



Plans sphériques de force t et applications en statistique

Frédéric Bertrand

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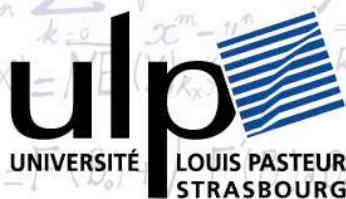
Par Frédéric BERTRAND

Plans sphériques de force t et applications en statistique

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Plans sphériques de force t et applications en statistique

par

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Mots-clés : Statistique mathématique – Statistique algébrique – Planification expérimentale – Invariance faible – Surface de réponse – Surface de variance – Caractérisations polynomiales – Bases de Gröbner – Programmation semi-définie positive

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Avant-propos

Notations

- $A \times B$ désigne le produit cartésien de deux ensembles A et B .
- Pour C une matrice réelle à p lignes et q colonnes, C' désigne la transposée de la matrice C .
- $\langle \mathbf{u}, \mathbf{v} \rangle$ est le produit scalaire canonique de deux vecteurs.
- $\langle \mathbf{u}, \mathbf{v} \rangle_E$ est le produit scalaire de deux vecteurs pour le produit scalaire défini par la matrice symétrique réelle définie positive E .
- V^\perp est l'orthogonal de la partie V pour le produit scalaire canonique.
- C^- est un pseudo inverse de la matrice réelle à p lignes et q colonnes C .
- C^+ est le pseudo inverse de Moore-Penrose de la matrice réelle à p lignes et q colonnes C .
- C^{-1} est l'inverse de la matrice carrée C lorsque celle-ci est inversible.
- $C^{1/2}$ désigne une racine carrée de la matrice C . Si C est symétrique réelle semi-définie positive alors $C^{1/2}$ est l'unique matrice symétrique réelle semi-définie positive dont le carré est égal à C .
- $C^{1/s}$, où s est un nombre réel positif, désigne une racine s -ème de la matrice C . Si C est symétrique réelle semi-définie positive alors $C^{1/s}$ est l'unique matrice symétrique réelle semi-définie positive dont la puissance s -ème est égal à C .
- $C \otimes D$ est le produit tensoriel des deux matrices C et D .
- Si C et D sont deux matrices symétriques réelles alors $C \leq D$ si et seulement si $D - C \in \mathcal{S}_k^+$. Il s'agit de l'ordre de Loewner.
- \sqrt{I} désigne le radical de l'idéal I .
- \bar{f}^F est le reste de la division euclidienne de f par le s -uplet (f_1, \dots, f_s) .
- \bar{f}^G est le reste de la division euclidienne de f par la base de Gröbner G .
- $\hat{\kappa}$ est un estimateur de la fonction κ des paramètres du modèle.
- $|A|$ désigne le cardinal de l'ensemble fini A .
- $n!$ désigne la factorielle de l'entier n .
- $\binom{n}{p}$ est le nombre de parties à p éléments d'un ensemble à n éléments.
- $\langle f_1, \dots, f_k \rangle$ désigne l'idéal engendré par les polynômes f_1, \dots, f_k .
- $\mathbb{E}[X]$ désigne l'espérance de la variable aléatoire X .
- $\text{Var}[X]$ désigne la variance de la variable aléatoire X .
- $\text{Cov}[X_1, X_2]$ désigne la covariance des variables aléatoires X_1 et X_2 .
- $\alpha = (\alpha_1, \dots, \alpha_v)$ où les $\alpha_1, \dots, \alpha_v$ sont v entiers positifs.

- γ_{ξ} est le cardinal effectif de $R(\xi)$, où ξ est un plan isovariant sur la boule \mathcal{B}_r , qui est égal à :

$$\gamma_{\xi} = \sum_{\rho \in R(\xi)} m_{\xi}(\rho).$$

- χ est un domaine expérimental, c'est-à-dire une partie d'un espace vectoriel réel de dimension v euclidien.
- η est une surface de réponse d'ordre d .
- $\lambda(A)$ est le vecteur $(\lambda_1(A), \dots, \lambda_k(A))'$ des valeurs propres de la matrice A symétrique réelle rangées dans l'ordre croissant : $\lambda_1(A) \leq \lambda_j(A) \leq \lambda_k(A)$.
- $\Phi : \mathcal{A} \rightarrow \mathbb{R}$ est un critère d'optimalité.
- $\Phi_p(A) = \left(\frac{1}{k} \sum_{j=1}^k \lambda_j(A)^p \right)^{\frac{-1}{p}}$ si $p \notin \{-\infty, 0\}$ est le critère d'optimalité Φ_p de Kiefer.
- $\Phi_0(A) = \left(\prod_{j=1}^k \lambda_j(A) \right)^{\frac{-1}{k}} = (\det(A))^{\frac{-1}{k}}$ est le critère de D -optimalité.
- $\Phi_{-1}(A) = \frac{1}{k} \sum_{j=1}^k \lambda_j(A)^{-1}$ est le critère de A -optimalité.
- $\Phi_{-\infty}(A) = (\min_{j=1, \dots, k} \lambda_j(A))^{-1}$ est le critère de E -optimalité.
- $\Phi_I(A) = \frac{1}{\text{vol}(\chi)} \int_{\chi} \mathbf{f}(\mathbf{x})' A^{-1} \mathbf{f}(\mathbf{x}) d\mathbf{x}$ est le critère de I -optimalité.
- $\mu_{\gamma}(\xi)$ désigne le moment d'ordre γ du modèle polynomial pour le plan ξ .
- $\theta = (\theta_1, \dots, \theta_k)$ est le vecteur des paramètres.
- σ^2 variance commune des variables aléatoires Y_1, \dots, Y_n représentant les valeurs de la réponse y aux points x_1, \dots, x_n du plan expérimental.
- τ désigne un ordre sur les monômes de $\mathbb{k}[x_1, \dots, x_v]$.
- ξ est un plan expérimental exact ou approché c'est-à-dire un couple (\mathbf{X}, \mathbf{w}) où \mathbf{X} est un sous-ensemble fini du domaine expérimental χ et \mathbf{w} est un r -uplet réel.
- $\xi(\mathbf{x}_i)$ le poids du point support \mathbf{x}_i du plan ξ .
- ξ^g est l'image du plan ξ par g une transformation bijective de χ sur χ .
- ξ_n est un plan expérimental exact de cardinal pour un nombre total d'essais égal à n .
- $\xi(\mathcal{S}_{\rho})$ est le poids total de tous les points du plan appartenant à la sphère de rayon ρ centrée en l'origine du domaine.
- $]a, b[$ désigne l'intervalle réel ouvert d'extrémités a et b .
- \mathcal{A} est un cône convexe inclus dans \mathcal{S}_k tel que $\mathcal{S}_k^{++} \subset \mathcal{A} \subset \mathcal{S}_k^+$.
- A désigne une matrice symétrique réelle définie positive telle que le groupe de matrices compact \mathcal{G} soit conjugué au sous-groupe \mathcal{K} du groupe orthogonal $\mathcal{O}_v(\mathbb{R})$: $\mathcal{G} = A\mathcal{K}A^{-1}$.
- A est un sous-ensemble fini de \mathbb{N}^v l'ensemble des degrés du modèle polynomial A .
- A_d est un sous-ensemble fini de \mathbb{N}^v appelé modèle polynomial complet de degré d .
- $\text{Aut}(H)$ désigne l'ensemble des automorphismes du groupe H .
- \mathcal{B}_r est la boule de rayon r centrée en $\mathbf{0}$.
- \mathbb{C} désigne le corps des nombres complexes.
- $\mathbb{C}P^v$ désigne un espace projectif sur le corps des nombres complexes.
- $\mathbb{C}[x_1, \dots, x_v]$ désigne l'algèbre des polynômes à v indéterminées à coefficients complexes.
- \mathcal{C}_b est un domaine expérimental de forme v -hypercube symétrique centré en $\mathbf{0}$:

$$\mathcal{C}_b = \left\{ \mathbf{x} = (x_1, \dots, x_v)' \in \mathbb{R}^v, |x_i| \leq b_i, 1 \leq i \leq v \right\}$$

où $b_i \in]0, +\infty[$, $1 \leq i \leq v$.

- $C_K(\xi)$ est la matrice d'information réduite du plan expérimental ξ .

- $\text{Conv}\{D\}$ est l'enveloppe convexe de D une partie d'un \mathbb{R} —espace vectoriel.
- $\det C$ désigne le déterminant de C une matrice carrée.
- $\text{Diag}(a_1, \dots, a_v)$ est égal à D une matrice diagonale d'ordre v dont le terme général D_{ij} est égal à 0 si $i \neq j$ et à a_i si $i = j$.
- E désigne une matrice symétrique réelle définie positive telle que le groupe de matrices compact \mathcal{G} soit un sous-groupe du groupe $\mathcal{O}_v(E)$.
- \mathcal{E}_H est un domaine expérimental de forme v —ellipsoïde centré en $\mathbf{0}$:

$$\mathcal{E}_H = \left\{ \mathbf{x} = (x_1, \dots, x_v)' \in \mathbb{R}^v, \mathbf{x}' H \mathbf{x} \leq 1 \right\}$$

où $H \in \mathcal{S}_k^{++}$.

- $\partial \mathcal{E}_H$ est la frontière de \mathcal{E}_H :

$$\partial \mathcal{E}_H = \left\{ \mathbf{x} = (x_1, \dots, x_v)' \in \mathbb{R}^v, \mathbf{x}' H \mathbf{x} = 1 \right\}.$$

- $\text{eff}(\boldsymbol{\xi} : \boldsymbol{\xi}^*) = \frac{\Phi(M(\boldsymbol{\xi}^*))}{\Phi(M(\boldsymbol{\xi}))}$ est l'efficacité relative d'un plan $\boldsymbol{\xi}$ par rapport à un plan $\boldsymbol{\xi}^*$.
- $\text{Est}_\tau(\boldsymbol{\xi})$ est l'ensemble des représentants de degré minimal des fonctions polynomiales estimables pour le plan $\boldsymbol{\xi}$ et l'ordre τ .
- $\mathbf{f} = (f_1, \dots, f_k)'$ est une fonction connue définie sur $\boldsymbol{\chi}$ et à valeurs dans \mathbb{R}^k . Il s'agit de la part déterministe d'un modèle statistique.
- $\mathbf{f}(\mathbf{x}_i)$ est la part déterministe du modèle évaluée en le point support \mathbf{x}_i du plan $\boldsymbol{\xi}$.
- \mathbf{f}_d est la part déterministe d'un modèle polynomial complet de degré d .
- G désigne une base de Gröbner.
- \mathcal{G} est un groupe de transformations bijectives de $\boldsymbol{\chi}$ sur $\boldsymbol{\chi}$.
- \mathcal{G}_{orth} est le groupe des endomorphismes orthogonaux de \mathbb{R}^v muni du produit scalaire canonique.
- $\mathcal{GL}_v(\mathbb{R})$ est le groupe des matrices réelles inversibles d'ordre v . Il s'agit d'un groupe topologique pour la topologie héritée de la topologie usuelle sur $\mathcal{M}_v(\mathbb{R})$.
- \mathbb{H} désigne le corps des quaternions.
- $\mathbb{H}\mathbf{P}^v$ désigne un espace projectif sur le corps des quaternions.
- I désigne un idéal de $\mathbb{R}[x_i, \dots, x_v]$.
- $\mathbb{I}_{\mathbb{K}}^n$ désigne l'ensemble des idéaux de $\mathbb{R}[x_i, \dots, x_v]$.
- $\mathbf{I}(V)$ désigne l'idéal annulateur d'une partie de \mathbb{k}^v .
- $\mathbf{I}(\boldsymbol{\xi})$ ou $\mathbf{I}(\mathbf{X})$ désignent l'idéal du plan $\boldsymbol{\xi}$.
- I_v est la matrice carrée identité d'ordre v .
- $\mathbf{i}_{M(\boldsymbol{\xi})}$ est la surface d'information généralisée de l'estimateur de la réponse moyenne $\boldsymbol{\eta}(\mathbf{x})$.
- Pour \mathbf{C} une matrice réelle à p lignes et q colonnes, $\text{Im}(\mathbf{C})$ désigne l'image de la matrice \mathbf{C} dans \mathbb{R}^q .
- \mathbb{K} ou \mathbb{k} désignent un corps de caractéristique nulle.
- $\mathbb{K}[x_1, \dots, x_v]$ désigne l'algèbre des polynômes à v indéterminées à coefficients dans le corps \mathbb{K} .
- $\mathbb{K}[V]$ désigne l'ensemble des fonctions polynomiales $\phi : V \rightarrow \mathbb{K}$.
- \mathcal{K} désigne un sous-groupe compact du groupe orthogonal $\mathcal{O}_v(\mathbb{R})$.
- Pour \mathbf{C} une matrice réelle à p lignes et q colonnes, $\text{Ker}(\mathbf{C})$ désigne le noyau de la matrice \mathbf{C} dans \mathbb{R}^p .
- $\text{LC}(f)$ désigne le coefficient du terme dominant du polynôme f .

- $\text{LM}(f)$ désigne le monôme dominant du polynôme f .
- $\text{LT}(f)$ désigne le terme dominant du polynôme f .
- $\mathcal{M} = \text{Conv}\{\mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})', \mathbf{x} \in \mathcal{X}\}$ est une partie convexe de \mathcal{S}_k .
- $\mathcal{M}_v(\mathbb{R})$ est le \mathbb{R} –espace vectoriel des matrices carrées réelles d'ordre v .
- $\mathcal{M}_{p,q}(\mathbb{R})$ ou $\mathcal{M}_{p,q}$ est le \mathbb{R} –espace vectoriel des matrices réelles à p lignes et q colonnes.
- $M(\boldsymbol{\xi})$ matrice des moments du plan $\boldsymbol{\xi}$.
- $\overline{M}_{d,\rho}$ est la matrice des moments d'un plan isovariant dont le support est inclus dans la sphère \mathcal{S}_ρ .
- $m_{\boldsymbol{\xi}}(\rho)$ est la multiplicité d'un rayon $\rho \in R(\boldsymbol{\xi})$ qui est égale à :

$$m_{\boldsymbol{\xi}}(\rho) = \begin{cases} 0, & \text{si } \rho = r \\ 1, & \text{si } 0 < \rho < r \\ \frac{1}{2}, & \text{si } \rho = 0 \end{cases}.$$

- M_0 est la matrice des moments de la distribution uniforme sur le domaine compact d'intérieur non vide d'intérieur vide $\mathcal{X} : M_0 = \int_{\mathcal{X}} \mathbf{f}(\mathbf{x})\mathbf{f}(\mathbf{x})' \frac{d\mathbf{x}}{\text{vol}(\mathcal{X})}$.
- $\text{MGF}^{A_d}(\boldsymbol{\xi})$ désigne la fonction génératrice des moments du plan $\boldsymbol{\xi}$ pour un modèle polynomial complet A_d de degré d .
- $\text{MGF}^A(\boldsymbol{\xi})$ désigne la fonction génératrice des moments du plan $\boldsymbol{\xi}$ pour un modèle polynomial A de degré d .
- $\text{MGF}_E^{A_d}(\boldsymbol{\xi})$ désigne la fonction E –génératrice des moments du modèle polynomial complet A_d de degré d .
- $\min_{x \in \Gamma} f(x)$ désigne le minimum de f pour x appartenant à Γ .
- $\text{multideg}(f)$ le degré multivarié du polynôme f .
- \mathbb{N} désigne les entiers naturels.
- $n(\boldsymbol{\xi})$ est la taille du plan exact $\boldsymbol{\xi}$.
- \mathbb{O} désigne les octaves de Cayley.
- $\mathbb{O}\mathbf{P}^1$ désigne la droite projective sur les octaves de Cayley.
- $\mathbb{O}\mathbf{P}^2$ désigne le plan projectif sur les octaves de Cayley.
- $\mathcal{O}_v(\mathbb{R})$, \mathcal{O}_v est le groupe orthogonal de \mathbb{R}^v muni du produit scalaire canonique.
- $\mathcal{O}_v(E)$ est le groupe orthogonal de \mathbb{R}^v muni du produit scalaire induit par la matrice symétrique réelle définie positive E .
- \mathbb{Q} désigne les nombres rationels.
- \mathcal{Q} est un groupe de matrices carrées d'ordre k .
- \mathcal{Q}_{orth} est le groupe de matrices carrées d'ordre k induit par le groupe \mathcal{G}_{orth} .
- \mathbb{R} désigne le corps des nombres réels.
- $\mathbb{R}\mathbf{P}^v$ désigne un espace projectif sur le corps des nombres réels.
- $\mathbb{R}[x_1, \dots, x_v]$ désigne l'algèbre des polynômes à v indéterminées à coefficients réels.
- $\mathbb{R}[x_1, \dots, x_v]/I$ désigne le quotient de l'algèbre des polynômes à v indéterminées à coefficients réels par l'idéal I .
- $\mathbb{R}[x_1, \dots, x_v]^{\mathcal{G}}$ désigne la \mathbb{R} –sous-algèbre de $\mathbb{R}[x_1, \dots, x_v]$ des polynômes invariants pour l'action induite par le groupe de matrices \mathcal{G} .
- $\mathbb{R}[f_1, \dots, f_k]$ désigne la \mathbb{R} –sous-algèbre des polynômes en les polynômes à coefficients réels f_1, \dots, f_k .
- r désigne généralement le cardinal de \mathbf{X} l'ensemble des points support du plan expérimental $\boldsymbol{\xi}$.
- $R(\boldsymbol{\xi})$ est l'ensemble, fini, des rayons $\rho \in [0, r]$ tels que $\boldsymbol{\xi}(\mathcal{S}_\rho) > 0$.

- $R^{\mathcal{G}}$ désigne l'opérateur de Reynolds associé au groupe de matrices compact \mathcal{G} .
- $\text{ri}(\mathcal{S})$ est l'intérieur relatif du segment \mathcal{S} .
- \mathcal{S}_k ou $\text{Sym}_k(\mathbb{R})$ désignent le \mathbb{R} —espace vectoriel des matrices symétriques réelles.
- \mathcal{S}_k^+ ou $\mathcal{S}_k^+(\mathbb{R})$ désignent le cône convexe des matrices symétriques réelles positives.
- \mathcal{S}_k^{++} ou $\mathcal{S}_k^{++}(\mathbb{R})$ désignent le cône convexe étoilé en $\mathbf{0}$ des matrices symétriques réelles positives.
- \mathcal{S}_r est la sphère de rayon r centrée en $\mathbf{0}$.
- \mathcal{S}^v est la sphère unité centrée en $\mathbf{0}$ de l'espace vectoriel \mathbb{R}^v muni du produit scalaire canonique.
- $\text{supp}(\xi)$ est le support du plan ξ .
- t est généralement un entier qui désigne la force d'un plan sphérique.
- \mathbf{t} désigne un élément de \mathbb{R}^v .
- $\text{tr}(M)$ est la trace de la matrice M .
- U_g est la matrice de g la transformation bijective de χ sur χ lorsque le groupe \mathcal{G} agit linéairement sur χ .
- $\mathbb{V}_{\mathbb{K}}^n$ désigne l'ensemble des variétés affines de \mathbb{K}^n .
- $\mathbf{V}(I)$ désigne la variété algébrique associée à l'idéal I .
- $V(\xi_n, \mathbf{c})$ est la fonction de variance normalisée.
- \mathcal{V}_b est l'ensemble des sommets de \mathcal{C}_b :

$$\mathcal{V}_b = \left\{ \mathbf{x} = (x_1, \dots, x_v)' \in \mathbb{R}^v, |x_i| = \pm b_i, 1 \leq i \leq v \right\}.$$

- $v_{M(\xi)} : \chi \rightarrow \mathbb{R}$ est la surface de variance extrapolée à χ de l'estimateur de la réponse moyenne $\widehat{\eta(\mathbf{x})}$. Si la matrice des moments $M(\xi)$ du plan expérimental ξ n'est pas singulière il s'agit de la surface de variance de l'estimateur de la réponse moyenne $\widehat{\eta(\mathbf{x})}$.
- $\mathbf{v}_{M(\xi)} : \chi \rightarrow \mathbb{R}$ est la surface de variance généralisée de l'estimateur de la réponse moyenne $\widehat{\eta(\mathbf{x})}$.
- $\text{Vect}(\mathbf{e}_1, \dots, \mathbf{e}_l)$ est le sous-espace vectoriel engendré par les vecteurs $\mathbf{e}_1, \dots, \mathbf{e}_l$.
- \mathbf{w} est un r —uplet réel (w_1, \dots, w_r) , les poids des points support du plan ξ .
- \mathbf{X} est un ensemble fini de r points $\mathbf{x}_1, \dots, \mathbf{x}_r$, les points support du plan ξ .
- y est une réponse univariée réelle c'est-à-dire une fonction définie sur χ et à valeurs réelles.
- $\widehat{\eta(\mathbf{x})}$ est l'estimateur de la réponse moyenne de η , une surface de réponse d'ordre d .
- \mathbb{Z} désigne les entiers relatifs.

Deuxième partie

Implémentation pratique et applications à la construction de plans expérimentaux

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Chapitre 10

Implémentation pratique

10.1. Description des programmes

Nous avons regroupé Les programmes, dont le code est reproduit les sections 10.2, 10.3, 10.4, 10.5, 10.6, 10.7, 10.8 et 10.9, par thème d'application.

10.2. Préambule

10.2.1. Gestion des répertoires

```
> with(FileTools);
```

Warning, the name Copy has been rebound

[AtEndOfFile, Binary, Copy, Exists, Flush, IsDirectory, IsExecutable, IsLockable, IsOpen, IsReadable, IsWritable, ListDirectory, Lock, ModificationTime, Position, Remove, Rename, Size, Status, TemporaryFilename, Text, Unlock]

```
>.Sockets[ ':-GetHostName' ]();
```

"TARTETATIN"

```
> if.Sockets[ ':-GetHostName' ]()="irmasrv4" then
  PATHPACKAGE:="~fbertan/Maple/"; PATHDATA:="~fbertan/Maple/MapleData/"
end if;
> if.Sockets[ ':-GetHostName' ]()="irmasrv2.nisirma" then
  PATHPACKAGE:=AbsolutePath(""); PATHDATA:=AbsolutePath("MapleData/")
end if;
> if.Sockets[ ':-GetHostName' ]()="TARTETATIN" then
  PATHPACKAGE:="D:\\Données\\Thèse\\Maple\\";
  PATHDATA:="D:\\Données\\Thèse\\Maple\\MapleData\\" end
if;
```

PATHPACKAGE := "D :\\Données\\ Thèse\\Maple\\"

PATHDATA := "D :\\Données\\ Thèse\\Maple\\MapleData\\"

```
> if.Sockets[ ':-GetHostName' ]()="TARTETATIN.u-strasbg.fr" then
  PATHPACKAGE:="D:\\Données\\Thèse\\Maple\\";
  PATHDATA:="D:\\Données\\Thèse\\Maple\\MapleData\\" end
if;
> if.Sockets[ ':-GetHostName' ]()="CHANTILLY" then
  PATHPACKAGE:="C:\\Données\\Thèse\\Maple\\";
  PATHDATA:="C:\\Données\\Thèse\\Maple\\MapleData\\" end
if;
> if.Sockets[ ':-GetHostName' ]()="CHANTILLY.u-strasbg.fr" then
  PATHPACKAGE:="C:\\Données\\Thèse\\Maple\\";
  PATHDATA:="C:\\Données\\Thèse\\Maple\\MapleData\\" end
if;
> if.Sockets[ ':-GetHostName' ]()="localhost" then
  PATHPACKAGE:="C:\\Données\\Thèse\\Maple\\";
  PATHDATA:="C:\\Données\\Thèse\\Maple\\MapleData\\" end
if;
> if.Sockets[ ':-GetHostName' ]()="Aceraline" then
  PATHPACKAGE:="C:\\FRED\\Maple\\Packages\\";
  PATHDATA:="C:\\FRED\\Maple\\MapleData\\" end if;
> if.Sockets[ ':-GetHostName' ]()="Athlon" then
  PATHPACKAGE:="L:\\Fred\\Maple\\";
  PATHDATA:="L:\\Fred\\Maple\\MapleData\\" end if;
> if.Sockets[ ':-GetHostName' ]()="Gericom" then
  PATHPACKAGE:="E:\\Fred\\Maple\\";
  PATHDATA:="E:\\Fred\\Maple\\MapleData\\" end if;
> if not (assigned(PATHPACKAGE)) then PATHPACKAGE:="C:\\Données\\Th
èse\\Maple\\"; PATHDATA:="C:\\Données\\Th
èse\\Maple\\MapleData\\" end if;
```

10.2.2. Chargement des boîtes à outils

```
> with(Ore_algebra);with(Groebner);
```


10.2 Préambule

```
[OA_Internals, Ore_to_DESol, Ore_to_RESol, Ore_to_diff, Ore_to_shift, annihilators,
applyopr, diff_algebra, dual_algebra, dual_polynomial, poly_algebra, qshift_algebra,
rand_skew_poly, reverse_algebra, reverse_polynomial, shift_algebra, skew_algebra,
skew_elim, skew_gcdex, skew_pdiv, skew_power, skew_prem, skew_product]
```

```
[MulMatrix, SetBasis, ToricIdealBasis, fglm_algo, gbasis, gsolve, hilbertdim, hilbertpoly,
hilbertseries, inter_reduce, is_finite, is_solvable, leadcoeff, leadmon, leadterm,
normalf, pretend_gbasis, reduce, spoly, termorder, testorder, univpoly]
```

```
> with(PolynomialIdeals);
```

Warning, the names Add and Multiply have been rebound

```
[<, >, Add, Contract, EliminationIdeal, Generators, GroebnerBasis, HilbertDimension,
IdealContainment, IdealInfo, IdealMembership, InterReduce, Intersect, IsMaximal,
IsPrimary, IsPrime, IsProper, IsRadical, IsZeroDimensional, LeadingCoefficient,
LeadingMonomial, LeadingTerm, MaximalIndependentSet, Multiply, NormalForm,
Operators, PolynomialIdeal, PrimaryDecomposition, PrimeDecomposition, Quotient,
Radical, RadicalMembership, Saturate, SimplifyIdeal, Spolynomial,
SuggestVariableOrder, TrailingMonomial, UnivariatePolynomial,
ZeroDimensionalDecomposition, in, subset]
```

```
> with(PolynomialTools);
```

Warning, the name MinimalPolynomial has been rebound

```
[CoefficientList, CoefficientVector, GcdFreeBasis, GreatestFactorialFactorization, Hurwitz,
IsSelfReciprocal, MinimalPolynomial, PDEToPolynomial, PolynomialToPDE,
ShiftEquivalent, ShiftlessDecomposition, Shorten, Shorter, Sort, Split, Splits, Translate]
```

```
> # with(numtheory);
```

```
> with(plottools);
```

```
[arc, arrow, circle, cone, cuboid, curve, cutin, cutout, cylinder, disk, dodecahedron, ellipse,
ellipticArc, hemisphere, hexahedron, homothety, hyperbola, icosahedron, line,
octahedron, parallelepiped, pieslice, point, polygon, project, rectangle, reflect, rotate,
scale, semitorus, sphere, stellate, tetrahedron, torus, transform, translate, vrm]l]
```

```
> with(SolveTools);
```

Warning, the name Basis has been rebound

```
[Basis, CancelInverses, Combine, Complexity, GreaterComplexity, Inequality, Linear,
PolynomialSystem, RationalCoefficients, SortByComplexity]
```

```
> with(plots);
```

Warning, the previous binding of the name arrow has been removed and
it now has an assigned value

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, graphplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

```
> read cat(PATHPACKAGE,"Convex/convex.m"); with(convex);
```

Convex version 1.1.0, Copyright (C) 1999-2004 Matthias Franz

This package is distributed under the GNU General Public License,

see <http://www-fourier.ujf-grenoble.fr/~franz/convex/> for more information.

Warning, these names have been redefined: *affinehull, ambientdim, arecompatible, boundary, codim, contains, containsrelint, convhull, corank, corners, crosspolytope, cube, cyclicpolytope, delaunay, dim, directsum, distance, domain, dotprod, draw, dual, edges, emptycomplex, emptypolyhedron, emptypolytope, facefan, faces, facets, fan, flagf, flagh, fullcone, fullpolyhedron, furthestdelaunay, fvector, genhpolynomial, genhvector, hilbertbasis, homology, hplanes, hspacenos, hspaces, hvector, image, incidencematrix, incidentfacets, incidentrays, intersection, isaffine, isbounded, iscomplete, iscontained, isempty, isface, isfulldim, islinear, ispointed, ispolytopal, isquasipolytopal, isregular, issimple, issimplex, issimplicial, issimplicial1, join, lensspace, lineality, linearhull, lines, maximal, maximum, minimal, minimum, minkowskisum, modz, newtonpolytope, normalfan, pcomplex, permutahedron, plotdata, polar, poshull, posorthant, pred, preimage, projspace, proximum, randompolytope, rank, raynos, rays, readpoly, recession, regularpart, regularsubdiv, relint, simplicialsubdiv, skeleton, stdsimplex, stellarsubdiv, succ, support, surface, torsion, transversalfan, traverse, traverse2, vertexnos, vertices, volume, voronoi, wprojspace, writepoly, zerocone, zerofan*

[affinehull, ambientdim, arecompatible, boundary, codim, contains, containsrelint, convhull, corank, corners, crosspolytope, cube, cyclicpolytope, delaunay, dim, directsum, distance, domain, dotprod, draw, dual, edges, emptycomplex, emptypolyhedron, emptypolytope, facefan, faces, facets, fan, flagf, flagh, fullcone, fullpolyhedron, furthestdelaunay, fvector, genhpolynomial, genhvector, hilbertbasis, homology, hplanes, hspacenos, hspaces, hvector, image, incidencematrix, incidentfacets, incidentrays, intersection, isaffine, isbounded, iscomplete, iscontained, isempty, isface, isfulldim, islinear, ispointed, ispolytopal, isquasipolytopal, isregular, issimple, issimplex, issimplicial, issimplicial1, join, lensspace, lineality, linearhull, lines, maximal, maximum, minimal, minimum, minkowskisum, modz, newtonpolytope, normalfan, pcomplex, permutahedron, plotdata, polar, poshull, posorthant, pred, preimage, projspace, proximum, randompolytope, rank, raynos, rays, readpoly, recession, regularpart, regularsubdiv, relint, simplicialsubdiv, skeleton, stdsimplex, stellarsubdiv, succ, support, surface, torsion, transversalfan, traverse, traverse2, vertexnos, vertices, volume, voronoi, wprojspace, writepoly, zerocone, zerofan]

10.2 Préambule

```
> with(combinat);

[Chi, bell, binomial, cartprod, character, choose, composition, conjpart, decodepart,
 encodepart, fibonacci, firstpart, graycode, inttovec, lastpart, multinomial, nextpart,
 numbc comb, numbcomp, numbp art, numbp perm, partition, permute, powerset, prevpart,
 randcomb, randpart, randperm, setpartition, stirling1, stirling2, subsets, vectoint]

> with(LinearAlgebra);
```

Warning, these names have been rebound: Add, Basis, Copy,
MinimalPolynomial, Multiply

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm,
BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column,
ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix,
ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation,
CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix,
Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues,
Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination,
GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape,
GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm,
HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix,
IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary,
JordanBlockMatrix, JordanForm, LA_Main, LUdecomposition, LeastSquares,
LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction,
MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower,
MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular,
Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix,
Permanent, Pivot, PopovForm, QRdecomposition, RandomMatrix, RandomVector,
Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension,
RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm,
SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix,
ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix,
VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply,
ZeroMatrix, ZeroVector, Zip]

10.2.3. Configuration

```
> interface(echo=1);interface(verboseproc=1);interface(rtablesize=25);i
nfolevel[Student[LinearAlgebra]]:=5;infolevel[hints]:=5;info_iop:=1;

1
1
25

infolevelStudent:-LinearAlgebra := 5

infolevelhints := 5

info_iop := 1
```

10.2.4. Conversion de types, tri

```

> SeqToSet:=proc(ListSeqToSet) local ValuesSeqToSet, ListSetSeqToSet,
PointSeqToSet, indiceSeqToSet; ValuesSeqToSet:=[];
ListSetSeqToSet:=[]; for PointSeqToSet in ListSeqToSet do if
member(PointSeqToSet[1],ValuesSeqToSet,'indiceSeqToSet') then
ListSetSeqToSet[indiceSeqToSet]:=[op(ListSetSeqToSet[indiceSeqToSet]),
PointSeqToSet] else
ValuesSeqToSet:=[op(ValuesSeqToSet),PointSeqToSet[1]];
ListSetSeqToSet:=[op(ListSetSeqToSet),[PointSeqToSet]] end if; end do;
return ListSetSeqToSet end proc;

SeqToSet := proc(ListSeqToSet)
local ValuesSeqToSet, ListSetSeqToSet, PointSeqToSet, indiceSeqToSet;
  ValuesSeqToSet := [];
  ListSetSeqToSet := [];
  for PointSeqToSet in ListSeqToSet do
    if member(PointSeqToSet1, ValuesSeqToSet, 'indiceSeqToSet') then
      ListSetSeqToSetindiceSeqToSet :=
        [op(ListSetSeqToSetindiceSeqToSet), PointSeqToSet]
    else
      ValuesSeqToSet := [op(ValuesSeqToSet), PointSeqToSet1];
      ListSetSeqToSet := [op(ListSetSeqToSet), [PointSeqToSet]]
    end if
  end do;
  return ListSetSeqToSet
end proc

> SortPoints:=proc(ListSortPoints) return
map(op,sort(SeqToSet(ListSortPoints),OrdreCardinal)); end proc;

SortPoints := proc(ListSortPoints)
  return map(op, sort(SeqToSet(ListSortPoints), OrdreCardinal))
end proc

> OrdreCardinal:=proc(Set1,Set2); if nops(Set1)>nops(Set2) then return
true elif nops(Set1)<=nops(Set2) then return false; else return FAIL
end if; end proc;

OrdreCardinal := proc(Set1, Set2)
  if nops(Set2) < nops(Set1) then return true
  elif nops(Set1) ≤ nops(Set2) then return false
  else return FAIL
  end if
end proc

> OrbToMatrix:=proc(OrbOrbToMatrix); return
Transpose(convert(convert(OrbOrbToMatrix,list),Matrix)); end proc;

OrbToMatrix := proc(OrbOrbToMatrix)
  return LinearAlgebra : - Transpose(convert(convert(OrbOrbToMatrix, list), Matrix))
end proc

```

```

> boolprod:=proc(lboolprod) local tempbool,boolboolprod,listboolprod;
  listboolprod:=lboolprod; if nops(listboolprod)>2 then
  tempbool:=listboolprod[1]; listboolprod:=subsop(1=NULL,listboolprod);
  for boolboolprod while listboolprod<>[] do tempbool:='and'(tempbool,
  listboolprod[1]);listboolprod:=subsop(1=NULL,listboolprod); end do
  else if nops(listboolprod)=2 then return
  'and'(lboolprod[1],lboolprod[2]) else return op(lboolprod) end if; end
  if; return tempbool; end proc;

  boolprod := proc(lboolprod)
local tempbool, boolboolprod, listboolprod;
  listboolprod := lboolprod;
  if 2 < nops(listboolprod) then
    tempbool := listboolprod1;
    listboolprod := subsop(1 = NULL, listboolprod);
    for boolboolprod while listboolprod ≠ [] do
      tempbool := 'and'(tempbool, listboolprod1);
      listboolprod := subsop(1 = NULL, listboolprod)
    end do
  else
    if nops(listboolprod) = 2 then return 'and'(lboolprod1, lboolprod2)
    else return op(lboolprod)
    end if
  end if;
  return tempbool
end proc

> booladd:=proc(lboolprod) local tempbool,boolboolprod,listboolprod;
  listboolprod:=lboolprod; if nops(listboolprod)>2 then
  tempbool:=listboolprod[1]; listboolprod:=subsop(1=NULL,listboolprod);
  for boolboolprod while listboolprod<>[] do tempbool:='or'(tempbool,
  listboolprod[1]);listboolprod:=subsop(1=NULL,listboolprod); end do
  else if nops(listboolprod)=2 then return
  'or'(lboolprod[1],lboolprod[2]) else return op(lboolprod) end if; end
  if; return tempbool; end proc;

  booladd := proc(lboolprod)
local tempbool, boolboolprod, listboolprod;
  listboolprod := lboolprod;
  if 2 < nops(listboolprod) then
    tempbool := listboolprod1;
    listboolprod := subsop(1 = NULL, listboolprod);
    for boolboolprod while listboolprod ≠ [] do
      tempbool := 'or'(tempbool, listboolprod1);
      listboolprod := subsop(1 = NULL, listboolprod)
    end do
  else
    if nops(listboolprod) = 2 then return 'or'(lboolprod1, lboolprod2)
    else return op(lboolprod)
    end if
  end if;
  return tempbool
end proc

```


10.3. Matrices et idéaux

10.3.1. Entrées sorties

```
> Fichier_Existant:= proc(A) local bool; description "Tests if a file
with the given name exists"; bool:=FAIL; try fopen( A, READ ): catch
"file or directory does not exist" : bool:= false ; catch :
bool:=false ; finally fclose(A); if bool=false then return false; else
return true; end if end end proc;
```

```
Fichier_Existant := proc(A)
local bool;
description "Tests if a file with the given name exists";
    bool := FAIL;
    try fopen(A, READ)
    catch "file or directory does not exist" : bool := false
    catch : bool := false
    finally fclose(A); if bool = false then return false else return true end if
    end try
end proc
```

```
> CreatePlan:=proc(PlanCreatePlan,NomPlanCreatePlan) local
NomFichierCreatePlan; global Plan; if not type(PlanCreatePlan,Matrix)
then error "Le plan doit etre donné sous forme matricielle"; else
;NomFichierCreatePlan:=cat(PATHDATA,NomPlanCreatePlan,".txt"); if
Fichier_Existant(NomFichierCreatePlan)=true then error "A file named
%1 has already being created",NomFichierCreatePlan ; else
Plan:=PlanCreatePlan; save Plan, NomFichierCreatePlan; end if; end if;
end proc;
```

```
CreatePlan := proc(PlanCreatePlan, NomPlanCreatePlan)
local NomFichierCreatePlan;
global Plan;
    if not type(PlanCreatePlan, Matrix) then
        error "Le plan doit etre donné sous forme matricielle"
    else
        NomFichierCreatePlan := cat(PATHDATA, NomPlanCreatePlan, ".txt" );
        if Fichier_Existant(NomFichierCreatePlan) = true then
            error "A file named %1 has already being created", NomFichierCreatePlan
        else Plan := PlanCreatePlan ; save Plan, NomFichierCreatePlan
        end if
    end if
end proc
```

```
> UpdatePlan:=proc(PlanUpdatePlan,NomPlanUpdatePlan) local
NomFichierUpdatePlan; global Plan;
NomFichierUpdatePlan:=cat(PATHDATA,NomPlanUpdatePlan,".txt"); if not
type(PlanUpdatePlan,Matrix) then error "Le plan doit etre donné sous
forme matricielle" else if Fichier_Existant(NomFichierUpdatePlan) then
Plan:= PlanUpdatePlan; save Plan, NomFichierUpdatePlan; print("Update
complete"); else error "A file named %1 doesn't
exist",NomFichierUpdatePlan; end if; end if; end proc;
```

```

UpdatePlan := proc(PlanUpdatePlan, NomPlanUpdatePlan)
local NomFichierUpdatePlan;
global Plan;
NomFichierUpdatePlan := cat(PATHDATA, NomPlanUpdatePlan, ".txt" );
if not type(PlanUpdatePlan, Matrix) then
error "Le plan doit etre donné sous forme matricielle"
else
if Fichier_Existant(NomFichierUpdatePlan) then
Plan := PlanUpdatePlan ;
save Plan, NomFichierUpdatePlan ;
print("Update complete")
else error "A file named %1 doesn't exist", NomFichierUpdatePlan
end if
end if
end proc

> ReadPlan:=proc(NomPlanReadPlan) local NomFichierReadPlan; global
Plan; NomFichierReadPlan:=cat(PATHDATA,NomPlanReadPlan,".txt"); if
Fichier_Existant(NomFichierReadPlan)=true then read
NomFichierReadPlan; else error "Such a design does not exist (no file
named %1 found)",NomFichierCreatePlan; end if ; end proc;

ReadPlan := proc(NomPlanReadPlan)
local NomFichierReadPlan;
global Plan;
NomFichierReadPlan := cat(PATHDATA, NomPlanReadPlan, ".txt");
if Fichier_Existant(NomFichierReadPlan) = true then read NomFichierReadPlan
else error
"Such a design does not exist (no file named %1 found)", NomFichierCreatePlan
end if
end proc

```

10.3.2. Matrice du plan et matrice des moments

```

> X_D_Set:=proc(DX_D_Set,ListIndetX_D_Set,SetX_D_Set) local
NomPlanX_D_Set,NomFichierX_D_Set,NX_D_Set,ListX_D_Set; global
Plan,XDX_D_Set; if type(SetX_D_Set,set) then
ListX_D_Set:=convert(SetX_D_Set,list) else ListX_D_Set:=SetX_D_Set end
if; if type(DX_D_Set,string) then
NomPlanX_D_Set:=cat(PATHDATA,DX_D_Set,".txt"); if
Fichier_Existant(NomPlanX_D_Set)=false then error "Plan inconnu" else
read NomPlanX_D_Set end if; NX_D_Set:=op(1,Plan); XDX_D_Set:=
convert([seq(subs(seq(op(aX_D_Set,ListIndetX_D_Set)=op(aX_D_Set,conver
t(Plan[iX_D_Set,1..NX_D_Set[2]],list)),aX_D_Set=1..nops(ListIndetX_D_S
et)),ListX_D_Set),iX_D_Set=1..NX_D_Set[1]]),Matrix) ; elif
type(DX_D_Set,Matrix) then NX_D_Set:=op(1,DX_D_Set); XDX_D_Set:=
convert([seq(subs(seq(op(aX_D_Set,ListIndetX_D_Set)=op(aX_D_Set,conver
t(DX_D_Set[iX_D_Set,1..NX_D_Set[2]],list)),aX_D_Set=1..nops(ListIndetX
_D_Set)),ListX_D_Set),iX_D_Set=1..NX_D_Set[1]]),Matrix) ; else error
"Le plan doit etre donné sous forme matricielle"; end if; return
XDX_D_Set; end proc;

```

```

X_D_Set := proc(DX_D_Set, ListIndetX_D_Set, SetX_D_Set)
local NomPlanX_D_Set, NomFichierX_D_Set, NX_D_Set, ListX_D_Set;
global Plan, XDX_D_Set;
  if type(SetX_D_Set, set) then ListX_D_Set := convert(SetX_D_Set, list)
  else ListX_D_Set := SetX_D_Set
  end if;
  if type(DX_D_Set, string) then
    NomPlanX_D_Set := cat(PATHDATA, DX_D_Set, ".txt");
    if Fichier_Existant(NomPlanX_D_Set) = false then error "Plan inconnu"
    else read NomPlanX_D_Set
    end if;
    NX_D_Set := op(1, Plan);
    XDX_D_Set := convert([seq(subs(seq(op(aX_D_Set, ListIndetX_D_Set) =
      op(aX_D_Set, convert(PlaniX_D_Set, 1..NX_D_Set2, list)),
      aX_D_Set = 1..nops(ListIndetX_D_Set)), ListX_D_Set), iX_D_Set = 1..NX_D_Set1)
      ], Matrix)
  elif type(DX_D_Set, Matrix) then
    NX_D_Set := op(1, DX_D_Set);
    XDX_D_Set := convert([seq(subs(seq(op(aX_D_Set, ListIndetX_D_Set) =
      op(aX_D_Set, convert(DX_D_SetiX_D_Set, 1..NX_D_Set2, list)),
      aX_D_Set = 1..nops(ListIndetX_D_Set)), ListX_D_Set), iX_D_Set = 1..NX_D_Set1)
      ], Matrix)
  else error "Le plan doit etre donné sous forme matricielle"
  end if;
  return XDX_D_Set
end proc

```

```

> X_D_Centre:=proc(DX_D_Centre,QX_D_Centre,PointsX_D_Centre) local
  NomPlanX_D_Centre,NomFichierX_D_Centre,NX_D_Centre; global
  Plan,XDX_D_Centre; if type(DX_D_Centre,string) then
    NomPlanX_D_Centre:=cat(PATHDATA,DX_D_Centre,".txt");
    NomFichierX_D_Centre:=cat(PATHDATA,DX_D_Centre,"_X_D_Centre_",convert(
    QX_D_Centre,string),".txt"); if
    Fichier_Existant(NomFichierX_D_Centre)=true then read
    NomFichierX_D_Centre else if Fichier_Existant(NomPlanX_D_Centre)=false
    then error "Plan inconnu" else read NomPlanX_D_Centre end if;
    NX_D_Centre:=op(1,Plan);
    XDX_D_Centre:=convert([seq(convert(K_X(convert(Plan[iX_D_Centre,1..NX_
    D_Centre[2]],list),QX_D_Centre),list),iX_D_Centre=1..NX_D_Centre[1]),s
    eq([1,seq(0,kX_D_Centre=2..binomial(NX_D_Centre[2]+QX_D_Centre,QX_D_Ce
    ntre)]),kX_D_Centre=1..PointsX_D_Centre)],Matrix); save XDX_D_Centre,
    NomFichierX_D_Centre; end if; elif type(DX_D_Centre,Matrix) then
    NX_D_Centre:=op(1,DX_D_Centre);
    XDX_D_Centre:=convert([seq(convert(K_X(convert(DX_D_Centre[iX_D_Centre
    ,1..NX_D_Centre[2]],list),QX_D_Centre),list),iX_D_Centre=1..NX_D_Centr
    e[1]),seq([1,seq(0,kX_D_Centre=2..binomial(NX_D_Centre[2]+QX_D_Centre,
    QX_D_Centre)]),kX_D_Centre=1..PointsX_D_Centre)],Matrix); else error
    "Le plan doit etre donné sous forme matricielle"; end if; return
    XDX_D_Centre; end proc;

```

```

X_D_Centre := proc(DX_D_Centre, QX_D_Centre, PointsX_D_Centre)
local NomPlanX_D_Centre, NomFichierX_D_Centre, NX_D_Centre;
global Plan, XDX_D_Centre;
  if type(DX_D_Centre, string) then
    NomPlanX_D_Centre := cat(PATHDATA, DX_D_Centre, ".txt");
    NomFichierX_D_Centre := cat(PATHDATA, DX_D_Centre, "_X_D_Centre_",
      convert(QX_D_Centre, string), ".txt");
    if Fichier_Existant(NomFichierX_D_Centre) = true then
      read NomFichierX_D_Centre
    else
      if Fichier_Existant(NomPlanX_D_Centre) = false then error "Plan inconnu"
      else read NomPlanX_D_Centre
      end if;
      NX_D_Centre := op(1, Plan);
      XDX_D_Centre := convert([seq(convert(
        K_X(convert(Plan[iX_D_Centre, 1..NX_D_Centre2], list), QX_D_Centre), list),
        iX_D_Centre = 1..NX_D_Centre1), seq([1, seq(0, kX_D_Centre =
        2..combinat : -binomial(NX_D_Centre2 + QX_D_Centre, QX_D_Centre)]),
        kX_D_Centre = 1..PointsX_D_Centre)], Matrix);
      save XDX_D_Centre, NomFichierX_D_Centre
    end if
  elif type(DX_D_Centre, Matrix) then
    NX_D_Centre := op(1, DX_D_Centre);
    XDX_D_Centre := convert([seq(convert(K_X(
      convert(DX_D_Centre[iX_D_Centre, 1..NX_D_Centre2], list), QX_D_Centre), list)
      , iX_D_Centre = 1..NX_D_Centre1), seq([1, seq(0, kX_D_Centre =
      2..combinat : -binomial(NX_D_Centre2 + QX_D_Centre, QX_D_Centre)]),
      kX_D_Centre = 1..PointsX_D_Centre)], Matrix)
    else error "Le plan doit etre donné sous forme matricielle"
    end if;
    return XDX_D_Centre
  end proc

> M_D_Centre:=proc(DM_D_Centre,QM_D_Centre,PointsM_D_Centre) local
NomPlanM_D_Centre,NomFichierM_D_Centre,NM_D_Centre,MDM_D_Centre;
global XDM_D_Centre,Plan; if type(DM_D_Centre,string) then
NomPlanM_D_Centre:=cat(PATHDATA,DM_D_Centre,".txt");
NomFichierM_D_Centre:=cat(PATHDATA,DM_D_Centre,"_M_D_Centre_",convert(
QM_D_Centre,string),".txt"); if
Fichier_Existant(NomFichierM_D_Centre)=true then read
NomFichierM_D_Centre else if Fichier_Existant(NomPlanM_D_Centre)=false
then error "Plan inconnu" else read NomPlanM_D_Centre end if;
NM_D_Centre:=op(1,Plan);
XDM_D_Centre:=X_D_Centre(Plan,QM_D_Centre,PointsM_D_Centre);
MDM_D_Centre:=Multiply(1/NM_D_Centre[1],Multiply(Transpose(XDM_D_Centr
e),XDM_D_Centre)); save MDM_D_Centre, NomFichierM_D_Centre; end if;
elif type(DM_D_Centre,Matrix) then NM_D_Centre:=op(1,DM_D_Centre);
XDM_D_Centre:=X_D_Centre(DM_D_Centre,QM_D_Centre,PointsM_D_Centre);
MDM_D_Centre:=Multiply(1/NM_D_Centre[1],Multiply(Transpose(XDM_D_Centr
e),XDM_D_Centre)); else error "Le plan doit etre donné sous forme
matricielle"; end if; return MDM_D_Centre; end proc;

```

```

M_D_Centre := proc(DM_D_Centre, QM_D_Centre, PointsM_D_Centre)
local
NomPlanM_D_Centre, NomFichierM_D_Centre, NM_D_Centre, MDM_D_Centre;
global XDM_D_Centre, Plan;
  iftype(DM_D_Centre, string) then
    NomPlanM_D_Centre := cat(PATHDATA, DM_D_Centre, ".txt");
    NomFichierM_D_Centre := cat(PATHDATA, DM_D_Centre, "_M_D_Centre_",
      convert(QM_D_Centre, string), ".txt");
    ifFichier_Existant(NomFichierM_D_Centre) = true then
      read NomFichierM_D_Centre
    else
      ifFichier_Existant(NomPlanM_D_Centre) = false then error "Plan inconnu"
      else read NomPlanM_D_Centre
      end if;
      NM_D_Centre := op(1, Plan);
      XDM_D_Centre := X_D_Centre(Plan, QM_D_Centre, PointsM_D_Centre);
      MDM_D_Centre := LinearAlgebra : -Multiply(1/NM_D_Centre1,
        LinearAlgebra : -Multiply(LinearAlgebra : -Transpose(XDM_D_Centre),
          XDM_D_Centre));
      save MDM_D_Centre, NomFichierM_D_Centre
    end if
  eliftype(DM_D_Centre, Matrix) then
    NM_D_Centre := op(1, DM_D_Centre);
    XDM_D_Centre := X_D_Centre(DM_D_Centre, QM_D_Centre, PointsM_D_Centre)
    ;
    MDM_D_Centre := LinearAlgebra : -Multiply(1/NM_D_Centre1,
      LinearAlgebra : -Multiply(LinearAlgebra : -Transpose(XDM_D_Centre),
        XDM_D_Centre))
  else error "Le plan doit etre donné sous forme matricielle"
  end if;
  return MDM_D_Centre
end proc

```

```

> M_D_Set:=proc(DM_D_Set,ListIndetM_D_Set,SetM_D_Set) local
NomPlanM_D_Set,NomFichierM_D_Set,NM_D_Set,MDM_D_Set; global
XDM_D_Set,Plan; if type(DM_D_Set,string) then
  NomPlanM_D_Set:=cat(PATHDATA,DM_D_Set,".txt"); if
  Fichier_Existant(NomPlanM_D_Set)=false then error "Plan inconnu" else
  read NomPlanM_D_Set end if; NM_D_Set:=op(1,Plan);
  XDM_D_Set:=X_D_Set(Plan,ListIndetM_D_Set,SetM_D_Set);
  MDM_D_Set:=Multiply(1/NM_D_Set[1],Multiply(Transpose(XDM_D_Set),XDM_D_
  Set)); elif type(DM_D_Set,Matrix) then NM_D_Set:=op(1,DM_D_Set);
  XDM_D_Set:=X_D_Set(DM_D_Set,ListIndetM_D_Set,SetM_D_Set);
  MDM_D_Set:=Multiply(1/NM_D_Set[1],Multiply(Transpose(XDM_D_Set),XDM_D_
  Set)); else error "Le plan doit etre donné sous forme matricielle";
end if; return MDM_D_Set; end proc;

```

```

M_D_Set := proc(DM_D_Set, ListIndetM_D_Set, SetM_D_Set)
local NomPlanM_D_Set, NomFichierM_D_Set, NM_D_Set, MDM_D_Set;
global XDM_D_Set, Plan;
  if type(DM_D_Set, string) then
    NomPlanM_D_Set := cat(PATHDATA, DM_D_Set, ".txt");
    if Fichier_Existant(NomPlanM_D_Set) = false then error "Plan inconnu"
    else read NomPlanM_D_Set
    end if;
    NM_D_Set := op(1, Plan);
    XDM_D_Set := X_D_Set(Plan, ListIndetM_D_Set, SetM_D_Set);
    MDM_D_Set := LinearAlgebra : -Multiply(1/NM_D_Set_1,
      LinearAlgebra : -Multiply(LinearAlgebra : -Transpose(XDM_D_Set), XDM_D_Set))
  elif type(DM_D_Set, Matrix) then
    NM_D_Set := op(1, DM_D_Set);
    XDM_D_Set := X_D_Set(DM_D_Set, ListIndetM_D_Set, SetM_D_Set);
    MDM_D_Set := LinearAlgebra : -Multiply(1/NM_D_Set_1,
      LinearAlgebra : -Multiply(LinearAlgebra : -Transpose(XDM_D_Set), XDM_D_Set))
  else error "Le plan doit etre donne sous forme matricielle"
  end if;
  return MDM_D_Set
end proc

> X_D_ModeleSat:=proc(DModeleSat,ListIndetModeleSat,EstSetModeleSat);
return X_D_Set(DModeleSat,ListIndetModeleSat,EstSetModeleSat); end
proc;

      X_D_ModeleSat := proc(DModeleSat, ListIndetModeleSat, EstSetModeleSat)
        return X_D_Set(DModeleSat, ListIndetModeleSat, EstSetModeleSat)
      end proc

> M_D_ModeleSat:=proc(DModeleSat,ListIndetModeleSat,EstSetModeleSat);
return M_D_Set(DModeleSat,ListIndetModeleSat,EstSetModeleSat); end
proc;

      M_D_ModeleSat := proc(DModeleSat, ListIndetModeleSat, EstSetModeleSat)
        return M_D_Set(DModeleSat, ListIndetModeleSat, EstSetModeleSat)
      end proc

> IsGLn:=proc(MatIsGLn) local DimIsGLn,BoolIsGLn;
DimIsGLn:=op(1,MatIsGLn);BoolIsGLn:=FAIL; if DimIsGLn[1]=DimIsGLn[2]
then if Rank(MatIsGLn)=DimIsGLn[1] then BoolIsGLn:=true; else
BoolIsGLn:=false; end if; else BoolIsGLn:=false; end if; return
BoolIsGLn end proc;

```

```

    IsGLn := proc(MatIsGLn)
local DimIsGLn, BoolIsGLn;
    DimIsGLn := op(1, MatIsGLn);
    BoolIsGLn := FAIL;
    if DimIsGLn1 = DimIsGLn2 then
        if LinearAlgebra : -Rank(MatIsGLn) = DimIsGLn1 then BoolIsGLn := true
        else BoolIsGLn := false
        end if
    else BoolIsGLn := false
    end if;
    return BoolIsGLn
end proc

> MD_IsGLn:=proc(DM_D_Centre,QM_D_Centre,PointsM_D_Centre) local
DimMD_IsGLn,BoolMD_IsGLn,MatMD_IsGLn;
MatMD_IsGLn:=M_D_Centre(DM_D_Centre,QM_D_Centre,PointsM_D_Centre);
DimMD_IsGLn:=op(1,MatMD_IsGLn);BoolMD_IsGLn:=FAIL; if
DimMD_IsGLn[1]=DimMD_IsGLn[2] then if Rank(MatMD_IsGLn)=DimMD_IsGLn[1]
then BoolMD_IsGLn:=true; else BoolMD_IsGLn:=false; end if; else
BoolMD_IsGLn:=false; end if; return BoolMD_IsGLn end proc;

MD_IsGLn := proc(DM_D_Centre, QM_D_Centre, PointsM_D_Centre)
local DimMD_IsGLn, BoolMD_IsGLn, MatMD_IsGLn;
    MatMD_IsGLn := M_D_Centre(DM_D_Centre, QM_D_Centre, PointsM_D_Centre);
    DimMD_IsGLn := op(1, MatMD_IsGLn);
    BoolMD_IsGLn := FAIL;
    if DimMD_IsGLn1 = DimMD_IsGLn2 then
        if LinearAlgebra : -Rank(MatMD_IsGLn) = DimMD_IsGLn1 then
            BoolMD_IsGLn := true
        else BoolMD_IsGLn := false
        end if
    else BoolMD_IsGLn := false
    end if;
    return BoolMD_IsGLn
end proc

> MatrixInverseRank:=proc(MatrixMatrixInverseRank) ; if
IsGLn(MatrixMatrixInverseRank) then return
MatrixInverse(MatrixMatrixInverseRank) else error "singular matrix"
end if; end proc;

MatrixInverseRank := proc(MatrixMatrixInverseRank)
    if IsGLn(MatrixMatrixInverseRank) then
        return LinearAlgebra : -MatrixInverse(MatrixMatrixInverseRank)
    else error "singular matrix"
    end if
end proc

```

10.3.3. Variance de l'estimateur BLUE

```

> K_X:=proc(ListK_X,QK_X) local iK_X,jK_X,ExposantsK_X,NK_X,MK_X
;NK_X:=nops(ListK_X); ExposantsK_X:=Exposants_Ord(QK_X,NK_X);
MK_X:=convert([seq(mul(ListK_X[iK_X]^ExposantsK_X[jK_X][iK_X],iK_X=1..
NK_X),jK_X=1..nops(ExposantsK_X))],Vector[row]); return MK_X; end
proc;

      K_X := proc(ListK_X, QK_X)
      local iK_X, jK_X, ExposantsK_X, NK_X, MK_X;
      NK_X := nops(ListK_X);
      ExposantsK_X := Exposants_Ord(QK_X, NK_X);
      MK_X := convert([seq(
          mul(ListK_X[iK_X]ExposantsK_X[jK_X][iK_X], iK_X = 1..NK_X),
          jK_X = 1..nops(ExposantsK_X))], Vectorrow);
      return MK_X
      end proc

> Var_Blue_D:=proc(DVar_Blue_D,QVar_Blue_D) local
NomPlanVar_Blue_D,NomFichierVar_Blue_D,NVar_Blue_D,XDVar_Blue_D,MdVar_
Blue_D,XVar_Blue_D; global Plan,VarVar_Blue_D; if
type(DVar_Blue_D,string) then
NomPlanVar_Blue_D:=cat(PATHDATA,DVar_Blue_D,".txt");
NomFichierVar_Blue_D:=cat(PATHDATA,DVar_Blue_D,"_Var_Blue_D",convert(
QVar_Blue_D,string),".txt"); if
Fichier_Existant(NomFichierVar_Blue_D)=true then read
NomFichierVar_Blue_D else if Fichier_Existant(NomPlanVar_Blue_D)=false
then error "Plan inconnu" else read NomPlanVar_Blue_D end if;
NVar_Blue_D:=op(1,Plan);
XVar_Blue_D:=K_X([seq(var[iVar_Blue_D],iVar_Blue_D=1..NVar_Blue_D[2])],
,QVar_Blue_D); try
VarVar_Blue_D:=sigma^2*(Multiply(XVar_Blue_D,MatrixVectorMultiply(Matr
ixInverseRank(M_D_Centre(Plan,QVar_Blue_D,0)),Transpose(XVar_Blue_D)))
); catch "singular matrix" : print("M_D is not full rank") catch:
print("Something went wrong while computing M_D") end try; save
VarVar_Blue_D, NomFichierVar_Blue_D; end if; elif
type(DVar_Blue_D,Matrix) then NVar_Blue_D:=op(1,DVar_Blue_D);
XVar_Blue_D:=K_X([seq(var[iVar_Blue_D],iVar_Blue_D=1..NVar_Blue_D[2])],
,QVar_Blue_D); try
VarVar_Blue_D:=sigma^2*(Multiply(XVar_Blue_D,MatrixVectorMultiply(Matr
ixInverseRank(M_D_Centre(Plan,QVar_Blue_D,0)),Transpose(XVar_Blue_D)))
); catch "singular matrix" : print("M_D is not full rank") catch:
print("Something went wrong while computing M_D") end try; else error
"Le plan doit etre donné sous forme matricielle"; end if; return
VarVar_Blue_D; end proc;

```



```

Var_Blue_D := proc(DVar_Blue_D, QVar_Blue_D)
local NomPlanVar_Blue_D, NomFichierVar_Blue_D, NVar_Blue_D, XDVar_Blue_D,
MDVar_Blue_D, XVar_Blue_D;
global Plan, VarVar_Blue_D;
  if type(DVar_Blue_D, string) then
    NomPlanVar_Blue_D := cat(PATHDATA, DVar_Blue_D, ".txt");
    NomFichierVar_Blue_D := cat(PATHDATA, DVar_Blue_D, "_Var_Blue_D_",
      convert(QVar_Blue_D, string), ".txt");
    if Fichier_Existant(NomFichierVar_Blue_D) = true then
      read NomFichierVar_Blue_D
    else
      if Fichier_Existant(NomPlanVar_Blue_D) = false then error "Plan inconnu"
      else read NomPlanVar_Blue_D
      end if;
      NVar_Blue_D := op(1, Plan);
      XVar_Blue_D :=
        K_X([seq(variVar_Blue_D, iVar_Blue_D = 1..NVar_Blue_D2), QVar_Blue_D]);
      try VarVar_Blue_D :=  $\sigma^2$  * LinearAlgebra : - Multiply(XVar_Blue_D,
        LinearAlgebra : - MatrixVectorMultiply(
          MatrixInverseRank(M_D_Centre(Plan, QVar_Blue_D, 0)),
          LinearAlgebra : - Transpose(XVar_Blue_D)))
      catch "singular matrix" : print("M_D is not full rank")
      catch : print( "Something went wrong while computing M_D")
      end try;
      save VarVar_Blue_D, NomFichierVar_Blue_D
    end if
  elif type(DVar_Blue_D, Matrix) then
    NVar_Blue_D := op(1, DVar_Blue_D);
    XVar_Blue_D :=
      K_X([seq(variVar_Blue_D, iVar_Blue_D = 1..NVar_Blue_D2), QVar_Blue_D]);
    try VarVar_Blue_D :=  $\sigma^2$  * LinearAlgebra : - Multiply(XVar_Blue_D,
      LinearAlgebra : - MatrixVectorMultiply(
        MatrixInverseRank(M_D_Centre(Plan, QVar_Blue_D, 0)),
        LinearAlgebra : - Transpose(XVar_Blue_D)))
    catch "singular matrix" : print("M_D is not full rank")
    catch : print( "Something went wrong while computing M_D")
    end try
  else error "Le plan doit etre donné sous forme matricielle"
  end if;
  return VarVar_Blue_D
end proc

```

```

> Var_Blue_D_Norm:=proc(DVar_Blue_D_Norm,QVar_Blue_D_Norm) local
  NomPlanVar_Blue_D_Norm,NomFichierVar_Blue_D_Norm,NVar_Blue_D_Norm,XDVar
  _Blue_D_Norm,MDVar_Blue_D_Norm,XVar_Blue_D_Norm; global
  Plan,VarVar_Blue_D_Norm; if type(DVar_Blue_D_Norm,string) then
  NomPlanVar_Blue_D_Norm:=cat(PATHDATA,DVar_Blue_D_Norm,".txt");
  NomFichierVar_Blue_D_Norm:=cat(PATHDATA,DVar_Blue_D_Norm,"_Var_Blue_D_
  Norm_",convert(QVar_Blue_D_Norm,string),".txt"); if
  Fichier_Existant(NomFichierVar_Blue_D_Norm)=true then read
  NomFichierVar_Blue_D_Norm else if
  Fichier_Existant(NomPlanVar_Blue_D_Norm)=false then error "Plan
  inconnu" else read NomPlanVar_Blue_D_Norm end if;
  NVar_Blue_D_Norm:=op(1,Plan);
  XVar_Blue_D_Norm:=K_X([seq(var[iVar_Blue_D_Norm],iVar_Blue_D_Norm=1..N
  Var_Blue_D_Norm[2])],QVar_Blue_D_Norm); try
  VarVar_Blue_D_Norm:=(Multiply(XVar_Blue_D_Norm,MatrixVectorMultiply(Ma
  trixInverseRank(M_D_Centre(Plan,QVar_Blue_D_Norm,0)),Transpose(XVar_Bl
  ue_D_Norm)))); catch "singular matrix" : print("M_D is not full rank")
  catch: print("Something went wrong while computing M_D") end try; save
  VarVar_Blue_D_Norm, NomFichierVar_Blue_D_Norm; end if; elif
  type(DVar_Blue_D_Norm,Matrix) then
  NVar_Blue_D_Norm:=op(1,DVar_Blue_D_Norm);
  XVar_Blue_D_Norm:=K_X([seq(var[iVar_Blue_D_Norm],iVar_Blue_D_Norm=1..N
  Var_Blue_D_Norm[2])],QVar_Blue_D_Norm); try
  VarVar_Blue_D_Norm:=(Multiply(XVar_Blue_D_Norm,MatrixVectorMultiply(Ma
  trixInverseRank(M_D_Centre(DVar_Blue_D_Norm,QVar_Blue_D_Norm,0)),Trans
  pose(XVar_Blue_D_Norm)))); catch "singular matrix": print("M_D is not
  full rank") catch: print("Something went wrong while computing M_D")
  end try; else error "Le plan doit etre donné sous forme matricielle";
  end if; return VarVar_Blue_D_Norm; end proc;

Var_Blue_D_Norm := proc(DVar_Blue_D_Norm, QVar_Blue_D_Norm)
local NomPlanVar_Blue_D_Norm, NomFichierVar_Blue_D_Norm,
  NVar_Blue_D_Norm, XDVar_Blue_D_Norm, MDVar_Blue_D_Norm,
  XVar_Blue_D_Norm;
global Plan, VarVar_Blue_D_Norm;
  if type(DVar_Blue_D_Norm, string) then
    NomPlanVar_Blue_D_Norm := cat(PATHDATA, DVar_Blue_D_Norm, ".txt");
    NomFichierVar_Blue_D_Norm := cat(PATHDATA, DVar_Blue_D_Norm,
      "_Var_Blue_D_Norm_", convert(QVar_Blue_D_Norm, string), ".txt");
    if Fichier_Existant(NomFichierVar_Blue_D_Norm) = true then
      read NomFichierVar_Blue_D_Norm
    else
      if Fichier_Existant(NomPlanVar_Blue_D_Norm) = false then error "Plan inconnu"
      else read NomPlanVar_Blue_D_Norm
      end if;
      NVar_Blue_D_Norm := op(1, Plan);
      XVar_Blue_D_Norm := K_X(
        [seq(var iVar_Blue_D_Norm, iVar_Blue_D_Norm = 1..NVar_Blue_D_Norm[2]),
        QVar_Blue_D_Norm];
      try VarVar_Blue_D_Norm := LinearAlgebra : -Multiply(XVar_Blue_D_Norm,
        LinearAlgebra : -MatrixVectorMultiply(
          MatrixInverseRank(M_D_Centre(Plan, QVar_Blue_D_Norm, 0)),
          LinearAlgebra : -Transpose(XVar_Blue_D_Norm)))

```

```

    catch "singular matrix" : print("M_D is not full rank")
    catch : print( "Something went wrong while computing M_D")
    end try;
    save VarVar_Blue_D_Norm, NomFichierVar_Blue_D_Norm
end if
elif type(DVar_Blue_D_Norm, Matrix) then
    NVar_Blue_D_Norm := op(1, DVar_Blue_D_Norm);
    XVar_Blue_D_Norm := K_X(
        [seq(variVar_Blue_D_Norm, iVar_Blue_D_Norm = 1..NVar_Blue_D_Norm2)],
        QVar_Blue_D_Norm);
    try VarVar_Blue_D_Norm := LinearAlgebra : -Multiply(XVar_Blue_D_Norm,
        LinearAlgebra : -MatrixVectorMultiply(
            MatrixInverseRank(M_D_Centre(DVar_Blue_D_Norm, QVar_Blue_D_Norm, 0)),
            LinearAlgebra : -Transpose(XVar_Blue_D_Norm)))
    catch "singular matrix" : print("M_D is not full rank")
    catch : print( "Something went wrong while computing M_D")
    end try
else error "Le plan doit etre donné sous forme matricielle"
end if;
return VarVar_Blue_D_Norm
end proc

> Var_Blue_D_Centre:=proc(DVar_Blue_D_Centre,QVar_Blue_D_Centre,PointsV
ar_Blue_D_Centre) local
NomPlanVar_Blue_D_Centre,NomFichierVar_Blue_D_Centre,NVar_Blue_D_Centr
e,XDVar_Blue_D_Centre,XVar_Blue_D_Centre; global
Plan,VarVar_Blue_D_Centre,MDVar_Blue_D_Centre; if
type(DVar_Blue_D_Centre,string) then
NomPlanVar_Blue_D_Centre:=cat(PATHDATA,DVar_Blue_D_Centre,".txt");
NomFichierVar_Blue_D_Centre:=cat(PATHDATA,DVar_Blue_D_Centre,"_Var_Blu
e_D_Centre_",convert(QVar_Blue_D_Centre,string),".txt"); if
Fichier_Existant(NomFichierVar_Blue_D_Centre)=true then read
NomFichierVar_Blue_D_Centre else if
Fichier_Existant(NomPlanVar_Blue_D_Centre)=false then error "Plan
inconnu" else read NomPlanVar_Blue_D_Centre end if;
NVar_Blue_D_Centre:=op(1,Plan);
XVar_Blue_D_Centre:=K_X([seq(var[iVar_Blue_D_Centre],iVar_Blue_D_Centr
e=1..NVar_Blue_D_Centre[2])],QVar_Blue_D_Centre); try
VarVar_Blue_D_Centre:=sigma^2*(Multiply(XVar_Blue_D_Centre,MatrixVecto
rMultiply(MatrixInverseRank(M_D_Centre(Plan,QVar_Blue_D_Centre,PointsV
ar_Blue_D_Centre)),Transpose(XVar_Blue_D_Centre)))); catch "singular
matrix" : print("M_D is not full rank") catch: print("Something went
wrong while computing M_D") end try; save VarVar_Blue_D_Centre,
NomFichierVar_Blue_D_Centre; end if; elif
type(DVar_Blue_D_Centre,Matrix) then
NVar_Blue_D_Centre:=op(1,DVar_Blue_D_Centre);
XVar_Blue_D_Centre:=K_X([seq(var[iVar_Blue_D_Centre],iVar_Blue_D_Centr
e=1..NVar_Blue_D_Centre[2])],QVar_Blue_D_Centre); try
VarVar_Blue_D_Centre:=sigma^2*(Multiply(XVar_Blue_D_Centre,MatrixVecto
rMultiply(MatrixInverseRank(M_D_Centre(DVar_Blue_D_Centre,QVar_Blue_D_
Centre,PointsVar_Blue_D_Centre)),Transpose(XVar_Blue_D_Centre))));
catch "singular matrix" : print("M_D is not full rank") catch:
print("Something went wrong while computing M_D") end try; else error
"Le plan doit etre donné sous forme matricielle"; end if; return
VarVar_Blue_D_Centre; end proc;

```

```

Var_Blue_D_Centre := proc(
  DVar_Blue_D_Centre, QVar_Blue_D_Centre, PointsVar_Blue_D_Centre)
local NomPlanVar_Blue_D_Centre, NomFichierVar_Blue_D_Centre,
  NVar_Blue_D_Centre, XDVar_Blue_D_Centre, XVar_Blue_D_Centre;
global Plan, VarVar_Blue_D_Centre, MDVar_Blue_D_Centre;
  if type(DVar_Blue_D_Centre, string) then
    NomPlanVar_Blue_D_Centre := cat(PATHDATA, DVar_Blue_D_Centre, ".txt");
    NomFichierVar_Blue_D_Centre := cat(PATHDATA, DVar_Blue_D_Centre,
      "_Var_Blue_D_Centre_", convert(QVar_Blue_D_Centre, string), ".txt");
    if Fichier_Existant(NomFichierVar_Blue_D_Centre) = true then
      read NomFichierVar_Blue_D_Centre
    else
      if Fichier_Existant(NomPlanVar_Blue_D_Centre) = false then error "Plan inconnu"
      else read NomPlanVar_Blue_D_Centre
      end if;
      NVar_Blue_D_Centre := op(1, Plan);
      XVar_Blue_D_Centre := K_X([
        seq(variVar_Blue_D_Centre, iVar_Blue_D_Centre = 1..NVar_Blue_D_Centre2)
        , QVar_Blue_D_Centre]);
      try VarVar_Blue_D_Centre :=  $\sigma^2$  * LinearAlgebra : -Multiply(XVar_Blue_D_Centre,
        LinearAlgebra : -MatrixVectorMultiply(MatrixInverseRank(
        M_D_Centre(Plan, QVar_Blue_D_Centre, PointsVar_Blue_D_Centre)),
        LinearAlgebra : -Transpose(XVar_Blue_D_Centre)))
      catch "singular matrix" : print("M_D is not full rank")
      catch : print("Something went wrong while computing M_D")
      end try;
      save VarVar_Blue_D_Centre, NomFichierVar_Blue_D_Centre
    end if
  elif type(DVar_Blue_D_Centre, Matrix) then
    NVar_Blue_D_Centre := op(1, DVar_Blue_D_Centre);
    XVar_Blue_D_Centre := K_X([
      seq(variVar_Blue_D_Centre, iVar_Blue_D_Centre = 1..NVar_Blue_D_Centre2)
      , QVar_Blue_D_Centre]);
    try VarVar_Blue_D_Centre :=  $\sigma^2$  * LinearAlgebra : -Multiply(XVar_Blue_D_Centre,
      LinearAlgebra : -MatrixVectorMultiply(MatrixInverseRank(M_D_Centre(
      DVar_Blue_D_Centre, QVar_Blue_D_Centre, PointsVar_Blue_D_Centre)),
      LinearAlgebra : -Transpose(XVar_Blue_D_Centre)))
    catch "singular matrix" : print("M_D is not full rank")
    catch : print("Something went wrong while computing M_D")
    end try
  else error "Le plan doit etre donné sous forme matricielle"
  end if;
  return VarVar_Blue_D_Centre
end proc

```

```

> Var_Norm_Blue_D_Centre:=proc(DVar_Norm_Blue_D_Centre,QVar_Norm_Blue_D
_Centre,PointsVar_Norm_Blue_D_Centre) local
NomPlanVar_Norm_Blue_D_Centre,NomFichierVar_Norm_Blue_D_Centre,NVar_No
rm_Blue_D_Centre,XDVar_Norm_Blue_D_Centre,XVar_Norm_Blue_D_Centre;
global Plan,VarVar_Norm_Blue_D_Centre,MDVar_Norm_Blue_D_Centre; if
type(DVar_Norm_Blue_D_Centre,string) then
NomPlanVar_Norm_Blue_D_Centre:=cat(PATHDATA,DVar_Norm_Blue_D_Centre,".
txt");
NomFichierVar_Norm_Blue_D_Centre:=cat(PATHDATA,DVar_Norm_Blue_D_Centre
,"_Var_Norm_Blue_D_Centre_",convert(QVar_Norm_Blue_D_Centre,string),".
txt"); if Fichier_Existant(NomFichierVar_Norm_Blue_D_Centre)=true then
read NomFichierVar_Norm_Blue_D_Centre else if
Fichier_Existant(NomPlanVar_Norm_Blue_D_Centre)=false then error "Plan
inconnu" else read NomPlanVar_Norm_Blue_D_Centre end if;
NVar_Norm_Blue_D_Centre:=op(1,Plan);
XVar_Norm_Blue_D_Centre:=K_X([seq(var[iVar_Norm_Blue_D_Centre],iVar_No
rm_Blue_D_Centre=1..NVar_Norm_Blue_D_Centre[2])],QVar_Norm_Blue_D_Cent
re); try
VarVar_Norm_Blue_D_Centre:=(Multiply(XVar_Norm_Blue_D_Centre,MatrixVec
torMultiply(MatrixInverseRank(M_D_Centre(Plan,QVar_Norm_Blue_D_Centre,
PointsVar_Norm_Blue_D_Centre)),Transpose(XVar_Norm_Blue_D_Centre)))));
catch "singular matrix" : print("M_D is not full rank") catch:
print("Something went wrong while computing M_D") end try; save
VarVar_Norm_Blue_D_Centre, NomFichierVar_Norm_Blue_D_Centre; end if;
elif type(DVar_Norm_Blue_D_Centre,Matrix) then
NVar_Norm_Blue_D_Centre:=op(1,DVar_Norm_Blue_D_Centre);
XVar_Norm_Blue_D_Centre:=K_X([seq(var[iVar_Norm_Blue_D_Centre],iVar_No
rm_Blue_D_Centre=1..NVar_Norm_Blue_D_Centre[2])],QVar_Norm_Blue_D_Cent
re); try
VarVar_Norm_Blue_D_Centre:=(Multiply(XVar_Norm_Blue_D_Centre,MatrixVec
torMultiply(MatrixInverseRank(M_D_Centre(DVar_Norm_Blue_D_Centre,QVar_
Norm_Blue_D_Centre,PointsVar_Norm_Blue_D_Centre)),Transpose(XVar_Norm_
Blue_D_Centre))))); catch "singular matrix" : print("M_D is not full
rank") catch: print("Something went wrong while computing M_D") end
try; else error "Le plan doit etre donné sous forme matricielle"; end
if; return VarVar_Norm_Blue_D_Centre; end proc;

Var_Norm_Blue_D_Centre := proc(DVar_Norm_Blue_D_Centre,
QVar_Norm_Blue_D_Centre, PointsVar_Norm_Blue_D_Centre)
local NomPlanVar_Norm_Blue_D_Centre, NomFichierVar_Norm_Blue_D_Centre,
NVar_Norm_Blue_D_Centre, XDVar_Norm_Blue_D_Centre,
XVar_Norm_Blue_D_Centre;
global Plan, VarVar_Norm_Blue_D_Centre, MDVar_Norm_Blue_D_Centre;
if type(DVar_Norm_Blue_D_Centre, string) then
    NomPlanVar_Norm_Blue_D_Centre :=
        cat(PATHDATA, DVar_Norm_Blue_D_Centre, ".txt");
    NomFichierVar_Norm_Blue_D_Centre := cat(PATHDATA,
        DVar_Norm_Blue_D_Centre, "_Var_Norm_Blue_D_Centre_",
        convert(QVar_Norm_Blue_D_Centre, string), ".txt");
    if Fichier_Existant(NomFichierVar_Norm_Blue_D_Centre) = true then
        read NomFichierVar_Norm_Blue_D_Centre
    else
        if Fichier_Existant(NomPlanVar_Norm_Blue_D_Centre) = false then
            error "Plan inconnu"
        else read NomPlanVar_Norm_Blue_D_Centre
        end if;

```

```

NVar_Norm_Blue_D_Centre := op(1, Plan);
XVar_Norm_Blue_D_Centre := K_X([seq(variVar_Norm_Blue_D_Centre,
iVar_Norm_Blue_D_Centre = 1..NVar_Norm_Blue_D_Centre2)],
QVar_Norm_Blue_D_Centre);
try VarVar_Norm_Blue_D_Centre := LinearAlgebra : -Multiply(
XVar_Norm_Blue_D_Centre, LinearAlgebra : -MatrixVectorMultiply(
MatrixInverseRank(
M_D_Centre(Plan, QVar_Norm_Blue_D_Centre, PointsVar_Norm_Blue_D_Centre)),
LinearAlgebra : -Transpose(XVar_Norm_Blue_D_Centre)))
catch "singular matrix" : print("M_D is not full rank")
catch : print("Something went wrong while computing M_D")
end try;
save VarVar_Norm_Blue_D_Centre, NomFichierVar_Norm_Blue_D_Centre
end if
elif type(DVar_Norm_Blue_D_Centre, Matrix) then
NVar_Norm_Blue_D_Centre := op(1, DVar_Norm_Blue_D_Centre);
XVar_Norm_Blue_D_Centre := K_X([seq(variVar_Norm_Blue_D_Centre,
iVar_Norm_Blue_D_Centre = 1..NVar_Norm_Blue_D_Centre2)],
QVar_Norm_Blue_D_Centre);
try VarVar_Norm_Blue_D_Centre := LinearAlgebra : -Multiply(
XVar_Norm_Blue_D_Centre, LinearAlgebra : -MatrixVectorMultiply(
MatrixInverseRank(M_D_Centre(DVar_Norm_Blue_D_Centre,
QVar_Norm_Blue_D_Centre, PointsVar_Norm_Blue_D_Centre)),
LinearAlgebra : -Transpose(XVar_Norm_Blue_D_Centre)))
catch "singular matrix" : print("M_D is not full rank")
catch : print("Something went wrong while computing M_D")
end try
else error "Le plan doit etre donné sous forme matricielle"
end if;
return VarVar_Norm_Blue_D_Centre
end proc

> Var_Norm_Blue_D_Centre_ListIndets:=proc(DVar_Norm_Blue_D_Centre_ListI
ndets,QVar_Norm_Blue_D_Centre_ListIndets,PointsVar_Norm_Blue_D_Centre_
ListIndets) local
NomPlanVar_Norm_Blue_D_Centre_ListIndets,NomFichierVar_Norm_Blue_D_Cen
tre_ListIndets,NVar_Norm_Blue_D_Centre_ListIndets,XDVar_Norm_Blue_D_Ce
ntre_ListIndets,XVar_Norm_Blue_D_Centre_ListIndets; global
Plan,VarVar_Norm_Blue_D_Centre_ListIndets,MDVar_Norm_Blue_D_Centre; if
type(DVar_Norm_Blue_D_Centre_ListIndets,string) then
NomPlanVar_Norm_Blue_D_Centre_ListIndets:=cat(PATHDATA,DVar_Norm_Blue_
D_Centre_ListIndets,".txt");
NomFichierVar_Norm_Blue_D_Centre_ListIndets:=cat(PATHDATA,DVar_Norm_Bl
ue_D_Centre_ListIndets,"_Var_Norm_Blue_D_Centre_ListIndets_",convert(Q
Var_Norm_Blue_D_Centre_ListIndets,string),".txt"); if
Fichier_Existant(NomFichierVar_Norm_Blue_D_Centre_ListIndets)=true
then read NomFichierVar_Norm_Blue_D_Centre_ListIndets else if
Fichier_Existant(NomPlanVar_Norm_Blue_D_Centre_ListIndets)=false then
error "Plan inconnu" else read
NomPlanVar_Norm_Blue_D_Centre_ListIndets end if;

```

```

NVar_Norm_Blue_D_Centre_ListIndets:=op(1,Plan);
XVar_Norm_Blue_D_Centre_ListIndets:=K_X([seq(var[iVar_Norm_Blue_D_Cent
re_ListIndets],iVar_Norm_Blue_D_Centre_ListIndets=1..NVar_Norm_Blue_D_
Centre_ListIndets[2])],QVar_Norm_Blue_D_Centre_ListIndets); try
VarVar_Norm_Blue_D_Centre_ListIndets:=(Multiply(XVar_Norm_Blue_D_Centr
e_ListIndets,MatrixVectorMultiply(MatrixInverseRank(M_D_Centre(Plan,QV
ar_Norm_Blue_D_Centre_ListIndets,PointsVar_Norm_Blue_D_Centre_ListInde
ts)),Transpose(XVar_Norm_Blue_D_Centre_ListIndets)))); catch "singular
matrix" : print("M_D is not full rank") catch: print("Something went
wrong while computing M_D") end try; save
VarVar_Norm_Blue_D_Centre_ListIndets,
NomFichierVar_Norm_Blue_D_Centre_ListIndets; end if; elif
type(DVar_Norm_Blue_D_Centre_ListIndets,Matrix) then
NVar_Norm_Blue_D_Centre_ListIndets:=op(1,DVar_Norm_Blue_D_Centre_ListI
ndets);
XVar_Norm_Blue_D_Centre_ListIndets:=K_X([seq(var[iVar_Norm_Blue_D_Cent
re_ListIndets],iVar_Norm_Blue_D_Centre_ListIndets=1..NVar_Norm_Blue_D_
Centre_ListIndets[2])],QVar_Norm_Blue_D_Centre_ListIndets); try
VarVar_Norm_Blue_D_Centre_ListIndets:=(Multiply(XVar_Norm_Blue_D_Centr
e_ListIndets,MatrixVectorMultiply(MatrixInverseRank(M_D_Centre(DVar_No
rm_Blue_D_Centre_ListIndets,QVar_Norm_Blue_D_Centre_ListIndets,PointsV
ar_Norm_Blue_D_Centre_ListIndets)),Transpose(XVar_Norm_Blue_D_Centre_L
istIndets)))); catch "singular matrix" : print("M_D is not full rank")
catch: print("Something went wrong while computing M_D") end try;
else error "Le plan doit etre donné sous forme matricielle"; end if;
return
[VarVar_Norm_Blue_D_Centre_ListIndets,
{seq(var[iVar_Norm_Blue_D_Centre_ListIndets],iVar_Norm_Blue_D_Centre_
ListIndets=1..NVar_Norm_Blue_D_Centre_ListIndets[2])}]; end proc;

Var_Norm_Blue_D_Centre_ListIndets := proc(DVar_Norm_Blue_D_Centre_ListIndets,
QVar_Norm_Blue_D_Centre_ListIndets, PointsVar_Norm_Blue_D_Centre_ListIndets)
local NomPlanVar_Norm_Blue_D_Centre_ListIndets,
NomFichierVar_Norm_Blue_D_Centre_ListIndets,
NVar_Norm_Blue_D_Centre_ListIndets, XDVar_Norm_Blue_D_Centre_ListIndets,
XVar_Norm_Blue_D_Centre_ListIndets;
global
Plan, VarVar_Norm_Blue_D_Centre_ListIndets, MDVar_Norm_Blue_D_Centre;
if type(DVar_Norm_Blue_D_Centre_ListIndets, string) then
    NomPlanVar_Norm_Blue_D_Centre_ListIndets :=
        cat(PATHDATA, DVar_Norm_Blue_D_Centre_ListIndets, ".txt");
    NomFichierVar_Norm_Blue_D_Centre_ListIndets := cat(PATHDATA,
        DVar_Norm_Blue_D_Centre_ListIndets, "_Var_Norm_Blue_D_Centre_ListIndets_",
        convert(QVar_Norm_Blue_D_Centre_ListIndets, string), ".txt");
    if Fichier_Existant(NomFichierVar_Norm_Blue_D_Centre_ListIndets) = true then
        read NomFichierVar_Norm_Blue_D_Centre_ListIndets
    else
        if Fichier_Existant(NomPlanVar_Norm_Blue_D_Centre_ListIndets) = false then
            error "Plan inconnu"
        else read NomPlanVar_Norm_Blue_D_Centre_ListIndets
        end if;
        NVar_Norm_Blue_D_Centre_ListIndets := op(1, Plan);
        XVar_Norm_Blue_D_Centre_ListIndets := K_X([seq(
            var[iVar_Norm_Blue_D_Centre_ListIndets, iVar_Norm_Blue_D_Centre_ListIndets
            = 1..NVar_Norm_Blue_D_Centre_ListIndets[2]),
            QVar_Norm_Blue_D_Centre_ListIndets];

```

```

    try VarVar_Norm_Blue_D_Centre_ListIndets := LinearAlgebra : -Multiply(
        XVar_Norm_Blue_D_Centre_ListIndets, LinearAlgebra : -MatrixVectorMultiply(
            MatrixInverseRank(M_D_Centre(Plan, QVar_Norm_Blue_D_Centre_ListIndets,
                PointsVar_Norm_Blue_D_Centre_ListIndets)),
            LinearAlgebra : -Transpose(XVar_Norm_Blue_D_Centre_ListIndets)))
    catch "singular matrix" : print("M_D is not full rank")
    catch : print( "Something went wrong while computing M_D")
    end try;
    save VarVar_Norm_Blue_D_Centre_ListIndets,
        NomFichierVar_Norm_Blue_D_Centre_ListIndets
    end if
    eliftype(DVar_Norm_Blue_D_Centre_ListIndets, Matrix) then
        NVar_Norm_Blue_D_Centre_ListIndets :=
            op(1, DVar_Norm_Blue_D_Centre_ListIndets);
        XVar_Norm_Blue_D_Centre_ListIndets := K_X([seq(
            var iVar_Norm_Blue_D_Centre_ListIndets, iVar_Norm_Blue_D_Centre_ListIndets
            = 1..NVar_Norm_Blue_D_Centre_ListIndets_2)],
            QVar_Norm_Blue_D_Centre_ListIndets);
        try VarVar_Norm_Blue_D_Centre_ListIndets := LinearAlgebra : -Multiply(
            XVar_Norm_Blue_D_Centre_ListIndets, LinearAlgebra : -MatrixVectorMultiply(
                MatrixInverseRank(M_D_Centre(DVar_Norm_Blue_D_Centre_ListIndets,
                    QVar_Norm_Blue_D_Centre_ListIndets, PointsVar_Norm_Blue_D_Centre_ListIndets))
                , LinearAlgebra : -Transpose(XVar_Norm_Blue_D_Centre_ListIndets)))
        catch "singular matrix" : print("M_D is not full rank")
        catch : print( "Something went wrong while computing M_D")
        end try
    else error "Le plan doit etre donné sous forme matricielle"
    end if;
    return [VarVar_Norm_Blue_D_Centre_ListIndets, {seq(
        var iVar_Norm_Blue_D_Centre_ListIndets, iVar_Norm_Blue_D_Centre_ListIndets
        = 1..NVar_Norm_Blue_D_Centre_ListIndets_2)}}]
end proc

> Var_Blue_D_Norm_Set:=proc(DVar_Blue_D_Norm_Set,ListIndetVar_Blue_D_Norm_Set,SetVar_Blue_D_Norm_Set) local
    NomPlanVar_Blue_D_Norm_Set,NomFichierVar_Blue_D_Norm_Set,NVar_Blue_D_Norm_Set,XDVar_Blue_D_Norm_Set,MDVar_Blue_D_Norm_Set,XVar_Blue_D_Norm_Set,ListVar_Blue_D_Norm_Set; global Plan,VarVar_Blue_D_Norm_Set; if
    type(SetVar_Blue_D_Norm_Set,set) then
        ListVar_Blue_D_Norm_Set:=convert(SetVar_Blue_D_Norm_Set,list) else
        ListVar_Blue_D_Norm_Set:=SetVar_Blue_D_Norm_Set end if; if
    type(DVar_Blue_D_Norm_Set,string) then
        NomPlanVar_Blue_D_Norm_Set:=cat(PATHDATA,DVar_Blue_D_Norm_Set,".txt");
        if Fichier_Existant(NomPlanVar_Blue_D_Norm_Set)=false then error "Plan inconnu" else read NomPlanVar_Blue_D_Norm_Set end if;
        NVar_Blue_D_Norm_Set:=op(1,Plan);
        XVar_Blue_D_Norm_Set:=convert(ListVar_Blue_D_Norm_Set,Vector[row]);
        try VarVar_Blue_D_Norm_Set:=(Multiply(XVar_Blue_D_Norm_Set,MatrixVectorMultiply(MatrixInverseRank(M_D_Set(Plan,ListIndetVar_Blue_D_Norm_Set,ListVar_Blue_D_Norm_Set)),Transpose(XVar_Blue_D_Norm_Set))))); catch
        "singular matrix" : print("M_D is not full rank") catch:
        print("Something went wrong while computing M_D") end try; elif

```



```

type(DVar_Blue_D_Norm_Set, Matrix) then NVar_Blue_D_Norm_Set := op(1, DVar_Blue_D_Norm_Set);
XVar_Blue_D_Norm_Set := convert(ListVar_Blue_D_Norm_Set, Vector[row]);
try VarVar_Blue_D_Norm_Set := (Multiply(XVar_Blue_D_Norm_Set, MatrixVectorMul
tiply(MatrixInverseRank(M_D_Set(DVar_Blue_D_Norm_Set, ListIndetVar_Blue
_D_Norm_Set, ListVar_Blue_D_Norm_Set)), Transpose(XVar_Blue_D_Norm_Set))
)); catch "singular matrix" : print("M_D is not full rank") catch:
print("Something went wrong while computing M_D") end try; else error
"Le plan doit etre donné sous forme matricielle"; end if; return
VarVar_Blue_D_Norm_Set; end proc;

Var_Blue_D_Norm_Set := proc(DVar_Blue_D_Norm_Set, ListIndetVar_Blue_D_Norm_Set,
SetVar_Blue_D_Norm_Set)
local NomPlanVar_Blue_D_Norm_Set, NomFichierVar_Blue_D_Norm_Set,
NVar_Blue_D_Norm_Set, XDVar_Blue_D_Norm_Set, MDVar_Blue_D_Norm_Set,
XVar_Blue_D_Norm_Set, ListVar_Blue_D_Norm_Set;
global Plan, VarVar_Blue_D_Norm_Set;
if type(SetVar_Blue_D_Norm_Set, set) then
    ListVar_Blue_D_Norm_Set := convert(SetVar_Blue_D_Norm_Set, list)
else ListVar_Blue_D_Norm_Set := SetVar_Blue_D_Norm_Set
end if;
if type(DVar_Blue_D_Norm_Set, string) then
    NomPlanVar_Blue_D_Norm_Set :=
        cat(PATHDATA, DVar_Blue_D_Norm_Set, ".txt");
    if Fichier_Existant(NomPlanVar_Blue_D_Norm_Set) = false then
        error "Plan inconnu"
    else read NomPlanVar_Blue_D_Norm_Set
    end if;
    NVar_Blue_D_Norm_Set := op(1, Plan);
    XVar_Blue_D_Norm_Set := convert(ListVar_Blue_D_Norm_Set, Vectorrow);
    try VarVar_Blue_D_Norm_Set := LinearAlgebra : -Multiply(XVar_Blue_D_Norm_Set,
        LinearAlgebra : -MatrixVectorMultiply(MatrixInverseRank(
            M_D_Set(Plan, ListIndetVar_Blue_D_Norm_Set, ListVar_Blue_D_Norm_Set)),
            LinearAlgebra : -Transpose(XVar_Blue_D_Norm_Set)))
    catch "singular matrix" : print("M_D is not full rank")
    catch : print("Something went wrong while computing M_D")
    end try
elif type(DVar_Blue_D_Norm_Set, Matrix) then
    NVar_Blue_D_Norm_Set := op(1, DVar_Blue_D_Norm_Set);
    XVar_Blue_D_Norm_Set := convert(ListVar_Blue_D_Norm_Set, Vectorrow);
    try VarVar_Blue_D_Norm_Set := LinearAlgebra : -Multiply(XVar_Blue_D_Norm_Set,
        LinearAlgebra : -MatrixVectorMultiply(MatrixInverseRank(M_D_Set(
            DVar_Blue_D_Norm_Set, ListIndetVar_Blue_D_Norm_Set,
            ListVar_Blue_D_Norm_Set)), LinearAlgebra : -Transpose(XVar_Blue_D_Norm_Set)))
    catch "singular matrix" : print("M_D is not full rank")
    catch : print("Something went wrong while computing M_D")
    end try
else error "Le plan doit etre donné sous forme matricielle"
end if;
return VarVar_Blue_D_Norm_Set
end proc

```

```

> Var_Blue_Modele_Sat:=proc(DModeleSat,ListIndetModeleSat,EstSetModeleS
at) return
Var_Blue_D_Norm_Set(DModeleSat,ListIndetModeleSat,EstSetModeleSat);
end proc;

Var_Blue_Modele_Sat := proc(DModeleSat, ListIndetModeleSat, EstSetModeleSat)
    return Var_Blue_D_Norm_Set(DModeleSat, ListIndetModeleSat, EstSetModeleSat)
end proc

> Inf_Norm_D_Obsolete:=proc(DInf_Norm_D_Obsolete,QInf_Norm_D_Obsolete)
local InfInf_Norm_D_Obsolete;
InfInf_Norm_D_Obsolete:=1/Var_Norm_Blue_D_Centre(DInf_Norm_D_Obsolete,
QInf_Norm_D_Obsolete,0); return InfInf_Norm_D_Obsolete; end proc;

Inf_Norm_D_Obsolete := proc(DInf_Norm_D_Obsolete, QInf_Norm_D_Obsolete)
local InfInf_Norm_D_Obsolete;
InfInf_Norm_D_Obsolete :=
    1/Var_Norm_Blue_D_Centre(DInf_Norm_D_Obsolete, QInf_Norm_D_Obsolete, 0);
return InfInf_Norm_D_Obsolete
end proc

> Inf_Norm_D_Centre:=proc(DInf_Norm_D_Centre,QInf_Norm_D_Centre) local
InfInf_Norm_D_Centre;
InfInf_Norm_D_Centre:=1/Var_Norm_Blue_D_Centre(DInf_Norm_D_Centre,QInf
_Norm_D_Centre,0); return InfInf_Norm_D_Centre; end proc;

Inf_Norm_D_Centre := proc(DInf_Norm_D_Centre, QInf_Norm_D_Centre)
local InfInf_Norm_D_Centre;
InfInf_Norm_D_Centre :=
    1/Var_Norm_Blue_D_Centre(DInf_Norm_D_Centre, QInf_Norm_D_Centre, 0);
return InfInf_Norm_D_Centre
end proc

```

10.3.4. Idéal du plan

```

> DesignToIdeal:=proc(DDesignToIdeal) local
NDesignToIdeal,XDesignToIdeal,PointsDesignToIdeal;
NDesignToIdeal:=op(1,DDesignToIdeal);
XDesignToIdeal:=Vector[row](NDesignToIdeal[2],symbol=u);
PointsDesignToIdeal:=Row(DDesignToIdeal,[1..NDesignToIdeal[1]]);
return
Intersect(seq(PolynomialIdeal(seq((PointsDesignToIdeal[iDesignToIdeal]
[jDesignToIdeal]-XDesignToIdeal[jDesignToIdeal]),jDesignToIdeal=1..NDe
signToIdeal[2])),iDesignToIdeal=1..NDesignToIdeal[1])); end proc;

DesignToIdeal := proc(DDesignToIdeal)
local NDesignToIdeal, XDesignToIdeal, PointsDesignToIdeal;
NDesignToIdeal := op(1, DDesignToIdeal);
XDesignToIdeal := Vectorrow(NDesignToIdeal2, symbol = u);
PointsDesignToIdeal :=
    LinearAlgebra : - Row(DDesignToIdeal, [1..NDesignToIdeal1]);
return PolynomialIdeals : - Intersect(seq(PolynomialIdeals : - PolynomialIdeal(seq(
    PointsDesignToIdealiDesignToIdealjDesignToIdeal
    - XDesignToIdealjDesignToIdeal, jDesignToIdeal = 1..NDesignToIdeal2)),
    iDesignToIdeal = 1..NDesignToIdeal1))
end proc

```

```

> DesignToIdealv2:=proc(DDesignToIdeal) local
  NDesignToIdeal,XDesignToIdeal,PointsDesignToIdeal,iDesignToIdeal,jDesi
  gnToIdeal;if type(DDesignToIdeal,Matrix) then
  NDesignToIdeal:=op(1,DDesignToIdeal);
  XDesignToIdeal:=Vector[row](NDesignToIdeal[2],symbol=u);
  PointsDesignToIdeal:=Row(DDesignToIdeal,[1..NDesignToIdeal[1]]);
  return
  EliminationIdeal(PolynomialIdeal(seq((seq(t[iDesignToIdeal]*(PointsDes
  ignToIdeal[iDesignToIdeal][jDesignToIdeal]-XDesignToIdeal[jDesignToIde
  al]),jDesignToIdeal=1..NDesignToIdeal[2])),iDesignToIdeal=1..NDesignTo
  Ideal[1]),(add(t[iDesignToIdeal],iDesignToIdeal=1..NDesignToIdeal[1]))
  -1), indets(DDesignToIdeal) union
  {seq(u[jDesignToIdeal],jDesignToIdeal=1..NDesignToIdeal[2])}); else
  error "Le plan doit etre donné sous forme matricielle"; end if; end
  proc;

DesignToIdealv2 := proc(DDesignToIdeal)
local NDesignToIdeal, XDesignToIdeal, PointsDesignToIdeal, iDesignToIdeal,
jDesignToIdeal;
  if type(DDesignToIdeal, Matrix) then
    NDesignToIdeal := op(1, DDesignToIdeal);
    XDesignToIdeal := Vector_row(NDesignToIdeal_2, symbol = u);
    PointsDesignToIdeal :=
      LinearAlgebra : - Row(DDesignToIdeal, [1..NDesignToIdeal_1]);
    return PolynomialIdeals : - EliminationIdeal(PolynomialIdeals : - PolynomialIdeal(seq(
      seq(t_iDesignToIdeal * (PointsDesignToIdeal_iDesignToIdeal_jDesignToIdeal
      - XDesignToIdeal_jDesignToIdeal), jDesignToIdeal = 1..NDesignToIdeal_2),
      iDesignToIdeal = 1..NDesignToIdeal_1),
      add(t_iDesignToIdeal, iDesignToIdeal = 1..NDesignToIdeal_1) - 1),
      indets(DDesignToIdeal) union
      {seq(u_jDesignToIdeal, jDesignToIdeal = 1..NDesignToIdeal_2)})
    else error "Le plan doit etre donné sous forme matricielle"
    end if
  end proc

> OrdreDesign:=proc(AOrdreDesign,BOrdreDesign) local
  TempOrdreDesign,iOrdreDesign,jOrdreDesign,DimOrdreDesign,AseqOrdreDesi
  gn,BseqOrdreDesign,KAseqOrdreDesign,KBseqOrdreDesign,ListOrdreDesign,k
  OrdreDesign; TempOrdreDesign:=FAIL;
  DimOrdreDesign:=op(1,AOrdreDesign); if
  Equal(AOrdreDesign,BOrdreDesign) then TempOrdreDesign:=false else if
  add(AOrdreDesign[iOrdreDesign],iOrdreDesign=1..DimOrdreDesign)<add(BOr
  dreDesign[iOrdreDesign],iOrdreDesign=1..DimOrdreDesign) then
  TempOrdreDesign:=true elif
  add(AOrdreDesign[iOrdreDesign],iOrdreDesign=1..DimOrdreDesign)>add(BOr
  dreDesign[iOrdreDesign],iOrdreDesign=1..DimOrdreDesign) then
  TempOrdreDesign:=false else
  AseqOrdreDesign:=convert(AOrdreDesign,list);BseqOrdreDesign:=convert(B
  OrdreDesign,list); for jOrdreDesign while nops(AseqOrdreDesign)>0 and
  (TempOrdreDesign=FAIL) do if
  max(op(AseqOrdreDesign))>max(op(BseqOrdreDesign)) then
  TempOrdreDesign:=false; elif
  max(op(AseqOrdreDesign))<max(op(BseqOrdreDesign)) then
  TempOrdreDesign:=true; else member(max(op(AseqOrdreDesign)),
  AseqOrdreDesign, 'KAseqOrdreDesign'); member(max(op(BseqOrdreDesign)),
  BseqOrdreDesign, 'KBseqOrdreDesign');
  AseqOrdreDesign:=subop(KAseqOrdreDesign=NULL, AseqOrdreDesign);
  BseqOrdreDesign:=subop(KBseqOrdreDesign=NULL, BseqOrdreDesign);end
  if; end do; end if; if AseqOrdreDesign=[] then

```

```

ListOrdreDesign:=convert(AOrdreDesign-BOrdreDesign,list); for
kOrdreDesign while nops(ListOrdreDesign)>0 and TempOrdreDesign=FAIL do
if ListOrdreDesign[1]>0 then TempOrdreDesign:=false elif
ListOrdreDesign[1]<0 then TempOrdreDesign:=true; else
ListOrdreDesign:=subsop(1=NULL, ListOrdreDesign); end if; end do; if
ListOrdreDesign=[] then TempOrdreDesign:=false; end if; end if; end
if; return TempOrdreDesign; end proc;

OrdreDesign := proc(AOrdreDesign, BOrdreDesign)
local TempOrdreDesign, iOrdreDesign, jOrdreDesign, DimOrdreDesign,
AseqOrdreDesign, BseqOrdreDesign, KaseqOrdreDesign, KBseqOrdreDesign,
ListOrdreDesign, kOrdreDesign;
    TempOrdreDesign := FAIL;
    DimOrdreDesign := op(1, AOrdreDesign);
    if LinearAlgebra : -Equal(AOrdreDesign, BOrdreDesign) then
        TempOrdreDesign := false
    else
        if add(AOrdreDesigniOrdreDesign, iOrdreDesign = 1..DimOrdreDesign) <
        add(BOrdreDesigniOrdreDesign, iOrdreDesign = 1..DimOrdreDesign) then
            TempOrdreDesign := true
        elif add(BOrdreDesigniOrdreDesign, iOrdreDesign = 1..DimOrdreDesign) <
        add(AOrdreDesigniOrdreDesign, iOrdreDesign = 1..DimOrdreDesign) then
            TempOrdreDesign := false
        else
            AseqOrdreDesign := convert(AOrdreDesign, list);
            BseqOrdreDesign := convert(BOrdreDesign, list);
            for jOrdreDesign while 0 < nops(AseqOrdreDesign) and TempOrdreDesign = FAIL
            do
                if max(op(BseqOrdreDesign)) < max(op(AseqOrdreDesign)) then
                    TempOrdreDesign := false
                elif max(op(AseqOrdreDesign)) < max(op(BseqOrdreDesign)) then
                    TempOrdreDesign := true
                else
                    member(max(op(AseqOrdreDesign)), AseqOrdreDesign, 'KaseqOrdreDesign');
                    member(max(op(BseqOrdreDesign)), BseqOrdreDesign, 'KBseqOrdreDesign');
                    AseqOrdreDesign := subsop(KaseqOrdreDesign = NULL, AseqOrdreDesign);
                    BseqOrdreDesign := subsop(KBseqOrdreDesign = NULL, BseqOrdreDesign)
                end if
            end do
        end if;
        if AseqOrdreDesign = [] then
            ListOrdreDesign := convert(AOrdreDesign - BOrdreDesign, list);
            for kOrdreDesign while 0 < nops(ListOrdreDesign) and TempOrdreDesign = FAIL
            do
                if 0 < ListOrdreDesign1 then TempOrdreDesign := false
                elif ListOrdreDesign1 < 0 then TempOrdreDesign := true
                else ListOrdreDesign := subsop(1 = NULL, ListOrdreDesign)
                end if
            end do;
    end if;

```

```

        if ListOrdreDesign = [] then TempOrdreDesign := false end if
    end if
end if;
return TempOrdreDesign
end proc

```

```

> Exposants_Ord:=proc(DegExposants_Ord,DimExposants_Ord) local
NomfichierExposants_Ord,IndiceExposants_Ord; global TempExposants_Ord;
NomfichierExposants_Ord:=cat(PATHDATA,"Exposants_Ord_Dim",convert(DimE
xposants_Ord,string), "_Deg",convert(DegExposants_Ord,string),".txt");
if Fichier_Existant(NomfichierExposants_Ord) = false then
TempExposants_Ord:=sort([seq(convert(inttovec(IndiceExposants_Ord,DimE
xposants_Ord),Vector),IndiceExposants_Ord=0..binomial(DimExposants_Ord
+DegExposants_Ord,DimExposants_Ord)-1)],OrdreDesign); save
TempExposants_Ord, NomfichierExposants_Ord; else read
NomfichierExposants_Ord; end if; return TempExposants_Ord; end
proc;

```

```

Exposants__Ord := proc(DegExposants__Ord, DimExposants__Ord)
local NomfichierExposants__Ord, IndiceExposants__Ord;
global TempExposants__Ord;
    NomfichierExposants__Ord := cat(PATHDATA, "Exposants__Ord_Dim",
        convert(DimExposants__Ord, string), "__Deg", convert(DegExposants__Ord, string),
        ".txt");
    if Fichier_Existant(NomfichierExposants__Ord) = false then
        TempExposants__Ord := sort([seq(
            convert(combinat : -inttovec(IndiceExposants__Ord, DimExposants__Ord), Vector),
            IndiceExposants__Ord = 0..
            combinat : -binomial(DegExposants__Ord + DimExposants__Ord, DimExposants__Ord)
            - 1)], OrdreDesign);
        save TempExposants__Ord, NomfichierExposants__Ord
    else read NomfichierExposants__Ord
    end if;
    return TempExposants__Ord
end proc

```

10.4. Isovariance

10.4.1. Équations d'isovariance

```

> Radial:=proc(ListRadial) local TempRadial;
TempRadial:=add(xRadial^2,xRadial=ListRadial); return TempRadial; end
proc;

Radial := proc(ListRadial)
local TempRadial;
    TempRadial := add(xRadial^2, xRadial = ListRadial); return TempRadial
end proc

```

```
> Eq_IsIso_D_Centre_ListIndets:=proc(DEq_IsIso_D_Centre_ListIndets,QEq_
IsIso_D_Centre_ListIndets,PointsEq_IsIso_D_Centre_ListIndets,IndetsIso
Eq_IsIso_D_Centre_ListIndets) local
VarEq_IsIso_D_Centre_ListIndets,Rat_IndetsEq_IsIso_D_Centre_ListIndets
,AlgebraEq_IsIso_D_Centre_ListIndets,TempEq_IsIso_D_Centre_ListIndets;
TempEq_IsIso_D_Centre_ListIndets:=Var_Norm_Blue_D_Centre_ListIndets(DE
q_IsIso_D_Centre_ListIndets,QEq_IsIso_D_Centre_ListIndets,PointsEq_IsI
so_D_Centre_ListIndets);
VarEq_IsIso_D_Centre_ListIndets:=op(1,TempEq_IsIso_D_Centre_ListIndets
);
Rat_IndetsEq_IsIso_D_Centre_ListIndets:=indets(op(1,TempEq_IsIso_D_Cen
tre_ListIndets)) minus IndetsIsoEq_IsIso_D_Centre_ListIndets;
AlgebraEq_IsIso_D_Centre_ListIndets:=poly_algebra(op(IndetsIsoEq_IsIso
_D_Centre_ListIndets),rational=Rat_IndetsEq_IsIso_D_Centre_ListIndets)
; return
EqInKfi_reduce([Radial(IndetsIsoEq_IsIso_D_Centre_ListIndets)],
{VarEq_IsIso_D_Centre_ListIndets
},AlgebraEq_IsIso_D_Centre_ListIndets),IndetsIsoEq_IsIso_D_Centre_Lis
tIndets,op(2,TempEq_IsIso_D_Centre_ListIndets); end proc;

Eq_IsIso_D_Centre_ListIndets := proc(DEq_IsIso_D_Centre_ListIndets,
QEq_IsIso_D_Centre_ListIndets, PointsEq_IsIso_D_Centre_ListIndets,
IndetsIsoEq_IsIso_D_Centre_ListIndets)
local VarEq_IsIso_D_Centre_ListIndets, Rat_IndetsEq_IsIso_D_Centre_ListIndets,
AlgebraEq_IsIso_D_Centre_ListIndets, TempEq_IsIso_D_Centre_ListIndets;
    TempEq_IsIso_D_Centre_ListIndets := Var_Norm_Blue_D_Centre_ListIndets(
        DEq_IsIso_D_Centre_ListIndets, QEq_IsIso_D_Centre_ListIndets,
        PointsEq_IsIso_D_Centre_ListIndets);
    VarEq_IsIso_D_Centre_ListIndets := op(1, TempEq_IsIso_D_Centre_ListIndets);
    Rat_IndetsEq_IsIso_D_Centre_ListIndets :=
        indets(op(1, TempEq_IsIso_D_Centre_ListIndets)) minus
        IndetsIsoEq_IsIso_D_Centre_ListIndets;
    AlgebraEq_IsIso_D_Centre_ListIndets := Ore_algebra : -poly_algebra(
        op(IndetsIsoEq_IsIso_D_Centre_ListIndets),
        rational = Rat_IndetsEq_IsIso_D_Centre_ListIndets);
    return EqInKfi_reduce([Radial(IndetsIsoEq_IsIso_D_Centre_ListIndets)],
        {VarEq_IsIso_D_Centre_ListIndets}, AlgebraEq_IsIso_D_Centre_ListIndets),
        IndetsIsoEq_IsIso_D_Centre_ListIndets, op(2, TempEq_IsIso_D_Centre_ListIndets)
end proc

> Eq_IsIso_D_Centre:=proc(DEq_IsIso_D_Centre,QEq_IsIso_D_Centre,PointsE
q_IsIso_D_Centre) local
VarEq_IsIso_D_Centre,IndetsIsoEq_IsIso_D_Centre,iEq_IsIso_D_Centre,Nom
PlanEq_IsIso_D_Centre,NEq_IsIso_D_Centre,Rat_IndetsEq_IsIso_D_Centre;
global Plan,AlgebraEq_IsIso_D_Centre; if
type(DEq_IsIso_D_Centre,string) then
NomPlanEq_IsIso_D_Centre:=cat(PATHDATA,DEq_IsIso_D_Centre,".txt"); if
Fichier_Existant(NomPlanEq_IsIso_D_Centre)=false then error "Plan
inconnu" else read NomPlanEq_IsIso_D_Centre end if;
NEq_IsIso_D_Centre[2]:=op(1,Plan);
IndetsIsoEq_IsIso_D_Centre:=[seq(var[iEq_IsIso_D_Centre],iEq_IsIso_D_C
entre=1..NEq_IsIso_D_Centre[2])];
VarEq_IsIso_D_Centre:=Var_Norm_Blue_D_Centre(DEq_IsIso_D_Centre,QEq_Is
Iso_D_Centre,PointsEq_IsIso_D_Centre); elif
type(DEq_IsIso_D_Centre,Matrix) then
NEq_IsIso_D_Centre:=op(1,DEq_IsIso_D_Centre);
IndetsIsoEq_IsIso_D_Centre:=[seq(var[iEq_IsIso_D_Centre],iEq_IsIso_D_C
entre=1..NEq_IsIso_D_Centre[2])];
```

```

VarEq_IsIso_D_Centre:=Var_Norm_Blue_D_Centre(DEq_IsIso_D_Centre,QEq_Is
Iso_D_Centre,PointsEq_IsIso_D_Centre); else error "Le plan doit etre
donné sous forme matricielle"; end if;
Rat_IndetsEq_IsIso_D_Centre:=indets(VarEq_IsIso_D_Centre) minus
convert(IndetsIsoEq_IsIso_D_Centre,set);
AlgebraEq_IsIso_D_Centre:=poly_algebra(op(IndetsIsoEq_IsIso_D_Centre),
op(Rat_IndetsEq_IsIso_D_Centre),rational=Rat_IndetsEq_IsIso_D_Centre);
return
EqInKfi_reduce([Radial(IndetsIsoEq_IsIso_D_Centre)],
{VarEq_IsIso_D_Centre},AlgebraEq_IsIso_D_Centre); end proc;

Eq_IsIso_D_Centre := proc(
DEq_IsIso_D_Centre, QEq_IsIso_D_Centre, PointsEq_IsIso_D_Centre)
local VarEq_IsIso_D_Centre, IndetsIsoEq_IsIso_D_Centre, iEq_IsIso_D_Centre,
NomPlanEq_IsIso_D_Centre, NEq_IsIso_D_Centre, Rat_IndetsEq_IsIso_D_Centre;
global Plan, AlgebraEq_IsIso_D_Centre;
if type(DEq_IsIso_D_Centre, string) then
  NomPlanEq_IsIso_D_Centre := cat(PATHDATA, DEq_IsIso_D_Centre, ".txt");
  if Fichier_Existant(NomPlanEq_IsIso_D_Centre) = false then error "Plan inconnu"
  else read NomPlanEq_IsIso_D_Centre
  end if;
  NEq_IsIso_D_Centre_2 := op(1, Plan);
  IndetsIsoEq_IsIso_D_Centre :=
    [seq(var iEq_IsIso_D_Centre, iEq_IsIso_D_Centre = 1..NEq_IsIso_D_Centre_2)];
  VarEq_IsIso_D_Centre := Var_Norm_Blue_D_Centre(DEq_IsIso_D_Centre,
    QEq_IsIso_D_Centre, PointsEq_IsIso_D_Centre)
elseif type(DEq_IsIso_D_Centre, Matrix) then
  NEq_IsIso_D_Centre := op(1, DEq_IsIso_D_Centre);
  IndetsIsoEq_IsIso_D_Centre :=
    [seq(var iEq_IsIso_D_Centre, iEq_IsIso_D_Centre = 1..NEq_IsIso_D_Centre_2)];
  VarEq_IsIso_D_Centre := Var_Norm_Blue_D_Centre(DEq_IsIso_D_Centre,
    QEq_IsIso_D_Centre, PointsEq_IsIso_D_Centre)
else error "Le plan doit etre donné sous forme matricielle"
end if;
Rat_IndetsEq_IsIso_D_Centre :=
  indets(VarEq_IsIso_D_Centre) minus convert(IndetsIsoEq_IsIso_D_Centre, set);
AlgebraEq_IsIso_D_Centre := Ore_algebra : -poly_algebra(
  op(IndetsIsoEq_IsIso_D_Centre), op(Rat_IndetsEq_IsIso_D_Centre),
  rational = Rat_IndetsEq_IsIso_D_Centre);
return EqInKfi_reduce([Radial(IndetsIsoEq_IsIso_D_Centre)],
  {VarEq_IsIso_D_Centre}, AlgebraEq_IsIso_D_Centre)
end proc

```

10.4.2. Fonction génératrice des moments

```
> F_Gen_Mom:=proc(DF_Gen_Mom,QF_Gen_Mom,PointsF_Gen_Mom) local
NomPlanF_Gen_Mom,NomFichierF_Gen_Mom,NF_Gen_Mom,iF_Gen_Mom,jF_Gen_Mom;
global Plan,F_GenF_Gen_Mom; if type(DF_Gen_Mom,string) then
NomPlanF_Gen_Mom:=cat(PATHDATA,DF_Gen_Mom,".txt");
NomFichierF_Gen_Mom:=cat(PATHDATA,DF_Gen_Mom,"_F_Gen_Mom_",convert(QF_
Gen_Mom,string),".txt"); if Fichier_Existant(NomFichierF_Gen_Mom)=true
then read NomFichierF_Gen_Mom else if
Fichier_Existant(NomPlanF_Gen_Mom)=false then error "Plan inconnu"
else read NomPlanF_Gen_Mom end if; NF_Gen_Mom:=op(1,Plan);
F_GenF_Gen_Mom:=1/NF_Gen_Mom[1]*add((1+add(Plan[iF_Gen_Mom,jF_Gen_Mom]
*xmom[jF_Gen_Mom],jF_Gen_Mom=1..NF_Gen_Mom[2]))^(2*QF_Gen_Mom),
iF_Gen_Mom=1..NF_Gen_Mom[1])); save F_GenF_Gen_Mom,
NomFichierF_Gen_Mom; end if; elif type(DF_Gen_Mom,Matrix) then
NF_Gen_Mom:=op(1,DF_Gen_Mom);
F_GenF_Gen_Mom:=1/NF_Gen_Mom[1]*add((1+add(DF_Gen_Mom[iF_Gen_Mom,jF_Ge
n_Mom]*xmom[jF_Gen_Mom],jF_Gen_Mom=1..NF_Gen_Mom[2]))^(2*QF_Gen_Mom),
iF_Gen_Mom=1..NF_Gen_Mom[1])); else error "Le plan doit etre donné sous
forme matricielle"; end if; return F_GenF_Gen_Mom; end proc;

F_Gen_Mom := proc(DF_Gen_Mom, QF_Gen_Mom, PointsF_Gen_Mom)
local NomPlanF_Gen_Mom, NomFichierF_Gen_Mom, NF_Gen_Mom, iF_Gen_Mom,
jF_Gen_Mom;
global Plan, F_GenF_Gen_Mom;
if type(DF_Gen_Mom, string) then
    NomPlanF_Gen_Mom := cat(PATHDATA, DF_Gen_Mom, ".txt");
    NomFichierF_Gen_Mom := cat(PATHDATA, DF_Gen_Mom, "_F_Gen_Mom_",
        convert(QF_Gen_Mom, string), ".txt");
    if Fichier_Existant(NomFichierF_Gen_Mom) = true then
        read NomFichierF_Gen_Mom
    else
        if Fichier_Existant(NomPlanF_Gen_Mom) = false then error "Plan inconnu"
        else read NomPlanF_Gen_Mom
        end if;
        NF_Gen_Mom := op(1, Plan);
        F_GenF_Gen_Mom := add((1 + add(
            Plan[iF_Gen_Mom, jF_Gen_Mom] * xmom[jF_Gen_Mom,
            jF_Gen_Mom = 1..NF_Gen_Mom[2]])^(2*QF_Gen_Mom),
            iF_Gen_Mom = 1..NF_Gen_Mom[1])/NF_Gen_Mom[1];
        save F_GenF_Gen_Mom, NomFichierF_Gen_Mom
    end if
elif type(DF_Gen_Mom, Matrix) then
    NF_Gen_Mom := op(1, DF_Gen_Mom);
    F_GenF_Gen_Mom := add((1 + add(
        DF_Gen_Mom[iF_Gen_Mom, jF_Gen_Mom] * xmom[jF_Gen_Mom,
        jF_Gen_Mom = 1..NF_Gen_Mom[2]])^(2*QF_Gen_Mom),
        iF_Gen_Mom = 1..NF_Gen_Mom[1])/NF_Gen_Mom[1]
    else error "Le plan doit etre donné sous forme matricielle"
    end if;
    return F_GenF_Gen_Mom
end proc
```



```

> EqInKr_reduce:=proc(ListIndetsEqInKr_reduce,ListEqInKr_reduce) local
AlgebraEqInKr_reduce,Rat_IndetsEqInKr_reduce;
Rat_IndetsEqInKr_reduce:=indets(ListEqInKr_reduce) minus
ListIndetsEqInKr_reduce;
AlgebraEqInKr_reduce:=poly_algebra(op(indets(ListEqInKr_reduce)),ratio
nal=Rat_IndetsEqInKr_reduce); return
EqInKfi_reduce([Radial(ListIndetsEqInKr_reduce)],ListEqInKr_reduce,Alg
ebraEqInKr_reduce); end proc;

EqInKr_reduce := proc(ListIndetsEqInKr_reduce, ListEqInKr_reduce)
local AlgebraEqInKr_reduce, Rat_IndetsEqInKr_reduce;
Rat_IndetsEqInKr_reduce :=
indets(ListEqInKr_reduce) minus ListIndetsEqInKr_reduce;
AlgebraEqInKr_reduce := Ore_algebra : -poly_algebra(
op(indets(ListEqInKr_reduce)), rational = Rat_IndetsEqInKr_reduce);
return EqInKfi_reduce([Radial(ListIndetsEqInKr_reduce)], ListEqInKr_reduce,
AlgebraEqInKr_reduce)
end proc

> EqInKr_normalf:=proc(ListIndetsEqInKr_normalf,ListEqInKr_normalf)
local AlgebraEqInKr_normalf,Rat_IndetsEqInKr_normalf;
Rat_IndetsEqInKr_normalf:=indets(ListEqInKr_normalf) minus
ListIndetsEqInKr_normalf;
AlgebraEqInKr_normalf:=poly_algebra(op(indets(ListEqInKr_normalf)),rat
ional=Rat_IndetsEqInKr_normalf); return
EqInKfi_normalf([Radial(ListIndetsEqInKr_normalf)],ListEqInKr_normalf,
AlgebraEqInKr_normalf); end proc;

EqInKr_normalf := proc(ListIndetsEqInKr_normalf, ListEqInKr_normalf)
local AlgebraEqInKr_normalf, Rat_IndetsEqInKr_normalf;
Rat_IndetsEqInKr_normalf :=
indets(ListEqInKr_normalf) minus ListIndetsEqInKr_normalf;
AlgebraEqInKr_normalf := Ore_algebra : -poly_algebra(
op(indets(ListEqInKr_normalf)), rational = Rat_IndetsEqInKr_normalf);
return EqInKfi_normalf([Radial(ListIndetsEqInKr_normalf)], ListEqInKr_normalf,
AlgebraEqInKr_normalf)
end proc

> EqF_Gen_Mom_Rad:=proc(DEqF_Gen_Mom_Rad,QEqF_Gen_Mom_Rad) local
NEqF_Gen_Mom_Rad,iEqF_Gen_Mom_Rad;
NEqF_Gen_Mom_Rad:=op(1,DEqF_Gen_Mom_Rad);
EqInKr_reduce({seq(xmom[i],i=1..NEqF_Gen_Mom_Rad[2])},
{F_Gen_Mom(DEqF_Gen_Mom_Rad,QEqF_Gen_Mom_Rad)}); end proc;

EqF_Gen_Mom_Rad := proc(DEqF_Gen_Mom_Rad, QEqF_Gen_Mom_Rad)
local NEqF_Gen_Mom_Rad, iEqF_Gen_Mom_Rad;
NEqF_Gen_Mom_Rad := op(1, DEqF_Gen_Mom_Rad);
EqInKr_reduce({seq(xmomi, i = 1..NEqF_Gen_Mom_Rad2)},
{F_Gen_Mom(DEqF_Gen_Mom_Rad, QEqF_Gen_Mom_Rad)});
end proc

> Eq_F_Gen_Mom_On_Inv:=proc(DEq_F_Gen_Mom_On_Inv,QEq_F_Gen_Mom_On_Inv)
local NEq_F_Gen_Mom_On_Inv,iEq_F_Gen_Mom_On_Inv;
NEq_F_Gen_Mom_On_Inv:=op(1,DEq_F_Gen_Mom_On_Inv);
EqInKr_reduce({seq(xmom[i],i=1..NEq_F_Gen_Mom_On_Inv[2])},
{F_Gen_Mom(DEq_F_Gen_Mom_On_Inv,QEq_F_Gen_Mom_On_Inv)}); end
proc;

```

```

Eq_F_Gen_Mom_On_Inv := proc(DEq_F_Gen_Mom_On_Inv, QEq_F_Gen_Mom_On_Inv)
local NEq_F_Gen_Mom_On_Inv, iEq_F_Gen_Mom_On_Inv;
    NEq_F_Gen_Mom_On_Inv := op(1, DEq_F_Gen_Mom_On_Inv);
    EqInKr_reduce({seq(xmomi, i = 1..NEq_F_Gen_Mom_On_Inv)},
        {F_Gen_Mom(DEq_F_Gen_Mom_On_Inv, QEq_F_Gen_Mom_On_Inv)})
end proc

```

10.4.3. Défaut d'isovariance

```

> CondIsoOrbite:=proc(GroupeCondIsoOrbite,Dim1CondIsoOrbite,Dim2CondIso
Orbite,Ordre1CondIsoOrbite,Ordre2CondIsoOrbite) local
NomGroupeCondIsoOrbite,TempCondIsoOrbite,QCondIsoOrbite,NomfichierCond
IsoOrbite,iCondIsoOrbite; global Conditions_IsovarianceCondIsoOrbite;
for iCondIsoOrbite from Dim1CondIsoOrbite by 1 to Dim2CondIsoOrbite do
    print(cat("Dimension de l'espace ambiant : ",
    convert(iCondIsoOrbite,string)));
    print(Vector(iCondIsoOrbite,symbol=u));
    NomGroupeCondIsoOrbite:=cat(GroupeCondIsoOrbite,"_",convert(iCondIsoOr
bite,string)); print(cat("Groupe utilisé : ",GroupeCondIsoOrbite));
    TempCondIsoOrbite:=OrbToMatrix(OrbPv2(Vector(iCondIsoOrbite,symbol=u),
    GroupeCondIsoOrbite));print(cat("Taille de l'orbite : ",
    convert(op(1,TempCondIsoOrbite)[1],string))); for QCondIsoOrbite
from Ordre1CondIsoOrbite by 1 to Ordre2CondIsoOrbite do
        NomfichierCondIsoOrbite:=
        cat(PATHDATA,NomGroupeCondIsoOrbite,"_Orbite_q_",convert(QCondIsoOrbit
e,string),"_Cond_Iso.txt"); if
        Fichier_Existant(NomfichierCondIsoOrbite) = false then
            print(cat("D'ordre : ",convert(QCondIsoOrbite,string)));
            Conditions_IsovarianceCondIsoOrbite:=(Radical(PolynomialIdeal(Eq_F_Gen
_Mom_On_Inv(TempCondIsoOrbite,QCondIsoOrbite))));
            print(Conditions_IsovarianceCondIsoOrbite); save
            Conditions_IsovarianceCondIsoOrbite, NomfichierCondIsoOrbite; else
            print(cat("D'ordre : ",convert(QCondIsoOrbite,string))); read
            NomfichierCondIsoOrbite; print(Conditions_IsovarianceCondIsoOrbite);
            end if; end do; end do; end proc;

CondIsoOrbite := proc(GroupeCondIsoOrbite, Dim1CondIsoOrbite, Dim2CondIsoOrbite,
Ordre1CondIsoOrbite, Ordre2CondIsoOrbite)
local NomGroupeCondIsoOrbite, TempCondIsoOrbite, QCondIsoOrbite,
NomfichierCondIsoOrbite, iCondIsoOrbite;
global Conditions_IsovarianceCondIsoOrbite;
    for iCondIsoOrbite from Dim1CondIsoOrbite to Dim2CondIsoOrbite do
        print(cat("Dimension de l'espace ambiant : ", convert(iCondIsoOrbite, string)));
        print(Vector(iCondIsoOrbite, symbol = u));
        NomGroupeCondIsoOrbite :=
            cat(GroupeCondIsoOrbite, "_", convert(iCondIsoOrbite, string));
        print(cat("Groupe utilisé : ", GroupeCondIsoOrbite));
        TempCondIsoOrbite :=
            OrbToMatrix(OrbPv2(Vector(iCondIsoOrbite, symbol = u), GroupeCondIsoOrbite));
        print(cat("Taille de l' orbite : ", convert(op(1, TempCondIsoOrbite)1, string)));
        for QCondIsoOrbite from Ordre1CondIsoOrbite to Ordre2CondIsoOrbite do
            NomfichierCondIsoOrbite := cat(PATHDATA, NomGroupeCondIsoOrbite,
            "_Orbite_q_", convert(QCondIsoOrbite, string), "_Cond_Iso.txt");

```

```

    if Fichier_Existant(NomFichierCondIsoOrbite) = false then
        print(cat("D'ordre : ", convert(QCondIsoOrbite, string)));
        Conditions_IsovarianceCondIsoOrbite := PolynomialIdeals : - Radical(
            PolynomialIdeals : - PolynomialIdeal(
                Eq_F_Gen_Mom_On_Inv(TempCondIsoOrbite, QCondIsoOrbite));
        print(Conditions_IsovarianceCondIsoOrbite);
        save Conditions_IsovarianceCondIsoOrbite, NomFichierCondIsoOrbite
    else
        print(cat("D'ordre : ", convert(QCondIsoOrbite, string)));
        read NomFichierCondIsoOrbite;
        print(Conditions_IsovarianceCondIsoOrbite)
    end if
end do
end do
end proc

> CondIso:=proc(NomPlanCondIso,Ordre1CondIso,Ordre2CondIso) local
    QCondIso,NomFichierCondIso,NomFichierPlanCondIso,iCondIso; global
    Conditions_IsovarianceCondIso,Plan; for QCondIso from Ordre1CondIso by
    1 to Ordre2CondIso do if type(NomPlanCondIso,string) then
        NomFichierPlanCondIso:=cat(PATHDATA,NomPlanCondIso,".txt");
        NomFichierCondIso:=cat(PATHDATA,NomPlanCondIso,"_CondIso_",convert(QCo
        ndIso,string),".txt"); if Fichier_Existant(NomFichierCondIso)=true
        then read NomFichierCondIso; print(Conditions_IsovarianceCondIso);
        else if Fichier_Existant(NomFichierPlanCondIso)=false then error "Plan
        inconnu" else read NomFichierPlanCondIso end if;
        iCondIso:=op(1,Plan)[2]; print(cat("Dimension de l'espace ambiant :
        ",convert(iCondIso,string))); print(cat("Taille du plan :
        ",convert(op(1,Plan)[1],string)));
        Conditions_IsovarianceCondIso:=(Radical(PolynomialIdeal(Eq_F_Gen_Mom_0
        n_Inv(Plan,QCondIso)))); save Conditions_IsovarianceCondIso,
        NomFichierCondIso; print(Conditions_IsovarianceCondIso); end if; elif
        type(NomPlanCondIso,Matrix) then
        Conditions_IsovarianceCondIso:=(Radical(PolynomialIdeal(Eq_F_Gen_Mom_0
        n_Inv(NomPlanCondIso,QCondIso))));
        print(Conditions_IsovarianceCondIso); else error "Le plan doit etre
        donné sous forme matricielle"; end if; end do; end proc;

CondIso := proc(NomPlanCondIso, Ordre1CondIso, Ordre2CondIso)
local QCondIso, NomFichierCondIso, NomFichierPlanCondIso, iCondIso;
global Conditions_IsovarianceCondIso, Plan;
for QCondIso from Ordre1CondIso to Ordre2CondIso do
    if type(NomPlanCondIso, string) then
        NomFichierPlanCondIso := cat(PATHDATA, NomPlanCondIso, ".txt");
        NomFichierCondIso :=
            cat(PATHDATA, NomPlanCondIso, "_CondIso_", convert(QCondIso, string), ".txt");
        if Fichier_Existant(NomFichierCondIso) = true then
            read NomFichierCondIso; print(Conditions_IsovarianceCondIso)
        else
            if Fichier_Existant(NomFichierPlanCondIso) = false then error "Plan inconnu"
            else read NomFichierPlanCondIso
            end if;
            iCondIso := op(1, Plan)[2];
            print(cat("Dimension de l'espace ambiant : ", convert(iCondIso, string)));

```

```

    print(cat("Taille du plan : ", convert(op(1, Plan)1, string)));
    Conditions_IsovarianceCondIso := PolynomialIdeals : -Radical(
        PolynomialIdeals : -PolynomialIdeal(Eq_F_Gen_Mom_On_Inv(Plan, QCondIso)));
    save Conditions_IsovarianceCondIso, NomFichierCondIso ;
    print(Conditions_IsovarianceCondIso)
end if
elif type(NomPlanCondIso, Matrix) then
    Conditions_IsovarianceCondIso := PolynomialIdeals : -Radical(
        PolynomialIdeals : -PolynomialIdeal(
            Eq_F_Gen_Mom_On_Inv(NomPlanCondIso, QCondIso)));
    print(Conditions_IsovarianceCondIso)
else error "Le plan doit etre donne sous forme matricielle"
end if
end do
end proc

```

10.5. Estimabilité

10.5.1. Algorithme de Möller, Marinari et Mora

```

> DiviseursPasAPas:=proc(SetPolDiviseursPasAPas) local
    DiviseursDiviseursPasAPas, TermsDiviseursPasAPas,
    IndetDiviseursPasAPas; DiviseursDiviseursPasAPas:={}; for
    TermsDiviseursPasAPas in SetPolDiviseursPasAPas do for
    IndetDiviseursPasAPas in indets(TermsDiviseursPasAPas) do
    DiviseursDiviseursPasAPas:=DiviseursDiviseursPasAPas union
    {TermsDiviseursPasAPas/IndetDiviseursPasAPas} end do end do; end
proc;

DiviseursPasAPas := proc(SetPolDiviseursPasAPas)
local DiviseursDiviseursPasAPas, TermsDiviseursPasAPas, IndetDiviseursPasAPas;
    DiviseursDiviseursPasAPas := {};
    for TermsDiviseursPasAPas in SetPolDiviseursPasAPas do for
        IndetDiviseursPasAPas in indets(TermsDiviseursPasAPas) do
            DiviseursDiviseursPasAPas :=
                DiviseursDiviseursPasAPas union {TermsDiviseursPasAPas/IndetDiviseursPasAPas}
        end do
    end do
end proc

> DiviseursIterative:=proc(SetDiviseurs) local TempDiviseurs, muetDiviseurs;
    TempDiviseurs:=SetDiviseurs; for muetDiviseurs while TempDiviseurs<>{1} do
    TempDiviseurs:=DiviseursPasAPas(TempDiviseurs); print(TempDiviseurs);
    end do; end proc;

DiviseursIterative := proc(SetDiviseurs)
local TempDiviseurs, muetDiviseurs;
    TempDiviseurs := SetDiviseurs;
    for muetDiviseurs while TempDiviseurs ≠ {1} do
        TempDiviseurs := DiviseursPasAPas(TempDiviseurs); print(TempDiviseurs)
    end do
end proc

```

```
> Diviseurs:=proc(SetDiviseurs) local TempDiviseurs,muetDiviseurs,
DiviseursSetDiviseurs ; DiviseursSetDiviseurs:={}; TempDiviseurs:=SetDiviseurs;
for muetDiviseurs while TempDiviseurs<>{1} do
TempDiviseurs:=DiviseursPasAPas(TempDiviseurs);
DiviseursSetDiviseurs:=DiviseursSetDiviseurs union TempDiviseurs; end do;
return DiviseursSetDiviseurs; end proc;
```

```
Diviseurs := proc(SetDiviseurs)
local TempDiviseurs, muetDiviseurs, DiviseursSetDiviseurs;
    DiviseursSetDiviseurs := {};
    TempDiviseurs := SetDiviseurs;
    for muetDiviseurs while TempDiviseurs  $\neq$  {1} do
        TempDiviseurs := DiviseursPasAPas(TempDiviseurs);
        DiviseursSetDiviseurs := DiviseursSetDiviseurs union TempDiviseurs
    end do;
    return DiviseursSetDiviseurs
end proc
```

```
> EstSetTrace:=proc(DiviseursEstSet,GeneratorsEstSet) local muetEstSet,
CompteurEstSet , AutresEstSet, TermsEstSet, IndetEstSet,TempEstSet,
TempSetEstSet; TempSetEstSet:=DiviseursEstSet; AutresEstSet:=DiviseursEstSet;
for muetEstSet while CompteurEstSet <> {} do CompteurEstSet:={};
for TermsEstSet in AutresEstSet do for IndetEstSet in indets(GeneratorsEstSet)
do TempEstSet:=TermsEstSet*IndetEstSet; print(TempEstSet); if not
member(TempEstSet,TempSetEstSet) then if not
member(true,map2(divide,TempEstSet,GeneratorsEstSet)) then
CompteurEstSet:=CompteurEstSet union {TempEstSet}; end if end if;
end do end do; AutresEstSet:=CompteurEstSet;
TempSetEstSet:=TempSetEstSet union CompteurEstSet;
Print(CompteurEstSet); end do; return TempSetEstSet; end proc;
```

```
EstSetTrace := proc(DiviseursEstSet, GeneratorsEstSet)
local muetEstSet, CompteurEstSet, AutresEstSet, TermsEstSet, IndetEstSet,
TempEstSet, TempSetEstSet;
    TempSetEstSet := DiviseursEstSet;
    AutresEstSet := DiviseursEstSet;
    for muetEstSet while CompteurEstSet  $\neq$  {} do
        CompteurEstSet := {};
        for TermsEstSet in AutresEstSet do for IndetEstSet in indets(GeneratorsEstSet) do
            TempEstSet := IndetEstSet * TermsEstSet;
            print(TempEstSet);
            if not member(TempEstSet, TempSetEstSet) then
                if not member(true, map2(divide, TempEstSet, GeneratorsEstSet)) then
                    CompteurEstSet := CompteurEstSet union {TempEstSet}
                end if
            end if
        end do
    end do;
    AutresEstSet := CompteurEstSet;
    TempSetEstSet := TempSetEstSet union CompteurEstSet;
    Print(CompteurEstSet)
end do;
    return TempSetEstSet
end proc
```

```

> EstSet:=proc(DiviseursEstSet,GeneratorsEstSet) local muetEstSet,
CompteurEstSet , AutresEstSet, TermsEstSet, IndetEstSet,TempEstSet,
TempSetEstSet; TempSetEstSet:=DiviseursEstSet;
AutresEstSet:=DiviseursEstSet; for muetEstSet while CompteurEstSet <>
{} do CompteurEstSet:={}; for TermsEstSet in AutresEstSet do for
IndetEstSet in indets(GeneratorsEstSet) do
TempEstSet:=TermsEstSet*IndetEstSet; if not
member(TempEstSet,TempSetEstSet) then if not
member(true,map2(divide,TempEstSet,GeneratorsEstSet)) then
CompteurEstSet:=CompteurEstSet union {TempEstSet}; end if end if;
end do end do; AutresEstSet:=CompteurEstSet;
TempSetEstSet:=TempSetEstSet union CompteurEstSet; end do; return
TempSetEstSet; end proc;

EstSet := proc(DiviseursEstSet, GeneratorsEstSet)
local muetEstSet, CompteurEstSet, AutresEstSet, TermsEstSet, IndetEstSet,
TempEstSet, TempSetEstSet;
    TempSetEstSet := DiviseursEstSet ;
    AutresEstSet := DiviseursEstSet ;
    for muetEstSet while CompteurEstSet ≠ {} do
        CompteurEstSet := {} ;
        for TermsEstSet in AutresEstSet do for IndetEstSet in indets(GeneratorsEstSet) do
            TempEstSet := IndetEstSet * TermsEstSet ;
            if not member(TempEstSet, TempSetEstSet) then
                if not member(true, map2(divide, TempEstSet, GeneratorsEstSet)) then
                    CompteurEstSet := CompteurEstSet union { TempEstSet }
                end if
            end if
        end do
    end do;
    AutresEstSet := CompteurEstSet ;
    TempSetEstSet := TempSetEstSet union CompteurEstSet
end do;
    return TempSetEstSet
end proc

```

10.6. Manipulation des objets algébriques

10.6.1. Utilitaires

```

> MultCoeffs:=proc(ListPoly,Indet); return map(coeffs,ListPoly,Indet);
end proc;

    MultCoeffs := proc(ListPoly, Indet) return map(coeffs, ListPoly, Indet) end proc

> MultCoeffsSet:=proc(ListPoly,Indet); return
map(coeffs,ListPoly,Indet); end proc;

    MultCoeffsSet := proc(ListPoly, Indet) return map(coeffs, ListPoly, Indet) end proc

> MultCoeffsVector:=proc(ListPoly,Indet); return
map(CoefficientVector,ListPoly,Indet); end proc;

    MultCoeffsVector := proc(ListPoly, Indet)
        return map(PolynomialTools : - CoefficientVector, ListPoly, Indet)
    end proc

```

```

> MultCoeffsList:=proc(ListPoly,Indet); return
  map(op,map(CoefficientList,ListPoly,Indet)); end proc;

  MultCoeffsList := proc(ListPoly, Indet)
    return map(op, map(PolynomialTools : -CoefficientList, ListPoly, Indet))
  end proc

> MultCoeffsListSet:=proc(ListPoly,Indet); return
  convert(map(op,convert(map(CoefficientList,ListPoly,Indet),set)),set);
  end proc;

  MultCoeffsListSet := proc(ListPoly, Indet)
    return convert(
      map(op, convert(map(PolynomialTools : -CoefficientList, ListPoly, Indet), set)), set)
  end proc

> IJpairs:=proc(NIJpairs) local aIJpairs, bIJpairs, NomfichierIJpairs;
  global TempIJpairs; NomfichierIJpairs := cat(PATHDATA, "IJpairs_",
  convert(NIJpairs,string), ".txt"); if
  Fichier_Existant(NomfichierIJpairs) = true then read
  NomfichierIJpairs; else TempIJpairs :=
  [seq(seq([aIJpairs,bIJpairs],aIJpairs=1..NIJpairs),bIJpairs=1..NIJpairs)];
  save TempIJpairs , NomfichierIJpairs; end if; return TempIJpairs;
  end proc;

  IJpairs := proc(NIJpairs)
    local aIJpairs, bIJpairs, NomfichierIJpairs;
    global TempIJpairs;
    NomfichierIJpairs := cat(PATHDATA, "IJpairs_", convert(NIJpairs, string), ".txt");
    if Fichier_Existant(NomfichierIJpairs) = true then read NomfichierIJpairs
    else
      TempIJpairs :=
        [seq(seq([aIJpairs, bIJpairs], aIJpairs = 1..NIJpairs), bIJpairs = 1..NIJpairs)];
      save TempIJpairs, NomfichierIJpairs
    end if;
    return TempIJpairs
  end proc

> IJensembles:=proc(NIJensembles) local aIJensembles, bIJensembles,
  NomfichierIJensembles ; global TempIJensembles ; NomfichierIJensembles
  := cat(PATHDATA, "IJensembles_", convert(NIJensembles,string),
  ".txt"); if Fichier_Existant(NomfichierIJensembles) = false then
  TempIJensembles:=[seq(seq(((aIJensembles,bIJensembles) -> if
  aIJensembles<=bIJensembles then return [aIJensembles,bIJensembles]
  else return NULL; end
  if)(aIJensembles,bIJensembles),aIJensembles=1..NIJensembles),bIJensembles=1..NIJensembles)];
  save TempIJensembles , NomfichierIJensembles ;
  else read NomfichierIJensembles end if; return TempIJensembles; end
  proc;

```

```

IJensembles := proc(NIJensembles)
local aIJensembles, bIJensembles, NomfichierIJensembles;
global TempIJensembles;
    NomfichierIJensembles :=
        cat(PATHDATA, "IJensembles_", convert(NIJensembles, string), ".txt");
    if Fichier_Existant(NomfichierIJensembles) = false then
        TempIJensembles := [seq(seq(proc(aIJensembles, bIJensembles)
            option operator, arrow;
            if aIJensembles ≤ bIJensembles then return [aIJensembles, bIJensembles]
            else return NULL
            end if
        end proc)(aIJensembles, bIJensembles), aIJensembles = 1..NIJensembles),
            bIJensembles = 1..NIJensembles)]];
        save TempIJensembles, NomfichierIJensembles
    else read NomfichierIJensembles
    end if;
    return TempIJensembles
end proc

> IJpairsdiff := proc(NIJpairsdiff) local aIJpairsdiff, bIJpairsdiff,
    NomfichierIJpairsdiff; global TempIJpairsdiff; NomfichierIJpairsdiff
    := cat(PATHDATA, "IJpairsdiff_", convert(NIJpairsdiff, string),
    ".txt"); if Fichier_Existant(NomfichierIJpairsdiff) = false then
    TempIJpairsdiff := [seq(seq((aIJpairsdiff, bIJpairsdiff) -> if
    aIJpairsdiff < bIJpairsdiff then return [aIJpairsdiff, bIJpairsdiff]
    else return NULL end
    if)(aIJpairsdiff, bIJpairsdiff), aIJpairsdiff = 1..NIJpairsdiff), bIJpairsd
    iff = 1..NIJpairsdiff)] ; save TempIJpairsdiff , NomfichierIJpairsdiff ;
    else read NomfichierIJpairsdiff end if; return TempIJpairsdiff; end
    proc;

    IJpairsdiff := proc(NIJpairsdiff)
    local aIJpairsdiff, bIJpairsdiff, NomfichierIJpairsdiff;
    global TempIJpairsdiff;
        NomfichierIJpairsdiff :=
            cat(PATHDATA, "IJpairsdiff_", convert(NIJpairsdiff, string), ".txt");
        if Fichier_Existant(NomfichierIJpairsdiff) = false then
            TempIJpairsdiff := [seq(seq(proc(aIJpairsdiff, bIJpairsdiff)
                option operator, arrow;
                if aIJpairsdiff ≠ bIJpairsdiff then return [aIJpairsdiff, bIJpairsdiff]
                else return NULL
                end if
            end proc)(aIJpairsdiff, bIJpairsdiff), aIJpairsdiff = 1..NIJpairsdiff),
                bIJpairsdiff = 1..NIJpairsdiff)]];
            save TempIJpairsdiff, NomfichierIJpairsdiff
        else read NomfichierIJpairsdiff
        end if;
        return TempIJpairsdiff
    end proc

```



```

> IJensemblesdiff:=proc(NIJensemblesdiff) local aIJensemblesdiff,
bIJensemblesdiff, NomfichierIJensemblesdiff; global
TempIJensemblesdiff; NomfichierIJensemblesdiff :=cat(PATHDATA,
"IJensemblesdiff_", convert(NIJensemblesdiff,string), ".txt"); if
Fichier_Existant(NomfichierIJensemblesdiff) = false then
TempIJensemblesdiff:=seq(seq(((aIJensemblesdiff,bIJensemblesdiff) ->
if aIJensemblesdiff < bIJensemblesdiff then return
[aIJensemblesdiff,bIJensemblesdiff] else return NULL end
if)(aIJensemblesdiff,bIJensemblesdiff),aIJensemblesdiff=1..NIJensemble
sdiff),bIJensemblesdiff=1..NIJensemblesdiff)] ; save
TempIJensemblesdiff , NomfichierIJensemblesdiff ; else read
NomfichierIJensemblesdiff end if; return TempIJensemblesdiff; end
proc;

IJensemblesdiff := proc(NIJensemblesdiff)
local aIJensemblesdiff, bIJensemblesdiff, NomfichierIJensemblesdiff;
global TempIJensemblesdiff;
  NomfichierIJensemblesdiff :=
    cat(PATHDATA, "IJensemblesdiff_", convert(NIJensemblesdiff, string), ".txt");
  if Fichier_Existant(NomfichierIJensemblesdiff) = false then
    TempIJensemblesdiff := [seq(seq(proc(aIJensemblesdiff, bIJensemblesdiff)
      option operator, arrow;
      if aIJensemblesdiff < bIJensemblesdiff then
        return [aIJensemblesdiff, bIJensemblesdiff]
      elsereturn NULL
      end if
    end proc)(aIJensemblesdiff, bIJensemblesdiff),
    aIJensemblesdiff = 1..NIJensemblesdiff), bIJensemblesdiff = 1..NIJensemblesdiff)];
  save TempIJensemblesdiff, NomfichierIJensemblesdiff
else read NomfichierIJensemblesdiff
end if;
  return TempIJensemblesdiff
end proc

