

UNIVERSITÉ DE LA ROCHELLE
Lab. de Maths & Applications

Mémoire présenté en vue de l'obtention du

Doctorat de Mathématiques

Spécialité : Mathématiques Appliquées

par

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**Sur les Systèmes Elliptiques Quasi-linéaires et Anisotropiques avec
Exposants Critiques de Sobolev**

soutenue publiquement le 13 juillet 2007 devant le jury composé de :

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Remerciements

Comme le veut la tradition, je vais tenter de satisfaire au difficile exercice de la page des remerciements. Non qu'exprimer ma gratitude envers les personnes en qui j'ai trouvé un soutien soit contre ma nature, bien au contraire. La difficulté tient plutôt dans le fait de n'oublier personne. C'est pourquoi, je remercie par avance ceux dont le nom n'apparaît pas dans cette page et qui m'ont aidé d'une manière ou d'une autre. Ils se reconnaîtront. Pour les autres, non merci. Ils se reconnaîtront aussi...

Cette thèse n'aurait jamais vu le jour sans la confiance, la patience et la générosité de mon directeur de thèse Abdallah El Hamidi que je tiens à remercier vivement. Je voudrais aussi le remercier pour le temps et la patience qu'il m'a accordé tout au long de ces années, d'avoir cru en mes capacités. De plus, les conseils qu'il m'a divulgué en période de rédaction ont toujours été clairs et succincts, me facilitant grandement la tâche et me permettant d'aboutir à la production de cette thèse.

Je tiens à remercier vivement Mokhtar Kirane pour ses conseils précieux durant les discussions que j'avais avec lui soit dans son bureau ou par téléphone depuis la Malaisie qui m'ont toujours été utiles. Ses qualités scientifiques et humaines, son encouragement et ses remarques ont largement contribué à l'aboutissement de cette thèse. Qu'il trouve ici l'expression de ma profonde gratitude.

Je remercie les professeurs Olivier Goubet et Vicentu Radulescu de m'avoir fait l'honneur d'être les rapporteurs de cette thèse. J'éprouve un profond respect pour leur travail et leur parcours, ainsi que pour leurs qualités humaines. Je les remercie aussi pour leur participation au jury de thèse. Ils ont également contribué par leurs nombreuses remarques et suggestions à améliorer la qualité de ce mémoire, et je leur en suis reconnaissant.

Mes remerciements s'adressent ensuite aux professeurs : Samir Adly, Claude-Michel Brauner, Mokhtar Kirane et Jean-Michel Rakotoson, d'avoir accepté de participer à mon jury de thèse.

Enfin, je remercie mes parents ; Lhoucine et Halima, mes frères ; Rédouane et Ous-sama, mes soeurs ; Fatima et Soumiya et mes camarades ; Amine et Hassan de leur soutien constant tout au long des années de la thèse.

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Chapitre 1

Introduction

Le but de cette thèse est de présenter des résultats récents concernant l'existence et la multiplicité des solutions positives de certaines classes de systèmes d'équations aux dérivées partielles elliptiques non linéaires faisant intervenir l'opérateur (p, q) -Laplacien du type suivant :

$$\begin{cases} \Delta_p u = f(x, u, v), & \text{dans } \Omega \\ \Delta_q v = g(x, u, v), & \text{dans } \Omega \end{cases} \quad (0.1)$$

et des systèmes anisotropiques de la forme suivante :

$$\begin{cases} \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = f(x, u, v), & \text{dans } \Omega \\ \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \right) = g(x, u, v), & \text{dans } \Omega \end{cases} \quad (0.2)$$

où $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ et $\Delta_q v = \nabla \cdot (|\nabla v|^{q-2} \nabla v)$ avec $p > 1$, $q > 1$, $p_i > 1$ et $q_i > 1$, Ω est un ouvert non vide de \mathbb{R}^N . D'autre part, les fonctions f et g sont de Caratheodory et sont soumises à certaines conditions de croissance pour garantir que la fonction d'Euler-Lagrange associée est bien définie sur un produit cartésien d'espaces de Sobolev adéquats.

Les systèmes d'équations non-linéaires elliptiques et anisotropiques présentent quelques nouveaux et intéressants phénomènes qu'on ne rencontre dans les cas scalaires. En général, les systèmes sont couplés, et même fortement couplés. Alors, les notions de superlinéarité ou sous-linéarité et même d'exposants critiques, au sens de Sobolev, doivent prendre en considération la nature de tels couplages.

L'opérateur p -Laplacien apparaît aussi bien en mathématiques pures ; par exemple en géométrie riemannienne, qu'en mathématiques appliquées. En effet, il intervient dans de nombreux domaines en sciences expérimentales : problèmes de réaction-diffusion non linéaires, dynamique des populations, écoulements de fluides non-newtoniens, écoulements dans les milieux poreux, élasticité non linéaire, extraction

de pétrole [23], etc ...

Dans la littérature on trouve de nombreux travaux dédiés à l'étude théorique de tels équations et systèmes d'équations. En fait, l'étude de ce type de problèmes a effectivement commencé au milieu des années 1980 par M. Ôtani [50] en une dimension puis par F. de Thélin [58] en dimension N , qui ont obtenu les premiers résultats sur une équation de la forme $-\Delta_p u = \lambda u^{\gamma-1}$. Ce dernier auteur [58] et W. M. Ni & J. Serrin [57] ont démontré indépendamment l'existence et l'unicité des solutions radiales dans \mathbb{R}^N , plus tard Ôtani [49] a généralisé ce résultat à des ouverts quelconques. En 1987, F. De Thélin [59] a étendu ces résultats pour des équations de type $\Delta_p u = g(x, u)$ où la fonction g est contrôlée par des fonctions polynomiales par rapport à u . On peut citer certains précurseurs de l'analyse des problèmes elliptiques aux valeurs propres G. Barles [10], S. Sakaguchi [56] et A. Anane [8], qui ont étudié les équations du type :

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{dans } \Omega \text{ domaine borné.}$$

Plus tard en 1990, P. Lindqvist [43] a établi différents résultats sur ce type d'équations qui font suite à l'article de A. Anane [8]. Par ailleurs, il y a d'autres résultats sur l'unicité qui ont été énoncés par J. I. Diaz et J. E. Saa [24] en 1987 pour des équations de la forme $-\Delta_p u = f(x, u)$ sous la condition $r \mapsto \frac{f(x, r)}{r^{p-1}}$ est décroissante. Le problème de bifurcation à la première valeur propre a été abordé par R. F. Manásevich et M. A. Del Pino [46], tandis que les problèmes de non résonance associés au p -Laplacien étaient étudiés par A. Anane et J. P. Gossez [9]. Plus tard, le cas non borné de ces équations a été abordé par P. Drábek [25], Drábek et Y. X. Huang [26] et A. Bechah, K. Chaïb et F. de Thélin [11], où les questions d'existence et d'unicité ont été résolues aussi bien pour des problèmes de valeurs propres que pour des problèmes non-linéaires.

Le cas des systèmes présente un nouveau défi et entraîne plusieurs complications liées au couplage. Les systèmes variationnels peuvent être traités en utilisant la théorie des points critiques, puisque les solutions faibles de ces systèmes sont précisément les points critiques des fonctionnelles d'Euler-Lagrange associées. Les espaces où ces fonctionnelles sont étudiées dépendent des conditions aux bords que les solutions doivent satisfaire. Cette méthode est appelée la méthode directe pour le calcul des variations, dont les origines remontent à Gauss et Thomson au milieu du 19^{ème} siècle et qui a été utilisée par Dirichlet et Riemann pour résoudre le problème de Dirichlet pour l'équation de Laplace. Cependant, en 1870 Weierstrass avait montré que la démonstration comportait des "trous" et manquait de rigueur mathématique et a été alors abandonnée. Il faut attendre le début du 20^{ème} lorsque Hilbert a ressuscité la méthode et développé la théorie nécessaire pour la justifier et depuis elle été connue sous le nom de *principe de Dirichlet*. De nos jours, les mêmes méthodes sont toujours utilisées pour résoudre des problèmes aux bords pour des classes plus générales d'équations et de systèmes elliptiques. Dans le cas simple du problème de

Dirichlet pour l'équation de Laplace, le point critique est un minimum de la fonctionnelle associée. Dans les années 1930, Ljusternik et Schnierelmann ont développé une théorie des points critiques de type min-max pour des fonctionnelles présentant une symétrie Z_2 . Plus tard, en 1973 Ambrosetti et Rabinowitz [7] ont établi plusieurs résultats sur les points critiques du type min-max pour des fonctionnelles sans symétrie. On peut distinguer deux sortes de systèmes variationnels : ceux qui sont du type gradient, s'il existe une fonction $F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ de classe C^1 telle que :

$$f = \frac{\partial F}{\partial u} \quad \text{et} \quad g = \frac{\partial F}{\partial v}, \quad (0.3)$$

et ceux du type hamiltonien s'il existe une fonction $H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ de classe C^1 telle que :

$$f = \frac{\partial H}{\partial v} \quad \text{et} \quad g = \frac{\partial H}{\partial u}, \quad (0.4)$$

où les fonctions f et g sont définies dans (0.1) et (0.2).

En ce qui concerne le système du type gradient (0.1) satisfaisant (0.3), on cherche les points critiques de la fonctionnelle

$$I(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q dx - \int_{\Omega} F(x, u, v) dx$$

qui sont solutions faibles du système (0.1). La fonctionnelle I est définie dans l'espace produit $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. On supposera que la fonction F doit satisfaire les conditions de croissance suivantes :

$$|F_u(x, u, v)| \leq C(|u|^\gamma + |u|^\alpha |v|^{\beta+1}) \quad \text{p. p. dans } \Omega \quad (0.5)$$

$$|F_v(x, u, v)| \leq C(|v|^\delta + |u|^{\alpha+1} |v|^\beta) \quad \text{p. p. dans } \Omega \quad (0.6)$$

avec les conditions de sous-criticalité ou de criticalité (au sens de Sobolev)

$$\frac{\alpha + 1}{p^*} + \frac{\beta + 1}{q^*} < 1 \quad \text{ou} \quad \frac{\alpha + 1}{p^*} + \frac{\beta + 1}{q^*} = 1,$$

où $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$ et $1 < p, q < N$ sont les exposants critiques des injections de Sobolev $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ et $W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega)$. Les exposants γ et δ vérifient $1 < \gamma < p^* - 1$ et $1 < \delta < q^* - 1$. Le fait d'imposer que F soit de classe C^1 et qu'elle satisfait les conditions (0.5) et (0.6) entraînent que I est aussi de classe C^1 .

C'est en 1990 où F. de Thélin [60] avait initié les travaux sur les systèmes faisant intervenir le p -Laplacien où il a montré l'existence et l'unicité de la première valeur propre du système

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} u |v|^{\beta+1} & \text{dans } \Omega \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v & \text{dans } \Omega \end{cases}$$

sous la condition de criticalité $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$. Le cas d'un système variationnel a été traité par P. Felmer, R. F. Manásevich et F. de Thélin [34] où les auteurs ont étudié l'existence et l'unicité de la solution positive d'un système variationnel du type :

$$\begin{cases} -\Delta_p u &= \frac{\partial H}{\partial u}(x, u, v) \quad \text{dans } \Omega \\ -\Delta_q v &= \frac{\partial H}{\partial v}(x, u, v) \quad \text{dans } \Omega, \end{cases}$$

généralisant ainsi les résultats obtenus dans le cas scalaire par J. I. Diaz et J. E. Saa [24]. Plus tard, ces résultats ont été étendus au cas du système dérivant d'un potentiel par F. de Thélin et J. Vélin [62], J. Chabrowski [19] et L. Boccardo and D. G. de Figueiredo [14] et ont commencé une approche du cas non variationnel en imposant des conditions sur la croissance des non-linéarités. Dans tout ce qui précède, les auteurs ont étudié ces systèmes dans des domaines bornés, pour ce qui est des problèmes non bornés de \mathbb{R}^N ont été abordés par J. Fleckinger, R. F. Manásevich, N. M. Stavrakakis et F. de Thélin [34] et A. Bechah, K. Chaïb et F. de Thélin [11]. Notons que l'étude des systèmes de p -Laplaciens dans \mathbb{R}^N ont été inspiré par l'étude générale faite par M. F. Bidaut-Véron [13] sur les Laplaciens classiques et par P. Clément, J. Fleckinger, R. F. Manásevich et F. de Thélin [21] et P. Clément, R. Manásevich et E. Mitidieri [22] qui ont étudié la question de l'existence des solutions pour les systèmes (p, q) -Laplacien purement non variationnels du type (0.1).

Actuellement, de nombreux travaux de recherche sont en cours sur les systèmes, en particulier les quatres chapitres de la présente thèse [1, 2, 3, 4]. L'historique qu'on vient de tracer ci-dessus est bien entendu loin d'être exhaustif.

Une des motivations de cette thèse est le fait que certains résultats sur les systèmes elliptiques faisant intervenir le p -Laplacien dans les domaines bornés ou non bornés méritaient d'être complétés et que dans certains travaux on imposait des conditions sur les exposants qui ne sont pas "naturelles" pour garantir l'existence ou la non-existence des solutions. La deuxième motivation est de généraliser certains résultats obtenus par T. Aubin et H. Brézis & L. Nirenberg concernant le niveau critique garantissant la compacité des suites minimisantes de Palais-Smale au cas d'équations scalaires plus générales et ensuite des systèmes elliptiques faisant intervenir le (p, q) -Laplacien.

Notre troisième travail a été motivé par la difficulté de démontrer des résultats de multiplicité dans le cas critique en domaine non borné.

Notre dernier travail a été motivé par un récent résultat important, dû à A. El Hamidi et J. M. Rakotoson [32] où ils ont généralisé le fameux principe de concentration-compacité de P. L. Lions au cas des opérateurs anisotropiques.

Dans le reste de cette introduction, nous décrivons brièvement les travaux présentés dans cette thèse.

Chapitre 2 : Dans leur article [6], A. Ambrosetti, H. Brézis et G. Cerami étudient l'existence et la multiplicité de l'équation suivante :

$$\begin{cases} -\Delta u = f_\lambda(u), & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$

avec f_λ est une fonction présentant une sous linéarité de type concave-convexe, en particulier $f_\lambda(u) = \lambda|u|^{p-1}u + |u|^{q-1}u$ sous la condition $1 < p < 2 < q < 2^*$. Ils ont montré, en utilisant la méthode des sous et sur solutions, l'existence d'une solution positive correspondant aux petites valeurs de $\lambda > 0$ et l'existence d'une deuxième solution avec le théorème du col. Ensuite, ils ont également démontré l'existence d'une infinité de solutions si la fonction f_λ est impaire. Ce résultat a été généralisé et amélioré par A. El Hamidi [29] au problème de Dirichlet et mixte suivant :

$$\begin{cases} -\Delta_p u = \lambda|u|^{q-1}u + |u|^{r-1}u, & x \in \Omega \\ \varepsilon|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + a(x)|u|^{p-2}u = 0, & x \in \partial\Omega, \end{cases} \quad (0.7)$$

avec $1 < q < p < r < p^*$ et $\varepsilon \in \{0, 1\}$. En utilisant la méthode de Nehari, introduite par Nehari en 1960. Cette méthode est équivalente à la méthode de stratification sphérique introduite plus tard par S. I. Pohozaev. A. El Hamidi a étudié la fonctionnelle énergie modifiée \tilde{E}_λ définie sur $\mathbb{R} \times W_0^{1,p}(\Omega)$ par :

$$\tilde{E}_\lambda(t, u) := E_\lambda(tu),$$

où E_λ est la fonctionnelle d'Euler-Lagrange associée au problème (0.7). Il a étudié la restriction E_λ^1 et E_λ^2 de E_λ à la variété de Nehari, qui est en fait constituée de deux ensembles disjoints dès que $0 < \lambda < \hat{\lambda}$, et a ensuite démontré que les suites minimisantes de E_λ^i , $i \in \{1, 2\}$, sont de Palais-Smale et convergent vers deux différentes solutions positives de l'équation (0.7). La première solution a une énergie négative tandis que la deuxième solution a une énergie qui change de signe en $\lambda_0 \in (0, \hat{\lambda})$.

Dans l'article [1], on a étudié le système elliptique variationnel sous critique suivant :

$$\begin{cases} -\Delta_p u = \lambda|u|^{p_1-2}u + (\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} & \text{dans } \Omega, \\ -\Delta_q v = \mu|v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v & \text{dans } \Omega, \end{cases} \quad (0.8)$$

avec $1 < p_1 < p < N$, $1 < \beta+1 < q < N$, $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$ et $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$. On a adopté les mêmes arguments développés par A. El Hamidi dans son article [29]. A la lumière du résultat de Y. Bozhkov and E. Mitidieri [16], on a introduit la fonctionnelle énergie modifiée définie par :

$$\tilde{I}_{\lambda,\mu}(s, u, t, v) = I_{\lambda,\mu}(su, tv)$$

où $I_{\lambda,\mu}$ est la fonctionnelle d'Euler-Lagrange associée au système (0.8). En explorant la variété de Nehari associée à $I_{\lambda,\mu}$ définie par tous les couples $(su, tv) \neq (0, 0)$ vérifiant $\partial \tilde{I}_{\lambda,\mu}(s, u, t, v)/\partial s = \partial \tilde{I}_{\lambda,\mu}(s, u, t, v)/\partial t = 0$, on a démontré que cette variété est encore composée de deux parties disjointes $\mathcal{N}_{\lambda,\mu}^1$ et $\mathcal{N}_{\lambda,\mu}^2$ dès que (λ, μ) appartient à un sous ensemble spécifique \mathcal{D} de \mathbb{R}^2 . L'étude de la restriction de la fonctionnelle $I_{\lambda,\mu}$ à $\mathcal{N}_{\lambda,\mu}^1$ et $\mathcal{N}_{\lambda,\mu}^2$ nous a permis ensuite de prouver que les suites minimisantes dans les deux parties de la variété de Nehari sont de Palais-Smale et convergent vers les deux solutions positives du système (0.8). En ce qui concerne le signe de leurs énergies, notons que la première est d'une énergie négative tandis que l'énergie de la seconde change de signe selon une fonction continue $\lambda_0(\mu)$ dont le graphe est situé dans l'ensemble \mathcal{D} .

Chapitre 3 : Dans cette partie, on s'intéressera à l'étude du système (0.8) dans le cas non borné et vérifiant la condition critique $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$. Notons que, par rapport au Chapitre 1, nous perdons la compacité des injections $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ et $W^{1,q}(\Omega) \subset L^{q^*}(\Omega)$. Dans la littérature, le principe de compacité par concentration de P. L. Lions [45] et le principe de compacité par concentration de Bianchi et al. [12] sont largement utilisés pour remédier à la perte de compacité des suites de Palais-Smale. Signalons que la décomposition de Struwe est aussi utile dans le cas des domaines bornés et les variétés compactes. J. Vélin et F. de Thélin [62] ont étudié le problème suivant :

$$\begin{cases} -\Delta_p = u|u|^{\alpha-1}|v|^{\beta+1}, & \text{dans } \Omega \\ -\Delta_q = |u|^{\alpha+1}|v|^{\beta-1}v, & \text{dans } \Omega. \end{cases}$$

Ils ont démontré un résultat d'existence de solutions sous l'hypothèse $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ et un résultat de non existence dans le cas $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$ et le domaine Ω est strictement étoilé en utilisant l'identité de Pohozaev [53]. Ensuite le système critique suivant

$$\begin{cases} -\Delta_p = u|u|^{\alpha-1}|v|^{\beta+1} + f, & \text{dans } \Omega \\ -\Delta_q = |u|^{\alpha+1}|v|^{\beta-1}v + g, & \text{dans } \Omega. \\ u = v = 0, & \text{sur } \partial\Omega \end{cases}$$

avec $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, a été discuté par J. Chabrowski [19] dans le cas $p = q$ et J. Velin [63] pour le cas $p \neq q$ et ont montré l'existence d'au moins une solution positive sous l'hypothèse $f \in W^{-1,p'}(\Omega) \setminus \{0\}$ et $g \in W^{-1,q'}(\Omega) \setminus \{0\}$ et $\|f\|_{-1,p'}, \|g\|_{-1,q'} < k$. Dans notre article [3], on s'est intéressé au système suivant

$$\begin{cases} -\Delta_p u = a(x)|u|^{p_1-2}u + u|u|^{\alpha-1}|v|^{\beta+1}, & \text{dans } \Omega \\ -\Delta_q v = b(x)|v|^{q-2}v + |u|^{\alpha+1}|v|^{\beta-1}v, & \text{dans } \Omega \\ u = v = 0, & \text{sur } \partial\Omega, \end{cases}$$

dans le cas de $\Omega = \mathbb{R}^N$ et $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, l'existence d'au moins une solution positive a été prouvé par contre pour ce qui est de la seconde solution positive comme dans l'article [1] on a pu la récupérer que pour $0 \leq \mu < \mu_1$ (μ_1 est la première valeur propre de $-\Delta_q v = \mu|v|^{q-2}v$) et λ suffisamment près de 0 en utilisant un théorème de convergence des gradients des suites minimisantes de Palais-Smale développé par A. El Hamidi et J. M. Rakotoson [31].

Chapitre 3 : L'une des difficultés majeures en analyse des problèmes elliptiques variationnels non linéaires faisant intervenir des non linéarités critiques est de récupérer la compacité des suites de Palais-Smale de la fonctionnelle d'Euler-Lagrange associée. Ce problème a été discuté par Brézis et Nirenberg dans leur fameux article [17]. Le principe de la compacité par concentration dû à P. L. Lions est largement utilisé pour surmonter ce type de problème. Il existe d'autres méthodes, basées sur la convergence presque partout du gradient des suites de Palais-Smale, qui nous permettent de récupérer la compacité. On peut citer par exemple l'article de L. Boccardo et F. Murat [15], J. M. Rakotoson [55] pour les domaines bornés et A. El Hamidi et J. M. Rakotoson [31] pour les domaines arbitraires.

Les auteurs dans [17] ont étudié le problème de valeur propre avec une perturbation critique :

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1}, & \text{dans } \Omega, \\ u > 0, & \text{dans } \Omega, \\ u = 0, & \text{sur } \partial\Omega, \end{cases}$$

avec Ω est un domaine borné de \mathbb{R}^N , $N \geq 3$ à bord régulier, $2^* = \frac{2N}{N-2}$ est l'exposant critique de Sobolev de l'injection $W^{1,2}(\Omega) \subset L^p(\Omega)$ et λ est un paramètre positif. Les auteurs ont introduit une condition importante sur le niveau correspondant à l'énergie des suites de Palais-Smale qui garantit leur compacité relative. En fait, soit (u_n) une suite de Palais-Smale pour la fonctionnelle d'Euler-Lagrange :

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega |u|^2 - \frac{1}{2^*} \int_\Omega |u|^{2^*}.$$

Ils ont montré que si une suite (u_n) de $(PS)_c$ satisfait :

$$\lim_{n \rightarrow +\infty} I_\lambda(u_n) < \frac{1}{N} S^{\frac{N}{2}}, \quad (0.9)$$

alors (u_n) est relativement compacte, ce qui entraîne l'existence des points critiques non triviaux de I_λ . Ici, S est la meilleur constante de Sobolev de l'injection $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. Dans ce chapitre, on donne une généralisation du condition (0.9) à l'équation semi-linéaire suivante :

$$(P_\lambda) \quad \begin{cases} -\Delta_p u = \lambda f(x, u) + |u|^{p^*-2}u, & \text{dans } \Omega, \\ u|_\Gamma = 0, \quad \text{et } \frac{\partial u}{\partial \nu}|_\Sigma = 0, \end{cases} \quad (0.10)$$

avec Ω est un domaine borné de R^N , $N \geq 3$ avec un bord régulier $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$ et Γ Σ sont des sous-variétés régulières de $\partial\Omega$ de dimension $(N - 1)$ avec des mesures positives telles que $\Gamma \cap \Sigma = \emptyset$. Ici, $\frac{\partial}{\partial \nu}$ dénote la différentielle normale extérieure et f est une perturbation sous critique de $|u|^{p^*-1}$. On a démontré que si une suite (u_n) de $(PS)_c$ telle que

$$c < c^* \equiv \inf_{u \in \mathcal{N}_0} I_0(u) + \inf_{v \in \mathcal{N}_\lambda \cup \{0\}} I_\lambda(v), \quad (0.11)$$

alors (u_n) est relativement compacte. Ici le niveau c^* est critique parce qu'on a réussi à construire une suite de $(PS)_{c^*}$ qui n'est pas relativement compacte. Les ensembles \mathcal{N}_λ et \mathcal{N}_0 dénotent respectivement les variétés de Nehari relatives aux problèmes (P_λ) et (P_0) .

Ensuite, dans la secondes partie du chapitre 2, on donne une condition analogue à (0.9) et (0.11) pour un système général avec des exposants critiques :

$$\begin{cases} -\Delta_p = \lambda f(x, u) + u|u|^{\alpha-1}|v|^{\beta+1}, & \text{dans } \Omega \\ -\Delta_q = \mu g(x, v) + |u|^{\alpha+1}|v|^{\beta-1}v, & \text{dans } \Omega \end{cases}$$

avec $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$ et des conditions aux bords de Dirichlet ou mixtes, où f et g sont des perturbations sous critiques de $|u|^{p^*-1}$ et $|v|^{q^*-1}$ respectivement. On dénote par $p^* = \frac{Np}{N-p}$ et $q^* = \frac{Nq}{N-q}$ sont respectivement les exposants critiques relatives aux injections de Sobolev $W^{1,p}(\Omega) \subset L^r(\Omega)$ et $W^{1,q}(\Omega) \subset L^r(\Omega)$. Notre approche fournit une condition générale basée sur la variété de Nehari, qui peut être étendue à une classe plus large de problèmes non linéaires critiques. L'optimalité de notre résultat est établie dans le cas spécial $p = q$ et obtenue par la constuction d'une suite de Palais-Smale qui n'est pas relativement compacte, cependant la question est encore ouverte dans le cas $p \neq q$. D. G. de Figueiredo [35], D. G. de Figueiredo et P. Felmer [36] et L. Boccardo et D. G. de Figueiredo [14] ont étudié une large classe de systèmes non linéaires elliptiques.

Chapitre 5 : Cette partie sera consacrée à l'étude d'un système anisotropique non-

linéaire de la forme :

$$\begin{cases} \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda a(x) |u|^{p-2} u + u |u|^{\alpha-1} |v|^{\beta+1}, & \text{dans } \Omega \\ \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \right) = \mu b(x) |v|^{q-2} v + |u|^{\alpha+1} |v|^{\beta-1} v, & \text{dans } \Omega, \\ u = v = 0, & \text{sur } \partial\Omega \end{cases} \quad (0.12)$$

avec

$$p_i, q_i > 1, \sum_{i=1}^N \frac{i-1}{p_i} > 1, \sum_{i=1}^N \frac{1}{q_i} > 1 \quad \text{et} \quad \frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$$

où

$$p^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i} - 1}, \quad \text{et} \quad q^* = \frac{N}{\sum_{i=1}^N \frac{1}{q_i} - 1}.$$

Ici, p^* et q^* sont respectivement les exposants critiques effectifs associés aux opérateurs

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial}{\partial x_i} \right|^{p_i-2} \frac{\partial}{\partial x_i} \right) \quad \text{et} \quad \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial}{\partial x_i} \right|^{q_i-2} \frac{\partial}{\partial x_i} \right).$$

Ce chapitre est motivé par les récents résultats de I. Fragalà et al. [38], C. O. Alves et A. El Hamidi [5] et A. El Hamidi et J. M. Rakotoson [30, 31]. En effet, les auteurs dans [38] ont considéré le même opérateur différentiel anisotropique comme (0.12) dans le cas scalaire et ont établi des résultats d'existence et de régularité dans le cas sous critique ainsi qu'un résultat de non existence dans les domaines étoilés a été montré.

Les auteurs dans [32] ont généralisé le principe de la compacité par concentration de P. L. Lions [45] au cas anisotropique. En utilisant ce principe généralisé, ils ont montré que la meilleur constant de Sobolev dans un certain cas critique est atteinte. Dans l'article [4] on a généralisé les résultats d'existence et de régularité des solutions positives obtenus par de C. O. Alves et A. El Hamidi [5] au cas du système (0.12) dans un domaine borné de \mathbb{R}^N , en utilisant les méthodes mini-max.

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Chapitre 2

The Nehari manifold for systems of nonlinear elliptic equations

Abstract

This paper deals with existence and multiplicity results of nonlocal positive solutions to the following system

$$\begin{cases} -\Delta_p u &= \lambda|u|^{p_1-2}u + (\alpha + 1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &= \mu|v|^{q-2}v + (\beta + 1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases}$$

together with Dirichlet or mixed boundary conditions, under some hypotheses on the parameters p, p_1, α, β and q . More precisely, the system considered corresponds to a perturbed eigenvalue equation combined with a second equation having concave and convex nonlinearities. The study is based on the extraction of Palais-Smale sequences in the Nehari manifold. The behaviour of the energy corresponding to these positive solutions, with respect to the real parameters λ and μ , is established.

2.1 Introduction

In this work, we consider the system of quasilinear elliptic equations

$$\begin{cases} -\Delta_p u &= \lambda|u|^{p_1-2}u + (\alpha + 1)u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &= \mu|v|^{q-2}v + (\beta + 1)|u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \quad (1.1)$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 & \text{and} & \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 & \text{and} & \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \quad (1.2)$$

where, Ω is a bounded domain in \mathbb{R}^N , with smooth boundary $\partial\Omega = \bar{\Gamma}_i \cap \bar{\Sigma}_i$, where Γ_i are smooth $(N-1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that

$\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1, 2\}$. Δ_p is the p -Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. It is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions.

Our aim here is to establish nonlocal existence and multiplicity results, with respect to the real parameters λ and μ , for Problem (1.1). Along this work, the following assumptions will hold

$$1 < p_1 < p < N, \quad q > 1, \quad \alpha > 1, \quad \beta > 1, \quad (1.3)$$

$$\frac{\alpha + 1}{p^*} + \frac{\beta + 1}{q^*} < 1, \quad (1.4)$$

$$\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} > 1 \quad \text{and} \quad \frac{\beta + 1}{q} < 1, \quad (1.5)$$

where

$$p^* = \frac{Np}{N-p}, \quad q^* = \frac{Nq}{N-q}$$

are the critical exponents for the p -Laplacian and q -Laplacian respectively. These assumptions mean that we are concerned with a subcritical and super-homogeneous system where the first equation is *concave-convex* and the second equation is only a perturbation of an eigenvalue equation. Also, the following assumptions concerning the real parameters λ and μ will hold

$$\lambda > 0, \quad \mu < \mu_1,$$

where μ_1 is the first eigenvalue of $-\Delta_q$ in Ω .

Problem (1.1), together with (1.2), is posed in the framework of the Sobolev space $W = W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega)$, where

$$W_{\Gamma_1}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0\}, \quad W_{\Gamma_2}^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_2} = 0\},$$

are respectively the closure of $C_0^1(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C_0^1(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. We can refer the reader to [9] for a complete description of this space in the case $p = 2$. Notice that $meas(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W_{\Gamma_1}^{1,p}(\Omega)$ and $W_{\Gamma_2}^{1,q}(\Omega)$, so W can be endowed with the norm

$$\|(u, v)\| = \|\nabla u\|_p + \|\nabla v\|_q$$

and $(W, \|\cdot\|)$ is a reflexive and separable Banach space.

Semilinear and quasilinear scalar elliptic equations with concave and convex nonlinearities are widely studied, we can refer the reader to [1, 4, 10, 18] and to the survey article [5]. For the nonlinear elliptic systems, we refer to [2, 3, 6, 8, 11, 14, 20, 21] and to the survey article [13]. In [15], the authors studied the existence of positive solutions to a perturbed eigenvalue problem involving the p -Laplacian operator. In

[6], the authors have generalized the results of [15] to a perturbed eigenvalue system involving p and q -Laplacian operators. Recently, in [10] the first author has considered a semilinear elliptic equation with concave and convex nonlinearities, and showed *nonlocal* existence and multiplicity results with respect to the parameter via the extraction of Palais-Smale sequences in the Nehari manifold.

In this paper, we extend this method to the system (1.1) where one equation contains concave and convex nonlinearities and the other one is simply a perturbation of an eigenvalue equation. We show that Problem (1.1) has at least two positive solutions when the pair of parameters (λ, μ) belongs to a subset of \mathbb{R}^2 which will be specified below.

For solutions of (1.1) we understand critical points of the Euler-Lagrange functional $I \in C^1(W, \mathbb{R})$ given by

$$I(u, v) = \frac{1}{p}P(u) - \frac{\lambda}{p_1}P_1(u) + \frac{1}{q}(Q(v) - \mu Q_1(v)) - R(u, v),$$

where $P(u) = \|\nabla u\|_p^p$, $P_1(u) = \|u\|_{p_1}^{p_1}$, $Q(v) = \|\nabla v\|_q^q$, $Q_1(v) = \|v\|_q^q$ and $R(u, v) = \int_{\Omega} |u|^{\alpha+1}|v|^{\beta+1}dx$.

Consider the "Nehari" manifold [16] associated to Problem (1.1) given by

$$\mathcal{N} = \{(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) / D_1I(u, v)(u) = D_2I(u, v)(v) = 0\},$$

where D_1I and D_2I are the derivatives of I with respect to the first variable and the second variable respectively.

