \frac{\|\varepsilon(t_1, \cdot)\|_{H^s}}{k_{\min}(t_1)^2}.$$

Finally, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} r_\delta(t_1, x_1) &\geq \delta + C \frac{1}{k_{\min}^2(t)} \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s} - C \frac{\|\varepsilon(t_1, \cdot)\|_{H^s}}{k_{\min}(t_1)^2} \\ &\geq \delta + 0 \quad (\text{thanks to the definition of } k_{\min}) \\ &> 0. \end{aligned}$$

Therefore, the function $u \mapsto r_\delta(u, x_1)$ is negative for $u < t_1$ in the neighborhood of t_1 . As t_1 is the first time for which r_δ vanishes and r_δ is necessarily positive on $]0, t_1[\times \mathbb{T}^3$ by continuity, we obtain a contradiction and as a consequence r_δ is a strictly positive function $[0, t[\times \mathbb{T}^3$ and a positive function by continuity at time t for all x . Eventually, letting δ tend to zero by continuity, we get

$$\begin{aligned} \frac{1}{k(u, x)} &\leq C \frac{u}{k_{\min}^2(t)} \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s} + \frac{1}{k_{\min}(0)} \\ &\leq C \frac{t}{k_{\min}^2(t)} \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s} + \frac{1}{k_{\min}(0)} \end{aligned}$$

(thanks to the definition of k_{\min}). Taking the maximum in space and time, one gets lemma 4.4. \blacksquare

In addition, we obtain the following estimates for k and ε (strict positivity of both k and ε is necessary for the inequalities).

Proposition 4.5 (a priori estimates on k and ε)

Let U , k and ε be solutions of (4.1)-(4.5) which belong to $C([0, T]; H^s(\mathbb{T}^3)) \cap C^1([0, T]; H^{s-2}(\mathbb{T}^3))$, with k and ε strictly positive and $s \geq 4 + 3/2$. We have:

$$\frac{d}{dt} \|k\|_{H^s}^2 \leq C Q_{4s+9}(U, k, \varepsilon), \quad \frac{d}{dt} \|\varepsilon\|_{H^s}^2 \leq C Q_{4s+9}(U, k, \varepsilon).$$

Proof of Proposition 4.5:

We only give a proof of the second inequality (the same tools are used to obtain the first result). In order to prove it, we only show that,

$$\frac{d}{dt} \|D^s \varepsilon\|_{L^2}^2 \leq C Q_{12}(U, k, \varepsilon),$$

since for lower order derivatives the same inequality is easier to prove (Sobolev embeddings give better results, especially in L^∞ -norm for lower order derivatives). Differentiating the ε equation s times in the direction $\alpha \in \mathbb{N}^3$, multiplying by $\partial^\alpha \varepsilon$ and integrating by parts the diffusive term, one gets:

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha \varepsilon\|_{L^2}^2 + (A) + (B) + (C) + (D) = 0,$$

with:

$$\begin{aligned}
- (A) &= \int \partial^\alpha (U \cdot \nabla \varepsilon) (\partial^\alpha \varepsilon) dx \\
- (B) &= - \int \frac{c_1}{2} \partial^\alpha (k |\nabla U + \nabla U^T|^2) (\partial^\alpha \varepsilon) dx \\
- (C) &= \int (\partial^\alpha \cdot \nabla \varepsilon) \partial^\alpha \left(c_\mu \frac{k^2}{\varepsilon} \nabla \varepsilon \right) dx \\
- (D) &= \int \partial^\alpha \varepsilon \partial^\alpha \left(\frac{\varepsilon^2}{k} \right) dx
\end{aligned}$$

The (A) term can be treated like the trilinear term of the Navier–Stokes equation, using that U is divergence free:

$$\int (U \cdot (\partial^\alpha \nabla \varepsilon)) (\partial^\alpha \varepsilon) dx = 0,$$

so that all the other terms of A contain derivatives of ε of order lesser than s , which, using Sobolev embedding, leads to $|(A)| \leq C \|U\|_{H^s} \|\varepsilon\|_{H^s}^2$.

By integrating by parts the (B) term, one gets that

$$|(B)| \leq C \|\varepsilon\|_{H^{s+1}} \|k |\nabla U + \nabla U^T|^2\|_{H^{s-1}}.$$

Using that H^s is an algebra since $s > 3/2$, we finally obtain

$$|(B)| \leq C \|D^{s+1} \varepsilon\|_{L^2} \|k\|_{H^s} \|U\|_{H^s}.$$

After some computations, one obtains the following result on (D):

$$|(D)| \leq C (\|\varepsilon\|_{H^s} + \|k\|_{H^s})^{2s+4} \leq C Q_{2s+4}(U, k, \varepsilon)(T).$$

Finally, there remains to study the diffusive term (C), whose most important part satisfies (using the positivity of k^2/ε):

$$\int \partial^\alpha \nabla \varepsilon \cdot \left(c_\mu \frac{k^2}{\varepsilon} \partial^\alpha (\nabla \varepsilon) \right) dx \geq c_\mu \frac{k_{\min}^2(t)}{\|\varepsilon(t, \cdot)\|_{H^s}} \|D^{s+1} \varepsilon\|_{L^2}^2.$$

The other terms of (C) are bounded by

$$\frac{1}{2} \frac{k_{\min}^2(t)}{\|\varepsilon\|_{H^s}} \|D^{s+1} \varepsilon\|_{L^2}^2 + 2 \frac{\|\varepsilon\|_{H^s}}{k_{\min}^2(t)} \left\| \frac{k^2}{\varepsilon} \right\|_{H^s} \|\varepsilon\|_{H^s}^2.$$

Since we already know that

$$\left\| \frac{k^2}{\varepsilon}(t, \cdot) \right\|_{H^s} \leq C \left(\|k(t)\|_{H^s} + \|\varepsilon(t)\|_{H^s} + \frac{1}{k_{\min}(t)} + \frac{1}{\varepsilon_{\min}(t)} \right)^{2s+3},$$

we obtain

$$(C) \geq \frac{1}{2} \frac{k_{\min}^2(t)}{\|\varepsilon(t, \cdot)\|_{H^s}} \|D^{s+1} \varepsilon\|_{L^2}^2 - C Q_{4s+9}(U, k, \varepsilon)(t).$$

Combining the results on (A), (B), (C) and (D), one gets

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha \varepsilon\|_{L^2}^2 \leq C Q_{4s+9}(U, k, \varepsilon)(t).$$

(Note that the (B) term is controlled by the (C) term, using again $ab \leq a^2 + b^2$).

This concludes the proof of proposition 4.5. ■

4.2.3 A priori estimates on k and ε

We establish an *a priori* estimate for the turbulent fields k and ε and then prove that solutions of the system can be controlled locally in time.

Proposition 4.6

Let $s \in \mathbb{N}$ such that $s \geq 4 + 3/2$ and U, k and ε be smooth solutions of (4.1)–(4.5) in $C([0, T]; H^s(\mathbb{T}^3)) \cap C^1([0, T]; H^{s-2}(\mathbb{T}^3))$. Then there exists a positive time T' ($T' \leq T$) such that k, ε remain strictly positive on $[0, T']$. T' depends only on the initial data of U, k and ε .

Proof of Proposition 4.6:

Thanks to the previous results we obtain

$$\frac{d}{dt} (\|\varepsilon\|_{H^s}^2 + \|k\|_{H^s}^2 + \|U\|_{H^s}^2) \leq C Q_{4s+9}(U, k, \varepsilon),$$

so that

$$\begin{aligned} \|\varepsilon\|_{H^s}^2(t) + \|k\|_{H^s}^2(t) + \|U\|_{H^s}^2(t) &\leq \|\varepsilon\|_{H^3}^2(0) + \|k\|_{H^3}^2(0) + \|U\|_{H^3}^2(0) \\ &\quad + C \int_0^t Q_{4s+9}(U, k, \varepsilon)(s) ds \end{aligned} \quad (4.10)$$

We introduce the following continuous increasing and positive function f :

$$\begin{aligned} f(t) &= 1 + \sup_{0 \leq t' \leq t} \|k(t', \cdot)\|_{H^s}^2 + \sup_{0 \leq t' \leq t} \|\varepsilon(t', \cdot)\|_{H^s}^2 + \sup_{0 \leq t' \leq t} \|U(t')\|_{H^s}^2 \\ &\quad + \frac{1}{k_{\min}(t)} + \frac{1}{\varepsilon_{\min}(t)}. \end{aligned}$$

Using (4.10) and the maximum principle to control $\frac{1}{k_{\min}(t)} + \frac{1}{\varepsilon_{\min}(t)}$, one gets:

$$f(t) \leq f(0) + C \left(\frac{t}{k_{\min}^2(t)} \sup_{0 \leq u \leq t} \|\varepsilon(u, \cdot)\|_{H^s} \right) \quad (4.11)$$

$$\begin{aligned} &\quad + \frac{t}{k_{\min}(t)\varepsilon_{\min}^2(t)} \sup_{0 \leq u \leq t} \|\varepsilon(u, \cdot)\|_{H^s}^2 + \int_0^t f^{4s+9} \\ &\leq f(0) + 3Ct f^{4s+9}(t) \end{aligned} \quad (4.12)$$

because

$$- \forall t' \leq t,$$

$$\begin{aligned} &\|k(t')\|_{H^s} + \|\varepsilon(t')\|_{H^s} + \|U(t')\|_{H^s} \\ &\leq 3 \left(1 + \sup_{0 \leq u \leq t} \|k(u)\|_{H^s}^2 + \sup_{0 \leq u \leq t} \|\varepsilon(u)\|_{H^s}^2 + \sup_{0 \leq u \leq t} \|U(u)\|_{H^s}^2 \right) \end{aligned}$$

$$- \forall t, \frac{1}{k_{min}(t)} - \frac{1}{k_{min}(0)} \leq C \frac{t}{k_{min}^2(t)} \sup_{0 \leq u \leq t} \|\varepsilon(u, \cdot)\|_{H^s}$$

$$- \forall t, \frac{1}{\varepsilon_{min}(t)} - \frac{1}{\varepsilon_{min}(0)} \leq C \frac{t}{k_{min}(t)\varepsilon_{min}^2(t)} \sup_{0 \leq u \leq t} \|\varepsilon(u, \cdot)\|_{H^s}^2$$

If we note $T'(M)$ the first time when $f(t)$ is equal to M we obtain that $T'(M) \geq \frac{(M - f(0))}{(3CM^{4s+9})}$. Then, studying the ratio $(M - f(0))(3CM^{4s+9})$, one concludes that f cannot blow up on $[0, T']$ with T' defined by $T' = \min(T, 1/(24Cf(0)^{4s+9}))$ (k, ε and U are defined on $[0, T]$). It completes the proof of proposition 4.6. ■

4.3 Existence and uniqueness of solutions for the whole system

Thanks to the ideas used to obtain the *a priori* estimates, we are able to prove Theorem 4.1.