```

10.6.2. Représentation algébrique des modèles possibles

```

> Polyn:=proc(Integ) description "K[x[1]...x[Integ]]" ; return
poly_algebra(seq(x[i],i=1..Integ),characteristic=0); end proc;

  Polyn := proc(Integ)
  description "K[x[1]...x[Integ]]";
    return Ore_algebra : -poly_algebra(seq(xi, i = 1..Integ), characteristic = 0)
  end proc

> Polyn_u:=proc(Integ) description "K[u[1]...u[Integ]]" ; return
poly_algebra(seq(u[i],i=1..Integ),characteristic=0); end proc;

  Polyn_u := proc(Integ)
  description "K[u[1]...u[Integ]]";
    return Ore_algebra : -poly_algebra(seq(ui, i = 1..Integ), characteristic = 0)
  end proc

```

```

> Modele:=proc(Integ) description "K[u[1]...u[Integ],x[1]...x[Integ],sigma]" ;
return poly_algebra(seq(u[i],i=1..Integ),seq(x[i],i=1..Integ),sigma,character
istic=0,rational=sigma,seq(rational=u[i],i=1..Integ)); end proc;

Modele := proc(Integ)
description "K[u[1]...u[Integ],x[1]...x[Integ],sigma]";
return Ore_algebra : -poly_algebra(seq(ui, i = 1..Integ), seq(xi, i = 1..Integ), σ,
characteristic = 0, rational = σ, seq(rational = ui, i = 1..Integ))
end proc

> ModeleXtd:=proc(Integ1,Integ2) description
"K[u[1]...u[Integ1],x[1]...x[Integ2],sigma]" ; return
poly_algebra(seq(u[i],i=1..Integ1),seq(x[i],i=1..Integ2),sigma,charact
eristic=0,rational=sigma,seq(rational=u[i],i=1..Integ1)); end proc;

ModeleXtd := proc(Integ1, Integ2)
description "K[u[1]...u[Integ1],x[1]...x[Integ2],sigma]";
return Ore_algebra : -poly_algebra(seq(ui, i = 1..Integ1), seq(xi, i = 1..Integ2), σ,
characteristic = 0, rational = σ, seq(rational = ui, i = 1..Integ1))
end proc

> CardModele:=proc(n,q) ; return binomial(n+q,q); end proc;
CardModele := proc(n, q) return combinat : -binomial(n + q, q) end proc

> CardModeleGraduated:=proc(n,q) local qtemp; return
seq(binomial(n+qtemp-1,qtemp),qtemp=0..q); end proc;

CardModeleGraduated := proc(n, q)
local qtemp;
return seq(combinat : -binomial(n + qtemp - 1, qtemp), qtemp = 0..q)
end proc

```

10.6.3. Décomposition dans une sous-algèbre de polynômes

```

> DecompInKfi:=proc(Gene,Poly,Algebra) local NDecompInKfi,MDecompInKfi,
AlgebraDecompInKfi,TermorderingDecompInKfi,OrderingDecompInKfi,
IdealDecompInKfi,IndetsDecompInKfi,TempDecompInKfi,IndDecompInKfi;
NDecompInKfi:=nops(GeneDecompInKfi);
IndetsDecompInKfi:=seq(IndetDecompInKfi[i],i=1..NDecompInKfi);
MDecompInKfi:=nops(AlgebraDecompInKfi["all_indets"]);
AlgebraDecompInKfi:=Algebra;AlgebraDecompInKfi["all_indets"]:=
{IndetsDecompInKfi,op(Algebra["all_indets"])};
AlgebraDecompInKfi["polynomial_indets"]:=
{IndetsDecompInKfi,op(Algebra["polynomial_indets"])};
AlgebraDecompInKfi["rational_indets"]:=Algebra["rational_indets"];
AlgebraDecompInKfi["comm_indets"]:= {IndetsDecompInKfi,op(Algebra["comm_indets"])};
TermorderingDecompInKfi:=termorder(AlgebraDecompInKfi,lexdeg([op(Algeb
ra["polynomial_indets"] minus {IndetsDecompInKfi})),[IndetsDecompInKfi]);
OrderingDecompInKfi:=lexdeg([op(Algebra["polynomial_indets"] minus
{IndetsDecompInKfi})),[IndetsDecompInKfi]);
IdealDecompInKfi:=seq(IndetDecompInKfi[i]-Gene[i],i=1..NDecompInKfi);
TempDecompInKfi:=map(normalf,ListPolyDecompInKfi,IdealDecompInKfi,Orde
ringDecompInKfi);
Algebra["all_indets"]:=AlgebraDecompInKfi["all_indets"] minus
{IndetsDecompInKfi};
Algebra["polynomial_indets"]:=AlgebraDecompInKfi["polynomial_indets"]
minus {IndetsDecompInKfi};
Algebra["rational_indets"]:=AlgebraDecompInKfi["rational_indets"];
Algebra["comm_indets"]:=AlgebraDecompInKfi["comm_indets"] minus {IndetsDecompInKfi};
return normalf(Poly,IdealDecompInKfi,OrderingDecompInKfi); end proc;

```

```

DecompInKfi := proc(Gene, Poly, Algebra)
local NDecompInKfi, MDecompInKfi, AlgebraDecompInKfi,
TermorderingDecompInKfi, OrderingDecompInKfi, IdealDecompInKfi,
IndetsDecompInKfi, TempDecompInKfi, IndDecompInKfi;
  NDecompInKfi := nops(GeneDecompInKfi);
  IndetsDecompInKfi := seq(IndetDecompInKfii, i = 1..NDecompInKfi);
  MDecompInKfi := nops(AlgebraDecompInKfi"all_indets");
  AlgebraDecompInKfi := Algebra;
  AlgebraDecompInKfi "all_indets" :=
    {IndetsDecompInKfi, op(Algebra"all_indets")};
  AlgebraDecompInKfi "polynomial_indets" :=
    {IndetsDecompInKfi, op(Algebra"polynomial_indets")};
  AlgebraDecompInKfi "rational_indets" := Algebra "rational_indets";
  AlgebraDecompInKfi "comm_indets" :=
    {IndetsDecompInKfi, op(Algebra"comm_indets")};
  TermorderingDecompInKfi := Groebner : -termorder(AlgebraDecompInKfi, lexdeg(
    [op(Algebra "polynomial_indets" minus {IndetsDecompInKfi}),
    [IndetsDecompInKfi]]));
  OrderingDecompInKfi := lexdeg(
    [op(Algebra "polynomial_indets" minus {IndetsDecompInKfi}),
    [IndetsDecompInKfi]]);
  IdealDecompInKfi := [seq(IndetDecompInKfii - Genei, i = 1..NDecompInKfi)];
  TempDecompInKfi := map(Groebner : -normalf, ListPolyDecompInKfi,
    IdealDecompInKfi, OrderingDecompInKfi);
  Algebra"all_indets" :=
    AlgebraDecompInKfi "all_indets" minus {IndetsDecompInKfi};
  Algebra"polynomial_indets" :=
    AlgebraDecompInKfi "polynomial_indets" minus {IndetsDecompInKfi};
  Algebra"rational_indets" := AlgebraDecompInKfi"rational_indets";
  Algebra"comm_indets" :=
    AlgebraDecompInKfi "comm_indets" minus {IndetsDecompInKfi};
  return Groebner : -normalf(Poly, IdealDecompInKfi, OrderingDecompInKfi)
end proc

```

```

> DecompInKfiInfo:=proc(Gene,Poly,Algebra) local NDecompInKfi,MDecompInKfi,
AlgebraDecompInKfi,TermorderingDecompInKfi,OrderingDecompInKfi,IdealDecompInKfi,
IndetsDecompInKfi,TempDecompInKfi,IndDecompInKfi; NDecompInKfi:=nops(Gene
DecompInKfi); IndetsDecompInKfi:=seq(IndetDecompInKfi[i],i=1..NDecompInKfi);
MDecompInKfi:=nops(AlgebraDecompInKfi["all_indets"]);
AlgebraDecompInKfi:=Algebra;AlgebraDecompInKfi["all_indets"]:=
{IndetsDecompInKfi,op(Algebra["all_indets"])};
AlgebraDecompInKfi["polynomial_indets"]:=
{IndetsDecompInKfi,op(Algebra["polynomial_indets"])};
AlgebraDecompInKfi["rational_indets"]:=Algebra["rational_indets"];
AlgebraDecompInKfi["comm_indets"]:= {IndetsDecompInKfi,op(Algebra["comm_indets"])};
TermorderingDecompInKfi:=termorder(AlgebraDecompInKfi,lexdeg([op(Algebra
["polynomial_indets"] minus {IndetsDecompInKfi}),[IndetsDecompInKfi]));

```

```

OrderingDecompInKfi:=lexdeg([op(Algebra["polynomial_indets"] minus
{IndetsDecompInKfi})),[IndetsDecompInKfi]);
IdealDecompInKfi:=seq(IndetDecompInKfi[i]-Gene[i],i=1..NDecompInKfi);
TempDecompInKfi:=map(normalf,ListPolyDecompInKfi,IdealDecompInKfi,OrderingDecompInKfi);
Algebra["all_indets"]:=AlgebraDecompInKfi["all_indets"] minus
{IndetsDecompInKfi};Algebra["polynomial_indets"]:=AlgebraDecomp
InKfi["polynomial_indets"] minus {IndetsDecompInKfi} ; Algebra["rational_indets"] :=
AlgebraDecompInKfi["rational_indets"] ; Algebra["comm_indets"] :=
AlgebraDecompInKfi["comm_indets"] minus {IndetsDecompInKfi}; return
(normalf(Poly,IdealDecompInKfi,OrderingDecompInKfi),seq(IndetDecompInK
fi[i]=Gene[i],i=1..NDecompInKfi)) ; end proc;

```

```

DecompInKfiInfo := proc(Gene, Poly, Algebra)
local NDecompInKfi, MDecompInKfi, AlgebraDecompInKfi,
TermorderingDecompInKfi, OrderingDecompInKfi, IdealDecompInKfi,
IndetsDecompInKfi, TempDecompInKfi, IndDecompInKfi;
NDecompInKfi := nops(GeneDecompInKfi);
IndetsDecompInKfi := seq(IndetDecompInKfii, i = 1..NDecompInKfi);
MDecompInKfi := nops(AlgebraDecompInKfi["all_indets"]);
AlgebraDecompInKfi := Algebra;
AlgebraDecompInKfi "all_indets" :=
{IndetsDecompInKfi, op(Algebra"all_indets")};
AlgebraDecompInKfi "polynomial_indets" :=
{IndetsDecompInKfi, op(Algebra"polynomial_indets")};
AlgebraDecompInKfi "rational_indets" := Algebra "rational_indets";
AlgebraDecompInKfi "comm_indets" :=
{IndetsDecompInKfi, op(Algebra"comm_indets")};
TermorderingDecompInKfi := Groebner : -termorder(AlgebraDecompInKfi, lexdeg(
[op(Algebra "polynomial_indets" minus {IndetsDecompInKfi})],
[IndetsDecompInKfi]));
OrderingDecompInKfi := lexdeg(
[op(Algebra "polynomial_indets" minus {IndetsDecompInKfi})],
[IndetsDecompInKfi]);
IdealDecompInKfi := [seq(IndetDecompInKfii - Genei, i = 1..NDecompInKfi)];
TempDecompInKfi := map(Groebner : -normalf, ListPolyDecompInKfi,
IdealDecompInKfi, OrderingDecompInKfi);
Algebra"all_indets" :=
AlgebraDecompInKfi "all_indets" minus {IndetsDecompInKfi};
Algebra"polynomial_indets" :=
AlgebraDecompInKfi "polynomial_indets" minus {IndetsDecompInKfi};
Algebra"rational_indets" := AlgebraDecompInKfi"rational_indets";
Algebra"comm_indets" :=
AlgebraDecompInKfi "comm_indets" minus {IndetsDecompInKfi};
return Groebner : -normalf(Poly, IdealDecompInKfi, OrderingDecompInKfi),
seq(IndetDecompInKfii = Genei, i = 1..NDecompInKfi)
end proc

```

```

> DecompInKfiSubs:=proc(Gene,Poly,Algebra) local
NDecompInKfi,MDecompInKfi,AlgebraDecompInKfi,TermorderingDecompInKfi,OrderingDecompInKfi,IdealDecompInKfi,IndetsDecompInKfi,TempDecompInKfi,IndDecompInKfi;
NDecompInKfi:=nops(GeneDecompInKfi);
IndetsDecompInKfi:=seq(IndetDecompInKfi[i],i=1..NDecompInKfi);
MDecompInKfi:=nops(AlgebraDecompInKfi["all_indets"]);
AlgebraDecompInKfi:=Algebra;AlgebraDecompInKfi["all_indets"]:=
{IndetsDecompInKfi,op(Algebra["all_indets"])};
AlgebraDecompInKfi["polynomial_indets"]:=
{IndetsDecompInKfi,op(Algebra["polynomial_indets"])};
AlgebraDecompInKfi["rational_indets"]:=Algebra["rational_indets"];
AlgebraDecompInKfi["comm_indets"]:=
{IndetsDecompInKfi,op(Algebra["comm_indets"])};
TermorderingDecompInKfi:=termorder(AlgebraDecompInKfi,lexdeg([op(Algebra["polynomial_indets"] minus {IndetsDecompInKfi})),[IndetsDecompInKfi]));
OrderingDecompInKfi:=lexdeg([op(Algebra["polynomial_indets"] minus {IndetsDecompInKfi})),[IndetsDecompInKfi]);
IdealDecompInKfi:=seq(IndetDecompInKfi[i]-Gene[i],i=1..NDecompInKfi);
;
TempDecompInKfi:=map(normalf,ListPolyDecompInKfi,IdealDecompInKfi,OrderingDecompInKfi);
Algebra["all_indets"]:=AlgebraDecompInKfi["all_indets"] minus {IndetsDecompInKfi};
Algebra["polynomial_indets"]:=AlgebraDecompInKfi["polynomial_indets"] minus {IndetsDecompInKfi};
Algebra["rational_indets"]:=AlgebraDecompInKfi["rational_indets"] minus {IndetsDecompInKfi};
Algebra["comm_indets"]:=AlgebraDecompInKfi["comm_indets"] minus {IndetsDecompInKfi};
return subs(seq(IndetDecompInKfi[i]=Gene[i],i=1..NDecompInKfi),normalf(Poly,IdealDecompInKfi,OrderingDecompInKfi)); end proc;

```

```

DecompInKfiSubs := proc(Gene, Poly, Algebra)
local NDecompInKfi, MDecompInKfi, AlgebraDecompInKfi,
TermorderingDecompInKfi, OrderingDecompInKfi, IdealDecompInKfi,
IndetsDecompInKfi, TempDecompInKfi, IndDecompInKfi;
NDecompInKfi := nops(GeneDecompInKfi);
IndetsDecompInKfi := seq(IndetDecompInKfii, i = 1..NDecompInKfi);
MDecompInKfi := nops(AlgebraDecompInKfi["all_indets"]);
AlgebraDecompInKfi := Algebra;
AlgebraDecompInKfi["all_indets"] :=
{IndetsDecompInKfi, op(Algebra["all_indets"])};
AlgebraDecompInKfi["polynomial_indets"] :=
{IndetsDecompInKfi, op(Algebra["polynomial_indets"])};
AlgebraDecompInKfi["rational_indets"] := Algebra["rational_indets"];
AlgebraDecompInKfi["comm_indets"] :=
{IndetsDecompInKfi, op(Algebra["comm_indets"])};
TermorderingDecompInKfi := Groebner : -termorder(AlgebraDecompInKfi, lexdeg(
[op(Algebra["polynomial_indets"] minus {IndetsDecompInKfi})),
[IndetsDecompInKfi]));
OrderingDecompInKfi := lexdeg(
[op(Algebra["polynomial_indets"] minus {IndetsDecompInKfi})),
[IndetsDecompInKfi]);

```

```

IdealDecompInKfi := [seq(IndetDecompInKfii - Genei, i = 1..NDecompInKfi)];
TempDecompInKfi := map(Groebner : -normalf, ListPolyDecompInKfi,
    IdealDecompInKfi, OrderingDecompInKfi);
Algebra"all_indets" :=
    AlgebraDecompInKfi "all_indets" minus {IndetsDecompInKfi};
Algebra"polynomial_indets" :=
    AlgebraDecompInKfi "polynomial_indets" minus {IndetsDecompInKfi};
Algebra"rational_indets" := AlgebraDecompInKfi"rational_indets";
Algebra"comm_indets" :=
    AlgebraDecompInKfi "comm_indets" minus {IndetsDecompInKfi};
return subs(seq(IndetDecompInKfii = Genei, i = 1..NDecompInKfi),
    Groebner : -normalf (Poly, IdealDecompInKfi, OrderingDecompInKfi))
end proc

> EqInKfi_normalf:=proc(GeneEqInKfi_normalf,ListPolyEqInKfi_normalf,Algebra) local
    NEqInKfi_normalf,MEqInKfi_normalf,AlgebraEqInKfi_normalf,TermorderingEqInKfi_normalf,OrderingEqInKfi_normalf,IdealEqInKfi_normalf,IndetsEqInKfi_normalf,TempEqInKfi_normalf,IndEqInKfi_normalf;
    NEqInKfi_normalf:=nops(GeneEqInKfi_normalf);
    IndetsEqInKfi_normalf:=seq(IndetEqInKfi_normalf[i],i=1..NEqInKfi_normalf); MEqInKfi_normalf:=nops(AlgebraEqInKfi_normalf["all_indets"]);
    AlgebraEqInKfi_normalf:=Algebra;AlgebraEqInKfi_normalf["all_indets"]:=
    {IndetsEqInKfi_normalf,op(Algebra["all_indets"])};
    AlgebraEqInKfi_normalf["polynomial_indets"]:=
    {IndetsEqInKfi_normalf,op(Algebra["polynomial_indets"])};
    AlgebraEqInKfi_normalf["rational_indets"]:=Algebra["rational_indets"];
    AlgebraEqInKfi_normalf["comm_indets"]:=
    {IndetsEqInKfi_normalf,op(Algebra["comm_indets"])};
    TermorderingEqInKfi_normalf:=termorder(AlgebraEqInKfi_normalf,lexdeg([op(Algebra["polynomial_indets"] minus {IndetsEqInKfi_normalf})], [IndetsEqInKfi_normalf]));
    OrderingEqInKfi_normalf:=lexdeg([op(Algebra["polynomial_indets"] minus {IndetsEqInKfi_normalf})], [IndetsEqInKfi_normalf]);
    IdealEqInKfi_normalf:=seq(IndetEqInKfi_normalf[i]-GeneEqInKfi_normalf[i],i=1..NEqInKfi_normalf);
    TempEqInKfi_normalf:=normalf(op(ListPolyEqInKfi_normalf),IdealEqInKfi_normalf,OrderingEqInKfi_normalf);
    Algebra["all_indets"]:=AlgebraEqInKfi_normalf["all_indets"] minus {IndetsEqInKfi_normalf};
    Algebra["polynomial_indets"]:=AlgebraEqInKfi_normalf["polynomial_indets"] minus {IndetsEqInKfi_normalf};
    Algebra["rational_indets"]:=AlgebraEqInKfi_normalf["rational_indets"];
    Algebra["comm_indets"]:=AlgebraEqInKfi_normalf["comm_indets"] minus {IndetsEqInKfi_normalf};
    for IndEqInKfi_normalf in Algebra["polynomial_indets"] do
        TempEqInKfi_normalf:=MultCoeffsList(TempEqInKfi_normalf,IndEqInKfi_normalf) end do;
    TempEqInKfi_normalf:=convert(subsop(1=NULL,TempEqInKfi_normalf),set);
    for IndEqInKfi_normalf in {IndetsEqInKfi_normalf} do
        TempEqInKfi_normalf:=MultCoeffsListSet(TempEqInKfi_normalf,IndEqInKfi_normalf) end do;
    return
    InterReduce(TempEqInKfi_normalf,tdeg(op(Algebra["rational_indets"])))
end proc;

```

```

EqInKfi_normalf := proc(GeneEqInKfi_normalf, ListPolyEqInKfi_normalf, Algebra)
localNEqInKfi_normalf, MEqInKfi_normalf, AlgebraEqInKfi_normalf,
TermorderingEqInKfi_normalf, OrderingEqInKfi_normalf, IdealEqInKfi_normalf,
IndetsEqInKfi_normalf, TempEqInKfi_normalf, IndEqInKfi_normalf;
  NEqInKfi_normalf := nops(GeneEqInKfi_normalf);
  IndetsEqInKfi_normalf := seq(IndetEqInKfi_normalfi, i = 1..NEqInKfi_normalf);
  MEqInKfi_normalf := nops(AlgebraEqInKfi_normalf“all_indets”);
  AlgebraEqInKfi_normalf := Algebra;
  AlgebraEqInKfi_normalf “all_indets” :=
    {IndetsEqInKfi_normalf, op(Algebra“all_indets”)};
  AlgebraEqInKfi_normalf “polynomial_indets” :=
    {IndetsEqInKfi_normalf, op(Algebra“polynomial_indets”)};
  AlgebraEqInKfi_normalf “rational_indets” := Algebra “rational_indets”;
  AlgebraEqInKfi_normalf “comm_indets” :=
    {IndetsEqInKfi_normalf, op(Algebra“comm_indets”)};
  TermorderingEqInKfi_normalf := Groebner : -termorder(AlgebraEqInKfi_normalf,
    lexdeg([op(Algebra“polynomial_indets” minus {IndetsEqInKfi_normalf})],
    [IndetsEqInKfi_normalf]));
  OrderingEqInKfi_normalf := lexdeg(
    [op(Algebra “polynomial_indets” minus {IndetsEqInKfi_normalf})],
    [IndetsEqInKfi_normalf]);
  IdealEqInKfi_normalf :=
    [seq(IndetEqInKfi_normalfi - GeneEqInKfi_normalfi, i = 1..NEqInKfi_normalf)];
  TempEqInKfi_normalf := [Groebner : -normalf(op(ListPolyEqInKfi_normalf),
    IdealEqInKfi_normalf, OrderingEqInKfi_normalf)];
  Algebra“all_indets” :=
    AlgebraEqInKfi_normalf “all_indets” minus {IndetsEqInKfi_normalf};
  Algebra“polynomial_indets” :=
    AlgebraEqInKfi_normalf “polynomial_indets” minus {IndetsEqInKfi_normalf};
  Algebra“rational_indets” := AlgebraEqInKfi_normalf “rational_indets” ;
  Algebra“comm_indets” :=
    AlgebraEqInKfi_normalf “comm_indets” minus {IndetsEqInKfi_normalf};
for IndEqInKfi_normalf in Algebra“polynomial_indets” do TempEqInKfi_normalf
  := MultCoeffsList(TempEqInKfi_normalf, IndEqInKfi_normalf)
end do;
  TempEqInKfi_normalf := convert(subsop(1 = NULL, TempEqInKfi_normalf), set);
for IndEqInKfi_normalf in {IndetsEqInKfi_normalf} do TempEqInKfi_normalf :=
  MultCoeffsListSet(TempEqInKfi_normalf, IndEqInKfi_normalf)
end do;
return PolynomialIdeals : -InterReduce(TempEqInKfi_normalf,
  tdeg(op(Algebra“rational_indets”)))
end proc

```

```
> EqInKfi_reduce:=proc(GeneEqInKfi_reduce,ListPolyEqInKfi_reduce,Algebra)
local
NEqInKfi_reduce,MEqInKfi_reduce,AlgebraEqInKfi_reduce,TermorderingEqInKfi_reduce,
OrderingEqInKfi_reduce,IdealEqInKfi_reduce,IndetsEqInKfi_reduce,TempEqInKfi_reduce,IndEqInKfi_reduce;
NEqInKfi_reduce:=nops(GeneEqInKfi_reduce);
IndetsEqInKfi_reduce:=seq(IndetEqInKfi_reduce[i],i=1..NEqInKfi_reduce);
MEqInKfi_reduce:=nops(AlgebraEqInKfi_reduce["all_indets"]);
AlgebraEqInKfi_reduce:=Algebra;AlgebraEqInKfi_reduce["all_indets"]:=
{IndetsEqInKfi_reduce,op(Algebra["all_indets"])};
AlgebraEqInKfi_reduce["polynomial_indets"]:=
{IndetsEqInKfi_reduce,op(Algebra["polynomial_indets"])};
AlgebraEqInKfi_reduce["rational_indets"]:=Algebra["rational_indets"];
AlgebraEqInKfi_reduce["comm_indets"]:=
{IndetsEqInKfi_reduce,op(Algebra["comm_indets"])};
TermorderingEqInKfi_reduce:=termorder(AlgebraEqInKfi_reduce,lexdeg([op(Algebra["polynomial_indets"]
minus {IndetsEqInKfi_reduce})),[IndetsEqInKfi_reduce]));
OrderingEqInKfi_reduce:=lexdeg([op(Algebra["polynomial_indets"] minus {IndetsEqInKfi_reduce})),[IndetsEqInKfi_reduce]);
IdealEqInKfi_reduce:=[seq(IndetEqInKfi_reduce[i]-GeneEqInKfi_reduce[i],i=1..NEqInKfi_reduce)];
TempEqInKfi_reduce:=[reduce(op(ListPolyEqInKfi_reduce),IdealEqInKfi_reduce,OrderingEqInKfi_reduce)];
Algebra["all_indets"]:=AlgebraEqInKfi_reduce["all_indets"] minus {IndetsEqInKfi_reduce};
Algebra["polynomial_indets"]:=AlgebraEqInKfi_reduce["polynomial_indets"] minus {IndetsEqInKfi_reduce};
Algebra["rational_indets"]:=AlgebraEqInKfi_reduce["rational_indets"];
Algebra["comm_indets"]:=AlgebraEqInKfi_reduce["comm_indets"] minus {IndetsEqInKfi_reduce};
for IndEqInKfi_reduce in Algebra["polynomial_indets"] do
TempEqInKfi_reduce:=MultCoeffsList(TempEqInKfi_reduce,IndEqInKfi_reduce) end do;
TempEqInKfi_reduce:=convert(subsop(1=NULL,TempEqInKfi_reduce),set);
for IndEqInKfi_reduce in {IndetsEqInKfi_reduce} do
TempEqInKfi_reduce:=MultCoeffsListSet(TempEqInKfi_reduce,IndEqInKfi_reduce) end do; return
InterReduce(TempEqInKfi_reduce,tdeg(op(Algebra["rational_indets"])))));
end proc;
```

```
EqInKfi_reduce := proc(GeneEqInKfi_reduce, ListPolyEqInKfi_reduce, Algebra)
localNEqInKfi_reduce, MEqInKfi_reduce, AlgebraEqInKfi_reduce,
TermorderingEqInKfi_reduce, OrderingEqInKfi_reduce, IdealEqInKfi_reduce,
IndetsEqInKfi_reduce, TempEqInKfi_reduce, IndEqInKfi_reduce;
NEqInKfi_reduce := nops(GeneEqInKfi_reduce);
IndetsEqInKfi_reduce := seq(IndetEqInKfi_reduce[i], i = 1..NEqInKfi_reduce);
MEqInKfi_reduce := nops(AlgebraEqInKfi_reduce["all_indets"]);
AlgebraEqInKfi_reduce := Algebra;
AlgebraEqInKfi_reduce["all_indets"] :=
{IndetsEqInKfi_reduce, op(Algebra["all_indets"])};
AlgebraEqInKfi_reduce["polynomial_indets"] :=
{IndetsEqInKfi_reduce, op(Algebra["polynomial_indets"])};
AlgebraEqInKfi_reduce["rational_indets"] := Algebra["rational_indets"];
AlgebraEqInKfi_reduce["comm_indets"] :=
{IndetsEqInKfi_reduce, op(Algebra["comm_indets"])};
TermorderingEqInKfi_reduce := Groebner : -termorder(AlgebraEqInKfi_reduce,
lexdeg([op(Algebra["polynomial_indets"] minus {IndetsEqInKfi_reduce})),
```



```

    [IndetsEqInKfi_reduce]));
    OrderingEqInKfi_reduce := lexdeg(
        [op(Algebra "polynomial_indets" minus {IndetsEqInKfi_reduce})],
        [IndetsEqInKfi_reduce]);
    IdealEqInKfi_reduce :=
        [seq(IndetEqInKfi_reducei - GeneEqInKfi_reducei, i = 1..NEqInKfi_reduce)];
    TempEqInKfi_reduce := [Groebner : -reduce(op(ListPolyEqInKfi_reduce),
        IdealEqInKfi_reduce, OrderingEqInKfi_reduce)];
    Algebra"all_indets" :=
        AlgebraEqInKfi_reduce "all_indets" minus {IndetsEqInKfi_reduce};
    Algebra"polynomial_indets" :=
        AlgebraEqInKfi_reduce "polynomial_indets" minus {IndetsEqInKfi_reduce};
    Algebra"rational_indets" := AlgebraEqInKfi_reduce"rational_indets" ;
    Algebra"comm_indets" :=
        AlgebraEqInKfi_reduce "comm_indets" minus {IndetsEqInKfi_reduce};
    for IndEqInKfi_reduce in Algebra"polynomial_indets" do
        TempEqInKfi_reduce := MultCoeffsList(TempEqInKfi_reduce, IndEqInKfi_reduce)
    end do;
    TempEqInKfi_reduce := convert(subsop(1 = NULL, TempEqInKfi_reduce), set);
    for IndEqInKfi_reduce in {IndetsEqInKfi_reduce} do TempEqInKfi_reduce :=
        MultCoeffsListSet(TempEqInKfi_reduce, IndEqInKfi_reduce)
    end do;
    return PolynomialIdeals : -InterReduce(TempEqInKfi_reduce,
        tdeg(op(Algebra"rational_indets")))
end proc

```

10.6.4. Solutions d'un système d'équations polynomiales

```

> IsRealSols:=proc(Solsirs) local tempsol; return
map('not',map(booladd,map(tempsol2->map(type,tempsol2,nonreal),map(tem
psol ->
map2(op,2,tempsol),convert(map(convert,map(evalf,Solsirs),list),list))
))); end proc;

IsRealSols := proc(Solsirs)
local tempsol;
return map('not',map(booladd,map(tempsol2 → map(type, tempsol2, nonreal),map(
tempsol → map2(op, 2, tempsol), convert(map(convert, map(evalf, Solsirs), list), list))))))
end proc

> WhichRealSols:=proc(Solswrs) local listwrs,boolwrs,tempwrs;
listwrs:=IsRealSols(Solswrs); if booladd(IsRealSols(Solswrs)) then
tempwrs:=NULL; for boolwrs while
member(true,listwrs,'indiceWhichRealSols') do
tempwrs:=tempwrs,indiceWhichRealSols ;
listwrs:=subsop(indiceWhichRealSols=false,listwrs);
unassign('indiceWhichRealSols'); end do; return
[tempwrs],[seq(op(tempwrs[Indicetempwrs],Solswrs),Indicetempwrs=1..nop
s([tempwrs]))]; end if; end proc;

```

```

WhichRealSols := proc(Solswrs)
local listwrs, boolwrs, tempwrs;
  listwrs := IsRealSols(Solswrs);
  if booladd(IsRealSols(Solswrs)) then
    tempwrs := NULL;
    for boolwrs while member(true, listwrs, 'indiceWhichRealSols') do
      tempwrs := tempwrs, indiceWhichRealSols;
      listwrs := subsop(indiceWhichRealSols = false, listwrs);
      unassign('indiceWhichRealSols')
    end do;
    return [tempwrs],
      [seq(op(tempwrsIndicetempwrs, Solswrs), Indicetempwrs = 1..nops([tempwrs]))]
  end if
end proc

> IsRealSolsApprox:=proc(Solsirs) local tempsol2; return map('not',map(
  booladd,map(tempsol2->map(tempsol3 -> ifabs(Im(tempsol3))<0.0000001
  then return false else true end if,tempsol2),map(tempsol -> map2(op,2,
  tempsol),convert(map(convert,map(evalf,Solsirs),list),list))))); end proc;

IsRealSolsApprox := proc(Solsirs)
local tempsol2;
  return map('not',map(booladd,map(tempsol2 → map(proc(tempsol3)
  option operator, arrow;
    ifabs(ℑ(tempsol3)) < 0.1 * 10(-6) then return false else true end if
  end proc, tempsol2),map(tempsol → map2(op, 2, tempsol),
  convert(map(convert, map(evalf, Solsirs), list), list))))))
end proc

> WhichRealSolsApprox:=proc(Solswrs) local listwrs, boolwrs, tempwrs;
  listwrs:=IsRealSolsApprox(Solswrs); if booladd(IsRealSolsApprox(Solswrs))
  then tempwrs:=NULL; for boolwrs while member(true,listwrs,
  'indiceWhichRealSols') do tempwrs:=tempwrs,indiceWhichRealSols;
  listwrs:=subsop(indiceWhichRealSols=false,listwrs); unassign('indice
  WhichRealSols'); end do; return [tempwrs], [seq(op(tempwrsIndicetempwrs
  ,Solswrs),Indicetempwrs=1..nops([tempwrs]))]; end if; end proc;

WhichRealSolsApprox := proc(Solswrs)
local listwrs, boolwrs, tempwrs;
  listwrs := IsRealSolsApprox(Solswrs);
  if booladd(IsRealSolsApprox(Solswrs)) then
    tempwrs := NULL;
    for boolwrs while member(true, listwrs, 'indiceWhichRealSols') do
      tempwrs := tempwrs, indiceWhichRealSols;
      listwrs := subsop(indiceWhichRealSols = false, listwrs);
      unassign('indiceWhichRealSols')
    end do;
    return [tempwrs],
      [seq(op(tempwrsIndicetempwrs, Solswrs), Indicetempwrs = 1..nops([tempwrs]))]
  end if
end proc

```


10.6.5. Idéal annulateur d'une variété de dimension 0

```

> IdealOfPoints:=proc(LPoints_BM,Minpol_BM) local
G_BM,O_BM,S_BM,M_BM,L_BM,t1_BM,t2_BM,i_BM,j_BM,k_BM,t_BM,trel_BM,LPointsTp_BM,Dim_BM,Npoints_BM,T_BM,a_BM,minpol_BM,Alg_BM,Ord_BM,Indets_BM,
reduce_mipol,reduce_mipol_i; unassign('x_');
G_BM:=[];O_BM:=[];S_BM:=[];L_BM:=[];M_BM:=[];if nargs>=2 then
minpol_BM:=Minpol_BM end if; a_BM:=indets(minpol_BM)[]; if
type(set,LPoints_BM) then LPointsTp_BM:=convert(LPoints_BM,list): else
if type(LPoints_BM,Matrix) then
LPointsTp_BM:=map(convert,convert((LPoints_BM),listlist),Vector) else
LPointsTp_BM:=LPoints_BM: end if: end if: Npoints_BM:=nops(%):
Dim_BM:=op(1,%%[1]): Indets_BM:=seq(x_[i_BM],i_BM=1..Dim_BM):
Alg_BM:=poly_algebra(%): Ord_BM:=tdeg(%):
T_BM:=termorder(%%,%,[%%]): if not (assigned(minpol_BM) and
has(LPointsTp_BM,a_BM)) then if not(hastype(LPointsTp_BM,radical))
then print(not(hastype(LPointsTp_BM,radical)));
while L_BM<>[] do
sort(L_BM,(t1_BM,t2_BM)->testorder(t1_BM,t2_BM,T_BM)): t_BM:=op(1,%);
L_BM:=subsop(1=NULL,%):
[seq(eval(t_BM,[seq(x_[j_BM]=LPointsTp_BM[k_BM][j_BM],j_BM=1..Dim_BM)]),
k_BM=1..Npoints_BM)]: if M_BM=[] then %: else
alginreducev3(ListListToEqs(M_BM,%)):
subs(%,%-add(map2('* ',aleph[i_BM],op(i_BM,M_BM)),
i_BM=1..nops(M_BM))): end if: if M_BM<>[] then trel_BM:=subs(%,
t_BM-add(map2('* ',aleph[i_BM],S_BM[i_BM]),i_BM=1..nops(M_BM))):
else trel_BM:=t_BM end if: if %=[seq(0,j_BM=1..Npoints_BM)] then
G_BM:=op(G_BM),trel_BM]; L_BM:=op(map(P_BM->if
not(divide(P_BM,t_BM,'q')) then return P_BM end if,L_BM))]; else
M_BM:=op(M_BM,%); S_BM:=op(S_BM),trel_BM];
O_BM:=op(O_BM),t_BM]; if G_BM<>[] then map(Pol_BM -> if
not(booladd(map2(divide,Pol_BM,Generators(PolynomialIdeal(map(Groebner
[leadterm],GroebnerBasis(<op(G_BM)>,Ord_BM),Ord_BM),select(type,[op(G_BM)],
equation)))))) then return Pol_BM end if,
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM)]: else
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM)]: end if; if L_BM<>[] then
map(Pol_BM-> if not(booladd(map2(divide,Pol_BM,L_BM))) then return
Pol_BM end if, %): else %: end if; L_BM:=op(L_BM),op(%); end if;
if info_iop>=3 then print(nops(O_BM)) end if; end do; else while
L_BM<>[] do sort(L_BM,(t1_BM,t2_BM)->testorder(t1_BM,t2_BM,T_BM)):
t_BM:=op(1,%); L_BM:=subsop(1=NULL,%):
[seq(eval(t_BM,[seq(x_[j_BM]=LPointsTp_BM[k_BM][j_BM],j_BM=1..Dim_BM)]),
k_BM=1..Npoints_BM)]: if M_BM=[] then %: else
alginreducev3(ListListToEqs(M_BM,%)):
map(radnormal,subs(%,%-add(map2('* ',aleph[i_BM],op(i_BM,M_BM)),
i_BM=1..nops(M_BM))), 'rationalized'): end if: if M_BM<>[] then
trel_BM:=map(radnormal,subs(%,t_BM-add(map2('* ',aleph[i_BM],
S_BM[i_BM]),i_BM=1..nops(M_BM))), 'rationalized'): else trel_BM:=t_BM
end if: if map(radnormal,%, 'rationalized')=[seq(0,j_BM=1..Npoints_BM)]
then G_BM:=op(G_BM),trel_BM]; L_BM:=op(map(P_BM->if
not(divide(P_BM,t_BM,'q')) then return P_BM end if,L_BM))]; else
M_BM:=op(M_BM,%); S_BM:=op(S_BM),trel_BM];
O_BM:=op(O_BM),t_BM]; if G_BM<>[] then map(Pol_BM -> if
not(booladd(map2(divide,Pol_BM,Generators(PolynomialIdeal(map(Groebner
[leadterm],GroebnerBasis(<op(G_BM)>,Ord_BM),Ord_BM),select(type,[op(G_BM)],
equation)))))) then return Pol_BM end if,
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM)]: else
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM)]: end if; if L_BM<>[] then
map(Pol_BM-> if not(booladd(map2(divide,Pol_BM,L_BM))) then return
Pol_BM end if, %): else %: end if; L_BM:=op(L_BM),op(%); end if;
if info_iop>=3 then print(nops(O_BM)) end if; end do; end if;
else reduce_mipol_i:=proc(x,m,a): expand(rem(x,m,a)): if %<>0 then
return %/icotent(%): else 0: end if: end proc:
reduce_mipol:=proc(x,m,a): expand(rem(x,m,a)): return %: end proc:
if not(hastype(LPointsTp_BM,radical)) then
while L_BM<>[] do
sort(L_BM,(t1_BM,t2_BM)->testorder(t1_BM,t2_BM,T_BM)): t_BM:=op(1,%);

```

```

L_BM:=subsop(1=NULL,%):
[seq(eval(t_BM,[seq(x_[j_BM]=LPointsTp_BM[k_BM][j_BM],j_BM=1..Dim_BM)]
),k_BM=1..Npoints_BM): if M_BM=[] then %: else
alginreducev3(ListListToEqs(M_BM,%)):
map(reduce_mipol_i,subs(%,%-add(map2('*','aleph[i_BM],op(i_BM,
M_BM)),i_BM=1..nops(M_BM))),minpol_BM,a_BM) end if: if M_BM<>[] then
trel_BM:=map(reduce_mipol,subs(%,%-add(map2('*','aleph[i_BM],
S_BM[i_BM]),i_BM=1..nops(M_BM))),minpol_BM,a_BM): else trel_BM:=t_BM
end if: if %=[seq(0,j_BM=1..Npoints_BM)] then G_BM:=op(G_BM),trel_BM];
L_BM:=op(map(P_BM->if not(divide(P_BM,t_BM,'q')) then return P_BM end
if,L_BM)); else M_BM:=op(M_BM,%); S_BM:=op(S_BM),trel_BM];
O_BM:=op(O_BM),t_BM]; if G_BM<>[] then map(Pol_BM -> if
not(booladd(map2(divide,Pol_BM,Generators(PolynomialIdeal(map(Groebner
[leadterm],GroebnerBasis(<op(G_BM)>,Ord_BM),Ord_BM),select(type,[op(G_
BM)],equation)))))) then return Pol_BM end if,
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM))]: else
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM)): end if; if L_BM<>[] then
map(Pol_BM-> if not(booladd(map2(divide,Pol_BM,L_BM))) then return
Pol_BM end if, %): else %: end if; L_BM:=op(L_BM),op(%); end if;
if info_iop>=3 then print(nops(O_BM)) end if; end do; else while
L_BM<>[] do sort(L_BM,(t1_BM,t2_BM)->testorder(t1_BM,t2_BM,T_BM)):
t_BM:=op(1,%); L_BM:=subsop(1=NULL,%):
[seq(eval(t_BM,[seq(x_[j_BM]=LPointsTp_BM[k_BM][j_BM],j_BM=1..Dim_BM)]
),k_BM=1..Npoints_BM): if M_BM=[] then %: else
alginreducev3(ListListToEqs(M_BM,%)):
map(radnormal,map(reduce_mipol_i,subs(%,%-add(map2('*','aleph[i_BM],
op(i_BM,M_BM)),i_BM=1..nops(M_BM))),minpol_BM,a_BM),'rationalized')
end if: if M_BM<>[] then trel_BM:=map(radnormal,map(reduce_mipol,
subs(%,%-add(map2('*','aleph[i_BM],S_BM[i_BM]),
i_BM=1..nops(M_BM))),minpol_BM,a_BM),'rationalized'):
else trel_BM:=t_BM end if: if %=[seq(0,j_BM=1..Npoints_BM)] then
G_BM:=op(G_BM),trel_BM]; L_BM:=op(map(P_BM->if not(divide(P_BM,
t_BM,'q')) then return P_BM end if,L_BM)); else M_BM:=op(M_BM),
%]; S_BM:=op(S_BM),trel_BM]; O_BM:=op(O_BM),t_BM]; if G_BM<>[]
then map(Pol_BM -> if not(booladd(map2(divide,Pol_BM,Generators(
PolynomialIdeal(map(Groebner[leadterm],GroebnerBasis(<op(G_BM)>,
Ord_BM),Ord_BM),select(type,[op(G_BM)],equation)))))) then return
Pol_BM end if, [seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM))]: else
[seq(x_[i_BM]*t_BM,i_BM=1..Dim_BM)): end if; if L_BM<>[] then
map(Pol_BM-> if not(booladd(map2(divide,Pol_BM,L_BM))) then return
Pol_BM end if, %): else %: end if; L_BM:=op(L_BM),op(%); end if;
if info_iop>=3 then print(nops(O_BM)) end if; end do; end if; end if;
return [G_BM,O_BM]; end proc;

```

IdealOfPoints := **proc**(*LPoints_BM*, *Minpol_BM*)

local *G_BM*, *O_BM*, *S_BM*, *M_BM*, *L_BM*, *t1_BM*, *t2_BM*, *i_BM*, *j_BM*, *k_BM*, *t_BM*,
trel_BM, *LPointsTp_BM*, *Dim_BM*, *Npoints_BM*, *T_BM*, *a_BM*, *minpol_BM*, *Alg_BM*,

Ord_BM, *Indets_BM*, *reduce_mipol*, *reduce_mipol_i*;

unassign('x_');

G_BM := [];

O_BM := [];

S_BM := [];

L_BM := [1];

M_BM := [];

if 2 ≤ nargs **then** *minpol_BM* := *Minpol_BM* **end if**;

a_BM := indets(*minpol_BM*);

if type(*set*, *LPoints_BM*) **then** *LPointsTp_BM* := convert(*LPoints_BM*, list)

else

if type(*LPoints_BM*, *Matrix*) **then**

LPointsTp_BM := map(convert, convert(*LPoints_BM*, listlist), Vector)

```

    else LPointsTp_BM := LPoints_BM
  end if
end if;
Npoints_BM := nops(%);
Dim_BM := op(1, '%%'1);
Indets_BM := seq(x_i_BM, i_BM = 1..Dim_BM);
Alg_BM := Ore_algebra : -poly_algebra(%);
Ord_BM := tdeg('%%');
T_BM := Groebner : -termorder('%%', %, ['%%%'']);
if not (assigned(minpol_BM) and has(LPointsTp_BM, a_BM)) then
  if not hastype(LPointsTp_BM, radical) then
    print( not hastype(LPointsTp_BM, radical));
    while L_BM ≠ [] do
      sort(L_BM, (t1_BM, t2_BM) → Groebner : -testorder(t1_BM, t2_BM, T_BM));
      t_BM := op(1, %);
      L_BM := subsop(1 = NULL, '%%');
      [seq(
        eval(t_BM, [seq(x_j_BM = LPointsTp_BMk_BM j_BM, j_BM = 1..Dim_BM)]),
        k_BM = 1..Npoints_BM);
      if M_BM = [] then %
      else
        alginreducev3(ListListToEqs(M_BM, %));
        subs(%,
          '%%' - add(map2(*', %i_BM, op(i_BM, M_BM)), i_BM = 1..nops(M_BM)))
      end if;
      if M_BM ≠ [] then trel_BM := subs('%%',
        t_BM - add(map2(*', %i_BM, S_BMi_BM), i_BM = 1..nops(M_BM)))
      else trel_BM := t_BM
      end if;
      if '%%' = [seq(0, j_BM = 1..Npoints_BM)] then
        G_BM := [op(G_BM), trel_BM];
        L_BM := [op(map(proc(P_BM)
          option operator, arrow;
          if not divide(P_BM, t_BM, 'q') then return P_BM end if
          end proc, L_BM))]
      else
        M_BM := [op(M_BM), '%%'];
        S_BM := [op(S_BM), trel_BM];
        O_BM := [op(O_BM), t_BM];
        if G_BM ≠ [] then map(proc(Pol_BM)
          option operator, arrow;
          if not booladd(map2(divide, Pol_BM, PolynomialIdeals : -Generators(
            PolynomialIdeals : -PolynomialIdeal(map(Groebner Groebner : -leadterm,
            PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, >'(op(G_BM)), Ord_BM),
            Ord_BM), select(type, [op(G_BM)], equation)))))) then return Pol_BM
          end if

```

```

        end proc, [seq( $x_{i\_BM} * t\_BM$ ,  $i\_BM = 1..Dim\_BM$ ))]
    else [seq( $x_{i\_BM} * t\_BM$ ,  $i\_BM = 1..Dim\_BM$ )]
    end if;
    if  $L\_BM \neq []$  then map(proc( $Pol\_BM$ )
        option operator, arrow;
        if not booladd(map2(divide,  $Pol\_BM$ ,  $L\_BM$ )) then return  $Pol\_BM$  end if
        end proc, %)
    else %
    end if;
     $L\_BM := [op(L\_BM), op(%)]$ 
end if;
if  $3 \leq info\_iop$  then print(nops( $O\_BM$ )) end if
end do
else while  $L\_BM \neq []$  do
    sort( $L\_BM$ , ( $t1\_BM$ ,  $t2\_BM$ )  $\rightarrow$  Groebner : -testorder( $t1\_BM$ ,  $t2\_BM$ ,  $T\_BM$ ));
     $t\_BM := op(1, %)$ ;
     $L\_BM := subsop(1 = NULL, '%%')$ ;
    [seq(
        eval( $t\_BM$ , [seq( $x_{j\_BM} = LPointsTp\_BM_{k\_BM j\_BM}$ ,  $j\_BM = 1..Dim\_BM$ )]),
         $k\_BM = 1..Npoints\_BM$ )]);
    if  $M\_BM = []$  then %
    else
        alginreducev3(ListListToEqs( $M\_BM$ , %));
        map(radnormal, subs(%,
            '%%' - add(map2('*',  $\aleph_{i\_BM}$ , op( $i\_BM$ ,  $M\_BM$ ))),  $i\_BM = 1..nops(M\_BM)$ ),
            'rationalized')
        end if;
    if  $M\_BM \neq []$  then  $trel\_BM := map(radnormal, subs('%%',
         $t\_BM$  - add(map2('*',  $\aleph_{i\_BM}$ ,  $S\_BM_{i\_BM}$ ),  $i\_BM = 1..nops(M\_BM)$ ),
        'rationalized'))
    else  $trel\_BM := t\_BM$ 
    end if;
    if map(radnormal, '%%', 'rationalized') = [seq(0,  $j\_BM = 1..Npoints\_BM$ )] then
         $G\_BM := [op(G\_BM), trel\_BM]$ ;
         $L\_BM := [op(map(proc( $P\_BM$ )
            option operator, arrow;
            if not divide( $P\_BM$ ,  $t\_BM$ , 'q') then return  $P\_BM$  end if
            end proc,  $L\_BM$ ))]
        else
             $M\_BM := [op(M\_BM), '%%']$ ;
             $S\_BM := [op(S\_BM), trel\_BM]$ ;
             $O\_BM := [op(O\_BM), t\_BM]$ ;
            if  $G\_BM \neq []$  then map(proc( $Pol\_BM$ )
                option operator, arrow;
                if not booladd(map2(divide,  $Pol\_BM$ , PolynomialIdeals : -Generators($$ 
```

```

    PolynomialIdeals : - PolynomialIdeal(map(Groebner Groebner:-leadterm,
    PolynomialIdeals : - GroebnerBasis(PolynomialIdeals : - ' <, > ' (op(G_BM)), Ord_BM),
    Ord_BM), select(type, [op(G_BM)], equation)))) then return Pol_BM
    end if
    end proc, [seq(x_{i_BM} * t_BM, i_BM = 1..Dim_BM))]
else [seq(x_{i_BM} * t_BM, i_BM = 1..Dim_BM)]
end if;
if L_BM ≠ [] then map(proc(Pol_BM)
    option operator, arrow;
    if not booladd(map2(divide, Pol_BM, L_BM)) then return Pol_BM end if
    end proc, %)
else %
end if;
L_BM := [op(L_BM), op(%)]
end if;
if 3 ≤ info_iop then print(nops(O_BM)) end if
end do
end if
else
    reduce_mipol_i := proc(x, m, a)
        expand(rem(x, m, a)); if % ≠ 0 then return %/icontent(%) else 0 end if
    end proc;
    reduce_mipol := proc(x, m, a) expand(rem(x, m, a)); return % end proc;
    if not hastype(LPointsTp_BM, radical) then while L_BM ≠ [] do
        sort(L_BM, (t1_BM, t2_BM) → Groebner : -testorder(t1_BM, t2_BM, T_BM));
        t_BM := op(1, %);
        L_BM := subsop(1 = NULL, '%%');
        [seq(
            eval(t_BM, [seq(x_{j_BM} = LPointsTp_BM_{k_BM j_BM}, j_BM = 1..Dim_BM)]),
            k_BM = 1..Npoints_BM)];
        if M_BM = [] then %
        else
            alginreducev3(ListListToEqs(M_BM, %));
            map(reduce_mipol_i, subs(%,
                '%%' - add(map2(' * ', %_{i_BM}, op(i_BM, M_BM)), i_BM = 1..nops(M_BM))),
                minpol_BM, a_BM)
        end if;
        if M_BM ≠ [] then trel_BM := map(reduce_mipol, subs('%%',
            t_BM - add(map2(' * ', %_{i_BM}, S_BM_{i_BM}), i_BM = 1..nops(M_BM))),
            minpol_BM, a_BM)
        else trel_BM := t_BM
        end if;
        if '%%' = [seq(0, j_BM = 1..Npoints_BM)] then
            G_BM := [op(G_BM), trel_BM];
            L_BM := [op(map(proc(P_BM)
                option operator, arrow;

```



```

        if not divide(P_BM, t_BM, 'q') then return P_BM end if
    end proc, L_BM))]
else
    M_BM := [op(M_BM), '%%'];
    S_BM := [op(S_BM), trel_BM];
    O_BM := [op(O_BM), t_BM];
    if G_BM ≠ [] then map(proc(Pol_BM)
        option operator, arrow;
        if not booladd(map2(divide, Pol_BM, PolynomialIdeals : -Generators(
            PolynomialIdeals : -PolynomialIdeal(map(Groebner Groebner:-leadterm,
            PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, >'(op(G_BM)), Ord_BM),
            Ord_BM), select(type, [op(G_BM)], equation)))))) then return Pol_BM
        end if
    end proc, [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)])
    else [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)]
    end if;
    if L_BM ≠ [] then map(proc(Pol_BM)
        option operator, arrow;
        if not booladd(map2(divide, Pol_BM, L_BM)) then return Pol_BM end if
    end proc, %)
    else %
    end if;
    L_BM := [op(L_BM), op(%)]
    end if;
    if 3 ≤ info_iop then print(nops(O_BM)) end if
end do
else while L_BM ≠ [] do
    sort(L_BM, (t1_BM, t2_BM) → Groebner : -testorder(t1_BM, t2_BM, T_BM));
    t_BM := op(1, %);
    L_BM := subsop(1 = NULL, '%%');
    [seq(
        eval(t_BM, [seq(x_j_BM = LPointsTp_BM_k_BM_j_BM, j_BM = 1..Dim_BM)]),
        k_BM = 1..Npoints_BM)];
    if M_BM = [] then %
    else
        alginreducev3(ListListToEqs(M_BM, %));
        map(radnormal, map(reduce_mipol_i, subs(%,
            '%%' - add(map2(*, N_i_BM, op(i_BM, M_BM)), i_BM = 1..nops(M_BM))),
            minpol_BM, a_BM), 'rationalized')
    end if;
    if M_BM ≠ [] then trel_BM := map(radnormal, map(reduce_mipol, subs('%%',
        t_BM - add(map2(*, N_i_BM, S_BM_i_BM), i_BM = 1..nops(M_BM))),
        minpol_BM, a_BM), 'rationalized')
    else trel_BM := t_BM
    end if;
    if '%%' = [seq(0, j_BM = 1..Npoints_BM)] then

```

```

G_BM := [op(G_BM), trel_BM];
L_BM := [op(map(proc(P_BM)
    option operator, arrow;
    if not divide(P_BM, t_BM, 'q') then return P_BM end if
end proc, L_BM))]
else
M_BM := [op(M_BM), '%%'];
S_BM := [op(S_BM), trel_BM];
O_BM := [op(O_BM), t_BM];
if G_BM ≠ [] then map(proc(Pol_BM)
    option operator, arrow;
    if not booladd(map2(divide, Pol_BM, PolynomialIdeals : -Generators(
        PolynomialIdeals : -PolynomialIdeal(map(Groebner Groebner:-leadterm,
        PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, > '(op(G_BM)), Ord_BM),
        Ord_BM), select(type, [op(G_BM)], equation)))))) then return Pol_BM
    end if
end proc, [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)])
else [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)]
end if;
if L_BM ≠ [] then map(proc(Pol_BM)
    option operator, arrow;
    if not booladd(map2(divide, Pol_BM, L_BM)) then return Pol_BM end if
end proc, %)
else %
end if;
L_BM := [op(L_BM), op(%)]
end if;
if 3 ≤ info_iop then print(nops(O_BM)) end if
end do
end if
end if;
return [G_BM, O_BM]
end proc

```

```

> IdealOfPoints_OptLeading_CardTest:=proc (LPoints_BM, Minpol_BM) local
  G_BM, O_BM, S_BM, M_BM, L_BM, t1_BM, t2_BM, i_BM, j_BM, k_BM, t_BM,
  trel_BM, LPointsTp_BM, Dim_BM, Npoints_BM, T_BM, a_BM, minpol_BM,
  Alg_BM, Ord_BM, Indets_BM, reduce_mipol, reduce_mipol_i, st2, st;
  unassign('x_'); G_BM := []; O_BM := []; S_BM := []; L_BM := [1]; M_BM
  := []; if 2 <= nargs then minpol_BM := Minpol_BM: end if;
  print(nargs,minpol_BM); a_BM := indets(minpol_BM)[]; if
  type(set,LPoints_BM) then LPointsTp_BM := convert(LPoints_BM,list)
  else if type(LPoints_BM,Matrix) then LPointsTp_BM :=
  map(convert,convert(LPoints_BM,listlist),Vector) else LPointsTp_BM :=
  LPoints_BM end if end if; Npoints_BM := nops(%); Dim_BM :=
  op(1,%%'[1]); Indets_BM := seq(x_[i_BM],i_BM = 1 .. Dim_BM); Alg_BM
  := Ore_algebra:-poly_algebra(%); Ord_BM := tdeg('%%'); T_BM :=
  Groebner:-termorder('%%',%,[%%]); if not (assigned(minpol_BM)
  and has(LPointsTp_BM,a_BM)) then if not hastype(LPointsTp_BM,radical)
  then print(not hastype(LPointsTp_BM,radical)); while L_BM <> [] do
  sort(L_BM,proc (t1_BM, t2_BM) options operator, arrow;
  Groebner:-testorder(t1_BM,t2_BM,T_BM) end proc); t_BM := op(1,%);
  L_BM := subsop(1 = NULL,%%'); [seq(eval(t_BM,[seq(x_[j_BM] =
  LPointsTp_BM[k_BM][j_BM],j_BM = 1 .. Dim_BM])),k_BM = 1 ..
  Npoints_BM)]; if M_BM = [] then % else
  alginreducev3(ListListToEqs(M_BM,%));
  subs(%%,'-add(map2('*',aleph[i_BM],op(i_BM,M_BM)),i_BM = 1 ..
  nops(M_BM))) end if; if M_BM <> [] then trel_BM :=
  subs(%%',t_BM-add(map2('*',aleph[i_BM],S_BM[i_BM]),i_BM = 1 ..
  nops(M_BM))) else trel_BM := t_BM end if; if '%%' = [seq(0,j_BM = 1
  .. Npoints_BM)] then G_BM := [op(G_BM), trel_BM]; L_BM := [op(map(proc
  (P_BM) options operator, arrow; if not divide(P_BM,t_BM,'q') then
  return P_BM end if end proc,L_BM))] else M_BM := [op(M_BM), '%%'];
  S_BM := [op(S_BM), trel_BM]; O_BM := [op(O_BM), t_BM]; if G_BM <> []
  then map(proc (Pol_BM) options operator, arrow; if not
  booladd(map2(divide,Pol_BM,PolynomialIdeals:-Generators(PolynomialIdeals:-
  PolynomialIdeal(map(Groebner[Groebner:-leadterm],PolynomialIdeals:-
  GroebnerBasis(PolynomialIdeals:-'<,>'(op(G_BM)),Ord_BM,Ord_BM),selec
  t(type,[op(G_BM)],equation)))) then return Pol_BM end if end
  proc,[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM))] else
  [seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM)] end if; if L_BM <> [] then
  map(proc (Pol_BM) options operator, arrow; if not
  booladd(map2(divide,Pol_BM,L_BM)) then return Pol_BM end if end
  proc,%) else % end if; L_BM := [op(L_BM), op(%)] end if; if 3 <=
  info_iop then print(nops(O_BM)) end if end do else while (L_BM <> []
  and nops(O_BM)<>Npoints_BM) do print(''); print(''); print(); st2
  := time(); sort(L_BM,proc (t1_BM, t2_BM) options operator, arrow;
  Groebner:-testorder(t1_BM,t2_BM,T_BM) end proc); t_BM := op(1,%);
  L_BM := subsop(1 = NULL,%%'); print(cat("Assign=",time()-st2));
  map(radnormal,[seq(eval(t_BM,[seq(x_[j_BM] =
  LPointsTp_BM[k_BM][j_BM],j_BM = 1 .. Dim_BM])),k_BM = 1 ..
  Npoints_BM)],'rationalized'); print(cat("Assign=",time()-st2)); if
  M_BM = [] then % else alginreducev3(ListListToEqs(M_BM,%));
  map(radnormal,subs(%%',
  %%'-add(map2('*',aleph[i_BM],op(i_BM,M_BM)),i_BM = 1 ..
  nops(M_BM))),'rationalized') end if; print(cat("Assign=",time()-st2));
  if M_BM <> [] then if optlevel>=1 then
  trel_BM:=subs(%%',
  %%',t_BM-add(map2('*',aleph[i_BM],S_BM[i_BM]),i_BM=1..nops(M_BM)))
  :
  else
  trel_BM:=map(radnormal,subs(%%',
  %%',t_BM-add(map2('*',aleph[i_BM],S_BM[i_BM]),i_BM=1..nops(M_BM))),
  'rationalized')
  :
  end if; print(cat("Here=",time()-st2)) else trel_BM := t_BM end if;
  print(cat("Assign=",time()-st2)); if
  map(radnormal,%%', 'rationalized') = [seq(0,j_BM = 1 .. Npoints_BM)]
  then print(cat("Assign=",time()-st2)); G_BM := [op(G_BM), trel_BM];
  L_BM := [op(map(proc (P_BM) options operator, arrow; if not
  divide(P_BM,t_BM,'q') then return P_BM end if end proc,L_BM))] else
  print(cat("Assign=",time()-st2)); M_BM := [op(M_BM), '%%']; S_BM :=
  [op(S_BM), trel_BM]; O_BM := [op(O_BM), t_BM];
  print(cat("Assign=",time()-st2)); st := time(); if G_BM <> [] then

```

```

map(proc (Pol_BM) options operator, arrow; if not
booladd(map2(divide,Pol_BM,PolynomialIdeals:-Generators(PolynomialIdea
ls:-PolynomialIdeal(map(Groebner[Groebner:-leadterm],PolynomialIdeals:
-GroebnerBasis(PolynomialIdeals:-'<,>'(op(G_BM)),Ord_BM),Ord_BM),selec
t(type,[op(G_BM)],equation)))) then return Pol_BM end if end
proc,[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM)]) else
[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM)] end if; if L_BM <> [] then
map(proc (Pol_BM) options operator, arrow; if not
booladd(map2(divide,Pol_BM,L_BM)) then return Pol_BM end if end
proc,%) else % end if; L_BM := [op(L_BM), op(%)]; print(cat("2eme
cas",time()-st)) end if; if 3 <= info_iop then
print(cat("|O_BM|=",nops(O_BM))) end if;
print(cat("Une boucle=",time()-st2)) end do end if else reduce_mipol_i
:= proc (x, m, a) expand(rem(x,m,a)); if % <> 0 then return
%/icontent(%) else 0 end if end proc; reduce_mipol := proc (x, m, a)
expand(rem(x,m,a)); return % end proc; if not
hastype(LPointsTp_BM,radical) then while L_BM <> [] do sort(L_BM,proc
(t1_BM, t2_BM) options operator, arrow;
Groebner:-testorder(t1_BM,t2_BM,T_BM) end proc); t_BM := op(1,%);
L_BM := subsop(1 = NULL, '%'); [seq(eval(t_BM,[seq(x_[j_BM] =
LPointsTp_BM[k_BM][j_BM],j_BM = 1 .. Dim_BM])),k_BM = 1 ..
Npoints_BM)]; if M_BM = [] then % else
alginreducev3(ListListToEqs(M_BM,%),minpol_BM);
map(reduce_mipol_i,subs(%, '%
%-add(map2('* ',aleph[i_BM],op(i_BM,M_BM)),i_BM = 1 ..
nops(M_BM))),minpol_BM,a_BM) end if; if M_BM <> [] then trel_BM :=
map(reduce_mipol_i,subs(%, '%
%',t_BM-add(map2('* ',aleph[i_BM],S_BM[i_BM]),i_BM = 1 ..
nops(M_BM))),minpol_BM,a_BM) else trel_BM := t_BM end if; if '% ' =
[seq(0,j_BM = 1 .. Npoints_BM)] then G_BM := [op(G_BM), trel_BM]; L_BM
:= [op(map(proc (P_BM) options operator, arrow; if not
divide(P_BM,t_BM,'q') then return P_BM end if end proc,L_BM))] else
M_BM := [op(M_BM), '%']; S_BM := [op(S_BM), trel_BM]; O_BM :=
[op(O_BM), t_BM]; if G_BM <> [] then map(proc (Pol_BM) options
operator, arrow; if not
booladd(map2(divide,Pol_BM,PolynomialIdeals:-Generators(PolynomialIdea
ls:-PolynomialIdeal(map(Groebner[Groebner:-leadterm],PolynomialIdeals:
-GroebnerBasis(PolynomialIdeals:-'<,>'(op(G_BM)),Ord_BM),Ord_BM),selec
t(type,[op(G_BM)],equation)))) then return Pol_BM end if end
proc,[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM)]) else
[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM)] end if; if L_BM <> [] then
map(proc (Pol_BM) options operator, arrow; if not
booladd(map2(divide,Pol_BM,L_BM)) then return Pol_BM end if end
proc,%) else % end if; L_BM := [op(L_BM), op(%)] end if; if 3 <=
info_iop then print(nops(O_BM)) end if end do else while L_BM <> []
do sort(L_BM,proc (t1_BM, t2_BM) options operator, arrow;
Groebner:-testorder(t1_BM,t2_BM,T_BM) end proc); t_BM := op(1,%);
L_BM := subsop(1 = NULL, '%'); [seq(eval(t_BM,[seq(x_[j_BM] =
LPointsTp_BM[k_BM][j_BM],j_BM = 1 .. Dim_BM])),k_BM = 1 ..
Npoints_BM)]; if M_BM = [] then % else
alginreducev3(ListListToEqs(M_BM,%),minpol_BM);
map(radnormal,map(reduce_mipol_i,subs(%, '%
%-add(map2('* ',aleph[i_BM],op(i_BM,M_BM)),i_BM = 1 ..
nops(M_BM))),minpol_BM,a_BM),'rationalized') end if; if M_BM <> []
then trel_BM :=
map(radnormal,map(reduce_mipol_i,subs(%, '%
%',t_BM-add(map2('* ',aleph[i_BM],S_BM[i_BM]),i_BM = 1 ..
nops(M_BM))),minpol_BM,a_BM),'rationalized') else trel_BM := t_BM end
if; if '% ' = [seq(0,j_BM = 1 .. Npoints_BM)] then G_BM :=
[op(G_BM), trel_BM]; L_BM := [op(map(proc (P_BM) options operator,
arrow; if not divide(P_BM,t_BM,'q') then return P_BM end if end
proc,L_BM))] else M_BM := [op(M_BM), '%']; S_BM := [op(S_BM),
trel_BM]; O_BM := [op(O_BM), t_BM]; if G_BM <> [] then map(proc
(Pol_BM) options operator, arrow; if not
booladd(map2(divide,Pol_BM,PolynomialIdeals:-Generators(PolynomialIdea
ls:-PolynomialIdeal(map(Groebner[Groebner:-leadterm],PolynomialIdeals:
-GroebnerBasis(PolynomialIdeals:-'<,>'(op(G_BM)),Ord_BM),Ord_BM),selec
t(type,[op(G_BM)],equation)))) then return Pol_BM end if end

```

```

proc,[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM))] else
[seq(x_[i_BM]*t_BM,i_BM = 1 .. Dim_BM)] end if; if L_BM <> [] then
map(proc (Pol_BM) options operator, arrow; if not
booladd(map2(divide,Pol_BM,L_BM)) then return Pol_BM end if end
proc,%) else % end if; L_BM := [op(L_BM), op(%)] end if; if 3 <=
info_iop then print(nops(O_BM)) end if end do end if end if; return
[G_BM, O_BM] end proc;

```

```

IdealOfPoints_OptLeading_CardTest := proc(LPoints_BM, Minpol_BM)
local G_BM, O_BM, S_BM, M_BM, L_BM, t1_BM, t2_BM, i_BM, j_BM, k_BM, t_BM,
trel_BM, LPointsTp_BM, Dim_BM, Npoints_BM, T_BM, a_BM, minpol_BM, Alg_BM,
Ord_BM, Indets_BM, reduce_mipol, reduce_mipol_i, st2, st;
  unassign('x_');
  G_BM := [];
  O_BM := [];
  S_BM := [];
  L_BM := [1];
  M_BM := [];
  if 2 ≤ nargs then minpol_BM := Minpol_BM end if;
  print(nargs, minpol_BM);
  a_BM := indets(minpol_BM);
  if type(set, LPoints_BM) then LPointsTp_BM := convert(LPoints_BM, list)
else
    if type(LPoints_BM, Matrix) then
      LPointsTp_BM := map(convert, convert(LPoints_BM, listlist), Vector)
    else LPointsTp_BM := LPoints_BM
    end if
  end if;
  Npoints_BM := nops(%);
  Dim_BM := op(1, '%%'1);
  Indets_BM := seq(x_i_BM, i_BM = 1..Dim_BM);
  Alg_BM := Ore_algebra : -poly_algebra(%);
  Ord_BM := tdeg('%%');
  T_BM := Groebner : -termorder('%%', %, ['%%%'']);
  if not (assigned(minpol_BM) and has(LPointsTp_BM, a_BM)) then
    if not hastype(LPointsTp_BM, radical) then
      print( not hastype(LPointsTp_BM, radical));
      while L_BM ≠ [] do
        sort(L_BM, (t1_BM, t2_BM) → Groebner : -testorder(t1_BM, t2_BM, T_BM));
        t_BM := op(1, %);
        L_BM := subsop(1 = NULL, '%%');
        [seq(
          eval(t_BM, [seq(x_j_BM = LPointsTp_BMk_BM j_BM, j_BM = 1..Dim_BM)]),
          k_BM = 1..Npoints_BM)];
        if M_BM = [] then %
        else
          alginreducev3(ListListToEqs(M_BM, %));
          subs(%,
            '%%' - add(map2(*, %i_BM, op(i_BM, M_BM)), i_BM = 1..nops(M_BM)));

```

```

end if
if  $M\_BM \neq []$  then  $trel\_BM := \text{subs}(' \% \%',$ 
     $t\_BM - \text{add}(\text{map2}('*', \mathbb{N}_{i\_BM}, S\_BM_{i\_BM}), i\_BM = 1..\text{nops}(M\_BM))$ 
else  $trel\_BM := t\_BM$ 
end if;
if  $' \% \%' = [\text{seq}(0, j\_BM = 1..Npoints\_BM)]$  then
     $G\_BM := [\text{op}(G\_BM), trel\_BM];$ 
     $L\_BM := [\text{op}(\text{map}(\text{proc}(P\_BM)$ 
        option operator, arrow;
        if not divide( $P\_BM, t\_BM, 'q'$ ) then return  $P\_BM$  end if
        end proc,  $L\_BM$ ))]
else
     $M\_BM := [\text{op}(M\_BM), ' \% \%'];$ 
     $S\_BM := [\text{op}(S\_BM), trel\_BM];$ 
     $O\_BM := [\text{op}(O\_BM), t\_BM];$ 
    if  $G\_BM \neq []$  then map(proc( $Pol\_BM$ )
        option operator, arrow;
        if not booladd(map2(divide,  $Pol\_BM, PolynomialIdeals : -Generators($ 
             $PolynomialIdeals : -PolynomialIdeal(\text{map}(Groebner_{Groebner:-leadterm},$ 
             $PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, > '(\text{op}(G\_BM)), Ord\_BM),$ 
             $Ord\_BM$ ), select(type,  $[\text{op}(G\_BM)], \text{equation})))$  then return  $Pol\_BM$ 
        end if
        end proc,  $[\text{seq}(x_{i\_BM} * t\_BM, i\_BM = 1..Dim\_BM)])$ 
    else  $[\text{seq}(x_{i\_BM} * t\_BM, i\_BM = 1..Dim\_BM)]$ 
    end if;
    if  $L\_BM \neq []$  then map(proc( $Pol\_BM$ )
        option operator, arrow;
        if not booladd(map2(divide,  $Pol\_BM, L\_BM$ )) then return  $Pol\_BM$  end if
        end proc,  $\%$ )
    else  $\%$ 
    end if;
     $L\_BM := [\text{op}(L\_BM), \text{op}(\%)]$ 
end if;
if  $3 \leq info\_iop$  then print( $\text{nops}(O\_BM)$ ) end if
end do
else while  $L\_BM \neq []$  and  $\text{nops}(O\_BM) \neq Npoints\_BM$  do
    print('“');
    print('“');
    print();
     $st2 := \text{time}();$ 
     $\text{sort}(L\_BM, (t1\_BM, t2\_BM) \rightarrow Groebner : -\text{testorder}(t1\_BM, t2\_BM, T\_BM));$ 
     $t\_BM := \text{op}(1, \%);$ 
     $L\_BM := \text{subsop}(1 = NULL, ' \% \%');$ 
    print(cat("Assign=",  $\text{time}() - st2$ ));
    map(radnormal,  $[\text{seq}$ 
        eval( $t\_BM, [\text{seq}(x_{j\_BM} = LPointsTp\_BM_{k\_BM j\_BM}, j\_BM = 1..Dim\_BM)]$ ),

```

```

    k_BM = 1..Npoints_BM], 'rationalized');
print(cat("Assign=", time() - st2));
if M_BM = [] then %
else
    alginreducev3(ListListToEqs(M_BM, %));
    map(radnormal, subs(% ,
        '%%' - add(map2(' * ', N_i_BM, op(i_BM, M_BM)), i_BM = 1..nops(M_BM))),
        'rationalized')
end if;
print(cat("Assign=", time() - st2));
if M_BM ≠ [] then
    if 1 ≤ optlevel then trel_BM := subs('%%',
        t_BM - add(map2(' * ', N_i_BM, S_BM_i_BM), i_BM = 1..nops(M_BM)))
    else trel_BM := map(radnormal, subs('%%',
        t_BM - add(map2(' * ', N_i_BM, S_BM_i_BM), i_BM = 1..nops(M_BM))),
        'rationalized')
    end if;
    print(cat("Here=", time() - st2))
else trel_BM := t_BM
end if;
print(cat("Assign=", time() - st2));
if map(radnormal, '%%', 'rationalized') = [seq(0, j_BM = 1..Npoints_BM)] then
    print(cat("Assign=", time() - st2));
    G_BM := [op(G_BM), trel_BM];
    L_BM := [op(map(proc(P_BM)
        option operator, arrow;
        if not divide(P_BM, t_BM, 'q') then return P_BM end if
        end proc, L_BM))]
else
    print(cat("Assign=", time() - st2));
    M_BM := [op(M_BM), '%%'];
    S_BM := [op(S_BM), trel_BM];
    O_BM := [op(O_BM), t_BM];
    print(cat("Assign=", time() - st2));
    st := time();
    if G_BM ≠ [] then map(proc(Pol_BM)
        option operator, arrow;
        if not booladd(map2(divide, Pol_BM, PolynomialIdeals : -Generators(
            PolynomialIdeals : -PolynomialIdeal(map(Groebner Groebner:-leadterm,
            PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, >'(op(G_BM)), Ord_BM),
            Ord_BM), select(type, [op(G_BM)], equation)))))) then return Pol_BM
        end if
        end proc, [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)])
    else [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)]
    end if;
    if L_BM ≠ [] then map(proc(Pol_BM)

```

```

    option operator, arrow;
    if not booladd(map2(divide, Pol_BM, L_BM)) then return Pol_BM end if
    end proc, %)
else %
end if;
L_BM := [op(L_BM), op(%)];
print(cat("2eme cas", time() - st))
end if;
if 3 ≤ info_iop then print(cat("|O_BM|=", nops(O_BM))) end if;
print(cat("Une boucle=", time() - st2))
end do
end if
else
    reduce_mipol_i := proc(x, m, a)
        expand(rem(x, m, a)); if % ≠ 0 then return %/icontent(%) else 0 end if
    end proc;
    reduce_mipol := proc(x, m, a) expand(rem(x, m, a)); return % end proc;
    if not hastype(LPointsTp_BM, radical) then while L_BM ≠ [] do
        sort(L_BM, (t1_BM, t2_BM) → Groebner : -testorder(t1_BM, t2_BM, T_BM));
        t_BM := op(1, %);
        L_BM := subsop(1 = NULL, '%%');
        [seq(
            eval(t_BM, [seq(x_j_BM = LPointsTp_BM_k_BM_j_BM, j_BM = 1..Dim_BM)]),
            k_BM = 1..Npoints_BM)];
        if M_BM = [] then %
        else
            alginreducev3(ListListToEqs(M_BM, %), minpol_BM);
            map(reduce_mipol_i, subs(%,
                '%%' - add(map2(' * ', N_i_BM, op(i_BM, M_BM)), i_BM = 1..nops(M_BM))),
                minpol_BM, a_BM)
        end if;
        if M_BM ≠ [] then trel_BM := map(reduce_mipol, subs('%%',
            t_BM - add(map2(' * ', N_i_BM, S_BM_i_BM), i_BM = 1..nops(M_BM))),
            minpol_BM, a_BM)
        else trel_BM := t_BM
        end if;
        if '%%' = [seq(0, j_BM = 1..Npoints_BM)] then
            G_BM := [op(G_BM), trel_BM];
            L_BM := [op(map(proc(P_BM)
                option operator, arrow;
                if not divide(P_BM, t_BM, 'q') then return P_BM end if
                end proc, L_BM))]
        else
            M_BM := [op(M_BM), '%%'];
            S_BM := [op(S_BM), trel_BM];
            O_BM := [op(O_BM), t_BM];

```



```

if  $G\_BM \neq []$  then map(proc( $Pol\_BM$ )
  option operator, arrow;
  if not booladd(map2(divide,  $Pol\_BM$ , PolynomialIdeals : - Generators(
    PolynomialIdeals : - PolynomialIdeal(map(Groebner Groebner:-leadterm,
    PolynomialIdeals : - GroebnerBasis(PolynomialIdeals : - '<, >'(op( $G\_BM$ )),  $Ord\_BM$ ),
     $Ord\_BM$ ), select(type, [op( $G\_BM$ )], equation))))))then return  $Pol\_BM$ 
  end if
  end proc, [seq( $x_{-i\_BM} * t\_BM$ ,  $i\_BM = 1..Dim\_BM$ )])
else [seq( $x_{-i\_BM} * t\_BM$ ,  $i\_BM = 1..Dim\_BM$ )]
end if;
if  $L\_BM \neq []$  then map(proc( $Pol\_BM$ )
  option operator, arrow;
  if not booladd(map2(divide,  $Pol\_BM$ ,  $L\_BM$ )) then return  $Pol\_BM$  end if
  end proc, %)
else %
end if;
 $L\_BM := [\mathbf{op}(L\_BM), \mathbf{op}(\%)]$ 
end if;
if  $3 \leq info\_iop$  then print(nops( $O\_BM$ )) end if
end do
else while  $L\_BM \neq []$  do
  sort( $L\_BM$ , ( $t1\_BM$ ,  $t2\_BM$ )  $\rightarrow$  Groebner : -testorder( $t1\_BM$ ,  $t2\_BM$ ,  $T\_BM$ ));
   $t\_BM := \mathbf{op}(1, \%)$ ;
   $L\_BM := \mathbf{subsop}(1 = NULL, '\% \%' )$ ;
  [seq(
    eval( $t\_BM$ , [seq( $x_{-j\_BM} = LPointsTp\_BM_{k\_BM j\_BM}$ ,  $j\_BM = 1..Dim\_BM$ )]),
     $k\_BM = 1..Npoints\_BM$ )]);
  if  $M\_BM = []$  then %
  else
    alginreducev3(ListListToEqs( $M\_BM$ , %), minpol\_BM);
    map(radnormal, map(reduce\_mipol,  $i$ , subs(%,
      '\% \%' - add(map2(' * ',  $\aleph_{i\_BM}$ , op( $i\_BM$ ,  $M\_BM$ ))),  $i\_BM = 1..nops(M\_BM)$ )),
      minpol\_BM,  $a\_BM$ ), 'rationalized')
  end if;
  if  $M\_BM \neq []$  then  $trel\_BM := \mathbf{map}(\mathbf{radnormal}, \mathbf{map}(\mathbf{reduce\_mipol}, \mathbf{subs}(\% \%,$ 
     $t\_BM - \mathbf{add}(\mathbf{map2}(' * ', \aleph_{i\_BM}, S\_BM_{i\_BM}), i\_BM = 1..nops(M\_BM))),$ 
     $minpol\_BM, a\_BM$ ), 'rationalized')
  else  $trel\_BM := t\_BM$ 
  end if;
  if '\% \%' = [seq(0,  $j\_BM = 1..Npoints\_BM$ )] then
     $G\_BM := [\mathbf{op}(G\_BM), trel\_BM]$ ;
     $L\_BM := [\mathbf{op}(\mathbf{map}(\mathbf{proc}(P\_BM)$ 
      option operator, arrow;
      if not divide( $P\_BM$ ,  $t\_BM$ , 'q') then return  $P\_BM$  end if
      end proc,  $L\_BM$ ))]
  else

```

```

M_BM := [op(M_BM), '%%'];
S_BM := [op(S_BM), trel_BM];
O_BM := [op(O_BM), t_BM];
if G_BM ≠ [] then map(proc(Pol_BM)
    option operator, arrow;
    if not booladd(map2(divide, Pol_BM, PolynomialIdeals : -Generators(
        PolynomialIdeals : -PolynomialIdeal(map(Groebner Groebner:-leadterm,
        PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, > '
        (op(G_BM)), Ord_BM),
        Ord_BM), select(type, [op(G_BM)], equation)))))) then return Pol_BM
    end if
    end proc, [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)])
else [seq(x_i_BM * t_BM, i_BM = 1..Dim_BM)]
end if;
if L_BM ≠ [] then map(proc(Pol_BM)
    option operator, arrow;
    if not booladd(map2(divide, Pol_BM, L_BM)) then return Pol_BM end if
    end proc, %)
else %
end if;
L_BM := [op(L_BM), op(%)]
end if;
if 3 ≤ info_iop then print(nops(O_BM)) end if
end do
end if
end if;
return [G_BM, O_BM]
end proc

> IdealOfPointsApprox:=proc(LPoints_BM, Approx_BM) local
G_BM, O_BM, S_BM, M_BM, t1_BM, t2_BM, i_BM, j_BM, k_BM, t_BM, trel_BM, LPoin
tsTp_BM, Dim_BM, Npoints_BM, T_BM, a_BM, minpol_BM; unassign('x_');
G_BM:=[]; O_BM:=[]; S_BM:=[]; L_BM:=[]; M_BM:=[]; if type(set, LPoints_BM)
then LPointsTp_BM:=convert(LPoints_BM, list): else LPointsTp_BM:=LPoints_BM:
end if: Npoints_BM:=nops(%): Dim_BM:=op(1, %)[1]:
T_BM:=termorder(poly_algebra(seq('x_'[i_BM], i_BM=1..Dim_BM)), tdeg(seq(
'x_'[i_BM], i_BM=1..Dim_BM)), [seq('x_'[i_BM], i_BM=1..Dim_BM)]): while
L_BM<>[] do sort(L_BM, (t1_BM, t2_BM)->testorder(t1_BM, t2_BM, T_BM)):
t_BM:=op(1, %): L_BM:=subsop(1=NULL, %):
[seq(eval(t_BM, [seq('x_'[j_BM]=LPointsTp_BM[k_BM][j_BM], j_BM=1..Dim_BM
)]), k_BM=1..Npoints_BM): if M_BM=[] then %: else
alginreducev3(ListListToEqs(M_BM, %)): subs(%, %)-add(map2('*',
aleph[i_BM], op(i_BM, M_BM)), i_BM=1..nops(M_BM)): end if: if M_BM<>[] then
trel_BM:=subs(%, t_BM-add(map2('*', aleph[i_BM], S_BM[i_BM]), i_BM=1..nops(M_BM))):
else trel_BM:=t_BM end if: if map(Approx -> if abs(Approx)<Approx_BM
then 0 else Approx end if, %)=[seq(0, j_BM=1..Npoints_BM)] then
G_BM:= [op(G_BM), trel_BM]; L_BM:= [op(map(P_BM->ifnot(divide(P_BM, t_BM, 'q'))
then return P_BM end if, L_BM)]]; else M_BM:= [op(M_BM), %];
S_BM:= [op(S_BM), trel_BM]; O_BM:= [op(O_BM), t_BM]; if G_BM<>[] then
map(Pol_BM -> if not(booladd(map2(divide, Pol_BM, Generators(PolynomialIdeal(
map(Groebner[leadterm], GroebnerBasis(<op(G_BM)>, T_BM), T_BM), select(
type, [op(G_BM)], equation)))))) then return Pol_BM end if,
[seq('x_'[i_BM]*t_BM, i_BM=1..Dim_BM)]): else
[seq('x_'[i_BM]*t_BM, i_BM=1..Dim_BM)]: end if; if L_BM<>[] then

```

```

map(Pol_BM-> if not(booladd(map2(divide,Pol_BM,L_BM))) then return
Pol_BM end if, %): else %: end if; L_BM:=[op(L_BM),op(%)]; if
info_iop>=3 then print(nops(O_BM)); end if; end if; end do; return
[G_BM,O_BM]; end proc;

IdealOfPointsApprox := proc(LPoints_BM, Approx_BM)
local G_BM, O_BM, S_BM, M_BM, L_BM, t1_BM, t2_BM, i_BM, j_BM, k_BM, t_BM,
trel_BM, LPointsTp_BM, Dim_BM, Npoints_BM, T_BM, a_BM, minpol_BM;
  unassign('x_');
  G_BM := [];
  O_BM := [];
  S_BM := [];
  L_BM := [1];
  M_BM := [];
  if type(set, LPoints_BM) then LPointsTp_BM := convert(LPoints_BM, list)
  else LPointsTp_BM := LPoints_BM
  end if;
  Npoints_BM := nops(%);
  Dim_BM := op(1, '%%'1);
  T_BM := Groebner : -termorder(
    Ore_algebra : -poly_algebra(seq('x_'i_BM, i_BM = 1..Dim_BM)),
    tdeg(seq('x_'i_BM, i_BM = 1..Dim_BM)), [seq('x_'i_BM, i_BM = 1..Dim_BM)])
  ;
  while L_BM ≠ [] do
    sort(L_BM, (t1_BM, t2_BM) → Groebner : -testorder(t1_BM, t2_BM, T_BM));
    t_BM := op(1, %);
    L_BM := subsop(1 = NULL, '%%');
    [seq(
      eval(t_BM, [seq('x_'j_BM = LPointsTp_BM_k_BMj_BM, j_BM = 1..Dim_BM)]),
      k_BM = 1..Npoints_BM)];
    if M_BM = [] then %
    else
      alglinreducev3(ListListToEqs(M_BM, %));
      subs(% ,
        '%%' - add(map2('* ', N_i_BM, op(i_BM, M_BM)), i_BM = 1..nops(M_BM)))
    end if;
    if M_BM ≠ [] then trel_BM := subs('%%',
      t_BM - add(map2('* ', N_i_BM, S_BM_i_BM), i_BM = 1..nops(M_BM)))
    else trel_BM := t_BM
    end if;
    if map(proc(Approx)
      option operator, arrow;
      if abs(Approx) < Approx_BM then 0 else Approx end if
    end proc, '%%') = [seq(0, j_BM = 1..Npoints_BM)] then
      G_BM := [op(G_BM), trel_BM];
      L_BM := [op(map(proc(P_BM)
        option operator, arrow;

```

```

        if not divide(P_BM, t_BM, 'q') then return P_BM end if
    end proc, L_BM))]
else
    M_BM := [op(M_BM), '%%'];
    S_BM := [op(S_BM), trel_BM];
    O_BM := [op(O_BM), t_BM];
    if G_BM ≠ [] then map(proc(Pol_BM)
        option operator, arrow;
        if not booladd(map2(divide, Pol_BM, PolynomialIdeals : -Generators(
            PolynomialIdeals : -PolynomialIdeal(map(Groebner Groebner:-leadterm,
            PolynomialIdeals : -GroebnerBasis(PolynomialIdeals : -' <, > '(op(G_BM)), T_BM),
            T_BM), select(type, [op(G_BM)], equation)))))) then return Pol_BM
        end if
    end proc, [seq('x_'i_BM * t_BM, i_BM = 1..Dim_BM)])
    else [seq('x_'i_BM * t_BM, i_BM = 1..Dim_BM)]
    end if;
    if L_BM ≠ [] then map(proc(Pol_BM)
        option operator, arrow;
        if not booladd(map2(divide, Pol_BM, L_BM)) then return Pol_BM end if
    end proc, %)
    else %
    end if;
    L_BM := [op(L_BM), op(%)];
    if 3 ≤ info_iop then print(nops(O_BM)) end if
end if
end do;
return [G_BM, O_BM]
end proc

```

10.7. Groupes

10.7.1. Outils

```

> Cartesian_product:=proc(ACartesian_product,BCartesian_product); return
[seq(seq([ICartesian_product, JCartesian_product], ICartesian_product =
ACartesian_product), JCartesian_product= BCartesian_product)]; end proc;

Cartesian_product := proc(ACartesian_product, BCartesian_product)
    return [seq(seq([ICartesian_product, JCartesian_product],
        ICartesian_product = ACartesian_product),
        JCartesian_product = BCartesian_product)]
end proc

> Matrixmember:=proc(MMatrixmember,SetMatrixmember) local
TempMatrixmember,XMatrixmember; TempMatrixmember:=false; for
XMatrixmember in SetMatrixmember while TempMatrixmember=false do if
Equal(XMatrixmember,MMatrixmember) then TempMatrixmember:=true end if;
end do; return TempMatrixmember; end proc;

```

```

Matrixmember := proc(MMatrixmember, SetMatrixmember)
local TempMatrixmember, XMatrixmember;
  TempMatrixmember := false;
  for XMatrixmember in SetMatrixmember while TempMatrixmember = false do
    if LinearAlgebra : -Equal(XMatrixmember, MMatrixmember) then
      TempMatrixmember := true
    end if
  end do;
  return TempMatrixmember
end proc

```

```

> Matrixunion:=proc(MMatrixmember,SetMatrixmember) local
  TempMatrixmember,XMatrixmember; TempMatrixmember:=false; for
  XMatrixmember in SetMatrixmember while TempMatrixmember=false do if
  Equal(XMatrixmember,MMatrixmember) then TempMatrixmember:=true end if;
end do; if TempMatrixmember=true then return SetMatrixmember else
return SetMatrixmember union {MMatrixmember} end if; end proc;

```

```

Matrixunion := proc(MMatrixmember, SetMatrixmember)
local TempMatrixmember, XMatrixmember;
  TempMatrixmember := false;
  for XMatrixmember in SetMatrixmember while TempMatrixmember = false do
    if LinearAlgebra : -Equal(XMatrixmember, MMatrixmember) then
      TempMatrixmember := true
    end if
  end do;
  if TempMatrixmember = true then return SetMatrixmember
  else return SetMatrixmember union {MMatrixmember}
  end if
end proc

```

```

> Matrixset:=proc(ListMatrixset) local TempMatrixset,MMatrixset;
  TempMatrixset:={}; for MMatrixset in ListMatrixset do if not
  Matrixmember(MMatrixset,TempMatrixset) then
  TempMatrixset:=TempMatrixset union {MMatrixset} end if; end do;
return TempMatrixset; end proc;

```

```

Matrixset := proc(ListMatrixset)
local TempMatrixset, MMatrixset;
  TempMatrixset := {};
  for MMatrixset in ListMatrixset do
    if not Matrixmember(MMatrixset, TempMatrixset) then
      TempMatrixset := TempMatrixset union {MMatrixset}
    end if
  end do;
  return TempMatrixset
end proc

```

10.7.2. Matrices remarquables d'ordre fini

```

> Transp:=proc(NTransp) local MTransp;
  MTransp:=IdentityMatrix(NTransp,compact=false); MTransp[1,1]:=0;
  MTransp[2,2]:=0; MTransp[1,2]:=1; MTransp[2,1]:=1; return MTransp; end
proc;

```

```
Transp := proc(NTransp)
local MTransp;
  MTransp := LinearAlgebra : -IdentityMatrix(NTransp, compact = false);
  MTransp1,1 := 0;
  MTransp2,2 := 0;
  MTransp1,2 := 1;
  MTransp2,1 := 1;
  return MTransp
end proc

> Transposition:=proc(ITransposition,JTransposition,NTransposition)
local MTransposition;
MTransposition:=IdentityMatrix(NTransposition,compact=false);
MTransposition[ITransposition,ITransposition]:=0;
MTransposition[JTransposition,JTransposition]:=0;
MTransposition[ITransposition,JTransposition]:=1;
MTransposition[JTransposition,ITransposition]:=1; return
MTransposition; end proc;

Transposition := proc(ITransposition, JTransposition, NTransposition)
local MTransposition;
  MTransposition := LinearAlgebra : -IdentityMatrix(NTransposition, compact = false);
  MTranspositionITransposition, ITransposition := 0;
  MTranspositionJTransposition, JTransposition := 0;
  MTranspositionITransposition, JTransposition := 1;
  MTranspositionJTransposition, ITransposition := 1;
  return MTransposition
end proc

> Transposition_SemiNeg:=proc(ITransposition_SemiNeg,JTransposition_SemiNeg,NTransposition_SemiNeg) local MTransposition_SemiNeg;
MTransposition_SemiNeg:=IdentityMatrix(NTransposition_SemiNeg,compact=false);
MTransposition_SemiNeg[ITransposition_SemiNeg,ITransposition_SemiNeg]:=0;
MTransposition_SemiNeg[JTransposition_SemiNeg,JTransposition_SemiNeg]:=0;
MTransposition_SemiNeg[ITransposition_SemiNeg,JTransposition_SemiNeg]:=1;
MTransposition_SemiNeg[JTransposition_SemiNeg,ITransposition_SemiNeg]:=-1; return MTransposition_SemiNeg; end proc;

Transposition_SemiNeg := proc(
ITransposition_SemiNeg, JTransposition_SemiNeg, NTransposition_SemiNeg)
local MTransposition_SemiNeg;
  MTransposition_SemiNeg :=
    LinearAlgebra : -IdentityMatrix(NTransposition_SemiNeg, compact = false);
  MTransposition_SemiNegITransposition_SemiNeg, ITransposition_SemiNeg := 0;
  MTransposition_SemiNegJTransposition_SemiNeg, JTransposition_SemiNeg := 0;
  MTransposition_SemiNegITransposition_SemiNeg, JTransposition_SemiNeg := 1;
  MTransposition_SemiNegJTransposition_SemiNeg, ITransposition_SemiNeg := -1;
  return MTransposition_SemiNeg
end proc
```

```

> Transposition_Neg := proc (ITransposition_Neg, JTransposition_Neg,
    NTransposition_Neg) local MTransposition_Neg; MTransposition_Neg :=
    LinearAlgebra:-IdentityMatrix(NTransposition_Neg, compact = false);
    MTransposition_Neg[ITransposition_Neg, ITransposition_Neg] := 0;
    MTransposition_Neg[JTransposition_Neg, JTransposition_Neg] := 0;
    MTransposition_Neg[ITransposition_Neg, JTransposition_Neg] := -1;
    MTransposition_Neg[JTransposition_Neg, ITransposition_Neg] := -1;
    return MTransposition_Neg end proc;

    Transposition_Neg := proc(ITransposition_Neg, JTransposition_Neg, NTransposition_Neg)
local MTransposition_Neg;
    MTransposition_Neg :=
        LinearAlgebra:-IdentityMatrix(NTransposition_Neg, compact = false);
    MTransposition_Neg[ITransposition_Neg, ITransposition_Neg] := 0;
    MTransposition_Neg[JTransposition_Neg, JTransposition_Neg] := 0;
    MTransposition_Neg[ITransposition_Neg, JTransposition_Neg] := -1;
    MTransposition_Neg[JTransposition_Neg, ITransposition_Neg] := -1;
    return MTransposition_Neg
end proc

> Neg:=proc(INeg, JNeg, NNeg) local MNeg;
    MNeg:=IdentityMatrix(NNeg, compact=false); MNeg[INeg, INeg]:=-1;
    MNeg[JNeg, JNeg]:=-1; return MNeg; end proc;

    Neg := proc(INeg, JNeg, NNeg)
local MNeg;
    MNeg := LinearAlgebra:-IdentityMatrix(NNeg, compact = false);
    MNeg[INeg, INeg] := -1;
    MNeg[JNeg, JNeg] := -1;
    return MNeg
end proc

> Cycle:=proc(NCycle) local MCycle, ICycle;
    MCycle:=Matrix(NCycle, NCycle, 0); for ICycle from 1 to NCycle-1 do
    MCycle[ICycle+1, ICycle]:=1; end do; MCycle[1, NCycle]:=1; return
    MCycle; end proc;

    Cycle := proc(NCycle)
local MCycle, ICycle;
    MCycle := Matrix(NCycle, NCycle, 0);
    for ICycle to NCycle - 1 do MCycle[ICycle+1, ICycle] := 1 end do;
    MCycle[1, NCycle] := 1;
    return MCycle
end proc

> Reflexion:=proc(KReflexion, NReflexion) local MReflexion;
    MReflexion:=IdentityMatrix(NReflexion, compact=false);
    MReflexion[KReflexion, KReflexion]:=-1; return MReflexion; end proc;

    Reflexion := proc(KReflexion, NReflexion)
local MReflexion;
    MReflexion := LinearAlgebra:-IdentityMatrix(NReflexion, compact = false);
    MReflexion[KReflexion, KReflexion] := -1;
    return MReflexion
end proc

```

```

> Reflexion_Gen:=proc(XReflexion) local MReflexion;
MReflexion:=IdentityMatrix(op(1,XReflexion),compact=false)-2*OuterProd
uctMatrix(XReflexion,Transpose(XReflexion))/Norm(XReflexion,2)^2;
return MReflexion; end proc;

Reflexion_Gen := proc(XReflexion)
local MReflexion;
MReflexion := LinearAlgebra : -IdentityMatrix(op(1, XReflexion), compact = false) - 2
* LinearAlgebra : -OuterProductMatrix(XReflexion,
LinearAlgebra : -Transpose(XReflexion))/LinearAlgebra : -Norm(XReflexion, 2)^2;
return MReflexion
end proc

```

10.7.3. Ensembles de générateurs de certains groupes de matrices

```

> Gen_Sn:=proc(NGen_Sn) return
Matrixunion(Cycle(NGen_Sn),{Transp(NGen_Sn)}); end proc;

Gen_Sn := proc(NGen_Sn)
return Matrixunion(Cycle(NGen_Sn), {Transp(NGen_Sn)})
end proc

> Gen_An:=proc(NGen_An) local TempGen_An,IndicesNGen_An,KGen_An;
TempGen_An:={}; IndicesNGen_An:=Cartesian_product(IJensemblesdiff(NGen_An),
IJensemblesdiff(NGen_An)); for KGen_An in IndicesNGen_An do
TempGen_An:=Matrixunion(Multiply(Transposition(KGen_An[1][1],KGen_An[1
][2],NGen_An),Transposition(KGen_An[2][1],KGen_An[2][2],NGen_An)),TempGen_An);
end do; return TempGen_An; end proc;

Gen_An := proc(NGen_An)
local TempGen_An, IndicesNGen_An, KGen_An;
TempGen_An := {};
IndicesNGen_An :=
Cartesian_product(IJensemblesdiff(NGen_An), IJensemblesdiff(NGen_An));
for KGen_An in IndicesNGen_An do TempGen_An := Matrixunion(
LinearAlgebra : -Multiply(Transposition(KGen_An11, KGen_An12, NGen_An),
Transposition(KGen_An21, KGen_An22, NGen_An)), TempGen_An)
end do;
return TempGen_An
end proc

> Gen_Bn:=proc(NGen_Bn) local TempGen_Bn,KGen_Bn ; TempGen_Bn:=Matrixunion(
Cycle(NGen_Bn),{Transp(NGen_Bn)}); for KGen_Bn in 1..NGen_Bn do TempGen_Bn:=
Matrixunion(Reflexion(KGen_Bn,NGen_Bn),TempGen_Bn) end do; return TempGen_Bn;
end proc;

```



```
Gen_Bn := proc(NGen_Bn)  
local TempGen_Bn, KGen_Bn;  
    TempGen_Bn := Matrixunion(Cycle(NGen_Bn), {Transp(NGen_Bn)});  
    for KGen_Bn in 1..NGen_Bn do  
        TempGen_Bn := Matrixunion(Reflexion(KGen_Bn, NGen_Bn), TempGen_Bn)  
    end do;  
    return TempGen_Bn  
end proc
```

```

> Gen_Bn_Plus:=proc(NGen_Bn_Plus) local TempGen_Bn_Plus,KGen_Bn_Plus;
TempGen_Bn_Plus:=Matrixunion(Transposition_SemiNeg(1,2,NGen_Bn_Plus),
{Neg(1,2,NGen_Bn_Plus)}); for KGen_Bn_Plus in
Cartesian_product([seq(i,i=1..NGen_Bn_Plus-1)],[seq(i,i=1..NGen_Bn_Plus-1)]) do
TempGen_Bn_Plus:=Matrixunion(Multiply(Transposition(KGen_Bn_Plus[1],KGen_Bn_Plus[1]+1,NGen_Bn_Plus),Transposition(KGen_Bn_Plus[2],KGen_Bn_Plus[2]+1,NGen_Bn_Plus)),TempGen_Bn_Plus); end do; return
TempGen_Bn_Plus; end proc;

Gen_Bn_Plus := proc(NGen_Bn_Plus)
local TempGen_Bn_Plus, KGen_Bn_Plus;
TempGen_Bn_Plus := Matrixunion(Transposition_SemiNeg(1, 2, NGen_Bn_Plus),
{Neg(1, 2, NGen_Bn_Plus)});
for KGen_Bn_Plus in Cartesian_product([seq(i, i = 1..NGen_Bn_Plus - 1)],
[seq(i, i = 1..NGen_Bn_Plus - 1)]) do TempGen_Bn_Plus := Matrixunion(
LinearAlgebra : - Multiply(
Transposition(KGen_Bn_Plus1, KGen_Bn_Plus1 + 1, NGen_Bn_Plus),
Transposition(KGen_Bn_Plus2, KGen_Bn_Plus2 + 1, NGen_Bn_Plus)),
TempGen_Bn_Plus)
end do;
return TempGen_Bn_Plus
end proc

> Gen_BD1959:=proc(NGen_BD1959) local TempGen_BD1959,KGen_BD1959;
TempGen_BD1959:={Cycle(NGen_BD1959)}; for KGen_BD1959 in
1..NGen_BD1959 do
TempGen_BD1959:=Matrixunion(Reflexion(KGen_BD1959,NGen_BD1959),TempGen_BD1959); end do ; return TempGen_BD1959; end proc;

Gen_BD1959 := proc(NGen_BD1959)
local TempGen_BD1959, KGen_BD1959;
TempGen_BD1959 := {Cycle(NGen_BD1959)};
for KGen_BD1959 in 1..NGen_BD1959 do TempGen_BD1959 :=
Matrixunion(Reflexion(KGen_BD1959, NGen_BD1959), TempGen_BD1959)
end do;
return TempGen_BD1959
end proc

> Gen_BD1959_Plus:=proc(NGen_BD1959_Plus) local
TempGen_BD1959_Plus,KGen_BD1959_Plus; if 'mod'(2,NGen_BD1959_Plus)=0
then TempGen_BD1959_Plus:={}; else
TempGen_BD1959_Plus:={Cycle(NGen_BD1959_Plus)}; end if; for
KGen_BD1959_Plus in IJpairs(NGen_BD1959_Plus) do
TempGen_BD1959_Plus:=Matrixunion(Multiply(Reflexion(KGen_BD1959_Plus[1],NGen_BD1959_Plus),Reflexion(KGen_BD1959_Plus[2],NGen_BD1959_Plus)),TempGen_BD1959_Plus); end do ; return TempGen_BD1959_Plus; end proc;

Gen_BD1959_Plus := proc(NGen_BD1959_Plus)
local TempGen_BD1959_Plus, KGen_BD1959_Plus;
if 2 mod NGen_BD1959_Plus = 0 then TempGen_BD1959_Plus := {}
else TempGen_BD1959_Plus := {Cycle(NGen_BD1959_Plus)}
end if;
for KGen_BD1959_Plus in IJpairs(NGen_BD1959_Plus) do TempGen_BD1959_Plus
:= Matrixunion(LinearAlgebra : - Multiply(
Reflexion(KGen_BD1959_Plus1, NGen_BD1959_Plus),

```

```

    Reflexion(KGen_BD1959_Plus2, NGen_BD1959_Plus)), TempGen_BD1959_Plus)
end do;
return TempGen_BD1959_Plus
end proc

> Gen_TwoPlusn:=proc(NGen_2Plusn) ; if NGen_2Plusn>2 then return
{Matrix([[1,0,0],[0,-1,0],[0,0,-1]]),convert(Matrix([[cos(2*Pi/NGen_2
Plusn),sin(2*Pi/NGen_2Plusn),0],[sin(2*Pi/NGen_2Plusn),-cos(2*Pi/NGen_
2Plusn),0],[0,0,1]]),'radical')} elif NGen_2Plusn=2 then return
{Matrix([[1,0,0],[0,-1,0],[0,0,-1]]),Matrix([[-1,0,0],[0,1,0],[0,0,-1
]])} else return {Matrix([[1,0,0],[0,1,0],[0,0,-1]])} end if; end
proc;

Gen_TwoPlusn := proc(NGen_2Plusn)
    if 2 < NGen_2Plusn then return {Matrix([[1, 0, 0], [0, -1, 0], [0, 0, -1]]), convert(
        Matrix([[cos(2 *  $\pi$ /NGen_2Plusn), sin(2 *  $\pi$ /NGen_2Plusn), 0],
        [sin(2 *  $\pi$ /NGen_2Plusn), -cos(2 *  $\pi$ /NGen_2Plusn), 0], [0, 0, 1]]), 'radical')}
    elif NGen_2Plusn = 2 then return
        {Matrix([[1, 0, 0], [0, -1, 0], [0, 0, -1]]), Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, -1]])}
    else return {Matrix([[1, 0, 0], [0, 1, 0], [0, 0, -1]])}
    end if
end proc

> Gen_Diedn := proc (NGen_Diedn) if 2 < NGen_Diedn then return
{Matrix([[1, 0], [0, -1]]), convert(Matrix([[cos(2*Pi/NGen_Diedn),
sin(2*Pi/NGen_Diedn)], [sin(2*Pi/NGen_Diedn),
-cos(2*Pi/NGen_Diedn)]),'radical')} elif NGen_Diedn = 2 then return
{Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))} else
return {Reflexion_Gen(Vector([0, 1]))} end if end proc;

Gen_Diedn := proc(NGen_Diedn)
    if 2 < NGen_Diedn then return {Matrix([[1, 0], [0, -1]]), convert(Matrix([
        cos(2 *  $\pi$ /NGen_Diedn), sin(2 *  $\pi$ /NGen_Diedn),
        sin(2 *  $\pi$ /NGen_Diedn), -cos(2 *  $\pi$ /NGen_Diedn)]), 'radical')}
    elif NGen_Diedn = 2 then
        return {Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))}
    else return {Reflexion_Gen(Vector([0, 1]))}
    end if
end proc

> Gen_Tetrahedron:=proc(); return
{MatrixMatrixMultiply(Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(ve
ctor([-1/2,sqrt(3)/2,0])),Reflexion_Gen(Vector([2/3*2^(1/2), 0,
-4/3]))}); end proc;

Gen_Tetrahedron := proc()
    return {LinearAlgebra : -MatrixMatrixMultiply(Reflexion_Gen(Vector([1, 0, 0])),
        Reflexion_Gen(Vector([-1/2, 1/2 * sqrt(3), 0])),
        Reflexion_Gen(Vector([2 * 2(1/2)/3, 0, -4/3]))}
end proc

> Gen_Tetrahedron_Plus:=proc() return {convert([[-2/3, 1/3*3^(1/2),
-1/3*2^(1/2)], [1/3*3^(1/2), 0, -1/3*2^(1/2)*3^(1/2)], [-1/3*2^(1/2),
-1/3*2^(1/2)*3^(1/2), -1/3]],Matrix),convert([[-1/2, -1/2*3^(1/2), 0],
[1/2*3^(1/2), -1/2, 0], [0, 0, 1]],Matrix)} end proc;

```

```

Gen_Tetrahedron_Plus := proc()
  return {convert([[−2/3, 3(1/2)/3, −2(1/2)/3], [3(1/2)/3, 0, −2(1/2) * 3(1/2)/3],
    [−2(1/2)/3, −2(1/2) * 3(1/2)/3, −1/3]], Matrix),
    convert([[−1/2, −3(1/2)/2, 0], [3(1/2)/2, −1/2, 0], [0, 0, 1]], Matrix)}
end proc

> Gen_TroisPlusQuatre:=proc(); return
{DiagonalMatrix(Vector([−1,−1,−1])),op(Gen_Tetrahedron_Plus())}; end
proc;

Gen_TroisPlusQuatre := proc()
  return
  {op(Gen_Tetrahedron_Plus()), LinearAlgebra : −DiagonalMatrix(Vector([−1, −1, −1]))}
end proc

```

10.7.4. Des générateurs aux groupes

```

> GentoGr:=proc(LGentoGr) local
  TempGentoGr,Temp1GentoGr,NGentoGr,MGentoGr,AGentoGr,BGentoGr,ZGentoGr;
  TempGentoGr:={
  };Temp1GentoGr:=LGentoGr;NGentoGr:=0;MGentoGr:=nops(LGentoGr); for
  ZGentoGr while NGentoGr<>MGentoGr do
  TempGentoGr:=Temp1GentoGr;NGentoGr:=MGentoGr; for AGentoGr in
  TempGentoGr do for BGentoGr in TempGentoGr do
  Temp1GentoGr:=Matrixunion(Map(radnormal,simplify(Multiply(AGentoGr,BGe
  ntoGr)), 'rationalized'),Temp1GentoGr); end do; end do;
  MGentoGr:=nops(Temp1GentoGr); end do; return Temp1GentoGr; end
  proc;

  GentoGr := proc(LGentoGr)
  local TempGentoGr, Temp1GentoGr, NGentoGr, MGentoGr, AGentoGr, BGentoGr,
  ZGentoGr;
    TempGentoGr := {};
    Temp1GentoGr := LGentoGr;
    NGentoGr := 0;
    MGentoGr := nops(LGentoGr);
    for ZGentoGr while NGentoGr ≠ MGentoGr do
      TempGentoGr := Temp1GentoGr;
      NGentoGr := MGentoGr;
      for AGentoGr in TempGentoGr do for BGentoGr in TempGentoGr do
        Temp1GentoGr := Matrixunion(LinearAlgebra : −Map(radnormal,
        simplify(LinearAlgebra : −Multiply(AGentoGr, BGentoGr)), 'rationalized'),
        Temp1GentoGr)
      end do
    end do;
    MGentoGr := nops(Temp1GentoGr)
  end do;
  return Temp1GentoGr
end proc

```

```

> RepGroupe:=proc(NomgroupeRepGroupe,NRepGroupe) local NomRepGroupe,
NomfichierRepGroupe; global TempRepGroupe; NomRepGroupe :=
cat("Gen_",NomgroupeRepGroupe); NomfichierRepGroupe :=
cat(PATHDATA,"Rep_",NomgroupeRepGroupe,"_",convert(NRepGroupe,string),
".txt"); if Fichier_Existant(NomfichierRepGroupe) = true then read
NomfichierRepGroupe else TempRepGroupe :=
GentoGr(eval(convert(NomRepGroupe,name)(NRepGroupe))); save
TempRepGroupe, NomfichierRepGroupe end if; return TempRepGroupe end
proc;

RepGroupe := proc(NomgroupeRepGroupe, NRepGroupe)
local NomRepGroupe, NomfichierRepGroupe;
global TempRepGroupe;
  NomRepGroupe := cat( "Gen_", NomgroupeRepGroupe);
  NomfichierRepGroupe := cat(PATHDATA, "Rep_", NomgroupeRepGroupe, "_",
    convert(NRepGroupe, string), ".txt");
  if Fichier_Existant(NomfichierRepGroupe) = true then read NomfichierRepGroupe
  else
    TempRepGroupe := GentoGr(eval(convert(NomRepGroupe, name)(NRepGroupe)));
    save TempRepGroupe, NomfichierRepGroupe
  end if;
  return TempRepGroupe
end proc

> Plus:=proc(GroupePlus) local matricePlus; return map(matricePlus ->
if simplify(Determinant(matricePlus))=1 then return matricePlus end
if, GroupePlus); end proc;

Plus := proc(GroupePlus)
local matricePlus;
  return map(proc(matricePlus)
    option operator, arrow;
    if simplify(LinearAlgebra : -Determinant(matricePlus)) = 1 then return matricePlus
    end if
  end proc, GroupePlus)
end proc

> Rned:=proc(GroupeRned,NRned) local IndiceRned,Indice1Rned, DimRned;
DimRned:=NRned-op(1,GroupeRned[1])[1]; return
{seq(DiagonalMatrix([GroupeRned[IndiceRned],Vector([seq(1,Indice1Rned
=1..DimRned)]])),IndiceRned=1..nops(GroupeRned)),seq(DiagonalMatrix([G
roupeRned[IndiceRned],Vector([seq(-1,Indice1Rned=1..DimRned)]])),Indic
eRned=1..nops(GroupeRned))}; end proc;

Rned := proc(GroupeRned, NRned)
local IndiceRned, Indice1Rned, DimRned;
  DimRned := NRned - op(1, GroupeRned1)1;
  return {seq(LinearAlgebra : -DiagonalMatrix(
    [GroupeRnedIndiceRned, Vector([seq(1, Indice1Rned = 1..DimRned)]))),
    IndiceRned = 1..nops(GroupeRned)), seq(LinearAlgebra : -DiagonalMatrix(
    [GroupeRnedIndiceRned, Vector([seq(-1, Indice1Rned = 1..DimRned)]))),
    IndiceRned = 1..nops(GroupeRned))}
end proc

```

```
> RnedInt:=proc(GroupeRned,NRned) local IndiceRned,Indice1Rned,
DimRned; DimRned:=NRned-op(1,GroupeRned[1])[1]; return
{seq(DiagonalMatrix([GroupeRned[IndiceRned],Vector([seq(1,Indice1Rned
=1..DimRned)]))],IndiceRned=1..nops(GroupeRned)),seq(DiagonalMatrix([-
1*GroupeRned[IndiceRned],Vector([seq(-1,Indice1Rned=1..DimRned)]))],In
diceRned=1..nops(GroupeRned))}; end proc;

RnedInt := proc(GroupeRned, NRned)
local IndiceRned, Indice1Rned, DimRned;
    DimRned := NRned - op(1, GroupeRned1)1;
    return {seq(LinearAlgebra : -DiagonalMatrix(
        [GroupeRnedIndiceRned, Vector([seq(1, Indice1Rned = 1..DimRned)]))],
        IndiceRned = 1..nops(GroupeRned)), seq(LinearAlgebra : -DiagonalMatrix(
        [- GroupeRnedIndiceRned, Vector([seq(-1, Indice1Rned = 1..DimRned)]))],
        IndiceRned = 1..nops(GroupeRned))}
end proc
```

10.7.5. Groupes finis de matrices

```
> Sn:=proc (NSn) local NomgroupeSn, NomSn, NomfichierSn; global TempSn;
TempSn:={} ;NomgroupeSn := "Sn"; NomSn := cat("Gen_",NomgroupeSn);
NomfichierSn :=
cat(PATHDATA,NomgroupeSn,"_",convert(NSn,string),".txt"); if
Fichier_Existant(NomfichierSn) = true then read NomfichierSn else
TempSn := GentoGr(eval(convert(NomSn,name)(NSn))); save TempSn,
NomfichierSn end if; if TempSn={} then return GROUPE else return
TempSn end if; end proc;

Sn := proc(NSn)
local NomgroupeSn, NomSn, NomfichierSn;
global TempSn;
    TempSn := {};
    NomgroupeSn := "Sn";
    NomSn := cat("Gen_", NomgroupeSn);
    NomfichierSn := cat(PATHDATA, NomgroupeSn, "_", convert(NSn, string), ".txt");
    if Fichier_Existant(NomfichierSn) = true then read NomfichierSn
    else
        TempSn := GentoGr(eval(convert(NomSn, name)(NSn))); save TempSn, NomfichierSn
    end if;
    if TempSn = {} then return GROUPE else return TempSn end if
end proc

> An:=proc (NAn) local NomgroupeAn, NomAn, NomfichierAn; global TempAn;
TempAn:={}; NomgroupeAn := "An"; NomAn := cat("Gen_",NomgroupeAn);
NomfichierAn :=
cat(PATHDATA,NomgroupeAn,"_",convert(NAn,string),".txt"); if
Fichier_Existant(NomfichierAn) = true then read NomfichierAn else
TempAn := GentoGr(eval(convert(NomAn,name)(NAn))); save TempAn,
NomfichierAn end if; if TempAn={} then return GROUPE else return
TempAn end if; end proc;
```

```

An := proc(NAn)
local NomgroupeAn, NomAn, NomfichierAn;
global TempAn;
  TempAn := {};
  NomgroupeAn := "An";
  NomAn := cat("Gen_", NomgroupeAn);
  NomfichierAn := cat(PATHDATA, NomgroupeAn, "_", convert(NAn, string), ".txt");
  if Fichier_Existant(NomfichierAn) = true then read NomfichierAn
  else
    TempAn := GentoGr(eval(convert(NomAn, name)(NAn)));
    save TempAn, NomfichierAn
  end if;
  if TempAn = {} then return GROUPE else return TempAn end if
end proc

```

```

> Bn:=proc (NBn) local NomgroupeBn, NomBn, NomfichierBn; global TempBn;
TempBn:={}; NomgroupeBn := "Bn"; NomBn := cat("Gen_",NomgroupeBn);
NomfichierBn :=
cat(PATHDATA,NomgroupeBn,"_",convert(NBn,string),".txt"); if
Fichier_Existant(NomfichierBn) = true then read NomfichierBn else
TempBn := GentoGr(eval(convert(NomBn,name)(NBn))); save TempBn,
NomfichierBn end if; if TempBn={} then return GROUPE else return
TempBn end if; end proc;

```

```

Bn := proc(NBn)
local NomgroupeBn, NomBn, NomfichierBn;
global TempBn;
  TempBn := {};
  NomgroupeBn := "Bn";
  NomBn := cat("Gen_", NomgroupeBn);
  NomfichierBn := cat(PATHDATA, NomgroupeBn, "_", convert(NBn, string), ".txt");
  if Fichier_Existant(NomfichierBn) = true then read NomfichierBn
  else
    TempBn := GentoGr(eval(convert(NomBn, name)(NBn)));
    save TempBn, NomfichierBn
  end if;
  if TempBn = {} then return GROUPE else return TempBn end if
end proc

```

```

> Bn_Plus:=proc (NBn_Plus) local NomgroupeBn_Plus, NomBn_Plus,
NomfichierBn_Plus; global TempBn_Plus; TempBn_Plus:={};
NomgroupeBn_Plus := "Bn_Plus"; NomBn_Plus :=
cat("Gen_",NomgroupeBn_Plus); NomfichierBn_Plus :=
cat(PATHDATA,NomgroupeBn_Plus,"_",convert(NBn_Plus,string),".txt"); if
Fichier_Existant(NomfichierBn_Plus) = true then read NomfichierBn_Plus
else TempBn_Plus := GentoGr(eval(convert(NomBn_Plus,name)(NBn_Plus)));
save TempBn_Plus, NomfichierBn_Plus end if; if TempBn_Plus={} then
return GROUPE else return TempBn_Plus end if; end proc;

```

```

Bn_Plus := proc(NBn_Plus)
local NomgroupeBn_Plus, NomBn_Plus, NomfichierBn_Plus;
global TempBn_Plus;
    TempBn_Plus := {};
    NomgroupeBn_Plus := "Bn_Plus";
    NomBn_Plus := cat("Gen_", NomgroupeBn_Plus);
    NomfichierBn_Plus :=
        cat(PATHDATA, NomgroupeBn_Plus, "_", convert(NBn_Plus, string), ".txt");
if Fichier_Existant(NomfichierBn_Plus) = true then read NomfichierBn_Plus
else
    TempBn_Plus := GentoGr(eval(convert(NomBn_Plus, name)(NBn_Plus)));
    save TempBn_Plus, NomfichierBn_Plus
end if;
if TempBn_Plus = {} then return GROUPE else return TempBn_Plus end if
end proc

> Dn:=proc (NDn) local NomgroupeDn, NomDn, NomfichierDn; global TempDn;
TempDn:={}; NomgroupeDn := "Dn"; NomDn := cat("Gen_",NomgroupeDn);
NomfichierDn :=
cat(PATHDATA,NomgroupeDn,"_",convert(NDn,string),".txt"); if
Fichier_Existant(NomfichierDn) = true then read NomfichierDn else
TempDn := GentoGr(eval(convert(NomDn,name)(NDn))); save TempDn,
NomfichierDn end if; if TempDn={} then return GROUPE else return
TempDn end if; end proc;

Dn := proc(NDn)
local NomgroupeDn, NomDn, NomfichierDn;
global TempDn;
    TempDn := {};
    NomgroupeDn := "Dn";
    NomDn := cat("Gen_", NomgroupeDn);
    NomfichierDn := cat(PATHDATA, NomgroupeDn, "_", convert(NDn, string), ".txt");
if Fichier_Existant(NomfichierDn) = true then read NomfichierDn
else
    TempDn := GentoGr(eval(convert(NomDn, name)(NDn)));
    save TempDn, NomfichierDn
end if;
if TempDn = {} then return GROUPE else return TempDn end if
end proc

> BD1959:=proc (NBD1959) local NomgroupeBD1959, NomBD1959,
NomfichierBD1959; global TempBD1959; TempBD1959:={}; NomgroupeBD1959
:= "BD1959"; NomBD1959 := cat("Gen_",NomgroupeBD1959);
NomfichierBD1959 :=
cat(PATHDATA,NomgroupeBD1959,"_",convert(NBD1959,string),".txt"); if
Fichier_Existant(NomfichierBD1959) = true then read NomfichierBD1959
else TempBD1959 := GentoGr(eval(convert(NomBD1959,name)(NBD1959)));
save TempBD1959, NomfichierBD1959 end if; if TempBD1959={} then
return GROUPE else return TempBD1959 end if; end proc;

```



```

BD1959 := proc(NBD1959)
local NomgroupeBD1959, NomBD1959, NomfichierBD1959;
global TempBD1959;
    TempBD1959 := {};
    NomgroupeBD1959 := "BD1959";
    NomBD1959 := cat( "Gen_", NomgroupeBD1959);
    NomfichierBD1959 :=
        cat(PATHDATA, NomgroupeBD1959, "_", convert(NBD1959, string), ".txt");
    if Fichier_Existant(NomfichierBD1959) = true then read NomfichierBD1959
    else
        TempBD1959 := GentoGr(eval(convert(NomBD1959, name)(NBD1959)));
        save TempBD1959, NomfichierBD1959
    end if;
    if TempBD1959 = {} then return GROUPE else return TempBD1959 end if
end proc

```

```

> BD1959_Plus:=proc (NBD1959_Plus) local NomgroupeBD1959_Plus,
NomBD1959_Plus, NomfichierBD1959_Plus; global TempBD1959_Plus;
TempBD1959_Plus:={}; NomgroupeBD1959_Plus := "BD1959_Plus";
NomBD1959_Plus := cat("Gen_",NomgroupeBD1959_Plus);
NomfichierBD1959_Plus :=
cat(PATHDATA,NomgroupeBD1959_Plus,"_",convert(NBD1959_Plus,string),".t
xt"); if Fichier_Existant(NomfichierBD1959_Plus) = true then read
NomfichierBD1959_Plus else TempBD1959_Plus :=
GentoGr(eval(convert(NomBD1959_Plus,name)(NBD1959_Plus))); save
TempBD1959_Plus, NomfichierBD1959_Plus end if; if TempBD1959_Plus={}
then return GROUPE else return TempBD1959_Plus end if; end proc;

```

```

BD1959_Plus := proc(NBD1959_Plus)
local NomgroupeBD1959_Plus, NomBD1959_Plus, NomfichierBD1959_Plus;
global TempBD1959_Plus;
    TempBD1959_Plus := {};
    NomgroupeBD1959_Plus := "BD1959_Plus";
    NomBD1959_Plus := cat( "Gen_", NomgroupeBD1959_Plus);
    NomfichierBD1959_Plus := cat(PATHDATA, NomgroupeBD1959_Plus, "_",
        convert(NBD1959_Plus, string), ".txt");
    if Fichier_Existant(NomfichierBD1959_Plus) = true then
        read NomfichierBD1959_Plus
    else
        TempBD1959_Plus :=
            GentoGr(eval(convert(NomBD1959_Plus, name)(NBD1959_Plus)));
        save TempBD1959_Plus, NomfichierBD1959_Plus
    end if;
    if TempBD1959_Plus = {} then return GROUPE
    else return TempBD1959_Plus
    end if
end proc

```

```

> Diedn:=proc (NDiedn) local NomgroupeDiedn, NomDiedn, NomfichierDiedn;
global TempDiedn; TempDiedn:={}; NomgroupeDiedn := "Diedn"; NomDiedn
:= cat("Gen_",NomgroupeDiedn); NomfichierDiedn :=
cat(PATHDATA,NomgroupeDiedn,"_",convert(NDiedn,string),".txt"); if
Fichier_Existant(NomfichierDiedn) = true then read NomfichierDiedn
else TempDiedn := GentoGr(eval(convert(NomDiedn,name)(NDiedn))); save
TempDiedn, NomfichierDiedn end if; if TempDiedn={} then return
GROUPE else return TempDiedn end if; end proc;

Diedn := proc(NDiedn)
local NomgroupeDiedn, NomDiedn, NomfichierDiedn;
global TempDiedn;
    TempDiedn := {};
    NomgroupeDiedn := "Diedn";
    NomDiedn := cat("Gen_", NomgroupeDiedn);
    NomfichierDiedn :=
        cat(PATHDATA, NomgroupeDiedn, "_", convert(NDiedn, string), ".txt");
    if Fichier_Existant(NomfichierDiedn) = true then read NomfichierDiedn
    else
        TempDiedn := GentoGr(eval(convert(NomDiedn, name)(NDiedn)));
        save TempDiedn, NomfichierDiedn
    end if;
    if TempDiedn = {} then return GROUPE else return TempDiedn end if
end proc

> TwoPlusn:=proc (N2Plusn) local Nomgroupe2Plusn, Nom2Plusn,
Nomfichier2Plusn; global Temp2Plusn; Temp2Plusn:={}; Nomgroupe2Plusn
:= "TwoPlusn"; Nom2Plusn := cat("Gen_",Nomgroupe2Plusn);
Nomfichier2Plusn :=
cat(PATHDATA,Nomgroupe2Plusn,"_",convert(N2Plusn,string),".txt"); if
Fichier_Existant(Nomfichier2Plusn) = true then read Nomfichier2Plusn
else Temp2Plusn := GentoGr(eval(convert(Nom2Plusn,name)(N2Plusn)));
save Temp2Plusn, Nomfichier2Plusn end if; if Temp2Plusn={} then
return GROUPE else return Temp2Plusn end if; end proc;

TwoPlusn := proc(N2Plusn)
local Nomgroupe2Plusn, Nom2Plusn, Nomfichier2Plusn;
global Temp2Plusn;
    Temp2Plusn := {};
    Nomgroupe2Plusn := "TwoPlusn";
    Nom2Plusn := cat("Gen_", Nomgroupe2Plusn);
    Nomfichier2Plusn :=
        cat(PATHDATA, Nomgroupe2Plusn, "_", convert(N2Plusn, string), ".txt");
    if Fichier_Existant(Nomfichier2Plusn) = true then read Nomfichier2Plusn
    else
        Temp2Plusn := GentoGr(eval(convert(Nom2Plusn, name)(N2Plusn)));
        save Temp2Plusn, Nomfichier2Plusn
    end if;
    if Temp2Plusn = {} then return GROUPE else return Temp2Plusn end if
end proc

```

```

> Tetrahedron:=proc() local NomgroupeTetrahedron,
  NomfichierTetrahedron; global TempTetrahedron; NomgroupeTetrahedron :=
  "Tetrahedron";
  NomfichierTetrahedron:=cat(PATHDATA,NomgroupeTetrahedron,"_3.txt"); if
  Fichier_Existant(NomfichierTetrahedron) = true then read
  NomfichierTetrahedron else TempTetrahedron :=
  GentoGr(Gen_Tetrahedron()); save TempTetrahedron,
  NomfichierTetrahedron end if; return TempTetrahedron end proc;

  Tetrahedron := proc()
  local NomgroupeTetrahedron, NomfichierTetrahedron;
  global TempTetrahedron;
    NomgroupeTetrahedron := "Tetrahedron";
    NomfichierTetrahedron := cat(PATHDATA, NomgroupeTetrahedron, "_3.txt");
    if Fichier_Existant(NomfichierTetrahedron) = true then read NomfichierTetrahedron
    else
      TempTetrahedron := GentoGr(Gen_Tetrahedron());
      save TempTetrahedron, NomfichierTetrahedron
    end if;
    return TempTetrahedron
  end proc

> TroisPlusQuatre:=proc() local NomgroupeTroisPlusQuatre,
  NomfichierTroisPlusQuatre; global TempTroisPlusQuatre;
  NomgroupeTroisPlusQuatre := "TroisPlusQuatre";
  NomfichierTroisPlusQuatre:=cat(PATHDATA,NomgroupeTroisPlusQuatre,"_3.t
  xt"); if Fichier_Existant(NomfichierTroisPlusQuatre) = true then read
  NomfichierTroisPlusQuatre else TempTroisPlusQuatre :=
  GentoGr(Gen_TroisPlusQuatre()); save TempTroisPlusQuatre,
  NomfichierTroisPlusQuatre end if; return TempTroisPlusQuatre end
  proc;

  TroisPlusQuatre := proc()
  local NomgroupeTroisPlusQuatre, NomfichierTroisPlusQuatre;
  global TempTroisPlusQuatre;
    NomgroupeTroisPlusQuatre := "TroisPlusQuatre";
    NomfichierTroisPlusQuatre :=
      cat(PATHDATA, NomgroupeTroisPlusQuatre, "_3.txt");
    if Fichier_Existant(NomfichierTroisPlusQuatre) = true then
      read NomfichierTroisPlusQuatre
    else
      TempTroisPlusQuatre := GentoGr(Gen_TroisPlusQuatre());
      save TempTroisPlusQuatre, NomfichierTroisPlusQuatre
    end if;
    return TempTroisPlusQuatre
  end proc

> Icosahedron:=proc() local NomgroupeIcosahedron,
  NomfichierIcosahedron; global TempIcosahedron; NomgroupeIcosahedron :=
  "Icosahedron";
  NomfichierIcosahedron:=cat(PATHDATA,NomgroupeIcosahedron,".txt"); if
  Fichier_Existant(NomfichierIcosahedron) = true then read
  NomfichierIcosahedron; end if; return TempIcosahedron end proc;

```

```

Icosahedron := proc()
local NomgroupeIcosahedron, NomfichierIcosahedron;
global TempIcosahedron;
    NomgroupeIcosahedron := "Icosahedron";
    NomfichierIcosahedron := cat(PATHDATA, NomgroupeIcosahedron, ".txt");
    if Fichier_Existant(NomfichierIcosahedron) = true then read NomfichierIcosahedron
    end if;
    return TempIcosahedron
end proc

> Diedn:=proc (NDiedn) local NomgroupeDiedn, NomDiedn, NomfichierDiedn;
global TempDiedn; TempDiedn:={}; NomgroupeDiedn := "Diedn"; NomDiedn
:= cat("Gen_",NomgroupeDiedn); NomfichierDiedn :=
cat(PATHDATA,NomgroupeDiedn,"_",convert(NDiedn,string),".txt"); if
Fichier_Existant(NomfichierDiedn) = true then read NomfichierDiedn
else TempDiedn := GentoGr(eval(convert(NomDiedn,name)(NDiedn))); save
TempDiedn, NomfichierDiedn end if; if TempDiedn={} then return
GROUPE else return TempDiedn end if; end proc;

    Diedn := proc(NDiedn)
local NomgroupeDiedn, NomDiedn, NomfichierDiedn;
global TempDiedn;
    TempDiedn := {};
    NomgroupeDiedn := "Diedn";
    NomDiedn := cat("Gen_" , NomgroupeDiedn);
    NomfichierDiedn :=
        cat(PATHDATA, NomgroupeDiedn, "_", convert(NDiedn, string), ".txt");
    if Fichier_Existant(NomfichierDiedn) = true then read NomfichierDiedn
    else
        TempDiedn := GentoGr(eval(convert(NomDiedn, name)(NDiedn)));
        save TempDiedn, NomfichierDiedn
    end if;
    if TempDiedn = {} then return GROUPE else return TempDiedn end if
end proc

> DiednPlus:=proc (NDiednPlus) local NomgroupeDiednPlus, NomDiednPlus,
NomfichierDiednPlus; global TempDiednPlus; TempDiednPlus:={};
NomgroupeDiednPlus := "DiednPlus"; NomDiednPlus :=
cat("Gen_",NomgroupeDiednPlus); NomfichierDiednPlus :=
cat(PATHDATA,NomgroupeDiednPlus,"_",convert(NDiednPlus,string),".txt")
; if Fichier_Existant(NomfichierDiednPlus) = true then read
NomfichierDiednPlus else TempDiednPlus :=
GentoGr(eval(convert(NomDiednPlus,name)(NDiednPlus))); save
TempDiednPlus, NomfichierDiednPlus end if; if TempDiednPlus={} then
return GROUPE else return TempDiednPlus end if; end proc;

    DiednPlus := proc(NDiednPlus)
local NomgroupeDiednPlus, NomDiednPlus, NomfichierDiednPlus;
global TempDiednPlus;
    TempDiednPlus := {};
    NomgroupeDiednPlus := "DiednPlus";
    NomDiednPlus := cat( "Gen_" , NomgroupeDiednPlus);
    NomfichierDiednPlus :=
        cat(PATHDATA, NomgroupeDiednPlus, "_", convert(NDiednPlus, string), ".txt");

```

```

    if Fichier_Existant(NomfichierDiednPlus) = true then read NomfichierDiednPlus
    else
        TempDiednPlus := GentoGr(eval(convert(NomDiednPlus, name)(NDiednPlus)));
        save TempDiednPlus, NomfichierDiednPlus
    end if;
    if TempDiednPlus = {} then return GROUPE else return TempDiednPlus end if
end proc

> Twon:=proc (NTwon) local NomgroupeTwon, NomTwon, NomfichierTwon;
global TempTwon; TempTwon:={}; NomgroupeTwon := "Twon"; NomTwon :=
cat("Gen_", NomgroupeTwon); NomfichierTwon :=
cat(PATHDATA, NomgroupeTwon, "_", convert(NTwon, string), ".txt"); if
Fichier_Existant(NomfichierTwon) = true then read NomfichierTwon else
TempTwon := GentoGr(eval(convert(NomTwon, name)(NTwon))); save
TempTwon, NomfichierTwon end if; if TempTwon={} then return GROUPE
else return TempTwon end if; end proc;

    Twon := proc(NTwon)
    local NomgroupeTwon, NomTwon, NomfichierTwon;
    global TempTwon;
        TempTwon := {};
        NomgroupeTwon := "Twon";
        NomTwon := cat("Gen_", NomgroupeTwon);
        NomfichierTwon :=
            cat(PATHDATA, NomgroupeTwon, "_", convert(NTwon, string), ".txt");
        if Fichier_Existant(NomfichierTwon) = true then read NomfichierTwon
        else
            TempTwon := GentoGr(eval(convert(NomTwon, name)(NTwon)));
            save TempTwon, NomfichierTwon
        end if;
        if TempTwon = {} then return GROUPE else return TempTwon end if
    end proc

> TwoPlusn:=proc (NTwoPlusn) local NomgroupeTwoPlusn, NomTwoPlusn,
NomfichierTwoPlusn; global TempTwoPlusn; TempTwoPlusn:={};
NomgroupeTwoPlusn := "TwoPlusn"; NomTwoPlusn :=
cat("Gen_", NomgroupeTwoPlusn); NomfichierTwoPlusn :=
cat(PATHDATA, NomgroupeTwoPlusn, "_", convert(NTwoPlusn, string), ".txt");
if Fichier_Existant(NomfichierTwoPlusn) = true then read
NomfichierTwoPlusn else TempTwoPlusn :=
GentoGr(eval(convert(NomTwoPlusn, name)(NTwoPlusn))); save
TempTwoPlusn, NomfichierTwoPlusn end if; if TempTwoPlusn={} then
return GROUPE else return TempTwoPlusn end if; end proc;

    TwoPlusn := proc(NTwoPlusn)
    local NomgroupeTwoPlusn, NomTwoPlusn, NomfichierTwoPlusn;
    global TempTwoPlusn;
        TempTwoPlusn := {};
        NomgroupeTwoPlusn := "TwoPlusn";
        NomTwoPlusn := cat("Gen_", NomgroupeTwoPlusn);
        NomfichierTwoPlusn :=
            cat(PATHDATA, NomgroupeTwoPlusn, "_", convert(NTwoPlusn, string), ".txt");

```

```

    if Fichier_Existant(NomfichierTwoPlusn) = true then read NomfichierTwoPlusn
    else
        TempTwoPlusn := GentoGr(eval(convert(NomTwoPlusn, name)(NTwoPlusn)));
        save TempTwoPlusn, NomfichierTwoPlusn
    end if;
    if TempTwoPlusn = {} then return GROUPE else return TempTwoPlusn end if
end proc

> TwonPlus:=proc (NTwonPlus) local NomgroupeTwonPlus, NomTwonPlus,
NomfichierTwonPlus; global TempTwonPlus; TempTwonPlus:={};
NomgroupeTwonPlus := "TwonPlus"; NomTwonPlus :=
cat("Gen_", NomgroupeTwonPlus); NomfichierTwonPlus :=
cat(PATHDATA, NomgroupeTwonPlus, "_", convert(NTwonPlus, string), ".txt");
if Fichier_Existant(NomfichierTwonPlus) = true then read
NomfichierTwonPlus else TempTwonPlus :=
GentoGr(eval(convert(NomTwonPlus, name)(NTwonPlus))); save
TempTwonPlus, NomfichierTwonPlus end if; if TempTwonPlus={} then
return GROUPE else return TempTwonPlus end if; end proc;

TwonPlus := proc(NTwonPlus)
local NomgroupeTwonPlus, NomTwonPlus, NomfichierTwonPlus;
global TempTwonPlus;
    TempTwonPlus := {};
    NomgroupeTwonPlus := "TwonPlus";
    NomTwonPlus := cat( "Gen_", NomgroupeTwonPlus);
    NomfichierTwonPlus :=
        cat(PATHDATA, NomgroupeTwonPlus, "_", convert(NTwonPlus, string), ".txt");
    if Fichier_Existant(NomfichierTwonPlus) = true then read NomfichierTwonPlus
    else
        TempTwonPlus := GentoGr(eval(convert(NomTwonPlus, name)(NTwonPlus)));
        save TempTwonPlus, NomfichierTwonPlus
    end if;
    if TempTwonPlus = {} then return GROUPE else return TempTwonPlus end if
end proc

> TwoPlusnPlus:=proc (NTwoPlusnPlus) local NomgroupeTwoPlusnPlus,
NomTwoPlusnPlus, NomfichierTwoPlusnPlus; global TempTwoPlusnPlus;
TempTwoPlusnPlus:={}; NomgroupeTwoPlusnPlus := "TwoPlusnPlus";
NomTwoPlusnPlus := cat("Gen_", NomgroupeTwoPlusnPlus);
NomfichierTwoPlusnPlus :=
cat(PATHDATA, NomgroupeTwoPlusnPlus, "_", convert(NTwoPlusnPlus, string),
.txt"); if Fichier_Existant(NomfichierTwoPlusnPlus) = true then read
NomfichierTwoPlusnPlus else TempTwoPlusnPlus :=
GentoGr(eval(convert(NomTwoPlusnPlus, name)(NTwoPlusnPlus))); save
TempTwoPlusnPlus, NomfichierTwoPlusnPlus end if; if
TempTwoPlusnPlus={} then return GROUPE else return TempTwoPlusnPlus
end if; end proc;

```

```

TwoPlusnPlus := proc(NTwoPlusnPlus)
local NomgroupeTwoPlusnPlus, NomTwoPlusnPlus, NomfichierTwoPlusnPlus;
global TempTwoPlusnPlus;
    TempTwoPlusnPlus := {};
    NomgroupeTwoPlusnPlus := "TwoPlusnPlus";
    NomTwoPlusnPlus := cat("Gen_", NomgroupeTwoPlusnPlus);
    NomfichierTwoPlusnPlus := cat(PATHDATA, NomgroupeTwoPlusnPlus, "_",
        convert(NTwoPlusnPlus, string), ".txt");
    if Fichier_Existant(NomfichierTwoPlusnPlus) = true then
        read NomfichierTwoPlusnPlus
    else
        TempTwoPlusnPlus :=
            GentoGr(eval(convert(NomTwoPlusnPlus, name)(NTwoPlusnPlus)));
        save TempTwoPlusnPlus, NomfichierTwoPlusnPlus
    end if;
    if TempTwoPlusnPlus = {} then return GROUPE
    else return TempTwoPlusnPlus
    end if
end proc

```

10.7.6. Orbite d'un point par un groupe de matrices

```

> OrbPGen:=proc(XOrbPGen,LOrbPGen) local
    MOrbPGen,OrbOrbPGen,GOrbPGen; OrbOrbPGen:={}
    }; GOrbPGen:=GentoGr(LOrbPGen); for MOrbPGen in GOrbPGen do
    OrbOrbPGen:=Matrixunion(Multiply(MOrbPGen,XOrbPGen),OrbOrbPGen); end do;
return OrbOrbPGen; end proc;

    OrbPGen := proc(XOrbPGen, LOrbPGen)
local MOrbPGen, OrbOrbPGen, GOrbPGen;
    OrbOrbPGen := {};
    GOrbPGen := GentoGr(LOrbPGen);
    for MOrbPGen in GOrbPGen do OrbOrbPGen :=
        Matrixunion(LinearAlgebra : - Multiply(MOrbPGen, XOrbPGen), OrbOrbPGen)
    end do;
    return OrbOrbPGen
end proc

> OrbPG:=proc(XOrbPG,LOrbPG) local MOrbPG,OrbOrbPG; OrbOrbPG:={}; for
    MOrbPG in LOrbPG do
    OrbOrbPG:=Matrixunion(Multiply(MOrbPG,XOrbPG),OrbOrbPG); end do;
return OrbOrbPG; end proc;

    OrbPG := proc(XOrbPG, LOrbPG)
local MOrbPG, OrbOrbPG;
    OrbOrbPG := {};
    for MOrbPG in LOrbPG do OrbOrbPG :=
        Matrixunion(LinearAlgebra : - Multiply(MOrbPG, XOrbPG), OrbOrbPG)
    end do;
    return OrbOrbPG
end proc

```

```

> OrbPGenMul:=proc(XOrbPGenMul,LOrbPGenMul) local
MOrbPGenMul,OrbOrbPGenMul,GOrbPGenMul;OrbOrbPGenMul:=[];GOrbPGenMul:=G
entoGr(LOrbPGenMul); for MOrbPGenMul in GOrbPGenMul do
OrbOrbPGenMul:=[op(OrbOrbPGenMul),Multiply(MOrbPGenMul,XOrbPGenMul)];
end do; return OrbOrbPGenMul; end proc;

OrbPGenMul := proc(XOrbPGenMul, LOrbPGenMul)
local MOrbPGenMul, OrbOrbPGenMul, GOrbPGenMul;
OrbOrbPGenMul := [];
GOrbPGenMul := GentoGr(LOrbPGenMul);
for MOrbPGenMul in GOrbPGenMul do OrbOrbPGenMul :=
[op(OrbOrbPGenMul), LinearAlgebra : - Multiply(MOrbPGenMul, XOrbPGenMul)]
end do;
return OrbOrbPGenMul
end proc

> OrbPGMul:=proc(XOrbPGMul,LOrbPGMul) local
MOrbPGMul,OrbOrbPGMul;OrbOrbPGMul:=[]; for MOrbPGMul in LOrbPGMul do
OrbOrbPGMul:=[op(OrbOrbPGMul),Multiply(MOrbPGMul,XOrbPGMul)]; end do;
return OrbOrbPGMul; end proc;

OrbPGMul := proc(XOrbPGMul, LOrbPGMul)
local MOrbPGMul, OrbOrbPGMul;
OrbOrbPGMul := [];
for MOrbPGMul in LOrbPGMul do OrbOrbPGMul :=
[op(OrbOrbPGMul), LinearAlgebra : - Multiply(MOrbPGMul, XOrbPGMul)]
end do;
return OrbOrbPGMul
end proc

> OrbPv3:=proc(XOrbPv3,GroupeOrbPv3) local
XorbiteOrbPv3,DimOrbPv3,NomGroupeOrbPv3,GOrbPv3,MOrbPv3,NomfichierOrbP
v3,KOrbPv3; global OrbOrbPv3;
DimOrbPv3:=op(1,XOrbPv3);XorbiteOrbPv3:=Vector(DimOrbPv3,symbol=Orbite
);NomGroupeOrbPv3:=cat(GroupeOrbPv3," ",convert(DimOrbPv3,string));GOr
bPv3:=RepGroupe(GroupeOrbPv3,DimOrbPv3);NomfichierOrbPv3:=cat(PATHDATA
,NomGroupeOrbPv3,"_Orbite.txt"); if Fichier_Existant(NomfichierOrbPv3)
= true then read NomfichierOrbPv3 else OrbOrbPv3:={}; for MOrbPv3 in
GOrbPv3 do
OrbOrbPv3:=Matrixunion(Multiply(MOrbPv3,XorbiteOrbPv3),OrbOrbPv3); end
do; save OrbOrbPv3, NomfichierOrbPv3; end if; return
Matrixset(subs(seq(XorbiteOrbPv3[KOrbPv3]=XOrbPv3[KOrbPv3],KOrbPv3=1..
DimOrbPv3),OrbOrbPv3)); end proc;

OrbPv3 := proc(XOrbPv3, GroupeOrbPv3)
local XorbiteOrbPv3, DimOrbPv3, NomGroupeOrbPv3, GOrbPv3, MOrbPv3,
NomfichierOrbPv3, KOrbPv3;
global OrbOrbPv3;
DimOrbPv3 := op(1, XOrbPv3);
XorbiteOrbPv3 := Vector(DimOrbPv3, symbol = Orbite);
NomGroupeOrbPv3 := cat(GroupeOrbPv3, " ", convert(DimOrbPv3, string));
GOrbPv3 := RepGroupe(GroupeOrbPv3, DimOrbPv3);
NomfichierOrbPv3 := cat(PATHDATA, NomGroupeOrbPv3, "_Orbite.txt");

```



```

if Fichier_Existant(NomfichierOrbPv3) = true then read NomfichierOrbPv3
else
  OrbOrbPv3 := {};
  for MOrbPv3 in GOrbPv3 do OrbOrbPv3 :=
    Matrixunion(LinearAlgebra : -Multiply(MOrbPv3, XorbiteOrbPv3), OrbOrbPv3)
  end do;
  save OrbOrbPv3, NomfichierOrbPv3
end if;
return Matrixset(subs(
  seq(XorbiteOrbPv3_KOrbPv3 = XOrbPv3_KOrbPv3, KOrbPv3 = 1..DimOrbPv3),
  OrbOrbPv3))
end proc

```

10.8. Représentations à l'aide du théorème de l'élément primitif

10.8.1. Polynômes minimaux

```

> MinCosn:=proc(NMinCosdn) local pMinCosdn;
  IsSelfReciprocal(cyclotomic(NMinCosdn,a), a, 'pMinCosdn'); return
  subs(a=2*a,pMinCosdn); end proc;

  MinCosn := proc(NMinCosdn)
    local pMinCosdn;
    PolynomialTools : -IsSelfReciprocal(cyclotomic(NMinCosdn, a), a, 'pMinCosdn');
    return subs(a = 2 * a, pMinCosdn)
  end proc

> MinDiedn:=proc(NMinDiedn) local
  _SinMinDiedn, val_cosMinDiedn, val_sinMinDiedn, MinPolyMinDiedn, FieldMinDiedn;
  if NMinDiedn=1 then return ["NULL",0,1] end if; if NMinDiedn=2
  then return ["NULL",0,1] end if; if NMinDiedn=3 then return
  [4*a^2-3,-1/2,a] end if; if NMinDiedn=6 then return [4*a^2-3,1/2,a]
  end if; if NMinDiedn=4 then return ["NULL",0,1] end if; if NMinDiedn=8
  then return [-1+2*a^2,a,a] end if; if 'mod'(NMinDiedn, 4)=0 then
  FieldMinDiedn:=evala(Primfield({RootOf(MinCosn(NMinDiedn,a))},{
  }));
  val_cosMinDiedn:=subs(op([1,1,1],FieldMinDiedn)=a,op(2,FieldMinDiedn[2,1]));
  MinPolyMinDiedn:=subs(_Z=a,op([1,1,1,1],FieldMinDiedn));
  val_sinMinDiedn:=subs(op([1,1,1],FieldMinDiedn)=a,op(2,op(1,op(5,evala(Algfield(RootOf(1-op(2,FieldMinDiedn[2,1])^2-_Sin^2,_Sin))))));
  else
  FieldMinDiedn:=evala(Primfield(
  {RootOf(MinCosn(NMinDiedn,a)),RootOf(_SinMinDiedn^2+RootOf(MinCosn(NMinDiedn,a))^2-1,_SinMinDiedn)},{}));
  val_cosMinDiedn:=subs(op([1,1,1],FieldMinDiedn)=a,op(2,FieldMinDiedn[2,1]));
  val_sinMinDiedn:=subs(op([1,1,1],FieldMinDiedn)=a,op(2,FieldMinDiedn[2,2]));
  MinPolyMinDiedn:=subs(_Z=a,op([1,1,1,1],FieldMinDiedn)); end
  if; return [MinPolyMinDiedn,val_cosMinDiedn,val_sinMinDiedn]; end
  proc;