An interesting and useful characterization of \mathcal{N} , [15, 18, 22, 10, 7] is the following

$$\mathcal{N} = \{(su, tv) / (s, u, t, v) \in \mathcal{Z}^* \text{ and } \partial_s I(su, tv) = \partial_t I(su, tv) = 0\},$$

where $\mathcal{Z}^* = (\mathbb{R} \setminus \{0\}) \times (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$ and I is considered as a functional of four variables (s, u, t, v) in $\mathcal{Z} := \mathbb{R} \times W_{\Gamma_1}^{1,p}(\Omega) \times \mathbb{R} \times W_{\Gamma_2}^{1,q}(\Omega)$. For this reason, we introduce the modified Euler-Lagrange functional \tilde{I} defined on \mathcal{Z} by

$$\tilde{I}(s, u, t, v) := I(su, tv).$$

2.2 Preliminary results

In this work, we are interested by nontrivial positive solutions $u \neq 0$ and $v \neq 0$ to Problem (1.1). Since the functional \tilde{I} is even in s and t , we limit our study for $s > 0$, $t > 0$ and for $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$.

Lemma 2.2.1. *For every $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$ there exists a unique $\lambda(u, v) > 0$ such that the real-valued function $(s, t) \in (0, +\infty)^2 \mapsto \tilde{I}(s, u, t, v)$ has exactly two critical points (resp. one critical point) for $0 < \lambda < \lambda(u, v)$ (resp. $\lambda = \lambda(u, v)$). This functional has no critical point for $\lambda > \lambda(u, v)$.*

Proof. Let (u, v) be an arbitrary element in $(W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\})$. Then

$$\tilde{I}(s, u, t, v) = \frac{s^p}{p}P(u) - \frac{\lambda}{p_1}s^{p_1}P_1(u) + \frac{t^q}{q}(Q(v) - \mu Q_1(v)) - s^{\alpha+1}t^{\beta+1}R(u, v).$$

A direct computation gives $\partial_t \tilde{I}(s, u, t, v) = 0$ if and only if

$$t = t(s) = \left[(\beta + 1) \frac{R(u, v)}{Q(v) - \mu Q_1(v)} \right]^{\frac{1}{q - (\beta + 1)}} s^{\frac{\alpha + 1}{q - (\beta + 1)}}, \quad (2.6)$$

and

$$\tilde{I}(s, u, t(s), v) = \frac{s^p}{p}P(u) - \frac{\lambda}{p_1}s^{p_1}P_1(u) - \frac{s^r}{r}A(u, v),$$

where

$$A(u, v) = (\alpha + 1)(\beta + 1) \frac{s^{\frac{\alpha + 1}{q - (\beta + 1)}} R(u, v)^{\frac{q}{q - (\beta + 1)}}}{(Q(v) - \mu Q_1(v))^{\frac{\beta + 1}{q - (\beta + 1)}}}$$

and $r = \frac{(\alpha + 1)q}{q - (\beta + 1)}$. It is easy to verify that $r > p$. Now consider the function $s \in (0, +\infty) \mapsto \tilde{I}(s, u, t(s), v)$ and let us write

$$\partial_s \tilde{I}(s, u, t(s), v) := s^{p_1 - 1} F_{\lambda, \mu}(s, u, v).$$

where $F_{\lambda, \mu}(s, u, v) := P(u)s^{p - p_1} - \lambda P_1(u) - A(u, v)s^{r - p_1}$. The function $s \in (0, +\infty) \mapsto F_{\lambda, \mu}(s, u, v)$ is increasing on $(0, \bar{s}_\mu(u, v))$, decreasing on $(\bar{s}_\mu(u, v), +\infty)$ and attains its unique maximum for $s = \bar{s}_\mu(u, v)$, where

$$\bar{s}_\mu(u, v) = \left[\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{1}{r - p}}. \quad (2.7)$$

So, the function $s \in (0, +\infty) \mapsto F_{\lambda, \mu}(s, u, v)$ has two positive zeros (resp. one positive zero) if $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) > 0$ (resp. $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) = 0$) and has no zero if $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) < 0$. On the other hand, a direct computation leads to

$$F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) = \frac{r - p}{r - p_1} \left[\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{p - p_1}{r - p_1}} P(u) - \lambda P_1(u).$$

Then, $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) > 0$ (resp. $F_{\lambda, \mu}(\bar{s}_\mu(u, v), u, v) < 0$) if $\lambda < \lambda(u, v)$ (resp. $\lambda > \lambda(u, v)$) and $F_{\lambda(u, v), \mu}(\bar{s}_\mu(u, v), u, v) = 0$, where

$$\lambda(u, v) = \hat{c} \frac{P(u)^{\frac{r - p_1}{r - p}}}{P_1(u)A(u, v)^{\frac{p - p_1}{r - p}}} \quad \text{and} \quad \hat{c} = \frac{r - p}{r - p_1} \left[\frac{p - p_1}{r - p_1} \right]^{\frac{p - p_1}{r - p}}. \quad (2.8)$$

Therefore, if $\lambda \in (0, \lambda(u, v))$, the function $s \in (0, +\infty) \mapsto \partial_s \tilde{I}(s, u, t(s), v)$ has two positive zeros denoted by $s_1(u, v, \lambda, \mu)$ and $s_2(u, v, \lambda, \mu)$ verifying $0 < s_1(u, v, \lambda, \mu) <$

$\bar{s}_\mu(u, v) < s_2(u, v, \lambda, \mu)$. Since $F_{\lambda, \mu}(s_1(u, v, \lambda, \mu), u, v) = F_{\lambda, \mu}(s_2(u, v, \lambda, \mu), u, v) = 0$, $\partial_s F_{\lambda, \mu}(s, u, v) > 0$ for $0 < s < \bar{s}_\mu(u, v)$ and $\partial_s F_{\lambda, \mu}(s, u, v) < 0$ for $s > \bar{s}_\mu(u, v)$ it follows that

$$\partial_{ss} \tilde{I}(s_1(u, v, \lambda, \mu), u, t(s_1(u, v, \lambda, \mu)), v) > 0, \quad (2.9)$$

$$\partial_{ss} \tilde{I}(s_2(u, v, \lambda, \mu), u, t(s_2(u, v, \lambda, \mu)), v) < 0. \quad (2.10)$$

This implies that the real-valued function $s \in (0, +\infty) \mapsto \tilde{I}(s, u, t(s), v)$ achieves its unique local minimum at $s = s_1(u, v, \lambda, \mu)$ and its unique local maximum at $s = s_2(u, v, \lambda, \mu)$, which ends the proof. \square

Hereafter, we will denote $t_i(u, v, \lambda, \mu) := t(s_i(u, v, \lambda, \mu))$, $i = 1, 2$. At this stage, we introduce the characteristic value

$$\widehat{\lambda}(\mu) := \inf \{ \lambda(u, v), (u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \}.$$

We claim that $\widehat{\lambda}(\mu)$ is great than a positive constant which depends only on $\mu, p, p_1, q, \alpha, \beta$ and Ω . Indeed, using the Hölder inequality, we get

$$R(u, v) \leq |\Omega|^\delta \|u\|_{p^*}^{\alpha+1} \|v\|_{q^*}^{\beta+1},$$

where $\delta > 1$ is such that $\frac{1}{p^*} + \frac{1}{q^*} + \frac{1}{\delta} = 1$. Using the continuous embedding $W_{\Gamma_2}^{1,q}(\Omega) \subset L^{q^*}(\Omega)$ we get

$$A(u, v) \leq c_1 \frac{P_*(u)^{\frac{r}{p^*}}}{(\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)}}},$$

where $P_*(u) = \|u\|_{p^*}^{p^*}$ and $c_1 = c_1(p, p_1, q, \alpha, \beta, \Omega)$. Using again the continuous embeddings $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p_1}(\Omega)$ and $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ we obtain

$$\lambda(u, v) \geq c_2 (\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

where $c_2 = c_2(p, p_1, q, \alpha, \beta, \Omega)$ and then

$$\widehat{\lambda}(\mu) \geq c_2 (\mu_1 - \mu)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

which achieves the claim. Now let us introduce

$$\mathcal{D} := \{(\lambda, \mu) \in (0, +\infty) \times (-\infty, \mu_1) : \lambda < \widehat{\lambda}(\mu)\}.$$

For every $(\lambda, \mu) \in \mathcal{D}$, the functionals $(u, v) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \mapsto \tilde{I}(s_i(u, v, \lambda, \mu), u, t_i(u, v, \lambda, \mu), v)$ $i = 1, 2$, are well defined and one can show easily that they are bounded below. Hence, for every $(\lambda, \mu) \in \mathcal{D}$, we define

$$\alpha_1(\lambda, \mu) := \inf \{ \tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v), (u, v) \in \widetilde{W} \} \quad (2.11)$$

$$\alpha_2(\lambda, \mu) := \inf \{ \tilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v), (u, v) \in \widetilde{W} \} \quad (2.12)$$

where

$$\widetilde{W} := (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}).$$

Our aim in the sequel is to show that $\alpha_1(\lambda, \mu)$ and $\alpha_2(\lambda, \mu)$ are in fact critical values of the Euler-Lagrange functional I for every $(\lambda, \mu) \in \mathcal{D}$. We start with the following

Lemma 2.2.2. *Let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11) (resp. of (2.12)) and let $(U_n^1, V_n^1) := (s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$ (resp. $(U_n^2, V_n^2) := (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$). Then it holds :*

- (i) $\limsup_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| < \infty$ (resp. $\limsup_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| < \infty$).
- (ii) $\liminf_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| > 0$ (resp. $\liminf_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| > 0$).

Proof. We show the assertion (i), let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11). Since $\partial_s \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$ and $\partial_t \widetilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$, it follows that

$$P(U_n^1) - \lambda P_1(U_n^1) - (\alpha + 1)R(U_n^1, V_n^1) = 0, \quad (2.13)$$

$$Q(V_n^1) - \mu Q_1(V_n^1) - (\beta + 1)R(U_n^1, V_n^1) = 0. \quad (2.14)$$

Suppose that there is a subsequence, still denoted by (U_n^1, V_n^1) , such that $\lim_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| = \infty$. We will distinguish three cases :

Case a) $\lim_{n \rightarrow +\infty} \|\nabla U_n^1\|_p = \infty$ and $\|\nabla V_n^1\|_q$ is bounded. By (2.14) we get that $R(U_n^1, V_n^1)$ is bounded. On the other hand, using the continuous embedding $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p_1}(\Omega)$, we have $P_1(U_n^1) = o_n(P(U_n^1))$, as n goes to $+\infty$. By (2.13) we get $R(U_n^1, V_n^1) = \frac{1}{\alpha+1}(1 + o_n(1))P(U_n^1)$ as n goes to $+\infty$ and hence $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$, which cannot hold true.

Case b) $\lim_{n \rightarrow +\infty} \|\nabla V_n^1\|_q = \infty$ and $\|\nabla U_n^1\|_p$ is bounded. By (2.13) we get $R(U_n^1, V_n^1)$ bounded. If $0 < \mu < \mu_1$, using the Sobolev and Young inequalities, for every $\varepsilon \in (0, 1)$, there is a positive constant C_ε such that

$$\|V_n^1\|_q^q \leq \frac{\varepsilon}{\mu} \|\nabla V_n^1\|_q^q + C_\varepsilon,$$

which gives $(\beta+1)R(U_n^1, V_n^1) + \mu C_\varepsilon \geq (1-\varepsilon)Q(V_n^1)$. Then $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$, which is impossible. If $\mu < 0$, then $Q(V_n^1) - \mu Q_1(V_n^1) = (\beta + 1)R(U_n^1, V_n^1) \geq Q(V_n^1)$ so $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$, which is also impossible.

Case c) $\lim_{n \rightarrow +\infty} \|\nabla U_n^1\|_p = \lim_{n \rightarrow +\infty} \|\nabla V_n^1\|_q = \infty$. As in the first case, we have

$$R(U_n^1, V_n^1) = \frac{1}{\alpha + 1}(1 + o_n(1))P(U_n^1), \text{ as } n \text{ goes to } +\infty.$$

Then $I(U_n^1, V_n^1) = \frac{1}{\alpha+1} \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 + o_n(1) \right) P(U_n^1)$ as n goes to $+\infty$. Hence, using the hypothese (1.5), $\lim_{n \rightarrow +\infty} I(U_n^1, V_n^1) = +\infty$, which is impossible. Consequently, $\limsup_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| < \infty$. We show in the same way that $\limsup_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| < \infty$.

Now, we show the assertion (ii), let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.11). Suppose that there is a subsequence, still denoted by (U_n^1, V_n^1) , such that $\lim_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| = 0$. By (2.13) we get $\lim_{n \rightarrow +\infty} I(U_n^1, V_n^1) = 0$ and this can not hold true because $I(U_n^1, V_n^1) < 0$ for every n .

Similarly, let $(u_n, v_n) \in \widetilde{W}$ be a minimizing sequence of (2.12). Suppose that there is a subsequence, still denoted by (U_n^2, V_n^2) , such that $\lim_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| = 0$. If $p > \alpha + 1$, by (2.10), we have

$$\partial_{ss}I(U_n^2, V_n^2) = (p-1)P(U_n^2) - \lambda(p_1-1)P_1(U_n^2) - \alpha(\alpha+1)R(U_n^2, V_n^2) < 0$$

Then $(p-1)P(U_n^2) - \lambda(p-1)P_1(U_n^2) - \alpha p R(U_n^2, V_n^2) < 0$, which implies that $(p - (\alpha + 1))R(U_n^2, V_n^2) < 0$ and this is impossible. Finally, if $p \leq \alpha + 1$, then $(p - p_1)P(U_n^2) < (\alpha + 1)^2 R(U_n^2, V_n^2)$. Since $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$, then there exist \tilde{p} and \tilde{q} satisfying $p < \tilde{p} < p^*$, $q < \tilde{q} < q^*$ and

$$\frac{\alpha+1}{\tilde{p}} + \frac{\beta+1}{\tilde{q}} = 1. \quad (2.15)$$

Therefore,

$$\begin{aligned} R(U_n^2, V_n^2) &\leq c(\Omega, p, q) \|U_n^2\|_{\tilde{p}}^{\alpha+1} \|V_n^2\|_{\tilde{q}}^{\beta+1} \\ &\leq c'(\Omega, p, q) \|\nabla U_n^2\|_p^{\alpha+1} \|\nabla V_n^2\|_q^{\beta+1} \end{aligned}$$

and consequently, $(p-p_1) \leq c'(\Omega, p, q)(\alpha+1)^2 \|\nabla U_n^2\|_p^{\alpha+1-p} \|\nabla V_n^2\|_q^{\beta+1}$ which converges to 0 as n goes to $+\infty$. This contradicts the fact $p > p_1$, which ends the proof. \square

2.3 Palais-Smale sequences in the Nehari Manifold

It is interesting to notice that for every $\gamma > 0$, $\delta > 0$, it holds

$$\begin{aligned} \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \tilde{I}(s, u, t, v), \\ \partial_t \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\delta} \partial_t \tilde{I}(s, u, t, v), \\ \partial_s \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma} \partial_s \tilde{I}(s, u, t, v), \\ \partial_{ss} \tilde{I}\left(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}\right) &= \frac{1}{\gamma^2} \partial_{ss} \tilde{I}(s, u, t, v). \end{aligned}$$

This implies that

$$s_1(u, v, \lambda, \mu) = \frac{1}{\gamma} s_1 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \delta > 0, \quad (3.16)$$

$$s_2(u, v, \lambda, \mu) = \frac{1}{\gamma} s_2 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \delta > 0, \quad (3.17)$$

$$t_1(u, v, \lambda, \mu) = \frac{1}{\delta} t_1 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \gamma > 0, \quad (3.18)$$

$$t_2(u, v, \lambda, \mu) = \frac{1}{\delta} t_2 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \gamma > 0. \quad (3.19)$$

It follows that

$$\alpha_1(\lambda, \mu) = \inf_{(u,v) \in \mathbb{S}_p \times \mathbb{S}_q} \{ \tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v) \}, \quad (3.20)$$

$$\alpha_2(\lambda, \mu) = \inf_{(u,v) \in \mathbb{S}_p \times \mathbb{S}_q} \{ \tilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v) \}, \quad (3.21)$$

where \mathbb{S}_p and \mathbb{S}_q are the unit spheres of $W_{\Gamma_1}^{1,p}(\Omega)$ and $W_{\Gamma_2}^{1,q}(\Omega)$ respectively. Make precise that $\mathbb{S}_p \times \mathbb{S}_q$ is a 2-codimensional and complete submanifold of W , we will denote it in the sequel by \mathbb{S} .

Lemma 2.3.1. *Let $(\lambda, \mu) \in \mathcal{D}$ and let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (3.20) (resp. of (3.21)). Then $(s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$, (resp. $(s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$) is a Palais-Smale sequence for the functional I .*

Proof. Let $(\lambda, \mu) \in \mathcal{D}$ and consider a minimizing sequence $(u_n, v_n) \in \mathbb{S}$ of (3.20). Let us set

$$\begin{aligned} U_n &= s_1(u_n, v_n, \lambda, \mu)u_n, \\ V_n &= t_1(u_n, v_n, \lambda, \mu)v_n. \end{aligned}$$

The sequence (U_n, V_n) is clearly bounded in W . On the other hand, the gradient (resp. the Hessian determinant) of \tilde{I} with respect to s and t at $(s, t) = (s_1(u_n, v_n, \lambda, \mu), t_1(u_n, v_n, \lambda, \mu))$ is equal to zero (resp. is strictly negative). So, the implicit function theorem implies that $s_1(u_n, v_n, \lambda, \mu)$ and $t_1(u_n, v_n, \lambda, \mu)$ are C^1 with respect to (u, v) , since \tilde{I} is.

We introduce now the functional \mathcal{I} defined on \mathbb{S} by

$$\mathcal{I}(u, v) = \tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v),$$

then

$$\alpha_1(\lambda, \mu) = \inf_{(u,v) \in \mathbb{S}} \mathcal{I}(u, v) = \lim_{n \rightarrow +\infty} \mathcal{I}(u_n, v_n).$$

Applying the Ekeland variational principle [12, 17, 19, 22] on the complete manifold $(\mathbb{S}, \|\cdot\|)$ to the functional \mathcal{I} we get

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) \leq \frac{1}{n} \|(\varphi_n, \psi_n)\|, \quad \forall (\varphi_n, \psi_n) \in T_{(u_n, v_n)}\mathbb{S},$$

where $T_{(u_n, v_n)}\mathbb{S}$ denotes the tangent space to \mathbb{S} at the point (u_n, v_n) . Recall that $T_{(u_n, v_n)}\mathbb{S} = T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$, where $T_{u_n}\mathbb{S}_p$ (resp. $T_{v_n}\mathbb{S}_q$) is the tangent space to \mathbb{S}_p (resp. \mathbb{S}_q) at the point u_n (resp. v_n).

Set

$$A_n := (u_n, v_n, \lambda, \mu), \quad \text{and} \quad B_n := (s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n).$$

For every $(\varphi_n, \psi_n) \in T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$, one has

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) = D_1\tilde{I}(B_n)(\varphi_n) + D_2\tilde{I}(B_n)(\psi_n)$$

where

$$\begin{aligned} D_1\tilde{I}(B_n)(\varphi_n) &= \partial_u s_1(A_n)(\varphi_n) \partial_s \tilde{I}(B_n) + \partial_u \tilde{I}(B_n)(\varphi_n) + \partial_u t_1(A_n)(\varphi_n) \partial_t \tilde{I}(B_n) \\ &= \partial_u \tilde{I}(B_n)(\varphi_n). \end{aligned}$$

Similarly, one has

$$D_2\tilde{I}(B_n)(\psi_n) = \partial_v \tilde{I}(B_n)(\psi_n).$$

Furthermore, consider the "fiber" maps

$$\begin{aligned} \pi : W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S}_p \\ u &\longmapsto \left(\|\nabla u\|_p, \frac{u}{\|\nabla u\|_p} \right) := (\pi_1(u), \pi_2(u)), \\ \tilde{\pi} : W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\} &\longrightarrow \mathbb{R} \times \mathbb{S}_q \\ v &\longmapsto \left(\|\nabla v\|_q, \frac{v}{\|\nabla v\|_q} \right) := (\tilde{\pi}_1(v), \tilde{\pi}_2(v)). \end{aligned}$$

Applying the Hölder inequality we get, for every $(u, \varphi) \in (W_{\Gamma_1}^{1,p}(\Omega) \setminus \{0\}) \times W_{\Gamma_1}^{1,p}(\Omega)$ and $(v, \psi) \in (W_{\Gamma_2}^{1,q}(\Omega) \setminus \{0\}) \times W_{\Gamma_2}^{1,q}(\Omega)$, the following estimates

$$\begin{aligned} |\pi'_1(u)(\varphi)| &\leq \|\nabla \varphi\|_p, \quad |\pi'_2(u)(\varphi)| \leq 2 \frac{\|\nabla \varphi\|_p}{\|\nabla u\|_p}, \\ |\tilde{\pi}'_1(v)(\psi)| &\leq \|\nabla \psi\|_q, \quad |\tilde{\pi}'_2(v)(\psi)| \leq 2 \frac{\|\nabla \psi\|_q}{\|\nabla v\|_q}. \end{aligned}$$

On one hand, from Lemma (2.2.2), there is a positive constant K such that $s_1(A_n) \geq K$ and $t_1(A_n) \geq K$, for every integer n . On the other hand, for every $(\varphi, \psi) \in W$,

$$\begin{aligned} D_1 I(U_n, V_n)(\varphi) &= \varphi_n^1 \partial_s \tilde{I}(B_n) + \partial_u \tilde{I}(B_n)(\varphi_n^2) + \varphi_n^1 \partial_t \tilde{I}(B_n) \\ &= \partial_u \tilde{I}(B_n)(\varphi_n^2). \end{aligned}$$

where $\varphi_n^1 = \pi_1'(u_n)(\varphi)$ and $\varphi_n^2 = \pi_2'(u_n)(\varphi)$. Then the following estimates hold true : $|\varphi_n^1| \leq \|\nabla\varphi\|_p$ and $\|\nabla\varphi_n^2\|_p \leq \frac{2}{K}\|\nabla\varphi\|_p$. In the same manner, we get

$$\begin{aligned} D_2I(U_n, V_n)(\psi) &= \psi_n^1 \partial_s \tilde{I}(B_n) + \partial_v \tilde{I}(B_n)(\psi_n^2) + \psi_n^1 \partial_t \tilde{I}(B_n) \\ &= \partial_v \tilde{I}(B_n)(\psi_n^2). \end{aligned}$$

where $\psi_n^1 = \tilde{\pi}_1'(v_n)(\psi)$ and $\psi_n^2 = \tilde{\pi}_2'(v_n)(\psi)$, with the estimates $|\psi_n^1| \leq \|\nabla\psi\|_q$ and $\|\nabla\psi_n^2\|_q \leq \frac{2}{K}\|\nabla\psi\|_q$. Therefore

$$\begin{aligned} |D_1I(U_n, V_n)(\varphi)| &\leq \frac{1}{n}\|\nabla\varphi_n^2\|_p \\ &\leq \frac{2}{nK}\|\nabla\varphi\|_p \end{aligned}$$

and

$$\begin{aligned} |D_2I(U_n, V_n)(\psi)| &\leq \frac{1}{n}\|\nabla\psi_n^2\|_q \\ &\leq \frac{2}{nK}\|\nabla\psi\|_q. \end{aligned}$$

We conclude easily that

$$\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_* = 0,$$

where $I'(U_n, V_n)(\varphi, \psi) = D_1I(U_n, V_n)(\varphi) + D_2I(U_n, V_n)(\psi)$ and $\|\cdot\|_*$ is the norm on the dual space of W .

The arguments are similar if $(u_n, v_n) \in \mathbb{S}$ is a minimizing sequence of (3.21). Hence, the lemma is proved. \square

Remark. For every $(u, v) \in \tilde{W}$ and $(\lambda, \mu) \in \mathcal{D}$, one has $\tilde{I}(s, u, t, v) = \tilde{I}(s, |u|, t, |v|)$, $s_i(|u|, |v|, \lambda, \mu) = s_i(u, v, \lambda, \mu)$, $i \in \{1, 2\}$ and consequently $t_i(|u|, |v|, \lambda, \mu) = t_i(u, v, \lambda, \mu)$, $i \in \{1, 2\}$. Therefore, every minimizing sequence $(u_n, v_n) \in \mathbb{S}_p \times \mathbb{S}_q$ of (3.20) or (3.21) can be considered as a sequence satisfying $u_n \geq 0$ and $v_n \geq 0$ in Ω .

2.4 Positive solutions and the behaviour of their energy

Theorem 2.4.1. *Let $(\lambda, \mu) \in \mathcal{D}$. Then Problem (1.1) has at least two nontrivial solutions (U^i, V^i) , $i \in \{1, 2\}$, such that $U^i \geq 0$ and $V^i \geq 0$ in Ω and $U^i \neq 0$, $V^i \neq 0$, for $i \in \{1, 2\}$.*

Proof. We will use the notations of the previous lemmas. Let $(\lambda, \mu) \in \mathcal{D}$ and consider a nonnegative minimizing sequence $(u_n, v_n) \in \mathbb{S}$ of (3.20). It is known from Lemma (2.3.1) that

$$\lim_{n \rightarrow +\infty} I(U_n, V_n) = \alpha_1(\lambda, \mu),$$

$$\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_* = 0$$

and that (U_n, V_n) is bounded in W . Passing if necessary to a subsequence, there are $U^1 \in W_{\Gamma_1}^{1,p}(\Omega)$ and $V^1 \in W_{\Gamma_2}^{1,q}(\Omega)$ such that

$$\begin{aligned} U_n &\rightharpoonup U^1 \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\ U_n &\rightarrow U^1 \text{ in } L^{p_1}(\Omega) \text{ and } L^{\tilde{p}}(\Omega), \\ V_n &\rightharpoonup V^1 \text{ in } W_{\Gamma_2}^{1,q}(\Omega), \\ V_n &\rightarrow V^1 \text{ in } L^{q_1}(\Omega) \text{ and } L^{\tilde{q}}(\Omega), \end{aligned}$$

where \tilde{p} and \tilde{q} are specified in (2.15). At this stage, we use the well known inequalities : $\forall (x, y) \in \mathbb{R}^N$

$$\begin{aligned} |x - y|^\gamma &\leq C (|x|^{\gamma-2}x - |y|^{\gamma-2}y) \cdot (x - y), \text{ if } \gamma \geq 2, \\ |x - y|^2 &\leq C(|x| - |y|)^{2-\gamma} (|x|^{\gamma-2}x - |y|^{\gamma-2}y) \cdot (x - y), \text{ if } \gamma < 2. \end{aligned}$$

where \cdot denotes the scalar product in \mathbb{R}^N .

In the case $p \geq 2$, we obtain

$$\begin{aligned} P(U_n - U^1) &\leq C \int_{\Omega} (|\nabla U_n|^{p-2} \nabla U_n - |\nabla U^1|^{p-2} \nabla U^1) \cdot (\nabla U_n - \nabla U^1) \\ &= C(D_1 I(U_n, V_n)(U_n - U^1) - D_1 I(U^1, V^1)(U_n - U^1) + \\ &\quad C\lambda \int_{\Omega} (|U_n|^{p_1-2} U_n - |U^1|^{p_1-2} U^1) (U_n - U^1) + \\ &\quad C(\alpha + 1) \int_{\Omega} (U_n |U_n|^{\alpha-1} |V_n|^{\beta+1} - U^1 |U^1|^{\alpha-1} |V^1|^{\beta+1}) (U_n - U^1). \end{aligned}$$

Since $\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_* = 0$, (V_n) is bounded, and using the fact that $U_n \rightarrow U^1$ in $L^{p_1}(\Omega)$ and in $L^{\tilde{p}}(\Omega)$, $V_n \rightarrow V^1$ in $L^{\tilde{q}}(\Omega)$, we conclude, by the Hölder inequality, that $P(U_n - U^1) \rightarrow 0$, as n goes to $+\infty$, which means that

$$U_n \longrightarrow U^1 \text{ in } W_{\Gamma_1}^{1,p}(\Omega).$$

In the case $p < 2$, a direct computation gives

$$\begin{aligned} \|\nabla U_n - \nabla U^1\|_p^2 &\leq C (\|\nabla U_n\|_p^{2-p} + \|\nabla U^1\|_p^{2-p}) \times \\ &\quad \int_{\Omega} (|\nabla U_n|^{p-2} \nabla U_n - |\nabla U^1|^{p-2} \nabla U^1) \cdot (\nabla U_n - \nabla U^1). \end{aligned}$$

Since $\|\nabla U_n - \nabla U^1\|_p$ is bounded, the same arguments used above show that $U_n \rightarrow U^1$ in $W_{\Gamma_1}^{1,p}(\Omega)$, as n goes to $+\infty$. In a similar way we get $V_n \rightarrow V^1$ in $W_{\Gamma_2}^{1,q}(\Omega)$, as n goes to $+\infty$.

Moreover, it is clear that (U^1, V^1) is a nontrivial solution of Problem (1.1) verifying $U^1 \geq 0$ and $V^1 \geq 0$ in Ω and $U^1 \neq 0$, $V^1 \neq 0$. On the other hand, there is a subsequence of (u_n, v_n) , still denoted by (u_n, v_n) such that

$$\begin{aligned} U_n &:= s_1(u_n, v_n, \lambda, \mu) u_n \longrightarrow U^1 \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\ V_n &:= t_1(u_n, v_n, \lambda, \mu) v_n \longrightarrow V^1 \text{ in } W_{\Gamma_2}^{1,q}(\Omega). \end{aligned}$$

According to Lemma (2.2.2), let $(s_1, t_1) \in (0, +\infty)^2$ such that

$$\begin{cases} s_1(u_n, v_n, \lambda, \mu) & \longrightarrow & s_1 & \text{in } \mathbb{R}, \\ t_1(u_n, v_n, \lambda, \mu) & \longrightarrow & t_1 & \text{in } \mathbb{R}, \\ u_n & \longrightarrow & u^1 = \frac{U^1}{s_1} & \text{in } W_{\Gamma_1}^{1,p}(\Omega), \\ v_n & \longrightarrow & v^1 = \frac{V^1}{t_1} & \text{in } W_{\Gamma_2}^{1,q}(\Omega), \end{cases}$$

with $u^1 = \frac{U^1}{s_1} \in \mathbb{S}_p$, $v^1 = \frac{V^1}{t_1} \in \mathbb{S}_q$, $s_1 = s_1(u^1, v^1, \lambda, \mu)$ and $t_1 = t_1(u^1, v^1, \lambda, \mu)$. Therefore, $\partial_{ss}\tilde{I}(s_1(u^1, v^1, \lambda, \mu), u^1, t_1(u^1, v^1, \lambda, \mu), v^1) > 0$.

Proceeding in the same manner with a nonnegative minimizing sequence $(\tilde{u}_n, \tilde{v}_n) \in \mathbb{S}$ of (3.21), we obtain a second nontrivial solution (U^2, V^2) of (1.1) verifying $U^2 \geq 0$ and $V^2 \geq 0$ in Ω and $U^2 \neq 0$, $V^2 \neq 0$.

Now, we have to show that $(U^1, V^1) \neq (U^2, V^2)$. Let $(s_2, t_2) \in (0, +\infty)^2$ such that

$$\begin{cases} s_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow & s_2 & \text{in } \mathbb{R}, \\ t_2(\tilde{u}_n, \tilde{v}_n, \lambda, \mu) & \longrightarrow & t_2 & \text{in } \mathbb{R}, \\ \tilde{u}_n & \longrightarrow & u^2 = \frac{U^2}{s_2} & \text{in } W_{\Gamma_1}^{1,p}(\Omega), \\ \tilde{v}_n & \longrightarrow & v^2 = \frac{V^2}{t_2} & \text{in } W_{\Gamma_2}^{1,q}(\Omega), \end{cases}$$

with $u^2 = \frac{U^2}{s_2} \in \mathbb{S}_p$, $v^2 = \frac{V^2}{t_2} \in \mathbb{S}_q$, $s_2 = s_2(u^2, v^2, \lambda, \mu)$ and $t_2 = t_2(u^2, v^2, \lambda, \mu)$. Therefore, $\partial_{ss}\tilde{I}(s_2(u^2, v^2, \lambda, \mu), u^2, t_2(u^2, v^2, \lambda, \mu), v^2) < 0$. Hence $(U^1, V^1) \neq (U^2, V^2)$, which ends the proof. \square

In the sequel, for every $(\lambda, \mu) \in \mathcal{D}$, the functions (u^1, v^1) and (u^2, v^2) will be denoted by $(u^1(\lambda, \mu), v^1(\lambda, \mu))$ and $(u^2(\lambda, \mu), v^2(\lambda, \mu))$ respectively. Similarly, the solutions (U^i, V^i) , $i \in \{1, 2\}$, will be denoted by $(U^i(\lambda, \mu), V^i(\lambda, \mu))$, $i \in \{1, 2\}$.

Theorem 2.4.2. *Let $(\lambda, \mu) \in \mathcal{D}$. Then*

- (i) $I(U^1, V^1) < 0$ for $\lambda \in]0, \hat{\lambda}(\mu)[$,
- (ii) $\begin{cases} I(U^2, V^2) > 0 & \text{for } \lambda \in]0, \lambda_0(\mu)[, \\ I(U^2, V^2) < 0 & \text{for } \lambda \in]\lambda_0(\mu), \hat{\lambda}(\mu)[, \end{cases}$

where

$$\lambda_0(\mu) := \frac{p_1}{r} \left(\frac{r}{p} \right)^{\frac{r-p_1}{r-p}} \hat{\lambda}(\mu).$$

Proof. In this proof, μ will be fixed in $(-\infty, \mu_1)$, so we will omit the dependence on μ in the expressions which will follow. However, the dependence on λ will be specified. In particular, the Euler-Lagrange functional I will be denoted by I_λ .