4.3.1 Existence:

We only give a sketch of proof for the existence. We use an iterative method to obtain the result. We note U^n , k^n and ε^n (k^n and ε^n are supposed to be strictly positive) the n -th iterate of U , k and ε which are defined on $[0, t_n]$: the time of existence depends on n ; we have to prove that it can be bounded below. U^0 , k^0 and ε^0 are the initial data defined above.

Iterative process

We obtain U^{n+1} , k^{n+1} and ε^{n+1} through the following iterative process:

$$\begin{aligned} \frac{\partial U^{n+1}}{\partial t} + U^n \cdot \nabla U^{n+1} + \nabla P^{n+1} - \nu \Delta U^{n+1} = \\ \nabla \cdot \left(-\frac{2}{3} k^n I + c_\mu \frac{(k^n)^2}{\varepsilon^n} (\nabla U^{n+1} + \nabla U^{n+1T}) \right), \end{aligned} \quad (4.13)$$

$$\nabla \cdot U^{n+1} = 0, \quad (4.14)$$

$$\begin{aligned} \frac{\partial k^{n+1}}{\partial t} + U^{n+1} \cdot \nabla k^{n+1} - \nabla \cdot \left(c_\mu \frac{(k^n)^2}{\varepsilon^n} \nabla k^{n+1} \right) = \\ \frac{c_\mu (k^n)^2}{2 \varepsilon^n} |\nabla U^{n+1} + \nabla U^{n+1T}|^2 - \varepsilon^n, \end{aligned} \quad (4.15)$$

$$\begin{aligned} \frac{\partial \varepsilon^{n+1}}{\partial t} + U^{n+1} \cdot \nabla \varepsilon^{n+1} - \nabla \cdot \left(c_\varepsilon \frac{(k^n)^2}{\varepsilon^n} \nabla \varepsilon^{n+1} \right) = \\ \frac{c_1}{2} k^n |\nabla U^{n+1} + \nabla U^{n+1T}|^2 - c_2 \frac{(\varepsilon^n)^2}{k^n}, \end{aligned} \quad (4.16)$$

$$U^{n+1}(0, x) = U^0(x), \quad (4.17)$$

$$k^{n+1}(0, x) = k^0(x), \quad (4.18)$$

$$\varepsilon^{n+1}(0, x) = \varepsilon^0(x). \quad (4.19)$$

Solving the iterative scheme

In order to solve the iterative scheme, we need to solve the following equations:

1. The first one is a linear parabolic equation with a Lagrange multiplier (P):

$$\begin{aligned} \frac{\partial U}{\partial t} + b \cdot \nabla U + \nabla P - \nu \Delta U = \nabla \cdot (a(\nabla U + \nabla U^T)), \\ \nabla \cdot U = 0, \\ U(0, x) = U^0(x). \end{aligned}$$

with a, b in $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$ and a strictly positive (we omit the term $\nabla \cdot (-2k^n I/3)$ as it can be considered as a part of the pressure). This equation has a unique solution in $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$ for some T depending on the data a and b (see [Lio96] p.69)

Then for the solution of (4.13), (4.14) and (4.17), we obtain an H^s estimate for all $s > 3 + 3/2$ thanks to Gagliardo-Nirenberg's inequality: there exist an integer $p(s)$ and a constant $C_{s,0}$ which depend on s such that:

$$\begin{aligned} \frac{d}{dt} (\|U^{n+1}\|_{H^s}^2) + \frac{\nu}{2} \|D^{(s+1)}U^{n+1}\|_{L^2}^2 \\ \leq C_{s,0} \left(1 + \|U^{n+1}\|_{H^s}^2 + \|\varepsilon^n\|_{H^s}^2 + \|k^n\|_{H^s}^2 + \frac{1}{k_{min}^n(t)} + \frac{1}{\varepsilon_{min}^n(t)} \right)^{p(s)}. \end{aligned} \quad (4.20)$$

We keep a part of diffusion because U^{n+1} appears in the equations of k^{n+1} and ε^{n+1} .

2. The others are linear parabolic equations with variable diffusion coefficients:

$$\begin{aligned} \frac{\partial k}{\partial t} + b \cdot \nabla k - \nabla \cdot (a \nabla k) &= c, \\ \frac{\partial \varepsilon}{\partial t} + b \cdot \nabla \varepsilon - \nabla \cdot \left(\frac{c_\varepsilon}{c_\mu} a \nabla \varepsilon \right) &= d, \end{aligned}$$

with a, b, c and d in $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$, b strictly positive and $\nabla \cdot a = 0$.

So we need to solve this parabolic system while conserving the strict positivity; the classic theory on parabolic systems give us a solution to the problem in H^s (see [Tay96b]). Finally, there exists an integer $q(s)$ and a constant $C_{s,1}$ which depend on s such that:

$$\begin{aligned} \frac{d}{dt} \|k^{n+1}\|_{H^s}^2 \leq C_{s,1} \left[\left(1 + \|k^{n+1}\|_{H^s}^2 + \|U^n\|_{H^s}^2 + \|\varepsilon^n\|_{H^s}^2 + \|k^n\|_{H^s}^2 \right. \right. \\ \left. \left. + \frac{1}{k_{min}^n(t)} + \frac{1}{\varepsilon_{min}^n(t)} \right)^{q(s)} + \|D^{(s+1)}U^{n+1}\|_{L^2}^2 \right], \end{aligned} \quad (4.21)$$

$$\begin{aligned} \frac{d}{dt} \|\varepsilon^{n+1}\|_{H^s}^2 \leq C_{s,1} \left[\left(1 + \|\varepsilon^{n+1}\|_{H^s}^2 + \|U^n\|_{H^s}^2 + \|\varepsilon^n\|_{H^s}^2 + \|k^n\|_{H^s}^2 \right. \right. \\ \left. \left. + \frac{1}{k_{min}^n(t)} + \frac{1}{\varepsilon_{min}^n(t)} \right)^{q(s)} + \|D^{(s+1)}U^{n+1}\|_{L^2}^2 \right]. \end{aligned} \quad (4.22)$$

The strict positivity can be controlled through maximum principle as it was done before if $U^n, k^n, \varepsilon^n, 1/k^n$ and $1/\varepsilon^n$ are bounded on $[0, t^n]$. So we obtain an H^s control of the solutions thanks to the three inequalities (4.20)-(4.21)-(4.22) and the maximum principle (the terms of order $s+1$ vanish).

There exists an integer r and a constant $C_{s,2}$ both depending on the Sobolev index s such that the quantity f_n defined as

$$f_n(t) = \sup_{0 \leq t' \leq t} \|U^n\|_{H^s}^2 + \sup_{0 \leq t' \leq t} \|\varepsilon^n\|_{H^s}^2 + \sup_{0 \leq t' \leq t} \|k^n\|_{H^s}^2 + \frac{1}{k_{min}^n} + \frac{1}{\varepsilon_{min}^n},$$

satisfies the following equation:

$$f_{n+1}(t) - f_{n+1}(0) \leq C_{s,2} t (f_n(t) + f_{n+1}(t))^{r(s)} \quad (4.23)$$

as long as f_n and f_{n+1} exist. If f_n is bounded on $[0, t_n]$ by M then on $[0, \min(t_n, t_{n+1})]$ we get ,

$$f_{n+1}(t) \leq f_1(0) + C_{s,2} t (M + f_{n+1}(t))^{r(s)} ,$$

($f_n(0)$ is independent of the iterate number n).

Using the same argument as in the *a priori* estimate 4.6, one gets that for all $t \leq T = (M - f_1(0))/(C(2M)^{r(s)})$, f_{n+1} is bounded by M . Finally, we ensure that the time of existence does not vanish when n goes to ∞ and the time of existence of the iterates is controlled. Moreover, for all n , we have that

- U^n , k^n and ε^n belong to $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$ where $T > 0$ is a time of existence for all iterates,
- they are bounded in $L^\infty([0, T]; H^s(\mathbb{T}^3))$,
- their (time) derivatives are bounded in $L^\infty([0, T]; H^{s-2}(\mathbb{T}^3))$
- k^n and ε^n are bounded below by a strictly positive given constant.

Passing to the limit

Finally, it can be proven (by decreasing the time T , if necessary) that the iterative scheme converges using that for this time,

$$\sum_{n \geq 1} \left(\|U^n - U^{n+1}\|_{L^\infty([0, T]; L^2)}^2 + \|\varepsilon^n - \varepsilon^{n+1}\|_{L^\infty([0, T]; L^2)}^2 + \|k^n - k^{n+1}\|_{L^\infty([0, T]; L^2)}^2 \right) < +\infty. \quad (4.24)$$

(see the proof of lemma 4.7).

The convergence of the series ensures that U_n , k_n and ε_n converge in $L^\infty([0, T]; L^2)$ and consequently in $\mathcal{C}([0, T]; L^2)$ since all the terms are continuous. They also converge in $\mathcal{C}([0, T]; H^{s'})$ for $s' < s$ since they are bounded in $L^\infty([0, T]; H^s)$ (one can prove this property using Gagliardo-Nirenberg inequality -see [Maj84]-). Using classical arguments on the regularity of Sobolev spaces (see [Maj84] or [Ser96]) we see that the limits of the sequences U_n , k_n and ε_n are solutions of the problem in $\mathcal{C}([0, T]; H^s(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; H^{s-2}(\mathbb{T}^3))$.