```

```

MinDiedn := proc(NMinDiedn)
local _SinMinDiedn, val_cosMinDiedn, val_sinMinDiedn, MinPolyMinDiedn,
FieldMinDiedn;
  if NMinDiedn = 1 then return ["NULL", 0, 1] end if ;
  if NMinDiedn = 2 then return ["NULL", 0, 1] end if ;
  if NMinDiedn = 3 then return [4 * a^2 - 3, -1/2, a] end if;
  if NMinDiedn = 6 then return [4 * a^2 - 3, 1/2, a] end if;
  if NMinDiedn = 4 then return ["NULL", 0, 1] end if ;
  if NMinDiedn = 8 then return [-1 + 2 * a^2, a, a] end if;
  if NMinDiedn mod 4 = 0 then
    FieldMinDiedn := evala(Primfield({RootOf(MinCosn(NMinDiedn, a)), {})));
    val_cosMinDiedn :=
      subs(op([1, 1, 1], FieldMinDiedn) = a, op(2, FieldMinDiedn_2, 1));
    MinPolyMinDiedn := subs(_Z = a, op([1, 1, 1, 1], FieldMinDiedn));
    val_sinMinDiedn := subs(op([1, 1, 1], FieldMinDiedn) = a, op(2, op(1,
      op(5, evala(Algfield(RootOf(1 - op(2, FieldMinDiedn_2, 1)^2 - _Sin^2, _Sin))))))
  else
    FieldMinDiedn := evala(Primfield({RootOf(MinCosn(NMinDiedn, a)),
      RootOf(_SinMinDiedn^2 + RootOf(MinCosn(NMinDiedn, a))^2 - 1, _SinMinDiedn)}
      , {})));
    val_cosMinDiedn :=
      subs(op([1, 1, 1], FieldMinDiedn) = a, op(2, FieldMinDiedn_2, 1));
    val_sinMinDiedn :=
      subs(op([1, 1, 1], FieldMinDiedn) = a, op(2, FieldMinDiedn_2, 2));
    MinPolyMinDiedn := subs(_Z = a, op([1, 1, 1, 1], FieldMinDiedn))
  end if;
  return [MinPolyMinDiedn, val_cosMinDiedn, val_sinMinDiedn]
end proc

```

10.8.2. Générateurs

```

> Gen_Diedn_alg := proc (NGen_Diedn_alg) local InfosDiedn; if 2 <
NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
[subs(cDiedn=InfosDiedn[2], sDiedn=InfosDiedn[3], {Matrix([[1, 0], [0,
-1]]), Matrix([[cDiedn, sDiedn], [sDiedn, -cDiedn]]))}, apply(Mipol ->
if Mipol="NULL" then return NULL else return Mipol end
if, InfosDiedn[1])]; elif NGen_Diedn_alg = 2 then return
[{Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))}]
else return [{Reflexion_Gen(Vector([0, 1]))}] end if end proc;

```

```

Gen_Diedn_alg := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3,
      {Matrix([[1, 0], [0, -1]]), Matrix([[cDiedn, sDiedn], [sDiedn, -cDiedn]])}), apply(
      proc(Mipol)
        option operator, arrow;
        if Mipol = "NULL" then return NULL else return Mipol end if
      end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then
    return [{Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))}]
  else return [{Reflexion_Gen(Vector([0, 1]))}]
  end if
end proc

```

```

> Gen_Diedn_alg_list := proc (NGen_Diedn_alg) local InfosDiedn; if 2 <
NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
[map(convert,subs(cDiedn=InfosDiedn[2],sDiedn=InfosDiedn[3],
{Matrix([[1, 0], [0, -1]]), Matrix([[cDiedn, sDiedn], [sDiedn,
-cDiedn]])}),listlist),apply(Mipol -> if Mipol="NULL" then return
NULL else return Mipol end if,InfosDiedn[1])]; elif NGen_Diedn_alg = 2
then return [map(convert,{Reflexion_Gen(Vector([0, 1])),
Reflexion_Gen(Vector([1, 0]))},listlist)] else return
[map(convert,{Reflexion_Gen(Vector([0, 1]))},listlist)] end if end
proc;

```

```

Gen_Diedn_alg_list := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [map(convert,subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3,
      {Matrix([[1, 0], [0, -1]]), Matrix([[cDiedn, sDiedn], [sDiedn, -cDiedn]])}), listlist),
    apply(proc(Mipol)
      option operator, arrow;
      if Mipol = "NULL" then return NULL else return Mipol end if
    end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then return [map(convert,
    {Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))}, listlist)]
  else return [map(convert, {Reflexion_Gen(Vector([0, 1]))}, listlist)]
  end if
end proc

```

```

> Gen_Diedn_Plus_alg := proc (NGen_Diedn_alg) local InfosDiedn; if 2 <
NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
[subs(cDiedn=InfosDiedn[2],sDiedn=InfosDiedn[3],{Matrix([[cDiedn,
sDiedn], [-sDiedn, cDiedn]])}),apply(Mipol -> if Mipol="NULL" then
return NULL else return Mipol end if,InfosDiedn[1])]; elif
NGen_Diedn_alg = 2 then return [{Reflexion_Gen(Vector([0, 1])),
Reflexion_Gen(Vector([1, 0]))}] else return
[{Reflexion_Gen(Vector([0, 1]))}] end if end proc;

```

```

Gen_Diedn_Plus_alg := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3,
      {Matrix([[cDiedn, sDiedn], [-sDiedn, cDiedn]]}), apply(proc(Mipol)
        option operator, arrow;
        if Mipol = "NULL" then return NULL else return Mipol end if
      end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then
    return [{Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))}]
  else return [{Reflexion_Gen(Vector([0, 1]))}]
  end if
end proc

> Gen_Diedn_Plus_alg_list := proc (NGen_Diedn_alg) local InfosDiedn; if
2 < NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
[map(convert,subs(cDiedn=InfosDiedn[2],sDiedn=InfosDiedn[3],{Matrix([[cDiedn,
sDiedn], [-sDiedn,cDiedn]])}),listlist),apply(Mipol -> if Mipol="NULL" then
return NULL else return Mipol end if,InfosDiedn[1])]; elif NGen_Diedn_alg = 2
then return [map(convert,{Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector(
[1, 0]))},listlist)] else return [map(convert,{Reflexion_Gen(Vector([0, 1]))},
listlist)] end if end proc;

Gen_Diedn_Plus_alg_list := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [map(convert,subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3,
      {Matrix([[cDiedn, sDiedn], [-sDiedn, cDiedn]]}), listlist), apply(proc(Mipol)
      option operator, arrow;
      if Mipol = "NULL" then return NULL else return Mipol end if
    end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then return [map(convert,
    {Reflexion_Gen(Vector([0, 1])), Reflexion_Gen(Vector([1, 0]))}, listlist)]
  else return [map(convert, {Reflexion_Gen(Vector([0, 1]))}, listlist)]
  end if
end proc

> Gen_2n_alg:=proc(kGen_2n_alg) local TempGen_2n_alg;
TempGen_2n_alg:=Gen_Diedn_alg(kGen_2n_alg); if nops(TempGen_2n_alg)=2
then return [Rned(TempGen_2n_alg[1],3),TempGen_2n_alg[2]] else return
[Rned(TempGen_2n_alg[1],3)]; end if; end proc;

Gen_2n_alg := proc(kGen_2n_alg)
local TempGen_2n_alg;
  TempGen_2n_alg := Gen_Diedn_alg(kGen_2n_alg);
  if nops(TempGen_2n_alg) = 2 then
    return [Rned(TempGen_2n_alg1, 3), TempGen_2n_alg2]
  else return [Rned(TempGen_2n_alg1, 3)]
  end if
end proc

```

```

> Gen_2n_alg_list:=proc(kGen_2n_alg) local TempGen_2n_alg;
  TempGen_2n_alg:=Gen_Diedn_alg(kGen_2n_alg); if nops(TempGen_2n_alg)=2
  then return
  [map(convert,Rned(TempGen_2n_alg[1],3),listlist),TempGen_2n_alg[2]]
  else return [map(convert,Rned(TempGen_2n_alg[1],3),listlist)]; end if;
end proc;

  Gen_2n_alg_list := proc(kGen_2n_alg)
local TempGen_2n_alg;
  TempGen_2n_alg := Gen_Diedn_alg(kGen_2n_alg);
  if nops(TempGen_2n_alg) = 2 then
    return [map(convert, Rned(TempGen_2n_alg1, 3), listlist), TempGen_2n_alg2]
  else return [map(convert, Rned(TempGen_2n_alg1, 3), listlist)]
  end if
end proc

> Gen_2Plusn_alg := proc (NGen_Diedn_alg) local InfosDiedn; if 2 <
  NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
  [subs(cDiedn=InfosDiedn[2],sDiedn=InfosDiedn[3],
  {Matrix([[1,0,0],[0,-1,0],[0,0,-1]]),Matrix([[cDiedn, sDiedn,0],
  [-sDiedn, cDiedn,0],[0,0,-1]])}),apply(Mipol -> if Mipol="NULL" then
  return NULL else return Mipol end if,InfosDiedn[1])]; elif
  NGen_Diedn_alg = 2 then return
  [
  {Matrix([[-1,0,0],[0,1,0],[0,0,-1]]),Matrix([[-1,0,0],[0,-1,0],[0,0,-
  1]])}] else return
  [
  {Matrix([[-1,0,0],[0,1,0],[0,0,-1]]),Matrix([[1,0,0],[0,1,0],[0,0,-1]
  ])}] end if end proc;

  Gen_2Plusn_alg := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3, {
    Matrix([[1, 0, 0], [0, -1, 0], [0, 0, -1]]),
    Matrix([[cDiedn, sDiedn, 0], [-sDiedn, cDiedn, 0], [0, 0, -1]])}), apply(proc(Mipol)
    option operator, arrow;
    if Mipol = "NULL" then return NULL else return Mipol end if
    end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then return
    [{Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, -1]]), Matrix([[-1, 0, 0], [0, -1, 0], [0, 0, -1]])}]
  else return
    [{Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, -1]]), Matrix([[1, 0, 0], [0, 1, 0], [0, 0, -1]])}]
  end if
end proc

> Gen_2Plusn_alg_list := proc (NGen_Diedn_alg) local InfosDiedn; if 2 <
  NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
  [map(convert,subs(cDiedn=InfosDiedn[2],sDiedn=InfosDiedn[3],
  {Matrix([[1,0,0],[0,-1,0],[0,0,-1]]),Matrix([[cDiedn, sDiedn,0],
  [-sDiedn, cDiedn,0],[0,0,-1]])}),listlist),apply(Mipol -> if
  Mipol="NULL" then return NULL else return Mipol end
  if,InfosDiedn[1])]; elif NGen_Diedn_alg = 2 then return
  [{[[[-1,0,0],[0,1,0],[0,0,-1]], [[-1,0,0],[0,-1,0],[0,0,-1]]}] else
  return [{[[[-1,0,0],[0,1,0],[0,0,-1]], [[1,0,0],[0,1,0],[0,0,-1]]}]
  end if end proc;

```

```

Gen_2Plusn_alg_list := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [map(convert, subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3, {
      Matrix([[1, 0, 0], [0, -1, 0], [0, 0, -1]]),
      Matrix([[cDiedn, sDiedn, 0], [-sDiedn, cDiedn, 0], [0, 0, -1]])), listlist), apply(proc(
        Mipol)
      option operator, arrow;
      if Mipol = "NULL" then return NULL else return Mipol end if
    end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then
    return [[[-1, 0, 0], [0, 1, 0], [0, 0, -1]], [[-1, 0, 0], [0, -1, 0], [0, 0, -1]]]
  else return [[[-1, 0, 0], [0, 1, 0], [0, 0, -1]], [[1, 0, 0], [0, 1, 0], [0, 0, -1]]]
  end if
end proc

> Gen_2nPlus_alg:=proc(kGen_2n_alg) local TempGen_2n_alg, GenGen_2n_alg;
  TempGen_2n_alg:=Gen_Diedn_alg(kGen_2n_alg); GenGen_2n_alg:=convert(Temp
  Gen_2n_alg[1], list); if kGen_2n_alg=1 then return
  [{Matrix([[1,0,0],[0,-1,0],[0,0,-1]])}] end if; if
  nops(TempGen_2n_alg)=2 then return
  [{DiagonalMatrix([GenGen_2n_alg[1], -1]), DiagonalMatrix([GenGen_2n_alg[
  2], -1]), DiagonalMatrix([Multiply(GenGen_2n_alg[1], GenGen_2n_alg[2]), 1]
  )}], TempGen_2n_alg[2]] else return
  [{DiagonalMatrix([GenGen_2n_alg[1], -1]), DiagonalMatrix([GenGen_2n_alg[
  2], -1]), DiagonalMatrix([Multiply(GenGen_2n_alg[1], GenGen_2n_alg[2]), 1]
  )}]; end if; end proc;

Gen_2nPlus_alg := proc(kGen_2n_alg)
local TempGen_2n_alg, GenGen_2n_alg;
  TempGen_2n_alg := Gen_Diedn_alg(kGen_2n_alg);
  GenGen_2n_alg := convert(TempGen_2n_alg1, list);
  if kGen_2n_alg = 1 then return [{Matrix([[1, 0, 0], [0, -1, 0], [0, 0, -1]])}] end if;
  if nops(TempGen_2n_alg) = 2 then return [{
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg1, -1]),
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg2, -1]),
    LinearAlgebra : -DiagonalMatrix(
      [LinearAlgebra : -Multiply(GenGen_2n_alg1, GenGen_2n_alg2), 1]),
    TempGen_2n_alg2]
  else return [{LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg1, -1]),
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg2, -1]),
    LinearAlgebra : -DiagonalMatrix(
      [LinearAlgebra : -Multiply(GenGen_2n_alg1, GenGen_2n_alg2), 1])}]
  end if
end proc

```

```

> Gen_2nPlus_alg_list:=proc(kGen_2n_alg) local
  TempGen_2n_alg, GenGen_2n_alg; TempGen_2n_alg:=Gen_Diedn_alg(kGen_2n_alg);
  GenGen_2n_alg:=convert(TempGen_2n_alg[1],list); if kGen_2n_alg=1 then
  return {[[1,0,0],[0,-1,0],[0,0,-1]]} end if; if nops(TempGen_2n_alg)=2
  then return [map(convert,{DiagonalMatrix([GenGen_2n_alg[1],-1]),
  DiagonalMatrix([GenGen_2n_alg[2],-1]),DiagonalMatrix([Multiply(
  GenGen_2n_alg[1],GenGen_2n_alg[2]),1])},listlist),TempGen_2n_alg[2])
  else return [map(convert,{DiagonalMatrix([GenGen_2n_alg[1],-1]),
  DiagonalMatrix([GenGen_2n_alg[2],-1]),DiagonalMatrix([Multiply(
  GenGen_2n_alg[1],GenGen_2n_alg[2]),1])},listlist)]; end if; end proc;

  Gen_2nPlus_alg_list := proc(kGen_2n_alg)
local TempGen_2n_alg, GenGen_2n_alg;
  TempGen_2n_alg := Gen_Diedn_alg(kGen_2n_alg);
  GenGen_2n_alg := convert(TempGen_2n_alg[1], list);
  if kGen_2n_alg = 1 then return {[[1, 0, 0], [0, -1, 0], [0, 0, -1]]} end if;
  if nops(TempGen_2n_alg) = 2 then return [map(convert, {
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg[1], -1]),
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg[2], -1]),
    LinearAlgebra : -DiagonalMatrix(
      [LinearAlgebra : -Multiply(GenGen_2n_alg[1], GenGen_2n_alg[2], 1)]}, listlist),
    TempGen_2n_alg[2])
  else return [map(convert, {
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg[1], -1]),
    LinearAlgebra : -DiagonalMatrix([GenGen_2n_alg[2], -1]),
    LinearAlgebra : -DiagonalMatrix(
      [LinearAlgebra : -Multiply(GenGen_2n_alg[1], GenGen_2n_alg[2], 1)]}, listlist)]
  end if
end proc

> Gen_2PlusnPlus_alg := proc (NGen_Diedn_alg) local InfosDiedn; if 2 <
  NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
  [subs(cDiedn=InfosDiedn[2], sDiedn=InfosDiedn[3], {Matrix([cDiedn,
  sDiedn, 0], [-sDiedn, cDiedn, 0], [0, 0, -1])})], apply(Mipol -> if
  Mipol="NULL" then return NULL else return Mipol end
  if, InfosDiedn[1])]; elif NGen_Diedn_alg = 2 then return
  [{Matrix([[-1, 0, 0], [0, -1, 0], [0, 0, -1]])}] else return
  [{Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, -1]])}] end if end proc;

  Gen_2PlusnPlus_alg := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [subs(cDiedn = InfosDiedn[2], sDiedn = InfosDiedn[3],
      {Matrix([cDiedn, sDiedn, 0], [-sDiedn, cDiedn, 0], [0, 0, -1])})], apply(proc(Mipol)
      option operator, arrow;
      if Mipol = "NULL" then return NULL else return Mipol end if
    end proc, InfosDiedn[1])
  elif NGen_Diedn_alg = 2 then return [{Matrix([[-1, 0, 0], [0, -1, 0], [0, 0, -1]])}]
  else return [{Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, -1]])}]
  end if
end proc

```

```

> Gen_2PlusnPlus_alg_list := proc (NGen_Diedn_alg) local InfosDiedn; if
2 < NGen_Diedn_alg then InfosDiedn:=MinDiedn(NGen_Diedn_alg); return
[map(convert,subs(cDiedn=InfosDiedn[2],sDiedn=InfosDiedn[3],
{Matrix([[cDiedn, sDiedn,0], [-sDiedn,
cDiedn,0],[0,0,-1]])}),listlist),apply(Mipol -> if Mipol="NULL" then
return NULL else return Mipol end if,InfosDiedn[1])]; elif
NGen_Diedn_alg = 2 then return {[[-1,0,0],[0,-1,0],[0,0,-1]]} else
return {[[-1,0,0],[0,1,0],[0,0,-1]]} end if end proc;

Gen_2PlusnPlus_alg_list := proc(NGen_Diedn_alg)
local InfosDiedn;
  if 2 < NGen_Diedn_alg then
    InfosDiedn := MinDiedn(NGen_Diedn_alg);
    return [map(convert,subs(cDiedn = InfosDiedn2, sDiedn = InfosDiedn3,
      {Matrix([[cDiedn, sDiedn, 0], [-sDiedn, cDiedn, 0], [0, 0, -1]])}), listlist), apply(proc(
        Mipol)
      option operator, arrow;
      if Mipol = "NULL" then return NULL else return Mipol end if
    end proc, InfosDiedn1)]
  elif NGen_Diedn_alg = 2 then return {[[-1, 0, 0], [0, -1, 0], [0, 0, -1]]}
  else return {[[-1, 0, 0], [0, 1, 0], [0, 0, -1]]}
  end if
end proc

> Gen_3Plus4_alg:=proc() ;return
[map(convert,
{map(L->map(rem,L,-10*a^2+a^4+1,a),op(expand(subs(sqrt(2)=1/2*a^3-9/2
*a,sqrt(3)=11/2*a-1/2*a^3,{[[-2/3, 1/3*sqrt(3), -1/3*sqrt(2)],
[1/3*sqrt(3), 0, -1/3*sqrt(2)*sqrt(3)], [-1/3*sqrt(2),
-1/3*sqrt(2)*sqrt(3), -1/3]]
}))),map(L->map(rem,L,-10*a^2+a^4+1,a),op(expand(subs(sqrt(2)=1/2*a^
3-9/2*a,sqrt(3)=11/2*a-1/2*a^3,{[[-1/2, -1/2*sqrt(3), 0],
[1/2*sqrt(3), -1/2, 0], [0, 0, 1]]}))),[[-1, 0, 0], [0, -1, 0], [0,
0, -1]]},Matrix),-10*a^2+a^4+1]; end proc;

Gen_3Plus4_alg := proc()
  return [map(convert, {[[-1, 0, 0], [0, -1, 0], [0, 0, -1]], map(
    L → map(rem, L, -10 * a2 + a4 + 1, a), op(expand(subs(sqrt(2) = 1/2 * a3 - 9/2 * a,
    sqrt(3) = 11/2 * a - 1/2 * a3, {[[-2/3, 1/3 * sqrt(3), -1/3 * sqrt(2)],
    [1/3 * sqrt(3), 0, -1/3 * sqrt(2) * sqrt(3)], [-1/3 * sqrt(2), -1/3 * sqrt(2) * sqrt(3), -1/3]]}))),
    map(L → map(rem, L, -10 * a2 + a4 + 1, a), op(expand(subs(
    sqrt(2) = 1/2 * a3 - 9/2 * a, sqrt(3) = 11/2 * a - 1/2 * a3,
    {[[-1/2, -1/2 * sqrt(3), 0], [1/2 * sqrt(3), -1/2, 0], [0, 0, 1]]}))))), Matrix),
    - 10 * a2 + a4 + 1]
  end proc

> Gen_3Plus4_alg_list:=proc() ;return
[map(L->map(rem,L,-10*a^2+a^4+1,a),op(expand(subs(sqrt(2)=1/2*a^3-9/2
*a,sqrt(3)=11/2*a-1/2*a^3,{[[-2/3, 1/3*sqrt(3), -1/3*sqrt(2)],
[1/3*sqrt(3), 0, -1/3*sqrt(2)*sqrt(3)], [-1/3*sqrt(2),
-1/3*sqrt(2)*sqrt(3), -1/3]]
}))),map(L->map(rem,L,-10*a^2+a^4+1,a),op(expand(subs(sqrt(2)=1/2*a^
3-9/2*a,sqrt(3)=11/2*a-1/2*a^3,{[[-1/2, -1/2*sqrt(3), 0],
[1/2*sqrt(3), -1/2, 0], [0, 0, 1]]}))),[[-1, 0, 0], [0, -1, 0], [0,
0, -1]]},-10*a^2+a^4+1]; end proc;

```



```

Gen_3Plus4_alg_list := proc()
  return [{map(L → map(rem, L, -10 * a2 + a4 + 1, a), op(expand(subs(
    sqrt(2) = 1/2 * a3 - 9/2 * a, sqrt(3) = 11/2 * a - 1/2 * a3,
    {[[-1/2, -1/2 * sqrt(3), 0], [1/2 * sqrt(3), -1/2, 0], [0, 0, 1]]}))), map(
    L → map(rem, L, -10 * a2 + a4 + 1, a), op(expand(subs(sqrt(2) = 1/2 * a3 - 9/2 * a,
    sqrt(3) = 11/2 * a - 1/2 * a3, {[[-2/3, 1/3 * sqrt(3), -1/3 * sqrt(2)],
    [1/3 * sqrt(3), 0, -1/3 * sqrt(2) * sqrt(3)], [-1/3 * sqrt(2), -1/3 * sqrt(2) * sqrt(3), -1/3]]}))),
    [[-1, 0, 0], [0, -1, 0], [0, 0, -1]]}, -10 * a2 + a4 + 1]
end proc

```

10.8.3. Outils

```

> Save_Gen:=proc(LSave_G,NomSave_G) local RepG; RepG:=LSave_G; save
RepG, cat(PATHDATA,NomSave_G,".txt"); print(cat("Generators saved in
file",PATHDATA,NomSave_G,".txt")); end proc;

Save_Gen := proc(LSave_G, NomSave_G)
  local RepG;
  RepG := LSave_G;
  save RepG, cat(PATHDATA, NomSave_G, ".txt");
  print(cat("Generators saved in file", PATHDATA, NomSave_G, ".txt"))
end proc

> Save_Minpol:=proc(MinpolSave_Minpol,NomSave_Minpol) local MINPOL;
MINPOL:=MinpolSave_Minpol; save MINPOL, cat(PATHDATA,NomSave_Minpol);
print(cat(PATHDATA,NomSave_Minpol)); end proc;

Save_Minpol := proc(MinpolSave_Minpol, NomSave_Minpol)
  local MINPOL;
  MINPOL := MinpolSave_Minpol;
  save MINPOL, cat(PATHDATA, NomSave_Minpol);
  print(cat(PATHDATA, NomSave_Minpol))
end proc

> ListListToEqs:=proc(ListEq,ListCont) local i_,j_; return
{seq(sum(aleph[i_] * ListEq[i_] [j_], i_=1..nops(ListEq))-ListCont[j_], j_
=1..nops(ListCont))}, {seq(aleph[i_], i_=1..nops(ListEq))}; end
proc;

ListListToEqs := proc(ListEq, ListCont)
  local i_, j_;
  return {seq(sum(ℵi_ * ListEqi_ j_, i_ = 1..nops(ListEq)) - ListContj_,
    j_ = 1..nops(ListCont))}, {seq(ℵi_, i_ = 1..nops(ListEq))}
end proc

```

```

> alglinreducev3:=proc(equations,unknowns,Minpol)
local i,k,t,eqn,eqns,var,vars,pivot,sol,sols,a,ans,x,minpol,solved;
if nargs>=3 then minpol:=Minpol
else if assigned(_known[group][minpol]) then
minpol:=_known[group][minpol] fi
fi;
a:=indets(minpol)[];
ans:=sum(x[i]*a^i, i=0..'if'(minpol=0,0,degree(minpol,a))-1);
if not(hastype(equations,radical)) then
eqns := equations minus {0};
for k while eqns <> {} do
if 2 < printlevel then
lprint('solve: SolveTools[Linear]: # equations
is:',nops(eqns))
fi;
eqn := eqns[1];
for i from 2 to nops(eqns) do
if nops(indets(eqn)) < nops(indets(eqns[i])) then eqn :=
eqns[i] fi
od;
vars := indets(eqn) intersect unknowns;
if vars = {} then break else var := vars[1]; fi;
eqns := eqns minus {eqn};
pivot := coeff(eqn,var,1);
for i from 2 to nops(vars) do
t := coeff(eqn,vars[i],1);
if length(t) < length(pivot) then pivot := t; var :=
vars[i] fi
od;
if 3 < printlevel then
lprint('solve: SolveTools[Linear]: best
unknown/equation',var,eqn)
fi;
if type(pivot,integer) then pivot:=1/pivot
else
{coeffs(rem(ans*pivot-1,minpol,a),a)};
SolveTools[Linear](%,{seq(x[i-1],
i=1..'if'(minpol=0,0,degree(minpol,a))));
pivot:=subs(%,ans)
end if;
eqn := -expand(rem(pivot*subs(var = 0,eqn),minpol,a));
sol[k] := var,eqn;
proc(x,m,a)
expand(rem(x,m,a));
if %>0 then %/icontent(%) fi
end;
eqns:=map(%,subs(var = eqn,eqns),minpol,a) minus {0}
;; od;
sols:={};
for i from k-1 by -1 to 1 do
if 2 < printlevel then
lprint('solve: Linear: backsubstitution at:',i)
fi;
var := sol[i][1];
eqn := sol[i][2];
eqn := expand(rem(subs(sols,eqn),minpol,a));
sols := sols union {var = eqn}
od;
map(x->x=x,unknowns minus map(lhs,sols)) union sols
else
eqns := map(radnormal,equations minus {0},'rationalized');
for k while eqns <> {} do
if 2 < printlevel then
lprint('solve: SolveTools[Linear]: # equations
is:',nops(eqns))
fi;
eqn := eqns[1];
for i from 2 to nops(eqns) do

```

```

if nops(indets(eqn)) < nops(indets(eqns[i])) then eqn :=
eqns[i] fi
od;
vars := indets(eqn) intersect unknowns;
if vars = {} then break else var := vars[1]; fi;
eqns := eqns minus {eqn};
pivot := coeff(eqn,var,1);
for i from 2 to nops(vars) do
t := coeff(eqn,vars[i],1);
if length(t) < length(pivot) then pivot := t; var :=
vars[i] fi
od;
if 3 < printlevel then
lprint('solve: SolveTools[Linear]: best
unknown/equation',var,eqn)
fi;
if type(pivot,integer) then pivot:=1/pivot
else
if hastype(pivot,radical) and not(has(pivot,a)) then
pivot:=radnormal(1/pivot,'rationalized') else
if not(hastype(pivot,radical)) and has(pivot,a) then
{coeffs(rem(ans*pivot-1,minpol,a),a)};
SolveTools[Linear](%,{seq(x[i-1],
i=1.. 'if'(minpol=0,0,degree(minpol,a))});
pivot:=subs(%,ans)
else
if hastype(pivot,radical) and has(pivot,a) then
{coeffs(rem(ans*pivot-1,minpol,a),a)};
SolveTools[Linear](%,{seq(x[i-1],
i=1.. 'if'(minpol=0,0,degree(minpol,a))});
pivot:=radnormal(subs(%,ans),'rationalized')
end if; end if; end if; end if;
eqn := -expand(rem(pivot*subs(var = 0,eqn),minpol,a));
sol[k] := var,eqn;
proc(x,m,a)
expand(rem(x,m,a));
if %>0 then %/iccontent(%) fi
end;
eqns:=map(radnormal,map(%,subs(var = eqn,eqns),minpol,a)
minus {0},'rationalized')
;; od;
sols:={};
for i from k-1 by -1 to 1 do
if 2 < printlevel then
lprint('solve: Linear: backsubstitution at:',i)
fi;
var := sol[i][1];
eqn := sol[i][2];
eqn :=
map(radnormal,expand(rem(subs(sols,eqn),minpol,a)),'rationalized');
sols := sols union {var = eqn}
od;
map(x->x=x,unknowns minus map(lhs,sols)) union sols
end if; end proc;

```

```

alginreducev3 := proc(equations, unknowns, Minpol)
local i, k, t, eqn, eqns, var, vars, pivot, sol, sols, a, ans, x, minpol, solved;
  if 3 ≤ nargs then minpol := Minpol
  else
    if assigned(__known_group_minpol) then minpol := __known_group_minpol end if
  end if;
  a := indets(minpol);
  ans := sum( $x_i * a^i$ , i = 0..‘if’(minpol = 0, 0, degree(minpol, a)) - 1);
  if not hastype(equations, radical) then
    eqns := equations minus {0};
    for k while eqns ≠ {} do
      if 2 < printlevel then lprint(‘solve : SolveTools[Linear] : # equations is : ‘, nops(eqns))
      end if;
      eqn := eqns1;
      for i from 2 to nops(eqns) do
        if nops(indets(eqn)) < nops(indets(eqnsi)) then eqn := eqnsi end if
      end do;
      vars := indets(eqn) intersect unknowns;
      if vars = {} then break else var := vars1 end if;
      eqns := eqns minus {eqn};
      pivot := coeff(eqn, var, 1);
      for i from 2 to nops(vars) do
        t := coeff(eqn, varsi, 1);
        if length(t) < length(pivot) then pivot := t; var := varsi end if
      end do;
      if 3 < printlevel then
        lprint(‘solve : SolveTools[Linear] : best unknown/equation’, var, eqn)
      end if;
      if type(pivot, integer) then pivot := 1/pivot
      else
        {coeffs(rem(ans * pivot - 1, minpol, a), a)};
        SolveTools_SolveTools:-Linear(% ,
          {seq( $x_{i-1}$ , i = 1..‘if’(minpol = 0, 0, degree(minpol, a)))));
        pivot := subs(% , ans)
      end if;
      eqn := -expand(rem(pivot * subs(var = 0, eqn), minpol, a));
      solk := var, eqn;
      proc(x, m, a) expand(rem(x, m, a)); if % ≠ 0 then %/icontent(%) end if end proc;
      eqns := map(% , subs(var = eqn, eqns), minpol, a) minus {0}
    end do;
    sols := {};
    for i from k - 1 by -1 to 1 do
      if 2 < printlevel then lprint(‘solve : Linear : backsubstitution at : ‘, i) end if;
      var := soli1;
      eqn := soli2;
      eqn := expand(rem(subs(sols, eqn), minpol, a));

```

```

    sols := sols union {var = eqn}
end do;
map(x → x = x, unknowns minus map(lhs, sols)) union sols
else
eqns := map(radnormal, equations minus {0}, 'rationalized');
for k while eqns ≠ {} do
    if 2 < printlevel then lprint('solve : SolveTools[Linear] : # equations is : ', nops(eqns))
    end if;
    eqn := eqns1;
    for i from 2 to nops(eqns) do
        if nops(indets(eqn)) < nops(indets(eqnsi)) then eqn := eqnsi end if
    end do;
    vars := indets(eqn) intersect unknowns;
    if vars = {} then break else var := vars1 end if;
    eqns := eqns minus {eqn};
    pivot := coeff(eqn, var, 1);
    for i from 2 to nops(vars) do
        t := coeff(eqn, varsi, 1);
        if length(t) < length(pivot) then pivot := t; var := varsi end if
    end do;
    if 3 < printlevel then
        lprint('solve : SolveTools[Linear] : best unknown/equation', var, eqn)
    end if;
    if type(pivot, integer) then pivot := 1/pivot
    else
        if hastype(pivot, radical) and not has(pivot, a) then
            pivot := radnormal(1/pivot, 'rationalized')
        else
            if not hastype(pivot, radical) and has(pivot, a) then
                {coeffs(rem(ans * pivot - 1, minpol, a), a)};
                SolveTools SolveTools:-Linear(% ,
                    {seq(xi-1, i = 1..if(minpol = 0, 0, degree(minpol, a))});
                pivot := subs(% , ans)
            else
                if hastype(pivot, radical) and has(pivot, a) then
                    {coeffs(rem(ans * pivot - 1, minpol, a), a)};
                    SolveTools SolveTools:-Linear(% ,
                        {seq(xi-1, i = 1..if(minpol = 0, 0, degree(minpol, a))});
                    pivot := radnormal(subs(% , ans), 'rationalized')
                end if
            end if
        end if;
    end if;
    eqn := -expand(rem(pivot * subs(var = 0, eqn), minpol, a));
    solk := var, eqn;
proc(x, m, a) expand(rem(x, m, a)); if % ≠ 0 then %/icontent(%) end if end proc;

```

```

eqns :=
  map(radnormal, map(%, subs(var = eqn, eqns), minpol, a) minus {0}, 'rationalized')
end do;
sols := {};
for i from k - 1 by -1 to 1 do
  if 2 < printlevel then lprint('solve : Linear : backsubstitution at : ', i) end if;
  var := soli1;
  eqn := soli2;
  eqn := map(radnormal, expand(rem(subs(sols, eqn), minpol, a)), 'rationalized');
  sols := sols union {var = eqn}
end do;
map(x → x = x, unknowns minus map(lhs, sols)) union sols
end if
end proc

```

10.8.4. Groupes

```

> Diedn_alg:=proc (NDiedn_alg) local NomgroupeDiedn_alg, NomDiedn_alg,
NomfichierDiedn_alg; global TempDiedn_alg; TempDiedn_alg:={};
NomgroupeDiedn_alg := "Diedn_alg"; NomDiedn_alg :=
cat("Gen_", NomgroupeDiedn_alg); NomfichierDiedn_alg :=
cat(PATHDATA, NomgroupeDiedn_alg, "_", convert(NDiedn_alg, string), ".txt")
; if Fichier_Existant(NomfichierDiedn_alg) = true then read
NomfichierDiedn_alg else TempDiedn_alg :=
GentoGr(eval(convert(NomDiedn_alg, name)(NDiedn_alg))); save
TempDiedn_alg, NomfichierDiedn_alg end if; if TempDiedn_alg={} then
return GROUPE else return TempDiedn_alg end if; end proc;

Diedn_alg := proc(NDiedn_alg)
local NomgroupeDiedn_alg, NomDiedn_alg, NomfichierDiedn_alg;
global TempDiedn_alg;
TempDiedn_alg := {};
NomgroupeDiedn_alg := "Diedn_alg";
NomDiedn_alg := cat( "Gen_", NomgroupeDiedn_alg);
NomfichierDiedn_alg :=
cat(PATHDATA, NomgroupeDiedn_alg, "_", convert(NDiedn_alg, string), ".txt");
if Fichier_Existant(NomfichierDiedn_alg) = true then read NomfichierDiedn_alg
else
TempDiedn_alg := GentoGr(eval(convert(NomDiedn_alg, name)(NDiedn_alg)));
save TempDiedn_alg, NomfichierDiedn_alg
end if;
if TempDiedn_alg = {} then return GROUPE else return TempDiedn_alg end if
end proc

> Diedn_alg:=proc(NTwon) ; read(cat(PATHDATA, "[", NTwon, "]_alg.txt"));
return GROUPE; end proc;

Diedn_alg := proc(NTwon)
read cat(PATHDATA, "[", NTwon, "]_alg.txt"); return GROUPE
end proc

```

```

> DiednPlus_alg:=proc(NTwon) ;
  read(cat(PATHDATA,"[",NTwon,""]+_alg.txt)); return GROUPE; end
proc;

  DiednPlus_alg := proc(NTwon)
    read cat(PATHDATA, "[", NTwon, ""+_alg.txt"); return GROUPE
  end proc

> Twon_alg:=proc(NTwon) ; read(cat(PATHDATA,"[2,",NTwon,""]+_alg.txt));
  return GROUPE; end proc;

  Twon_alg := proc(NTwon)
    read cat(PATHDATA, "[2,", NTwon, ""+_alg.txt"); return GROUPE
  end proc

> TwoPlusn_alg:=proc(NTwon) ;
  read(cat(PATHDATA,"[2+,",NTwon,""]+_alg.txt)); return GROUPE; end
proc;

  TwoPlusn_alg := proc(NTwon)
    read cat(PATHDATA, "[2+,", NTwon, ""+_alg.txt"); return GROUPE
  end proc

> TwonPlus_alg:=proc(NTwon) ;
  read(cat(PATHDATA,"[2,",NTwon,""]+_alg.txt)); return GROUPE; end
proc;

  TwoPlusnPlus_alg := proc(NTwon)
    read cat(PATHDATA, "[2,", NTwon, ""+_alg.txt"); return GROUPE
  end proc

> TwoPlusnPlus_alg:=proc(NTwon) ;
  read(cat(PATHDATA,"[2+,",NTwon,""]+_alg.txt)); return GROUPE; end
proc;

  TwoPlusnPlus_alg := proc(NTwon)
    read cat(PATHDATA, "[2+,", NTwon, ""+_alg.txt"); return GROUPE
  end proc

```

10.9. Compléments

```

> FarrGao:=proc(ListOrderedPoints,Ordering) local G,n,k,l,Vali,j,h ;
  G:={1}; n:=nops(ListOrderedPoints); for k from 1 by 1 to n do l:=1;
  for l while eval(apply(G[j],ListOrderedPoints[k]))=0 do l:=l+1; end
  do; Vali:=eval(apply(G[l],ListOrderedPoints[k])); for j from i+1 by 1
  to nops(G) do
  G[j]:=G[j]-eval(apply(G[j],ListOrderedPoints[k]))/Vali*G[i]; end do;
  G:=subsop(i=NULL,G); for j from 1 by 1 to m do if
  map(divide,map(LeadingTerm,G),x[j]*LeadingTerm(g[i], Ordering)) then
  h:=normal((x[j]-a[j])*g[i],G); G:=InsertOrder(h,G); end if; end do;
  end do; return G; end proc;

```

```

FarrGao := proc(ListOrderedPoints, Ordering)
local G, n, k, l, Vali, j, h;
  G := {1};
  n := nops(ListOrderedPoints);
  for k to n do
    l := 1;
    for l while eval(apply(Gj, ListOrderedPointsk)) = 0 do l := l + 1 end do;
    Vali := eval(apply(Gl, ListOrderedPointsk));
    for j from i + 1 to nops(G) do
      Gj := Gj - eval(apply(Gj, ListOrderedPointsk)) * Gi / Vali
    end do;
    G := subsop(i = NULL, G);
    for j to m do
      ifmap(divide, map(PolynomialIdeals : -LeadingTerm, G),
        xj * PolynomialIdeals : -LeadingTerm(gi, Ordering))then
        h := normal((xj - aj) * gi, G); G := InsertOrder(h, G)
      end if
    end do
  end do;
  return G
end proc

```


> TroisPlusQuatre();

$$\left\{ \begin{aligned} & \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{3}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix}, \\ & \begin{bmatrix} -\frac{5}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{5}{6} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}, \\ & \begin{bmatrix} -\frac{1}{6} & -\frac{\sqrt{3}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{5}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \\ & \begin{bmatrix} \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ 0 & -1 & 0 \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ & \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \\ & \begin{bmatrix} \frac{5}{6} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ & \begin{bmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} & -\frac{2\sqrt{2}}{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & -\frac{\sqrt{3}}{6} & -\frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}, \\ & \left. \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ 0 & 1 & 0 \\ -\frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix} \right\}
 \end{aligned} \right\}$$

> evala(Primfield({RootOf(X^2-2,X),RootOf(X^2-3,X)},{}));

$$\begin{aligned} & [[\%1 = \text{RootOf}(_Z^2 - 2) + \text{RootOf}(_Z^2 - 3)], \\ & [\text{RootOf}(_Z^2 - 2) = \frac{1}{2} \%1^3 - \frac{9}{2} \%1, \text{RootOf}(_Z^2 - 3) = \frac{11}{2} \%1 - \frac{1}{2} \%1^3]] \\ & \%1 := \text{RootOf}(-10_Z^2 + _Z^4 + 1) \end{aligned}$$

> evala(Primfield({RootOf(X^2-2),RootOf(X^2-3)},{}));

$$\begin{aligned} & [[\%1 = \text{RootOf}(_Z^2 - 2) + \text{RootOf}(_Z^2 - 3)], \\ & [\text{RootOf}(_Z^2 - 2) = \frac{1}{2} \%1^3 - \frac{9}{2} \%1, \text{RootOf}(_Z^2 - 3) = \frac{11}{2} \%1 - \frac{1}{2} \%1^3]] \\ & \%1 := \text{RootOf}(-10_Z^2 + _Z^4 + 1) \end{aligned}$$

> Gen_3Plus4_alg_list();

$$\begin{aligned} & [\{[-1, 0, 0], [0, -1, 0], [0, 0, -1]], \\ & [[\frac{-2}{3}, \frac{11}{6}a - \frac{1}{6}a^3, -\frac{1}{6}a^3 + \frac{3}{2}a], [\frac{11}{6}a - \frac{1}{6}a^3, 0, -\frac{a^2}{6} + \frac{5}{6}], [-\frac{1}{6}a^3 + \frac{3}{2}a, -\frac{a^2}{6} + \frac{5}{6}, \frac{-1}{3}]], \\ & [[\frac{-1}{2}, -\frac{11}{4}a + \frac{1}{4}a^3, 0], [\frac{11}{4}a - \frac{1}{4}a^3, \frac{-1}{2}, 0], [0, 0, 1]], -10a^2 + a^4 + 1] \end{aligned}$$

Chapitre 11

Une étude de cas

11.1. Cas de la dimension 3

11.1.1. Les résultats de Hardin et Sloane

Dans ce chapitre nous nous intéressons à une série de dispositifs qui ont été construits ou étudiés par Hardin et Sloane dans leur article [HS96].

L'objectif des deux auteurs a été de déterminer, dans le cas de la dimension 3, quelle était la force maximale possible pour un plan sphérique de cardinal donné ainsi que de proposer un dispositif possédant cette propriété. Les inégalités obtenues par Delsarte, Seidel et Goethals, [DGS77], donnent une première majoration de la force possible d'un plan, voir la proposition 4.3.1. Toutefois les résultats d'existence plus précis, voir la section 4.4, obtenus par Bannai montrent que les « tight designs » sont rares même en dimension 3.

Les résultats obtenus par Hardin et Sloane sont résumés dans le tableau 7.1 :

- La première colonne N correspond au cardinal des plans sphériques considérés.
- La deuxième colonne $\tau_3(N)$ correspond à la force maximale qu'il a été possible d'obtenir avec un plan sphérique de cardinal N .
- La troisième colonne « Proof » indique comment le dispositif maximisant la force et décrit par les colonnes suivantes a été obtenu. V1 indique qu'il existe une preuve de nature algébrique de la force de ce dispositif. Une valeur entre crochet fait référence à l'article où l'existence d'un dispositif de même taille et de même force a été prouvée. V2 indique que la preuve de la force de ce dispositif est basée sur une méthode par intervalles pour s'assurer que les équations polynomiales définissant les conditions de force $\tau_3(N)$ ont bien une solution. Enfin V3 indique que la discrétance Δ du plan est inférieure à 10^{-26} .
- La quatrième colonne « Group » correspond au groupe de symétries utilisé pour construire les orbites dont le plan sphérique est l'union.
- La cinquième colonne « Order » est égale à l'ordre du groupe apparaissant dans la colonne « Group » .
- La sixième colonne « Orbits (Description) » précise le cardinal des orbites à assembler pour obtenir le plan sphérique. Si la configuration obtenue correspond à un polytope remarquable, son nom est indiqué entre parenthèses.

Rappelons qu'un t -plan sphérique de \mathbb{R}^3 muni de sa structure euclidienne canonique est un ensemble de N points $\mathcal{P} = \{\mathbf{P}_1, \dots, \mathbf{P}_N\}$ de la sphère unité $\mathcal{S}^3 = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\}$ tel que l'égalité

$$\int_{\mathcal{S}^3} f(\mathbf{x}) d\mu(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N f(\mathbf{P}_i), \quad (11.1.1)$$

où μ est la mesure de probabilité uniforme sur \mathcal{S}^3 , soit vérifiée pour tout polynôme de $\mathbb{R}[\mathbf{x}]$ de degré inférieur ou égal à t .

Étant donné un polynôme f , nous notons $\Delta_f(\mathcal{P})$ la valeur absolue de la différence entre le membre de gauche et le membre de droite de l'égalité 11.1.1.

Si nous appelons \mathcal{F}_d l'ensemble des monômes unitaires de degré total égal à d , il suffit de vérifier la validité de l'équation 11.1.1 sur l'ensemble $\mathcal{F} = \bigcup_{1 \leq d \leq t} \mathcal{F}_d$, ce qui justifie l'introduction de la discrèpance, notée $\Delta(\mathcal{P})$, définie par :

$$\Delta(\mathcal{P}) = \sum_{f \in \mathcal{F}} \Delta_f(\mathcal{P})^2. \quad (11.1.2)$$

Hardin et Sloane ont mis en libre disposition les coordonnées approchées de ces plans sphériques. Toutefois comme nous l'avons précisé dans les chapitres précédents ces informations ne peuvent être utilisées qu'à des fins de vérification de nos calculs.

TAB. 11.1: « Conjectured values of $\tau_3(N)$, the largest t for which an N -point configuration on the sphere in 3 dimensions forms a spherical t -designs. »

N	$\tau_3(N)$	Proof	Group	Order	Orbits (Description)
1	0	V1	∞	∞	1 (single point)
2	1	V1	∞	∞	2 (two antipodal points)
3	1	V1	[2,3]	12	3 (equilateral triangle)
4	2	V1	[3,3]	24	4 (regular tetrahedron)
5	1	V1	[2,3]	12	3 + 2 (triangular bipyramid)
6	3	V1	[3,4]	48	6 (regular octahedron)
7	2	[23]	[3]	6	$3^2 + 1$
8	3	V1	[3,4]	48	8 (cube)
9	2	[23]	[2,3]	12	6 + 3 (triangular bipyramid)
10	3	[2]	[2 ⁺ ,10]	20	10 (pentagonal prism)
11	3	[2]	[2,3] ⁺	6	6 + 3 + 2
12	5	V1	[3,5]	120	12 (regular icosahedron)
13	3	[2]	[4]	8	$4^3 + 1$
14	4	[16]	[2,3] ⁺	6	$6^2 + 2$
15	3	[2]	[2,5]	20	10 + 5
16	5	[16]	[3,3] ⁺	12	12 + 4 (hexakis truncated tetrahedron)

11.1 Cas de la dimension 3

N	$\tau_3(N)$	Proof	Group	Order	Orbits (Description)
17	4	[16]	$[2,3]^+$	6	$6^2 + 3 + 2$
18	5	[25]	$[2^+,6]$	12	$12 + 6$
19	4	[16]	$[3]$	6	$6^2 + 3^2 + 1$
20	5	V1	$[3,5]$	120	20 (regular dodecahedron)
21	4	[16]	$[2,3]$	12	$12 + 6 + 3$
22	5	[25]	$[2^+,10]$	20	$10^2 + 2$
23	5	V2	$[2,3]^+$	6	$6^3 + 3 + 2$
24	7	[22]	$[3,4]^+$	24	24 (improved snub cube)
25	5	V1	$[2,5]^+$	10	$10^2 + 5$
26	6	V3	$[2,3]^+$	6	$6^4 + 2$
27	5	[25]	$[2,3]$	12	$12^2 + 3$
28	6	V3	$[2^+,4]$	8	$8^3 + 4$
29	6	V3	$[2^+]$	2	$2^{14} + 1$
30	7	V1	$[3,4]^+$	24	$24 + 6$ (tetrakis snub cube)
31	6	V3	$[5^+]$	5	$5^6 + 1$
32	7	V1	$[3,4]^+$	24	$24 + 8$ (snub cube + cube)
33	6	V3	$[2,3]^+$	6	
34	7	V3	$[2,4]^+$	8	
35	6	V3	$[2,5]^+$	10	$10^3 + 5$
36	8	V3	$[3,3]^+$	12	12^3 (three snub tetrahedra)
37	7	V3	$[3^+]$	3	
38	7	V3	$[3,4]^+$	24	$24 + 8 + 6$
39	7	V3	$[2,3]^+$	6	
40	8	V3	$[3,3]^+$	12	$12^3 + 4$
41	7	V3	$[2,3]^+$	6	
42	8	V3	$[2,4]^+$	8	
43	7	V3	$[6^+]$	6	
44	8	V3	$[3,3]^+$	12	$12^3 + 4^2$
45	8	V3	$[2^+]$	2	
46	8	V3	$[2,4]^+$	8	
47	8	V3	$[2,3]^+$	6	
48	9	V1	$[3,4]^+$	24	24^2 (two snub cubes)
49	8	V3	$[4^+]$	4	
50	9	V3	$[2,6]^+$	12	$12^4 + 2$
51	8	V3	$[2,3]^+$	6	
52	9	V3	$[3,3]^+$	12	$12^4 + 4$
53	8	V3	$[2,3]^+$	6	
54	9	V3	$[3,4]^+$	24	$24^2 + 6$
55	9	V3	$[2^+]$	2	
56	9	V3	$[3^+,4]$	24	$24^2 + 8$
57	9	V3	$[2,3]^+$	6	
58	9	V3	$[2,4]^+$	8	
59	9	V3	$[2,3]^+$	6	
60	10	V3	$[3,3]^+$	12	12^5 (five snub tetrahedra)

N	$\tau_3(N)$	Proof	Group	Order	Orbits (Description)
61	9	V3	$[6^+]$	6	
62	10	V3	$[2,3]^+$	6	
63	9	V3	$[2,7]^+$	14	$14^4 + 7$
64	10	V3	$[3,3]^+$	12	$12^5 + 4$
65	10	V3	$[2^+]$	2	
66	10	V3	$[2,4]^+$	8	
67	10	V3	$[2^+]$	2	
68	10	V3	$[2^+, 4]$	8	
69	10	V3	$[4^+]$	4	
70	11	V3	$[2,5]^+$	10	10^7
71	10	V3	$[2,3^+]$	6	
72	11	V3	$[3,5]^+$	60	$60 + 12$ (pentakis truncated icosahedron)
73	10	V3	$[4^+]$	4	
74	11	V3	$[2,6]^+$	12	$12^6 + 2$
75	11	V3	$[2^+]$	2	
76	11	V3	$[3,3]^+$	12	$12^6 + 4$
77	11	V3	$[4^+]$	4	
78	11	V3	$[3,4]^+$	24	$24^3 + 6$
79	11	V3	$[2^+]$	2	
80	11	V3	$[3,5]^+$	60	$60 + 20$ (hexakis truncated icosahedron)
81	11	V3	$[4^+]$	4	
82	11	V3	$[2^+, 10^+]$	10	$10^8 + 2$
83	11	V3	$[2,3]^+$	6	
84	12	V3	$[3,3]^+$	12	12^7 (seven snub tetrahedra)
85	11	V3	$[2,5]^+$	10	
86	12	V3	$[2,2]^+$	4	
87	12	V3	$[1^+]$	1	
88	12	V3	$[3,3]^+$	12	$12^7 + 4$
89	12	V3	$[2^+]$	2	
90	12	V3	$[2,4]^+$	8	
91	12	V3	$[2^+]$	2	
92	12	V3	$[3,3]^+$	12	$12^7 + 4^2$
93	12	V3	$[4^+]$	4	
94	13	V3	$[2^+, 2^+]$	2	
95	12	V3	$[2^+]$	2	
96	13	V3	$[3,3]^+$	12	12^8 (eight snub tetrahedra)
97	12	V3	$[4^+]$	4	
98	13	V3	$[2,4]^+$	8	
99	12	V3	$[2]$	4	
100	13	V3	$[3,3]^+$	12	$12^8 + 4$

Nous résumons ici les informations concernant la nature de la preuve de la force des dispositifs de la liste précédente.

- Pour des plans de cardinal N de 1 à 22, 24, 25, 27, 30, 32 et 48 les résultats obtenus l'ont été par des preuves algébriques.
- L'existence du dispositif à 23 points de force 5 proposé repose sur l'utilisation d'une méthode par intervalle.
- Tous les autres dispositifs, $N = 28, 29, 31, 33$ à 47, 49 à 100, ont été retenus pour leur discrédance inférieure à 10^{-26} .

11.1.2. Utilisation des informations de Hardin et Sloane

Rappelons que nous avons caractérisé la nature radiale de la variance de l'estimateur de la réponse moyenne à l'aide de la fonction génératrice des moments. Un plan est un $2t$ –plan sphérique si et seulement si la partie polynomiale limitée au degré $2t$ de la fonction génératrice des moments est radiale. Tous les plans du tableau 11.1 dont la force est supérieure à $2t$ sont donc des plans pour lesquels la partie polynomiale limitée au degré $2t$ de la fonction génératrice des moments est radiale.

Théorème 11.1.1. *Les dispositifs de cardinal $N=4, 6$ à 100 sont des dispositifs isovariants lorsqu'ils sont analysés avec un modèle polynomial d'ordre 1.*

Les dispositifs de cardinal $N=12, 14, 16$ à 100, augmentés d'au moins un point au centre, sont des dispositifs fortement isovariants lorsqu'ils sont analysés avec un modèle polynomial d'ordre 2.

Démonstration 11.1.1. En premier, les dispositifs dont le support est un plan sphérique de force supérieure ou égale à deux et analysés à l'aide d'un modèle d'ordre 1 sont isovariants.

Considérons en second lieu un dispositif expérimental analysé à l'aide d'un modèle d'ordre 2. Tous les plans sphériques pour lesquels $\tau_3(N) \geq 2$, augmentés d'au moins un point au centre, sont des dispositifs isovariants.

Enfin précisons que certains des dispositifs proposés par Hardin et Sloane sont de force impaire $2t + 1$ ce qui n'est pas utile pour les considérations d'isovariance. Considérer un dispositif de force paire $2t$ de même cardinal permettra l'ajout d'une recherche d'optimalité alphabétique.

11.2. Présentation des résultats

Il s'agit d'un résumé de la partie calculatoire de la thèse. Nous suivons la démarche proposée dans le chapitre 5 qui permet d'aboutir à la construction explicite de chacun des plans isovariants ainsi que l'étude de leurs propriétés. Nous appliquons également certaines des idées que nous avons rappelées dans le chapitre 6 : nous menons en particulier l'analyse des confusions d'effets sur la plupart des exemples proposés.

11.3. N1

```
> N1:=Matrix([[1,0,0]]);
                                      $N1 := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ 
> UpdatePlan(N1,"N1");
                                     "Update complete"
> pointplot3d(N1,symbol=box,scaling=constrained,shading=Z);
```

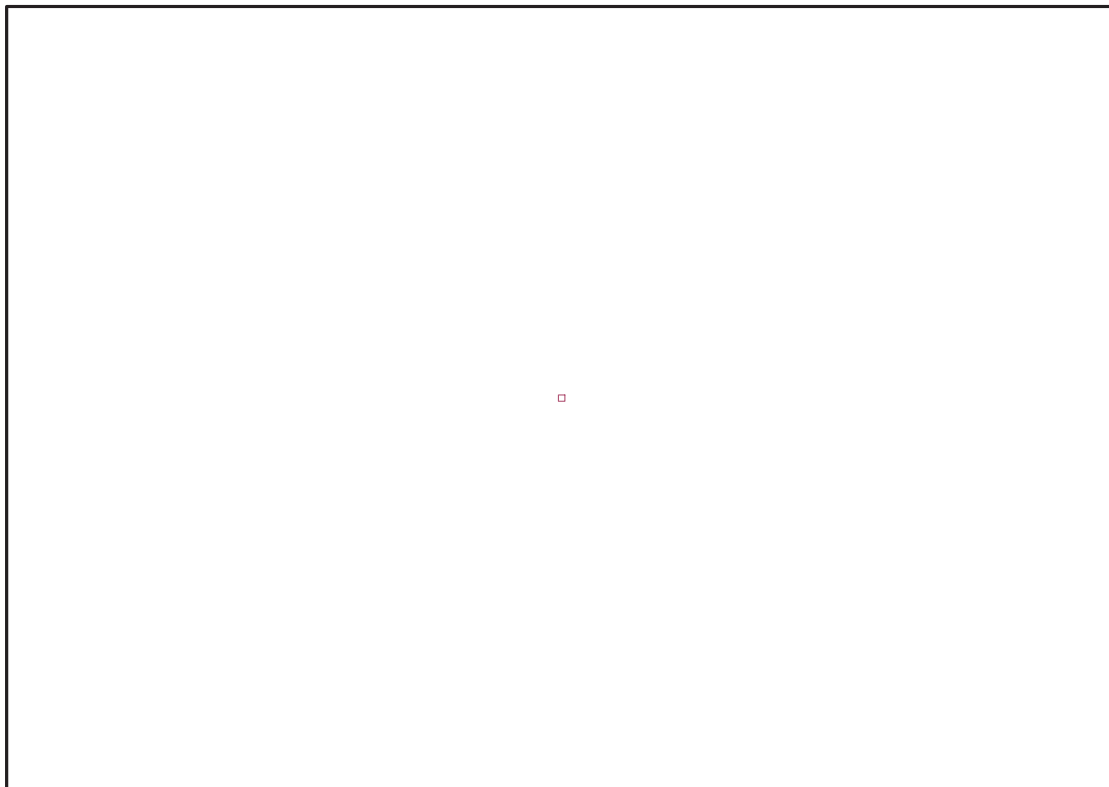


FIG. 11.1 – Graphique : Support du plan N1.

```
> IdealN1:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N1));
> Gens:=Generators(IdealN1);
                                      $IdealN1 := \langle -1_2 + y, -1_3 + z, -1_1 + x \rangle$ 
                                      $Gens := \{-1_3 + z, -1_1 + x, -1_2 + y\}$ 
> info_iop:=3;
                                      $info\_iop := 3$ 
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N1),1
istlist),Vector)));
                                     true
                                     1
                                     1
                                     1
```

$$\begin{array}{c} 1 \\ [[z, y, x - 1], [1]] \end{array}$$

11.3.1. tdeg(x,y,z)

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),tdeg(x,y,z,t));

$$Frontiere_tdeg := [z, y, x]$$

> Div:=Diviseurs({op(Frontiere_tdeg)});nops(Div);

$$\begin{array}{c} Div := \{1\} \\ 1 \end{array}$$

> EstSet_N1_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N1_tdeg:=sort(EstSet_N1_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N1_tdeg);

$$\begin{array}{c} EstSet_N1_tdeg := [1] \\ 1 \end{array}$$

> X_D_ModeleSat(N1,[x,y,z],EstSet_N1_tdeg);

$$\begin{bmatrix} 1 \end{bmatrix}$$

> M_D_ModeleSat(N1,[x,y,z],EstSet_N1_tdeg);

$$\begin{bmatrix} 1 \end{bmatrix}$$

> Var_Blue_Modele_Sat(N1,[x,y,z],EstSet_N1_tdeg);

$$1$$

11.3.2. plex(x,y,z)

> Frontiere_plex:=map(LeadingMonomial,{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z));

$$Frontiere_plex := \{x, z, y\}$$

> Div:=Diviseurs({op(Frontiere_plex)});nops(Div);

$$\begin{array}{c} Div := \{1\} \\ 1 \end{array}$$

> EstSet_N1_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N1_plex:=sort(EstSet_N1_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N1_plex);

$$\begin{array}{c} EstSet_N1_plex := [1] \\ 1 \end{array}$$

> X_D_ModeleSat(N1,[x,y,z],EstSet_N1_plex);

$$\begin{bmatrix} 1 \end{bmatrix}$$

> M_D_ModeleSat(N1,[x,y,z],EstSet_N1_plex);

$$\begin{bmatrix} 1 \end{bmatrix}$$

> Var_Blue_Modele_Sat(N1,[x,y,z],EstSet_N1_plex);

$$1$$

11.3.3. Variance

Modèle de degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N1,1,0);  
      "M_D is not full rank"  
Var := VarVar_Norm_Blue_D_Centre
```

11.4. N2

```
> N2:=Matrix([[1,0,0],[-1,0,0]]);  

$$N2 := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$
  
> UpdatePlan(N2,"N2");  
"Update complete"  
> pointplot3d(N2,symbol=box,scaling=constrained,shading=Z);
```

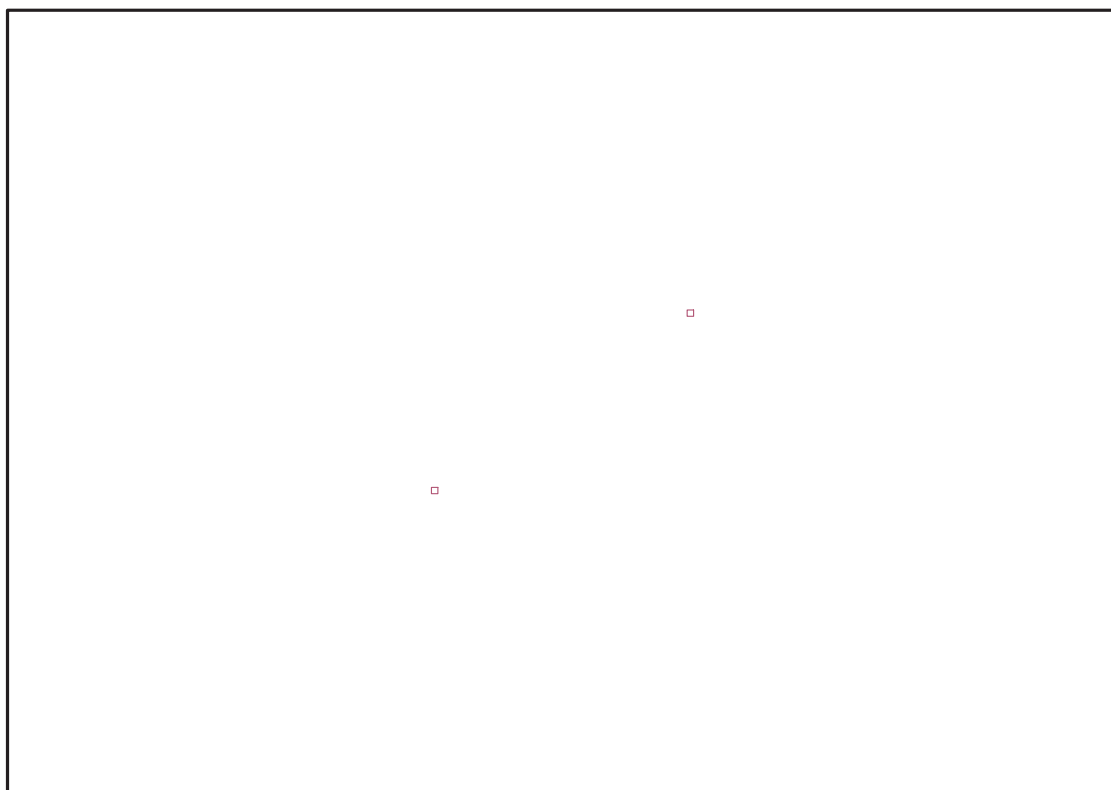


FIG. 11.2 – Graphique : Support du plan N2.

```
> Graph:=convhull(op(convert(N2,listlist)));  
Graph := POLYTOPE(3, 1, 2, 2)  
> plotdes:=draw(Graph):  
> display(plotdes,scaling=constrained,shading=Z);  
  
> IdealN2:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N2));  
> Gens:=Generators(IdealN2);  

$$IdealN2 := \langle y, z, x^2 - 1 \rangle$$

```

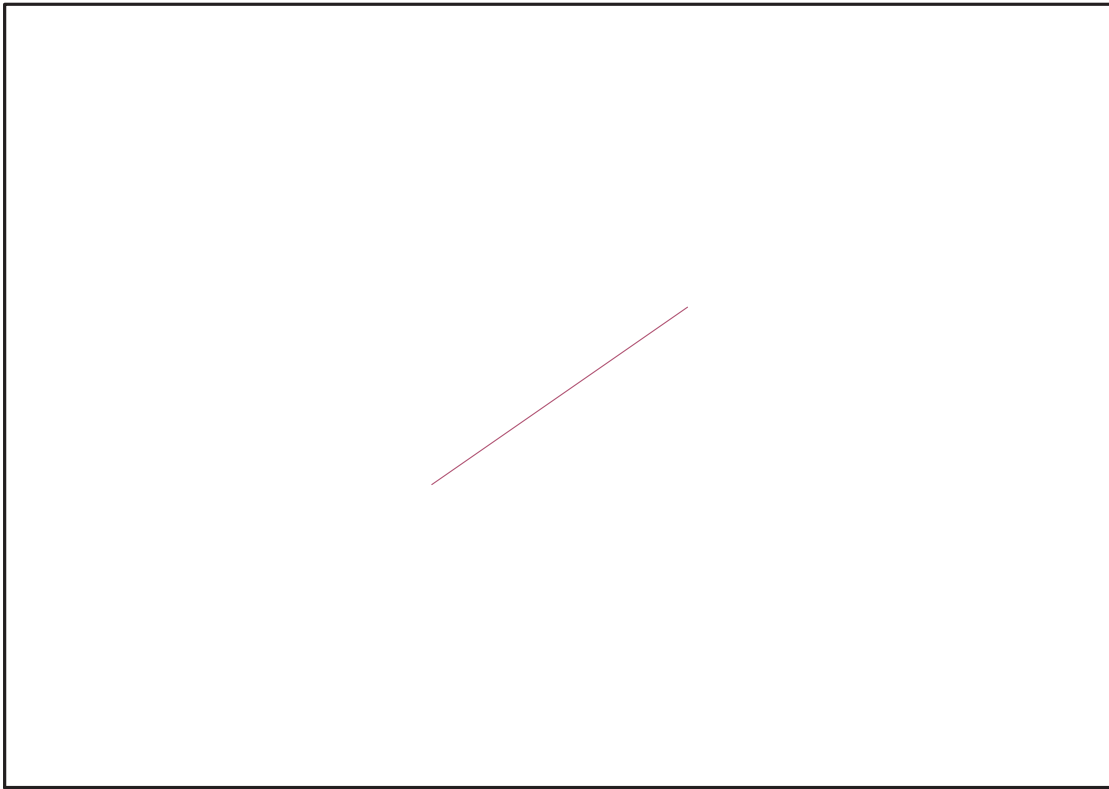


FIG. 11.3 – Graphique : Enveloppe convexe du support du plan N2.

$$Gens := \{z, x^2 - 1, y\}$$

```
> info_iop:=3;
```

info_iop := 3

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N2),1
istlist),Vector)));
```

true

1

1

1

2

2

$[[z, y, x^2 - 1], [1, x]]$

11.4.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t));
```

Frontiere_tdeg := $[z, y, x^2]$

```
> Div:=Diviseurs({op(Frontiere_tdeg)});nops(Div);
```

Div := $\{1, x\}$

2

```
> EstSet_N2_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N2_plex:=sort(EstSet_N2_tdeg,(t1
,t2)->testorder(t1,t2,T));nops(EstSet_N2_tdeg);
```

$$\text{EstSet_N2_plex} := [1, x]$$

$$2$$

```
> X_D_ModeleSat(N2,[x,y,z],EstSet_N2_tdeg);
```

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

```
> M_D_ModeleSat(N2,[x,y,z],EstSet_N2_tdeg);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> Var_Blue_Modele_Sat(N2,[x,y,z],EstSet_N2_tdeg);
```

$$1 + x^2$$

11.4.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z));
```

$$\text{Frontiere_plex} := \{z, y, x^2\}$$

```
> Div:=Diviseurs({op(Frontiere_plex)});nops(Div);
```

$$\text{Div} := \{1, x\}$$

$$2$$

```
> EstSet_N2_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,
z):T:=termorder(A,plex(x,y,z)):EstSet_N2_plex:=sort(EstSet_N2_plex,(t1
,t2)->testorder(t1,t2,T));nops(EstSet_N2_plex);
```

$$\text{EstSet_N2_plex} := [1, x]$$

$$2$$

```
> X_D_ModeleSat(N2,[x,y,z],EstSet_N2_tdeg);
```

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

```
> M_D_ModeleSat(N2,[x,y,z],EstSet_N2_tdeg);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> Var_Blue_Modele_Sat(N2,[x,y,z],EstSet_N2_tdeg);
```

$$1 + x^2$$

11.4.3. Variance

Modèle de degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N2,1,0);
```

“M_D is not full rank”

$$\text{Var} := \text{VarVar_Norm_Blue_D_Centre}$$

11.5. N3

```
> GrN3_Total:=GentoGr(
  {Reflexion_Gen(Vector([1/2,sqrt(3)/2,0])),Reflexion_Gen(Vector([-1/2,
    sqrt(3)/2,0])),Reflexion_Gen(Vector([0,0,1]))}):

> GrN3_Gen:=GentoGr(
  {MatrixMatrixMultiply(Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(
    Vector([-1/2,sqrt(3)/2,0])),Reflexion_Gen(Vector([0,0,1]))}):
```

$$GrN3_Gen := \left\{ \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> N3:=OrbToMatrix(OrbPG(Vector([1,0,0]),GrN3_Gen));
```

$$N3 := \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

```
> CreatePlan(N3,"N3");
```

```
> pointplot3d(N3,symbol=box,scaling=constrained,shading=Z);
```

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N3,listlist)))],'rat
  ional',5))));
```

$$Graph := \text{POLYTOPE}(3, 2, 3, 3)$$

```
> plotdes:=draw(Graph):display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN3:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N3));
```

```
> Gens:=Generators(IdealN3);
```

$$IdealN3 := \langle 2yx + y, z, 2y^2 + x - 1, 2x^2 - x - 1 \rangle$$

$$Gens := \{2yx + y, z, 2x^2 - x - 1, 2y^2 + x - 1\}$$

```
> info_iop:=3;
```

$$info_iop := 3$$

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N3),l
  istlist),Vector)));
```

$$\begin{matrix} 1 \\ 1 \end{matrix}$$



FIG. 11.4 – Graphique : Support du plan N3.

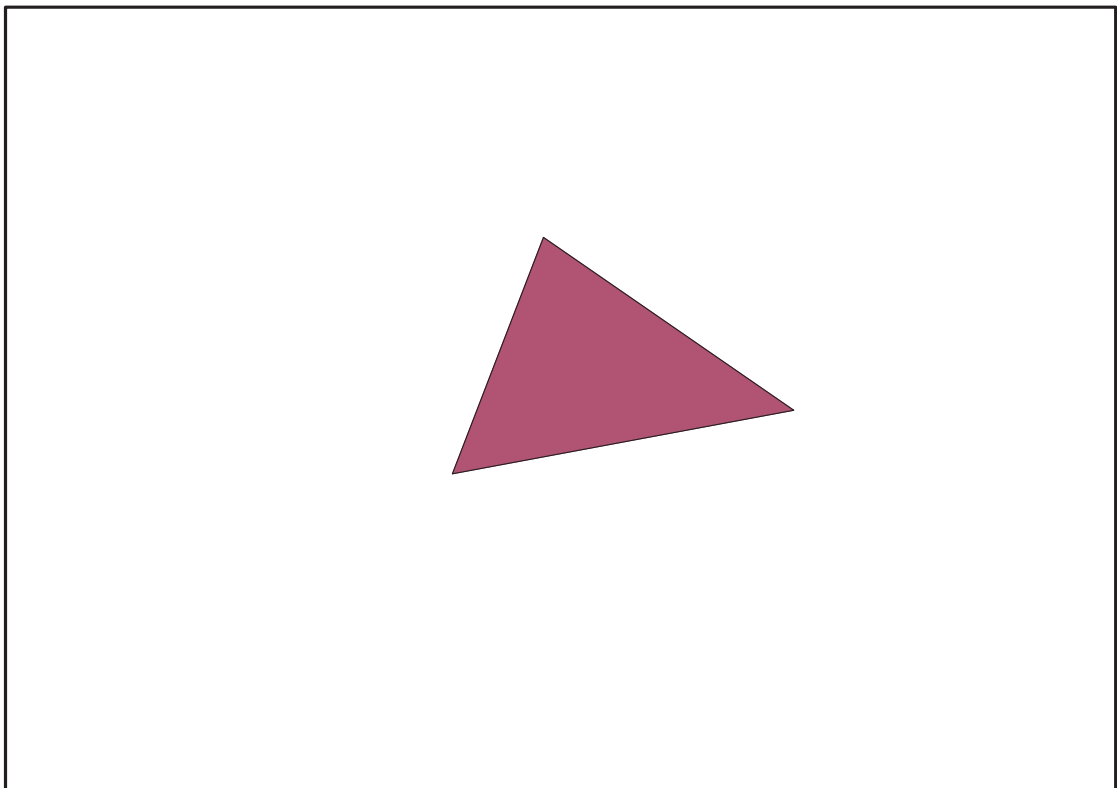


FIG. 11.5 – Graphique : Enveloppe convexe du support du plan N3.

$$\begin{array}{c} 2 \\ 3 \\ 3 \\ 3 \\ 3 \end{array} \quad \left[\left[z, y^2 - \frac{1}{2} + \frac{x}{2}, yx + \frac{1}{2}y, x^2 - \frac{1}{2} - \frac{1}{2}x \right], [1, y, x] \right]$$

11.5.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),tdeg(x,y,z,t)):
> Div:=Diviseurs({op(Frontiere_tdeg)}):nops(Div):
EstSet_N3_tdeg:=op(EstSet(Div,Frontiere_tdeg)):A:=poly_algebra(x,y,z):
T:=termorder(A,tdeg(x,y,z)):EstSet_N3_tdeg:=sort(EstSet_N3_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N3_tdeg);
```

$$\begin{array}{c} EstSet_N3_tdeg := [1, y, x] \\ 3 \end{array}$$

```
> X_D_ModeleSat(N3,[x,y,z],EstSet_N3_tdeg);
```

$$\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & \frac{-1}{2} \\ 1 & 0 & 1 \\ 1 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$

```
> M_D_ModeleSat(N3,[x,y,z],EstSet_N3_tdeg);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N3,[x,y,z],EstSet_N3_tdeg));
```

$$1 + 2y^2 + 2x^2$$

11.5.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
> EstSet_N3_plex:=op(EstSet(Div,Frontiere_plex)):A:=poly_algebra(x,y,z):
T:=termorder(A,plex(x,y,z)):EstSet_N3_plex:=sort(EstSet_N3_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N3_plex);
```

$$\begin{array}{c} EstSet_N3_plex := [1, y, y^2] \\ 3 \end{array}$$

```
> X_D_ModeleSat(N3,[x,y,z],EstSet_N3_plex);
```

$$\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & \frac{3}{4} \\ 1 & 0 & 0 \\ 1 & \frac{\sqrt{3}}{2} & \frac{3}{4} \end{bmatrix}$$

```
> M_D_ModeleSat(N3,[x,y,z],EstSet_N3_plex);
```

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{3}{8} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N3,[x,y,z],EstSet_N3_plex));
```

$$3 - 6y^2 + 8y^4$$

11.5.3. Variance

Modèle de degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N3,1,0);
```

“M_D is not full rank”

$$Var := VarVar_Norm_Blue_D_Centre$$

11.6. N4

```
> GrN4:=GentoGr(  
  {MatrixMatrixMultiply(Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(Ve  
    ctor([-1/2,sqrt(3)/2,0]))),Reflexion_Gen(Vector([2/3*2^(1/2), 0,  
    -4/3]))});
```

$$\begin{aligned}
GrN_4 := & \left\{ \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{3}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{5}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \right. \\
& \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \\
& \begin{bmatrix} -\frac{1}{6} & -\frac{\sqrt{3}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{5}{6} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \\
& \begin{bmatrix} -\frac{1}{6} & \frac{\sqrt{3}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
& \begin{bmatrix} \frac{5}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \\
& \begin{bmatrix} -\frac{2}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ 0 & -1 & 0 \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
& \begin{bmatrix} -\frac{2}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} & -\frac{\sqrt{3}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{5}{6} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \end{bmatrix}, \\
& \left. \begin{bmatrix} \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ 0 & 1 & 0 \\ \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \end{bmatrix} \right\}
\end{aligned}$$

```
> N4Coord:=Matrix([[2*sqrt(2)/3,0,-1/3],[-sqrt(2)/3,sqrt(6)/3,-1/3],[-sqrt(2)/3,-sqrt(6)/3,-1/3],[0,0,1]]):N4:=OrbToMatrix(OrbPG(Vector([0,0,1]),GrN4));
```

$$N_4 := \begin{bmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & \frac{-1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & \frac{-1}{3} \\ \frac{2\sqrt{2}}{3} & 0 & \frac{-1}{3} \end{bmatrix}$$

```
> CreatePlan(N4,"N4");
> pointplot3d(N4,symbol=box,scaling=constrained,shading=Z);
```

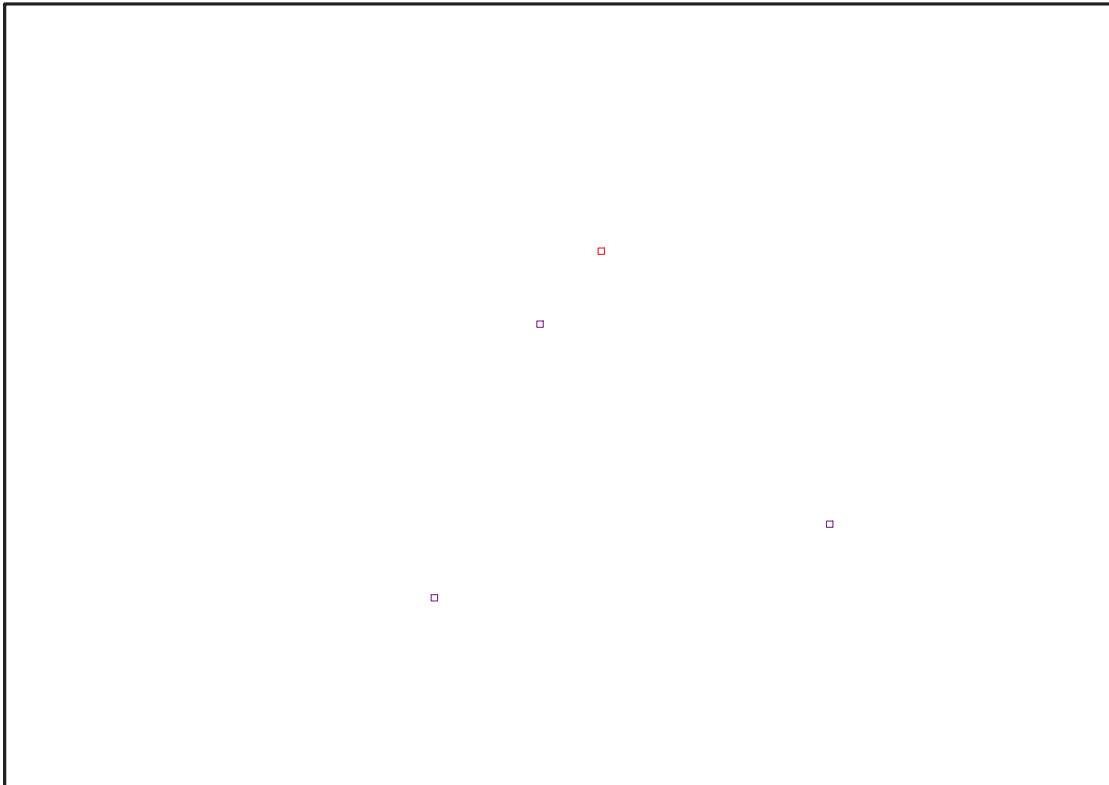


FIG. 11.6 – Graphique : Support du plan N4.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N4,listlist)))],'rational',5))));
```

$Graph := \text{POLYTOPE}(3, 3, 4, 4)$

```
> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z);
```

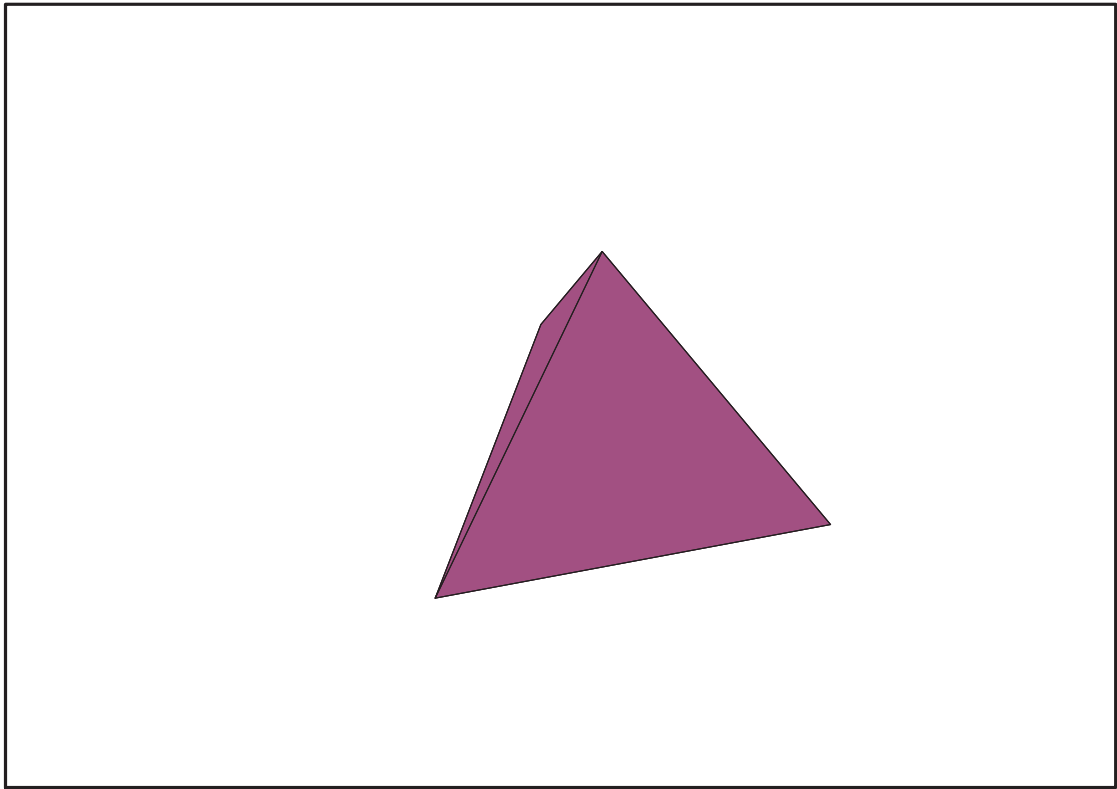


FIG. 11.7 – Graphique : Enveloppe convexe du support du plan N4.

```

> IdealN4:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N4));
> Gens:=Generators(IdealN4);

      IdealN4 :=
      ⟨3zy + y, 3yx + y√2, 3x² + z - √2x - 1, x + 3zx, 3z² - 2z - 1, 3y² + z + √2x - 1⟩

      Gens := {3z² - 2z - 1, 3x² + z - √2x - 1, 3y² + z + √2x - 1, 3yx + y√2, 3zy + y, x + 3zx
      }

> info_iop:=3;

                        info_iop := 3

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N4),1
istlist),Vector)));

1
2
3
4
4
4
4
4
4
4

[[z² - 1/3 - 2/3 z, zy + 1/3 y, zx + 1/3 x, y² - 1/3 + √2x/3 + z/3, yx + 1/3 y√2, x² - 1/3 + 1/3 z - 1/3 √2x],
[1, z, y, x]]

```


11.6.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)):
> Div:=Diviseurs({op(Frontiere_tdeg)}):nops(Div):
EstSet_N4_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,z
):T:=termorder(A,tdeg(x,y,z)):EstSet_N4_tdeg:=sort(EstSet_N4_tdeg,(t1,
t2)->testorder(t1,t2,T));nops(EstSet_N4_tdeg);
```

$$\text{EstSet_N4_tdeg} := [1, z, y, x]$$

4

```
> X_D_ModeleSat(N4,[x,y,z],EstSet_N4_tdeg);
```

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ 1 & \frac{-1}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \\ 1 & 1 & 0 & 0 \\ 1 & \frac{-1}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{\sqrt{2}}{3} \end{bmatrix}$$

```
> M_D_ModeleSat(N4,[x,y,z],EstSet_N4_tdeg);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N4,[x,y,z],EstSet_N4_tdeg));
```

$$1 + 3z^2 + 3y^2 + 3x^2$$

11.6.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):Div:=Diviseurs(
{op(Frontiere_plex)}):nops(Div):
EstSet_N4_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,z
):T:=termorder(A,plex(x,y,z)):EstSet_N4_plex:=sort(EstSet_N4_plex,(t1,
t2)->testorder(t1,t2,T));nops(EstSet_N4_plex);
```

$$\text{EstSet_N4_plex} := [1, z, y, y^2]$$

4

```
> X_D_ModeleSat(N4,[x,y,z],EstSet_N4_plex);
```

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 & 0 \\ 1 & \frac{-1}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & \frac{2}{3} \\ 1 & 1 & 0 & 0 \\ 1 & \frac{-1}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & \frac{2}{3} \end{bmatrix}$$

```
> M_D_ModeleSat(N4,[x,y,z],EstSet_N4_plex);
```

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{-1}{9} \\ 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{-1}{9} & 0 & \frac{2}{9} \end{bmatrix}$$

> expand(Var_Blue_Modele_Sat(N4,[x,y,z],EstSet_N4_plex));

$$\frac{5}{2} - 3z - 6y^2 + \frac{9}{2}z^2 + 9zy^2 + \frac{27}{2}y^4$$

11.6.3. Variance

Modèle de degré 1

> Var:=expand(Var_Norm_Blue_D_Centre(N4,1,0));

$$Var := 1 + 3\,var_3^2 + 3\,var_2^2 + 3\,var_1^2$$

11.7. N5

```
> GrN5_Total:=GentoGr(
  {Reflexion_Gen(Vector([1/2,sqrt(3)/2,0])),Reflexion_Gen(Vector([-1/2,
    sqrt(3)/2,0])),Reflexion_Gen(Vector([0,0,1]))}):

> GrN5_Gen:=GentoGr(
  {MatrixMatrixMultiply(Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(
    Vector([-1/2,sqrt(3)/2,0])),Reflexion_Gen(Vector([0,0,1]))}):
```

$$GrN5_Gen := \left\{ \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

```
> N5:=OrbToMatrix(OrbPG(Vector([1,0,0]),GrN5_Gen) union
  OrbPG(Vector([0,0,-1]),GrN5_Gen));
```

$$N5 := \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> UpdatePlan(N5,"N5");
```

“Update complete”

```
> pointplot3d(N5,symbol=box,scaling=constrained,shading=Z);
```

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N5,listlist)))]),'rat
  ional',5)));
```

$$Graph := \text{POLYTOPE}(3, 3, 5, 6)$$

```
> plotdes:=draw(Graph);
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN5:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N5));
```

```
> Gens:=Generators(IdealN5);
```

$$IdealN5 := \langle 2x^3 - x^2 - x, 2yx + y, z^2 + 2x^2 - x - 1, zx, zy, y^2 - x^2 + x \rangle$$

$$Gens := \{2yx + y, zx, 2x^3 - x^2 - x, y^2 - x^2 + x, z^2 + 2x^2 - x - 1, zy\}$$

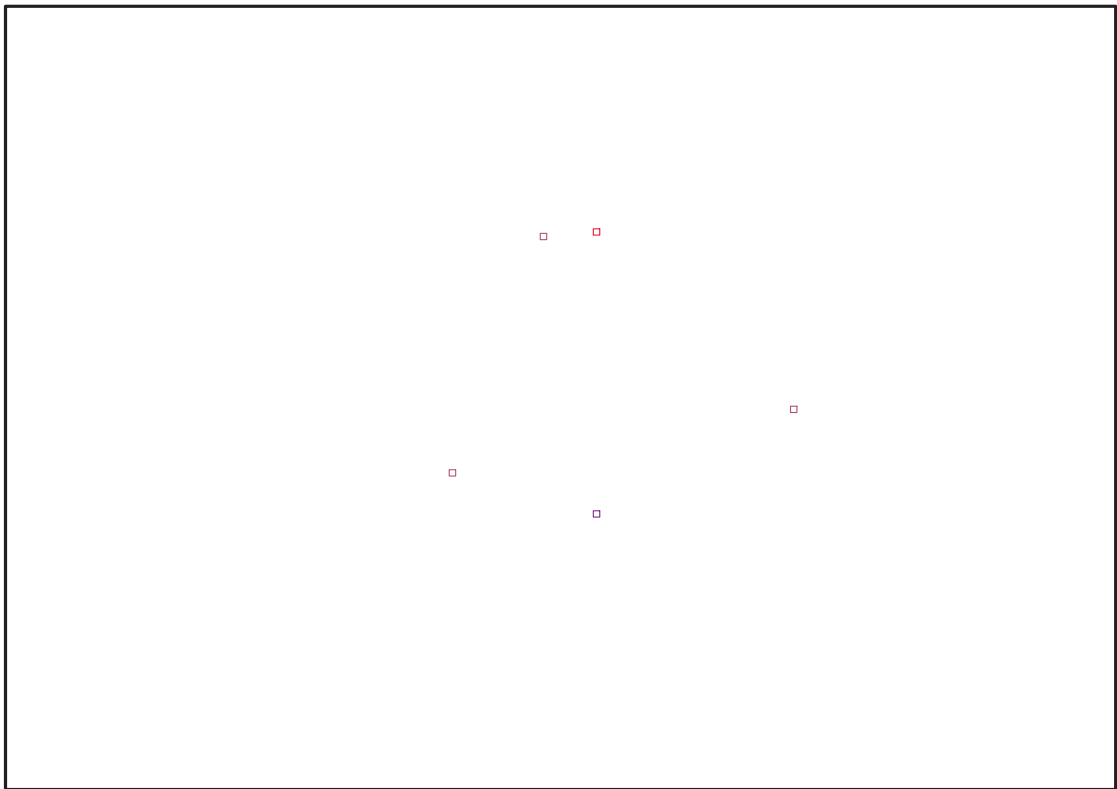


FIG. 11.8 – Graphique : Support du plan N5.

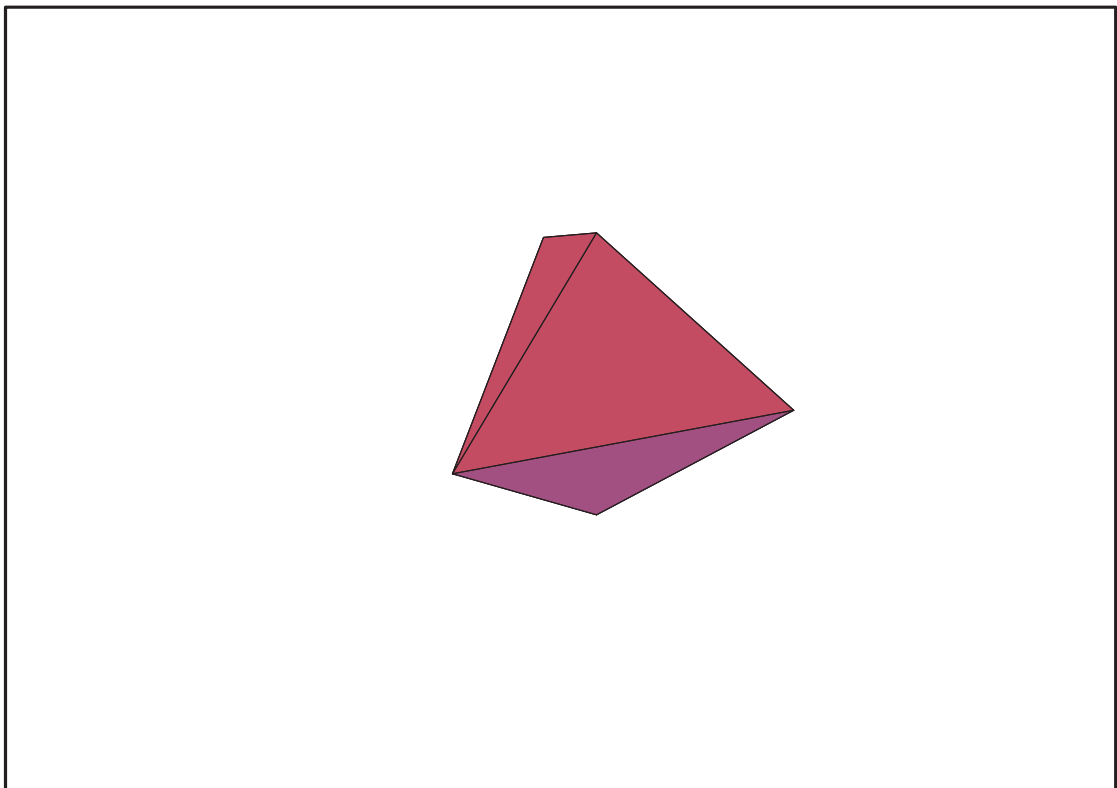


FIG. 11.9 – Graphique : Enveloppe convexe du support du plan N5.

11.7.1. tdeg(x,y,z)

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),tdeg(x,y,z,t));

$$Frontiere_tdeg := [zy, zx, y^2, yx, x^2, z^3]$$

> Div:=Diviseurs({op(Frontiere_tdeg)});nops(Div);

$$Div := \{1, x, z, y, z^2\}$$

> EstSet_N5_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N5_tdeg:=sort(EstSet_N5_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N5_tdeg);

$$EstSet_N5_tdeg := [1, z, y, x, z^2]$$

> X_D_ModeleSat(N5,[x,y,z],EstSet_N5_tdeg);

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

> M_D_ModeleSat(N5,[x,y,z],EstSet_N5_tdeg);

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & \frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{10} & 0 \\ \frac{2}{5} & 0 & 0 & 0 & \frac{2}{5} \end{bmatrix}$$

> expand(Var_Blue_Modele_Sat(N5,[x,y,z],EstSet_N5_tdeg));

$$\frac{5}{3} - \frac{5}{6} z^2 + \frac{10}{3} y^2 + \frac{10}{3} x^2 + \frac{25}{6} z^4$$

11.7.2. plex(x,y,z)

> Frontiere_plex:=map(LeadingMonomial,{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z));

$$Frontiere_plex := \{x, y^3, zy, z^3\}$$

> Div:=Diviseurs({op(Frontiere_plex)});nops(Div);

$$Div := \{1, y^2, z, y, z^2\}$$

> EstSet_N5_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N5_plex:=sort(EstSet_N5_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N5_plex);

$$EstSet_N5_plex := [1, z, z^2, y, y^2]$$

5

```
> X_D_ModeleSat(N5,[x,y,z],EstSet_N5_plex);
```

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{3}{4} \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> M_D_ModeleSat(N5,[x,y,z],EstSet_N5_plex);
```

$$\begin{bmatrix} 1 & 0 & \frac{2}{5} & 0 & \frac{3}{10} \\ 0 & \frac{2}{5} & 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{10} & 0 \\ \frac{3}{10} & 0 & 0 & 0 & \frac{9}{40} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N5,[x,y,z],EstSet_N5_plex));
```

$$5 - \frac{15}{2} z^2 - 10 y^2 + \frac{15}{2} z^4 + \frac{40}{3} z^2 y^2 + \frac{40}{3} y^4$$

11.7.3. Variance

Modèle de degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N5,1,0);
```

$$Var := 1 + \frac{5}{2} var_3^2 + \frac{10}{3} var_2^2 + \frac{10}{3} var_1^2$$

```
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
```

$$1 - \frac{5}{6} var_3^2 + \frac{10}{3} IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

```
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N5,2,0):
```

“M_D is not full rank”

11.8. N6

```
> GrN6_Total:=Bn(3):
> GrN6_Gen:=Bn(3):
> N6:=OrbToMatrix(OrbPG(Vector([1,0,0]),GrN6_Gen));
```

$$N6 := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> Var:=Var_Norm_Blue_D_Centre(N6,1,0);
      Var := 1 + 3 var32 + 3 var22 + 3 var12
> UpdatePlan(N6,"N6");
      "Update complete"
> pointplot3d(N6,symbol=box,scaling=constrained,shading=Z);
```

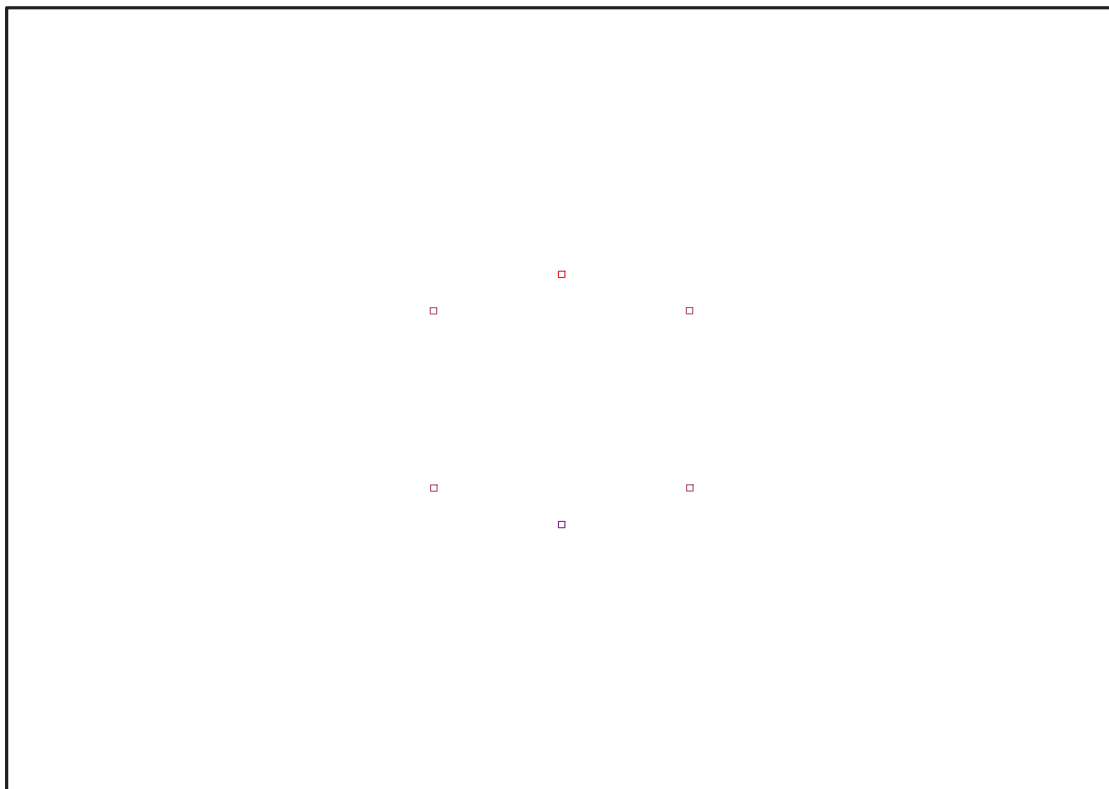


FIG. 11.10 – Graphique : Support du plan N6.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N6,listlist)))],'rational',7))));
      Graph := POLYTOPE(3, 3, 6, 8)
```

```
> plotdes:=draw(Graph):
> display({plotdes},scaling=constrained,shading=Z);
```

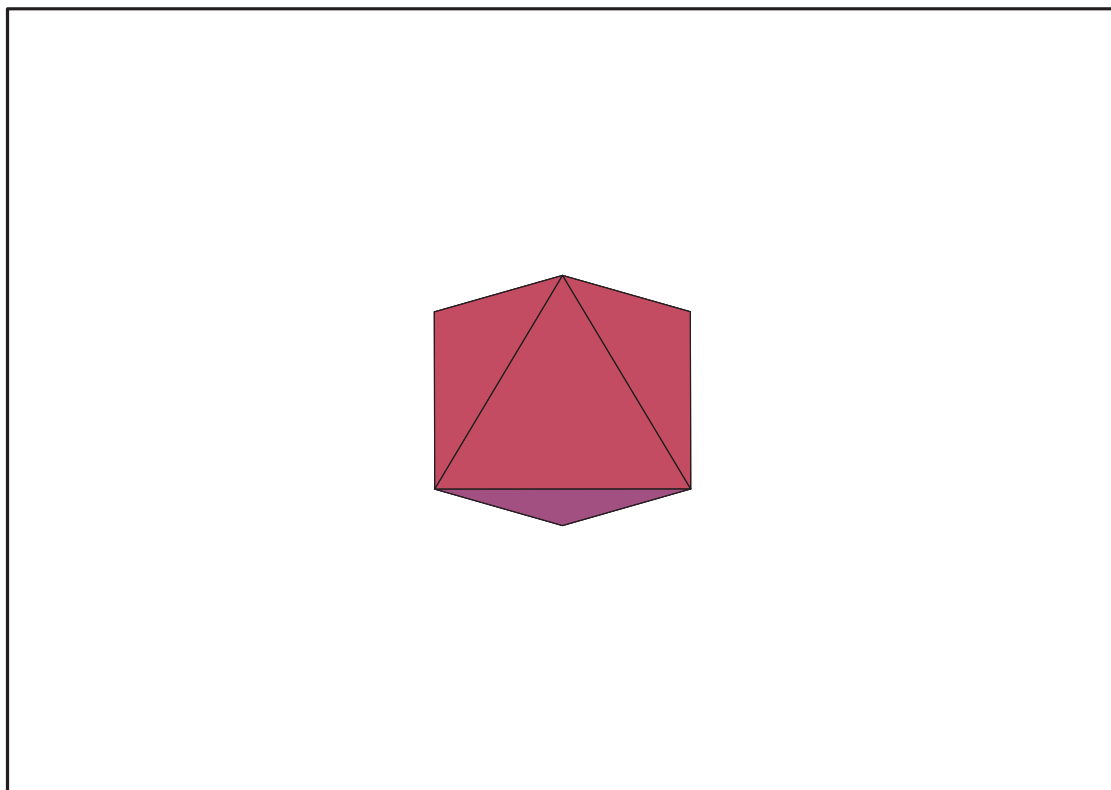


FIG. 11.11 – Graphique : Enveloppe convexe du support du plan N6.

```
> IdealN6:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N6));
> Gens:=Generators(IdealN6);
      IdealN6 := <x3 - x, z x, y x, z y, y3 - y, z2 + y2 + x2 - 1>
      Gens := {z2 + y2 + x2 - 1, y3 - y, z y, z x, x3 - x, y x}

> info_iop:=3;
      info_iop := 3

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N6),1
istlist),Vector)));
      true
      1
      2
      3
      4
      5
      5
      5
      6
      6
```


$$\begin{array}{c} 6 \\ 6 \\ 6 \end{array} \quad [[z y, z x, y x, z^2 + y^2 + x^2 - 1, z^3 - z, y^3 - y], [1, z, y, x, z^2, y^2]]$$

11.8.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),tdeg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div): EstSet_N6_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]: A:=poly_algebra(x,y,z): T:=termorder(A,tdeg(x,y,z)): EstSet_N6_tdeg:=sort(EstSet_N6_tdeg,(t1,t2)->testorder(t1,t2,T)); nops(EstSet_N6_tdeg);
```

$$\begin{array}{c} EstSet_N6_tdeg := [1, z, y, x, z^2, y^2] \\ 6 \end{array}$$

```
> X_D_ModeleSat(N6,[x,y,z],EstSet_N6_tdeg);
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

```
> M_D_ModeleSat(N6,[x,y,z],EstSet_N6_tdeg);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N6,[x,y,z],EstSet_N6_tdeg));
```

$$3 + 3x^2 - 3z^2 + 6z^4 + 6z^2y^2 - 3y^2 + 6y^4$$

11.8.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)): Div:=Diviseurs({op(Frontiere_plex)}): nops(Div): EstSet_N6_plex:=[op(EstSet(Div,Frontiere_plex))]: A:=poly_algebra(x,y,z): T:=termorder(A,plex(x,y,z)): EstSet_N6_plex:=sort(EstSet_N6_plex,(t1,t2)->testorder(t1,t2,T)); nops(EstSet_N6_plex);
```

$$\begin{array}{c} EstSet_N6_plex := [1, z, z^2, y, y^2, x] \\ 6 \end{array}$$

```
> N6;untrace(X_D_ModeleSat);
```

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$X_D_ModeleSat$

> $X_D_ModeleSat(N6, [x, y, z], EstSet_N6_plex);$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

> $M_D_ModeleSat(N6, [x, y, z], EstSet_N6_plex);$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

> $expand(Var_Blue_Modele_Sat(N6, [x, y, z], EstSet_N6_plex));$

$$3 + 3x^2 - 3z^2 + 6z^4 + 6z^2y^2 - 3y^2 + 6y^4$$

11.8.3. Variance

Modèle de degré 1

> $Var := Var_Norm_Blue_D_Centre(N6, 1, 0);$

$$Var := 1 + 3var_3^2 + 3var_2^2 + 3var_1^2$$

> $DecompInKfiInfo(\{add(y^2, y = seq(var[i], i=1..3))\}, Var, poly_algebra(seq(var[i], i=1..3)));$

$$1 + 3IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

Modèle de degré 2

> $Var := Var_Norm_Blue_D_Centre(N6, 2, 0):$

“M_D is not full rank”

11.9. N7

```
> GrN7_Total:=GentoGr(
  {Reflexion_Gen(Vector([1/2,sqrt(3)/2,0])),Reflexion_Gen(Vector([-1/2,
  sqrt(3)/2,0]))});
```

```
> GrN7_Gen:=GentoGr(
  {MatrixMatrixMultiply(Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(
  Vector([-1/2,sqrt(3)/2,0]))});
```

$$GrN7_Gen := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> N7:=OrbToMatrix(OrbPG(Vector([a,0,z[1]]),GrN7_Gen) union
  OrbPG(Vector([b,0,z[2]]),GrN7_Gen) union {Vector([0,0,1])});
```

$$N7 := \begin{bmatrix} -\frac{a}{2} & \frac{\sqrt{3}a}{2} & z_1 \\ b & 0 & z_2 \\ -\frac{b}{2} & -\frac{\sqrt{3}b}{2} & z_2 \\ -\frac{b}{2} & \frac{\sqrt{3}b}{2} & z_2 \\ -\frac{a}{2} & -\frac{\sqrt{3}a}{2} & z_1 \\ a & 0 & z_1 \\ 0 & 0 & 1 \end{bmatrix}$$

```
> Var:=Var_Norm_Blue_D_Centre(N7,1,0);
```

$$\begin{aligned} Var &:= \frac{7}{6} \frac{3z_2^2 + 1 + 3z_1^2}{\%1} - \frac{7}{6} \frac{(3z_2 + 1 + 3z_1) var_3}{\%1} + \left(-\frac{7}{6} \frac{3z_2 + 1 + 3z_1}{\%1} + \frac{49}{6} \frac{var_3}{\%1} \right) var_3 \\ &+ \frac{14}{3} \frac{var_2^2}{b^2 + a^2} + \frac{14}{3} \frac{var_1^2}{b^2 + a^2} \\ \%1 &:= 2z_2^2 + 1 + 2z_1^2 - z_2 - 3z_2z_1 - z_1 \end{aligned}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},[Var]);
```

$$\begin{aligned} Cond &:= [-4 + 12z_2z_1 + 7b^2 + 7a^2 - 8z_2^2 + 4z_2 - 8z_1^2 + 4z_1, \\ &6z_2^3 - 3z_2^2z_1 - 3z_2z_1^2 + 6z_1^3 - z_2^2 - 9z_2z_1 - z_1^2 + 2z_2 + 2z_1 + 1] \end{aligned}$$

```
> gsolve([op(Cond),a^2+z[1]^2-1,b^2+z[2]^2-1],[a,b,z[1],z[2]],
  {a,b,z[1],z[2],a-1,b-1,z[1]-1,z[2]-1});
```

$$\begin{aligned} &\{[6z_2^2 + 2z_2 - 1, 6b^2 - 2z_2 - 5, 18a^2 + 6z_2 - 13, 3z_2 + 1 + 3z_1], \text{plex}(z_1, a, b, z_2), \\ &\{a, b, b-1, z_2, a-1, z_2-1, 3z_2+1, 3z_2+4\}, [\\ &[9z_1^2 + 24z_1 + 23, 3z_2 + 3z_1 + 8, 9b^2 + 24z_1 + 32, 9a^2 - 24z_1 - 32], \text{plex}(a, b, z_2, z_1), \\ &\{a, b, z_1-1, b-1, z_1, a-1, -3z_1-8, -11-3z_1\}]\} \end{aligned}$$

```
> Sols:=convert(map(L->solve(convert(L[1],set),{a,b,z[1],z[2]}),
  %),list);
```

```

Sols := [{a =  $\frac{1}{3}$  RootOf( $\_Z^2 + 9 \%2^2 - 14$ , label =  $\_L8$ ),  $z_1 = -3 \%2^2 + \frac{13}{6}$ ,  $b = \%2$ ,
 $z_2 = 3 \%2^2 - \frac{5}{2}$ },
{a =  $\frac{2}{3}$  RootOf( $\%1^2 + \_Z^2$ , label =  $\_L18$ ),  $z_1 = -\frac{1}{6} \%1^2 - \frac{4}{3}$ ,  $b = \frac{2}{3} \%1$ ,  $z_2 = \frac{1}{6} \%1^2 - \frac{4}{3}$ }]
%1 := RootOf( $\_Z^4 + 28$ , label =  $\_L16$ )
%2 := RootOf( $36 \_Z^4 - 56 \_Z^2 + 21$ )

> IsRealSols(Sols);

[true, false]

> res:= [allvalues({z[2] = -3*RootOf(36*_Z^4-56*_Z^2+21)^2+13/6, a =
RootOf(36*_Z^4-56*_Z^2+21), z[1] = 3*RootOf(36*_Z^4-56*_Z^2+21)^2-5/2,
b = 1/3*RootOf( $\_Z^2+9*$ RootOf(36*_Z^4-56*_Z^2+21)^2-14,label =
_L1)}))];

res := [{b = %4, %8, a = %1, %7}, {b = %6, %8, a = %1, %7}, {b = %4, %8, %7, a = %5},
{b = %6, %8, %7, a = %5}, {b = %5, %3, %2, a = %6}, {%3, %2, a = %6, b = %1},
{b = %5, a = %4, %3, %2}, {a = %4, %3, %2, b = %1}]

%1 :=  $-\frac{\sqrt{28+2\sqrt{7}}}{6}$ 
%2 :=  $z_1 = -\frac{1}{6} - \frac{\sqrt{7}}{6}$ 
%3 :=  $z_2 = -\frac{1}{6} + \frac{\sqrt{7}}{6}$ 
%4 :=  $\frac{\sqrt{28-2\sqrt{7}}}{6}$ 
%5 :=  $\frac{\sqrt{28+2\sqrt{7}}}{6}$ 
%6 :=  $-\frac{\sqrt{28-2\sqrt{7}}}{6}$ 
%7 :=  $z_2 = -\frac{1}{6} - \frac{\sqrt{7}}{6}$ 
%8 :=  $z_1 = -\frac{1}{6} + \frac{\sqrt{7}}{6}$ 

> res:= [{b =  $\frac{1}{6}*(28-2*7^{(1/2)})^{(1/2)}$ , z[1] =  $-\frac{1}{6}+\frac{1}{6}*7^{(1/2)}$ , a =
 $-\frac{1}{6}*(28+2*7^{(1/2)})^{(1/2)}$ , z[2] =  $-\frac{1}{6}-\frac{1}{6}*7^{(1/2)}$ }, {b =
 $-\frac{1}{6}*(28-2*7^{(1/2)})^{(1/2)}$ , z[1] =  $-\frac{1}{6}+\frac{1}{6}*7^{(1/2)}$ , a =
 $-\frac{1}{6}*(28+2*7^{(1/2)})^{(1/2)}$ , z[2] =  $-\frac{1}{6}-\frac{1}{6}*7^{(1/2)}$ }]

res := [{b =  $\frac{\sqrt{28-2\sqrt{7}}}{6}$ ,  $z_1 = -\frac{1}{6} + \frac{\sqrt{7}}{6}$ ,  $a = -\frac{\sqrt{28+2\sqrt{7}}}{6}$ ,  $z_2 = -\frac{1}{6} - \frac{\sqrt{7}}{6}$ },
{b =  $-\frac{\sqrt{28-2\sqrt{7}}}{6}$ ,  $z_1 = -\frac{1}{6} + \frac{\sqrt{7}}{6}$ ,  $a = -\frac{\sqrt{28+2\sqrt{7}}}{6}$ ,  $z_2 = -\frac{1}{6} - \frac{\sqrt{7}}{6}$ }]

> unassign('a','b','z');assign(res[1]);
N7_1:=N7;UpdatePlan(N7,"N7_1");

```

$$N7_1 := \begin{bmatrix} -\frac{\sqrt{28-2\sqrt{7}}}{12} & -\frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12} & -\frac{1}{6}-\frac{\sqrt{7}}{6} \\ -\frac{\sqrt{28-2\sqrt{7}}}{12} & \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12} & -\frac{1}{6}-\frac{\sqrt{7}}{6} \\ 0 & 0 & 1 \\ \frac{\sqrt{28+2\sqrt{7}}}{12} & -\frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12} & -\frac{1}{6}+\frac{\sqrt{7}}{6} \\ \frac{\sqrt{28+2\sqrt{7}}}{12} & \frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12} & -\frac{1}{6}+\frac{\sqrt{7}}{6} \\ \frac{\sqrt{28-2\sqrt{7}}}{6} & 0 & -\frac{1}{6}-\frac{\sqrt{7}}{6} \\ -\frac{\sqrt{28+2\sqrt{7}}}{6} & 0 & -\frac{1}{6}+\frac{\sqrt{7}}{6} \end{bmatrix}$$

“Update complete”

```
> pointplot3d(N7_1,symbol=box,shading=Z,scaling=constrained,shading=Z);
```

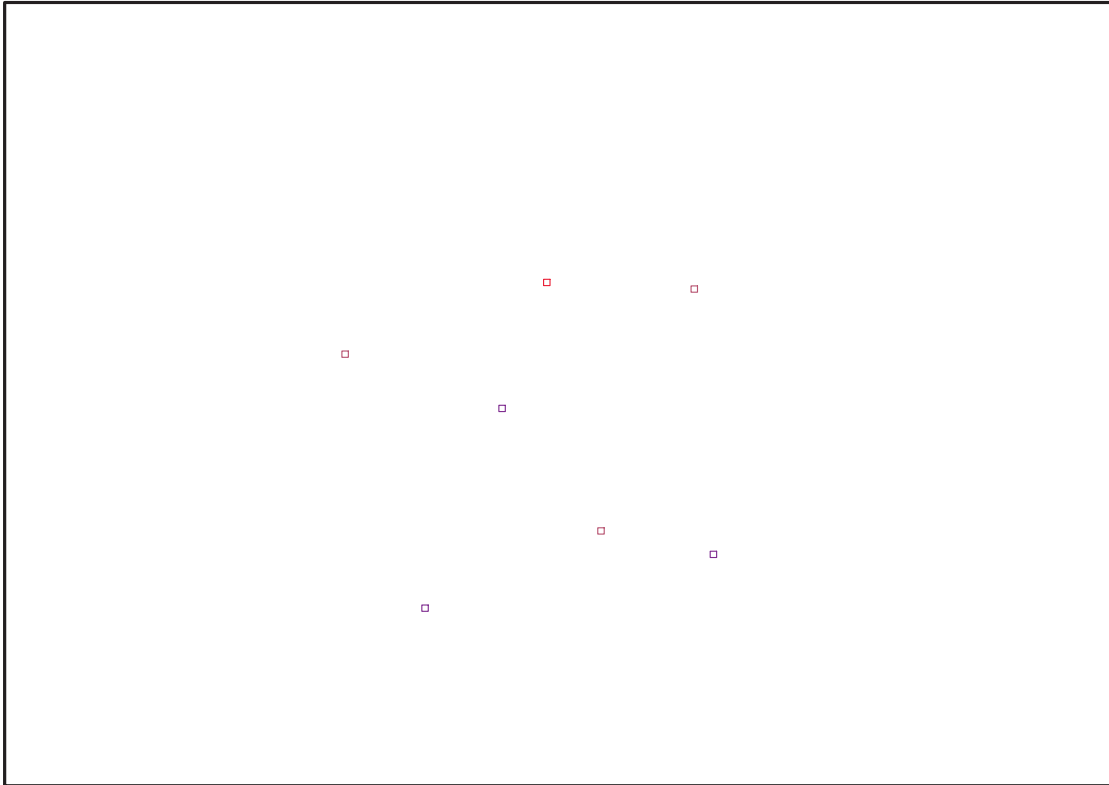


FIG. 11.12 – Graphique : Support du plan N7.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N7_1,listlist)))],'rational',7))));
```

```
Graph := POLYTOPE(3, 3, 7, 10)
```

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

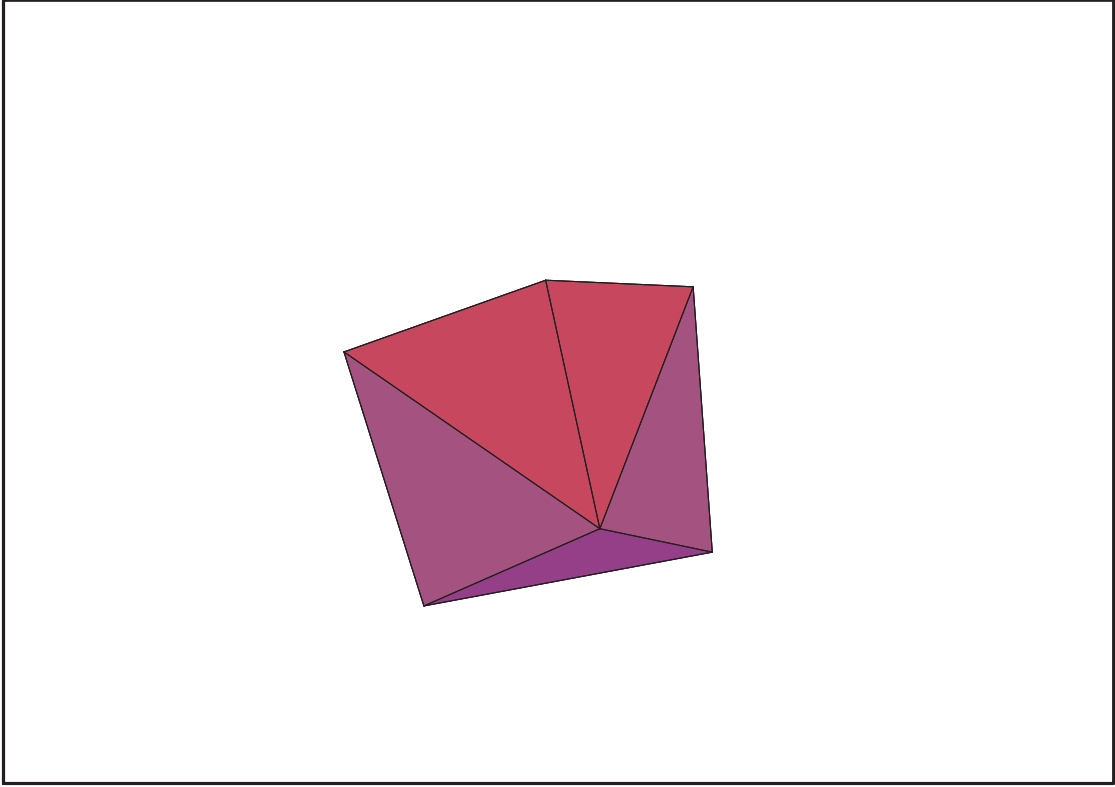


FIG. 11.13 – Graphique : Enveloppe convexe du support du plan N7.

```
> IdealN7_1:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N7_1));
```

$$\begin{aligned}
IdealN7_1 := & \langle 144x^2y + 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} - y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}}, \\
& 12yz + 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} - y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 30y, \\
& 1008x^3 + 15xz\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 210xz + 30x^2\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 30x^2\sqrt{28+2\sqrt{7}}\sqrt{7} - 42x^2\sqrt{28-2\sqrt{7}} + 42\sqrt{28+2\sqrt{7}}x^2 + 9z\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 9z\sqrt{28+2\sqrt{7}}\sqrt{7} - x\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} - 420x - 9\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - 9\sqrt{28+2\sqrt{7}}\sqrt{7}, 84z^2 + 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} + 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 168x^2 \\
& + x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} - 7x\sqrt{28-2\sqrt{7}} + 7x\sqrt{28+2\sqrt{7}} - 84, 84y^2 \\
& - 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 84x^2 - x\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - x\sqrt{28+2\sqrt{7}}\sqrt{7} + 7x\sqrt{28-2\sqrt{7}} - 7x\sqrt{28+2\sqrt{7}}, 504x^2z \\
& - 15xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 15\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 21xz\sqrt{28-2\sqrt{7}} \\
& + 21\sqrt{28+2\sqrt{7}}xz - 336x^2 - 126z + x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} \\
& + 14x\sqrt{28-2\sqrt{7}} - 14x\sqrt{28+2\sqrt{7}} + 126 \rangle
\end{aligned}$$

```
> Gens:=Generators(IdealN7_1);
```

$$\begin{aligned}
Gens := \{ & 144x^2y + 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} - y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}}, \\
& 12yz + 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} - y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 30y, \\
& 1008x^3 + 15xz\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 210xz + 30x^2\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 30x^2\sqrt{28+2\sqrt{7}}\sqrt{7} - 42x^2\sqrt{28-2\sqrt{7}} + 42\sqrt{28+2\sqrt{7}}x^2 + 9z\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 9z\sqrt{28+2\sqrt{7}}\sqrt{7} - x\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} - 420x - 9\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - 9\sqrt{28+2\sqrt{7}}\sqrt{7}, 84z^2 + 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} + 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 168x^2 \\
& + x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} - 7x\sqrt{28-2\sqrt{7}} + 7x\sqrt{28+2\sqrt{7}} - 84, 84y^2 \\
& - 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 84x^2 - x\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - x\sqrt{28+2\sqrt{7}}\sqrt{7} + 7x\sqrt{28-2\sqrt{7}} - 7x\sqrt{28+2\sqrt{7}}, 504x^2z \\
& - 15xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 15\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 21xz\sqrt{28-2\sqrt{7}} \\
& + 21\sqrt{28+2\sqrt{7}}xz - 336x^2 - 126z + x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} \\
& + 14x\sqrt{28-2\sqrt{7}} - 14x\sqrt{28+2\sqrt{7}} + 126 \}
\end{aligned}$$

> info_iop:=3;

info_iop := 3

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N7_1),listlist),Vector)));

1
2
3
4
5
6
7
7
7
7
7
7
7

$$\begin{aligned}
& \left[-\frac{1}{2} + y^2 - \frac{(21\sqrt{3} - 7 + 3\sqrt{3}\sqrt{7} + 11\sqrt{7})z}{336} - \frac{\sqrt{28-2\sqrt{7}}(9\sqrt{7} + \sqrt{3}\sqrt{7} + 14\sqrt{3})zx}{252} \right. \\
& - \frac{\sqrt{28-2\sqrt{7}}(3\sqrt{7} + 21 + 5\sqrt{3}\sqrt{7} + 7\sqrt{3})x}{504} + \frac{(3\sqrt{3}\sqrt{7} + 49 - 213\sqrt{7} - 315\sqrt{3})z}{336} \\
& + \frac{(35 - 25\sqrt{7} - 35\sqrt{3} + \sqrt{3}\sqrt{7})z^2}{56} - \frac{(-4\sqrt{7} + 3\sqrt{3} + 2)z}{24} - \frac{\%1 z^2}{56} \\
& + \frac{(35 - 25\sqrt{7} - 35\sqrt{3} + \sqrt{3}\sqrt{7})z}{168} + \frac{x\sqrt{28-2\sqrt{7}}}{12} + \frac{(-2 + 9\sqrt{3} + 4\sqrt{7})z}{8} - \frac{\%1 z}{168}, \\
& yx - \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}(1 + 2\sqrt{7})y}{108} - \frac{\sqrt{28-2\sqrt{7}}(9\sqrt{7} + \sqrt{3}\sqrt{7} + 14\sqrt{3})yz}{252} \\
& + \frac{\sqrt{28-2\sqrt{7}}(-9\sqrt{7} + 63 + 13\sqrt{3}\sqrt{7} - 7\sqrt{3})y}{1512} - \frac{3z}{8} + \frac{(-\sqrt{3} + 2\sqrt{3}\sqrt{7} + 18)z}{24} \\
& + \frac{(-7\sqrt{3} + 5\sqrt{3}\sqrt{7} + 21 - 3\sqrt{7})z}{112} - \frac{(-35\sqrt{3} + 43\sqrt{3}\sqrt{7} - 9\sqrt{7} + 189)z}{336}, -\frac{1}{2} + x^2 \\
& + \frac{(21\sqrt{3} - 7 + 3\sqrt{3}\sqrt{7} + 11\sqrt{7})z}{336} + \frac{\sqrt{28-2\sqrt{7}}(9\sqrt{7} + \sqrt{3}\sqrt{7} + 14\sqrt{3})zx}{252} \\
& + \frac{\sqrt{28-2\sqrt{7}}(3\sqrt{7} + 21 + 5\sqrt{3}\sqrt{7} + 7\sqrt{3})x}{504} - \frac{(-21 - 35\sqrt{3} - 25\sqrt{7} + \sqrt{3}\sqrt{7})z}{168} \\
& - \frac{(-21 - 35\sqrt{3} - 25\sqrt{7} + \sqrt{3}\sqrt{7})z^2}{56} - \frac{(2 + 27\sqrt{3} + 12\sqrt{7})z}{24} \\
& - \frac{(3\sqrt{3}\sqrt{7} + 49 - 213\sqrt{7} - 315\sqrt{3})z}{336} + \frac{(-4\sqrt{7} + 3\sqrt{3} + 2)z}{24} + \frac{\%1 z^2}{56} \\
& - \frac{x\sqrt{28-2\sqrt{7}}}{12} + \frac{\%1 z}{168}, \frac{1}{6} - \frac{1}{2}z + z^3 - \frac{2}{3}z^2, -\frac{\sqrt{3}\sqrt{28-2\sqrt{7}}(\sqrt{7} - 21)z}{504} \\
& + \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}(-7 + \sqrt{7})z}{504} + z^2y - \frac{(-1 + \sqrt{7})y}{18} - \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}(-7 + 4\sqrt{7})z}{504} \\
& + \frac{yz}{3} + \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}(-21 + 4\sqrt{7})z}{504} + \frac{(\sqrt{7} - 4)y}{18}, \\
& \frac{\sqrt{28-2\sqrt{7}}(2\sqrt{3}\sqrt{7} - 8\sqrt{7} + 7\sqrt{3} - 14)z}{504} \\
& - \frac{\sqrt{28-2\sqrt{7}}(91\sqrt{3} - 72\sqrt{7} + 74\sqrt{3}\sqrt{7} + 630)z}{4536} + \frac{xz}{3} - \frac{(4 + \sqrt{7})x}{18} + z^2x \\
& + \frac{\sqrt{28-2\sqrt{7}}(49\sqrt{3} - 18\sqrt{7} + 630 + 71\sqrt{3}\sqrt{7})z}{4536} \\
& - \frac{\sqrt{28-2\sqrt{7}}(5\sqrt{3}\sqrt{7} - 6\sqrt{7} + 7\sqrt{3} - 42)z}{1512} + \frac{(1 + \sqrt{7})x}{18}], [1, z, y, x, z^2, yz, xz] \\
& \%1 := \sqrt{3}\sqrt{7} + 7 - 25\sqrt{7} - 35\sqrt{3}
\end{aligned}$$


```

> Gens :=
{144*x^2*y+12*x*(28-2*7^(1/2))^(1/2)*y-12*x*y*(28+2*7^(1/2))^(1/2)-y*
(28+2*7^(1/2))^(1/2)*(28-2*7^(1/2))^(1/2),
12*y*z+12*x*(28-2*7^(1/2))^(1/2)*y-12*x*y*(28+2*7^(1/2))^(1/2)-y*(28+2
*7^(1/2))^(1/2)*(28-2*7^(1/2))^(1/2)+30*y,
1008*x^3+15*x*z*(28+2*7^(1/2))^(1/2)*(28-2*7^(1/2))^(1/2)+210*x*z+30*x
^2*(28-2*7^(1/2))^(1/2)*7^(1/2)+30*x^2*(28+2*7^(1/2))^(1/2)*7^(1/2)-42
*x^2*(28-2*7^(1/2))^(1/2)+42*(28+2*7^(1/2))^(1/2)*x^2+9*z*(28-2*7^(1/2)
)^(1/2)*7^(1/2)+9*z*(28+2*7^(1/2))^(1/2)*7^(1/2)-x*(28+2*7^(1/2))^(1/2)
*(28-2*7^(1/2))^(1/2)-420*x-9*(28-2*7^(1/2))^(1/2)*7^(1/2)-9*(28+2*7
^(1/2))^(1/2)*7^(1/2),
84*z^2+6*x*z*7^(1/2)*(28-2*7^(1/2))^(1/2)+6*(28+2*7^(1/2))^(1/2)*x*z*7
^(1/2)+168*x^2+x*(28-2*7^(1/2))^(1/2)*7^(1/2)+x*(28+2*7^(1/2))^(1/2)*7
^(1/2)-7*x*(28-2*7^(1/2))^(1/2)+7*x*(28+2*7^(1/2))^(1/2)-84,
84*y^2-6*x*z*7^(1/2)*(28-2*7^(1/2))^(1/2)-6*(28+2*7^(1/2))^(1/2)*x*z*7
^(1/2)-84*x^2-x*(28-2*7^(1/2))^(1/2)*7^(1/2)-x*(28+2*7^(1/2))^(1/2)*7
^(1/2)+7*x*(28-2*7^(1/2))^(1/2)-7*x*(28+2*7^(1/2))^(1/2),
504*x^2*z-15*x*z*7^(1/2)*(28-2*7^(1/2))^(1/2)-15*(28+2*7^(1/2))^(1/2)*
x*z*7^(1/2)-21*x*z*(28-2*7^(1/2))^(1/2)+21*(28+2*7^(1/2))^(1/2)*x*z-33
6*x^2-126*z+x*(28-2*7^(1/2))^(1/2)*7^(1/2)+x*(28+2*7^(1/2))^(1/2)*7^(1
/2)+14*x*(28-2*7^(1/2))^(1/2)-14*x*(28+2*7^(1/2))^(1/2)+126};
    
```

$$\begin{aligned}
 Gens := & \{144x^2y + 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} - y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}}, \\
 & 12yz + 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} - y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 30y, \\
 & 1008x^3 + 15xz\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 210xz + 30x^2\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & + 30x^2\sqrt{28+2\sqrt{7}}\sqrt{7} - 42x^2\sqrt{28-2\sqrt{7}} + 42\sqrt{28+2\sqrt{7}}x^2 + 9z\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & + 9z\sqrt{28+2\sqrt{7}}\sqrt{7} - x\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} - 420x - 9\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & - 9\sqrt{28+2\sqrt{7}}\sqrt{7}, 84z^2 + 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} + 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 168x^2 \\
 & + x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} - 7x\sqrt{28-2\sqrt{7}} + 7x\sqrt{28+2\sqrt{7}} - 84, 84y^2 \\
 & - 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 84x^2 - x\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & - x\sqrt{28+2\sqrt{7}}\sqrt{7} + 7x\sqrt{28-2\sqrt{7}} - 7x\sqrt{28+2\sqrt{7}}, 504x^2z \\
 & - 15xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 15\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 21xz\sqrt{28-2\sqrt{7}} \\
 & + 21\sqrt{28+2\sqrt{7}}xz - 336x^2 - 126z + x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} \\
 & + 14x\sqrt{28-2\sqrt{7}} - 14x\sqrt{28+2\sqrt{7}} + 126\}
 \end{aligned}$$

```

> unassign('a','b','z');assign(res[2]);
N7_bis:=N7;UpdatePlan(N7,"N7_bis");
    
```

$$N7_bis := \begin{bmatrix} \frac{\sqrt{28-2\sqrt{7}}}{12} & \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12} & -\frac{1}{6} - \frac{\sqrt{7}}{6} \\ \frac{\sqrt{28-2\sqrt{7}}}{12} & -\frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12} & -\frac{1}{6} - \frac{\sqrt{7}}{6} \\ 0 & 0 & 1 \\ \frac{\sqrt{28+2\sqrt{7}}}{12} & -\frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12} & -\frac{1}{6} + \frac{\sqrt{7}}{6} \\ \frac{\sqrt{28+2\sqrt{7}}}{12} & \frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12} & -\frac{1}{6} + \frac{\sqrt{7}}{6} \\ -\frac{\sqrt{28-2\sqrt{7}}}{6} & 0 & -\frac{1}{6} - \frac{\sqrt{7}}{6} \\ -\frac{\sqrt{28+2\sqrt{7}}}{6} & 0 & -\frac{1}{6} + \frac{\sqrt{7}}{6} \end{bmatrix}$$

“Update complete”

```
> pointplot3d(N7_1,symbol=box,shading=Z,scaling=constrained,shading=Z);
```

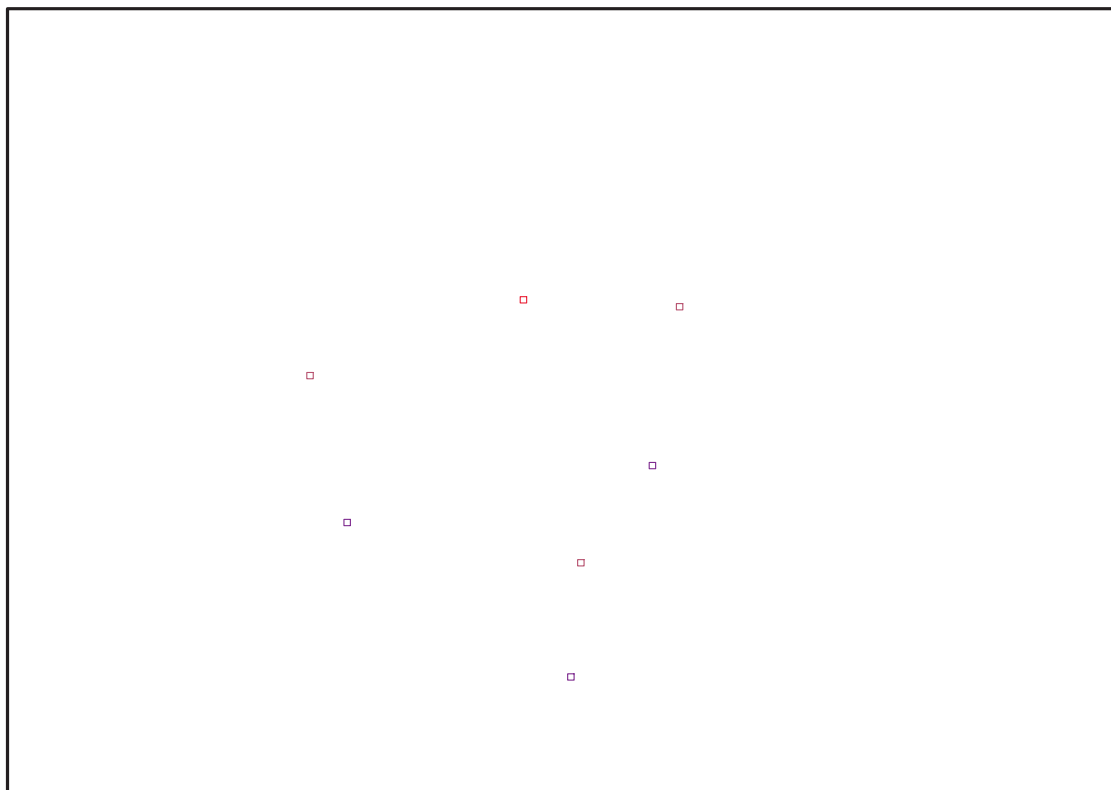


FIG. 11.14 – Graphique : Support du plan N7bis.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N7_bis,listlist)))]),  
  'rational',7)));
```

$Graph := \text{POLYTOPE}(3, 3, 7, 9)$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN7_bis:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N7_bis));
```

```
> Gens_bis:=Generators(IdealN7_bis);
```

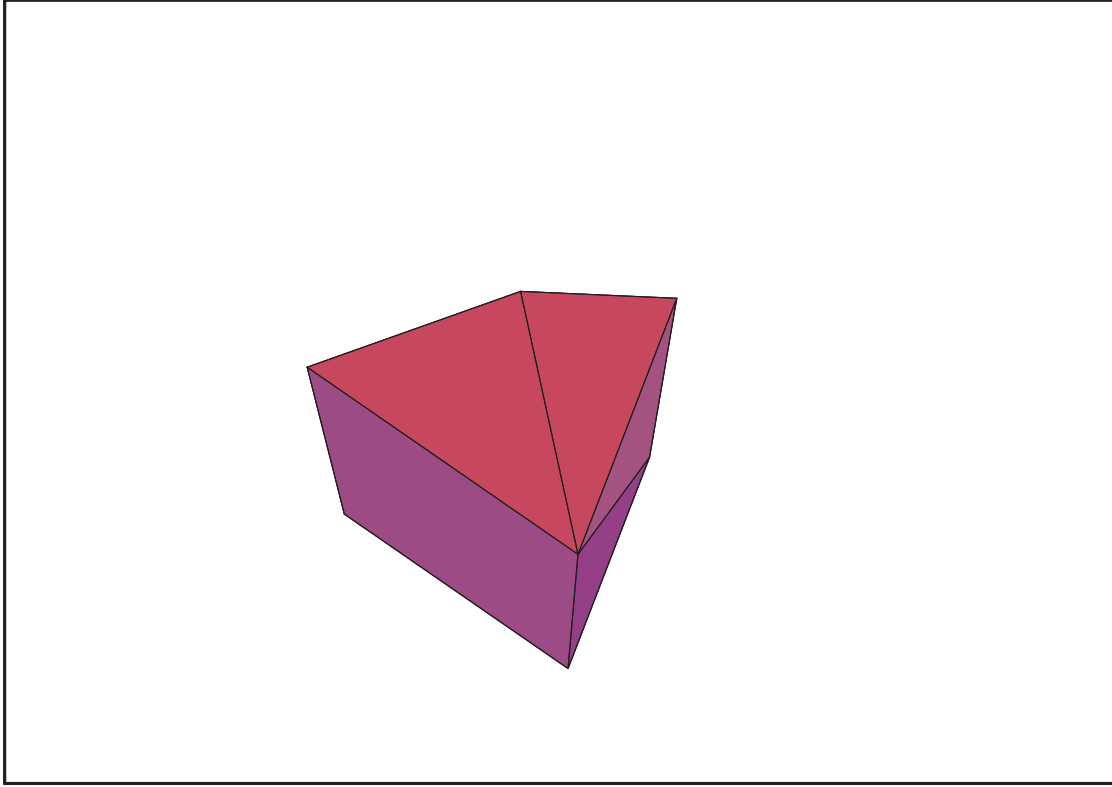


FIG. 11.15 – Graphique : Enveloppe convexe du support du plan N7bis.

$$\begin{aligned}
 IdealN7_bis := \langle & \\
 & 144x^2y - 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} + y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}}, \\
 & 12yz - 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} + y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 30y, \\
 & 84y^2 + 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 84x^2 + x\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & \quad - x\sqrt{28+2\sqrt{7}}\sqrt{7} - 7x\sqrt{28-2\sqrt{7}} - 7x\sqrt{28+2\sqrt{7}}, 84z^2 - 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} \\
 & \quad + 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 168x^2 - x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} \\
 & \quad + 7x\sqrt{28-2\sqrt{7}} + 7x\sqrt{28+2\sqrt{7}} - 84, 504x^2z + 15xz\sqrt{7}\sqrt{28-2\sqrt{7}} \\
 & \quad - 15\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 21xz\sqrt{28-2\sqrt{7}} + 21\sqrt{28+2\sqrt{7}}xz - 336x^2 - 126z \\
 & \quad - x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} - 14x\sqrt{28-2\sqrt{7}} - 14x\sqrt{28+2\sqrt{7}} + 126, \\
 & 1008x^3 - 15xz\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 210xz - 30x^2\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & \quad + 30x^2\sqrt{28+2\sqrt{7}}\sqrt{7} + 42x^2\sqrt{28-2\sqrt{7}} + 42\sqrt{28+2\sqrt{7}}x^2 - 9z\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & \quad + 9z\sqrt{28+2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} - 420x + 9\sqrt{28-2\sqrt{7}}\sqrt{7} \\
 & \quad - 9\sqrt{28+2\sqrt{7}}\sqrt{7} \rangle
 \end{aligned}$$

$$\begin{aligned}
Gens_bis := & \{144x^2y - 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} + y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}}, \\
& 12yz - 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} + y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 30y, \\
& 84y^2 + 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 84x^2 + x\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - x\sqrt{28+2\sqrt{7}}\sqrt{7} - 7x\sqrt{28-2\sqrt{7}} - 7x\sqrt{28+2\sqrt{7}}, 84z^2 - 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} \\
& + 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 168x^2 - x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} \\
& + 7x\sqrt{28-2\sqrt{7}} + 7x\sqrt{28+2\sqrt{7}} - 84, 504x^2z + 15xz\sqrt{7}\sqrt{28-2\sqrt{7}} \\
& - 15\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 21xz\sqrt{28-2\sqrt{7}} + 21\sqrt{28+2\sqrt{7}}xz - 336x^2 - 126z \\
& - x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} - 14x\sqrt{28-2\sqrt{7}} - 14x\sqrt{28+2\sqrt{7}} + 126, \\
& 1008x^3 - 15xz\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 210xz - 30x^2\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 30x^2\sqrt{28+2\sqrt{7}}\sqrt{7} + 42x^2\sqrt{28-2\sqrt{7}} + 42\sqrt{28+2\sqrt{7}}x^2 - 9z\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 9z\sqrt{28+2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} - 420x + 9\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - 9\sqrt{28+2\sqrt{7}}\sqrt{7}\}
\end{aligned}$$

> info_iop:=3;

info_iop := 3

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N7_bis),listlist),Vector)));

1
2
3
4
5
6
7
7
7
7
7
7
7

$$\begin{aligned}
 & \left[-\frac{1}{2} - \frac{(4\sqrt{7} + 3\sqrt{3} + 2)z}{24} + y^2 - \frac{\sqrt{28 - 2\sqrt{7}}(14\sqrt{3} + \sqrt{3}\sqrt{7} - 9\sqrt{7})zx}{252} \right. \\
 & - \frac{\sqrt{3}\sqrt{28 - 2\sqrt{7}}(1 + 2\sqrt{7})x}{108} + \frac{\sqrt{28 - 2\sqrt{7}}(-7\sqrt{3} + 13\sqrt{3}\sqrt{7} + 9\sqrt{7} - 63)x}{1512} \\
 & + \frac{(-1 + 2\sqrt{7})z}{8} + \frac{\%1z}{168} - \frac{(-35 + 5\sqrt{7} - 7\sqrt{3} + \sqrt{3}\sqrt{7})z}{168} \\
 & + \frac{(7 + 3\sqrt{3}\sqrt{7} + 11\sqrt{7} - 21\sqrt{3})z}{336} + \frac{\%1z^2}{56} - \frac{(-35 + 5\sqrt{7} - 7\sqrt{3} + \sqrt{3}\sqrt{7})z^2}{56} \\
 & - \frac{(3\sqrt{3}\sqrt{7} - 63\sqrt{3} + 39\sqrt{7} - 7)z}{336}, yx - \frac{\sqrt{28 - 2\sqrt{7}}(14\sqrt{3} + \sqrt{3}\sqrt{7} - 9\sqrt{7})yz}{252} \\
 & - \frac{\sqrt{28 - 2\sqrt{7}}(-9 + 4\sqrt{3}\sqrt{7} + 2\sqrt{3})y}{108} \\
 & + \frac{\sqrt{28 - 2\sqrt{7}}(41\sqrt{3}\sqrt{7} + 7\sqrt{3} - 189 + 9\sqrt{7})y}{1512} - \frac{(2\sqrt{3} + 9)z}{24} \\
 & - \frac{(15\sqrt{3}\sqrt{7} - 49\sqrt{3} - 63 + 9\sqrt{7})z}{336} + \frac{(-189 + 9\sqrt{7} + 43\sqrt{3}\sqrt{7} - 35\sqrt{3})z}{336} \\
 & - \frac{(2\sqrt{3}\sqrt{7} - 18 - \sqrt{3})z}{24}, -\frac{1}{2} + \frac{(4\sqrt{7} + 3\sqrt{3} + 2)z}{24} + x^2 \\
 & + \frac{\sqrt{28 - 2\sqrt{7}}(14\sqrt{3} + \sqrt{3}\sqrt{7} - 9\sqrt{7})zx}{252} + \frac{\sqrt{3}\sqrt{28 - 2\sqrt{7}}(1 + 2\sqrt{7})x}{108} \\
 & - \frac{\sqrt{28 - 2\sqrt{7}}(-7\sqrt{3} + 13\sqrt{3}\sqrt{7} + 9\sqrt{7} - 63)x}{1512} - \frac{\%1z}{168} \\
 & - \frac{(7 + 3\sqrt{3}\sqrt{7} + 11\sqrt{7} - 21\sqrt{3})z}{336} - \frac{\%1z^2}{56} + \frac{(21 + 5\sqrt{7} - 7\sqrt{3} + \sqrt{3}\sqrt{7})z}{168} \\
 & - \frac{(5 + 6\sqrt{7})z}{24} + \frac{(21 + 5\sqrt{7} - 7\sqrt{3} + \sqrt{3}\sqrt{7})z^2}{56} + \frac{(3\sqrt{3}\sqrt{7} - 63\sqrt{3} + 39\sqrt{7} - 7)z}{336}, \\
 & \frac{1}{6} - \frac{1}{2}z + z^3 - \frac{2}{3}z^2, -\frac{\sqrt{3}\sqrt{28 - 2\sqrt{7}}(-21 + \sqrt{7})z}{504} \\
 & + \frac{\sqrt{28 - 2\sqrt{7}}(-28\sqrt{7} - 14 - 63\sqrt{3} + 3\sqrt{3}\sqrt{7})z}{1512} + z^2y \\
 & + \frac{\sqrt{3}\sqrt{28 - 2\sqrt{7}}(-21 + 4\sqrt{7})z}{504} - \frac{\sqrt{28 - 2\sqrt{7}}(-14 - 28\sqrt{7} - 63\sqrt{3} + 12\sqrt{3}\sqrt{7})z}{1512} \\
 & - \frac{(3\sqrt{7} - 1)y}{18} + \frac{yz}{3} + \frac{(3\sqrt{7} - 4)y}{18}, \frac{xz}{3} \\
 & - \frac{\sqrt{28 - 2\sqrt{7}}(-14\sqrt{3} - 9\sqrt{7} + 26\sqrt{3}\sqrt{7} + 63)z}{4536} + z^2x \\
 & + \frac{\sqrt{28 - 2\sqrt{7}}(-9\sqrt{7} - 63 + 26\sqrt{3}\sqrt{7} - 14\sqrt{3})z}{4536} \\
 & + \frac{\sqrt{28 - 2\sqrt{7}}(20\sqrt{3}\sqrt{7} + 63 - 98\sqrt{3} - 36\sqrt{7})z}{4536} \\
 & - \frac{\sqrt{28 - 2\sqrt{7}}(20\sqrt{3}\sqrt{7} - 98\sqrt{3} - 36\sqrt{7} - 63)z}{4536} - \frac{(-1 + \sqrt{7})x}{18} + \frac{(\sqrt{7} - 4)x}{18}], \\
 & [1, z, y, x, z^2, yz, xz] \\
 & \%1 := -7\sqrt{3} + \sqrt{3}\sqrt{7} + 5\sqrt{7} - 7
 \end{aligned}$$

```

> Gens_bis :=
{144*x^2*y-12*x*(28-2*7^(1/2))^(1/2)*y-12*x*y*(28+2*7^(1/2))^(1/2)+y*
(28+2*7^(1/2))^(1/2)*(28-2*7^(1/2))^(1/2),
12*y*z-12*x*(28-2*7^(1/2))^(1/2)*y-12*x*y*(28+2*7^(1/2))^(1/2)+y*(28+2
*7^(1/2))^(1/2)*(28-2*7^(1/2))^(1/2)+30*y,
84*y^2+6*x*z*7^(1/2)*(28-2*7^(1/2))^(1/2)-6*(28+2*7^(1/2))^(1/2)*x*z*7
^(1/2)-84*x^2+x*(28-2*7^(1/2))^(1/2)*7^(1/2)-x*(28+2*7^(1/2))^(1/2)*7^(
1/2)-7*x*(28-2*7^(1/2))^(1/2)-7*x*(28+2*7^(1/2))^(1/2),
84*z^2-6*x*z*7^(1/2)*(28-2*7^(1/2))^(1/2)+6*(28+2*7^(1/2))^(1/2)*x*z*7
^(1/2)+168*x^2-x*(28-2*7^(1/2))^(1/2)*7^(1/2)+x*(28+2*7^(1/2))^(1/2)*7
^(1/2)+7*x*(28-2*7^(1/2))^(1/2)+7*x*(28+2*7^(1/2))^(1/2)-84,
504*x^2*z+15*x*z*7^(1/2)*(28-2*7^(1/2))^(1/2)-15*(28+2*7^(1/2))^(1/2)*
x*z*7^(1/2)+21*x*z*(28-2*7^(1/2))^(1/2)+21*(28+2*7^(1/2))^(1/2)*x*z-33
6*x^2-126*z-x*(28-2*7^(1/2))^(1/2)*7^(1/2)+x*(28+2*7^(1/2))^(1/2)*7^(1
/2)-14*x*(28-2*7^(1/2))^(1/2)-14*x*(28+2*7^(1/2))^(1/2)+126,
1008*x^3-15*x*z*(28+2*7^(1/2))^(1/2)*(28-2*7^(1/2))^(1/2)+210*x*z-30*x
^2*(28-2*7^(1/2))^(1/2)*7^(1/2)+30*x^2*(28+2*7^(1/2))^(1/2)*7^(1/2)+42
*x^2*(28-2*7^(1/2))^(1/2)+42*(28+2*7^(1/2))^(1/2)*x^2-9*x*z*(28-2*7^(1/
2))^(1/2)*7^(1/2)+9*x*z*(28+2*7^(1/2))^(1/2)*7^(1/2)+x*(28+2*7^(1/2))^(1
/2)*(28-2*7^(1/2))^(1/2)-420*x+9*(28-2*7^(1/2))^(1/2)*7^(1/2)-9*(28+2*7
^(1/2))^(1/2)*7^(1/2)};