(ii) Let (u, v) be an arbitrary element of \widetilde{W} . We denote

$$\widetilde{I}_\lambda(s, u, t(s), v) = \frac{s^p}{p}P(u) - \frac{\lambda}{p_1}s^{p_1}P_1(u) - \frac{s^r}{r}A(u, v),$$

and write

$$\widetilde{I}_\lambda(s, u, t(s), v) = s^{p_1}\widetilde{G}_\lambda(s, u, v),$$

where

$$\widetilde{G}_\lambda(s, u, v) = s^{p-p_1}\frac{P(u)}{p} - \lambda\frac{P_1(u)}{p_1} - s^{r-p_1}\frac{A(u, v)}{r}.$$

It follows that

$$\partial_s\widetilde{I}_\lambda(s, u, t(s), v) = p_1s^{p_1-1}\widetilde{G}_\lambda(s, u, v) + s^{p_1}\partial_s\widetilde{G}_\lambda(s, u, v),$$

with

$$\partial_s\widetilde{G}_\lambda(s, u, v) = s^{p-p_1-1}\left\{\frac{p-p_1}{p}P(u) - \frac{r-p_1}{r}s^{r-p}A(u, v)\right\}.$$

The real valued function $s \mapsto \widetilde{G}_\lambda(s, u, v)$ is increasing on $]0, s_0(u, v)[$, decreasing on $]s_0(u, v), +\infty[$ and attains its unique maximum for $s = s_0(u, v)$, where

$$s_0(u, v) = \left(\frac{r}{p}\right)^{\frac{1}{r-p}}\bar{s}_\mu(u, v), \quad (4.22)$$

and $\bar{s}_\mu(u, v)$ is defined in (2.7). On the other hand, a direct computation gives

$$\widetilde{G}_\lambda(s_0(u, v), u, v) = \left(\frac{p-p_1}{r-p_1}\frac{P(u)}{A(u, v)}\right)^{\frac{r-p_1}{r-p}}R(u, v) - \lambda P_1(u).$$

Similarly, $\widetilde{G}_\lambda(s_0(u, v), u, v) > 0$ (resp. $\widetilde{G}_\lambda(s_0(u, v), u, v) < 0$) if $\lambda < \lambda_0(u, v)$ (resp. $\lambda > \lambda_0(u, v)$) and $\widetilde{G}_{\lambda_0(u, v)}(s_0(u, v), u, v) = 0$, where

$$\lambda_0(u, v) = \frac{p_1}{r}\left(\frac{r}{p}\right)^{\frac{r-p_1}{r-p}}\lambda(u, v), \quad (4.23)$$

with $\lambda(u, v)$ given by (2.8). Thus, we get

$$\begin{cases} \widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) > 0 & \text{if } \lambda < \lambda_0(u, v), \\ \widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) = 0 & \text{if } \lambda = \lambda_0(u, v), \\ \widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) < 0 & \text{if } \lambda > \lambda_0(u, v). \end{cases} \quad (4.24)$$

First, since the function

$$\begin{aligned}]0, 1[&\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{\ln t}{1-t} \end{aligned}$$

is increasing, then for every real numbers x, y such that $0 < x < y < 1$, one has

$$\ln \left[\frac{1}{x} \right] > \frac{1-x}{1-y} \ln \left[\frac{1}{y} \right] = \ln \left[\left(\frac{1}{y} \right)^{\frac{1-x}{1-y}} \right],$$

and consequently

$$0 < x \left(\frac{1}{y} \right)^{\frac{1-x}{1-y}} < 1.$$

In the particular case $x = p_1/r$ and $y = p/r$ we get

$$0 < \frac{p_1}{r} \left(\frac{r}{p} \right)^{\frac{r-p_1}{r-p}} < 1,$$

and therefore $0 < \lambda_0(u, v) < \lambda(u, v)$.

Moreover, for every $(u, v) \in \widetilde{W}$, one has $\widetilde{G}_{\lambda_0(u, v)}(s, u, v) < 0$ for $s \in]0, +\infty[\setminus \{s_0(u, v)\}$ and $\widetilde{G}_{\lambda_0(u, v)}(s_0(u, v), u, v) = 0$. Hence, the real valued function $s \longmapsto \widetilde{I}_{\lambda_0(u, v)}(s, u, t(s), v)$, ($s > 0$), attains its unique maximum at $s = s_0(u, v)$ and we obtain the following interesting identity

$$s_2(u, v, \lambda_0(u, v), \mu) = s_0(u, v). \quad (4.25)$$

We will set

$$t_0(u, v) := t_2(u, v, \lambda_0(u, v), \mu).$$

On the other hand, it is clear that the functional $\lambda_0(u, v)$ is weakly lower semi-continuous on \widetilde{W} . Thus, the value

$$\widehat{\lambda}_0 := \inf_{(u, v) \in \widetilde{W}} \lambda_0(u, v) \quad (4.26)$$

is achieved on \widetilde{W} . Since $\lambda_0(u, v)$ is 0-homogeneous in u and v , we can assume that there is some $(u^*, v^*) \in \mathbb{S}_p \times \mathbb{S}_q$ such that $\widehat{\lambda}_0 = \lambda_0(u^*, v^*)$.

Now, let λ be such that $0 < \lambda < \widehat{\lambda}_0$. Then, for every $(u, v) \in \widetilde{W}$ one has $0 < \lambda < \lambda_0(u, v)$ and consequently $\widetilde{I}_\lambda(s_0(u, v), u, t(s_0(u, v)), v) > 0$ holds from (4.24). But, $s \longmapsto \widetilde{I}_\lambda(s, u, t(s), v)$, ($s > 0$) attains its unique maximum for $s = s_2(u, v, \lambda)$, hence $\widetilde{I}_\lambda(s_2(u, v, \lambda), u, t_2(u, v, \lambda), v) > 0$, for every $(u, v) \in \widetilde{W}$. In particular, we have

$$\widetilde{I}_\lambda(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) > 0,$$

i.e. $I_\lambda(U^2(\lambda), V^2(\lambda)) > 0$.

If $\lambda = \widehat{\lambda}_0$, then

$$\begin{aligned}
I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0))) \\
&= \inf_{(u,v) \in \mathbb{S}_p \times \mathbb{S}_q} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u, v, \widehat{\lambda}_0), u, t_2(u, v, \widehat{\lambda}_0), v) \\
&\leq \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*), u^*, t_2(u^*, v^*), v^*) \\
&= \widetilde{I}_{\lambda_0(u^*, v^*)}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*) \\
&= 0
\end{aligned}$$

which implies that $I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) \leq 0$. In addition, it is known from (4.24) that

$$\begin{aligned}
\widetilde{I}_{\widehat{\lambda}_0}(s_0(u, v), u, t_0(u, v), v) &\geq 0, \\
\widetilde{I}_{\widehat{\lambda}_0}(s_1(u, v, \widehat{\lambda}_0), u, t_1(u, v, \widehat{\lambda}_0), v) &< 0,
\end{aligned}$$

for every $(u, v) \in \widetilde{W}$. Then

$$s_0(u, v) > s_1(u, v, \widehat{\lambda}_0), \quad \forall (u, v) \in \widetilde{W}.$$

It follows that

$$\begin{aligned}
\widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0))) &\geq \\
\widetilde{I}_{\widehat{\lambda}_0}(s_0(u^1(\widehat{\lambda}_0), v^1(\widehat{\lambda}_0)), u^1(\widehat{\lambda}_0), t_0(u^1(\widehat{\lambda}_0), v^1(\widehat{\lambda}_0)), v^1(\widehat{\lambda}_0))) &\geq 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_{\widehat{\lambda}_0}(U^2(\widehat{\lambda}_0), V^2(\widehat{\lambda}_0)) &= \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), u^2(\widehat{\lambda}_0), t_2(u^2(\widehat{\lambda}_0), v^2(\widehat{\lambda}_0), \widehat{\lambda}_0), v^2(\widehat{\lambda}_0))) \\
&= 0.
\end{aligned}$$

Finally, assume that $\widehat{\lambda}_0 < \lambda < \widehat{\lambda}$. Since, for every $s \in]0, +\infty[$ and $(u, v) \in \widetilde{W}$, the real valued function $\lambda \mapsto \widetilde{I}_\lambda(s, u, t(s), v)$ is decreasing, it follows that

$$\widetilde{I}_\lambda(s, u, t(s), v) < \widetilde{I}_{\widehat{\lambda}_0}(s, u, t(s), v), \quad \text{for every } s > 0 \text{ and } (u, v) \in \widetilde{W}. \quad (4.27)$$

In addition, we have

$$\begin{aligned}
\widetilde{I}_\lambda(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda))) &= \\
\inf_{(u,v) \in \mathbb{S}_p \times \mathbb{S}_q} \widetilde{I}_\lambda(s_2(u, v, \lambda), u, t_2(u, v, \lambda), v) &\leq \\
\widetilde{I}_\lambda(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) &< \\
\widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) &
\end{aligned}$$

where the last inequality follows from (4.27). Moreover, the real valued function $s \mapsto \widetilde{I}_{\widehat{\lambda}_0}(s, u^*, t(s), v^*)$, ($s > 0$), achieves its unique maximum at $s = s_0(u^*, v^*)$.

Thus,

$$\begin{aligned} \widetilde{I}_{\widehat{\lambda}_0}(s_2(u^*, v^*, \lambda), u^*, t_2(u^*, v^*, \lambda), v^*) &\leq \widetilde{I}_{\widehat{\lambda}_0}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*) \\ &= \widetilde{I}_{\widehat{\lambda}_0(u^*, v^*)}(s_0(u^*, v^*), u^*, t_0(u^*, v^*), v^*) \\ &= 0. \end{aligned}$$

Hence $\widetilde{I}_{\widehat{\lambda}}(s_2(u^2(\lambda), v^2(\lambda), \lambda), u^2(\lambda), t_2(u^2(\lambda), v^2(\lambda), \lambda), v^2(\lambda)) < 0$, which ends the proof. \square

The following result shows the subtle link existing between the characteristic value $\widehat{\lambda}_0$ defined by (4.26) and Problem (1.1).

Theorem 2.4.3. *If (u, v) is a solution of (4.26) then $(s_0(u, v)u, t_0(u, v)v)$ is a solution of the system (1.1) when $\lambda = \widehat{\lambda}_0$.*

Proof. Let (u, v) be a solution of (4.26). In order to simplify the notations, we set $U := s_0(u, v)u$ and $V := t_0(u, v)v$. Thus, for $\lambda = \widehat{\lambda}_0 = \lambda_0(u, v)$ we have :

$$I_{\widehat{\lambda}_0, \mu}(U, V) = \frac{s_0(u, v)^p}{p} P(u) - \widehat{\lambda}_0 \frac{s_0(u, v)^{p_1}}{p_1} P_1(u) - \frac{s_0(u, v)^r}{r} A(u, v)$$

and for every $\varphi \in W_0^{1,p}(\Omega)$:

$$D_1 I_{\widehat{\lambda}_0, \mu}(U, V)(\varphi) = \frac{1}{p} P'(U)(\varphi) - \frac{\widehat{\lambda}_0}{p_1} P'_1(U)(\varphi) - \frac{1}{r} D_1 A(U, V)(\varphi),$$

where

$$\begin{cases} P'(U)(\varphi) &= s_0(u, v)^{p-1} P'(u)(\varphi), \\ P'_1(U)(\varphi) &= s_0(u, v)^{p_1-1} P'_1(u)(\varphi), \\ D_1 A(U, V)(\varphi) &= s_0(u, v)^{r-1} D_1 A(u, v)(\varphi). \end{cases}$$

We calculate now,

$$\begin{aligned} \widehat{\lambda}_0 P'_1(U)(\varphi) &= \lambda_0(u, v) s_0(u, v)^{p_1-1} P'_1(u)(\varphi) \\ &= \frac{p_1}{r} \left(\frac{r}{p} \right)^{\frac{r-p_1}{r-p}} \left(\frac{p-p_1}{r-p_1} \right)^{\frac{p-p_1}{r-p}} \frac{P(u)}{P_1(u)} \left(\frac{P(u)}{A(u, v)} \right)^{\frac{p-1}{r-p}} \\ &\times \frac{r-p}{r-p_1} \left(\frac{r}{p} \right)^{\frac{p_1-1}{r-p}} \left(\frac{p-p_1}{r-p_1} \frac{P(u)}{A(u, v)} \right)^{\frac{p_1-1}{r-p}} P'_1(u)(\varphi) \\ &= \frac{r-p}{r-p_1} \frac{p_1}{r} \frac{r}{p} \left(\frac{p-p_1}{r-p_1} \right)^{\frac{p-1}{r-p}} \left(\frac{r}{p} \right)^{\frac{p-1}{r-p}} \frac{P(u)}{P_1(u)} \frac{P(u)}{A(u, v)}^{\frac{p-1}{r-p}} P'_1(u)(\varphi) \\ &= \frac{r-p}{r-p_1} \frac{p_1}{p} \frac{P(u)}{P_1(u)} \left(\left(\frac{r}{p} \right)^{\frac{1}{r-p}} \left(\frac{p-p_1}{r-p_1} \right)^{\frac{1}{r-p}} \frac{P(u)}{A(u, v)} \right)^{p-1} P'_1(u)(\varphi) \\ &= \frac{p_1}{p} \frac{r-p}{r-p_1} P(u) s_0(u, v)^{p-1} \frac{P'_1(u)(\varphi)}{P_1(u)}. \end{aligned}$$

In addition, one has

$$\begin{aligned}
D_1 A(U, V)(\varphi) &= s_0(u, v)^{r-1} D_1(u, v)(\varphi) \\
&= \left(\frac{r p - p_1}{p r - p_1} \frac{P(u)}{A(u, v)} \right)^{\frac{p-1}{r-p}} \frac{r p - p_1}{p r - p_1} \frac{P(u)}{A(u, v)} D_1 A(u, v)(\varphi) \\
&= \frac{r p - p_1}{p r - p_1} P(u) s_0(u, v)^{p-1} \frac{D_1 A(u, v)(\varphi)}{A(u, v)}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
D_1 I_{\widehat{\lambda}_0, \mu}(U, V)(\varphi) &= \left[\frac{P'(u)(\varphi)}{P(u)} - \frac{r-p}{r-p_1} \frac{P'_1(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p_1} \frac{D_1 A(u, v)(\varphi)}{A(u, v)} \right] \\
&\quad \times \frac{P(u) s_0(u, v)^{p-1}}{p} \\
&= K \left(\frac{r-p_1}{r-p} \frac{P'(u)(\varphi)}{P(u)} - \frac{P'_1(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p} \frac{D_1 A(u, v)(\varphi)}{A(u, v)} \right),
\end{aligned}$$

where $K := \frac{r-p}{r-p_1} \frac{P(u)}{p} s_0(u, v)^{p-1}$. On the other hand, a direct computation gives :

$$D_1 \lambda_0(u, v)(\varphi) = \widehat{\lambda}_0 \left(\frac{r-p_1}{r-p} \frac{P'(u)(\varphi)}{P(u)} - \frac{P'_1(u)(\varphi)}{P_1(u)} - \frac{p-p_1}{r-p} \frac{D_1 A(u, v)(\varphi)}{A(u, v)} \right),$$

which is equal to zero by assumption. Hence $D_1 I_{\widehat{\lambda}_0, \mu}(U, V)(\varphi) = 0$ since it is proportional to $D_1 \lambda_0(u, v)(\varphi)$.

Moreover, for every $\psi \in W_0^{1,q}(\Omega)$, we get

$$D_2 \lambda_0(u, v)(\psi) = -\frac{p-p_1}{r-p_1} \lambda_0(u, v) \frac{D_2 A(u, v)(\psi)}{A(u, v)},$$

which is also equal to zero by assumption. This implies that $D_2 A(u, v)(\psi) = 0$, since $\lambda_0(u, v) = \widehat{\lambda}_0 \neq 0$. Then

$$D_2 I_{\widehat{\lambda}_0, \mu}(U, V)(\psi) = -\frac{s_0(u, v)^r}{r} D_2 A(u, v)(\psi) = 0.$$

which implies that $(s_0(u, v)u, t_0(u, v)v)$ is well a solution of the problem (1.1) with $\lambda = \widehat{\lambda}_0$. \square

Acknowledgments

The authors are very grateful to Prof. Claudianor O. Alves for the interesting discussions and references on this subject.

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Chapitre 3

Nehari manifold for a critical system in \mathbb{R}^N

Abstract

In this paper, we are interested in existence and multiplicity results of non local solutions to the following critical system :

$$\begin{cases} -\Delta_p u &= \lambda a(x)|u|^{p_1-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \text{ in } \mathbb{R}^N, \\ -\Delta_q v &= \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}|v|^{\beta-1}v \text{ in } \mathbb{R}^N, \end{cases}$$

under some conditions for the parameters $a, b, p, p_1, \alpha, \beta, q, \lambda$ and μ in the critical case : $\frac{\alpha+1}{p^} + \frac{\beta+1}{q^*} = 1$. We show these results by developing variational tools. The study consists in the extraction of Palais-Smale sequences in the Nehari manifold. A compactness principle due to A. El Hamidi and J-M. ROKOTOSON allows us to obtain convergence results for the gradients in our unbounded case.*

3.1 Introduction

We consider the system of quasilinear elliptic equations :

$$\begin{cases} -\Delta_p u &= \lambda a(x)|u|^{p_1-2}u + u|u|^{\alpha-1}|v|^{\beta+1} \text{ in } \mathbb{R}^N, \\ -\Delta_q v &= \mu b(x)|v|^{q-2}v + |u|^{\alpha+1}|v|^{\beta-1}v \text{ in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

We are interested in establishing nonlocal existence and multiplicity results for Problem (1.1). Of course *nonlocal* solutions means *with respect to the real parameters λ and μ* . Throughout this paper, the following assumptions will hold :

$$a \geq 0, a \not\equiv 0, a \in L^{\frac{p^*}{p^*-p_1}}(\mathbb{R}^N) \text{ and } b \geq 0, b \not\equiv 0, b \in L^{\frac{N}{q}}(\mathbb{R}^N), \quad (1.2)$$

$$1 < p_1 < p < N, \quad 1 < q < N, \quad \alpha > -1, \quad \beta > -1, \quad (1.3)$$

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1, \quad (1.4)$$

$$q > \beta + 1, \quad (1.5)$$

where

$$p^* = \frac{Np}{N-p}, \quad q^* = \frac{Nq}{N-q},$$

are the critical exponents for the p -Laplacian and q -Laplacian respectively. These assumptions mean that we are concerned with a critical system where the first equation is *concave-convex* and the second equation is only a critical perturbation of an eigenvalue equation. Also, the following assumptions concerning the real parameters λ and μ will hold

$$\lambda > 0, \quad \mu < \mu_1,$$

where μ_1 is the first eigenvalue of the equation

$$-\Delta_q v = \mu b(x)v|v|^{q-2} \text{ in } \mathbb{R}^N.$$

Thus

$$\mu_1 = \inf_{\psi \in \mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \psi|^q dx}{\int_{\mathbb{R}^N} b(x)|\psi|^q dx},$$

where the space $\mathcal{D}^{1,q}(\mathbb{R}^N)$ is the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{\frac{1}{q}}.$$

One can prove that $\mu_1 > 0$ and μ_1 is achieved. Indeed, on one hand, by integrability of b , we claim that the functional

$$\begin{aligned} Q_b : \mathcal{D}^{1,q}(\mathbb{R}^N) &\longrightarrow \mathbb{R} \\ v &\longmapsto \int_{\mathbb{R}^N} b(x)|v|^q dx. \end{aligned}$$

is weakly continuous. It is clear that the functional Q_b is well defined since q^*/q and N/q are conjugate exponents. Now, let $u_n \rightharpoonup u$ in $\mathcal{D}^{1,q}(\mathbb{R}^N)$ weakly. We are going to prove that $|u_n|^q \rightharpoonup |u|^q$ in $L^{q^*/q}(\mathbb{R}^N)$. Since $\| |u_n|^q \|_{q^*/q} = \|u_n\|_{q^*}^q$ is bounded we can assume, up to a subsequence, that $|u_n|^q \rightharpoonup v$ in $L^{q^*/q}(\mathbb{R}^N)$. The claim is complete if we show that $v = |u|^q$ because then the limit does not depend of the subsequence. Choose any increasing sequence $(K_n)_{n \in \mathbb{N}}$ of open relatively subsets, with regular boundaries, of \mathbb{R}^N covering $\mathbb{R}^N : \mathbb{R}^N = \cup_{n=0}^{\infty} K_n$. By using the compact/continuous embeddings

$$\begin{aligned} \mathcal{D}^{1,q}(K_n) &\hookrightarrow L^q(K_n) \subset L^1(K_n) \\ u_n \rightharpoonup u &\implies u_n \longrightarrow u \implies |u_n|^q \longrightarrow |u|^q \end{aligned}$$

and

$$L^{q^*/q}(K_n) \subset L^1(K_n)$$

$$|u_n|^q \rightharpoonup v \implies |u_n|^q \rightharpoonup v.$$

Thus, $v = |u|^q$ a.e. on each K_n . Using the diagonal process of Cantor, we conclude that $v = |u|^q$ a.e. in \mathbb{R}^N and the claim is achieved.

On the other hand, let (ψ_n) be a minimizing nonnegative sequence of μ_1 , (with $Q_b(\psi_n) = 1$, which is possible by homogeneity arguments), there is a nonnegative function $\psi \in \mathcal{D}^{1,q}(\mathbb{R}^N)$ such that, up to a subsequence,

$$\psi_n \rightharpoonup \psi \text{ in } \mathcal{D}^{1,q}(\mathbb{R}^N) \text{ weakly.}$$

Using the claim proved above, we get as $n \rightarrow +\infty$

$$Q_b(\psi_n) \rightarrow Q_b(\psi) = 1.$$

But,

$$\int_{\mathbb{R}^N} |\nabla \psi|^q dx \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |\nabla \psi_n|^q dx = \mu_1$$

then μ_1 is achieved by ψ . Finally, suppose that $\mu_1 = 0$, then $\int_{\mathbb{R}^N} |\nabla \psi|^q dx = 0$ which implies that ψ is a constant function which is positive since $Q_b(\psi) = 1$. But positive constant functions do not belong to $\mathcal{D}^{1,q}(\mathbb{R}^N)$. Notice that ψ satisfies, in the weak sense, the equation

$$-\Delta_q \psi = \mu_1 b(x) \psi |\psi|^{q-2} \text{ in } \mathbb{R}^N.$$

We denote by S_p (resp. S_q) the best Sobolev's constant for the continuous embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ (resp. $\mathcal{D}^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$).

Problem (1.1) is well posed in the framework of the space $W := \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,q}(\mathbb{R}^N)$, where

$$\begin{aligned} \mathcal{D}^{1,p}(\mathbb{R}^N) &= \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}, \\ \mathcal{D}^{1,q}(\mathbb{R}^N) &= \{v \in L^{q^*}(\mathbb{R}^N) : |\nabla v| \in L^q(\mathbb{R}^N)\}, \end{aligned}$$

which are, as mentioned above, respectively the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norms of

$$\begin{aligned} \|u\|_{1,p} &:= \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{1}{p}}, \\ \|v\|_{1,q} &:= \left(\int_{\mathbb{R}^N} |\nabla v|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

The space W is endowed by the following norm :

$$\|(u, v)\| = \|u\|_{1,p} + \|v\|_{1,q}$$

which gives to $(W, \|\cdot\|)$ Banach space properties, reflexivity and separability.

For solutions of (1.1) we mean critical points of the Euler-Lagrange functional $I \in C^1(W, \mathbb{R})$ given by

$$I(u, v) := (\alpha + 1) \left(\frac{1}{p} P(u) - \frac{\lambda}{p_1} P_{1,a}(u) \right) + \frac{\beta + 1}{q} \{Q(v) - \mu Q_b(v)\} - R(u, v),$$

where

$$\begin{aligned} P(u) &= \|u\|_p^p, & P_{1,a}(u) &= \int_{\mathbb{R}^N} a(x)|u|^{p_1} dx, \\ Q(v) &= \|v\|_q^q, & Q_b(v) &= \int_{\mathbb{R}^N} b(x)|v|^q dx, \\ R(u, v) &= \int_{\mathbb{R}^N} |u|^{\alpha+1}|v|^{\beta+1} dx. \end{aligned}$$

Remark that the functional I is bounded neither above nor below on W . For this reason we introduce the Nehari manifold corresponding to I , which contains all critical points of I and on which I is bounded below, as we will see in the sequel. For each $(u, v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$, the Nehari manifold associated to the functional I is defined by

$$\mathcal{N}_{\lambda,\mu} := \{(u, v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\}) : I'(u, v)(u, v) = 0\}.$$

This manifold can be characterized as follows

$$\mathcal{N}_{\lambda,\mu} = \{(s, u, t, v) \in \mathcal{Z}^* : \partial_s I(su, tv) = 0 \text{ and } \partial_t I(su, tv) = 0\},$$

where $\mathcal{Z}^* = (\mathbb{R} \setminus \{0\}) \times (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$ and I is considered as a functional of four variables $(s, u, t, v) \in \mathcal{Z} := \mathbb{R} \times \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathbb{R} \times \mathcal{D}^{1,q}(\mathbb{R}^N)$. This suggests the introduction of the modified Euler-Lagrange functional \tilde{I} defined on \mathcal{Z} by

$$\tilde{I}(s, u, t, v) := I(su, tv).$$

3.2 Some properties of minimizing sequences

Troughout this paper, we will be interested by positive solutions; $u > 0$ and $v > 0$ to the problem (1.1). As the functional \tilde{I} is even in s and t we can limit our study for $s > 0$, $t > 0$ and $(u, v) \in \tilde{W} := (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$. In the first lemma, we establish some properties of the modified functional $(s, t) \mapsto \tilde{I}(s, \cdot, t, \cdot)$:

Lemma 3.2.1. *For every $(u, v) \in \tilde{W}$, and for all $\mu < \mu_1$ there exists a unique $\lambda(u, v, \mu) > 0$ such that the real-valued function $(s, t) \in (0, +\infty)^2 \mapsto \tilde{I}(s, u, t, v)$ has exactly two critical points (resp. one critical point) for $0 < \lambda < \lambda(u, v, \mu)$ (resp. $\lambda = \lambda(u, v, \mu)$). This functional has no critical point for $\lambda > \lambda(u, v, \mu)$.*

Proof. Let $(u, v) \in \widetilde{W}$, $(s, t) \in (0, +\infty)^2$ and $\mu < \mu_1$ be arbitrary elements. We have

$$\begin{aligned} \widetilde{I}(s, u, t, v) &= (\alpha + 1) \left(\frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_{1,a}(u) \right) - s^{\alpha+1} t^{\beta+1} R(u, v) \\ &\quad + (\beta + 1) \frac{t^q}{q} (Q(v) - \mu Q_b(v)) \end{aligned}$$

Then $\partial_t \widetilde{I}(s, u, t, v) = 0$ if and only if

$$t = t(s) = \left[\frac{R(u, v)}{Q(v) - \mu Q_b(v)} \right]^{\frac{1}{q - (\beta+1)}} s^{\frac{\alpha+1}{q - (\beta+1)}}, \quad (2.6)$$

and consequently

$$\widetilde{I}(s, u, t(s), v) = (\alpha + 1) \left(\frac{s^p}{p} P(u) - \frac{\lambda}{p_1} s^{p_1} P_{1,a}(u) - \frac{s^r}{r} A(u, v) \right),$$

where

$$A(u, v) = \frac{R(u, v)^{\frac{q}{q - (\beta+1)}}}{(Q(v) - \mu Q_b(v))^{\frac{\beta+1}{q - (\beta+1)}}}$$

and $r = \frac{(\alpha+1)q}{q - (\beta+1)} > p$. Let us write $\partial_s \widetilde{I}(s, u, t(s), v) = s^{p_1-1} F(s, u, v)$, where $F(s, u, v) := P(u) s^{p-p_1} - \lambda P_{1,a}(u) - A(u, v) s^{r-p_1}$. The function $s \in (0, +\infty) \mapsto F_{\lambda, \mu}(s, u, v)$ is increasing on $(\overline{s}_\mu(u, v), +\infty)$ and attains its unique maximum for $s = \overline{s}_\mu(u, v)$, where

$$\overline{s}_\mu(u, v) = \left[\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{1}{r-p}}. \quad (2.7)$$

Thus, the function $s \in (0, +\infty) \mapsto F_{\lambda, \mu}(s, u, v)$ has two positive zeros (resp. one positive zero) if $F_{\lambda, \mu}(\overline{s}_\mu(u, v), u, v) > 0$ (resp. $F_{\lambda, \mu}(\overline{s}_\mu(u, v), u, v) = 0$) and has no zero if $F_{\lambda, \mu}(\overline{s}_\mu(u, v), u, v) < 0$. On the other hand, a direct computation leads to

$$F_{\lambda, \mu}(\overline{s}_\mu(u, v), u, v) = \frac{r - p}{r - p_1} \left[\frac{p - p_1}{r - p_1} \frac{P(u)}{A(u, v)} \right]^{\frac{p-p_1}{r-p_1}} P(u) - \lambda P_{1,a}(u).$$

Then, $F_{\lambda, \mu}(\overline{s}_\mu(u, v), u, v) > 0$ (resp. $F_{\lambda, \mu}(\overline{s}_\mu(u, v), u, v) < 0$) if $\lambda < \lambda(u, v)$ (resp. $\lambda > \lambda(u, v)$) and $F_{\lambda(u, v), \mu}(\overline{s}_\mu(u, v), u, v) = 0$, where

$$\lambda(u, v) = \widehat{c} \frac{P(u)^{\frac{r-p_1}{r-p}}}{P_{1,a}(u) A(u, v)^{\frac{p-p_1}{r-p}}} \quad \text{and} \quad \widehat{c} = \frac{r - p}{r - p_1} \left[\frac{p - p_1}{r - p_1} \right]^{\frac{p-p_1}{r-p}}. \quad (2.8)$$

Therefore, if $\lambda \in (0, \lambda(u, v))$, the function $s \in (0, +\infty) \mapsto \partial_s \widetilde{I}(s, u, t(s), v)$ has two positive zeros denoted by $s_1(u, v, \lambda, \mu)$ and $s_2(u, v, \lambda, \mu)$ verifying $0 < s_1(u, v, \lambda, \mu) <$

$\bar{s}_\mu(u, v) < s_2(u, v, \lambda, \mu)$. Since $F_{\lambda, \mu}(s_1(u, v, \lambda, \mu), u, v) = F_{\lambda, \mu}(s_2(u, v, \lambda, \mu), u, v) = 0$, $\partial_s F_{\lambda, \mu}(s, u, v) > 0$ for $0 < s < \bar{s}_\mu(u, v)$ and $\partial_s F_{\lambda, \mu}(s, u, v) < 0$ for $s > \bar{s}_\mu(u, v)$ it follows that

$$\partial_{ss} \tilde{I}(s_1(u, v, \lambda, \mu)u, t_1(u, v, \lambda, \mu), v) > 0, \quad (2.9)$$

$$\partial_{ss} \tilde{I}(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v) < 0. \quad (2.10)$$

This implies that the real-valued function $s \in (0, +\infty) \mapsto \tilde{I}(s, u, t(s), v)$ achieves its unique local minimum at $s = s_1(u, v, \lambda, \mu)$ and its unique local maximum at $s = s_2(u, v, \lambda, \mu)$, which ends the proof. \square

Hereafter, we will denote $t_i(u, v, \lambda, \mu) = t(s_i(u, v, \lambda, \mu))$, $i = 1, 2$. Notice that for every $(u, v) \in \widetilde{W}$, $\mu < \mu_1$ and $\lambda \in (0, \lambda(u, v, \mu))$, the points $(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v)$ and $(s_2(u, v, \lambda, \mu), u, t_2(u, v, \lambda, \mu), v)$ belong to the Nehari manifold $\mathcal{N}_{\lambda, \mu}$.

At this stage, we introduce the characteristic value, for all $\mu < \mu_1$,

$$\widehat{\lambda}(\mu) := \inf_{\substack{(u, v) \in W \\ u \neq 0, v \neq 0}} \lambda(u, v, \mu).$$

We consider the spaces

$$\begin{aligned} L_a^{p_1}(\mathbb{R}^N) &:= \left\{ u : \mathbb{R}^N \longrightarrow \mathbb{R}, \text{ measurable} : \int_{\mathbb{R}^N} a(x)|u(x)|^{p_1} dx < +\infty \right\}, \\ L_b^q(\mathbb{R}^N) &:= \left\{ u : \mathbb{R}^N \longrightarrow \mathbb{R}, \text{ measurable} : \int_{\mathbb{R}^N} b(x)|u(x)|^q dx < +\infty \right\}, \end{aligned}$$

endowed with their seminorms

$$\begin{aligned} \|u\|_{p_1, a} &:= \left(\int_{\mathbb{R}^N} a(x)|u(x)|^{p_1} dx \right)^{1/p_1} \\ \|u\|_{q, b} &:= \left(\int_{\mathbb{R}^N} b(x)|u(x)|^q dx \right)^{1/q}. \end{aligned}$$

It is clear that, under these notations, the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L_b^q(\mathbb{R}^N)$ are continuous.