Concerning the pressure term, it can be recovered from the Navier–Stokes equations as usual (see [CF88] for instance: the series of pressure is also bounded since U , k and ε are bounded). Therefore, theorem 4.1 is finally proven.

We now prove that the series is indeed convergent:

Lemma 4.7 (Convergence of the series)

For T small enough, the infinite sum (4.24) is finite.

Proof of Lemma 4.7: If we note $(\alpha_n)_{n \geq 0} = \|U^n - U^{n+1}\|_{L^2}^2 + \|\varepsilon^n - \varepsilon^{n+1}\|_{L^2}^2 + \|k^n - k^{n+1}\|_{L^2}^2$, the terms $(\alpha_n)_{n \geq 0}$ satisfy the following inequality:

$$\frac{d}{dt} \alpha_{n+1} \leq C(\alpha_{n+1} + \alpha_n)^{r(s)}, \quad (4.25)$$

with C depending on s and the initial data of k , U and ε , and r strictly positive integer depending on s (the inequality is once again obtained using *a priori* estimates). The sequence $(\alpha_n)_{n \geq 0}$ is bounded since the sequences U_n , k_n and ε_n are bounded in $\mathcal{C}([0, T]; H^s(\mathbb{T}^3))$. We obtain that there exists another constant C such that

$$\frac{d}{dt} \alpha_{n+1} \leq C (\alpha_{n+1} + \alpha_n), \quad (4.26)$$

thanks to Sobolev embedding. Finally, using that $\alpha_n(0) = 0$, for all n , we obtain

$$(\alpha_{n+1})(t) \leq \int_0^t \alpha_n(s) C \exp(C(t-s)) ds,$$

and that the sequence

$$\beta_n = \|U^n - U^{n+1}\|_{L^\infty([0, T]; L^2)}^2 + \|\varepsilon^n - \varepsilon^{n+1}\|_{L^\infty([0, T]; L^2)}^2 + \|k^n - k^{n+1}\|_{L^\infty([0, T]; L^2)}^2$$

is controlled by

$$\beta_{n+1} \leq \beta_n (\exp(CT) - 1).$$

Then, for T small enough, β_n is clearly a convergent series and lemma 4.7 is proved. \blacksquare

4.3.2 Uniqueness

As the solutions are regular enough (they belong indeed to the functional space $\mathcal{C}([0, T]; H^2(\mathbb{T}^3)) \cap \mathcal{C}^1([0, T]; L^2(\mathbb{T}^3))$), the $U \cdot \nabla U$ term does not prevent us from proving uniqueness. So adapting the proofs of uniqueness for parabolic equations (see [Tay96b]) and Navier–Stokes equations (see [Lio96]), we get to uniqueness.

4.4 Study of a simplified k - ε model

We now simplify the model by assuming that the system is initially at rest so that $U = 0$ - as it is the case for instance in the early development of a Rayleigh-Taylor or Richtmyer-Meshkov mixing layer (see [DLY99]). Consequently we only take account of the following simplified k - ε equations:

$$\frac{\partial k}{\partial t} - \nabla_x \cdot \left(c_\mu \frac{k^2}{\varepsilon} \nabla k \right) + \varepsilon = 0, \quad k(0, x) = k^0(x) \in H^7(\mathbb{T}^3), \quad (4.27)$$

$$\frac{\partial \varepsilon}{\partial t} - \nabla \cdot \left(c_\varepsilon \frac{k^2}{\varepsilon} \nabla \varepsilon \right) + c_2 \frac{\varepsilon^2}{k} = 0, \quad \varepsilon(0, x) = \varepsilon^0(x) \in H^7(\mathbb{T}^3). \quad (4.28)$$

We make a dimensional analysis which leads us to compute an asymptotic expansion.

4.4.1 Dimensional analysis

We analyze the different terms of the equations by making a change of variables:

- $x \mapsto \tilde{x} = \frac{x}{L}$ with L : typical length scale
- $t \mapsto \tilde{t} = \frac{t}{T}$ with T : typical time scale
- $\varepsilon \mapsto \tilde{\varepsilon} = \frac{\varepsilon}{\varepsilon^0}$ with ε^0 : typical rate of dissipation of turbulent energy scale
- $k \mapsto \tilde{k} = \frac{k}{k^0}$ with k^0 : typical turbulent energy scale

Thanks to the change of variable, we obtain:

$$\frac{\partial k}{\partial t} - \eta \nabla \cdot \left(\frac{k^2}{\varepsilon} \nabla k \right) + A \varepsilon = 0, \quad (4.29)$$

$$\frac{\partial \varepsilon}{\partial t} - \eta \nabla \cdot \left(\frac{c_\varepsilon k^2}{c_\mu \varepsilon} \nabla \varepsilon \right) + c_2 A \frac{\varepsilon^2}{k} = 0, \quad (4.30)$$

with: $A = \frac{\varepsilon^0 T}{k^0}$ and $\eta = c_\mu \frac{(k^0)^2 T}{\varepsilon^0 L^2}$.

For instance we have the following numerical data (in c.g.s. system) for Rayleigh-Taylor instabilities in dense hot plasma (see [TCB⁺00]):

$$\begin{array}{lll} k^0 = 10^{16} \text{ cm}^2/\text{s}^2 & T = 10^{-3} \text{ s} & A = 0.5 \\ \varepsilon^0 = 5 \cdot 10^{18} \text{ cm}^2/\text{s}^3 & L = 10^7 \text{ cm} & \eta \sim 10^{-5} \end{array}$$

As it can be noticed, η is small for the physical applications we study. This is why we expand ε and k in formal series (see next subsection).

We can also write η as: $c_\mu A \left(\frac{k^0}{\varepsilon^0 L} \right)^2$ with $\frac{(k^0)^{\frac{3}{2}}}{\varepsilon^0}$ representing the typical length of turbulent vortices (see [LB03]). So a small η is equivalent to neglect diffusion of vortices because they are too small. Another equivalent approach consists in saying (writing η as $\frac{c_\mu}{A} \times \left(T \frac{\sqrt{k^0}}{L} \right)^2$) that the typical time of creation of the vortices $\frac{L}{\sqrt{k^0}}$ is so large that vortices cannot diffuse turbulence.

4.4.2 Bounds for k and ε

From now on, we suppose that k and ε satisfy (4.29)-(4.30) and that they belong to $C^1([0, T] ; H^5(\mathbb{T}^3))$ (the smoothness comes from the parabolic behavior of the system; the time of existence a priori depends on η). We obtain bounds for the solutions of the system (4.29)-(4.30) which are independent of η and which allow us to control nonlinear terms.

Proposition 4.8 (Maximum principle for k and ε)

Let k^0, ε^0 belong to $H^7(\mathbb{T}^3)$ and be strictly positive. Let k and ε be strictly positive solutions of (4.7) and (4.8) and belong to $C^1([0, T(\eta)]; H^5(\mathbb{T}^3))$. Then we get that $\forall x \in \mathbb{T}^3$ and $\forall t \in [0, T]$

$$\begin{aligned} k(t, x) &\leq k_{max}(0), \\ \varepsilon(t, x) &\leq \varepsilon_{max}(0), \\ k(t, x) &\geq k_{min}(0) - A \varepsilon_{max}(0) t, \\ \varepsilon(t, x) &\geq \varepsilon_{min}(0) / \left(1 - c_2 \frac{\varepsilon_{min}(0)}{\varepsilon_{max}(0)} \log_2 \left(1 - \frac{t A \varepsilon_{max}(0)}{k_{min}(0)} \right) \right). \end{aligned}$$

Proof of Proposition 4.8:

We only prove the result for the lower bound for ε (we admit the three other results). Let's introduce $\delta > 1$ and the following function:

$$r_\delta(t, x) = \left(1 - c_2 \frac{\varepsilon_{min}(0)}{\varepsilon_{max}(0)} \log_2 \left(1 - \frac{t A \varepsilon_{max}(0)}{k_{min}(0)} \right) \right) \frac{\delta}{\varepsilon_{min}(0)} - \frac{1}{\varepsilon(t, x)}.$$

First note that

$$r_\delta(0, x) = \frac{\delta}{\varepsilon_{min}(0)} - \frac{1}{\varepsilon(0, x)} > 0 \quad \text{since } \delta > 1.$$

Moreover, r_δ belongs to $\mathcal{C}^1([0, T], \mathcal{C}^2(\mathbb{T}^3))$ since ε has $C^1([0, T]; H^5(\mathbb{T}^5))$ regularity. Since $[0, T] \times \mathbb{T}^3$ is a compact set, r_δ admits a minimum on $[0, T] \times \mathbb{T}^3$; and there exists s and x for which the minimum is reached. This minimum is also a local minimum of ε in space. As a consequence $\nabla \varepsilon(s, x) = 0$ and $\Delta \varepsilon(s, x) \geq 0$ (\mathbb{T}^3 is an open set).

So the ε equation at point (s, x) gives us:

$$\frac{\partial \varepsilon}{\partial t} - \eta \frac{k^2}{\varepsilon} \Delta \varepsilon + c_2 A \frac{\varepsilon^2}{k} = 0,$$

and we get, using the strict positivity of ε and k :

$$\frac{\partial \varepsilon(s, x)}{\partial t} \geq -c_2 A \frac{\varepsilon^2}{k} \quad \text{and} \quad \frac{\partial 1/\varepsilon(s, x)}{\partial t} \leq c_2 A \frac{1}{k}.$$

Finally,

$$\begin{aligned} \frac{\partial r_\delta}{\partial t}(s, x) &\geq \left(c_2 A \frac{\delta}{k_{min}(0)} / \left(1 - \frac{t A \varepsilon_{max}(0)}{k_{min}(0)} \right) \right) - c_2 \frac{A}{k} \\ &> c_2 A \frac{1}{k_{min}(0) - t A \varepsilon_{max}(0)} - c_2 \frac{A}{k} \quad (\text{since } \delta > 1) \\ &> 0 \quad (\text{using the inequality controlling the lower bound of } k), \end{aligned}$$

so the minimum can only be reached in $s = 0$, since if $s > 0$, for $s' < s$ such as $s' - s$ is small enough, $r_\delta(s', x) < r_\delta(s, x)$.