```

$$\begin{aligned}
Gens_bis := & \{144x^2y - 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} + y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}}, \\
& 12yz - 12x\sqrt{28-2\sqrt{7}}y - 12xy\sqrt{28+2\sqrt{7}} + y\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 30y, \\
& 84y^2 + 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} - 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} - 84x^2 + x\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - x\sqrt{28+2\sqrt{7}}\sqrt{7} - 7x\sqrt{28-2\sqrt{7}} - 7x\sqrt{28+2\sqrt{7}}, 84z^2 - 6xz\sqrt{7}\sqrt{28-2\sqrt{7}} \\
& + 6\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 168x^2 - x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} \\
& + 7x\sqrt{28-2\sqrt{7}} + 7x\sqrt{28+2\sqrt{7}} - 84, 504x^2z + 15xz\sqrt{7}\sqrt{28-2\sqrt{7}} \\
& - 15\sqrt{28+2\sqrt{7}}xz\sqrt{7} + 21xz\sqrt{28-2\sqrt{7}} + 21\sqrt{28+2\sqrt{7}}xz - 336x^2 - 126z \\
& - x\sqrt{28-2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{7} - 14x\sqrt{28-2\sqrt{7}} - 14x\sqrt{28+2\sqrt{7}} + 126, \\
& 1008x^3 - 15xz\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} + 210xz - 30x^2\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 30x^2\sqrt{28+2\sqrt{7}}\sqrt{7} + 42x^2\sqrt{28-2\sqrt{7}} + 42\sqrt{28+2\sqrt{7}}x^2 - 9z\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& + 9z\sqrt{28+2\sqrt{7}}\sqrt{7} + x\sqrt{28+2\sqrt{7}}\sqrt{28-2\sqrt{7}} - 420x + 9\sqrt{28-2\sqrt{7}}\sqrt{7} \\
& - 9\sqrt{28+2\sqrt{7}}\sqrt{7}\}
\end{aligned}$$

11.9.1. tdeg(x,y,z)

```

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N7_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,z
):T:=termorder(A,tdeg(x,y,z)):EstSet_N7_tdeg:=sort(EstSet_N7_tdeg,(t1,
t2)->testorder(t1,t2,T));nops(EstSet_N7_tdeg);

```

$$EstSet_N7_tdeg := [1, z, y, x, z^2, yz, xz]$$

7

```

> Frontiere_tdegbis:=map(LeadingMonomial,GroebnerBasis(Gens_bis,tdeg(x,
y,z)),tdeg(x,y,z,t)): Divbis:=Diviseurs({op(Frontiere_tdegbis)}):
nops(Divbis):
EstSet_N7_tdegbis:=[op(EstSet(Divbis,Frontiere_tdegbis))]:A:=poly_alge
bra(x,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N7_tdegbis:=sort(EstSet_
N7_tdegbis,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N7_tdegbis);

```

$$EstSet_N7_tdegbis := [1, z, y, x, z^2, yz, xz]$$

7

> X_D_ModeleSat(N7_1,[x,y,z],EstSet_N7_tdeg);

$$\begin{aligned}
 & \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\sqrt{28-2\sqrt{7}}}{12}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \right. \\
 & \left. \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{28-2\sqrt{7}}}{12} \right] \\
 & \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, -\frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\sqrt{28-2\sqrt{7}}}{12}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, \right. \\
 & \left. -\frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{28-2\sqrt{7}}}{12} \right] \\
 & [1, 1, 0, 0, 1, 0, 0] \\
 & \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, -\frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\sqrt{28+2\sqrt{7}}}{12}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, \right. \\
 & \left. -\frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{28+2\sqrt{7}}}{12} \right] \\
 & \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, \frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\sqrt{28+2\sqrt{7}}}{12}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \right. \\
 & \left. \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{28+2\sqrt{7}}}{12} \right] \\
 & \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, 0, -\frac{\sqrt{28-2\sqrt{7}}}{6}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, 0, -\frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{28-2\sqrt{7}}}{6} \right] \\
 & \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, 0, -\frac{\sqrt{28+2\sqrt{7}}}{6}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, 0, -\frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{28+2\sqrt{7}}}{6} \right]
 \end{aligned}$$

> X_D_ModeleSat(N7_bis,[x,y,z],EstSet_N7_tdeg);

$$\begin{aligned}
& \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\sqrt{28-2\sqrt{7}}}{12}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \right. \\
& \left. \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{28-2\sqrt{7}}}{12} \right] \\
& \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, -\frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\sqrt{28-2\sqrt{7}}}{12}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, \right. \\
& \left. -\frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{28-2\sqrt{7}}}{12} \right] \\
& [1, 1, 0, 0, 1, 0, 0] \\
& \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, -\frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\sqrt{28+2\sqrt{7}}}{12}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, \right. \\
& \left. -\frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{28+2\sqrt{7}}}{12} \right] \\
& \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, \frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\sqrt{28+2\sqrt{7}}}{12}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \right. \\
& \left. \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{28+2\sqrt{7}}}{12} \right] \\
& \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, 0, -\frac{\sqrt{28-2\sqrt{7}}}{6}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, 0, -\frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{28-2\sqrt{7}}}{6} \right] \\
& \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, 0, -\frac{\sqrt{28+2\sqrt{7}}}{6}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, 0, -\frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{28+2\sqrt{7}}}{6} \right]
\end{aligned}$$

```
> M_D_ModelleSat(N7_1,[x,y,z],EstSet_N7_tdeg);
```


$$\left[\begin{array}{c} 1, 0, 0, 0, \%1, 0, 0 \\ 0, \%1, 0, 0, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3}{7} + \frac{1}{7}, 0, 0 \\ 0, 0, \frac{1}{3}, 0, 0, \%2, 0 \\ 0, 0, 0, \frac{1}{3}, 0, 0, \%2 \\ \%1, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3}{7} + \frac{1}{7}, 0, 0, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^4}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^4}{7} + \frac{1}{7}, 0, 0 \\ 0, 0, \%2, 0, 0, \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2 (28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2 (28 + 2\sqrt{7})}{168}, 0 \\ 0, 0, 0, \%2, 0, 0, \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2 (28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2 (28 + 2\sqrt{7})}{168} \end{array} \right]$$

$$\%1 := \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2}{7} + \frac{1}{7}$$

$$\%2 := \frac{(28 - 2\sqrt{7})(-\frac{1}{6} - \frac{\sqrt{7}}{6})}{168} + \frac{(28 + 2\sqrt{7})(-\frac{1}{6} + \frac{\sqrt{7}}{6})}{168}$$

> M_D_ModeleSat(N7_bis, [x,y,z], EstSet_N7_tdeg);

$$\left[\begin{array}{c} 1, 0, 0, 0, \%1, 0, 0 \\ 0, \%1, 0, 0, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3}{7} + \frac{1}{7}, 0, 0 \\ 0, 0, \frac{1}{3}, 0, 0, \%2, 0 \\ 0, 0, 0, \frac{1}{3}, 0, 0, \%2 \\ \%1, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3}{7} + \frac{1}{7}, 0, 0, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^4}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^4}{7} + \frac{1}{7}, 0, 0 \\ 0, 0, \%2, 0, 0, \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2 (28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2 (28 + 2\sqrt{7})}{168}, 0 \\ 0, 0, 0, \%2, 0, 0, \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2 (28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2 (28 + 2\sqrt{7})}{168} \end{array} \right]$$

$$\%1 := \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2}{7} + \frac{1}{7}$$

$$\%2 := \frac{(28 - 2\sqrt{7})(-\frac{1}{6} - \frac{\sqrt{7}}{6})}{168} + \frac{(28 + 2\sqrt{7})(-\frac{1}{6} + \frac{\sqrt{7}}{6})}{168}$$

> expand(Var_Blue_Modele_Sat(N7_1, [x,y,z], EstSet_N7_tdeg));

$$\frac{7}{3} + \frac{4}{3}z - \frac{14}{3}z^2 - 4z^3 + \frac{28}{9}y^2 + \frac{8}{3}zy^2 + \frac{28}{9}x^2 + \frac{8}{3}x^2z + 12z^4 + 16y^2z^2 + 16z^2x^2$$

> expand(Var_Blue_Modele_Sat(N7_bis, [x,y,z], EstSet_N7_tdegbis));

$$\frac{7}{3} + \frac{4}{3}z - \frac{14}{3}z^2 - 4z^3 + \frac{28}{9}y^2 + \frac{8}{3}zy^2 + \frac{28}{9}x^2 + \frac{8}{3}x^2z + 12z^4 + 16y^2z^2 + 16z^2x^2$$

11.9.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
  {op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
```

```
> EstSet_N7_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,
z):T:=termorder(A,plex(x,y,z)):EstSet_N7_plex:=sort(EstSet_N7_plex,(t1
,t2)->testorder(t1,t2,T));nops(EstSet_N7_plex);
```

$$EstSet_N7_plex := [1, z, z^2, y, yz, y^2, zy^2]$$

7

```
> Frontiere_plexbis:=map(LeadingMonomial,
  {op(GroebnerBasis(Gens_bis,plex(x,y,z)))},plex(x,y,z)):
Divbis:=Diviseurs({op(Frontiere_plex)}):nops(Divbis):
```

```
> EstSet_N7_plexbis:=[op(EstSet(Divbis,Frontiere_plexbis))]:A:=poly_alg
ebra(x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N7_plexbis:=sort(EstSet
_N7_plexbis,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N7_plexbis);
```

$$EstSet_N7_plexbis := [1, z, z^2, y, yz, y^2, zy^2]$$

7

```
> X_D_ModeleSat(N7_1,[x,y,z],EstSet_N7_plex);
```

$$\begin{aligned}
 & \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, \frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{7}{12} - \frac{\sqrt{7}}{24}, \right. \\
 & \left. \frac{(28-2\sqrt{7})\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)}{48} \right] \\
 & \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, -\frac{\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, -\frac{\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28-2\sqrt{7}}}{12}, \frac{7}{12} - \frac{\sqrt{7}}{24}, \right. \\
 & \left. \frac{(28-2\sqrt{7})\left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)}{48} \right] \\
 & [1, 1, 1, 0, 0, 0, 0] \\
 & \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, -\frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, -\frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{7}{12} + \frac{\sqrt{7}}{24}, \right. \\
 & \left. \frac{(28+2\sqrt{7})\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)}{48} \right] \\
 & \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, \frac{\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)\sqrt{3}\sqrt{28+2\sqrt{7}}}{12}, \frac{7}{12} + \frac{\sqrt{7}}{24}, \right. \\
 & \left. \frac{(28+2\sqrt{7})\left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)}{48} \right] \\
 & \left[1, -\frac{1}{6} - \frac{\sqrt{7}}{6}, \left(-\frac{1}{6} - \frac{\sqrt{7}}{6}\right)^2, 0, 0, 0, 0 \right] \\
 & \left[1, -\frac{1}{6} + \frac{\sqrt{7}}{6}, \left(-\frac{1}{6} + \frac{\sqrt{7}}{6}\right)^2, 0, 0, 0, 0 \right]
 \end{aligned}$$

```
> M_D_ModeleSat(N7_1,[x,y,z],EstSet_N7_plex);
```

$$\begin{aligned}
& \left[1, 0, \%1, 0, 0, \frac{1}{3}, \%2 \right] \\
& \left[0, \%1, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3}{7} + \frac{1}{7}, 0, 0, \right. \\
& \left. \frac{2(-\frac{1}{6} - \frac{\sqrt{7}}{6})(\frac{7}{12} - \frac{\sqrt{7}}{24})}{7} + \frac{2(-\frac{1}{6} + \frac{\sqrt{7}}{6})(\frac{7}{12} + \frac{\sqrt{7}}{24})}{7}, \%3 \right] \\
& \left[\%1, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3}{7} + \frac{1}{7}, \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^4}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^4}{7} + \frac{1}{7}, 0, 0, \right. \\
& \frac{2(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2(\frac{7}{12} - \frac{\sqrt{7}}{24})}{7} + \frac{2(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2(\frac{7}{12} + \frac{\sqrt{7}}{24})}{7}, \\
& \left. \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3(28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3(28 + 2\sqrt{7})}{168} \right] \\
& \left[0, 0, 0, \frac{1}{3}, \%2, 0, 0 \right] \\
& [0, 0, 0, \%2, \%3, 0, 0] \\
& \left[\frac{1}{3}, \frac{2(-\frac{1}{6} - \frac{\sqrt{7}}{6})(\frac{7}{12} - \frac{\sqrt{7}}{24})}{7} + \frac{2(-\frac{1}{6} + \frac{\sqrt{7}}{6})(\frac{7}{12} + \frac{\sqrt{7}}{24})}{7}, \frac{2(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2(\frac{7}{12} - \frac{\sqrt{7}}{24})}{7} + \right. \\
& \frac{2(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2(\frac{7}{12} + \frac{\sqrt{7}}{24})}{7}, 0, 0, \frac{2(\frac{7}{12} - \frac{\sqrt{7}}{24})^2}{7} + \frac{2(\frac{7}{12} + \frac{\sqrt{7}}{24})^2}{7}, \\
& \left. \frac{(\frac{7}{12} - \frac{\sqrt{7}}{24})(28 - 2\sqrt{7})(-\frac{1}{6} - \frac{\sqrt{7}}{6})}{168} + \frac{(\frac{7}{12} + \frac{\sqrt{7}}{24})(28 + 2\sqrt{7})(-\frac{1}{6} + \frac{\sqrt{7}}{6})}{168} \right] \\
& \left[\%2, \%3, \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^3(28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^3(28 + 2\sqrt{7})}{168}, 0, 0, \right. \\
& \frac{(\frac{7}{12} - \frac{\sqrt{7}}{24})(28 - 2\sqrt{7})(-\frac{1}{6} - \frac{\sqrt{7}}{6})}{168} + \frac{(\frac{7}{12} + \frac{\sqrt{7}}{24})(28 + 2\sqrt{7})(-\frac{1}{6} + \frac{\sqrt{7}}{6})}{168}, \\
& \left. \frac{(28 - 2\sqrt{7})^2(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2}{8064} + \frac{(28 + 2\sqrt{7})^2(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2}{8064} \right] \\
& \%1 := \frac{3(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2}{7} + \frac{3(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2}{7} + \frac{1}{7} \\
& \%2 := \frac{(28 - 2\sqrt{7})(-\frac{1}{6} - \frac{\sqrt{7}}{6})}{168} + \frac{(28 + 2\sqrt{7})(-\frac{1}{6} + \frac{\sqrt{7}}{6})}{168} \\
& \%3 := \frac{(-\frac{1}{6} - \frac{\sqrt{7}}{6})^2(28 - 2\sqrt{7})}{168} + \frac{(-\frac{1}{6} + \frac{\sqrt{7}}{6})^2(28 + 2\sqrt{7})}{168}
\end{aligned}$$

```
> expand(Var_Blue_Modele_Sat(N7_1,[x,y,z],EstSet_N7_plex));
```

$$\frac{36}{7}z + \frac{1856}{21}y^4z^2 - \frac{808}{63}zy^2 + \frac{180}{7}z^4 + \frac{64}{63}y^4z + \frac{2816}{189}y^4 + \frac{416}{7}y^2z^3 - \frac{272}{21}y^2z^2 + \frac{47}{7} - \frac{748}{63}y^2 - \frac{132}{7}z^3 - \frac{82}{7}z^2$$

```
> expand(Var_Blue_Modele_Sat(N7_bis,[x,y,z],EstSet_N7_plexbis));
```

$$\frac{36}{7}z + \frac{1856}{21}y^4z^2 - \frac{808}{63}zy^2 + \frac{180}{7}z^4 + \frac{64}{63}y^4z + \frac{2816}{189}y^4 + \frac{416}{7}y^2z^3 - \frac{272}{21}y^2z^2 + \frac{47}{7} - \frac{748}{63}y^2 - \frac{132}{7}z^3 - \frac{82}{7}z^2$$

11.9.3. Variance

Modèle de degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N7_1,1,0);
```

$$Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$$

```
> Varbis:=Var_Norm_Blue_D_Centre(N7_bis,1,0);
```

$$Varbis := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$$

```
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
```

$$1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

```
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Varbis,poly_algebra(seq(var[i],i=1..3)));
```

$$1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

```
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N7_1,2,0):
```

“M_D is not full rank”

```
> Var:=Var_Norm_Blue_D_Centre(N7_bis,2,0):
```

“M_D is not full rank”

11.10. N8

```
> GrN8:=Bn(3);
```

$$GrN8 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

```
> Norm(Vector([alpha,alpha,alpha]),2);alpha:=1/sqrt(3);N8:=OrbToMatrix(
OrbPG(Vector([alpha,alpha,alpha]),GrN8));
```

$$N8 := \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

```
> UpdatePlan(N8,"N8");
```

“Update complete”

```
> pointplot3d(N8,symbol=box,scaling=constrained,shading=Z);
```

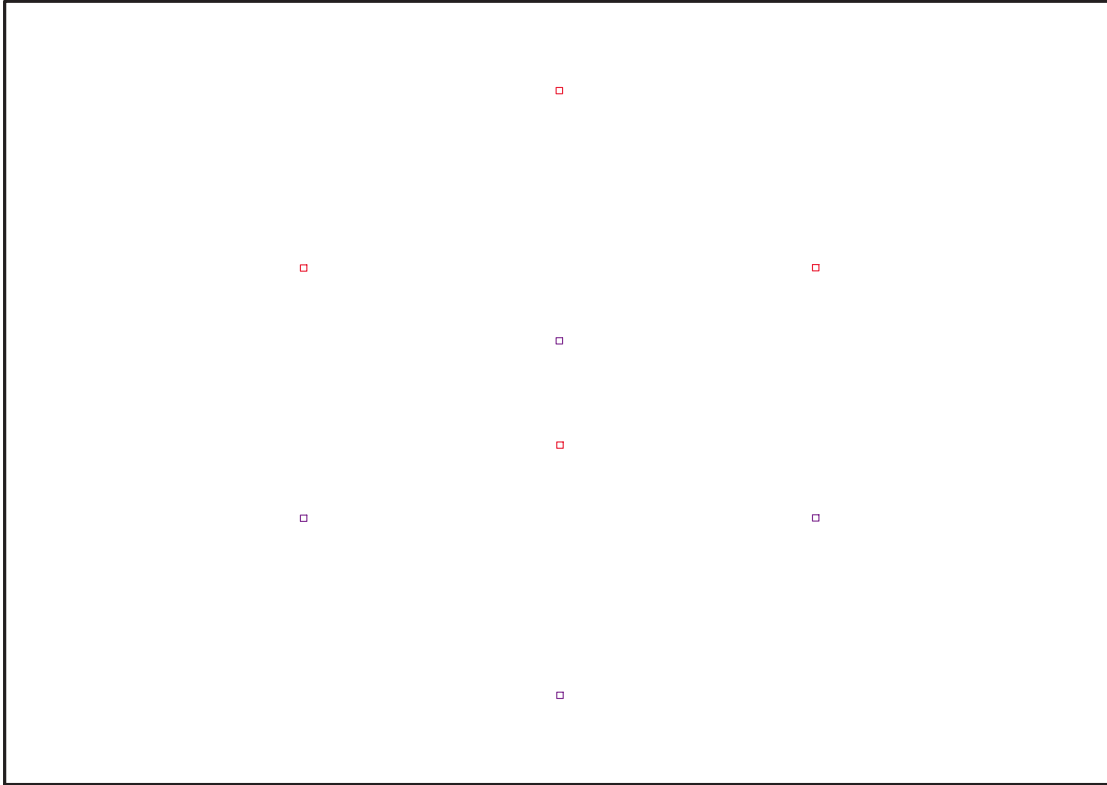


FIG. 11.16 – Graphique : Support du plan N8.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N8,listlist)))]),'rational',5)));
```

$Graph := \text{POLYTOPE}(3, 3, 8, 6)$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN8:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N8));
```

```
> Gens:=Generators(IdealN8);
```

$IdealN8 := \langle 3x^2 - 1, 3z^2 - 1, 3y^2 - 1 \rangle$

$Gens := \{3z^2 - 1, 3y^2 - 1, 3x^2 - 1\}$

```
> info_iop:=3;
```

$info_iop := 3$

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N8),listlist),Vector)));
```

$[[z^2 - \frac{1}{3}, y^2 - \frac{1}{3}, x^2 - \frac{1}{3}], [1, z, y, x, yz, xz, xy, xyz]]$

```
> Gens := {3*z^2-1, 3*y^2-1, 3*x^2-1};
```

$Gens := \{3z^2 - 1, 3y^2 - 1, 3x^2 - 1\}$

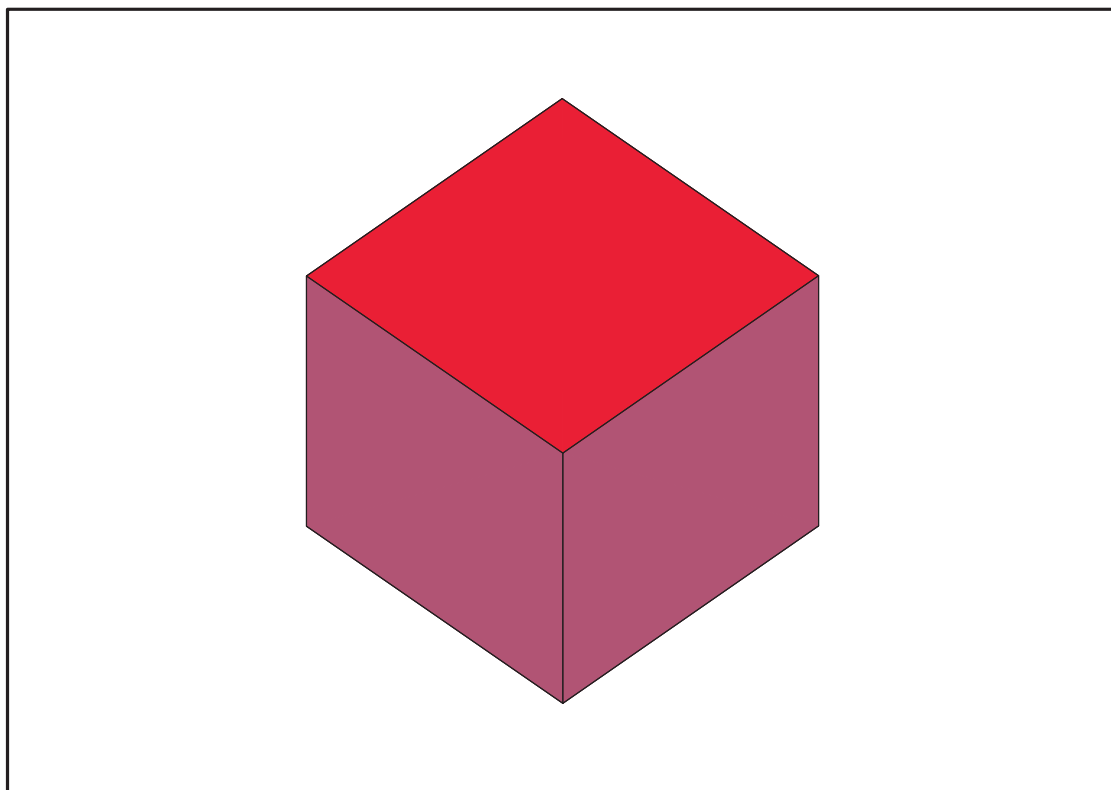


FIG. 11.17 – Graphique : Enveloppe convexe du support du plan N8.

11.10.1. $tdeg(x,y,z)$

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N8_tdeg:=op(EstSet(Div,Frontiere_tdeg)): A:=poly_algebra(x,y,z
):T:=termorder(A,tdeg(x,y,z)): EstSet_N8_tdeg:=sort(EstSet_N8_tdeg,(t1,
t2)->testorder(t1,t2,T)); nops(EstSet_N8_tdeg);
```

$$EstSet_N8_tdeg := [1, z, y, x, yz, xz, yx, zyx]$$

8

```
> X_D_ModeleSat(N8,[x,y,z],EstSet_N8_tdeg);
```


$$\begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} & -\frac{\sqrt{3}}{9} \\ 1 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{9} \\ 1 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} & -\frac{\sqrt{3}}{9} \\ 1 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{9} \end{bmatrix}$$

```
> M_D_ModeleSat(N8,[x,y,z],EstSet_N8_tdeg);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{27} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N8,[x,y,z],EstSet_N8_tdeg));
```

$$1 + 3z^2 + 3y^2 + 3x^2 + 9y^2z^2 + 9z^2x^2 + 9y^2x^2 + 27x^2y^2z^2$$

11.10.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
```

```
> EstSet_N8_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,
z):T:=termorder(A,plex(x,y,z)):EstSet_N8_plex:=sort(EstSet_N8_plex,(t1
,t2)->testorder(t1,t2,T));nops(EstSet_N8_plex);
```

$$EstSet_N8_plex := [1, z, y, yz, x, xz, yx, zyx]$$

8

```
> X_D_ModeleSat(N8,[x,y,z],EstSet_N8_plex);
```

$$\begin{bmatrix} 1 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{3}}{9} \\ 1 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{9} \\ 1 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{3}}{9} \\ 1 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{1}{3} & -\frac{\sqrt{3}}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{1}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{1}{3} & \frac{\sqrt{3}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{\sqrt{3}}{9} \\ 1 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{1}{3} & \frac{\sqrt{3}}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{\sqrt{3}}{9} \end{bmatrix}$$

> M_D_ModeleSat(N8,[x,y,z],EstSet_N8_plex);

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{27} \end{bmatrix}$$

> expand(Var_Blue_Modele_Sat(N8,[x,y,z],EstSet_N8_plex));

$$1 + 3z^2 + 3y^2 + 3x^2 + 9y^2z^2 + 9z^2x^2 + 9y^2x^2 + 27x^2y^2z^2$$

11.10.3. Variance

Modèle de degré 1

> Var:=Var_Norm_Blue_D_Centre(N8,1,0);

$$Var := 1 + 3var_3^2 + 3var_2^2 + 3var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
,Var,poly_algebra(seq(var[i],i=1..3)));

$$1 + 3 \text{IndetDecompInKfi}_1, \text{IndetDecompInKfi}_1 = var_1^2 + var_2^2 + var_3^2$$

Modèle de degré 2

> Var:=Var_Norm_Blue_D_Centre(N8,2,1):

“M_D is not full rank”

11.11. N9

```
> unassign('s','t','u','x');
```

```
> GrN9:=GentoGr(
{MatrixMatrixMultiply(Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(
Vector([-1/2,sqrt(3)/2,0])))});
```

$$GrN9 := \left\{ \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> GrN9_Tot:=GentoGr(
{Reflexion_Gen(Vector([1,0,0])),Reflexion_Gen(Vector([-1/2,sqrt(3)/2,
0])),Reflexion_Gen(Vector([0,0,1]))});
```

$$GrN9_Tot := \left\{ \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\left. \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> N9:=OrbToMatrix(OrbPG(Vector([s,0,x[1]]),GrN9) union
OrbPG(Vector([t,0,x[2]]),GrN9) union
OrbPG(Vector([u,0,x[3]]),GrN9));
```

$$N9 := \begin{bmatrix} t & 0 & x_2 \\ -\frac{t}{2} & -\frac{\sqrt{3}t}{2} & x_2 \\ -\frac{u}{2} & -\frac{\sqrt{3}u}{2} & x_3 \\ -\frac{s}{2} & \frac{\sqrt{3}s}{2} & x_1 \\ -\frac{s}{2} & -\frac{\sqrt{3}s}{2} & x_1 \\ u & 0 & x_3 \\ -\frac{u}{2} & \frac{\sqrt{3}u}{2} & x_3 \\ s & 0 & x_1 \\ -\frac{t}{2} & \frac{\sqrt{3}t}{2} & x_2 \end{bmatrix}$$

```
> UpdatePlan(N9,"N9");
```

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N9,1,0);

$$\begin{aligned} Var := & \frac{3}{2} \frac{x_2^2 + x_3^2 + x_1^2}{\%1} - \frac{3}{2} \frac{(x_2 + x_3 + x_1) var_3}{\%1} + \left(-\frac{3}{2} \frac{x_2 + x_3 + x_1}{\%1} + \frac{9}{2} \frac{var_3}{\%1}\right) var_3 + \frac{6 var_2^2}{t^2 + u^2 + s^2} \\ & + \frac{6 var_1^2}{t^2 + u^2 + s^2} \\ \%1 := & x_2^2 + x_3^2 + x_1^2 - x_2 x_3 - x_2 x_1 - x_3 x_1 \end{aligned}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

$$\begin{aligned} Cond := & [3 s^2 + 3 u^2 + 3 t^2 - 4 x_2^2 - 4 x_3^2 - 4 x_1^2 + 4 x_3 x_1 + 4 x_2 x_3 + 4 x_2 x_1, \\ & x_2^3 + x_1^3 - 3 x_2 x_3 x_1 + x_3^3] \end{aligned}$$

> Cond_gsolve:=gsolve([op(Cond),s^2+x[1]^2-1,t^2+x[2]^2-1,u^2+x[3]^2-1],{s,t,u,x[1],x[2],x[3]}):

> Sols:=[op(map(L->solve(convert(L[1],set) union {-1<=s, s<=1, -1<= t, t <= 1, -1<=u, u<=1}),Cond_gsolve))];

$$\begin{aligned} Sols := & [\{s = -1, x_2 = \%2, x_3 = -\%2, t = \%2, u = \%2, x_1 = 0\}, \\ & \{s = 1, x_2 = \%2, x_3 = -\%2, t = \%2, u = \%2, x_1 = 0\}, \\ & \{s = 1, x_2 = \%1, x_3 = -\%1, t = \%1, u = \%1, x_1 = 0\}, \\ & \{s = -1, x_2 = \%1, x_3 = -\%1, t = \%1, u = \%1, x_1 = 0\}] \\ \%1 := & \text{RootOf}(2_Z^2 - 1, 0.7071067812) \\ \%2 := & \text{RootOf}(2_Z^2 - 1, -0.7071067812) \end{aligned}$$

> IsRealSols(Sols);

$$[true, true, true, true]$$

> Res:=allvalues(op(2,[WhichRealSols(Sols)]));

$$\begin{aligned} Res := & [\{s = -1, x_1 = 0, x_2 = -\frac{\sqrt{2}}{2}, x_3 = \frac{\sqrt{2}}{2}, t = -\frac{\sqrt{2}}{2}, u = -\frac{\sqrt{2}}{2}\}, \\ & \{s = 1, x_1 = 0, x_2 = -\frac{\sqrt{2}}{2}, x_3 = \frac{\sqrt{2}}{2}, t = -\frac{\sqrt{2}}{2}, u = -\frac{\sqrt{2}}{2}\}, \\ & \{s = 1, x_2 = \frac{\sqrt{2}}{2}, x_3 = -\frac{\sqrt{2}}{2}, t = \frac{\sqrt{2}}{2}, u = \frac{\sqrt{2}}{2}, x_1 = 0\}, \\ & \{s = -1, x_2 = \frac{\sqrt{2}}{2}, x_3 = -\frac{\sqrt{2}}{2}, t = \frac{\sqrt{2}}{2}, u = \frac{\sqrt{2}}{2}, x_1 = 0\}] \end{aligned}$$

On obtient ainsi 2 solutions possibles : les trois orbites alignées ou l'orbite centrale en décalage par rapport aux deux autres.

> unassign('s','t','u','x');assign(Res[1]);

> N9:=OrbToMatrix(OrbPG(Vector([-sqrt(2)/2,0,-sqrt(2)/2]),GrN9) union
OrbPG(Vector([-1,0,0]),GrN9) union
OrbPG(Vector([-sqrt(2)/2,0,sqrt(2)/2]),GrN9));;

$$N9 := \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}\sqrt{3}}{4} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}\sqrt{3}}{4} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

```
> UpdatePlan(N9,"N9");
"Update complete"
> pointplot3d(N9,symbol=box,scaling=constrained,shading=Z);
```

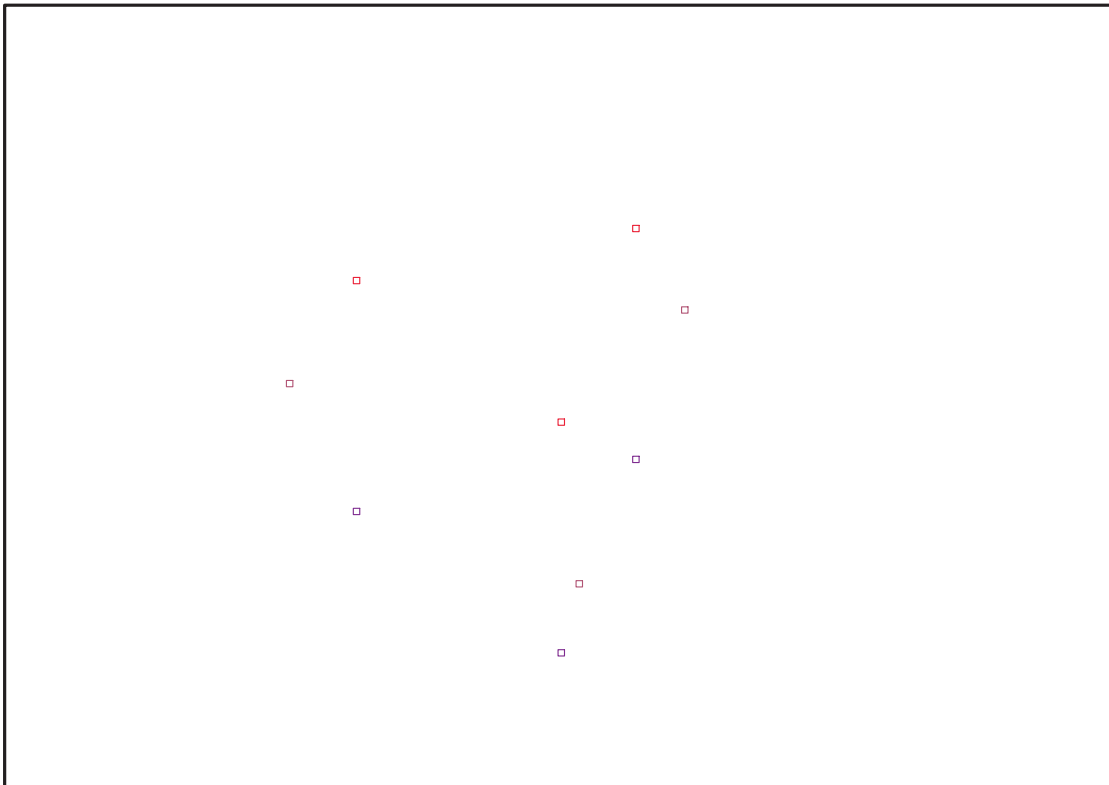


FIG. 11.18 – Graphique : Support du plan N9.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N9,listlist)))],'rational',9))));
```

```

Graph := POLYTOPE(3, 3, 9, 12)

> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z);

```

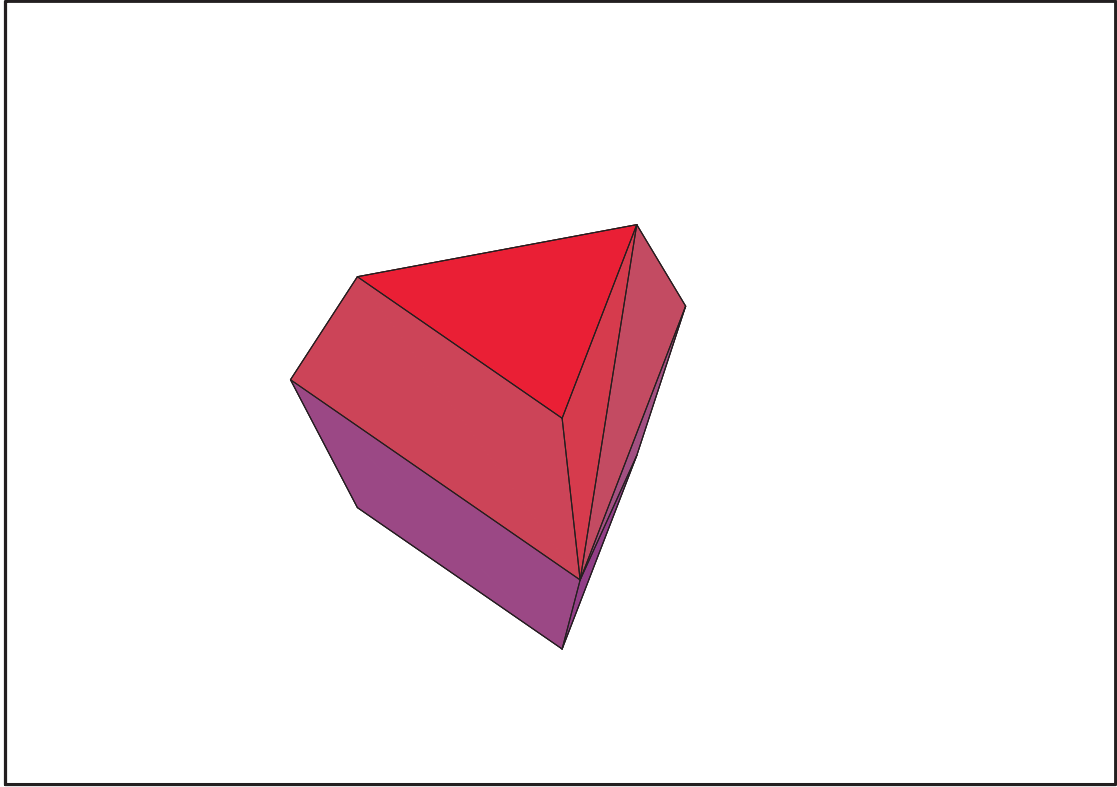


FIG. 11.19 – Graphique : Enveloppe convexe du support du plan N9.

```

> ReadPlan("N9"):N9:=Plan:
> unassign('x','y','z','u','r','s','t');
> IdealN9:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N9));
> Gens:=Generators(IdealN9);

IdealN9 := <8y3 - 6xy√2 - 12xy + 3y√2, 8yx2 - 4xy + y√2 - 2xy√2, z2 + y2 + x2 - 1,
4y2z - xz√2 - z, 4zx2 + xz√2 - z, 4xyz - yz√2,
8xy2 - 6y2 + 2√2x2 - 2x2 + x√2 - √2 + 2,
8x3 - 4√2y2 - 2y2 + 2√2x2 + 10x2 + 3x√2 + √2 - 2>

Gens := {z2 + y2 + x2 - 1, 8yx2 - 4xy + y√2 - 2xy√2, 8y3 - 6xy√2 - 12xy + 3y√2,
4xyz - yz√2, 4zx2 + xz√2 - z, 4y2z - xz√2 - z,
8xy2 - 6y2 + 2√2x2 - 2x2 + x√2 - √2 + 2,
8x3 - 4√2y2 - 2y2 + 2√2x2 + 10x2 + 3x√2 + √2 - 2}

> info_iop:=3;

info_iop := 3

```

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N9),1
istlist),Vector)));
```

1
2
3
4
5
6
7
8
9
9
9
9
9
9
9
9

$$\begin{aligned} & \left[-1 + \frac{3(-2 + \sqrt{2})z}{8} + y^2 + \frac{(8 + 5\sqrt{2})x}{4} - \frac{x\sqrt{2}}{4} + x^2 + \frac{(4 + \sqrt{2})z}{7} - \frac{(\sqrt{2} - 1)z^2}{4} + \frac{(3 + \sqrt{2})z^2}{4} \right. \\ & - \frac{(29\sqrt{2} - 10)z}{56} - (2 + \sqrt{2})x, -\frac{1}{2}z + z^3, \\ & \frac{(5 + 3\sqrt{2})y}{7} + (2 + \sqrt{2})xy - \frac{(24 + 13\sqrt{2})y}{14} + z^2y, -1 - \frac{\sqrt{2}z^2}{4} + \frac{(4 + 3\sqrt{2})z^2}{4} \\ & - \frac{(8 + 5\sqrt{2})x}{2} - \frac{(1 + 2\sqrt{2})z}{4} + z^2x - \frac{3z}{4} - \frac{(1 + \sqrt{2})x}{2} + \frac{(7 + 5\sqrt{2})x}{2} + \frac{(2 + \sqrt{2})z}{2} \\ & + (2 + \sqrt{2})y^2 - \frac{\sqrt{2}}{2}, -\frac{xz\sqrt{2}}{4} - \frac{(4 + 15\sqrt{2})z}{28} + y^2z + \frac{3(-1 + 5\sqrt{2})z}{28}, xyz - \frac{1}{4}yz\sqrt{2}, \\ & -\frac{3(5 + 3\sqrt{2})y}{28} + y^3 - \frac{3(2 + \sqrt{2})xy}{4} + \frac{3(13\sqrt{2} + 10)y}{56}, xy^2 - \frac{(1 + 3\sqrt{2})x}{8} + \frac{\sqrt{2}z^2}{16} \\ & - \frac{3(3\sqrt{2} - 2)z}{56} + \frac{3z}{16} + \frac{3x\sqrt{2}}{8} + \frac{3(6\sqrt{2} - 11)z}{112} + \frac{(1 + \sqrt{2})x}{8} - \frac{(5\sqrt{2} - 4)z^2}{16} \\ & \left. - \frac{(2 + \sqrt{2})y^2}{4} + \frac{\sqrt{2}}{8} \right], [1, z, y, x, z^2, yz, xz, y^2, xy]] \end{aligned}$$

```
> Gens := {z^2+y^2+x^2-1, 8*y*x^2-4*x*y+y*2^(1/2)-2*x*y*2^(1/2),
8*y^3-6*x*y*2^(1/2)-12*x*y+3*y*2^(1/2), 4*x*y*z-y*z*2^(1/2),
4*z*x^2+x*z*2^(1/2)-z, 4*y^2*z-x*z*2^(1/2)-z,
8*x*y^2-6*y^2+2*2^(1/2)*x^2-2*x^2+x*2^(1/2)-2^(1/2)+2,
8*x^3-4*2^(1/2)*y^2-2*y^2+2*2^(1/2)*x^2+10*x^2+3*x*2^(1/2)+2^(1/2)-2
};
```

$$\begin{aligned} Gens := & \{z^2 + y^2 + x^2 - 1, 8yx^2 - 4xy + y\sqrt{2} - 2xy\sqrt{2}, 8y^3 - 6xy\sqrt{2} - 12xy + 3y\sqrt{2}, \\ & 4xyz - yz\sqrt{2}, 4zx^2 + xz\sqrt{2} - z, 4y^2z - xz\sqrt{2} - z, \\ & 8xy^2 - 6y^2 + 2\sqrt{2}x^2 - 2x^2 + x\sqrt{2} - \sqrt{2} + 2, \\ & 8x^3 - 4\sqrt{2}y^2 - 2y^2 + 2\sqrt{2}x^2 + 10x^2 + 3x\sqrt{2} + \sqrt{2} - 2\} \end{aligned}$$

```
> unassign('s','t','u','x');assign(Res[2]);
```

```
> N9bis:=OrbToMatrix(OrbPG(Vector([sqrt(2)/2,0,sqrt(2)/2]),GrN9) union
OrbPG(Vector([-1,0,0]),GrN9) union
OrbPG(Vector([sqrt(2)/2,0,-sqrt(2)/2]),GrN9));;
```

$$N9bis := \begin{bmatrix} -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}\sqrt{3}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}\sqrt{3}}{4} & -\frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

```
> UpdatePlan(N9bis,"N9bis");
```

“Update complete”

```
> pointplot3d(N9bis,symbol=box,scaling=constrained,shading=Z);
```

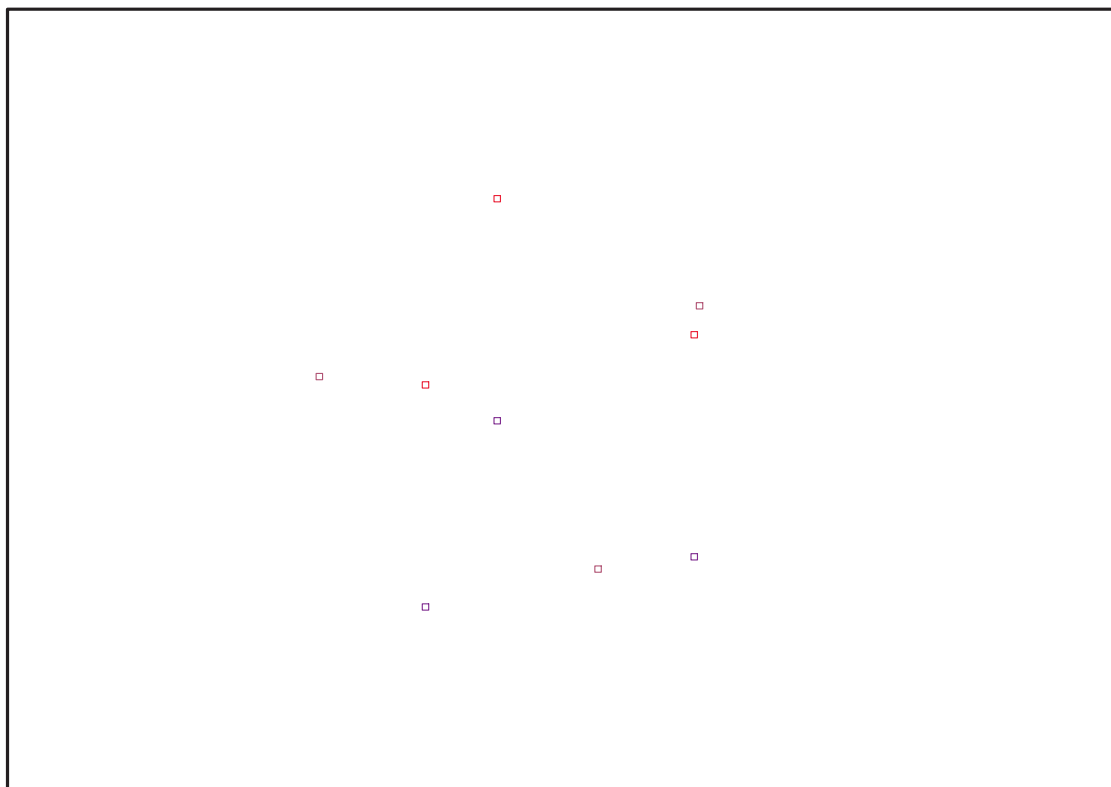


FIG. 11.20 – Graphique : Support du plan N9bis.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N9bis,listlist)))]),'
rational',11)));
```


$Graph := \text{POLYTOPE}(3, 3, 9, 14)$

```
> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z);
```

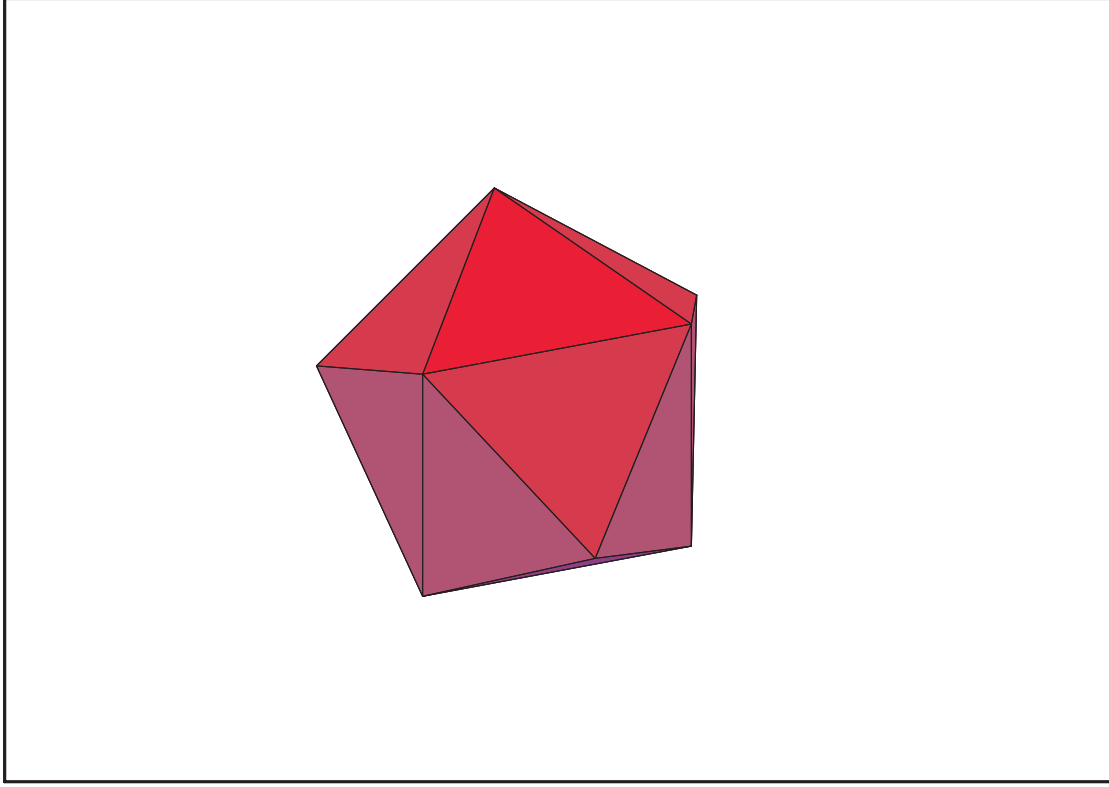


FIG. 11.21 – Graphique : Enveloppe convexe du support du plan N9bis.

```
> ReadPlan("N9bis"):N9bis:=Plan:
> unassign('x','y','z','u','r','s','t');
> IdealN9bis:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N9bis));
> Gensbis:=Generators(IdealN9bis);

IdealN9bis := < z2 + y2 + x2 - 1, 4xy z + y z √2,
8x3 + 4√2y2 - 2y2 - 2√2x2 + 10x2 - 3x√2 - √2 - 2,
8xy2 - 6y2 - 2√2x2 - 2x2 - x√2 + √2 + 2, 4y2z + xz√2 - z, 4zx2 - xz√2 - z,
8y3 + 6xy√2 - 12xy - 3y√2, 8yx2 + 2xy√2 - 4xy - y√2 >

Gensbis := {8xy2 + 2xy√2 - 4xy - y√2, 8y3 + 6xy√2 - 12xy - 3y√2, z2 + y2 + x2 - 1,
4zx2 - xz√2 - z, 4y2z + xz√2 - z, 8xy2 - 6y2 - 2√2x2 - 2x2 - x√2 + √2 + 2,
8x3 + 4√2y2 - 2y2 - 2√2x2 + 10x2 - 3x√2 - √2 - 2, 4xyz + yz√2}

> info_iop:=3;

info_iop := 3
```

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N9bis
),listlist),Vector)));
```

1
2
3
4
5
6
7
8
9
9
9
9
9
9
9
9

$$\begin{aligned}
& \left[-1 + y^2 - \frac{(2 + 3\sqrt{2})z^2}{4} + x^2 + \frac{3(\sqrt{2} + 2)z^2}{4}, -\frac{1}{2}z + z^3, \frac{\sqrt{3}(16\sqrt{2} + 7)z}{24} - \frac{5(1 + \sqrt{2})y}{6} \right. \\
& + \frac{(8\sqrt{2} - 1)y}{6} - (-2 + \sqrt{2})xy - \frac{\sqrt{3}(1 + \sqrt{2})z}{3} - \frac{\sqrt{3}(8\sqrt{2} - 1)z}{24} + z^2y, \\
& -(-2 + \sqrt{2})y^2 + z^2x + \frac{x}{2} + \frac{(\sqrt{2} - 3)x}{2} - \frac{(-1 + 2\sqrt{2})z^2}{2} + \frac{(1 + \sqrt{2})z^2}{2} + \frac{\sqrt{2}}{2} - 1, \\
& \frac{xz\sqrt{2}}{4} + \frac{(7 + 8\sqrt{2})z}{16} + y^2z - \frac{(11 + 8\sqrt{2})z}{16}, xyz + \frac{1}{4}yz\sqrt{2}, \frac{\sqrt{3}(5 + 8\sqrt{2})z}{32} \\
& - \frac{\sqrt{3}(16\sqrt{2} + 13)z}{32} + \frac{5(1 + \sqrt{2})y}{8} - \frac{(5 + 8\sqrt{2})y}{8} + y^3 + \frac{3(-2 + \sqrt{2})xy}{4} \\
& + \frac{\sqrt{3}(1 + \sqrt{2})z}{4}, \\
& \left. xy^2 - \frac{(\sqrt{2} - 3)x}{8} + \frac{(-2 + \sqrt{2})y^2}{4} + \frac{(-1 + 2\sqrt{2})z^2}{8} - \frac{3x}{8} + \frac{3z^2}{8} - \frac{\sqrt{2}}{8} \right], \\
& [1, z, y, x, z^2, yz, xz, y^2, xy]]
\end{aligned}$$

```
> Gensbis := {8*y*x^2+2*x*y*2^(1/2)-4*x*y-y*2^(1/2),
8*y^3+6*x*y*2^(1/2)-12*x*y-3*y*2^(1/2), z^2+y^2+x^2-1,
4*z*x^2-x*z*2^(1/2)-z, 4*y^2*z+x*z*2^(1/2)-z,
8*x*y^2-6*y^2-2*2^(1/2)*x^2-2*x^2-x*2^(1/2)+2^(1/2)+2,
8*x^3+4*2^(1/2)*y^2-2*y^2-2*2^(1/2)*x^2+10*x^2-3*x*2^(1/2)-2^(1/2)-2,
4*x*y*z+y*z*2^(1/2)};
```

$$\begin{aligned}
& Gensbis := \{8yx^2 + 2xy\sqrt{2} - 4xy - y\sqrt{2}, 8y^3 + 6xy\sqrt{2} - 12xy - 3y\sqrt{2}, z^2 + y^2 + x^2 - 1, \\
& 4zx^2 - xz\sqrt{2} - z, 4y^2z + xz\sqrt{2} - z, 8xy^2 - 6y^2 - 2\sqrt{2}x^2 - 2x^2 - x\sqrt{2} + \sqrt{2} + 2, \\
& 8x^3 + 4\sqrt{2}y^2 - 2y^2 - 2\sqrt{2}x^2 + 10x^2 - 3x\sqrt{2} - \sqrt{2} - 2, 4xyz + yz\sqrt{2}\}
\end{aligned}$$

11.11.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),tdeg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div): EstSet_N9_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N9_tdeg:=sort(EstSet_N9_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N9_tdeg);
```

$$EstSet_N9_tdeg := [1, z, y, x, z^2, yz, xz, y^2, xy]$$

9

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gensbis,tdeg(x,y,z)),tdeg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div): EstSet_N9_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N9_tdeg:=sort(EstSet_N9_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N9_tdeg);
```

$$EstSet_N9_tdeg := [1, z, y, x, z^2, yz, xz, y^2, xy]$$

9

```
> X_D_ModeleSat(N9,[x,y,z],EstSet_N9_tdeg);
```

$$\begin{bmatrix} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{3}{8} & \frac{\sqrt{3}}{8} \\ 1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{3}{8} & -\frac{\sqrt{3}}{8} \\ 1 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{3}{8} & \frac{\sqrt{3}}{8} \\ 1 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{3}{8} & -\frac{\sqrt{3}}{8} \\ 1 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & -\frac{\sqrt{3}}{4} \end{bmatrix}$$

```
> M_D_ModeleSat(N9,[x,y,z],EstSet_N9_tdeg);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} + \frac{\sqrt{2}}{24} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{12} + \frac{\sqrt{2}}{24} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{12} + \frac{\sqrt{2}}{24} & \frac{1}{12} & 0 & 0 & \frac{3}{16} & 0 \\ 0 & 0 & \frac{1}{12} + \frac{\sqrt{2}}{24} & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix}$$

> expand(Var_Blue_Modelle_Sat(N9,[x,y,z],EstSet_N9_tdeg));

$$\begin{aligned} & \frac{12xz^2\sqrt{2}}{-3+2\sqrt{2}} + \frac{48xy^2\sqrt{2}}{-3+2\sqrt{2}} + 12x^2z^2 + 12z^2y^2 + \frac{78y^2}{-3+2\sqrt{2}} - \frac{33}{-3+2\sqrt{2}} - \frac{96y^4}{-3+2\sqrt{2}} \\ & + \frac{84z^2}{-3+2\sqrt{2}} + \frac{6\sqrt{2}}{-3+2\sqrt{2}} - \frac{18x^2}{-3+2\sqrt{2}} - \frac{78z^4}{-3+2\sqrt{2}} - \frac{24x}{-3+2\sqrt{2}} + 3z^2 - \frac{24z^2\sqrt{2}}{-3+2\sqrt{2}} \\ & - \frac{12x\sqrt{2}}{-3+2\sqrt{2}} - \frac{96z^2y^2}{-3+2\sqrt{2}} + \frac{36z^4\sqrt{2}}{-3+2\sqrt{2}} + \frac{24xz^2}{-3+2\sqrt{2}} + \frac{96xy^2}{-3+2\sqrt{2}} - \frac{96x^2y^2}{-3+2\sqrt{2}} \end{aligned}$$

> expand(Var_Blue_Modelle_Sat(N9bis,[x,y,z],EstSet_N9_tdeg));

$$\begin{aligned} & \frac{12xz^2\sqrt{2}}{3+2\sqrt{2}} + \frac{48xy^2\sqrt{2}}{3+2\sqrt{2}} + 12x^2z^2 + 12z^2y^2 - \frac{96xy^2}{3+2\sqrt{2}} - \frac{24z^2\sqrt{2}}{3+2\sqrt{2}} - \frac{12x\sqrt{2}}{3+2\sqrt{2}} - \frac{24xz^2}{3+2\sqrt{2}} \\ & + \frac{96z^2y^2}{3+2\sqrt{2}} + \frac{36z^4\sqrt{2}}{3+2\sqrt{2}} + \frac{96x^2y^2}{3+2\sqrt{2}} + \frac{33}{3+2\sqrt{2}} + 3z^2 + \frac{24x}{3+2\sqrt{2}} - \frac{78y^2}{3+2\sqrt{2}} \\ & - \frac{84z^2}{3+2\sqrt{2}} + \frac{6\sqrt{2}}{3+2\sqrt{2}} + \frac{78z^4}{3+2\sqrt{2}} + \frac{18x^2}{3+2\sqrt{2}} + \frac{96y^4}{3+2\sqrt{2}} \end{aligned}$$

11.11.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
> EstSet_N9_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,
z):T:=termorder(A,plex(x,y,z)):EstSet_N9_plex:=sort(EstSet_N9_plex,(t1
,t2)->testorder(t1,t2,T));nops(EstSet_N9_plex);
```

$$EstSet_N9_plex := [1, z, z^2, y, yz, z^2y, y^2, y^2z, z^2y^2]$$

9

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gensbis,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
> EstSet_N9_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,
z):T:=termorder(A,plex(x,y,z)):EstSet_N9_plex:=sort(EstSet_N9_plex,(t1
,t2)->testorder(t1,t2,T));nops(EstSet_N9_plex);
```

$$EstSet_N9_plex := [1, z, z^2, y, yz, z^2y, y^2, y^2z, z^2y^2]$$

9

> X_D_ModeleSat(N9,[x,y,z],EstSet_N9_plex);

$$\begin{bmatrix} 1 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 1 & \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{2}\sqrt{3}}{8} & \frac{3}{8} & \frac{3\sqrt{2}}{16} & \frac{3}{16} \\ 1 & \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{2}\sqrt{3}}{8} & \frac{3}{8} & \frac{3\sqrt{2}}{16} & \frac{3}{16} \\ 1 & -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{\sqrt{2}\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{2}\sqrt{3}}{8} & \frac{3}{8} & -\frac{3\sqrt{2}}{16} & \frac{3}{16} \\ 1 & -\frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{\sqrt{2}\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{2}\sqrt{3}}{8} & \frac{3}{8} & -\frac{3\sqrt{2}}{16} & \frac{3}{16} \\ 1 & \frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\sqrt{2}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & \frac{3}{4} & 0 & 0 \end{bmatrix}$$

> M_D_ModeleSat(N9,[x,y,z],EstSet_N9_plex);

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{12} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{3}{16} & 0 & \frac{1}{32} \\ 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & 0 \\ \frac{1}{12} & 0 & \frac{1}{24} & 0 & 0 & 0 & \frac{1}{32} & 0 & \frac{1}{64} \end{bmatrix}$$

> expand(Var_Blue_Modele_Sat(N9,[x,y,z],EstSet_N9_plex));

$$9 - 27z^2 - 18y^2 + 36z^2y^2 + 54z^4 - 144z^4y^2 + 24y^4 + 288z^4y^4$$

> expand(Var_Blue_Modele_Sat(N9bis,[x,y,z],EstSet_N9_plex));

$$9 - 27z^2 - 18y^2 + 36z^2y^2 + 54z^4 - 144z^4y^2 + 24y^4 + 288z^4y^4$$

11.11.3. Variance

Modèle de degré 1

> Var:=Var_Norm_Blue_D_Centre(N9,1,0);

```

Var := 1 + 3 var32 + 3 var22 + 3 var12
> Varbis:=Var_Norm_Blue_D_Centre(N9bis,1,0);
Varbis := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Varbis,poly_algebra(seq(var[i],i=1..3)));
1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):

```

Modèle de degré 2

```

> Var:=radnormal(expand(Var_Norm_Blue_D_Centre(N9,2,1)));
Var :=  $\frac{3}{17}(7 + 4\sqrt{2})(21 - 12\sqrt{2} - 168 var_1 var_2^2 + 136 var_1^2 var_2^2 - 32 var_2^2 var_3^2 \sqrt{2} - 32 var_1^2 var_3^2 \sqrt{2} - 108 var_1 var_2^2 \sqrt{2} + 56 var_2^2 var_3^2 + 56 var_1^2 var_3^2 + 36 var_1^3 \sqrt{2} + 36 var_1^2 \sqrt{2} - 24 var_3^4 \sqrt{2} + 20 var_3^2 \sqrt{2} + 36 var_2^2 \sqrt{2} + 68 var_2^4 + 68 var_1^4 - 35 var_3^2 - 12 var_2^2 - 12 var_1^2 + 42 var_3^4 + 56 var_1^3)$ 
> Varbis:=radnormal(expand(Var_Norm_Blue_D_Centre(N9bis,2,1)));
Varbis :=  $\frac{3}{17}(-7 + 4\sqrt{2})(-21 - 12\sqrt{2} + 168 var_1 var_2^2 - 136 var_1^2 var_2^2 - 32 var_2^2 var_3^2 \sqrt{2} - 32 var_1^2 var_3^2 \sqrt{2} - 108 var_1 var_2^2 \sqrt{2} - 56 var_2^2 var_3^2 - 56 var_1^2 var_3^2 + 36 var_1^3 \sqrt{2} + 36 var_1^2 \sqrt{2} - 24 var_3^4 \sqrt{2} + 20 var_3^2 \sqrt{2} + 36 var_2^2 \sqrt{2} - 68 var_2^4 - 68 var_1^4 + 35 var_3^2 + 12 var_2^2 + 12 var_1^2 - 42 var_3^4 - 56 var_1^3)$ 
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},subs(sqrt(2)=s,Var),poly_algebra(seq(var[i],i=1..3),alg_relations=
{s^2-2}));

$$\frac{2046}{17} var_3^4 - \frac{12144}{17} var_1 var_2^2 - \frac{3036}{17} var_1 var_3^2 - \frac{3696}{17} var_3^2 \%1 - \frac{1287}{17} var_3^2 + \frac{3036}{17} var_1 \%1 + 132 \%1^2 + \frac{792}{17} \%1 + \frac{297}{17}, \%1 = var_1^2 + var_2^2 + var_3^2$$


$$\%1 := IndetDecompInKfi_1$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},subs(sqrt(2)=s,Varbis),poly_algebra(seq(var[i],i=1..3)));

$$\frac{414}{17} var_3^4 - \frac{720}{17} var_1 var_2^2 - \frac{180}{17} var_1 var_3^2 - \frac{432}{17} var_3^2 \%1 - \frac{63}{17} var_3^2 + \frac{180}{17} var_1 \%1 + 36 \%1^2 - \frac{432}{17} \%1 + \frac{297}{17}, \%1 = var_1^2 + var_2^2 + var_3^2$$


$$\%1 := IndetDecompInKfi_1$$


```

11.12. N10

```
> GrN10:=Plus(Rned(Diedn(5),3));unassign('eta','delta','nu');
```

$$GrN10 := \left\{ \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%3 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%2 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%1 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\left. \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%4 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

$$\%1 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%2 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%3 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%4 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

```
> N10:=Map(radnormal,OrbToMatrix(OrbPG(Vector([eta,nu,delta]),GrN10)));
```

$$N10 := \begin{bmatrix} \frac{(\sqrt{5}-1)(\%2 + \%1 + 4\eta)}{16} & \frac{(\sqrt{5}-1)(4\nu - \%4 - \%3)}{16} & \delta \\ -\frac{(\sqrt{5}-1)(\%2 + \%1 - 4\eta)}{16} & -\frac{(\sqrt{5}-1)(4\nu + \%4 + \%3)}{16} & -\delta \\ \frac{(1+\sqrt{5})(-3\sqrt{2}\sqrt{5+\sqrt{5}}\nu + \%1 - 4\eta)}{16} & \frac{(1+\sqrt{5})(4\nu + \%3 - 3\sqrt{2}\sqrt{5+\sqrt{5}}\eta)}{16} & -\delta \\ -\frac{(1+\sqrt{5})(\%1 - 3\sqrt{2}\sqrt{5+\sqrt{5}}\nu + 4\eta)}{16} & -\frac{(1+\sqrt{5})(4\nu + 3\sqrt{2}\sqrt{5+\sqrt{5}}\eta - \%3)}{16} & \delta \\ -\frac{(1+\sqrt{5})(\%1 - 3\sqrt{2}\sqrt{5+\sqrt{5}}\nu + 4\eta)}{16} & \frac{(1+\sqrt{5})(4\nu + 3\sqrt{2}\sqrt{5+\sqrt{5}}\eta - \%3)}{16} & -\delta \\ \frac{(1+\sqrt{5})(-3\sqrt{2}\sqrt{5+\sqrt{5}}\nu + \%1 - 4\eta)}{16} & -\frac{(1+\sqrt{5})(4\nu + \%3 - 3\sqrt{2}\sqrt{5+\sqrt{5}}\eta)}{16} & \delta \\ \frac{\eta}{16} & \frac{\nu}{16} & \delta \\ \frac{(\sqrt{5}-1)(\%2 + \%1 + 4\eta)}{16} & -\frac{(\sqrt{5}-1)(4\nu - \%4 - \%3)}{16} & -\delta \\ -\frac{(\sqrt{5}-1)(\%2 + \%1 - 4\eta)}{16} & \frac{(\sqrt{5}-1)(4\nu + \%4 + \%3)}{16} & \delta \\ \eta & -\nu & -\delta \end{bmatrix}$$

$$\%1 := \nu \sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{5}$$

$$\%2 := \sqrt{2} \sqrt{5 + \sqrt{5}} \nu$$

$$\%3 := \eta \sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{5}$$

$$\%4 := \sqrt{2} \sqrt{5 + \sqrt{5}} \eta$$

> UpdatePlan(N10,"N10");

“Update complete”

> Var:=(Var_Norm_Blue_D_Centre(N10,1,0));

$$Var := 1 + \frac{var_3^2}{\delta^2} + \frac{2 var_2^2}{\eta^2 + \nu^2} + \frac{2 var_1^2}{\eta^2 + \nu^2}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},[Var]);

$$Cond := [\eta^2 + \nu^2 - 2\delta^2]$$

> Cond_gsolved:=convert(gsolve([op(Cond),delta^2+nu^2+eta^2-1],{delta,eta}),list);

$$Cond_gsolved := [[[3\delta^2 - 1, 3\eta^2 + 3\nu^2 - 2], \text{plex}(\eta, \delta), \{\}]]$$

> Sols:=map(L->solve({op(L[1])},{delta,eta}),Cond_gsolved);

$$Sols := [\{\eta = \text{RootOf}(3_Z^2 + 3\nu^2 - 2, \text{label} = _L11), \delta = \text{RootOf}(3_Z^2 - 1, \text{label} = _L12)\}]$$

> allvalues(Sols);

$$\begin{aligned} & [\{\delta = \frac{\sqrt{3}}{3}, \eta = \sqrt{-\nu^2 + \frac{2}{3}}\}, \{\delta = \frac{\sqrt{3}}{3}, \eta = -\sqrt{-\nu^2 + \frac{2}{3}}\}, \{\delta = -\frac{\sqrt{3}}{3}, \eta = \sqrt{-\nu^2 + \frac{2}{3}}\}, \\ & \{\delta = -\frac{\sqrt{3}}{3}, \eta = -\sqrt{-\nu^2 + \frac{2}{3}}\}] \end{aligned}$$

nu peut varier entre 0 et sqrt(2)/sqrt(3).

> n:=0; eta := sqrt(2)*sqrt(3)/3; delta := sqrt(3)/3;

$$\begin{aligned} n &:= 0 \\ \eta &:= \frac{\sqrt{2}\sqrt{3}}{3} \\ \delta &:= \frac{\sqrt{3}}{3} \end{aligned}$$

```
> N10:=Map(radnormal,OrbToMatrix(OrbPG(Vector([sqrt(2)*sqrt(3)/3,0,sqrt(3)/3]),GrN10))));
```

$$N10 := \begin{bmatrix} \%1 & \frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{6} & -\frac{\sqrt{3}}{3} \\ \%1 & -\frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{6} & -\frac{\sqrt{3}}{3} \\ \%2 & \frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{12} - \frac{\sqrt{5+\sqrt{5}}\sqrt{3}\sqrt{5}}{12} & \frac{\sqrt{3}}{3} \\ \%2 & -\frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{12} + \frac{\sqrt{5+\sqrt{5}}\sqrt{3}\sqrt{5}}{12} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} \\ \%1 & \frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{6} & \frac{\sqrt{3}}{3} \\ \%1 & -\frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{6} & \frac{\sqrt{3}}{3} \\ \%2 & \frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{12} - \frac{\sqrt{5+\sqrt{5}}\sqrt{3}\sqrt{5}}{12} & -\frac{\sqrt{3}}{3} \\ \%2 & -\frac{\sqrt{5+\sqrt{5}}\sqrt{3}}{12} + \frac{\sqrt{5+\sqrt{5}}\sqrt{3}\sqrt{5}}{12} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$$

$$\begin{aligned} \%1 &:= -\frac{\sqrt{2}\sqrt{3}}{12} + \frac{\sqrt{2}\sqrt{3}\sqrt{5}}{12} \\ \%2 &:= -\frac{\sqrt{2}\sqrt{3}}{12} - \frac{\sqrt{2}\sqrt{3}\sqrt{5}}{12} \end{aligned}$$

```
> UpdatePlan(N10,"N10");
```

“Update complete”

```
> pointplot3d(N10,symbol=box,scaling=constrained,shading=Z);
```

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N10,listlist)))],'rational',5))));
```

Graph := POLYTOPE(3, 3, 10, 7)

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

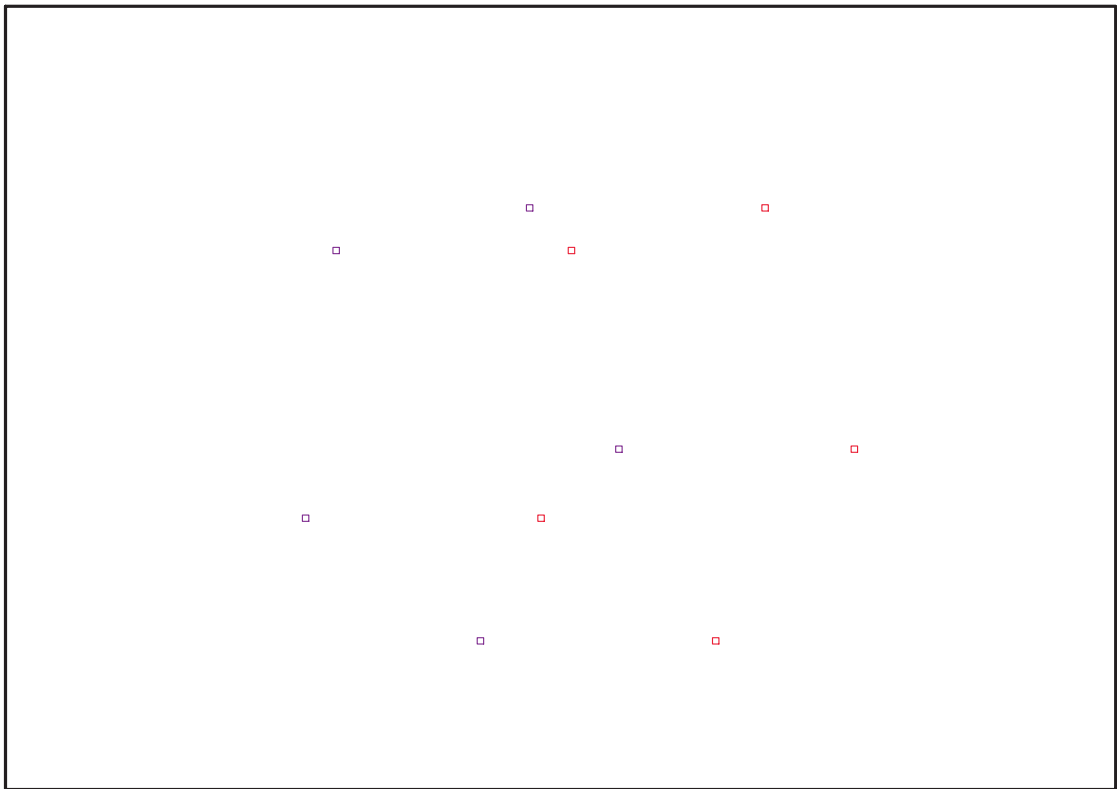


FIG. 11.22 – Graphique : Support du plan N10.

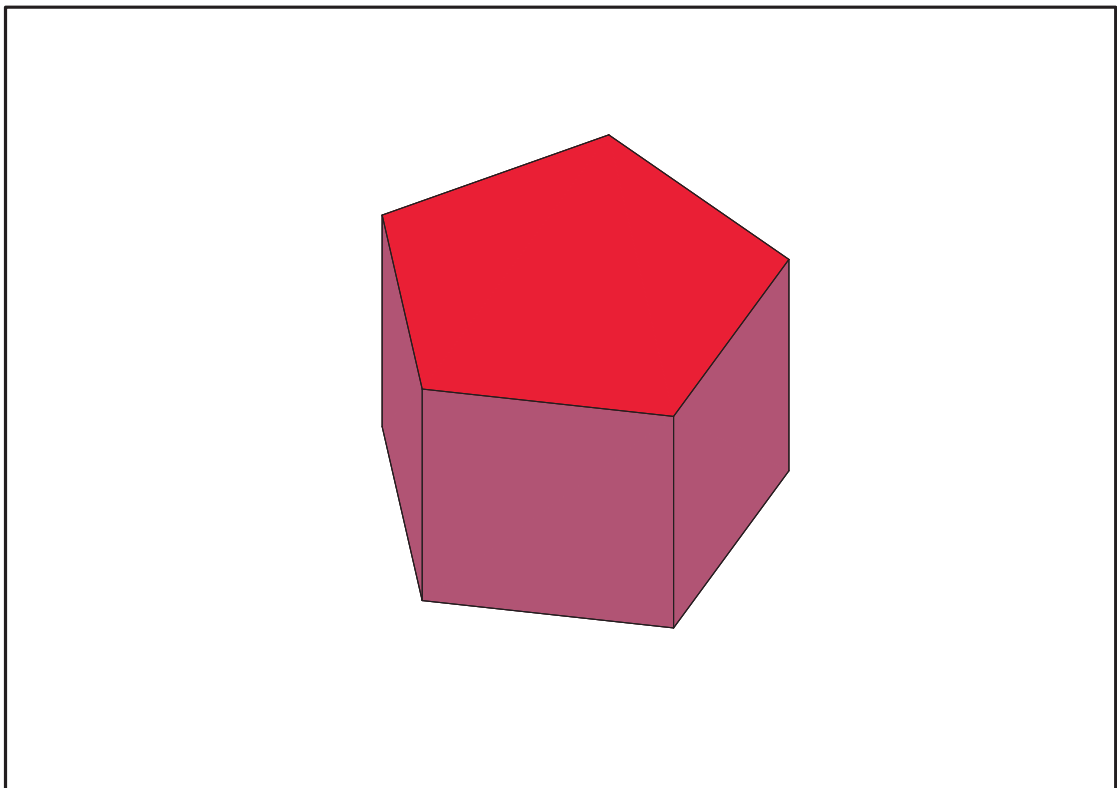


FIG. 11.23 – Graphique : Enveloppe convexe du support du plan N10.

```

> IdealN10:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N10));
> Gens:=Generators(IdealN10);

IdealN10 :=
<3 z^2 - 1, 6 x^2 y + sqrt(2) x y sqrt(3) - y, 18 x^3 - 3 x^2 sqrt(2) sqrt(3) - 9 x + sqrt(2) sqrt(3), 3 y^2 + 3 x^2 - 2>
Gens := {18 x^3 - 3 x^2 sqrt(2) sqrt(3) - 9 x + sqrt(2) sqrt(3), 3 z^2 - 1, 6 x^2 y + sqrt(2) x y sqrt(3) - y, 3 y^2 + 3 x^2 - 2}

> info_iop:=3;

info_iop := 3

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N10),
listlist),Vector))));

1
2
3
4
4
5
6
7
8
8
9
10
10
10

[[z^2 - 1/3, -2/3 + x^2 + y^2, -sqrt(2) x y sqrt(3) / 6 - sqrt(5 + sqrt(5)) (6 sqrt(5) - 5) z / 24 + sqrt(5 + sqrt(5)) (-5 + 3 sqrt(5)) z / 8 + y^3
+ sqrt(5 + sqrt(5)) (-19 + 15 sqrt(5)) z / 48 + (-1 + 2 sqrt(5)) y / 2 - sqrt(5) y - sqrt(5 + sqrt(5)) (-13 + 7 sqrt(5)) z / 16, sqrt(2) sqrt(3) / 18
+ sqrt(2) sqrt(5 + sqrt(5)) (-9 + sqrt(5)) y / 48 - sqrt(2) sqrt(5 + sqrt(5)) (-7 + sqrt(5)) y / 24 + sqrt(2) sqrt(5 + sqrt(5)) (3 + sqrt(5)) y / 48
- sqrt(2) sqrt(5 + sqrt(5)) y / 6 - sqrt(2) sqrt(3) y^2 / 6 + sqrt(2) (-10 + 3 sqrt(5)) z / 24 - sqrt(2) (-15 + sqrt(5)) z / 48 - sqrt(2) (sqrt(5) - 5) z / 24
- sqrt(2) (5 + 3 sqrt(5)) z / 48 + (7 + 3 sqrt(5)) x / 24 - (11 + 3 sqrt(5)) x / 24 + x y^2],
[1, z, y, x, y z, z x, y^2, x y, z y^2, z x y]]

> Gens := {18*x^3-3*x^2*2^(1/2)*3^(1/2)-9*x+2^(1/2)*3^(1/2), 3*z^2-1,
6*x^2*y+2^(1/2)*x*y*3^(1/2)-y, 3*y^2+3*x^2-2};
Gens := {18 x^3 - 3 x^2 sqrt(2) sqrt(3) - 9 x + sqrt(2) sqrt(3), 3 z^2 - 1, 3 y^2 + 3 x^2 - 2, 6 x^2 y + sqrt(2) x y sqrt(3) - y}

```

11.12.1. tdeg(x,y,z)

```

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N10_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N10_tdeg:=sort(EstSet_N10_tdeg,(
t1,t2)->testorder(t1,t2,T));nops(EstSet_N10_tdeg);

EstSet_N10_tdeg := [1, z, y, x, z y, x z, y^2, x y, y^2 z, x y z]
10

```

> X_D_ModelSat(N10,[x,y,z],EstSet_N10_tdeg):

> radnormal(simplify(M_D_ModelSat(N10,[x,y,z],EstSet_N10_tdeg)));

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{54} \end{bmatrix}$$

> expand(Var_Blue_ModelSat(N10,[x,y,z],EstSet_N10_tdeg));

$$3 - 9y^2 + 9z^2 - 27z^2y^2 + 3x^2 + 9z^2x^2 + 18y^4 + 18x^2y^2 + 54y^4z^2 + 54x^2y^2z^2$$

11.12.2. plex(x,y,z)

> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

> EstSet_N10_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y
,z):T:=termorder(A,plex(x,y,z)):EstSet_N10_plex:=sort(EstSet_N10_plex,
(t1,t2)->testorder(t1,t2,T));nops(EstSet_N10_plex);

$$EstSet_N10_plex := [1, z, y, zy, y^2, y^2z, y^3, y^3z, y^4, y^4z]$$

10

> X_D_ModelSat(N10,[x,y,z],EstSet_N10_plex):

> radnormal(simplify(M_D_ModelSat(N10,[x,y,z],EstSet_N10_plex)));

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & \frac{1}{18} \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{5}{54} & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & \frac{5}{162} \\ 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{5}{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{18} & 0 & 0 & 0 & \frac{5}{162} & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 & \frac{5}{54} & 0 & 0 & 0 & \frac{35}{648} & 0 \\ 0 & \frac{1}{18} & 0 & 0 & 0 & \frac{5}{162} & 0 & 0 & 0 & \frac{35}{1944} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N10,[x,y,z],EstSet_N10_plex));
5 - 756 y^6 - 90 z^2 y^2 + 648 y^8 - 2268 y^6 z^2 + 1944 y^8 z^2 + 810 y^4 z^2 + 270 y^4 + 15 z^2 - 30 y^2
```

11.12.3. Variance

Modèle de degré 1

```
> Var:=radnormal(Var_Norm_Blue_D_Centre(N10,1,0),'rationalized');
Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de degré 2

```
> Var:=radnormal(expand(Var_Norm_Blue_D_Centre(N10,2,1)),'rationalized'
);
"M_D is not full rank"
```

11.13. N10twisted

```
> unassign('eta','delta','nu');
```

```
> delta := (1/3)^(1/2); eta := (2/3-nu^2)^(1/2);
```

$$\delta := \frac{\sqrt{3}}{3}$$

$$\eta := \sqrt{\frac{2}{3} - \nu^2}$$

```
> GrN10twisted:=Plus(Rned(Diedn(5),3)):
```

```
> N10twisted:=radnormal(OrbToMatrix(OrbPG(Vector([(2/3-1/3*nu^2)^(1/2),
nu,(1/3*nu^2+1/3)^(1/2)]),GrN10twisted)));
```

$$N10twisted := \begin{bmatrix} \%1 - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\nu}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{\frac{2}{3}-\frac{\nu^2}{3}}}{4} + (-\frac{1}{4} + \frac{\sqrt{5}}{4})\nu & \sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%3 + \%2\nu & \%2\sqrt{\frac{2}{3}-\frac{\nu^2}{3}} + (\frac{1}{4} + \frac{\sqrt{5}}{4})\nu & -\sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \sqrt{\frac{2}{3}-\frac{\nu^2}{3}} & -\nu & -\sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%1 + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\nu}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{\frac{2}{3}-\frac{\nu^2}{3}}}{4} + (-\frac{\sqrt{5}}{4} + \frac{1}{4})\nu & -\sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%3 + \%4\nu & \%4\sqrt{\frac{2}{3}-\frac{\nu^2}{3}} + (\frac{1}{4} + \frac{\sqrt{5}}{4})\nu & -\sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \sqrt{\frac{2}{3}-\frac{\nu^2}{3}} & \nu & \sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%3 + \%4\nu & \%2\sqrt{\frac{2}{3}-\frac{\nu^2}{3}} + (-\frac{1}{4} - \frac{\sqrt{5}}{4})\nu & \sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%3 + \%2\nu & \%4\sqrt{\frac{2}{3}-\frac{\nu^2}{3}} + (-\frac{1}{4} - \frac{\sqrt{5}}{4})\nu & \sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%1 - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\nu}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{\frac{2}{3}-\frac{\nu^2}{3}}}{4} + (-\frac{\sqrt{5}}{4} + \frac{1}{4})\nu & -\sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \\ \%1 + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\nu}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{\frac{2}{3}-\frac{\nu^2}{3}}}{4} + (-\frac{1}{4} + \frac{\sqrt{5}}{4})\nu & \sqrt{\frac{\nu^2}{3} + \frac{1}{3}} \end{bmatrix}$$

$$\%1 := (-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{\frac{2}{3}-\frac{\nu^2}{3}}$$

$$\%2 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%3 := (-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{\frac{2}{3}-\frac{\nu^2}{3}}$$

$$\%4 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

```
> UpdatePlan(N10twisted,"N10twisted");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N10twisted,1,0);
```

$$Var := 1 + \frac{3 var_3^2}{\nu^2 + 1} + \frac{3 var_2^2}{\nu^2 + 1} + \frac{3 var_1^2}{\nu^2 + 1}$$

Pour toute valeur de ν permise la surface de variance est radiale.

```
> nu:=sqrt(2)/sqrt(3);
```

$$\nu := \frac{\sqrt{2} \sqrt{3}}{3}$$

```
> pointplot3d(N10twisted,symbol=box,scaling=constrained,shading=Z);
```

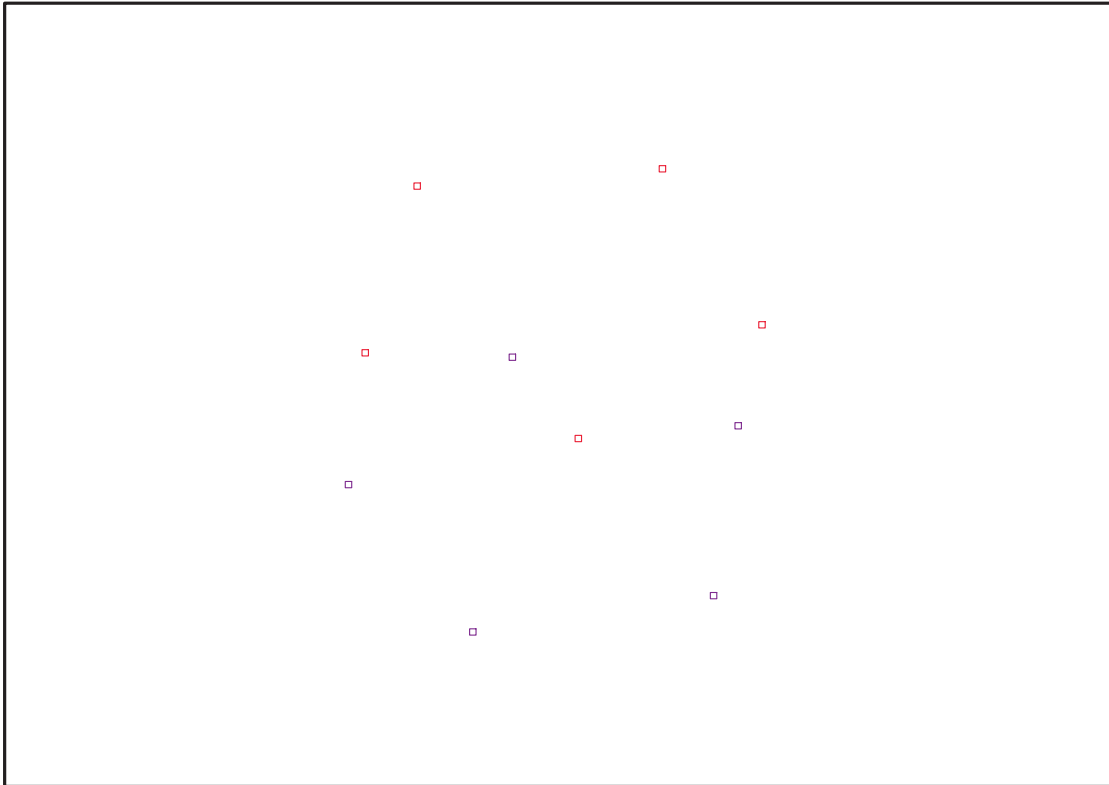


FIG. 11.24 – Graphique : Support du plan N10twisted.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N10twisted,listlist)
))], 'rational', 5))));
```

$Graph := \text{POLYTOPE}(3, 3, 10, 12)$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

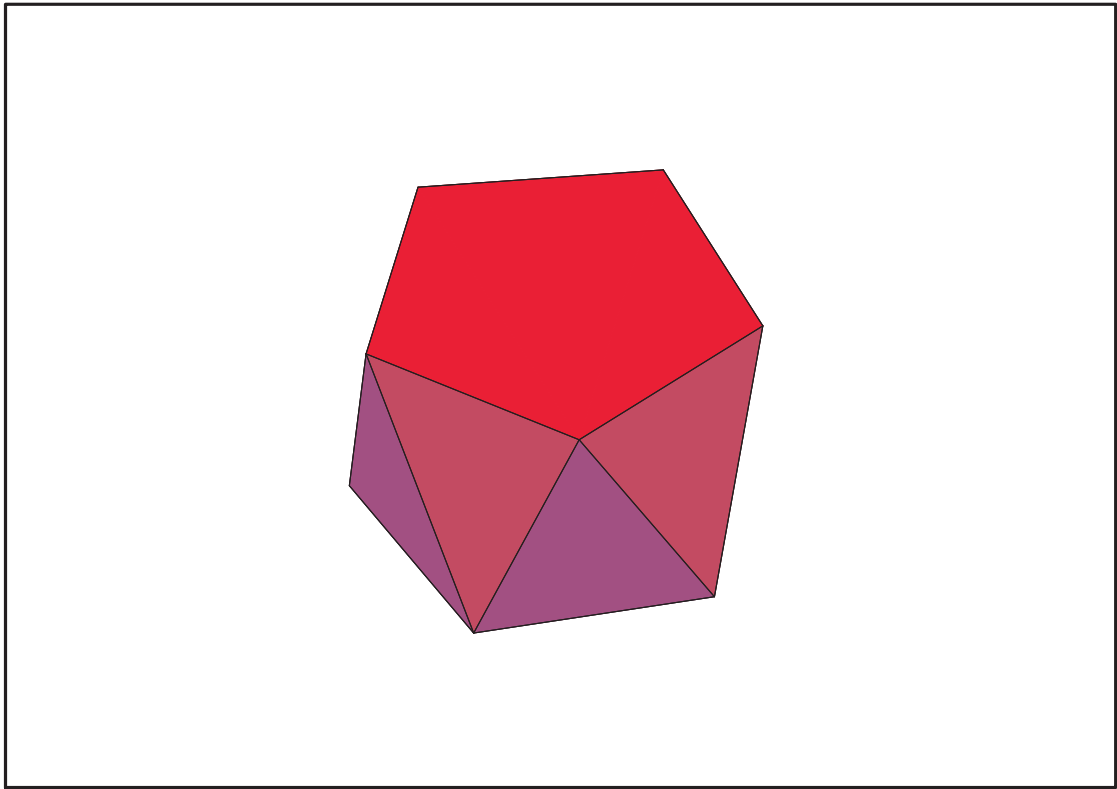


FIG. 11.25 – Graphique : Enveloppe convexe du support du plan N10.

```

> IdealN10twisted:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N10twisted
));
> Gens:=Generators(IdealN10twisted);
> info_iop:=3;
                                info_iop := 3
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N10tw
isted),listlist),Vector)));
                                1
                                2
                                3
                                4
                                4
                                5
                                6
                                7
                                8
                                8
                                9
                                10
                                10
                                10
> Gens := {18*x^3-3*x^2*2^(1/2)*3^(1/2)-9*x+2^(1/2)*3^(1/2), 3*z^2-1,
6*x^2*y+2^(1/2)*x*y*3^(1/2)-y, 3*y^2+3*x^2-2};
      Gens := {18 x^3 - 3 x^2 sqrt(2) sqrt(3) - 9 x + sqrt(2) sqrt(3), 3 z^2 - 1, 3 y^2 + 3 x^2 - 2, 6 x^2 y + sqrt(2) x y sqrt(3) - y}

```


11.13.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N10twisted_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algeb
ra(x,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N10twisted_tdeg:=sort(Est
Set_N10twisted_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N10twiste
d_tdeg);
```

$$EstSet_N10twisted_tdeg := [1, z, y, x, zy, xz, y^2, xy, y^2z, xyz]$$

10

```
> X_D_ModeleSat(N10twisted,[x,y,z],EstSet_N10twisted_tdeg):
> radnormal(simplify(M_D_ModeleSat(N10twisted,[x,y,z],EstSet_N10twisted
_tdeg)));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{5}{9} & 0 & 0 & 0 \\ 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{81} & 0 \\ 0 & 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{25}{81} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{81} & 0 & 0 & 0 & 0 \\ \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & \frac{25}{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{162} & 0 & 0 \\ 0 & \frac{25}{81} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{125}{486} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{125}{1458} \end{bmatrix}$$

```
> expand(Var_Blue_Modele_Sat(N10twisted,[x,y,z],EstSet_N10twisted_tdeg)
);
```

$$3 - \frac{27}{5}y^2 + \frac{27}{5}z^2 - \frac{243}{25}z^2y^2 + \frac{9}{5}x^2 + \frac{81}{25}z^2x^2 + \frac{162}{25}y^4 + \frac{162}{25}x^2y^2 + \frac{1458}{125}y^4z^2 + \frac{1458}{125}x^2y^2z^2$$

11.13.2. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
> EstSet_N10twisted_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_alge
bra(x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N10twisted_plex:=sort(Es
tSet_N10twisted_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N10twist
ed_plex);
```

$$EstSet_N10twisted_plex := [1, z, y, zy, y^2, y^2z, y^3, y^3z, y^4, y^4z]$$

10

```
> X_D_ModeleSat(N10twisted,[x,y,z],EstSet_N10twisted_plex):
> radnormal(simplify(M_D_ModeleSat(N10twisted,[x,y,z],EstSet_N10twisted
_plex)));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{5}{9} & 0 & 0 & 0 & \frac{25}{54} & 0 \\ 0 & \frac{5}{9} & 0 & 0 & 0 & \frac{25}{81} & 0 & 0 & 0 & \frac{125}{486} \\ 0 & 0 & \frac{5}{9} & 0 & 0 & 0 & \frac{25}{54} & 0 & 0 & \%1 \\ 0 & 0 & 0 & \frac{25}{81} & 0 & 0 & 0 & \frac{125}{486} & \%1 & 0 \\ \frac{5}{9} & 0 & 0 & 0 & \frac{25}{54} & 0 & 0 & \%1 & \frac{625}{1458} & 0 \\ 0 & \frac{25}{81} & 0 & 0 & 0 & \frac{125}{486} & \%1 & 0 & 0 & \frac{3125}{13122} \\ 0 & 0 & \frac{25}{54} & 0 & 0 & \%1 & \frac{625}{1458} & 0 & 0 & \%2 \\ 0 & 0 & 0 & \frac{125}{486} & \%1 & 0 & 0 & \frac{3125}{13122} & \%2 & 0 \\ \frac{25}{54} & 0 & 0 & \%1 & \frac{625}{1458} & 0 & 0 & \%2 & \frac{21875}{52488} & 0 \\ 0 & \frac{125}{486} & \%1 & 0 & 0 & \frac{3125}{13122} & \%2 & 0 & 0 & \frac{109375}{472392} \end{bmatrix}$$

$$\%1 := -\frac{31\sqrt{2}\sqrt{3}\sqrt{5}}{2916}$$

$$\%2 := -\frac{1085\sqrt{2}\sqrt{3}\sqrt{5}}{52488}$$

> expand(Var_Blue_Modele_Sat(N10twisted,[x,y,z],EstSet_N10twisted_plex)
);

$$\begin{aligned} & -\frac{255150}{121}y^6 + \frac{131220}{121}y^8 + \frac{180792}{605}\sqrt{2}\sqrt{3}\sqrt{5}y^5z + \frac{236196}{121}y^8z^2 + \frac{6371}{121} - \frac{50625}{121}z^2y^2 \\ & + \frac{151875}{121}y^4 - \frac{45198}{121}\sqrt{2}\sqrt{3}\sqrt{5}y^3z + \frac{11160}{121}\sqrt{2}\sqrt{3}\sqrt{5}zy + \frac{273375}{121}y^4z^2 \\ & - \frac{459270}{121}y^6z^2 + \frac{57339}{605}z^2 - \frac{28125}{121}y^2 \end{aligned}$$

11.13.3. Variance

Modèle de degré 1

> Var:=Var_Norm_Blue_D_Centre(N10twisted,1,0);

$$Var := 1 + \frac{9}{5}var_3^2 + \frac{9}{5}var_2^2 + \frac{9}{5}var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
,Var,poly_algebra(seq(var[i],i=1..3)));

$$1 + \frac{9}{5}IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

> animate(plot3d,[Var,var[1]==-2..2,var[2]==-2..2],var[3]==-2..2):

Modèle de degré 2

> Var:=radnormal(expand(Var_Norm_Blue_D_Centre(N10twisted,2,1)), 'ration
alized');

“M_D is not full rank”

11.14. N11

```
> unassign('alpha','beta','delta');
```

```
> GrN11:=Plus(Rned(Diedn(3),3));
```

$$GrN11 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> N11:=OrbToMatrix(OrbPG(Vector([1,0,0]),GrN11) union
OrbPG(Vector([alpha,delta,beta]),GrN11) union
OrbPG(Vector([0,0,1]),GrN11));
```

$$N11 := \begin{bmatrix} \alpha & \delta & \beta \\ \alpha & -\delta & -\beta \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\delta}{2} & \beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\delta}{2} & \beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\delta}{2} & -\beta \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\delta}{2} & -\beta \\ 0 & 0 & 1 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

```
> UpdatePlan(N11,"N11");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N11,1,0);
```

$$Var := 1 + \frac{11}{2} \frac{var_3^2}{1+3\beta^2} + \frac{22}{3} \frac{var_2^2}{1+2\alpha^2+2\delta^2} + \frac{22}{3} \frac{var_1^2}{1+2\alpha^2+2\delta^2}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});
```

$$Cond := [1 - 6\alpha^2 - 6\delta^2 + 12\beta^2]$$

```
> Cond_gsoloved:=convert(gsolve([op(Cond),alpha^2+beta^2+delta^2-1],
{alpha,beta,delta}),list);
```

$$Cond_gsoloved := [[[18\alpha^2 + 18\delta^2 - 13, 18\beta^2 - 5], \text{plex}(\beta, \delta, \alpha), \{\}]]$$

```
> Sols:=map(L->solve({op(L[1])},{alpha,beta}),Cond_gsoloved);
```

```

Sols := [{
  beta = 1/3 RootOf(2__Z^2 - 5, label = __L40), alpha = 1/3 RootOf(2__Z^2 + 18 delta^2 - 13, label = __L41)
}]
delta:=0

```

```

> delta:=0; beta := sqrt(5)/3/sqrt(2); alpha := sqrt(13)/sqrt(2)/3;
      delta := 0
      beta :=  $\frac{\sqrt{5}\sqrt{2}}{6}$ 
      alpha :=  $\frac{\sqrt{13}\sqrt{2}}{6}$ 

```

```

> N11:=OrbToMatrix(SortPoints(OrbPG(Vector([1,0,0]),GrN11) union
OrbPG(Vector([alpha,delta,beta]),GrN11) union
OrbPG(Vector([0,0,1]),GrN11)));

```

$$N11 := \begin{bmatrix} -\frac{\sqrt{13}\sqrt{2}}{12} & -\frac{\sqrt{13}\sqrt{3}\sqrt{2}}{12} & -\frac{\sqrt{5}\sqrt{2}}{6} \\ -\frac{\sqrt{13}\sqrt{2}}{12} & \frac{\sqrt{13}\sqrt{3}\sqrt{2}}{12} & \frac{\sqrt{5}\sqrt{2}}{6} \\ -\frac{\sqrt{13}\sqrt{2}}{12} & \frac{\sqrt{13}\sqrt{3}\sqrt{2}}{12} & -\frac{\sqrt{5}\sqrt{2}}{6} \\ -\frac{\sqrt{13}\sqrt{2}}{12} & -\frac{\sqrt{13}\sqrt{3}\sqrt{2}}{12} & \frac{\sqrt{5}\sqrt{2}}{6} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{13}\sqrt{2}}{6} & 0 & \frac{\sqrt{5}\sqrt{2}}{6} \\ \frac{\sqrt{13}\sqrt{2}}{6} & 0 & -\frac{\sqrt{5}\sqrt{2}}{6} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

```

> pointplot3d(N11,symbol=box,scaling=constrained,shading=Z);

```

```

> Graph:=convhull(op(map(convert,[evalf(op(convert(N11,listlist)))]),'rational',5)));

```

$Graph := \text{POLYTOPE}(3, 3, 11, 16)$

```

> plotdes:=draw(Graph):

```

```

> display(plotdes,scaling=constrained,shading=Z);

```

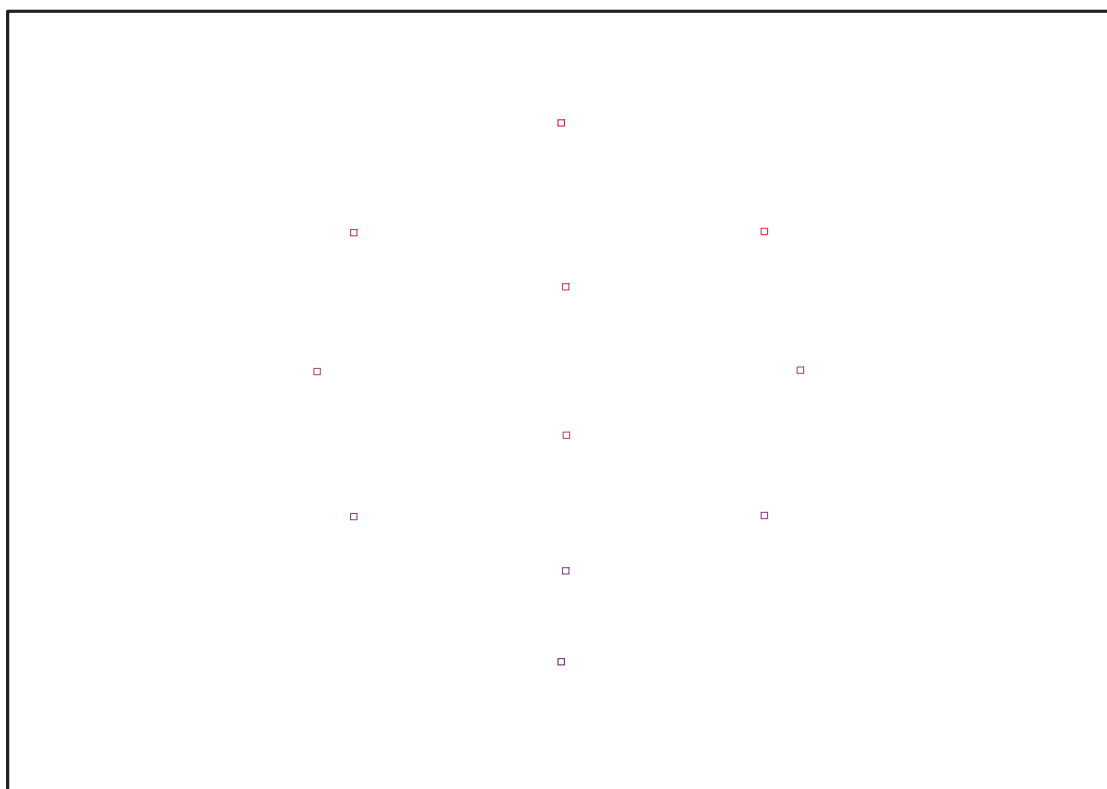


FIG. 11.26 – Graphique : Support du plan N11.

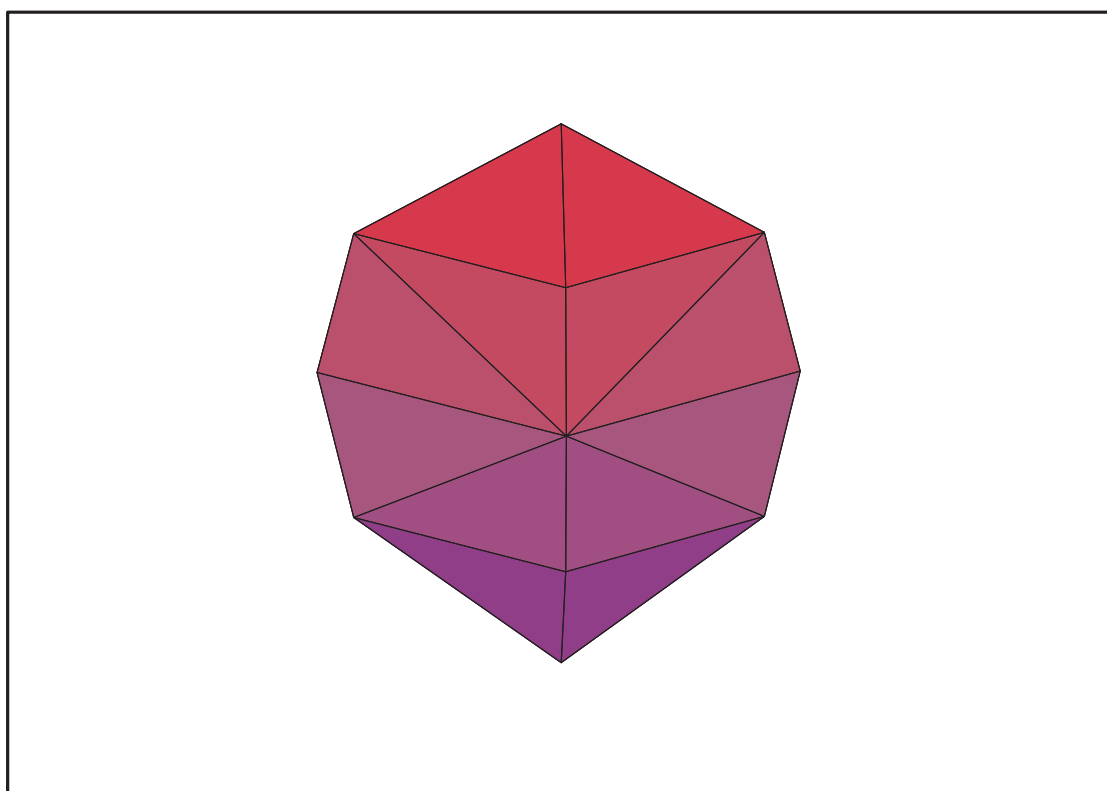


FIG. 11.27 – Graphique : Enveloppe convexe du plan N11.

```

> IdealN11:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N11));

> Gens:=Generators(IdealN11);

IdealN11 := < z^2 + y^2 + x^2 - 1, 36 x^3 z - 3 z sqrt(13) sqrt(2) x^2 - 13 z x, 6 z y^2 - 6 z x^2 + z x sqrt(13) sqrt(2),
8 y^3 + 2 sqrt(13) sqrt(2) y x + 12 y x + y sqrt(13) sqrt(2), 24 y x^2 + 2 sqrt(13) sqrt(2) y x + 12 y x + y sqrt(13) sqrt(2),
12 y z x + y z sqrt(13) sqrt(2), 6 x y^2 + 6 x^3 + sqrt(13) sqrt(2) y^2 + 6 y^2 - sqrt(13) sqrt(2) x^2 - 6 x^2 + x sqrt(13) sqrt(2),
72 x^4 - 18 sqrt(13) sqrt(2) x^3 - 108 x^3 - 15 sqrt(13) sqrt(2) y^2 - 62 y^2 + 24 sqrt(13) sqrt(2) x^2 + 62 x^2
- 6 x sqrt(13) sqrt(2) - 26 x >

Gens := {6 z y^2 - 6 z x^2 + z x sqrt(13) sqrt(2), 36 x^3 z - 3 z sqrt(13) sqrt(2) x^2 - 13 z x, z^2 + y^2 + x^2 - 1,
24 y x^2 + 2 sqrt(13) sqrt(2) y x + 12 y x + y sqrt(13) sqrt(2), 8 y^3 + 2 sqrt(13) sqrt(2) y x + 12 y x + y sqrt(13) sqrt(2),
12 y z x + y z sqrt(13) sqrt(2), 6 x y^2 + 6 x^3 + sqrt(13) sqrt(2) y^2 + 6 y^2 - sqrt(13) sqrt(2) x^2 - 6 x^2 + x sqrt(13) sqrt(2),
72 x^4 - 18 sqrt(13) sqrt(2) x^3 - 108 x^3 - 15 sqrt(13) sqrt(2) y^2 - 62 y^2 + 24 sqrt(13) sqrt(2) x^2 + 62 x^2
- 6 x sqrt(13) sqrt(2) - 26 x}

> Gens := {6*z*y^2-6*z*x^2+z*x*13^(1/2)*2^(1/2),
36*x^3*z-3*z*13^(1/2)*2^(1/2)*x^2-13*z*x, z^2+y^2+x^2-1,
24*y*x^2+2*13^(1/2)*2^(1/2)*y*x+12*y*x+y*13^(1/2)*2^(1/2),
8*y^3+2*13^(1/2)*2^(1/2)*y*x+12*y*x+y*13^(1/2)*2^(1/2),
12*y*z*x+y*z*13^(1/2)*2^(1/2),
6*x*y^2+6*x^3+13^(1/2)*2^(1/2)*y^2+6*y^2-13^(1/2)*2^(1/2)*x^2-6*x^2+x*
13^(1/2)*2^(1/2),
72*x^4-18*13^(1/2)*2^(1/2)*x^3-108*x^3-15*13^(1/2)*2^(1/2)*y^2-62*y^2+
24*13^(1/2)*2^(1/2)*x^2+62*x^2-6*x*13^(1/2)*2^(1/2)-26*x};

Gens := {z^2 + y^2 + x^2 - 1, 8 y^3 + 2 sqrt(13) sqrt(2) y x + 12 x y + y sqrt(13) sqrt(2),
24 y x^2 + 2 sqrt(13) sqrt(2) y x + 12 x y + y sqrt(13) sqrt(2), 6 z y^2 - 6 z x^2 + z x sqrt(13) sqrt(2),
36 x^3 z - 3 z sqrt(13) sqrt(2) x^2 - 13 z x,
6 y^2 x + 6 x^3 + sqrt(13) sqrt(2) y^2 + 6 y^2 - sqrt(13) sqrt(2) x^2 - 6 x^2 + x sqrt(13) sqrt(2), 72 x^4 - 18 sqrt(13) sqrt(2) x^3
- 108 x^3 - 15 sqrt(13) sqrt(2) y^2 - 62 y^2 + 24 sqrt(13) sqrt(2) x^2 + 62 x^2 - 6 x sqrt(13) sqrt(2) - 26 x,
12 z x y + y z sqrt(13) sqrt(2)}

```

11.14.1. tdeg(x,y,z)

```

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N11_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N11_tdeg:=sort(EstSet_N11_tdeg,(
t1,t2)->testorder(t1,t2,T));nops(EstSet_N11_tdeg);

```

$$EstSet_N11_tdeg := [1, z, y, x, z^2, yz, zx, y^2, xy, z^3, y^2x]$$

11

```

> X_D_ModeleSat(N11,[x,y,z],EstSet_N11_tdeg);

```

$$\begin{aligned}
 & \left[1, \frac{\sqrt{5}\sqrt{2}}{6}, -\frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, -\frac{\sqrt{13}\sqrt{2}}{12}, \frac{5}{18}, -\frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, -\frac{\sqrt{5}\sqrt{13}}{36}, \frac{13}{24}, \frac{13\sqrt{3}}{72}, \right. \\
 & \left. \frac{5\sqrt{5}\sqrt{2}}{108}, -\frac{13\sqrt{13}\sqrt{2}}{288} \right] \\
 & \left[1, \frac{\sqrt{5}\sqrt{2}}{6}, \frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, -\frac{\sqrt{13}\sqrt{2}}{12}, \frac{5}{18}, \frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, -\frac{\sqrt{5}\sqrt{13}}{36}, \frac{13}{24}, -\frac{13\sqrt{3}}{72}, \right. \\
 & \left. \frac{5\sqrt{5}\sqrt{2}}{108}, -\frac{13\sqrt{13}\sqrt{2}}{288} \right] \\
 & \left[1, -\frac{\sqrt{5}\sqrt{2}}{6}, -\frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, -\frac{\sqrt{13}\sqrt{2}}{12}, \frac{5}{18}, \frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, \frac{\sqrt{5}\sqrt{13}}{36}, \frac{13}{24}, \frac{13\sqrt{3}}{72}, \right. \\
 & \left. -\frac{5\sqrt{5}\sqrt{2}}{108}, -\frac{13\sqrt{13}\sqrt{2}}{288} \right] \\
 & \left[1, -\frac{\sqrt{5}\sqrt{2}}{6}, \frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, -\frac{\sqrt{13}\sqrt{2}}{12}, \frac{5}{18}, -\frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, \frac{\sqrt{5}\sqrt{13}}{36}, \frac{13}{24}, -\frac{13\sqrt{3}}{72}, \right. \\
 & \left. -\frac{5\sqrt{5}\sqrt{2}}{108}, -\frac{13\sqrt{13}\sqrt{2}}{288} \right] \\
 & [1, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0] \\
 & [1, -1, 0, 0, 1, 0, 0, 0, 0, -1, 0] \\
 & \left[1, \frac{\sqrt{5}\sqrt{2}}{6}, 0, \frac{\sqrt{13}\sqrt{2}}{6}, \frac{5}{18}, 0, \frac{\sqrt{5}\sqrt{13}}{18}, 0, 0, \frac{5\sqrt{5}\sqrt{2}}{108}, 0 \right] \\
 & \left[1, -\frac{\sqrt{5}\sqrt{2}}{6}, 0, \frac{\sqrt{13}\sqrt{2}}{6}, \frac{5}{18}, 0, -\frac{\sqrt{5}\sqrt{13}}{18}, 0, 0, -\frac{5\sqrt{5}\sqrt{2}}{108}, 0 \right] \\
 & \left[1, 0, \frac{\sqrt{3}}{2}, \frac{-1}{2}, 0, 0, 0, \frac{3}{4}, -\frac{\sqrt{3}}{4}, 0, \frac{-3}{8} \right] \\
 & \left[1, 0, -\frac{\sqrt{3}}{2}, \frac{-1}{2}, 0, 0, 0, \frac{3}{4}, \frac{\sqrt{3}}{4}, 0, \frac{-3}{8} \right] \\
 & [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
 \end{aligned}$$

```
> M_D_ModeleSat(N11,[x,y,z],EstSet_N11_tdeg);
```

$$\begin{aligned}
& \left[\begin{array}{c}
1, 0, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, 0, 0, \%1 \\
0, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, \frac{133}{594}, 0 \\
0, 0, \frac{1}{3}, 0, 0, 0, 0, 0, 0, \%1, 0, 0 \\
0, 0, 0, \frac{1}{3}, 0, 0, 0, \%1, 0, 0, \frac{331}{4752} \\
\frac{1}{3}, 0, 0, 0, \frac{133}{594}, 0, 0, \frac{65}{1188}, 0, 0, -\frac{65\sqrt{13}\sqrt{2}}{14256} \\
0, 0, 0, 0, 0, \frac{65}{1188}, 0, 0, 0, 0, 0 \\
0, 0, 0, 0, 0, 0, \frac{65}{1188}, 0, 0, 0, 0 \\
\frac{1}{3}, 0, 0, \%1, \frac{65}{1188}, 0, 0, \frac{331}{1584}, 0, 0, -\frac{169\sqrt{13}\sqrt{2}}{19008} - \frac{9}{176} \\
0, 0, \%1, 0, 0, 0, 0, 0, \frac{331}{4752}, 0, 0 \\
0, \frac{133}{594}, 0, 0, 0, 0, 0, 0, 0, \frac{2069}{10692}, 0 \\
\%1, 0, 0, \frac{331}{4752}, -\frac{65\sqrt{13}\sqrt{2}}{14256}, 0, 0, -\frac{169\sqrt{13}\sqrt{2}}{19008} - \frac{9}{176}, 0, 0, \frac{5113}{114048}
\end{array} \right] \\
\%1 &:= -\frac{13\sqrt{13}\sqrt{2}}{792} - \frac{3}{44}
\end{aligned}$$

> expand(Var_Blue_Modele_Sat(N11,[x,y,z],EstSet_N11_tdeg));

$$\begin{aligned}
& \frac{1188z^2x^2}{65} + \frac{2112y^2x}{13\%2} - \frac{22033462y^2}{13\%2\%1} - \frac{20000574z^2}{13\%2\%1} + \frac{1936242xz^2}{13\%2\%1} + \frac{1614624z^2y^2}{\%2\%1} \\
& - \frac{11582362xy^2}{39\%2\%1} + \frac{223608z^2y^2x}{13\%2} + \frac{22759z^2}{1690} - \frac{26334z^4}{845} + \frac{19602z^6}{845} + \frac{1188z^2y^2}{65} \\
& - \frac{7557x^2}{26\%1} - \frac{7282y^2}{39\%1} - \frac{199518x\sqrt{13}\sqrt{2}}{13\%2\%1} + \frac{3922149z^2\sqrt{13}\sqrt{2}}{13\%2\%1} \\
& + \frac{56164306y^2\sqrt{13}\sqrt{2}}{169\%2\%1} - \frac{4752xy^2}{13\%1} + \frac{8844y^2x\sqrt{13}\sqrt{2}}{169\%2} - \frac{88xy^2\sqrt{13}\sqrt{2}}{\%1} \\
& + \frac{389136y^2x^2}{13\%2} - \frac{1005268y^2x^2\sqrt{13}\sqrt{2}}{169\%2} - \frac{379467xz^2\sqrt{13}\sqrt{2}}{13\%2\%1} \\
& + \frac{9852502xy^2\sqrt{13}\sqrt{2}}{169\%2\%1} + \frac{19830393z^4}{13\%2\%1} - \frac{53504506z^2y^2\sqrt{13}\sqrt{2}}{169\%2\%1} \\
& - \frac{3888918z^4\sqrt{13}\sqrt{2}}{13\%2\%1} - \frac{583704z^2y^2x\sqrt{13}\sqrt{2}}{169\%2} + \frac{871952378y^4}{507\%2\%1} \\
& - \frac{56985676y^4\sqrt{13}\sqrt{2}}{169\%2\%1} - \frac{1945680y^4x}{169\%2} + \frac{324280y^4x\sqrt{13}\sqrt{2}}{169\%2} - \frac{11616x^2y^2}{13\%1} \\
& - \frac{12063216y^4x^2}{169\%2} + \frac{2334816y^4x^2\sqrt{13}\sqrt{2}}{169\%2} + \frac{1015993x}{13\%2\%1} - \frac{8987}{26\%1} \\
\%1 &:= -31 + 6\sqrt{13}\sqrt{2} \\
\%2 &:= -1897 + 372\sqrt{13}\sqrt{2}
\end{aligned}$$

11.14.2. $\text{plex}(x,y,z)$

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

> EstSet_N11_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y
,z):T:=termorder(A,plex(x,y,z)):EstSet_N11_plex:=sort(EstSet_N11_plex,
(t1,t2)->testorder(t1,t2,T));nops(EstSet_N11_plex);
```

$$\text{EstSet_N11_plex} := [1, z, z^2, z^3, z^4, y, yz, z^2y, y^2, zy^2, z^2y^2]$$

11

```
> X_D_ModeleSat(N11,[x,y,z],EstSet_N11_plex);
```

$$\begin{aligned} & \left[1, \frac{\sqrt{5}\sqrt{2}}{6}, \frac{5}{18}, \frac{5\sqrt{5}\sqrt{2}}{108}, \frac{25}{324}, -\frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, -\frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, -\frac{5\sqrt{3}\sqrt{13}\sqrt{2}}{216}, \frac{13}{24}, \right. \\ & \left. \frac{13\sqrt{5}\sqrt{2}}{144}, \frac{65}{432} \right] \\ & \left[1, \frac{\sqrt{5}\sqrt{2}}{6}, \frac{5}{18}, \frac{5\sqrt{5}\sqrt{2}}{108}, \frac{25}{324}, \frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, \frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, \frac{5\sqrt{3}\sqrt{13}\sqrt{2}}{216}, \frac{13}{24}, \right. \\ & \left. \frac{13\sqrt{5}\sqrt{2}}{144}, \frac{65}{432} \right] \\ & \left[1, -\frac{\sqrt{5}\sqrt{2}}{6}, \frac{5}{18}, -\frac{5\sqrt{5}\sqrt{2}}{108}, \frac{25}{324}, -\frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, \frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, -\frac{5\sqrt{3}\sqrt{13}\sqrt{2}}{216}, \frac{13}{24}, \right. \\ & \left. -\frac{13\sqrt{5}\sqrt{2}}{144}, \frac{65}{432} \right] \\ & \left[1, -\frac{\sqrt{5}\sqrt{2}}{6}, \frac{5}{18}, -\frac{5\sqrt{5}\sqrt{2}}{108}, \frac{25}{324}, \frac{\sqrt{3}\sqrt{13}\sqrt{2}}{12}, -\frac{\sqrt{3}\sqrt{13}\sqrt{5}}{36}, \frac{5\sqrt{3}\sqrt{13}\sqrt{2}}{216}, \frac{13}{24}, \right. \\ & \left. -\frac{13\sqrt{5}\sqrt{2}}{144}, \frac{65}{432} \right] \\ & [1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0] \\ & [1, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0] \\ & \left[1, \frac{\sqrt{5}\sqrt{2}}{6}, \frac{5}{18}, \frac{5\sqrt{5}\sqrt{2}}{108}, \frac{25}{324}, 0, 0, 0, 0, 0, 0 \right] \\ & \left[1, -\frac{\sqrt{5}\sqrt{2}}{6}, \frac{5}{18}, -\frac{5\sqrt{5}\sqrt{2}}{108}, \frac{25}{324}, 0, 0, 0, 0, 0, 0 \right] \\ & \left[1, 0, 0, 0, 0, \frac{\sqrt{3}}{2}, 0, 0, \frac{3}{4}, 0, 0 \right] \\ & \left[1, 0, 0, 0, 0, -\frac{\sqrt{3}}{2}, 0, 0, \frac{3}{4}, 0, 0 \right] \\ & [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \end{aligned}$$

```
> M_D_ModeleSat(N11,[x,y,z],EstSet_N11_plex);
```

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{133}{594} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{65}{1188} \\ 0 & \frac{1}{3} & 0 & \frac{133}{594} & 0 & 0 & 0 & 0 & 0 & \frac{65}{1188} & 0 \\ \frac{1}{3} & 0 & \frac{133}{594} & 0 & \frac{2069}{10692} & 0 & 0 & 0 & \frac{65}{1188} & 0 & \frac{325}{21384} \\ 0 & \frac{133}{594} & 0 & \frac{2069}{10692} & 0 & 0 & 0 & 0 & 0 & \frac{325}{21384} & 0 \\ \frac{133}{594} & 0 & \frac{2069}{10692} & 0 & \frac{35617}{192456} & 0 & 0 & 0 & \frac{325}{21384} & 0 & \frac{1625}{384912} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{65}{1188} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{65}{1188} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{65}{1188} & 0 & \frac{325}{21384} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{65}{1188} & 0 & \frac{325}{21384} & 0 & 0 & 0 & \frac{331}{1584} & 0 & \frac{845}{28512} \\ 0 & \frac{65}{1188} & 0 & \frac{325}{21384} & 0 & 0 & 0 & 0 & 0 & \frac{845}{28512} & 0 \\ \frac{65}{1188} & 0 & \frac{325}{21384} & 0 & \frac{1625}{384912} & 0 & 0 & 0 & \frac{845}{28512} & 0 & \frac{4225}{513216} \end{bmatrix}$$

> expand(Var_Blue_Modelle_Sat(N11,[x,y,z],EstSet_N11_plex));

$$\begin{aligned} & 11 - 22y^2 - \frac{105501}{1690}z^2 + \frac{3106653}{8450}z^4 - \frac{92928}{845}z^2y^4 - \frac{225456}{325}z^4y^2 + \frac{3145824}{4225}z^4y^4 + \frac{88}{3}y^4 \\ & - \frac{2513808}{4225}z^6 + \frac{265628}{2535}z^2y^2 + \frac{1224234}{4225}z^8 + \frac{3145824}{4225}z^6y^2 \end{aligned}$$

11.14.3. Variance

Modèle de Degré 1

> Var:=Var_Norm_Blue_D_Centre(N11,1,0);

$$Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
,Var,poly_algebra(seq(var[i],i=1..3))});

$$1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

> animate(plot3d,[Var,var[1]==-2..2,var[2]==-2..2],var[3]==-2..2):

Modèle de Degré 2

> Var:=Var_Norm_Blue_D_Centre(N11,2,0);

“M_D is not full rank”

> Var:=radnormal(expand(Var_Norm_Blue_D_Centre(N11,2,1)),’rationalized’
);Var_subs:=subs(sqrt(13)=t,sqrt(2)=s,%):

$$\begin{aligned}
 Var := & \frac{1}{1610348187449625} (1052393 + 194568 \sqrt{13} \sqrt{2}) (-97452885024 \text{var}_2^2 \text{var}_3^2 \sqrt{13} \sqrt{2} \\
 & + 237721280940 \text{var}_1 \text{var}_2^2 \sqrt{13} \sqrt{2} - 97452885024 \text{var}_1^2 \text{var}_3^2 \sqrt{13} \sqrt{2} \\
 & + 887447910420 \text{var}_1^2 \text{var}_2^2 + 1158039816120 \text{var}_1 \text{var}_2^2 \\
 & + 83515085940 \text{var}_1^2 \sqrt{13} \sqrt{2} + 83515085940 \text{var}_2^2 \sqrt{13} \sqrt{2} \\
 & + 48298587480 \text{var}_3^2 \sqrt{13} \sqrt{2} - 40520731680 \text{var}_3^4 \sqrt{13} \sqrt{2} \\
 & - 79240426980 \text{var}_1^3 \sqrt{13} \sqrt{2} - 27962340120 \sqrt{13} \sqrt{2} + 527109977124 \text{var}_1^2 \text{var}_3^2 \\
 & + 527109977124 \text{var}_2^2 \text{var}_3^2 + 151244659995 - 80831757380 \text{var}_1^2 \\
 & - 80831757380 \text{var}_2^2 - 261240776355 \text{var}_3^2 + 219171366180 \text{var}_3^4 \\
 & - 386013272040 \text{var}_1^3 + 443723955210 \text{var}_1^4 + 443723955210 \text{var}_2^4)
 \end{aligned}$$

```

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var_subs,poly_algebra(seq(var[i],i=1..3),alg_relations=
{t^2-13,s^2-2})):eval(%,{s=sqrt(2),t=sqrt(13)});

```

$$\begin{aligned}
 & 11 + \frac{127952}{325} \text{var}_1 \text{var}_2^2 \sqrt{13} \sqrt{2} + \frac{31988}{325} \text{var}_1 \sqrt{13} \sqrt{2} \text{var}_3^2 - \frac{31988}{325} \text{var}_1 \sqrt{13} \sqrt{2} \%1 \\
 & - \frac{34848}{325} \text{var}_3^2 \sqrt{13} \sqrt{2} \%1 + \frac{651552}{325} \text{var}_1 \text{var}_2^2 - \frac{14564}{325} \text{var}_3^2 \sqrt{13} \sqrt{2} \\
 & + \frac{17424}{325} \text{var}_3^4 \sqrt{13} \sqrt{2} + \frac{14564}{325} \sqrt{13} \sqrt{2} \%1 + \frac{17424}{325} \sqrt{13} \sqrt{2} \%1^2 + \frac{204292}{975} \%1 \\
 & - \frac{11793936}{21775} \text{var}_3^2 \%1 + \frac{162888}{325} \text{var}_1 \text{var}_3^2 - \frac{162888}{325} \text{var}_1 \%1 + \frac{6314358}{21775} \%1^2 \\
 & - \frac{222817}{975} \text{var}_3^2 + \frac{448206}{1675} \text{var}_3^4, \%1 = \text{var}_1^2 + \text{var}_2^2 + \text{var}_3^2 \\
 \%1 & := \text{IndetDecompInKfi}_1
 \end{aligned}$$

11.15. N11+1

> unassign('alpha','beta','delta');

> GrN11:=Plus(Rned(Diedn(3),3));

$$GrN11 := \left\{ \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

> N11bis:=OrbToMatrix(OrbPG(Vector([1,0,0]),GrN11) union
OrbPG(Vector([alpha,delta,beta]),GrN11) union
OrbPG(Vector([0,0,1]),GrN11));

$$N11bis := \begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\delta}{2} & \beta \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \alpha & \delta & \beta \\ \alpha & -\delta & -\beta \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\delta}{2} & -\beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\delta}{2} & -\beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\delta}{2} & \beta \end{bmatrix}$$

> UpdatePlan(N11bis,"N11bis");

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N11bis,1,0);

$$Var := 1 + \frac{11}{2} \frac{var_3^2}{3\beta^2 + 1} + \frac{22}{3} \frac{var_2^2}{2\alpha^2 + 2\delta^2 + 1} + \frac{22}{3} \frac{var_1^2}{2\alpha^2 + 2\delta^2 + 1}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

$$Cond := [6\alpha^2 + 6\delta^2 - 1 - 12\beta^2]$$

> Cond_gsolved:=convert(gsolve([op(Cond),alpha^2+beta^2+delta^2-1],
{alpha,beta,delta}),list);

$$Cond_gsolved := [[[-5 + 18\beta^2, 18\alpha^2 + 18\delta^2 - 13], \text{plex}(\alpha, \beta, \delta), \{\}]]$$

> Sols:=map(L->solve({op(L[1])},{alpha,beta,delta}),Cond_gsolved);

$$\begin{aligned} \text{Sols} &:= [\{\delta = \delta, \beta = \frac{1}{3} \text{RootOf}(-5 + 2_Z^2, \text{label} = _L3), \\ &\alpha = \frac{1}{3} \text{RootOf}(2_Z^2 - 13 + 18\delta^2, \text{label} = _L2)\}] \end{aligned}$$

> allvalues(Sols);

$$\begin{aligned} &\left[\left\{ \delta = \delta, \beta = \frac{\sqrt{10}}{6}, \alpha = \frac{\sqrt{\frac{13}{2} - 9\delta^2}}{3} \right\}, \left\{ \alpha = -\frac{\sqrt{\frac{13}{2} - 9\delta^2}}{3}, \delta = \delta, \beta = \frac{\sqrt{10}}{6} \right\}, \right. \\ &\left. \left[\left\{ \delta = \delta, \beta = -\frac{\sqrt{10}}{6}, \alpha = \frac{\sqrt{\frac{13}{2} - 9\delta^2}}{3} \right\}, \left\{ \alpha = -\frac{\sqrt{\frac{13}{2} - 9\delta^2}}{3}, \delta = \delta, \beta = -\frac{\sqrt{10}}{6} \right\} \right] \right] \end{aligned}$$

> alpha := 1/3*(-9*delta^2+13/2)^(1/2); delta := delta; beta := 1/6*10^(1/2);

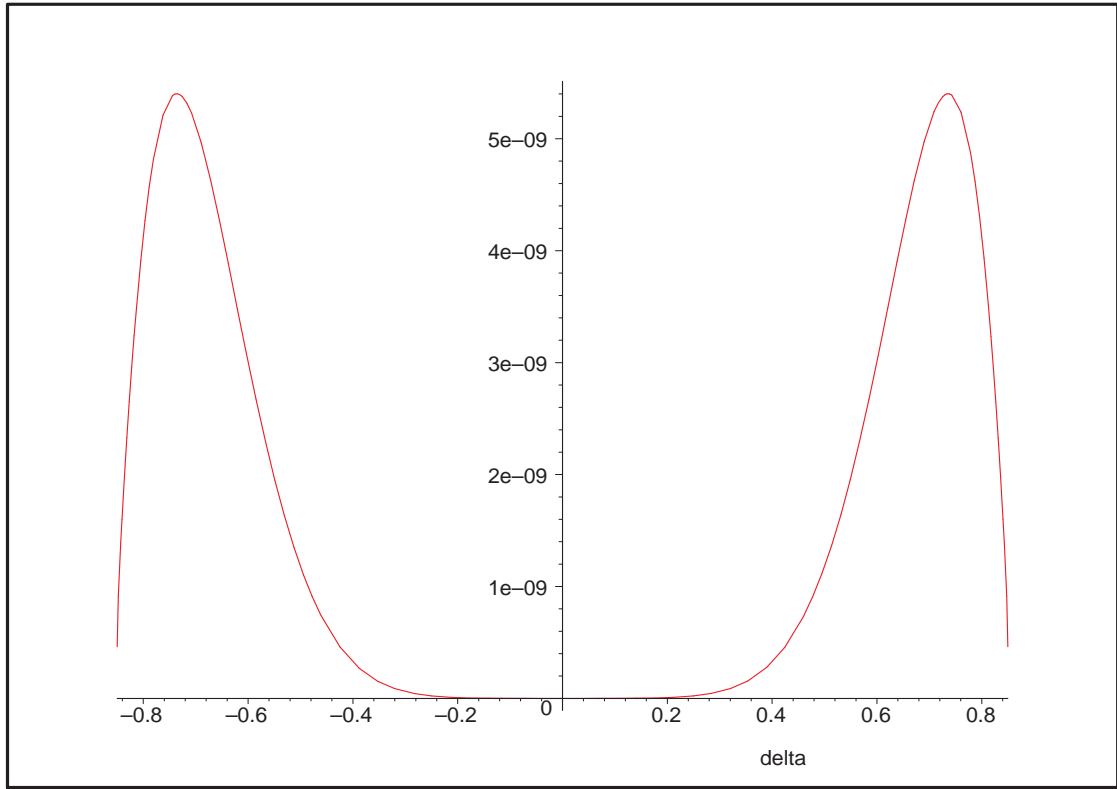
$$\begin{aligned} \alpha &:= \frac{\sqrt{\frac{13}{2} - 9\delta^2}}{3} \\ \delta &:= \delta \\ \beta &:= \frac{\sqrt{10}}{6} \end{aligned}$$

> M_D:=radnormal(simplify(M_D_Centre(N11bis,2,1)));

$$\begin{aligned} M_D &:= \begin{bmatrix} \frac{12}{11}, 0, 0, 0, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ 0, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, \frac{1}{3}, 0, 0, 0, \%1, 0, 0, 0 \\ 0, 0, 0, \frac{1}{3}, 0, 0, 0, 0, \%1, -\frac{\sqrt{-36\delta^2+26}\delta^2}{11} + \frac{13\sqrt{-36\delta^2+26}}{792} + \frac{3}{44} \\ 0, 0, 0, 0, \frac{65}{1188}, 0, 0, 0, \frac{1}{11}\delta^3\sqrt{10} - \frac{13}{264}\delta\sqrt{10}, \%2 \\ 0, 0, 0, 0, 0, \frac{65}{1188}, \%2, 0, 0, 0 \\ 0, 0, \%1, 0, 0, \%2, \frac{331}{4752}, 0, 0, 0 \\ \frac{1}{3}, 0, 0, 0, 0, 0, 0, \frac{133}{594}, \frac{65}{1188}, \frac{65}{1188} \\ \frac{1}{3}, 0, 0, \%1, \frac{1}{11}\delta^3\sqrt{10} - \frac{13}{264}\delta\sqrt{10}, 0, 0, \frac{65}{1188}, \frac{331}{1584}, \frac{331}{4752} \\ \frac{1}{3}, 0, 0, -\frac{\sqrt{-36\delta^2+26}\delta^2}{11} + \frac{13\sqrt{-36\delta^2+26}}{792} + \frac{3}{44}, \%2, 0, 0, \frac{65}{1188}, \frac{331}{4752}, \frac{331}{1584} \end{bmatrix} \\ \%1 &:= \frac{\sqrt{-36\delta^2+26}\delta^2}{11} - \frac{13\sqrt{-36\delta^2+26}}{792} - \frac{3}{44} \\ \%2 &:= -\frac{1}{11}\delta^3\sqrt{10} + \frac{13}{264}\delta\sqrt{10} \end{aligned}$$

> Var:=radnormal(simplify(Var_Norm_Blue_D_Centre(N11bis,2,1))):

> plot(Determinant(M_D),delta=-sqrt(13)/sqrt(18)..sqrt(13)/sqrt(18));

FIG. 11.28 – Graphique : Valeur du déterminant de M_d en fonction de δ .

```
> solve(
{radnormal(simplify(diff(Determinant(M_D),delta))),-sqrt(13)/sqrt(18)
<=delta,delta<=sqrt(13)/sqrt(18)},delta);
```

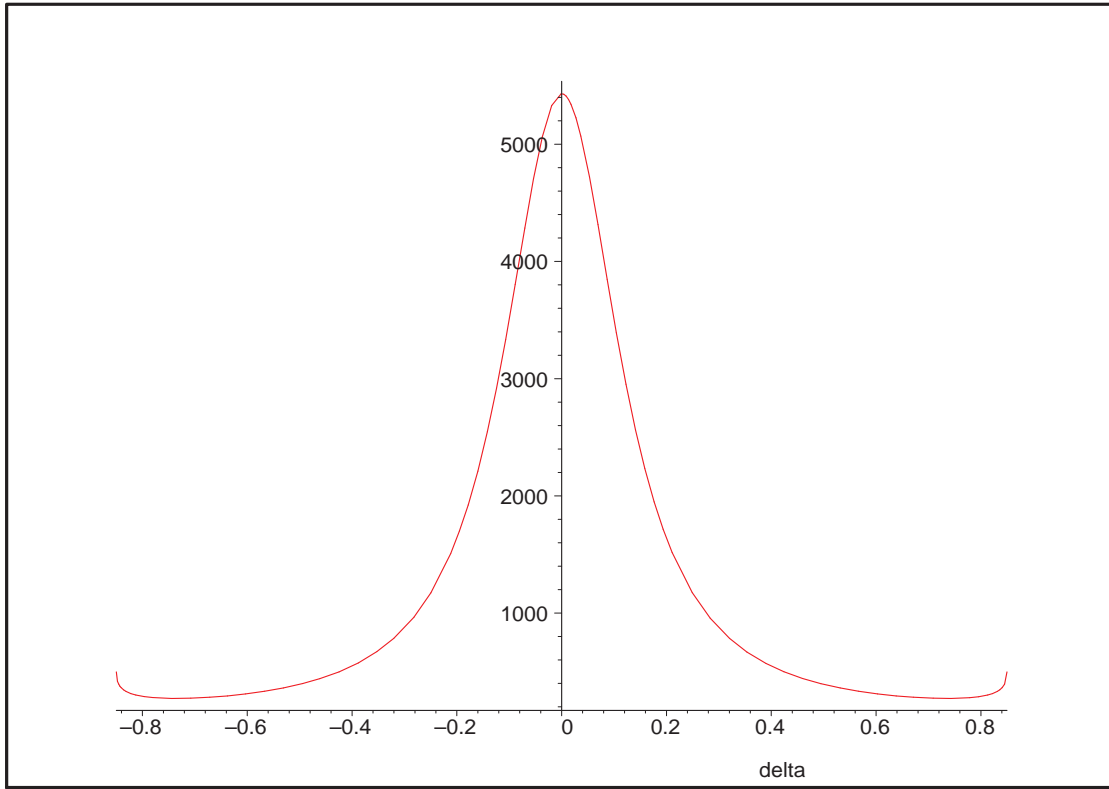
$$\{\delta = 0\}, \{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, -1.471960144)\},$$

$$\{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, 1.471960144)\}$$

```
> Meanvalue:=value(Doubleint(subs(var[3]=sqrt(1-var[1]^2+var[2]^2),Var)
,var[1]=-1..1,var[2]=-1..1))+value(Doubleint(subs(var[3]=-sqrt(1-var[1]^2+var[2]^2),Var),var[1]=-1..1,var[2]=-1..1));
```

$$\begin{aligned} \text{Meanvalue} := & 32(-10653663106944 \delta^6 - 4808722295520 \delta^4 \sqrt{-36 \delta^2 + 26} \\ & - 7635893341284 \delta^4 + 332459358660 \sqrt{-36 \delta^2 + 26} \delta^2 + 56024481876480 \delta^8 \\ & - 9833694824448 \delta^8 \sqrt{-36 \delta^2 + 26} + 2810035703124 \delta^2 \\ & + 12428697625344 \delta^6 \sqrt{-36 \delta^2 + 26} + 31824557752320 \delta^{12} - 68953208463360 \delta^{10} \\ & + 34032114402 \sqrt{-36 \delta^2 + 26} + 191841141427) / (3015(-5377477248 \delta^6 \\ & + 578716164 \delta^4 + 46268820 \delta^2 + 714025 + 8707129344 \delta^{12} - 18865446912 \delta^{10} \\ & + 15328175616 \delta^8)) \end{aligned}$$

```
> plot(Meanvalue,delta=-sqrt(13)/sqrt(18)..sqrt(13)/sqrt(18));
```


 FIG. 11.29 – Graphique : Variance moyenne en fonction de δ .