Remark 3.2.1. *Let $(u_n)_n$ be a sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, then we have the assertion :*

$$u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N) \text{ weakly} \implies \lim_{n \rightarrow +\infty} \|u_n - u\|_{p_1, a} = 0.$$

We will write this property by

$$u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N) \text{ weakly} \implies u_n \rightarrow u \text{ in } L_a^{p_1}(\mathbb{R}^N) \text{ strongly,}$$

and that the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L_a^{p_1}(\mathbb{R}^N)$ is compact.

Similarly,

$$v_n \rightharpoonup v \text{ in } \mathcal{D}^{1,q}(\mathbb{R}^N) \text{ weakly} \implies v_n \rightarrow v \text{ in } L_b^q(\mathbb{R}^N) \text{ strongly,}$$

that is, the embedding $\mathcal{D}^{1,q}(\mathbb{R}^N) \hookrightarrow L_b^q(\mathbb{R}^N)$ is compact.

Indeed, fix $(u_n)_n \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ and a real number $\varepsilon > 0$. It is clear that there is a constant $C > 0$ such that $\|u_n - u\|_{p^*} \leq C$, for every $n \in \mathbb{N}$. Since $a \in L^{\frac{p^*}{p^*-p_1}}(\mathbb{R}^N)$, there is a compact $K(a, \varepsilon, N) \subset \mathbb{R}^N$ such that

$$\|a\|_{L^{\frac{p^*}{p^*-p_1}}(\mathbb{R}^N \setminus K)} \leq (\varepsilon/C)^{p_1}.$$

On the other hand, by standard compact Sobolev embeddings, there is $n_1 \in \mathbb{N}$ such that

$$\int_K a(x)|u_n - u|^{p_1} dx \leq \varepsilon^{p_1}, \quad \forall n \geq n_1.$$

Therefore, using the Hölder inequality, we get

$$\int_{\mathbb{R}^N} a(x)|u_n - u|^{p_1} dx \leq 2\varepsilon^{p_1}, \quad \forall n \geq n_1,$$

which ends the claim. The argumentation is the same for the sequence $(v_n)_n \subset \mathcal{D}^{1,q}(\mathbb{R}^N)$.

We prove that $\widehat{\lambda}(\mu)$ is greater than a nonnegative constant which depends only on μ, p, p_1, q, α and β . Indeed, using the Hölder inequality, we get

$$R(u, v) \leq \|u\|_{p^*}^{\alpha+1} \|v\|_{q^*}^{\beta+1}.$$

On the other hand we have, also by the Hölder inequality

$$P_{1,a}(u) := \int_{\mathbb{R}^N} a(x)|u|^{p_1} dx \leq \|a\|_{\frac{p^*}{p^*-p_1}} \|u\|_{p^*}^{p_1}.$$

Using the continuous embedding $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L^{q^*}(\mathbb{R}^N)$, we obtain

$$A(u, v) \leq c_1 \frac{P_*(u)^{\frac{r}{p^*}}}{(\mu - \mu_1)^{\frac{\beta+1}{q-(\beta+1)}}},$$

where $P_*(u) = \|u\|_{p^*}^{p^*}$ and $c_1 = c_1(b, p, p_1, q, \alpha, \beta)$. Then, using the continuous embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$ and $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$ we obtain

$$\lambda(u, v, \mu) \geq c_2 (\mu - \mu_1)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

where $c_2 = c_2(a, b, p, p_1, q, \alpha, \beta)$ and $L_a^{p_1}(\mathbb{R}^N) := \{u \text{ measurable} / a|u|^{p_1} \in L^1(\mathbb{R}^N)\}$. Consequently

$$\widehat{\lambda}(\mu) \geq c_2 (\mu - \mu_1)^{\frac{\beta+1}{q-(\beta+1)} \frac{p-p_1}{r-p}},$$

which achieves the claim. We now introduce the subset of \mathbb{R}^2 defined by

$$\mathcal{D} := \{(\lambda, \mu) \in (0, +\infty) \times (-\infty, \mu_1) : \lambda < \widehat{\lambda}(\mu)\}.$$

For every $(\lambda, \mu) \in \mathcal{D}$ and $(u, v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$, we have $\partial_s \tilde{I}(s_1(u, v), u, t_1(u, v), v) = 0$ and (resp. $\partial_s \tilde{I}(s_2(u, v), u, t_2(u, v), v) = 0$), it follows that the functional $(u, v) \mapsto \tilde{I}(s_1(u, v), u, t_1(u, v), v)$ (resp. $(u, v) \mapsto \tilde{I}(s_2(u, v), u, t_2(u, v), v)$) is bounded below on $(u, v) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\})$. Thus, for every $(\lambda, \mu) \in \mathcal{D}$, we define

$$\alpha_1(\lambda, \mu) = \inf \{ \tilde{I}(s_1(u, v), u, t_1(u, v), v) : (u, v) \in \widetilde{W} \}, \quad (2.11)$$

$$\alpha_2(\lambda, \mu) = \inf \{ \tilde{I}(s_2(u, v), u, t_2(u, v), v) : (u, v) \in \widetilde{W} \}. \quad (2.12)$$

Remark 3.2.2. *It is interesting to notice that for every $\gamma > 0$, and $\delta > 0$, it holds that*

$$\begin{aligned} \tilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \tilde{I}(s, u, t, v), \\ \partial_t \tilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \frac{1}{\delta} \partial_t \tilde{I}(s, u, t, v), \\ \partial_s \tilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \frac{1}{\gamma} \partial_s \tilde{I}(s, u, t, v) \text{ and} \\ \partial_{ss} \tilde{I}(\gamma s, \frac{u}{\gamma}, \delta t, \frac{v}{\delta}) &= \frac{1}{\gamma^2} \partial_{ss} \tilde{I}(s, u, t, v). \end{aligned}$$

It follows that

$$s_1(u, v, \lambda, \mu) = \frac{1}{\gamma} s_1 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \delta > 0, \quad (2.13)$$

$$s_2(u, v, \lambda, \mu) = \frac{1}{\gamma} s_2 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \delta > 0, \quad (2.14)$$

$$t_1(u, v, \lambda, \mu) = \frac{1}{\delta} t_1 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \gamma > 0, \quad (2.15)$$

$$t_2(u, v, \lambda, \mu) = \frac{1}{\delta} t_2 \left(\frac{u}{\gamma}, \frac{v}{\delta}, \lambda, \mu \right), \quad \forall \gamma > 0. \quad (2.16)$$

Therefore,

$$\alpha_1(\lambda, \mu) = \inf_{(u, v) \in \mathbb{S}_p \times \mathbb{S}_q} \tilde{I}(s_1(u, v), u, t_1(u, v), v), \quad (2.17)$$

$$\alpha_2(\lambda, \mu) = \inf_{(u, v) \in \mathbb{S}_p \times \mathbb{S}_q} \tilde{I}(s_2(u, v), u, t_2(u, v), v), \quad (2.18)$$

where \mathbb{S}_p and \mathbb{S}_q are the unit spheres of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N)$ respectively. Specify that $\mathbb{S}_p \times \mathbb{S}_q$ is a 2-codimensional and complete submanifold of W , and we will denote it by \mathbb{S} .

Our aim in this part is to show that $\alpha_1(\lambda, \mu)$ is in fact a critical value of the Euler-Lagrange functional I for every $(\lambda, \mu) \in \mathcal{D}$. We start with characterizing the minimizing sequences of $\alpha_1(\lambda, \mu)$ and $\alpha_2(\lambda, \mu)$, for every $(\lambda, \mu) \in \mathcal{D}$.

Lemma 3.2.2. *Let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17) (resp. of (2.18)) and let $(U_n^1, V_n^1) := (s_1(u_n, v_n, \lambda, \mu)u_n, t_1(u_n, v_n, \lambda, \mu)v_n)$ (resp. $(U_n^2, V_n^2) := (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$). Then it holds :*

- (i) $\limsup_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| < \infty$ (resp. $\limsup_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| < \infty$).
- (ii) $\liminf_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| > 0$ (resp. $\liminf_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| > 0$).

Proof. We start by checking the point (i), let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17). Since $\partial_s \tilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$ and $\partial_t \tilde{I}(s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n) = 0$, it follows that

$$\begin{aligned} P(U_n^1) - \lambda P_{1,a}(U_n^1) - R(U_n^1, V_n^1) &= 0, \\ Q(V_n^1) - \mu Q_b(V_n^1) - R(U_n^1, V_n^1) &= 0. \end{aligned}$$

Suppose that there exists a subsequence of (U_n^1, V_n^1) , still denoted by (U_n^1, V_n^1) , such that

$$\lim_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| = \infty.$$

We can distinguish three cases :

case a) $\lim_{n \rightarrow +\infty} \|\nabla U_n^1\|_p = \infty$ and $\|\nabla V_n^1\|_q$ is bounded. By (2.19) we get $R(U_n^1, V_n^1)$ is bounded. On the other hand, the continuous embedding of $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$ enables us to have $P_{1,a}(U_n^1) = o(P(U_n^1))$, as n goes to $+\infty$. By (2.19) we get $R(U_n^1, V_n^1) = (1+o(1))P(U_n^1)$, as n goes to $+\infty$ and consequently $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$, which leads to a contradiction.

case b) $\lim_{n \rightarrow +\infty} \|\nabla V_n^1\|_q = \infty$ and $\|\nabla U_n^1\|_p$ is bounded. We obtain by (2.19) the fact $R(U_n^1, V_n^1)$ is bounded. On the other hand, if $0 < \mu < \mu_1$, by Sobolev and Young's inequalities, for $\varepsilon \in (0, 1)$, there is a constant $C_\varepsilon > 0$ such that

$$\|V_n^1\|_q^q \leq \frac{\varepsilon}{\mu} \|\nabla V_n^1\|_q^q + C_\varepsilon$$

which gives $R(U_n^1, V_n^1) + \mu C_\varepsilon \geq (1 - \varepsilon)Q(V_n^1)$. Then $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$, which is impossible. If $\mu \leq 0$, then $Q(V_n^1) - \mu Q_b(V_n^1) = R(U_n^1, V_n^1) \geq Q(V_n^1)$ so $\lim_{n \rightarrow +\infty} R(U_n^1, V_n^1) = +\infty$, which can not hold not.

case c) $\lim_{n \rightarrow +\infty} \|\nabla U_n^1\|_p = \lim_{n \rightarrow +\infty} \|\nabla V_n^1\|_q = \infty$. As in the first case, we have

$$R(U_n^1, V_n^1) = (1 + o(1))P(U_n^1), \quad \text{as } n \text{ goes to } +\infty.$$

Then $I(U_n^1, V_n^1) = (\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 + o_n(1))P(U_n^1)$ as n goes to $+\infty$. Hence, $\lim_{n \rightarrow +\infty} I(U_n^1, V_n^1) = +\infty$, which contradicts the hypothesis $I(U_n^1, V_n^1) \leq 0$ for every $n \in \mathbb{N}$. The first assertion for (2.18) follows by the same arguments.

Now, let us show the second assertion of the lemma. Let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17). Suppose that there is a subsequence, still denoted by (U_n^1, V_n^1) ,

such that $\lim_{n \rightarrow +\infty} \|(U_n^1, V_n^1)\| = 0$. By (2.19), we get $\lim_{n \rightarrow +\infty} I(U_n^1, V_n^1) = 0$ and this can not hold true because $I(U_n^1, V_n^1) \leq 0$ for every $n \in \mathbb{N}$.

Similarly, let $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.18). Suppose that there is a subsequence, still denoted by (U_n^2, V_n^2) , such that $\lim_{n \rightarrow +\infty} \|(U_n^2, V_n^2)\| = 0$. If $p > \alpha + 1$, by (2.10), we have

$$\partial_{ss} \tilde{I}(U_n^2, V_n^2) = (\alpha + 1)((p - 1)P(U_n^2) - \lambda(p_1 - 1)P_{1,a}(U_n^2) - \alpha R(U_n^2, V_n^2)) < 0.$$

Then $(p - 1)P(U_n^2) - \lambda(p_1 - 1)P_{1,a}(U_n^2) - \alpha p R(U_n^2, V_n^2) < 0$, which implies that $(p - (\alpha + 1))R(U_n^2, V_n^2) < 0$ and this is impossible. Finally, if $p \leq \alpha + 1$, then $(p - p_1)P(U_n^2) < (\alpha + 1 + p)R(U_n^2, V_n^2)$. Since $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, then

$$\begin{aligned} R(U_n^2, V_n^2) &\leq \|U_n^2\|_{p^*}^{\alpha+1} \|V_n^2\|_{q^*}^{\beta+1} \\ &\leq c(p, q) \|\nabla U_n^2\|_p^{\alpha+1} \|\nabla V_n^2\|_q^{\beta+1}, \end{aligned}$$

and consequently, $(p - p_1) \leq c(\alpha, p, q) \|\nabla U_n^2\|_p^{\alpha+1-p} \|\nabla V_n^2\|_q^{\beta+1}$, which converges to 0 as n goes to $+\infty$. This contradicts the fact $p > p_1$, which ends the proof. \square

Lemma 3.2.3. *Let $(u_n, v_n) \subset \mathbb{S}$ be a minimizing sequence of $\alpha_1(\lambda, \mu)$ (resp. for $\alpha_2(\lambda, \mu)$), then the sequences (U_n^1, V_n^1) and (resp. (U_n^2, V_n^2)) is a Palais-Smale for the functional I , where (U_n^1, V_n^1) and (U_n^2, V_n^2) are defined bellow.*

Proof. Let $(\lambda, \mu) \in \mathcal{D}$ and $(u_n, v_n) \in \mathbb{S}$ be a minimizing sequence of (2.17). Let us set

$$\begin{aligned} U_n &= s_1(u_n, v_n, \lambda, \mu)u_n, \\ V_n &= t_1(u_n, v_n, \lambda, \mu)v_n. \end{aligned}$$

By the previous lemma, it is clear that the sequence (U_n, V_n) is bounded in W . On the other hand, the functional $(s, t) \mapsto \tilde{I}(\cdot, s, \cdot, t)$ has null gradient and its Hessian determinant is strictly positive. So, the implicit functions theorem allows to confirm that the functions $(u, v) \mapsto s_1(u, v, \lambda, \mu)$ and $(u, v) \mapsto t_1(u, v, \lambda, \mu)$ are $\mathcal{C}^1(W, \mathbb{R})$, since $(u, v) \mapsto \tilde{I}(\cdot, u, \cdot, v)$ is.

We introduce now the functional \mathcal{I} defined on \mathbb{S} by

$$\mathcal{I}(u, v) = \tilde{I}(s_1(u, v, \lambda, \mu), u, t_1(u, v, \lambda, \mu), v),$$

Then

$$\alpha_1(\lambda, \mu) = \inf_{(u,v) \in \mathbb{S}} \mathcal{I}(u, v) = \lim_{n \rightarrow +\infty} \mathcal{I}(u_n, v_n).$$

By the Eklund variational principle on the complete manifold $(\mathbb{S}, \|\cdot\|)$ to the functional \mathcal{I} we get

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) \leq \frac{1}{n} \|(\varphi_n, \psi_n)\|, \quad \forall (\varphi_n, \psi_n) \in T_{(u_n, v_n)} \mathbb{S},$$

where $T_{(u_n, v_n)}\mathbb{S}$ designs the tangent space to \mathbb{S} at the point (u_n, v_n) . We know that the space $T_{(u_n, v_n)}\mathbb{S} = T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$ where $T_{u_n}\mathbb{S}_p$ and $T_{v_n}\mathbb{S}_q$ are respectively the tangent spaces to \mathbb{S}_p and \mathbb{S}_q at the respective points u_n and v_n .

In order to reduce the notations, we set

$$A_n := (u_n, v_n, \lambda, \mu), \quad \text{and} \quad B_n := (s_1(u_n, v_n, \lambda, \mu), u_n, t_1(u_n, v_n, \lambda, \mu), v_n).$$

For every $(\varphi_n, \psi_n) \in T_{u_n}\mathbb{S}_p \times T_{v_n}\mathbb{S}_q$, one has

$$\mathcal{I}'(u_n, v_n)(\varphi_n, \psi_n) = D_1\tilde{I}(B_n)(\varphi_n) + D_2\tilde{I}(B_n)(\psi_n)$$

where

$$\begin{aligned} D_1\tilde{I}(B_n)(\varphi_n) &= \partial_s s_1(A_n)(\varphi_n) \partial_s \tilde{I}(B_n) + \partial_u \tilde{I}(B_n)(\varphi_n) + \partial_u t_1(A_n)(\varphi_n) \partial_t \tilde{I}(B_n) \\ &= \partial_u \tilde{I}(B_n)(\varphi_n). \end{aligned}$$

With the same manner, one has

$$D_2\tilde{I}(B_n)(\psi_n) = \partial_v \tilde{I}(B_n)(\psi_n).$$

Furthermore, consider the following "fiber" maps

$$\begin{aligned} \pi : \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} &\rightarrow \mathbb{R} \times \mathbb{S}_p \\ u &\mapsto \left(\|u\|_p, \frac{u}{\|u\|_p} \right) := (\pi_1(u), \pi_2(u)), \\ \tilde{\pi} : \mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\} &\rightarrow \mathbb{R} \times \mathbb{S}_q \\ v &\mapsto \left(\|v\|_q, \frac{v}{\|v\|_q} \right) := (\tilde{\pi}_1(v), \tilde{\pi}_2(v)). \end{aligned}$$

Applying the Hölder inequality we get

$$|\pi'_1(u)(\varphi)| \leq \|\varphi\|_p, \quad |\pi'_2(u)(\varphi)| \leq 2 \frac{\|\varphi\|_p}{\|u\|_p}, \quad \forall (u, \varphi) \in (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}) \times \mathcal{D}^{1,p}(\mathbb{R}^N)$$

and

$$|\tilde{\pi}'_1(v)(\psi)| \leq \|\psi\|_q, \quad |\tilde{\pi}'_2(v)(\psi)| \leq 2 \frac{\|\psi\|_q}{\|v\|_q}, \quad \forall (v, \psi) \in (\mathcal{D}^{1,q}(\mathbb{R}^N) \setminus \{0\}) \times \mathcal{D}^{1,q}(\mathbb{R}^N).$$

From Lemma 3.2.2, there exists a nonnegative constant K such that $s_1(A_n) \geq K$ and $t_1(A_n) \geq K$ for any integer n . On the other hand, for every $(\varphi, \psi) \in W$

$$\begin{aligned} D_1 I(U_n, V_n)(\varphi) &= \varphi_n^1 \partial_s \tilde{I}(B_n) + \partial_u \tilde{I}(B_n)(\varphi_n^2) + \varphi_n^1 \partial_t \tilde{I}(B_n) \\ &= \partial_u \tilde{I}(B_n)(\varphi_n^2), \end{aligned}$$

where $\varphi_n^1 = \pi'_1(u_n)(\varphi)$ and $\varphi_n^2 = \pi'_2(u_n)(\varphi)$. Then we have the following estimates $|\varphi_n^1| \leq \|\varphi\|_p$ and $\|\varphi_n^2\|_p \leq \frac{2}{K}\|\varphi\|_p$. In the same way, we get

$$\begin{aligned} D_2I(U_n, V_n)(\psi) &= \psi_n^1 \partial_s \tilde{I}(B_n) + \partial_v \tilde{I}(B_n)(\psi_n^2) + \psi_n^1 \partial_t \tilde{I}(B_n) \\ &= \partial_v \tilde{I}(B_n)(\psi_n^2), \end{aligned}$$

where $\psi_n^1 = \tilde{\pi}'_2(v_n)(\psi)$ and $\psi_n^2 = \tilde{\pi}'_1(v_n)(\psi)$. Then we have the following estimates $|\psi_n^1| \leq \|\psi\|_p$ and $\|\psi_n^2\|_p \leq \frac{2}{K}\|\psi\|_q$. Therefore

$$\begin{aligned} |D_1I(U_n, V_n)(\varphi)| &\leq \frac{1}{n}\|\varphi_n^2\|_p \\ &\leq \frac{2}{nK}\|\varphi\|_p \end{aligned}$$

and

$$\begin{aligned} |D_2I(U_n, V_n)(\psi)| &\leq \frac{1}{n}\|\psi_n^2\|_p \\ &\leq \frac{2}{nK}\|\psi\|_q \end{aligned}$$

Finally, we have

$$\lim_{n \rightarrow +\infty} \|I'(U_n, V_n)\|_{W^*} = 0,$$

where $I'(U_n, V_n)(\varphi, \psi) = D_1I(U_n, V_n)(\varphi) + D_2I(U_n, V_n)(\psi)$ and $\|\cdot\|_{W^*}$ designs the norm of the dual space of W , which achieves the first claim. The second one follows with similar arguments. \square

3.3 Existence and multiplicity results of solutions to the problem

In this section, we will show that there is at least one solution to the system (1.1) and two solutions in the case $p = q$ obtained by considering minimizing sequences of (2.17) and (2.18) under some supplementary conditions on (λ, μ) which belongs to \mathcal{D} . We begin by stating the following lemma, due to A. El Hamidi and J.M. Rakotoson [8]

Lemma 3.3.1. [8] *Let $\widehat{\Psi}$ be a Caratheodory function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ into \mathbb{R}^N satisfying the usual Leray-Lions growth and monotonicity conditions. Let (u_n) be a bounded sequence of $W_{loc}^{1,p}(\mathbb{R}^N) = \left\{v \in L_{loc}^p(\mathbb{R}^N), |\nabla v| \in L_{loc}^p(\mathbb{R}^N)\right\}$, with $1 < p < +\infty$, (f_n) be a bounded sequence of $L_{loc}^1(\mathbb{R}^N)$ and (g_n) be a sequence of $W_{loc}^{-1,p'}(\mathbb{R}^N)$ tending strongly to zero.*

Assume that (u_n) satisfies :

$$(H1) \quad \int_{\mathbb{R}^N} \widehat{\Psi}(x, u_n(x), \nabla u_n(x)) \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^N} f_n \varphi \, dx + \langle g_n, \varphi \rangle,$$

$$\forall \varphi \in W_{comp}^{1,p}(\mathbb{R}^N) = \left\{ v \in W^{1,p}(\mathbb{R}^N), \text{ with compact support} \right\}, \varphi \text{ bounded.}$$

Then

1. there exists a function u such that $u_n(x) \rightarrow u(x)$ a.e. in \mathbb{R}^N ,
2. $u \in W_{loc}^{1,p}(\mathbb{R}^N)$,
3. there exists a subsequence, still denoted (u_n) , such that

$$\nabla u_n(x) \rightarrow \nabla u(x) \text{ a.e. in } \mathbb{R}^N.$$

At this stage, we state and show the following

Lemma 3.3.2. *Let $(\lambda, \mu) \in \mathcal{D}$ and $(u_n, v_n)_n \subset W$ be a $(P.S)_c$ sequences of I such that*

$$c := \lim_{n \rightarrow +\infty} I(u_n, v_n) < \inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) + \alpha_1(\lambda, \mu). \quad (3.19)$$

Then the sequence (u_n, v_n) is relatively compact.

Proof. Let $(\lambda, \mu) \in \mathcal{D}$ and $(u_n, v_n)_n \subset W$ be a $(P.S)_c$ sequence of I satisfying the condition (3.19).

On one hand, We claim that (u_n, v_n) is bounded in W . Since (u_n, v_n) is Palais-Smale sequence of I , then we have

$$I(u_n, v_n) = c + o_n(1) \quad (3.20)$$

$$P(u_n) - \lambda P_{1,a}(u_n) = R(u_n, v_n) + o(\|u_n\|_{p^*}) \quad (3.21)$$

$$Q(v_n) - \mu Q_b(v_n) = R(u_n, v_n) + o(\|v_n\|_{q^*}) \quad (3.22)$$

Then we can apply the result of the lemma 3.2.2 to prove that (u_n, v_n) is bounded in W . At this stage, we can assume, up to a subsequence, that

$$u_n \rightharpoonup u \text{ in } \mathcal{D}^{1,p}(\mathbb{R}^N),$$

$$v_n \rightharpoonup v \text{ in } \mathcal{D}^{1,q}(\mathbb{R}^N),$$

$$u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N,$$

$$v_n \rightarrow v \text{ a.e. in } \mathbb{R}^N.$$

It is clear that $(u, v) \in \mathcal{N}_{\lambda, \mu} \cup \{(0, 0)\}$.

Notice that $I'(u_n, v_n)(\varphi, 0) \rightarrow 0$ and $I'(u_n, v_n)(0, \psi) \rightarrow 0$ for every $(\varphi, \psi) \in \mathcal{D}^{1,p}(\mathbb{R}^N) \times \mathcal{D}^{1,q}(\mathbb{R}^N)$ imply the hypothesis (H_1) for the sequences $(u_n)_n$ and $(v_n)_n$, in the special case where $\widehat{\Psi}$ corresponds to the p -Laplacian or the q -Laplacian respectively. Consequently, Lemma 3.3.1 show that, up to subsequences :

$$\nabla u_n \rightarrow \nabla u \text{ a.e. in } \mathbb{R}^N,$$

$$\nabla v_n \rightarrow \nabla v \text{ a.e. in } \mathbb{R}^N.$$

Let us set $x_n := u_n - u$ and $y_n := v_n - v$. Using the Brézis-Lieb lemma [4], so we obtain the following decompositions

$$\begin{aligned} P(x_n) &= P(u_n) - P(u) + o_n(1), \\ Q(y_n) &= Q(v_n) - Q(v) + o_n(1), \\ P_{1,a}(x_n) &= P_{1,a}(u_n) - P_{1,a}(u) + o_n(1), \\ Q_b(y_n) &= Q_b(v_n) - Q_b(v) + o_n(1), \\ R(x_n, y_n) &= R(u_n, v_n) - R(u, v) + o_n(1). \end{aligned}$$

Using the compactness of the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L_a^{p_1}(\mathbb{R}^N)$, and $\mathcal{D}^{1,q}(\mathbb{R}^N) \hookrightarrow L_b^q(\mathbb{R}^N)$ we get $P_{1,a}(x_n) = o_n(1)$ and $Q_b(y_n) = o_n(1)$.

It follows that

$$\begin{aligned} P(x_n) &= R(x_n, y_n) + o_n(1), \\ Q(y_n) &= R(x_n, y_n) + o_n(1), \\ I_{0,0}(x_n, y_n) &= c - I(u, v) + o_n(1). \end{aligned}$$

Notice that the Nehari Manifold associated to $I_{0,0}$ is given by

$$\mathcal{N}_{0,0} = \{(s_0(u, v)u, t_0(u, v)v); (u, v) \in W, u \neq 0, v \neq 0\},$$

where

$$s_0(u, v) = \left[\frac{P(u)Q(v)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{R(u, v)^{\frac{r}{\alpha+1}}} \right]^{\frac{1}{r-p}} \quad \text{and} \quad t_0(u, v) = t(s_0(u, v)),$$

and $s \rightarrow t(s)$ is defined by (2.6). Let l be the common limit of $P(x_n)$, $Q(y_n)$ and $R(x_n, y_n)$. Suppose that $l \neq 0$, we get then

$$I_{0,0}(s_0(x_n, y_n)x_n, t_0(x_n, y_n)y_n) = (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) K(x_n, y_n) \quad (3.23)$$

$$\geq \inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v), \quad (3.24)$$

where

$$K(x_n, y_n) = \left[\frac{P(x_n)^{(\alpha+1)} Q(y_n)^{(\beta+1)\frac{r}{q}}}{R(x_n, y_n)^p} \right]^{\frac{r}{(\alpha+1)(r-p)}},$$

which tends to l as n tends to $+\infty$.

Therefore

$$\lim_{n \rightarrow +\infty} I_{0,0}(s_0(x_n, y_n)x_n, t_0(x_n, y_n)y_n) = l(\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right).$$

On the other hand,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_{\lambda, \mu}(x_n, y_n) &= l \left(\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} - 1 \right) \\ &= l(\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right). \end{aligned}$$

Hence, we obtain

$$l(\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) = c - I(u, v),$$

and consequently

$$\begin{aligned} c &\geq \inf_{(u,v) \in \mathcal{N}_{0,0}} I(u, v) + I(u, v) \\ &\geq \inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) + \alpha_1(\lambda, \mu), \end{aligned}$$

which contradicts the hypothesis (3.19), then $l = 0$, which achieves the proof. \square

Theorem 3.3.1. *The system (1.1) has at least one solution, for every $(\lambda, \mu) \in \mathcal{D}$.*

Proof. Using the Hölder inequality in $R(u, v)$, we get

$$\begin{aligned} \inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) &= \inf_{(u,v) \in \mathbb{S}} (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) \left[\frac{P(u)^{(\alpha+1)} Q(v)^{(\beta+1) \frac{p}{q}}}{R(u, v)^p} \right]^{\frac{r}{(\alpha+1)(r-p)}} \\ &\geq (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) \left[S_p S_q^{\frac{p(\beta+1)}{q(\alpha+1)}} \right]^{\frac{r}{r-p}} > 0, \end{aligned} \quad (3.25)$$

where S_p and S_q are the best Sobolev constants in the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L^{q^*}(\mathbb{R}^N)$ respectively. Consequently,

$$\forall (\lambda, \mu) \in \mathcal{D}, \quad \alpha_1(\lambda, \mu) < \inf_{(u,v) \in \mathcal{N}_{0,0}} I(u, v) + \alpha_1(\lambda, \mu)$$

We set $U_n^1 := s_1(u_n, v_n, \lambda, \mu)u_n$ and $V_n^1 := t_1(u_n, v_n, \lambda, \mu)v_n$, where (u_n, v_n) is a minimizing sequence of (2.11). By Lemma 3.2.3 the sequence (U_n^1, V_n^1) is of Palais-smale type whose level is $\alpha_1(\lambda, \mu)$. Then, according to Lemma 3.3.2 there is a subsequence, still denoted (U_n^1, V_n^1) , and (U^1, V^1) such that

$$(U_n^1, V_n^1) \longrightarrow (U^1, V^1) \text{ strongly in } W.$$

Now, since $\lim_{n \rightarrow +\infty} \|I'(U_n^1, V_n^1)\|_{W^*} = 0$, we have for every $(\varphi, \psi) \in W$

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla U_n^1|^{p-2} \nabla U_n^1 \cdot \nabla \varphi \, dx = \lambda \int_{\mathbb{R}^N} A_n \varphi \, dx + \int_{\mathbb{R}^N} X_n^1 \varphi \, dx + o_n(1), \\ \int_{\mathbb{R}^N} |\nabla V_n^1|^{q-2} \nabla V_n^1 \cdot \nabla \psi \, dx = \mu \int_{\mathbb{R}^N} B_n \psi \, dx + \int_{\mathbb{R}^N} Y_n^1 \psi \, dx + o_n(1), \end{cases} \quad (3.26)$$

where $A_n := a |U_n^1|^{p_1-2} U_n^1$, $B_n := b |V_n^1|^{q-2} V_n^1$, $X_n^1 := |U_n^1|^{\alpha-1} U_n^1 |V_n^1|^{\beta+1}$ and $Y_n^1 := |U_n^1|^{\alpha+1} |V_n^1|^{\beta-1} V_n^1$. On one hand, the continuity of the embeddings $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$ and $\mathcal{D}^{1,q}(\mathbb{R}^N) \subset L_b^q(\mathbb{R}^N)$ implies that $\int_{\mathbb{R}^N} A_n \varphi \, dx \longrightarrow \int_{\mathbb{R}^N} a |U^1|^{p_1-2} U^1 \varphi \, dx$ and $\int_{\mathbb{R}^N} B_n \psi \, dx \longrightarrow \int_{\mathbb{R}^N} b |V^1|^{q-2} V^1 \psi \, dx$ as n tends to $+\infty$. On the other hand, since $A_n \varphi \longrightarrow a |U^1|^{p_1-2} U^1 \varphi$ and $B_n \psi \longrightarrow b |V^1|^{q-2} V^1 \psi$ a.e. in \mathbb{R}^N as n goes to $+\infty$, the inequalities $\int_{\mathbb{R}^N} A_n \varphi \, dx \leq \|u_n\|_{p^*}^\alpha \|v_n\|_{q^*}^{\beta+1} \|\varphi\|_{p^*}$, $\int_{\mathbb{R}^N} B_n \psi \, dx \leq \|u_n\|_{p^*}^{\alpha+1} \|v_n\|_{q^*}^\beta \|\psi\|_{q^*}$

and the Lebesgue theorem imply that we can pass to the limit under integral sign in (3.26) to obtain for all $(\varphi, \psi) \in W$

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla U^1|^{p-2} \nabla U^1 \cdot \nabla \varphi \, dx &= \lambda \int_{\mathbb{R}^N} a |U^1|^{p_1-2} U^1 \varphi \, dx + \int_{\mathbb{R}^N} X^1 \varphi \, dx, \\ \int_{\mathbb{R}^N} b |\nabla V^1|^{q-2} \nabla V^1 \cdot \nabla \psi \, dx &= \mu \int_{\mathbb{R}^N} b |V^1|^{q-2} V^1 \psi \, dx + \int_{\mathbb{R}^N} Y^1 \psi \, dx, \end{cases}$$

where $X^1 := |U^1|^{\alpha-1} U^1 |V^1|^{\beta+1}$ and $Y^1 := |U^1|^{\alpha+1} |V^1|^{\beta-1} V^1$. Hence (U^1, V^1) is a weak solution to the problem (1.1). \square

Remark 3.3.1. *In the scalar case, we obtain the analogous of Theorem 3.3.2 with the same arguments. We note here in this special case, direct computations give*

$$\inf_{u \in \mathcal{N}_0} I_0(u) = \frac{1}{N} S_p^{\frac{N}{p}} \quad \text{and} \quad \inf_{u \in \mathcal{N}_\lambda \cup \{0\}} I_\lambda(u) = 0,$$

which generalize the famous Brézis-Nirenberg condition for the entire space.