As a consequence, r_δ is strictly positive for all t and letting δ tends to zero, one gets proposition 4.8. ■

Remark 4.2 (Control of the time of existence for k and ε at fixed A)

Let us observe that we also obtain a time T_{min} independent of η -whose value is $k_{min}/(A\varepsilon_{max})$ - for which k and ε remain positive strictly, so that if A is fixed, the time of existence for k and ε is independent of η in system (4.29)-(4.30).

4.4.3 Asymptotic analysis of the system

The values used in physics lead to make an asymptotic expansion in η -series (η tends to zero) in order to approach the real solution; we write k and ε as:

$$k = \sum_{n=0}^{\infty} k_n \eta^n, \quad \varepsilon = \sum_{n=0}^{\infty} \varepsilon_n \eta^n.$$

We first establish properties for the differential systems obtained by expanding in η -series. Then, the exact solution is compared with the truncated series. Moreover, we show that the more η decreases the more the exact solution remains positive. Replacing k and ε by their expansion we get the following systems (we limit our study to second order but it could be extended further; although theorem (4.2) concerns only zero-th and first order systems, we need the second order, in order to prove it).

Zero-th order system

$$\begin{aligned} \frac{\partial k_0}{\partial t} + A\varepsilon_0 &= 0, & k_0(0, \cdot) &= k^0(\cdot), \\ \frac{\partial \varepsilon_0}{\partial t} + c_2 A \frac{\varepsilon_0^2}{k_0} &= 0, & \varepsilon_0(0, \cdot) &= \varepsilon^0(\cdot). \end{aligned}$$

First order system

$$\begin{aligned} \frac{\partial k_1}{\partial t} - \nabla \cdot \left(\frac{k_0^2}{\varepsilon_0} \nabla k_0 \right) + A\varepsilon_1 &= 0, & k_1(0, \cdot) &= 0, \\ \frac{\partial \varepsilon_1}{\partial t} - \nabla \cdot \left(\frac{c_\varepsilon k_0^2}{c_\mu \varepsilon_0} \nabla \varepsilon_0 \right) + c_2 A \left(\frac{2\varepsilon_0 \varepsilon_1}{k_0} - \frac{\varepsilon_0^2 k_1}{k_0^2} \right) &= 0, & \varepsilon_1(0, \cdot) &= 0. \end{aligned}$$

Second order system

$$\begin{aligned} \frac{\partial k_2}{\partial t} - \nabla \cdot \left(\frac{k_0^2}{\varepsilon_0} \nabla k_1 + \frac{2k_0 k_1}{\varepsilon_0} \nabla k_0 - \frac{k_0^2 \varepsilon_1}{\varepsilon_0^2} \nabla k_0 \right) + A\varepsilon_2 &= 0, \\ \frac{\partial \varepsilon_2}{\partial t} - \frac{c_\varepsilon}{c_\mu} \nabla \cdot \left(\frac{k_0^2}{\varepsilon_0} \nabla \varepsilon_1 + \frac{2k_0 k_1}{\varepsilon_0} \nabla \varepsilon_0 - \frac{k_0^2 \varepsilon_1}{\varepsilon_0^2} \nabla \varepsilon_0 \right) \\ + c_2 A \left(\frac{2\varepsilon_0 \varepsilon_2}{k_0} - \frac{\varepsilon_0^2 k_2}{k_0^2} - 2 \frac{\varepsilon_0 \varepsilon_1 k_1}{k_0^2} + \frac{\varepsilon_0^2 k_1^2}{k_0^3} + \frac{\varepsilon_1^2}{k_0} \right) &= 0, \\ k_2(0, \cdot) &= 0, & \varepsilon_2(0, \cdot) &= 0. \end{aligned}$$

After some computations we obtain the following solutions of these systems.

Proposition 4.9 (Solutions of the systems)

Let k^0, ε^0 belong to H^7 and be strictly positive. Let's define k_0 and ε_0 as:

$$k_0(t, x) = k^0(x) \left(1 + (c_2 - 1) A \frac{\varepsilon^0(x)}{k^0(x)} t \right)^{\frac{1}{1-c_2}} \quad (4.31)$$

$$\varepsilon_0(t, x) = \varepsilon^0(x) \left(1 + (c_2 - 1) A \frac{\varepsilon^0(x)}{k^0(x)} t \right)^{\frac{c_2}{1-c_2}} \quad (4.32)$$

Then, k_0 et ε_0 are solutions of the zero-order system, belong to $C^\infty([0, \infty[; H^7(\mathbb{R}))$ and remain strictly positive.

A solution k_1 and ε_1 of the first order system exists, is unique and belongs to $C^\infty([0, \infty[; H^5(\mathbb{R}))$. Moreover, its growth and the growth of its derivatives is at worst polynomial in time.

A solution k_2 and ε_2 of the second order system exists, is unique and belongs to $C^\infty([0, \infty[; H^3(\mathbb{R}))$. Moreover its growth and the growth of its derivatives is at worst polynomial in time.

Proof of Proposition 4.9:

For the zero-order system it is an immediate consequence of computing the equations with k_0 and ε_0 given by (4.31) and (4.32).

For the first order system, let us define \mathcal{B} , the matrix

$$\mathcal{B} = \begin{pmatrix} 0 & A \\ -c_2 A \frac{\varepsilon_0^2}{k_0} & c_2 A \frac{2\varepsilon_0}{k_0^2} \end{pmatrix} \quad \text{and} \quad v(x) = \begin{pmatrix} k_1(x) \\ \varepsilon_1(x) \end{pmatrix}.$$

Using the properties of k_0 and ε_0 , we see that $\mathcal{B} \in C^\infty([0, \infty[; H^7(\mathbb{R}))$.

Besides, v satisfies

$$\frac{dv(x)}{dt} + \mathcal{B}(t, x)v(x) = f(t, x), \quad v(0, x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(f depends uniquely on k_0 and ε_0 and belongs to $C^\infty([0, \infty[; H^5(\mathbb{R}))$).

The classical theory of ODE's gives us existence and uniqueness of k_1 and ε_1 . Moreover as the growth of \mathcal{B} and f is at worst polynomial (they both depend on ε_0 and k_0) it ensures that the growth of k_1, ε_1 and their derivatives is at worst polynomial.

The proof of the properties of the second order system is similar. ■

4.4.4 A priori estimates

We now compare k and ε with their second order expansion with respect to η in H^2 -norm. Let us define

$$K = k - \sum_{n=0}^2 \eta^n k_n \quad \text{and} \quad E = \varepsilon - \sum_{n=0}^2 \eta^n \varepsilon_n.$$

We obtain the following equations for K and E :

$$\frac{\partial K}{\partial t} + \eta \nabla \cdot (F1) + AE = 0, \quad K(0, \cdot) = 0, \quad (4.33)$$

$$\frac{\partial E}{\partial t} + \frac{c_\varepsilon}{c_\mu} \eta \nabla \cdot (F2) + c_2 A \times (F3) = 0, \quad E(0, \cdot) = 0, \quad (4.34)$$

with:

$$(F1) = \left(\frac{k_0^2}{\varepsilon_0} \nabla k_0 + \eta \left(\frac{k_0^2}{\varepsilon_0} \nabla k_1 + \left(\frac{2k_0 k_1}{\varepsilon_0} - \frac{k_0^2 \varepsilon_1}{\varepsilon_0^2} \right) \nabla k_0 \right) - \frac{k^2}{\varepsilon} \nabla k \right),$$

$$(F2) = \left(\frac{k_0^2}{\varepsilon_0} \nabla \varepsilon_0 + \eta \left(\frac{k_0^2}{\varepsilon_0} \nabla \varepsilon_1 + \frac{2k_0 k_1}{\varepsilon_0} \nabla \varepsilon_0 - \frac{k_0^2 \varepsilon_1}{\varepsilon_0^2} \nabla \varepsilon_0 \right) - \frac{k^2}{\varepsilon} \nabla \varepsilon \right),$$

$$(F3) = \left(\frac{\varepsilon^2}{k} - \frac{\varepsilon_0^2}{k_0} + \eta \left(\frac{\varepsilon_0^2 k_1}{k_0^2} - \frac{2\varepsilon_0 \varepsilon_1}{k_0} \right) + \eta^2 \left(\frac{\varepsilon_0^2 k_2}{k_0^2} - \frac{\varepsilon_1^2}{k_0} - \frac{2\varepsilon_0 \varepsilon_2}{k_0} + 2 \frac{\varepsilon_0 \varepsilon_1 k_1}{k_0^2} - \frac{\varepsilon_0^2 k_1^2}{k_0^3} \right) \right).$$

We define $T \mapsto B(T)$ by

$$B(T)^{-1} = \inf_{0 \leq t \leq T, x \in \mathbb{T}^3} (k(t, x), k_0(t, x), \varepsilon(t, x), \varepsilon_0(t, x)),$$

which is bounded since k and ε are bounded and strictly positive on $[0, T]$.