```
> radnormal(simplify(diff(Meanvalue,delta)));solve(
{radnormal(simplify(diff(Meanvalue,delta))),-sqrt(13)/sqrt(18)<=delta
,delta<=sqrt(13)/sqrt(18)},delta);
```

$$\begin{aligned}
 & -76032(31655509370784 \delta^6 - 4245395579100 \delta^4 - 6350784860160 \delta^8 \sqrt{-36 \delta^2 + 26} \\
 & + 17123233452 \sqrt{-36 \delta^2 + 26} \delta^2 - 1001961499968 \delta^4 \sqrt{-36 \delta^2 + 26} \\
 & + 4128010159104 \delta^6 \sqrt{-36 \delta^2 + 26} + 3517357768704 \delta^{10} \sqrt{-36 \delta^2 + 26} \\
 & - 111087451897344 \delta^8 + 58159706813 + 11348941838 \sqrt{-36 \delta^2 + 26} \\
 & + 76805898468 \delta^2 - 187333887836160 \delta^{12} + 202945045155840 \delta^{10} \\
 & + 69169435508736 \delta^{14}) \delta / (5(845 + 27378 \delta^2 - 101088 \delta^4 + 93312 \delta^6)(\\
 & -5377477248 \delta^6 + 578716164 \delta^4 + 46268820 \delta^2 + 714025 + 8707129344 \delta^{12} \\
 & - 18865446912 \delta^{10} + 15328175616 \delta^8) \sqrt{-36 \delta^2 + 26}) \\
 & \{\delta = 0\}, \{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, -1.471960144)\}, \\
 & \{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, 1.471960144)\}
 \end{aligned}$$

```
> delta := 1/2*sqrt(13/6); beta := 1/6*10^(1/2);alpha :=
1/3*(-9*delta^2+13/2)^(1/2);
```

$$\begin{aligned}
 \delta &:= \frac{\sqrt{78}}{12} \\
 \beta &:= \frac{\sqrt{10}}{6} \\
 \alpha &:= \frac{\sqrt{13} \sqrt{8}}{24}
 \end{aligned}$$

```
> N11bis:=Map(radnormal,OrbToMatrix(OrbPG(Vector([1,0,0]),GrN11) union
OrbPG(Vector([1/3*(-9*delta^2+13/2)^(1/2),1/2*sqrt(13/6),1/6*10^(1/2)]
),GrN11) union OrbPG(Vector([0,0,1]),GrN11) union
{Vector([0,0,0])}));
```

$$N11bis := \begin{bmatrix} \frac{\sqrt{13}\sqrt{2}}{12} & -\frac{\sqrt{78}}{12} & -\frac{\sqrt{10}}{6} \\ -\frac{\sqrt{13}\sqrt{2}}{6} & 0 & -\frac{\sqrt{10}}{6} \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{13}\sqrt{2}}{12} & \frac{\sqrt{78}}{12} & \frac{\sqrt{10}}{6} \\ \frac{\sqrt{13}\sqrt{2}}{12} & \frac{\sqrt{13}\sqrt{3}\sqrt{2}}{12} & -\frac{\sqrt{10}}{6} \\ -\frac{\sqrt{13}\sqrt{2}}{6} & 0 & \frac{\sqrt{10}}{6} \\ \frac{\sqrt{13}\sqrt{2}}{12} & -\frac{\sqrt{13}\sqrt{3}\sqrt{2}}{12} & \frac{\sqrt{10}}{6} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

```
> pointplot3d((N11bis),symbol=box,scaling=constrained,shading=Z,orientation=[35,84]);
```

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N11bis,listlist)))]),
'rational',5)));
```

$$Graph := \text{POLYTOPE}(3, 3, 11, 18)$$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z,orientation=[35,84]);
```

```
> info_iop:=3;
```

$$info_iop := 3$$

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N11bis),listlist),Vector))) : gens:=op(1,%):op(2,%);
```

$$[1, z, y, x, z^2, yz, zx, y^2, xy, x^2, z^3, y^2x]$$

11.15.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(gens,tdeg(x,y,z)),tdeg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N11_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]: A:=poly_algebra(x,y,z): T:=termorder(A,tdeg(x,y,z)): EstSet_N11_tdeg:=sort(EstSet_N11_tdeg,(t1,t2)->testorder(t1,t2,T)); nops(EstSet_N11_tdeg);
```

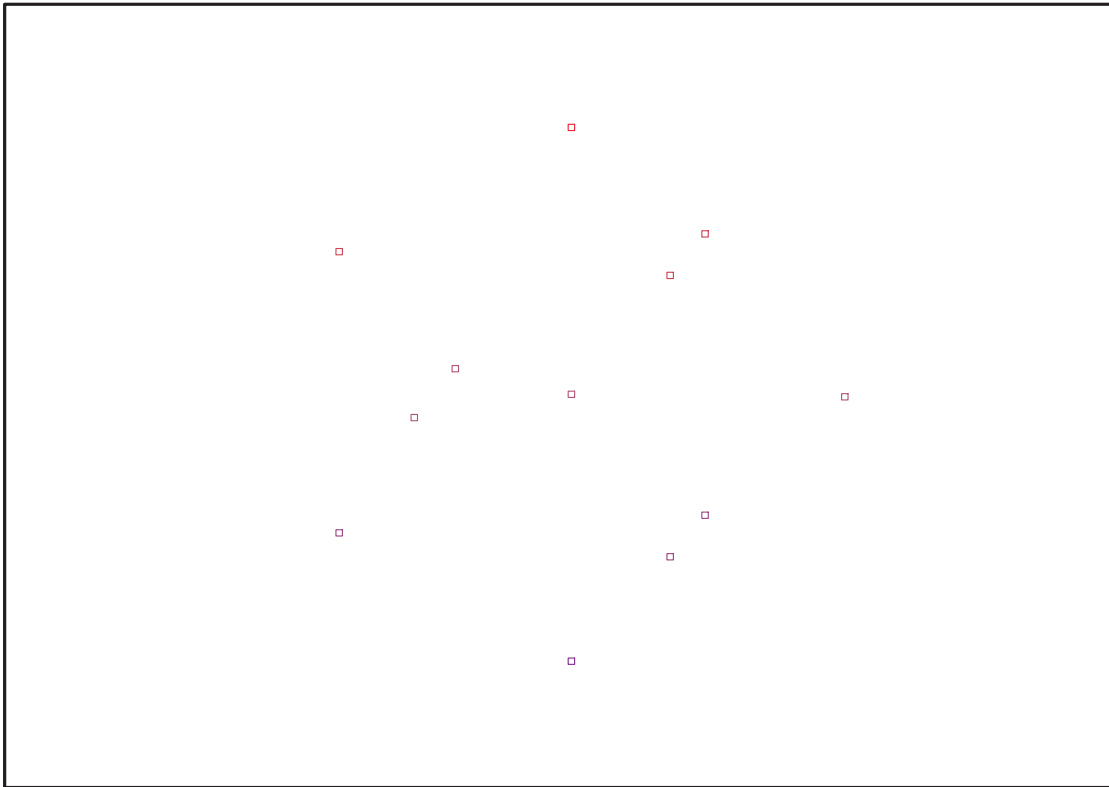



FIG. 11.30 – Graphique : Support du plan N11bis.

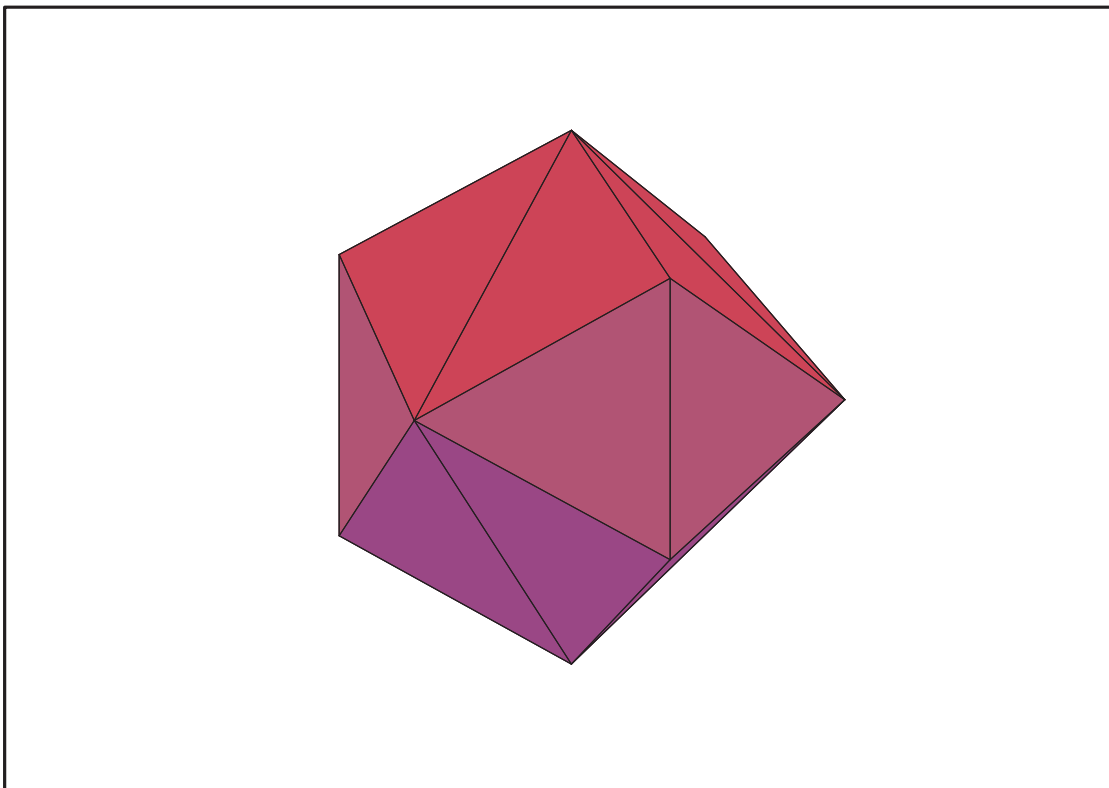


FIG. 11.31 – Graphique : Enveloppe convexe du plan N11bis.

$$\text{EstSet_N11_tdeg} := [1, z, y, x, z^2, yz, zx, y^2, xy, x^2, z^3, y^2x]$$

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> X_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_tdeg):

> simplify(M_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_tdeg));

$$\left[\begin{array}{l} 1, 0, 0, 0, \frac{11}{36}, 0, 0, \frac{11}{36}, 0, \frac{11}{36}, 0, \%1 \\ 0, \frac{11}{36}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{133}{648}, 0 \\ 0, 0, \frac{11}{36}, 0, 0, 0, 0, 0, \%1, 0, 0, 0 \\ 0, 0, 0, \frac{11}{36}, 0, 0, 0, \%1, 0, -\frac{13\sqrt{13}\sqrt{2}}{864} + \frac{1}{16}, 0, \frac{331}{5184} \\ \frac{11}{36}, 0, 0, 0, \frac{133}{648}, 0, 0, \frac{65}{1296}, 0, \frac{65}{1296}, 0, \frac{65\sqrt{13}\sqrt{2}}{15552} \\ 0, 0, 0, 0, 0, \frac{65}{1296}, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, \frac{65}{1296}, 0, 0, 0, 0, 0 \\ \frac{11}{36}, 0, 0, \%1, \frac{65}{1296}, 0, 0, \frac{331}{1728}, 0, \frac{331}{5184}, 0, \frac{169\sqrt{13}\sqrt{2}}{20736} - \frac{3}{64} \\ 0, 0, \%1, 0, 0, 0, 0, 0, \frac{331}{5184}, 0, 0, 0 \\ \frac{11}{36}, 0, 0, -\frac{13\sqrt{13}\sqrt{2}}{864} + \frac{1}{16}, \frac{65}{1296}, 0, 0, \frac{331}{5184}, 0, \frac{331}{1728}, 0, \frac{169\sqrt{13}\sqrt{2}}{62208} - \frac{1}{64} \\ 0, \frac{133}{648}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{2069}{11664}, 0 \\ \%1, 0, 0, \frac{331}{5184}, \frac{65\sqrt{13}\sqrt{2}}{15552}, 0, 0, \frac{169\sqrt{13}\sqrt{2}}{20736} - \frac{3}{64}, 0, \frac{169\sqrt{13}\sqrt{2}}{62208} - \frac{1}{64}, 0, \frac{5113}{124416} \end{array} \right]$$

$$\%1 := \frac{13\sqrt{13}\sqrt{2}}{864} - \frac{1}{16}$$

> radnormal(simplify(expand(Var_Blue_Model_Sat(N11bis,[x,y,z],EstSet_N11_tdeg))));

$$\begin{aligned} & 2(1634\sqrt{13}\sqrt{2} - 8659)(-43901130 - 198014310y^2x^2 + 34055847z^2 + 41243800y^2 \\ & + 15327195x^2 - 92582028z^6 + 44319600z^2xy^2 + 71760160xy^2\sqrt{13}\sqrt{2} \\ & - 315128305y^4 - 8284380\sqrt{13}\sqrt{2} - 340658110xy^2 + 58526181z^4 \\ & + 196391520x^3y^2 - 5805150z^2x - 1846650z^2x\sqrt{13}\sqrt{2} + 114763350x^3 \\ & + 5358600z^2xy^2\sqrt{13}\sqrt{2} + 12201180x^2y^2\sqrt{13}\sqrt{2} - 28486926z^2x^2\sqrt{13}\sqrt{2} \\ & + 112320840y^4x\sqrt{13}\sqrt{2} + 55186560x^2y^4\sqrt{13}\sqrt{2} - 38557800x^3y^2\sqrt{13}\sqrt{2} \\ & + 6426522z^2\sqrt{13}\sqrt{2} + 11044206z^4\sqrt{13}\sqrt{2} - 154940526z^2x^2 \\ & + 21043880y^2\sqrt{13}\sqrt{2} - 574152480x^2y^4 - 616902720y^4x + 35962060y^4\sqrt{13}\sqrt{2} \\ & - 23534940x^3\sqrt{13}\sqrt{2} - 17470728z^6\sqrt{13}\sqrt{2} + 23534940x^2\sqrt{13}\sqrt{2} \\ & - 144391026y^2z^2 - 130090545x^4 - 29500926y^2z^2\sqrt{13}\sqrt{2})/4697714125 \end{aligned}$$

11.15.2. plex(x,y,z)

> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

```
> EstSet_N11_plex:= [op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N11_plex:=sort(EstSet_N11_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N11_plex);
```

$EstSet_N11_plex := [1, z, z^2, z^3, z^4, y, yz, z^2y, y^2, zy^2, y^2z^2, x]$

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```
> X_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_plex):
```

```
> simplify(M_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_plex));
```

$$\left[\begin{array}{l} 1, 0, \frac{11}{36}, 0, \frac{133}{648}, 0, 0, 0, \frac{11}{36}, 0, \frac{65}{1296}, 0 \\ 0, \frac{11}{36}, 0, \frac{133}{648}, 0, 0, 0, 0, 0, \frac{65}{1296}, 0, 0 \\ \frac{11}{36}, 0, \frac{133}{648}, 0, \frac{2069}{11664}, 0, 0, 0, \frac{65}{1296}, 0, \frac{325}{23328}, 0 \\ 0, \frac{133}{648}, 0, \frac{2069}{11664}, 0, 0, 0, 0, 0, \frac{325}{23328}, 0, 0 \\ \frac{133}{648}, 0, \frac{2069}{11664}, 0, \frac{35617}{209952}, 0, 0, 0, \frac{325}{23328}, 0, \frac{1625}{419904}, 0 \\ 0, 0, 0, 0, 0, \frac{11}{36}, 0, \frac{65}{1296}, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, \frac{65}{1296}, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, \frac{65}{1296}, 0, \frac{325}{23328}, 0, 0, 0, 0 \\ \frac{11}{36}, 0, \frac{65}{1296}, 0, \frac{325}{23328}, 0, 0, 0, \frac{331}{1728}, 0, \frac{845}{31104}, -\frac{1}{16} + \frac{13\sqrt{13}\sqrt{2}}{864} \\ 0, \frac{65}{1296}, 0, \frac{325}{23328}, 0, 0, 0, 0, 0, \frac{845}{31104}, 0, 0 \\ \frac{65}{1296}, 0, \frac{325}{23328}, 0, \frac{1625}{419904}, 0, 0, 0, \frac{845}{31104}, 0, \frac{4225}{559872}, \frac{65\sqrt{13}\sqrt{2}}{15552} \\ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{16} + \frac{13\sqrt{13}\sqrt{2}}{864}, 0, \frac{65\sqrt{13}\sqrt{2}}{15552}, \frac{11}{36} \end{array} \right]$$

```
> radnormal(simplify(expand(Var_Blue_Modele_Sat(N11bis,[x,y,z],EstSet_N11_plex))));
```

$$\begin{aligned} & 12 - \frac{57546}{845} z^2 - 40 y^2 + 24 x^2 + \frac{3155112}{4225} z^8 - \frac{6381504}{4225} z^6 - \frac{139104}{325} z^4 y^2 \sqrt{13} \sqrt{2} \\ & + \frac{2592}{65} z^2 x \sqrt{13} \sqrt{2} + \frac{108864}{325} z^6 y^2 \sqrt{13} \sqrt{2} - \frac{2592}{65} z^4 x \sqrt{13} \sqrt{2} - \frac{6912}{65} y^4 z^2 \sqrt{13} \sqrt{2} \\ & + \frac{124416}{325} y^4 z^4 \sqrt{13} \sqrt{2} - \frac{5184}{65} y^2 z^2 x \sqrt{13} \sqrt{2} - \frac{432}{5} x z^4 - \frac{910656}{325} y^2 z^4 - \frac{1152}{5} z^2 x y^2 \\ & + 64 y^4 + 64 x y^2 + \frac{3514122}{4225} z^4 + \frac{12462336}{4225} z^4 y^4 - \frac{296064}{845} z^2 y^4 + \frac{2317248}{845} z^6 y^2 \\ & + \frac{552}{5} z^2 x - 24 x - \frac{1296}{65} z^2 \sqrt{13} \sqrt{2} + \frac{36288}{325} z^4 \sqrt{13} \sqrt{2} + \frac{207456}{845} y^2 z^2 \\ & - \frac{53136}{325} z^6 \sqrt{13} \sqrt{2} + \frac{23328}{325} z^8 \sqrt{13} \sqrt{2} + \frac{6048}{65} y^2 z^2 \sqrt{13} \sqrt{2} \end{aligned}$$

11.15.3. Variance

Modèle de Degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N11bis,1,0);
```

$$Var := 1 + \frac{36}{11} var_3^2 + \frac{36}{11} var_2^2 + \frac{36}{11} var_1^2$$

```
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
```

$$1 + \frac{36}{11} IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

```
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de Degré 2

```
> Var:=simplify(radnormal(simplify(Var_Norm_Blue_D_Centre(N11bis,2,0)))
);
```

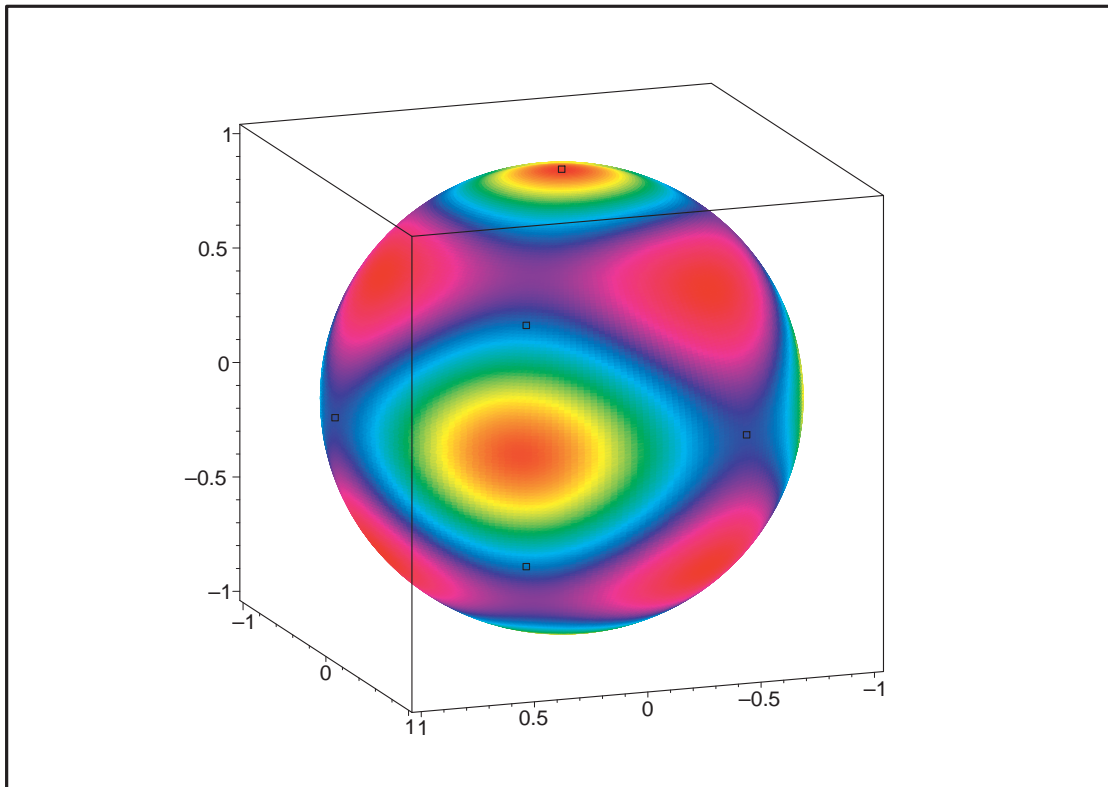
$$\begin{aligned} Var := & -\frac{4}{5904610020648625}(-1052393 + 194568\sqrt{13}\sqrt{2})(-261240776355 var_3^2 \\ & - 80831757380 var_2^2 - 80831757380 var_1^2 + 151244659995 - 386013272040 var_1^3 \\ & + 219171366180 var_3^4 + 443723955210 var_2^4 + 443723955210 var_1^4 \\ & + 27962340120\sqrt{13}\sqrt{2} + 527109977124 var_1^2 var_3^2 + 527109977124 var_2^2 var_3^2 \\ & + 1158039816120 var_2^2 var_1 + 887447910420 var_2^2 var_1^2 \\ & + 79240426980 var_1^3\sqrt{13}\sqrt{2} - 48298587480 var_3^2\sqrt{13}\sqrt{2} \\ & - 83515085940 var_1^2\sqrt{13}\sqrt{2} - 83515085940 var_2^2\sqrt{13}\sqrt{2} \\ & + 40520731680 var_3^4\sqrt{13}\sqrt{2} + 97452885024 var_1^2 var_3^2\sqrt{13}\sqrt{2} \\ & - 237721280940 var_2^2 var_1\sqrt{13}\sqrt{2} + 97452885024 var_2^2 var_3^2\sqrt{13}\sqrt{2}) \end{aligned}$$

```
> simplify(eval(DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},subs(sqrt(13)=t,subs(sqrt(2)=s,Var)),poly_algebra(seq(var[i],i=1..3
),alg_relations={t^2-13,s^2-2})),{s=sqrt(2),t=sqrt(13)}[1]));
```

$$\begin{aligned} & 12 - \frac{891268}{3575} var_3^2 + \frac{177696}{325} var_1 var_3^2 - \frac{177696}{325} var_1 \%1 + \frac{488952}{1675} var_3^4 \\ & - \frac{12866112}{21775} var_3^2 \%1 + \frac{38016}{325} var_3^2\sqrt{13}\sqrt{2} \%1 - \frac{34896}{325} var_1\sqrt{13}\sqrt{2} var_3^2 \\ & + \frac{34896}{325} var_1\sqrt{13}\sqrt{2} \%1 + \frac{710784}{325} var_2^2 var_1 - \frac{19008}{325} \sqrt{13}\sqrt{2} \%1^2 \\ & - \frac{15888}{325} \sqrt{13}\sqrt{2} \%1 + \frac{15888}{325} var_3^2\sqrt{13}\sqrt{2} - \frac{19008}{325} var_3^4\sqrt{13}\sqrt{2} \\ & - \frac{139584}{325} var_2^2 var_1\sqrt{13}\sqrt{2} + \frac{75772296}{239525} \%1^2 + \frac{74288}{325} \%1 \\ & \%1 := IndetDecompInKfi_1 \end{aligned}$$

11.15.4. Etude de la variance pour N11 et N11bis et un modèle de degré 2

```
> SupN11bis:=pointplot3d((N11bis),symbol=box,scaling=constrained,shadin
g=Z,color=black):
> VarN11bis:=plot3d(1,theta=-Pi..Pi,phi=0..Pi,coords=spherical,axes=BOX
ED,color=evalf(simplify(subs(lambda=1,subs(var[2]=lambda*sin(phi)*sin(
theta),subs(var[1]=lambda*sin(phi)*cos(theta),(subs(var[3]=sqrt(1-var[
1]^2-var[2]^2),radnormal(simplify(Var_Norm_Blue_D_Centre(N11bis,2,0)))
)))))),numpoints=50000,style=PATCHNOGRID,orientation=[70,76]):
> display(SupN11bis,VarN11bis);
```


 FIG. 11.32 – Graphique : Variance du plan N11bis pour A_2 .

```
> SupN11bis2:=pointplot3d((10*N11bis),symbol=box,scaling=constrained,shading=Z,color=black):

> VarN11bis2:=plot3d(evalf(simplify(subs(lambda=1,subs(var[2]=lambda*sin(phi)*sin(theta),subs(var[1]=lambda*sin(phi)*cos(theta),(subs(var[3]=sqrt(1-var[1]^2-var[2]^2),radnormal(simplify(Var_Norm_Blue_D_Centre(N11bis,2,0))))))))),theta=-Pi..Pi,phi=0..Pi,coords=spherical,axes=BOXED,numpoints=5000,style=PATCHNOGRID,color=evalf(simplify(subs(lambda=1,subs(var[2]=lambda*sin(phi)*sin(theta),subs(var[1]=lambda*sin(phi)*cos(theta),(subs(var[3]=sqrt(1-var[1]^2-var[2]^2),radnormal(simplify(Var_Norm_Blue_D_Centre(N11bis,2,0)))))))))),orientation=[14,78]):

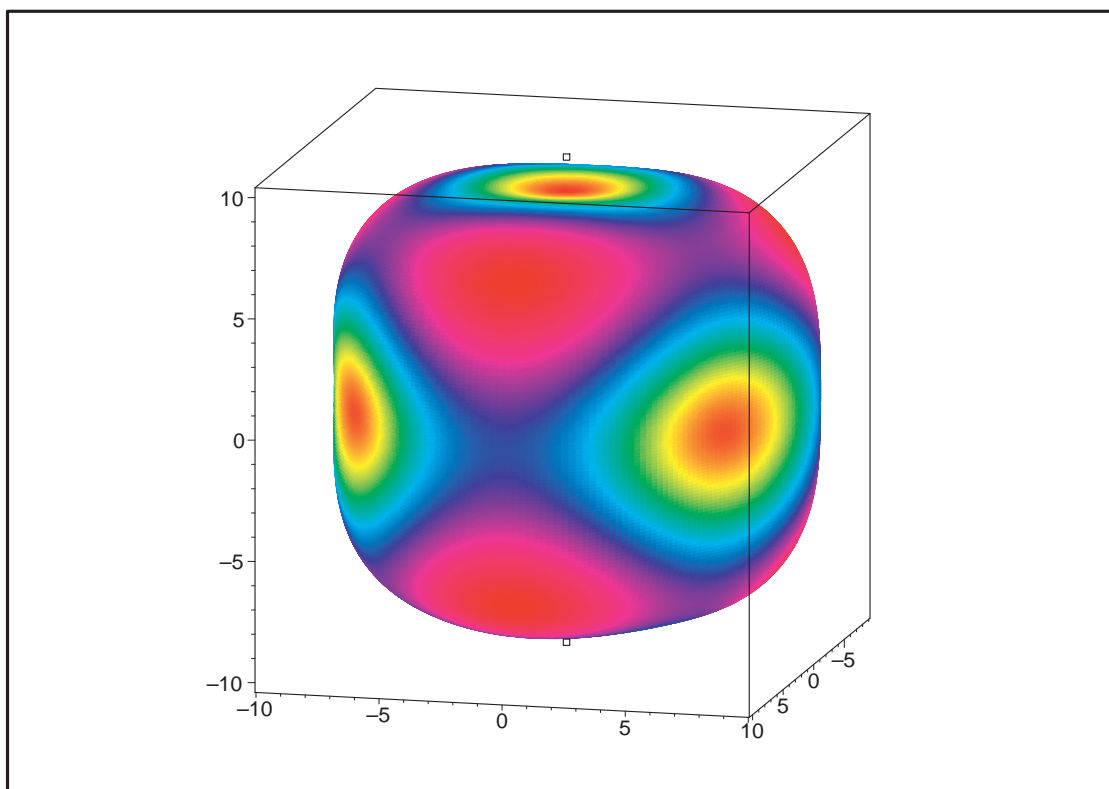
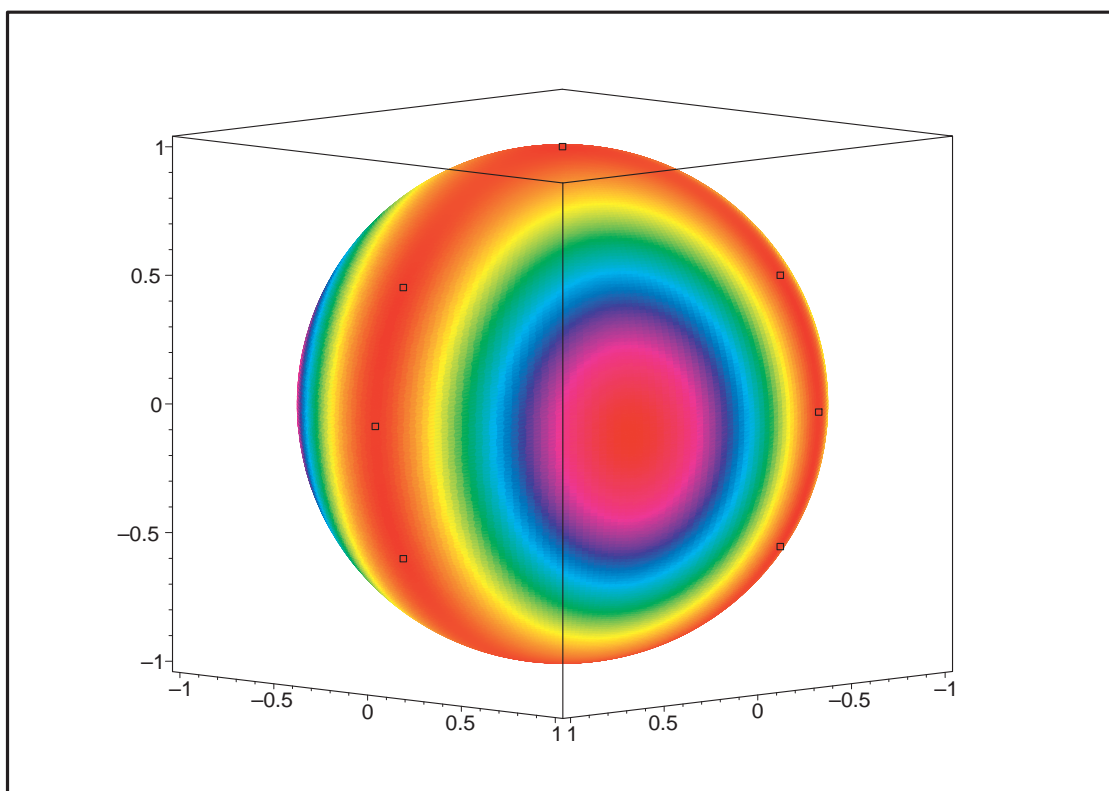
> display(SupN11bis2,VarN11bis2);

> SupN11:=pointplot3d((N11),symbol=box,scaling=constrained,shading=Z,color=black):

> VarN11:=plot3d(1,theta=-Pi..Pi,phi=0..Pi,coords=spherical,axes=BOXED,color=evalf(simplify(subs(lambda=1,subs(var[2]=lambda*sin(phi)*sin(theta),subs(var[1]=lambda*sin(phi)*cos(theta),(subs(var[3]=sqrt(1-var[1]^2-var[2]^2),radnormal(simplify(Var_Norm_Blue_D_Centre(N11,2,1))))))))),numpoints=50000,style=PATCHNOGRID,orientation=[45,83]):

> display(SupN11,VarN11);

> SupN11_2:=pointplot3d((1000*N11),symbol=box,scaling=constrained,shading=Z,color=black):
```

FIG. 11.33 – Graphique : Variance du plan N11bis pour A_2 .FIG. 11.34 – Graphique : Variance du plan N11 pour A_2 .

```
> VarN11_2:=plot3d(evalf(simplify(subs(lambda=1,subs(var[2]=lambda*sin(
phi)*sin(theta),subs(var[1]=lambda*sin(phi)*cos(theta),(subs(var[3]=sq
rt(1-var[1]^2-var[2]^2),radnormal(simplify(Var_Norm_Blue_D_Centre(N11,
2,1))))))))),theta=-Pi..Pi,phi=0..Pi,coords=spherical,axes=BOXED,nump
oints=5000,style=PATCHNOGRID,color=evalf(simplify(subs(lambda=1,subs(v
ar[2]=lambda*sin(phi)*sin(theta),subs(var[1]=lambda*sin(phi)*cos(theta
),(subs(var[3]=sqrt(1-var[1]^2-var[2]^2),radnormal(simplify(Var_Norm_B
lue_D_Centre(N11,2,1)))))))))),orientation=[45,45]):

> display(SupN11_2,VarN11_2);
```

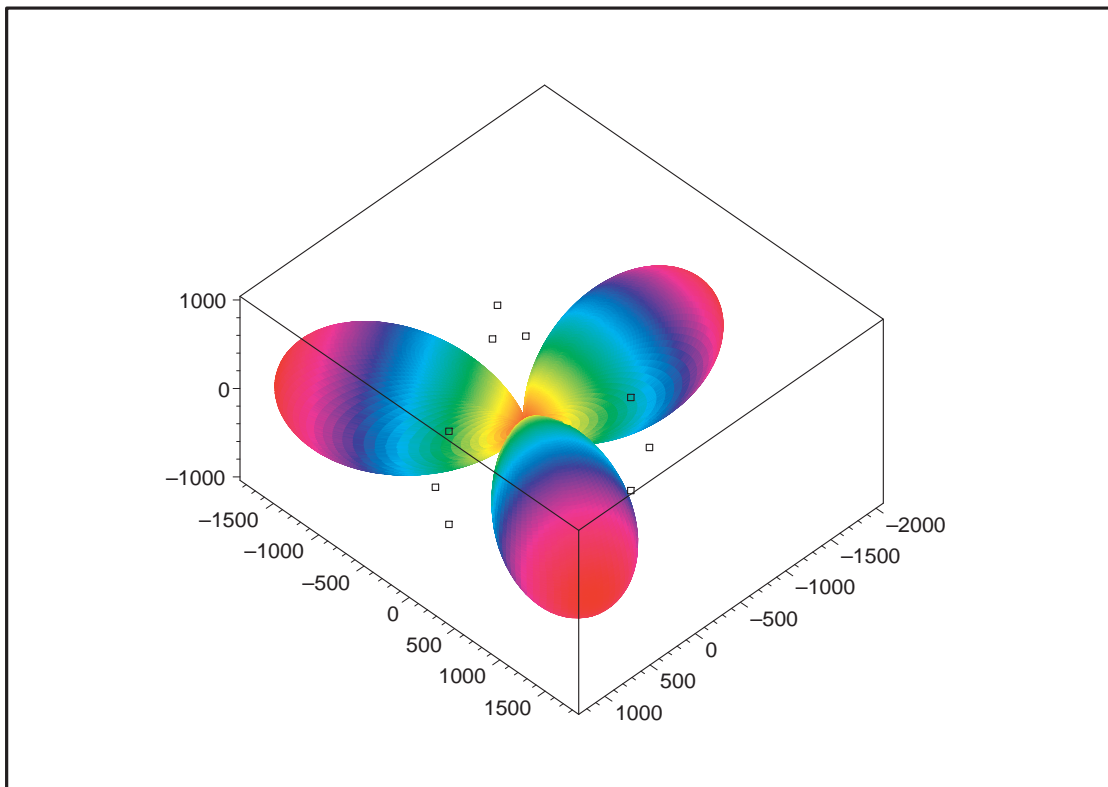


FIG. 11.35 – Graphique : Variance du plan N11 pour A_2 .

11.15.5. Variance pour le modèle tdeg saturé.

```
> unassign('alpha','beta','delta');

> N11bis:=Map(radnormal,OrbToMatrix(OrbPG(Vector([1,0,0]),GrN11) union
OrbPG(Vector([1/3*(-9*delta^2+13/2)^(1/2),delta,1/6*10^(1/2)]),GrN11)
union OrbPG(Vector([0,0,1]),GrN11) union {Vector([0,0,0])}));
```

$$N11bis := \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6} + \frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6} - \frac{\delta}{2} & \frac{\sqrt{10}}{6} \\ \frac{1}{2} & \frac{0}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{0}{2} & \frac{0}{2} & -1 \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6} - \frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6} + \frac{\delta}{2} & -\frac{\sqrt{10}}{6} \\ \frac{\sqrt{\frac{13}{2}-9\delta^2}}{3} & -\delta & -\frac{\sqrt{10}}{6} \\ \frac{\sqrt{\frac{13}{2}-9\delta^2}}{3} & \delta & \frac{\sqrt{10}}{6} \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6} + \frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6} + \frac{\delta}{2} & -\frac{\sqrt{10}}{6} \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6} - \frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6} - \frac{\delta}{2} & \frac{\sqrt{10}}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

> EstSet_N11_tdeg;

$$[1, z, y, x, z^2, yz, zx, y^2, xy, x^2, z^3, y^2x]$$

> M_D_tdeg:=radnormal(simplify(M_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_tdeg))):

> Var_tdeg:=radnormal(simplify((Var_Blue_Modele_Sat(N11bis,[x,y,z],EstSet_N11_tdeg)))):

> Determinant(M_D_tdeg);

$$\begin{aligned}
& -\frac{421019018376125 \delta^2}{22358488275953674912530432} + \frac{21362017873625 \delta^{10} \sqrt{-36 \delta^2 + 26}}{29112614942648014209024} \\
& -\frac{1942001624875 \delta^{12} \sqrt{-36 \delta^2 + 26}}{2426051245220667850752} + \frac{11491133875 \delta^{14} \sqrt{-36 \delta^2 + 26}}{33695156183620386816} \\
& + \frac{11913645213864875}{10866225302113486007489789952} + \frac{8896110594383822875 \sqrt{-36 \delta^2 + 26} \delta^2}{2347104665256512977617794629632} \\
& -\frac{9498393904942691125 \sqrt{-36 \delta^2 + 26}}{4224788397461723359712030333376} \\
& -\frac{6998083263377375 \delta^2 (-36 \delta^2 + 26)^{(3/2)}}{782368221752170992539264876544} + \frac{968204187149875 (-36 \delta^2 + 26)^{(3/2)} \delta^4}{10866225302113486007489789952} \\
& -\frac{18262649556125 (-36 \delta^2 + 26)^{(3/2)} \delta^6}{37729948965671826414895104} + \frac{31865398986123625 (-36 \delta^2 + 26)^{(3/2)}}{84495767949234467194240606666752} \\
& -\frac{130390722151625 \delta^8 \sqrt{-36 \delta^2 + 26}}{349351379311776170508288} + \frac{13001404029784625 \delta^6 \sqrt{-36 \delta^2 + 26}}{100613197241791537106386944} \\
& -\frac{69528392295509875 \delta^4 \sqrt{-36 \delta^2 + 26}}{2414716733802996890553286656} + \frac{1589830214375 \delta^8 (-36 \delta^2 + 26)^{(3/2)}}{1048054137935328511524864} \\
& + \frac{1881455875 \delta^{12} (-36 \delta^2 + 26)^{(3/2)}}{1213025622610333925376} + \frac{153515375 \delta^{12}}{25999348907114496} \\
& -\frac{157938759875 \delta^{10}}{39934999921327865856} + \frac{319172745125 \delta^8}{159739999685311463424} - \frac{17850625 \delta^{14}}{2567836929097728} \\
& + \frac{274625 \delta^{16}}{53496602689536} - \frac{21125 \delta^{18}}{13374150672384} - \frac{413644860926375 \delta^6}{621069118776490969792512} \\
& + \frac{542657018580625 \delta^4}{3726414712658945818755072} - \frac{24458926375 \delta^{10} (-36 \delta^2 + 26)^{(3/2)}}{9704204980882671403008}
\end{aligned}$$

> solve(Determinant(M_D_tdeg),delta);

$$\begin{aligned}
& \frac{1}{12} I \sqrt{2 (3887 + 1014 \sqrt{10})^{(1/3)} + \frac{338}{(3887 + 1014 \sqrt{10})^{(1/3)}}} - 52, \\
& \frac{-1}{12} I \sqrt{2 (3887 + 1014 \sqrt{10})^{(1/3)} + \frac{338}{(3887 + 1014 \sqrt{10})^{(1/3)}}} - 52
\end{aligned}$$

> info_iop:=3;

info_iop := 3

> radnormal(simplify(diff(Determinant(M_D_tdeg),delta)));solve(
{radnormal(simplify(diff(Determinant(M_D_tdeg),delta))),-sqrt(13)/sqrt(18)}<=delta,delta<=sqrt(13)/sqrt(18)},delta);

$$\begin{aligned}
& -21125 \delta (-37496123817984 \delta^{10} \sqrt{-36 \delta^2 + 26} - 8458976717568 \delta^6 \sqrt{-36 \delta^2 + 26} \\
& + 20929109909760 \delta^8 \sqrt{-36 \delta^2 + 26} + 2114728756452 \delta^4 \sqrt{-36 \delta^2 + 26} \\
& + 51502670069760 \delta^{12} \sqrt{-36 \delta^2 + 26} - 308254874460 \sqrt{-36 \delta^2 + 26} \delta^2 \\
& - 43465989685248 \delta^{14} \sqrt{-36 \delta^2 + 26} + 19929894361 \sqrt{-36 \delta^2 + 26} \\
& + 1642026955608 \delta^2 - 101623636686 - 11855459748600 \delta^4 \\
& + 15045919506432 \delta^{16} \sqrt{-36 \delta^2 + 26} + 81498730659840 \delta^{14} + 49950337469760 \delta^6 \\
& - 220725728870400 \delta^{12} + 239119539609600 \delta^{10} - 138674572508160 \delta^8) / (\\
& 11179244137976837456265216 \sqrt{-36 \delta^2 + 26})
\end{aligned}$$

$$\{\delta = 0\}, \{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, -1.471960144)\},$$
$$\{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, 1.471960144)\}$$

```
> Meanvalue_tdeg:=value(Doubleint(subs(z=sqrt(1-x^2+y^2),Var_tdeg),x=-1
..1,y=-1..1))+value(Doubleint(subs(z=-sqrt(1-x^2+y^2),Var_tdeg),x=-1..
1,y=-1..1));
```

$$\begin{aligned}
& \text{Meanvalue_tdeg} := 8(160716625862733866250 \delta^2 \\
& + 225484820267700712680 \delta^4 \sqrt{-36 \delta^2 + 26} + 2019341027808327499380 \delta^4 \\
& + 27837192239611326000 \sqrt{-36 \delta^2 + 26} \delta^2 \\
& - 19773218236797935320320 \delta^8 \sqrt{-36 \delta^2 + 26} - 39020995722425333345943552 \delta^{22} \\
& + 9004845166713538464448512 \delta^{24} - 79358442067652930244182016 \delta^{18} \\
& + 51716044013556831796592640 \delta^{16} - 20065086698077883714863104 \delta^{14} \\
& + 73977304390431361135017984 \delta^{20} \\
& + 2294711780560380509552640 \delta^{18} \sqrt{-36 \delta^2 + 26} \\
& - 1170085846801684906229760 \delta^{12} \sqrt{-36 \delta^2 + 26} \\
& + 2624629093138890898145280 \delta^{14} \sqrt{-36 \delta^2 + 26} + 1046401650968629845240 \delta^6 \\
& + 606738267020317250 \sqrt{-36 \delta^2 + 26} + 3252574626039734125 \\
& - 3336390154630553240862720 \delta^{16} \sqrt{-36 \delta^2 + 26} \\
& - 668903838867803225456640 \sqrt{-36 \delta^2 + 26} \delta^{20} \\
& + 272019015403053579048960 \delta^{10} \sqrt{-36 \delta^2 + 26} \\
& - 2840731336564535737440 \delta^6 \sqrt{-36 \delta^2 + 26} + 3907885151455028596629504 \delta^{12} \\
& - 39444147386655977092608 \delta^{10} - 99081759925364014232928 \delta^8) / (12675(\\
& 509831700625 + 66074188401000 \delta^2 + 2967239322192600 \delta^4 \\
& - 328527772790298771456 \delta^{22} + 75814101413145870336 \delta^{24} \\
& - 671990546568428126208 \delta^{18} + 447927903787023728640 \delta^{16} \\
& - 185883572194224832512 \delta^{14} + 622833902581608087552 \delta^{20} \\
& + 45873721662406560 \delta^6 + 44925257821255299072 \delta^{12} \\
& - 4832573614372168704 \delta^{10} - 140817254020311024 \delta^8)) - 8(\\
& -160716625862733866250 \delta^2 - 2019341027808327499380 \delta^4 \\
& + 39020995722425333345943552 \delta^{22} - 9004845166713538464448512 \delta^{24} \\
& - 606738267020317250 \sqrt{-36 \delta^2 + 26} + 79358442067652930244182016 \delta^{18} \\
& - 51716044013556831796592640 \delta^{16} + 20065086698077883714863104 \delta^{14} \\
& - 73977304390431361135017984 \delta^{20} - 3252574626039734125 \\
& - 272019015403053579048960 \delta^{10} \sqrt{-36 \delta^2 + 26} - 1046401650968629845240 \delta^6 \\
& + 668903838867803225456640 \sqrt{-36 \delta^2 + 26} \delta^{20} \\
& - 2294711780560380509552640 \delta^{18} \sqrt{-36 \delta^2 + 26} \\
& - 2624629093138890898145280 \delta^{14} \sqrt{-36 \delta^2 + 26} \\
& + 3336390154630553240862720 \delta^{16} \sqrt{-36 \delta^2 + 26} \\
& - 27837192239611326000 \sqrt{-36 \delta^2 + 26} \delta^2 \\
& + 1170085846801684906229760 \delta^{12} \sqrt{-36 \delta^2 + 26} \\
& + 2840731336564535737440 \delta^6 \sqrt{-36 \delta^2 + 26} \\
& - 225484820267700712680 \delta^4 \sqrt{-36 \delta^2 + 26} \\
& + 19773218236797935320320 \delta^8 \sqrt{-36 \delta^2 + 26} - 3907885151455028596629504 \delta^{12} \\
& + 39444147386655977092608 \delta^{10} + 99081759925364014232928 \delta^8) / (12675(\\
& 509831700625 + 66074188401000 \delta^2 + 2967239322192600 \delta^4 \\
& - 328527772790298771456 \delta^{22} + 75814101413145870336 \delta^{24} \\
& - 671990546568428126208 \delta^{18} + 447927903787023728640 \delta^{16} \\
& - 185883572194224832512 \delta^{14} + 622833902581608087552 \delta^{20} \\
& + 45873721662406560 \delta^6 + 44925257821255299072 \delta^{12} \\
& - 4832573614372168704 \delta^{10} - 140817254020311024 \delta^8))
\end{aligned}$$

```
> plot(Meanvalue_tdeg,delta=-sqrt(13)/sqrt(18)..sqrt(13)/sqrt(18));
```

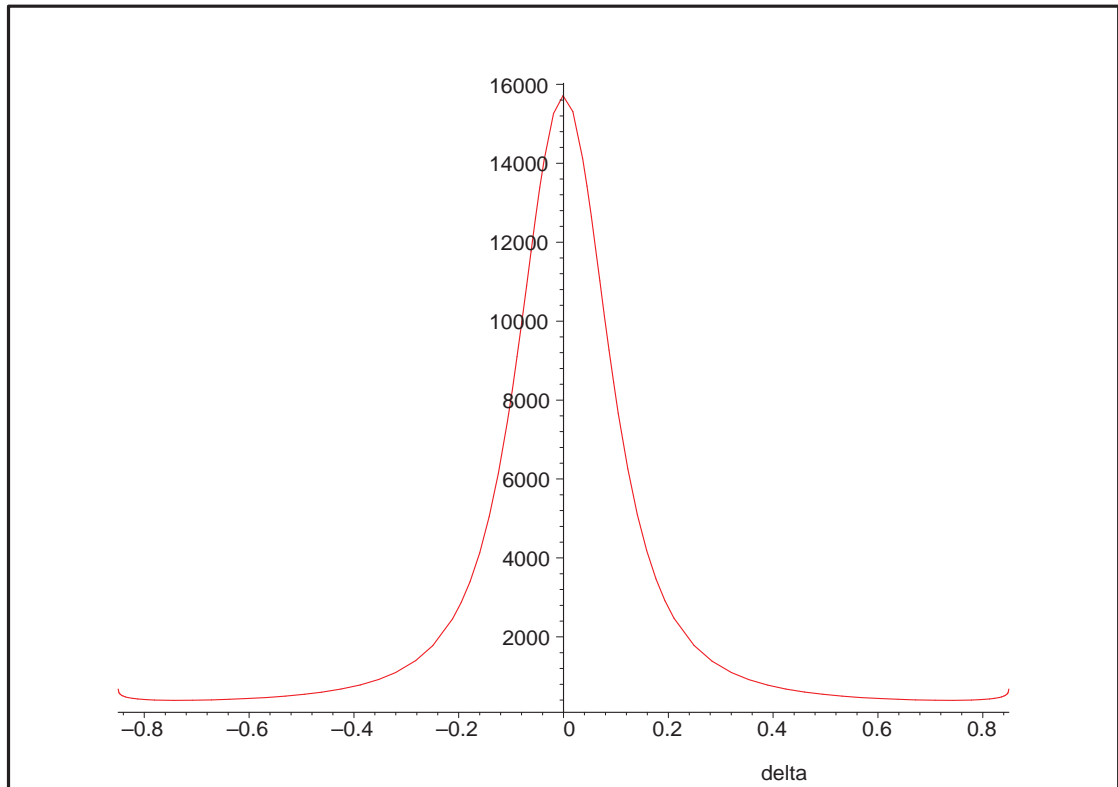


FIG. 11.36 – Graphique : Valeur moyenne de la variance en fonction de δ pour tdeg..

```
> solve(
{radnormal(simplify(diff(Meanvalue_tdeg,delta))),-sqrt(13)/sqrt(18)<=
delta,delta<=sqrt(13)/sqrt(18)},delta);
```

$$\{\delta = 0\}, \{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, -1.471960144)\},$$

$$\{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, 1.471960144)\}$$

```
> SupN11bis_tdeg:=pointplot3d((subs(delta=sqrt(13/6)/2,N11bis)),symbol=
box,scaling=constrained,shading=Z,color=black):
> VarN11bis_tdeg:=plot3d(1,theta=-Pi..Pi,phi=0..Pi,coords=spherical,axe
s=BOXED,color=evalf(simplify(subs(lambda=1,subs(y=lambda*sin(phi))*sin(
theta),subs(x=lambda*sin(phi))*cos(theta),(subs(z=sqrt(1-x^2-y^2),2/469
7714125*(1634*13^(1/2)*2^(1/2)-8659)*(-43901130-198014310*y^2*x^2+3405
5847*z^2+41243800*y^2+15327195*x^2-92582028*z^6+44319600*z^2*x*y^2+717
60160*x*y^2*13^(1/2)*2^(1/2)-315128305*y^4-8284380*13^(1/2)*2^(1/2)-34
0658110*x*y^2+58526181*z^4+196391520*x^3*y^2-5805150*z^2*x-1846650*z^2
*x*13^(1/2)*2^(1/2)+114763350*x^3+5358600*z^2*x*y^2*13^(1/2)*2^(1/2)+1
2201180*x^2*y^2*13^(1/2)*2^(1/2)-28486926*z^2*x^2*13^(1/2)*2^(1/2)+112
320840*y^4*x*13^(1/2)*2^(1/2)+55186560*x^2*y^4*13^(1/2)*2^(1/2)-385578
00*x^3*y^2*13^(1/2)*2^(1/2)+6426522*z^2*13^(1/2)*2^(1/2)+11044206*z^4*
13^(1/2)*2^(1/2)-154940526*z^2*x^2+21043880*y^2*13^(1/2)*2^(1/2)-57415
2480*x^2*y^4-616902720*y^4*x+35962060*y^4*13^(1/2)*2^(1/2)-23534940*x^
3*13^(1/2)*2^(1/2)-17470728*z^6*13^(1/2)*2^(1/2)+23534940*x^2*13^(1/2)
*2^(1/2)-144391026*y^2*z^2-130090545*x^4-29500926*y^2*z^2*13^(1/2)*2^(
1/2)))))),numpoints=50000,style=PATCHNOGRID,orientation=[68,79]):
```

```
> display(SupN11bis_tdeg,VarN11bis_tdeg);
```

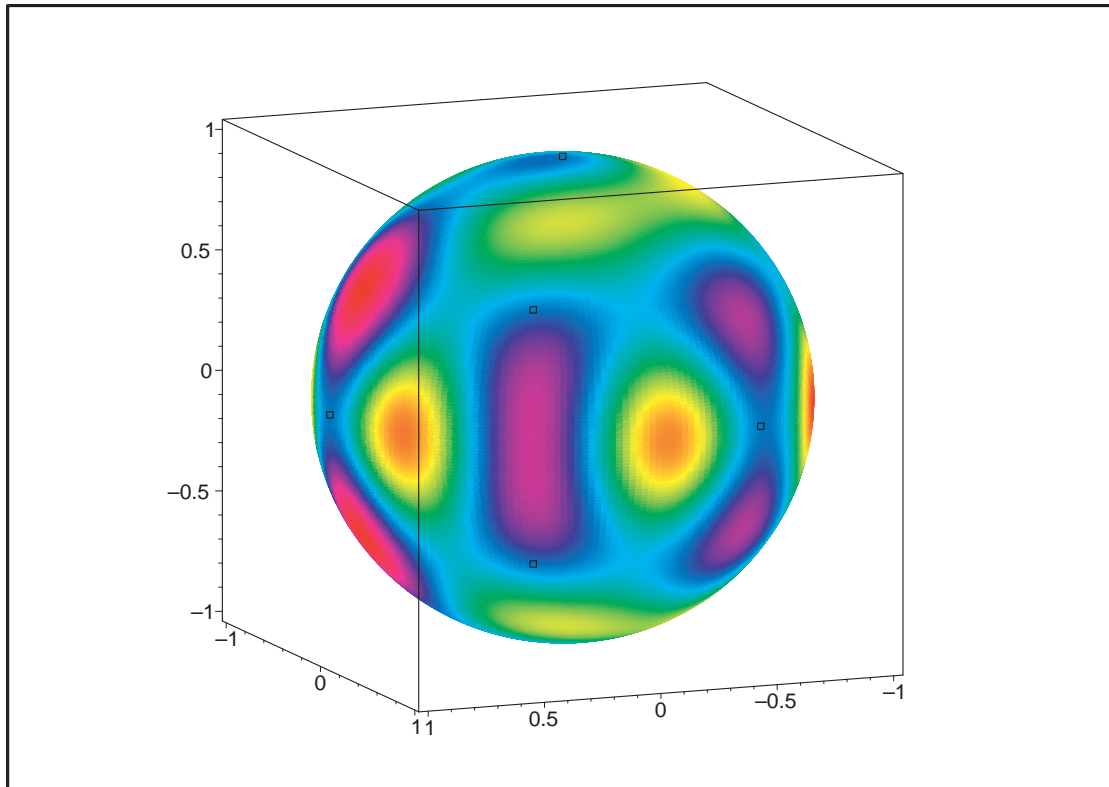


FIG. 11.37 – Graphique : Valeur de la variance pour δ optimal et un modèle tdeg saturé.

```
> SupN11bis_2_tdeg:=pointplot3d(15*(subs(delta=sqrt(13/6)/2,N11bis)),sy
mbol=box,scaling=constrained,shading=Z,color=black):
```

```

> VarN11bis_2_tdeg:=plot3d(evalf(simplify(subs(lambda=1,subs(y=lambda*sin(phi)*sin(theta),subs(x=lambda*sin(phi)*cos(theta),subs(z=sqrt(1-x^2-y^2),2/4697714125*(1634*13^(1/2)*2^(1/2)-8659)*(-43901130-198014310*y^2*x^2+34055847*z^2+41243800*y^2+15327195*x^2-92582028*z^6+44319600*z^2*x*y^2+71760160*x*y^2*13^(1/2)*2^(1/2)-315128305*y^4-8284380*13^(1/2)*2^(1/2)-340658110*x*y^2+58526181*z^4+196391520*x^3*y^2-5805150*z^2*x-1846650*z^2*x*13^(1/2)*2^(1/2)+114763350*x^3+5358600*z^2*x*y^2*13^(1/2)*2^(1/2)+12201180*x^2*y^2*13^(1/2)*2^(1/2)-28486926*z^2*x^2*13^(1/2)*2^(1/2)+112320840*y^4*x*13^(1/2)*2^(1/2)+55186560*x^2*y^4*13^(1/2)*2^(1/2)-38557800*x^3*y^2*13^(1/2)*2^(1/2)+6426522*z^2*13^(1/2)*2^(1/2)+11044206*z^4*13^(1/2)*2^(1/2)-154940526*z^2*x^2+21043880*y^2*13^(1/2)*2^(1/2)-574152480*x^2*y^4-616902720*y^4*x+35962060*y^4*13^(1/2)*2^(1/2)-23534940*x^3*13^(1/2)*2^(1/2)-17470728*z^6*13^(1/2)*2^(1/2)+23534940*x^2*13^(1/2)*2^(1/2)-144391026*y^2*z^2-130090545*x^4-29500926*y^2*z^2*13^(1/2)*2^(1/2)))))),theta=-Pi..Pi,phi=0..Pi,coords=spherical,axes=BOXED,numpoints=5000,style=PATCHNOGRID,color=evalf(simplify(subs(lambda=1,subs(y=lambda*sin(phi)*sin(theta),subs(x=lambda*sin(phi)*cos(theta),subs(z=sqrt(1-x^2-y^2),2/4697714125*(1634*13^(1/2)*2^(1/2)-8659)*(-43901130-198014310*y^2*x^2+34055847*z^2+41243800*y^2+15327195*x^2-92582028*z^6+44319600*z^2*x*y^2+71760160*x*y^2*13^(1/2)*2^(1/2)-315128305*y^4-8284380*13^(1/2)*2^(1/2)-340658110*x*y^2+58526181*z^4+196391520*x^3*y^2-5805150*z^2*x-1846650*z^2*x*13^(1/2)*2^(1/2)+114763350*x^3+5358600*z^2*x*y^2*13^(1/2)*2^(1/2)+12201180*x^2*y^2*13^(1/2)*2^(1/2)-28486926*z^2*x^2*13^(1/2)*2^(1/2)+112320840*y^4*x*13^(1/2)*2^(1/2)+55186560*x^2*y^4*13^(1/2)*2^(1/2)-38557800*x^3*y^2*13^(1/2)*2^(1/2)+6426522*z^2*13^(1/2)*2^(1/2)+11044206*z^4*13^(1/2)*2^(1/2)-154940526*z^2*x^2+21043880*y^2*13^(1/2)*2^(1/2)-574152480*x^2*y^4-616902720*y^4*x+35962060*y^4*13^(1/2)*2^(1/2)-23534940*x^3*13^(1/2)*2^(1/2)-17470728*z^6*13^(1/2)*2^(1/2)+23534940*x^2*13^(1/2)*2^(1/2)-144391026*y^2*z^2-130090545*x^4-29500926*y^2*z^2*13^(1/2)*2^(1/2)))))),orientation=[25,74]):

> display(SupN11bis_2_tdeg,VarN11bis_2_tdeg);

```

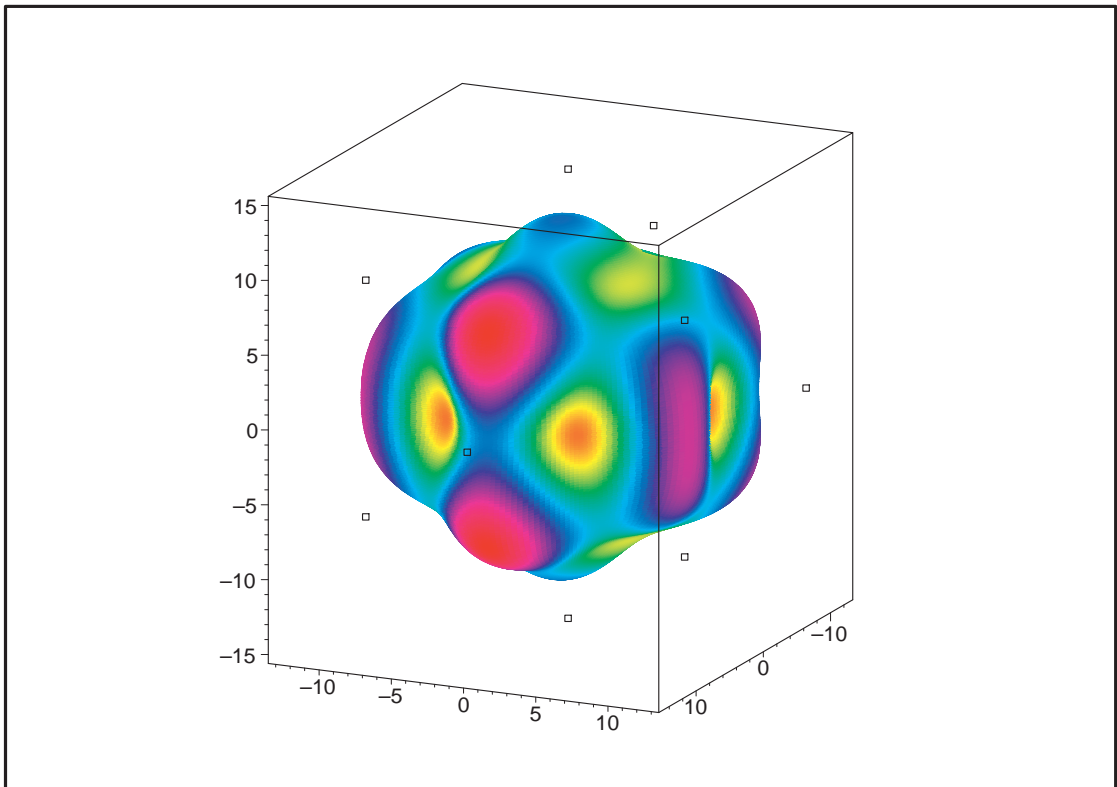


FIG. 11.38 – Graphique : Valeur de la variance pour δ optimal et un modèle tdeg saturé.

11.15.6. Optimalité pour le modèle plex saturé.

```
> unassign('alpha','beta','delta');

> N11bis:=Map(radnormal,OrbToMatrix(OrbPG(Vector([1,0,0]),GrN11) union
OrbPG(Vector([1/3*(-9*delta^2+13/2)^(1/2),delta,1/6*10^(1/2)]),GrN11)
union OrbPG(Vector([0,0,1]),GrN11) union {Vector([0,0,0])}));
```

$$N11bis := \begin{bmatrix} -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6}-\frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6}-\frac{\delta}{2} & \frac{\sqrt{10}}{6} \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6}+\frac{\sqrt{3}\delta}{2} & \frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6}+\frac{\delta}{2} & -\frac{\sqrt{10}}{6} \\ \frac{\sqrt{\frac{13}{2}-9\delta^2}}{3} & \delta & \frac{\sqrt{10}}{6} \\ \frac{\sqrt{\frac{13}{2}-9\delta^2}}{3} & -\delta & -\frac{\sqrt{10}}{6} \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6}-\frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6}+\frac{\delta}{2} & -\frac{\sqrt{10}}{6} \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{\frac{13}{2}-9\delta^2}}{6}+\frac{\sqrt{3}\delta}{2} & -\frac{\sqrt{3}\sqrt{\frac{13}{2}-9\delta^2}}{6}-\frac{\delta}{2} & \frac{\sqrt{10}}{6} \\ 1 & 0 & 0 \end{bmatrix}$$

```
> EstSet_N11_plex;

[1, z, z^2, z^3, z^4, y, y z, z^2 y, y^2, z y^2, y^2 z^2, x]

> M_D_plex:=radnormal(simplify(M_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_
plex))):

> Var_tdeg:=radnormal(simplify((Var_Blue_Modele_Sat(N11bis,[x,y,z],EstS
et_N11_plex)))):

> Determinant(M_D_plex);
```

$$\begin{aligned}
& -\frac{20711912837890625}{15842956490481462598920113750016}\delta^2 - \frac{3554115220703125}{55010265591949522912917061632}\delta^6 \\
& + \frac{55783203125}{589530452588622287732736}\delta^{12} - \frac{725181640625}{3537182715531733726396416}\delta^{10} \\
& + \frac{9427361328125}{56594923448507739622342656}\delta^8 + \frac{68508634771484375}{5280985496827154199640037916672}\delta^4 \\
& + \frac{269254866892578125}{5133117902915993882050116855005184}
\end{aligned}$$

```

> solve(Determinant(M_D_plex),delta);


$$\frac{\sqrt{26}}{6}, -\frac{\sqrt{26}}{6}, \frac{\sqrt{26}}{6}, -\frac{\sqrt{26}}{6}, \frac{\sqrt{26}}{12}, -\frac{\sqrt{26}}{12}, \frac{\sqrt{26}}{12}, -\frac{\sqrt{26}}{12}, \frac{\sqrt{26}}{12}, -\frac{\sqrt{26}}{12}, \frac{\sqrt{26}}{12}, -\frac{\sqrt{26}}{12}$$


> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((eval(N11bis,[delta=sqrt(26)/6])),listlist),Vector))) : gens_1:=op(1,%) : op(2,%%);


$$[1, z, y, x, z^2, yz, zx, y^2, xy, x^2, z^3, y^2x]$$


> Frontiere_plex:=map(LeadingMonomial, {op(GroebnerBasis(gens_1,plex(x,y,z)))},plex(x,y,z)): Div:=Diviseurs({op(Frontiere_plex)}): nops(Div):

> EstSet_N11_plex:=[op(EstSet(Div,Frontiere_plex))]: A:=poly_algebra(x,y,z): T:=termorder(A,plex(x,y,z)): EstSet_N11_plex:=sort(EstSet_N11_plex,(t1,t2)->testorder(t1,t2,T)); nops(EstSet_N11_plex);


$$EstSet\_N11\_plex := [1, z, z^2, z^3, z^4, y, yz, z^2y, y^2, x, zx, z^2x]$$


> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((eval(N11bis,[delta=sqrt(26)/12])),listlist),Vector))) : gens_2:=op(1,%) : op(2,%%);


$$[1, z, y, x, z^2, yz, zx, y^2, xy, x^2, z^3, y^2x]$$


> Frontiere_plex:=map(LeadingMonomial, {op(GroebnerBasis(gens_2,plex(x,y,z)))},plex(x,y,z)): Div:=Diviseurs({op(Frontiere_plex)}): nops(Div):

> EstSet_N11_plex:=[op(EstSet(Div,Frontiere_plex))]: A:=poly_algebra(x,y,z): T:=termorder(A,plex(x,y,z)): EstSet_N11_plex:=sort(EstSet_N11_plex,(t1,t2)->testorder(t1,t2,T)); nops(EstSet_N11_plex);


$$EstSet\_N11\_plex := [1, z, z^2, z^3, z^4, y, yz, z^2y, y^2, x, zx, z^2x]$$

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> plot(Determinant(M_D_plex),delta=-sqrt(13)/sqrt(18)..sqrt(13)/sqrt(18));

> solve(
{radnormal(simplify(diff(Determinant(M_D_plex),delta)),-sqrt(13)/sqrt(18)<=delta,delta<=sqrt(13)/sqrt(18))},delta);


$$\begin{aligned}
& \{\delta = 0\}, \{\delta = \text{RootOf}(-13 + 18\_Z^2, -0.8498365856)\}, \\
& \{\delta = \text{RootOf}(-13 + 18\_Z^2, 0.8498365856)\}, \\
& \{\delta = \text{RootOf}(24\_Z^2 - 13, -0.7359800722)\}, \\
& \{\delta = \text{RootOf}(24\_Z^2 - 13, 0.7359800722)\}, \%2, \%1, \%2, \%1, \%2, \%1 \\
& \%1 := \{\delta = \text{RootOf}(72\_Z^2 - 13, 0.4249182928)\} \\
& \%2 := \{\delta = \text{RootOf}(72\_Z^2 - 13, -0.4249182928)\}
\end{aligned}$$


```

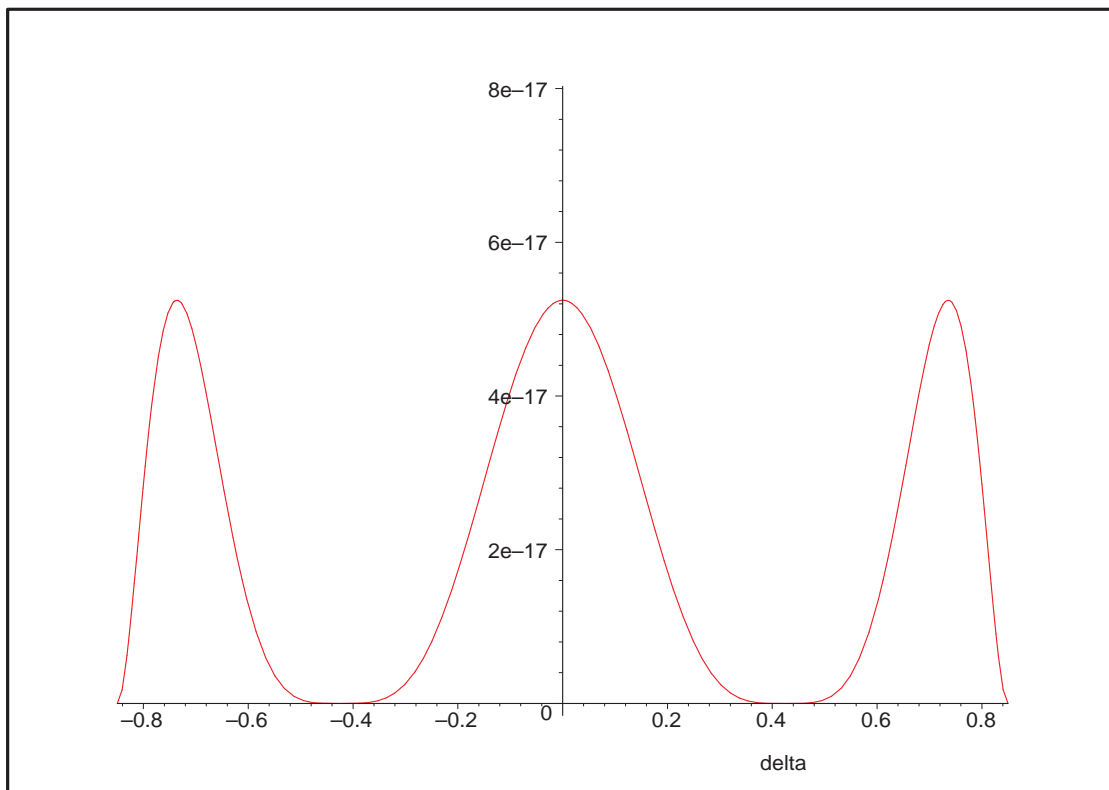


FIG. 11.39 – Graphique : Valeur du déterminant de M_D en fonction de δ pour plex.

```
> eval(Determinant(M_D_plex),[delta=0]);eval(Determinant(M_D_plex),[del
ta=sqrt(13/24)]);
```

$$\frac{269254866892578125}{5133117902915993882050116855005184}$$

$$\frac{269254866892578125}{5133117902915993882050116855005184}$$

```
> Meanvalue_plex:=value(student[Doubleint](subs(z=sqrt(1-x^2+y^2),Var_t
deg),x=-1..1,y=-1..1))+value(student[Doubleint](subs(z=-sqrt(1-x^2+y^2
),Var_tdeg),x=-1..1,y=-1..1));
```

$$\begin{aligned} \text{Meanvalue_plex} := & 32(6480181583694 \delta^2 - 1491444804096 \sqrt{-36 \delta^2 + 26} \delta^2 \\ & + 22086299362176 \delta^6 - 23926824309024 \delta^4 + 269288645184 \sqrt{-36 \delta^2 + 26} \\ & - 2641440284899) / (2218125 (-2197 + 93312 \delta^6 - 101088 \delta^4 + 27378 \delta^2)) \end{aligned}$$

```
> solve(-2197+93312*delta^6-101088*delta^4+27378*delta^2,delta);
```

$$\frac{\sqrt{26}}{6}, -\frac{\sqrt{26}}{6}, \frac{\sqrt{26}}{12}, -\frac{\sqrt{26}}{12}, \frac{\sqrt{26}}{12}, -\frac{\sqrt{26}}{12}$$

```
> plot(Meanvalue_plex,delta=-sqrt(13)/sqrt(18)..sqrt(13)/sqrt(18),y=500
0..50000);
```

```
> solve(
{radnormal(simplify(diff(Meanvalue_plex,delta))),-sqrt(13)/sqrt(18)<=
delta,delta<=sqrt(13)/sqrt(18)},delta);
```

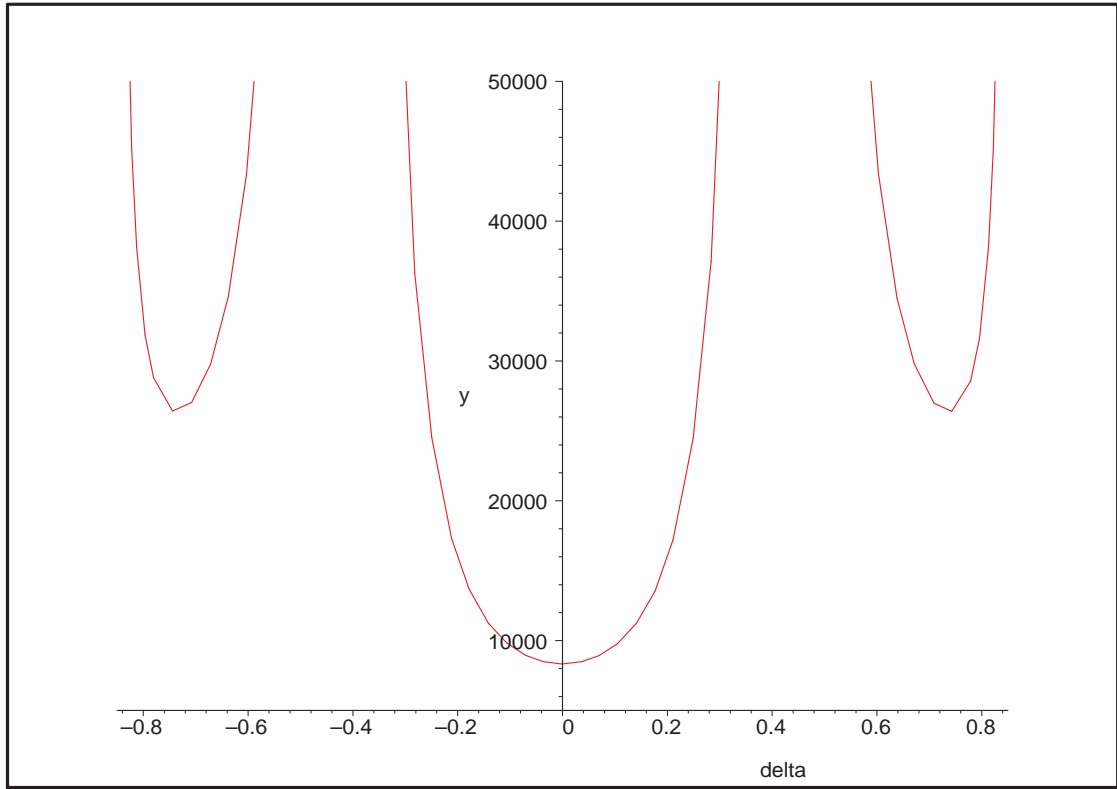


FIG. 11.40 – Graphique : Valeur moyenne de la variance en fonction de δ pour plex.

$$\{\delta = 0\}, \{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, -1.471960144)\},$$

$$\{\delta = \frac{1}{2} \text{RootOf}(6_Z^2 - 13, 1.471960144)\}$$

```
> SupN11bis_plex:=pointplot3d((subs(delta=sqrt(13/6)/2,N11bis)),symbol=
box,scaling=constrained,shading=Z,color=black):

> VarN11bis_plex:=plot3d(1,theta=-Pi..Pi,phi=0..Pi,coords=spherical,axe
s=BOXED,color=evalf(simplify(subs(lambda=1,subs(y=lambda*sin(phi)*sin(
theta),subs(x=lambda*sin(phi)*cos(theta),(subs(z=sqrt(1-x^2-y^2),12-57
546/845*z^2-40*y^2+24*x^2+3155112/4225*z^8-6381504/4225*z^6-139104/325
*z^4*y^2*13^(1/2)*2^(1/2)+2592/65*z^2*x*13^(1/2)*2^(1/2)+108864/325*z^
6*y^2*13^(1/2)*2^(1/2)-2592/65*z^4*x*13^(1/2)*2^(1/2)-6912/65*y^4*z^2*
13^(1/2)*2^(1/2)+124416/325*y^4*z^4*13^(1/2)*2^(1/2)-5184/65*y^2*z^2*x
*13^(1/2)*2^(1/2)-432/5*x*z^4-910656/325*y^2*z^4-1152/5*z^2*x*y^2+64*y
^4+64*x*y^2+3514122/4225*z^4+12462336/4225*z^4*y^4-296064/845*z^2*y^4+
2317248/845*z^6*y^2+552/5*z^2*x-24*x-1296/65*z^2*13^(1/2)*2^(1/2)+3628
8/325*z^4*13^(1/2)*2^(1/2)+207456/845*y^2*z^2-53136/325*z^6*13^(1/2)*2
^(1/2)+23328/325*z^8*13^(1/2)*2^(1/2)+6048/65*y^2*z^2*13^(1/2)*2^(1/2)
))))),numpoints=50000,style=PATCHNOGRID,orientation=[-71,82]):

> display(SupN11bis_plex,VarN11bis_plex);

> SupN11bis_2_plex:=pointplot3d(50*(subs(delta=sqrt(13/6)/2,N11bis)),sy
mbol=box,scaling=constrained,shading=Z,color=black):
```

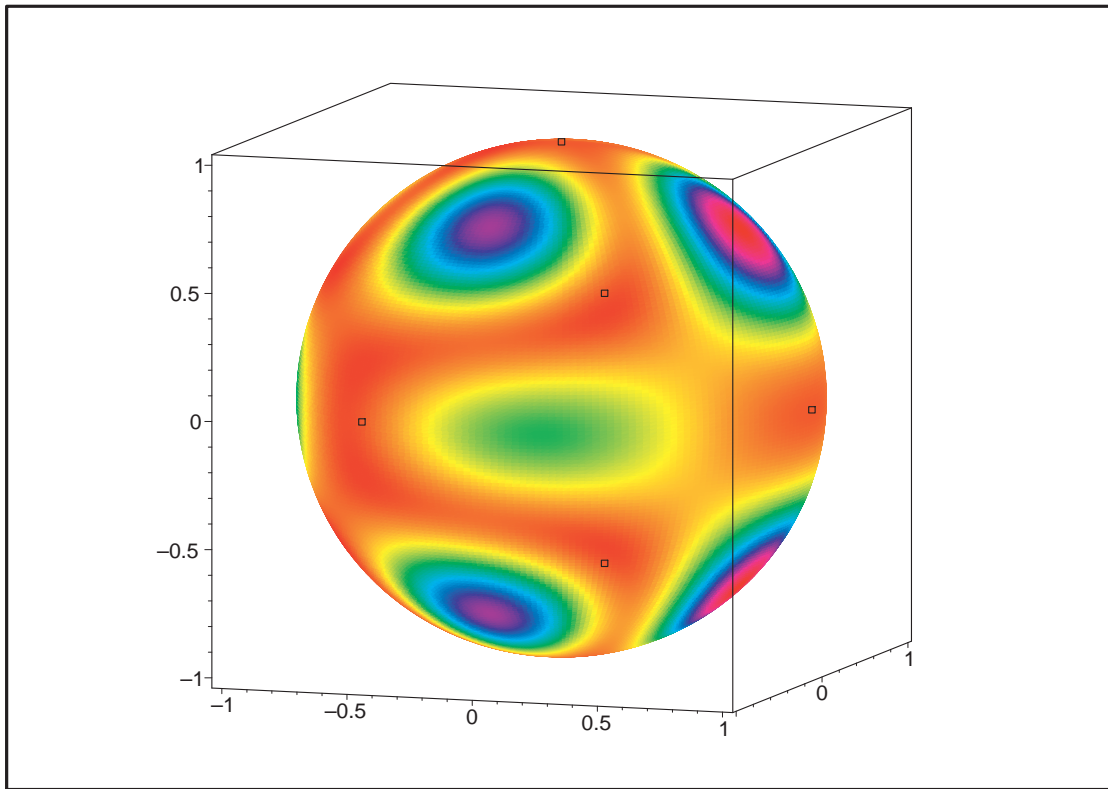


FIG. 11.41 – Graphique : Valeur de la variance pour δ optimal et un modèle plex saturé.

```
> VarN11bis_2_plex:=plot3d(evalf(simplify(subs(lambda=1,subs(y=lambda*
in(phi)*sin(theta),subs(x=lambda*sin(phi)*cos(theta),(subs(z=sqrt(1-x^
2-y^2),12-57546/845*z^2-40*y^2+24*x^2+3155112/4225*z^8-6381504/4225*z^
6-139104/325*z^4*y^2*13^(1/2)*2^(1/2)+2592/65*z^2*x*13^(1/2)*2^(1/2)+1
08864/325*z^6*y^2*13^(1/2)*2^(1/2)-2592/65*z^4*x*13^(1/2)*2^(1/2)-6912
/65*y^4*z^2*13^(1/2)*2^(1/2)+124416/325*y^4*z^4*13^(1/2)*2^(1/2)-5184/
65*y^2*z^2*x*13^(1/2)*2^(1/2)-432/5*x*z^4-910656/325*y^2*z^4-1152/5*z^
2*x*y^2+64*y^4+64*x*y^2+3514122/4225*z^4+12462336/4225*z^4*y^4-296064/
845*z^2*y^4+2317248/845*z^6*y^2+552/5*z^2*x-24*x-1296/65*z^2*13^(1/2)*
2^(1/2)+36288/325*z^4*13^(1/2)*2^(1/2)+207456/845*y^2*z^2-53136/325*z^
6*13^(1/2)*2^(1/2)+23328/325*z^8*13^(1/2)*2^(1/2)+6048/65*y^2*z^2*13^(
1/2)*2^(1/2)))))),theta=-Pi..Pi,phi=0..Pi,coords=spherical,axes=BOXE
D,numpoints=5000,style=PATCHNOGRID,color=evalf(simplify(subs(lambda=1,
subs(y=lambda*sin(phi)*sin(theta),subs(x=lambda*sin(phi)*cos(theta),(s
ubs(z=sqrt(1-x^2-y^2),12-57546/845*z^2-40*y^2+24*x^2+3155112/4225*z^8-
6381504/4225*z^6-139104/325*z^4*y^2*13^(1/2)*2^(1/2)+2592/65*z^2*x*13^
(1/2)*2^(1/2)+108864/325*z^6*y^2*13^(1/2)*2^(1/2)-2592/65*z^4*x*13^(1/
2)*2^(1/2)-6912/65*y^4*z^2*13^(1/2)*2^(1/2)+124416/325*y^4*z^4*13^(1/2
)*2^(1/2)-5184/65*y^2*z^2*x*13^(1/2)*2^(1/2)-432/5*x*z^4-910656/325*y^
2*z^4-1152/5*z^2*x*y^2+64*y^4+64*x*y^2+3514122/4225*z^4+12462336/4225*
z^4*y^4-296064/845*z^2*y^4+2317248/845*z^6*y^2+552/5*z^2*x-24*x-1296/6
5*z^2*13^(1/2)*2^(1/2)+36288/325*z^4*13^(1/2)*2^(1/2)+207456/845*y^2*z
^2-53136/325*z^6*13^(1/2)*2^(1/2)+23328/325*z^8*13^(1/2)*2^(1/2)+6048/
65*y^2*z^2*13^(1/2)*2^(1/2)))))),orientation=[122,78]):

> display(SupN11bis_2_plex,VarN11bis_2_plex);

> M_D:=radnormal(simplify(M_D_Centre(N11bis,2,1)));
```

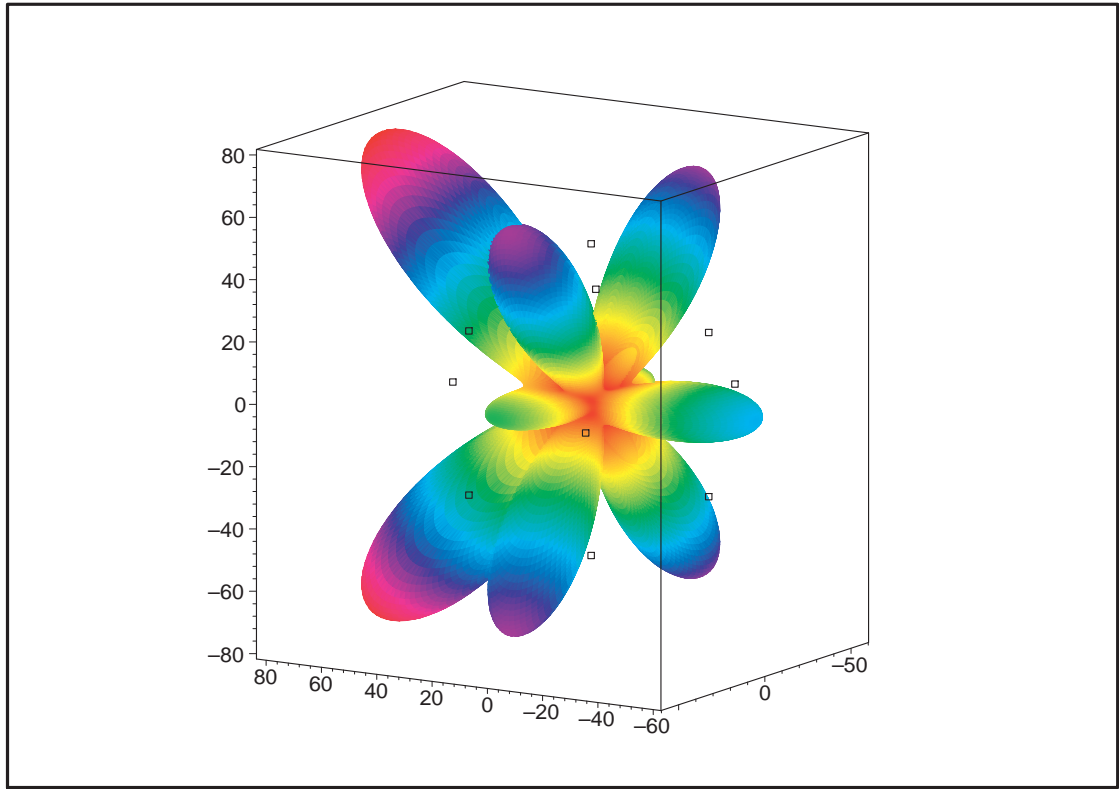


FIG. 11.42 – Graphique : Valeur de la variance pour δ optimal et un modèle plex saturé.

$$\begin{aligned}
 M_D := & \begin{bmatrix} \frac{13}{12}, 0, 0, 0, 0, 0, 0, \frac{11}{36}, \frac{11}{36}, \frac{11}{36} \\ 0, \frac{11}{36}, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, \frac{11}{36}, 0, 0, 0, \%1, 0, 0, 0 \\ 0, 0, 0, \frac{11}{36}, 0, 0, 0, 0, \%1, \frac{1}{16} - \frac{\sqrt{-36\delta^2 + 26\delta^2}}{12} + \frac{13\sqrt{-36\delta^2 + 26}}{864} \\ 0, 0, 0, 0, \frac{65}{1296}, 0, 0, 0, \frac{1}{12}\delta^3\sqrt{10} - \frac{13}{288}\delta\sqrt{10}, \%2 \\ 0, 0, 0, 0, 0, \frac{65}{1296}, \%2, 0, 0, 0 \\ 0, 0, \%1, 0, 0, \%2, \frac{331}{5184}, 0, 0, 0 \\ \frac{11}{36}, 0, 0, 0, 0, 0, 0, \frac{133}{648}, \frac{65}{1296}, \frac{65}{1296} \\ \frac{11}{36}, 0, 0, \%1, \frac{1}{12}\delta^3\sqrt{10} - \frac{13}{288}\delta\sqrt{10}, 0, 0, \frac{65}{1296}, \frac{331}{1728}, \frac{331}{5184} \\ \frac{11}{36}, 0, 0, \frac{1}{16} - \frac{\sqrt{-36\delta^2 + 26\delta^2}}{12} + \frac{13\sqrt{-36\delta^2 + 26}}{864}, \%2, 0, 0, \frac{65}{1296}, \frac{331}{5184}, \frac{331}{1728} \end{bmatrix} \\
 \%1 := & -\frac{1}{16} + \frac{\sqrt{-36\delta^2 + 26\delta^2}}{12} - \frac{13\sqrt{-36\delta^2 + 26}}{864} \\
 \%2 := & -\frac{1}{12}\delta^3\sqrt{10} + \frac{13}{288}\delta\sqrt{10}
 \end{aligned}$$

```
> LinearAlgebra[Trace](M_D);Determinant(M_D);exp(1/s*ln(add(M_D[i,i]^s,
i=1..10)));
```

$$\begin{aligned} & \frac{1077811}{373248} - \frac{1}{2} \delta^6 + \frac{13}{24} \delta^4 - \frac{169}{1152} \delta^2 \\ & - \frac{157938759875 \delta^{10}}{39934999921327865856} + \frac{319172745125 \delta^8}{159739999685311463424} + \frac{153515375 \delta^{12}}{25999348907114496} \\ & - \frac{130390722151625 \delta^8 \sqrt{-36 \delta^2 + 26}}{349351379311776170508288} + \frac{13001404029784625 \delta^6 \sqrt{-36 \delta^2 + 26}}{100613197241791537106386944} \\ & - \frac{69528392295509875 \delta^4 \sqrt{-36 \delta^2 + 26}}{2414716733802996890553286656} - \frac{18262649556125 (-36 \delta^2 + 26)^{(3/2)} \delta^6}{37729948965671826414895104} \\ & + \frac{968204187149875 (-36 \delta^2 + 26)^{(3/2)} \delta^4}{10866225302113486007489789952} - \frac{6998083263377375 (-36 \delta^2 + 26)^{(3/2)} \delta^2}{782368221752170992539264876544} \\ & - \frac{9498393904942691125 \sqrt{-36 \delta^2 + 26}}{4224788397461723359712030333376} - \frac{421019018376125 \delta^2}{22358488275953674912530432} \\ & + \frac{8896110594383822875 \sqrt{-36 \delta^2 + 26} \delta^2}{2347104665256512977617794629632} + \frac{21362017873625 \delta^{10} \sqrt{-36 \delta^2 + 26}}{29112614942648014209024} \\ & - \frac{21125 \delta^{18}}{13374150672384} + \frac{274625 \delta^{16}}{53496602689536} - \frac{17850625 \delta^{14}}{2567836929097728} \\ & - \frac{413644860926375 \delta^6}{621069118776490969792512} + \frac{542657018580625 \delta^4}{3726414712658945818755072} \\ & + \frac{31865398986123625 (-36 \delta^2 + 26)^{(3/2)}}{84495767949234467194240606666752} + \frac{11913645213864875}{10866225302113486007489789952} \\ & - \frac{1942001624875 \sqrt{-36 \delta^2 + 26} \delta^{12}}{2426051245220667850752} + \frac{11491133875 \sqrt{-36 \delta^2 + 26} \delta^{14}}{33695156183620386816} \\ & + \frac{1881455875 (-36 \delta^2 + 26)^{(3/2)} \delta^{12}}{1213025622610333925376} + \frac{1589830214375 (-36 \delta^2 + 26)^{(3/2)} \delta^8}{1048054137935328511524864} \\ & - \frac{24458926375 (-36 \delta^2 + 26)^{(3/2)} \delta^{10}}{9704204980882671403008} \\ & e^{\left(\frac{\ln(1+3(\frac{11}{36})^s + (\frac{133}{648})^s + 2(\frac{65}{1296})^s + 2(\frac{331}{1728})^s + (\frac{331}{5184})^s + (\frac{2069}{11664})^s + \%1^s)(\frac{1}{s})}{s} \right)} \%1^s \end{aligned}$$

```
> M_D:=radnormal(simplify(M_D_ModeleSat(N11bis,[x,y,z],EstSet_N11_tdeg)
)):M_D[12,12];
```

$$-\frac{1}{2} \delta^6 + \frac{13}{24} \delta^4 - \frac{169}{1152} \delta^2 + \frac{5113}{124416}$$

```
> s:=1;unassign('s');diff(add(M_D[i,i]^s,i=1..12)^(1/s),delta);solve({
%,-sqrt(13)/sqrt(18)<=delta,delta<=sqrt(13)/sqrt(18)},delta);evalf(
%);evalf(solve(
{-1/2*delta^6+13/24*delta^4-169/1152*delta^2+5113/124416
},delta)):evalf(sqrt(13)/sqrt(18));
```

$s := 1$

$$\begin{aligned} & (1 + 3(\frac{11}{36})^s + (\frac{133}{648})^s + 2(\frac{65}{1296})^s + 2(\frac{331}{1728})^s + (\frac{331}{5184})^s + (\frac{2069}{11664})^s + \%1^s)(\frac{1}{s}) \%1^s \\ & (-3 \delta^5 + \frac{13}{6} \delta^3 - \frac{169}{576} \delta) / (\%1 \\ & (1 + 3(\frac{11}{36})^s + (\frac{133}{648})^s + 2(\frac{65}{1296})^s + 2(\frac{331}{1728})^s + (\frac{331}{5184})^s + (\frac{2069}{11664})^s + \%1^s)) \\ & \%1 := -\frac{1}{2} \delta^6 + \frac{13}{24} \delta^4 - \frac{169}{1152} \delta^2 + \frac{5113}{124416} \end{aligned}$$

$$\begin{aligned}
& \{\delta = 0\}, \{\delta = \text{RootOf}(24_Z^2 - 13, -0.7359800722)\}, \\
& \{\delta = \text{RootOf}(24_Z^2 - 13, 0.7359800722)\}, \{\delta = \text{RootOf}(72_Z^2 - 13, -0.4249182928)\}, \\
& \{\delta = \text{RootOf}(72_Z^2 - 13, 0.4249182928)\} \\
& \quad \{\delta = 0.\}, \{\delta = -0.7359800722\}, \{\delta = 0.7359800722\}, \{\delta = -0.4249182928\}, \\
& \quad \{\delta = 0.4249182928\} \\
& \qquad \qquad \qquad 0.8498365855
\end{aligned}$$

Indep de s -> D opt limite donc logique que les critères Phi_t coïncident.

```

> solve(
  {diff(exp(1/s*ln(add(M_D[i,i]^s,i=1..12))),delta),-sqrt(13)/sqrt(18)<
    =delta,delta<=sqrt(13)/sqrt(18)},delta);

  {\delta = 0}, {\delta = RootOf(24_Z^2 - 13, -0.7359800722)},
  {\delta = RootOf(24_Z^2 - 13, 0.7359800722)}, {\delta = RootOf(72_Z^2 - 13, -0.4249182928)},
  {\delta = RootOf(72_Z^2 - 13, 0.4249182928)}

```

11.16. N12

```
> unassign('alpha','beta');
```

```
> TempGr:=Diedn(5):GrN12:=
{seq(DiagonalMatrix([TempGr[Indice],Vector([1])]),Indice=1..nops(Temp
Gr)),seq(DiagonalMatrix([-1*TempGr[Indice],Vector([-1])]),Indice=1..no
ps(TempGr))};
```

$$GrN12 := \left\{ \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%3 & 0 \\ \%1 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%4 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%1 & 0 \\ \%1 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%2 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%1 & 0 \\ \%3 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%3 & 0 \\ \%3 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%3 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%2 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%1 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%4 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left. \right\}$$

$$\%1 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%2 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%3 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%4 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

```
> N12:=OrbToMatrix(OrbPG(Vector([alpha,0,beta]),GrN12) union
{Vector([0,0,1]),Vector([0,0,-1])});
```

$$N12 := \begin{bmatrix} (-\frac{\sqrt{5}}{4} + \frac{1}{4})\alpha & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & -\beta \\ (\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & (\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & -\beta \\ (\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & (-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & -\beta \\ -\alpha & 0 & -\beta \\ (-\frac{1}{4} - \frac{\sqrt{5}}{4})\alpha & (-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & \beta \\ (-\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & \beta \\ (-\frac{\sqrt{5}}{4} + \frac{1}{4})\alpha & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & -\beta \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ (-\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & \beta \\ \alpha & 0 & \beta \\ (-\frac{1}{4} - \frac{\sqrt{5}}{4})\alpha & (\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & \beta \end{bmatrix}$$

```
> UpdatePlan(N12,"N12");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N12,1,0);
```

$$Var := 1 + \frac{6 \text{ var}_3^2}{5\beta^2 + 1} + \frac{12 \text{ var}_2^2}{5\alpha^2} + \frac{12 \text{ var}_1^2}{5\alpha^2}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});
```

$$Cond := [5\alpha^2 - 10\beta^2 - 2]$$

```
> gsolve([op(Cond),alpha^2+beta^2-1],[alpha,beta]);
```

$$\{[[5\beta^2 - 1, 5\alpha^2 - 4], \text{plex}(\alpha, \beta), \{\}]\}$$

```
> beta := 1/sqrt(5); alpha := 2/sqrt(5);
```

$$\beta := \frac{\sqrt{5}}{5}$$

$$\alpha := \frac{2\sqrt{5}}{5}$$

```
> N12:=OrbToMatrix(SortPoints(simplify(OrbPG(Vector([alpha,0,beta]),GrN
12) union {Vector([0,0,1]),Vector([0,0,-1])})));
```


$$N12 := \begin{bmatrix} -\frac{(1+\sqrt{5})\sqrt{5}}{10} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20} & \frac{\sqrt{5}}{5} \\ -\frac{(1+\sqrt{5})\sqrt{5}}{10} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20} & \frac{\sqrt{5}}{5} \\ \frac{(\sqrt{5}-1)\sqrt{5}}{10} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{(\sqrt{5}-1)\sqrt{5}}{10} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ -\frac{(\sqrt{5}-1)\sqrt{5}}{10} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ -\frac{(\sqrt{5}-1)\sqrt{5}}{10} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ \frac{(1+\sqrt{5})\sqrt{5}}{10} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20} & -\frac{\sqrt{5}}{5} \\ \frac{(1+\sqrt{5})\sqrt{5}}{10} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20} & -\frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} & 0 & -\frac{\sqrt{5}}{5} \end{bmatrix}$$

> UpdatePlan(N12,"N12");

“Update complete”

> convert(evalf(convert(N12,listlist)),rational,5);

$$\begin{aligned} & \left[\left[\frac{-89}{123}, \frac{51}{97}, \frac{72}{161} \right], \left[\frac{-89}{123}, \frac{-51}{97}, \frac{72}{161} \right], \left[\frac{55}{199}, \frac{-131}{154}, \frac{72}{161} \right], \left[\frac{55}{199}, \frac{131}{154}, \frac{72}{161} \right], \left[\frac{-55}{199}, \frac{-131}{154}, \frac{-72}{161} \right], \right. \\ & \left[\frac{-55}{199}, \frac{131}{154}, \frac{-72}{161} \right], [0, 0, -1], [0, 0, 1], \left[\frac{89}{123}, \frac{51}{97}, \frac{-72}{161} \right], \left[\frac{89}{123}, \frac{-51}{97}, \frac{-72}{161} \right], \left[\frac{144}{161}, 0, \frac{72}{161} \right], \\ & \left. \left[\frac{-144}{161}, 0, \frac{-72}{161} \right] \right] \end{aligned}$$

> pointplot3d(N12,symbol=box,scaling=constrained,shading=Z);

> Graph:=convhull(op(map(convert,[evalf(op(convert(N12,listlist)))],'rational',5))));

$$Graph := \text{POLYTOPE}(3, 3, 12, 20)$$

> plotdes:=draw(Graph):

> display({plotdes},scaling=constrained,shading=Z);

> unassign('alpha','beta','eta','delta');

> IdealN12:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N12));

> Gens:=Generators(IdealN12);

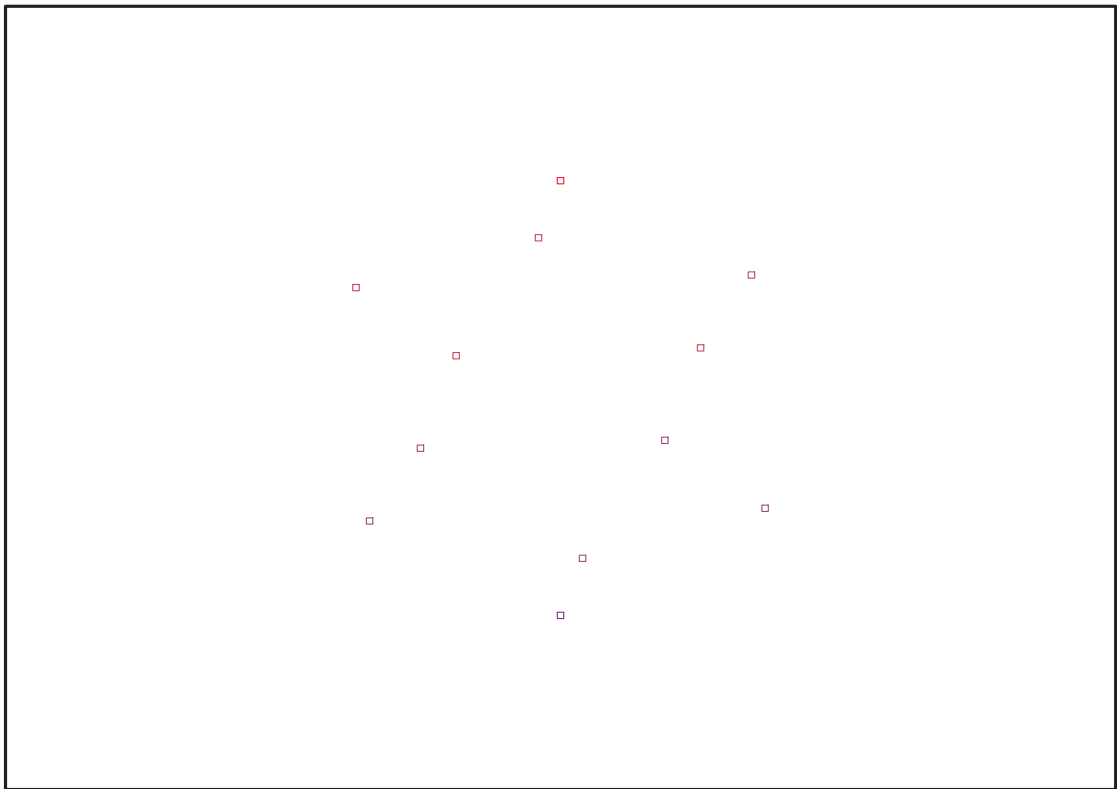


FIG. 11.43 – Graphique : Support du plan N12.

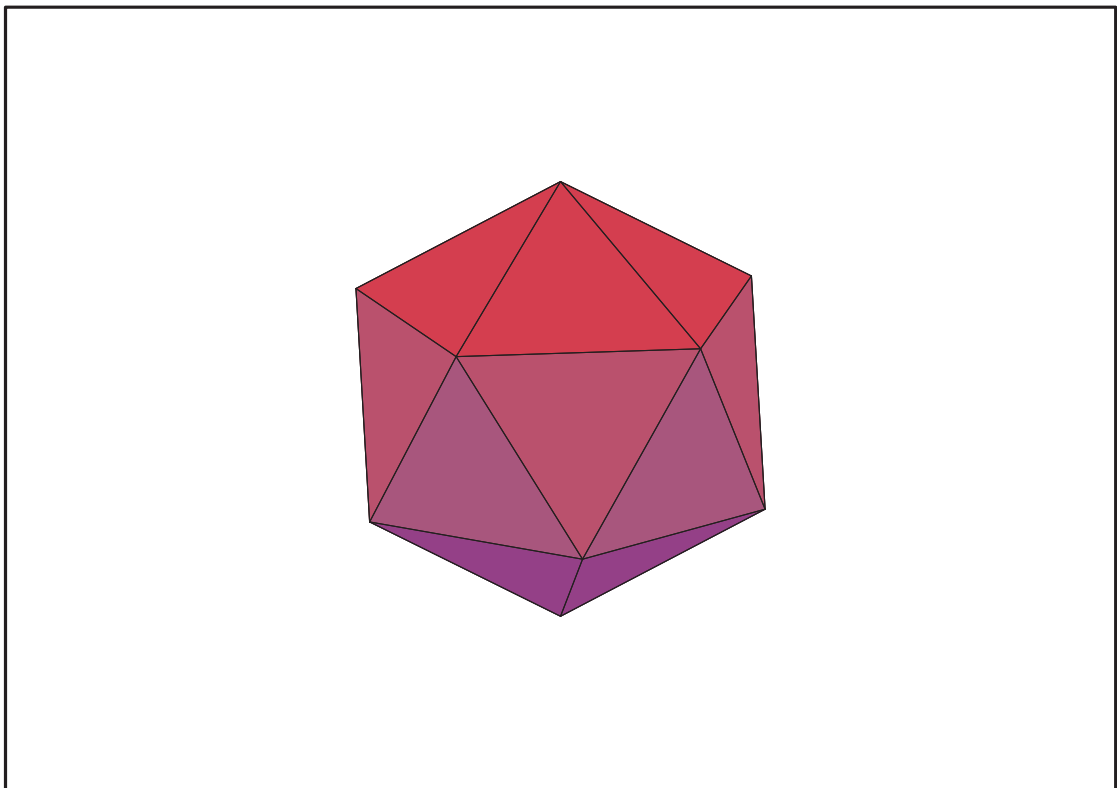


FIG. 11.44 – Graphique : Enveloppe convexe du support du plan N12.

$$\begin{aligned} \text{IdealN12} := & \langle z^2 + y^2 + x^2 - 1, 5xy^2 + 5x^3 - 4x, 10x^4 + y^2 - 2xz - 7x^2, \\ & 5y^2z - 5zx^2 + 10x^3 - 6x, 10zx^3 + y^2 - 6xz - x^2, 5y xz + 5yx^2 - y, \\ & 5y^3 + 5yx^2 - 4y, 5yx^3 + yz - 2yx \rangle \end{aligned}$$

$$\begin{aligned} \text{Gens} := & \{10x^4 + y^2 - 2xz - 7x^2, z^2 + y^2 + x^2 - 1, 10zx^3 + y^2 - 6xz - x^2, \\ & 5y^2z - 5zx^2 + 10x^3 - 6x, 5xy^2 + 5x^3 - 4x, 5yx^3 + yz - 2yx, 5y xz + 5yx^2 - y, \\ & 5y^3 + 5yx^2 - 4y\} \end{aligned}$$

```
> info_iop:=3;
```

info_iop := 3

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N12),
listlist),Vector)));
```

1
2
3
4
5
6
7
8
9
9
10
10
10
11
12
12
12
12

$$\begin{aligned}
& [-1 + y^2 + \frac{(5 + \sqrt{5})z^2}{8} - \frac{(-3 + \sqrt{5})z^2}{8} + x^2, z^2 y - \frac{1}{5}y, z^2 x - \frac{1}{5}x, \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(1 + 3\sqrt{5})z}{40} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(5\sqrt{5} - 9)z}{40} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}\sqrt{5}z}{20} - yxz + \frac{(3\sqrt{5} - 1)y}{10} + y^3 \\
& - \frac{(3\sqrt{5} + 5)y}{10} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}z}{4}, \frac{z}{2} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(25 + 7\sqrt{5})y}{400} \\
& - \frac{3\sqrt{2}\sqrt{5 + \sqrt{5}}(11\sqrt{5} + 25)y}{80} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(39\sqrt{5} + 85)y}{100} \\
& + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(3\sqrt{5} + 5)y}{20} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(\sqrt{5} - 1)y}{20} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(9\sqrt{5} + 5)y}{200} \\
& + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(8 + 3\sqrt{5})y}{10} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(145 + 67\sqrt{5})y}{100} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(5\sqrt{5} + 7)y}{20} + \frac{2\sqrt{2}\sqrt{5 + \sqrt{5}}(1 + \sqrt{5})y}{5} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}\sqrt{5}y}{25} \\
& + \frac{13\sqrt{2}\sqrt{5 + \sqrt{5}}(3 + \sqrt{5})y}{80} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(55 + 23\sqrt{5})y}{100} \\
& - \frac{7\sqrt{2}\sqrt{5 + \sqrt{5}}(7 + 3\sqrt{5})y}{40} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(15 + 7\sqrt{5})y}{25} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(31 + 17\sqrt{5})y}{20} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(45 + 11\sqrt{5})y}{400} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(525 + 223\sqrt{5})y}{400} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(29\sqrt{5} + 67)y}{80} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(14 + 5\sqrt{5})y}{10} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(13\sqrt{5} + 35)y}{20} \\
& + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(17\sqrt{5} + 39)y}{40} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(89 + 31\sqrt{5})y}{80} \\
& + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(21\sqrt{5} + 55)y}{80} - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(-13 + \sqrt{5})y}{80} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(3\sqrt{5} + 10)y}{10} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(131\sqrt{5} + 305)y}{400} \\
& + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(4\sqrt{5} + 11)y}{10} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(3\sqrt{5} - 1)y}{40} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}(17\sqrt{5} + 5)y}{200} \\
& - \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}yz}{5} + \frac{(11\sqrt{5} + 23)xz}{40} - \frac{(17\sqrt{5} + 25)xz}{40} - \frac{(19\sqrt{5} + 39)xz}{40} \\
& + \frac{3(15 + 7\sqrt{5})xz}{40} + \frac{(2 + \sqrt{5})xz}{5} - \frac{(5 + \sqrt{5})xz}{10} + \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}y}{5} \\
& - \frac{\sqrt{2}(5 + \sqrt{5})^{(3/2)}y}{100} + xy^2 - y^2z + \frac{z^2}{2} + \frac{(9\sqrt{5} + 11)x}{10} - \frac{(5 + \sqrt{5})x}{10} + \frac{(7\sqrt{5} + 5)x}{10} \\
& - \frac{(5\sqrt{5} + 7)x}{10} + \frac{(11\sqrt{5} + 23)x}{40} - \frac{(19\sqrt{5} + 39)x}{40} - \frac{(17\sqrt{5} + 25)x}{40} - \frac{(11\sqrt{5} + 15)x}{10} \\
& + \frac{3(15 + 7\sqrt{5})x}{40} - \frac{2(1 + \sqrt{5})x}{5} + \frac{3(2 + \sqrt{5})x}{5} + \frac{2(\sqrt{5} - 1)y^2}{5} - \frac{(4\sqrt{5} - 3)y^2}{5} \\
& + \frac{(-3 + \sqrt{5})z^2}{8} + \frac{(2\sqrt{5} - 1)y^2}{5} - \frac{(25 + 7\sqrt{5})z}{20} + \frac{(-9 + 2\sqrt{5})z}{10} + \frac{(35 + 27\sqrt{5})z}{20} \\
& + \frac{3(7 + 3\sqrt{5})z}{16} - \frac{(-5 + 3\sqrt{5})z}{20} - \frac{(3 + \sqrt{5})z^2}{10} - \frac{(1 + \sqrt{5})z^2}{8} + \frac{(151 + 59\sqrt{5})z}{80}
\end{aligned}$$

$$\begin{aligned}
 & -\frac{(8+3\sqrt{5})z^2}{20} - \frac{(\sqrt{5}+17)z}{16} - \frac{(19\sqrt{5}+39)z}{16} + \frac{(33\sqrt{5}+145)z}{80} - \frac{(39\sqrt{5}+95)z}{80} \\
 & + \frac{(9\sqrt{5}-35)z}{80} + \frac{(14+5\sqrt{5})z^2}{20} + \frac{(45+11\sqrt{5})z}{20} + \frac{(\sqrt{5}+7)z}{4} + \frac{(9\sqrt{5}+11)z}{20} \\
 & - \frac{(35\sqrt{5}+87)z}{80} + \frac{(9\sqrt{5}+5)z}{20} + \frac{(9+\sqrt{5})z}{16} + \frac{(9\sqrt{5}+5)z^2}{40} - \frac{(55+23\sqrt{5})z}{40} \\
 & - \frac{3(2\sqrt{5}+5)z^2}{20} - \frac{(17\sqrt{5}+5)z^2}{40} - \frac{(17+5\sqrt{5})z}{16} + \frac{2\sqrt{5}z^2}{5} - \frac{(11\sqrt{5}+15)z}{20} \\
 & - \frac{17(5+\sqrt{5})z}{80} - \frac{(-35+\sqrt{5})z}{20} - \frac{(5\sqrt{5}+7)z}{20} + \frac{3(2\sqrt{5}+5)z}{10} - \frac{(8+3\sqrt{5})z}{10} \\
 & - \frac{(11+3\sqrt{5})z}{4} - \frac{(9\sqrt{5}+35)z}{20} + \frac{7(2+\sqrt{5})z}{10} - \frac{(17\sqrt{5}+25)z}{80} - \frac{(3\sqrt{5}+5)z}{20} \\
 & + \frac{(2\sqrt{5}+15)z^2}{20} + \frac{(11\sqrt{5}+23)z}{16} - \frac{9(1+\sqrt{5})z}{20} + \frac{3(15+7\sqrt{5})z}{80} + \frac{(7\sqrt{5}+5)z}{20} \\
 & - \frac{4\sqrt{5}z}{5} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}yz}{5} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(5\sqrt{5}+7)yz}{80} \\
 & - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(1+\sqrt{5})yz}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(9+\sqrt{5})yz}{40} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(-15+\sqrt{5})yz}{40} \\
 & - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-5)yz}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(15+7\sqrt{5})yz}{40} \\
 & - \frac{3\sqrt{2}\sqrt{5+\sqrt{5}}(3\sqrt{5}+5)yz}{80} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(9\sqrt{5}+11)yz}{80} \\
 & + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}+17)yz}{40} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(13\sqrt{5}-5)yz}{80} - \frac{z^3}{2}, \frac{1}{5} + z^4 - \frac{6}{5}z^2], \\
 & [1, z, y, x, z^2, yz, xz, y^2, yx, z^3, y^2z, yxz]]
 \end{aligned}$$

```

> Gens := {10*x^4+y^2-2*x*z-7*x^2, z^2+y^2+x^2-1,
10*z*x^3+y^2-6*x*z-x^2, 5*y^2*z-5*z*x^2+10*x^3-6*x, 5*x*y^2+5*x^3-4*x,
5*y*x^3+y*z-2*y*x, 5*y*x*z+5*y*x^2-y, 5*y^3+5*y*x^2-4*y};

```

$$\begin{aligned}
 Gens := \{ & z^2 + y^2 + x^2 - 1, 5y^2z - 5x^2z + 10x^3 - 6x, 10x^3z + y^2 - 6xz - x^2, \\
 & 10x^4 + y^2 - 2xz - 7x^2, 5y^3 + 5yx^2 - 4y, 5yxz + 5yx^2 - y, 5yx^3 + yz - 2xy, \\
 & 5xy^2 + 5x^3 - 4x \}
 \end{aligned}$$

11.16.1. tdeg(x,y,z)

```

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N12_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N12_tdeg:=sort(EstSet_N12_tdeg,(
t1,t2)->testorder(t1,t2,T));nops(EstSet_N12_tdeg);

```

$$EstSet_N12_tdeg := [1, z, y, x, z^2, yz, xz, y^2, yx, z^3, y^2z, yxz]$$

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```

> X_D_ModeleSat(N12,[x,y,z],EstSet_N12_tdeg);

```

$$\begin{aligned}
& \left[1, -\frac{\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, -\frac{(\sqrt{5}-1)\sqrt{5}}{10}, \frac{1}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{\sqrt{5}}{10} - \frac{1}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \%1 \right. \\
& \left. , -\frac{\sqrt{5}}{25}, -\frac{(5+\sqrt{5})\sqrt{5}}{50}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{100} \right] \\
& \left[1, -\frac{\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, -\frac{(\sqrt{5}-1)\sqrt{5}}{10}, \frac{1}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{\sqrt{5}}{10} - \frac{1}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \%2 \right. \\
& \left. , -\frac{\sqrt{5}}{25}, -\frac{(5+\sqrt{5})\sqrt{5}}{50}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{100} \right] \\
& [1, -1, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0] \\
& [1, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0] \\
& \left[1, \frac{\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, -\frac{(1+\sqrt{5})\sqrt{5}}{10}, \frac{1}{5}, \%1, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \%3, \right. \\
& \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})}{40}, \frac{\sqrt{5}}{25}, \frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, \\
& \left. \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})\sqrt{5}}{200} \right] \\
& \left[1, \frac{\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, -\frac{(1+\sqrt{5})\sqrt{5}}{10}, \frac{1}{5}, \%2, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \%3, \right. \\
& -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})}{40}, \frac{\sqrt{5}}{25}, \frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, \\
& \left. -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})\sqrt{5}}{200} \right] \\
& \left[1, \frac{\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{(\sqrt{5}-1)\sqrt{5}}{10}, \frac{1}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{\sqrt{5}}{10} - \frac{1}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \%2, \right. \\
& \frac{\sqrt{5}}{25}, \frac{(5+\sqrt{5})\sqrt{5}}{50}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{100} \left. \right] \\
& \left[1, \frac{\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{(\sqrt{5}-1)\sqrt{5}}{10}, \frac{1}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{\sqrt{5}}{10} - \frac{1}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \%1, \right. \\
& \frac{\sqrt{5}}{25}, \frac{(5+\sqrt{5})\sqrt{5}}{50}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{100} \left. \right] \\
& \left[1, -\frac{\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, \frac{(1+\sqrt{5})\sqrt{5}}{10}, \frac{1}{5}, \%1, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \%3, \right. \\
& \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})}{40}, -\frac{\sqrt{5}}{25}, -\frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, \\
& \left. -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})\sqrt{5}}{200} \right] \\
& \left[1, -\frac{\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, \frac{(1+\sqrt{5})\sqrt{5}}{10}, \frac{1}{5}, \%2, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \%3, \right. \\
& -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})}{40}, -\frac{\sqrt{5}}{25}, -\frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, \\
& \left. \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)(1+\sqrt{5})\sqrt{5}}{200} \right]
\end{aligned}$$

$$\begin{aligned} & \left[1, -\frac{\sqrt{5}}{5}, 0, -\frac{2\sqrt{5}}{5}, \frac{1}{5}, 0, \frac{2}{5}, 0, 0, -\frac{\sqrt{5}}{25}, 0, 0 \right] \\ & \left[1, \frac{\sqrt{5}}{5}, 0, \frac{2\sqrt{5}}{5}, \frac{1}{5}, 0, \frac{2}{5}, 0, 0, \frac{\sqrt{5}}{25}, 0, 0 \right] \\ \%1 &:= -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{20} \\ \%2 &:= \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{20} \\ \%3 &:= \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{40} \end{aligned}$$

```
> M_D_ModeleSat(N12,[x,y,z],EstSet_N12_tdeg);
```

$$\begin{aligned}
& \left[1, 0, 0, 0, \frac{1}{3}, 0, 0, \%1, 0, 0, 0, 0 \right] \\
& \left[0, \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{5}, \%2, 0 \right] \\
& [0, 0, \%1, 0, 0, 0, 0, 0, 0, 0, 0, \%3] \\
& \left[0, 0, 0, \frac{(\sqrt{5}-1)^2}{60} + \frac{2}{15} + \frac{(1+\sqrt{5})^2}{60}, 0, 0, 0, 0, 0, 0, \%3, 0 \right] \\
& \left[\frac{1}{3}, 0, 0, 0, \frac{1}{5}, 0, 0, \%2, 0, 0, 0, 0 \right] \\
& [0, 0, 0, 0, 0, 0, \%2, 0, 0, \%3, 0, 0] \\
& \left[0, 0, 0, 0, 0, 0, \frac{(\frac{\sqrt{5}}{10} - \frac{1}{10})^2}{3} + \frac{2}{75} + \frac{(-\frac{1}{10} - \frac{\sqrt{5}}{10})^2}{3}, \right. \\
& \left. \frac{(\frac{\sqrt{5}}{10} - \frac{1}{10})(\frac{1}{2} + \frac{\sqrt{5}}{10})}{3} + \frac{(-\frac{1}{10} - \frac{\sqrt{5}}{10})(5+\sqrt{5})(\sqrt{5}-1)^2}{120}, 0, 0, 0, 0 \right] \\
& \left[\%1, 0, 0, 0, 0, \%2, 0, \frac{(\frac{\sqrt{5}}{10} - \frac{1}{10})(\frac{1}{2} + \frac{\sqrt{5}}{10})}{3} + \frac{(-\frac{1}{10} - \frac{\sqrt{5}}{10})(5+\sqrt{5})(\sqrt{5}-1)^2}{120}, \right. \\
& \left. \frac{(\frac{1}{2} + \frac{\sqrt{5}}{10})^2}{3} + \frac{(5+\sqrt{5})^2(\sqrt{5}-1)^4}{4800}, 0, 0, 0, 0 \right] \\
& \left[0, 0, 0, 0, 0, 0, \%3, 0, 0, \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{600} + \frac{(5+\sqrt{5})(\sqrt{5}-1)^2(1+\sqrt{5})^2}{2400}, 0, 0, 0 \right] \\
& \left[0, \frac{1}{5}, 0, 0, 0, 0, 0, 0, 0, \frac{13}{75}, \frac{1}{150} + \frac{\sqrt{5}}{750} + \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{3000}, 0 \right] \\
& \left[0, \%2, 0, \%3, 0, 0, 0, 0, 0, \frac{1}{150} + \frac{\sqrt{5}}{750} + \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{3000}, \right. \\
& \left. \frac{(5+\sqrt{5})^2}{1500} + \frac{(5+\sqrt{5})^2(\sqrt{5}-1)^4}{24000}, 0 \right] \\
& \left[0, 0, \%3, 0, 0, 0, 0, 0, 0, 0, 0, \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{3000} + \frac{(5+\sqrt{5})(\sqrt{5}-1)^2(1+\sqrt{5})^2}{12000} \right] \\
& \%1 := \frac{1}{6} + \frac{\sqrt{5}}{30} + \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{120} \\
& \%2 := \frac{1}{30} + \frac{\sqrt{5}}{150} + \frac{(5+\sqrt{5})(\sqrt{5}-1)^2}{600} \\
& \%3 := \frac{(5+\sqrt{5})(\sqrt{5}-1)}{300} - \frac{(5+\sqrt{5})(\sqrt{5}-1)^2(1+\sqrt{5})}{1200}
\end{aligned}$$

> expand(Var_Blue_Modelle_Sat(N12,[x,y,z],EstSet_N12_tdeg));

$$\begin{aligned}
& 6 + \frac{27}{2} z^2 - 12 y^2 - 45 z^4 - 45 y^2 z^2 + 3 x^2 + 15 x^2 z^2 + 15 y^4 + 15 x^2 y^2 + \frac{75}{2} z^6 + 75 z^4 y^2 \\
& + 75 y^4 z^2 + 75 y^2 x^2 z^2
\end{aligned}$$

11.16.2. $\text{plex}(x,y,z)$

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

> EstSet_N12_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y
,z):T:=termorder(A,plex(x,y,z)):EstSet_N12_plex:=sort(EstSet_N12_plex,
(t1,t2)->testorder(t1,t2,T));nops(EstSet_N12_plex);
      EstSet_N12_plex := [1, z, z2, z3, y, y z, y2, y2 z, y3, y3 z, y4, z y4]
```

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```
> X_D_ModeleSat(N12,[x,y,z],EstSet_N12_plex);
```

$$\begin{aligned}
& \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, -\frac{(5+\sqrt{5})\sqrt{5}}{50}, \right. \\
& \left. \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, -\frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
& \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, -\frac{(5+\sqrt{5})\sqrt{5}}{50}, \right. \\
& \left. -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, -\frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
& [1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0] \\
& [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0] \\
& \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{20}, \%1, \right. \\
& \left. \frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3\sqrt{5}}{800}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3}{800}, \right. \\
& \left. \%2, \frac{\sqrt{5}(5+\sqrt{5})^2(\sqrt{5}-1)^4}{8000} \right] \\
& \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{20}, \%1, \right. \\
& \left. \frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3\sqrt{5}}{800}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3}{800}, \right. \\
& \left. \%2, \frac{\sqrt{5}(5+\sqrt{5})^2(\sqrt{5}-1)^4}{8000} \right] \\
& \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \frac{(5+\sqrt{5})\sqrt{5}}{50}, \right. \\
& \left. \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, \frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
& \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \frac{(5+\sqrt{5})\sqrt{5}}{50}, \right. \\
& \left. -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, \frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
& \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{20}, \%1, \right. \\
& \left. -\frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3\sqrt{5}}{800}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3}{800}, \right. \\
& \left. \%2, -\frac{\sqrt{5}(5+\sqrt{5})^2(\sqrt{5}-1)^4}{8000} \right] \\
& \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{5}}{20}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{20}, \%1, \right. \\
& \left. -\frac{(5+\sqrt{5})(\sqrt{5}-1)^2\sqrt{5}}{200}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3\sqrt{5}}{800}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}(\sqrt{5}-1)^3}{800}, \right. \\
& \left. \%2, -\frac{\sqrt{5}(5+\sqrt{5})^2(\sqrt{5}-1)^4}{8000} \right]
\end{aligned}$$

$$\begin{aligned} & \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, 0, 0, 0, 0, 0, 0, 0, 0 \right] \\ & \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, 0, 0, 0, 0, 0, 0, 0, 0 \right] \\ \%1 &:= \frac{(5 + \sqrt{5})(\sqrt{5} - 1)^2}{40} \\ \%2 &:= \frac{(5 + \sqrt{5})^2(\sqrt{5} - 1)^4}{1600} \end{aligned}$$

```
> M_D_ModeleSat(N12,[x,y,z],EstSet_N12_plex);
```

$$\begin{aligned} & \left[1, 0, \frac{1}{3}, 0, 0, 0, \%1, 0, 0, 0, \%2, 0 \right] \\ & \left[0, \frac{1}{3}, 0, \frac{1}{5}, 0, 0, 0, \%3, 0, 0, 0, \%4 \right] \\ & \left[\frac{1}{3}, 0, \frac{1}{5}, 0, 0, 0, \%3, 0, 0, 0, \%4, 0 \right] \\ & \left[0, \frac{1}{5}, 0, \frac{13}{75}, 0, 0, 0, \frac{1}{150} + \frac{\sqrt{5}}{750} + \frac{(5 + \sqrt{5})(\sqrt{5} - 1)^2}{3000}, 0, 0, 0, \right. \\ & \quad \left. \frac{(5 + \sqrt{5})^2}{7500} + \frac{(5 + \sqrt{5})^2(\sqrt{5} - 1)^4}{120000} \right] \\ & [0, 0, 0, 0, \%1, 0, 0, 0, \%2, 0, 0, 0] \\ & [0, 0, 0, 0, 0, \%3, 0, 0, 0, \%4, 0, 0] \\ & \left[\%1, 0, \%3, 0, 0, 0, \frac{(\frac{1}{2} + \frac{\sqrt{5}}{10})^2}{3} + \frac{(5 + \sqrt{5})^2(\sqrt{5} - 1)^4}{4800}, 0, 0, 0, \right. \\ & \quad \left. \frac{(\frac{1}{2} + \frac{\sqrt{5}}{10})(5 + \sqrt{5})^2}{300} + \frac{(5 + \sqrt{5})^3(\sqrt{5} - 1)^6}{192000}, 0 \right] \\ & \left[0, \%3, 0, \frac{1}{150} + \frac{\sqrt{5}}{750} + \frac{(5 + \sqrt{5})(\sqrt{5} - 1)^2}{3000}, 0, 0, 0, \%4, 0, 0, 0, \%5 \right] \\ & \left[0, 0, 0, 0, \%2, 0, 0, 0, \frac{(5 + \sqrt{5})^3}{3000} + \frac{(5 + \sqrt{5})^3(\sqrt{5} - 1)^6}{192000}, 0, 0, 0 \right] \\ & [0, 0, 0, 0, 0, \%4, 0, 0, 0, \%5, 0, 0] \\ & \left[\%2, 0, \%4, 0, 0, 0, \frac{(\frac{1}{2} + \frac{\sqrt{5}}{10})(5 + \sqrt{5})^2}{300} + \frac{(5 + \sqrt{5})^3(\sqrt{5} - 1)^6}{192000}, 0, 0, 0, \right. \\ & \quad \left. \frac{(5 + \sqrt{5})^4}{30000} + \frac{(5 + \sqrt{5})^4(\sqrt{5} - 1)^8}{7680000}, 0 \right] \\ & \left[0, \%4, 0, \frac{(5 + \sqrt{5})^2}{7500} + \frac{(5 + \sqrt{5})^2(\sqrt{5} - 1)^4}{120000}, 0, 0, 0, \%5, 0, 0, 0, \right. \\ & \quad \left. \frac{(5 + \sqrt{5})^4}{150000} + \frac{(5 + \sqrt{5})^4(\sqrt{5} - 1)^8}{38400000} \right] \\ & \%1 := \frac{1}{6} + \frac{\sqrt{5}}{30} + \frac{(5 + \sqrt{5})(\sqrt{5} - 1)^2}{120} \\ & \%2 := \frac{(5 + \sqrt{5})^2}{300} + \frac{(5 + \sqrt{5})^2(\sqrt{5} - 1)^4}{4800} \\ & \%3 := \frac{1}{30} + \frac{\sqrt{5}}{150} + \frac{(5 + \sqrt{5})(\sqrt{5} - 1)^2}{600} \\ & \%4 := \frac{(5 + \sqrt{5})^2}{1500} + \frac{(5 + \sqrt{5})^2(\sqrt{5} - 1)^4}{24000} \\ & \%5 := \frac{(5 + \sqrt{5})^3}{15000} + \frac{(5 + \sqrt{5})^3(\sqrt{5} - 1)^6}{960000} \end{aligned}$$

> expand(Var_Blue_Modele_Sat(N12,[x,y,z],EstSet_N12_plex));

$$240y^4 + \frac{39}{4} + \frac{99}{4}z^2 - 45y^2 + \frac{225}{4}z^6 - 150y^2z^2 + 375z^4y^2 + 1125y^4z^2 + 1875z^2y^8 \\ - 375z^4y^4 - 2625z^2y^6 + 375y^8 - 525y^6 - \frac{315}{4}z^4$$

11.16.3. Variance

Modèle de Degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N12,1,0);
                               Var := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
                               1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N12,2,0):
                               "M_D is not full rank"
> Var:=expand(Var_Norm_Blue_D_Centre(N12,2,1));
                               Var := 12 - 21 var32 - 21 var22 - 21 var12 + 36 var22 var32 + 36 var12 var32 + 36 var12 var22
                               + 18 var34 + 18 var24 + 18 var14
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
                               12 + 18 IndetDecompInKfi12 - 21 IndetDecompInKfi1,
                               IndetDecompInKfi1 = var12 + var22 + var32
```

11.17. N12+1 point au centre

```
> unassign('alpha','beta');
```

```
> TempGr:=Diedn(5):GrN12Plus1:=
{seq(DiagonalMatrix([TempGr[Indice],Vector([1])]),Indice=1..nops(Temp
Gr)),seq(DiagonalMatrix([-1*TempGr[Indice],Vector([-1])]),Indice=1..no
ps(TempGr))};
```

$$GrN12Plus1 := \left\{ \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%4 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%3 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%4 & 0 \\ \%1 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%2 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%3 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%1 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%4 & 0 \\ \%4 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%1 & 0 \\ \%4 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%2 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%1 & 0 \\ \%1 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix} \left. \right\}$$

$$\%1 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%2 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%3 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%4 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

```
> N12Plus1:=OrbToMatrix(OrbPG(Vector([alpha,0,beta]),GrN12Plus1) union
{Vector([0,0,0]),Vector([0,0,1]),Vector([0,0,-1])});
```

$$N12Plus1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ (\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & (\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & -\beta \\ (\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & (-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & -\beta \\ (-\frac{1}{4} - \frac{\sqrt{5}}{4})\alpha & (-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & \beta \\ (-\frac{\sqrt{5}}{4} + \frac{1}{4})\alpha & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & -\beta \\ (-\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & \beta \\ (-\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & \beta \\ (-\frac{1}{4} - \frac{\sqrt{5}}{4})\alpha & (\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\alpha & \beta \\ -\alpha & 0 & -\beta \\ \alpha & 0 & \beta \\ (-\frac{\sqrt{5}}{4} + \frac{1}{4})\alpha & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & -\beta \end{bmatrix}$$

```
> UpdatePlan(N12Plus1,"N12Plus1");
"Update complete"
```

```
> Var:=Var_Norm_Blue_D_Centre(N12Plus1,1,0);
```

$$Var := 1 + \frac{13}{2} \frac{var_3^2}{1+5\beta^2} + \frac{13}{5} \frac{var_2^2}{\alpha^2} + \frac{13}{5} \frac{var_1^2}{\alpha^2}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});
```

$$Cond := [5\alpha^2 - 2 - 10\beta^2]$$

```
> gsolve([op(Cond),alpha^2+beta^2-1],[alpha,beta]);
```

$$\{[[5\beta^2 - 1, 5\alpha^2 - 4], \text{plex}(\alpha, \beta), \{\}]\}$$

```
> beta := 1/sqrt(5); alpha := 2/sqrt(5);
```

$$\beta := \frac{\sqrt{5}}{5}$$

$$\alpha := \frac{2\sqrt{5}}{5}$$

```
> N12Plus1;
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5} & \frac{2(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5} & \frac{2(-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}}{5} & \frac{2(-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{2(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}}{5} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ \frac{2(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{2(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}}{5} & \frac{2(\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8})\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} & 0 & -\frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \\ \frac{2(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}}{5} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \end{bmatrix}$$

> UpdatePlan(N12Plus1,"N12Plus1");

“Update complete”

> convert(evalf(convert(N12Plus1,listlist)),rational,5);

$$\begin{aligned} & [[0, 0, 0], [0, 0, 1], [0, 0, -1], [\frac{89}{123}, \frac{-51}{97}, \frac{-72}{161}], [\frac{89}{123}, \frac{51}{97}, \frac{-72}{161}], [\frac{-89}{123}, \frac{51}{97}, \frac{72}{161}], [\frac{-55}{199}, \frac{131}{154}, \frac{-72}{161}], \\ & [\frac{55}{199}, \frac{131}{154}, \frac{72}{161}], [\frac{55}{199}, \frac{-131}{154}, \frac{72}{161}], [\frac{-89}{123}, \frac{-51}{97}, \frac{72}{161}], [\frac{-144}{161}, 0, \frac{-72}{161}], [\frac{144}{161}, 0, \frac{72}{161}], \\ & [\frac{-55}{199}, \frac{-131}{154}, \frac{-72}{161}]] \end{aligned}$$

> pointplot3d(N12Plus1,symbol=box,scaling=constrained,shading=Z);

> Graph:=convhull(op(map(convert,[evalf(op(convert(N12Plus1,listlist)))],'rational',5))));

Graph := POLYTOPE(3, 3, 12, 20)

> plotdes:=draw(Graph):

> display({plotdes},scaling=constrained,shading=Z);

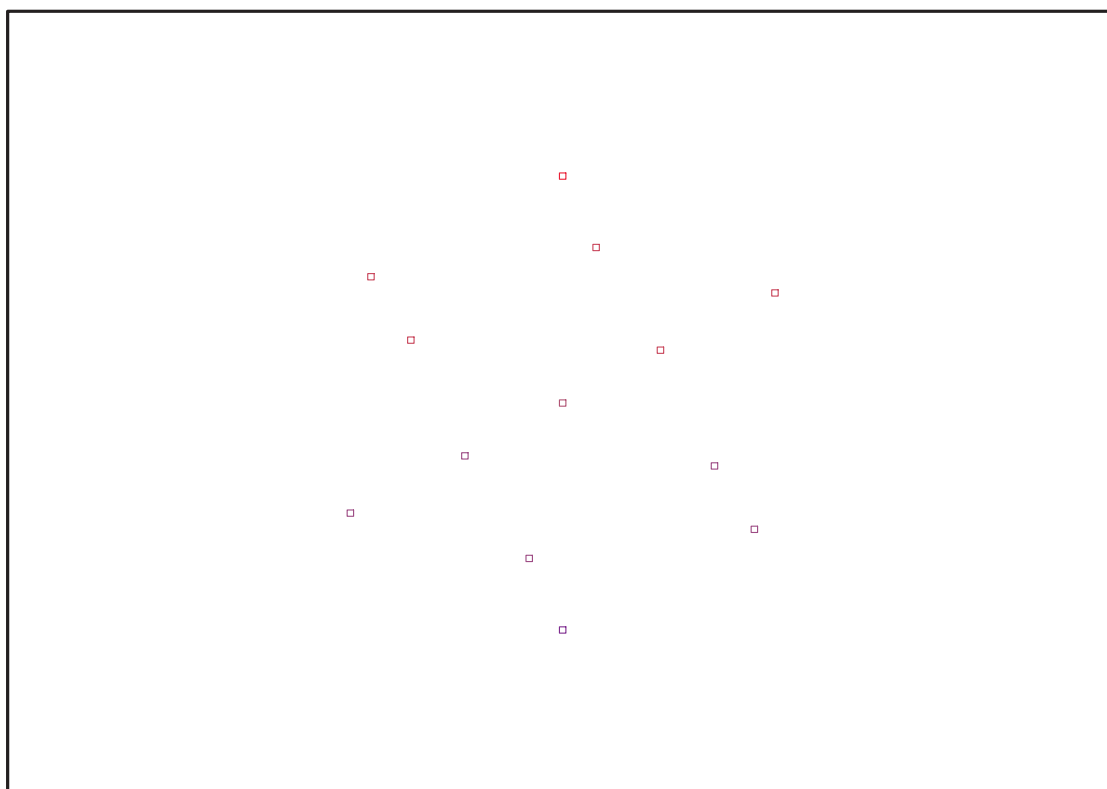


FIG. 11.45 – Graphique : Support du plan N_{12+1} .