Proposition 3.3.1. *Let $p = q > 1$ and (λ, μ) be in \mathcal{D} . Then,*

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) = \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

Proof. Assume that $p = q > 1$, then

$$p^* = \alpha + \beta + 2 \quad \text{and} \quad (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) = \frac{p}{N-p}.$$

By the inequality (3.25), we conclude that

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) \geq \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

On the other hand, let $(u_n) \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a minimizing sequence of S_p . Then using the identity (3.23), we get

$$\begin{aligned} \inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) &\leq I_{0,0}(s_0(u_n, u_n)u_n, t_0(u_n, u_n)u_n) \\ &= \frac{p}{N-p} \left[\frac{P(u_n)}{P_*(u_n)^{\frac{p}{p^*}}} \right]^{\frac{r p^*}{(\alpha+1)(r-p)}} \\ &= \frac{p}{N-p} \left[\frac{P(u_n)}{P_*(u_n)^{\frac{p}{p^*}}} \right]^{\frac{N}{p}}, \end{aligned}$$

making tend n to $+\infty$ the right hand of the last quantity goes to $\frac{p}{N-p} S_p^{\frac{N}{p}}$, which achieves the proof. \square

Theorem 3.3.2. *If $p = q > 1$, the system (1.1) has another nontrivial nonnegative solution different from the solution established in Theorem 3.3.1.*

We start by stating and showing the following

Lemma 3.3.3. *Let $p > 1$, $q > 1$, $(\lambda, \mu) \in \mathcal{D}$ and (u_n, v_n) in W be a Palais-Smale sequence for $I_{\lambda, \mu}$ such that $(u_n, v_n) \rightharpoonup (u, v)$. Then there exists a constant $K > 0$ depending on p , p_1 , a and N such that*

$$I(u, v) \geq -K\lambda^{\frac{p}{p-p_1}}.$$

Proof. Let $(u_n, v_n) \in W$ be a Palais-Smale sequence for I converging weakly to (u, v) in W . It is clear that $(u, v) \in \mathcal{N}_{\lambda, \mu} \cup \{(0, 0)\}$ and if $u = 0$ or $v = 0$ then $(u, v) = (0, 0)$. If $(u, v) = (0, 0)$, the proof is achieved. We assume now that $(u, v) \in \mathcal{N}_{\lambda, \mu}$. Since (u_n, v_n) is a Palais-Smale sequence, then

$$\begin{cases} P(u_n) - \lambda P_{1,a}(u_n) &= R(u_n, v_n) + o_n(1), \\ Q(v_n) - \mu Q_b(v_n) &= R(u_n, v_n) + o_n(1). \end{cases}$$

It follows that

$$I(u_n, v_n) = (\alpha + 1) \left(\left(\frac{1}{p} - \frac{1}{r} \right) P(u_n) - \lambda \left(\frac{1}{p_1} - \frac{1}{r} \right) P_{1,a}(u_n) \right) + o_n(1).$$

We introduce the following function

$$f(t, u) := (\alpha + 1) \left(t^p \left(\frac{1}{p} - \frac{1}{r} \right) P(u) - t^{p_1} \lambda \left(\frac{1}{p_1} - \frac{1}{r} \right) P_{1,a}(u) \right).$$

Then

$$\frac{\partial f}{\partial t}(t, u) = 0 \iff t = t(u) := \left[\frac{\lambda p_1 \left(\left(\frac{1}{p_1} - \frac{1}{r} \right) P_{1,a}(u) \right)^{\frac{1}{p-p_1}}}{p \left(\frac{1}{p} - \frac{1}{r} \right) P(u)} \right]^{\frac{1}{p-p_1}},$$

and

$$f(t(u), u) = -\lambda^{\frac{p}{p-p_1}} \left(1 - \frac{p_1}{p} \right) \left(\frac{p_1}{p} \right)^{\frac{p_1}{p-p_1}} \frac{\left(\frac{1}{p_1} - \frac{1}{r} \right)^{\frac{p_1}{p-p_1}} P_{1,a}(u)^{\frac{p}{p-p_1}}}{\left(\frac{1}{p} - \frac{1}{r} \right)^{\frac{p}{p-p_1}} P(u)^{\frac{p_1}{p-p_1}}}.$$

If S_{p_1} denotes the best constant of the continuous embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L_a^{p_1}(\mathbb{R}^N)$, we get

$$\forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}, \quad \frac{P_{1,a}(u)^{\frac{p}{p-p_1}}}{P(u)^{\frac{p_1}{p-p_1}}} \leq \|a\|_{L^{\frac{p}{p^*-p_1}}}^{\frac{p}{p-p_1}} S_p^{\frac{pp_1}{p-p_1}}.$$

So there exists a constant $K > 0$ such that

$$\forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}, f(t, u) \geq f(t(u), u) \geq -K\lambda^{\frac{p}{p-p_1}}.$$

Therefore, for every $(u, v) \in \mathcal{N}_{\lambda, \mu}$, one has

$$I(u, v) = f(t(u), u) \geq -K\lambda^{\frac{p}{p-p_1}} + o_n(1),$$

which ends the proof. \square

Lemma 3.3.4. *If $p = q > 1$ and $(\lambda, \mu) \in \mathcal{D}$, then the functional I satisfies the Palais-Smale condition on the interval $\left(-\infty, \frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}\right)$.*

Proof. Let (u_n, v_n) be a Palais-Smale sequence such that

$$\lim_{n \rightarrow \infty} I(u_n, v_n) = c < \frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}.$$

By Standard arguments one can prove that (u_n, v_n) is bounded in W , so one can extract a subsequence of (u_n, v_n) , still denoted (u_n, v_n) , such that

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u, v) \text{ in } W, \\ u_n &\rightarrow u \text{ in } L_a^{p_1}(\mathbb{R}^N), \\ v_n &\rightarrow v \text{ in } L_b^q(\mathbb{R}^N), \\ u_n(x) &\rightarrow u(x) \text{ a.e } x \text{ in } \mathbb{R}^N \\ v_n(x) &\rightarrow v(x) \text{ a.e } x \text{ in } \mathbb{R}^N. \end{aligned}$$

Let $x_n := u_n - u$ and $y_n := v_n - v$, applying again Lemma 3.3.1, we get $\nabla x_n \rightarrow 0$ and $\nabla y_n \rightarrow 0$ almost everywhere in \mathbb{R}^N . So by Brézis-Lieb lemma, it follows

$$\begin{aligned} P(x_n) &= P(u_n) - P(u) + o_n(1), \\ Q(y_n) &= Q(v_n) - Q(v) + o_n(1), \\ R(x_n, y_n) &= R(u_n, v_n) - R(u, v) + o_n(1). \end{aligned}$$

So

$$\begin{aligned} P(x_n) &= R(x_n, y_n) + o_n(1), \\ Q(y_n) &= R(x_n, y_n) + o_n(1), \\ I_{0,0}(x_n, y_n) &= c - I(u, v) + o_n(1). \end{aligned}$$

Let l be the common limit of $P(x_n)$, $Q(y_n)$ and $R(x_n, y_n)$. If $l \neq 0$, we get

$$\begin{aligned} I_{0,0}(s_0(x_n, y_n)x_n, t_0(x_n, y_n)y_n) &= \frac{N}{p^*}K(x_n, y_n), \\ &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w), \end{aligned}$$

where

$$K(x_n, y_n) = \left[\frac{P(x_n)^{\frac{\alpha+1}{p}} P(y_n)^{\frac{\beta+1}{p}}}{R(x_n, y_n)} \right]^{\frac{N}{p}}.$$

Direct computations show that

$$K(x_n, y_n) \rightarrow l,$$

so

$$\lim_{n \rightarrow \infty} I_{0,0}(s_0(x_n, y_n)x_n, t_0(x_n, y_n)y_n) = \frac{N}{p^*}l.$$

On the other hand we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_{0,0}(x_n, y_n) &= l \left(\frac{\alpha+1}{p} + \frac{\beta+1}{p} - 1 \right), \\ &= \frac{N}{p^*}l. \end{aligned}$$

Hence, we obtain

$$\frac{N}{p^*}l = c - I(u, v).$$

Using the lemma 3.3.3 , we have

$$\begin{aligned} c &= \frac{N}{p^*}l + I(u, v), \\ &\geq \frac{N}{p^*}l - K\lambda^{\frac{p}{p-p_1}}, \end{aligned}$$

which cannot hold true, and $l = 0$. □

Lemma 3.3.5. *Let $p = q > 1$. There exists $v \in W^{1,p}(\mathbb{R}^N)$ and $\lambda^* > 0$ such that for $(\lambda, \mu) \in (0, \lambda^*) \times (0, +\infty)$, we have*

$$\sup_{s \geq 0} I_{\lambda, \mu}(sv, sv) < \frac{N}{p^*}l - K\lambda^{\frac{p}{p-p_1}}.$$

In particular,

$$\alpha_2(\lambda, \mu) < \frac{N}{p^*}l - K\lambda^{\frac{p}{p-p_1}}.$$

Proof. Let's consider the following family of functions given by

$$w_\varepsilon = C_N \varepsilon^{(N-p)/p^2} (\varepsilon + |x|^{p'})^{(p-N)/p}$$

which attains the best constant S_p of the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ in a neighborhood of the origin. We define $u_\varepsilon(x) = \phi(x)w_\varepsilon(x)$. Taking $v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{p^*}}$ and using the following estimates

$$P(v_\varepsilon) = \begin{cases} \frac{S_p}{2^{p/N}} - C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) & \text{if } N \geq p^2, \\ \frac{S_p}{2^{p/N}} - C\varepsilon^{(p-1)/p}f(\varepsilon) + O(\varepsilon^{(N-p)/p}) & \text{if } N < p^2, \end{cases}$$

where C is a positive constant and $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = +\infty$. Let $\delta_2 > 0$ be such that

$$\frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}} > 0, \quad \forall \lambda \in (0, \delta_2).$$

$$\begin{aligned} I_{\lambda,\mu}(sv_\varepsilon, sv_\varepsilon) &= \frac{s^p}{p}(p^*P(v_\varepsilon) - (\beta+1)\mu Q_b(v_\varepsilon)) - \frac{\alpha+1}{p_1}\lambda s^{p_1}P_{1,a}(v_\varepsilon) - s^{p^*}P_*(v_\varepsilon), \\ &\leq \frac{s^p p^*}{p}P(v_\varepsilon) - \frac{\alpha+1}{p_1}\lambda s^{p_1}P_{1,a}(v_\varepsilon) - s^{p^*}P_*(v_\varepsilon), \\ &\leq \frac{s^p p^*}{p}P(v_\varepsilon) - \frac{\alpha+1}{p_1}\lambda s^{p_1}P_{1,a}(v_\varepsilon) - s^{p^*} \equiv J(s, v_\varepsilon). \end{aligned}$$

As the function $s \mapsto J(s, v_\varepsilon)$ is continuous, $\lim_{s \rightarrow +\infty} J(s, v_\varepsilon) = -\infty$, and

$$\sup_{s \geq 0} \left\{ \frac{s^p p^*}{p}P(v_\varepsilon) - \frac{\alpha+1}{p_1}\lambda s^{p_1}P_{1,a}(v_\varepsilon) - s^{p^*} \right\} > 0,$$

then there exists $s_0 \in (0, +\infty)$ such that :

$$\sup_{0 \leq s \leq s_0} J(s, v_\varepsilon) < \frac{N}{p^*}S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}, \quad \forall \lambda \in (0, \delta_2).$$

If $N \geq p^2$, we have

$$\begin{aligned} J(s, v_\varepsilon) &\leq P(v_\varepsilon)s^p - s^{p^*} - \frac{\alpha+1}{p_1}\lambda s_0^{p_1}P_{1,a}(v_\varepsilon), \\ &\leq \frac{S_p}{2^{\frac{p}{N}}}s^p - s^{p^*} - C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) \\ &\quad - \frac{\alpha+1}{p_1}\lambda s_0^{p_1}P_{1,a}(v_\varepsilon). \end{aligned}$$

Therefore, for all $\lambda \in (0, \delta_3)$

$$\sup_{s \geq s_0} J(s, v_\varepsilon) \leq \frac{N}{2p^*}S_p^{\frac{N}{p}} - C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) - K\lambda^{\frac{p}{p-p_1}},$$

where $\delta_3 = \left(\frac{(\alpha+1)s_0^{p_1} P_{1,\alpha}(v_\varepsilon)}{2Kp_1} \right)^{\frac{p-p_1}{p_1}}$.

As

$$\frac{N-p}{p} - \frac{p-1}{p} \geq \frac{(p-1)^2}{p} > 0,$$

one can fix $\varepsilon > 0$ such that

$$-C\varepsilon^{(p-1)/p} + o(\varepsilon^{(p-1)/p}) + O(\varepsilon^{(N-p)/p}) < 0.$$

If we set $\lambda^* = \min\{\delta_2, \delta_3\}$, we obtain

$$\sup_{s \geq 0} I_{\lambda,\mu}(sv_\varepsilon, sv_\varepsilon) \leq \sup_{s \geq 0} J(s, v_\varepsilon) < \frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}, \quad \forall \lambda \in (0, \lambda^*),$$

and finally

$$\alpha_2(\lambda, \mu) < \frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}, \quad \forall \lambda \in (0, \lambda^*).$$

The case $N < p^2$ can be proved by following the same steps. □

Theorem 3.3.3. *If $p = q > 1$ and $(\lambda, \mu) \in \mathcal{D}_+ \equiv \mathcal{D} \cap ((0, \lambda^*) \times (0, +\infty))$. Then Problem (1.1) has at least two nonnegative solutions.*

Proof. The first solution (U^1, V^1) corresponding to the level $\alpha_1(\lambda, \mu)$ has been proved in the above. Now, to obtain the second solution, we take the minimizing sequence $(U_n^2, V_n^2) \equiv (s_2(u_n, v_n, \lambda, \mu)u_n, t_2(u_n, v_n, \lambda, \mu)v_n)$ such that

$$I(U_n^2, V_n^2) \rightarrow \alpha_2(\lambda, \mu), \quad \|I'(U_n^2, V_n^2)\|_* \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Notice that If $(\lambda, \mu) \in \mathcal{D}_+$, one has

$$\alpha_2(\lambda, \mu) < \frac{N}{p^*} S_p^{\frac{N}{p}} - K\lambda^{\frac{p}{p-p_1}}.$$

Then, we can extract a subsequence of (U_n^2, V_n^2) , still denoted (U_n^2, V_n^2) , and two nonnegative and nontrivial functions belonging to W such that

$$\begin{aligned} U_n^2 &\rightarrow U^2 \text{ in } W^{1,p}(\Omega), \text{ as } n \rightarrow +\infty \\ V_n^2 &\rightarrow V^2 \text{ in } W^{1,q}(\Omega), \text{ as } n \rightarrow +\infty. \end{aligned}$$

So (U^2, V^2) is a solution of Problem (1.1) satisfying

$$\partial_{ss} \tilde{I}(U^2, V^2) < 0 \text{ and } \partial_{ss} \tilde{I}(U^1, V^1) > 0,$$

which imply $(U^1, V^1) \neq (U^2, V^2)$. □

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Chapitre 4

On local compactness in quasilinear elliptic problems

Abstract

One of the major difficulties in nonlinear elliptic problems involving critical nonlinearities is the compactness of Palais-Smale sequences. In their celebrated work [7], Brézis and Nirenberg introduced the notion of critical level for these sequences in the case of a critical perturbation of the Laplacian homogeneous eigenvalue problem. In this paper, we give a natural and general formula of the critical level for a large class of nonlinear elliptic critical problems. The sharpness of our formula is established by the construction of suitable Palais-Smale sequences which are not relatively compact.

4.1 Introduction

In nonlinear elliptic variational problems involving critical nonlinearities, one of the major difficulties is to recover the compactness of Palais-Smale sequences of the associated Euler-Lagrange functional. Such questions were first studied, in our knowledge, by Brézis and Nirenberg in their well-known work [7]. The concentration-compactness principle due to Lions [12] is widely used to overcome these difficulties. Other methods, based on the convergence almost everywhere of the gradients of Palais-Smale sequences, can be also used to recover the compactness. We refer the reader to the papers by Boccardo and Murat [5] and by J. M. Rakotoson [14] for bounded domains. For arbitrary domains, we refer to the recent work by A. El Hamidi and J. M. Rakotoson [9].

In [7], the authors studied the critical perturbation of the eigenvalue problem :

$$\begin{cases} -\Delta u &= \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary, $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent of the embedding $W^{1,2}(\Omega) \subset L^p(\Omega)$, and λ is a positive parameter. The authors introduced an important condition on the level corresponding to the energy of Palais-Smale sequences which guarantees their relative compactness. Indeed, let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega |u|^2 - \frac{1}{2^*} \int_\Omega |u|^{2^*}.$$

More precisely, the authors showed that if

$$\lim_{n \rightarrow +\infty} I_\lambda(u_n) < \frac{1}{N} S^{\frac{N}{2}} \tag{1.2}$$

then (u_n) est relatively compact, which implies the existence of nontrivial critical points of I_λ . Here, S denotes the best Sobolev constant in the embedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. In this work, we begin by giving the generalization of condition (1.2) for the quasilinear equation

$$\begin{aligned} -\Delta_p u &= \lambda f(x, u) + |u|^{p^*-2} u \text{ in } \Omega, \\ u|_\Gamma &= 0 \text{ and } \frac{\partial u}{\partial \nu}|_\Sigma = 0, \end{aligned} \tag{1.3}$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$, where Γ and Σ are smooth $(N - 1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Δ_p is the p -Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. Here, f is a subcritical perturbation of $|u|^{p^*-1}$.

The sharpness of our result is established by the construction of suitable Palais-Smale sequences (corresponding to the critical level) which are not relatively compact. Then we give the analogous condition to (1.2) for a general system with critical exponents

$$\begin{cases} -\Delta_p u &= \lambda f(x, u) + u|u|^{\alpha-1}|v|^{\beta+1} \text{ in } \Omega \\ -\Delta_q v &= \mu g(x, v) + |u|^{\alpha+1}|v|^{\beta-1}v \text{ in } \Omega \end{cases}$$

together with Dirichlet or mixed boundary conditions, where f and g are subcritical perturbations of $|u|^{p^*-1}$ and $|v|^{q^*-1}$ respectively, $p^* = \frac{Np}{N-p}$ (resp. $q^* = \frac{Nq}{N-q}$) is the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subset L^r(\Omega)$ (resp. $W^{1,q}(\Omega) \subset L^r(\Omega)$). Our approach provides a general condition based on the Nehari manifold, which can be extended to a large class of critical nonlinear problems. In this work, we confine ourselves to systems involving (p, q) -Laplacian operators and critical nonlinearities. The sharpness of our result is established, in the special case $p = q$, by the construction of suitable Palais-Smale sequences which are not relatively compact. The question of sharpness corresponding to the case $p \neq q$ is still open.

For a more complete description of nonlinear elliptic systems, we refer the reader to the papers by De Figueiredo [10] and by De Figueiredo & Felmer [11] and the references therein.

4.2 A general local compactness result

For the reader's convenience, we start with the scalar case and to render the paper self-contained we will recall or show some well-known facts.

4.2.1 The scalar case

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary $\partial\Omega$. Let $f(x, u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function which is measurable in x , continuous in u and satisfying the growth condition at infinity

$$|f(x, u)| = o(u^{p^*-1}) \text{ as } u \rightarrow +\infty, \text{ uniformly in } x. \quad (2.4)$$

This situation occurs, for example, in the special cases $f(x, u) = u$ or $f(x, u) = u^{q-1}$, $1 < q < p^*$.

Consider the problem

$$\begin{aligned} -\Delta_p u &= \lambda f(x, u) + |u|^{p^*-2}u \text{ in } \Omega, \\ u|_\Gamma &= 0 \text{ and } \frac{\partial u}{\partial \nu}|_\Sigma = 0, \end{aligned} \quad (2.5)$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$, where Γ and Σ are smooth $(N-1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Problem (2.5) is posed in the framework of the Sobolev space

$$W_\Gamma^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_\Gamma = 0\},$$

which is the closure of $C_0^1(\Omega \cap \Gamma, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$. Notice that $meas(\Gamma) > 0$ implies that the Poincaré inequality is still available in $W_\Gamma^{1,p}(\Omega)$, so it can be endowed with the norm

$$\|u\| = \|\nabla u\|_p$$

and $(W_\Gamma^{1,p}(\Omega), \|\cdot\|)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional is given by

$$J_\lambda(u) := \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p^*} \|u\|_{p^*}^{p^*} - \lambda \int_\Omega F(x, u(x)) dx$$

the corresponding Euler-Lagrange functional, where $F(x, u) := \int_0^u f(x, s) ds$.

We recall here that the Nahari manifold associated to the functional J_λ is given by :

$$\mathcal{N}_{J_\lambda} = \{u \in W_\Gamma^{1,p}(\Omega) \setminus \{0\} : J'_\lambda(u)(u) = 0\},$$

and it is clear that \mathcal{N}_{J_λ} contains all nontrivial critical points of J_λ . This manifold can be characterized more explicitly by the following

$$\mathcal{N}_{J_\lambda} = \left\{ tu, (t, u) \in (\mathbb{R} \setminus \{0\}) \times (W_\Gamma^{1,p}(\Omega) \setminus \{0\}) : \frac{d}{dt} J_\lambda(tu) = 0 \right\},$$

where $t \mapsto J_\lambda(tu)$ is a function defined from \mathbb{R} to itself, for every u given in $W_\Gamma^{1,p}(\Omega) \setminus \{0\}$. We define the critical level associated to Problem (2.5) by :

$$c^*(\lambda) := \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + \inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w). \quad (2.6)$$

At this stage, we can state and show our first result

Theorem 4.2.1. *Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence of J_λ such that*

$$\lim_{n \rightarrow +\infty} J_\lambda(u_n) < c^*(\lambda). \quad (2.7)$$

Then (u_n) is relatively compact.

Proof. Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence for J_λ of level $c \in \mathbb{R}$ ((PS) $_c$ for short) satisfying the condition (2.7). We claim that (u_n) is bounded in $W_\Gamma^{1,p}(\Omega)$. Indeed, on has one hand

$$\frac{1}{p} \|\nabla u_n\|_p^p - \frac{1}{p^*} \|u_n\|_{p^*}^{p^*} - \lambda \int_\Omega F(x, u_n) dx = c + o_n(1), \quad (2.8)$$

and

$$\|\nabla u_n\|_p^p - \|u_n\|_{p^*}^{p^*} - \lambda \int_\Omega f(x, u_n) u_n dx = o_n(\|\nabla u_n\|_p). \quad (2.9)$$

Then,

$$\left(\frac{1}{p} - \frac{1}{p^*}\right) \|u_n\|_{p^*}^{p^*} + \frac{\lambda}{p} \int_\Omega f(x, u_n) u_n dx - \lambda \int_\Omega F(x, u_n) dx = c + o_n(1) + o_n(\|\nabla u_n\|_p).$$

Now, let $\varepsilon > 0$, using the growth condition (2.4), there exists $c_1(\varepsilon) > 0$ such that

$$|f(x, u)| \leq \varepsilon |u|^{p^*-1} + c_1 \quad \text{and} \quad |F(x, u)| \leq \frac{\varepsilon}{p^*} |u|^{p^*} + c_1, \quad \text{a.e. } x \in \Omega \quad \text{and for every } u \in \mathbb{R}.$$

Applying the Hölder and the Young inequalities to the last relations, it follows

$$\|u_n\|_{p^*}^{p^*} \leq \varepsilon \|\nabla u_n\|_p + c_2(|\Omega|, \lambda, \varepsilon). \quad (2.10)$$

Combining (2.10) and (2.8), we deduce that (u_n) is in fact bounded in $W_\Gamma^{1,p}(\Omega)$. So passing, if necessary to a subsequence, we can consider that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_\Gamma^{1,p}(\Omega), \\ u_n &\rightarrow u \text{ a.e. in } \Omega. \end{aligned}$$

On the other hand, the growth condition (2.4) implies also that, for almost every $x \in \Omega$, the functions $s \mapsto F(x, s)$ and $s \mapsto sf(x, s)$ satisfy the conditions of the Brézis-Lieb Lemma (see Theorem 2 in [6]). Thus, we get the identities

$$\begin{aligned} \int_\Omega F(x, v_n) dx &= \int_\Omega F(x, u_n) - \int_\Omega F(x, u) + o_n(1), \\ \int_\Omega f(x, v_n) v_n dx &= \int_\Omega f(x, u_n) u_n - \int_\Omega f(x, u) u + o_n(1). \end{aligned}$$

Moreover, let $\varepsilon > 0$, there is $c_1(\varepsilon) > 0$ such that

$$\left| \int_{\Omega} f(x, v_n) v_n dx \right| \leq \varepsilon \|v_n\|_{p^*}^{p^*} + c_1 \|v_n\|_1.$$

Let $C > 0$ (which is independent of n and ε), such that $\|v_n\|_{p^*}^{p^*} \leq C$. Since (v_n) converges strongly to 0 in $L^1(\Omega)$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $\|v_n\|_1 \leq \varepsilon/c_1$, for every $n \geq n_0(\varepsilon)$, and consequently

$$\left| \int_{\Omega} f(x, v_n) v_n dx \right| \leq \varepsilon(1 + C), \quad \forall n \geq n_0(\varepsilon).$$

In the same way, rewriting $F(x, v_n) = \int_0^{v_n} f(x, s) ds$ and using the same arguments as above, we deduce that

$$\int_{\Omega} F(x, v_n) dx = o_n(1) \tag{2.11}$$

$$\int_{\Omega} f(x, v_n) v_n dx = o_n(1). \tag{2.12}$$

Applying once again the Brézis-Lieb Lemma, we conclude that $u \in \mathcal{N}_{J_\lambda} \cup \{0\}$ and

$$\|v_n\|^p - \|v_n\|_{p^*}^{p^*} = o_n(1), \tag{2.13}$$

$$J_0(v_n) := \frac{1}{p} \|v_n\|^p - \frac{1}{p^*} \|v_n\|_{p^*}^{p^*} = c - J_\lambda(u) + o_n(1). \tag{2.14}$$

A direct computation gives

$$\mathcal{N}_{J_0} = \{t_0(u)u : u \in W_\Gamma^{1,p}(\Omega) \setminus \{0\}\},$$

where

$$t_0(u) := \left(\frac{\|u\|^p}{\|u\|_{p^*}^{p^*}} \right)^{\frac{1}{p^*-p}}.$$

Now, let b be the common limit of $\|v_n\|^p$ and $\|v_n\|_{p^*}^{p^*}$. Suppose that $b \neq 0$. On one hand we have

$$\begin{aligned} J_0(t_0(v_n)v_n) &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \left(\frac{\|v_n\|^p}{\|v_n\|_{p^*}^{p^*}} \right)^{\frac{p^*}{p^*-p}} \\ &\geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w). \end{aligned}$$

Then

$$\lim_{n \rightarrow +\infty} J_0(t_0(v_n)v_n) = \frac{b}{N} \geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w).$$

On the other hand, the identity (2.14) leads to

$$\frac{b}{N} = c - J_\lambda(u).$$

It follows then

$$\begin{aligned} c &\geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + J_\lambda(u) \\ &\geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + \inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w), \end{aligned}$$

which contradicts the condition (2.7). This achieves the proof. \square

4.2.2 Sharpness of the critical level formula in the scalar case

To show the sharpness of the critical level formula (2.7), it suffices to carry out a Palais-Smale sequence for J_λ of level $c^*(\lambda)$ which contains no convergent subsequence.

Consider, for a given $\varepsilon > 0$, the extremal function

$$\Phi_\varepsilon(x) = C_N \varepsilon^{\frac{N-p}{p^2}} \left(\varepsilon + |x|^{\frac{p}{p-1}} \right)^{\frac{p-N}{p}} \quad \text{with } C_N := \left(N \left(\frac{N-p}{p-1} \right)^{p-1} \right)^{(N-p)/p^2}$$

which attains the best constant S of the Sobolev embedding

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N).$$

Without loss of generality, we can consider that $0 \in \Sigma$. Moreover, the set $\partial\Omega$ satisfies the following property (see more details in Adimurthi, Pacella and Yadava [1]) :

There exist $\delta > 0$, an open neighborhood \mathcal{V} of 0 and a diffeomorphism $\Psi : B_\delta(0) \rightarrow \mathcal{V}$ which has a jacobian determinant equal to one at 0, with $\Psi(B_\delta^+) = \mathcal{V} \cap \Omega$, where $B_\delta^+ = B_\delta(0) \cap \{x \in \mathbb{R}^N : x_N > 0\}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.

We define the sequence defined by

$$\psi_n(x) := \varphi(x) \Phi_{1/n}(x), \quad \text{for } n \in \mathbb{N}^*. \tag{2.15}$$

It is well known that the sequence $(\psi_n) \subset W_\Gamma^{1,p}(\Omega)$ is a Palais-Smale sequence for J_0 of level $\inf_{w \in \mathcal{N}_{J_0}} J_0(w)$, which satisfies

$$\begin{aligned} \psi_n &\rightarrow 0 \text{ a.e. in } \Omega, \\ \nabla \psi_n &\rightarrow 0 \text{ a.e. in } \Omega, \\ \|\psi_n\|_{p^*}^{p^*} &\rightarrow \left[N \inf_{w \in \mathcal{N}_{J_0}} J_0(w) \right]^{p/N} := \ell \text{ as } n \rightarrow +\infty, \\ \|\nabla \psi_n\|_p^p &\rightarrow \left[N \inf_{w \in \mathcal{N}_{J_0}} J_0(w) \right]^{p/N} := \ell \text{ as } n \rightarrow +\infty. \end{aligned}$$

Now, let (u_n) be a Palais-Smale sequence of J_λ of level $\inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w)$. We will not go into further details concerning which subcritical terms $f(u)$ allow the existence of such sequences, but in the literature, this occurs for various classes of subcritical terms. Applying Theorem 4.2.1, there exists a subsequence, still denoted by (u_n) , which converges to some $u \in W_\Gamma^{1,p}(\Omega)$. Then

$$\begin{aligned} \|u_n + \psi_n\|_{p^*} &\leq C, \\ u_n + \psi_n &\rightarrow u \text{ a.e. in } \Omega, \\ \|\nabla u_n + \nabla \psi_n\|_p &\leq C, \\ \nabla u_n + \nabla \psi_n &\rightarrow \nabla u \text{ a.e. in } \Omega. \end{aligned}$$

where C a positive constant independent of n . We apply the Brézis-Lieb Lemma to the sequence $(u_n + \psi_n)$ and get

$$\|u_n + \psi_n\|_{p^*}^{p^*} = \|(u_n - u) + \psi_n\|_{p^*}^{p^*} + \|u\|_{p^*}^{p^*} + o_n(1).$$

Moreover, one has

$$-\|u_n - u\|_{p^*} + \|\psi_n\|_{p^*} - \ell^{1/p^*} \leq \|(u_n - u) + \psi_n\|_{p^*} - \ell^{1/p^*} \leq \|u_n - u\|_{p^*} + \|\psi_n\|_{p^*} - \ell^{1/p^*}$$

which implies that

$$\|(u_n - u) + \psi_n\|_{p^*} - \ell^{1/p^*} = o_n(1).$$

Therefore, we conclude that

$$\|u_n + \psi_n\|_{p^*}^{p^*} = \|u\|_{p^*}^{p^*} + \ell + o_n(1).$$

The same arguments applied to the sequence $(\nabla u_n + \nabla \psi_n)$ give

$$\|\nabla u_n + \nabla \psi_n\|_p^p = \|\nabla u\|_p^p + \ell + o_n(1).$$

Finally, using the fact that

$$|\psi_n|^{p^*} \xrightarrow{*} \ell \delta_0 \text{ weakly } * \text{ in } \mathcal{M}^+(\Omega) \quad (2.16)$$

$$|\nabla \psi_n|^p \xrightarrow{*} \ell \delta_0 \text{ weakly } * \text{ in } \mathcal{M}^+(\Omega) \quad (2.17)$$

where δ_0 is the Dirac measure concentrated at the origin and $\mathcal{M}^+(\Omega)$ is the space of positive finite measures [20]), we get that the sequence $(u_n + \psi_n)$ is a Palais-Smale sequence of J_λ of level $c^*(\lambda)$.

We hence constructed a Palais-Smale sequence $(u_n + \psi_n)$ of J_λ of level $c^*(\lambda)$ which can not be relatively compact in $W_\Gamma^{1,p}(\Omega)$. This justifies the sharpness of the critical level formula (2.7).