We assess the norm of the different terms in order to obtain energy estimates for the whole system. For the sake of simplicity, we suppose that $\eta < 1$ (η tends to zero..). We get:

Lemma 4.10

Let k^0, ε^0 belong to H^7 and be strictly positive. Let k and ε be strictly positive solutions of (4.7) and (4.8) and belong to $C^1([0, T(\eta)]; H^5(\mathbb{T}^3))$. There exists a positive function f depending in a polynomial way of

time ² but independent of η such that:

$$\begin{aligned}
|(F1)(t, x)| + |(F2)(t, x)| &\leq B(T)^5 f(T) \left[(\|K\|_{H^2} + \|E\|_{H^2})^6 + \eta^2 \right], \\
|D^1(F1)(t, x)| + |D^1(F2)(t, x)| &\leq B(T)^5 f(T) \left[\eta^2 \right. \\
&\quad \left. + \left(1 + |D^2 K(t, x)| + |D^2 E(t, x)| \right) \right. \\
&\quad \left. \times \left(\|K\|_{H^2} + \|E\|_{H^2} \right)^7 \right], \\
|(F3)(t, x)| + |D^1(F3)(t, x)| &\leq B(T)^5 f(T) \left[\left(\|K\|_{H^2} + \|E\|_{H^2} \right)^6 + \eta^3 \right], \\
|D^2(F3)(t, x)| &\leq B(T)^5 f(T) \left[\eta^3 \right. \\
&\quad \left. + \left(1 + |D^2 K(t, x)| + |D^2 E(t, x)| \right) \right. \\
&\quad \left. \times \left(\|K\|_{H^2} + \|E\|_{H^2} \right)^7 \right].
\end{aligned}$$

Proof of Lemma 4.10:

We prove the result for $|(F3)|$. Using a Taylor expansion, we get

$$\frac{1}{k} = \frac{1}{k_0} - \frac{k - k_0}{k_0^2} + \frac{(k - k_0)^2}{k_0(t, x)^3} + c(t, x)(k - k_0)^3 \text{ with } |c(t, x)| \leq B(T)^3.$$

We put this formula in $(F3)$ and replace k by $K + k_0 + \eta k_1 + \eta^2 k_2$, ε by $E + \varepsilon_0 + \eta \varepsilon_1 + \eta^2 \varepsilon_2$. All zero-th, first and second order terms independent of K and E vanish. The remaining terms which do not depend on K and E are preceded by η^3 at least and their growth in time is at worst polynomial, thanks to results obtained on $k_0, k_1, k_2, \varepsilon_0, \varepsilon_1, \varepsilon_2$ (it gives us the bound $\eta^3 B(T)^6 f(T)$).

Besides, E^2 , E , $E^2 K^3$, EK^3 , K^3 , $E^2 K^2$, EK^2 , K^2 , $E^2 K$, EK and K terms appear: these terms are multiplied by functions depending on $k_0 - \varepsilon_2$ and η which can be bounded independently of η as $\eta \leq 1$ and whose growth is at worst polynomial, thanks to results obtained on $k_0, \dots, \varepsilon_2$ (it gives us the other bound: $B(T)^5 f(T) \left[(\|K\|_{H^2} + \|E\|_{H^2})^6 \right]$).

Thus, we get lemma 4.10. ■

Proposition 4.11 (H^2 estimates)

Let k^0, ε^0 belong to H^7 and be strictly positive. Let k and ε be strictly positive solutions of (4.7) and (4.8) and belong to $C^1([0, T(\eta)]; H^5(\mathbb{T}^3))$. There exists f depending in a polynomial way of time but independent of η such that:

$$\frac{d}{dt} \|D^2 K\|_{L^2}^2 \leq B(T)^5 f(t) \left[(\|K\|_{H^2}^2 + \|E\|_{H^2}^2)^9 + \eta^3 \right], \tag{4.35}$$

$$\frac{d}{dt} \|D^2 E\|_{L^2}^2 \leq B(T)^5 f(t) \left[(\|K\|_{H^2}^2 + \|E\|_{H^2}^2)^9 + \eta^3 \right], \tag{4.36}$$

$$\frac{d}{dt} (\|K\|_{H^2}^2 + \|E\|_{H^2}^2) \leq B(T)^5 f(t) \left[(\|K\|_{H^2}^2 + \|E\|_{H^2}^2)^9 + \eta^3 \right]. \tag{4.37}$$

² by this we mean $f \leq C(1+t)^n$ for some integer n

Proof of Proposition 4.11:

First, one can easily see that using equations (4.33) and (4.34):

$$\begin{aligned}\frac{d}{dt}\|K\|_{L^2}^2 &\leq |\nabla(F1)| \int |K| + A \int |KE|, \\ \frac{d}{dt}\|E\|_{L^2}^2 &\leq |\nabla(F2)| \int |K| + A \int |E| \times |(F3)|, \\ \frac{d}{dt}\|D^1K\|_{L^2}^2 &\leq |\nabla(F1)| \int |\Delta K| + A \int |\nabla K \cdot \nabla E|, \\ \frac{d}{dt}\|D^1E\|_{L^2}^2 &\leq \left(\frac{c_\varepsilon}{c_\mu} |\nabla(F2)| \int |\Delta E| + c_2 A |\nabla(F3)| \int |\nabla E| \right),\end{aligned}$$

Thanks to the bounds on (F1), (F2) and (F3) and their derivatives, as soon as one is able to prove (4.35) and (4.36) one obtains inequality (4.37).

We prove only the result in L^2 -norm for D^2K (inequality (4.35)), since the other proofs are very similar. We differentiate two times in $\alpha \in \mathbb{N}^3$ ($|\alpha| = 2$) the K equation and we multiply it by $\partial^\alpha K$. Integrating by parts the second term one gets,

$$\frac{d}{dt} \int |\partial^\alpha K|^2 = -\eta \int (\partial^\alpha(F1)) \cdot \partial^\alpha \nabla K - A \int \partial^\alpha K \partial^\alpha E.$$

Studying precisely $\partial^\alpha(F1)$, terms in K of third order (in derivatives) are those coming from $\partial^\alpha(\frac{k^2}{\varepsilon} \nabla k)$ and especially $\frac{k^2}{\varepsilon} \partial^\alpha \nabla k$.

Using again the mean value theorem on $1/\varepsilon$, $1/\varepsilon^2$, $1/\varepsilon^3$, we obtain that $\partial^\alpha F1$ contains $k^2/\varepsilon (\partial^\alpha \nabla K)$, linear terms in ΔK and ΔE - these functions are only multiplied by first derivatives of E and K , E and K and independent functions of η - other terms in ∇K , ∇E , K and E and finally terms in $\eta^2 f(t)$ ($f(t)$ can be bounded independently of E , K and η thanks to bounds on terms $k_0, k_1, k_2, \varepsilon_0, \varepsilon_1, \varepsilon_2$).

Making use of the diffusive term $k^2/\varepsilon (\partial^\alpha \nabla K)$ to control all terms of strictly inferior order via Sobolev embeddings, we get the result on $\partial_\alpha K$. This proof holds for all second derivatives, so that the H^2 -estimate is obtained and it ends the proof of proposition 4.11. \blacksquare

We now prove **theorem 4.2**.

First we recall that at fixed A , the time of existence $T(\eta)$ for k and ε is controlled by

$$T(\eta) \geq T_{min}, \tag{4.38}$$

where $T_{min} = k_{min}/(A\varepsilon_{max})$, so that we can consider the solutions of system (4.29)-(4.30) on $[0, T_{min}[$, for all η (cf. remark 4.2 p.140).

Let us define $S(t) = \|K\|_{H^2}^2(t) + \|E\|_{H^2}^2(t)$. Note that $S(0) = 0$. Thanks to Proposition 4.11, there exists f function of time bounded by $C(1+t)^n$, such that we have:

$$S'(t) \leq B(T_{min})^5 f(t) [S(t)^9 + \eta^3] \leq H(T_{min}) [S(t)^9 + \eta^3],$$

with $H(T_{min}) = B(T_{min})^5 \sup_{0 \leq t \leq T_{min}} f(t)$.

Let T_1 be the first time such that $S(T_1) = 1$. For all $t \leq T_1$, $S^9(t) \leq S(t)$ and $S'(t) \leq H(T) [S(t) + \eta^3]$, so that

$$\forall t \leq T_1, S(t) \leq \eta^3 (\exp(H(T_{min})t) - 1) \quad (\text{using Gronwall lemma.}), \quad (4.39)$$

So we get:

$$\begin{aligned} \forall t \leq T_1, \|K\|_{H^2}(t) &\leq \sqrt{\exp(H(T_{min})t) - 1} \eta^{\frac{3}{2}}, \\ \forall t \leq T_1, \|E\|_{H^2}(t) &\leq \sqrt{\exp(H(T_{min})t) - 1} \eta^{\frac{3}{2}}. \end{aligned}$$

Then, using the Sobolev embedding $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$, one gets:

$$\begin{aligned} \forall t \leq T_1, \|k - k_0 - \eta k_1 - \eta^2 k_2\|_\infty(t) &\leq C \sqrt{\exp(H(T_{min})t) - 1} \eta^{\frac{3}{2}} \leq C \eta^{\frac{3}{2}} \\ \forall t \leq T_1, \|\varepsilon - \varepsilon_0 - \eta \varepsilon_1 - \eta^2 \varepsilon_2\|_\infty(t) &\leq C \sqrt{\exp(H(T_{min})t) - 1} \eta^{\frac{3}{2}} \leq C \eta^{\frac{3}{2}} \end{aligned}$$

Using the fact that k_2 and ε_2 are bounded by polynomial functions of time which are bounded on $[0, T_{min}]$ (see prop 4.9), one gets theorem 4.2.

From this theorem we obtain the following result,

Corollary 4.12 (Strict positivity)

The theorem holds for $T_1 = T_{min}$, for η small enough.

Proof of Corollary 4.12:

In order to prove the corollary, using inequality (4.39) at time T_1 one gets:

$$1 \leq \eta^3 (\exp(H(T_{min})T_1) - 1)$$

So that:

$$T_1 \geq \frac{1}{H(T_{min})} \ln \left(1 + \frac{1}{\eta^3} \right)$$

Hence when η is small enough, $T_1 = T$ and it ends the proof of corollary 4.12. ■

The main consequence of the corollary is that we can increase the time T' as much as we want in theorem 4.2 -letting η tend to zero-, because k_0, k_1, ε_0 and ε_1 are bounded on all compact set of $[0, +\infty[$.