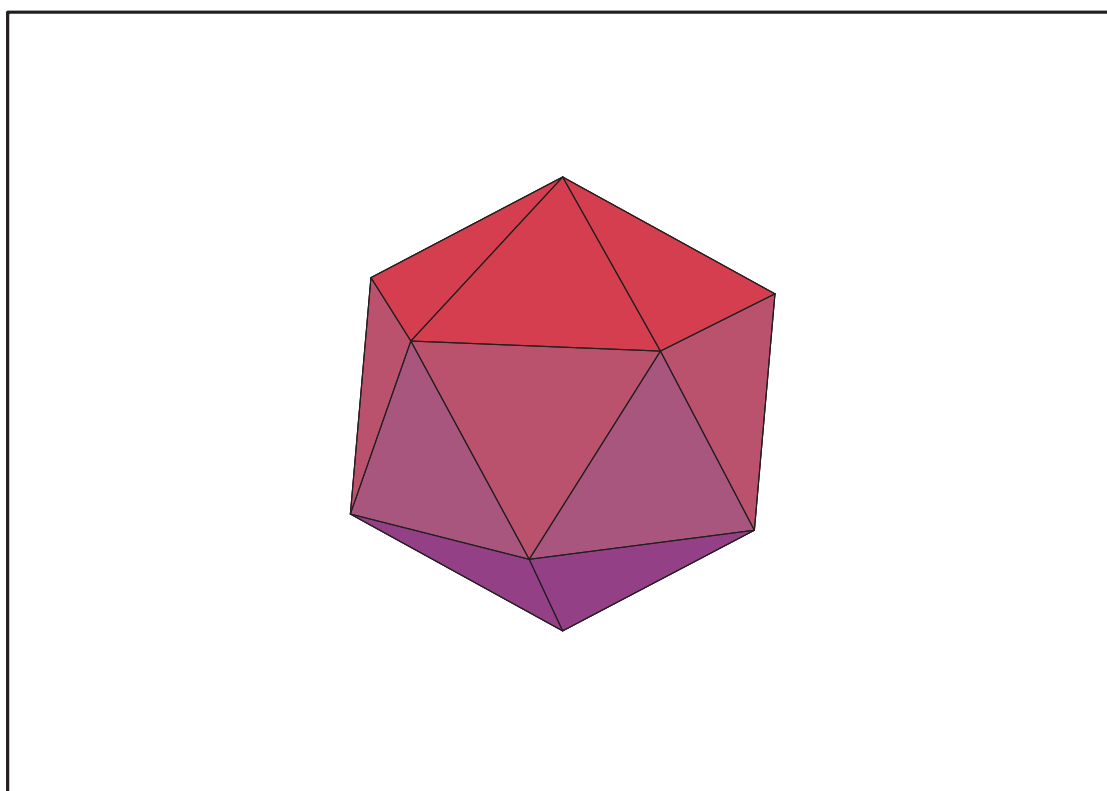


FIG. 11.46 – Graphique : Enveloppe convexe du support du plan N_{12+1} .

```

> IdealN12Plus1:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N12Plus1));

> Gens:=Generators(IdealN12Plus1);

      IdealN12Plus1 := <5 y x^2 + 5 z x y - y, 5 y^2 x + 5 x^3 - 4 x, 5 y^3 + 5 y x^2 - 4 y,
      10 x^4 + y^2 - 2 z x - 7 x^2, 5 z y^2 - 5 z x^2 + 10 x^3 - 6 x, 10 x^3 z + y^2 - 6 z x - x^2,
      5 x^3 y + y z - 2 x y, 5 z^2 x - x, 5 z^3 + 10 z x^2 - 10 x^3 - 5 z + 6 x, 5 z^2 y - y>

      Gens := {10 x^3 z + y^2 - 6 z x - x^2, 5 z y^2 - 5 z x^2 + 10 x^3 - 6 x, 10 x^4 + y^2 - 2 z x - 7 x^2,
      5 x^3 y + y z - 2 x y, 5 y x^2 + 5 z x y - y, 5 y^3 + 5 y x^2 - 4 y, 5 y^2 x + 5 x^3 - 4 x,
      5 z^3 + 10 z x^2 - 10 x^3 - 5 z + 6 x, 5 z^2 y - y, 5 z^2 x - x}

> info_iop:=3;

      info_iop := 3

> Gens := {10*x^3*z+y^2-6*z*x-x^2, 5*z*y^2-5*z*x^2+10*x^3-6*x,
      10*x^4+y^2-2*z*x-7*x^2, 5*x^3*y+y*z-2*x*y, 5*y*x^2+5*z*x*y-y,
      5*y^3+5*y*x^2-4*y, 5*y^2*x+5*x^3-4*x, 5*z^3+10*z*x^2-10*x^3-5*z+6*x,
      5*z^2*y-y, 5*z^2*x-x};

      Gens := {10 x^3 z + y^2 - 6 z x - x^2, 5 z y^2 - 5 z x^2 + 10 x^3 - 6 x, 10 x^4 + y^2 - 2 z x - 7 x^2,
      5 x^3 y + y z - 2 x y, 5 y x^2 + 5 z x y - y, 5 y^3 + 5 y x^2 - 4 y, 5 y^2 x + 5 x^3 - 4 x,
      5 z^3 + 10 z x^2 - 10 x^3 - 5 z + 6 x, 5 z^2 y - y, 5 z^2 x - x}

```

11.17.1. tdeg(x,y,z)

```

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N12_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N12_tdeg:=sort(EstSet_N12_tdeg,(
t1,t2)->testorder(t1,t2,T));nops(EstSet_N12_tdeg);

      EstSet_N12_tdeg := [1, z, y, x, z^2, y z, z x, y^2, x y, x^2, z^3, z y^2, z x y]

```

13

```

> X_D_ModeleSat(N12Plus1,[x,y,z],EstSet_N12_tdeg);

```

$$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

$$[1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0]$$

$$[1, -1, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0]$$

$$\left[1, -\frac{\sqrt{5}}{5}, \frac{2\%1\sqrt{5}}{5}, \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \right. \\ \left. \frac{4\%1^2}{5}, \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{5}, \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{5}, -\frac{\sqrt{5}}{25}, -\frac{4\sqrt{5}\%1^2}{25}, -\frac{4\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{25} \right]$$

$$\left[1, -\frac{\sqrt{5}}{5}, \frac{2\%2\sqrt{5}}{5}, \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \right. \\ \left. \frac{4\%2^2}{5}, \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{5}, \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{5}, -\frac{\sqrt{5}}{25}, -\frac{4\sqrt{5}\%2^2}{25}, -\frac{4\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{25} \right]$$

$$\left[1, \frac{\sqrt{5}}{5}, \frac{2\%2\sqrt{5}}{5}, \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \right. \\ \left. \frac{4\%2^2}{5}, \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{5}, \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{5}, \frac{\sqrt{5}}{25}, \frac{4\sqrt{5}\%2^2}{25}, \frac{4\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{25} \right]$$

$$\left[1, -\frac{\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{2(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, -\frac{1}{10} + \frac{\sqrt{5}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \right. \\ \left. \frac{(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{5}, \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{5}, -\frac{\sqrt{5}}{25}, -\frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\ \left. -\frac{\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{25} \right]$$

$$\left[1, \frac{\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{2(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, -\frac{1}{10} + \frac{\sqrt{5}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \right. \\ \left. \frac{(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{5}, \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{5}, \frac{\sqrt{5}}{25}, \frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\ \left. \frac{\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{25} \right]$$

$$\left[1, \frac{\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{2(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, -\frac{1}{10} + \frac{\sqrt{5}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \right. \\ \left. -\frac{(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{5}, \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{5}, \frac{\sqrt{5}}{25}, \frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\ \left. -\frac{\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{25} \right]$$

$$\begin{aligned}
 & \left[1, \frac{\sqrt{5}}{5}, \frac{2\%1\sqrt{5}}{5}, \frac{2(-\frac{1}{4}-\frac{\sqrt{5}}{4})\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, -\frac{1}{10} - \frac{\sqrt{5}}{10}, \right. \\
 & \left. \frac{4\%1^2}{5}, \frac{4(-\frac{1}{4}-\frac{\sqrt{5}}{4})\%1}{5}, \frac{4(-\frac{1}{4}-\frac{\sqrt{5}}{4})^2}{5}, \frac{\sqrt{5}}{25}, \frac{4\sqrt{5}\%1^2}{25}, \frac{4\sqrt{5}(-\frac{1}{4}-\frac{\sqrt{5}}{4})\%1}{25} \right] \\
 & \left[1, -\frac{\sqrt{5}}{5}, 0, -\frac{2\sqrt{5}}{5}, \frac{1}{5}, 0, \frac{2}{5}, 0, 0, \frac{4}{5}, -\frac{\sqrt{5}}{25}, 0, 0 \right] \\
 & \left[1, \frac{\sqrt{5}}{5}, 0, \frac{2\sqrt{5}}{5}, \frac{1}{5}, 0, \frac{2}{5}, 0, 0, \frac{4}{5}, \frac{\sqrt{5}}{25}, 0, 0 \right] \\
 & \left[1, -\frac{\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{2(-\frac{\sqrt{5}}{4}+\frac{1}{4})\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, -\frac{1}{10} + \frac{\sqrt{5}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \right. \\
 & \left. -\frac{(-\frac{\sqrt{5}}{4}+\frac{1}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{5}, \frac{4(-\frac{\sqrt{5}}{4}+\frac{1}{4})^2}{5}, -\frac{\sqrt{5}}{25}, -\frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\
 & \left. \frac{\sqrt{5}(-\frac{\sqrt{5}}{4}+\frac{1}{4})\sqrt{2}\sqrt{5+\sqrt{5}}}{25} \right] \\
 & \%1 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\
 & \%2 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}
 \end{aligned}$$

> M_D_ModeleSat(N12Plus1, [x,y,z], EstSet_N12_tdeg);

$$\begin{aligned}
 & \left[1, 0, \frac{4\%2\sqrt{5}}{65} + \frac{4\%1\sqrt{5}}{65}, \right. \\
 & \frac{4(\frac{1}{4}+\frac{\sqrt{5}}{4})\sqrt{5}}{65} + \frac{4(-\frac{1}{4}-\frac{\sqrt{5}}{4})\sqrt{5}}{65} + \frac{4(-\frac{\sqrt{5}}{4}+\frac{1}{4})\sqrt{5}}{65} + \frac{4(-\frac{1}{4}+\frac{\sqrt{5}}{4})\sqrt{5}}{65}, \frac{4}{13}, 0, 0, \\
 & \left. \frac{2}{13} + \frac{8\%2^2}{65} + \frac{8\%1^2}{65} + \frac{2\sqrt{5}}{65}, \%3, \%4, 0, 0, \%5 \right] \\
 & \left[0, \frac{4}{13}, 0, 0, 0, 0, 0, 0, \%5, \right. \\
 & -\frac{8\sqrt{5}(\frac{1}{4}+\frac{\sqrt{5}}{4})^2}{325} + \frac{8\sqrt{5}(-\frac{1}{4}-\frac{\sqrt{5}}{4})^2}{325} - \frac{8\sqrt{5}(-\frac{\sqrt{5}}{4}+\frac{1}{4})^2}{325} + \frac{8\sqrt{5}(-\frac{1}{4}+\frac{\sqrt{5}}{4})^2}{325}, \frac{12}{65}, \%6 \\
 & \left. , \%7 \right]
 \end{aligned}$$

$$\begin{aligned}
 & \left[\frac{4\%2\sqrt{5}}{65} + \frac{4\%1\sqrt{5}}{65}, 0, \frac{2}{13} + \frac{8\%2^2}{65} + \frac{8\%1^2}{65} + \frac{2\sqrt{5}}{65}, \%3, \frac{4\%2\sqrt{5}}{325} + \frac{4\%1\sqrt{5}}{325}, \right. \\
 & \frac{2\%2\sqrt{5}\%9}{65} + \frac{2\%1\sqrt{5}\%8}{65} + \frac{2\%1\sqrt{5}\%9}{65} + \frac{2\%2\sqrt{5}\%8}{65}, \\
 & \frac{4\%2\sqrt{5}(-\frac{1}{10} - \frac{\sqrt{5}}{10})}{65} + \frac{4\%1\sqrt{5}(-\frac{1}{10} - \frac{\sqrt{5}}{10})}{65}, \frac{16\%2^3\sqrt{5}}{325} + \frac{16\%1^3\sqrt{5}}{325}, \\
 & \frac{8\%2^2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%1^2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%1^2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} \\
 & + \frac{2(5 + \sqrt{5})\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{325} + \frac{2(5 + \sqrt{5})\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%2^2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325}, \%10 \\
 & \left. , 0, 0, \%11 \right] \\
 & \left[\frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{65} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}}{65} + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}}{65} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{65}, 0, \%3, \%4, \right. \\
 & \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{325} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}}{325} + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}}{325} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{325}, \\
 & \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}\%9}{65} + \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}\%8}{65} + \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}\%9}{65} + \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}\%8}{65} \\
 & , \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}(-\frac{1}{10} - \frac{\sqrt{5}}{10})}{65} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}(-\frac{1}{10} - \frac{\sqrt{5}}{10})}{65} \\
 & + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}(-\frac{1}{10} + \frac{\sqrt{5}}{10})}{65} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}(-\frac{1}{10} + \frac{\sqrt{5}}{10})}{65}, \frac{8\%2^2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} \\
 & + \frac{8\%1^2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%1^2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}(\frac{1}{2} + \frac{\sqrt{5}}{10})}{65} \\
 & + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}(\frac{1}{2} + \frac{\sqrt{5}}{10})}{65} + \frac{8\%2^2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325}, \%10, \\
 & \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3\sqrt{5}}{325} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3\sqrt{5}}{325} + \frac{16(-\frac{\sqrt{5}}{4} + \frac{1}{4})^3\sqrt{5}}{325} + \frac{16(-\frac{1}{4} + \frac{\sqrt{5}}{4})^3\sqrt{5}}{325}, 0, \\
 & \%11, -\frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2\%2}{325} - \frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2\%1}{325} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2\%1}{325} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2\%2}{325} \left. \right]
 \end{aligned}$$

$$\begin{aligned}
 & \left[\frac{4}{13}, 0, \frac{4\%2\sqrt{5}}{325} + \frac{4\%1\sqrt{5}}{325}, \right. \\
 & \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{325} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}}{325} + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})\sqrt{5}}{325} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}}{325}, \frac{12}{65}, 0, 0, \\
 & \%6, \%7, \frac{8}{325} + \frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} + \frac{8(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{325} + \frac{8(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325}, 0, 0, \\
 & \%12 \left. \right] \\
 & \left[0, 0, \frac{2\%2\sqrt{5}\%9}{65} + \frac{2\%1\sqrt{5}\%8}{65} + \frac{2\%1\sqrt{5}\%9}{65} + \frac{2\%2\sqrt{5}\%8}{65}, \right. \\
 & \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}\%9}{65} + \frac{2(\frac{1}{4} + \frac{\sqrt{5}}{4})\sqrt{5}\%8}{65} + \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}\%9}{65} + \frac{2(-\frac{1}{4} - \frac{\sqrt{5}}{4})\sqrt{5}\%8}{65} \\
 & , 0, \frac{2}{65} + \frac{2\%9^2}{13} + \frac{2\%8^2}{13} + \frac{2\sqrt{5}}{325}, \frac{2\%9(-\frac{1}{10} - \frac{\sqrt{5}}{10})}{13} + \frac{2\%8(-\frac{1}{10} - \frac{\sqrt{5}}{10})}{13}, \\
 & \frac{4\%9\%2^2}{65} + \frac{4\%8\%1^2}{65} + \frac{4\%9\%1^2}{65} + \frac{4\%8\%2^2}{65}, \frac{4\%9(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{65} \\
 & + \frac{4\%8(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{65} + \frac{4\%9(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{65} - \frac{2(5 + \sqrt{5})(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{325} \\
 & + \frac{2(5 + \sqrt{5})(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{4\%8(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{65}, \\
 & \frac{4\%9(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{4\%8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{4\%9(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{65} + \frac{4\%8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{65}, 0, \\
 & -\frac{4\%9\sqrt{5}\%2^2}{325} - \frac{4\%8\sqrt{5}\%1^2}{325} + \frac{4\%9\sqrt{5}\%1^2}{325} + \frac{4\%8\sqrt{5}\%2^2}{325}, \\
 & -\frac{4\%9\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{325} - \frac{4\%8\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{325} + \frac{4\%9\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{325} \\
 & + \frac{2(5 + \sqrt{5})\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{1625} + \frac{2(5 + \sqrt{5})\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{4\%8\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{325} \left. \right]
 \end{aligned}$$

$$\begin{aligned}
& \left[0, 0, \frac{4\%2\sqrt{5}\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)}{65} + \frac{4\%1\sqrt{5}\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)}{65}, \frac{4\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\sqrt{5}\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)}{65} \right. \\
& + \frac{4\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)\sqrt{5}\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)}{65} + \frac{4\left(-\frac{\sqrt{5}}{4}+\frac{1}{4}\right)\sqrt{5}\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)}{65} \\
& + \frac{4\left(-\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\sqrt{5}\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)}{65}, 0, \frac{2\%9\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)}{13} + \frac{2\%8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)}{13}, \\
& \frac{8}{325} + \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)^2}{13} + \frac{4\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)^2}{13}, \\
& \frac{8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\%2^2}{65} + \frac{8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\%1^2}{65} + \frac{4\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)}{13}, \\
& \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\%2}{65} + \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\%1}{65} \\
& + \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)\%1}{65} + \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)\%2}{65}, \frac{16}{325} \\
& + \frac{8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)^2}{65} + \frac{8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)^2}{65} + \frac{8\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)\left(-\frac{\sqrt{5}}{4}+\frac{1}{4}\right)^2}{65} \\
& + \frac{8\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)\left(-\frac{1}{4}+\frac{\sqrt{5}}{4}\right)^2}{65}, 0, 0, -\frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\sqrt{5}\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\%2}{325} \\
& - \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\sqrt{5}\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\%1}{325} + \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\sqrt{5}\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)\%1}{325} \\
& \left. + \frac{4\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\sqrt{5}\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)\%2}{325} \right] \\
& \left[\frac{2}{13} + \frac{8\%2^2}{65} + \frac{8\%1^2}{65} + \frac{2\sqrt{5}}{65}, 0, \frac{16\%2^3\sqrt{5}}{325} + \frac{16\%1^3\sqrt{5}}{325}, \frac{8\%2^2\sqrt{5}\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)}{325} \right. \\
& + \frac{8\%1^2\sqrt{5}\left(\frac{1}{4}+\frac{\sqrt{5}}{4}\right)}{325} + \frac{8\%1^2\sqrt{5}\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)}{325} + \frac{4\left(-\frac{\sqrt{5}}{4}+\frac{1}{4}\right)\sqrt{5}\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)}{65} \\
& + \frac{4\left(-\frac{1}{4}+\frac{\sqrt{5}}{4}\right)\sqrt{5}\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)}{65} + \frac{8\%2^2\sqrt{5}\left(-\frac{1}{4}-\frac{\sqrt{5}}{4}\right)}{325}, \%6, \\
& \frac{4\%9\%2^2}{65} + \frac{4\%8\%1^2}{65} + \frac{4\%9\%1^2}{65} + \frac{4\%8\%2^2}{65}, \\
& \frac{8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\%2^2}{65} + \frac{8\left(-\frac{1}{10}-\frac{\sqrt{5}}{10}\right)\%1^2}{65} + \frac{4\left(-\frac{1}{10}+\frac{\sqrt{5}}{10}\right)\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)}{13}, \\
& \left. \frac{32\%2^4}{325} + \frac{32\%1^4}{325} + \frac{4\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)^2}{13} \right],
\end{aligned}$$

$$\begin{aligned}
 & \frac{16\%2^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{16\%1^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{16\%1^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} + \frac{16\%2^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325}, \\
 & \frac{16\%2^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{16\%1^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{16\%1^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} \\
 & + \frac{8(\frac{1}{2} + \frac{\sqrt{5}}{10})(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{65} + \frac{8(\frac{1}{2} + \frac{\sqrt{5}}{10})(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{16\%2^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325}, 0, 0, \\
 & \%13 \Bigg] \\
 & \left[\%3, \%5, \frac{8\%2^2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%1^2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%1^2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} \right. \\
 & + \frac{2(5 + \sqrt{5})\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{325} + \frac{2(5 + \sqrt{5})\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%2^2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325}, \%10 \\
 & , \%7, \frac{4\%9(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{65} + \frac{4\%8(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{65} + \frac{4\%9(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{65} \\
 & - \frac{2(5 + \sqrt{5})(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{325} + \frac{2(5 + \sqrt{5})(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{4\%8(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{65}, \\
 & \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{65} + \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{65} \\
 & + \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{65} + \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{65}, \\
 & \frac{16\%2^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{16\%1^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{16\%1^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} + \frac{16\%2^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325}, \\
 & \frac{16\%2^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{16\%1^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{16\%1^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} \\
 & + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2(5 + \sqrt{5})}{325} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2(5 + \sqrt{5})}{325} + \frac{16\%2^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325}, \\
 & \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3\%2}{325} + \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3\%1}{325} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3\%1}{325} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3\%2}{325}, \\
 & \%12, \%13, \%14 \Bigg] \\
 & \left[\%4, -\frac{8\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{8\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} - \frac{8\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{325} + \frac{8\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325}, \right.
 \end{aligned}$$

$$\begin{aligned}
 & \%10, \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \sqrt{5}}{325} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \sqrt{5}}{325} + \frac{16(-\frac{\sqrt{5}}{4} + \frac{1}{4})^3 \sqrt{5}}{325} + \frac{16(-\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \sqrt{5}}{325} \\
 & , \frac{8}{325} + \frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} + \frac{8(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{325} + \frac{8(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325}, \\
 & \frac{4\%9(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{4\%8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{4\%9(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{65} + \frac{4\%8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{65}, \frac{16}{325} \\
 & + \frac{8(-\frac{1}{10} - \frac{\sqrt{5}}{10})(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{8(-\frac{1}{10} - \frac{\sqrt{5}}{10})(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{65} + \frac{8(-\frac{1}{10} + \frac{\sqrt{5}}{10})(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{65} \\
 & + \frac{8(-\frac{1}{10} + \frac{\sqrt{5}}{10})(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65}, \frac{16\%2^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{16\%1^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} \\
 & + \frac{16\%1^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} + \frac{8(\frac{1}{2} + \frac{\sqrt{5}}{10})(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{65} + \frac{8(\frac{1}{2} + \frac{\sqrt{5}}{10})(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} \\
 & + \frac{16\%2^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325}, \\
 & \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \%2}{325} + \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \%1}{325} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \%1}{325} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \%2}{325}, \\
 & \frac{32}{325} + \frac{32(\frac{1}{4} + \frac{\sqrt{5}}{4})^4}{325} + \frac{32(-\frac{1}{4} - \frac{\sqrt{5}}{4})^4}{325} + \frac{32(-\frac{\sqrt{5}}{4} + \frac{1}{4})^4}{325} + \frac{32(-\frac{1}{4} + \frac{\sqrt{5}}{4})^4}{325}, \\
 & -\frac{8\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{1625} + \frac{8\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{1625} - \frac{8\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{1625} + \frac{8\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{1625}, \%14, \\
 & -\frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \sqrt{5} \%2}{1625} - \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \sqrt{5} \%1}{1625} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \sqrt{5} \%1}{1625} \\
 & + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \sqrt{5} \%2}{1625} \Big] \\
 & \left[0, \frac{12}{65}, 0, 0, 0, 0, 0, 0, \%12, \right. \\
 & -\frac{8\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{1625} + \frac{8\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{1625} - \frac{8\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{1625} + \frac{8\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{1625}, \frac{4}{25}, \\
 & \frac{2}{325} + \frac{8\%2^2}{1625} + \frac{8\%1^2}{1625} + \frac{2\sqrt{5}}{1625}, \\
 & \left. \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%2}{1625} + \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%1}{1625} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%1}{1625} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%2}{1625} \right] \\
 & \left[0, \%6, 0, \%11, 0, -\frac{4\%9\sqrt{5}\%2^2}{325} - \frac{4\%8\sqrt{5}\%1^2}{325} + \frac{4\%9\sqrt{5}\%1^2}{325} + \frac{4\%8\sqrt{5}\%2^2}{325} \right. \\
 & , 0, 0, \%13, \%14, \frac{2}{325} + \frac{8\%2^2}{1625} + \frac{8\%1^2}{1625} + \frac{2\sqrt{5}}{1625}, \frac{32\%2^4}{1625} + \frac{32\%1^4}{1625} + \frac{(5 + \sqrt{5})^2}{1625},
 \end{aligned}$$

$$\begin{aligned}
 & , 0, 0, \%13, \%14, \frac{2}{325} + \frac{8\%2^2}{1625} + \frac{8\%1^2}{1625} + \frac{2\sqrt{5}}{1625}, \frac{32\%2^4}{1625} + \frac{32\%1^4}{1625} + \frac{(5+\sqrt{5})^2}{1625}, \\
 & \left[\frac{16\%2^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{16\%1^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{16\%1^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{1625} + \frac{16\%2^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{1625} \right] \\
 & \left[\%5, \%7, \%11, \right. \\
 & -\frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2 \%2}{325} - \frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2 \%1}{325} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2 \%1}{325} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2 \%2}{325}, \%12, \\
 & -\frac{4\%9\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%2}{325} - \frac{4\%8\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%1}{325} + \frac{4\%9\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%1}{325} \\
 & + \frac{2(5+\sqrt{5})\sqrt{5}(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{1625} + \frac{2(5+\sqrt{5})\sqrt{5}(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{4\%8\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%2}{325}, \\
 & -\frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%2}{325} - \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%1}{325} \\
 & + \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%1}{325} + \frac{4(-\frac{1}{10} - \frac{\sqrt{5}}{10})\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%2}{325}, \%13, \%14, \\
 & -\frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \sqrt{5} \%2}{1625} - \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^3 \sqrt{5} \%1}{1625} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \sqrt{5} \%1}{1625} \\
 & + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^3 \sqrt{5} \%2}{1625}, \\
 & \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%2}{1625} + \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%1}{1625} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%1}{1625} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%2}{1625}, \\
 & \frac{16\%2^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{16\%1^3(\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{16\%1^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{1625} + \frac{16\%2^3(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{1625}, \\
 & \frac{16\%2^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{1625} + \frac{16\%1^2(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{1625} + \frac{16\%1^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{1625} \\
 & + \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2 (5+\sqrt{5})}{1625} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2 (5+\sqrt{5})}{1625} + \frac{16\%2^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{1625} \left. \right] \\
 & \%1 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\
 & \%2 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\
 & \%3 := \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%2}{65} + \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4}) \%1}{65} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%1}{65} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4}) \%2}{65}
 \end{aligned}$$

$$\begin{aligned}
 \%4 &:= \frac{8}{65} + \frac{8(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} + \frac{8(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{65} + \frac{8(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2}{65} + \frac{8(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{65} \\
 \%5 &:= -\frac{4\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{325} - \frac{4\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{325} + \frac{4\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{325} \\
 &+ \frac{4\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{325} \\
 \%6 &:= \frac{2}{65} + \frac{8\%2^2}{325} + \frac{8\%1^2}{325} + \frac{2\sqrt{5}}{325} \\
 \%7 &:= \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{325} + \frac{4(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{325} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{325} + \frac{4(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{325} \\
 \%8 &:= \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20} \\
 \%9 &:= -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20} \\
 \%10 &:= \frac{8\%2\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{8\%1\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})^2}{325} + \frac{8\%1\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} \\
 &+ \frac{8\%2\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2}{325} \\
 \%11 &:= -\frac{8\%2^2(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} - \frac{8\%1^2(\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%1^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} - \frac{2(5+\sqrt{5})(-\frac{\sqrt{5}}{4} + \frac{1}{4})}{325} \\
 &+ \frac{2(5+\sqrt{5})(-\frac{1}{4} + \frac{\sqrt{5}}{4})}{325} + \frac{8\%2^2(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{325} \\
 \%12 &:= -\frac{4\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%2}{1625} - \frac{4\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})\%1}{1625} + \frac{4\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%1}{1625} \\
 &+ \frac{4\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})\%2}{1625} \\
 \%13 &:= -\frac{16\%2^3\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} - \frac{16\%1^3\sqrt{5}(\frac{1}{4} + \frac{\sqrt{5}}{4})}{1625} + \frac{16\%1^3\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{1625} \\
 &+ \frac{16\%2^3\sqrt{5}(-\frac{1}{4} - \frac{\sqrt{5}}{4})}{1625} \\
 \%14 &:= -\frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^2\%2^2\sqrt{5}}{1625} - \frac{16(\frac{1}{4} + \frac{\sqrt{5}}{4})^2\%1^2\sqrt{5}}{1625} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2\%1^2\sqrt{5}}{1625} \\
 &- \frac{4(-\frac{\sqrt{5}}{4} + \frac{1}{4})^2(5+\sqrt{5})\sqrt{5}}{1625} + \frac{4(-\frac{1}{4} + \frac{\sqrt{5}}{4})^2(5+\sqrt{5})\sqrt{5}}{1625} + \frac{16(-\frac{1}{4} - \frac{\sqrt{5}}{4})^2\%2^2\sqrt{5}}{1625}
 \end{aligned}$$

```
> expand(Var_Blue_Modele_Sat(N12Plus1,[x,y,z],EstSet_N12_tdeg));
```

$$\begin{aligned}
 & 13 + \frac{325}{4} z^2 x^2 y^2 + 39 z^2 x^2 + 39 x^2 y^2 - \frac{91}{4} x^2 - \frac{91}{4} y^2 + \frac{39}{8} z^2 + \frac{39}{2} x^4 - \frac{91}{2} z^4 + \frac{325}{4} z^2 y^4 \\
 & + \frac{325}{4} z^4 y^2 + \frac{39}{2} y^4 + \frac{325}{8} z^6 - \frac{169}{4} z^2 y^2
 \end{aligned}$$

11.17.2. plex(x,y,z)

```

> Frontiere_plex:=map(LeadingMonomial,
  {op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

> EstSet_N12_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y
,z):T:=termorder(A,plex(x,y,z)):EstSet_N12_plex:=sort(EstSet_N12_plex,
(t1,t2)->testorder(t1,t2,T));nops(EstSet_N12_plex);
      EstSet_N12_plex := [1, z, z^2, z^3, z^4, y, y z, y^2, z y^2, y^3, z y^3, y^4, z y^4]

```

13

```

> X_D_ModelerSat(N12Plus1,[x,y,z],EstSet_N12_plex);

```

$$\begin{aligned}
 & [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\
 & [1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0] \\
 & [1, -1, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0] \\
 & \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{1}{25}, \frac{2\%1\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, \frac{4\%1^2}{5}, \right. \\
 & \quad \left. -\frac{4\sqrt{5}\%1^2}{25}, \frac{8\%1^3\sqrt{5}}{25}, -\frac{8\%1^3}{25}, \frac{16\%1^4}{25}, -\frac{16\sqrt{5}\%1^4}{125} \right] \\
 & \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{1}{25}, \frac{2\%2\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, \frac{4\%2^2}{5}, \right. \\
 & \quad \left. -\frac{4\sqrt{5}\%2^2}{25}, \frac{8\%2^3\sqrt{5}}{25}, -\frac{8\%2^3}{25}, \frac{16\%2^4}{25}, -\frac{16\sqrt{5}\%2^4}{125} \right] \\
 & \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{1}{25}, \frac{2\%2\sqrt{5}}{5}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, \frac{4\%2^2}{5}, \frac{4\sqrt{5}\%2^2}{25}, \right. \\
 & \quad \left. \frac{8\%2^3\sqrt{5}}{25}, \frac{8\%2^3}{25}, \frac{16\%2^4}{25}, \frac{16\sqrt{5}\%2^4}{125} \right] \\
 & \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{1}{25}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, -\frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\
 & \quad \left. \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, -\frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
 & \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{1}{25}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\
 & \quad \left. \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, \frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
 & \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{1}{25}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, \frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\
 & \quad \left. -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, \frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
 & \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{1}{25}, \frac{2\%1\sqrt{5}}{5}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20}, \frac{4\%1^2}{5}, \frac{4\sqrt{5}\%1^2}{25}, \right. \\
 & \quad \left. \frac{8\%1^3\sqrt{5}}{25}, \frac{8\%1^3}{25}, \frac{16\%1^4}{25}, \frac{16\sqrt{5}\%1^4}{125} \right] \\
 & \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{1}{25}, 0, 0, 0, 0, 0, 0, 0, 0 \right] \\
 & \left[1, \frac{\sqrt{5}}{5}, \frac{1}{5}, \frac{\sqrt{5}}{25}, \frac{1}{25}, 0, 0, 0, 0, 0, 0, 0, 0 \right] \\
 & \left[1, -\frac{\sqrt{5}}{5}, \frac{1}{5}, -\frac{\sqrt{5}}{25}, \frac{1}{25}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{10}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{10}, \frac{1}{2} + \frac{\sqrt{5}}{10}, -\frac{\sqrt{5}(5+\sqrt{5})}{50}, \right. \\
 & \quad \left. -\frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}\sqrt{5}}{100}, \frac{\sqrt{2}(5+\sqrt{5})^{(3/2)}}{100}, \frac{(5+\sqrt{5})^2}{100}, -\frac{\sqrt{5}(5+\sqrt{5})^2}{500} \right] \\
 & \%1 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\
 & \%2 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}
 \end{aligned}$$

> M_D_ModeleSat(N12Plus1,[x,y,z],EstSet_N12_plex);

$$\begin{aligned}
 & \left[1, 0, \frac{4}{13}, 0, \frac{12}{65}, \frac{4\%2\sqrt{5}}{65} + \frac{4\%1\sqrt{5}}{65}, 0, \frac{2}{13} + \frac{8\%2^2}{65} + \frac{8\%1^2}{65} + \frac{2\sqrt{5}}{65}, 0, \%3, 0, \right. \\
 & \left. \%4, 0 \right] \\
 & \left[0, \frac{4}{13}, 0, \frac{12}{65}, 0, 0, 0, 0, \%5, 0, \%6, 0, \%7 \right] \\
 & \left[\frac{4}{13}, 0, \frac{12}{65}, 0, \frac{4}{25}, \frac{4\%2\sqrt{5}}{325} + \frac{4\%1\sqrt{5}}{325}, 0, \%5, 0, \%6, 0, \%7, 0 \right] \\
 & \left[0, \frac{12}{65}, 0, \frac{4}{25}, 0, 0, 0, 0, \%8, 0, \%9, 0, \%10 \right] \\
 & \left[\frac{12}{65}, 0, \frac{4}{25}, 0, \frac{252}{1625}, \frac{4\%2\sqrt{5}}{1625} + \frac{4\%1\sqrt{5}}{1625}, 0, \%8, 0, \%9, 0, \%10, 0 \right] \\
 & \left[\frac{4\%2\sqrt{5}}{65} + \frac{4\%1\sqrt{5}}{65}, 0, \frac{4\%2\sqrt{5}}{325} + \frac{4\%1\sqrt{5}}{325}, 0, \frac{4\%2\sqrt{5}}{1625} + \frac{4\%1\sqrt{5}}{1625}, \right. \\
 & \frac{2}{13} + \frac{8\%2^2}{65} + \frac{8\%1^2}{65} + \frac{2\sqrt{5}}{65}, \\
 & \frac{2\%2\sqrt{5}\%12}{65} + \frac{2\%1\sqrt{5}\%11}{65} + \frac{2\%1\sqrt{5}\%12}{65} + \frac{2\%2\sqrt{5}\%11}{65}, \%3, 0, \%4, 0, \\
 & \left. \%13, 0 \right] \\
 & \left[0, 0, 0, 0, 0, \frac{2\%2\sqrt{5}\%12}{65} + \frac{2\%1\sqrt{5}\%11}{65} + \frac{2\%1\sqrt{5}\%12}{65} + \frac{2\%2\sqrt{5}\%11}{65}, \right. \\
 & \frac{2}{65} + \frac{2\%12^2}{13} + \frac{2\%11^2}{13} + \frac{2\sqrt{5}}{325}, \\
 & \frac{4\%12\%2^2}{65} + \frac{4\%11\%1^2}{65} + \frac{4\%12\%1^2}{65} + \frac{4\%11\%2^2}{65}, \\
 & -\frac{4\%12\sqrt{5}\%2^2}{325} - \frac{4\%11\sqrt{5}\%1^2}{325} + \frac{4\%12\sqrt{5}\%1^2}{325} + \frac{4\%11\sqrt{5}\%2^2}{325}, \\
 & \frac{8\%12\%2^3\sqrt{5}}{325} + \frac{8\%11\%1^3\sqrt{5}}{325} + \frac{8\%12\%1^3\sqrt{5}}{325} + \frac{8\%11\%2^3\sqrt{5}}{325}, \\
 & -\frac{8\%12\%2^3}{325} - \frac{8\%11\%1^3}{325} + \frac{8\%12\%1^3}{325} + \frac{(5+\sqrt{5})^2}{1625} + \frac{8\%11\%2^3}{325}, \\
 & \frac{16\%12\%2^4}{325} + \frac{16\%11\%1^4}{325} + \frac{16\%12\%1^4}{325} + \frac{16\%11\%2^4}{325}, \\
 & \left. -\frac{16\%12\sqrt{5}\%2^4}{1625} - \frac{16\%11\sqrt{5}\%1^4}{1625} + \frac{16\%12\sqrt{5}\%1^4}{1625} + \frac{16\%11\sqrt{5}\%2^4}{1625} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \left[\frac{2}{13} + \frac{8\%2^2}{65} + \frac{8\%1^2}{65} + \frac{2\sqrt{5}}{65}, 0, \%5, 0, \%8, \%3, \right. \\
 & \frac{4\%12\%2^2}{65} + \frac{4\%11\%1^2}{65} + \frac{4\%12\%1^2}{65} + \frac{4\%11\%2^2}{65}, \\
 & \frac{32\%2^4}{325} + \frac{32\%1^4}{325} + \frac{4(\frac{1}{2} + \frac{\sqrt{5}}{10})^2}{13}, 0, \%13, 0, \\
 & \left. \frac{128\%2^6}{1625} + \frac{128\%1^6}{1625} + \frac{(\frac{1}{2} + \frac{\sqrt{5}}{10})(5 + \sqrt{5})^2}{325}, 0 \right] \\
 & \left[0, \%5, 0, \%8, 0, 0, \right. \\
 & -\frac{4\%12\sqrt{5}\%2^2}{325} - \frac{4\%11\sqrt{5}\%1^2}{325} + \frac{4\%12\sqrt{5}\%1^2}{325} + \frac{4\%11\sqrt{5}\%2^2}{325}, 0, \%7, 0, \\
 & \left. \frac{64\%2^5\sqrt{5}}{8125} + \frac{64\%1^5\sqrt{5}}{8125}, 0, \frac{128\%2^6}{8125} + \frac{128\%1^6}{8125} + \frac{(5 + \sqrt{5})^3}{16250} \right] \\
 & \left[\%3, 0, \%6, 0, \%9, \%4, \right. \\
 & \frac{8\%12\%2^3\sqrt{5}}{325} + \frac{8\%11\%1^3\sqrt{5}}{325} + \frac{8\%12\%1^3\sqrt{5}}{325} + \frac{8\%11\%2^3\sqrt{5}}{325}, \%13, 0, \\
 & \left. \frac{128\%2^6}{1625} + \frac{128\%1^6}{1625} + \frac{(5 + \sqrt{5})^3}{3250}, 0, \frac{256\%2^7\sqrt{5}}{8125} + \frac{256\%1^7\sqrt{5}}{8125}, 0 \right] \\
 & \left[0, \%6, 0, \%9, 0, 0, \right. \\
 & -\frac{8\%12\%2^3}{325} - \frac{8\%11\%1^3}{325} + \frac{8\%12\%1^3}{325} + \frac{(5 + \sqrt{5})^2}{1625} + \frac{8\%11\%2^3}{325}, 0, \\
 & \frac{64\%2^5\sqrt{5}}{8125} + \frac{64\%1^5\sqrt{5}}{8125}, 0, \frac{128\%2^6}{8125} + \frac{128\%1^6}{8125} + \frac{(5 + \sqrt{5})^3}{16250}, 0, \\
 & \left. \frac{256\%2^7\sqrt{5}}{40625} + \frac{256\%1^7\sqrt{5}}{40625} \right] \\
 & \left[\%4, 0, \%7, 0, \%10, \%13, \right. \\
 & \frac{16\%12\%2^4}{325} + \frac{16\%11\%1^4}{325} + \frac{16\%12\%1^4}{325} + \frac{16\%11\%2^4}{325}, \\
 & \frac{128\%2^6}{1625} + \frac{128\%1^6}{1625} + \frac{(\frac{1}{2} + \frac{\sqrt{5}}{10})(5 + \sqrt{5})^2}{325}, 0, \frac{256\%2^7\sqrt{5}}{8125} + \frac{256\%1^7\sqrt{5}}{8125}, 0, \\
 & \left. \frac{512\%2^8}{8125} + \frac{512\%1^8}{8125} + \frac{(5 + \sqrt{5})^4}{32500}, 0 \right] \\
 & \left[0, \%7, 0, \%10, 0, 0, \right. \\
 & -\frac{16\%12\sqrt{5}\%2^4}{1625} - \frac{16\%11\sqrt{5}\%1^4}{1625} + \frac{16\%12\sqrt{5}\%1^4}{1625} + \frac{16\%11\sqrt{5}\%2^4}{1625}, 0, \\
 & \frac{128\%2^6}{8125} + \frac{128\%1^6}{8125} + \frac{(5 + \sqrt{5})^3}{16250}, 0, \frac{256\%2^7\sqrt{5}}{40625} + \frac{256\%1^7\sqrt{5}}{40625}, 0, \\
 & \left. \frac{512\%2^8}{40625} + \frac{512\%1^8}{40625} + \frac{(5 + \sqrt{5})^4}{162500} \right]
 \end{aligned}$$

$$\begin{aligned}
 \%1 &:= -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\
 \%2 &:= \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\
 \%3 &:= \frac{16\%2^3\sqrt{5}}{325} + \frac{16\%1^3\sqrt{5}}{325} \\
 \%4 &:= \frac{32\%2^4}{325} + \frac{32\%1^4}{325} + \frac{(5+\sqrt{5})^2}{325} \\
 \%5 &:= \frac{2}{65} + \frac{8\%2^2}{325} + \frac{8\%1^2}{325} + \frac{2\sqrt{5}}{325} \\
 \%6 &:= \frac{16\%2^3\sqrt{5}}{1625} + \frac{16\%1^3\sqrt{5}}{1625} \\
 \%7 &:= \frac{32\%2^4}{1625} + \frac{32\%1^4}{1625} + \frac{(5+\sqrt{5})^2}{1625} \\
 \%8 &:= \frac{2}{325} + \frac{8\%2^2}{1625} + \frac{8\%1^2}{1625} + \frac{2\sqrt{5}}{1625} \\
 \%9 &:= \frac{16\%2^3\sqrt{5}}{8125} + \frac{16\%1^3\sqrt{5}}{8125} \\
 \%10 &:= \frac{32\%2^4}{8125} + \frac{32\%1^4}{8125} + \frac{(5+\sqrt{5})^2}{8125} \\
 \%11 &:= \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20} \\
 \%12 &:= -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{20} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{20} \\
 \%13 &:= \frac{64\%2^5\sqrt{5}}{1625} + \frac{64\%1^5\sqrt{5}}{1625}
 \end{aligned}$$

> expand(Var_Blue_Modele_Sat(N12Plus1,[x,y,z],EstSet_N12_plex));

$$\begin{aligned}
 &13 + \frac{65}{2}y^2 - \frac{1677}{16}z^2 + \frac{11947}{16}z^4 + \frac{6825}{4}z^2y^4 + \frac{1625}{2}z^4y^2 + \frac{8125}{4}z^2y^8 - \frac{1625}{2}z^4y^4 \\
 &- \frac{11375}{4}z^2y^6 + \frac{715}{4}y^4 - \frac{19695}{16}z^6 - 650z^2y^2 + \frac{9425}{16}z^8 + \frac{1625}{4}y^8 - \frac{2275}{4}y^6
 \end{aligned}$$

11.17.3. Variance

Modèle de Degré 1

> Var:=Var_Norm_Blue_D_Centre(N12Plus1,1,0);

$$Var := 1 + \frac{13}{4}var_3^2 + \frac{13}{4}var_2^2 + \frac{13}{4}var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
,Var,poly_algebra(seq(var[i],i=1..3)));

$$1 + \frac{13}{4}IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):

Modèle de Degré 2

> Var:=Var_Norm_Blue_D_Centre(N12Plus1,2,0);

$$\begin{aligned}
 Var := & 13 - \frac{39}{4} var_3^2 - \frac{39}{4} var_2^2 - \frac{39}{4} var_1^2 + \frac{65}{4} var_2^2 var_3^2 + \frac{65}{4} var_1^2 var_3^2 + \frac{65}{4} var_1^2 var_2^2 \\
 & + (-13 + \frac{39}{2} var_3^2 + \frac{91}{8} var_2^2 + \frac{91}{8} var_1^2) var_3^2 \\
 & + (-13 + \frac{91}{8} var_3^2 + \frac{39}{2} var_2^2 + \frac{91}{8} var_1^2) var_2^2 \\
 & + (-13 + \frac{91}{8} var_3^2 + \frac{91}{8} var_2^2 + \frac{39}{2} var_1^2) var_1^2
 \end{aligned}$$

```

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));

```

$$\begin{aligned}
 & 13 + \frac{39}{2} IndetDecompInKfi_1^2 - \frac{91}{4} IndetDecompInKfi_1, \\
 IndetDecompInKfi_1 &= var_1^2 + var_2^2 + var_3^2
 \end{aligned}$$

11.18. N13

```
> unassign('alpha','beta','eta','delta','nu','mu');
```

```
> TempGr:=Diedn(4):GrN13:=
{seq(DiagonalMatrix([TempGr[Indice],Vector([1])]),Indice=1..nops(Temp
Gr))};
```

$$GrN13 := \left\{ \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> N13:=OrbToMatrix(OrbPG(Vector([alpha,0,beta]),GrN13) union
OrbPG(Vector([eta,0,delta]),GrN13) union
OrbPG(Vector([mu,0,nu]),GrN13) union {Vector([0,0,1])});
```

$$N13 := \begin{bmatrix} -\eta & 0 & \delta \\ \alpha & 0 & \beta \\ 0 & \eta & \delta \\ 0 & -\alpha & \beta \\ 0 & -\mu & \nu \\ -\mu & 0 & \nu \\ \eta & 0 & \delta \\ -\alpha & 0 & \beta \\ 0 & -\eta & \delta \\ 0 & 0 & 1 \\ 0 & \alpha & \beta \\ 0 & \mu & \nu \\ \mu & 0 & \nu \end{bmatrix}$$

```
> CreatePlan(N13,"N13");
```

```
> Var:=expand(Var_Norm_Blue_D_Centre(N13,1,0));
```

$$\begin{aligned} Var &:= \frac{13 \delta^2}{\%1} + \frac{13 \beta^2}{\%1} + \frac{13 \nu^2}{\%1} + \frac{13}{4 \%1} - \frac{26 var_3 \delta}{\%1} - \frac{26 var_3 \beta}{\%1} - \frac{26 var_3 \nu}{\%1} - \frac{13 var_3}{2 \%1} \\ &+ \frac{169 var_3^2}{4 \%1} + \frac{13 var_2^2}{2 \eta^2 + \alpha^2 + \mu^2} + \frac{13 var_1^2}{2 \eta^2 + \alpha^2 + \mu^2} \\ \%1 &:= 9 \delta^2 + 9 \beta^2 + 9 \nu^2 + 3 - 8 \delta \beta - 8 \delta \nu - 2 \delta - 8 \beta \nu - 2 \beta - 2 \nu \end{aligned}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},[Var]);
```

$$\begin{aligned} Cond &:= [-13 \eta^2 - 13 \mu^2 + 18 \beta^2 - 13 \alpha^2 + 18 \delta^2 + 18 \nu^2 - 4 \delta - 4 \beta - 4 \nu - 16 \delta \beta - 16 \delta \nu \\ &- 16 \beta \nu + 6, 4 \eta^2 \delta + 4 \delta \alpha^2 + 4 \delta \mu^2 + 4 \eta^2 \beta + 4 \alpha^2 \beta + 4 \mu^2 \beta + 4 \eta^2 \nu + 4 \alpha^2 \nu \\ &+ 4 \mu^2 \nu + \eta^2 + \alpha^2 + \mu^2] \end{aligned}$$

```
> Cond_g solved:=convert(gsolve([op(Cond),alpha^2+beta^2-1,mu^2+nu^2-1,d
elta^2+eta^2-1],[alpha,beta,mu,nu,delta,eta]),list);
```

```

Cond_gsolved := [[[
9216 μ4 - 9216 μ2 β2 - 10752 μ2 + 9216 β4 + 4608 β3 + 2688 β2 - 1776 β + 2905,
α2 + β2 - 1, 6 η2 + 6 μ2 - 6 β2 - 7, 96 β ν + 24 ν - 96 μ2 + 96 β2 + 24 β + 59,
96 μ2 ν - 59 ν + 24 μ2 + 96 β3 + 24 β2 - 37 β - 24, μ2 + ν2 - 1, 4 δ + 4 β + 4 ν + 1],
plex(δ, ν, η, α, μ, β), {}], [[8 β2 + 28 β + 37 + 8 δ β + 28 δ + 8 δ2,
8 μ2 + 8 δ β + 28 β + 28 δ + 53, 8 α2 - 8 δ β - 28 β - 8 δ2 - 28 δ - 45, δ2 + η2 - 1,
2 δ + 2 β + 7 + 2 ν], plex(ν, η, α, μ, β, δ), {-4 β - 1 - 4 δ}], [
[16 δ2 + 4 δ + 61, 4 β + 4 δ + 1, 16 μ2 + 153, 4 α2 + δ - 19, 16 η2 - 4 δ - 77, 4 ν + 13],
plex(ν, η, α, μ, β, δ), {-7 - 2 δ}], [[β2 - 3 β + 3 + δ β - 3 δ + δ2,
μ2 + δ β - 3 β - 3 δ + 5, α2 - δ β + 3 β - δ2 + 3 δ - 4, δ2 + η2 - 1, δ + β - 3 + ν],
plex(ν, η, α, μ, β, δ), {}]]

> Sols:=map(L->solve({op(L[1])} union
{-1<=alpha,alpha<=1,-1<=beta,beta<=1,-1<=delta,delta<=1,-1<=mu,mu<=1,
-1<=nu,nu<=1,-1<=eta,eta<=1},{alpha,beta,mu,nu,delta,eta
}),Cond_gsolved):

> IsRealSols(Sols);
[true, true, true, true, true, true, true, true, true]

> WhichRealSols(Sols):

> allvalues(Sols);

```

$$\begin{aligned}
& \{ \alpha = -\frac{\sqrt{15}}{4}, \beta = \frac{-1}{4}, \eta = -\frac{\sqrt{354}}{24}, \mu = -\frac{\sqrt{354}}{24}, \nu = -\frac{\sqrt{222}}{24}, \delta = \frac{\sqrt{222}}{24} \}, \\
& \{ \eta = \frac{\sqrt{354}}{24}, \mu = \frac{\sqrt{354}}{24}, \alpha = -\frac{\sqrt{15}}{4}, \beta = \frac{-1}{4}, \nu = -\frac{\sqrt{222}}{24}, \delta = \frac{\sqrt{222}}{24} \}, \\
& \{ \alpha = -\frac{\sqrt{15}}{4}, \delta = -\frac{\sqrt{222}}{24}, \nu = \frac{\sqrt{222}}{24}, \beta = \frac{-1}{4}, \eta = -\frac{\sqrt{354}}{24}, \mu = -\frac{\sqrt{354}}{24} \}, \\
& \{ \eta = \frac{\sqrt{354}}{24}, \mu = \frac{\sqrt{354}}{24}, \alpha = -\frac{\sqrt{15}}{4}, \delta = -\frac{\sqrt{222}}{24}, \nu = \frac{\sqrt{222}}{24}, \beta = \frac{-1}{4} \}, \\
& \{ \beta = \frac{-1}{4}, \eta = -\frac{\sqrt{354}}{24}, \mu = -\frac{\sqrt{354}}{24}, \alpha = \frac{\sqrt{15}}{4}, \nu = -\frac{\sqrt{222}}{24}, \delta = \frac{\sqrt{222}}{24} \}, \\
& \{ \eta = \frac{\sqrt{354}}{24}, \mu = \frac{\sqrt{354}}{24}, \beta = \frac{-1}{4}, \alpha = \frac{\sqrt{15}}{4}, \nu = -\frac{\sqrt{222}}{24}, \delta = \frac{\sqrt{222}}{24} \}, \\
& \{ \delta = -\frac{\sqrt{222}}{24}, \nu = \frac{\sqrt{222}}{24}, \beta = \frac{-1}{4}, \eta = -\frac{\sqrt{354}}{24}, \mu = -\frac{\sqrt{354}}{24}, \alpha = \frac{\sqrt{15}}{4} \}, \\
& \{ \eta = \frac{\sqrt{354}}{24}, \mu = \frac{\sqrt{354}}{24}, \delta = -\frac{\sqrt{222}}{24}, \nu = \frac{\sqrt{222}}{24}, \beta = \frac{-1}{4}, \alpha = \frac{\sqrt{15}}{4} \}, \\
& \{ \delta = 1, \nu = 1, \mu = 0, \eta = 0, \alpha = 0, \beta = 1 \}
\end{aligned}$$

```

> nu := -1/4*sqrt(37/6) ; mu := -1/4*sqrt(59/6); alpha := 1/4*sqrt(15);
beta := -1/4; delta := 1/4*sqrt(37/6); eta := 1/4*sqrt(59/6);

```

$$\begin{aligned}
\nu &:= -\frac{\sqrt{222}}{24} \\
\mu &:= -\frac{\sqrt{354}}{24} \\
\alpha &:= \frac{\sqrt{15}}{4} \\
\beta &:= \frac{-1}{4}
\end{aligned}$$

$$\delta := \frac{\sqrt{222}}{24}$$

$$\eta := \frac{\sqrt{354}}{24}$$

```
> N13:=OrbToMatrix(SortPoints(simplify(OrbPG(Vector([alpha,0,beta]),GrN
13) union OrbPG(Vector([eta,0,delta]),GrN13) union
OrbPG(Vector([mu,0,nu]),GrN13) union {Vector([0,0,1])})));
```

$$N13 := \begin{bmatrix} 0 & \frac{\sqrt{354}}{24} & -\frac{\sqrt{222}}{24} \\ 0 & 0 & 1 \\ 0 & -\frac{\sqrt{354}}{24} & -\frac{\sqrt{222}}{24} \\ 0 & \frac{\sqrt{354}}{24} & \frac{\sqrt{222}}{24} \\ 0 & -\frac{\sqrt{15}}{4} & \frac{-1}{4} \\ 0 & -\frac{\sqrt{354}}{24} & \frac{\sqrt{222}}{24} \\ 0 & \frac{\sqrt{15}}{4} & \frac{-1}{4} \\ \frac{\sqrt{354}}{24} & 0 & \frac{\sqrt{222}}{24} \\ \frac{\sqrt{354}}{24} & 0 & -\frac{\sqrt{222}}{24} \\ -\frac{\sqrt{354}}{24} & 0 & -\frac{\sqrt{222}}{24} \\ -\frac{\sqrt{354}}{24} & 0 & \frac{\sqrt{222}}{24} \\ -\frac{\sqrt{15}}{4} & 0 & \frac{-1}{4} \\ \frac{\sqrt{15}}{4} & 0 & \frac{-1}{4} \end{bmatrix}$$

```
> pointplot3d(N13,symbol=box,scaling=constrained,shading=Z);
```

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N13,listlist)))],'ra
tional',5))));
```

$$Graph := \text{POLYTOPE}(3, 3, 13, 13)$$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN13:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N13));
```

```
> Gens:=Generators(IdealN13);
```

$$\begin{aligned} \text{IdealN13} := & \langle xy, z^2 + y^2 + x^2 - 1, 1536x^5 - 2384x^3 + 885x, 384x^3z + 96x^3 - 236xz - 59x, \\ & 384y^3z + 96y^3 - 236yz - 59y, \\ & 384y^4 + 384x^4 + 288y^2z + 288x^2z - 524y^2 - 524x^2 - 177z + 177 \rangle \end{aligned}$$

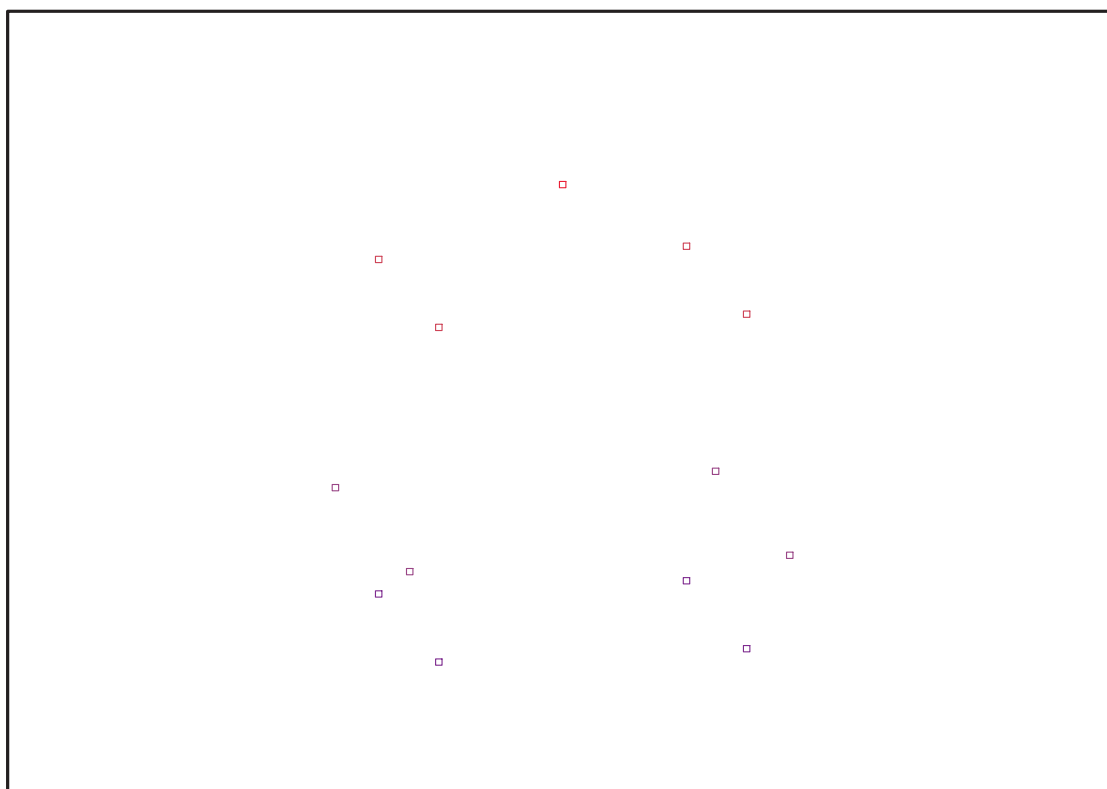


FIG. 11.47 – Graphique : Support du plan N13.

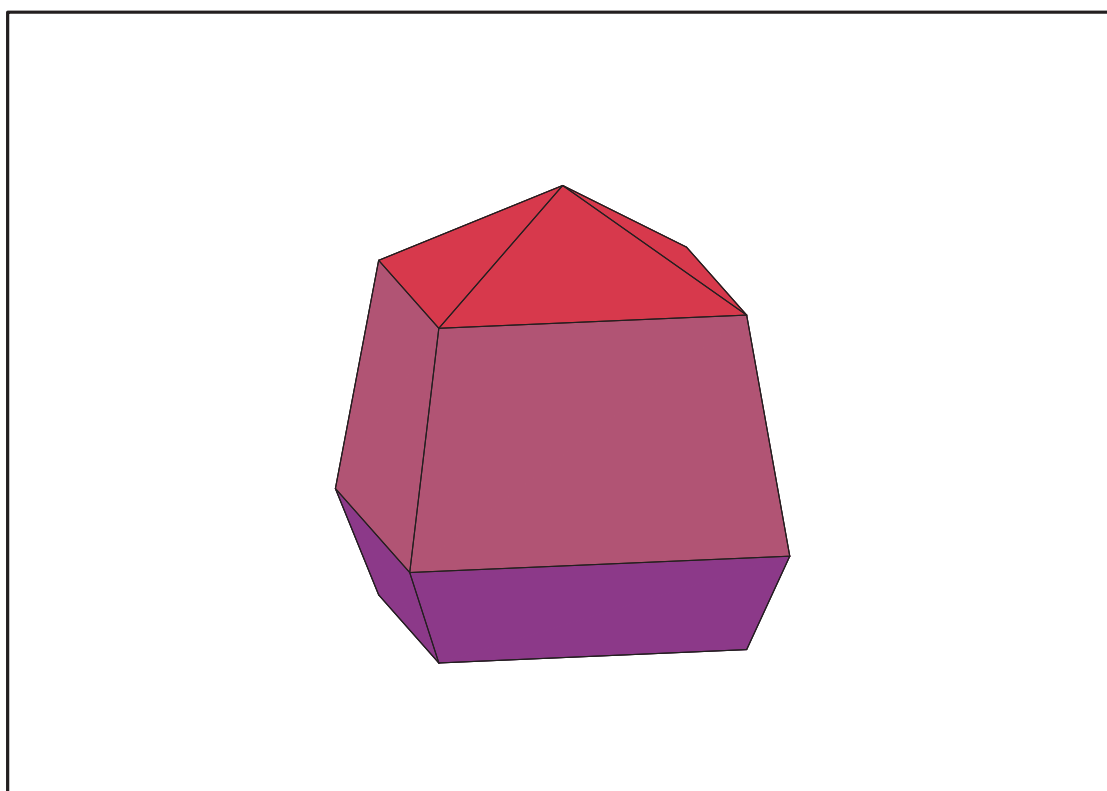


FIG. 11.48 – Graphique : Enveloppe convexe du support du plan N13.

$$\begin{aligned} \text{Gens} := & \{384y^3z + 96y^3 - 236yz - 59y, 384x^3z + 96x^3 - 236xz - 59x, z^2 + y^2 + x^2 - 1, \\ & 1536x^5 - 2384x^3 + 885x, xy, \\ & 384y^4 + 384x^4 + 288y^2z + 288x^2z - 524y^2 - 524x^2 - 177z + 177\} \end{aligned}$$

> info_iop:=3;

$$\text{info_iop} := 3$$

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N13),
listlist),Vector)));

1
2
3
4
5
6
7
8
8
8
9
10
11
12
12
12
12
12
13

> Gens:={384*y^3*z+96*y^3-236*y*z-59*y, 384*x^3*z+96*x^3-236*x*z-59*x,
z^2+y^2+x^2-1, 1536*x^5-2384*x^3+885*x, x*y,
384*y^4+384*x^4+288*y^2*z+288*x^2*z-524*y^2-524*x^2-177*z+177};

$$\begin{aligned} \text{Gens} := & \{384y^3z + 96y^3 - 236yz - 59y, 384x^3z + 96x^3 - 236xz - 59x, \\ & 1536x^5 - 2384x^3 + 885x, z^2 + y^2 + x^2 - 1, \\ & 384y^4 + 384x^4 + 288yz^2 + 288x^2z - 524y^2 - 524x^2 - 177z + 177, xy\} \end{aligned}$$

11.18.1. tdeg(x,y,z)

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),tdeg(x,y,z)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N13_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N13_tdeg:=sort(EstSet_N13_tdeg,(
t1,t2)->testorder(t1,t2,T));nops(EstSet_N13_tdeg);

$$\text{EstSet_N13_tdeg} := [1, z, y, x, z^2, yz, zx, y^2, z^3, yz^2, z^2x, zy^2, z^2y^2]$$

13

> X_D_ModeleSat(N13,[x,y,z],EstSet_N13_tdeg);

$$\begin{aligned}
& \left[1, -\frac{\sqrt{222}}{24}, \frac{\sqrt{354}}{24}, 0, \frac{37}{96}, -\frac{\sqrt{222}\sqrt{354}}{576}, 0, \frac{59}{96}, -\frac{37\sqrt{222}}{2304}, \frac{37\sqrt{354}}{2304}, 0, \right. \\
& \left. -\frac{59\sqrt{222}}{2304}, \frac{2183}{9216} \right] \\
& [1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0] \\
& \left[1, -\frac{\sqrt{222}}{24}, -\frac{\sqrt{354}}{24}, 0, \frac{37}{96}, \frac{\sqrt{222}\sqrt{354}}{576}, 0, \frac{59}{96}, -\frac{37\sqrt{222}}{2304}, -\frac{37\sqrt{354}}{2304}, 0, \right. \\
& \left. -\frac{59\sqrt{222}}{2304}, \frac{2183}{9216} \right] \\
& \left[1, \frac{\sqrt{222}}{24}, \frac{\sqrt{354}}{24}, 0, \frac{37}{96}, \frac{\sqrt{222}\sqrt{354}}{576}, 0, \frac{59}{96}, \frac{37\sqrt{222}}{2304}, \frac{37\sqrt{354}}{2304}, 0, \frac{59\sqrt{222}}{2304}, \right. \\
& \left. \frac{2183}{9216} \right] \\
& \left[1, \frac{-1}{4}, -\frac{\sqrt{15}}{4}, 0, \frac{1}{16}, \frac{\sqrt{15}}{16}, 0, \frac{15}{16}, \frac{-1}{64}, -\frac{\sqrt{15}}{64}, 0, \frac{-15}{64}, \frac{15}{256} \right] \\
& \left[1, \frac{\sqrt{222}}{24}, -\frac{\sqrt{354}}{24}, 0, \frac{37}{96}, -\frac{\sqrt{222}\sqrt{354}}{576}, 0, \frac{59}{96}, \frac{37\sqrt{222}}{2304}, -\frac{37\sqrt{354}}{2304}, 0, \frac{59\sqrt{222}}{2304} \right. \\
& \left. , \frac{2183}{9216} \right] \\
& \left[1, \frac{-1}{4}, \frac{\sqrt{15}}{4}, 0, \frac{1}{16}, -\frac{\sqrt{15}}{16}, 0, \frac{15}{16}, \frac{-1}{64}, \frac{\sqrt{15}}{64}, 0, \frac{-15}{64}, \frac{15}{256} \right] \\
& \left[1, \frac{\sqrt{222}}{24}, 0, \frac{\sqrt{354}}{24}, \frac{37}{96}, 0, \frac{\sqrt{222}\sqrt{354}}{576}, 0, \frac{37\sqrt{222}}{2304}, 0, \frac{37\sqrt{354}}{2304}, 0, 0 \right] \\
& \left[1, -\frac{\sqrt{222}}{24}, 0, \frac{\sqrt{354}}{24}, \frac{37}{96}, 0, -\frac{\sqrt{222}\sqrt{354}}{576}, 0, -\frac{37\sqrt{222}}{2304}, 0, \frac{37\sqrt{354}}{2304}, 0, 0 \right] \\
& \left[1, -\frac{\sqrt{222}}{24}, 0, -\frac{\sqrt{354}}{24}, \frac{37}{96}, 0, \frac{\sqrt{222}\sqrt{354}}{576}, 0, -\frac{37\sqrt{222}}{2304}, 0, -\frac{37\sqrt{354}}{2304}, 0, 0 \right] \\
& \left[1, \frac{\sqrt{222}}{24}, 0, -\frac{\sqrt{354}}{24}, \frac{37}{96}, 0, -\frac{\sqrt{222}\sqrt{354}}{576}, 0, \frac{37\sqrt{222}}{2304}, 0, -\frac{37\sqrt{354}}{2304}, 0, 0 \right] \\
& \left[1, \frac{-1}{4}, 0, -\frac{\sqrt{15}}{4}, \frac{1}{16}, 0, \frac{\sqrt{15}}{16}, 0, \frac{-1}{64}, 0, -\frac{\sqrt{15}}{64}, 0, 0 \right] \\
& \left[1, \frac{-1}{4}, 0, \frac{\sqrt{15}}{4}, \frac{1}{16}, 0, -\frac{\sqrt{15}}{16}, 0, \frac{-1}{64}, 0, \frac{\sqrt{15}}{64}, 0, 0 \right]
\end{aligned}$$

> M_D_ModeleSat(N13,[x,y,z],EstSet_N13_tdeg);

$$\begin{aligned}
& 1, 0, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{3}, \frac{15}{208}, 0, 0, \frac{-15}{416}, \frac{2453}{29952} \\
& 0, \frac{1}{3}, 0, 0, \frac{15}{208}, 0, 0, \frac{-15}{416}, \frac{2539}{14976}, 0, 0, \frac{2453}{29952}, \frac{-15}{6656} \\
& 0, 0, \frac{1}{3}, 0, 0, \frac{-15}{416}, 0, 0, 0, \frac{2453}{29952}, 0, 0, 0 \\
& 0, 0, 0, \frac{1}{3}, 0, 0, \frac{-15}{416}, 0, 0, 0, \frac{2453}{29952}, 0, 0 \\
& \frac{1}{3}, \frac{15}{208}, 0, 0, \frac{2539}{14976}, 0, 0, \frac{2453}{29952}, \frac{255}{3328}, 0, 0, \frac{-15}{6656}, \frac{82391}{2875392} \\
& 0, 0, \frac{-15}{416}, 0, 0, \frac{2453}{29952}, 0, 0, 0, \frac{-15}{6656}, 0, 0, 0 \\
& 0, 0, 0, \frac{-15}{416}, 0, 0, \frac{2453}{29952}, 0, 0, 0, \frac{-15}{6656}, 0, 0 \\
& \frac{1}{3}, \frac{-15}{416}, 0, 0, \frac{2453}{29952}, 0, 0, \frac{7531}{29952}, \frac{-15}{6656}, 0, 0, \frac{-225}{6656}, \frac{153097}{2875392} \\
& \frac{15}{208}, \frac{2539}{14976}, 0, 0, \frac{255}{3328}, 0, 0, \frac{-15}{6656}, \frac{161353}{1437696}, 0, 0, \frac{82391}{2875392}, \frac{-15}{106496} \\
& 0, 0, \frac{2453}{29952}, 0, 0, \frac{-15}{6656}, 0, 0, 0, \frac{82391}{2875392}, 0, 0, 0 \\
& 0, 0, 0, \frac{2453}{29952}, 0, 0, \frac{-15}{6656}, 0, 0, 0, \frac{82391}{2875392}, 0, 0 \\
& \frac{-15}{416}, \frac{2453}{29952}, 0, 0, \frac{-15}{6656}, 0, 0, \frac{-225}{6656}, \frac{82391}{2875392}, 0, 0, \frac{153097}{2875392}, \frac{-225}{106496} \\
& \frac{2453}{29952}, \frac{-15}{6656}, 0, 0, \frac{82391}{2875392}, 0, 0, \frac{153097}{2875392}, \frac{-15}{106496}, 0, 0, \frac{-225}{106496}, \frac{4911289}{276037632}
\end{aligned}$$

> expand(Var_Blue_Modelle_Sat(N13,[x,y,z],EstSet_N13_tdeg));

$$\begin{aligned}
& \frac{19336974592}{752679225} y^4 - \frac{70685576}{83631025} z + \frac{14976}{1829} x^2 z + \frac{1032404598784}{3094347925} y^4 z^4 - \frac{1437696}{67673} x^2 z^3 \\
& - \frac{276037632}{3992707} y^4 z^3 + \frac{35357367552}{3094347925} z^2 y^2 + \frac{1665696448}{250893075} z y^2 - \frac{526679296}{10489315} x^2 z^2 \\
& - \frac{105810861312}{3094347925} z^4 - \frac{113494056}{19963535} z^2 + \frac{138885931392}{3094347925} z^3 - \frac{597026456}{83631025} y^2 + \frac{9607624}{850485} x^2 \\
& + \frac{2875392}{107911} z y^4 + \frac{614129373}{83631025} + \frac{1315651584}{10489315} x^2 z^4 - \frac{342746247168}{3094347925} z^5 \\
& - \frac{310505807872}{3094347925} z^3 y^2 + \frac{667338866688}{3094347925} z^5 y^2 - \frac{983525490688}{9283043775} y^4 z^2 \\
& - \frac{4414685184}{83631025} z^4 y^2 + \frac{347381858304}{3094347925} z^6
\end{aligned}$$

11.18.2. plex(x,y,z)

> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

> EstSet_N13_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N13_plex:=sort(EstSet_N13_plex,(t1,t2)->testorder(t1,t2,T)):nops(EstSet_N13_plex);

$EstSet_N13_plex := [1, z, z^2, z^3, y, yz, yz^2, y^2, zy^2, z^2y^2, x, zx, z^2x]$

13

> X_D_ModelleSat(N13,[x,y,z],EstSet_N13_plex);

$$\begin{aligned}
 & \left[1, -\frac{\sqrt{222}}{24}, \frac{37}{96}, -\frac{37\sqrt{222}}{2304}, \frac{\sqrt{354}}{24}, -\frac{\sqrt{222}\sqrt{354}}{576}, \frac{37\sqrt{354}}{2304}, \frac{59}{96}, -\frac{59\sqrt{222}}{2304}, \frac{2183}{9216}, \right. \\
 & \left. 0, 0, 0 \right] \\
 & [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\
 & \left[1, -\frac{\sqrt{222}}{24}, \frac{37}{96}, -\frac{37\sqrt{222}}{2304}, -\frac{\sqrt{354}}{24}, \frac{\sqrt{222}\sqrt{354}}{576}, -\frac{37\sqrt{354}}{2304}, \frac{59}{96}, -\frac{59\sqrt{222}}{2304}, \frac{2183}{9216} \right. \\
 & \left. , 0, 0, 0 \right] \\
 & \left[1, \frac{\sqrt{222}}{24}, \frac{37}{96}, \frac{37\sqrt{222}}{2304}, \frac{\sqrt{354}}{24}, \frac{\sqrt{222}\sqrt{354}}{576}, \frac{37\sqrt{354}}{2304}, \frac{59}{96}, \frac{59\sqrt{222}}{2304}, \frac{2183}{9216}, 0, 0, \right. \\
 & \left. 0 \right] \\
 & \left[1, \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, -\frac{\sqrt{15}}{4}, \frac{\sqrt{15}}{16}, -\frac{\sqrt{15}}{64}, \frac{15}{16}, \frac{-15}{64}, \frac{15}{256}, 0, 0, 0 \right] \\
 & \left[1, \frac{\sqrt{222}}{24}, \frac{37}{96}, \frac{37\sqrt{222}}{2304}, -\frac{\sqrt{354}}{24}, -\frac{\sqrt{222}\sqrt{354}}{576}, -\frac{37\sqrt{354}}{2304}, \frac{59}{96}, \frac{59\sqrt{222}}{2304}, \frac{2183}{9216}, 0 \right. \\
 & \left. , 0, 0 \right] \\
 & \left[1, \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, \frac{\sqrt{15}}{4}, -\frac{\sqrt{15}}{16}, \frac{\sqrt{15}}{64}, \frac{15}{16}, \frac{-15}{64}, \frac{15}{256}, 0, 0, 0 \right] \\
 & \left[1, \frac{\sqrt{222}}{24}, \frac{37}{96}, \frac{37\sqrt{222}}{2304}, 0, 0, 0, 0, 0, 0, \frac{\sqrt{354}}{24}, \frac{\sqrt{222}\sqrt{354}}{576}, \frac{37\sqrt{354}}{2304} \right] \\
 & \left[1, -\frac{\sqrt{222}}{24}, \frac{37}{96}, -\frac{37\sqrt{222}}{2304}, 0, 0, 0, 0, 0, 0, \frac{\sqrt{354}}{24}, -\frac{\sqrt{222}\sqrt{354}}{576}, \frac{37\sqrt{354}}{2304} \right] \\
 & \left[1, -\frac{\sqrt{222}}{24}, \frac{37}{96}, -\frac{37\sqrt{222}}{2304}, 0, 0, 0, 0, 0, 0, -\frac{\sqrt{354}}{24}, \frac{\sqrt{222}\sqrt{354}}{576}, -\frac{37\sqrt{354}}{2304} \right] \\
 & \left[1, \frac{\sqrt{222}}{24}, \frac{37}{96}, \frac{37\sqrt{222}}{2304}, 0, 0, 0, 0, 0, 0, -\frac{\sqrt{354}}{24}, -\frac{\sqrt{222}\sqrt{354}}{576}, -\frac{37\sqrt{354}}{2304} \right] \\
 & \left[1, \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, 0, 0, 0, 0, 0, 0, -\frac{\sqrt{15}}{4}, \frac{\sqrt{15}}{16}, -\frac{\sqrt{15}}{64} \right] \\
 & \left[1, \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, 0, 0, 0, 0, 0, 0, \frac{\sqrt{15}}{4}, -\frac{\sqrt{15}}{16}, \frac{\sqrt{15}}{64} \right]
 \end{aligned}$$

> M_D_ModeleSat(N13,[x,y,z],EstSet_N13_plex);

$$\begin{aligned}
& 1, 0, \frac{1}{3}, \frac{15}{208}, 0, 0, 0, \frac{1}{3}, \frac{-15}{416}, \frac{2453}{29952}, 0, 0, 0 \\
& 0, \frac{1}{3}, \frac{15}{208}, \frac{2539}{14976}, 0, 0, 0, \frac{-15}{416}, \frac{2453}{29952}, \frac{-15}{6656}, 0, 0, 0 \\
& \frac{1}{3}, \frac{15}{208}, \frac{2539}{14976}, \frac{255}{3328}, 0, 0, 0, \frac{2453}{29952}, \frac{-15}{6656}, \frac{82391}{2875392}, 0, 0, 0 \\
& \frac{15}{208}, \frac{2539}{14976}, \frac{255}{3328}, \frac{161353}{1437696}, 0, 0, 0, \frac{-15}{6656}, \frac{82391}{2875392}, \frac{-15}{106496}, 0, 0, 0 \\
& 0, 0, 0, 0, \frac{1}{3}, \frac{-15}{416}, \frac{2453}{29952}, 0, 0, 0, 0, 0, 0 \\
& 0, 0, 0, 0, \frac{-15}{416}, \frac{2453}{29952}, \frac{-15}{6656}, 0, 0, 0, 0, 0, 0 \\
& 0, 0, 0, 0, \frac{2453}{29952}, \frac{-15}{6656}, \frac{82391}{2875392}, 0, 0, 0, 0, 0, 0 \\
& \frac{1}{3}, \frac{-15}{416}, \frac{2453}{29952}, \frac{-15}{6656}, 0, 0, 0, \frac{7531}{29952}, \frac{-225}{6656}, \frac{153097}{2875392}, 0, 0, 0 \\
& \frac{-15}{416}, \frac{2453}{29952}, \frac{-15}{6656}, \frac{82391}{2875392}, 0, 0, 0, \frac{-225}{6656}, \frac{153097}{2875392}, \frac{-225}{106496}, 0, 0, 0 \\
& \frac{2453}{29952}, \frac{-15}{6656}, \frac{82391}{2875392}, \frac{-15}{106496}, 0, 0, 0, \frac{153097}{2875392}, \frac{-225}{106496}, \frac{4911289}{276037632}, 0, 0, 0 \\
& 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{3}, \frac{-15}{416}, \frac{2453}{29952} \\
& 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{-15}{416}, \frac{2453}{29952}, \frac{-15}{6656} \\
& 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{2453}{29952}, \frac{-15}{6656}, \frac{82391}{2875392}
\end{aligned}$$

> expand(Var_Blue_Modele_Sat(N13,[x,y,z],EstSet_N13_plex));

$$\begin{aligned}
& \frac{19336974592}{752679225} y^4 - \frac{70685576}{83631025} z + \frac{14976}{1829} x^2 z + \frac{1032404598784}{3094347925} y^4 z^4 - \frac{1437696}{67673} x^2 z^3 \\
& - \frac{276037632}{3992707} y^4 z^3 + \frac{35357367552}{3094347925} z^2 y^2 + \frac{1665696448}{250893075} z y^2 - \frac{526679296}{10489315} x^2 z^2 \\
& - \frac{983525490688}{9283043775} y^4 z^2 - \frac{105810861312}{3094347925} z^4 - \frac{113494056}{19963535} z^2 + \frac{138885931392}{3094347925} z^3 \\
& - \frac{597026456}{83631025} y^2 + \frac{9607624}{850485} x^2 + \frac{2875392}{107911} z y^4 + \frac{614129373}{83631025} + \frac{1315651584}{10489315} x^2 z^4 \\
& - \frac{342746247168}{3094347925} z^5 - \frac{310505807872}{3094347925} z^3 y^2 + \frac{667338866688}{3094347925} z^5 y^2 - \frac{4414685184}{83631025} z^4 y^2 \\
& + \frac{347381858304}{3094347925} z^6
\end{aligned}$$

11.18.3. Variance

Modèle de Degré 1

> Var:=Var_Norm_Blue_D_Centre(N13,1,0);

$$Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
,Var,poly_algebra(seq(var[i],i=1..3)));

$$1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

> animate(plot3d,[Var,var[1]==-2..2,var[2]==-2..2],var[3]==-2..2):

Modèle de Degré 2

```
> Var:=expand(Var_Norm_Blue_D_Centre(N13,2,1));  
"M_D is not full rank"
```

11.19. N14

```
> unassign('alpha','beta','eta','delta','nu','mu');
```

```
> GrN14:=Plus(Rned(Diedn(3),3));
```

$$GrN14 := \left\{ \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

11.19.1. 1er essai

```
> N14_1:=OrbToMatrix(OrbPG(Vector([alpha,0,beta]),GrN14) union  
OrbPG(Vector([delta,0,eta]),GrN14) union  
OrbPG(Vector([0,0,1]),GrN14));
```

$$N14_1 := \begin{bmatrix} \alpha & 0 & -\beta \\ -\frac{\delta}{2} & -\frac{\sqrt{3}\delta}{2} & -\eta \\ \delta & 0 & -\eta \\ -\frac{\delta}{2} & -\frac{\sqrt{3}\delta}{2} & \eta \\ -\frac{\alpha}{2} & -\frac{\sqrt{3}\alpha}{2} & \beta \\ 0 & 0 & -1 \\ -\frac{\delta}{2} & \frac{\sqrt{3}\delta}{2} & -\eta \\ -\frac{\alpha}{2} & \frac{\sqrt{3}\alpha}{2} & \beta \\ -\frac{\delta}{2} & \frac{\sqrt{3}\delta}{2} & \eta \\ -\frac{\alpha}{2} & -\frac{\sqrt{3}\alpha}{2} & -\beta \\ \alpha & 0 & \beta \\ 0 & 0 & 1 \\ -\frac{\alpha}{2} & \frac{\sqrt{3}\alpha}{2} & -\beta \\ \delta & 0 & \eta \end{bmatrix}$$

```
> UpdatePlan(N14_1,"N14_1");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N14_1,1,0);
```

$$Var := 1 + \frac{7 \, var_3^2}{1 + 3 \, \eta^2 + 3 \, \beta^2} + \frac{14}{3} \frac{var_2^2}{\delta^2 + \alpha^2} + \frac{14}{3} \frac{var_1^2}{\delta^2 + \alpha^2}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},[Var]);
      Cond := [3α² + 3δ² - 2 - 6β² - 6η²]

> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2-1,delta^2+eta^2-1],
{alpha,beta,delta,eta},
{alpha,beta,delta,eta,alpha-1,beta-1,delta-1,eta-1});

      Cond_gsolve := {[[δ² + η² - 1, 9β² + 9η² - 4, 9α² - 9η² - 5], plex(α, β, δ, η),
{α, β, α - 1, β - 1, δ - 1, η - 1, δ, η}]}

> Cond:=[op(map(mem -> if mem<>0 then return mem end
if,eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,N14_1),2),radic=sqrt(3)))),
alpha^2+beta^2-1,delta^2+eta^2-1];

      Cond := [3α² + 3δ² - 2 - 6β² - 6η², 3δ³ - 3αδ² + 2α + 6αβ² + 6αη²,
-18δ⁴ + 27δ²β² - 54η²β² + 45η²δ² - 18η⁴ - 18β² + 12δ² - 24η² + 2,
18β⁴ - 9δ⁴ + 36η²δ² - 18η⁴ + 6δ² - 12η² - 2, α² + β² - 1, δ² + η² - 1]

> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2-1,delta^2+eta^2-1],
{alpha,beta,delta,eta},
{alpha,beta,delta,eta,alpha-1,beta-1,delta-1,eta-1});

      Cond_gsolve := {}
```

Pas de solutions.

11.19.2. 2ème essai

```
> N14_2:=OrbToMatrix(OrbPG(Vector([alpha,nu,beta]),GrN14) union
OrbPG(Vector([delta,0,eta]),GrN14) union
OrbPG(Vector([0,0,1]),GrN14));
```

$$N14_2 := \begin{bmatrix} \alpha & -\nu & -\beta \\ -\frac{\delta}{2} & \frac{\sqrt{3}\delta}{2} & \eta \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\nu}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\nu}{2} & \beta \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\nu}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\nu}{2} & -\beta \\ \alpha & \nu & \beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\nu}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\nu}{2} & \beta \\ \delta & 0 & -\eta \\ 0 & 0 & -1 \\ -\frac{\delta}{2} & -\frac{\sqrt{3}\delta}{2} & -\eta \\ \delta & 0 & \eta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\nu}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\nu}{2} & -\beta \\ -\frac{\delta}{2} & -\frac{\sqrt{3}\delta}{2} & \eta \\ 0 & 0 & 1 \\ -\frac{\delta}{2} & \frac{\sqrt{3}\delta}{2} & -\eta \end{bmatrix}$$

```
> UpdatePlan(N14_2,"N14_2");
```

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N14_2,1,0);

$$Var := 1 + \frac{7 var_3^2}{1 + 3 \eta^2 + 3 \beta^2} + \frac{14}{3} \frac{var_2^2}{\alpha^2 + \nu^2 + \delta^2} + \frac{14}{3} \frac{var_1^2}{\alpha^2 + \nu^2 + \delta^2}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

$$Cond := [-6 \beta^2 - 6 \eta^2 - 2 + 3 \alpha^2 + 3 \nu^2 + 3 \delta^2]$$

> Cond_gsolve:=gsolve([op(Cond),alpha^2+nu^2+beta^2-1,delta^2+eta^2-1],
{alpha,beta,delta,eta,nu},
{alpha,beta,delta,eta,nu,nu-1,alpha-1,beta-1,delta-1,eta-1});

$$Cond_gsolved := \{[\delta^2 + \eta^2 - 1, 9 \beta^2 + 9 \eta^2 - 4, 9 \alpha^2 + 9 \nu^2 - 9 \eta^2 - 5], \text{plex}(\alpha, \nu, \beta, \delta, \eta), \\ \{\alpha, \nu, \beta, \alpha - 1, \beta - 1, \delta - 1, \eta - 1, \delta, \eta, \nu - 1\}\}$$

> Sols:=map(L->solve(
{op(convert(L[1],set)),alpha<>0,beta<>0,delta<>0,eta<>0,nu<>0,nu<1,alpha<1,beta<1,delta<1,eta<1},{alpha,beta,nu,delta,eta
}),Cond_gsolve);

$$Sols := \{\{\alpha = \frac{1}{3} \text{RootOf}(_Z^2 + 9 \nu^2 - 9 \eta^2 - 5, \text{label} = _L34), \delta = \text{RootOf}(_Z^2 + \eta^2 - 1), \\ \beta = \frac{1}{3} \text{RootOf}(_Z^2 + 9 \eta^2 - 4, \text{label} = _L33), \nu = \nu, \eta = \eta\}\}$$

> Cond:=[op(map(mem->if mem<>0 then return mem end
if,convert(convert(eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,N14_2),2),r
adic=sqrt(3)),set),list)))];

$$Cond := [6 \alpha^2 \nu \beta + 2 \nu^3 \beta - 6 \nu \beta^3 - 6 \eta^2 \nu \beta - 2 \nu \beta + 3 \nu \beta \delta^2, \\ -18 \delta^4 + 27 \delta^2 \beta^2 - 54 \eta^2 \beta^2 + 45 \eta^2 \delta^2 - 18 \eta^4 - 18 \beta^2 + 12 \delta^2 - 24 \eta^2 + 2, \\ -6 \beta^2 - 6 \eta^2 - 2 + 3 \alpha^2 + 3 \nu^2 + 3 \delta^2, 18 \beta^4 - 9 \delta^4 + 36 \eta^2 \delta^2 - 18 \eta^4 + 6 \delta^2 - 12 \eta^2 - 2, \\ 128 - 216 \alpha^2 \nu^2 - 576 \eta^2 \delta^2 + 192 \eta^2 - 96 \delta^2 - 54 \alpha^2 + 576 \eta^4 - 108 \nu^4 + 36 \delta^4 \\ - 162 \alpha^2 \beta^2 - 162 \alpha^2 \eta^2 + 81 \delta^2 \alpha^2 + 27 \alpha \delta^3, 32 + 108 \alpha^2 \nu^2 - 54 \nu^2 \beta^2 - 54 \eta^2 \nu^2 \\ - 144 \eta^2 \delta^2 + 48 \eta^2 - 24 \delta^2 - 36 \alpha^2 + 144 \eta^4 + 9 \delta^4 - 18 \nu^2 - 108 \alpha^2 \beta^2 - 108 \alpha^2 \eta^2 \\ + 54 \delta^2 \alpha^2 - 27 \alpha \delta^3 + 27 \nu^2 \delta^2, \delta^3 + 4 \alpha^3 + 3 \alpha \delta^2 - 6 \alpha \beta^2 - 6 \alpha \eta^2 - 2 \alpha, \\ 3 \delta^3 - 3 \alpha \delta^2 + 6 \alpha \beta^2 + 6 \alpha \eta^2 - 12 \alpha \nu^2 + 2 \alpha, \\ 3 \nu \beta \delta^2 + 4 \nu^3 \beta - 6 \nu \beta^3 - 6 \eta^2 \nu \beta - 2 \nu \beta]$$

> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,delta^2+eta^2-1],
{alpha,beta,delta,eta,nu},
{alpha,beta,delta,eta,nu,nu-1,alpha-1,beta-1,delta-1,eta-1});

$$Cond_gsolved := \{\}$$

Pas de solution.

> TempGr:=Diedn(3):GrN14:=Plus(
{seq(DiagonalMatrix([TempGr[Indice],Vector([1])]),Indice=1..nops(Died
n(3))),seq(DiagonalMatrix([TempGr[Indice],Vector([-1])]),Indice=1..nop
s(Diedn(3)))});

$$GrN14 := \left\{ \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> unassign('alpha','beta','eta','delta','nu','mu');
> N14:=OrbToMatrix(OrbPG(Vector([alpha,nu,beta]),GrN14) union
OrbPG(Vector([delta,mu,eta]),GrN14) union
OrbPG(Vector([0,0,1]),GrN14));
```

$$N14 := \begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{3}\nu}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\nu}{2} & -\beta \\ \alpha & -\nu & -\beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\nu}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\nu}{2} & -\beta \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\nu}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\nu}{2} & \beta \\ \alpha & \nu & \beta \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\nu}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\nu}{2} & \beta \\ -\frac{\delta}{2} + \frac{\sqrt{3}\mu}{2} & \frac{\sqrt{3}\delta}{2} + \frac{\mu}{2} & -\eta \\ \delta & -\mu & -\eta \\ -\frac{\delta}{2} - \frac{\sqrt{3}\mu}{2} & -\frac{\sqrt{3}\delta}{2} + \frac{\mu}{2} & -\eta \\ 0 & 0 & 1 \\ -\frac{\delta}{2} - \frac{\sqrt{3}\mu}{2} & \frac{\sqrt{3}\delta}{2} - \frac{\mu}{2} & \eta \\ -\frac{\delta}{2} + \frac{\sqrt{3}\mu}{2} & -\frac{\sqrt{3}\delta}{2} - \frac{\mu}{2} & \eta \\ \delta & \mu & \eta \\ 0 & 0 & -1 \end{bmatrix}$$

```
> UpdatePlan(N14,"N14");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N14,1,0);
```

$$Var := 1 + \frac{7 var_3^2}{1 + 3\eta^2 + 3\beta^2} + \frac{14}{3} \frac{var_2^2}{\mu^2 + \nu^2 + \delta^2 + \alpha^2} + \frac{14}{3} \frac{var_1^2}{\mu^2 + \nu^2 + \delta^2 + \alpha^2}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},[Var]);
```

$$Cond := [3\mu^2 + 3\nu^2 + 3\delta^2 + 3\alpha^2 - 2 - 6\eta^2 - 6\beta^2]$$

```
> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,delta^2+eta^2+mu^2-1],{alpha,beta,delta,eta,mu,nu},
{alpha,beta,delta,eta,mu,nu,alpha-1,beta-1,delta-1,eta-1,mu-1,nu-1
});
```

$Cond_gsolved := \{[\delta^2 + \eta^2 + \mu^2 - 1, 9\beta^2 + 9\eta^2 - 4, 9\alpha^2 + 9\nu^2 - 9\eta^2 - 5],$
 $plex(\alpha, \nu, \beta, \delta, \eta, \mu), \{\alpha, \nu, \beta, \alpha - 1, \beta - 1, \delta - 1, \eta - 1, \delta, \eta, \mu, \mu - 1, \nu - 1\}]\}$

```
> Cond:=[op(map(mem->if mem<>0 then return mem end
if,eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,N14),2),radic=sqrt(3))))];

Cond := [3 μ2 + 3 ν2 + 3 δ2 + 3 α2 - 2 - 6 η2 - 6 β2, 3 μ2 + 3 ν2 + 3 δ2 + 3 α2 - 2 - 6 η2 - 6 β2,
-3 μ2 δ + δ3 + 4 α3 + 3 α δ2 + 3 α μ2 - 2 α - 6 α η2 - 6 α β2,
3 α δ2 - δ3 + 2 α3 + 6 α ν2 - 6 α β2 - 6 α η2 + 3 α μ2 + 3 μ2 δ - 2 α,
-9 μ2 η2 + 9 μ2 β2 - 12 β4 + 9 δ2 β2 - 18 η2 β2 - 9 η2 δ2 + 6 η4 - 6 β2 + 2,
-μ3 η + 2 ν3 β + 6 α2 ν β + 3 δ2 μ η - 2 ν β - 6 η2 ν β - 6 ν β3 + 3 ν β δ2 + 3 μ2 β ν,
-9 μ2 η2 + 9 μ2 β2 - 12 β4 + 9 δ2 β2 - 18 η2 β2 - 9 η2 δ2 + 6 η4 - 6 β2 + 2,
3 ν β δ2 - 3 δ2 μ η + 4 ν3 β - 6 ν β3 - 6 η2 ν β + 3 μ2 β ν + μ3 η - 2 ν β,
9 μ4 + 18 δ2 μ2 - 36 μ2 η2 - 18 β4 + 9 δ4 - 36 η2 δ2 + 18 η4 - 6 μ2 - 6 δ2 + 12 η2 + 2,
14 + 54 μ2 β2 + 18 α2 ν2 - 54 μ2 η2 - 18 ν2 β2 - 18 η2 ν2 - 90 η2 δ2 + 36 η2 - 12 δ2
- 9 μ4 - 18 μ2 ν2 - 18 α2 + 78 η4 + 9 δ4 - 6 ν2 - 54 α2 β2 - 54 α2 η2 + 18 δ2 β2
+ 18 δ2 α2 + 18 α4 - 30 β4, 4 - 18 μ2 β2 + 36 δ2 μ2 - 18 μ2 - 90 μ2 η2 - 18 ν2 β2
- 18 η2 ν2 - 54 η2 δ2 + 24 η2 - 6 δ2 + 27 μ4 + 18 μ2 ν2 + 6 α2 + 36 η4 + 9 ν4 + 9 δ4
- 6 ν2 + 18 α2 β2 + 18 α2 η2 + 18 δ2 β2 - 18 δ2 α2 - 9 α4 - 36 β4]
```

```
> convert(Cond,set);

{14 + 54 μ2 β2 + 18 α2 ν2 - 54 μ2 η2 - 18 ν2 β2 - 18 η2 ν2 - 90 η2 δ2 + 36 η2 - 12 δ2 - 9 μ4
- 18 μ2 ν2 - 18 α2 + 78 η4 + 9 δ4 - 6 ν2 - 54 α2 β2 - 54 α2 η2 + 18 δ2 β2 + 18 δ2 α2
+ 18 α4 - 30 β4, 3 μ2 + 3 ν2 + 3 δ2 + 3 α2 - 2 - 6 η2 - 6 β2, 4 - 18 μ2 β2 + 36 δ2 μ2
- 18 μ2 - 90 μ2 η2 - 18 ν2 β2 - 18 η2 ν2 - 54 η2 δ2 + 24 η2 - 6 δ2 + 27 μ4 + 18 μ2 ν2
+ 6 α2 + 36 η4 + 9 ν4 + 9 δ4 - 6 ν2 + 18 α2 β2 + 18 α2 η2 + 18 δ2 β2 - 18 δ2 α2 - 9 α4
- 36 β4, -3 μ2 δ + δ3 + 4 α3 + 3 α δ2 + 3 α μ2 - 2 α - 6 α η2 - 6 α β2,
3 α δ2 - δ3 + 2 α3 + 6 α ν2 - 6 α β2 - 6 α η2 + 3 α μ2 + 3 μ2 δ - 2 α,
-9 μ2 η2 + 9 μ2 β2 - 12 β4 + 9 δ2 β2 - 18 η2 β2 - 9 η2 δ2 + 6 η4 - 6 β2 + 2,
-μ3 η + 2 ν3 β + 6 α2 ν β + 3 δ2 μ η - 2 ν β - 6 η2 ν β - 6 ν β3 + 3 ν β δ2 + 3 μ2 β ν,
3 ν β δ2 - 3 δ2 μ η + 4 ν3 β - 6 ν β3 - 6 η2 ν β + 3 μ2 β ν + μ3 η - 2 ν β,
9 μ4 + 18 δ2 μ2 - 36 μ2 η2 - 18 β4 + 9 δ4 - 36 η2 δ2 + 18 η4 - 6 μ2 - 6 δ2 + 12 η2 + 2}
```

```
> Cond_gsolved:=gsolve([op(
%),alpha^2+beta^2+nu^2-1,delta^2+eta^2+mu^2-1],
{alpha,beta,delta,eta,mu,nu},
{alpha,beta,delta,eta,mu,nu,alpha-1,beta-1,delta-1,eta-1,mu-1,nu-1
});
```


$$\begin{aligned}
\text{Cond_gsolved} := \{ & [[4251528000 \mu^{12} - 9920232000 \mu^{10} + 8597534400 \mu^8 - 3495555000 \mu^6 \\
& + 691737165 \mu^4 - 61003530 \mu^2 + 1789333, 45610740 \eta^2 + 52435512000 \mu^{10} \\
& - 107327462400 \mu^8 + 75744849600 \mu^6 - 22210773480 \mu^4 + 2620994355 \mu^2 \\
& - 106087079, 45610740 \delta^2 - 52435512000 \mu^{10} + 107327462400 \mu^8 \\
& - 75744849600 \mu^6 + 22210773480 \mu^4 - 2575383615 \mu^2 + 60476339, 5067860 \beta^2 \\
& - 5826168000 \mu^{10} + 11925273600 \mu^8 - 8416094400 \mu^6 + 2467863720 \mu^4 \\
& - 291221595 \mu^2 + 9535071, 263528720 \nu^3 + 227220552000 \mu^{10} \nu \\
& - 465085670400 \mu^8 \nu + 328227681600 \mu^6 \nu - 96246685080 \mu^4 \nu \\
& + 11357642205 \mu^2 \nu - 569514309 \nu + 1228691592000 \mu^{11} \beta \eta \\
& - 2597106240000 \mu^9 \beta \eta + 1918670846400 \mu^7 \beta \eta - 601930488600 \mu^5 \beta \eta \\
& + 80242414245 \mu^3 \beta \eta - 4181858205 \beta \mu \eta, 4150577340 \alpha \\
& - 105454900512000 \mu^{10} \delta \nu^2 + 212645684160000 \mu^8 \delta \nu^2 \\
& - 146725522070400 \mu^6 \delta \nu^2 + 41491223373600 \mu^4 \delta \nu^2 - 4359072446820 \mu^2 \delta \nu^2 \\
& + 107880574620 \delta \nu^2 + 379066236480000 \mu^{11} \beta \eta \delta \nu \\
& - 781436515104000 \mu^9 \beta \eta \delta \nu + 543979228464000 \mu^7 \beta \eta \delta \nu \\
& - 146115513532800 \beta \mu^5 \eta \delta \nu + 12336892709400 \beta \mu^3 \eta \delta \nu \\
& - 4421704140 \beta \mu \eta \delta \nu + 54981232488000 \mu^{10} \delta - 110045892902400 \mu^8 \delta \\
& + 75060756388800 \mu^6 \delta - 20817595988280 \mu^4 \delta + 2132910011505 \mu^2 \delta \\
& - 52721481449 \delta], \text{plex}(\alpha, \nu, \beta, \delta, \eta, \mu), \{\nu, \beta, 105454900512000 \mu^{10} \delta \nu^2 \\
& - 212645684160000 \mu^8 \delta \nu^2 + 146725522070400 \mu^6 \delta \nu^2 - 41491223373600 \mu^4 \delta \nu^2 \\
& + 4359072446820 \mu^2 \delta \nu^2 - 107880574620 \delta \nu^2 - 379066236480000 \mu^{11} \beta \eta \delta \nu \\
& + 781436515104000 \mu^9 \beta \eta \delta \nu - 543979228464000 \mu^7 \beta \eta \delta \nu \\
& + 146115513532800 \beta \mu^5 \eta \delta \nu - 12336892709400 \beta \mu^3 \eta \delta \nu \\
& + 4421704140 \beta \mu \eta \delta \nu - 54981232488000 \mu^{10} \delta + 110045892902400 \mu^8 \delta \\
& - 75060756388800 \mu^6 \delta + 20817595988280 \mu^4 \delta - 2132910011505 \mu^2 \delta \\
& + 52721481449 \delta, \beta - 1, \delta - 1, \eta - 1, \delta, -4150577340 + 105454900512000 \mu^{10} \delta \nu^2 \\
& - 212645684160000 \mu^8 \delta \nu^2 + 146725522070400 \mu^6 \delta \nu^2 - 41491223373600 \mu^4 \delta \nu^2 \\
& + 4359072446820 \mu^2 \delta \nu^2 - 107880574620 \delta \nu^2 - 379066236480000 \mu^{11} \beta \eta \delta \nu \\
& + 781436515104000 \mu^9 \beta \eta \delta \nu - 543979228464000 \mu^7 \beta \eta \delta \nu \\
& + 146115513532800 \beta \mu^5 \eta \delta \nu - 12336892709400 \beta \mu^3 \eta \delta \nu \\
& + 4421704140 \beta \mu \eta \delta \nu - 54981232488000 \mu^{10} \delta + 110045892902400 \mu^8 \delta \\
& - 75060756388800 \mu^6 \delta + 20817595988280 \mu^4 \delta - 2132910011505 \mu^2 \delta \\
& + 52721481449 \delta, \eta, \mu, -1574640 \mu^6 + 1564920 \mu^4 - 392364 \mu^2 + 21749, 3 \eta - 1, \\
& 3 \eta + 1, 5067860 \nu^2 + 5826168000 \mu^{10} - 11925273600 \mu^8 + 8416094400 \mu^6 \\
& - 2467863720 \mu^4 + 291221595 \mu^2 - 14602931, \mu - 1, \nu - 1\} \}
\end{aligned}$$

```
> Sols:=map(L->solve({op(L[1])},{alpha,beta,eta,delta,nu,mu
  }),Cond_gsolved):
```

```
> assign(op(1,Sols));
```

```
> N14:=(OrbToMatrix(OrbPG(Vector([alpha,nu,beta]),GrN14) union
  OrbPG(Vector([delta,mu,eta]),GrN14) union
  OrbPG(Vector([0,0,1]),GrN14))):
```

```
> UpdatePlan(Map(evalf,N14),"N14");
```

“Update complete”

```
> pointplot3d(N14,symbol=box,scaling=constrained,shading=Z,orientation=[32,74]);
```

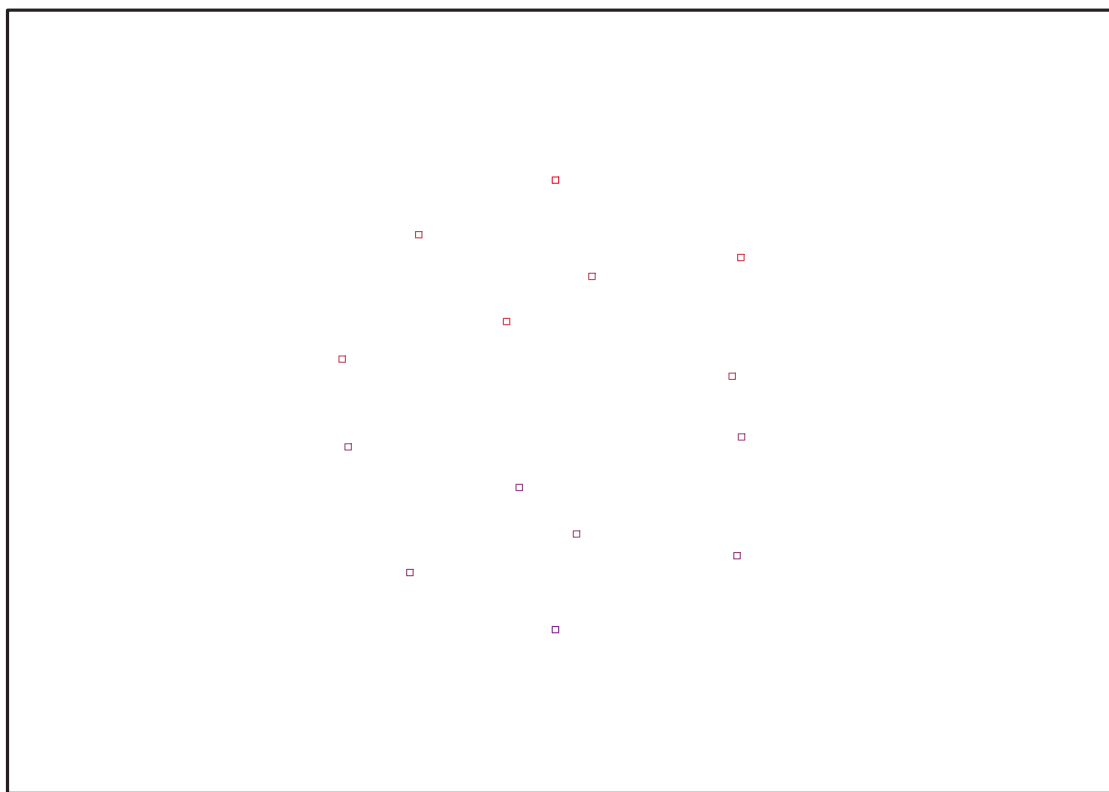


FIG. 11.49 – Graphique : Support du plan N14.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N14,listlist)))]),'rational',5)));
```

$Graph := \text{POLYTOPE}(3, 3, 14, 24)$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z,orientation=[32,74]);
```

```
> map(convert,[evalf(op(convert(N14,listlist)))],'rational',10);
```

$$\begin{aligned} & \left[\left[\frac{-18943}{21539}, \frac{13961}{37889}, \frac{-30157}{100105} \right], \left[\frac{21327}{176794}, \frac{-40445}{42759}, \frac{-30157}{100105} \right], \left[\frac{31967}{42126}, \frac{18681}{32353}, \frac{-30157}{100105} \right], \right. \\ & \left[\frac{-11003}{61935}, \frac{29806}{38015}, \frac{-10338}{17383} \right], \left[\frac{31967}{42126}, \frac{-18681}{32353}, \frac{30157}{100105} \right], \left[\frac{21327}{176794}, \frac{40445}{42759}, \frac{30157}{100105} \right], \\ & \left[\frac{-18943}{21539}, \frac{-13961}{37889}, \frac{30157}{100105} \right], [0, 0, 1], [0, 0, -1], \left[\frac{-11003}{61935}, \frac{-29806}{38015}, \frac{10338}{17383} \right], \\ & \left[\frac{17139}{22321}, \frac{14499}{60875}, \frac{10338}{17383} \right], \left[\frac{-18935}{32083}, \frac{63455}{116243}, \frac{10338}{17383} \right], \left[\frac{-18935}{32083}, \frac{-63455}{116243}, \frac{-10338}{17383} \right], \\ & \left. \left[\frac{17139}{22321}, \frac{-14499}{60875}, \frac{-10338}{17383} \right] \right] \end{aligned}$$

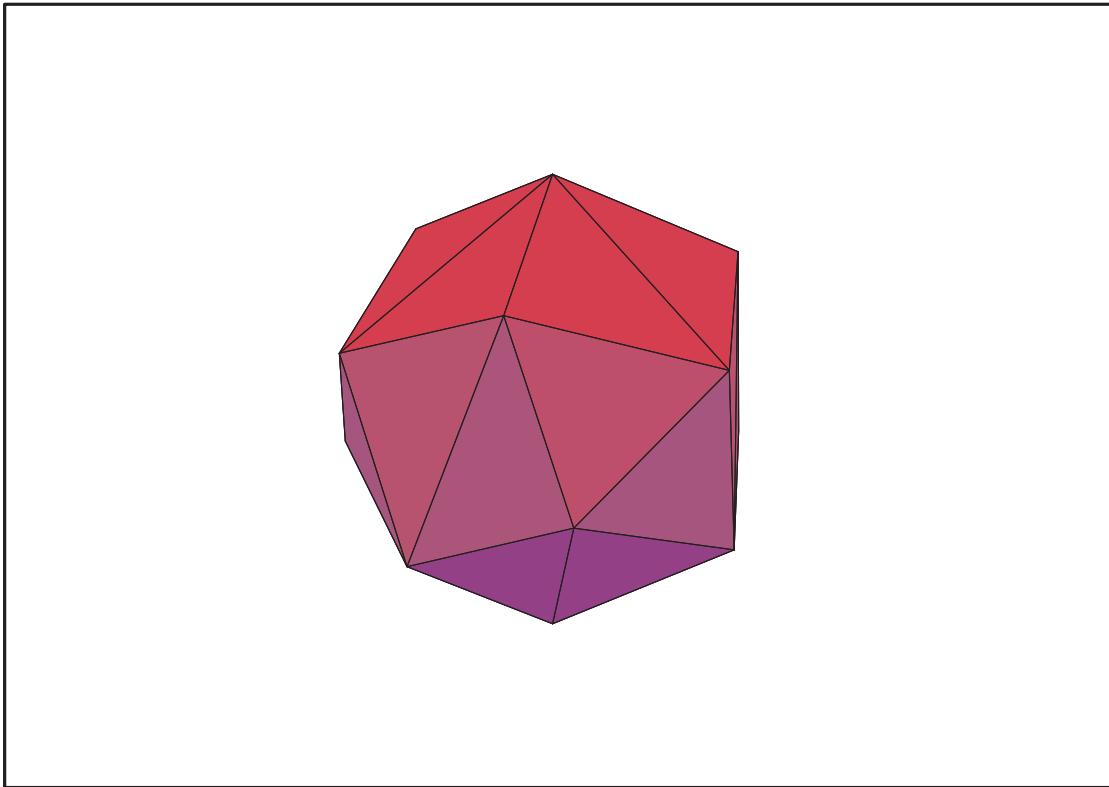


FIG. 11.50 – Graphique : Enveloppe convexe du support du plan N14.

11.20. N15

```
> unassign('alpha','beta','eta','delta','nu','mu');
```

```
> GrN15:=Rned(Diedn(5),3):
```

```
> N15:=OrbToMatrix(OrbPG(Vector([alpha,0,beta]),GrN15) union  
OrbPG(Vector([1,0,0]),GrN15));
```

$$N15 := \begin{bmatrix} \%1 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & \beta \\ \%2 & \%3\alpha & \beta \\ \alpha & 0 & -\beta \\ \%1 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & -\beta \\ \%2 & \%4\alpha & -\beta \\ \%1 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & -\beta \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%2 & \%3\alpha & -\beta \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ \%1 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} & \beta \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ 1 & 0 & 0 \\ \alpha & 0 & \beta \\ \%2 & \%4\alpha & \beta \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \end{bmatrix}$$

$$\%1 := \left(-\frac{1}{4} + \frac{\sqrt{5}}{4}\right)\alpha$$

$$\%2 := \left(-\frac{1}{4} - \frac{\sqrt{5}}{4}\right)\alpha$$

$$\%3 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%4 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

> UpdatePlan(N15,"N15");

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N15,1,0);

$$Var := 1 + \frac{3}{2} \frac{var_3^2}{\beta^2} + \frac{6}{1+2\alpha^2} \frac{var_2^2}{\beta^2} + \frac{6}{1+2\alpha^2} \frac{var_1^2}{\beta^2}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

$$Cond := [1 + 2\alpha^2 - 4\beta^2]$$

> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2-1],[alpha,beta]);

$$Cond_gsolved := \{[2\beta^2 - 1, 2\alpha^2 - 1], \text{plex}(\alpha, \beta), \{\}\}$$

> Sols:=map(L->solve({op(convert(L[1],set))},{alpha,beta}),Cond_gsolve);

$$Sols := \{\{\alpha = \text{RootOf}(2_Z^2 - 1, \text{label} = _L424), \beta = \text{RootOf}(2_Z^2 - 1, \text{label} = _L425)\}\}$$

> allvalues(Sols);

$$\{\{\alpha = \frac{\sqrt{2}}{2}, \beta = \frac{\sqrt{2}}{2}\}\}, \{\{\alpha = \frac{\sqrt{2}}{2}, \beta = -\frac{\sqrt{2}}{2}\}\}, \{\{\alpha = -\frac{\sqrt{2}}{2}, \beta = \frac{\sqrt{2}}{2}\}\}, \{\{\alpha = -\frac{\sqrt{2}}{2}, \beta = -\frac{\sqrt{2}}{2}\}\}$$

> beta := 1/sqrt(2); alpha := 1/sqrt(2);

$$\beta := \frac{\sqrt{2}}{2}$$

$$\alpha := -\frac{\sqrt{2}}{2}$$

> N15:=simplify(OrbToMatrix(SortPoints(OrbPG(Vector([alpha,0,beta])),GrN15) union OrbPG(Vector([1,0,0]),GrN15))));

$$N15 := \begin{bmatrix} -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}}{4} & \frac{\sqrt{2}}{2} \\ -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}}{4} & \frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & \frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & -\frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & -\frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & \frac{\sqrt{2}}{2} \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

> UpdatePlan(N15,"N15");

Error, (in UpdatePlan) A file named D:\Données\Thèse\Maple\MapleData\N15.txt doesn't exist

> pointplot3d(N15,symbol=box,scaling=constrained,shading=Z);

> Graph:=convhull(op(map(convert,[evalf(op(convert(N15,listlist)))],'rational',5))));

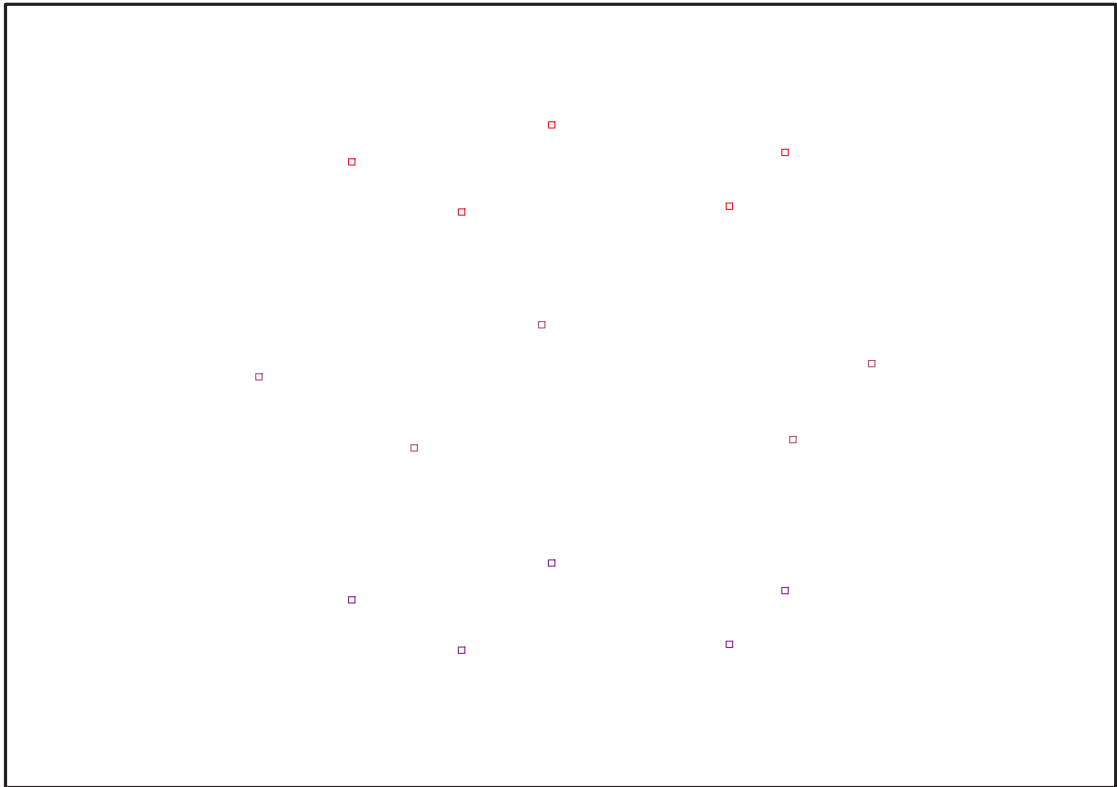


FIG. 11.51 – Graphique : Support du plan N15.

$$Graph := \text{POLYTOPE}(3, 3, 15, 20)$$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> beta := 1/sqrt(2); alpha := -1/sqrt(2);
```

$$\beta := \frac{\sqrt{2}}{2}$$
$$\alpha := -\frac{\sqrt{2}}{2}$$

```
> N15bis:=simplify(OrbToMatrix(SortPoints(OrbPG(Vector([alpha,0,beta]),  
GrN15) union OrbPG(Vector([1,0,0]),GrN15))));
```

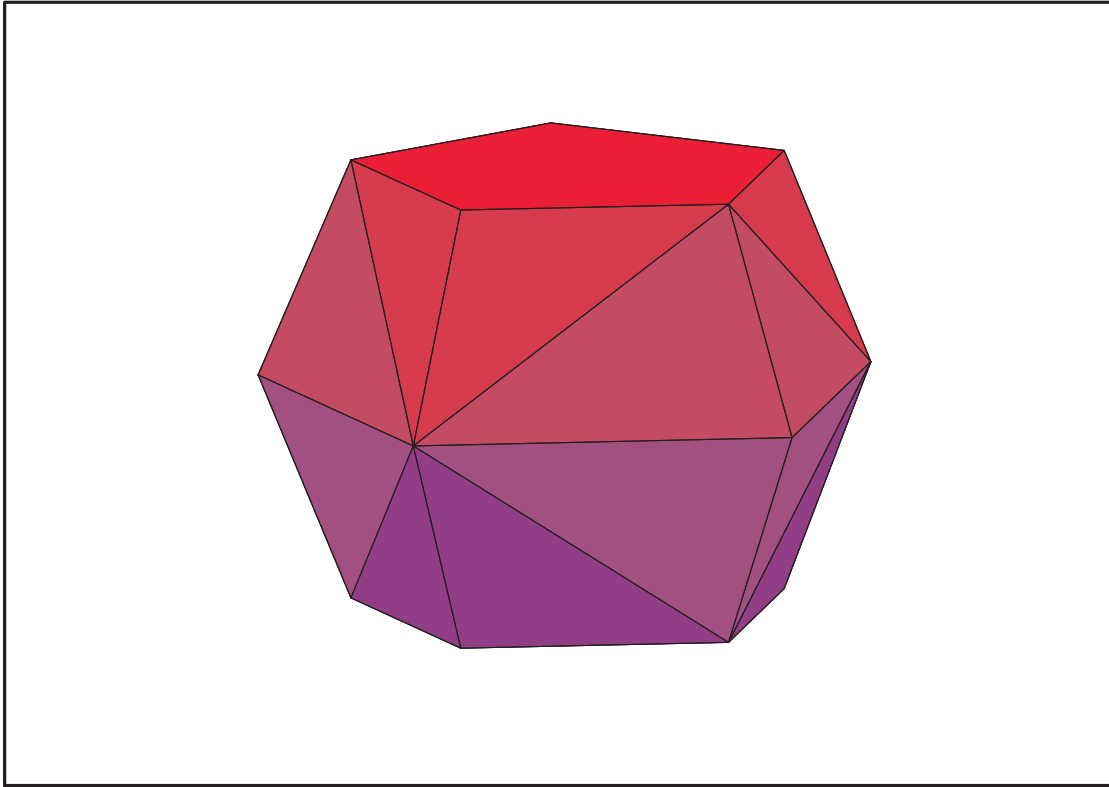


FIG. 11.52 – Graphique : Enveloppe convexe du support du plan N15.

$$N15bis := \begin{bmatrix} -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}}{4} & \frac{\sqrt{2}}{2} \\ -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{(\sqrt{5}-1)\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}}{4} & \frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & \frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & -\frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & -\frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & \frac{\sqrt{2}}{2} \\ \frac{(1+\sqrt{5})\sqrt{2}}{8} & \frac{\sqrt{5+\sqrt{5}}(\sqrt{5}-1)}{8} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

```
> CreatePlan(N15bis,"N15bis");  
> pointplot3d(N15bis,symbol=box,scaling=constrained,shading=Z);
```

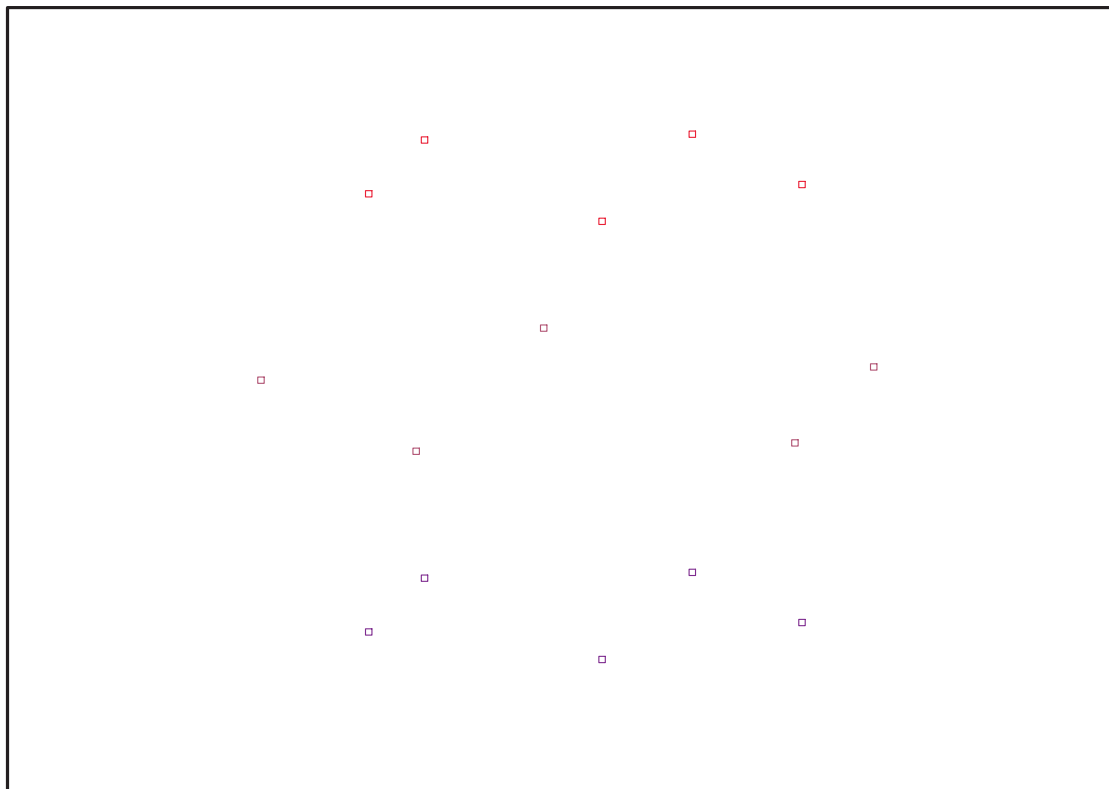


FIG. 11.53 – Graphique : Support du plan N15bis.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N15,listlist)))],'ra  
tional',5))));
```

$Graph := \text{POLYTOPE}(3, 3, 15, 22)$

```
> plotdes:=draw(Graph):  
> display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN15:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N15));
```

```
> Gens:=Generators(IdealN15);
```

```
> info_iop:=3;
```

```
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N15),  
listlist),Vector)));
```

$info_iop := 3$

1
2
3
4

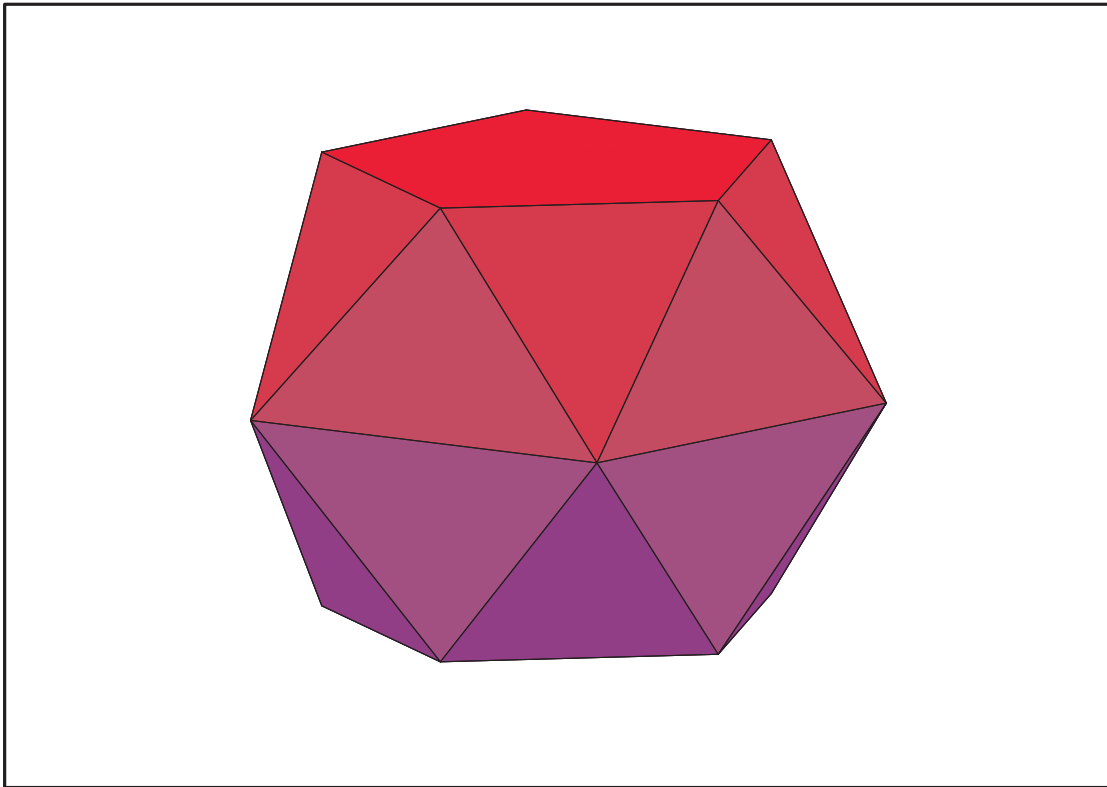


FIG. 11.54 – Graphique : Enveloppe convexe du support du plan N15bis.

5
6
7
8
9
9
9
10
11
12
13
14
15
15
15
15
15
15
15

```
> EstSetTdeg:=[1, z, y, x, z^2, y*z, z*x, y^2, x*y, y*z^2, z^2*x,
z*y^2, y*z*x, y^3, x*y^2];
EstSetTdeg := [1, z, y, x, z^2, y z, z x, y^2, x y, y z^2, z^2 x, z y^2, y z x, y^3, x y^2]
```

11.20.1. tdeg(x,y,z)

```

> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N15_tdeg:=[op(EstSet(Div,Frontiere_tdeg))]:A:=poly_algebra(x,y,
z):T:=termorder(A,tdeg(x,y,z)):EstSet_N15_tdeg:=sort(EstSet_N15_tdeg,(
t1,t2)->testorder(t1,t2,T));nops(EstSet_N15_tdeg);

      EstSet_N15_tdeg := [1, z, y, x, z2, y z, z x, y2, x y, y z2, z2 x, z y2, y z x, y3, x y2]
                                15

> X_D_ModelleSat(N15,[x,y,z],EstSet_N15_tdeg);
      [14 x 14 Matrix  Data Type : anything  Storage : rectangular  Order : Fortran_order]

> M_D_ModelleSat(N15,[x,y,z],EstSet_N15_tdeg);
      [14 x 14 Matrix  Data Type : float[8]  Storage : rectangular  Order : Fortran_order]

> expand(Var_Blue_Modelle_Sat(N15,[x,y,z],EstSet_N15_tdeg));
      "M_D is not full rank"
      VarVar_Blue_D_Norm_Set

```

11.20.2. plex(x,y,z)

```

> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
> EstSet_N15_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(x,y,
z):T:=termorder(A,plex(x,y,z)):EstSet_N15_plex:=sort(EstSet_N15_plex,
(t1,t2)->testorder(t1,t2,T));nops(EstSet_N15_plex);

```

Warning, computation interrupted

```

> X_D_ModelleSat(N15,[x,y,z],EstSet_N15_plex);
      [11 x 11 Matrix  Data Type : anything  Storage : rectangular  Order : Fortran_order]

> M_D_ModelleSat(N15,[x,y,z],EstSet_N15_plex);
      [11 x 11 Matrix  Data Type : anything  Storage : rectangular  Order : Fortran_order]

> expand(Var_Blue_Modelle_Sat(N15,[x,y,z],EstSet_N15_plex));
      11 -  $\frac{225456}{325} y^2 z^4 + \frac{1224234}{4225} z^8 + \frac{3145824}{4225} z^6 y^2 - \frac{2513808}{4225} z^6 - \frac{92928}{845} z^2 y^4 + \frac{3145824}{4225} y^4 z^4$ 
      - 22 y2 -  $\frac{105501}{1690} z^2 + \frac{265628}{2535} y^2 z^2 + \frac{88}{3} y^4 + \frac{3106653}{8450} z^4$ 

```

11.20.3. Variance

Modèle de Degré 1

```

> Var:=Var_Norm_Blue_D_Centre(N15,1,0);
      Var := 1 + 3 var32 + 3 var22 + 3 var12

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
      1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32

> animate(plot3d,[Var,var[1]==-2..2,var[2]==-2..2],var[3]==-2..2):

```

Modèle de Degré 2

```
> Var:=(expand(Var_Norm_Blue_D_Centre(N15,2,1)));

Var := 15. - 27. var32 - 27. var22 - 27. var12 + 36. var22 var32 + 36. var12 var32
+ 44. var12 var22 + 24. var34 + 22. var24 + 22. var14

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));

10.00000000 var34 - 8.000000000 var32 %1 + 22.00000000 %12 - 27.00000000 %1
+ 15.00000000, %1 = var12 + var22 + var32
%1 := IndetDecompInKfi1
```

11.21. N16

```

> unassign('alpha','beta');

> GrN16:=Plus(Tetrahedron());

> N16:=OrbToMatrix(OrbPG(Vector([alpha,0,beta]),GrN16) union
OrbPG(Vector([0,0,1]),GrN16));nops(OrbPG(Vector([alpha,0,beta]),GrN16)
union OrbPG(Vector([0,0,1]),GrN16));

```

$$N16 := \begin{bmatrix} \frac{2\sqrt{2}}{3} & 0 & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \\ 0 & 0 & 1 \\ -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}\sqrt{3}}{3} & -\frac{1}{3} \\ \frac{5\alpha}{6} - \frac{\beta\sqrt{2}}{3} & -\frac{\sqrt{3}\alpha}{6} - \frac{\sqrt{2}\sqrt{3}\beta}{3} & \%1 \\ -\frac{\alpha}{6} - \frac{\beta\sqrt{2}}{3} & \frac{\sqrt{3}\alpha}{6} + \frac{\sqrt{2}\sqrt{3}\beta}{3} & \frac{2\sqrt{2}\alpha}{3} - \frac{\beta}{3} \\ -\frac{\alpha}{6} + \frac{2\beta\sqrt{2}}{3} & \frac{\sqrt{3}\alpha}{2} & \%1 \\ -\frac{\alpha}{6} - \frac{\beta\sqrt{2}}{3} & -\frac{\sqrt{3}\alpha}{6} - \frac{\sqrt{2}\sqrt{3}\beta}{3} & \frac{2\sqrt{2}\alpha}{3} - \frac{\beta}{3} \\ -\frac{\alpha}{6} + \frac{2\beta\sqrt{2}}{3} & -\frac{\sqrt{3}\alpha}{2} & \%1 \\ -\frac{\alpha}{2} & \frac{\sqrt{3}\alpha}{2} & \beta \\ -\frac{\alpha}{2} & -\frac{\sqrt{3}\alpha}{2} & \beta \\ -\frac{2\alpha}{3} - \frac{\beta\sqrt{2}}{3} & -\frac{\sqrt{3}\alpha}{3} + \frac{\sqrt{2}\sqrt{3}\beta}{3} & \%1 \\ \alpha & 0 & \beta \\ -\frac{2\alpha}{3} - \frac{\beta\sqrt{2}}{3} & \frac{\sqrt{3}\alpha}{3} - \frac{\sqrt{2}\sqrt{3}\beta}{3} & \%1 \\ \frac{5\alpha}{6} - \frac{\beta\sqrt{2}}{3} & \frac{\sqrt{3}\alpha}{6} + \frac{\sqrt{2}\sqrt{3}\beta}{3} & \%1 \\ \frac{\alpha}{3} + \frac{2\beta\sqrt{2}}{3} & 0 & \frac{2\sqrt{2}\alpha}{3} - \frac{\beta}{3} \end{bmatrix}$$

$$\%1 := -\frac{\sqrt{2}\alpha}{3} - \frac{\beta}{3}$$

16

```

> UpdatePlan(N16,"N16");

```

“Update complete”

```

> Var:=Var_Norm_Blue_D_Centre(N16,1,0);

```

$$Var := 1 + \frac{12 var_3^2}{1 + 3\alpha^2 + 3\beta^2} + \frac{12 var_2^2}{1 + 3\alpha^2 + 3\beta^2} + \frac{12 var_1^2}{1 + 3\alpha^2 + 3\beta^2}$$

```

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

```

$Cond := []$

```
> Cond:=map(mem -> if mem<>0 then return mem end
if,eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,sqrt(2)=radic2,N16),2),[rad
ic=sqrt(3),radic2=sqrt(2)]));
```

$$\begin{aligned} Cond := & [3\alpha^3\sqrt{2} - 9\alpha^2\beta + 2 + 6\beta^3, \\ & -1260\beta^3\sqrt{2}\alpha - 72 - 5076\beta^4 - 420\sqrt{2}\alpha + 7938\alpha^2\beta^2 - 1341\alpha^4 - 1620\beta, \\ & 450\beta^4\sqrt{2} - 783\alpha^2\beta^2\sqrt{2} + 126\beta\sqrt{2} - 54\alpha\beta^3 + 99\alpha^3\beta + 24\sqrt{2} - 18\alpha, \\ & 31500\beta^3\sqrt{2}\alpha^2 + 72\alpha + 126900\beta^4\alpha + 10500\sqrt{2}\alpha^2 - 10368\alpha^2\beta^2\sqrt{2} \\ & - 198450\beta^2\alpha^3 + 33525\alpha^5 - 2304\beta\sqrt{2} + 26400\beta^3\sqrt{2} + 40500\alpha\beta - 5184\alpha\beta^3 \\ & - 39600\beta\sqrt{2}\alpha^2 - 33696\alpha^3\beta + 11104\sqrt{2} + 26400\alpha^3, \\ & -2016\beta^3\sqrt{2}\alpha - 264 - 10800\beta^4 + 17388\alpha^2\beta^2 - 2313\alpha^4 - 3336\beta - 672\sqrt{2}\alpha, \\ & -77928\beta\sqrt{2} - 1101600\beta\sqrt{2}\alpha^2 - 350676\alpha^2\beta^2\sqrt{2} + 322728\sqrt{2} + 734400\beta^3\sqrt{2} \\ & + 734400\alpha^3 - 1139697\alpha^3\beta - 175338\alpha\beta^3 - 58446\alpha, -420657792 \\ & - 40700859264\alpha^4\beta - 18132894144\alpha^2\beta^2 + 13350569472\beta^4 - 2087223552\alpha^2 \\ & + 6500896560\alpha^4 + 6974136576\beta^3\sqrt{2}\alpha - 2782964736\sqrt{2}\alpha\beta + 5107676928\sqrt{2}\alpha \\ & - 43831694592\beta^3\alpha^2 + 25046682624\beta^5 + 4870847616\beta + 8348894208\beta^2] \end{aligned}$$

```
> Sols:=solve(convert(Cond,set) union
{-1<=alpha,alpha<=1,-1<=beta,beta<=1},{alpha,beta});
```

$$\begin{aligned} Sols := & \left\{ \alpha = \%3, \beta = \frac{2(-45\%3^4 + 72\%3^3\sqrt{2} + 8)}{288\%3^3\sqrt{2} + 2187\%3^6 + 72\%3^2 + 16} \right\}, \\ & \left\{ \alpha = \%2, \beta = \frac{2(-45\%2^4 + 72\%2^3\sqrt{2} + 8)}{288\%2^3\sqrt{2} + 2187\%2^6 + 72\%2^2 + 16} \right\} \\ \%1 := & 1594323_Z^{12} + 944784_Z^9\sqrt{2} - 157464_Z^8 + 264384_Z^6 + 31104_Z^5\sqrt{2} \\ & - 15552_Z^4 + 14592_Z^3\sqrt{2} + 4608_Z^2 + 2048 \\ \%2 := & \text{RootOf}(\%1, -0.4842650455) \\ \%3 := & \text{RootOf}(\%1, -0.9288778279) \end{aligned}$$

```
> alpha:=convert(RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+2
64384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+
2048,-.9288778279),radical);
```

$$\alpha := \text{RootOf}(1594323_Z^{12} + 944784_Z^9\sqrt{2} - 157464_Z^8 + 264384_Z^6 + 31104_Z^5\sqrt{2} - 15552_Z^4 + 14592_Z^3\sqrt{2} + 4608_Z^2 + 2048, index = 7)$$

```
> alpha2:=convert(RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+
264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2
+2048,-.4842650455),radical);
```

$$\alpha_2 := \text{RootOf}(1594323_Z^{12} + 944784_Z^9\sqrt{2} - 157464_Z^8 + 264384_Z^6 + 31104_Z^5\sqrt{2} - 15552_Z^4 + 14592_Z^3\sqrt{2} + 4608_Z^2 + 2048, index = 6)$$

```
> beta:=convert(-2*(45*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.9288778279)^4-72*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.9288778279)^3*2^(1/2)-8)/(288*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.9288778279)^3*2^(1/2)+2187*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.9288778279)^6+72*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.9288778279)^2+16),radical);
```

$$\beta := -\frac{2(45\%1^4 - 72\%1^3\sqrt{2} - 8)}{288\%1^3\sqrt{2} + 2187\%1^6 + 72\%1^2 + 16}$$

$$\%1 := \text{RootOf}(1594323_Z^{12} + 944784_Z^9\sqrt{2} - 157464_Z^8 + 264384_Z^6 + 31104_Z^5\sqrt{2} - 15552_Z^4 + 14592_Z^3\sqrt{2} + 4608_Z^2 + 2048, \text{index} = 7)$$

```
> beta2:=convert(-2*(45*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.4842650455)^4-72*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.4842650455)^3*2^(1/2)-8)/(288*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.4842650455)^3*2^(1/2)+2187*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.4842650455)^6+72*RootOf(1594323*_Z^12+944784*_Z^9*2^(1/2)-157464*_Z^8+264384*_Z^6+31104*_Z^5*2^(1/2)-15552*_Z^4+14592*_Z^3*2^(1/2)+4608*_Z^2+2048,-.4842650455)^2+16),radical);
```

$$\beta_2 := -\frac{2(45\%1^4 - 72\%1^3\sqrt{2} - 8)}{288\%1^3\sqrt{2} + 2187\%1^6 + 72\%1^2 + 16}$$

$$\%1 := \text{RootOf}(1594323_Z^{12} + 944784_Z^9\sqrt{2} - 157464_Z^8 + 264384_Z^6 + 31104_Z^5\sqrt{2} - 15552_Z^4 + 14592_Z^3\sqrt{2} + 4608_Z^2 + 2048, \text{index} = 6)$$

```
> evalf([alpha,beta]);evalf([alpha2,beta2]);
[-0.9288778279, -0.1852327407]
[-0.4842650455, -0.8140101636]
```

```
> N16:=simplify(OrbToMatrix(SortPoints(OrbPG(Vector([alpha,0,beta]),GrN16) union OrbPG(Vector([0,0,1]),GrN16)))):
```

```
> UpdatePlan(N16,"N16");
```

“Update complete”

```
> pointplot3d(N16,symbol=box,scaling=constrained,shading=Z);
```

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N16,listlist))]),'rational',5))):
```

$$\text{Graph} := \text{POLYTOPE}(3, 3, 16, 28)$$

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

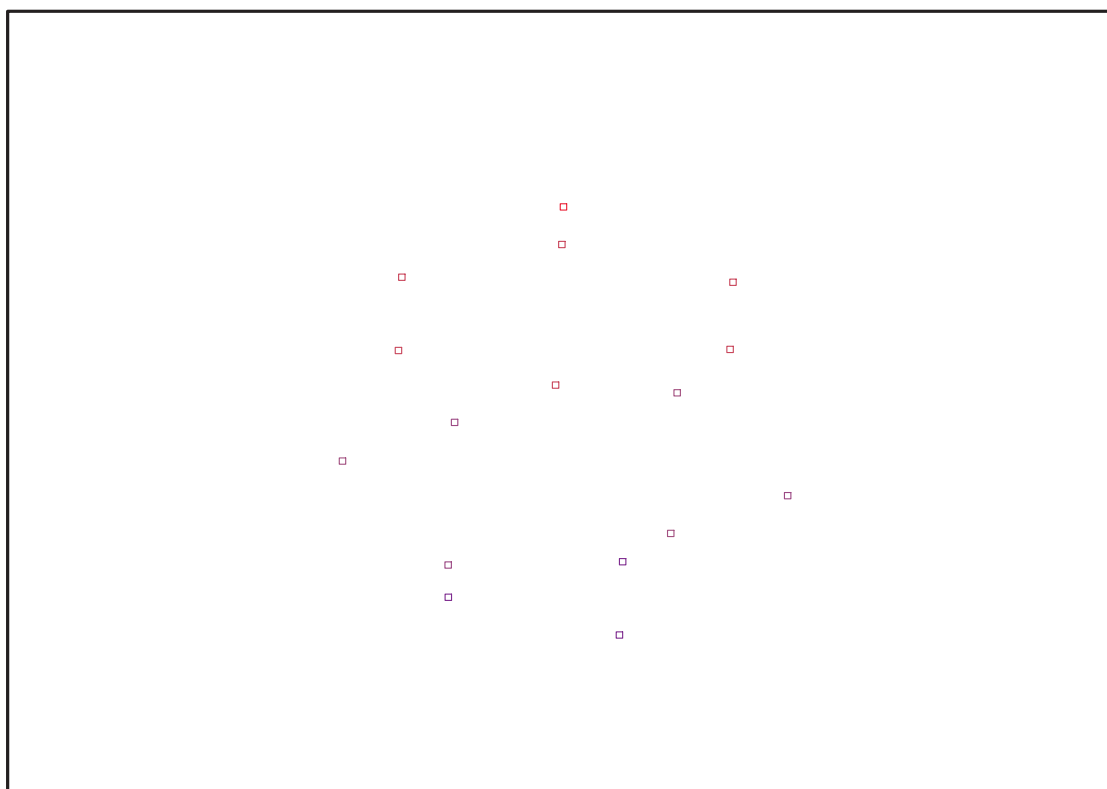


FIG. 11.55 – Graphique : Support du plan N16.

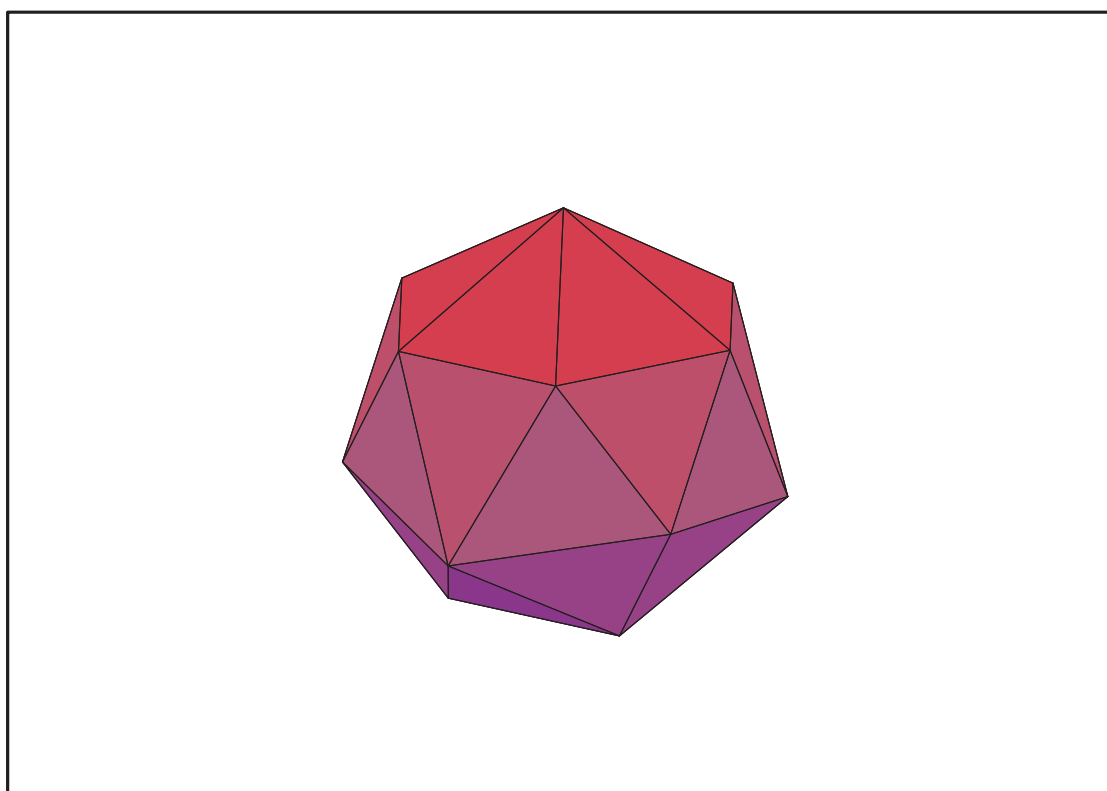


FIG. 11.56 – Graphique : Enveloppe convexe du support du plan N16.

```
> IdealN16:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N16));

> Gens:=Generators(IdealN16);

> info_iop:=3;
                                info_iop := 3

> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N16),
listlist),Vector)));
                                [1, z, y, x, z^2, y z, z x, y^2, x y, x^2, z^3, y z^2, z^2 x, z y^2, z x y, y^3]

> EstSetTdeg:=[1, z, y, x, z^2, y*z, z*x, y^2, x*y, x^2, z^3, y*z^2,
z^2*x, z*y^2, z*x*y, y^3];
                                EstSetTdeg := [1, z, y, x, z^2, y z, z x, y^2, x y, x^2, z^3, y z^2, z^2 x, z y^2, z x y, y^3]
```

L'expression complexe des coordonnées des points du plan pose problème puisque la confusion entre 1 et $x^2+y^2+z^2$ n'est pas détectée. Nous proposons une manière de remédier à cette nouvelle difficulté.

11.21.1. Variance

Modèle de Degré 1

```
> Var:=evalf(Var_Norm_Blue_D_Centre(N16,1,0));
                                Var := 1. + 3.250820579 var_3^2 + 3.250820579 var_2^2 + 3.250820579 var_1^2

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
                                1.000000000 + 3.250820579 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de Degré 2

L'approximation numérique de M_D la rend inversible, ce qui ne peut pas être le cas sans la présence d'un point au centre.

```
> Var:=Var_Norm_Blue_D_Centre(Map(evalf,N16),2,0);
```


$$\begin{aligned}
Var := & 430.1776459 + 0.596181102157441640 \cdot 10^{-8} var_3 - 0.114857079358671446 \cdot 10^{-6} var_1 \\
& - 0.352228196370709097 \cdot 10^{-6} var_1 var_3 - 465.059855370317678 var_3^2 \\
& - 465.059855040689740 var_2^2 - 465.059855368153990 var_1^2 + (0.5961811022 \cdot 10^{-8} \\
& + 3.25082067805647236 var_3 + 0.445660917050829340 \cdot 10^{-9} var_1 \\
& + 0.711419468553220501 \cdot 10^{-9} var_1 var_3 - 0.915697243327594246 \cdot 10^{-8} var_3^2 \\
& - 0.519335068219176328 \cdot 10^{-8} var_2^2 - 0.489835762714629598 \cdot 10^{-8} var_1^2) var_3 + (\\
& 3.25082067648117868 var_2 + 0.207036658707020700 \cdot 10^{-8} var_2 var_3 \\
& + 0.260491425003194278 \cdot 10^{-8} var_1 var_2) var_2 + (-0.1148570794 \cdot 10^{-6} \\
& + 0.445660917050829288 \cdot 10^{-9} var_3 + 3.25082067820923326 var_1 \\
& + 0.324905581601755584 \cdot 10^{-8} var_1 var_3 + 0.123926020660647941 \cdot 10^{-6} var_3^2 \\
& + 0.124718283265983024 \cdot 10^{-6} var_2^2 + 0.125316695433863368 \cdot 10^{-6} var_1^2) var_1 + (\\
& 0.207036658707020700 \cdot 10^{-8} var_2 + 17.5721147789632362 var_2 var_3 \\
& + 0.656923833792263516 \cdot 10^{-8} var_1 var_2) var_2 var_3 + (-0.3522281964 \cdot 10^{-6} \\
& + 0.711419468553220501 \cdot 10^{-9} var_3 + 0.324905581601755626 \cdot 10^{-8} var_1 \\
& + 17.5721147954468542 var_1 var_3 + 0.380196553565887061 \cdot 10^{-6} var_3^2 \\
& + 0.380804065704055213 \cdot 10^{-6} var_2^2 + 0.386686341119311678 \cdot 10^{-6} var_1^2) var_1 var_3 \\
& + (0.260491425003194320 \cdot 10^{-8} var_2 + 0.656923833792263516 \cdot 10^{-8} var_2 var_3 \\
& + 17.5721147783524750 var_1 var_2) var_1 var_2 + (-465.0598554 \\
& - 0.915697243327593418 \cdot 10^{-8} var_3 + 0.123926020660647914 \cdot 10^{-6} var_1 \\
& + 0.380196553565887061 \cdot 10^{-6} var_1 var_3 + 509.799436435742506 var_3^2 \\
& + 501.013378693743334 var_2^2 + 501.013379047928993 var_1^2) var_3^2 + (\\
& -465.0598550 - 0.519335068219176908 \cdot 10^{-8} var_3 \\
& + 0.124718283265982998 \cdot 10^{-6} var_1 + 0.380804065704055213 \cdot 10^{-6} var_1 var_3 \\
& + 501.013378693743334 var_3^2 + 509.799435716426670 var_2^2 \\
& + 501.013378691381433 var_1^2) var_2^2 + (-465.0598554 \\
& - 0.489835762714629928 \cdot 10^{-8} var_3 + 0.125316695433863342 \cdot 10^{-6} var_1 \\
& + 0.386686341119311678 \cdot 10^{-6} var_1 var_3 + 501.013379047928993 var_3^2 \\
& + 501.013378691381433 var_2^2 + 509.799436431788990 var_1^2) var_1^2
\end{aligned}$$

11.22. N17

```
> unassign('alpha','beta','nu','delta','eta','mu');
```

```
> GrN17:=Plus(Rned(Diedn(3),3));
```

$$GrN17 := \left\{ \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

```
> OrbPG(Vector([alpha,beta,nu]),GrN17);OrbPG(Vector([delta,eta,mu]),GrN17);OrbPG(Vector([1,0,0]),GrN17);OrbPG(Vector([0,0,1]),GrN17);
```

$$\left\{ \begin{bmatrix} \alpha \\ -\beta \\ -\nu \end{bmatrix}, \begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} \\ -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} \\ -\nu \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \\ \nu \end{bmatrix}, \begin{bmatrix} -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} \\ -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} \\ \nu \end{bmatrix}, \begin{bmatrix} -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} \\ \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} \\ \nu \end{bmatrix}, \begin{bmatrix} -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} \\ \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} \\ -\nu \end{bmatrix} \right\} \\ \left\{ \begin{bmatrix} -\frac{\delta}{2} - \frac{\sqrt{3}\eta}{2} \\ -\frac{\sqrt{3}\delta}{2} + \frac{\eta}{2} \\ -\mu \end{bmatrix}, \begin{bmatrix} \delta \\ \eta \\ \mu \end{bmatrix}, \begin{bmatrix} -\frac{\delta}{2} - \frac{\sqrt{3}\eta}{2} \\ \frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} \\ \mu \end{bmatrix}, \begin{bmatrix} -\frac{\delta}{2} + \frac{\sqrt{3}\eta}{2} \\ -\frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} \\ \mu \end{bmatrix}, \begin{bmatrix} -\frac{\delta}{2} + \frac{\sqrt{3}\eta}{2} \\ \frac{\sqrt{3}\delta}{2} + \frac{\eta}{2} \\ -\mu \end{bmatrix}, \begin{bmatrix} \delta \\ -\eta \\ -\mu \end{bmatrix} \right\} \\ \left\{ \begin{bmatrix} \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\ \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

```
> N17:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN17) union
OrbPG(Vector([delta,eta,mu]),GrN17) union OrbPG(Vector([1,0,0]),GrN17)
union OrbPG(Vector([0,0,1]),GrN17));
```

$$N17 := \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \alpha & \beta & \nu \\ \alpha & -\beta & -\nu \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ -\frac{\delta}{2} + \frac{\sqrt{3}\eta}{2} & -\frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} & \mu \\ \delta & -\eta & -\mu \\ -\frac{\delta}{2} - \frac{\sqrt{3}\eta}{2} & -\frac{\sqrt{3}\delta}{2} + \frac{\eta}{2} & -\mu \\ \delta & \eta & \mu \\ -\frac{\delta}{2} + \frac{\sqrt{3}\eta}{2} & \frac{\sqrt{3}\delta}{2} + \frac{\eta}{2} & -\mu \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\delta}{2} - \frac{\sqrt{3}\eta}{2} & \frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} & \mu \end{bmatrix}$$

> UpdatePlan(N17, "N17");

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N17,1,0);

$$\begin{aligned} Var := & 1 + \frac{17}{2} \frac{var_3^2}{3\nu^2 + 1 + 3\mu^2} + \frac{34}{3} \frac{var_2^2}{2\alpha^2 + 2\delta^2 + 2\eta^2 + 2\beta^2 + 1} \\ & + \frac{34}{3} \frac{var_1^2}{2\alpha^2 + 2\delta^2 + 2\eta^2 + 2\beta^2 + 1} \end{aligned}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

$$Cond := [-12\nu^2 - 12\mu^2 + 6\alpha^2 + 6\delta^2 + 6\eta^2 + 6\beta^2 - 1]$$

> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,delta^2+eta^2+mu^2-1],{alpha,beta,nu,delta,eta,mu});

$$\begin{aligned} Cond_gsolve := & \{[\delta^2 + \eta^2 + \mu^2 - 1, 18\nu^2 + 18\mu^2 - 11, 18\alpha^2 + 18\beta^2 - 18\mu^2 - 7], \\ & plex(\alpha, \beta, \nu, \eta, \mu, \delta), \{\}\} \end{aligned}$$

> Cond:=eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,sqrt(2)=radic2,N17),2),[radic=sqrt(3),radic2=sqrt(2)]);

$$\begin{aligned}
\text{Cond} := & [-12\nu^2 - 12\mu^2 + 6\alpha^2 + 6\delta^2 + 6\eta^2 + 6\beta^2 - 1, \\
& -12\nu^2 - 12\mu^2 + 6\alpha^2 + 6\delta^2 + 6\eta^2 + 6\beta^2 - 1, \\
& -1 - \alpha - 2\delta^3 + 6\delta\eta^2 + 12\alpha\beta^2 - 12\nu^2\alpha + 4\alpha^3 + 6\alpha\eta^2 + 6\alpha\delta^2 - 12\mu^2\alpha, \\
& -6\delta\eta^2 + 8\alpha^3 + 1 + 2\delta^3 + 6\alpha\eta^2 - \alpha + 6\alpha\delta^2 - 12\mu^2\alpha - 12\nu^2\alpha, 0, \\
& 18\nu^2\delta^2 - 18\mu^2\delta^2 - 24\nu^4 + 18\eta^2\nu^2 - 36\nu^2\mu^2 - 18\mu^2\eta^2 + 12\mu^4 - 3\nu^2 + 4, \\
& -6\delta^2\eta\mu + 8\beta^3\nu + 2\eta^3\mu - \nu\beta + 6\nu\beta\eta^2 - 12\nu\beta\mu^2 - 12\nu^3\beta + 6\nu\beta\delta^2, \\
& 18\nu^2\delta^2 - 18\mu^2\delta^2 - 24\nu^4 + 18\eta^2\nu^2 - 36\nu^2\mu^2 - 18\mu^2\eta^2 + 12\mu^4 - 3\nu^2 + 4, \\
& 6\nu\beta\eta^2 - 2\eta^3\mu + 12\alpha^2\beta\nu + 4\beta^3\nu - 12\nu^3\beta - 12\nu\beta\mu^2 + 6\nu\beta\delta^2 + 6\delta^2\eta\mu - \nu\beta \\
& , 22 + 72\beta^2\eta^2 - 72\beta^2\mu^2 - 72\eta^2\nu^2 + 36\nu^2 + 60\mu^2 + 72\nu^2\alpha^2 + 72\nu^2\delta^2 - 72\nu^2\beta^2 \\
& + 72\mu^2\alpha^2 - 216\mu^2\delta^2 - 360\mu^2\eta^2 - 36\alpha^4 - 72\alpha^2\delta^2 + 36\beta^4 + 108\eta^4 - 36\eta^2 \\
& - 144\nu^4 + 144\mu^4 + 144\delta^2\eta^2 + 36\delta^4 - 24\beta^2 - 12\alpha^2 - 24\delta^2, 91 + 144\alpha^2\beta^2 \\
& - 144\beta^2\eta^2 - 144\beta^2\mu^2 + 432\eta^2\nu^2 + 72\nu^2 + 144\mu^2 - 432\nu^2\alpha^2 + 144\nu^2\delta^2 \\
& - 144\nu^2\beta^2 - 432\mu^2\alpha^2 - 720\mu^2\delta^2 - 432\mu^2\eta^2 + 144\alpha^4 + 144\alpha^2\delta^2 - 72\eta^4 - 36\eta^2 \\
& - 240\nu^4 + 624\mu^4 + 72\delta^4 - 48\beta^2 - 72\alpha^2 - 60\delta^2, 72\eta^4 - 288\mu^2\eta^2 + 144\delta^2\eta^2 \\
& - 144\nu^4 + 144\mu^4 - 288\mu^2\delta^2 + 72\delta^4 - 12\eta^2 + 24\mu^2 - 12\delta^2 + 19]
\end{aligned}$$

> subs(0=NULL,convert(Cond,set));

$$\begin{aligned}
& \{-12\nu^2 - 12\mu^2 + 6\alpha^2 + 6\delta^2 + 6\eta^2 + 6\beta^2 - 1, 91 + 144\alpha^2\beta^2 - 144\beta^2\eta^2 - 144\beta^2\mu^2 \\
& + 432\eta^2\nu^2 + 72\nu^2 + 144\mu^2 - 432\nu^2\alpha^2 + 144\nu^2\delta^2 - 144\nu^2\beta^2 - 432\mu^2\alpha^2 \\
& - 720\mu^2\delta^2 - 432\mu^2\eta^2 + 144\alpha^4 + 144\alpha^2\delta^2 - 72\eta^4 - 36\eta^2 - 240\nu^4 + 624\mu^4 \\
& + 72\delta^4 - 48\beta^2 - 72\alpha^2 - 60\delta^2, 22 + 72\beta^2\eta^2 - 72\beta^2\mu^2 - 72\eta^2\nu^2 + 36\nu^2 + 60\mu^2 \\
& + 72\nu^2\alpha^2 + 72\nu^2\delta^2 - 72\nu^2\beta^2 + 72\mu^2\alpha^2 - 216\mu^2\delta^2 - 360\mu^2\eta^2 - 36\alpha^4 - 72\alpha^2\delta^2 \\
& + 36\beta^4 + 108\eta^4 - 36\eta^2 - 144\nu^4 + 144\mu^4 + 144\delta^2\eta^2 + 36\delta^4 - 24\beta^2 - 12\alpha^2 \\
& - 24\delta^2, -6\delta^2\eta\mu + 8\beta^3\nu + 2\eta^3\mu - \nu\beta + 6\nu\beta\eta^2 - 12\nu\beta\mu^2 - 12\nu^3\beta + 6\nu\beta\delta^2, \\
& 18\nu^2\delta^2 - 18\mu^2\delta^2 - 24\nu^4 + 18\eta^2\nu^2 - 36\nu^2\mu^2 - 18\mu^2\eta^2 + 12\mu^4 - 3\nu^2 + 4, \\
& -6\delta\eta^2 + 8\alpha^3 + 1 + 2\delta^3 + 6\alpha\eta^2 - \alpha + 6\alpha\delta^2 - 12\mu^2\alpha - 12\nu^2\alpha, \\
& -1 - \alpha - 2\delta^3 + 6\delta\eta^2 + 12\alpha\beta^2 - 12\nu^2\alpha + 4\alpha^3 + 6\alpha\eta^2 + 6\alpha\delta^2 - 12\mu^2\alpha, \\
& 6\nu\beta\eta^2 - 2\eta^3\mu + 12\alpha^2\beta\nu + 4\beta^3\nu - 12\nu^3\beta - 12\nu\beta\mu^2 + 6\nu\beta\delta^2 + 6\delta^2\eta\mu - \nu\beta \\
& , 72\eta^4 - 288\mu^2\eta^2 + 144\delta^2\eta^2 - 144\nu^4 + 144\mu^4 - 288\mu^2\delta^2 + 72\delta^4 - 12\eta^2 + 24\mu^2 \\
& - 12\delta^2 + 19\}
\end{aligned}$$

> Cond_ksolved:=gsolve([op(
%),alpha^2+beta^2+nu^2-1,delta^2+eta^2+mu^2-1],[beta,eta,delta,alpha,
mu,nu]));

```

Cond_gsolved := {[[3240 ν4 - 1980 ν2 + 227, 18 ν2 + 18 μ2 - 11, 2609280 α6
+ 3913920 ν2 α4 - 3913920 α4 - 4276800 ν2 α3 + 1632960 α3 - 2038500 ν2 α2
+ 1364889 α2 + 2472120 ν2 α - 999990 α - 267300 ν2 + 112781,
24 δ3 - 18 ν2 δ - 7 δ + 24 α3 + 18 ν2 α - 18 α + 3, 18 η2 + 18 δ2 - 18 ν2 - 7,
461357406187 β - 591143790528000 ν3 δ2 η μ α + 92839304736000 ν3 δ2 η μ α2
+ 5735035726200 α2 ν η μ + 87095101996800 ν3 α η μ - 33956134448400 α ν η μ
- 440798423040000 α4 ν3 δ2 η μ + 255488573260800 α4 ν δ2 η μ
- 12644044272000 ν3 α2 η μ + 830442461184000 α5 ν δ2 η μ
+ 2228480916480000 ν3 δ2 η μ α3 + 404609416704000 α5 ν3 η μ
- 123630655104000 α5 ν η μ - 351223452000000 ν3 α3 η μ
+ 119103775056000 α3 ν η μ + 46327462444800 α4 ν3 η μ
- 32559941692800 α4 ν η μ - 2448880128000000 α5 ν3 δ2 η μ
- 42263127648000 α2 ν δ2 η μ + 242763382540800 α ν δ2 η μ
+ 107301136752000 ν3 δ2 η μ - 55131455489760 ν δ2 η μ - 13042420315560 ν3 η μ
+ 7239428391960 ν η μ - 823587108480000 α3 ν δ2 η μ], plex(β, η, δ, α, μ, ν),
{ν, μ, -809 + 3960 ν2}]}

```

```

> SolsMuNu:=allvalues(solve(
{3240*nu^4+227-1980*nu^2,18*nu^2+18*mu^2-11},{mu,nu}));

SolsMuNu := {μ = %2, ν = %1}, {μ = %4, ν = %1}, {ν = %3, μ = %2}, {ν = %3, μ = %4},
{μ = %3, ν = %4}, {μ = %1, ν = %4}, {ν = %2, μ = %3}, {ν = %2, μ = %1}

%1 := - $\frac{\sqrt{275 + 5\sqrt{755}}}{30}$ 
%2 :=  $\frac{\sqrt{275 - 5\sqrt{755}}}{30}$ 
%3 :=  $\frac{\sqrt{275 + 5\sqrt{755}}}{30}$ 
%4 := - $\frac{\sqrt{275 - 5\sqrt{755}}}{30}$ 

> assign(SolsMuNu[1]);

> SolsAlpha:=[solve(
{2609280*alpha^6+3913920*alpha^4*nu^2-3913920*alpha^4-4276800*nu^2*al
pha^3+1632960*alpha^3-2038500*nu^2*alpha^2+1364889*alpha^2+2472120*nu^
2*alpha-999990*alpha-267300*nu^2+112781},{alpha})]:

> assign(SolsAlpha[6]);

> SolsDelta:=[solve(
{24*delta^3-18*delta*nu^2-7*delta+24*alpha^3+18*nu^2*alpha-18*alpha+3
},{delta})]:

> assign(SolsDelta[3]);

> SolsEta:=(solve({18*eta^2+18*delta^2-18*nu^2-7},{eta}))]:

> assign(SolsEta[2]);

> SolsBeta:=(solve({beta^2+nu^2+alpha^2-1},{beta}))]:

> assign(SolsBeta[2]);

```

```
> N17:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN17) union
OrbPG(Vector([delta,eta,mu]),GrN17) union OrbPG(Vector([1,0,0]),GrN17)
union OrbPG(Vector([0,0,1]),GrN17)):

> UpdatePlan(N17,"N17");
                                "Update complete"

> evalf(Cond);


$$\begin{aligned} & [-0.16 \cdot 10^{-7} - 0.4186107980 \cdot 10^{-8} I, -0.16 \cdot 10^{-7} - 0.4186107980 \cdot 10^{-8} I, \\ & -0.36 \cdot 10^{-8} + 0.5360796557 \cdot 10^{-8} I, 0.11 \cdot 10^{-8} - 0.5898443648 \cdot 10^{-8} I, 0., \\ & -0.8 \cdot 10^{-8} - 0.2514407944 \cdot 10^{-8} I, -0.624 \cdot 10^{-8} - 0.6625623063 \cdot 10^{-9} I, \\ & -0.8 \cdot 10^{-8} - 0.2514407944 \cdot 10^{-8} I, -0.66 \cdot 10^{-8} - 0.2083132463 \cdot 10^{-8} I, \\ & 0.63 \cdot 10^{-6} + 0.3058626010 \cdot 10^{-7} I, -0.1 \cdot 10^{-6} + 0.535807126 \cdot 10^{-7} I, \\ & 0.4113 \cdot 10^{-6} + 0.1551570354 \cdot 10^{-7} I] \end{aligned}$$


> pointplot3d(Map(Re,evalf(N17)),symbol=box,scaling=constrained,shading
=Z,orientation=[32,76]);
```

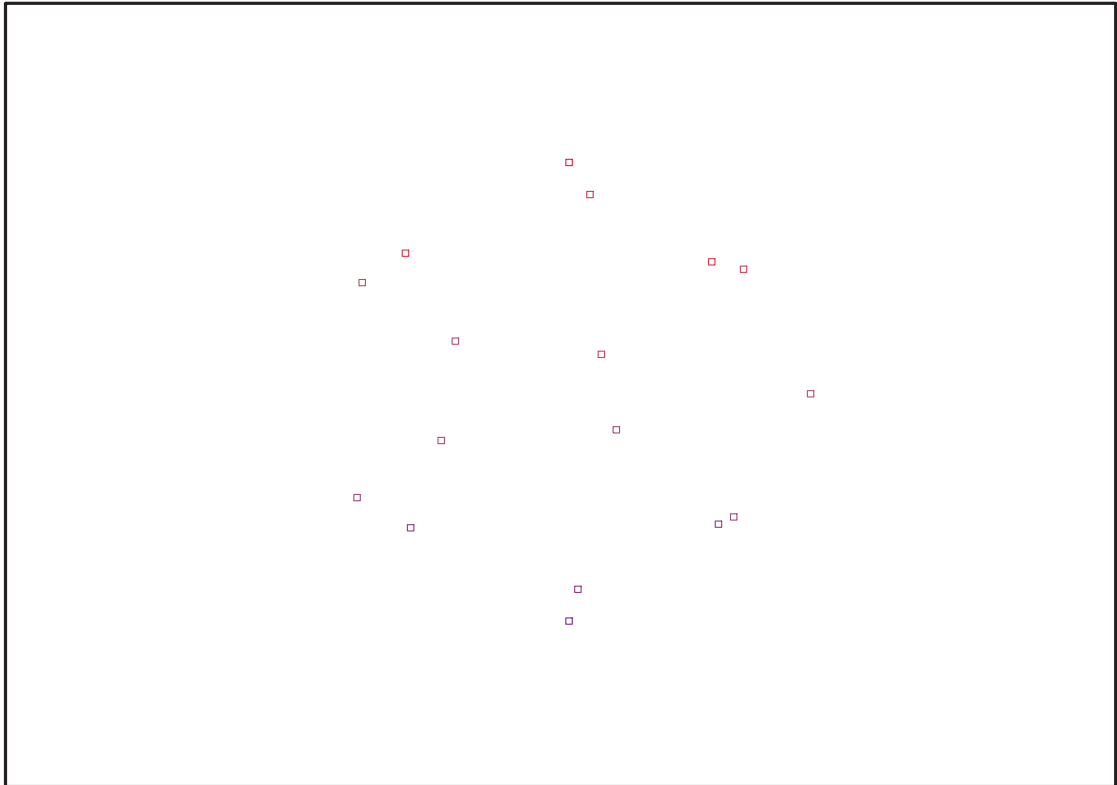


FIG. 11.57 – Graphique : Support du plan N17.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N17)),1
istlist)))],'rational',5))));
                                Graph := POLYTOPE(3, 3, 17, 30)

> plotdes:=draw(Graph):

> display(plotdes,scaling=constrained,shading=Z,orientation=[32,76]);
```

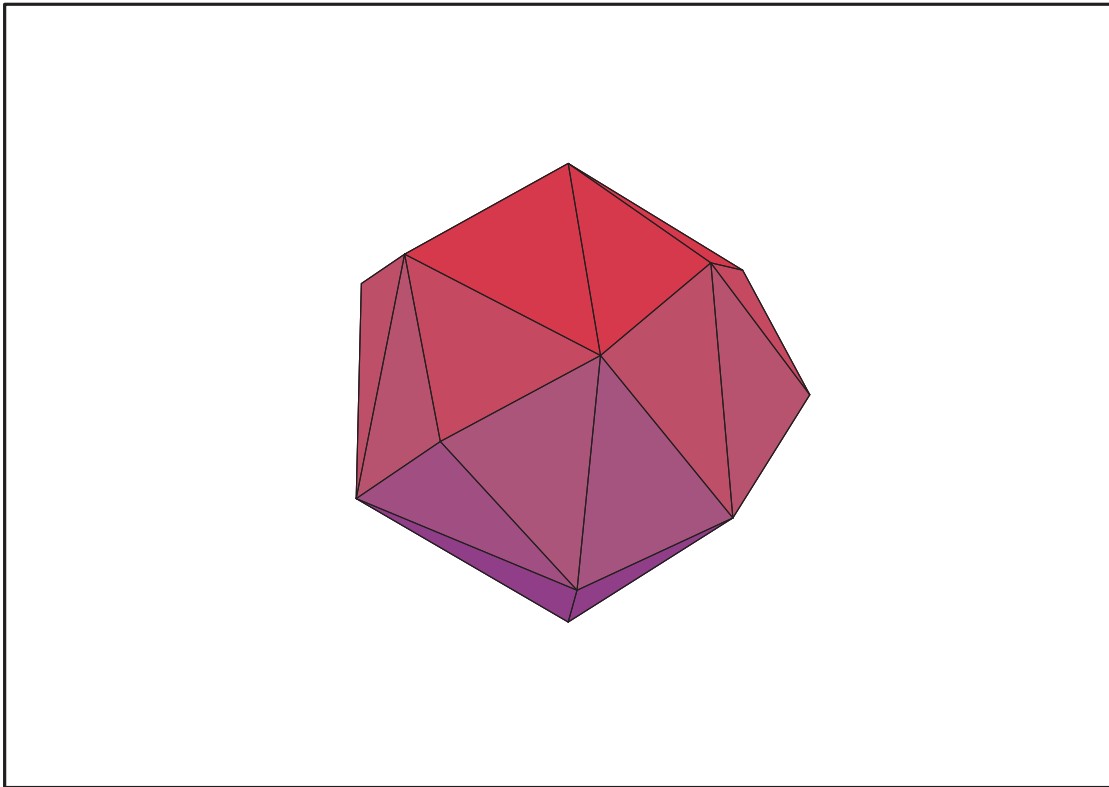


FIG. 11.58 – Graphique : Enveloppe convexe du plan N17.