Remark 4.2.1. *If we are interested by the homogeneous Dirichlet conditions, i.e. if $\Sigma = \emptyset$, the same arguments developed above are still valid, it suffices to assume that the origin $0 \in \Omega$ and consider $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.*

4.2.3 The system case

Now, consider the system

$$\begin{cases} -\Delta_p u &= \lambda f(x, u) + u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v &= \mu g(x, v) + |u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \quad (2.18)$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 & \text{and} & \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 & \text{and} & \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \quad (2.19)$$

where, Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary $\partial\Omega = \bar{\Gamma}_i \cup \bar{\Sigma}_i$, where Γ_i and Σ_i are smooth $(N-1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1, 2\}$. Δ_p is the p -Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. Also, it is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions. We assume here that

$$1 < p < N, \quad 1 < q < N, \quad (2.20)$$

and the critical condition

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1. \quad (2.21)$$

Indeed, this condition represents the maximal growth such that the integrability of the product term $|u|^{\alpha+1}|v|^{\beta+1}$ (which will appear in the Euler-Lagrange functional) can be guaranteed by suitable Hölder estimates.

The functions f and g are two caratheodory functions which satisfy the growth conditions

$$|f(x, u)| = o(u^{p^*-1}) \text{ as } u \rightarrow +\infty, \text{ uniformly in } x, \quad (2.22)$$

$$|g(x, v)| = o(v^{q^*-1}) \text{ as } v \rightarrow +\infty, \text{ uniformly in } x. \quad (2.23)$$

Problem (2.18), together with (2.19), is posed in the framework of the Sobolev space $W = W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega)$, where

$$W_{\Gamma_1}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0\}, \quad W_{\Gamma_2}^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_2} = 0\},$$

which are respectively the closure of $C_0^1(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C_0^1(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. Notice that $meas(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W_{\Gamma_1}^{1,p}(\Omega)$ and $W_{\Gamma_2}^{1,q}(\Omega)$, so W can be endowed with the norm

$$\|(u, v)\| = \|\nabla u\|_p + \|\nabla v\|_q$$

and $(W, \|\cdot\|)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional $I_{\lambda,\mu} \in C^1(W, \mathbb{R})$ is given by

$$I_{\lambda,\mu}(u, v) = (\alpha+1) \left(\frac{P(u)}{p} - \lambda \int_{\Omega} F(x, u) \right) + (\beta+1) \left(\frac{Q(v)}{q} - \mu \int_{\Omega} G(x, v) \right) - R(u, v),$$

where $P(u) = \|\nabla u\|_p^p$, $Q(v) = \|\nabla v\|_q^q$, $F(x, u) = \int_0^u f(x, s) ds$, $G(x, v) = \int_0^v g(x, t) dt$, and $R(u, v) = \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} dx$. Notice that $R(u, v) \leq \|u\|_{p^*}^{\alpha+1} \|v\|_{q^*}^{\beta+1} < +\infty$. Consider the Nehari manifold associated to Problem (2.18) given by

$$\mathcal{N}_{\lambda,\mu} = \{(u, v) \in W \setminus \{(0, 0)\} / D_1 I_{\lambda,\mu}(u, v)(u) = D_2 I_{\lambda,\mu}(u, v)(v) = 0\},$$

where $D_1 I_{\lambda,\mu}$ and $D_2 I_{\lambda,\mu}$ are the derivative of $I_{\lambda,\mu}$ with respect to the first variable and the second variable respectively.

An interesting and useful characterization of $\mathcal{N}_{\lambda,\mu}$ is the following

$$\mathcal{N}_{\lambda,\mu} = \{(su, tv) / (s, u, t, v) \in \mathcal{Z}^* \text{ and } \partial_s I_{\lambda,\mu}(su, tv) = \partial_t I_{\lambda,\mu}(su, tv) = 0\},$$

where

$$\mathcal{Z}^* = \{(s, u, t, v); (s, t) \in \mathbb{R}^2, (u, v) \in W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega), (su, tv) \neq (0, 0)\}$$

and $I_{\lambda,\mu}$ is considered as a functional of four variables (s, u, t, v) in $\mathcal{Z} := \mathbb{R} \times W_{\Gamma_1}^{1,p}(\Omega) \times \mathbb{R} \times W_{\Gamma_2}^{1,q}(\Omega)$.

Definition 4.2.1. *Let λ and μ be two real parameters. A sequence $(u_n, v_n) \in W$ is a Palais-Smale sequence of the functional $I_{\lambda,\mu}$ if*

- *there exists $c \in \mathbb{R}$ such that $\lim_{n \rightarrow +\infty} I_{\lambda,\mu}(u_n, v_n) = c$* (2.24)

- *$DI_{\lambda,\mu}(u_n, v_n)$ converges strongly in the dual W' of W* (2.25)

where $DI_{\lambda,\mu}(u_n, v_n)$ denotes the Gâteaux derivative of $I_{\lambda,\mu}$.

The last condition (2.25) implies that

$$D_1 I_{\lambda,\mu}(u_n, v_n)(u_n) = o(\|u_n\|_{p^*}) \tag{2.26}$$

$$D_2 I_{\lambda,\mu}(u_n, v_n)(v_n) = o(\|v_n\|_{q^*}). \tag{2.27}$$

where $D_1 I_{\lambda,\mu}(u_n, v_n)$ (resp. $D_2 I_{\lambda,\mu}(u_n, v_n)$) denotes the Gâteaux derivative of $I_{\lambda,\mu}$ with respect to its first (resp. second) variable.

We introduce the critical level corresponding to Problem (2.18) by

$$c^*(\lambda, \mu) := \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w \in \mathcal{N}_{\lambda,\mu} \cup \{(0,0)\}} I_{\lambda,\mu}(w). \tag{2.28}$$

Then we have the following

Theorem 4.2.2. *Let λ and μ be two real parameters and (u_n, v_n) be a Palais-Smale sequence of $I_{\lambda, \mu}$ such that*

$$c := \lim_{n \rightarrow +\infty} I_{\lambda, \mu}(u_n, v_n) < c^*(\lambda, \mu). \quad (2.29)$$

Then (u_n, v_n) relatively compact.

Proof. Let λ and μ be two real parameters and (u_n, v_n) be a Palais-Smale sequence of $I_{\lambda, \mu}$ satisfying the condition (2.29). We claim that (u_n, v_n) is bounded in W . Indeed, on one hand conditions (2.24), (2.26) and (2.27) can be rewritten as the following

$$I_{\lambda, \mu}(u_n, v_n) = c + o_n(1) \quad (2.30)$$

$$P(u_n) - \lambda \int_{\Omega} f(x, u_n) u_n dx = R(u_n, v_n) + o(\|u_n\|_{p^*}) \quad (2.31)$$

$$Q(v_n) - \mu \int_{\Omega} f(x, v_n) v_n dx = R(u_n, v_n) + o(\|v_n\|_{q^*}). \quad (2.32)$$

Using (2.21), one gets

$$\begin{aligned} R(u_n, v_n) &= \frac{\alpha + 1}{p^*} \left(P(u_n) - \lambda \int_{\Omega} f(x, u_n) u_n \right) + o(\|u_n\|_{p^*}) \\ &+ \frac{\beta + 1}{q^*} \left(Q(v_n) - \mu \int_{\Omega} g(x, v_n) v_n \right) + o(\|v_n\|_{q^*}). \end{aligned} \quad (2.33)$$

Suppose that there is a subsequence, still denoted by (u_n, v_n) in W which is unbounded, *i.e.* $\|\nabla u_n\|_p + \|\nabla v_n\|_q$ tends to $+\infty$ as n goes to $+\infty$.

If

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_p = +\infty,$$

then using (2.22) one has

$$\begin{aligned} \int_{\Omega} |f(x, u_n) u_n| &= o(P(u_n)), \\ \int_{\Omega} |F(x, u_n)| &= o(P(u_n)), \end{aligned}$$

since (2.22) implies that for every $\varepsilon > 0$, there exists $c_1(\varepsilon) > 0$ such that

$$|f(x, s)| \leq \varepsilon |s|^{p^*-1} + c_1 \quad \text{and} \quad |F(x, s)| \leq \frac{\varepsilon}{p^*} |s|^{p^*} + c_1, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}.$$

Similarly, if

$$\lim_{n \rightarrow +\infty} \|\nabla v_n\|_q = +\infty,$$

then using (2.23) it follows

$$\begin{aligned} \int_{\Omega} |g(x, v_n)v_n| &= o(Q(v_n)), \\ \int_{\Omega} |G(x, v_n)| &= o(Q(v_n)). \end{aligned}$$

On one hand, suppose that

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_p = \lim_{n \rightarrow +\infty} \|\nabla v_n\|_q = +\infty.$$

Substituting (2.33) in (2.30), we obtain

$$\begin{aligned} c + o_n(1) &= (\alpha + 1) \left(\frac{1}{p} - \frac{1}{p^*} + o(P(u_n))^{\frac{p^*-p}{p}} \right) P(u_n) \\ &+ (\beta + 1) \left(\frac{1}{q} - \frac{1}{q^*} + o(Q(v_n))^{\frac{q^*-q}{q}} \right) Q(v_n) \xrightarrow{n \rightarrow +\infty} +\infty \end{aligned}$$

which can not hold true. On the other hand, suppose that

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_p = +\infty \text{ and the sequence } \|\nabla v_n\|_q \text{ is bounded,}$$

then (2.31) implies that $R(u_n, v_n)$ is unbounded while (2.32) implies, on the contrary, that $R(u_n, v_n)$ is bounded. The case

$$\lim_{n \rightarrow +\infty} \|\nabla v_n\|_q = +\infty \text{ and the sequence } \|\nabla u_n\|_p \text{ is bounded,}$$

leads to a contradiction with the same argument, which achieves the claim.

At this stage, we can assume, up to a subsequence, that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\ v_n &\rightharpoonup v \text{ in } W_{\Gamma_2}^{1,q}(\Omega), \\ u_n &\rightarrow u \text{ a.e. in } \Omega, \\ v_n &\rightarrow v \text{ a.e. in } \Omega. \end{aligned}$$

It is clear that

$$(u, v) \in \mathcal{N}_{\lambda, \mu} \cup \{(0, 0)\}.$$

Let us set

$$X_n = u_n - u \text{ and } Y_n = v_n - v.$$

Using again the growth conditions (2.22) and (2.23), we show easily that the functions, which are defined on $\Omega \times \mathbb{R} : (x, s) \mapsto sf(x, s), (x, s) \mapsto sg(x, s), (x, s) \mapsto$

$F(x, s)$ and $(x, s) \mapsto G(x, s)$ satisfy the conditions of the Brézis-Lieb lemma [6]. Then, we have the decompositions

$$\begin{aligned} \int_{\Omega} F(x, X_n) &= \int_{\Omega} F(x, u_n) - \int_{\Omega} F(x, u) + o_n(1), \\ \int_{\Omega} f(x, X_n)X_n &= \int_{\Omega} f(x, u_n)u_n - \int_{\Omega} f(x, u)u + o_n(1), \\ \int_{\Omega} G(x, Y_n) &= \int_{\Omega} G(x, v_n) - \int_{\Omega} G(x, v) + o_n(1), \\ \int_{\Omega} g(x, Y_n)Y_n &= \int_{\Omega} g(x, v_n)v_n - \int_{\Omega} g(x, v)v + o_n(1). \end{aligned}$$

Moreover, let $\varepsilon > 0$, then there is $c_1(\varepsilon) > 0$ such that

$$\left| \int_{\Omega} f(x, X_n)X_n dx \right| \leq \varepsilon \|X_n\|_{p^*}^{p^*} + c_1 \|X_n\|_1.$$

Let C be a positive constant such that $\|X_n\|_{p^*}^{p^*} \leq C$. Since X_n converges to 0 in $L^1(\Omega)$, there exists $n_0(\varepsilon) \in \mathbb{N}$ verifying $\|X_n\|_1 \leq \varepsilon/c_1$, for every $n \geq n_0(\varepsilon)$, thus

$$\left| \int_{\Omega} f(x, X_n)X_n dx \right| \leq \varepsilon(1 + C), \quad \forall n \geq n_0(\varepsilon).$$

In the same manner, writing $F(x, X_n) = \int_0^{X_n} f(x, s) ds$ and using the same arguments as above, we get

$$\int_{\Omega} F(x, X_n) = o_n(1) \quad \text{and} \quad \int_{\Omega} f(x, X_n)X_n = o_n(1).$$

Similarly, it follows that

$$\int_{\Omega} G(x, Y_n) = o_n(1) \quad \text{and} \quad \int_{\Omega} g(x, Y_n)Y_n = o_n(1).$$

Applying a slightly modified version of the Brézis-Lieb lemma [13], one has

$$R(X_n, Y_n) = R(u_n, v_n) - R(u, v) + o_n(1).$$

It follows that

$$\begin{aligned} P(X_n) - R(X_n, Y_n) &= o_n(1), \\ Q(Y_n) - R(X_n, Y_n) &= o_n(1), \\ I_{0,0}(X_n, Y_n) &= c - I_{\lambda,\mu}(u, v) + o_n(1). \end{aligned}$$

Notice that the Nehari manifold associated to $I_{0,0}$ is given by

$$\mathcal{N}_{0,0} = \{(s_0(u, v)u, t_0(u, v)v); (u, v) \in W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega), u \not\equiv 0, v \not\equiv 0\},$$

where

$$s_0(u, v) = \left[\frac{P(u)Q(v)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{R(u, v)^{\frac{r}{\alpha+1}}} \right]^{\frac{1}{r-p}}, \quad t_0(u, v) = t(s_0(u, v)),$$

and

$$r = \frac{(\alpha+1)q}{q - (\beta+1)} > p, \quad t(s) = \left[\frac{R(u, v)}{Q(v)} \right]^{\frac{r}{q(\alpha+1)}} s^{\frac{r}{q}}.$$

Let ℓ be the common limit of $P(X_n)$, $Q(Y_n)$ and $R(X_n, Y_n)$. We claim that $\ell = 0$. By contradiction, suppose that $\ell \neq 0$, then on one hand we get

$$\begin{aligned} I_{0,0}(s_0(X_n, Y_n)X_n, t_0(X_n, Y_n)Y_n) &= (\alpha+1) \left(\frac{1}{p} - \frac{1}{r} \right) K(X_n, Y_n), \quad (2.34) \\ &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w), \end{aligned}$$

where

$$K(X_n, Y_n) = \left[\frac{P(X_n)^{(\alpha+1)} Q(Y_n)^{\frac{(\beta+1)q}{r}}}{R(X_n, Y_n)^p} \right]^{\frac{r}{(\alpha+1)(r-p)}}.$$

A direct computation shows that

$$\lim_{n \rightarrow +\infty} K(X_n, Y_n) = \ell,$$

therefore

$$\lim_{n \rightarrow +\infty} I_{0,0}(s_0(X_n, Y_n)X_n, t_0(X_n, Y_n)Y_n) = \ell(\alpha+1) \left(\frac{1}{p} - \frac{1}{r} \right).$$

On the other hand,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_{0,0}(X_n, Y_n) &= \ell \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 \right) \\ &= \ell(\alpha+1) \left(\frac{1}{p} - \frac{1}{r} \right). \end{aligned}$$

Hence, we obtain

$$\ell(\alpha+1) \left(\frac{1}{p} - \frac{1}{r} \right) = c - I_{\lambda,\mu}(u, v),$$

and consequently

$$\begin{aligned} c &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) + I_{\lambda,\mu}(u, v) \\ &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w \in \mathcal{N}_{\lambda,\mu} \cup \{(0,0)\}} I_{\lambda,\mu}(w). \end{aligned}$$

This leads to a contradiction with (2.29), then $\ell = 0$, which achieves the proof. \square

Remark 4.2.2. 1) In the scalar case, we obtain the analogous of Theorem 4.2.2, the proof follows easily with the same arguments. We note here that if we consider the special case (1.1), direct computations show that

$$\inf_{w \in \mathcal{N}_0} I_0(w) = \frac{1}{N} S^{\frac{N}{2}} \quad \text{and} \quad \inf_{w \in \mathcal{N}_\lambda \cup \{0\}} I_\lambda(w) = 0,$$

which recovers the famous Brézis-Nirenberg condition (1.2).

2) It is clear that our condition (2.7) or (2.29) can be extended to a large class of quasilinear or semilinear differential operators : Leray-Lions type operators, fourth-order operators.

3) Using the Hölder inequality in the denominator $R(u, v)$, we get

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) \geq (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) \left[S_p S_q^{\frac{p(\beta+1)}{q(\alpha+1)}} \right]^{\frac{r}{r-p}}, \quad (2.35)$$

where S_p (resp. S_q) denotes the best Sobolev constant in the embedding $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ (resp. $W_{\Gamma_2}^{1,q}(\Omega) \subset L^{q^*}(\Omega)$).

We end this note by the following interesting relation arising in the special case $p = q$ and $\Gamma_1 = \Gamma_2$.

Proposition 4.2.1. Assume that $p = q > 1$. Then,

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) = \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

Proof. In the special case $p = q$, direct computations give

$$p^* = \alpha + \beta + 2 \quad \text{and} \quad (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) = \frac{p}{N-p}.$$

Then, using (2.35), we conclude that

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) \geq \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

On the other hand, let $(u_n) \subset W_{\Gamma_1}^{1,p}(\Omega)$ be a minimizing sequence of S_p . Then using the identity (2.34), we get

$$\begin{aligned} \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) &\leq I_{0,0}(s_0(u_n, u_n)u_n, t_0(u_n, u_n)u_n) = \frac{p}{N-p} \left[\frac{\|\nabla u_n\|_p^p}{\|u_n\|_{p^*}^p} \right]^{\frac{rp^*}{(\alpha+1)(r-p)}} \\ &= \frac{p}{N-p} \left[\frac{\|\nabla u_n\|_p^p}{\|u_n\|_{p^*}^p} \right]^{\frac{N}{p}}. \end{aligned}$$

It is clear that the last quantity goes to $\frac{p}{N-p} S_p^{\frac{N}{p}}$ as $n \rightarrow \infty$, which achieves the proof. \square

Remark 4.2.3. *For the sharpness of the critical level (2.29), we define the sequence $\psi_n(x) := \varphi(x)\Phi_{1/n}(x)$ as in (2.15). We consider then a Palais-Smale sequence (u_n, v_n) for $J_{\lambda, \mu}$ of level $\inf_{w \in \mathcal{N}_{\lambda, \mu} \cup \{(0,0)\}} I_{\lambda, \mu}(w)$. Following the same arguments developed in the scalar case and using Proposition 4.2.1, we prove that the sequence $(u_n + \psi_n, v_n + \psi_n)$ is a Palais-Smale sequence for $J_{\lambda, \mu}$ of level $c^*(\lambda, \mu)$ and which can not be relatively compact in W . This implies the sharpness of the critical level formula (2.29).*

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Chapitre 5

Existence and Regularity Results for an anisotropic system involving critical exponents

Abstract

In this paper, we establish some existence and regularity results of positive solutions of a critical anisotropic system by using variational methods

$$(P_{\lambda,\mu}) \left\{ \begin{array}{l} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda a(x) |u|^{p-2} u + u |u|^{\alpha-1} |v|^{\beta+1} \text{ in } \Omega, \\ - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \right) = \mu b(x) |v|^{q-2} v + |u|^{\alpha+1} |v|^{\beta-1} \text{ in } \Omega, \\ u \geq 0 \text{ and } v \geq 0 \text{ in } \Omega, \\ u = 0 \text{ and } v = 0 \text{ on } \partial\Omega \end{array} \right.$$

where Ω is a bounded open domain of \mathbb{R}^N , λ and μ are positive parameters, p^* and q^* are respectively the critical exponents for these classes of problem. The functions a and b belong to spaces which will be specified later.

5.1 Introduction

In this paper, we are interested in existence results of nonlocal solutions to the following critical system :

$$\left\{ \begin{array}{l} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda a(x) |u|^{p-2} u + u |u|^{\alpha-1} |v|^{\beta+1} \text{ in } \Omega, \\ - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \right) = \mu b(x) |v|^{q-2} v + |u|^{\alpha+1} |v|^{\beta-1} \text{ in } \Omega, \\ u \geq 0 \text{ and } v \geq 0 \text{ in } \Omega, \\ u = 0 \text{ and } v = 0 \text{ on } \partial\Omega \end{array} \right. \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N , $\lambda \geq 0$, $\mu \geq 0$ are real parameters and the exponents p_i , α , q_i , β satisfy the following conditions

$$p_i > 1, q_i > 1, \sum_{i=1}^N \frac{1}{p_i} > 1, \sum_{i=1}^N \frac{1}{q_i} > 1$$

and

$$\frac{\alpha + 1}{p^*} + \frac{\beta + 1}{q^*} = 1,$$

where p^* and q^* are defined by

$$p^* := \frac{N}{\sum_{i=1}^N \frac{1}{p_i} - 1} \text{ and } q^* := \frac{N}{\sum_{i=1}^N \frac{1}{q_i} - 1}.$$

We assume in the sequel that

$$\max\{p_1, p_2, \dots, p_N\} < p^* \text{ and } \max\{q_1, q_2, \dots, q_N\} < q^*,$$

p^* and q^* are the effective critical exponents associated to the operators

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial}{\partial x_i} \right|^{p_i-2} \frac{\partial}{\partial x_i} \right) \quad \text{and} \quad \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial}{\partial x_i} \right|^{q_i-2} \frac{\partial}{\partial x_i} \right),$$

respectively [8, 7].

The functions a and b are assumed to be nontrivial, nonnegative, $a \in L^{\frac{p^*}{p^*-p}}(\Omega)$ and $b \in L^{\frac{q^*}{q^*-q}}(\Omega)$.

In this work, we deal with the nonlocal existence, with respect to λ and μ , of nonnegative, nontrivial solutions to Problem (1.1). Consider the Euler-Lagrange functional associated to Problem (1.1) defined by

$$I(u, v) := (\alpha + 1) \left(\sum_{i=1}^N \frac{P_i(u)}{p_i} - \frac{\lambda}{p} P_a(u) \right) + (\beta + 1) \left(\sum_{i=1}^N \frac{Q_i(v)}{q_i} - \frac{\mu}{q} Q_b(v) \right) - R(u, v),$$

where $P_i(u) := \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx$, $P_a(u) := \int_{\Omega} a(x)|u|^p dx$, $Q_i(v) := \int_{\Omega} \left| \frac{\partial v}{\partial x_i} \right|^{q_i} dx$,
 $Q_b(v) := \int_{\Omega} b(x)|v|^q dx$ and $R(u, v) := \int_{\Omega} |u|^{\alpha+1}|v|^{\beta+1} dx$.

The functional I is of class $C^1(W; \mathbb{R})$, where $W := W_0^{1, \vec{p}}(\Omega) \times W_0^{1, \vec{q}}(\Omega)$, $W_0^{1, \vec{p}}(\Omega)$ and $W_0^{1, \vec{q}}(\Omega)$ are respectively the completions of the space $\mathcal{D}(\Omega)$ with respect of the norms :

$$\|u\|_{\vec{p}} := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i} \quad \text{and} \quad \|v\|_{\vec{q}} := \sum_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{q_i}.$$

The spaces $W_0^{1, \vec{p}}(\Omega)$ and $W_0^{1, \vec{q}}(\Omega)$ can also be seen as

$$W_0^{1, \vec{p}}(\Omega) = \left\{ u \in L^{p^+}(\Omega) \quad : \quad \left| \frac{\partial u}{\partial x_i} \right| \in L^{p_i}(\Omega), \quad i = 1, \dots, N, \quad u|_{\partial\Omega} = 0, \right\},$$

$$W_0^{1, \vec{q}}(\Omega) = \left\{ v \in L^{q^+}(\Omega) \quad : \quad \left| \frac{\partial v}{\partial x_i} \right| \in L^{q_i}(\Omega), \quad i = 1, \dots, N, \quad v|_{\partial\Omega} = 0, \right\},$$

where $\vec{p} := (p_1, p_2, \dots, p_N)$ and $\vec{q} := (q_1, q_2, \dots, q_N)$, the space W is endowed with norm

$$\|(u, v)\| := \|u\|_{\vec{p}} + \|v\|_{\vec{q}}.$$

We introduce the modified Euler-Lagrange functional \tilde{I} defined on $\mathcal{Z} := \mathbb{R} \times W_0^{1, \vec{p}}(\Omega) \times \mathbb{R} \times W_0^{1, \vec{q}}(\Omega)$ by

$$\tilde{I}(s, u, t, v) := I(su, tv).$$

In the sequel, we set $p_- = \min\{p_1, p_2, \dots, p_N\} = p_{i_0}$, $p_+ = \max\{p_1, p_2, \dots, p_N\} = p_{i_1}$, $q_- = \min\{q_1, q_2, \dots, q_N\} = q_{j_0}$, $q_+ = \max\{q_1, q_2, \dots, q_N\} = q_{j_1}$, $P_-(u) = P_{i_0}(u)$, $P_+(u) = P_{i_1}(u)$, $Q_-(v) = Q_{j_0}(v)$ and $Q_+(v) = Q_{j_1}(v)$.

5.2 Preliminary results

Under the following assumptions

$$\begin{cases} p < p_- \leq p_+ < \alpha + 1, \\ q < q_- \leq q_+ < \beta + 1, \\ \lambda > 0, \\ \mu > 0. \end{cases} \quad (2.2)$$

We have the following lemmas

Lemma 5.2.1. *There exist $\lambda^* > 0$, $\mu^* > 0$ and $r > 0$, $\rho > 0$ such that*

$$I(u, v) \geq r, \quad \forall (\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*), \quad \forall (u, v) \in W, \quad \text{such that} \quad \|(u, v)\| = \rho.$$

Proof. Let $(u, v) \in W$ such that

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{p_i} \leq 1 \quad \text{and} \quad \left\| \frac{\partial v}{\partial x_i} \right\|_{q_i} \leq 1, \quad \forall i \in \{1, \dots, N\}.$$

Let $\varepsilon > 0$ fixed, by Young's inequality, there exists a positive constant C_ε such that

$$R(u, v) \leq (\alpha + 1)\varepsilon \|u\|_{p^*}^{\alpha+1} + (\beta + 1)C_\varepsilon \|v\|_{q^*}^{\beta+1}.$$

Since

$$I(u, v) = (\alpha + 1) \left(\sum_{i=1}^N \frac{P_i(u)}{p_i} - \lambda \frac{P_a(u)}{p} \right) + (\beta + 1) \left(\sum_{i=1}^N \frac{Q_i(v)}{q_i} - \mu \frac{Q_b(v)}{q} \right) - R(u, v)$$

it follows that

$$\begin{aligned} I(u, v) &\geq (\alpha + 1) \left(\left(\frac{1}{p_1} + \dots + \frac{1}{p_N} \right) \sum_{i=1}^N P_i(u)^{p_+/p_i} - \lambda \frac{P_a(u)}{p} - \varepsilon \|u\|_{p^*}^{\alpha+1} \right) \\ &\quad + (\beta + 1) \left(\left(\frac{1}{q_1} + \dots + \frac{1}{q_N} \right) \sum_{i=1}^N Q_i(v)^{q_+/q_i} - \mu \frac{Q_b(v)}{q} - C_\varepsilon \|v\|_{q^*}^{\beta+1} \right). \end{aligned}$$

Now, we calculate, for a fixed $j \in \{1, \dots, N\}$

$$\begin{aligned} \frac{1}{p^*} - \frac{1}{p_j^*} &= \frac{1}{N} \left(\sum_{i=1}^N \frac{1}{p_i} - 1 \right) - \left(\frac{1}{p_j} - \frac{1}{N} \right) \\ &= \sum_{\substack{i=1 \\ i \neq j}}^N \frac{1}{p_i} > 0, \end{aligned}$$

then $p^* < p_j^*$ for all $j \in \{1, \dots, N\}$, where $p_j^* := p_j N / (N - p_j)$ is the critical exponent of the compact embedding $W^{1,p_j}(\Omega) \hookrightarrow L^r(\Omega)$. As $p^* < p_j^*$, we use then Sobolev inequalities;

$$\|u\|_{p^*} \leq c_{1,j} \|\nabla u\|_{p_j}$$

and then

$$\|u\|_{p^*} \leq c_1 \|u\|_{\vec{p}}.$$

By the same way

$$\|v\|_{q^*} \leq c_2 \|v\|_{\vec{q}}.$$

Consequently, there exist positive constants h_1, h_2, h_3, k_1, k_2 and k_3 such that

$$I(u, v) \geq \left(h_1 \|u\|_{\vec{p}}^{p^+} - h_2 \lambda \|u\|_{\vec{p}}^p - h_3 \|u\|_{\vec{p}}^{\alpha+1} \right) + \left(k_1 \|v\|_{\vec{q}}^{q^+} - k_2 \mu \|v\|_{\vec{q}}^q - k_3 \|v\|_{\vec{q}}^{\beta+1} \right).$$

Since $1 < p < p^+ < \alpha + 1$, there exist $\lambda^* > 0$, $r_1 > 0$ and $\rho_1 > 0$ and such that

$$h_1 \|u\|_{\vec{p}}^{p^+} - h_2 \lambda \|u\|_{\vec{p}}^p - h_3 \|u\|_{\vec{p}}^{\alpha+1} \geq r_1, \quad \forall u \in W_0^{1,\vec{p}}(\Omega) : \|u\|_{\vec{p}} = \rho_1, \quad \forall \lambda \in (0, \lambda^*).$$

Similarly, Since $1 < q < q^+ < \beta + 1$, there exist $\mu^* > 0$, $r_2 > 0$ and $\rho_2 > 0$ and such that

$$k_1 \|v\|_{\bar{q}}^{q^+} - k_2 \mu \|v\|_{\bar{q}}^q - k_3 \|v\|_{\bar{q}}^{\beta+1} \geq r_2, \quad \forall v \in W_0^{1,\bar{q}}(\Omega) : \|v\|_{\bar{q}} = \rho_2, \quad \forall \mu \in (0, \mu^*).$$

Therefore, for all $(u, v) \in W$ such that $\|(u, v)\| = \rho := \rho_1 + \rho_2$ and for all $(\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*)$, one has

$$I(u, v) \geq r := r_1 + r_2 > 0,$$

which achieves the proof. \square

Lemma 5.2.2. *The functional I is bounded from below in*

$$\bar{B}_\rho(0) = \{(u, v) \in W; \|(u, v)\| \leq \rho\}.$$

Moreover,

$$\inf_{(u,v) \in \bar{B}_\rho(0)} I(u, v) < 0, \quad \forall (\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*). \quad (2.3)$$

Proof. It is easy to check that I is bounded from below in $\bar{B}_\rho(0)$. To prove (2.3), fix $(\phi, \psi) \in \widetilde{W}$ and let $s > 0$ and $t > 0$. Then

$$I(s\phi, t\psi) \leq (\alpha+1) \left(\sum_{i=1}^N s^{p_i} \frac{P_i(\phi)}{p_i} - \lambda s^p \frac{P_a(\phi)}{p} \right) + (\beta+1) \left(\sum_{i=1}^N t^{q_i} \frac{Q_i(\psi)}{q_i} - \mu t^p \frac{Q_b(\psi)}{q} \right).$$

As $p < p_-$ and $q < q_-$, the last inequality implies for s_0 and t_0 sufficiently small

$$I(s_0\phi, t_0\psi) < 0 \quad \text{and} \quad (s_0\phi, t_0\psi) \in \bar{B}_\rho(0)$$

from where follows the lemma. \square

For every $(\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*)$, we introduce

$$\alpha(\lambda, \mu) := \inf_{(u,v) \in \bar{B}_\rho(0)} I(u, v).$$

Applying Ekeland's principle to the functional I on the metric space $(\bar{B}_\rho(0), d)$ endowed with the metric d given by

$$d((u_1, v_1), (u_2, v_2)) = \|(u_1 - u_2, v_1 - v_2)\| = \|u_1 - u_2\|_{\bar{p}} + \|v_1 - v_2\|_{\bar{q}},$$

there exists a sequence $(u_n, v_n) \subset \bar{B}_\rho(0)$ such that

$$I(u_n, v_n) \longrightarrow \alpha(\lambda, \mu) \quad \text{as} \quad n \rightarrow +\infty \quad (2.4)$$

and

$$I(u, v) - I(u_n, v_n) \leq \frac{1}{n} \|(u - u_n, v - v_n)\|, \quad \text{for all} \quad (u, v) \neq (u_n, v_n). \quad (2.5)$$

Using the differentiability of I over W , from the previous inequality it follows that

$$I'(u_n, v_n) \longrightarrow 0, \quad \text{as } n \rightarrow +\infty. \quad (2.6)$$

From (2.5) and (2.6)

$$I(u_n, v_n) \longrightarrow \alpha(\lambda, \mu) \quad \text{and} \quad I'(u_n, v_n) \longrightarrow 0, \quad \text{as } n \rightarrow +\infty, \quad (2.7)$$

then (u_n, v_n) is a bounded $(PS)_{\alpha(\lambda, \mu)}$ sequence to I . Hereafter, we will denote by $(u^*, v^*) \in W$ the weak limit of (u_n, v_n) , up to a subsequence. Moreover, by definition of I we can assume that the sequence (u_n, v_n) is a sequence of nonnegative functions.