BIBLIOGRAPHY

- [ABKW89] D. Arnett, J. Bahcall, R. Kirschner, and S. Woosley. *Ann. Rev. Astron. Astrophys.*, 27:629, 1989.
- [Bre83] H. Brezis. *Analyse fonctionnelle*. Masson, 1983.
- [CF88] P. Constantin and C. Foias. *Navier Stokes equations*. The University of Chicago Press, 1988.
- [Cou89] J. Cousteix. *Turbulence et couche limite*. Cepadues, Septembre 1989.
- [DLY99] S. B. Dalziel, P. F. Linden, and D. L. Youngs. Self-similarity and internal structure of turbulence induced by Rayleigh-Taylor instability. *J. Fluid Mech.*, 399:1–48, 1999.
- [GHP91] E. Guyon, J.-P. Hulin, and L. Petit. *Hydrodynamique physique*. Intereditions, Editions du C.N.R.S., 1991.
- [KMC90] R. I. Klein, C. F. McKee, and P. Colella. In the evolution of interstellar medium. *L. Blitz, Ed., Astronomical Society of the Pacific Conference Series*, 12:117, 1990.
- [KMC94] R. I. Klein, C. F. McKee, and P. Colella. The hydrodynamic interaction of shock waves with interstellar clouds. I. Nonradiative shocks in small clouds. *Ap. J.*, 420:213, 1994.
- [KSN⁺89] T. Kumagai, T. Shigezama, K. Nomoto, M. Itoh, J. Nishimura, and S. Tsuruta. Gamma-rays, x-rays and optical light from the cobalt and the neutron star in sn 1987a. *Ap. J.*, 345:412–422, 1989.
- [LB03] A. Llor and P. Bailly. A new turbulent two-field concept for modeling Rayleigh-Taylor, Richtmyer-Meshkov, and Kelvin-Helmholtz mixing layers. *Laser and particle beams*, 21-3:305–315, 2003.
- [Lio96] P.-L. Lions. *Mathematical topics in fluid mechanics . vol. 1 , Incompressible models*. Clarendon Press, 1996.
- [LM93] R. Lewandowski and B. Mohammadi. Existence and positivity results for the ϕ - θ and a modified k - ϵ two-equation turbulence models. *Math. Models Methods Appl. Sci.*, 3(2):195–215, 1993.
- [LS72] B.E. Launder and D.B. Spalding. *Mathematical models of turbulence*. Academic press, 1972.
- [Maj84] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Springer-Verlag, 1984.
- [MP94] B. Mohammadi and O. Pironneau. *Analysis of the k - ϵ Turbulence Model*. Masson, 1994.

-
- [Ser96] D. Serre. *Systèmes de lois de conservation I*. Diderot, 1996.
- [Tay96a] M. Taylor. *Partial differential equations, basic theory*. Springer, 1996.
- [Tay96b] M. Taylor. *Partial differential equations III, non linear equations*. Springer, 1996.
- [TCB⁺00] F. Thais, J.P. Chièze, P. Baclet, A. Benuzzi, S.Bouquet, C. Cherfils, F. Mucchielli, L. Polès, C. Reverdin, R. Teyssier, J.P. Thébault, and P. Troussel. *The Astrolab experiment: Rayleigh-Taylor instabilities in supernova*. Elsevier, 2000.
- [Tem84] T. Temam. *Navier Stokes equation*. Elsevier Science Publisher, 1984.

APPENDIX

A. APPENDIX OF CHAPTER 1

We prove here a result about the work of pressure on particles for the compressible spray (section 1.7 p.60).

A.1 Work of pressure

Lemma A.1 (Work of the pressure)

- $p(\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v)) = \iint_{u_p, e_p} p m_p f \frac{d}{dt} \left(\frac{1}{\rho_p} \right) de_p du_p,$
- $\iint_{u_p, e_p} \partial_{e_p}(f m_p \Phi) e_p du_p de_p = p(\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v)) - \iint_{u_p, e_p} 4\pi r \lambda N u (T_g - T_p) f du_p de_p.$

Proof of Lemma A.1:

Note that:

$$\begin{aligned}
 p\partial_t(1 - \alpha) &= p \left[\iint_{u_p, e_p} \frac{m_p}{\rho_p} \partial_t f de_p du_p + \iint_{u_p, e_p} m_p f \partial_t \left(\frac{1}{\rho_p} \right) de_p du_p \right] \\
 &= p \left[\iint_{u_p, e_p} \frac{m_p}{\rho_p} (-\nabla_x \cdot (f u_p) - \nabla_{u_p} \cdot (f \Gamma) - \partial_{e_p}(f \phi)) de_p du_p \right. \\
 &\quad \left. + \iint_{u_p, e_p} m_p f \partial_t \left(\frac{1}{\rho_p} \right) de_p du_p \right] \\
 &= p \left[\iint_{u_p, e_p} \frac{m_p}{\rho_p} (-\nabla_x \cdot (f u_p) - \partial_{e_p}(f \phi)) de_p du_p \right. \\
 &\quad \left. + \iint_{u_p, e_p} m_p f \partial_t \left(\frac{1}{\rho_p} \right) de_p du_p \right] \\
 &\quad \left(\text{as } \iint_{u_p, e_p} \frac{m_p}{\rho_p} \nabla_{u_p} \cdot (f \Gamma) du_p de_p = 0 \right).
 \end{aligned}$$

Integrating by parts, we obtain:

$$p\partial_t(1 - \alpha) = p \left[\iint_{u_p, e_p} -\frac{m_p}{\rho_p} \nabla_x \cdot (f u_p) + (f m_p \phi) \partial_{e_p} \left(\frac{1}{\rho_p} \right) de_p du_p + \iint_{u_p, e_p} m_p f \partial_t \left(\frac{1}{\rho_p} \right) de_p du_p \right].$$

We also have:

$$\begin{aligned}
p\nabla_x \cdot (1 - \alpha)v &= p\nabla_x \cdot \left(v \iint_{u_p, e_p} \frac{m_p}{\rho_p} f de_p du_p \right) \\
&= p\nabla_x \cdot \left(\iint_{u_p, e_p} \frac{m_p}{\rho_p} f u_p de_p du_p \right) \\
&\quad \text{(thanks to the definition of } v \text{)} \\
&= p \left[\iint_{u_p, e_p} \frac{m_p}{\rho_p} \nabla_x \cdot (f u_p) de_p du_p + \iint_{u_p, e_p} \nabla_x \left(\frac{1}{\rho_p} \right) \cdot f m_p u_p de_p du_p \right].
\end{aligned}$$

Then, combining the last two identities, one obtains:

$$\begin{aligned}
p [\partial_t(1 - \alpha) + \nabla_x \cdot ((1 - \alpha)v)] &= \iint_{u_p, e_p} p m_p f \frac{d}{dt} \left(\frac{1}{\rho_p} \right) de_p du_p \\
&= \iint_{u_p, e_p} (4\pi r \lambda N u(T_g - T_p) - m_p \Phi) f de_p du_p \\
&= \iint_{u_p, e_p} ((4\pi r \lambda N u(T_g - T_p)) f + \partial_{e_p} (f m_p \Phi) e_p) de_p du_p \\
&\quad \text{(by integrating by parts) .}
\end{aligned}$$

This ends the proof of lemma A.1. ■

A.2 Equation satisfied by the temperature of a Maxwellian distribution in the inelastic homogeneous case

We look for a solution to the inelastic spatially homogeneous problem

$$\partial_t f(t, u_p, e_p) = Q(f, f)(t, u_p, e_p) \quad (Q: \text{inelastic collision kernel with } \beta < 1) \quad (\text{A.1})$$

under the following form $f(t, e_p, u_p) = \frac{Z(t)}{((2\pi T(t)))^{\frac{3}{2}}} \exp\left(-\frac{(u_p)^2}{2T(t)}\right) \delta_{e_p=e}(e_p)$.

$$\text{We recall that } T(t) = \frac{\iint_{u_p, e_p} \frac{1}{3} f u_p^2 de_p du_p}{\iint_{u_p, e_p} f de_p du_p}.$$

Using the inelastic spatially homogeneous Boltzmann equation, one gets

$$\begin{aligned} \frac{d}{dt} T(t) &= -\frac{2}{3} \langle Q(f, f), \frac{1}{2} u_p^2 \rangle / \iint_{u_p, e_p} f de_p du_p \\ &= -\frac{2}{3} \iiint_{u_p, e_p, u_{p_*}, e_{p_*}} \frac{1 - \beta^2}{8} f f_* 4\pi r^2 |u_p - u_{p_*}|^3 du_{p_*} de_{p_*} du_p de_p / Z(t), \end{aligned}$$

using equation (1.68).

There remains to compute $\iiint_{u_p, e_p, u_{p_*}, e_{p_*}} f f_* 4\pi r^2 |u_p - u_{p_*}|^3 du_{p_*} de_{p_*} du_p de_p$. Using the change of variable $V = \frac{u_p + u_{p_*}}{\sqrt{2}T}, V_* = \frac{u_p - u_{p_*}}{\sqrt{2}T}$, one gets

$$\begin{aligned} &\iiint_{u_p, e_p, u_{p_*}, e_{p_*}} f f_* 4\pi r^2 |u_p - u_{p_*}|^3 du_{p_*} de_{p_*} du_p de_p \\ &= \frac{4\pi r^2 Z^2}{(\sqrt{(2\pi)})^6} T^{3/2} \iiint_{u_p, e_p, u_{p_*}, e_{p_*}} \exp\left(-\frac{V_*^2 + V^2}{2}\right) \sqrt{2}^3 |V_*|^3 dV dV_* \\ &= 4\pi r^2 \frac{32 Z^2}{\sqrt{\pi}} T^{3/2}, \end{aligned} \quad (\text{A.2})$$

after a spherical change of variables for V_* , and using the Euler Γ function. If we note that the fraction of volume $1 - \alpha$ occupied by the particles is equal to

$$1 - \alpha = \frac{4}{3} Z \pi r^3,$$

we finally end up with the following equation

$$\frac{d}{dt} T(t) = -\frac{3}{r} (1 - \alpha) \frac{8}{3\sqrt{(\pi)}} T^{3/2}. \quad (\text{A.3})$$