11.23. N18

```
> unassign('alpha','beta');
```

```
> GrN18:=TwoPlusn(6);
```

$$GrN18 := \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$

$$\left. \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

```
> N18:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN18) union
OrbPG(Vector([0,delta,mu]),GrN18));
```


$$N18 := \begin{bmatrix} \frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ 0 & -\delta & -\mu \\ \frac{\sqrt{3}\delta}{2} & -\frac{\delta}{2} & \mu \\ -\alpha & \beta & \nu \\ \alpha & -\beta & -\nu \\ \alpha & \beta & \nu \\ -\frac{\sqrt{3}\delta}{2} & -\frac{\delta}{2} & \mu \\ \frac{\sqrt{3}\delta}{2} & \frac{\delta}{2} & -\mu \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ \frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ -\frac{\sqrt{3}\delta}{2} & \frac{\delta}{2} & -\mu \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ -\alpha & -\beta & -\nu \\ 0 & \delta & \mu \end{bmatrix}$$

```
> UpdatePlan(N18,"N18");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N18,1,0);
```

$$Var := 1 + \frac{3 var_3^2}{2 \nu^2 + \mu^2} + \frac{6 var_2^2}{2 \alpha^2 + 2 \beta^2 + \delta^2} + \frac{6 var_1^2}{2 \alpha^2 + 2 \beta^2 + \delta^2}$$

```
> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});
```

$$Cond := [2 \alpha^2 + 2 \beta^2 + \delta^2 - 4 \nu^2 - 2 \mu^2]$$

```
> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,delta^2+mu^2-1],
{alpha,beta,nu,mu,delta});
```

$$Cond_gsolved := \{[[-\delta^2 + 2 \nu^2, \delta^2 + \mu^2 - 1, 2 \alpha^2 + 2 \beta^2 + \delta^2 - 2], \text{plex}(\alpha, \mu, \nu, \delta, \beta), \{\}]\}$$

```
> Cond:=convert(convert(map(mem-> if mem<>0 then return mem end
if,eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,N18),2),[radic=sqrt(3)])),s
et),list);
```

$$\begin{aligned}
\text{Cond} := & [-6\mu^2 + 6\alpha^2 - 12\nu^2 + 6\beta^2 + 3\delta^2, -9\beta^3\nu + 3\mu\delta\beta^2 + 3\mu\delta\alpha^2 - 3\mu^3\delta - 6\mu\delta\nu^2 \\
& + 3\alpha^2\beta\nu + 6\nu\beta\mu^2 + 12\nu^3\beta - 3\nu\beta\delta^2, -18\nu^2\beta^2 + 14\beta^4 - 4\alpha^2\mu^2 + 10\nu^2\alpha^2 \\
& + 16\nu^2\mu^2 - 4\nu^4 - 3\delta^2\alpha^2 + 4\mu^2\delta^2 + 2\delta^2\nu^2 + 14\beta^2\alpha^2 - 18\beta^2\mu^2 + 4\delta^2\beta^2, 14\beta^4 \\
& + 14\alpha^2\mu^2 - 8\nu^2\alpha^2 - 56\nu^2\mu^2 - 40\nu^4 - 3\delta^2\alpha^2 - 14\mu^2\delta^2 + 20\delta^2\nu^2 + 14\beta^2\alpha^2 \\
& - 36\nu^2\beta^2 + 4\delta^2\beta^2, 56\beta^4 + 2\alpha^2\mu^2 + 4\nu^2\alpha^2 - 26\nu^2\mu^2 - 16\nu^4 - 12\delta^2\alpha^2 + 16\mu^2\delta^2 \\
& + 35\delta^2\nu^2 - 9\delta^4 + 56\beta^2\alpha^2 - 54\beta^2\mu^2 - 108\nu^2\beta^2 + 16\delta^2\beta^2, \\
& 3\delta^2\nu^2 - 6\nu^2\mu^2 - 3\mu^2\delta^2 - 8\nu^4 + 2\mu^4, 2\alpha^4 - 2\alpha^2\mu^2 - 4\nu^2\alpha^2 + 6\nu^2\mu^2 - 2\mu^2\delta^2 \\
& - \delta^2\nu^2 + \delta^4 + 2\beta^2\mu^2 + 4\nu^2\beta^2 - 2\delta^2\beta^2 - 2\beta^4, \\
& -6\nu\beta\mu^2 - 12\nu^3\beta + 3\nu\beta\delta^2 + 8\beta^3\nu + \delta^3\mu]
\end{aligned}$$

```
> Cond_ksolved:=gsolve({-delta^2+2*nu^2, delta^2+mu^2-1,
2*alpha^2+2*beta^2+delta^2-2,op(Cond)}, {mu,nu,delta,alpha,beta});
```

$$\begin{aligned}
\text{Cond_ksolved} := & \{[[\\
& 10215\beta^4 + 270\beta^2 - 1 + 313200\beta^8 - 96800\beta^6 + 216000\beta^{12} - 432000\beta^{10}, \\
& 13572720\beta^6 - 4574880\beta^4 + 514917\beta^2 + 20297\delta^2 - 6625 + 7387200\beta^{10} \\
& - 16718400\beta^8, 40594\nu^2 + 7387200\beta^{10} - 16718400\beta^8 + 13572720\beta^6 - 4574880\beta^4 \\
& + 514917\beta^2 - 6625, 81188\mu - 4158216000\beta^{11}\delta\nu + 8487720000\beta^9\delta\nu \\
& - 6255543600\beta^7\delta\nu + 1952164800\delta\beta^5\nu - 204254825\delta\beta^3\nu - 6430005\delta\nu\beta, \\
& 40594\alpha^2 - 7387200\beta^{10} + 16718400\beta^8 - 13572720\beta^6 + 4574880\beta^4 - 474323\beta^2 \\
& - 33969], \text{plex}(\alpha, \mu, \nu, \delta, \beta), \{4\beta^2 - 3\}]\}
\end{aligned}$$

```
> Sols:=convert(map(L->solve(
{op(convert(L[1],set)), -1<=alpha, alpha<=1, -1<=beta, beta<=1, -1<=delta,
delta<=1, -1<=nu, nu<=1, -1<=mu, mu<=1}, {nu,mu,delta,alpha,beta
}),Cond_ksolved),radical):
```

```
> evalf(Sols);
```

$\{\{\mu = -0.8688898735, \nu = -0.3500211746, \delta = 0.4950046922, \beta = -0.8384520810, \alpha = -0.4177119643\}, \{\beta = 0.05747686793, \mu = 0.8688903009, \nu = 0.3500211746, \delta = 0.4950046922, \alpha = 0.9349767842\}, \{\mu = -0.8688894159, \nu = -0.3500211746, \beta = 0.7809752131, \alpha = 0.5172648199, \delta = 0.4950046922\}, \{\delta = -0.4950046922, \nu = 0.3500211746, \beta = 0.8384520810, \alpha = 0.4177119643, \mu = 0.8688898735\}, \{\mu = 0.8688903009, \nu = -0.3500211746, \alpha = -0.9349767842, \beta = -0.05747686793, \delta = 0.4950046922\}, \{\mu = -0.8688903009, \beta = -0.05747686793, \nu = 0.3500211746, \delta = 0.4950046922, \alpha = 0.9349767842\}, \{\beta = 0.7809752131, \alpha = 0.5172648199, \nu = 0.3500211746, \delta = 0.4950046922, \mu = 0.8688894159\}, \{\nu = -0.3500211746, \beta = 0.7809752131, \delta = -0.4950046922, \alpha = 0.5172648199, \mu = 0.8688894159\}, \{\mu = -0.8688898735, \nu = 0.3500211746, \delta = 0.4950046922, \beta = 0.8384520810, \alpha = -0.4177119643\}, \{\mu = -0.8688898735, \nu = -0.3500211746, \delta = -0.4950046922, \beta = 0.8384520810, \alpha = 0.4177119643\}, \{\mu = 0.8688903009, \delta = -0.4950046922, \alpha = -0.9349767842, \beta = -0.05747686793, \nu = 0.3500211746\}, \{\nu = -0.3500211746, \beta = 0.7809752131, \delta = -0.4950046922, \alpha = -0.5172648199, \mu = 0.8688894159\}, \{\nu = 0.3500211746, \delta = 0.4950046922, \beta = -0.8384520810, \alpha = 0.4177119643, \mu = 0.8688898735\}, \{\mu = -0.8688898735, \nu = -0.3500211746, \delta = 0.4950046922, \beta = -0.8384520810, \alpha = 0.4177119643\}, \{\mu = -0.8688894159, \nu = -0.3500211746, \delta = -0.4950046922, \alpha = 0.5172648199, \beta = -0.7809752131\}, \{\beta = 0.05747686793, \delta = -0.4950046922, \alpha = -0.9349767842, \mu = -0.8688903009, \nu = 0.3500211746\}, \{\mu = -0.8688894159, \nu = -0.3500211746, \beta = 0.7809752131, \alpha = -0.5172648199, \delta = 0.4950046922\}, \{\nu = -0.3500211746, \alpha = -0.5172648199, \beta = -0.7809752131, \delta = 0.4950046922, \mu = 0.8688894159\}, \{\beta = 0.05747686793, \mu = 0.8688903009, \nu = -0.3500211746, \delta = -0.4950046922, \alpha = 0.9349767842\}, \{\beta = 0.7809752131, \alpha = -0.5172648199, \nu = 0.3500211746, \delta = 0.4950046922, \mu = 0.8688894159\}, \{\beta = 0.05747686793, \delta = -0.4950046922, \mu = -0.8688903009, \nu = 0.3500211746, \alpha = 0.9349767842\}, \{\delta = -0.4950046922, \nu = 0.3500211746, \beta = 0.8384520810, \mu = 0.8688898735, \alpha = -0.4177119643\}, \{\beta = 0.05747686793, \nu = -0.3500211746, \alpha = -0.9349767842, \mu = -0.8688903009, \delta = 0.4950046922\}, \{\delta = -0.4950046922, \alpha = -0.5172648199, \beta = -0.7809752131, \nu = 0.3500211746, \mu = 0.8688894159\}, \{\mu = -0.8688894159, \alpha = -0.5172648199, \beta = -0.7809752131, \nu = 0.3500211746, \delta = 0.4950046922\}, \{\beta = 0.05747686793, \nu = -0.3500211746, \mu = -0.8688903009, \delta = 0.4950046922, \alpha = 0.9349767842\}, \{\nu = -0.3500211746, \delta = -0.4950046922, \alpha = -0.9349767842, \mu = -0.8688903009, \beta = -0.05747686793\}, \{\nu = -0.3500211746, \alpha = 0.5172648199, \beta = -0.7809752131, \delta = 0.4950046922, \mu = 0.8688894159\}, \{\alpha = -0.9349767842, \mu = -0.8688903009, \beta = -0.05747686793, \nu = 0.3500211746, \delta = 0.4950046922\}, \{\beta = 0.05747686793, \mu = 0.8688903009, \nu = -0.3500211746, \delta = -0.4950046922, \alpha = -0.9349767842\}, \{\beta = 0.05747686793, \mu = 0.8688903009, \alpha = -0.9349767842, \nu = 0.3500211746, \delta = 0.4950046922\}, \{\nu = -0.3500211746, \delta = 0.4950046922, \beta = 0.8384520810, \mu = 0.8688898735, \alpha = -0.4177119643\}, \{\nu = -0.3500211746, \delta = 0.4950046922, \beta = 0.8384520810, \alpha = 0.4177119643, \mu = 0.8688898735\}, \{\mu = -0.8688894159, \nu = -0.3500211746, \delta = -0.4950046922, \alpha = -0.5172648199, \beta = -0.7809752131\}, \{\mu = 0.8688903009, \nu = -0.3500211746, \beta = -0.05747686793, \delta = 0.4950046922, \alpha = 0.9349767842\}, \{\mu = -0.8688898735, \delta = -0.4950046922, \nu = 0.3500211746, \beta = -0.8384520810, \alpha = -0.4177119643\}, \{\mu = -0.8688898735, \nu = -0.3500211746, \delta = -0.4950046922, \beta = 0.8384520810, \alpha = 0.4177119643\}\}$

```

$$\alpha = -0.4177119643, \{\mu = -0.8688898735, \delta = -0.4950046922, \nu = 0.3500211746,$$

$$\beta = -0.8384520810, \alpha = 0.4177119643, \{\mu = -0.8688894159, \beta = 0.7809752131,$$

$$\delta = -0.4950046922, \alpha = -0.5172648199, \nu = 0.3500211746, \{\mu = -0.8688894159,$$

$$\beta = 0.7809752131, \delta = -0.4950046922, \alpha = 0.5172648199, \nu = 0.3500211746, \{$$

$$\mu = -0.8688894159, \alpha = 0.5172648199, \beta = -0.7809752131, \nu = 0.3500211746,$$

$$\delta = 0.4950046922, \{\nu = 0.3500211746, \delta = 0.4950046922, \beta = -0.8384520810,$$

$$\mu = 0.8688898735, \alpha = -0.4177119643, \{\mu = 0.8688903009, \delta = -0.4950046922,$$

$$\beta = -0.05747686793, \nu = 0.3500211746, \alpha = 0.9349767842, \{\nu = -0.3500211746,$$

$$\delta = -0.4950046922, \beta = -0.8384520810, \alpha = 0.4177119643, \mu = 0.8688898735, \{$$

$$\mu = -0.8688898735, \nu = 0.3500211746, \delta = 0.4950046922, \beta = 0.8384520810,$$

$$\alpha = 0.4177119643, \{\delta = -0.4950046922, \alpha = 0.5172648199, \beta = -0.7809752131,$$

$$\nu = 0.3500211746, \mu = 0.8688894159, \{\nu = -0.3500211746, \delta = -0.4950046922,$$

$$\beta = -0.8384520810, \mu = 0.8688898735, \alpha = -0.4177119643, \{\nu = -0.3500211746,$$

$$\delta = -0.4950046922, \mu = -0.8688903009, \beta = -0.05747686793, \alpha = 0.9349767842\}\}$$
  
> op(1,Sols);  

$$\{\alpha = \text{RootOf}(-7387200 \%1^{10} + 16718400 \%1^8 - 13572720 \%1^6 + 4574880 \%1^4$$

$$- 474323 \%1^2 + 40594 \_Z^2 - 33969, 0.5172648481), \mu = \frac{5}{81188} \text{RootOf}(\$$

$$7387200 \%1^{10} - 16718400 \%1^8 + 13572720 \%1^6 - 4574880 \%1^4 + 514917 \%1^2$$

$$+ 20297 \_Z^2 - 6625, -0.4950046320) \text{RootOf}(7387200 \%1^{10} - 16718400 \%1^8$$

$$+ 13572720 \%1^6 - 4574880 \%1^4 + 514917 \%1^2 + 40594 \_Z^2 - 6625, -0.3500211320)$$

$$\%1(831643200 \%1^{10} - 1697544000 \%1^8 + 1251108720 \%1^6 - 390432960 \%1^4$$

$$+ 40850965 \%1^2 + 1286001), \nu = \text{RootOf}(7387200 \%1^{10} - 16718400 \%1^8$$

$$+ 13572720 \%1^6 - 4574880 \%1^4 + 514917 \%1^2 + 40594 \_Z^2 - 6625, -0.3500211320),$$

$$\beta = \%1, \delta = \text{RootOf}(7387200 \%1^{10} - 16718400 \%1^8 + 13572720 \%1^6 - 4574880 \%1^4$$

$$+ 514917 \%1^2 + 20297 \_Z^2 - 6625, -0.4950046320)\}$$

$$\%1 := \text{RootOf}(\$$

$$10215 \_Z^4 + 270 \_Z^2 - 1 + 313200 \_Z^8 - 96800 \_Z^6 + 216000 \_Z^{12} - 432000 \_Z^{10},$$

$$0.7809752131)$$
  
> assign(%);  
> N18:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN18) union  
OrbPG(Vector([0,delta,mu]),GrN18))):  
> UpdatePlan(N18,"N18");  
"Update complete"  
> pointplot3d(Map(Re,evalf(N18)),symbol=box,scaling=constrained,shading  
=Z);  
  
> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N18)),1  
istlist)))],'rational',5))));  
Graph := POLYTOPE(3, 3, 18, 28)  
> plotdes:=draw(Graph);
```

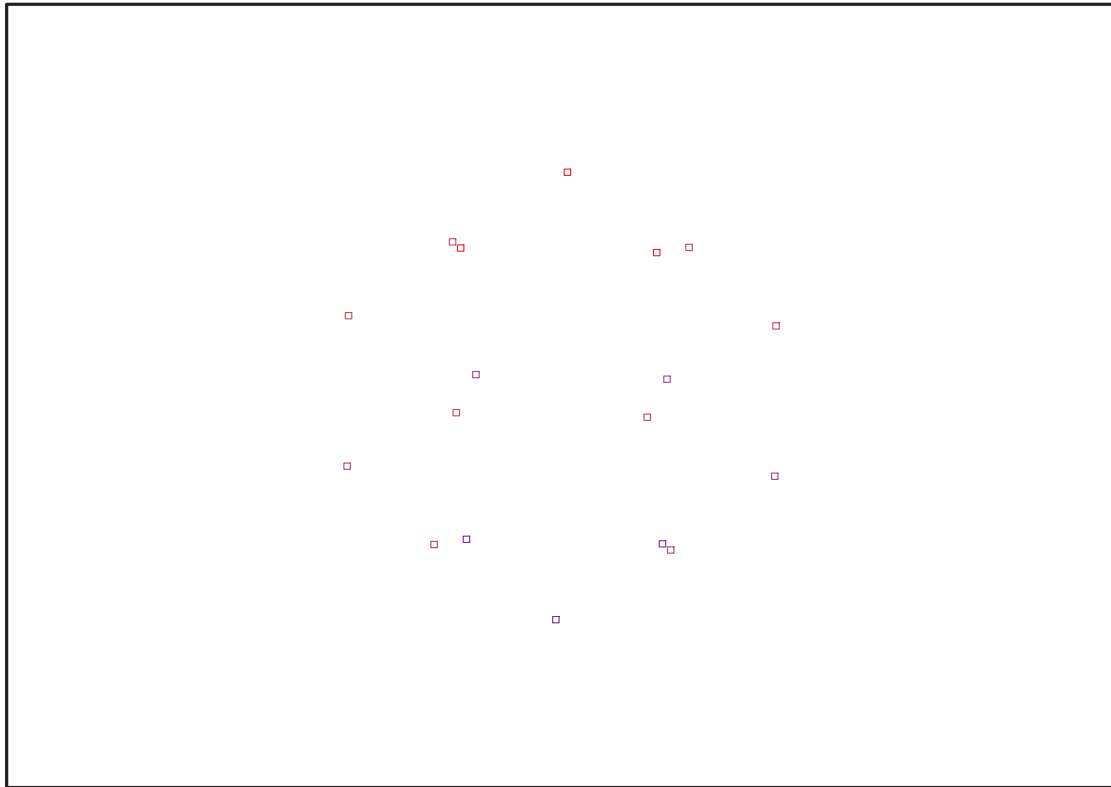


FIG. 11.59 – Graphique : Support du plan N18.

```
> display(plotdes,scaling=constrained,shading=Z);
```

11.23.1. Variance

Modèle de Degré 1

```
> Var:=(Var_Norm_Blue_D_Centre(N18,1,0));
      Var := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
  },Var,poly_algebra(seq(var[i],i=1..3)));
      1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(Map(Re,evalf(N18)),2,0);
      "M_D is not full rank"
> Var:=(expand(Var_Norm_Blue_D_Centre(N18,2,1))):
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
  },Var,poly_algebra(seq(var[i],i=1..3),alg_relations={t^2-13,s^2-2}
  )));
```

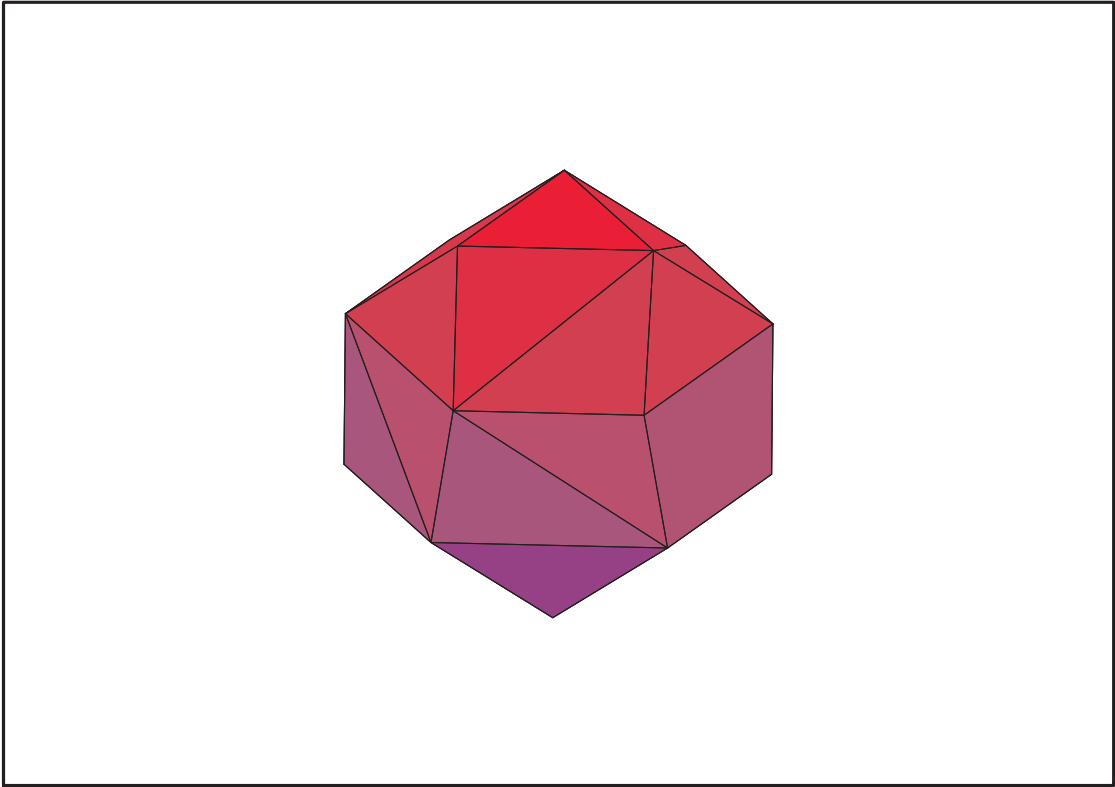


FIG. 11.60 – Graphique : Enveloppe convexe du support du plan N18.

$$18 + 24 \operatorname{IndetDecompInKf}_1^2 - 33 \operatorname{IndetDecompInKf}_1,$$
$$\operatorname{IndetDecompInKf}_1 = \operatorname{var}_1^2 + \operatorname{var}_2^2 + \operatorname{var}_3^2$$

11.24. N19

```
> unassign('alpha','beta','mu','nu','delta','eta','psi','zeta','phi','kappa');
```

```
> TempGr:=Diedn(3):GrN19:=
{seq(DiagonalMatrix([TempGr[IndiceRned],Vector([1])]),IndiceRned=1..nops(Diedn(3)))};
```

$$GrN19 := \left\{ \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

```
> N19:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN19) union
OrbPG(Vector([eta,delta,mu]),GrN19) union
OrbPG(Vector([psi/2,sqrt(3)/2*psi,kappa]),GrN19) union
OrbPG(Vector([phi,0,zeta]),GrN19) union
OrbPG(Vector([0,0,1]),GrN19));
```

$$N19 := \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} & \frac{\sqrt{3}\eta}{2} + \frac{\delta}{2} & \mu \\ -\frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} & -\frac{\delta}{2} + \frac{\sqrt{3}\eta}{2} & \mu \\ -\frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} & -\frac{\sqrt{3}\eta}{2} + \frac{\delta}{2} & \mu \\ -\frac{\phi}{2} & -\frac{\sqrt{3}\phi}{2} & \zeta \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & \nu \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \alpha & -\beta & \nu \\ \phi & 0 & \zeta \\ \frac{\sqrt{3}\delta}{2} - \frac{\eta}{2} & -\frac{\delta}{2} - \frac{\sqrt{3}\eta}{2} & \mu \\ \frac{\psi}{2} & -\frac{\sqrt{3}\psi}{2} & \kappa \\ \eta & -\delta & \mu \\ \frac{\psi}{2} & \frac{\sqrt{3}\psi}{2} & \kappa \\ -\psi & 0 & \kappa \\ \eta & \delta & \mu \\ -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \alpha & \beta & \nu \\ -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2} & \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & \nu \\ -\frac{\phi}{2} & \frac{\sqrt{3}\phi}{2} & \zeta \end{bmatrix}$$

> UpdatePlan(N19,"N19");

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N19,1,0);

$$\begin{aligned} Var &:= \frac{19(6\mu^2 + 1 + 6\nu^2 + 3\kappa^2 + 3\zeta^2)}{6\%1} - \frac{19(6\mu + 1 + 6\nu + 3\kappa + 3\zeta)var_3}{6\%1} \\ &+ \left(-\frac{19(6\mu + 1 + 6\nu + 3\kappa + 3\zeta)}{6\%1} + \frac{361}{6}\frac{var_3}{\%1}\right)var_3 \\ &+ \frac{38}{3}\frac{var_2^2}{2\beta^2 + 2\delta^2 + \psi^2 + \phi^2 + 2\eta^2 + 2\alpha^2} + \frac{38}{3}\frac{var_1^2}{2\beta^2 + 2\delta^2 + \psi^2 + \phi^2 + 2\eta^2 + 2\alpha^2} \\ \%1 &:= 13\mu^2 + 3 + 13\nu^2 + 8\kappa^2 + 8\zeta^2 - \kappa - \zeta - 2\nu - 2\mu - 12\mu\nu - 6\mu\kappa - 6\mu\zeta - 6\nu\kappa \\ &- 6\nu\zeta - 3\kappa\zeta \end{aligned}$$

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});

$$\begin{aligned} \text{Cond} := & [-52\mu^2 + 4\zeta - 52\nu^2 + 8\mu + 8\nu - 32\zeta^2 + 4\kappa - 32\kappa^2 + 38\alpha^2 + 38\eta^2 + 19\phi^2 \\ & + 19\psi^2 + 38\delta^2 + 38\beta^2 - 12 + 48\mu\nu + 24\mu\kappa + 24\mu\zeta + 24\nu\kappa + 24\nu\zeta + 12\kappa\zeta, 3 \\ & + 6\nu\mu^2 + 15\kappa\zeta^2 + 3\kappa\nu^2 + 6\mu\nu^2 + 30\nu\zeta^2 + 30\mu\zeta^2 + 15\kappa^2\zeta + 30\nu\kappa^2 + 30\mu\kappa^2 \\ & + 78\nu^3 + 24\zeta^3 + 24\kappa^3 + 78\mu^3 + 8\kappa + 8\zeta + 3\kappa\mu^2 - 54\kappa\nu\zeta - 54\kappa\mu\zeta - 108\kappa\mu\nu \\ & + 3\zeta\nu^2 - 108\zeta\mu\nu + 3\zeta\mu^2 + \nu^2 + 5\kappa^2 + 5\zeta^2 + 16\nu + 16\mu - 36\mu\nu - 18\mu\kappa \\ & - 18\mu\zeta - 18\nu\kappa - 18\nu\zeta - 9\kappa\zeta + \mu^2] \end{aligned}$$

```
> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,eta^2+delta^2+mu^2-1,phi^2+zeta^2-1,psi^2+kappa^2-1],
{alpha,eta,psi,phi,beta,delta,nu,mu,zeta,kappa});

Cond_gsolve := {[[18ζ² - 108ζ + 631, 54κ² + 36κζ + 12κ + 336ζ - 1385, 6μ + 19,
54κν + 54νζ - 324ν + 36κζ - 159κ - 159ζ - 308,
108ν² - 36κζ - 12κ - 12ζ + 251, 18φ² + 108ζ - 649,
54ψ² - 36κζ - 12κ - 336ζ + 1331, 36δ² + 36η² + 325,
108α² + 108β² + 36κζ + 12κ + 12ζ - 359], %1,
{2ν + κ + ζ - 6, 3κ + 6ν + 3ζ + 1}], [[
18ν² - 5 + 4ν + 4μ + 2κ + 12κν + 12κμ + 24νμ + 6κ² + 18μ²,
9φ² + 24νμ + 12κν + 4ν + 12κμ + 4μ - 3κ² + 2κ + 2, ψ² + κ² - 1,
η² + δ² + μ² - 1,
18α² + 18β² - 24νμ - 12κν - 4ν - 18μ² - 12κμ - 4μ - 6κ² - 2κ - 13,
3κ + 6ν + 6μ + 3ζ + 1], plex(ζ, α, β, δ, ψ, η, φ, ν, μ, κ), {}], [[
3ζ² - 12ζ + 24 + 4μζ + 2κζ - 24μ - 12κ + 8μ² + 4κμ + 3κ²,
κ + 2ν - 6 + 2μ + ζ, φ² + ζ² - 1, ψ² + κ² - 1,
8δ² + 8η² - 4κμ - 4μζ + 24μ - 3κ² - 2κζ + 12κ - 3ζ² + 12ζ - 32,
8α² + 8β² + 4κμ + 4μζ - 24μ - κ² + 2κζ - 12κ - ζ² - 12ζ + 40], %1,
{3ζ + 3κ + 6μ + 1}], [[
27ζ² + 120ζ + 292 + 36μζ + 18κζ + 240μ + 120κ + 72μ² + 36κμ + 27κ²,
3κ + 6ν + 20 + 6μ + 3ζ, φ² + ζ² - 1, ψ² + κ² - 1, 72δ² + 72η² - 36κμ - 36μζ
- 240μ - 27κ² - 18κζ - 120κ - 27ζ² - 120ζ - 364,
72α² + 72β² + 36κμ + 36μζ + 240μ - 9κ² + 18κζ + 120κ - 9ζ² + 120ζ + 436],
%1, {}], [[27κ² + 18κζ + 6κ + 27ζ² + 6ζ + 254, 3ζ + 3κ + 6μ + 1, 6ν - 19,
φ² + ζ² - 1, 27ψ² - 18κζ - 6κ - 27ζ² - 6ζ - 281,
108δ² + 108η² + 36κζ + 12κ + 12ζ - 359, 36α² + 36β² + 325], %1, {6 - ζ - κ}],
[[18ζ² + 120ζ + 707, 54κ² + 36κζ + 12κ - 348ζ - 1613, 6μ - 19,
54κν + 54νζ + 360ν + 36κζ + 183κ + 183ζ - 194,
108ν² - 36κζ - 12κ - 12ζ + 251, 18φ² - 120ζ - 725,
54ψ² - 36κζ - 12κ + 348ζ + 1559, 36δ² + 36η² + 325,
108α² + 108β² + 36κζ + 12κ + 12ζ - 359], %1,
{6ν + 3κ + 3ζ + 20, 3κ + 6ν + 3ζ + 1, 3ζ + 20 + 3κ}]]
%1 := plex(α, β, δ, ψ, η, φ, ν, μ, κ, ζ)

> Cond:=convert(convert(map(mem->if mem<>0 then return mem end
if,eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,N19),2),[radic=sqrt(3)])),s
et) union
{alpha^2+beta^2+nu^2-1,eta^2+delta^2+mu^2-1,phi^2+zeta^2-1,psi^2+kapp
a^2-1},list);
```

$$\begin{aligned}
\text{Cond} := & [9\alpha\zeta\phi^2 + 3\phi^3\zeta + 24\alpha^3\nu - 9\alpha\zeta\psi^2 - 18\alpha\nu\psi^2 + 3\psi^3\zeta + 6\psi^3\nu - 3\mu\phi^3 \\
& - 9\mu\alpha\phi^2 - 24\mu\alpha^3 + 36\alpha\mu\zeta^2 + 72\alpha\nu\mu\zeta + 108\alpha\nu^2\mu - 27\alpha\mu\psi^2 + 9\mu\psi^3 \\
& + 72\alpha\zeta\mu^2 + 144\alpha\nu\mu^2 + 108\alpha\mu^3 - 3\alpha\psi^2 + \psi^3 + 12\alpha\mu\zeta + 24\alpha\mu\nu + 24\mu^2\alpha \\
& + 8\alpha\mu, \eta^2 + \delta^2 + \mu^2 - 1, 162\nu\zeta^2 + 162\nu^3 + 324\zeta\nu^2 + 162\mu\zeta^2 + 648\zeta\mu\nu \\
& + 648\mu\nu^2 + 324\zeta\mu^2 + 648\nu\mu^2 + 162\mu^3 + 27\zeta^2 + 108\nu\zeta + 108\nu^2 + 108\mu\zeta \\
& + 216\mu\nu + 108\mu^2 + 9\zeta + 18\nu + 18\mu - 8, -200 - 3240\nu\mu^2 + 1215\nu\psi^2 \\
& + 1215\mu\psi^2 + 486\zeta\psi^2 - 5184\mu\nu^2 + 486\mu\zeta^2 - 1944\nu^3 - 228\zeta - 3888\nu^3\zeta \\
& + 4374\nu^2\psi^2 - 17496\nu^3\mu - 15552\nu^2\mu^2 - 972\zeta^4 - 1944\mu^4 - 1296\zeta\nu^2 + 81\psi^2 \\
& - 1620\zeta\mu\nu + 648\zeta\mu^2 - 4860\nu^4 - 432\nu^2 + 729\zeta^2\psi^2 + 2916\zeta^2\mu^2 + 3645\zeta\mu\psi^2 \\
& - 264\nu + 3645\nu\zeta\psi^2 - 7776\nu^2\zeta\mu - 1944\nu\zeta\mu^2 - 408\mu - 1188\mu\nu - 54\mu\zeta \\
& - 216\nu\zeta + 1944\zeta\mu^3 - 8748\nu\mu^3 + 9477\nu\mu\psi^2 - 729\phi^2\nu\zeta - 729\phi^2\mu\zeta \\
& + 729\phi^2\mu\nu - 108\mu^2 + 4374\psi^2\mu^2 + 729\zeta^2\phi^2, 6\mu + 1 + 6\nu + 3\kappa + 3\zeta, -800 \\
& - 20736\nu\mu^2 + 5832\eta\zeta^2\alpha + 17496\eta\nu^2\alpha - 1458\eta\alpha\psi^2 + 17496\eta\mu^2\alpha \\
& + 1944\eta\alpha\zeta + 3888\eta\alpha\nu + 3888\eta\alpha\mu + 11664\eta\alpha\nu\zeta + 11664\eta\alpha\mu\zeta \\
& + 23328\eta\alpha\mu\nu - 20736\mu\nu^2 - 7776\nu^3 - 7776\mu^3 - 912\zeta - 15552\nu^3\zeta \\
& - 46656\nu^3\mu - 62208\nu^2\mu^2 + 2106\alpha^4 + 1458\delta^4 + 972\eta^2\delta^2 + 729\phi^4 - 3888\zeta^4 \\
& - 19440\mu^4 + 972\alpha^2\beta^2 - 2268\eta\alpha^3 + 324\eta\phi^3 - 324\eta\psi^3 + 1296\eta\alpha - 5184\zeta\nu^2 \\
& - 10368\zeta\mu\nu - 5184\zeta\mu^2 - 19440\nu^4 + 1458\beta^4 - 1728\nu^2 + 2106\eta^4 - 1056\nu \\
& - 31104\nu^2\zeta\mu - 31104\nu\zeta\mu^2 - 1056\mu - 3456\mu\nu - 864\mu\zeta - 864\nu\zeta - 15552\zeta\mu^3 \\
& + 324\alpha\phi^3 - 2268\alpha\eta^3 - 324\alpha\psi^3 - 46656\nu\mu^3 + 729\psi^4 - 4860\alpha\eta\delta^2 - 1728\mu^2 \\
& - 1458\eta\alpha\phi^2 - 4860\eta\alpha\beta^2, \psi^2 + \kappa^2 - 1, \phi^2 + \zeta^2 - 1, 18\beta^2 + 18\delta^2 + 9\phi^2 + 18\alpha^2 \\
& - 36\zeta^2 - 72\nu\zeta - 108\nu^2 + 9\psi^2 - 72\mu\zeta - 144\mu\nu - 108\mu^2 + 18\eta^2 - 12\zeta - 24\nu \\
& - 24\mu - 8, -16 + 324\nu\mu^2 + 162\nu\eta^2 + 81\nu\phi^2 + 243\nu\psi^2 + 162\nu\delta^2 - 162\mu\eta^2 \\
& + 162\mu\psi^2 - 162\mu\delta^2 - 81\zeta\phi^2 + 81\zeta\psi^2 + 324\mu\zeta^2 - 648\nu^3 + 324\mu^3 + 18\zeta \\
& + 27\psi^2 + 648\zeta\mu\nu + 648\zeta\mu^2 + 54\zeta^2 - 36\nu + 36\mu + 216\mu\nu + 216\mu\zeta + 108\nu\zeta \\
& + 216\mu^2, 24\alpha^3 + 3\phi^3 - 18\eta\delta^2 + 6\eta^3 - 3\psi^3 + 18\alpha\delta^2 + 9\alpha\phi^2 - 36\zeta^2\alpha - 72\alpha\nu\zeta \\
& - 108\nu^2\alpha + 9\alpha\psi^2 - 72\alpha\mu\zeta - 144\alpha\mu\nu - 108\mu^2\alpha + 18\alpha\eta^2 - 12\alpha\zeta - 24\alpha\nu \\
& - 24\alpha\mu - 8\alpha, -32 + 2592\nu\mu^2 - 162\nu\alpha^2 + 162\nu\psi^2 - 162\mu\eta^2 + 162\mu\psi^2 \\
& - 162\mu\delta^2 - 162\nu\beta^2 + 162\zeta\alpha^2 + 162\zeta\eta^2 + 162\zeta\psi^2 + 162\zeta\delta^2 + 162\zeta\beta^2 \\
& + 2592\mu\nu^2 + 648\nu^3 - 324\zeta^3 + 648\mu^3 - 36\zeta + 324\zeta\nu^2 + 27\psi^2 + 1296\zeta\mu\nu \\
& + 324\zeta\mu^2 + 432\nu^2 + 72\nu + 72\mu + 864\mu\nu + 216\mu\zeta + 216\nu\zeta + 432\mu^2, -6\alpha^3 \\
& + 18\phi\alpha^2 + 9\phi\psi^2 + 3\psi^3 + 18\phi\eta^2 - 6\eta^3 + 6\phi^3 - 36\zeta^2\phi + 18\phi\beta^2 + 18\alpha\beta^2 \\
& - 72\phi\nu\zeta - 108\nu^2\phi + 18\phi\delta^2 - 72\phi\mu\zeta - 144\phi\mu\nu - 108\mu^2\phi + 18\eta\delta^2 - 12\phi\zeta \\
& - 24\phi\nu - 24\phi\mu - 8\phi, -5440 - 134784\nu\mu^2 - 7776\nu\alpha^2 + 42768\nu\psi^2 + 7776\nu\delta^2 \\
& - 7776\mu\alpha^2 - 15552\mu\eta^2 + 46656\mu\psi^2 - 3888\mu\phi^2 - 7776\mu\delta^2 - 3888\zeta\alpha^2 \\
& - 3888\zeta\eta^2 - 5832\zeta\phi^2 + 17496\zeta\psi^2 + 2916\phi^2\alpha^2 + 1458\phi^2\psi^2 - 17496\phi^2\mu^2 \\
& + 2916\phi^2\eta^2 - 11664\eta\zeta^2\alpha - 34992\eta\nu^2\alpha + 2916\eta\alpha\psi^2 - 34992\eta\mu^2\alpha \\
& - 3888\eta\alpha\zeta - 7776\eta\alpha\nu - 7776\eta\alpha\mu - 23328\eta\alpha\nu\zeta - 23328\eta\alpha\mu\zeta \\
& - 46656\eta\alpha\mu\nu + 46656\delta^2\mu\nu - 228096\mu\nu^2 + 38880\mu\zeta^2 - 85536\nu^3 + 7776\zeta^3 \\
& + 38880\mu^3 - 6000\zeta - 171072\nu^3\zeta + 139968\nu^2\psi^2 - 793152\nu^3\mu - 684288\nu^2\mu^2 \\
& + 7776\alpha^4 + 2187\phi^4 - 19440\zeta^4 + 42768\mu^4 + 7776\eta\alpha^3 + 972\eta\phi^3 - 972\eta\psi^3 \\
& - 2592\eta\alpha - 57024\zeta\nu^2 + 1944\psi^2 - 1296\phi^2 - 67392\zeta\mu\nu + 59616\zeta\mu^2 \\
& - 213840\nu^4 - 19008\nu^2 + 5184\zeta^2 + 7776\eta^4 - 11664\alpha^2\zeta^2 + 2916\alpha^2\psi^2]
\end{aligned}$$

$$\begin{aligned}
& -34992\alpha^2\mu^2 + 5832\alpha^2\eta^2 + 23328\zeta^2\psi^2 + 46656\zeta^3\mu + 209952\zeta^2\mu^2 - 11664\zeta^2\eta^2 \\
& - 46656\zeta\mu\eta^2 + 139968\zeta\mu\psi^2 - 11616\nu + 34992\delta^2\nu^2 - 5832\beta^2\delta^2 \\
& + 128304\nu\zeta\psi^2 - 342144\nu^2\zeta\mu - 62208\nu\zeta\mu^2 + 23328\delta^2\nu\zeta - 23328\delta^2\mu\zeta \\
& - 10464\mu - 2592\eta^2 - 2592\alpha^2 - 53568\mu\nu + 8640\mu\zeta - 9504\nu\zeta + 202176\zeta\mu^3 \\
& + 972\alpha\phi^3 + 7776\alpha\eta^3 - 972\alpha\psi^3 - 373248\nu\mu^3 + 2187\psi^4 + 361584\nu\mu\psi^2 \\
& - 34992\phi^2\nu\zeta - 69984\phi^2\mu\zeta + 34992\phi^2\mu\nu - 23328\alpha^2\nu\zeta - 23328\alpha^2\mu\zeta \\
& - 46656\alpha^2\mu\nu + 12096\mu^2 + 169128\psi^2\mu^2 + 2916\psi^2\eta^2 - 81648\mu^2\delta^2 \\
& - 116640\mu^2\eta^2 - 34992\nu^2\alpha^2 + 2916\eta\alpha\phi^2, -152 - 16848\nu\mu^2 + 1215\nu\psi^2 \\
& + 972\mu\psi^2 + 729\zeta\psi^2 - 1458\delta^2\mu\nu - 12960\mu\nu^2 - 2916\nu^3 + 486\zeta^3 - 6804\mu^3 \\
& - 426\zeta - 9720\nu^3\zeta + 4374\nu^2\psi^2 - 17496\nu^3\mu - 38880\nu^2\mu^2 - 972\zeta^4 - 10692\mu^4 \\
& - 3240\zeta\nu^2 + 81\psi^2 - 8424\zeta\mu\nu - 5184\zeta\mu^2 - 4860\nu^4 - 1080\nu^2 + 1458\zeta^2\psi^2 \\
& + 2916\zeta^3\mu - 1458\zeta\mu\eta^2 + 4374\zeta\mu\psi^2 - 372\nu + 5832\nu\zeta\psi^2 - 19440\nu^2\zeta\mu \\
& - 28188\nu\zeta\mu^2 + 1458\nu\zeta\eta^2 + 1458\delta^2\nu\zeta - 1458\delta^2\mu\zeta - 84\mu - 2808\mu\nu \\
& - 864\mu\zeta - 1188\nu\zeta - 12636\zeta\mu^3 - 34992\nu\mu^3 - 1458\nu\mu\eta^2 + 7290\nu\mu\psi^2 \\
& - 1728\mu^2 + 2916\psi^2\mu^2 + 1458\mu^2\delta^2 + 1458\mu^2\eta^2, 1296\alpha\zeta\mu^2 + 648\alpha\mu\zeta^2 \\
& - 162\alpha\zeta\psi^2 + 162\alpha\zeta\phi^2 - 486\alpha\mu\psi^2 + 2592\alpha\nu\mu^2 + 1944\alpha\nu^2\mu - 324\alpha\nu\psi^2 \\
& - 162\mu\alpha\phi^2 + 1296\alpha\nu\mu\zeta - 16\phi + 1944\alpha\mu^3 - 162\eta\delta^2\mu - 162\phi\mu\delta^2 \\
& + 324\phi\nu\mu^2 + 648\phi\zeta\mu^2 + 324\phi\zeta^2\mu + 162\nu\eta\delta^2 + 162\nu\phi\delta^2 - 648\phi\nu^3 \\
& + 324\phi\mu^3 + 648\phi\zeta\mu\nu + 216\alpha\mu\zeta + 432\alpha\mu\nu + 27\psi^3 + 54\zeta^2\phi + 189\psi^3\nu \\
& + 81\psi^3\zeta + 216\mu\psi^3 - 54\mu\phi^3 - 432\mu\alpha^3 + 27\phi\psi^2 + 162\nu\phi\eta^2 + 162\phi\psi^2\mu \\
& + 243\phi\psi^2\nu + 432\alpha^3\nu + 54\eta^3\mu + 54\nu\phi^3 - 54\alpha\psi^2 + 144\alpha\mu + 432\mu^2\alpha \\
& + 216\mu^2\phi - 54\nu\eta^3 + 18\phi\zeta - 36\phi\nu + 36\phi\mu + 108\phi\nu\zeta + 216\phi\mu\zeta + 216\phi\mu\nu \\
& + 81\phi\zeta\psi^2 - 162\phi\eta^2\mu, \alpha^2 + \beta^2 + \nu^2 - 1, -864\nu\mu^2 - 432\nu\alpha^2 - 432\nu\eta^2 \\
& + 216\nu\phi^2 - 756\nu\psi^2 - 216\nu\delta^2 - 432\mu\alpha^2 - 432\mu\eta^2 - 756\mu\psi^2 + 216\mu\phi^2 \\
& - 216\mu\delta^2 + 216\mu\beta^2 + 216\nu\beta^2 - 216\zeta\alpha^2 - 216\zeta\eta^2 + 108\zeta\phi^2 - 324\zeta\psi^2 \\
& - 108\zeta\delta^2 + 108\zeta\beta^2 + 972\phi^2\nu^2 + 972\phi^2\mu^2 + 1296\eta\zeta^2\alpha + 3888\eta\nu^2\alpha \\
& - 324\eta\alpha\psi^2 + 3888\eta\mu^2\alpha + 432\eta\alpha\zeta + 864\eta\alpha\nu + 864\eta\alpha\mu + 2592\eta\alpha\nu\zeta \\
& + 2592\eta\alpha\mu\zeta + 5184\eta\alpha\mu\nu - 1296\delta^2\mu\nu - 216\mu\zeta^2 + 72\beta^2 - 864\mu^3 \\
& - 2916\nu^2\psi^2 + 2592\nu^3\mu - 1944\nu^2\eta^2 + 378\alpha^4 + 324\delta^4 + 1134\eta^2\delta^2 - 1296\mu^4 \\
& + 810\alpha^2\beta^2 - 756\eta\alpha^3 - 54\eta\phi^3 + 54\eta\psi^3 + 288\eta\alpha - 72\delta^2 - 108\psi^2 + 72\phi^2 \\
& - 432\zeta\mu\nu - 864\zeta\mu^2 + 378\eta^4 - 648\alpha^2\zeta^2 + 324\alpha^2\psi^2 - 1944\alpha^2\mu^2 + 648\alpha^2\eta^2 \\
& - 648\zeta^2\psi^2 - 1296\zeta^2\mu^2 - 648\zeta^2\eta^2 - 1296\zeta\mu\eta^2 - 2268\zeta\mu\psi^2 - 324\delta^2\zeta^2 \\
& - 972\delta^2\nu^2 - 2268\nu\zeta\psi^2 - 2592\nu\zeta\mu^2 - 1296\nu\zeta\eta^2 - 648\delta^2\nu\zeta - 648\delta^2\mu\zeta \\
& + 64\mu - 144\eta^2 - 144\alpha^2 + 144\mu\nu - 72\mu\zeta - 2592\zeta\mu^3 - 54\alpha\phi^3 - 756\alpha\eta^3 \\
& + 54\alpha\psi^3 - 1296\nu\mu^3 + 162\psi^4 - 2592\nu\mu\eta^2 - 5508\nu\mu\psi^2 + 972\phi^2\nu\zeta \\
& + 972\phi^2\mu\zeta + 972\phi^2\mu\nu - 1296\alpha^2\nu\zeta - 1296\alpha^2\mu\zeta - 2592\alpha^2\mu\nu - 324\alpha\eta\delta^2 \\
& - 144\mu^2 - 2916\psi^2\mu^2 + 324\psi^2\eta^2 - 972\mu^2\delta^2 - 1944\mu^2\eta^2 + 972\nu^2\beta^2 - 1944\nu^2\alpha^2 \\
& - 324\eta\alpha\phi^2 + 486\alpha^2\delta^2 + 648\beta^2\nu\zeta + 648\beta^2\mu\zeta + 1296\beta^2\mu\nu - 243\phi^2\beta^2 \\
& - 81\phi^2\delta^2 + 324\beta^2\zeta^2 + 81\beta^2\psi^2 + 972\beta^2\mu^2 + 162\beta^2\eta^2 + 243\delta^2\psi^2 - 324\eta\alpha\beta^2]
\end{aligned}$$

```

> ReducedCond:=inter_reduce(Cond,termorder(poly_algebra(alpha,eta,beta,
delta,phi,nu,mu,zeta,psi,kappa),plex(alpha,eta,psi,beta,delta,phi,nu,m
u,zeta,kappa)));

```

$$\begin{aligned}
\text{ReducedCond} := & [-31 + 27\zeta^2 + 36\mu\zeta + 72\mu^2 + 6\zeta + 12\mu + 27\kappa^2 + 36\mu\kappa + 18\kappa\zeta + 6\kappa, \\
& -35 + 81\kappa\zeta^2 + 162\mu\zeta^2 + 81\kappa^2\zeta + 162\mu\kappa^2 - 81\zeta^3 - 81\kappa^3 + 9\kappa + 9\zeta + 324\kappa\mu^2 \\
& + 324\kappa\mu\zeta + 324\zeta\mu^2 + 27\kappa^2 + 27\zeta^2 + 18\mu + 108\mu\kappa + 108\mu\zeta + 54\kappa\zeta + 108\mu^2, \\
& -64 + 243\kappa\zeta^2 + 486\mu\zeta^2 + 243\kappa^2\zeta + 486\mu\kappa^2 + 81\zeta^3 + 81\kappa^3 + 648\mu^3 - 126\kappa \\
& - 126\zeta + 810\kappa\mu^2 + 648\kappa\mu\zeta + 810\zeta\mu^2 + 81\kappa^2 + 81\zeta^2 - 252\mu + 216\mu\kappa \\
& + 216\mu\zeta + 108\kappa\zeta + 270\mu^2, -98 - 11664\kappa\mu^2\zeta - 486\mu^2\kappa^2 - 2430\kappa\zeta^2 \\
& - 2997\mu\zeta^2 - 2268\kappa^2\zeta - 2349\mu\kappa^2 - 405\zeta^3 - 243\kappa^3 + 1296\mu^3 + 639\kappa + 849\zeta \\
& + 3888\kappa\mu^3 - 3888\kappa\mu^2 - 1701\zeta^4 + 7776\mu^4 - 6804\kappa\mu\zeta - 4536\zeta\mu^2 - 2187\kappa^4 \\
& + 1809\kappa^2 + 1755\zeta^2 - 243\zeta^3\mu - 2430\zeta^2\mu^2 - 30\mu - 2106\mu\kappa - 2214\mu\zeta + 648\kappa\zeta \\
& + 3888\zeta\mu^3 - 2700\mu^2 - 7047\kappa^2\zeta\mu - 8019\kappa\zeta^2\mu + 729\kappa^3\mu - 729\kappa^3\zeta \\
& - 3402\kappa^2\zeta^2 - 1701\kappa\zeta^3, 6\mu + 1 + 6\nu + 3\kappa + 3\zeta, \phi^2 + \zeta^2 - 1, \psi^2 + \kappa^2 - 1, 61\phi \\
& + 288\eta\delta^2\mu - 810\phi\zeta\mu^2 - 468\phi\zeta^2\mu - 648\phi\mu^3 - 18\eta\kappa + 24\eta\delta^2 - 108\phi\zeta\kappa \\
& - 78\zeta^2\phi + 6\eta\mu^2 + 72\eta\mu^3 - 648\phi\mu\zeta\kappa - 81\phi\kappa^2 + 18\mu\psi - 18\mu\psi\kappa^2 + 3\psi - 6\eta \\
& - 810\mu^2\phi\kappa + 27\psi\kappa - 234\phi\kappa\zeta^2 + 117\phi\kappa - 18\eta\zeta - 27\psi\kappa^3 - 270\mu^2\phi \\
& + 72\eta\delta^2\kappa + 72\eta\delta^2\zeta - 54\phi\zeta^3 + 99\phi\zeta + 234\phi\mu - 216\phi\mu\zeta + 9\psi\zeta - 9\psi\zeta\kappa^2 \\
& - 486\phi\mu\kappa^2 - 3\psi\kappa^2 - 243\phi\zeta\kappa^2 + 18\eta\mu^2\kappa + 18\eta\mu^2\zeta - 72\eta\mu - 216\phi\mu\kappa \\
& - 81\phi\kappa^3, \eta^2 + \delta^2 + \mu^2 - 1, 72\alpha\beta^2 - 90\mu^2\alpha - 36\alpha\mu\zeta - 36\alpha\mu\kappa - 12\alpha\mu \\
& - 36\zeta^2\alpha - 18\alpha\zeta\kappa - 6\alpha\zeta - 36\alpha\kappa^2 - 6\alpha\kappa + 29\alpha + 72\eta\delta^2 + 18\eta\mu^2 - 18\eta \\
& - 9\psi\kappa^2 + 9\psi + 9\zeta^2\phi - 9\phi, -1080\alpha\zeta\mu^2 - 756\alpha\mu\zeta^2 - 108\alpha\zeta\kappa - 1512\alpha\mu^3 \\
& + 58\alpha + 27\phi\zeta^2\mu - 756\alpha\mu\kappa^2 - 234\alpha\mu\zeta - 108\zeta^2\alpha + 36\alpha\beta^2 - 108\alpha\kappa^2 \\
& + 108\alpha\beta^2\kappa + 108\alpha\beta^2\zeta - 297\alpha\zeta^3 + 27\mu\psi - 27\mu\psi\kappa^2 - 324\alpha\zeta\kappa^2 \\
& - 702\alpha\mu\zeta\kappa - 1080\mu^2\alpha\kappa + 243\alpha\zeta + 573\alpha\mu - 27\psi\kappa + 243\alpha\kappa + 27\psi\kappa^3 \\
& + 432\alpha\mu\beta^2 - 360\mu^2\alpha - 27\phi\zeta^3 - 297\alpha\kappa^3 + 27\phi\zeta - 27\phi\mu - 234\alpha\mu\kappa \\
& - 324\alpha\kappa\zeta^2, -17669 + 40824\kappa\mu^2\zeta - 46656\mu^2\kappa^2 + 2160\kappa\beta^2 + 4320\mu\beta^2 \\
& + 2160\zeta\beta^2 - 2592\eta\zeta^2\alpha - 5184\eta\mu^2\alpha - 432\eta\alpha\zeta - 864\eta\alpha\mu - 2592\eta\alpha\mu\zeta \\
& + 1296\kappa\zeta^2 + 3888\mu\zeta^2 + 648\kappa^2\zeta + 1296\mu\kappa^2 - 10440\beta^2 - 4968\zeta^3 - 5616\kappa^3 \\
& - 6048\mu^3 + 2940\kappa + 2100\zeta - 18144\kappa\mu^3 + 13608\kappa\mu^2 - 5184\delta^4 - 20250\zeta^4 \\
& - 93960\mu^4 + 792\eta\alpha + 22032\kappa\mu\zeta + 6480\delta^2 + 16200\zeta\mu^2 - 18306\kappa^4 - 5184\beta^4 \\
& - 2592\eta\alpha\kappa^2 + 31104\kappa^2 + 31320\zeta^2 - 432\eta\alpha\kappa - 29808\zeta^3\mu - 38880\zeta^2\mu^2 \\
& + 12960\beta^2\kappa^2 - 648\eta\psi\kappa^2 + 4200\mu - 2592\eta\alpha\mu\kappa + 31752\mu\kappa + 32184\mu\zeta \\
& + 15876\kappa\zeta - 18144\zeta\mu^3 + 12960\mu\beta^2\kappa + 6480\zeta\beta^2\kappa + 5184\alpha\eta\delta^2 + 81324\mu^2 \\
& - 6480\mu^2\delta^2 + 648\alpha\psi - 648\alpha\psi\kappa^2 - 1296\eta\alpha\zeta\kappa - 648\alpha\phi + 12960\beta^2\mu\zeta \\
& + 648\eta\phi\zeta^2 + 648\alpha\phi\zeta^2 + 12960\beta^2\zeta^2 + 32400\beta^2\mu^2 + 3888\kappa^2\zeta\mu + 7776\kappa\zeta^2\mu \\
& - 33696\kappa^3\mu - 16848\kappa^3\zeta - 24300\kappa^2\zeta^2 - 12960\kappa\zeta^3 - 648\eta\phi + 5184\eta\alpha\beta^2 \\
& + 648\eta\psi]
\end{aligned}$$

```

> Cond_gsolve:=gsolve(ReducedCond,
  {alpha,eta,beta,psi,delta,phi,nu,mu,zeta,kappa});

```

$$\begin{aligned}
\text{Cond_gsolved} := & \{ [137749507200000 \nu^{12} + 91833004800000 \nu^{11} - 190043301600000 \nu^{10} \\
& - 113459110560000 \nu^9 + 99098236296000 \nu^8 + 49729860576000 \nu^7 \\
& - 23574334932000 \nu^6 - 9037158530400 \nu^5 + 2429532364320 \nu^4 + 605658954960 \nu^3 \\
& - 96421318920 \nu^2 - 9600201720 \nu + 269577679, 882237344002756566 \mu \\
& + 184914375299378064000000 \nu^{11} + 87339111681328646400000 \nu^{10} \\
& - 188663594377202568000000 \nu^9 - 70095471715232269200000 \nu^8 \\
& + 63847635491870416080000 \nu^7 + 16088118532231145868000 \nu^6 \\
& - 7646043205351995145200 \nu^5 - 908221854751171119000 \nu^4 \\
& + 266325750825546657600 \nu^3 - 16397994046603147440 \nu^2 \\
& + 6489335551993867416 \nu + 504999696376840859, \\
& 1884324789218430808783997176712938562272200600000 \nu \eta^2 \\
& - 26696865707621524095576897588625178198667455220 \nu \\
& - 25464621982489374623624854567453057408996147200000000 \eta^6 \nu^{11} \\
& + 213651654813630292013197152510706635369407201280000000 \eta^6 \nu^9 \\
& + 331109068074270001821665536934807144347355873280000000 \eta^6 \nu^{10} \\
& - 331914062902099319527889075660913715761818926080000000 \eta^6 \nu^8 \\
& + 110413790890671980834138288926659328703568889344000000 \eta^6 \nu^6 \\
& - 166980675212704666889656232268791287812318336000000000 \eta^6 \nu^7 \\
& + 1694348762368999880727186777585816553188737825040 \nu^3 \\
& - 13248014476754187673365261466241594381107659571200000 \eta^6 \nu^4 \\
& + 41535952459016730657410497944925505150016741273600000 \eta^6 \nu^5 \\
& - 3287557893969552897385381216436897564526363724800000 \eta^6 \nu^3 \\
& + 5616846024124343561010149648747026508518544012880 \nu^4 \\
& + 7508569301222986379309540097155626627895948342400000 \eta^2 \nu^{10} \\
& - 8373630869524671271562594887334213393168985024000000 \eta^2 \nu^{11} \\
& - 8173146274432089807899651246120356004974851136800000 \eta^2 \nu^8 \\
& + 14860748440592486695461046016882372682683228284800000 \eta^2 \nu^9 \\
& - 8207168280193083236402546586750763984598792009280000 \eta^2 \nu^7 \\
& - 255700290976652905249013832371209609941497275380 \nu^2 \\
& - 53184483616668097594714804910255399480560292100 \eta^2 \\
& - 83042901476689409241861100845222451540058619456000000 \eta^4 \nu^9 \\
& - 89013113185669928581311446070202958825325835200000000 \eta^4 \nu^{10} \\
& + 26796408319213519323918230427493271666724526080000000 \eta^4 \nu^{11} \\
& - 30829065704970537817296295047683629201871430868800000 \eta^4 \nu^6 \\
& + 56165079554061550856270620715585438956798129036800000 \eta^4 \nu^7 \\
& + 90866129671392136440056284020016350506439817382400000 \eta^4 \nu^8 \\
& + 997829940213560367528087757753154025326832324192000 \eta^4 \nu^3 \\
& + 3760779514115617491335535213081585120648847841184000 \eta^4 \nu^4 \\
& - 13088327606059748281070920045039454751951091692480000 \eta^4 \nu^5 \\
& - 164053936954582144547999605730702667515516617584000 \eta^4 \nu^2 \\
& - 16816905953675229278064935528775559490257381104000 \eta^4 \nu \\
& - 192674442271602653794201910456258616102063909120000 \nu^9 \\
& + 17772814202614516410814562931098669080710521834400 \eta^2 \nu^2 \\
& - 117549558565249912622872311715454241032333198400000 \nu^{10} \\
& + 493397520169293755787755198122382913575669758800 \eta^4 \\
& + 100115192226182666470536873923623779671626848000000 \nu^{11}
\end{aligned}$$

$$\begin{aligned}
& + 110668138544325532871057351502121203325049335296000 \nu^7 \\
& - 382327610204042139173705743459753328811877552320000 \eta^2 \nu^4 \\
& + 1702362037410218923812350415292379995835511533136000 \eta^2 \nu^5 \\
& + 2963222758793631674286862783451165402276329368432000 \eta^2 \nu^6 \\
& - 121235613657885845788084492074450762025534038984000 \eta^2 \nu^3 \\
& + 564341386448335919887060294495349930084109930240000 \eta^6 \nu^2 \\
& + 58150156221331592070083978144702807427032663040000 \eta^6 \nu \\
& - 1841052561017886706136161758406760612258006400000 \eta^6 \\
& + 125323723693367704491149393746336762729281297144000 \nu^8 \\
& - 23509609656163851746414386362612433584598539760800 \nu^5 \\
& + 1751952264209188123192549574616369293819904000000000 \eta^{10} \nu^{11} \\
& - 44457946811723218573257696215873776489526176644000 \nu^6 \\
& + 743715432261681591040898764375704652416858031 \\
& - 499421341789405949709985419479747913248784384000000000 \eta^8 \nu^{10} \\
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& - 2512283984478142250103374881631129423369699328000000 \eta^{10} \nu^3 \\
& - 267458175333057022709240151247507901507321856000000000 \eta^{10} \nu^8 \\
& - 10477345405016720774999488405741188578137128960000000 \eta^{10} \nu^4 \\
& + 30365623700460681018295112758951118925158154240000000 \eta^{10} \nu^5 \\
& + 1294486946190898426393015165777439363793715200000 \eta^{12} \\
& - 3099506755926746897975550649320427946692608000000 \eta^{10} \\
& + 87975582507368913703779228522690162785468252160000000 \eta^{10} \nu^6 \\
& + 2698374741466479094707772931469248249652838400000000 \eta^{10} \nu^{10} \\
& + 19607104364853117273255033114809609778280162406400000 \eta^8 \nu^4 \\
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& + 4751561692451994068533874594862581187954551193600000 \eta^8 \nu^3 \\
& - 87292381708104809570148213411342069803274086400000 \eta^8 \nu \\
& + 3338475651354562530806941170841307642820664320000 \eta^8 \\
& + 48083962790356253237564595747093523253914828800000 \eta^{10} \nu \\
& + 436845554153752516225908350281696457922075033600000 \eta^{10} \nu^2 \\
& - 825688845673378136741203608478411489415479142400000 \eta^8 \nu^2 \\
& + 227738357971261236040762152396245328912721797120000000 \eta^8 \nu^7 \\
& + 12520504385628033496959108998141064949039104000000000 \eta^{10} \nu^9 \\
& + 497187565394246313685584258690984986801700618240000000 \eta^8 \nu^8 \\
& - 267093405035578166025840220263019343583626035200000000 \eta^8 \nu^9, \\
& - 2147760834258288588 \zeta + 98387964279618884640 \zeta \nu^2 \\
& - 73608831541079708196 \nu + 4283910046889713175760 \nu^3 \\
& + 16335253490056873342200 \nu^4 + 420572830291393615200000 \nu^8 \zeta \\
& + 2646712032008269698 \zeta^2 - 697187136970177824354 \nu^2 \\
& - 33642589247946665100 \nu \zeta - 353217082881133850400000 \nu^9 \\
& - 420364766130800012640000 \nu^{10} - 1597954504953279945600 \zeta \nu^3 \\
& + 97237382045545756800000 \nu^{11} + 245713282960109963616000 \nu^7 \\
& + 2729841890269375051 + 423522308364905289240000 \nu^8 \\
& - 57300938743228807680000 \nu^5 + 45876259232111970871200 \zeta \nu^5
\end{aligned}$$

$$\begin{aligned}
& - 96528711193386875208000 \nu^6 \zeta - 383085812951222496480000 \nu^7 \zeta \\
& - 1109486251796268384000000 \zeta \nu^{11} + 1131981566263215408000000 \zeta \nu^9 \\
& - 524034670087971878400000 \zeta \nu^{10} - 140301659271092969365200 \nu^6 \\
& + 5449331128507026714000 \nu^4 \zeta, 1323356016004134849 \delta^2 \\
& + 1323356016004134849 \eta^2 + 5970078404203521600000 \nu^{11} \\
& + 91951756332521661360000 \nu^{10} + 42665770277772951600000 \nu^9 \\
& - 91140969374774395740000 \nu^8 - 39103326994847399208000 \nu^7 \\
& + 29979191554145869451400 \nu^6 + 10347608093427194529600 \nu^5 \\
& - 3570335494506521910900 \nu^4 - 856103940008291490120 \nu^3 \\
& + 148862629542184544244 \nu^2 + 16385436618777370102 \nu \\
& - 1056210190498188195, -1191798660045217320103900805492145556111 \backslash \\
& 42770828563905319310785879390978564228311148748\alpha + 325084982519 \backslash \\
& 835097011856876242839845105522083417487633308272292503109745386 \backslash \\
& 655722952454956440\nu^2 \eta^3 - 24188788828822909760241781213212781702 \backslash \\
& 43739137167759433405261176290474938973919390685782200000\nu^5 \eta^3 - \\
& 591921876545704795753340256424480652099218449869145943431640290 \backslash \\
& 29989170410720824225328544000\nu^7 \eta + 9057397502470615914518622673 \backslash \\
& 524736418099571303990936974244652072129608942858848885025549600 \backslash \\
& 00\nu^8 \eta + 3582868506068306887373268082993205155576236072393968620 \backslash \\
& 61386400033788039096429086843967680000\nu^9 \eta - 7693457923662890264 \backslash \\
& 402538540484156955652110769212360823720355277267727302545264810 \backslash \\
& 30385280000\nu^{10} \eta - 438100418708860166870130266276663064131154362 \backslash \\
& 456468362704102104756763775687323875379340800000\nu^{11} \eta - 17252862 \backslash \\
& 640341021429061004056783871647654570342514771680335771239047440 \backslash \\
& 502023305235222062400\nu^5 \eta + 568836656160476523736931206016236538 \backslash \\
& 19260474491293742479867835461208922747148472001494878600\nu^4 \eta - 3 \backslash \\
& 674943400550699821602844101713195798335799623364200931971847860 \backslash \\
& 99719698605096362722391799200\nu^6 - 4944949440568251601161852844 \backslash \\
& 718199700040899899552748841544386147890649562952095023128582269 \backslash \\
& 60\eta^3 \nu^3 - 773711132328333943927002292833387236617656402466765890 \backslash \\
& 7566564269980434753802596752695682913600\eta^3 \nu^4 + 1191798660045217 \backslash \\
& 320103900805492145556111427708285639053193107858793909785642283 \backslash \\
& 11148748\alpha \nu^2 + 6834604880518709215335909169199653448884352958161 \backslash \\
& 637398421302120981188750381336400210969\eta + 47671946401808692804 \backslash \\
& 156032219685822244457108331425562127724314351756391425691324459 \backslash \\
& 4992\alpha \beta^2 - 72229532557658849483025774657530989096444058196589661 \backslash \\
& 3336609091142302420529845452849702784\eta^3 + 4892726315228398126982 \backslash \\
& 406052971173997839542904056710643807086694789919053875829703175 \backslash \\
& 1879128000\nu^6 \eta^3 + 3431100748964362148960491847995918182961441840 \backslash \\
& 6782059404400108454336441958496410051366606032000\nu^7 \eta^3 - 1187313 \backslash \\
& 339971364881730190406593525871672674117846273654848805961647490 \backslash \\
& 71193060744938686794400000\nu^8 \eta^3 - 102342361788972321997300804704 \backslash \\
& 084540240760326660259992456214021317654225482555880968385124800 \backslash \\
& 000\nu^9 \eta^3 + 99719506487973798191384524710983161978309151155444791 \backslash \\
& 406433072466925674610377145738771420800000\nu^{10} \eta^3 + 95680607530557 \backslash
\end{aligned}$$

$$\begin{aligned}
& 113241160912733783728767438108009595848707175131017716125780152 \backslash \\
& 702443774662400000 \nu^{11} \eta^3 + 22637465101472757504174394449978556666 \backslash \\
& 478753293189069763328123883913916100647143108634020640 \nu \eta^3 + 544 \backslash \\
& 370576481080399012406415488776243112305615243788456759846069429 \backslash \\
& 6713340116213900929908440 \eta \nu^3 - 17956635152688054914023103323633 \backslash \\
& 3621913793837425908433188966918166805553142945751401265708 \eta \nu - \\
& 246614764918355689832062563244511700398417508114219411324214363 \backslash \\
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& 867175994367199856012625407086072214951124479720894885268446208 \backslash \\
& 000 \eta^7 \nu^4 - 19663538867199294922691824986551567783679462455861588 \backslash \\
& 323104561251695797425968721106140160000000 \eta^7 \nu^5 - 35504768488712 \backslash \\
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& 24833863533926400 \eta^7 \nu^3 - 877120958883998239809216526391958609678 \backslash \\
& 678320433928005981428480982887318398308679627316736000000 \eta^7 \nu^8 + \\
& 110479338642916074966294635093588976610716732881409290078784295 \backslash \\
& 95217943455918486284144107520000 \eta^9 \nu^5 + 183523847769744097956994 \backslash \\
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& 490899645039942628805639853041594394140611102431821970597120000 \backslash \\
& 0 \eta^5 \nu^8 + 4858162752845491044023776802228752932634963322840642411 \backslash \\
& 85858246142926328845152513671551744000000 \eta^5 \nu^9 - 163903846648384 \backslash \\
& 729447196891972356017501763264711008861425522247912052518408615 \backslash \\
& 238845086384640000 \eta^5 \nu^7 - 77661594243806404757030142524770164004 \backslash \\
& 6616520313635041191266557222518258986389390247115776000000 \eta^7 \nu^9 - \\
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& 164501503711225015258316800000000 \eta^9 \nu^{11} + 41712704804349999910924 \backslash \\
& 713251672032909854228572103605271356732927832373859209018523374 \backslash \\
& 3872000000 \eta^9 \nu^9 - 3898283120622465309917318614675396319855464389 \backslash \\
& 53514606997820360819146371161175793453891584000000 \eta^9 \nu^{10} + 465257 \backslash \\
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& 964336815114849624064000000 \eta^9 \nu^8 - 19243367168379578248111377850 \backslash \\
& 996933212543155263982370881655755503464174654822084199077814272 \backslash \\
& 0000 \eta^9 \nu^6 - 1425446787161772214799378418930304332055121057017637 \backslash \\
& 72837390830264305684872503452288179404800000 \eta^9 \nu^7 + 306364805705 \backslash \\
& 932626790741598616203833743311134523678129556092474195889323969 \backslash \\
& 00048326378106880000 \eta^9 \nu^4 - 563164809515704107091781932260500052 \backslash \\
& 7671929201543946884618266239421049067740800225837009920 \eta^7 + 3186 \backslash \\
& 366341654571519572562424046785037520582163352735084955122229480 \backslash \\
& 107669344598343164620800 \eta^9 - 15247926644235813358887448989885847 \backslash \\
& 66242243292685719515468909430698474635194203676859226589440 \eta^5 \nu^2 + \\
& 166530748959035946590395985932756259489642840943038953545250270 \backslash \\
& 957833788229651104290830105600 \eta^7 \nu + 242980417254376247037990465 \backslash
\end{aligned}$$

$$\begin{aligned}
& 320756620983535883563551370103153487218561917066515022697895237 \backslash \\
& 1200\eta^7 \nu^2 + 3428935269123615760825071437820986668523397700618191 \backslash \\
& 909178329473571053573932721397049948320\eta^5 - 10560985774414800108 \backslash \\
& 573782237065267877849964173238194107402981621118877452331006514 \backslash \\
& 6975901760\eta^5 \nu - 87733883166881971538076311572848248509324570870 \backslash \\
& 967078647986496220318233244071939155481190400\eta^9 \nu - 130604384972 \backslash \\
& 700971008772906758340592339500951207451128432807971290178299461 \backslash \\
& 2487408585868902400\eta^9 \nu^2 + 2278143339551605206737895296479157355 \backslash \\
& 820238715906526751530903812984229666389556791445911897600\eta^5 \nu^3 - \\
& 228717988921944857696712412663637438705715655906463388120692048 \backslash \\
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& 639574493888541083520447737563939977743555525033052007130483739 \backslash \\
& 931456000\eta^5 \nu^5 + 36219155907136992110768253856049521407675705106 \backslash \\
& 584253260352563882424214356120123950619019616000\eta^5 \nu^4 + 26335715 \backslash \\
& 814662402497836368951261361393491342420633233753458693363449478 \backslash \\
& 4868701725781436590080000\eta^7 \nu^7 + 3620842218786935610541787694802 \backslash \\
& 480588970944105278244700235129155661957691891629729619638528000 \backslash \\
& 00\eta^7 \nu^6 + 735926548103905420492998303054071339789290049016759886 \backslash \\
& 286547567560531390825487268173723648000000\eta^7 \nu^{10} + 72105878552249 \backslash \\
& 125543358923725311345726686140161707924019645581978189640916372 \backslash \\
& 4093710667776000000\eta^7 \nu^{11}, 441118672001378283 \kappa + 441118672001378283 \zeta \\
& - 184914375299378064000000 \nu^{11} - 87339111681328646400000 \nu^{10} \\
& + 188663594377202568000000 \nu^9 + 70095471715232269200000 \nu^8 \\
& - 63847635491870416080000 \nu^7 - 16088118532231145868000 \nu^6 \\
& + 7646043205351995145200 \nu^5 + 908221854751171119000 \nu^4 \\
& - 266325750825546657600 \nu^3 + 16397994046603147440 \nu^2 \\
& - 5607098207991110850 \nu - 357960139043048098, 97261463459468224162 \backslash \\
& 171453956121777689543236312295278280672793780817474312537213143 \backslash \\
& 453564821435277094985792321120\nu^2 \eta^3 - 98592273083964905513779875 \backslash \\
& 417897324965428381804798029860882739723685691297559264283360864 \backslash \\
& 5100164996294094823136000\nu^5 \eta^3 - 1065456686012978908186963673660 \backslash \\
& 898477638280418001099303621900065945521601911627549920592689879 \backslash \\
& 19066496946595516000\nu^7 \eta + 1520130916715149486614321095005368690 \backslash \\
& 439958361027023017613321202547796969323715409769176200636023504 \backslash \\
& 06043118600000\nu^8 \eta + 2475201071760321367243108113585756382664346 \backslash \\
& 587158396080723287340918517329940514086310413583250471640817349 \backslash \\
& 26640000\nu^9 \eta - 1130966949194031472689177248375260677697329226048 \backslash \\
& 675516579047602447047221449045309531834731257326345662889198400 \backslash \\
& 00\nu^{10} \eta - 188054080627918566382689649381728008147578019997021948 \backslash \\
& 733081031392294110972391451975579359863622328259246782400000\nu^{11} \eta + \\
& 147563456180386013491968983008644398840483789387672324073539448 \backslash \\
& 57487832555588401123875614365750590350885064578400\nu^5 \eta + 1236285 \backslash
\end{aligned}$$

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 7 $\phi - 11315885022958980496697884896243186284295506016763623660936$
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 276890238493305625276819767810400 $\zeta \nu^2 \eta^3 + 6903588752489394750523$
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194407606780781755467677422788568086424029174369459723189896913\
947292464332403355750400000\zeta \eta^7 \nu^7 + 1636472303541606666685638022\
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9427584000\zeta \eta^9 \nu^2 - 137698870591633239053214493400218600973387335\
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60000\zeta \eta^9 \nu - 911436185811173473083138943802530091797897244318488\
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860097338733593352182816277008888702064739287604412728136207651 \backslash

693290803189760000 $\zeta \eta^9 \nu - 91143618581117347308313894380253009179 \backslash$
78972443184884841906222092042986797566131814092491966915581804 \backslash

99143254400 $\zeta \eta^5 \nu + 669186163702759755964225324341845015343809077 \backslash$
695922021469832935529665719889607700692699591622284119322945461 \backslash

76000 $\zeta \eta^7 \nu^2 + 18841050120706817705024297047388344940762236264488 \backslash$
864600860433401103684928645657449385730014766568412552834406400 $\zeta \eta^7 \nu$
 $- 552483981790540779555514594806368642118556308905930161395063 \backslash$

5163756650987188479298285011064690468359866024032000 $\zeta \eta^5 \nu^2 + 970 \backslash$
394533755634113133644488450908776313685500659397792437352449514 \backslash

457596772910159599771717410958263473956413440000 $\zeta \eta^9 \nu^4 - 9104902 \backslash$
255077069326931238616699852470807243365819054252205415014132462 \backslash

469113332124116706685702202676447582617600000 $\zeta \eta^9 \nu^7 - 7793526473 \backslash$
580296547473051840667136342439311051816020048565935807786492938 \backslash

388617891295310202787308561919664128000000 $\zeta \eta^9 \nu^6 + 2273763288960 \backslash$
637361481277540860012109117602050794417275043868008519097550128 \backslash

1749100258993251110705243888336896000000 $\zeta \eta^9 \nu^8 - 223720742353890 \backslash$
929248740099579070678472452482827344262202662795119060998181329 \backslash

62574726547863839577649524572160000000 $\zeta \eta^9 \nu^{10}$ + 3289460121968902\
 355186267675076457668041725886433613019720695772578409861520719\
 5015416830758561802950227394560000000 $\zeta \eta^9 \nu^9$ - 333222690760749873\
 426106292096061756537270096635664768168944490247761451679690045\
 36577556146284898980264017920000000 $\zeta \eta^9 \nu^{11}$ - 7347852939669633743\
 370978819802001958724629134943133764107940789961450372018058643\
 0507024847887329707458041856000000 $\zeta \eta^7 \nu^9$ - 181116792162670128119\
 289065707417538094909181625053579458482672009826274222770238277\
 77210941619485263766730152960000 $\zeta \eta^5 \nu^7$ + 50779619151778562740707\
 232230599972951066876953126623523944944565514491602001524884480\
 727289332747832396921600000000 $\zeta \eta^5 \nu^9$ + 2984202019028640914788696\
 926021781648638238074067501389310609486959299834930484543355048\
 2323087085587557906521600000 $\zeta \eta^5 \nu^8$ - 457285614306195171611029820\
 023883493854530862555648286348939638144945128819393549400015367\
 64007343083811182592000000 $\zeta \eta^5 \nu^{11}$ - 2749262457144628329186588909\
 800070679966471433405954544596559780603129369923007344543517318\
 3790734133512736512000000 $\zeta \eta^5 \nu^{10}$ + 25613984887759239508336250132\
 429401548920138930884698307238264080237904931152758906885541857\
 9822440394994761728000 $\zeta \eta^9 \nu^3$ - 743644079075256955018709983314190\
 306961205756867927533024137570883042923986573983640867222784812\
 87448617369600000 $\zeta \eta^9 \nu^5$ - 46321771737179210661240016675234451326\
 704994331934618556808851526454165579675197962551874749605466191\
 409411072000000 $\zeta \eta^7 \nu^8$ - 2970625178202408538196392472007657545481\
 372567429421005225756742570774716858351343710032056477328279838\
 48340992000 $\zeta \eta^7 \nu^3$ - 16703312251023423392268314509032796027119880\
 556538180026610056792860333179964487765642298993701808088785031\
 37280000 $\zeta \eta^7 \nu^5$ - 23364603812939450403225078052269509007369387730\
 34034385914196391551034333117537104605772004758972869842345625\
 60000 $\zeta \eta^7 \nu^4$ + 31798517721450054628780247993483454795760602205760\
 6051927583883936102383940831971090674491926732378523973163120 $\zeta \eta \nu^3$ +
 159823995839413492413992992298571595163299409007423475818666315\
 0455728650707485278448482543797103386735704030320 $\zeta \nu \eta^3$ + 9883461\
 199733456406859402889163598166899358175201378383649460929991964\
 137242564904441460670872569300148608320000000 $\zeta \nu^{11} \eta^3$ + 557482979\
 536695282520879571926210530369921784906853926875857790569592406\
 6405298044100694773537476048329709536000000 $\zeta \nu^{10} \eta^3$ + 73797997352\
 694389191118680198332254398363187711094477189389187768595042911\
 926908193142436787278818831263929680 $\eta \zeta \nu^2$ - 20611987478602182454\
 193213369539213172795290740751371014314107425634893484880680344\
 856295798746377780201756760 $\eta \zeta \nu$ + 17665288696578747319879648018\
 486369181885999458839324757262808683585676707263601542397859875\
 6386811453447190510 η^3 - 1435932375657996238481880327484492649912\
 643109396860024728645545267806860439126892265908351566705157383\
 3882186456000 $\nu^6 \eta^3$ - 1198278256529830465068641300577160491954454\
 275513612742062386173406745365327012180376717956493177924125679

$$\begin{aligned} & \backslash \\ & 2432400000\nu^7\eta^3 + 3485348366187957829170961508638058777073235751\backslash \\ & 503681089118160871792714719312739438810581516937213237591503416\backslash \\ & 5600000\nu^8\eta^3 + 3412561384935812274823236869690867289037798085514\backslash \\ & 719852079715525102911421784589698235092722905060501004613866080\backslash \\ & 0000\nu^9\eta^3 - 2931877231409739508183026798230771293433360822917071\backslash \\ & 832536338543883861818112275143541926308841996646023345768320000\backslash \\ & 0\nu^{10}\eta^3 - 3105166069264340881807585044908152844181241044880673215\backslash \\ & 0352575040320770847495764329508722900479571672014037664000000\nu^{11}\eta^3 - \\ & 635793123163102306537154485659001729133469028929401821471260951\backslash \\ & 4907710810571964201447648546960697929772165740300\nu\eta^3 - 17541050\backslash \\ & 044536004814050531353970119530658523938714097380707819951144513\backslash \\ & 68356932577601854435009020634871193805480\nu\eta^3 + 5696968247464571\backslash \\ & 007712783833468568054096548383445151464015292481140515668761549\backslash \\ & 4043979428024518452920549092130\nu\eta + 711911509665135910826128753\backslash \\ & 778882512663341773411952464340846189798984681302295114914866980\backslash \\ & 827396046415506122070\nu\eta^2 + 165423181061763522614504904355689253\backslash \\ & 005147443951278480593414818988793388238674374037697202661194584\backslash \\ & 05217257345280000\eta^7\nu^4 + 765410404284167662540429410743421256497\backslash \\ & 902481999904040279994151302797254564418938485390875730334437567\backslash \\ & 3827566080000\eta^7\nu^5 + 9361925061481826987509702327193755468168307\backslash \\ & 353683946788384800624058252226879901894869142146365482577624466\backslash \\ & 39872000\eta^7\nu^3 + 250844581373471016388681899379950418940337131180\backslash \\ & 366613796537182700006056463119501014388589794101307884385523200\backslash \\ & 000000\eta^7\nu^8 - 37771560391522492263785761293121798325946898711499\backslash \\ & 582000697822449620812491571940596404102578204383185152481280000\backslash \\ & 00\eta^9\nu^5 - 517739611415590549797115963500007311979218234463129909\backslash \\ & 527490366835627119082182802788033014135495467267751911424000\eta^9\nu^3 + \\ & 134943289500063298177209672916429254960871068410786269671845814\backslash \\ & 184252750188444741412053540487155076029490471296000000\eta^5\nu^{10} + 14\backslash \\ & 390341000312273662178744569645862174135963182666691866774053749\backslash \\ & 1243466305096192475980103396930794405553321984000000\eta^5\nu^{11} - 1609\backslash \\ & 228177288045878551112139444529354484383075794234482756141610877\backslash \\ & 18256995080417682549759479655113435158878342400000\eta^5\nu^8 - 158927\backslash \\ & 073605197641810465198477949692601012391011681883704923696922035\backslash \\ & 645669846402499964335738957869203885693952000000\eta^5\nu^9 + 56111273\backslash \\ & 163582220405755713924860306102652461680987134141516869263804870\backslash \\ & 120309181243410072399880343077643407365120000\eta^5\nu^7 + 24748162302\backslash \\ & 992992853919653527143353192507813034769055278149733192138247979\backslash \\ & 4107672129595186172636466444731796480000000\eta^7\nu^9 + 1154616198773\backslash \\ & 758226011821503853660716583404439461885138776345701351961158299\backslash \end{aligned}$$

\

$$56981212692295416624779258947010560000000\eta^9\nu^{11} - 127603256569689\backslash$$

$$837001229769167887616630050304387145541938694349274885668863941\backslash$$

$$818521469739542318966590193049600000000\eta^9\nu^9 + 10861287921203032\backslash$$

$$703743002315108931162160658753764869179493439214633024348110623\backslash$$

$$7588745339113204760503447961600000000\eta^9\nu^{10} - 1301035881955935363\backslash$$

$$418007511467549271273481103142899877816627285761448395902864557\backslash$$

$$05875109197491076691774484480000000\eta^9\nu^8 + 540036469037663722648\backslash$$

$$288410071137536556076390455669174080462305030437856723755503136\backslash$$

$$23125072701496494766628249600000\eta^9\nu^6 + 447904434185118825326022\backslash$$

$$833235892740977501491952918533198421143461876619812700698677880\backslash$$

$$33332300913527461418188800000\eta^9\nu^7 - 860709599051617386064938639\backslash$$

$$338042543524457130477267961194337395338893843873263346172820776\backslash$$

$$6035917407915359150080000\eta^9\nu^4 + 9712114078028631599916271737996\backslash$$

$$667951171548088843808050087378703682342647773194621779309070355\backslash$$

$$28677228021843200\eta^7 - 304305047113786257284602152119960605318668\backslash$$

$$080663567687743507295921783134933197900322838300708664769450594\backslash$$

$$099200\eta^9 + 43973319111891197544154663751681207037976607952128472\backslash$$

$$2973786648281735989995459018962120410371806599308025737067200\eta^5\nu^2 -$$

$$472634397270786054727832242333100424096998856492282620360152913\backslash$$

$$49039269851935536108480362415626972827269678284800\eta^7\nu - 6865452\backslash$$

$$530373054682241780996027415158845862448853754667044519010957683\backslash$$

$$95321961530411018170971511627886429090099200\eta^7\nu^2 - 755637910602\backslash$$

$$725645169441585184008113260804331336498063325460469538926224683\backslash$$

$$801534863914647247380168200313515520\eta^5 + 29618367260848814745789\backslash$$

$$998029138701484112871612309908869378914638049324637932367544383\backslash$$

$$978847178543923510166427200\eta^5\nu + 257579419179963600695265867265\backslash$$

$$453192794893481547349110836659107486758315309344327233508852069\backslash$$

$$05301854488791040000\eta^9\nu + 3549453418902583577752010293769906011\backslash$$

$$683084613024518510638145195535434007668527963931310517598152210\backslash$$

$$72093389824000\eta^9\nu^2 - 582244117416880579919118888960125515057699\backslash$$

$$356253234653697889727608889671277589433433899937413699950049754\backslash$$

$$229641600\eta^5\nu^3 + 66511673597986564287399294823279109128108574773\backslash$$

$$944018915926468056333246964940983598952594509909035803608977267\backslash$$

$$520000\eta^5\nu^6 - 49829101585077279758135390304352391959583660304813\backslash$$

$$520751048191717150893510866424338232363965255052054528918288000\backslash$$

$$00\eta^5\nu^5 - 105689881196562466446790849281229187100747086978951887\backslash$$

$$85589868509871547526205226312993085006852665025420966800064000\eta^5\nu^4 -$$

$$743270165374270938364069442674932131289689018456453552972416452\backslash$$

$$50309777912699553148756598740700075645391700800\zeta\eta^5 + 4529419159\backslash$$

$$179350015911637848185607700402000610094481064961746757312056297\backslash$$

$$67294541754699360524752364071213708800\zeta\eta^7 - 5775935976767599609\backslash$$

$$466729599157473377596228776235789270237144765603298733685098880\backslash$$

$$21078021121994989499585536000\zeta\eta^9 - 2702051001730633696337991001\backslash$$

$$277501566215094320646840516119456528854390407948299081022824576\backslash$$

$$\begin{aligned}
& 730972700932256040\zeta\eta^3 - 143377180980883620732082478825631892078\backslash \\
& 136467989435361400354749840115676366670144489917935059425569924\backslash \\
& 1933900\eta\zeta - 873310012432464189699294566599533725122262701753484\backslash \\
& 474182174378725232612835657167457381997004942581715963660800000\backslash \\
& 00\eta^7\nu^7 - 103911882037839943756436281524935930664728358066650645\backslash \\
& 395341423126076329447661134126883519299072224030944318479360000\eta^7\nu^6 \\
& - 2098637932651894510008970172328388975591886764391035133292722\backslash \\
& 54045735968809732571614480258628526535138254118912000000\eta^7\nu^{10} - \\
& 223777900645461865967137188845535297450832489378021962067631537\backslash \\
& 220978749623832264315592146191404331342819000320000000\eta^7\nu^{11}], \\
& \text{plex}(\psi, \phi, \kappa, \alpha, \beta, \delta, \zeta, \eta, \mu, \nu), \{-107461411275663388800000\zeta\nu^{11} \\
& - 1655131613985389904480000\zeta\nu^{10} - 767983864999913128800000\zeta\nu^9 \\
& + 1640537448745939123320000\nu^8\zeta + 703859885907253185744000\nu^7\zeta \\
& - 539625447974625650125200\nu^6\zeta - 186256945681689501532800\zeta\nu^5 \\
& + 64266038901117394396200\nu^4\zeta + 15409870920149246822160\zeta\nu^3 \\
& - 2703347740047396223674\zeta\nu^2 - 294937859137992661836\nu\zeta \\
& - 7455336891115309470\zeta + 3848383689348533647680000\nu^{11} \\
& + 1507075848187833930480000\nu^{10} - 4096156177115514512640000\nu^9 \\
& - 1149778757544509182860000\nu^8 + 1471743746270846535189600\nu^7 \\
& + 232965561754194664048200\nu^6 - 196942904138956198111200\nu^5 \\
& - 7157485696073773871880\nu^4 + 9175970354904788081040\nu^3 \\
& - 742414071757520206719\nu^2 + 569474288997510024\nu + 7724504913684364961, \\
& 2147760834258288588\zeta - 98387964279618884640\zeta\nu^2 \\
& + 73608831541079708196\nu - 4283910046889713175760\nu^3 \\
& - 16335253490056873342200\nu^4 - 420572830291393615200000\nu^8\zeta \\
& + 697187136970177824354\nu^2 + 33642589247946665100\nu\zeta \\
& + 353217082881133850400000\nu^9 + 420364766130800012640000\nu^{10} \\
& + 1597954504953279945600\zeta\nu^3 - 97237382045545756800000\nu^{11} \\
& - 245713282960109963616000\nu^7 - 423522308364905289240000\nu^8 \\
& + 57300938743228807680000\nu^5 - 45876259232111970871200\zeta\nu^5 \\
& + 96528711193386875208000\nu^6\zeta + 383085812951222496480000\nu^7\zeta \\
& + 1109486251796268384000000\zeta\nu^{11} - 1131981566263215408000000\zeta\nu^9 \\
& + 524034670087971878400000\zeta\nu^{10} + 140301659271092969365200\nu^6 \\
& - 5449331128507026714000\nu^4\zeta - 5376553922277644749\}]]
\end{aligned}$$

```

> map(indets,[op(1..9,op(1,op(1,Cond_g solved)))]);
    [{\nu}, {\nu, \mu}, {\nu, \eta}, {\zeta, \nu}, {\nu, \delta, \eta}, {\alpha, \beta, \nu, \eta}, {\kappa, \zeta, \nu}, {\phi, \zeta, \nu, \eta}, {\zeta, \nu, \psi, \eta}]

> map(indets,evalf(Cond));
    [{\phi, \alpha, \zeta, \nu, \psi, \mu}, {\delta, \mu, \eta}, {\zeta, \nu, \mu}, {\phi, \zeta, \nu, \psi, \mu}, {\kappa, \zeta, \nu, \mu}, %1, {\kappa, \psi}, {\phi, \zeta}, %1,
    {\phi, \zeta, \nu, \delta, \psi, \mu, \eta}, {\phi, \alpha, \zeta, \nu, \delta, \psi, \mu, \eta}, {\alpha, \zeta, \beta, \nu, \delta, \psi, \mu, \eta}, %1, %1,
    {\zeta, \nu, \delta, \psi, \mu, \eta}, {\phi, \alpha, \zeta, \nu, \delta, \psi, \mu, \eta}, {\alpha, \beta, \nu}, %1]
    %1 := {\phi, \alpha, \zeta, \beta, \nu, \delta, \psi, \mu, \eta}

> map(indets,evalf(ReducedCond));

```

```

[{\kappa, \zeta, \mu}, {\kappa, \zeta, \mu}, {\kappa, \zeta, \mu}, {\kappa, \zeta, \mu}, {\kappa, \zeta, \nu, \mu}, {\phi, \zeta}, {\kappa, \psi}, {\phi, \kappa, \zeta, \delta, \psi, \mu, \eta},
{\delta, \mu, \eta}, {\phi, \alpha, \kappa, \zeta, \beta, \delta, \psi, \mu, \eta}, {\phi, \alpha, \kappa, \zeta, \beta, \psi, \mu}, {\phi, \alpha, \kappa, \zeta, \beta, \delta, \psi, \mu, \eta}]

> Sols_alpha:=solve({op(1..1,Cond)},{alpha}):

> Sols_beta:=solve({op(6..6,op(1,op(1,Cond_gsolve))))},{beta}):

> fsolve({op(1..5,op(1,op(1,Cond_gsolve)))) union
{op(7..7,op(1,op(1,Cond_gsolve))))},{mu,nu,eta,delta,kappa,zeta
},fulldigits,nu=-1..1,mu=-1..1,eta=-1..1,delta=-1..1,zeta=-1..1,kappa
=-1..1);

      {\delta = -0.91714785297834405191699935762843996310868817200300,
      \kappa = 0.73637962873394729580800450596577811320233327484881,
      \zeta = -0.87205606433941227010122337018088512325594431711422,
      \nu = 0.29164074859563564442927140211611205478028286944243,
      \eta = 0.079834964849326977833458085771344578788147872108576,
      \mu = -0.39046919745956982394932863667522521642014401497639}

> assign(%);

> [fsolve({op(8..8,op(1,op(1,Cond_gsolve))))},{phi}):assign(op(1,
%));phi;

      0.48940598754904383291473170400091475604911164333066

> [fsolve({op(9..9,op(1,op(1,Cond_gsolve))))},{psi}):assign(op(1,
%));psi;

      0.67656857921843661549659267943605190892053036385797

> [fsolve({op(1..1,Cond)},{alpha}):assign(op(1,%));alpha;

      -0.76774720843848208484129676885263329261038930410290

> [fsolve({op(6..6,op(1,op(1,Cond_gsolve))))},{beta}):assign(op(2,
%));beta;

      0.57053474713946668518159662560801862393869829912172

> evalf(Cond);

      [0.3 10-48, 0., 0.20 10-47, 0.3530 10-43, 0., -0.141460 10-42, 0.722 10-46, 0.218 10-46,
      -0.4936 10-45, -0.539 10-45, 0.2929 10-45, 0.4629 10-44, -0.1562 10-45, 0.893834 10-42,
      0.376881 10-43, -0.163556 10-44, -0.7442 10-46, -0.421815 10-43]

> unassign('alpha','beta','mu','nu','delta','eta','psi','zeta','phi','k
appa');

> assign([solve({op(1..1,op(1,op(1,Cond_gsolve)))) union
{-1<=nu,nu<=1},{nu}}][9]):evalf(nu);

> assign([solve({op(2..2,op(1,op(1,Cond_gsolve)))) union
{-1<=mu,mu<=1},{mu}}][1]):evalf(mu);

> assign([allvalues(solve({op(3..3,op(1,op(1,Cond_gsolve))))},{eta
}))][3]):evalf(eta);

> assign([solve({op(4..4,op(1,op(1,Cond_gsolve)))) union
{-1<=zeta,zeta<=1},{zeta}}][1]):evalf(zeta);

> assign([solve({op(5..5,op(1,op(1,Cond_gsolve))))},{delta
}][2]):evalf(delta);

> assign([solve({op(7..7,op(1,op(1,Cond_gsolve)))) union
{-1<=kappa,kappa<=1},{kappa}}][1]):evalf(kappa);

> assign([solve({op(8..8,op(1,op(1,Cond_gsolve)))) union
{-1<=phi,phi<=1},{phi}}][1]):evalf(phi);

```

```
> assign([solve({op(9..9,op(1,op(1,Cond_gsolved))))} union  
{-1<=psi,psi<=1},{psi}}][1]);evalf(psi);  
> assign(Sols_alpha[1]);assign(Sols_beta[1]);  
  
> N19:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN19) union  
OrbPG(Vector([eta,delta,mu]),GrN19) union  
OrbPG(Vector([psi/2,sqrt(3)/2*psi,kappa]),GrN19) union  
OrbPG(Vector([phi,0,zeta]),GrN19) union  
OrbPG(Vector([0,0,1]),GrN19));  
  
> UpdatePlan(N19,"N19");  
                                     "Update complete"  
  
> pointplot3d(evalf(N19),symbol=box,scaling=constrained,shading=Z,orien  
tation=[14,86]);
```

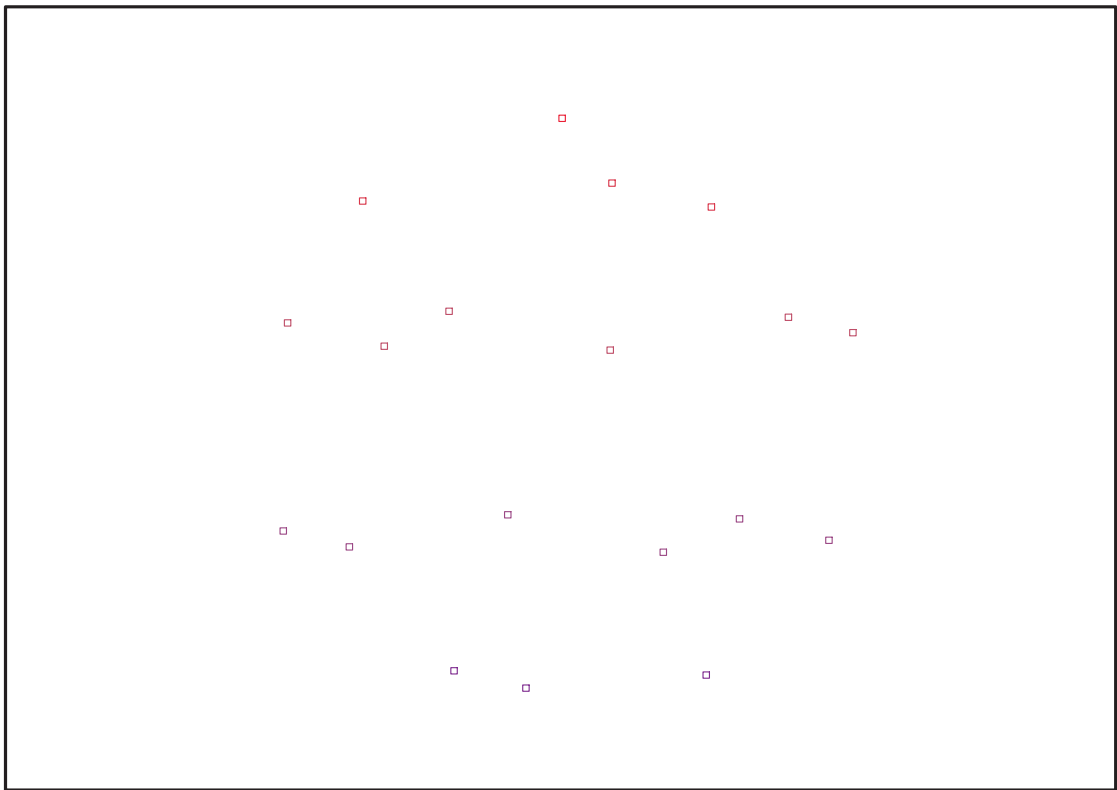


FIG. 11.61 – Graphique : Support du plan N19.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(evalf(N19),listlist)  
))], 'rational', 5))));  
                                     Graph := POLYTOPE(3, 3, 19, 30)  
  
> plotdes:=draw(Graph):  
  
> display(plotdes,scaling=constrained,shading=Z,orientation=[14,86]);
```

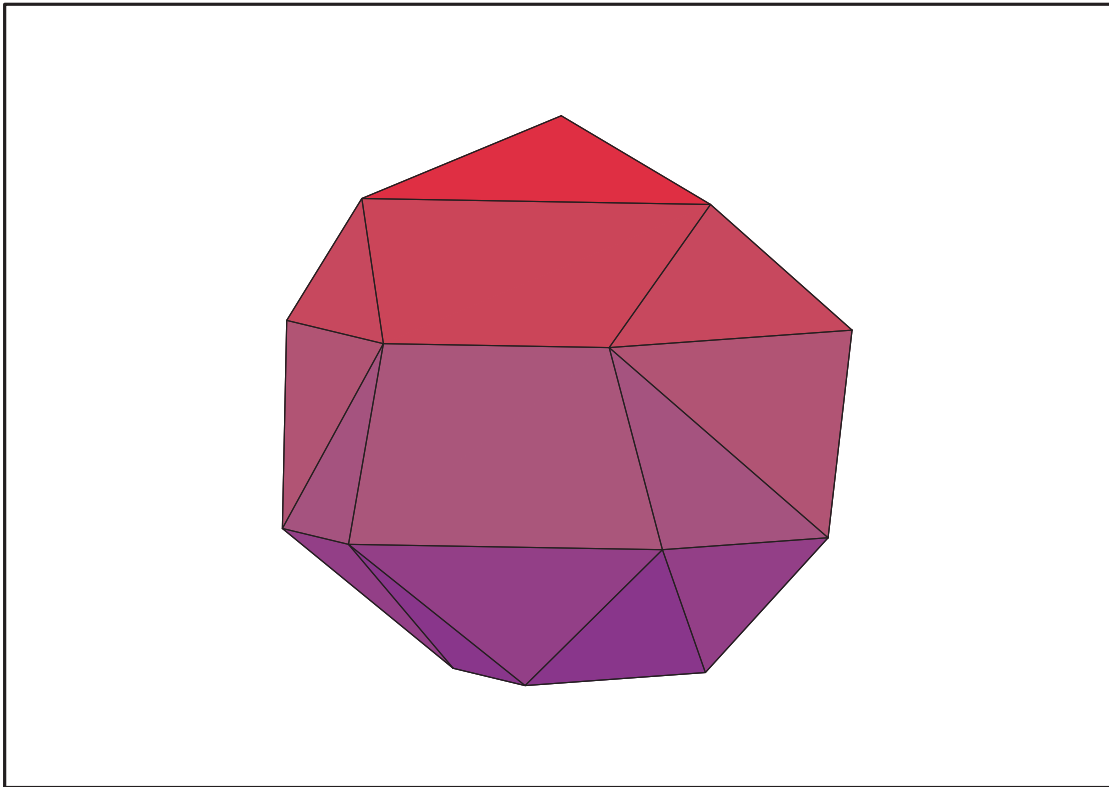


FIG. 11.62 – Graphique : Enveloppe convexe du plan N19.

11.25. N20

```
> unassign('alpha','beta','nu','mu');
```

```
> GrN20:=TwoPlusn(10);
```

$$\begin{aligned}
GrN20 := & \left\{ \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%3 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%2 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%1 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \right. \\
& \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%4 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\
& \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%4 & 0 \\ \%4 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%2 & 0 \\ \%4 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
& \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%4 & 0 \\ \%2 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\
& \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%3 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%1 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
& \left. \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%2 & 0 \\ \%2 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \\
\%1 := & \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4} \\
\%2 := & -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8} \\
\%3 := & -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4} \\
\%4 := & \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8}
\end{aligned}$$

```

> N20:=OrbToMatrix(OrbPG(Vector([0,alpha,nu]),GrN20) union
OrbPG(Vector([0,beta,mu]),GrN20));

```


$$N20 := \begin{bmatrix} -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\beta}{4} & (-\frac{1}{4}-\frac{\sqrt{5}}{4})\beta & \mu \\ \%1\beta & (-\frac{1}{4}+\frac{\sqrt{5}}{4})\beta & \mu \\ \%1\beta & (-\frac{\sqrt{5}}{4}+\frac{1}{4})\beta & -\mu \\ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (\frac{1}{4}+\frac{\sqrt{5}}{4})\alpha & -\nu \\ 0 & -\alpha & -\nu \\ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (\frac{1}{4}+\frac{\sqrt{5}}{4})\alpha & -\nu \\ 0 & \alpha & \nu \\ \%1\alpha & (-\frac{\sqrt{5}}{4}+\frac{1}{4})\alpha & -\nu \\ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (-\frac{1}{4}-\frac{\sqrt{5}}{4})\alpha & \nu \\ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (-\frac{1}{4}-\frac{\sqrt{5}}{4})\alpha & \nu \\ \%2\alpha & (-\frac{1}{4}+\frac{\sqrt{5}}{4})\alpha & \nu \\ \%2\alpha & (-\frac{\sqrt{5}}{4}+\frac{1}{4})\alpha & -\nu \\ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\beta}{4} & (-\frac{1}{4}-\frac{\sqrt{5}}{4})\beta & \mu \\ \%1\alpha & (-\frac{1}{4}+\frac{\sqrt{5}}{4})\alpha & \nu \\ 0 & -\beta & -\mu \\ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\beta}{4} & (\frac{1}{4}+\frac{\sqrt{5}}{4})\beta & -\mu \\ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\beta}{4} & (\frac{1}{4}+\frac{\sqrt{5}}{4})\beta & -\mu \\ 0 & \beta & \mu \\ \%2\beta & (-\frac{1}{4}+\frac{\sqrt{5}}{4})\beta & \mu \\ \%2\beta & (-\frac{\sqrt{5}}{4}+\frac{1}{4})\beta & -\mu \end{bmatrix}$$

$$\%1 := -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

$$\%2 := \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

> UpdatePlan(N20,"N20");

“Update complete”

> Var:=Var_Norm_Blue_D_Centre(N20,1,0);

$$Var := 1 + \frac{2 \text{var}_3^2}{\mu^2 + \nu^2} + \frac{4 \text{var}_2^2}{\beta^2 + \alpha^2} + \frac{4 \text{var}_1^2}{\beta^2 + \alpha^2}$$

-
- ```

> Cond:=eval(EqInKr_reduce({var[1],var[2],var[3]
},[subs(sqrt(5)=radic3,Var)]),[radic3=sqrt(5)]);
 Cond := [$\beta^2 + \alpha^2 - 2\mu^2 - 2\nu^2$]

> Cond_gsolve:=eval(gsolve([subs(sqrt(5)=radic3,op(Cond)),alpha^2+nu^2
-1,beta^2+mu^2-1],[alpha,nu,beta,mu]),[radic3=sqrt(5)]);
 Cond_gsolve := {[$3\nu^2 + 1 - 3\beta^2, \beta^2 + \mu^2 - 1, 3\alpha^2 + 3\beta^2 - 4$], plex(α, μ, ν, β), {}}
```
- 
- ```

> Cond:=[op(convert(convert(map(mem->if mem<>0 then return mem end
if,simplify(eval(EqF_Gen_Mom_Rad(subs(sqrt(5)=radic3,subs((1/3+2/15*sq
rt(5))^(1/2)=radic4,sqrt(2)=radic,sqrt(5-sqrt(5))=radic2,N20)),2),[rad
ic=sqrt(2),radic3=sqrt(5),radic2=sqrt(5-sqrt(5)),radic4=(1/3+2/15*sqrt
(5))^(1/2)]))),set),list)),alpha^2+nu^2-1,beta^2+mu^2-1];
      Cond := [ $-1680\beta^4 - 2880\alpha^4 + 35280\nu^4 + 16200\beta^2\mu^2 - 3120\mu^4 + 960\alpha^4\sqrt{5} - 240\beta^4\sqrt{5}$ 
 $- 3960\beta^2\sqrt{5}\mu^2 + 7560\nu^2\beta^2\sqrt{5} - 600\alpha^2\mu^2\sqrt{5} - 600\alpha^2\nu^2\sqrt{5} - 600\alpha^2\nu^2$ 
 $- 600\alpha^2\mu^2 - 22200\nu^2\beta^2 + 45600\mu^2\nu^2 - 9840\sqrt{5}\nu^4 + 1680\sqrt{5}\mu^4$ 
 $- 13920\sqrt{5}\mu^2\nu^2, 30\beta^2\sqrt{5}\mu^2 - 30\beta^2\mu^2 - 30\nu^2\beta^2\sqrt{5} + 30\nu^2\beta^2 + 20\mu^4 - 20\sqrt{5}\mu^4$ 
 $- 60\mu^2\nu^2 + 60\sqrt{5}\mu^2\nu^2 - 40\nu^4 + 40\sqrt{5}\nu^4, 510\beta^4 - 150\beta^4\sqrt{5} + 150\alpha^4 - 30\alpha^4\sqrt{5}$ 
 $+ 360\alpha^2\mu^2 - 120\alpha^2\mu^2\sqrt{5} - 880\mu^4 + 240\sqrt{5}\mu^4 + 360\alpha^2\nu^2 - 120\alpha^2\nu^2\sqrt{5}$ 
 $- 880\nu^4 + 240\sqrt{5}\nu^4 - 360\beta^2\mu^2 + 120\beta^2\sqrt{5}\mu^2 - 360\nu^2\beta^2 + 120\nu^2\beta^2\sqrt{5},$ 
 $10\alpha^2 + 10\beta^2 - 20\mu^2 - 20\nu^2, 10\alpha^3\nu + 10\nu\alpha\beta^2 - 20\nu\alpha\mu^2 - 20\nu^3\alpha,$ 
 $40\nu^4 + 30\beta^2\mu^2 - 20\mu^4 - 30\nu^2\beta^2 + 60\mu^2\nu^2, 270\beta^4 - 1080\nu^2\beta^2 + 90\alpha^4$ 
 $- 120\alpha^2\mu^2 - 280\mu^4 - 120\alpha^2\nu^2 + 2400\mu^2\nu^2 + 920\nu^4 + 120\beta^2\mu^2,$ 
 $20\mu^2 + 20\nu^2 + 10\alpha^2\sqrt{5} - 10\alpha^2 + 10\beta^2\sqrt{5} - 10\beta^2 - 20\mu^2\sqrt{5} - 20\nu^2\sqrt{5},$ 
 $\alpha^2 + \nu^2 - 1, \beta^2 + \mu^2 - 1]$ 

> Cond_gsolve:=eval(gsolve(subs(sqrt(5)=radic3,subs((1/3+2/15*sqrt(5))
^(1/2)=radic4,sqrt(2)=radic,sqrt(5-sqrt(5))=radic2,[3*nu^2+1-3*beta^2,
beta^2+mu^2-1,
3*alpha^2+3*beta^2-4,op(Cond)])),[alpha,beta,nu,mu]),[radic=sqrt(2),ra
dic3=sqrt(5),radic2=sqrt(5-sqrt(5)),radic4=(1/3+2/15*sqrt(5))^(1/2)]);
      Cond_gsolve := {[ $45\mu^4 - 30\mu^2 + 1, 3\nu^2 - 2 + 3\mu^2, \beta^2 + \mu^2 - 1, 3\alpha^2 - 3\mu^2 - 1$ ],
plex( $\alpha, \beta, \nu, \mu$ ), {}}
```
-
- ```

> simplify(radnormal(convert([solve({-30*mu^2+45*mu^4+1,
3*nu^2-2+3*mu^2, beta^2+mu^2-1,
3*alpha^2-3*mu^2-1},{alpha,nu,beta,mu}]),radical)));

```

$\{ \{ \%6, \%7, \%1, \%5 \}, \{ \%6, \%4, \%7, \%1 \}, \{ \%7, \%2, \%1, \%5 \}, \{ \%4, \%7, \%2, \%1 \},$   
 $\{ \%6, \%3, \%1, \%5 \}, \{ \%6, \%4, \%3, \%1 \}, \{ \%3, \%2, \%1, \%5 \}, \{ \%4, \%3, \%2, \%1 \} \}$

$$\%1 := \mu = \frac{\sqrt{75-30\sqrt{5}}}{15}$$

$$\%2 := \alpha = -\frac{\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

$$\%3 := \nu = \frac{2\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

$$\%4 := \beta = -\frac{\sqrt{75-30\sqrt{5}}}{5} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

$$\%5 := \beta = \frac{\sqrt{75-30\sqrt{5}}}{5} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

$$\%6 := \alpha = \frac{\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

$$\%7 := \nu = -\frac{2\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

> Première solution :

```
> assign({mu = 1/15*(75-30*5^(1/2))^(1/2), alpha =
1/15*(75-30*5^(1/2))^(1/2)+1/15*(75-30*5^(1/2))^(1/2)*5^(1/2), beta =
1/5*(75-30*5^(1/2))^(1/2)+1/15*(75-30*5^(1/2))^(1/2)*5^(1/2), nu =
-2/15*(75-30*5^(1/2))^(1/2)-1/15*(75-30*5^(1/2))^(1/2)*5^(1/2)});
```

> alpha ; nu ; beta ; mu ;

$$\begin{aligned} & \frac{\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15} \\ & -\frac{2\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15} \\ & \frac{\sqrt{75-30\sqrt{5}}}{5} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15} \\ & \frac{\sqrt{75-30\sqrt{5}}}{15} \end{aligned}$$

```
> N20bis:=simplify(OrbToMatrix(OrbPG(Vector([0,alpha,nu]),GrN20) union
OrbPG(Vector([0,beta,mu]),GrN20)));
```

$$\begin{aligned}
N20bis := & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, \frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%1 \right] \\
& \left[ 0, -\frac{\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%1 \right] \\
& \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, \frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%1 \right] \\
& \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%2, \%1 \right] \\
& \left[ 0, \frac{\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%3 \right] \\
& \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%3 \right] \\
& \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \frac{(1+\sqrt{5})\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right] \\
& \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%4 \right] \\
& \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%4, \%3 \right] \\
& \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%2, \%1 \right] \\
& \left[ 0, \frac{\sqrt{75-30\sqrt{5}}}{5} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%4 \right] \\
& \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%3 \right] \\
& \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%4, \%3 \right] \\
& \left[ 0, -\frac{\sqrt{75-30\sqrt{5}}}{5} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%2 \right] \\
& \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
& \quad \left. -\frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right] \\
& \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
& \quad \left. \frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%4 \right] \\
& \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
& \quad \left. -\frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right]
\end{aligned}$$

$$\begin{aligned} & \left[ \frac{\sqrt{2} \sqrt{5 - \sqrt{5}} \sqrt{75 - 30 \sqrt{5}} (3 + \sqrt{5})}{60}, -\frac{(1 + \sqrt{5}) \sqrt{75 - 30 \sqrt{5}} (3 + \sqrt{5})}{60}, \%4 \right] \\ & \left[ -\frac{\sqrt{2} \sqrt{5 - \sqrt{5}} \sqrt{75 - 30 \sqrt{5}} (3 + \sqrt{5})}{60}, \frac{(1 + \sqrt{5}) \sqrt{75 - 30 \sqrt{5}} (3 + \sqrt{5})}{60}, \%2 \right] \\ & \left[ \frac{\sqrt{2} \sqrt{5 - \sqrt{5}} \sqrt{75 - 30 \sqrt{5}} \sqrt{5}}{30} + \frac{\sqrt{2} \sqrt{5 - \sqrt{5}} \sqrt{75 - 30 \sqrt{5}}}{15}, \right. \\ & \left. \frac{(\sqrt{5} - 1) \sqrt{75 - 30 \sqrt{5}} (3 + \sqrt{5})}{60}, \%4 \right] \\ \%1 &:= \frac{2 \sqrt{75 - 30 \sqrt{5}}}{15} + \frac{\sqrt{75 - 30 \sqrt{5}} \sqrt{5}}{15} \\ \%2 &:= -\frac{\sqrt{75 - 30 \sqrt{5}}}{15} \\ \%3 &:= -\frac{2 \sqrt{75 - 30 \sqrt{5}}}{15} - \frac{\sqrt{75 - 30 \sqrt{5}} \sqrt{5}}{15} \\ \%4 &:= \frac{\sqrt{75 - 30 \sqrt{5}}}{15} \end{aligned}$$

```
> CreatePlan(N20bis,"N20bis");
> pointplot3d(N20bis,symbol=box,scaling=constrained,shading=Z);
```

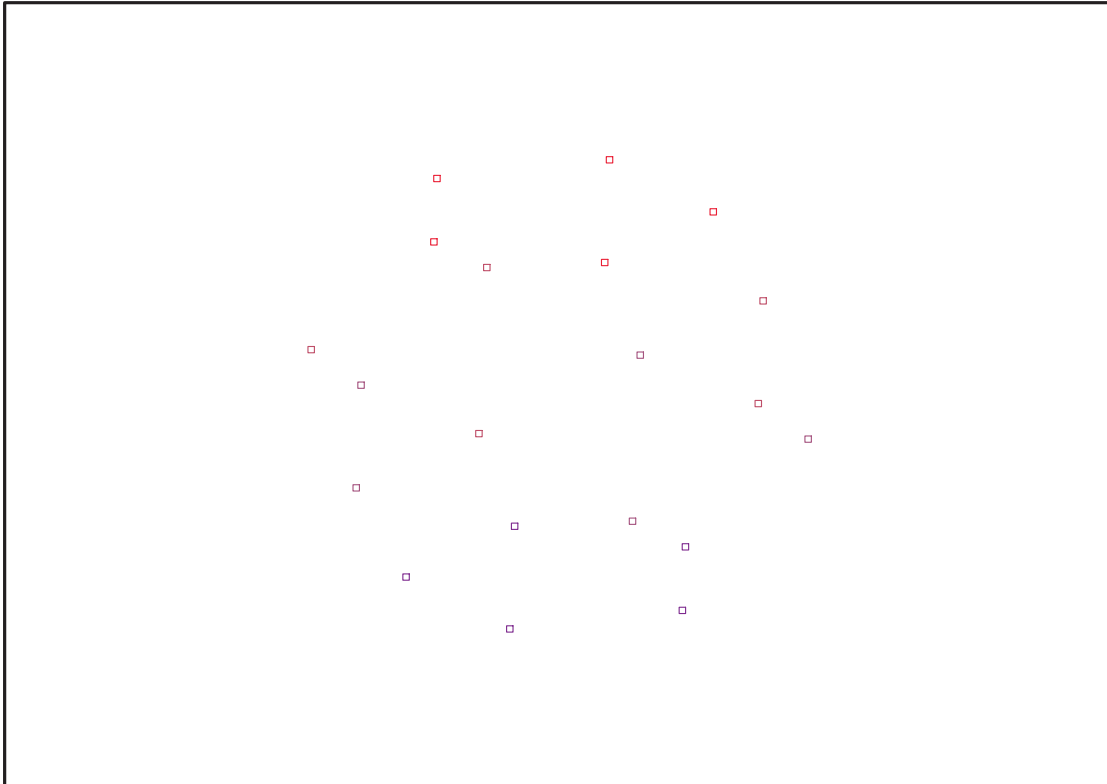


FIG. 11.63 – Graphique : Support du plan N20bis.