Theorem 5.2.1. *The weak limit (u^*, v^*) of (u_n, v_n) satisfies*

$$I'(u^*, v^*) = 0 \quad \text{and} \quad (u^*, v^*) \in \widetilde{W}, \quad \text{for all } (\lambda, \mu) \in (0, \lambda^*) \times (0, \mu^*).$$

Proof. Using the properties of (u_n, v_n) , one can easily show that

$$I(u_n, v_n) - \frac{1}{p^*} D_1 I(u_n, v_n) u_n - \frac{1}{q^*} D_1 I(u_n, v_n) v_n = \alpha(\lambda, \mu) + o_n(1),$$

where $D_1 I(u_n, v_n)$ and $D_2 I(u_n, v_n)$ are respectively the first and second partial Gâteaux derivatives in u and v of the functional I . Thus

$$\begin{aligned} \alpha(\lambda, \mu) + o_n(1) &= (\alpha + 1) \left(\sum_{i=1}^N \left(\frac{1}{p_i} - \frac{1}{p^*} \right) P_i(u_n) - \lambda \left(\frac{1}{p} - \frac{1}{p^*} \right) P_a(u_n) \right) \\ &\quad + (\beta + 1) \left(\sum_{i=1}^N \left(\frac{1}{q_i} - \frac{1}{q^*} \right) Q_i(v_n) - \mu \left(\frac{1}{q} - \frac{1}{q^*} \right) Q_b(v_n) \right). \end{aligned} \quad (2.8)$$

As it is said before, one has

$$\begin{aligned} u_n &\rightharpoonup u^* \quad \text{in } W_0^{1, \vec{p}}(\Omega), \\ u_n &\longrightarrow u^* \quad \text{in } L^p(\Omega), \\ v_n &\rightharpoonup v^* \quad \text{in } W_0^{1, \vec{q}}(\Omega), \\ v_n &\longrightarrow v^* \quad \text{in } L^q(\Omega). \end{aligned}$$

By the relation (2.8) and using the compact embeddings of $W_0^{1, \vec{p}}(\Omega) \hookrightarrow L^p(\Omega)$ and $W_0^{1, \vec{q}}(\Omega) \hookrightarrow L^q(\Omega)$, we obtain

$$\alpha(\lambda, \mu) + o_n(1) \geq -K_1 \|u_n\|_p^p - K_2 \|v_n\|_q^q$$

where K_1 and K_2 are two positive constants. Then we get, as n goes to $+\infty$:

$$0 > \alpha(\lambda, \mu) > -K_1 \|u^*\|_p^p - K_2 \|v^*\|_q^q,$$

and consequently $u^* \neq 0$ and $v^* \neq 0$. Finally, we use the result due to El Hamidi and Rakotoson [6] and obtain

$$\begin{aligned} \nabla u_n(x) &\longrightarrow \nabla u^*(x) \quad \text{a.e. in } \Omega, \\ \nabla v_n(x) &\longrightarrow \nabla v^*(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Therefore, the weak limit (u^*, v^*) is a nonnegative and nontrivial solution to (1.1). \square

Under the following assumptions

$$\left\{ \begin{array}{l} p < \alpha + 1 \\ q < \beta + 1 \\ \max\{p_+, q_+\} < \min\{p, q\} \\ \lambda \geq 0 \\ \mu \geq 0 \end{array} \right. \quad (2.9)$$

We have the following lemmas

Lemma 5.2.3. *For every $\lambda > 0$ and $\mu > 0$, the functional I satisfies the following properties :*

a) *There exist $r > 0$ and $\rho > 0$ such that*

$$I(u, v) \geq r, \quad \forall (u, v) \in W, \quad \text{such that } \|(u, v)\| = \rho.$$

b) *There exists $(e_{\lambda, \mu}, f_{\lambda, \mu}) \in W$ such that $\|(e_{\lambda, \mu}, f_{\lambda, \mu})\| \geq \rho$ such that*

$$I(e_{\lambda, \mu}, f_{\lambda, \mu}) < 0.$$

Proof.

a) Using the same arguments in the proof of Lemma 5.2.1, we have for every $(u, v) \in W$ and $\|(u, v)\| \leq 1$, there exist positive constants h_1, h_2, h_3, k_1, k_2 and k_3 such that

$$I(u, v) \geq \left(h_1 \|u\|_{\bar{p}}^{p_+} - h_2 \lambda \|u\|_{\bar{p}}^p - h_3 \|u\|_{\bar{p}}^{\alpha+1} \right) + \left(k_1 \|v\|_{\bar{q}}^{q_+} - k_2 \mu \|v\|_{\bar{q}}^q - k_3 \|v\|_{\bar{q}}^{\alpha+1} \right).$$

As $p_+ < p < \alpha + 1$ and $q_+ < q < \beta + 1$, there exist $r_1, r_2, \rho_1 > 0$ and $\rho_2 > 0$ such that

$$h_1 \|u\|_{\bar{p}}^{p_+} - h_2 \lambda \|u\|_{\bar{p}}^p - h_3 \|u\|_{\bar{p}}^{\alpha+1} \geq r_1, \quad \text{for all } \|u\|_{\bar{p}} \leq \rho_1$$

and

$$k_1 \|v\|_{\bar{q}}^{q_+} - k_2 \mu \|v\|_{\bar{q}}^q - k_3 \|v\|_{\bar{q}}^{\alpha+1} \geq r_2, \quad \text{for all } \|v\|_{\bar{q}} \leq \rho_2.$$

Let us introduce $\rho := \min\{\rho_1, \rho_2\}$, so if $\|(u, v)\| = \rho$ then we have either $\|u\|_{\bar{p}} \leq \rho \leq \rho_1$ and $\|v\|_{\bar{q}} \leq \rho \leq \rho_2$. Consequently, if $\|(u, v)\| = \rho$ we have $I(u, v) \geq r := r_1 + r_2$.

b) Let $(\varphi, \psi) \in W$ such that $\varphi \neq 0$, $\psi \neq 0$ and $R(\varphi, \psi) \neq 0$ then

$$\begin{aligned} I(s\varphi, s\psi) &= (\alpha + 1) \left(\sum_{i=1}^N \frac{s^{p_i}}{p_i} P_i(\varphi) - \lambda \frac{s^p}{p} P_a(\varphi) \right) - s^{\alpha+\beta+2} R(\varphi, \psi) \\ &+ (\beta + 1) \left(\sum_{i=1}^N \frac{s^{q_i}}{q_i} Q_i(\psi) - \mu \frac{s^q}{q} Q_b(\psi) \right). \end{aligned}$$

We know that $\alpha + \beta + 2 > \max\{p_+, q_+\}$, then for s sufficiently large, we have

$$I(s\varphi, s\psi) < 0.$$

By a lemma in [6], the equation $\frac{\partial I}{\partial s}(su, sv) = 0$ has only one solution $s_{\lambda, \mu} > 0$ such that $\frac{\partial I}{\partial s}(s_{\lambda, \mu}u, s_{\lambda, \mu}v) = 0$. and $s_{\lambda, \mu}$ satisfies the condition

$$I(s_{\lambda, \mu}\varphi, s_{\lambda, \mu}\psi) = \max_{s \geq 0} I(s\varphi, s\psi).$$

By considering $(u_{\lambda, \mu}, v_{\lambda, \mu}) := (s_{\lambda, \mu}\varphi, s_{\lambda, \mu}\psi)$ we have the following results :

- (i) $I(u_{\lambda, \mu}, v_{\lambda, \mu}) = \max_{s \geq 0} I(su_{\lambda, \mu}, sv_{\lambda, \mu})$
- (ii) $I(u_{\lambda, \mu}, v_{\lambda, \mu}) \longrightarrow 0$ as $\lambda \rightarrow +\infty$ or $\mu \rightarrow +\infty$.

Let us show (i). The function $I(su_{\lambda, \mu}, sv_{\lambda, \mu})$ has a unique maximum value at some point $s_0 > 0$. Since $u_{\lambda, \mu} \neq 0$ and $v_{\lambda, \mu} \neq 0$, we have

$$\frac{(\alpha + 1)\lambda s_0^p P_a(u_{\lambda, \mu}) + (\beta + 1)\mu s_0^q Q_b(v_{\lambda, \mu}) + (\alpha + \beta + 2)s_0^{\alpha+\beta+2} R(u_{\lambda, \mu}, v_{\lambda, \mu})}{\sum_{i=1}^N [(\alpha + 1)P_i(u_{\lambda, \mu})s_0^{p_i} + (\beta + 1)Q_i(v_{\lambda, \mu})s_0^{q_i}]} = 1.$$

On the other hand $(u_{\lambda, \mu}, v_{\lambda, \mu})$ satisfies

$$\frac{(\alpha + 1)\lambda P_a(u_{\lambda, \mu}) + (\beta + 1)\mu Q_b(v_{\lambda, \mu}) + (\alpha + \beta + 2)R(u_{\lambda, \mu}, v_{\lambda, \mu})}{\sum_{i=1}^N [(\alpha + 1)P_i(u_{\lambda, \mu}) + (\beta + 1)Q_i(v_{\lambda, \mu})]} = 1.$$

We study the variation of the function

$$\theta : s \mapsto \frac{(\alpha + 1)\lambda s^p P_a(u_{\lambda, \mu}) + (\beta + 1)\mu s^q Q_b(v_{\lambda, \mu}) + (\alpha + \beta + 2)s^{\alpha+\beta+2} R(u_{\lambda, \mu}, v_{\lambda, \mu})}{\sum_{i=1}^N [(\alpha + 1)P_i(u_{\lambda, \mu})s^{p_i} + (\beta + 1)Q_i(v_{\lambda, \mu})s^{q_i}]}$$

The function θ is of the form $\theta(s) = \frac{s^a}{Bs^b + Cs^c}$ with $B > 0$, $C > 0$, $a > b > 0$ and $a > c > 0$. For every $s > 0$ we have

$$\theta'(s) = \frac{B(a-b)s^{a+b-1} + C(a-c)s^{a+c-1}}{(Bs^b + Cs^c)^2} > 0,$$

then the function θ is increasing on $(0, +\infty)$. Therefore it follows that $s_0 = 1$. We show the claim (ii), notice that

$$\frac{s_{\lambda,\mu}^p P_a(\varphi)}{\sum_{i=1}^N [(\alpha+1)P_i(\varphi)s_{\lambda,\mu}^{p_i} + (\beta+1)Q_i(\psi)s_{\lambda,\mu}^{q_i}]} \leq \frac{1}{\lambda},$$

and

$$\frac{s_{\lambda,\mu}^q Q_b(\psi)}{\sum_{i=1}^N [(\alpha+1)P_i(\varphi)s_{\lambda,\mu}^{p_i} + (\beta+1)Q_i(\psi)s_{\lambda,\mu}^{q_i}]} \leq \frac{1}{\mu}.$$

Thus,

$$\frac{s_{\lambda,\mu}^p P_a(\varphi)}{\sum_{i=1}^N [(\alpha+1)P_i(\varphi)s_{\lambda,\mu}^{p_i} + (\beta+1)Q_i(\psi)s_{\lambda,\mu}^{q_i}]} \longrightarrow 0 \quad \text{as } \lambda \longrightarrow +\infty$$

and

$$\frac{s_{\lambda,\mu}^q Q_b(\psi)}{\sum_{i=1}^N [(\alpha+1)P_i(\varphi)s_{\lambda,\mu}^{p_i} + (\beta+1)Q_i(\psi)s_{\lambda,\mu}^{q_i}]} \longrightarrow 0 \quad \text{as } \mu \longrightarrow +\infty,$$

as the function θ satisfies : $\theta(0) = 0$, θ is continuous on \mathbb{R}^+ and increasing then $\theta(s) \rightarrow 0 \iff s \rightarrow 0$. Thus, we have either

$$s_{\lambda,\mu} \longrightarrow 0 \quad \text{as } \lambda \rightarrow +\infty \quad \text{or} \quad \mu \rightarrow +\infty.$$

Recalling that

$$0 \leq I(u_{\lambda,\mu}, v_{\lambda,\mu}) \leq (\alpha+1) \sum_{i=1}^N \frac{s_{\lambda,\mu}^{p_i}}{p_i} P_i(\varphi) + (\beta+1) \frac{s_{\lambda,\mu}^{q_i}}{q_i} Q_i(\psi) - s_{\lambda,\mu}^{\alpha+\beta+2} R(\varphi, \psi)$$

it follows that

$$I(u_{\lambda,\mu}, v_{\lambda,\mu}) \longrightarrow 0 \quad \text{as } \lambda \rightarrow +\infty \quad \text{or} \quad \mu \rightarrow +\infty,$$

which proves the claim. □

Hereafter, we fix $(e_{\lambda,\mu}, f_{\lambda,\mu})$ with $e_{\lambda,\mu} = s_{\lambda,\mu}^* u_{\lambda,\mu}$ and $f_{\lambda,\mu} = s_{\lambda,\mu}^* v_{\lambda,\mu}$ such that

$$\|(e_{\lambda,\mu}, f_{\lambda,\mu})\| \geq r \quad \text{and} \quad I(e_{\lambda,\mu}, f_{\lambda,\mu}) < 0.$$

Lemma 5.2.4. *If $c_{\lambda,\mu}$ is the minimax value obtained by the Mountain Pass Theorem applied to the functional I , then we get*

$$c_{\lambda,\mu} \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty \quad \text{or} \quad \mu \longrightarrow +\infty.$$

Proof. The maximum value $c_{\lambda,\mu}$ is given by

$$c_{\lambda,\mu} = \inf_{(\gamma,\xi) \in \Gamma} \max_{s \in [0,1]} I(\gamma(s), \xi(s)),$$

where the set of all paths linking $(0, 0)$ and $(e_{\lambda,\mu}, f_{\lambda,\mu})$ is defined by

$$\Gamma = \left\{ (\gamma, \xi) \in \mathcal{C}(W; \mathbb{R}), \quad (\gamma(0), \xi(0)) = (0, 0) \quad \text{and} \quad (\gamma(1), \xi(1)) = (e_{\lambda,\mu}, f_{\lambda,\mu}) \right\}.$$

Let's consider $\gamma(s) = se_{\lambda,\mu}$ and $\xi(s) = sf_{\lambda,\mu}$, so then $(\gamma, \xi) \in \Gamma$ and

$$\max_{s \in [0,1]} I(\gamma(s), \xi(s)) = \max_{s \geq 0} I(su_{\lambda,\mu}, sv_{\lambda,\mu}) = I(u_{\lambda,\mu}, v_{\lambda,\mu})$$

then

$$0 \leq c_{\lambda,\mu} \leq I(u_{\lambda,\mu}, v_{\lambda,\mu}),$$

from where

$$c_{\lambda,\mu} \longrightarrow 0 \quad \text{as} \quad \lambda \longrightarrow +\infty \quad \text{or} \quad \mu \longrightarrow +\infty,$$

which ends the proof. □

Hereafter, we shall denote by $S_{\vec{p}} > 0$ and $S_{\vec{q}} > 0$ the positive constants, see [7] :

$$S_{\vec{p}} = \inf_{u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N), \|u\|_{p^*} = 1} \left\{ \sum_{i=1}^N \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} \right\}$$

and

$$S_{\vec{q}} = \inf_{v \in \mathcal{D}^{1,\vec{q}}(\mathbb{R}^N), \|v\|_{q^*} = 1} \left\{ \sum_{i=1}^N \frac{1}{q_i} \left\| \frac{\partial v}{\partial x_i} \right\|_{q_i}^{q_i} \right\}.$$

The following lemma is an immediate consequence of the last one, that is,

Lemma 5.2.5. *There exists $\lambda^* > 0$ and $\mu^* > 0$ such that for all $(\lambda, \mu) \in [\lambda^*, +\infty[\times [\mu^*, +\infty[$ we have*

$$0 < c_{\lambda, \mu} < d_1 \min \left\{ \left(S_{\bar{p}}^{\frac{\alpha+1}{p_+}} S_{\bar{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{d_1}}, \left(S_{\bar{p}}^{\frac{\alpha+1}{p_+}} S_{\bar{q}}^{\frac{\beta+1}{q_-}} \right)^{\frac{1}{d_2}}, \left(S_{\bar{p}}^{\frac{\alpha+1}{p_-}} S_{\bar{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{d_3}}, \left(S_{\bar{p}}^{\frac{\alpha+1}{p_-}} S_{\bar{q}}^{\frac{\beta+1}{q_-}} \right)^{\frac{1}{d_4}} \right\}$$

where $d_1 = \frac{\alpha+1}{p_+} + \frac{\beta+1}{q_+} - 1$, $d_2 = \frac{\alpha+1}{p_+} + \frac{\beta+1}{q_-} - 1$, $d_3 = \frac{\alpha+1}{p_-} + \frac{\beta+1}{q_+} - 1$ and $d_4 = \frac{\alpha+1}{p_-} + \frac{\beta+1}{q_-} - 1$.

Related to the Mountain Pass level $c_{\lambda, \mu}$, there exists a sequence $(u_n, v_n) \subset W$ satisfying

$$I(u_n, v_n) \longrightarrow c_{\lambda, \mu} \quad \text{and} \quad I'(u_n, v_n) \longrightarrow 0 \quad \text{in} \quad W^*.$$

Using standard arguments, we have that (u_n, v_n) is bounded in W , hence we can assume that there exists $(u, v) \in W$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1, \bar{p}}(\Omega)$$

and

$$v_n \rightharpoonup v \quad \text{in} \quad W_0^{1, \bar{q}}(\Omega).$$

Lemma 5.2.6. *The weak limit (u, v) is such that $u \neq 0$ and $v \neq 0$.*

Proof. By applying the result of A. El Hamidi and Rakotoson in [5], we prove that we can extract a subsequence of (u_n, v_n) , still denoted (u_n, v_n) , such that

$$\nabla u_n(x) \longrightarrow \nabla u(x) \quad \text{a.e. in} \quad \Omega,$$

and

$$\nabla v_n(x) \longrightarrow \nabla v(x) \quad \text{a.e. in} \quad \Omega,$$

then $I'(u, v) = 0$.

To prove now that $u \neq 0$ and $v \neq 0$, we assume by contradiction that $u = 0$ and we set $x_n := u_n$ and $y_n := v_n - v$. Using similar arguments as above, we get

$$\begin{cases} \sum_{i=1}^N P_i(u_n) - \lambda P_a(u_n) = R(u_n, v_n) + o_n(1) \\ \sum_{i=1}^N Q_i(v_n) - \mu Q_b(v_n) = R(u_n, v_n) + o_n(1). \end{cases} \quad (2.10)$$

Using Brézis-Lieb Lemma [3], we obtain

$$\sum_{i=1}^N P_i(x_n) = R(x_n, y_n) + o_n(1), \quad (2.11)$$

$$\sum_{i=1}^N Q_i(y_n) = R(x_n, y_n) + o_n(1), \quad (2.12)$$

$$I_{0,0}(x_n, y_n) = c_{\lambda, \mu} - I(0, v) + o_n(1). \quad (2.13)$$

On the other hand, we have

$$I(0, v) = (\beta + 1) \left(\sum_{i=1}^N \frac{1}{q_i} Q_i(v) - \mu \frac{1}{q} Q_b(v) \right),$$

with

$$\sum_{i=1}^N Q_i(v) - \mu Q_b(v) = 0.$$

Therefore

$$I(0, v) = (\beta + 1) \sum_{i=1}^N \left(\frac{1}{q_i} - \frac{1}{q} \right) Q_i(v),$$

and consequently $I(0, v) \geq 0$, since $q_+ < q$. Now, if $\lim_{n \rightarrow +\infty} R(x_n, y_n) = L$ then it follows by (2.11) and (2.12) that

$$\sum_{i=1}^N P_i(x_n) \longrightarrow L \quad \text{and} \quad \sum_{i=1}^N Q_i(y_n) \longrightarrow L.$$

Then

$$I_{0,0}(x_n, y_n) = (\alpha + 1) \sum_{i=1}^N \left(\frac{1}{p_i} - \frac{1}{p^*} \right) P_i(x_n) + (\beta + 1) \sum_{i=1}^N \left(\frac{1}{q_i} - \frac{1}{q^*} \right) Q_i(y_n) + o_n(1),$$

and

$$\lim_{n \rightarrow +\infty} I_{0,0}(x_n, y_n) \geq \left[(\alpha + 1) \left(\frac{1}{p_+} - \frac{1}{p^*} \right) + (\beta + 1) \left(\frac{1}{q_+} - \frac{1}{q^*} \right) \right] L.$$

By the relation (2.13) we get

$$\begin{aligned} c_{\lambda, \mu} &= \lim_{n \rightarrow +\infty} I_{0,0}(x_n, y_n) + I(0, v) \\ &\geq \lim_{n \rightarrow +\infty} I_{0,0}(x_n, y_n) \\ &\geq \left[(\alpha + 1) \left(\frac{1}{p_+} - \frac{1}{p^*} \right) + (\beta + 1) \left(\frac{1}{q_+} - \frac{1}{q^*} \right) \right] L \\ &= \left(\frac{\alpha + 1}{p_+} + \frac{\beta + 1}{q_+} - 1 \right) L. \end{aligned}$$

We use the same arguments as in [2] and [7], to prove that we have either

$$S_{\bar{p}} \|x_n\|_{p^*}^{p_+} \leq \sum_{i=1}^N \left\| \frac{\partial x_n}{\partial x_i} \right\|_{p_i}^{p_i} \quad \text{if} \quad \|x_n\|_{p^*} \leq 1 \quad (2.14)$$

$$S_{\bar{q}} \|y_n\|_{q^*}^{q_+} \leq \sum_{i=1}^N \left\| \frac{\partial y_n}{\partial x_i} \right\|_{q_i}^{q_i} \quad \text{if} \quad \|y_n\|_{p^*} \leq 1 \quad (2.15)$$

or

$$S_{\bar{p}} \|x_n\|_{p^*}^{p^-} \leq \sum_{i=1}^N \left\| \frac{\partial x_n}{\partial x_i} \right\|_{p_i}^{p_i} \quad \text{if } \|x_n\|_{p^*} \geq 1$$

$$S_{\bar{q}} \|y_n\|_{q^*}^{q^-} \leq \sum_{i=1}^N \left\| \frac{\partial y_n}{\partial x_i} \right\|_{q_i}^{q_i} \quad \text{if } \|y_n\|_{p^*} \geq 1.$$

From the previous inequalities, if $\|x_n\|_{p^*} \leq 1$ and $\|y_n\|_{p^*} \leq 1$ we obtain

$$R(x_n, y_n) \leq \|x_n\|_{p^*}^{\alpha+1} \|y_n\|_{q^*}^{\beta+1}$$

$$\leq \left(\frac{1}{S_{\bar{p}}} \sum_{i=1}^N P_i(x_n) \right)^{\frac{\alpha+1}{p_+}} \left(\frac{1}{S_{\bar{q}}} \sum_{i=1}^N Q_i(y_n) \right)^{\frac{\beta+1}{q_+}}$$

as $n \rightarrow +\infty$ we have

$$L \geq \left(S_{\bar{p}}^{\frac{\alpha+1}{p_+}} S_{\bar{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{\frac{\alpha+1}{p_+} + \frac{\beta+1}{q_+} - 1}}$$

and

$$c_{\lambda, \mu} \geq \left(\frac{\alpha+1}{p_+} + \frac{\beta+1}{q_+} - 1 \right) \left(S_{\bar{p}}^{\frac{\alpha+1}{p_+}} S_{\bar{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{\frac{\alpha+1}{p_+} + \frac{\beta+1}{q_+} - 1}}.$$

Then

$$c_{\lambda, \mu} \geq d_1 \min \left\{ \left(S_{\bar{p}}^{\frac{\alpha+1}{p_+}} S_{\bar{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{d_1}}, \left(S_{\bar{p}}^{\frac{\alpha+1}{p_+}} S_{\bar{q}}^{\frac{\beta+1}{q_-}} \right)^{\frac{1}{d_2}}, \left(S_{\bar{p}}^{\frac{\alpha+1}{p_-}} S_{\bar{q}}^{\frac{\beta+1}{q_+}} \right)^{\frac{1}{d_3}}, \left(S_{\bar{p}}^{\frac{\alpha+1}{p_-}} S_{\bar{q}}^{\frac{\beta+1}{q_-}} \right)^{\frac{1}{d_4}} \right\},$$

which cannot hold true. Thus $L = 0$ and $c_{\lambda, \mu} = I(0, v) > 0$, in this case $\lim_{\lambda \rightarrow +\infty} c_{\lambda, \mu} = I(0, v) > 0$ which leads to a contradiction with the fact that $\lim_{\lambda \rightarrow +\infty} c_{\lambda, \mu} = 0$. Then $u \neq 0$ and $v \neq 0$. \square

5.3 Regularity of Weak Solutions

In this section, we show that every weak solution $(u, v) \in W_0^{1, \bar{p}}(\Omega) \times W_0^{1, \bar{q}}(\Omega)$ of the following problem

$$(P) \quad \begin{cases} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = f(x, u, v) & \text{in } \Omega, \\ - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \right) = g(x, u, v) & \text{in } \Omega, \\ u \geq 0 \quad \text{and} \quad v \geq 0 & \text{in } \Omega, \\ u = 0 \quad \text{and} \quad v = 0 & \text{on } \partial\Omega \end{cases}$$

is a *strong solution*, under some hypothesis on the functions f and g .

Lemma 5.3.1. *Suppose that the functions f and g satisfy the following growth conditions :*

(h₁) *There exist $c_1, c_2 \geq 0$ and $p \in (1, p^*)$ such that*

$$|f(x, u, v)| \leq c_1 u^{p-1} + c_2 u^\alpha v^{\beta+1}, \quad \forall u \geq 0 \quad \text{and} \quad v \geq 0,$$

(h₂) *There exist $c'_1, c'_2 \geq 0$ and $q \in (1, p^*)$ such that*

$$|g(x, u, v)| \leq c'_1 v^{q-1} + c'_2 u^{\alpha+1} v^\beta, \quad \forall u \geq 0 \quad \text{and} \quad v \geq 0.$$

Then every weak solution $(u, v) \in W_0^{1, \vec{p}}(\Omega) \times W_0^{1, \vec{q}}(\Omega)$ of $(P_{\lambda, \mu})$ belongs to $L^r(\Omega) \times L^s(\Omega)$ for all $1 \leq r < +\infty$ and $1 \leq s < +\infty$.

Proof. We will use similar arguments developed by Fragala, Gazzola and Kawohl [8].

Let (u, v) be a weak solution to (P) . The assertion that $(u, v) \in L^r(\Omega) \times L^s(\Omega)$ for all $1 \leq r < +\infty$ and $1 \leq s < +\infty$ may be equivalently reformulated as

$$(u, v) \in L^{(1+a)p^*}(\Omega) \times L^{(1+b)q^*}(\Omega) \quad \text{for all } (a, b) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+. \quad (3.16)$$

To prove (3.16) it is enough to show that $(u^{a+1}, v^{b+1}) \in W_0^{1, \vec{p}}(\Omega) \times W_0^{1, \vec{q}}(\Omega)$, which is equivalent to

$$\lim_{L \rightarrow +\infty} \sum_{i=1}^N \left(\int_{\Omega} |\partial_{x_i}(u \min[u^a, L]|^{p_i} dx \right)^{\frac{1}{p_i}} < +\infty, \quad (3.17)$$

$$\lim_{L \rightarrow +\infty} \sum_{i=1}^N \left(\int_{\Omega} |\partial_{x_i}(v \min[v^b, L]|^{q_i} dx \right)^{\frac{1}{q_i}} < +\infty. \quad (3.18)$$

For each L there exist indexes j and k such that

$$\sum_{i=1}^N \left(\int_{\Omega} |\partial_i(u \cdot \min[u^a, L]|^{p_i} dx \right)^{\frac{1}{p_i}} \leq C \left(\int_{\Omega} |\partial_j(u \cdot \min[u^a, L]|^{p_j} dx \right)^{\frac{1}{p_j}}, \quad (3.19)$$

$$\sum_{i=1}^N \left(\int_{\Omega} |\partial_i(v \cdot \min[v^b, L]|^{q_i} dx \right)^{\frac{1}{q_i}} \leq C' \left(\int_{\Omega} |\partial_k(v \cdot \min[v^b, L]|^{q_k} dx \right)^{\frac{1}{q_k}} \quad (3.20)$$

where C and C' are positive constants independent of L .

For these indexes j and k , and for every $L > 0$, set $\phi_L := \min[u^{ap_j}, L^{p_j}]$ and $\psi_L := v \min[v^{bq_k}, L^{q_k}]$ such that $(\phi_L, \psi_L) \in W_0^{1, \vec{p}}(\Omega) \times W_0^{1, \vec{q}}(\Omega)$. Note that for every $1 \leq i \leq N$ and for almost every $x \in \Omega$,

$$|\partial_i u|^{p_i-2} \partial_i u \partial_i \phi_L \geq \min[u^{ap_j}, L^{p_j}] |\partial_i u|^{p_i},$$

$$|\partial_i v|^{q_i-2} \partial_i v \partial_i \psi_L \geq \min[v^{bq_k}, L^{q_k}] |\partial_i v|^{q_i},$$

$$|\partial_i(u \cdot \min[u^a, L])|^{p_i} \leq (a+1) \min[u^{ap_i}, L^{p_i}] |\partial_i u|^{p_i}, \quad (3.21)$$

and

$$|\partial_i(v \cdot \min[v^b, L])|^{q_i} \leq (b+1) \min[v^{bq_i}, L^{q_i}] |\partial_i v|^{q_i}. \quad (3.22)$$

As (u, v) is a weak solution to (P) , we rewrite, for every $k \geq 1$, $\Omega \equiv \Omega_{1,k} \cup \Omega_{2,k} \cup \Omega_{3,k} \cup \Omega_{4,k}$, where

$$\Omega_{1,k} := \{x \in \Omega \text{ such that } u < k \text{ and } v < k\},$$

$$\Omega_{2,k} := \{x \in \Omega \text{ such that } u < k \text{ and } v \geq k\},$$

$$\Omega_{3,k} := \{x \in \Omega \text{ such that } u \geq k \text{ and } v < k\},$$

$$\Omega_{4,k} := \{x \in \Omega \text{ such that } u \geq k \text{ and } v \geq k\}.$$

Then we have

$$\sum_{i=1}^N \int_{\Omega} \min[u^{ap_j}, L^{p_j}] |\partial_i u|^{p_i} dx \leq \sum_{i=1}^N \int_{\Omega} |\partial_i u|^{p_i-2} \partial_i u \partial_i \phi_L dx = \int_{\Omega} f(x, u, v) \phi_L dx,$$

and

$$\sum_{i=1}^N \int_{\Omega} \min[v^{bq_k}, L^{q_k}] |\partial_i v|^{q_i} dx \leq \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{q_i-2} \partial_i v \partial_i \psi_L dx = \int_{\Omega} g(x, u, v) \psi_L dx.$$

Using $(h1)$, for $L \geq k^a \geq 1$, it follows that

$$|f(x, u, v)| \phi_L \leq \tilde{C}_{k,1} \text{ in } \Omega_1 \quad (3.23)$$

$$|g(x, u, v)| \psi_L \leq \tilde{C}_{k,2} \text{ in } \Omega_1 \quad (3.24)$$

hence

$$\int_{\Omega_1} f(x, u, v) \phi_L dx \leq C_{k,1} = \tilde{C}_{k,1} |\Omega|,$$

$$\begin{aligned} \int_{\Omega_{2,k}} f(x, u, v) \phi_L dx &\leq \int_{\Omega_2} [c_1 u^p + c_2 u^{\alpha+1} v^{\beta+1}] \min[u^{ap_j}, L^{p_j}] dx, \\ &\leq C_{2,k} + c_2 \int_{\Omega_2} u^{\alpha+1} v^{\beta+1} \min[u^{ap_j}, L^{p_j}] dx, \\ &\leq C_{2,k} + c_2 \left(\int_{\Omega_{2,k}} (u^{\alpha+1} \min[u^{ap_j}, L^{p_j}])^{p^*/(\alpha+1)} dx \right)^{\frac{\alpha+1}{p^*}} \times \\ &\quad \times \left(\int_{\Omega_{2,k}} v^{q^*} dx \right)^{\frac{\beta+1}{q^*}} \\ &\leq C_{2,k} + \varepsilon_k C'_{2,k} \end{aligned}$$

where ε_k tends to 0 as k tends to $+\infty$.

$$\begin{aligned}
\int_{\Omega_{3,k}} f(x, u, v) \phi_L dx &\leq c_1 \int_{\Omega_{3,k}} u^{p^*} \min[u^{ap_j}, L^{p_j}] dx + \int_{\Omega_{3,k}} u^{\alpha+1} v^{\beta+1} \min[u^{ap_j}, L^{p_j}] dx \\
&\leq c_1 \left(\int_{\Omega_{3,k}} u^{p^*} dx \right)^{\frac{p^*-p_j}{p^*}} \cdot \int_{\Omega_{3,k}} (u^{p_j} \cdot \min[u^{ap_j}, L^{p_j}])^{p^*/p_j} dx \\
&+ c_2 \int_{\Omega_{3,k}} u^{p^*} \min[u^{ap_j}, L^{p_j}] v^{\beta+1} dx \\
&\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{p_j/p^*} \\
&+ c_2 \left(\int_{\Omega_{3,k}} u^{p^*-p_j} u^{p_j} \cdot (\min[u^{ap_j}, L^{p_j}])^{p^*/(\alpha+1)} dx \right)^{\frac{\alpha+1}{p^*}} \\
&\quad \times \left(\int_{\Omega_{3,k}} v^{q^*} dx \right)^{\frac{\beta+1}{q^*}} \\
&\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{p_j/p^*} + c_2 \|u\|_{p^*, \Omega_{3,k}}^{(\alpha+1)(p^*-p_j)/p^*} \|v\|_{q^*, \Omega_{3,k}}^{\beta+1} \times \\
&\quad \times \left(\int_{\Omega_{3,k}} (u^{p_j} \cdot (\min[u^{ap_j}, L^{p_j}])^{p^*/(\alpha+1)})^{\frac{p^*}{p_j}} dx \right)^{\frac{(\alpha+1)p_j}{p^*2}} \\
&\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{p_j/p^*} + \varepsilon''_k \|v\|_{q^*}^{\beta+1} \times \\
&\quad \times \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{\frac{(\alpha+1)p_j}{p^*2}} \\
&\leq \varepsilon'_k \left[\sum_{i=1}^N \left(\int_{\Omega} |\partial_i (u \cdot \min[u^a, L])|^{p_i} dx \right)^{1/p_i} \right]^{p_j} \\
&+ \varepsilon''_k \left[\sum_{i=1}^N \left(\int_{\Omega} |\partial_i (u \cdot \min[u^a, L])|^{p_i} dx \right)^{\frac{1}{p_i}} \right]^{(\alpha+1)p_i/p^*}
\end{aligned}$$

where ε'_k and ε''_k tend to 0 as k goes to $+\infty$.

$$\begin{aligned} \int_{\Omega_{4,k}} f(x, u, v) \phi_L dx &\leq \int_{\Omega_{4,k}} [c_1 u^p + c_2 u^{\alpha+1} v^{\beta+1}] \min[u^{ap_j}, L^{p_j}] dx \\ &\leq \varepsilon'_k \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{p_j/p^*} + c_2 \|u\|_{p^*, \Omega_{4,k}}^{(\alpha+1)(p^*-p_j)/p^*} \times \\ &\quad \times \|v\|_{q^*, \Omega_{4,k}}^{\beta+1} \left(\int_{\Omega_{3,k}} (u^{p_j} \cdot (\min[u^{ap_j}, L^{p_j}])^{p^*/(\alpha+1)})^{\frac{p^*}{p_j}} dx \right)^{\frac{(\alpha+1)p_j}{p^*2}} \\ &\leq \varepsilon_k^1 \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{p_j/p^*} + \\ &\quad + \varepsilon_k^2 \left(\int_{\Omega} (u \cdot \min[u^a, L])^{p^*} dx \right)^{\frac{(\alpha+1)p_j}{p^*2}} \\ &\leq \varepsilon_k^1 \left[\sum_{i=1}^N \left(\int_{\Omega} |\partial_i(u \cdot \min[u^a, L])|^{p_i} dx \right)^{1/p_i} \right]^{p_j} + \\ &\quad + \varepsilon_k^2 \left[\sum_{i=1}^N \left(\int_{\Omega} |\partial_i(u \cdot \min[u^a, L])|^{p_i} dx \right)^{\frac{1}{p_i}} \right]^{(\alpha+1)p_i/p^*} \end{aligned}$$

where ε_k^1 and ε_k^2 tend to 0 as k goes to $+\infty$.