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- [ABKW89] D. Arnett, J. Bahcall, R. Kirschner, and S. Woosley. *Ann. Rev. Astron. Astrophys.*, 27:629, 1989.
- [AG91] S. Alinhac and P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Interéditions/Éditions du CNRS, 1991.
- [AO89] A.A. Amsden and P.J. O'Rourke. The T.A.B. method for numerical calculation of spray droplet breakup. Technical report, Los Alamos National Laboratory, 1989.
- [AOB89] A.A. Amsden, P.J. O'Rourke, and T.D. Butler. Kiva II, a computer program for chemical reactive flows with spray. Technical report, Los Alamos National Laboratory, 1989.
- [Bar01] C. Baranger. Collisions, coalescences et fragmentations des gouttelettes dans un spray: écriture précise des équations relatives au modèle T.A.B. Master's thesis, CMLA, Ecole normale supérieure de Cachan, 2001.
- [Bar04a] C. Baranger. Modelling of oscillations, breakup and collisions for droplets: the establishment of kernels for the T.A.B. model. *Math. Models Methods Appl. Sci.*, 14(5):775–794, 2004.
- [Bar04b] C. Baranger. *Modélisation, étude mathématique et simulation des collisions dans les fluides complexes*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2004.
- [BCG00] A. V. Bobylev, J. A. Carrillo, and I. M. Gamba. On some properties of kinetic and hydrodynamic equations for inelastic interactions. *J. Statist. Phys.*, 98(3-4):743–773, 2000.
- [BD06] C. Baranger and L. Desvillettes. Coupling Euler and Vlasov equations in the context of sprays : local smooth solutions. *J. Hyperbolic Differ. Equ.*, 3:1–26, 2006.
- [BDM03] L. Boudin, L. Desvillettes, and R. Motte. A modeling of compressible droplets in a fluid. *Commun. Math. Sci.*, 1(4):657–669, 2003.
- [BG90] M. Bonnet and S. Gauthier. A $k - \varepsilon$ model for turbulent mixing in shock-tube flows induced by Rayleigh-Taylor instability. *Phys. Fluid A*, 10:2698–2700, 1990.
- [BGP04] A. V. Bobylev, I. M. Gamba, and V. Panferov. Moment inequalities and high-energy tails for the boltzmann equations with inelastic interactions. *J. Statist. Phys.*, 116(5-6):1651–1682, 2004.
- [Bir94] G.A. Bird. *Molecular gas dynamics and the direct simulation of gas flows*. Oxford Science publications, 1994.

- [Bou98] M. Boucker. *Modélisation numérique multidimensionnelle d'écoulements diphasiques liquide-gaz en régimes transitoire et permanent: méthodes et applications*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 1998.
- [Bou00] L. Boudin. *Etude mathématique des équations aux dérivées partielles cinétiques et hyperboliques de la physique*. PhD thesis, Université d'Orléans, 2000.
- [Bra05] J.-P. Braeunig. Une méthode de résolution de modèle de turbulence $k - \varepsilon$ préservant les supports compacts. Rapport CEA, 2005.
- [Bre83] H. Brezis. *Analyse fonctionnelle*. Masson, 1983.
- [BT85] N. Bellomo and G. Toscani. On the Cauchy problem for the nonlinear Boltzmann equation: global existence, uniqueness and asymptotic stability. *J. Math. Phys.*, 26(2):334–338, 1985.
- [Car57] T. Carleman. *Problèmes mathématiques dans la théorie cinétique des gaz*. Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.
- [CC70] S. Chapman and T.G. Cowling. *The mathematical theory of non uniform gases*. Cambridge Mathematical Library, 1970.
- [Cer75] C. Cercignani. *Theory and application of the Boltzmann equation*. Elsevier, New York, 1975.
- [Cer88] C. Cercignani. *The Boltzmann Equation and its Applications*. Springer-Verlag, 1988.
- [Cer03] C. Cercignani. Recent results in the kinetic theory of granular materials. In *Perspectives and problems in nonlinear science*, pages 217–228. Springer, New York, 2003.
- [Ces97] N. Cesco. *Etude et modélisation de l'écoulement diphasique à l'intérieur des propulseurs à poudre*. PhD thesis, Ecole nationale supérieure d'aéronautique et de l'espace (Toulouse), 1997.
- [CF88] P. Constantin and C. Foias. *Navier Stokes equations*. The University of Chicago Press, 1988.
- [CGW78] R. Clift, J.R. Grace, and M.E. Weber. *Bubbles, drops and particles*. Academic press, 1978.
- [CIP94] C. Cercignani, R. Illner, and M. Pulvirenti. *The mathematical theory of dilute gases*, volume 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.
- [Cou89] J. Cousteix. *Turbulence et couche limite*. Cepadues, Septembre 1989.
- [CP83] R. Caflisch and G. C. Papanicolaou. Dynamic theory of suspensions with Brownian effects. *SIAM J. Appl. Math.*, 43(4):885–906, 1983.
- [Des93] L. Desvillettes. *Equations cinétiques*, 1993. Ecole nationale des ponts et chaussées.
- [Des97] L. Desvillettes. Sur un modèle de type Borgnakke-Larsen conduisant à des lois d'énergie non linéaires en température pour les gaz parfaits polyatomiques. *Ann. Fac. Sci. Toulouse Math. (6)*, 6(2):257–262, 1997.
- [DG02] B. Desjardins and E. Grenier. On Rayleigh Taylor and Richtmyer instabilities, 2002.

- [DL89] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.
- [DLY99] S. B. Dalziel, P. F. Linden, and D. L. Youngs. Self-similarity and internal structure of turbulence induced by Rayleigh-Taylor instability. *J. Fluid Mech.*, 399:1–48, 1999.
- [DMS05] L. Desvillettes, R. Monaco, and F. Salvarani. A kinetic model allowing to obtain the energy law of polytropic gases in the presence of chemical reactions. *Eur. J. Mech. B Fluids*, 24(2):219–236, 2005.
- [DMV03] G. Dufour, M. Massot, and P. Villedieu. Étude d’un modèle de fragmentation secondaire pour les brouillards de gouttelettes. *C. R. Math. Acad. Sci. Paris*, 336(5):447–452, 2003.
- [DR99] K. Domelevo and J.-M. Roquejoffre. Existence and stability of travelling wave solutions in a kinetic model of two-phase flows. *Comm. Partial Differential Equations*, 24(1-2):61–108, 1999.
- [Duf05] G. Dufour. *Modélisation multi-fluide eulérienne pour les écoulements diphasiques à inclusions dispersées*. PhD thesis, Université Paul Sabatier Toulouse III, 2005.
- [DV05] K. Domelevo and P. Villedieu. A hierarchy of models for turbulent dispersed two-phase flows derived from a kinetic equation for the joint particle-gas pdf. *Phys. Fluid*, 2005.
- [FKD⁺76] A. Favre, L.S.G. Kovasnay, R. Dumas, J. Gaviglio, and M. Coantic. *La turbulence en mécanique des fluides*. Gauthier Villars, 1976.
- [FM05] N. Fournier and S. Mischler. A spatially homogeneous Boltzmann equation for elastic, inelastic and coalescing collisions. *J. Math. Pures Appl. (9)*, 84(9):1173–1234, 2005.
- [GBWQ04] P.-G. De Gennes, F. Brochard-Wyart, and D. Quéré. *Capillarity and Wetting phenomena*. Springer, 2004.
- [GHP91] E. Guyon, J.-P. Hulin, and L. Petit. *Hydrodynamique physique*. Interéditions, Editions du C.N.R.S., 1991.
- [GHS04] T. Gallouët, J.-M. Hérard, and N. Seguin. Numerical modeling of two-phase flows using the two-fluid two-pressure approach. *Math. Models Methods Appl. Sci.*, 14(5):663–700, 2004.
- [GPV04] I. M. Gamba, V. Panferov, and C. Villani. On the Boltzmann equation for diffusively excited granular media. *Comm. Math. Phys.*, 246(3):503–541, 2004.
- [Gra58] H. Grad. *Principles of the kinetic theory of gases*. Springer, 1958.
- [GSR04] F. Golse and L. Saint-Raymond. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.*, 155(1):81–161, 2004.
- [Guo02] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys.*, 231(3):391–434, 2002.
- [Guo03] Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.*, 169(4):305–353, 2003.
- [Haf83] P.K. Haff. Grain flow as a fluid mechanical phenomenon. *J. Fluid. Mech.*, 134, 1983.

- [Ham98] K. Hamdache. Global existence and large time behavior of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.* 15,1, 1998.
- [HF92] L.P. Hsiang and G.M. Faeth. Near-limit drop deformation and secondary breakup. *International Journal of Multiphase Flow*, 18(5):635–652, 1992.
- [HF95] L.P. Hsiang and G.M. Faeth. Drop deformation and breakup due to shock-wave and steady disturbances. *International Journal of Multiphase Flow*, 21(5):545–560, 1995.
- [HG03] Hyung Ju Hwang and Yan Guo. On the dynamical Rayleigh-Taylor instability. *Arch. Ration. Mech. Anal.*, 167(3):235–253, 2003.
- [Hor90] L. Hormander. *The analysis of Linear Partial Differential Operators I*. Springer-Verlag, 1990. 2nd Edition.
- [Hyl99] J. Hylkema. *Modélisation cinétique et simulation numérique d'un brouillard dense de gouttelettes. Application aux propulseurs à poudre*. PhD thesis, Ecole nationale supérieure de l'aéronautique et de l'espace (Toulouse), 1999.
- [IS84] R. Illner and M. Shinbrot. The Boltzmann equation: global existence for a rare gas in an infinite vacuum. *Comm. Math. Phys.*, 95(2):217–226, 1984.
- [JUL92] Y.J. Jiang, A. Umemura, and C.K. Law. An experimental investigation on the collision behaviour of hydrocarbon droplets. *J. Fluid. Mech.*, 172, 1992.
- [KG05] S. Kamnis and S. Gu. Numerical modelling of droplet impingement. *J. Phys. D.*, 38:3664–3673, 2005.
- [KMC90] R. I. Klein, C. F. McKee, and P. Colella. In the evolution of interstellar medium. *L. Blitz, Ed., Astronomical Society of the Pacific Conference Series*, 12:117, 1990.
- [KMC94] R. I. Klein, C. F. McKee, and P. Colella. The hydrodynamic interaction of shock waves with interstellar clouds. I. Nonradiative shocks in small clouds. *Ap. J.*, 420:213, 1994.
- [KMN79] S. Kawashima, A. Matsumura, and T. Nishida. On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation. *Comm. Math. Phys.*, 70(2):97–124, 1979.
- [KSN⁺89] T. Kumagai, T. Shigezuma, K. Nomoto, M. Itoh, J. Nishimura, and S. Tsuruta. Gamma-rays, x-rays and optical light from the cobalt and the neutron star in sn 1987a. *Ap. J.*, 345:412–422, 1989.
- [Lag00] F. Lagoutière. *Modélisation mathématique et résolution numérique de problèmes de fluides compressibles à plusieurs constituants*. PhD thesis, PARIS VI, 2000.
- [Lan48] I. Langmuir. The production of rain by a chain reaction in cumulous clouds at temperatures above freezing. *J. Atm. Sci.*, 5:175–192, 1948.
- [Lau02] F. Laurent. *Modélisation mathématique et numérique de la combustion de brouillards de gouttes polydispersés*. PhD thesis, Université Claude Bernard (Lyon), 2002.

- [Lax73] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.
- [LB03] A. Llor and P. Bailly. A new turbulent two-field concept for modeling Rayleigh-Taylor, Richtmyer-Meshkov, and Kelvin-Helmholtz mixing layers. *Laser and particle beams*, 21-3:305–315, 2003.
- [Lio96] P.-L. Lions. *Mathematical topics in fluid mechanics . vol. 1 , Incompressible models*. Clarendon Press, 1996.
- [Lio98] P.-L. Lions. *Mathematical topics in fluid mechanics . vol. 2 , Compressible models*. Clarendon Press, 1998.
- [LL59] L.D. Landau and E.M. Lifschitz. *Fluid Mechanics*. Pergamon Press, 1959.
- [Llo01] A. Llor. *Modèles hydrodynamiques statistiques pour les écoulements d'instabilités de mélange en régime développé: critères théoriques d'évaluation "0D" et comparaison des approches mono et bifluides*. C.E.A./ DAM, Septembre 2001.
- [Llo05] A. Llor. *Statistical Hydrodynamic Models for developed Mixing Instability flows*. Springer, 2005.
- [LM93] R. Lewandowski and B. Mohammadi. Existence and positivity results for the ϕ - θ and a modified k - ϵ two-equation turbulence models. *Math. Models Methods Appl. Sci.*, 3(2):195–215, 1993.
- [LM01] F. Laurent and M. Massot. Multi-fluid modeling of laminar polydisperse spray flames: origin, assumptions and comparison of sectional and sampling methods. *Combustion theory and modelling*, 5(4):537–572, 2001.
- [LM03] P. Laurençot and S. Mischler. Convergence to equilibrium for the continuous coagulation-fragmentation equation. *Bull. Sci. Math.*, 127(3):179–190, 2003.
- [LMV04] F. Laurent, M. Massot, and P. Villedieu. Eulerian multi-fluid modeling for the numerical simulation of coalescence in polydisperse dense liquid sprays. *J. Comput. Phys.*, 194(2):505–543, 2004.
- [LS72] B.E. Launder and D.B. Spalding. *Mathematical models of turbulence*. Academic press, 1972.
- [Maj84] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Springer-Verlag, 1984.
- [Mat03] J. Mathiaud. Différents aspects du modèle k-epsilon. Master's thesis, Ecole normale supérieure de Lyon, 2003.
- [Mau04] B. Mauroy. *Hydrodynamique dans les poumons, relations entre flux et géométries*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2004.

- [MM06] S. Mischler and C. Mouhot. Cooling process for inelastic Boltzmann equations for hard spheres, part II: Self-similar solutions and tail behavior. *J. Statis. Phys.*, To appear, 2006.
- [MMR06] S. Mischler, C. Mouhot, and M. Rodriguez Ricard. Cooling process for inelastic Boltzmann equations for hard spheres, part I: The Cauchy problem. *J. Statis. Phys.*, To appear, 2006.
- [Mot00] R. Motte. A numerical method for solving particle-fluid equations. In *Trends in Numerical and Physical Modeling for Industrial Multiphase Flows*, 2000.
- [MP94] B. Mohammadi and O. Pironneau. *Analysis of the k - ε Turbulence Model*. Masson, 1994.
- [MP97a] J-P. Minier and J. Pozorski. Derivation of a PDF model for turbulent flows based on principles from statistical physics. *Phys. Fluids*, 9(6):1748–1753, 1997.
- [MP97b] S. Mischler and B. Perthame. Boltzmann equation with infinite energy: renormalized solutions and distributional solutions for small initial data and initial data close to a Maxwellian. *SIAM J. Math. Anal.*, 28(5):1015–1027, 1997.
- [MP01] J-P. Minier and E. Peirano. The pdf approach to turbulent polydispersed two-phase flows. *Phys. Rep.*, 352(1-3):1–214, 2001.
- [MR83] R. Maxey and J. Riley. Equation of motion of a small rigid sphere in a non uniform flow. *Physics of Fluids*, 26(4):883–889, 1983.
- [MV01] M. Massot and P. Villedieu. Modélisation multi-fluide eulérienne pour la simulation de brouillards denses polydispersés. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):869–874, 2001.
- [O’R81] P. O’Rourke. *Collective drop effects on vaporizing liquid sprays*. PhD thesis, Princeton University, 1981.
- [Pop94] S.B Pope. Lagrangian pdf methods for turbulent flows. *Annual review of fluid mechanics*, 1994.
- [PR04] L. Pareschi and G. Russo. *An introduction to the numerical analysis of the Boltzmann equation in Summer School on methods and models of kinetic theory*. Rivista di Matematica della Università di Parma, 2004.
- [Ram00] D. Ramos. *Quelques résultats mathématiques et simulations numériques d’écoulements régis par des modèles bifluïdes*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2000.
- [Ree91] M.W. Reeks. On a kinetic equation for the transport of particles in turbulent flows. *Phys. Fluid A*, 3, 1991.
- [Ree01] M.W. Reeks. An introduction to the pdf approach for modelling dispersed particle flows. In *Trends in Numerical and Physical Modeling for Industrial Multiphase Flows*, 2001.
- [Rov06] J.M. Rovarch. *Un solveur volume fini 3D pour les écoulements diphasiques*. PhD thesis, CMLA, Ecole normale supérieure de Cachan, 2006.
- [RPMK04] J. Réveillon, C. Pera, M. Massot, and R. Knikker. Eulerian analysis of the dispersion of evaporating sprays in a statistically stationary turbulent flow. *Journal of Turbulence*, 1:1–27, 2004.

- [Sai95] L. Sainsaulieu. *Contribution à la modélisation mathématique et numérique des écoulements diphasiques constitués d'un nuage de particules dans un écoulement de gaz*. Habilitation à diriger des recherches. Université Paris VI, 1995.
- [Sch66] L. Schwartz. *Théorie des distributions*. Hermann, 1966.
- [Sch91] Jack Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335, 1991.
- [Ser96a] D. Serre. *Systèmes de lois de conservation I*. Diderot, 1996.
- [Ser96b] D. Serre. *Systèmes de lois de conservation II*. Diderot, 1996.
- [SF00] O. Simonin and P. Fevrier. Statistical and continuum modelling of turbulent reactive particulate flows. part 2: Application of a two-phase second-moment transport model for prediction of turbulent gas-particle flows. *Theoretical and Experimental Modeling of Particulate Flows, Lecture Series 2000-06, Von Karman Institute for Fluid Dynamics, Rhode Saint Genèse (Belgium).*, 2000.
- [Sim00] O. Simonin. Statistical and continuum modelling of turbulent reactive particulate flows. part 1: Theoretical derivation of dispersed eulerian modelling from probability density function kinetic equation. *Theoretical and Experimental Modeling of Particulate Flows, Lecture Series 2000-06, Von Karman Institute for Fluid Dynamics, Rhode Saint Genèse (Belgium).*, 2000.
- [SN00] N.R. Smirnov and V.F. Nikitin. Ignition and combustion of turbulized dust-air mixtures. *Combustion and flame*, 123(3), October 2000.
- [SR96] D. Stanton and C. Rutland. Modeling fuel film interaction and wall interaction in diesel engines. *SAE 960628*, 1996.
- [SR98] D. Stanton and C. Rutland. Multi-dimensional modeling of heat and mass transfer of fuel films resulting from impinging sprays. *SAE 980132*, 1998.
- [Tam85] Y. Tambour. A lagrangian sectional approach for simulating droplet size distribution of vaporizing fuel in a turbulent jet. *Combustion and flame*, 60:15–28, 1985.
- [Tay96a] M. Taylor. *Partial differential equations, basic theory*. Springer, 1996.
- [Tay96b] M. Taylor. *Partial differential equations III, non linear equations*. Springer, 1996.
- [TCB⁺00] F. Thais, J.P. Chièze, P. Baclet, A. Benuzzi, S. Bouquet, C. Cherfils, F. Mucchielli, L. Polès, C. Reverdin, R. Teyssier, J.P. Thébault, and P. Troussel. *The Astrolab experiment: Rayleigh-Taylor instabilities in supernova*. Elsevier, 2000.
- [Tem84] T. Temam. *Navier Stokes equation*. Elsevier Science Publisher, 1984.
- [UA82] S. Ukai and K. Asano. On the Cauchy problem of the Boltzmann equation with a soft potential. *Publ. Res. Inst. Math. Sci.*, 18(2):477–519 (57–99), 1982.
- [Van83] D. Vandromme. *Contribution à la modélisation et à la prédiction d'écoulements turbulents à masse volumique variable*. PhD thesis, University of Lille, 1983.

-
- [Ver96] O. Vermorel. *Etude numérique et modélisation de la modulation de la turbulence dans un écoulement de nappe chargée en particules*. PhD thesis, Ecole Polytechnique (Palaiseau), 1996.
- [VH97] P. Villedieu and J. Hylkema. Une méthode particulière aléatoire reposant sur une équation cinétique pour la simulation numérique des sprays denses de gouttelettes liquides. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(3):323–328, 1997.
- [Vil02] C. Villani. *A review of mathematical topics in collisional kinetic theory*. Elsevier Science, 2002.
- [VS04] P. Villedieu and O. Simonin. Modeling of coalescence in turbulent gas-droplet flows. *Commun. Math. Sci.*, 2(suppl. 1):13–33, 2004.
- [Wil85] F.A. Williams. *Combustion theory*. Benjamin Cummings, 1985.
- [WO03] K. Willis and M. Orme. Binary droplet collisions in vacuum environment: an experimental investigation of the role of viscosity. *Exp. Fluids*, 34:28–41, 2003.