Inserting (3.19) and (3.20) in the last inequalities, we then obtain,

$$\begin{aligned} \int_{\Omega} |\partial_j(u \cdot \min[u^a, L])|^{p_j} dx &\leq C'_k + \varepsilon_{k,1} \int_{\Omega} |\partial_j(u \cdot \min[u^a, L])|^{p_j} dx \\ &\quad + \varepsilon_{k,2} \left[\int_{\Omega} |\partial_i(u \cdot \min[u^a, L])|^{p_i} dx \right]^{(\alpha+1)/p^*}. \end{aligned}$$

Choosing k sufficiently large, that are $\varepsilon_{k,1}$ and $\varepsilon_{k,2}$ sufficiently small, the last inequality ensures that the integral $\int_{\Omega} |\partial_j(u \cdot \min[u^a, L])|^{p_j} dx$ is bounded for L large enough, from where follows (3.17). By the same way we can prove (3.18), and we can conclude then that every weak solution $(u, v) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$ of (P) belongs to $L^r(\Omega) \times L^s(\Omega)$, for all $r \geq 1$ and $s \geq 1$. \square

Proposition 5.3.1. *Under the conditions (h_1) and (h_2) , every nonnegative solution $(u, v) \in W_0^{1,\vec{p}}(\Omega) \times W_0^{1,\vec{q}}(\Omega)$ of (P) belongs to $L^\infty(\Omega) \times L^\infty(\Omega)$.*

Proof. For $u \geq 0$ and $v \geq 0$ such that (u, v) is a solution of (P), we set $A_\tau = \{x \in \Omega, u(x) \geq \tau\}$ and $B_\tau = \{x \in \Omega, v(x) \geq \tau\}$, $|A_\tau|$ and $|B_\tau|$ their Lebesgue measures. Recall that Cavalieri principle, based on Fubini theorem, gives :

$$\int_k^{+\infty} |A_\tau| d\tau = \int_{\Omega} (u - k)_+ dx, \quad \text{for all } k \geq 0$$

and

$$\int_k^{+\infty} |B_\tau| d\tau = \int_\Omega (v - k)_+ dx, \quad \text{for all } k \geq 0.$$

Let $\varphi_k = (u - k)_+$, for $k > 0$ fixed. Choosing this function as a test function, combining the Cavalieri principle and Hölder inequality, one gets

$$\begin{aligned} \sum_{i=1}^N P_i(\varphi_k) &= \int_\Omega (\lambda a u^{p-1} + u^\alpha v^{\beta+1}) \varphi_k dx \\ &\leq c_1 \lambda \int_\Omega |u|^{p-1} \varphi_k dx + c_2 \int_\Omega |u|^\alpha |v|^{\beta+1} \varphi_k dx \\ &\leq c \left(\int_{[u \leq 1]} \varphi_k dx + \int_{[u \leq 1]} |v|^{\beta+1} \varphi_k dx + \int_{[u \geq 1]} |u|^\alpha |v|^{\beta+1} \varphi_k dx \right), \\ &\leq c \left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{(1-\frac{\alpha+1}{p^*})(1-\frac{1}{p^*})} \right) \|\varphi_k\|_{p^*}. \end{aligned} \quad (3.25)$$

Since $\lim_{n \rightarrow +\infty} \|\varphi_k\|_{p^*} = 0$, then for $k \geq k_0 > 0$, $\|\varphi_k\|_{p^*} \leq 1$. Relations (2.14) and (3.25) give

$$\begin{aligned} S_{\bar{p}} \|\varphi_k\|_{p^*}^{p^+} &\leq \sum_{i=1}^N P_i(\varphi_k) \\ &\leq c \left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{(1-\frac{\alpha+1}{p^*})(1-\frac{1}{p^*})} \right) \|\varphi_k\|_{p^*}. \end{aligned}$$

Thus, for every $k \leq k_0$, we have :

$$\begin{aligned} \|\varphi_k\|_{p^*} &\leq c \left(|A_k|^{1-\frac{1}{p^*}} + |A_k|^{(1-\frac{\alpha+1}{p^*})(1-\frac{1}{p^*})} \right)^{\frac{1}{p^+-1}} \\ &\leq c \left(|A_k|^{(1-\frac{1}{p^*})\frac{1}{p^+-1}} + |A_k|^{(1-\frac{\alpha+1}{p^*})(1-\frac{1}{p^*})\frac{1}{p^+-1}} \right). \end{aligned} \quad (3.26)$$

Using Cavalieri's principle, Hölder inequality and Relation (3.26), one has for all $k \geq k_0$:

$$\begin{aligned} \int_k^{+\infty} |A_\tau| d\tau &= \int_\Omega (u - k)_+ dx \\ &\leq |A_k|^{1-\frac{1}{p^*}} \|\varphi_k\|_{p^*} \\ &\leq c \left(|A_k|^{1+\frac{1}{p^*}\frac{p^*-1}{p^+-1}} + |A_k|^{1+\theta} \right), \end{aligned} \quad (3.27)$$

where $\theta := -\frac{1}{p^*} + \left(1 - \frac{\alpha+1}{p^*}\right) \left(1 - \frac{1}{p^*}\right) \frac{1}{p^+-1} > 0$.

Since

$$\gamma := \frac{1}{p^*} \frac{p^* - 1}{p^+ - 1} \geq \theta,$$

then

$$\int_k^{+\infty} |A_\tau| d\tau \leq c + |A_k|^{1+\gamma}.$$

This inequality is of Gronwall type, which shows that there exists $c_\lambda > 0$ such that

$$\|u\|_\infty \leq c_\lambda.$$

One can prove by exactly the same way the fact that there exists $c_\mu > 0$ such that $\|v\|_\infty \leq c_\mu$. \square

5.4 On the weak sub and supersolutions

In this section, we will use some classical tools concerning sub and super solutions for a class of systems involving the anisotropic operators considered above. Notice that the standard laplacian operator and the anisotropic operator were studied respectively in [12] and [2].

Let us consider the following problem

$$\left\{ \begin{array}{l} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = f(x, u, v) \quad \text{in } \Omega \\ - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \right) = g(x, u, v) \quad \text{in } \Omega, \\ u \geq 0 \quad \text{and} \quad v \geq 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{and} \quad v = 0 \quad \text{on } \partial\Omega \end{array} \right. \quad (4.28)$$

where Ω is a smooth and bounded domain in \mathbb{R}^N , both $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory satisfying the following property that for each fixed $A > 0$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$(H) \quad \left\{ \begin{array}{l} |f(x, s, t)| \leq C_1, \quad \forall (x, s, t) \in \Omega \times [-A, A] \times [-A, A] \\ |g(x, s, t)| \leq C_2, \quad \forall (x, s, t) \in \Omega \times [-A, A] \times [-A, A]. \end{array} \right.$$

Definition 5.4.1. *The couple $(u, v) \in W$ is a (weak) sub-solution to (4.28) if $u \leq 0$ and $v \leq 0$ on $\partial\Omega$ and for all $(\varphi, \psi) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with $\varphi(x) \geq 0$ and $\psi(x) \geq 0$ for all $x \in \Omega$, we have simultaneously*

$$\left\{ \begin{array}{l} \sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx - \int_{\Omega} f(x, u, v) \varphi dx \leq 0 \\ \sum_{i=1}^N \int_{\Omega} \left| \frac{\partial v}{\partial x_i} \right|^{q_i-2} \frac{\partial v}{\partial x_i} \frac{\partial \psi}{\partial x_i} dx - \int_{\Omega} g(x, u, v) \psi dx \leq 0. \end{array} \right.$$

Similarly, $(u, v) \in W$ is a (weak) super-solution to (4.28) if in the above the reverse inequalities hold.

Theorem 5.4.1. *Suppose (H) holds, $(\underline{u}, \underline{v}) \in W$ is sub-solution while $(\bar{u}, \bar{v}) \in W$ is a super-solution to Problem (4.28) and assume that there exist $\underline{c}, \bar{c}, \underline{c}', \bar{c}' \in \mathbb{R}$ there holds $\underline{c} \leq \underline{u} \leq \bar{u} \leq \bar{c}$ and $\underline{c}' \leq \underline{v} \leq \bar{v} \leq \bar{c}'$ almost everywhere in Ω . Then, there exists a weak solution $(u, v) \in W_0^{1, \bar{p}}(\Omega) \times W_0^{1, \bar{q}}(\Omega)$ of (4.28), satisfying the condition $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$ almost everywhere in Ω .*

Proof. Let $F(x, u, v) = \int_0^u f(x, s, v) ds$ and $G(x, u, v) = \int_0^v g(x, u, t) dt$ denote respectively primitives of f and g . Let us define $J : W_0^{1, \bar{p}}(\Omega) \times W_0^{1, \bar{q}}(\Omega) \rightarrow \mathbb{R}$ the Euler-Lagrange functional associated to (4.28) given by

$$J(u, v) := \sum_{i=1}^N \int_{\Omega} \left(\frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \frac{1}{q_i} \left| \frac{\partial v}{\partial x_i} \right|^{q_i} \right) dx - \int_{\Omega} (F(x, u, v) + G(x, u, v)) dx$$

We introduce the closed and convex subset \mathcal{M} of $W_0^{1, \bar{p}}(\Omega) \times W_0^{1, \bar{q}}(\Omega)$ defined by

$$\mathcal{M} = \left\{ (u, v) \in W_0^{1, \bar{p}}(\Omega) \times W_0^{1, \bar{q}}(\Omega) : \underline{u} \leq u \leq \bar{u} \text{ and } \underline{v} \leq v \leq \bar{v} \text{ a.e. in } \Omega \right\}.$$

Since $\underline{u}, \underline{v}, \bar{u}$ and $\bar{v} \in L^\infty$ by assumption, also $\mathcal{M} \in L^\infty \times L^\infty$ and consequently there exists $\underline{c} > 0$ and $\bar{c} > 0$ such that $|F(x, u(x), v(x))| \leq \bar{c}$ and $|G(x, u(x), v(x))| \leq \bar{c}$ for all $(u, v) \in \mathcal{M}$ and for almost all $x \in \Omega$. Consequently

$$J(u, v) \geq \sum_{i=1}^N \int_{\Omega} \left(\frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} + \frac{1}{q_i} \left| \frac{\partial v}{\partial x_i} \right|^{q_i} \right) dx - 2\bar{c} \text{meas}(\Omega)$$

on \mathcal{M} , which implies that J is coercive on \mathcal{M} . We claim now that the functional J is weakly lower semi-continuous on \mathcal{M} . Indeed, let $(u_n, v_n), (u, v) \in \mathcal{M}$ such that $u_n \rightharpoonup u$ in $W_0^{1, \bar{p}}(\Omega)$ and $v_n \rightharpoonup v$ in $W_0^{1, \bar{q}}(\Omega)$. We may assume that, up to a subsequence, $u_n \rightarrow u$ and $v_n \rightarrow v$ pointwise almost everywhere; moreover $|F(x, u_n(x), v_n(x))| \leq \bar{c}$ and $|G(x, u_n(x), v_n(x))| \leq \bar{c}$ uniformly. Hence we may appeal to Lebesgue's theorem on dominated convergence which implies that

$$\begin{aligned} \int_{\Omega} F(x, u_n, v_n) dx &\longrightarrow \int_{\Omega} F(x, u, v) dx \text{ and} \\ \int_{\Omega} G(x, u_n, v_n) dx &\longrightarrow \int_{\Omega} G(x, u, v) dx, \quad \text{as } n \text{ tends to } +\infty, \end{aligned}$$

these end the claim, since the functionals

$$u \in W_0^{1, \bar{p}}(\Omega) \mapsto \sum_{i=1}^N \int_{\Omega} \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx$$

and

$$v \in W_0^{1, \bar{q}}(\Omega) \mapsto \sum_{i=1}^N \int_{\Omega} \frac{1}{q_i} \left| \frac{\partial v}{\partial x_i} \right|^{q_i} dx$$

are clearly weakly lower semi-continuous on the whole space. As the space W is reflexive then there exists $(u, v) \in \mathcal{M}$ such that $J(u, v) = \inf_{(w,s) \in \mathcal{M}} J(w, s)$. We claim that (u, v) solves weakly Problem (4.28), that is $J'(u, v) = 0$. Indeed, fix $(\varphi, \psi) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ and $\varepsilon > 0$ and consider the couple $(u_\varepsilon, v_\varepsilon) \in \mathcal{M}$ defined on Ω by :

$$u_\varepsilon(x) = \begin{cases} \bar{u}(x) & \text{if } u(x) + \varepsilon\varphi(x) \geq \bar{u}(x), \\ u(x) + \varepsilon\varphi(x) & \text{if } \underline{u}(x) \leq u(x) + \varepsilon\varphi(x) \leq \bar{u}(x), \\ \underline{u}(x) & \text{if } u(x) + \varepsilon\varphi(x) \leq \underline{u}(x). \end{cases}$$

and

$$v_\varepsilon(x) = \begin{cases} \bar{v}(x) & \text{if } v(x) + \varepsilon\psi(x) \geq \bar{v}(x), \\ v(x) + \varepsilon\psi(x) & \text{if } \underline{v}(x) \leq v(x) + \varepsilon\psi(x) \leq \bar{v}(x), \\ \underline{v}(x) & \text{if } v(x) + \varepsilon\psi(x) \leq \underline{v}(x). \end{cases}$$

The functions u_ε and v_ε can be characterised by $u_\varepsilon = (u + \varepsilon\varphi) - (\bar{\varphi}_\varepsilon - \underline{\varphi}_\varepsilon)$ and $v_\varepsilon = (v + \varepsilon\psi) - (\bar{\psi}_\varepsilon - \underline{\psi}_\varepsilon)$, where $\bar{\varphi}_\varepsilon = \max\{0, u + \varepsilon\varphi - \bar{u}\} \geq 0$, $\underline{\varphi}_\varepsilon = -\min\{0, u + \varepsilon\varphi - \underline{u}\} \geq 0$, $\bar{\psi}_\varepsilon = \max\{0, v + \varepsilon\psi - \bar{v}\} \geq 0$ and $\underline{\psi}_\varepsilon = -\min\{0, v + \varepsilon\psi - \underline{v}\} \geq 0$. Note that $\bar{\varphi}_\varepsilon, \underline{\varphi}_\varepsilon, \bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon \in W \cap L^\infty(\Omega) \times L^\infty(\Omega)$. Since (u, v) minimizes J on \mathcal{M} and J is differentiable, then

$$0 \leq D_1J(u, v)(u_\varepsilon - u) = \varepsilon D_1J(u, v)(\varphi) + D_1J(u, v)(\underline{\varphi}_\varepsilon) - D_1J(u, v)(\bar{\varphi}_\varepsilon),$$

where $D_1J(u, v)$ denotes the first derivative in u of J , so that

$$D_1J(u, v)(\varphi) \leq \frac{1}{\varepsilon} \left(D_1J(u, v)(\bar{\varphi}_\varepsilon) - D_1J(u, v)(\underline{\varphi}_\varepsilon) \right). \quad (4.29)$$

Using the fact that (\bar{u}, \bar{v}) is a super-solution to (4.28), we get

$$\begin{aligned} D_1J(u, v)(\bar{\varphi}_\varepsilon) &= D_1J(\bar{u}, \bar{v})(\bar{\varphi}_\varepsilon) + [D_1J(u, v) - D_1J(\bar{u}, \bar{v})](\bar{\varphi}_\varepsilon) \\ &\geq [D_1J(u, v) - D_1J(\bar{u}, \bar{v})](\bar{\varphi}_\varepsilon) \\ &= \sum_{i=1}^N \int_{\Omega_\varepsilon} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \bar{u}}{\partial x_i} \right) \frac{\partial}{\partial x_i} (u - \bar{u} + \varepsilon\varphi) dx - \\ &\quad - \int_{\Omega_\varepsilon} [f(x, u, v) - f(x, \bar{u}, \bar{v})] (u - \bar{u} + \varepsilon\varphi) dx \\ &\geq \varepsilon \sum_{i=1}^N \int_{\Omega_\varepsilon} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \bar{u}}{\partial x_i} \right) \frac{\partial}{\partial x_i} (\varphi) dx - \\ &\quad - \varepsilon \int_{\Omega_\varepsilon} |f(x, u, v) - f(x, \bar{u}, \bar{v})| |\varphi| dx \end{aligned}$$

where $\Omega_\varepsilon = \{x \in \Omega : u(x) + \varepsilon\varphi(x) \geq \bar{u}(x) \text{ and } v(x) + \varepsilon\psi(x) \geq \bar{v}(x)\}$. Notice that $meas(\Omega_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus,

$$D_1J(u, v)(\bar{\varphi}_\varepsilon) \geq o(\varepsilon),$$

where $o(\varepsilon)/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Similarly, we conclude that

$$D_1 J(u, v)(\underline{\varphi}_\varepsilon) \leq o(\varepsilon),$$

and consequently, with (4.29), we get

$$D_1 J(u, v)(\varphi) \geq 0$$

for every $\varphi \in \mathcal{D}(\Omega)$. This implies, by reversing the sign of φ , that $D_1 J(u, v)(\varphi) = 0$ for every $\varphi \in \mathcal{D}(\Omega)$. Using the density of $\mathcal{D}(\Omega)$ in $W_0^{1,\vec{p}}(\Omega)$. The proof of $D_2 J(u, v)(\psi) = 0$ where $D_2 J(u, v)$ is the seconde derivative of J in the seconde variable, follows the same steps as in the above. \square

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Chapitre 6

Annexe

Dans cette partie, on va rappeler quelques définitions, théorèmes et outils essentiels que nous utilisons au cours de cette thèse.

6.1 Condition de Palais-Smale

Pour exprimer la compacité de certaines suites minimisantes, on a souvent recours à la *condition de Palais-Smale* (en abrégé (PS)).

Définition 6.1.1. *Soit W un espace de Banach et $J : W \rightarrow \mathbb{R}$ une fonction de classe \mathcal{C}^1 . On dit que la suite $(u_n) \subset W$ est une suite de Palais-Smale s'il existe $c \in \mathbb{R}$ tel que*

$$J(u_n) \rightarrow c \text{ dans } \mathbb{R} \text{ et } J'(u_n) \rightarrow 0 \text{ dans } W'.$$

On dit que J vérifie la condition de Palais-Smale au niveau $c \in \mathbb{R}$ si toute suite $(u_n)_n$ de Palais-Smale pour la fonctionnelle J (de niveau c) contient une sous-suite $(u_{n_k})_k$ convergente.

On dispose généralement de deux théorèmes importants permettant la construction de suites de Palais-Smale : le théorème du col, dû à Ambrosetti-Rabinowitz et le principe variationnel d'Ekeland.

6.2 Théorème du col et "Mountain Pass Geometry"

Théorème 6.2.1. *Soient W un espace de Banach et $I \in \mathcal{C}^1(W, \mathbb{R})$. Supposons qu'il existe $u_0 \in W$, $u_1 \in W$ et un voisinage ouvert Ω de u_0 tel que $u_1 \notin \overline{\Omega}$ et*

$$c_0 := \max\{I(u_0), I(u_1)\} < c_1 := \max_{\partial\Omega} I(v).$$

Soit

$$H = \{h \in \mathcal{C}([0, 1], W) : h(0) = u_0, h(1) = u_1\},$$

$$c := \inf_{h \in H} \sup_{s \in [0,1]} I(h(s)).$$

Alors,

$$\inf_{\partial\Omega} I(v) \leq c < +\infty,$$

et pour tout $\varepsilon \in (0, c_1 - c_0]$ et pour chaque $h \in H$ tel que

$$\max_{s \in [0,1]} I(h(s)) \leq c + \varepsilon,$$

il existe $v \in W$ satisfaisant :

- i. $c - \varepsilon \leq I(v) \leq c + \varepsilon$
- ii. $\text{dist}(v, h([0, 1])) \leq \sqrt{\varepsilon}$
- iii. $\|I'(v)\| \leq \sqrt{\varepsilon}$.

Un autre théorème minimax de type géométrique démontré par P. H. Rabinowitz.

Théorème 6.2.2. (Point selle géométrique)

Supposons que $W = U \oplus V$ un espace de Banach et U et V deux sous-espaces fermés, tels que $\dim U < +\infty$. Pour un $\rho > 0$, on considère les ensembles suivants :

$$M := \{u \in U : \|u\| \leq \rho\}, \quad M_0 := \{u \in U : \|u\| = \rho\}.$$

On définit l'espace suivant

$$H := \{h \in \mathcal{C}(M, W) : h|_{M_0} = \text{id}\},$$

et soit $I \in \mathcal{C}^1(W, \mathbb{R})$ telle que

$$c_0 := \max_{u \in M_0} I(u) < c_1 := \inf_{v \in V} I(v).$$

Soit

$$c := \inf_{h \in H} \max_{s \in M} I(h(s)).$$

Si $c_1 \leq c$, alors pour tout $\varepsilon \in]0, c_1 - c_0]$ et pour tout $h \in H$ tel que

$$\max_{s \in M} I(h(s)) \leq c + \varepsilon,$$

il existe $u \in W$ tel que

1. $c - \varepsilon \leq I(u) \leq \max_{s \in M} I(h(s))$
2. $\text{dist}(u, h(M)) \leq \sqrt{\varepsilon}$
3. $\|I'(u)\| \leq \sqrt{\varepsilon}$.

Principe Variationnel d'Ekeland

En général, une fonctionnelle bornée et semi-continue inférieurement J n'atteint pas nécessairement son infimum. Par exemple, la fonction analytique $f(x) = \arctan x$ n'atteint ni son infimum ni son supremum sur la droite réelle.

Théorème 6.2.3. *Soit (M, d) un espace métrique complet et soit $J : M \rightarrow \mathbb{R} \cup +\infty$ semi-continue inférieurement, bornée inférieurement et $\neq +\infty$. Alors pour chaque $\varepsilon, \delta > 0$ et tout $u \in M$ avec*

$$J(u) \leq \inf_M + \varepsilon,$$

il existe un élément $v \in M$ minimisant strictement la fonctionnelle

$$J_v(w) \equiv J(w) + \frac{\varepsilon}{\delta} d(v, w).$$

En plus on a

$$J(v) \leq J(u) \quad \text{dès que} \quad d(u, v) \leq \delta.$$

Comme corollaire du principe d'Ekeland, on a

Corollaire *Si W est un espace de Banach et $J \in \mathcal{C}^1(W)$ bornée inférieurement alors il existe une suite minimisante (v_n) pour J dans W telle que*

$$J(v_n) \longrightarrow \inf_W J, \quad DJ(v_n) \longrightarrow 0 \quad \text{dans} \quad W' \quad \text{quand} \quad n \rightarrow \infty.$$

Niveau critique

Pour résoudre le problème de Yamabe, en géométrie riemannienne, Thierry Aubin a introduit la notion de niveau critique associée au problème de Yamabe, notée c^* , et a montré que si une suite minimisante de l'énergie correspondante a un niveau plus bas que le niveau critique alors le problème de Yamabe possède des solutions.

Dans un article de Brézis et Nirenberg, les auteurs ont montré que si $(u_n) \subset W$ est une suite de Palais-Smale pour la fonctionnelle J telle que $J(u_n) \rightarrow c < c^*$ quand n tend vers $+\infty$, alors (u_n) est relativement compacte. Le niveau c^* est appelé critique dans le sens où l'on peut construire une suite $(w_n) \subset W$ de Palais-Smale, avec $J(w_n) \rightarrow c^*$, qui n'admet aucune sous-suite convergente dans W .

6.3 Théorème de Concentration-Compacité

Cette méthode introduite par P. L. Lions est la méthode la plus générale pour traiter les problèmes de minimisation qui interviennent dans les domaines les plus variés (équations aux dérivées partielles, calculs des variations, analyse harmonique, etc ...).

Définition 6.3.1. Soit Ω un ouvert de \mathbb{R}^N , on définit les ensembles suivants :

$$\begin{aligned}\mathcal{K}(\Omega) &:= \{u \in \mathcal{C}(\Omega) : u \text{ est à support compact de } \Omega\} \\ \mathcal{BC}(\Omega) &:= \left\{ u \in \mathcal{C}(\Omega) \mid \|u\|_\infty := \sup_{x \in \Omega} |u(x)| < \infty \right\}.\end{aligned}$$

L'espace $\mathcal{C}_0(\Omega)$ est l'adhérence de $\mathcal{K}(\Omega)$ pour la norme uniforme. Une mesure finie définie sur Ω est une application linéaire continue sur $\mathcal{C}_0(\Omega)$. La norme d'une mesure finie μ est définie par

$$\|\mu\| := \sup_{\substack{u \in \mathcal{C}_0(\Omega) \\ \|u\|_\infty = 1}} | \langle \mu, u \rangle |.$$

On dénote par $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}^+(\Omega)$) l'espace de mesures finies (resp. mesures finies positives) sur Ω . Une suite de mesures finies (μ_n) converge faiblement vers μ dans $\mathcal{M}(\Omega)$, dans ce cas on écrit

$$\mu_n \rightharpoonup \mu,$$

si on a

$$\langle \mu_n, u \rangle \longrightarrow \langle \mu, u \rangle, \quad \forall u \in \mathcal{C}_0(\Omega).$$

Théorème 6.3.1. (lemme de concentration-compacité)

Soit $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ une suite telle que

$$\begin{aligned}u_n &\rightharpoonup u \quad \text{dans } \mathcal{D}^{1,2}(\mathbb{R}^N), \\ |\nabla(u_n - u)|^2 &\rightharpoonup \mu \quad \text{dans } \mathcal{M}(\mathbb{R}^N), \\ |u_n - u|^{2^*} &\rightharpoonup \nu \quad \text{dans } \mathcal{M}(\mathbb{R}^N), \\ u_n &\rightarrow u \quad \text{p.p. dans } \mathbb{R}^N\end{aligned}$$

et on définit

$$\mu_\infty := \lim_{R \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \int_{|x| \geq R} |\nabla u_n|^2, \quad \nu_\infty := \lim_{R \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{2^*}.$$

On a alors

$$\begin{aligned}- \|\nu\|^{2/2^*} &\leq S^{-1} \|\mu\|, \\ - \nu_\infty^{2/2^*} &\leq S^{-1} \mu_\infty, \\ - \overline{\lim}_{n \rightarrow \infty} \|\nabla u_n\|_2^2 &= \|\nabla u\|_2^2 + \|\mu\| + \mu_\infty \\ - \overline{\lim}_{n \rightarrow \infty} \|u_n\|_{2^*}^{2^*} &= \|u\|_{2^*}^{2^*} + \|\nu\| + \nu_\infty.\end{aligned}$$

En outre, si $u = 0$ et $\|\nu\|^{2/2^*} = S^{-1} \|\mu\|$, alors μ et ν sont concentrées en un seul point.

6.4 Lemme de Brézis-Lieb

Théorème 6.4.1. (Lemme de Brézis-Lieb) Soit $j : \mathbb{C} \rightarrow \mathbb{C}$ une fonction continue telle que $j(0) = 0$ et vérifie la condition suivante :

Pour tout $\varepsilon > 0$ suffisamment petit il existe deux fonctions continues et positives φ_ε et ψ_ε telles que

$$(P) \quad \forall a, b \in \mathbb{C}, \quad |j(a+b) - j(a)| \leq \varepsilon \varphi_\varepsilon(a) + \psi_\varepsilon(b).$$

Supposons que j satisfait l'hypothèse du dessus et soit $f_n = f + g_n$ une suite de fonctions mesurables de Ω vers \mathbb{C} telles que :

i) $g_n \rightarrow 0$ presque partout,

ii) $j(f) \in L^1$,

iii) il existe une constante C indépendante de ε et n telle que $\int \varphi_\varepsilon(g_n(x)) d\mu(x) \leq C < \infty$,

iv) $\int \psi_\varepsilon(f(x)) d\mu(x) < \infty$ pour tout $\varepsilon > 0$, où μ est la mesure de Lebesgues dans \mathbb{R}^N .

Alors quand $n \rightarrow \infty$,

$$\int |j(f + g_n) - j(g_n) - j(f)| d\mu \rightarrow 0.$$

Proposition 6.4.1. Les fonctionnelles considérées dans les chapitres § 1, 2, 3 et 4 de la forme $s \mapsto sf(x, s)$ ou $s \mapsto F(x, s) = \int_0^s f(x, t) dt$ satisfont les conditions du lemme de Brézis-Lieb où f est une fonctionnelle de Carathéodory et vérifie pour tout $\varepsilon > 0$ il existe $C_\varepsilon > 0$ vérifiant

$$|f(x, u)| \leq \varepsilon |u|^{p^*-1} + C_\varepsilon,$$

uniformément en $x \in \mathbb{R}^N$.

Preuve. Soit $\varepsilon > 0$, alors

$$\begin{aligned} |(a+b)f(x, a+b) - af(x, a)| &\leq |(a+b)f(a+b)| + |af(a)| \\ &< \varepsilon |a+b|^{p^*} + C_\varepsilon |a+b| + \varepsilon |a|^{p^*} + C_\varepsilon |a| \\ &< \varepsilon c(p) [|a|^{p^*} + |b|^{p^*}] + \varepsilon |a|^{p^*} + 2C_\varepsilon |a| + C_\varepsilon |b| \\ &< \varepsilon [(c(p) + 1)|a|^{p^*} + (2C_\varepsilon/\varepsilon)|a|] + \varepsilon |b|^{p^*} + C_\varepsilon |b|. \end{aligned}$$

Avec les mêmes notations du lemme de Brézis-Lieb, on prend

$$\begin{aligned} \varphi_\varepsilon(a) &= (c(p) + 1)|a|^{p^*} + (2C_\varepsilon/\varepsilon)|a| \\ \psi_\varepsilon(b) &= \varepsilon |b|^{p^*} + C_\varepsilon |b|, \end{aligned}$$

D'où la relation (P) du lemme de Brézis-Lieb. D'autre part, pour $\varepsilon > 0$

$$\begin{aligned}
 |F(x, a+b) - F(x, a)| &= \left| \int_a^{a+b} f(x, t) dt \right| \\
 &\leq \int_a^{a+b} [\varepsilon |t|^{p^*-1} + C_\varepsilon] dt \\
 &< \frac{\varepsilon}{p^*} ||a+b|^{p^*} - |a|^{p^*}| + C_\varepsilon |b| \\
 &< \frac{\varepsilon}{p^*} (c(p) + 1) |a|^{p^*} + \frac{\varepsilon c(p)}{p^*} |b|^{p^*} + C_\varepsilon |b|.
 \end{aligned}$$

Donc

$$\begin{aligned}
 \varphi_\varepsilon(a) &= \frac{1}{p^*} (c(p) + 1) |a|^{p^*} \\
 \text{et } \psi_\varepsilon(b) &= \frac{\varepsilon c(p)}{p^*} |b|^{p^*} + C_\varepsilon |b|.
 \end{aligned}$$