```

> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N20bis))
,listlist)))],'rational',5)));
 Graph := POLYTOPE(3, 3, 20, 32)

> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z);

```

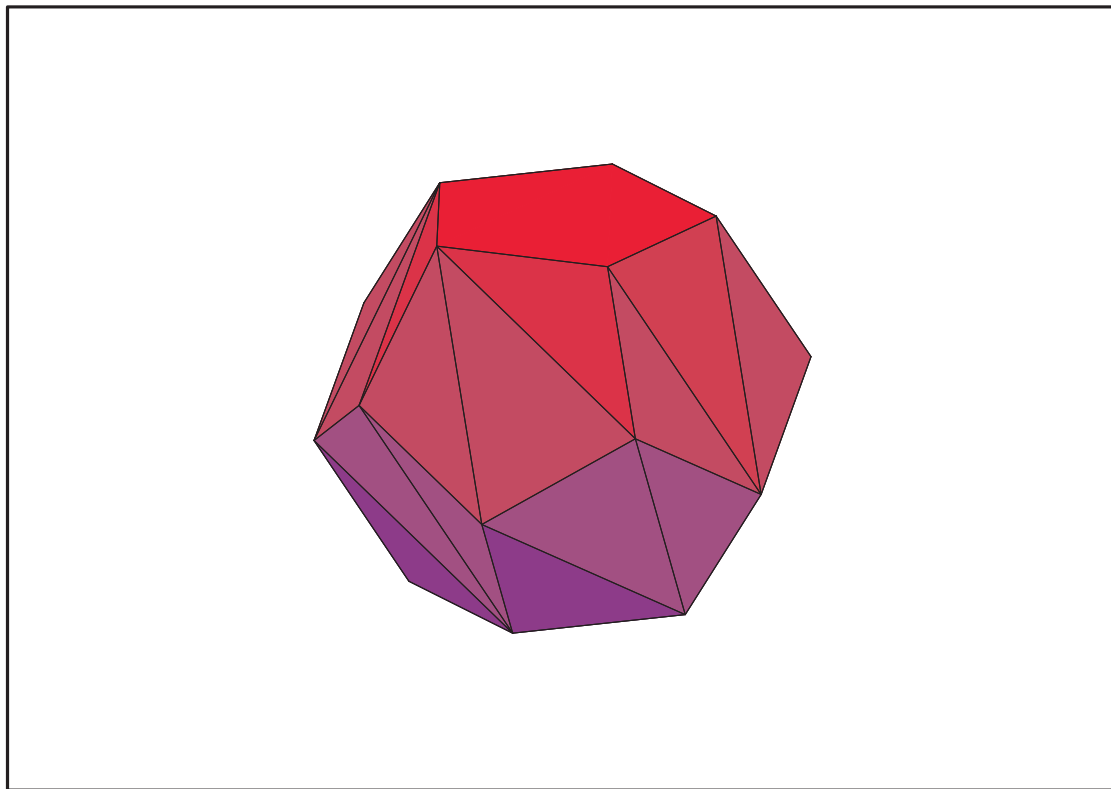


FIG. 11.64 – Graphique : Enveloppe convexe du support du plan N20bis.

> Dodécaèdre :

```

> assign({'beta' =
-1/5*(75-30*5^(1/2))^(1/2)-1/15*(75-30*5^(1/2))^(1/2)*5^(1/2), 'mu' =
1/15*(75-30*5^(1/2))^(1/2), 'alpha' =
1/15*(75-30*5^(1/2))^(1/2)+1/15*(75-30*5^(1/2))^(1/2)*5^(1/2), 'nu' =
-2/15*(75-30*5^(1/2))^(1/2)-1/15*(75-30*5^(1/2))^(1/2)*5^(1/2)});

```

> alpha ; nu ; beta ; mu ;

$$\begin{aligned}
& \frac{\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15} \\
& - \frac{2\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15} \\
& - \frac{\sqrt{75-30\sqrt{5}}}{5} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15} \\
& \frac{\sqrt{75-30\sqrt{5}}}{15}
\end{aligned}$$

```
> N20:=simplify(OrbToMatrix(OrbPG(Vector([0,alpha,nu]),GrN20) union
OrbPG(Vector([0,beta,mu]),GrN20)));
```

$$\begin{aligned}
 N20 := & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%1 \right] \\
 & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
 & \left. \frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%1 \right] \\
 & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
 & \left. -\frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right] \\
 & \left[ 0, \frac{\sqrt{75-30\sqrt{5}}}{5} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%1 \right] \\
 & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
 & \left. -\frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right] \\
 & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{30} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{15}, \right. \\
 & \left. \frac{(\sqrt{5}-1)\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%1 \right] \\
 & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%1, \%3 \right] \\
 & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \frac{(1+\sqrt{5})\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right] \\
 & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%1 \right] \\
 & \left[ 0, -\frac{\sqrt{75-30\sqrt{5}}}{5} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%2 \right] \\
 & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \frac{(1+\sqrt{5})\sqrt{75-30\sqrt{5}}(3+\sqrt{5})}{60}, \%2 \right] \\
 & \left[ 0, -\frac{\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%3 \right] \\
 & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, \frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%3 \right] \\
 & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%1, \%3 \right] \\
 & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%4 \right]
 \end{aligned}$$

$$\begin{aligned} & \left[ 0, \frac{\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}, \%4 \right] \\ & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%2, \%4 \right] \\ & \left[ -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, \frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%3 \right] \\ & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}(1+\sqrt{5})}{60}, -\frac{(1+\sqrt{5})^2\sqrt{75-30\sqrt{5}}}{60}, \%4 \right] \\ & \left[ \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}\sqrt{5}}{60} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{75-30\sqrt{5}}}{20}, \%2, \%4 \right] \end{aligned}$$

$$\%1 := -\frac{\sqrt{75-30\sqrt{5}}}{15}$$

$$\%2 := \frac{\sqrt{75-30\sqrt{5}}}{15}$$

$$\%3 := \frac{2\sqrt{75-30\sqrt{5}}}{15} + \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

$$\%4 := -\frac{2\sqrt{75-30\sqrt{5}}}{15} - \frac{\sqrt{75-30\sqrt{5}}\sqrt{5}}{15}$$

> UpdatePlan(N20,"N20");

“Update complete”

> pointplot3d(N20,symbol=box,scaling=constrained,scaling=constrained,shading=Z);

> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N20)),listlist))]),'rational',10))));

*Graph* := POLYTOPE(3, 3, 20, 32)

> plotdes:=draw(Graph):

> display(plotdes,scaling=constrained,shading=Z);

> IdealN20:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N20));

> Gens:=Generators(IdealN20);

> info\_iop:=3;

> info\_iop:=3;

> subs(x\_[1]=x,x\_[2]=y,x\_[3]=z,IdealOfPoints(map(convert,convert((N20),listlist),Vector))));

*EstSet\_N20\_tdeg* :=

$[1, z, y, x, z^2, yz, zx, y^2, xy, z^3, yz^2, z^2x, zy^2, zxy, y^3, y^2x, z^3y, z^3x, z^2y^2, z^2xy]$

> EstSet\_N20\_tdeg:=[1, z, y, x, z^2, y\*z, z\*x, y^2, x\*y, z^3, y\*z^2, z^2\*x, z\*y^2, z\*x\*y, y^3, y^2\*x, z^3\*y, z^3\*x, z^2\*y^2, z^2\*x\*y];



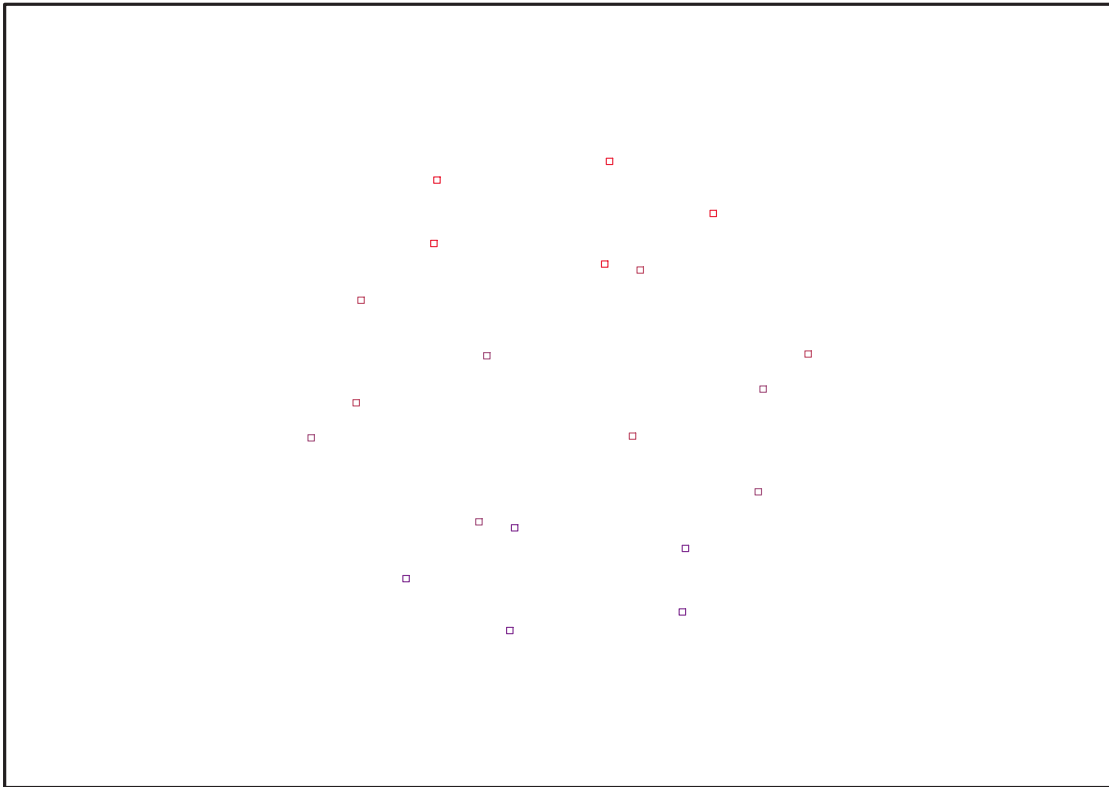


FIG. 11.65 – Graphique : Support du plan N20.

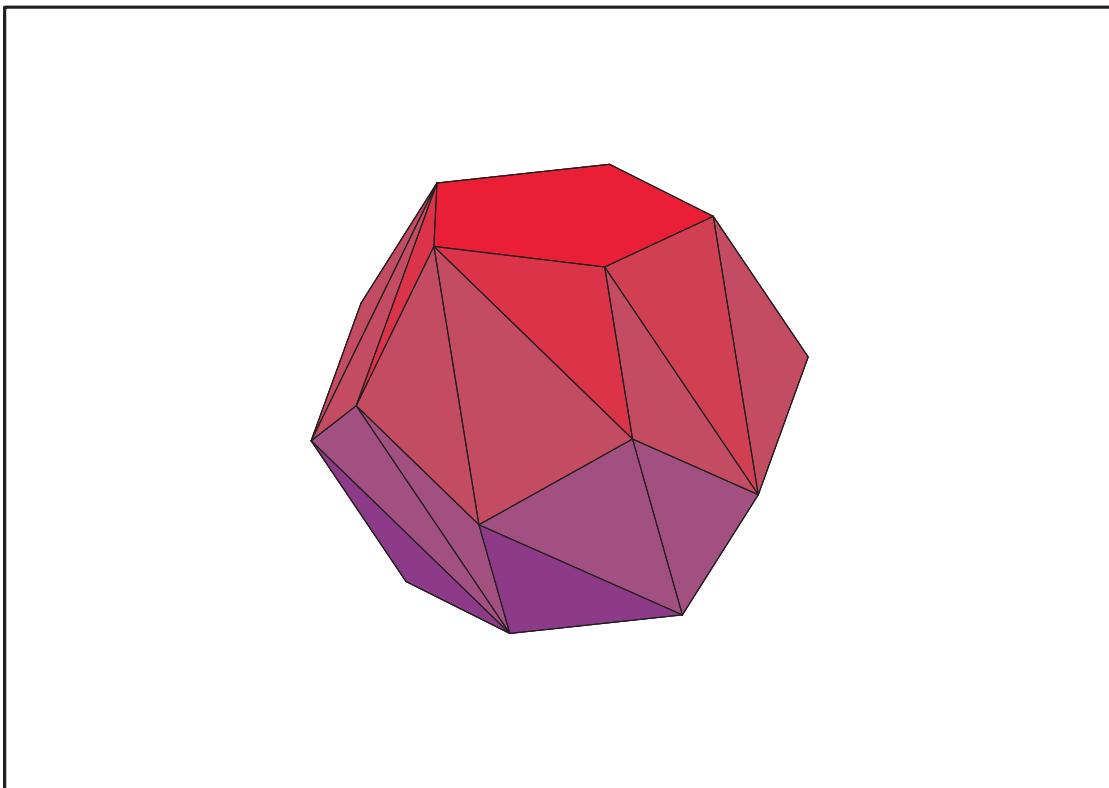


FIG. 11.66 – Graphique : Enveloppe convexe du support du plan N20.

*EstSet\_N20\_tdeg* :=

$[1, z, y, x, z^2, yz, zx, y^2, xy, z^3, yz^2, z^2x, zy^2, zxy, y^3, y^2x, z^3y, z^3x, z^2y^2, z^2xy]$

### 11.25.1. tdeg(x,y,z)

> X\_D\_ModeleSat(N20,[x,y,z],EstSet\_N20\_tdeg):

> M\_D\_ModeleSat(N20,[x,y,z],EstSet\_N20\_tdeg):

> expand(Var\_Blue\_Modele\_Sat(N20,[x,y,z],EstSet\_N20\_tdeg));

$$\begin{aligned} & -\frac{1215}{4}z^4 - \frac{405}{2}z^4y^2 + \frac{1575}{16}z^2x^2 - \frac{765}{16}y^2x^2 + \frac{105}{8}x^2 - \frac{675}{2}zy^3 + \frac{435}{4}z^2 + \frac{6075}{16}z^4x^2y^2 \\ & - \frac{405}{2}z^3y - \frac{3825}{16}z^2y^2 + \frac{405}{4}yz + \frac{27}{4} - 270zx^2y^3 - \frac{2925}{16}y^4 - \frac{1485}{4}z^4x^2 + \frac{405}{4}z^5y \\ & + \frac{945}{4}z^6 + \frac{3105}{8}y^4z^2 + \frac{675}{2}z^3y^3 + 135y^4x^2 + 135y^6 + 270zy^5 + \frac{6075}{16}z^6x^2 \\ & + \frac{6075}{16}z^6y^2 - \frac{135}{2}z^3x^2y + \frac{6075}{16}z^4y^4 + \frac{2025}{8}x^2y^2z^2 + \frac{495}{8}y^2 + \frac{135}{2}zx^2y \end{aligned}$$

### 11.25.2. Variance

#### Modèle de Degré 1

> Var:=(Var\_Norm\_Blue\_D\_Centre(N20,1,0));

$$Var := 1 + 3var_3^2 + 3var_2^2 + 3var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))  
,Var,poly\_algebra(seq(var[i],i=1..3)));

$$1 + 3 \text{IndetDecompInKfi}_1, \text{IndetDecompInKfi}_1 = var_1^2 + var_2^2 + var_3^2$$

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):

#### Modèle de Degré 2

> Var:=Var\_Norm\_Blue\_D\_Centre(N20,2,1);

$$\begin{aligned} Var := & 20 - 17var_3^2 - 17var_2^2 - 17var_1^2 + 15var_2^2var_3^2 + 15var_1^2var_3^2 + 15var_1^2var_2^2 \\ & + (-20 + 26var_3^2 + \frac{37}{2}var_2^2 + \frac{37}{2}var_1^2)var_3^2 \\ & + (-20 + \frac{37}{2}var_3^2 + 26var_2^2 + \frac{37}{2}var_1^2)var_2^2 \\ & + (-20 + \frac{37}{2}var_3^2 + \frac{37}{2}var_2^2 + 26var_1^2)var_1^2 \end{aligned}$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))  
,Var,poly\_algebra(seq(var[i],i=1..3),alg\_relations={t^2-13,s^2-2})));

$$\begin{aligned} & 20 + 26 \text{IndetDecompInKfi}_1^2 - 37 \text{IndetDecompInKfi}_1, \\ & \text{IndetDecompInKfi}_1 = var_1^2 + var_2^2 + var_3^2 \end{aligned}$$

> seq((DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))  
,Var\_Norm\_Blue\_D\_Centre(N20,2,k),poly\_algebra(seq(var[i],i=1..3)))),  
k=1..10);

```

20 + 26 %12 - 37 %1, %2, 10 + 16 %12 - 17 %1, %2, $\frac{20}{3} + \frac{38}{3} %1^2 - \frac{31}{3} %1, %2,$
5 + 11 %12 - 7 %1, %2, 4 + 10 %12 - 5 %1, %2, $\frac{10}{3} + \frac{28}{3} %1^2 - \frac{11}{3} %1, %2,$
 $\frac{20}{7} + \frac{62}{7} %1^2 - \frac{19}{7} %1, %2, \frac{5}{2} + \frac{17}{2} %1^2 - 2 %1, %2, \frac{20}{9} + \frac{74}{9} %1^2 - \frac{13}{9} %1, %2,$
2 + 8 %12 - %1, %2
%1 := IndetDecompInKfi1
%2 := %1 = var12 + var22 + var32

```

**11.26. N21**

```
> unassign('alpha','beta','nu','delta','eta');
```

```
> GrN21:=Rned(Diedn(3),3);
```

$$GrN21 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\left. \begin{bmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

```
> N21:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN21) union
OrbPG(Vector([delta,0,eta]),GrN21) union
OrbPG(Vector([1,0,0]),GrN21));
```

$$N21 := \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \%1 & -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & \nu \\ -\frac{\delta}{2} & -\frac{\sqrt{3}\delta}{2} & -\eta \\ \alpha & -\beta & -\nu \\ -\frac{\delta}{2} & \frac{\sqrt{3}\delta}{2} & -\eta \\ \alpha & \beta & \nu \\ \%1 & \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & -\nu \\ \%1 & \frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \delta & 0 & \eta \\ \alpha & -\beta & \nu \\ 1 & 0 & 0 \\ -\frac{\delta}{2} & -\frac{\sqrt{3}\delta}{2} & \eta \\ \%2 & -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & -\nu \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \%2 & -\frac{\sqrt{3}\alpha}{2} - \frac{\beta}{2} & \nu \\ \alpha & \beta & -\nu \\ \delta & 0 & -\eta \\ \%2 & \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & \nu \\ \%2 & \frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ \%1 & -\frac{\sqrt{3}\alpha}{2} + \frac{\beta}{2} & -\nu \\ -\frac{\delta}{2} & \frac{\sqrt{3}\delta}{2} & \eta \end{bmatrix}$$

$$\%1 := -\frac{\alpha}{2} - \frac{\sqrt{3}\beta}{2}$$

$$\%2 := -\frac{\alpha}{2} + \frac{\sqrt{3}\beta}{2}$$

> UpdatePlan(N21,"N21");

“Update complete”

> Var:=Var\_Norm\_Blue\_D\_Centre(N21,1,0);

$$Var := 1 + \frac{7}{2} \frac{var_3^2}{2\nu^2 + \eta^2} + \frac{14 var_2^2}{4\alpha^2 + 4\beta^2 + 2\delta^2 + 1} + \frac{14 var_1^2}{4\alpha^2 + 4\beta^2 + 2\delta^2 + 1}$$

> Cond:=EqInKr\_reduce({var[1],var[2],var[3]},{Var});

$$Cond := [4\alpha^2 + 4\beta^2 + 2\delta^2 + 1 - 8\nu^2 - 4\eta^2]$$

> Cond\_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,delta^2+eta^2-1],{alpha,beta,nu,eta,delta});

*Cond\_gsolved* :=

$\{[-6\delta^2 - 1 + 12\nu^2, \delta^2 + \eta^2 - 1, 12\alpha^2 + 12\beta^2 + 6\delta^2 - 11], \text{plex}(\alpha, \eta, \nu, \beta, \delta), \{\}\}$

```
> Cond:=convert(convert(map(mem->if mem<>0 then return mem end
if,simplify(eval(EqF_Gen_Mom_Rad(subs(sqrt(3)=radic,N21),2),[radic=sqr
t(3)]))),set),list);
```

$$\begin{aligned} \text{Cond} := & [16\eta^4 - 64\nu^4 - 48\eta^2\delta^2 + 12\delta^4 - 8\eta^2 + 4\delta^2 + 5, \\ & 4\eta^2\alpha + 8\alpha\nu^2 - 2\alpha\delta^2 + 2\delta^3 - 16\alpha\beta^2 - \alpha + 1, \\ & 2\delta^3 + 16\alpha^3 + 1 - 24\alpha\nu^2 + 6\alpha\delta^2 + 3\alpha - 12\eta^2\alpha, \\ & -24\nu^2 + 12\beta^2 + 6\delta^2 + 12\alpha^2 + 3 - 12\eta^2, 32 - 384\nu^4 + 12\alpha - 12\beta^2 - 192\alpha^2\beta^2 \\ & - 24\alpha^2 + 192\alpha^2\nu^2 + 96\nu^2\beta^2 - 192\eta^2\delta^2 + 96\eta^2\alpha^2 + 48\eta^2\beta^2 - 32\eta^2 + 72\delta^4 \\ & - 48\alpha^2\delta^2 + 16\delta^2 - 24\beta^2\delta^2 + 24\alpha\delta^3, \\ & 48\nu^2\eta^2 - 24\eta^2\delta^2 - 24\nu^2\delta^2 + 12\delta^4 - 8\eta^2 - 12\nu^2 + 4\delta^2 + 5, -64 - 432\beta^4 + 768\nu^4 \\ & + 3\alpha - 84\beta^2 - 480\alpha^2\beta^2 + 21\alpha^2 - 168\alpha^2\nu^2 + 672\nu^2\beta^2 + 384\eta^2\delta^2 - 84\eta^2\alpha^2 \\ & + 336\eta^2\beta^2 + 64\eta^2 - 144\delta^4 + 42\alpha^2\delta^2 - 32\delta^2 - 168\beta^2\delta^2 + 6\alpha\delta^3] \end{aligned}$$

```
> Cond_gsolved:=gsolve(map(mem->if mem<>0 then return mem end
if,[-6*delta^2-1+12*nu^2, delta^2+eta^2-1,
12*alpha^2+12*beta^2+6*delta^2-11,op(convert(Cond,set))]),
{alpha,beta,delta,eta,nu});
```

$$\begin{aligned} \text{Cond\_gsolved} := & \{[540\delta^4 - 660\delta^2 + 113, 207360\beta^6 + 155520\beta^4\delta^2 - 285120\beta^4 \\ & - 71280\beta^2\delta^2 + 91908\beta^2 + 3240\delta^3 + 11688\delta^2 - 8551, -6\delta^2 - 1 + 12\nu^2, \delta^2 + \eta^2 - 1, \\ & 30384703\alpha + 637113600\delta^3\beta^4 - 809092800\beta^4\delta^2 - 338620320\delta\beta^4 + 670334400\beta^4 \\ & - 454979700\delta^3\beta^2 + 727987500\beta^2\delta^2 + 304676250\delta\beta^2 - 662272650\beta^2 \\ & + 58271940\delta^3 - 110715390\delta^2 - 46336441\delta + 106182840], \text{plex}(\alpha, \eta, \nu, \beta, \delta), \\ & \{-6\delta^2 - 48\beta^2 + 11, 2\delta^3 + 1\}]\} \end{aligned}$$

```
> Sols:=solve({540*delta^4-660*delta^2+113,
207360*beta^6+155520*beta^4*delta^2-285120*beta^4-71280*beta^2*delta^2
+91908*beta^2+3240*delta^3+11688*delta^2-8551, -6*delta^2-1+12*nu^2,
delta^2+eta^2-1,
30384703*alpha+637113600*delta^3*beta^4-809092800*beta^4*delta^2-33862
0320*delta*beta^4+670334400*beta^4-454979700*delta^3*beta^2+727987500*
beta^2*delta^2+304676250*delta*beta^2-662272650*beta^2+58271940*delta^
3-110715390*delta^2-46336441*delta+106182840} union
{-1<=alpha,alpha<=1,-1<=beta,beta<=1,-1<=delta,delta<=1,-1<=eta,eta<=
1,-1<=nu,nu<=1,-6*delta^2-48*beta^2+11<>0,
2*delta^3+1<>0},{alpha,nu,beta,eta,delta}):
```

```
> evalf(Sols);
```



$\eta = 0.8911239162, \beta = -0.8933985598, \alpha = 0.124726072, \delta = 0.4537600313,$   
 $\nu = -0.4316044675\}, \{\eta = 0.8911239162, \alpha = 0.711342803, \delta = 0.4537600313,$   
 $\beta = -0.5547152407, \nu = -0.4316044675\}, \{\eta = 0.8911239162, \delta = 0.4537600313,$   
 $\alpha = -0.836068893, \beta = -0.3386833191, \nu = -0.4316044675\}, \{\eta = 0.8911239162,$   
 $\delta = 0.4537600313, \alpha = -0.836068893, \beta = 0.3386833191, \nu = -0.4316044675\}, \{\eta = 0.8911239162, \alpha = 0.711342803, \delta = 0.4537600313, \beta = 0.5547152407,$   
 $\nu = -0.4316044675\}, \{\eta = 0.8911239162, \alpha = 0.124726072, \delta = 0.4537600313,$   
 $\beta = 0.8933985598, \nu = -0.4316044675\}, \{\nu = 0.4316044675, \eta = 0.8911239162,$   
 $\beta = -0.8933985598, \alpha = 0.124726072, \delta = 0.4537600313\}, \{\nu = 0.4316044675,$   
 $\eta = 0.8911239162, \alpha = 0.711342803, \delta = 0.4537600313, \beta = -0.5547152407\}, \{\nu = 0.4316044675, \eta = 0.8911239162, \delta = 0.4537600313, \alpha = -0.836068893,$   
 $\beta = -0.3386833191\}, \{\nu = 0.4316044675, \eta = 0.8911239162, \delta = 0.4537600313,$   
 $\alpha = -0.836068893, \beta = 0.3386833191\}, \{\nu = 0.4316044675, \eta = 0.8911239162,$   
 $\alpha = 0.711342803, \delta = 0.4537600313, \beta = 0.5547152407\}, \{\nu = 0.4316044675,$   
 $\eta = 0.8911239162, \alpha = 0.124726072, \delta = 0.4537600313, \beta = 0.8933985598\}$

> nops([Sols]);

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Ces dispositifs sont tous identiques à une rotation d'axe (Oz) près.

> assign(op(1,[Sols]));

> N21:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN21) union  
OrbPG(Vector([delta,0,eta]),GrN21) union  
OrbPG(Vector([1,0,0]),GrN21))):

> UpdatePlan(N21,"N21");

"Update complete"

> pointplot3d(Map(Re,evalf(N21)),symbol=box,scaling=constrained,shading  
=Z);

> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N21)),1  
istlist))]),'rational',5))));

$Graph := \text{POLYTOPE}(3, 3, 21, 33)$

> plotdes:=draw(Graph):

> display(plotdes,scaling=constrained,shading=Z);



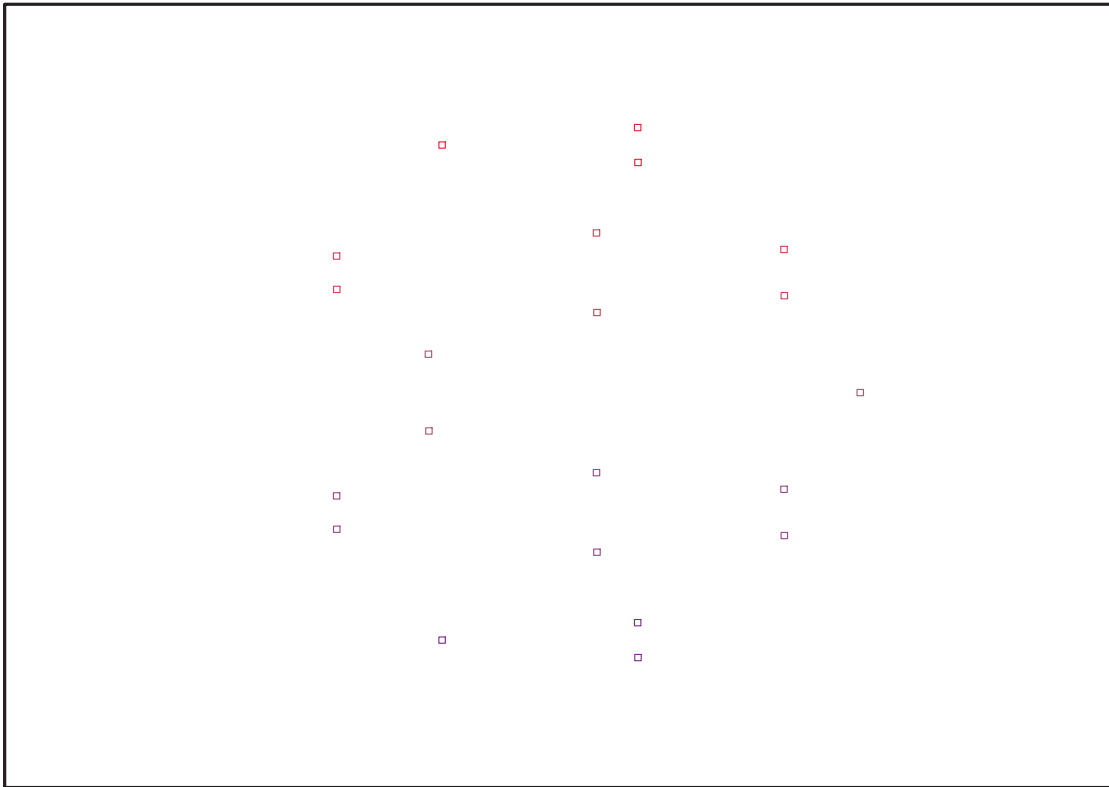


FIG. 11.67 – Graphique : Support du plan N21.

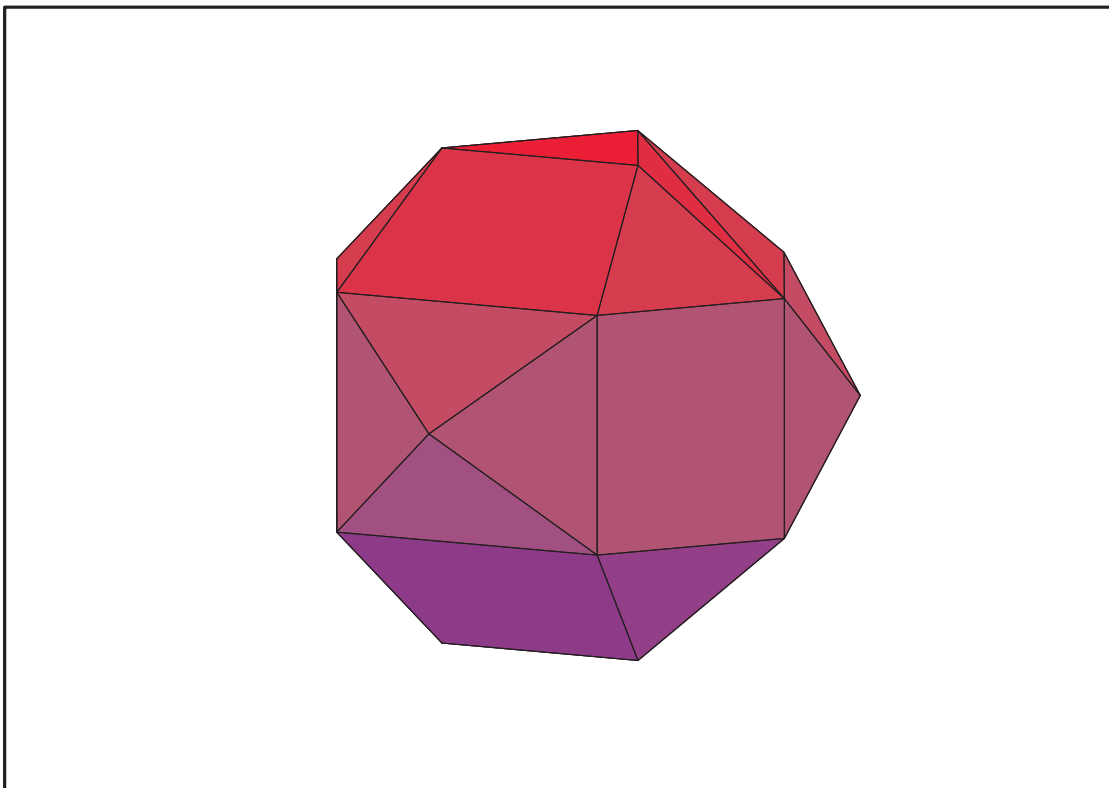


FIG. 11.68 – Graphique : Enveloppe convexe du support du plan N21.

### 11.26.1. Variance

#### Modèle de Degré 1

```
> Var:=Var_Norm_Blue_D_Centre(N21,1,0);
 $Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
 },Var,poly_algebra(seq(var[i],i=1..3)));
 $1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

#### Modèle de Degré 2

```
> Var:=expand(Var_Norm_Blue_D_Centre(N21,2,1)):
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
 },Var,poly_algebra(seq(var[i],i=1..3)));
 $21 + 27 IndetDecompInKfi_1^2 - 39 IndetDecompInKfi_1,$
 $IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$
```

## 11.27. N22

```
> unassign('alpha','beta','nu','delta','mu','eta');
```

```
> GrN22:=TwoPlusn(10);
```

$$GrN22 := \left\{ \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%1 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%3 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%4 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%3 & 0 \\ \%4 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%2 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%4 & 0 \\ \%4 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%4 & 0 \\ \%3 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{5}}{4} + \frac{1}{4} & \%3 & 0 \\ \%3 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%1 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%2 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \Bigg\}$$

$$\%1 := \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}$$

$$\%2 := -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}$$

$$\%3 := -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

$$\%4 := \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$

```
> N22:=OrbToMatrix(OrbPG(Vector([0,alpha,nu]),GrN22) union
OrbPG(Vector([0,delta,mu]),GrN22) union
OrbPG(Vector([0,0,1]),GrN22));
```

$$\begin{aligned}
N22 := & \begin{bmatrix}
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & -\nu \\
0 & \alpha & \nu \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (-\frac{1}{4} - \frac{\sqrt{5}}{4})\alpha & \nu \\
\%1\alpha & (-\frac{\sqrt{5}}{4} + \frac{1}{4})\alpha & -\nu \\
0 & -\alpha & -\nu \\
\%2\alpha & (-\frac{\sqrt{5}}{4} + \frac{1}{4})\alpha & -\nu \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & -\nu \\
\%2\alpha & (-\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & \nu \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\alpha}{4} & (-\frac{1}{4} - \frac{\sqrt{5}}{4})\alpha & \nu \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\delta}{4} & (\frac{1}{4} + \frac{\sqrt{5}}{4})\delta & -\mu \\
0 & \delta & \mu \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\delta}{4} & (-\frac{1}{4} - \frac{\sqrt{5}}{4})\delta & \mu \\
\%1\delta & (-\frac{\sqrt{5}}{4} + \frac{1}{4})\delta & -\mu \\
\%1\delta & (-\frac{1}{4} + \frac{\sqrt{5}}{4})\delta & \mu \\
\%2\delta & (-\frac{1}{4} + \frac{\sqrt{5}}{4})\delta & \mu \\
0 & -\delta & -\mu \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\delta}{4} & (-\frac{1}{4} - \frac{\sqrt{5}}{4})\delta & \mu \\
0 & 0 & -1 \\
\%1\alpha & (-\frac{1}{4} + \frac{\sqrt{5}}{4})\alpha & \nu \\
\%2\delta & (-\frac{\sqrt{5}}{4} + \frac{1}{4})\delta & -\mu \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\delta}{4} & (\frac{1}{4} + \frac{\sqrt{5}}{4})\delta & -\mu \\
0 & 0 & 1
\end{bmatrix} \\
\%1 := & -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} - \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8} \\
\%2 := & \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{5}}{8} + \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{8}
\end{aligned}$$

> UpdatePlan(N22,"N22");

“Update complete”

> Var:=Var\_Norm\_Blue\_D\_Centre(N22,1,0);

$$Var := 1 + \frac{11 \, var_3^2}{5 \, \mu^2 + 1 + 5 \, \nu^2} + \frac{22}{5} \frac{var_2^2}{\alpha^2 + \delta^2} + \frac{22}{5} \frac{var_1^2}{\alpha^2 + \delta^2}$$

```

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},[Var]);
 Cond := [-2 - 10 ν2 - 10 μ2 + 5 δ2 + 5 α2]

> Cond_gsolve:=gsolve([op(Cond),alpha^2+nu^2-1,delta^2+mu^2-1],
{alpha,nu,mu,delta});
 Cond_gsolve := {[15 ν2 + 7 - 15 δ2, δ2 + μ2 - 1, 15 α2 + 15 δ2 - 22], plex(α, μ, ν, δ), {}}

> Cond:=convert(convert(map(mem->if mem<>0 then return mem end
if,simplify(eval(EqF_Gen_Mom_Rad(subs(sqrt(5)=radic3,subs(sqrt(2)=radi
c,sqrt(5-sqrt(5))=radic2,N22))),2),[radic=sqrt(2),radic3=sqrt(5),radic2
=sqrt(5-sqrt(5))])),set),list);

 Cond := [10 α2 + 10 δ2 - 4 - 20 μ2 - 20 ν2,
4 + 20 μ2 + 20 ν2 + 10 α2 √5 - 10 α2 + 10 δ2 √5 - 10 δ2 - 4 √5 - 20 μ2 √5 - 20 ν2 √5,
-4 + 40 ν4 - 20 μ4 + 12 ν2 + 30 δ2 μ2 + 60 μ2 ν2 - 30 ν2 δ2, -176 + 150 α4 - 880 ν4
- 880 μ4 - 360 δ2 μ2 + 120 ν2 δ2 √5 + 120 δ2 √5 μ2 + 24 δ2 √5 - 24 α2 √5 + 72 α2
+ 360 α2 ν2 + 360 α2 μ2 - 360 ν2 δ2 - 120 α2 μ2 √5 - 120 α2 ν2 √5 + 510 δ4 + 48 √5
+ 240 μ4 √5 + 240 ν4 √5 - 72 δ2 - 150 δ4 √5 - 30 α4 √5, 270 δ4 - 1080 ν2 δ2 + 90 α4
- 120 α2 μ2 - 280 μ4 - 120 α2 ν2 + 2400 μ2 ν2 + 920 ν4 - 96 δ2 + 120 δ2 μ2 - 24 α2
+ 240 μ2 + 480 ν2 - 152, 880 - 2880 α4 + 14480 ν4 - 80 √5 + 7280 μ4 + 2880 ν2
+ 600 δ2 μ2 - 120 α2 - 600 α2 ν2 + 1440 μ2 - 4800 μ2 ν2 √5 + 3000 ν2 δ2 √5
+ 600 δ2 √5 μ2 - 600 α2 μ2 + 14400 μ2 ν2 - 3760 ν4 √5 - 6600 ν2 δ2 - 960 ν2 √5
- 600 α2 ν2 √5 - 600 α2 μ2 √5 - 480 μ2 √5 + 360 δ2 √5 - 120 α2 √5 - 1680 δ4
- 1360 μ4 √5 - 600 δ2 - 240 δ4 √5 + 960 α4 √5, 30 δ2 √5 μ2 - 30 δ2 μ2 - 30 ν2 δ2 √5
+ 30 ν2 δ2 + 20 μ4 - 20 μ4 √5 - 60 μ2 ν2 + 60 μ2 ν2 √5 - 40 ν4 + 40 ν4 √5 - 12 ν2
+ 12 ν2 √5 + 4 - 4 √5, -20 ν α μ2 - 20 ν3 α + 10 α3 ν + 10 ν α δ2 - 4 ν α]

> Cond_gsolve:=gsolve([15*nu^2+7-15*delta^2, delta^2+mu^2-1,
15*alpha^2+15*delta^2-22,op(subs(sqrt(5)=radic3,convert(Cond,set))),ra
dic3^2-5],{alpha,nu,mu,delta,radic3});

 Cond_gsolve := {[
[22 + 45 δ4 - 66 δ2, 15 ν2 + 7 - 15 δ2, δ2 + μ2 - 1, radic32 - 5, 15 α2 + 15 δ2 - 22],
plex(α, radic3, μ, ν, δ), {α, ν}]}

> Sols:=allvalues(solve({22+45*delta^4-66*delta^2,
15*nu^2+7-15*delta^2, delta^2+mu^2-1,
15*alpha^2+15*delta^2-22},{alpha,nu,delta,mu})));

```

$Sols := [\{\alpha = \%1, \mu = \%6, \delta = \%2, \nu = \%7\}, \{\alpha = \%8, \mu = \%6, \delta = \%2, \nu = \%7\},$   
 $\{\alpha = \%1, \delta = \%2, \mu = \%3, \nu = \%7\}, \{\alpha = \%8, \delta = \%2, \mu = \%3, \nu = \%7\},$   
 $\{\alpha = \%1, \nu = \%4, \mu = \%6, \delta = \%2\}, \{\alpha = \%8, \nu = \%4, \mu = \%6, \delta = \%2\},$   
 $\{\alpha = \%1, \nu = \%4, \delta = \%2, \mu = \%3\}, \{\alpha = \%8, \nu = \%4, \delta = \%2, \mu = \%3\},$   
 $\{\alpha = \%1, \mu = \%6, \delta = \%5, \nu = \%7\}, \{\alpha = \%8, \mu = \%6, \delta = \%5, \nu = \%7\},$   
 $\{\alpha = \%1, \delta = \%5, \mu = \%3, \nu = \%7\}, \{\alpha = \%8, \delta = \%5, \mu = \%3, \nu = \%7\},$   
 $\{\alpha = \%1, \nu = \%4, \mu = \%6, \delta = \%5\}, \{\alpha = \%8, \nu = \%4, \mu = \%6, \delta = \%5\},$   
 $\{\alpha = \%1, \nu = \%4, \delta = \%5, \mu = \%3\}, \{\alpha = \%8, \nu = \%4, \delta = \%5, \mu = \%3\},$   
 $\{\mu = \%7, \nu = \%6, \alpha = \%5, \delta = \%8\}, \{\mu = \%7, \nu = \%6, \alpha = \%2, \delta = \%8\},$   
 $\{\mu = \%7, \nu = \%3, \alpha = \%5, \delta = \%8\}, \{\mu = \%7, \nu = \%3, \alpha = \%2, \delta = \%8\},$   
 $\{\mu = \%4, \nu = \%6, \alpha = \%5, \delta = \%8\}, \{\mu = \%4, \nu = \%6, \alpha = \%2, \delta = \%8\},$   
 $\{\mu = \%4, \nu = \%3, \alpha = \%5, \delta = \%8\}, \{\mu = \%4, \nu = \%3, \alpha = \%2, \delta = \%8\},$   
 $\{\mu = \%7, \nu = \%6, \alpha = \%5, \delta = \%1\}, \{\mu = \%7, \nu = \%6, \alpha = \%2, \delta = \%1\},$   
 $\{\mu = \%7, \nu = \%3, \alpha = \%5, \delta = \%1\}, \{\mu = \%7, \nu = \%3, \alpha = \%2, \delta = \%1\},$   
 $\{\mu = \%4, \nu = \%6, \alpha = \%5, \delta = \%1\}, \{\mu = \%4, \nu = \%6, \alpha = \%2, \delta = \%1\},$   
 $\{\mu = \%4, \nu = \%3, \alpha = \%5, \delta = \%1\}, \{\mu = \%4, \nu = \%3, \alpha = \%2, \delta = \%1\}]$

$$\%1 := \frac{\sqrt{165 - 15\sqrt{11}}}{15}$$

$$\%2 := -\frac{\sqrt{165 + 15\sqrt{11}}}{15}$$

$$\%3 := -\frac{\sqrt{60 - 15\sqrt{11}}}{15}$$

$$\%4 := -\frac{\sqrt{60 + 15\sqrt{11}}}{15}$$

$$\%5 := \frac{\sqrt{165 + 15\sqrt{11}}}{15}$$

$$\%6 := \frac{\sqrt{60 - 15\sqrt{11}}}{15}$$

$$\%7 := \frac{\sqrt{60 + 15\sqrt{11}}}{15}$$

$$\%8 := -\frac{\sqrt{165 - 15\sqrt{11}}}{15}$$

> evalf(Sols);

```
{\alpha = 0.7156989687, \mu = 0.2134440145, \delta = -0.9769552974, \nu = 0.6984088960},
{\mu = 0.2134440145, \delta = -0.9769552974, \nu = 0.6984088960, \alpha = -0.7156989687},
{\alpha = 0.7156989687, \delta = -0.9769552974, \nu = 0.6984088960, \mu = -0.2134440145},
{\delta = -0.9769552974, \nu = 0.6984088960, \alpha = -0.7156989687, \mu = -0.2134440145},
{\alpha = 0.7156989687, \mu = 0.2134440145, \delta = -0.9769552974, \nu = -0.6984088960},
{\mu = 0.2134440145, \delta = -0.9769552974, \alpha = -0.7156989687, \nu = -0.6984088960},
{\alpha = 0.7156989687, \delta = -0.9769552974, \mu = -0.2134440145, \nu = -0.6984088960},
{\delta = -0.9769552974, \alpha = -0.7156989687, \mu = -0.2134440145, \nu = -0.6984088960},
{\alpha = 0.7156989687, \mu = 0.2134440145, \nu = 0.6984088960, \delta = 0.9769552974},
{\mu = 0.2134440145, \nu = 0.6984088960, \alpha = -0.7156989687, \delta = 0.9769552974},
{\alpha = 0.7156989687, \nu = 0.6984088960, \mu = -0.2134440145, \delta = 0.9769552974},
{\nu = 0.6984088960, \alpha = -0.7156989687, \mu = -0.2134440145, \delta = 0.9769552974},
{\alpha = 0.7156989687, \mu = 0.2134440145, \nu = -0.6984088960, \delta = 0.9769552974},
{\mu = 0.2134440145, \alpha = -0.7156989687, \nu = -0.6984088960, \delta = 0.9769552974},
{\alpha = 0.7156989687, \mu = -0.2134440145, \nu = -0.6984088960, \delta = 0.9769552974},
{\alpha = -0.7156989687, \mu = -0.2134440145, \nu = -0.6984088960, \delta = 0.9769552974},
{\mu = 0.6984088960, \nu = 0.2134440145, \alpha = 0.9769552974, \delta = -0.7156989687},
{\mu = 0.6984088960, \nu = 0.2134440145, \delta = -0.7156989687, \alpha = -0.9769552974},
{\mu = 0.6984088960, \alpha = 0.9769552974, \delta = -0.7156989687, \nu = -0.2134440145},
{\mu = 0.6984088960, \delta = -0.7156989687, \alpha = -0.9769552974, \nu = -0.2134440145},
{\mu = -0.6984088960, \nu = 0.2134440145, \alpha = 0.9769552974, \delta = -0.7156989687},
{\mu = -0.6984088960, \nu = 0.2134440145, \delta = -0.7156989687, \alpha = -0.9769552974},
{\mu = -0.6984088960, \alpha = 0.9769552974, \delta = -0.7156989687, \nu = -0.2134440145},
{\mu = -0.6984088960, \delta = -0.7156989687, \alpha = -0.9769552974, \nu = -0.2134440145},
{\mu = 0.6984088960, \nu = 0.2134440145, \alpha = 0.9769552974, \delta = 0.7156989687},
{\mu = 0.6984088960, \nu = 0.2134440145, \delta = 0.7156989687, \alpha = -0.9769552974},
{\mu = 0.6984088960, \alpha = 0.9769552974, \delta = 0.7156989687, \nu = -0.2134440145},
{\mu = 0.6984088960, \delta = 0.7156989687, \alpha = -0.9769552974, \nu = -0.2134440145},
{\mu = -0.6984088960, \nu = 0.2134440145, \alpha = 0.9769552974, \delta = 0.7156989687},
{\mu = -0.6984088960, \nu = 0.2134440145, \delta = 0.7156989687, \alpha = -0.9769552974},
{\mu = -0.6984088960, \alpha = 0.9769552974, \delta = 0.7156989687, \nu = -0.2134440145},
{\mu = -0.6984088960, \delta = 0.7156989687, \alpha = -0.9769552974, \nu = -0.2134440145}}
```

```
> nops(%);
```

32

```
> Première solution :
```

```
> Sols[1];
```

$$\{\nu = \frac{\sqrt{60+15\sqrt{11}}}{15}, \mu = \frac{\sqrt{60-15\sqrt{11}}}{15}, \alpha = \frac{\sqrt{165-15\sqrt{11}}}{15}, \delta = -\frac{\sqrt{165+15\sqrt{11}}}{15}\}$$

```
> assign(Sols[1]);
```

```
> N22:=OrbToMatrix(SortPoints(simplify(OrbPG(Vector([0,alpha,nu])),GrN22
) union OrbPG(Vector([0,delta,mu]),GrN22) union
OrbPG(Vector([0,0,1]),GrN22))));
```

$$\begin{aligned}
N22 := & \left[ \begin{array}{ccc}
0 & 0 & -1 \\
0 & 0 & 1 \\
0 & -\frac{\sqrt{165-15\sqrt{11}}}{15} & \%1 \\
0 & \frac{\sqrt{165-15\sqrt{11}}}{15} & \%2 \\
0 & \frac{\sqrt{165+15\sqrt{11}}}{15} & \%3 \\
0 & -\frac{\sqrt{165+15\sqrt{11}}}{15} & \%4 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%3 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%4 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%4 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%3 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%1 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%2 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%1 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%2 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%3 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%4 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%3 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%4 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%1 \\
\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%2 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%2 \\
-\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%1
\end{array} \right]
\end{aligned}$$



$$\begin{aligned}\%1 &:= -\frac{\sqrt{60+15\sqrt{11}}}{15} \\ \%2 &:= \frac{\sqrt{60+15\sqrt{11}}}{15} \\ \%3 &:= -\frac{\sqrt{60-15\sqrt{11}}}{15} \\ \%4 &:= \frac{\sqrt{60-15\sqrt{11}}}{15}\end{aligned}$$

```
> UpdatePlan(N22,"N22");
```

“Update complete”

```
> pointplot3d(Map(Re,evalf(N22)),symbol=box,scaling=constrained,shading
=Z);
```

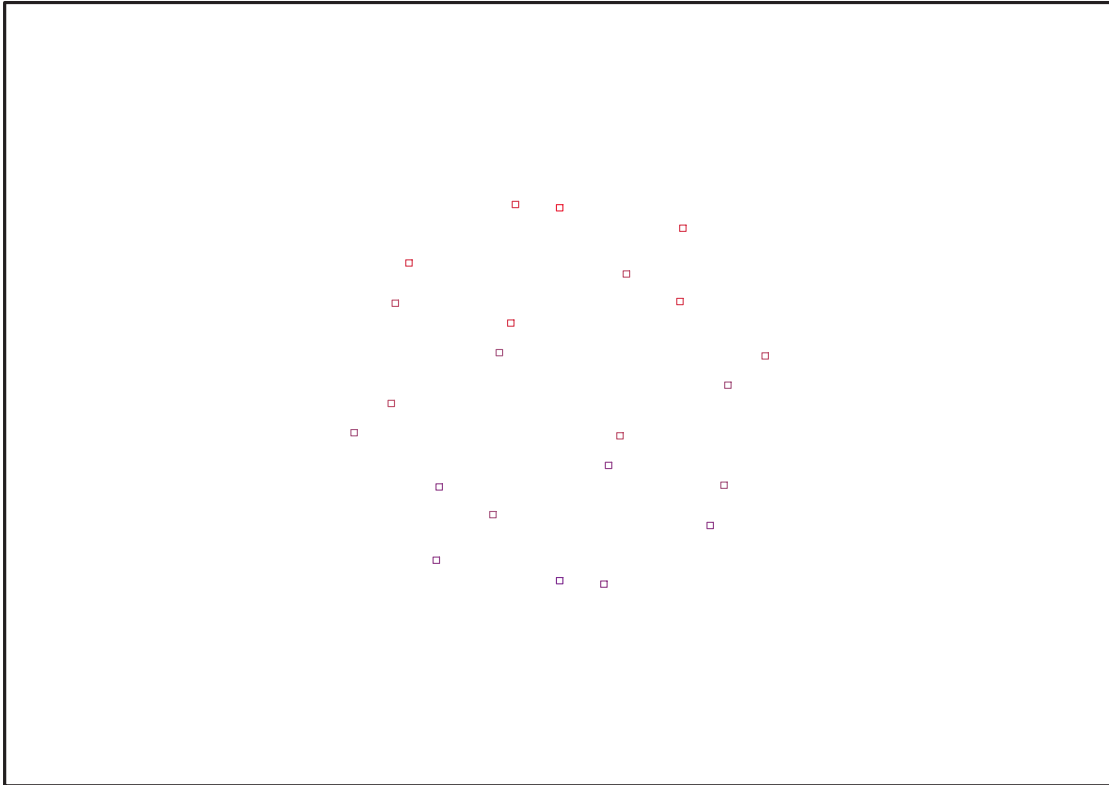


FIG. 11.69 – Graphique : Support du plan N22.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N22)),1
istlist))]),'rational',5))));
```

```
Graph := POLYTOPE(3, 3, 22, 40)
```

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

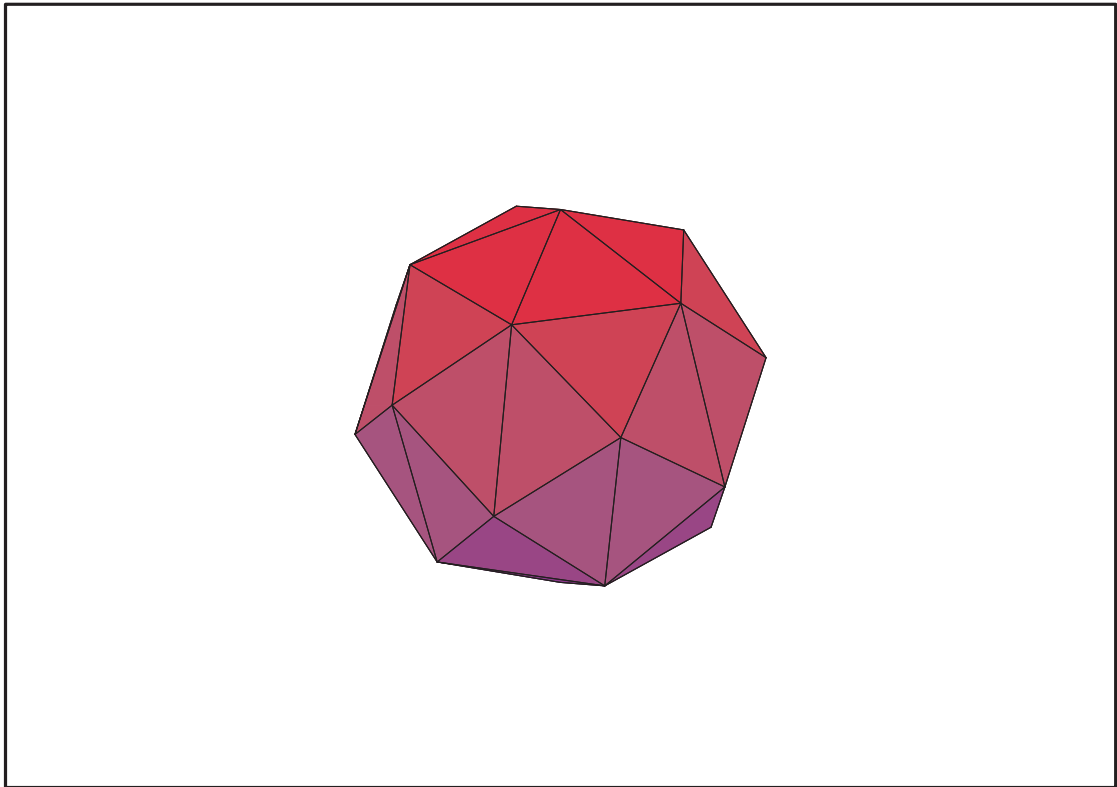


FIG. 11.70 – Graphique : Enveloppe convexe du support du plan N22.

> Deuxième solution :

> Sols[2];

$$\left\{ \alpha = -\frac{\sqrt{165 - 15\sqrt{11}}}{15}, \mu = \frac{\sqrt{60 - 15\sqrt{11}}}{15}, \delta = -\frac{\sqrt{165 + 15\sqrt{11}}}{15}, \nu = \frac{\sqrt{60 + 15\sqrt{11}}}{15} \right\}$$

> assign(Sols[2]);

> N22bis:=OrbToMatrix(SortPoints(simplify(OrbPG(Vector([0,alpha,nu]),Gr  
N22) union OrbPG(Vector([0,delta,mu]),GrN22) union  
OrbPG(Vector([0,0,1]),GrN22))));

$$\begin{aligned}
 N22bis := & \left[ \begin{array}{ccc}
 0 & -\frac{\sqrt{165+15\sqrt{11}}}{15} & \%1 \\
 0 & \frac{\sqrt{165+15\sqrt{11}}}{15} & \%2 \\
 0 & \frac{\sqrt{165-15\sqrt{11}}}{15} & \%3 \\
 0 & -\frac{\sqrt{165-15\sqrt{11}}}{15} & \%4 \\
 0 & 0 & 1 \\
 0 & 0 & -1 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%4 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%3 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%3 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%4 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%2 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%1 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%2 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%1 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%4 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165-15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{60} & \%3 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%4 \\
 -\frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165-15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165-15\sqrt{11}}}{60} & \%3 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & -\frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%2 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}\sqrt{165+15\sqrt{11}}}{60} & \frac{(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{60} & \%1 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & -\frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%1 \\
 \frac{\sqrt{2}\sqrt{5-\sqrt{5}}(1+\sqrt{5})\sqrt{165+15\sqrt{11}}}{120} & \frac{(\sqrt{5}-1)\sqrt{165+15\sqrt{11}}}{60} & \%2
 \end{array} \right]
 \end{aligned}$$

$$\begin{aligned}\%1 &:= \frac{\sqrt{60 - 15\sqrt{11}}}{15} \\ \%2 &:= -\frac{\sqrt{60 - 15\sqrt{11}}}{15} \\ \%3 &:= -\frac{\sqrt{60 + 15\sqrt{11}}}{15} \\ \%4 &:= \frac{\sqrt{60 + 15\sqrt{11}}}{15}\end{aligned}$$

```
> UpdatePlan(N22bis,"N22bis");
```

“Update complete”

```
> pointplot3d(Map(Re,evalf(N22bis)),symbol=box,scaling=constrained,shading=Z);
```

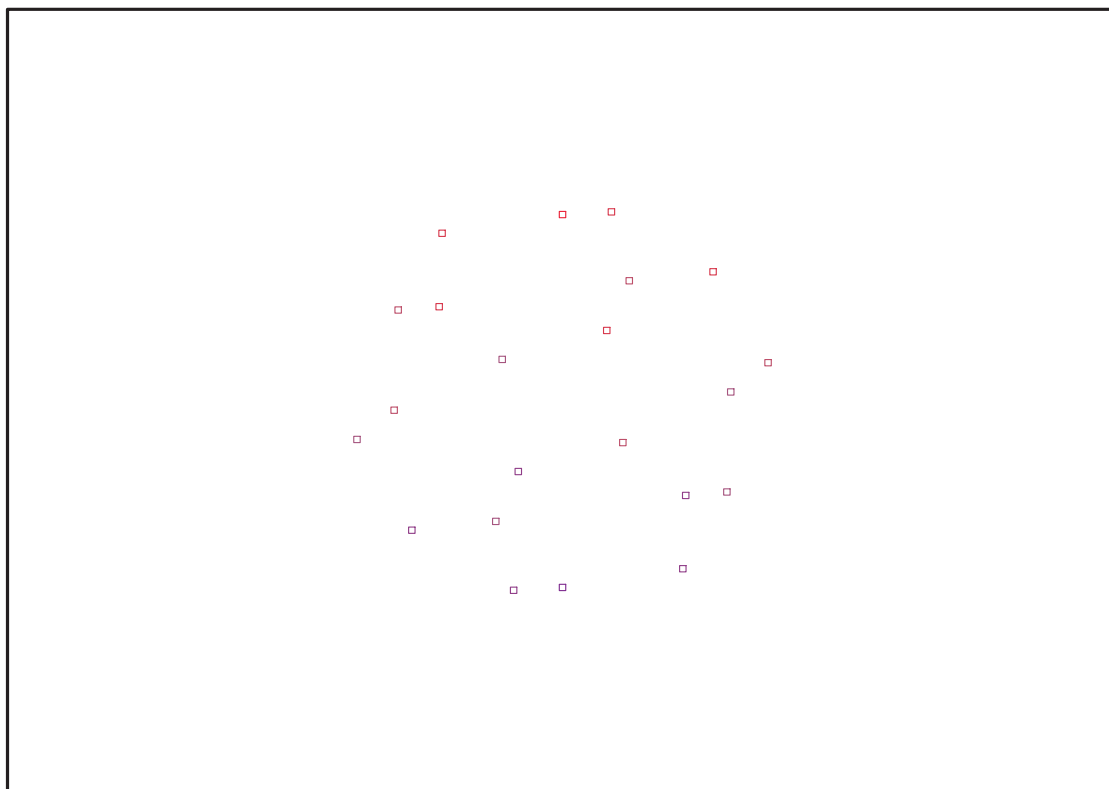


FIG. 11.71 – Graphique : Support du plan N22bis.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(Map(Re,evalf(N22bis)),listlist))]),'rational',5))));
```

*Graph* := POLYTOPE(3, 3, 22, 40)

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> Gens:=Generators(IdealN22bis);
```

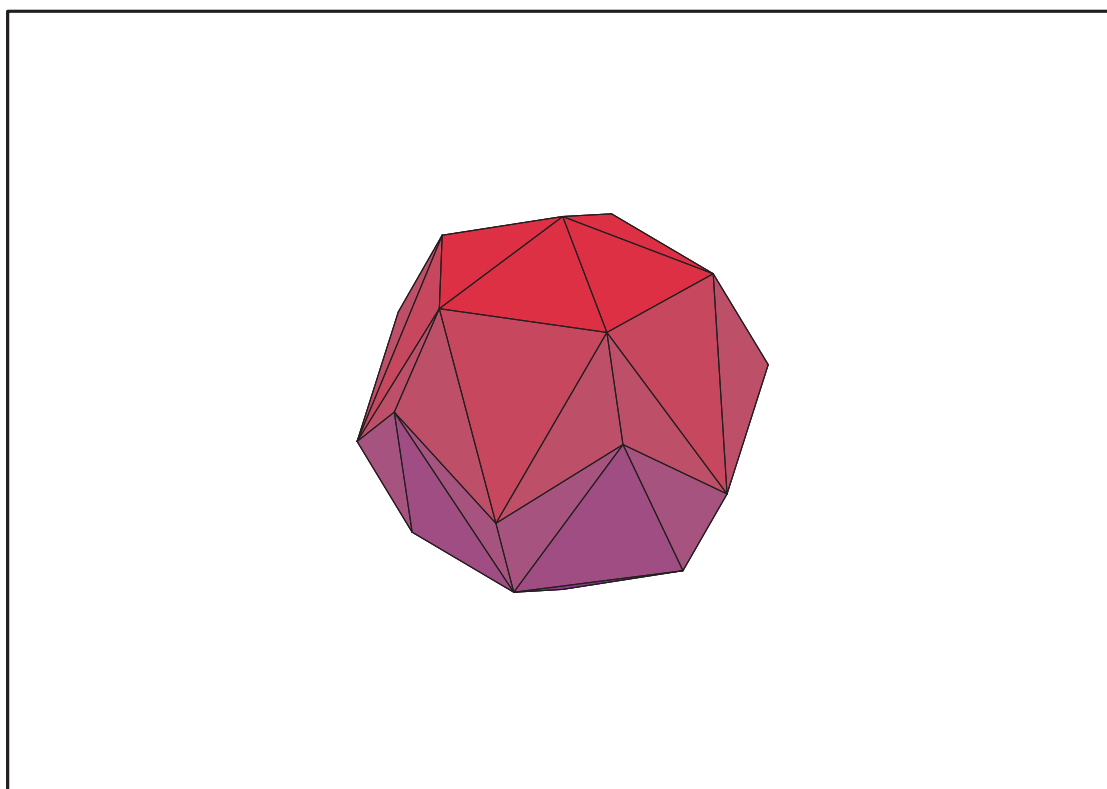


FIG. 11.72 – Graphique : Enveloppe convexe du support du plan N22bis.

### 11.27.1. Variance

#### Modèle de Degré 1

```
> ReadPlan("N22");Plan;N22:=Plan;
 [22 x 3 Matrix Data Type : anything Storage : rectangular Order : Fortran_order]
N22 := [22 x 3 Matrix Data Type : anything Storage : rectangular Order : Fortran_order]

> Var:=(Var_Norm_Blue_D_Centre((N22)),1,0);
 Var := 1 + 3 var32 + 3 var22 + 3 var12

> Varbis:=(Var_Norm_Blue_D_Centre((N22bis)),1,0);
 Varbis := 1 + 3 var32 + 3 var22 + 3 var12

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
 },Var,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
 },Varbis,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

#### Modèle de Degré 2

```
> Var:=expand(Var_Norm_Blue_D_Centre((N22),2,1));
```

```

Var := 22 - 41 var32 - 41 var22 - 41 var12 + 56 var22 var32 + 56 var12 var32 + 56 var12 var22
+ 28 var34 + 28 var24 + 28 var14
> Varbis:=expand(Var_Norm_Blue_D_Centre((N22bis),2,1));

Varbis := 22 - 41 var32 - 41 var22 - 41 var12 + 56 var22 var32 + 56 var12 var32 + 56 var12 var22
+ 28 var34 + 28 var24 + 28 var14
> (DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3),alg_relations={t^2-13,s^2-2
})));

22 + 28 IndetDecompInKfi12 - 41 IndetDecompInKfi1,
IndetDecompInKfi1 = var12 + var22 + var32
> (DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Varbis,poly_algebra(seq(var[i],i=1..3),alg_relations={t^2-13,s^2-2
})));

22 + 28 IndetDecompInKfi12 - 41 IndetDecompInKfi1,
IndetDecompInKfi1 = var12 + var22 + var32
> -Id appartient toujours à TwoPlusn donc les plans sont à symétriecentrale et de
fait automatiquement de force 4+1=5.

```

## 11.28. N24 5-plan

```
> unassign('alpha','beta','nu');
```

```
> GrN24F5:=Plus(Bn(3));
```

$$GrN24 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

```
> N24F5:=OrbToMatrix(OrbPG(Vector([alpha,0,nu]),GrN24F5));
```

$$N24F5 := \begin{bmatrix} 0 & -\alpha & -\nu \\ -\nu & 0 & \alpha \\ 0 & \nu & \alpha \\ -\alpha & 0 & -\nu \\ \nu & \alpha & 0 \\ -\alpha & -\nu & 0 \\ 0 & -\nu & -\alpha \\ \nu & 0 & -\alpha \\ \alpha & -\nu & 0 \\ 0 & \alpha & \nu \\ \alpha & 0 & \nu \\ 0 & -\alpha & \nu \\ -\alpha & 0 & \nu \\ -\nu & -\alpha & 0 \\ 0 & \nu & -\alpha \\ 0 & -\nu & \alpha \\ \alpha & \nu & 0 \\ \nu & -\alpha & 0 \\ 0 & \alpha & -\nu \\ -\alpha & \nu & 0 \\ -\nu & \alpha & 0 \\ \alpha & 0 & -\nu \\ -\nu & 0 & -\alpha \\ \nu & 0 & \alpha \end{bmatrix}$$

```
> UpdatePlan(N24F5,"N24F5");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N24F5,1,0);
```

$$Var := 1 + \frac{3 \, var_3^2}{\alpha^2 + \nu^2} + \frac{3 \, var_2^2}{\alpha^2 + \nu^2} + \frac{3 \, var_1^2}{\alpha^2 + \nu^2}$$

```

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});
 Cond := []

> Cond_gsolve:=gsolve([op(Cond),alpha^2+nu^2-1],[alpha,beta,nu]);
 Cond_gsolve := {[[alpha^2 + nu^2 - 1], plex(beta, alpha, nu), {}]}

> Sols:=convert(map(L->solve({op(convert(L[1],set))},{alpha,nu
}),Cond_gsolve),radical);
 Sols := {{alpha = sqrt(1 - nu^2), nu = nu}, {alpha = -sqrt(1 - nu^2), nu = nu}}

> Cond_4:=op(convert(convert(map(mem->if mem<>0 then return mem end
if,simplify(EqF_Gen_Mom_Rad(N24F5,2))),set),list),alpha^2+nu^2-1];
 Cond_4 := [alpha^4 + nu^4 - 3 alpha^2 nu^2, alpha^2 + nu^2 - 1]

> Cond_4_gsolve:=gsolve(map(mem->if mem<>0 then return mem end
if,Cond_4),[alpha,nu]);
 Cond_4_gsolve := {[[-5 nu^2 + 1 + 5 nu^4, -5 nu^3 + 3 nu + alpha], plex(alpha, nu), {-2 nu^2 + 1}],
 [[-5 nu^2 + 1 + 5 nu^4, 5 nu^3 - 3 nu + alpha], plex(alpha, nu), {}]}

> convert(map(L->allvalues(solve({op(convert(L[1],set))},{alpha,nu
})),Cond_4_gsolve),radical);

```

$$\begin{aligned}
& \left\{ \left\{ \nu = \frac{\sqrt{50-10\sqrt{5}}}{10}, \alpha = -\frac{(-\frac{1}{2} - \frac{\sqrt{5}}{2})\sqrt{50-10\sqrt{5}}}{10} \right\}, \right. \\
& \left\{ \alpha = \frac{(-\frac{1}{2} - \frac{\sqrt{5}}{2})\sqrt{50-10\sqrt{5}}}{10}, \nu = -\frac{\sqrt{50-10\sqrt{5}}}{10} \right\}, \\
& \left\{ \alpha = -\frac{(-\frac{1}{2} + \frac{\sqrt{5}}{2})\sqrt{50+10\sqrt{5}}}{10}, \nu = -\frac{\sqrt{50+10\sqrt{5}}}{10} \right\}, \\
& \left\{ \nu = -\frac{\sqrt{50+10\sqrt{5}}}{10}, \alpha = \frac{(-\frac{1}{2} + \frac{\sqrt{5}}{2})\sqrt{50+10\sqrt{5}}}{10} \right\}, \\
& \left\{ \alpha = \frac{(-\frac{1}{2} - \frac{\sqrt{5}}{2})\sqrt{50-10\sqrt{5}}}{10}, \nu = \frac{\sqrt{50-10\sqrt{5}}}{10} \right\}, \\
& \left\{ \alpha = -\frac{(-\frac{1}{2} - \frac{\sqrt{5}}{2})\sqrt{50-10\sqrt{5}}}{10}, \nu = -\frac{\sqrt{50-10\sqrt{5}}}{10} \right\}, \\
& \left\{ \alpha = \frac{(-\frac{1}{2} + \frac{\sqrt{5}}{2})\sqrt{50+10\sqrt{5}}}{10}, \nu = \frac{\sqrt{50+10\sqrt{5}}}{10} \right\}, \\
& \left. \left\{ \alpha = -\frac{(-\frac{1}{2} + \frac{\sqrt{5}}{2})\sqrt{50+10\sqrt{5}}}{10}, \nu = \frac{\sqrt{50+10\sqrt{5}}}{10} \right\} \right\}
\end{aligned}$$



```
> Cond_6:=convert(convert(map(mem->if mem<>0 then return mem end
if,[op(EqF_Gen_Mom_Rad(N24F5,3)),alpha^2+nu^2-1]),set),list);
```

$$Cond\_6 := [\alpha^2 + \nu^2 - 1, -\nu^6 + 4\alpha^2\nu^4, \alpha^4 + \nu^4 - 3\alpha^2\nu^2]$$

```
> Cond_6_gsolve:=gsolve(map(mem->if mem<>0 then return mem end
if,Cond_6),[alpha,nu]);
```

$$Cond\_6\_gsolved := \{\}$$

> Snub cube : alpha -> -alpha ne change pas le dispositif puisqu'il est Bn invariant donc de force 5. En effet, par construction, il est Bn+invariant et comme une coordonnées de l'orbote est nulle un changement de signe d'un autre coordonnées et de celle-ci est dans Bn+ donc la classe le dispositif inchangé.

```
> alpha := -(-1/2-1/2*5^(1/2))*(1/2-1/10*5^(1/2))^(1/2); nu :=
(1/2-1/10*5^(1/2))^(1/2);
```

$$\alpha := -\left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right) \sqrt{-\frac{\sqrt{5}}{10} + \frac{1}{2}}$$

$$\nu := \sqrt{-\frac{\sqrt{5}}{10} + \frac{1}{2}}$$

```
> N24F5:=simplify(OrbToMatrix(OrbPG(Vector([alpha,0,nu]),GrN24)));
```

$$N24F5 := \begin{bmatrix} \%1 & \%2 & 0 \\ \%3 & \%4 & 0 \\ \%4 & \%1 & 0 \\ \%4 & 0 & \%3 \\ \%1 & 0 & \%4 \\ 0 & \%1 & \%2 \\ \%2 & 0 & \%1 \\ 0 & \%3 & \%2 \\ 0 & \%2 & \%1 \\ \%1 & 0 & \%2 \\ \%2 & \%3 & 0 \\ \%1 & \%4 & 0 \\ 0 & \%2 & \%3 \\ \%2 & \%1 & 0 \\ 0 & \%4 & \%1 \\ 0 & \%1 & \%4 \\ \%3 & \%2 & 0 \\ \%4 & 0 & \%1 \\ \%2 & 0 & \%3 \\ \%4 & \%3 & 0 \\ \%3 & 0 & \%2 \\ \%3 & 0 & \%4 \\ 0 & \%4 & \%3 \\ 0 & \%3 & \%4 \end{bmatrix}$$

$$\%1 := -\frac{\sqrt{50-10\sqrt{5}}}{10}$$

$$\%2 := -\frac{(1+\sqrt{5})\sqrt{50-10\sqrt{5}}}{20}$$

$$\%3 := \frac{\sqrt{50-10\sqrt{5}}}{10}$$

$$\%4 := \frac{(1+\sqrt{5})\sqrt{50-10\sqrt{5}}}{20}$$

```
> UpdatePlan(N24F5, "N24F5");
 "Update complete"
> radnormal(eval(Cond));radnormal(eval(Cond_4));radnormal(eval(Cond_6))
;
 []
 [0, 0]
 [0, 0, $\frac{1}{5}$]
> pointplot3d((N24F5),symbol=box,scaling=constrained,shading=Z);
```

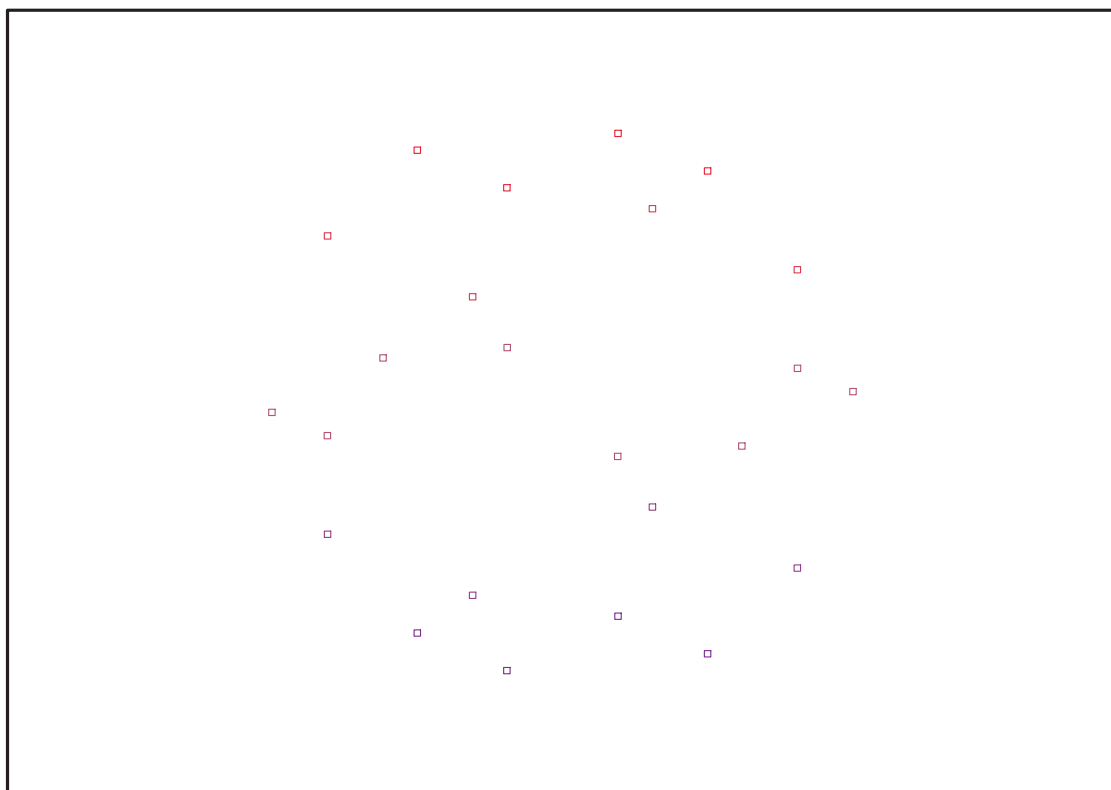


FIG. 11.73 – Graphique : Support du plan N24F5.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N24F5,listlist)))]),'
rational',5)));
 Graph := POLYTOPE(3, 3, 24, 14)
> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z);

> info_iop:=3;
 info_iop := 3
> subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,convert((N24F5
),listlist),Vector)));
```

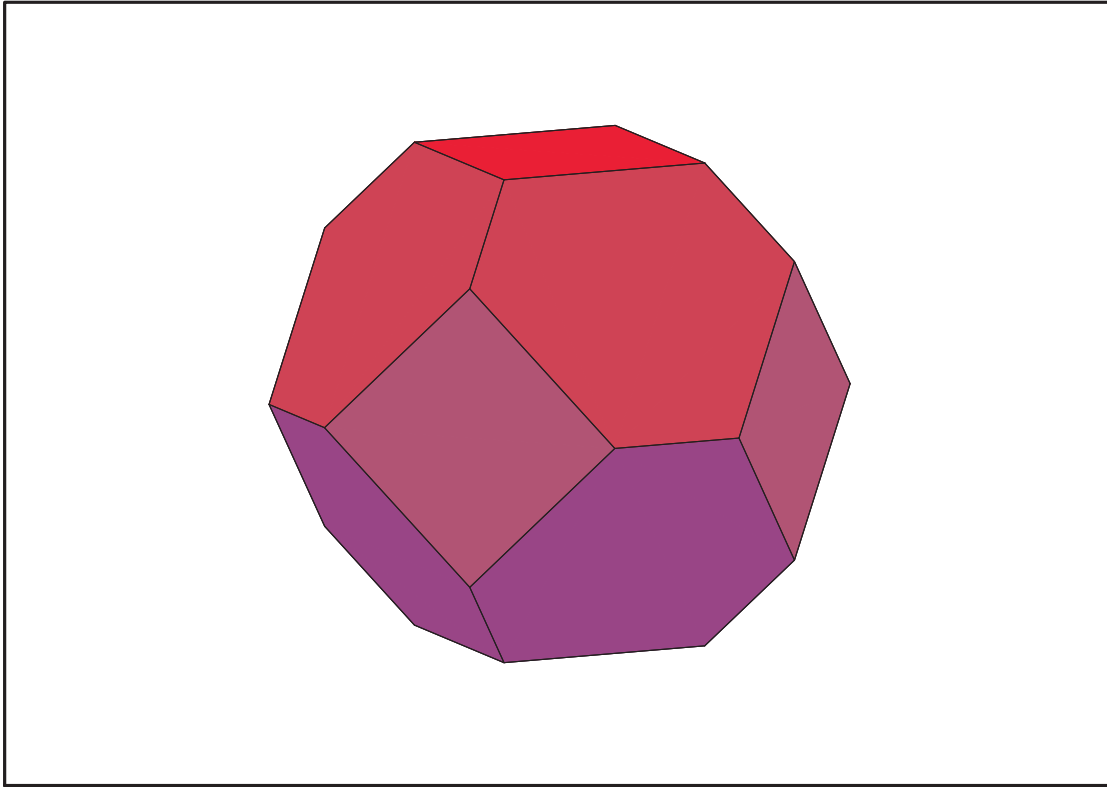


FIG. 11.74 – Graphique : Enveloppe convexe du support du plan N25F5.

$[1, z, y, x, z^2, yz, zx, y^2, xy, z^3, z^2y, z^2x, zy^2, y^3, y^2x, z^4, z^3y, z^3x, z^2y^2, xy^3, yz^4, z^4x, z^3y^2, z^4y^2]$

### 11.28.1. tdeg(x,y,z)

```

> EstSet_N24_tdeg:= [1, z, y, x, z^2, y*z, z*x, y^2, x*y, z^3, z^2*y,
 z^2*x, z*y^2, y^3, y^2*x, z^4, z^3*y, z^3*x, z^2*y^2, x*y^3, y*z^4,
 z^4*x, z^3*y^2, z^4*y^2];

```

$EstSet\_N24\_tdeg := [1, z, y, x, z^2, yz, zx, y^2, xy, z^3, z^2y, z^2x, zy^2, y^3, y^2x, z^4, z^3y, z^3x, z^2y^2, xy^3, yz^4, z^4x, z^3y^2, z^4y^2]$

```

> X_D_ModeleSat(N24F5, [x,y,z], EstSet_N24_tdeg):

```

```

> M_D_ModeleSat(N24F5, [x,y,z], EstSet_N24_tdeg):

```

```

> expand(Var_Blue_Modele_Sat(N24F5, [x,y,z], EstSet_N24_tdeg));

```

$$\begin{aligned}
 & 18 - 120z^2 - 1350z^6 + 2700z^6y^2 + 30x^2 - 30y^2x^2 + 1050z^4x^2 - 210z^2x^2 - 120y^4 + 150y^6 \\
 & - 600y^4z^2 + 2400y^4z^4 + 1500z^8x^2 - 2100z^6x^2 - 4500z^6y^4 + 4500z^8y^4 - 1500z^8y^2 \\
 & + 900z^8 - 150x^2y^4 + 300x^2y^6 + 630z^2y^2 - 2100z^4y^2 + 660z^4 + 600y^2x^2z^2 \\
 & - 600z^4x^2y^2
 \end{aligned}$$

## 11.28.2. Variance

### Modèle de Degré 1

```
> ReadPlan("N24F5");Plan:N24F5:=Plan:
> Var:=Var_Norm_Blue_D_Centre(N24F5,1,0);

$$Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$$

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
 },Var,poly_algebra(seq(var[i],i=1..3)));

$$1 + 3 IndetDecompInKfi_1, IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

### Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N24,2,0);
 "M_D is not full rank"
> Var:=expand(Var_Norm_Blue_D_Centre(N24,2,1)):
> (DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
 },Var,poly_algebra(seq(var[i],i=1..3),alg_relations={t^2-13,s^2-2
 })))));

$$24 + 30 IndetDecompInKfi_1^2 - 45 IndetDecompInKfi_1,$$

$$IndetDecompInKfi_1 = var_1^2 + var_2^2 + var_3^2$$

```

## 11.29. N24bis 7-plan

```
> unassign('alpha','beta','nu');
```

```
> GrN24:=Plus(Bn(3));
```

$$GrN24 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

```
> N24:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN24));
```

$$N24 := \begin{bmatrix} -\beta & -\nu & \alpha \\ -\nu & \beta & \alpha \\ -\beta & \nu & -\alpha \\ \nu & -\alpha & -\beta \\ -\alpha & -\beta & \nu \\ -\beta & \alpha & \nu \\ \nu & \alpha & \beta \\ -\nu & -\beta & -\alpha \\ \alpha & -\beta & -\nu \\ -\alpha & -\nu & -\beta \\ \beta & \alpha & -\nu \\ -\beta & -\alpha & -\nu \\ \beta & -\alpha & \nu \\ \nu & -\beta & \alpha \\ -\nu & \alpha & -\beta \\ -\alpha & \nu & \beta \\ \beta & \nu & \alpha \\ \beta & -\nu & -\alpha \\ \nu & \beta & -\alpha \\ \alpha & \nu & -\beta \\ \alpha & \beta & \nu \\ -\nu & -\alpha & \beta \\ \alpha & -\nu & \beta \\ -\alpha & \beta & -\nu \end{bmatrix}$$

```
> UpdatePlan(N24,"N24");
```

“Update complete”

```
> Var:=Var_Norm_Blue_D_Centre(N24,1,0);
```

$$Var := 1 + \frac{3 \, var_3^2}{\alpha^2 + \beta^2 + \nu^2} + \frac{3 \, var_2^2}{\alpha^2 + \beta^2 + \nu^2} + \frac{3 \, var_1^2}{\alpha^2 + \beta^2 + \nu^2}$$

```

> Cond:=EqInKr_reduce({var[1],var[2],var[3]},{Var});
 Cond := []

> Cond_gsolve:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1],{alpha,beta,nu
});
 Cond_gsolve := {[alpha^2 + beta^2 + nu^2 - 1], plex(beta, alpha, nu), {}}

> Sols:=convert(map(L->solve({op(convert(L[1],set))},{alpha,nu
}),Cond_gsolve),radical);
 Sols := {{alpha = -sqrt(-beta^2 - nu^2 + 1), nu = nu}, {alpha = sqrt(-beta^2 - nu^2 + 1), nu = nu}}

> Cond_4:=convert(convert(map(mem->if mem<>0 then return mem end
if,simplify(EqF_Gen_Mom_Rad(N24,2))),set),list);
 Cond_4 := [-3*beta^2*alpha^2 - 3*nu^2*beta^2 - 3*alpha^2*nu^2 + alpha^4 + nu^4 + beta^4]

> Cond_4_gsolve:=gsolve([op(Cond),op(Cond_4),alpha^2+beta^2+nu^2-1],
{alpha,beta,nu});
 Cond_4_gsolve :=
 {[[5*nu^4 - 5*nu^2 + 1 + 5*alpha^2*nu^2 - 5*alpha^2 + 5*alpha^4, alpha^2 + beta^2 + nu^2 - 1], plex(beta, alpha, nu), {}}]

> Sols:=convert(map(L->solve({op(convert(L[1],set))},{alpha,nu,beta
}),Cond_4_gsolve),radical);
 Sols := {{beta = (sqrt(50 - 50*nu^2 + 10*sqrt(-75*nu^4 + 50*nu^2 + 5)))/10, alpha = sqrt(-nu^2/2 + 1/2 - (sqrt(-75*nu^4 + 50*nu^2 + 5))/10),
nu = nu}}

> plot([1/10*(50-50*nu^2+10*(-75*nu^4+50*nu^2+5)^(1/2))^(1/2),(-1/2*nu^
2+1/2-1/10*(-75*nu^4+50*nu^2+5)^(1/2))^(1/2)],nu=-1..1);

```

On montre facilement qu'il y a une infinité de solutions pour lesquelles alpha, beta et nu sont compris entre -1 et 1 et donc de dispositifs de force 4. On retrouve en particulier le plan N24F5 en résolvant beta=0.

```

> Cond_6:=convert(convert(map(mem->if mem<>0 then return mem end
if,EqF_Gen_Mom_Rad(N24,3)),set),list);
 Cond_6 := [-alpha^6 + 4*alpha^4*beta^2 - 12*beta^2*nu^2*alpha^2 + 3*alpha^4*nu^2 + 4*nu^4*beta^2 + 3*alpha^2*nu^4 - nu^6,
-3*beta^2*alpha^2 - 3*nu^2*beta^2 - 3*alpha^2*nu^2 + alpha^4 + nu^4 + beta^4]

> Cond_6_gsolve:=gsolve([op(Cond),op(Cond_4),op(Cond_6),alpha^2+beta^2
+nu^2-1],{alpha,beta,nu});
 Cond_6_gsolve := {[
[105*nu^6 - 105*nu^4 + 21*nu^2 - 1, 5*nu^4 - 5*nu^2 + 1 + 5*alpha^2*nu^2 - 5*alpha^2 + 5*alpha^4, alpha^2 + beta^2 + nu^2 - 1],
plex(beta, alpha, nu), {}}]

> SolsBis:=map(L->((solve({op(convert(L[1],set))} union
{-1<=alpha,alpha<=1,-1<=beta,beta<=1,-1<=nu,nu<=1},{alpha,beta,nu
}))),Cond_6_gsolve):

> AllSolsBis:=allvalues(SolsBis):

> AllSols2Bis:=map(op,[AllSolsBis]):

```

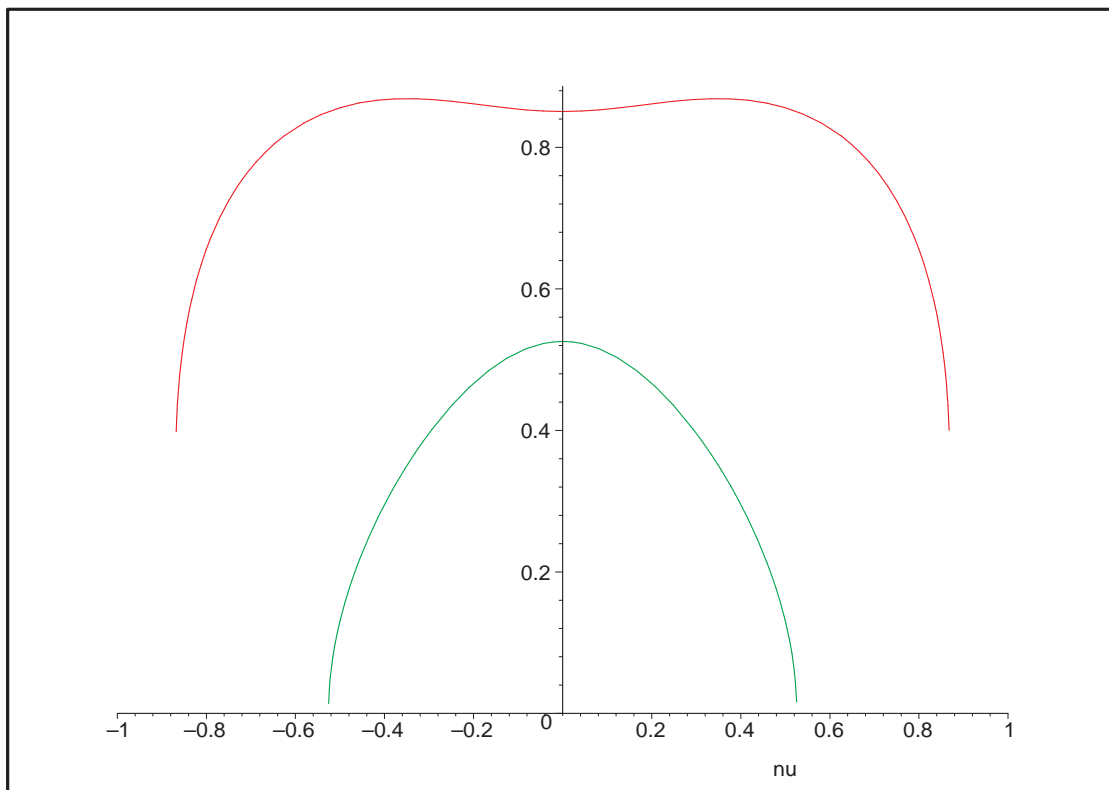


FIG. 11.75 – Graphique : Infinité de plans de force 5.

```
> evalf(map(L->map(K->map(Re,K),L),AllSols2Bis));

[{R(beta) = -0.4225186539, R(nu) = 0.2666354012, R(alpha) = 0.8662468180},
 {R(alpha) = 0.2666354020 + 0. I, R(nu) = 0.8662468188, R(beta) = 0.4225186534},
 {R(nu) = 0.8662468188, R(beta) = 0.4225186534, R(alpha) = -0.2666354020 - 0. I},
 {R(alpha) = 0.2666354014, R(nu) = -0.4225186537, R(beta) = 0.8662468183},
 {R(beta) = -0.4225186534, R(nu) = 0.8662468188, R(alpha) = -0.2666354020 - 0. I},
 {R(alpha) = 0.2666354020 + 0. I, R(beta) = -0.4225186534, R(nu) = 0.8662468188},
 {R(beta) = -0.8662468183, R(alpha) = 0.2666354014, R(nu) = -0.4225186537},
 {R(nu) = 0.2666354012, R(beta) = 0.4225186539, R(alpha) = 0.8662468180}]

> 2 dispositifs distincts, image l'un de l'autre par -I_3.

> Snub cube 1 (improved) :

> assign(AllSols2Bis[2]);

> N24:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN24)));

> UpdatePlan(N24,"N24");

 "Update complete"

> pointplot3d(Map(Re,N24),symbol=box,scaling=constrained,shading=Z);
```

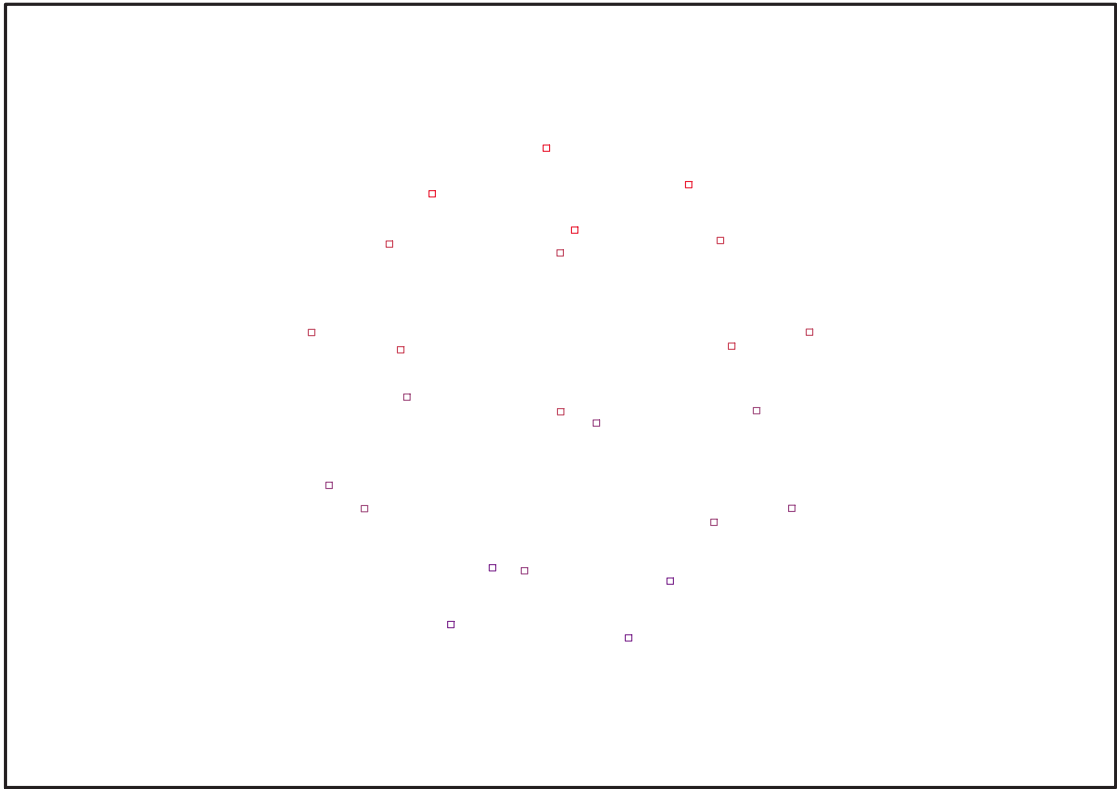


FIG. 11.76 – Graphique : Support du plan N24.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N24,listlist)))]),'ra
tional',5)));
```

```
Graph := POLYTOPE(3, 3, 24, 38)
```

```
> plotdes:=draw(Graph):
```

```
> display(plotdes,scaling=constrained,shading=Z);
```

```
> IdealN24bis:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N24));
```

```
> Gens:=Generators(IdealN24);
```

### 11.29.1. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N24_tdeg:=op(EstSet(Div,Frontiere_tdeg)): A:=poly_algebra(x,y,
z): T:=termorder(A,tdeg(x,y,z)): EstSet_N24_tdeg:=sort(EstSet_N24_tdeg,(
t1,t2)->testorder(t1,t2,T)); nops(EstSet_N24_tdeg);
```

```
EstSet_N14_tdeg := [1, z, y, x, z^2, y z, x z, y^2, x y, x^2, z^3, y z^2, x z^2, y^2 z]
```

14

```
> Problème du à l'approximation puisque x^2,y^2 et z^2 sont estimables.
```



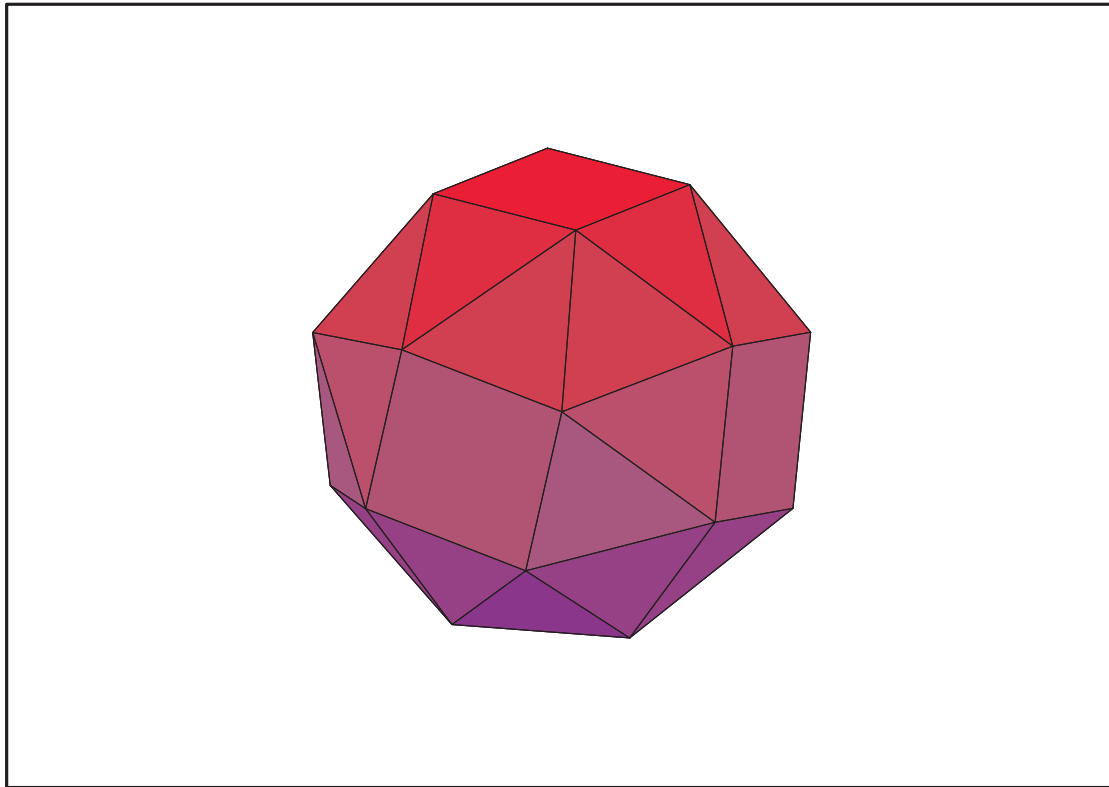


FIG. 11.77 – Graphique : Enveloppe convexe du support du plan N24.

```
> X_D_ModeleSat(N24,[x,y,z],EstSet_N24_tdeg);
 [14 x 14 Matrix Data Type : anything Storage : rectangular Order : Fortran_order]

> M_D_ModeleSat(N24,[x,y,z],EstSet_N24_tdeg);
 [14 x 14 Matrix Data Type : float[8] Storage : rectangular Order : Fortran_order]

> expand(Var_Blue_Modele_Sat(N24bis,[x,y,z],EstSet_N24bis_tdeg));
 "M_D is not full rank"
 VarVar_Blue_D_Norm_Set
```

### 11.29.2. $\text{plex}(x,y,z)$

```
> Frontiere_plex:=map(LeadingMonomial,
 {op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
 Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):

> EstSet_N24bis_plex:=[op(EstSet(Div,Frontiere_plex))]:A:=poly_algebra(
 x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N24bis_plex:=sort(EstSet_N24
 bis_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N24bis_plex);
```

Warning, computation interrupted

```
> X_D_ModeleSat(N24bis,[x,y,z],EstSet_N24bis_plex);
 [11 x 11 Matrix Data Type : anything Storage : rectangular Order : Fortran_order]

> M_D_ModeleSat(N24bis,[x,y,z],EstSet_N24bis_plex);
 [11 x 11 Matrix Data Type : anything Storage : rectangular Order : Fortran_order]
```

```
> expand(Var_Blue_Modele_Sat(N24bis,[x,y,z],EstSet_N24bis_plex));
```

$$11 - \frac{225456}{325} y^2 z^4 + \frac{1224234}{4225} z^8 + \frac{3145824}{4225} z^6 y^2 - \frac{2513808}{4225} z^6 - \frac{92928}{845} z^2 y^4 + \frac{3145824}{4225} y^4 z^4 \\ - 22 y^2 - \frac{105501}{1690} z^2 + \frac{265628}{2535} y^2 z^2 + \frac{88}{3} y^4 + \frac{3106653}{8450} z^4$$

### 11.29.3. Variance

#### Modèle de Degré 1

```
> Var:=simplify(radnormal(Var_Norm_Blue_D_Centre(N24,1,0)));
```

Warning, computation interrupted

```
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
```

$$1 + 3 \operatorname{IndetDecompInKfi}_1, \operatorname{IndetDecompInKfi}_1 = \operatorname{var}_1^2 + \operatorname{var}_2^2 + \operatorname{var}_3^2$$

```
> animate(plot3d,[Var,var[1]==-2..2,var[2]==-2..2],var[3]==-2..2):
```

#### Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N24,2,0);
```

“M\_D is not full rank”

```
> Var:=simplify(radnormal(expand(Var_Norm_Blue_D_Centre(N24,2,1)))):
```

```
> (DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3),alg_relations={t^2-13,s^2-2
})));
```

$$24 + 30 \operatorname{IndetDecompInKfi}_1^2 - 45 \operatorname{IndetDecompInKfi}_1, \\ \operatorname{IndetDecompInKfi}_1 = \operatorname{var}_1^2 + \operatorname{var}_2^2 + \operatorname{var}_3^2$$

```
> Snub cube 2 (improved) : image du précédent par sigma_x, sigma_y ousigma_z
```

```
> unassign('alpha','beta','nu');
```

```
> assign(AllSols2Bis[3]);
```

```
> N24bis:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN24))):
```

```
> UpdatePlan(N24bis,"N24bis");
```

“Update complete”

```
> pointplot3d(Map(Re,N24bis),symbol=box,scaling=constrained,shading=Z);
```

```
> Graph:=convhull(op(map(convert,[(op(convert(Map(Re,evalf(N24bis)),lis
tlist))], 'rational',5))));
```

$$\text{Graph} := \text{POLYTOPE}(3, 3, 24, 38)$$

```
> display(plotdes,scaling=constrained,shading=Z);
```

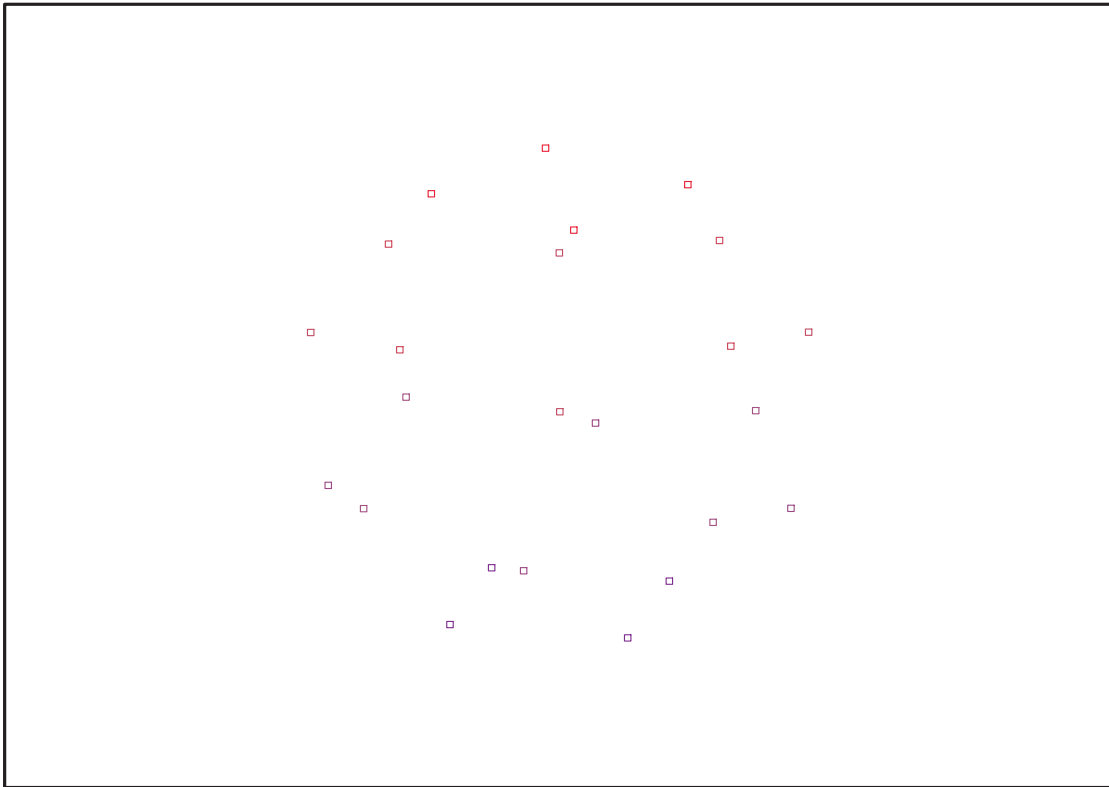


FIG. 11.78 – Graphique : Support du plan N24bis.

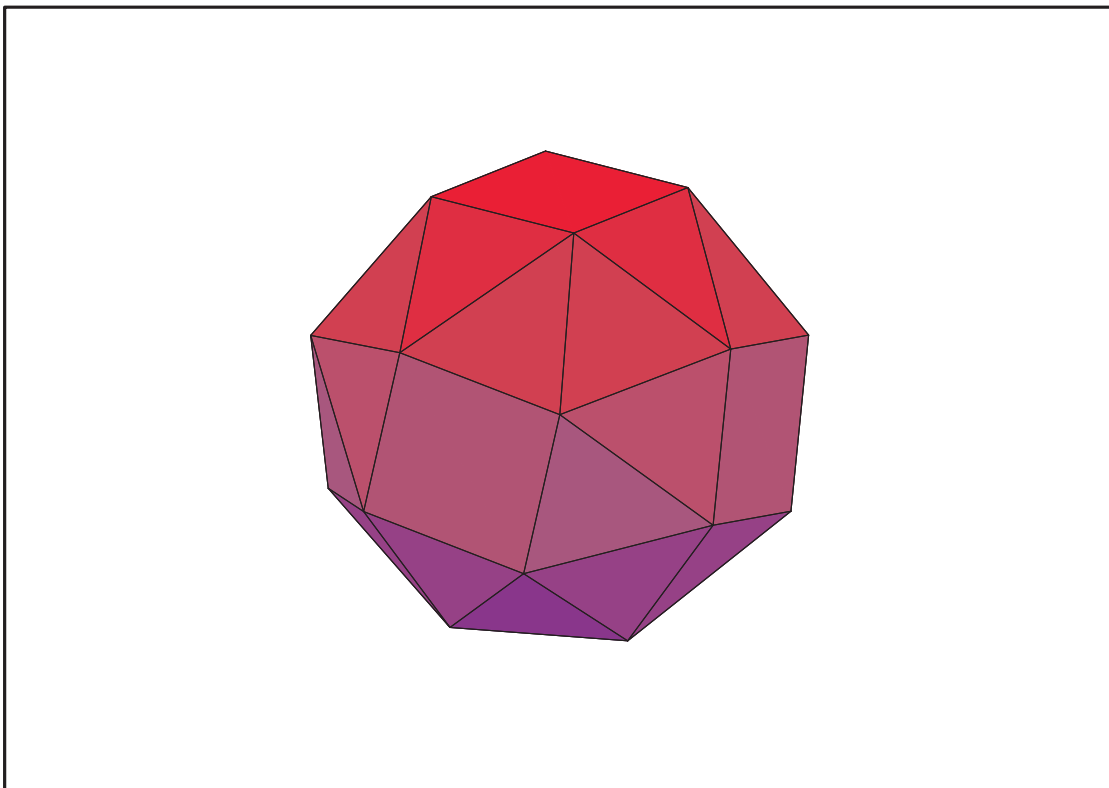


FIG. 11.79 – Graphique : Enveloppe convexe du support du plan N24bis.

```
> IdealN24bis:=subs(u[1]=x,u[2]=y,u[3]=z,DesignToIdealv2(N24bis));

Error, (in unknown) invalid input: degree received -.4842650455, which
is not valid for its 2nd argument, x

> Gens:=Generators(IdealN24bis);

> info_iop:=3;subs(x_[1]=x,x_[2]=y,x_[3]=z,IdealOfPoints(map(convert,co
nvert((N24),listlist),Vector))):

 info_iop := 3
 1

Warning, computation interrupted
```

### 11.29.4. tdeg(x,y,z)

```
> Frontiere_tdeg:=map(LeadingMonomial,GroebnerBasis(Gens,tdeg(x,y,z)),t
deg(x,y,z,t)): Div:=Diviseurs({op(Frontiere_tdeg)}): nops(Div):
EstSet_N24bis_tdeg:=op(EstSet(Div,Frontiere_tdeg)): A:=poly_algebra(x
,y,z):T:=termorder(A,tdeg(x,y,z)):EstSet_N24bis_tdeg:=sort(EstSet_N24b
is_tdeg,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N24bis_tdeg);

 EstSet_N14_tdeg := [1, z, y, x, z^2, y z, x z, y^2, x y, x^2, z^3, y z^2, x z^2, y^2 z]
 14

> Problème du à l'approximation puisque x^2,y^2 et z^2 sont estimables.

> X_D_ModeleSat(N24bis,[x,y,z],EstSet_N24bis_tdeg);
[14 x 14 Matrix Data Type: anything Storage: rectangular Order: Fortran_order]

> M_D_ModeleSat(N24bis,[x,y,z],EstSet_N24bis_tdeg);
[14 x 14 Matrix Data Type: float[8] Storage: rectangular Order: Fortran_order]

> expand(Var_Blue_Modele_Sat(N24bis,[x,y,z],EstSet_N24bis_tdeg));
 "M_D is not full rank"
 VarVar_Blue_D_Norm_Set
```

### 11.29.5. plex(x,y,z)

```
> Frontiere_plex:=map(LeadingMonomial,
{op(GroebnerBasis(Gens,plex(x,y,z)))},plex(x,y,z)):
Div:=Diviseurs({op(Frontiere_plex)}):nops(Div):
> EstSet_N24bis_plex:=op(EstSet(Div,Frontiere_plex)):A:=poly_algebra(
x,y,z):T:=termorder(A,plex(x,y,z)):EstSet_N24bis_plex:=sort(EstSet_N24
bis_plex,(t1,t2)->testorder(t1,t2,T));nops(EstSet_N24bis_plex);

Warning, computation interrupted

> X_D_ModeleSat(N24bis,[x,y,z],EstSet_N24bis_plex);
[11 x 11 Matrix Data Type: anything Storage: rectangular Order: Fortran_order]

> M_D_ModeleSat(N24bis,[x,y,z],EstSet_N24bis_plex);
[11 x 11 Matrix Data Type: anything Storage: rectangular Order: Fortran_order]
```

```
> expand(Var_Blue_Modele_Sat(N24bis,[x,y,z],EstSet_N24bis_plex));
```

$$11 - \frac{225456}{325} y^2 z^4 + \frac{1224234}{4225} z^8 + \frac{3145824}{4225} z^6 y^2 - \frac{2513808}{4225} z^6 - \frac{92928}{845} z^2 y^4 + \frac{3145824}{4225} y^4 z^4 \\ - 22 y^2 - \frac{105501}{1690} z^2 + \frac{265628}{2535} y^2 z^2 + \frac{88}{3} y^4 + \frac{3106653}{8450} z^4$$

## 11.29.6. Variance

### Modèle de Degré 1

```
> ReadPlan("N24ter");Plan;N24ter:=Plan:
```

```
[24 x 3 Matrix Data Type: anything Storage: rectangular Order: Fortran_order]
```

```
> Var:=simplify(radnormal(Var_Norm_Blue_D_Centre(N24ter,1,0)));
```

$$Var := 1 + 3 var_3^2 + 3 var_2^2 + 3 var_1^2$$

```
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
```

$$1 + 3 \text{IndetDecompInKfi}_1, \text{IndetDecompInKfi}_1 = var_1^2 + var_2^2 + var_3^2$$

```
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

### Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N24ter,2,0);
```

```
"M_D is not full rank"
```

```
> Var:=simplify(radnormal(expand(Var_Norm_Blue_D_Centre(N24ter,2,1)))):
```

```
> (DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3),alg_relations={t^2-13,s^2-2
})));
```

$$24 + 30 \text{IndetDecompInKfi}_1^2 - 45 \text{IndetDecompInKfi}_1, \\ \text{IndetDecompInKfi}_1 = var_1^2 + var_2^2 + var_3^2$$

## 11.30. N25

```
> unassign('alpha','beta','nu','delta','eta','mu','psi','phi','theta');
```

```
> GrN25:=Plus(Rned(Diedn(5),3));
```

$$GrN25 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%3 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%4 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%2 & -\frac{\sqrt{5}}{4} + \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%3 & 0 \\ \%2 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%1 & -\frac{1}{4} - \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%4 & 0 \\ \%4 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{5}}{4} & \%2 & 0 \\ \%3 & -\frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ \%1 & \frac{1}{4} + \frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix} \Bigg\}$$

$$\%1 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

$$\%2 := \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%3 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\%4 := -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}$$

```
> N25:=OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN25) union
OrbPG(Vector([delta,eta,mu]),GrN25) union
OrbPG(Vector([1,0,0]),GrN25));
```

$$N25 := \begin{bmatrix} \delta & -\eta & -\mu \\ \delta & \eta & \mu \\ \%2 + \%1\beta & \%3\alpha + (-\frac{1}{4} - \frac{\sqrt{5}}{4})\beta & \nu \\ \%4 + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\eta}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\delta}{4} + (-\frac{\sqrt{5}}{4} + \frac{1}{4})\eta & -\mu \\ \%5 - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\beta}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} + (-\frac{\sqrt{5}}{4} + \frac{1}{4})\beta & -\nu \\ \%2 + \%3\beta & \%1\alpha + (-\frac{1}{4} - \frac{\sqrt{5}}{4})\beta & \nu \\ \%2 + \%1\beta & \%1\alpha + (\frac{1}{4} + \frac{\sqrt{5}}{4})\beta & -\nu \\ \%6 + \%3\eta & \%3\delta + (\frac{1}{4} + \frac{\sqrt{5}}{4})\eta & -\mu \\ \%4 + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\eta}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\delta}{4} + (-\frac{1}{4} + \frac{\sqrt{5}}{4})\eta & \mu \\ \alpha & \beta & \nu \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%1 & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \%3 & 0 \\ \alpha & -\beta & -\nu \\ \%6 + \%3\eta & \%1\delta + (-\frac{1}{4} - \frac{\sqrt{5}}{4})\eta & \mu \\ \%6 + \%1\eta & \%1\delta + (\frac{1}{4} + \frac{\sqrt{5}}{4})\eta & -\mu \\ \%4 - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\eta}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\delta}{4} + (-\frac{1}{4} + \frac{\sqrt{5}}{4})\eta & \mu \\ \%4 - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\eta}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\delta}{4} + (-\frac{\sqrt{5}}{4} + \frac{1}{4})\eta & -\mu \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ \%5 + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\beta}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} + (-\frac{1}{4} + \frac{\sqrt{5}}{4})\beta & \nu \\ \%5 + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\beta}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} + (-\frac{\sqrt{5}}{4} + \frac{1}{4})\beta & -\nu \\ \%2 + \%3\beta & \%3\alpha + (\frac{1}{4} + \frac{\sqrt{5}}{4})\beta & -\nu \\ \%5 - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\beta}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\alpha}{4} + (-\frac{1}{4} + \frac{\sqrt{5}}{4})\beta & \nu \\ \%6 + \%1\eta & \%3\delta + (-\frac{1}{4} - \frac{\sqrt{5}}{4})\eta & \mu \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}\%1 &:= \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\ \%2 &:= \left(-\frac{1}{4} - \frac{\sqrt{5}}{4}\right)\alpha \\ \%3 &:= -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} \\ \%4 &:= \left(-\frac{1}{4} + \frac{\sqrt{5}}{4}\right)\delta \\ \%5 &:= \left(-\frac{1}{4} + \frac{\sqrt{5}}{4}\right)\alpha \\ \%6 &:= \left(-\frac{1}{4} - \frac{\sqrt{5}}{4}\right)\delta\end{aligned}$$

> UpdatePlan(N25,"N25");

“Update complete”

> Var:=Var\_Norm\_Blue\_D\_Centre(N25,1,0);

$$Var := 1 + \frac{5}{2} \frac{var_3^2}{\mu^2 + \nu^2} + \frac{10 var_2^2}{2\alpha^2 + 2\delta^2 + 2\eta^2 + 1 + 2\beta^2} + \frac{10 var_1^2}{2\alpha^2 + 2\delta^2 + 2\eta^2 + 1 + 2\beta^2}$$

> Cond:=EqInKr\_reduce({var[1],var[2],var[3]},{Var});

$$Cond := [2\delta^2 + 2\eta^2 + 2\beta^2 + 2\alpha^2 + 1 - 4\mu^2 - 4\nu^2]$$

> Cond\_gsolved:=gsolve([op(Cond),alpha^2+beta^2+nu^2-1,delta^2+eta^2+mu^2-1],{alpha,beta,nu,delta,eta,mu});

$$\begin{aligned}Cond\_gsolved &:= \\ &\{[[6\mu^2 - 5 + 6\nu^2, 6\alpha^2 + 6\beta^2 - 6\mu^2 - 1, \delta^2 + \eta^2 + \mu^2 - 1], \text{plex}(\eta, \delta, \alpha, \nu, \beta, \mu), \{\}]\}\end{aligned}$$

> Sols:=convert(map(L->solve({op(convert(L[1],set))}, {alpha,beta,nu,delta,eta,mu}),Cond\_gsolved),radical);

$$Sols := \{ \{ \eta = \eta, \alpha = \sqrt{-\beta^2 + 1 - \nu^2}, \nu = \nu, \delta = \sqrt{-\eta^2 + \frac{1}{6} + \nu^2}, \beta = \beta, \mu = \sqrt{\frac{5}{6} - \nu^2} \} \}$$

> Cond\_4:=convert(convert(map(mem->if mem<>0 then return mem end if, simplify(eval(EqF\_Gen\_Mom\_Rad(subs(sqrt(5)=radic3,subs(sqrt(2)=radic,sqrt(5+sqrt(5))=radic2,N25))),2),[radic=sqrt(2),radic3=sqrt(5),radic2=sqrt(5+sqrt(5))])),set),list);



$$\begin{aligned}
\text{Cond\_4} := & [160 \eta^3 \mu - 320 \mu \sqrt{5} \eta \nu^2 + 160 \mu \sqrt{5} \eta \delta^2 + 160 \delta^2 \eta \mu + 160 \mu \sqrt{5} \eta \alpha^2 \\
& + 160 \eta \mu \alpha^2 + 160 \mu \sqrt{5} \eta \beta^2 + 80 \mu \sqrt{5} \eta + 80 \eta \mu - 320 \eta \mu \nu^2 + 160 \eta \mu \beta^2 \\
& + 160 \mu \sqrt{5} \eta^3 - 320 \mu^3 \sqrt{5} \eta - 320 \eta \mu^3, \\
& 240 \nu^2 \beta^2 - 240 \beta^2 \mu^2 + 240 \nu^2 \alpha^2 - 240 \mu^2 \alpha^2 - 160 \nu^4 + 480 \nu^2 \mu^2 - 120 \mu^2 + 320 \mu^4, \\
& 480 \nu^2 \alpha^2 - 480 \mu^2 \alpha^2 - 240 \mu^2 + 480 \nu^2 \beta^2 - 480 \beta^2 \mu^2 - 320 \nu^4 + 960 \nu^2 \mu^2 + 640 \mu^4, \\
& 40 + 80 \delta^2 + 80 \alpha^2 - 160 \mu^2 - 160 \nu^2 + 80 \eta^2 + 80 \beta^2, 40 \delta + 80 \delta \alpha^2 + 80 \delta^3 \\
& - 160 \delta \nu^2 + 40 \delta \sqrt{5} + 80 \delta \sqrt{5} \beta^2 + 80 \delta \beta^2 + 80 \delta \sqrt{5} \eta^2 + 80 \delta \eta^2 + 80 \delta^3 \sqrt{5} \\
& - 160 \delta \sqrt{5} \mu^2 - 160 \delta \sqrt{5} \nu^2 - 160 \delta \mu^2 + 80 \delta \sqrt{5} \alpha^2, -80 \delta^2 \eta \mu + 160 \mu \sqrt{5} \eta \nu^2 \\
& - 80 \mu \sqrt{5} \eta \delta^2 - 80 \mu \sqrt{5} \eta \alpha^2 - 80 \mu \sqrt{5} \eta^3 - 80 \eta^3 \mu - 80 \mu \sqrt{5} \eta \beta^2 - 80 \eta \mu \beta^2 \\
& + 160 \eta \mu \nu^2 + 160 \mu^3 \sqrt{5} \eta - 40 \mu \sqrt{5} \eta + 160 \eta \mu^3 - 40 \eta \mu - 80 \eta \mu \alpha^2, 49920 \\
& - 675840 \mu^2 \alpha^2 - 153600 \mu^2 \delta^2 - 15360 \eta^2 \alpha^2 - 107520 \eta^2 \beta^2 - 30720 \eta^2 \delta^2 \\
& - 153600 \nu^2 \delta^2 - 337920 \mu^2 + 7680 \sqrt{5} + 92160 \eta^2 \mu^2 + 23040 \delta^2 \sqrt{5} + 122880 \nu^2 \beta^2 \\
& - 61440 \nu^2 \alpha^2 + 107520 \delta^2 \alpha^2 + 153600 \alpha^4 + 53760 \delta^2 + 107520 \alpha^2 - 184320 \nu^2 \\
& - 7680 \eta^2 + 61440 \beta^4 - 20480 \nu^4 - 76800 \eta^4 + 215040 \alpha^2 \beta^2 - 30720 \sqrt{5} \nu^2 \\
& + 61440 \beta^2 + 15360 \beta^2 \delta^2 + 46080 \delta^4 + 92160 \eta^2 \nu^2 + 1228800 \nu^2 \mu^2 + 15360 \delta^4 \sqrt{5} \\
& + 593920 \mu^4 - 30720 \sqrt{5} \mu^2 + 15360 \beta^2 \sqrt{5} + 30720 \alpha^4 \sqrt{5} + 30720 \alpha^2 \sqrt{5} \\
& - 61440 \sqrt{5} \mu^2 \alpha^2 - 30720 \delta^2 \sqrt{5} \mu^2 - 30720 \delta^2 \sqrt{5} \nu^2 - 46080 \eta^4 \sqrt{5} \\
& + 92160 \eta^2 \sqrt{5} \nu^2 + 92160 \eta^2 \sqrt{5} \mu^2 + 46080 \alpha^2 \sqrt{5} \delta^2 - 61440 \sqrt{5} \nu^2 \alpha^2 \\
& - 30720 \eta^2 \sqrt{5} \delta^2 - 46080 \eta^2 \sqrt{5} \beta^2 + 30720 \alpha^2 \sqrt{5} \beta^2 - 491520 \beta^2 \mu^2 \\
& + 15360 \beta^2 \sqrt{5} \delta^2 - 15360 \eta^2 \sqrt{5} \alpha^2 - 7680 \sqrt{5} \eta^2, -320 \delta \nu^2 + 160 \delta \beta^2 \\
& + 160 \delta \sqrt{5} \alpha^2 + 160 \delta \alpha^2 + 160 \delta \eta^2 + 160 \delta \sqrt{5} \eta^2 + 160 \delta \sqrt{5} \beta^2 - 320 \delta \sqrt{5} \mu^2 \\
& - 320 \delta \sqrt{5} \nu^2 + 160 \delta^3 \sqrt{5} + 160 \delta^3 + 80 \delta \sqrt{5} + 80 \delta - 320 \delta \mu^2, \\
& 160 \delta^2 + 160 \beta^2 + 160 \eta^2 - 320 \mu^2 - 320 \nu^2 + 80 + 160 \alpha^2, -2240 - 48640 \mu^2 \alpha^2 \\
& - 17920 \mu^2 \delta^2 + 17920 \eta^2 \beta^2 + 15360 \eta^2 \delta^2 - 17920 \nu^2 \delta^2 - 24320 \mu^2 - 1600 \sqrt{5} \\
& - 53760 \eta^2 \mu^2 - 3840 \delta^2 \sqrt{5} + 17920 \nu^2 \beta^2 + 1280 \beta^4 \sqrt{5} + 53760 \nu^2 \alpha^2 - 17920 \delta^2 \alpha^2 \\
& - 11520 \alpha^4 - 8960 \delta^2 - 14080 \alpha^2 + 1280 \nu^2 + 6400 \beta^4 + 10240 \nu^4 + 16640 \eta^4 \\
& - 5120 \alpha^2 \beta^2 + 6400 \sqrt{5} \nu^2 - 5120 \beta^2 - 1280 \delta^4 - 53760 \eta^2 \nu^2 + 204800 \nu^2 \mu^2 \\
& - 1280 \delta^4 \sqrt{5} + 112640 \mu^4 + 6400 \sqrt{5} \mu^2 - 2560 \beta^2 \sqrt{5} - 6400 \alpha^4 \sqrt{5} - 6400 \alpha^2 \sqrt{5} \\
& + 12800 \sqrt{5} \mu^2 \alpha^2 - 2560 \beta^2 \sqrt{5} \nu^2 + 2560 \delta^2 \sqrt{5} \mu^2 + 2560 \delta^2 \sqrt{5} \nu^2 + 6400 \eta^4 \sqrt{5} \\
& - 12800 \eta^2 \sqrt{5} \nu^2 - 12800 \eta^2 \sqrt{5} \mu^2 - 7680 \alpha^2 \sqrt{5} \delta^2 + 12800 \sqrt{5} \nu^2 \alpha^2 \\
& - 2560 \beta^2 \sqrt{5} \mu^2 + 5120 \eta^2 \sqrt{5} \delta^2 + 7680 \eta^2 \sqrt{5} \beta^2 - 5120 \alpha^2 \sqrt{5} \beta^2 - 84480 \beta^2 \mu^2, \\
& 7200 - 92160 \mu^2 \alpha^2 - 19200 \mu^2 \delta^2 + 1920 \eta^2 \alpha^2 - 17280 \eta^2 \beta^2 - 3840 \eta^2 \delta^2 \\
& - 19200 \nu^2 \delta^2 - 46080 \mu^2 + 1440 \sqrt{5} + 11520 \eta^2 \mu^2 + 2880 \delta^2 \sqrt{5} + 23040 \nu^2 \beta^2 \\
& - 1920 \beta^4 \sqrt{5} - 15360 \nu^2 \alpha^2 + 17280 \delta^2 \alpha^2 + 23040 \alpha^4 + 8640 \delta^2 + 17280 \alpha^2 \\
& - 26880 \nu^2 + 960 \eta^2 + 3840 \beta^4 - 2560 \nu^4 - 9600 \eta^4 + 26880 \alpha^2 \beta^2 - 5760 \sqrt{5} \nu^2 \\
& + 7680 \beta^2 - 1920 \beta^2 \delta^2 + 5760 \delta^4 + 11520 \eta^2 \nu^2 + 153600 \nu^2 \mu^2 + 74240 \mu^4 \\
& - 5760 \sqrt{5} \mu^2 + 1920 \beta^2 \sqrt{5} + 5760 \alpha^4 \sqrt{5} + 5760 \alpha^2 \sqrt{5} - 11520 \sqrt{5} \mu^2 \alpha^2 \\
& + 3840 \beta^2 \sqrt{5} \nu^2 - 3840 \eta^4 \sqrt{5} + 7680 \eta^2 \sqrt{5} \nu^2 + 7680 \eta^2 \sqrt{5} \mu^2 + 5760 \alpha^2 \sqrt{5} \delta^2 \\
& - 11520 \sqrt{5} \nu^2 \alpha^2 + 3840 \beta^2 \sqrt{5} \mu^2 - 3840 \eta^2 \sqrt{5} \delta^2 - 5760 \eta^2 \sqrt{5} \beta^2 + 3840 \alpha^2 \sqrt{5} \beta^2 \\
& - 53760 \beta^2 \mu^2 - 1920 \beta^2 \sqrt{5} \delta^2 + 1920 \eta^2 \sqrt{5} \alpha^2 + 960 \sqrt{5} \eta^2]
\end{aligned}$$

```

> Cond_gsolved:=gsolve(subs(sqrt(5)=radic,[op(Cond),op(Cond_4),alpha^2+
beta^2+nu^2-1,delta^2+eta^2+mu^2-1,radic^2-5])),
{alpha,beta,nu,delta,eta,mu,radic});

```

```

Cond_gsolved := {[[%5, radic2 - 5, -7 - 3μ2 + 6β2, %4, 2α2 - μ2 + 2, %2, η], %1,
{δ, μ, 3μ2 - 8, radic + 1}], [[%5, radic2 - 5, 18β2 - 5 - 15μ2, %4, 18α2 - 3μ2 + 2,
18δ2 + 15μ2 - 10, 18η2 + 3μ2 - 8], %1, {η, δ, μ, radic + 1}], [
[%5, radic2 - 5, 5 - 15μ2 + 6β2, %4, 2α2 + 3μ2 - 2, %7, δ], %6,
{η, μ, radic + 1, -2 + 3μ2}], [
[%5, radic2 - 5, 5 - 30μ2 + 18β2, %4, 9α2 + 6μ2 - 4, %7, δ], %6,
{η, μ, radic + 1, -2 + 3μ2, 2 - 3μ2}], [[%5, radic2 - 5, %4, %3, δ2 + η2 + μ2 - 1], %1,
{η, -12δ2 + 6β2 - 15μ2 + 5, δ, -18δ2 + 18β2 - 30μ2 + 5, -2δ2 + 4β2 - 5μ2, μ,
radic + 1}], [[%5, radic2 - 5, %4, %3, %7, δ], %6,
{η, -5μ2 + 4β2, μ, radic + 1, 5 - 30μ2 + 18β2, 5 - 15μ2 + 6β2}], [
[%5, radic2 - 5, %4, %3, 12δ2 - 6β2 + 15μ2 - 5, 12η2 + 6β2 - 3μ2 - 7], %1,
{18β2 - 5 - 15μ2, η, δ, μ, radic + 1}], [
[%5, radic2 - 5, %4, %3, 2δ2 - 4β2 + 5μ2, 2η2 + 4β2 - 3μ2 - 2], %1,
{η, δ, 5 - 18β2 + 15μ2, μ, radic + 1}], [
[%5, radic2 - 5, -3μ2 - 2 + 4β2, %4, 12α2 - 3μ2 + 4, %2, η], %1,
{δ, μ, radic + 1, -3μ2 + 8}], [
[%5, radic2 - 5, -5μ2 + 4β2, %4, 12α2 + 3μ2 - 2, %7, δ], %6,
{η, μ, radic + 1, 2 - 3μ2}], [
[%5, radic2 - 5, %4, %3, 18δ2 - 18β2 + 30μ2 - 5, 18η2 + 18β2 - 12μ2 - 13], %1,
{18β2 - 5 - 15μ2, η, δ, 5 - 18β2 + 15μ2, μ, radic + 1}], [
[%5, radic2 - 5, -13 + 18β2 - 12μ2, %4, 9α2 - 3μ2 + 5, %2, η], %1,
{δ, μ, 3μ2 - 8, radic + 1, -3μ2 + 8}], [[%5, radic2 - 5, %4, %3, %2, η], %1,
{δ, 13 - 18β2 + 12μ2, 2 - 4β2 + 3μ2, μ, radic + 1, 3μ2 + 7 - 6β2}}]
%1 := plex(η, δ, α, ν, β, radic, μ)
%2 := δ2 + μ2 - 1
%3 := 6α2 + 6β2 - 6μ2 - 1
%4 := 6μ2 - 5 + 6ν2
%5 := -60μ2 + 72μ4 + 7
%6 := plex(δ, η, α, ν, β, radic, μ)
%7 := -1 + η2 + μ2

```

```

> is_finite(subs(sqrt(5)=radic,[op(Cond),op(Cond_4),alpha^2+beta^2+nu^2
-1,delta^2+eta^2+mu^2-1,radic^2-5]),{alpha,beta,nu,delta,eta,mu,radic
});

```

false

```

> Cond_6:=map(mem->if mem<>0 then return mem end
if,convert(convert(map(mem->if mem<>0 then return mem end if,
simplify(eval(EqF_Gen_Mom_Rad(subs(sqrt(5)=radic3,subs(sqrt(2)=radic,s
qrt(5+sqrt(5))=radic2,N25))),3),[radic=sqrt(2),radic3=sqrt(5),radic2=sq
rt(5+sqrt(5))])),set),list));

```

$$\begin{aligned}
\text{Cond\_6} := & [160\eta^3\mu - 320\mu\sqrt{5}\eta\nu^2 + 160\mu\sqrt{5}\eta\delta^2 + 160\delta^2\eta\mu + 160\mu\sqrt{5}\eta\alpha^2 \\
& + 160\eta\mu\alpha^2 + 160\mu\sqrt{5}\eta\beta^2 + 80\mu\sqrt{5}\eta + 80\eta\mu - 320\eta\mu\nu^2 + 160\eta\mu\beta^2 \\
& + 160\mu\sqrt{5}\eta^3 - 320\mu^3\sqrt{5}\eta - 320\eta\mu^3, \\
& 240\nu^2\beta^2 - 240\beta^2\mu^2 + 240\nu^2\alpha^2 - 240\mu^2\alpha^2 - 160\nu^4 + 480\nu^2\mu^2 - 120\mu^2 + 320\mu^4, \\
& 480\nu^2\alpha^2 - 480\mu^2\alpha^2 - 240\mu^2 + 480\nu^2\beta^2 - 480\beta^2\mu^2 - 320\nu^4 + 960\nu^2\mu^2 + 640\mu^4, \\
& 40 + 80\delta^2 + 80\alpha^2 - 160\mu^2 - 160\nu^2 + 80\eta^2 + 80\beta^2, 40\delta + 80\delta\alpha^2 + 80\delta^3 \\
& - 160\delta\nu^2 + 40\delta\sqrt{5} + 80\delta\sqrt{5}\beta^2 + 80\delta\beta^2 + 80\delta\sqrt{5}\eta^2 + 80\delta\eta^2 + 80\delta^3\sqrt{5} \\
& - 160\delta\sqrt{5}\mu^2 - 160\delta\sqrt{5}\nu^2 - 160\delta\mu^2 + 80\delta\sqrt{5}\alpha^2, -80\delta^2\eta\mu + 160\mu\sqrt{5}\eta\nu^2 \\
& - 80\mu\sqrt{5}\eta\delta^2 - 80\mu\sqrt{5}\eta\alpha^2 - 80\mu\sqrt{5}\eta^3 - 80\eta^3\mu - 80\mu\sqrt{5}\eta\beta^2 - 80\eta\mu\beta^2 \\
& + 160\eta\mu\nu^2 + 160\mu^3\sqrt{5}\eta - 40\mu\sqrt{5}\eta + 160\eta\mu^3 - 40\eta\mu - 80\eta\mu\alpha^2, 49920 \\
& - 675840\mu^2\alpha^2 - 153600\mu^2\delta^2 - 15360\eta^2\alpha^2 - 107520\eta^2\beta^2 - 30720\eta^2\delta^2 \\
& - 153600\nu^2\delta^2 - 337920\mu^2 + 7680\sqrt{5} + 92160\eta^2\mu^2 + 23040\delta^2\sqrt{5} + 122880\nu^2\beta^2 \\
& - 61440\nu^2\alpha^2 + 107520\delta^2\alpha^2 + 153600\alpha^4 + 53760\delta^2 + 107520\alpha^2 - 184320\nu^2 \\
& - 7680\eta^2 + 61440\beta^4 - 20480\nu^4 - 76800\eta^4 + 215040\alpha^2\beta^2 - 30720\sqrt{5}\nu^2 \\
& + 61440\beta^2 + 15360\beta^2\delta^2 + 46080\delta^4 + 92160\eta^2\nu^2 + 1228800\nu^2\mu^2 + 15360\delta^4\sqrt{5} \\
& + 593920\mu^4 - 30720\sqrt{5}\mu^2 + 15360\beta^2\sqrt{5} + 30720\alpha^4\sqrt{5} + 30720\alpha^2\sqrt{5} \\
& - 61440\sqrt{5}\mu^2\alpha^2 - 30720\delta^2\sqrt{5}\mu^2 - 30720\delta^2\sqrt{5}\nu^2 - 46080\eta^4\sqrt{5} \\
& + 92160\eta^2\sqrt{5}\nu^2 + 92160\eta^2\sqrt{5}\mu^2 + 46080\alpha^2\sqrt{5}\delta^2 - 61440\sqrt{5}\nu^2\alpha^2 \\
& - 30720\eta^2\sqrt{5}\delta^2 - 46080\eta^2\sqrt{5}\beta^2 + 30720\alpha^2\sqrt{5}\beta^2 - 491520\beta^2\mu^2 \\
& + 15360\beta^2\sqrt{5}\delta^2 - 15360\eta^2\sqrt{5}\alpha^2 - 7680\sqrt{5}\eta^2, -320\delta\nu^2 + 160\delta\beta^2 \\
& + 160\delta\sqrt{5}\alpha^2 + 160\delta\alpha^2 + 160\delta\eta^2 + 160\delta\sqrt{5}\eta^2 + 160\delta\sqrt{5}\beta^2 - 320\delta\sqrt{5}\mu^2 \\
& - 320\delta\sqrt{5}\nu^2 + 160\delta^3\sqrt{5} + 160\delta^3 + 80\delta\sqrt{5} + 80\delta - 320\delta\mu^2, \\
& 160\delta^2 + 160\beta^2 + 160\eta^2 - 320\mu^2 - 320\nu^2 + 80 + 160\alpha^2, -2240 - 48640\mu^2\alpha^2 \\
& - 17920\mu^2\delta^2 + 17920\eta^2\beta^2 + 15360\eta^2\delta^2 - 17920\nu^2\delta^2 - 24320\mu^2 - 1600\sqrt{5} \\
& - 53760\eta^2\mu^2 - 3840\delta^2\sqrt{5} + 17920\nu^2\beta^2 + 1280\beta^4\sqrt{5} + 53760\nu^2\alpha^2 - 17920\delta^2\alpha^2 \\
& - 11520\alpha^4 - 8960\delta^2 - 14080\alpha^2 + 1280\nu^2 + 6400\beta^4 + 10240\nu^4 + 16640\eta^4 \\
& - 5120\alpha^2\beta^2 + 6400\sqrt{5}\nu^2 - 5120\beta^2 - 1280\delta^4 - 53760\eta^2\nu^2 + 204800\nu^2\mu^2 \\
& - 1280\delta^4\sqrt{5} + 112640\mu^4 + 6400\sqrt{5}\mu^2 - 2560\beta^2\sqrt{5} - 6400\alpha^4\sqrt{5} - 6400\alpha^2\sqrt{5} \\
& + 12800\sqrt{5}\mu^2\alpha^2 - 2560\beta^2\sqrt{5}\nu^2 + 2560\delta^2\sqrt{5}\mu^2 + 2560\delta^2\sqrt{5}\nu^2 + 6400\eta^4\sqrt{5} \\
& - 12800\eta^2\sqrt{5}\nu^2 - 12800\eta^2\sqrt{5}\mu^2 - 7680\alpha^2\sqrt{5}\delta^2 + 12800\sqrt{5}\nu^2\alpha^2 \\
& - 2560\beta^2\sqrt{5}\mu^2 + 5120\eta^2\sqrt{5}\delta^2 + 7680\eta^2\sqrt{5}\beta^2 - 5120\alpha^2\sqrt{5}\beta^2 - 84480\beta^2\mu^2, \\
& 7200 - 92160\mu^2\alpha^2 - 19200\mu^2\delta^2 + 1920\eta^2\alpha^2 - 17280\eta^2\beta^2 - 3840\eta^2\delta^2 \\
& - 19200\nu^2\delta^2 - 46080\mu^2 + 1440\sqrt{5} + 11520\eta^2\mu^2 + 2880\delta^2\sqrt{5} + 23040\nu^2\beta^2 \\
& - 1920\beta^4\sqrt{5} - 15360\nu^2\alpha^2 + 17280\delta^2\alpha^2 + 23040\alpha^4 + 8640\delta^2 + 17280\alpha^2 \\
& - 26880\nu^2 + 960\eta^2 + 3840\beta^4 - 2560\nu^4 - 9600\eta^4 + 26880\alpha^2\beta^2 - 5760\sqrt{5}\nu^2 \\
& + 7680\beta^2 - 1920\beta^2\delta^2 + 5760\delta^4 + 11520\eta^2\nu^2 + 153600\nu^2\mu^2 + 74240\mu^4 \\
& - 5760\sqrt{5}\mu^2 + 1920\beta^2\sqrt{5} + 5760\alpha^4\sqrt{5} + 5760\alpha^2\sqrt{5} - 11520\sqrt{5}\mu^2\alpha^2 \\
& + 3840\beta^2\sqrt{5}\nu^2 - 3840\eta^4\sqrt{5} + 7680\eta^2\sqrt{5}\nu^2 + 7680\eta^2\sqrt{5}\mu^2 + 5760\alpha^2\sqrt{5}\delta^2 \\
& - 11520\sqrt{5}\nu^2\alpha^2 + 3840\beta^2\sqrt{5}\mu^2 - 3840\eta^2\sqrt{5}\delta^2 - 5760\eta^2\sqrt{5}\beta^2 + 3840\alpha^2\sqrt{5}\beta^2 \\
& - 53760\beta^2\mu^2 - 1920\beta^2\sqrt{5}\delta^2 + 1920\eta^2\sqrt{5}\alpha^2 + 960\sqrt{5}\eta^2, 33280\eta\mu\delta\beta^2 \\
& - 66560\eta\mu\delta\nu^2 + 33280\eta\mu\delta\alpha^2 + 33280\delta^3\eta\mu + 3840\mu\sqrt{5}\eta\delta + 7680\mu\sqrt{5}\eta^3\delta \\
& + 16640\delta\eta\mu + 7680\mu\sqrt{5}\eta\delta^3 + 33280\eta^3\mu\delta - 66560\mu^3\eta\delta - 15360\mu^3\eta\delta\sqrt{5} \\
& + 7680\eta\mu\delta\beta^2\sqrt{5} + 7680\mu\sqrt{5}\eta\delta\alpha^2 - 15360\eta\mu\delta\nu^2\sqrt{5}, 480\mu\sqrt{5}\eta\nu^2\delta^2 \\
& + 480\eta\mu\nu^2\delta^2 + 240\mu\sqrt{5}\eta\nu^2 + 240\eta\mu\nu^2 - 480\mu^3\sqrt{5}\eta\delta^2 - 480\eta\mu^3\delta^2
\end{aligned}$$

$$\begin{aligned}
& + 480 \mu \sqrt{5} \eta^3 \nu^2 - 480 \mu^3 \sqrt{5} \eta^3 - 640 \mu \sqrt{5} \eta \nu^4 - 960 \mu^3 \sqrt{5} \eta \nu^2 + 320 \mu^5 \sqrt{5} \eta \\
& + 480 \eta^3 \mu \nu^2 - 640 \eta \mu \nu^4 - 480 \eta^3 \mu^3 - 960 \eta \mu^3 \nu^2 + 320 \eta \mu^5, -1480320 \mu^2 \alpha^2 \\
& - 817920 \mu^2 \delta^2 + 284160 \alpha^2 \nu^4 - 478080 \mu^2 + 284160 \nu^4 \beta^2 - 223232 \nu^6 \\
& - 1290240 \mu^2 \alpha^2 \beta^2 - 1635840 \mu^2 \delta^2 \alpha^2 + 4492800 \mu^2 \nu^2 \alpha^2 + 1221120 \mu^2 \nu^2 \beta^2 \\
& - 2073600 \mu^2 \nu^2 \delta^2 + 1635840 \mu^2 \eta^2 \beta^2 + 9668608 \mu^6 - 5345280 \mu^2 \eta^2 \nu^2 \\
& + 376320 \mu^2 \nu^2 \mu^2 - 172800 \beta^4 \nu^2 - 253440 \mu^2 \beta^2 \sqrt{5} \nu^2 - 460800 \mu^2 \alpha^2 \sqrt{5} \beta^2 \\
& + 1175040 \mu^2 \sqrt{5} \nu^2 \alpha^2 - 691200 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 691200 \mu^2 \eta^2 \sqrt{5} \delta^2 - 1397760 \mu^4 \\
& + 691200 \sqrt{5} \nu^2 \mu^2 + 84480 \beta^2 \sqrt{5} \nu^4 + 69120 \sqrt{5} \nu^4 \alpha^2 - 168960 \mu^4 \beta^2 \sqrt{5} \\
& + 1244160 \mu^4 \sqrt{5} \alpha^2 + 691200 \mu^2 \eta^4 \sqrt{5} - 1382400 \mu^4 \eta^2 \sqrt{5} - 1382400 \mu^2 \eta^2 \sqrt{5} \nu^2 \\
& + 691200 \mu^2 \eta^2 \sqrt{5} \beta^2 - 172800 \sqrt{5} \mu^2 + 126720 \beta^4 \sqrt{5} \mu^2 - 126720 \beta^4 \sqrt{5} \nu^2 \\
& - 639360 \sqrt{5} \mu^2 \alpha^2 - 345600 \delta^2 \sqrt{5} \mu^2 - 172800 \alpha^4 \nu^2 - 282240 \beta^2 \sqrt{5} \mu^2 \\
& - 662400 \beta^2 \mu^2 - 587520 \mu^2 \alpha^4 \sqrt{5} - 103680 \alpha^4 \sqrt{5} \nu^2 - 2073600 \mu^4 \delta^2 \\
& - 345600 \alpha^2 \nu^2 \beta^2 - 230400 \alpha^2 \sqrt{5} \beta^2 \nu^2 - 5345280 \eta^2 \mu^4 + 1635840 \eta^4 \mu^2 \\
& + 691200 \sqrt{5} \mu^4 + 172800 \mu^2 \beta^4 + 1635840 \eta^2 \mu^2 \delta^2 - 6481920 \mu^4 \beta^2 \\
& + 16875520 \mu^4 \nu^2 + 2498560 \mu^2 \nu^4 - 1463040 \mu^2 \alpha^4 - 3210240 \mu^4 \alpha^2, -648000 \mu^2 \alpha^2 \\
& - 324000 \mu^2 \delta^2 + 172800 \alpha^2 \nu^4 + 2246400 \alpha^2 \mu^2 \nu^2 \beta^2 - 731520 \alpha^4 \mu^2 \beta^2 - 123120 \mu^2 \\
& - 1036800 \beta^2 \mu^2 \nu^2 \delta^2 - 12960 \eta^2 \mu^2 - 2672640 \beta^2 \mu^2 \eta^2 \nu^2 - 86400 \alpha^4 \nu^2 \beta^2 \\
& - 1605120 \alpha^2 \mu^4 \beta^2 - 645120 \beta^4 \mu^2 \alpha^2 + 610560 \beta^4 \mu^2 \nu^2 - 172800 \beta^4 \alpha^2 \nu^2 \\
& - 1036800 \beta^2 \mu^4 \delta^2 + 1249280 \beta^2 \mu^2 \nu^4 + 8437760 \beta^2 \mu^4 \nu^2 - 2672640 \beta^2 \eta^2 \mu^4 \\
& + 142080 \nu^4 \beta^4 + 172800 \nu^4 \beta^2 - 86400 \beta^6 \nu^2 + 23040 \nu^6 - 1206720 \mu^2 \alpha^2 \beta^2 \\
& - 648000 \mu^2 \delta^2 \alpha^2 + 1105920 \mu^2 \nu^2 \alpha^2 - 53760 \mu^2 \nu^2 \beta^2 + 259200 \mu^2 \nu^2 \delta^2 \\
& + 648000 \mu^2 \eta^2 \beta^2 - 956160 \mu^6 - 984960 \mu^2 \eta^2 \nu^2 - 383040 \beta^2 \mu^2 \delta^2 + 86400 \beta^6 \mu^2 \\
& + 63360 \beta^6 \mu^2 \sqrt{5} - 63360 \beta^6 \nu^2 \sqrt{5} + 794880 \nu^2 \mu^2 - 155520 \beta^4 \nu^2 \\
& - 817920 \beta^2 \mu^2 \delta^2 \alpha^2 + 817920 \beta^2 \eta^2 \mu^2 \delta^2 + 190080 \mu^2 \beta^2 \sqrt{5} \nu^2 \\
& - 475200 \mu^2 \alpha^2 \sqrt{5} \beta^2 + 466560 \mu^2 \sqrt{5} \nu^2 \alpha^2 - 233280 \mu^2 \alpha^2 \sqrt{5} \delta^2 \\
& + 207360 \mu^2 \eta^2 \sqrt{5} \delta^2 - 25920 \eta^2 \alpha^2 \mu^2 + 915840 \mu^4 + 817920 \beta^2 \eta^4 \mu^2 \\
& + 817920 \beta^4 \mu^2 \eta^2 + 233280 \sqrt{5} \nu^2 \mu^2 + 34560 \beta^2 \sqrt{5} \nu^4 + 224640 \mu^4 \beta^2 \sqrt{5} \\
& + 466560 \mu^4 \sqrt{5} \alpha^2 + 207360 \mu^2 \eta^4 \sqrt{5} - 414720 \mu^4 \eta^2 \sqrt{5} - 414720 \mu^2 \eta^2 \sqrt{5} \nu^2 \\
& + 233280 \mu^2 \eta^2 \sqrt{5} \beta^2 - 58320 \sqrt{5} \mu^2 - 63360 \beta^4 \sqrt{5} \mu^2 - 51840 \beta^4 \sqrt{5} \nu^2 \\
& - 146880 \beta^2 \sqrt{5} \mu^2 \delta^2 - 25920 \eta^2 \sqrt{5} \alpha^2 \mu^2 + 42240 \beta^4 \sqrt{5} \nu^4 - 84480 \beta^4 \mu^4 \sqrt{5} \\
& - 233280 \sqrt{5} \mu^2 \alpha^2 - 116640 \delta^2 \sqrt{5} \mu^2 - 155520 \alpha^4 \nu^2 - 12960 \eta^2 \sqrt{5} \mu^2 \\
& - 164160 \beta^2 \sqrt{5} \mu^2 + 345600 \beta^4 \mu^2 \eta^2 \sqrt{5} + 34560 \alpha^2 \beta^2 \sqrt{5} \nu^4 + 622080 \alpha^2 \mu^4 \beta^2 \sqrt{5} \\
& - 115200 \beta^4 \alpha^2 \sqrt{5} \nu^2 - 691200 \beta^2 \mu^4 \eta^2 \sqrt{5} - 550080 \beta^2 \mu^2 - 126720 \beta^4 \mu^2 \sqrt{5} \nu^2 \\
& - 233280 \mu^2 \alpha^4 \sqrt{5} - 77760 \mu^2 \delta^4 + 259200 \mu^4 \delta^2 - 311040 \alpha^2 \nu^2 \beta^2 \\
& - 51840 \alpha^2 \sqrt{5} \beta^2 \nu^2 + 345600 \beta^2 \mu^2 \eta^4 \sqrt{5} - 230400 \beta^4 \mu^2 \alpha^2 \sqrt{5} \\
& - 293760 \alpha^4 \mu^2 \sqrt{5} \beta^2 - 51840 \alpha^4 \sqrt{5} \beta^2 \nu^2 - 984960 \eta^2 \mu^4 + 544320 \eta^4 \mu^2 \\
& + 233280 \sqrt{5} \mu^4 + 142080 \nu^4 \alpha^2 \beta^2 - 227520 \mu^2 \beta^4 + 466560 \eta^2 \mu^2 \delta^2 - 215040 \mu^4 \beta^2 \\
& - 1635840 \mu^4 \nu^2 - 472320 \mu^2 \nu^4 - 691200 \beta^2 \mu^2 \eta^2 \sqrt{5} \nu^2 + 587520 \alpha^2 \mu^2 \beta^2 \sqrt{5} \nu^2 \\
& - 570240 \mu^2 \alpha^4 - 3240960 \beta^4 \mu^4 + 4834304 \beta^2 \mu^6 - 111616 \beta^2 \nu^6 + 1831680 \mu^4 \alpha^2 \\
& + 345600 \beta^2 \mu^2 \eta^2 \sqrt{5} \delta^2 - 345600 \beta^2 \mu^2 \alpha^2 \sqrt{5} \delta^2, -266400 + 18345600 \mu^2 \alpha^2 \\
& + 8611200 \mu^2 \delta^2 - 172800 \beta^6 - 8832000 \alpha^2 \nu^4 - 172800 \alpha^4 \delta^2 + 5990400 \eta^2 \nu^4 \\
& + 1958400 \nu^2 \beta^2 \delta^2 + 230400 \eta^2 \beta^2 \nu^2 + 57600 \delta^2 \beta^4 - 1296000 \eta^2 \alpha^2 - 201600 \eta^2 \beta^2 \\
& - 1468800 \eta^2 \delta^2 + 432000 \nu^2 \delta^2 + 4219200 \mu^2 - 151200 \sqrt{5} + 1065600 \eta^2 \mu^2 \\
& - 172800 \delta^2 \sqrt{5} - 316800 \nu^2 \beta^2 - 60480 \beta^4 \sqrt{5} + 345600 \eta^4 \sqrt{5} \beta^2 + 259200 \nu^2 \alpha^2
\end{aligned}$$

$$\begin{aligned}
& + 86400 \delta^2 \alpha^2 - 388800 \alpha^4 - 7449600 \nu^4 \beta^2 + 1094400 \beta^4 \alpha^2 + 86400 \delta^2 + 129600 \alpha^2 \\
& + 43200 \nu^2 - 216000 \eta^2 + 1728000 \delta^2 \alpha^2 \beta^2 + 115200 \eta^2 \alpha^2 \beta^2 - 1728000 \eta^2 \alpha^2 \delta^2 \\
& - 230400 \delta^2 \eta^2 \beta^2 + 72000 \beta^4 + 1900800 \nu^4 - 100800 \eta^4 + 604800 \alpha^2 \beta^2 - 716800 \nu^6 \\
& - 864000 \eta^2 \delta^4 + 129600 \sqrt{5} \nu^2 + 17280000 \mu^2 \alpha^2 \beta^2 + 18086400 \mu^2 \delta^2 \alpha^2 \\
& - 39628800 \mu^2 \nu^2 \alpha^2 - 11980800 \mu^2 \nu^2 \beta^2 - 2304000 \mu^2 \nu^2 \delta^2 - 16128000 \mu^2 \eta^2 \beta^2 \\
& + 2662400 \mu^6 + 302400 \beta^2 + 432000 \beta^2 \delta^2 + 36403200 \mu^2 \eta^2 \nu^2 + 1267200 \beta^2 \mu^2 \delta^2 \\
& + 1728000 \eta^4 \beta^2 + 230400 \eta^4 \alpha^2 - 388800 \delta^4 + 720000 \eta^2 \nu^2 - 403200 \alpha^6 \\
& - 18048000 \nu^2 \mu^2 + 1728000 \alpha^2 \nu^2 \delta^2 + 2995200 \eta^2 \alpha^2 \nu^2 + 1843200 \beta^4 \nu^2 \\
& - 129600 \delta^4 \sqrt{5} - 5414400 \eta^4 \nu^2 + 8755200 \mu^2 \beta^2 \sqrt{5} \nu^2 - 4377600 \mu^2 \alpha^2 \sqrt{5} \beta^2 \\
& + 4147200 \mu^2 \sqrt{5} \nu^2 \alpha^2 - 345600 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 1036800 \mu^2 \eta^2 \sqrt{5} \delta^2 \\
& + 3686400 \eta^2 \alpha^2 \mu^2 - 16953600 \mu^4 + 2073600 \sqrt{5} \nu^2 \mu^2 + 998400 \beta^2 \sqrt{5} \nu^4 \\
& + 3456000 \sqrt{5} \nu^4 \alpha^2 + 5606400 \mu^4 \beta^2 \sqrt{5} + 691200 \mu^4 \sqrt{5} \alpha^2 + 69120 \mu^2 \eta^4 \sqrt{5} \\
& - 1382400 \mu^4 \eta^2 \sqrt{5} - 4147200 \mu^2 \eta^2 \sqrt{5} \nu^2 - 1105920 \mu^2 \eta^2 \sqrt{5} \beta^2 \\
& - 2764800 \eta^2 \sqrt{5} \nu^4 + 760320 \eta^4 \sqrt{5} \nu^2 + 138240 \eta^4 \sqrt{5} \delta^2 + 138240 \eta^4 \sqrt{5} \alpha^2 \\
& + 43200 \sqrt{5} \mu^2 - 3271680 \beta^4 \sqrt{5} \mu^2 + 645120 \beta^4 \sqrt{5} \nu^2 - 129600 \beta^2 \sqrt{5} \\
& + 1728000 \eta^2 \sqrt{5} \delta^2 \nu^2 - 138240 \eta^2 \sqrt{5} \delta^2 \beta^2 - 648000 \alpha^4 \sqrt{5} - 388800 \alpha^2 \sqrt{5} \\
& - 1728000 \beta^2 \sqrt{5} \mu^2 \delta^2 + 1555200 \eta^6 + 691200 \eta^2 \sqrt{5} \alpha^2 \mu^2 + 276480 \eta^2 \sqrt{5} \nu^2 \beta^2 \\
& + 207360 \eta^2 \sqrt{5} \alpha^2 \beta^2 + 86400 \sqrt{5} \mu^2 \alpha^2 - 1728000 \eta^2 \alpha^4 - 259200 \beta^2 \sqrt{5} \nu^2 \\
& + 259200 \delta^2 \sqrt{5} \mu^2 - 432000 \delta^2 \sqrt{5} \nu^2 + 864000 \alpha^4 \beta^2 + 4838400 \alpha^4 \nu^2 \\
& + 285120 \eta^4 \sqrt{5} - 259200 \eta^2 \sqrt{5} \nu^2 + 86400 \eta^2 \sqrt{5} \mu^2 - 432000 \alpha^2 \sqrt{5} \delta^2 \\
& + 432000 \sqrt{5} \nu^2 \alpha^2 - 345600 \eta^2 \sqrt{5} \alpha^4 - 172800 \delta^4 \sqrt{5} \eta^2 - 2448000 \beta^2 \sqrt{5} \mu^2 \\
& - 86400 \eta^2 \sqrt{5} \delta^2 + 362880 \eta^2 \sqrt{5} \beta^2 - 432000 \alpha^2 \sqrt{5} \beta^2 + 10627200 \beta^2 \mu^2 \\
& - 172800 \alpha^6 \sqrt{5} - 345600 \eta^2 \sqrt{5} \alpha^2 \delta^2 + 921600 \eta^4 \delta^2 - 34560 \beta^4 \sqrt{5} \delta^2 \\
& + 311040 \alpha^2 \sqrt{5} \beta^4 - 1382400 \alpha^4 \sqrt{5} \nu^2 - 172800 \alpha^4 \sqrt{5} \delta^2 + 2073600 \mu^2 \delta^4 \\
& - 4608000 \mu^4 \delta^2 + 2995200 \alpha^2 \nu^2 \beta^2 + 86400 \beta^2 \sqrt{5} \delta^2 - 259200 \eta^2 \sqrt{5} \alpha^2 \\
& - 1843200 \alpha^2 \sqrt{5} \beta^2 \nu^2 + 2304000 \nu^4 \delta^2 - 1728000 \alpha^2 \sqrt{5} \delta^2 \nu^2 - 1036800 \beta^2 \sqrt{5} \nu^2 \delta^2 \\
& + 311040 \eta^6 \sqrt{5} - 34560 \beta^6 \sqrt{5} + 30412800 \eta^2 \mu^4 - 19008000 \eta^4 \mu^2 - 230400 \delta^6 \\
& + 633600 \sqrt{5} \nu^4 + 979200 \sqrt{5} \mu^4 + 345600 \alpha^2 \sqrt{5} \delta^2 \beta^2 + 3225600 \mu^2 \beta^4 \\
& - 43200 \sqrt{5} \eta^2 + 172800 \alpha^4 \sqrt{5} \beta^2 - 13017600 \eta^2 \mu^2 \delta^2 - 5452800 \mu^4 \beta^2 \\
& - 4710400 \mu^4 \nu^2 - 7475200 \mu^2 \nu^4 + 17740800 \mu^2 \alpha^4 - 35404800 \mu^4 \alpha^2 \\
& - 1497600 \delta^2 \eta^2 \nu^2, 3200 \eta \mu \delta \beta^2 - 6400 \eta \mu \delta \nu^2 + 3200 \eta \mu \delta \alpha^2 + 3200 \delta^3 \eta \mu \\
& + 960 \mu \sqrt{5} \eta \delta + 1920 \mu \sqrt{5} \eta^3 \delta + 1600 \delta \eta \mu + 1920 \mu \sqrt{5} \eta \delta^3 + 3200 \eta^3 \mu \delta \\
& - 6400 \mu^3 \eta \delta - 3840 \mu^3 \eta \delta \sqrt{5} + 1920 \eta \mu \delta \beta^2 \sqrt{5} + 1920 \mu \sqrt{5} \eta \delta \alpha^2 \\
& - 3840 \eta \mu \delta \nu^2 \sqrt{5}, 60 \beta^3 \sqrt{5} \nu^3 + 60 \beta^3 \nu^3 - 60 \sqrt{5} \beta^3 \nu \mu^2 - 60 \mu^2 \beta^3 \nu \\
& + 60 \sqrt{5} \alpha^2 \nu^3 \beta - 60 \sqrt{5} \beta \nu \alpha^2 \mu^2 + 60 \beta \nu^3 \alpha^2 - 60 \mu^2 \alpha^2 \beta \nu - 40 \sqrt{5} \beta \nu^5 \\
& + 120 \sqrt{5} \beta \nu^3 \mu^2 - 30 \sqrt{5} \beta \nu \mu^2 + 80 \sqrt{5} \beta \nu \mu^4 - 40 \beta \nu^5 + 120 \beta \nu^3 \mu^2 - 30 \beta \nu \mu^2 \\
& + 80 \beta \nu \mu^4, -11520 \eta^3 \mu + 34560 \delta^2 \eta \mu + 38400 \alpha \mu \eta^3 \delta + 38400 \alpha \mu \eta \delta^3 \\
& - 76800 \alpha \mu^3 \eta \delta + 15360 \mu \sqrt{5} \eta \nu^2 - 3840 \mu \sqrt{5} \eta \beta^2 - 3840 \mu \sqrt{5} \eta \delta^2 \\
& - 19200 \mu \sqrt{5} \eta \alpha^2 + 19200 \alpha \delta \eta \mu + 153600 \mu^3 \sqrt{5} \eta \nu^2 - 51200 \mu \sqrt{5} \eta \nu^4 \\
& - 30720 \eta \mu \nu^2 - 11520 \eta \mu \beta^2 + 38400 \alpha^3 \mu \eta \delta + 102400 \mu^5 \sqrt{5} \eta \\
& + 184320 \eta \mu \nu^2 \beta^2 - 76800 \eta \mu \nu^2 \delta^2 + 107520 \eta \mu \delta^2 \alpha^2 - 61440 \eta \mu \nu^2 \alpha^2 \\
& - 3840 \mu \sqrt{5} \eta^3 - 23040 \mu^3 \sqrt{5} \eta - 3840 \mu \sqrt{5} \eta - 30720 \mu^3 \sqrt{5} \eta \alpha^2 \\
& - 92160 \mu^3 \sqrt{5} \eta \beta^2 + 15360 \mu \sqrt{5} \eta^3 \delta^2 - 15360 \mu \sqrt{5} \eta^3 \alpha^2 + 15360 \mu \sqrt{5} \eta^3 \beta^2 \\
& - 76800 \eta \mu^3 \delta^2 + 15360 \eta^3 \mu \alpha^2 + 102400 \eta \mu^5 - 15360 \mu^3 \sqrt{5} \eta^3 + 7680 \mu \sqrt{5} \eta^5
\end{aligned}$$

$$\begin{aligned}
& -15360\beta\nu\alpha^4 + 23040\eta\mu\delta^4 - 15360\mu\sqrt{5}\eta^3\nu^2 + 30720\eta\mu^3\beta^2 + 153600\eta\mu^3\nu^2 \\
& + 107520\eta^3\mu\nu^2 - 51200\eta\mu\nu^4 - 53760\eta\mu\beta^4 + 69120\eta\mu\alpha^4 - 107520\eta^3\mu\beta^2 \\
& - 215040\eta\mu^3\alpha^2 - 7680\sqrt{5}\alpha\beta^2\delta\eta\mu + 15360\alpha\mu\eta\sqrt{5}\delta\nu^2 - 69120\eta\mu^3 \\
& + 7680\mu\sqrt{5}\eta\delta^4 - 23040\mu\sqrt{5}\eta\alpha^4 + 7680\mu\sqrt{5}\eta\beta^4 + 7680\eta\mu \\
& - 76800\delta\eta\mu\nu^2\alpha + 107520\eta^3\mu^3 - 3072\beta^5\nu - 56832\eta^5\mu - 15360\mu^3\sqrt{5}\eta\delta^2 \\
& + 49920\eta\mu\alpha^2 - 3840\sqrt{5}\alpha\delta\eta\mu + 15360\beta^2\eta\mu\alpha^2 - 15360\beta^2\sqrt{5}\eta\mu\alpha^2 \\
& - 15360\beta^2\delta^2\eta\mu + 15360\beta^2\sqrt{5}\delta^2\eta\mu + 122880\mu\sqrt{5}\eta\nu^2\alpha^2 - 15360\mu\sqrt{5}\eta\delta^2\alpha^2 \\
& + 61440\mu\sqrt{5}\eta\nu^2\beta^2 - 7680\alpha^3\mu\eta\sqrt{5}\delta - 7680\alpha\mu\eta\sqrt{5}\delta^3 + 15360\eta^3\mu\delta^2 \\
& + 30720\beta^3\nu\alpha^2 - 7680\sqrt{5}\alpha\delta\eta^3\mu - 15360\mu\sqrt{5}\eta\nu^2\delta^2 + 38400\alpha\mu\eta\delta\beta^2 \\
& + 15360\sqrt{5}\alpha\eta\mu^3\delta, 3072 - 399360\beta^2\nu^2\alpha - 56320\delta - 215040\nu^2\alpha + 57600\alpha \\
& - 117760\delta\eta^2\alpha^2 - 215040\delta\eta^2\nu^2 + 215040\delta\eta^2\beta^2 + 15360\eta^2\alpha - 225280\delta\alpha^2 \\
& + 491520\nu^4\alpha + 107520\delta\beta^4 - 117760\delta\alpha^2\beta^2 + 138240\alpha^3 - 151040\delta^3 \\
& + 15360\alpha\beta^2 - 30720\beta^4\alpha + 225280\delta\nu^2 - 302080\delta^3\alpha^2 - 225280\delta\alpha^4 \\
& + 153600\delta^3\nu^2 + 30720\delta^3\beta^2 + 7680\delta\sqrt{5} + 450560\alpha^2\delta\nu^2 - 58880\delta\beta^2 \\
& + 450560\alpha^2\delta\mu^2 - 215040\beta^2\delta\mu^2 - 30720\delta\sqrt{5}\eta^2\beta^2 + 30720\delta\sqrt{5}\eta^2\nu^2 \\
& + 30720\delta\sqrt{5}\eta^2\mu^2 - 184320\sqrt{5}\alpha\mu^2\delta^2 - 307200\sqrt{5}\alpha\nu^2\mu^2 + 276480\sqrt{5}\alpha\beta^2\mu^2 \\
& - 153600\sqrt{5}\alpha\eta^2\beta^2 + 184320\sqrt{5}\alpha\eta^2\mu^2 + 184320\sqrt{5}\alpha\eta^2\nu^2 - 184320\sqrt{5}\alpha\nu^2\delta^2 \\
& - 215040\delta\beta^2\nu^2 - 30720\delta^2\alpha\beta^2 + 107520\alpha\delta^2 - 15360\delta\sqrt{5}\eta^4 - 30720\delta^3\sqrt{5}\eta^2 \\
& - 122880\sqrt{5}\alpha\eta^4 + 491520\sqrt{5}\alpha\nu^4 + 61440\sqrt{5}\alpha\beta^4 - 430080\sqrt{5}\alpha\mu^4 \\
& + 30720\delta^3\sqrt{5}\mu^2 + 30720\delta^3\sqrt{5}\nu^2 + 7680\delta^3\sqrt{5} - 58880\delta\eta^2 + 7680\delta\sqrt{5}\eta^2 \\
& + 7680\delta\sqrt{5}\beta^2 - 30720\delta\sqrt{5}\mu^2 - 30720\delta\sqrt{5}\nu^2 + 30720\alpha^3\eta^2 - 890880\sqrt{5}\alpha^3\nu^2 \\
& + 153600\sqrt{5}\alpha^3\delta^2 + 30720\sqrt{5}\alpha^3\mu^2 - 215040\sqrt{5}\alpha\nu^2 + 15360\sqrt{5}\alpha\mu^2 \\
& + 76800\sqrt{5}\alpha\delta^2 + 15360\delta^3\sqrt{5}\alpha^2 + 76800\sqrt{5}\alpha\beta^2 + 245760\sqrt{5}\alpha^3\beta^2 \\
& - 30720\delta^3\sqrt{5}\beta^2 + 61440\sqrt{5}\alpha\delta^4 + 61440\beta^2\alpha^3 + 138240\eta^4\delta + 190464\alpha^5 \\
& + 225280\delta\mu^2 - 15360\delta\sqrt{5}\beta^4 - 70656\delta^5 + 30720\delta\sqrt{5}\alpha^4 - 645120\sqrt{5}\alpha\beta^2\nu^2 \\
& + 57600\sqrt{5}\alpha + 15360\delta\sqrt{5}\eta^2\alpha^2 - 61440\sqrt{5}\alpha\delta^2\eta^2 + 30720\delta\sqrt{5}\beta^2\mu^2 \\
& + 30720\delta\sqrt{5}\beta^2\nu^2 - 61440\delta\sqrt{5}\alpha^2\mu^2 - 61440\delta\sqrt{5}\alpha^2\nu^2 - 30720\sqrt{5}\alpha^3\eta^2 \\
& - 15360\sqrt{5}\delta^5 + 15360\delta\sqrt{5}\alpha^2\beta^2 + 30720\beta^2\sqrt{5}\alpha\delta^2 + 122880\alpha\delta^4 - 890880\nu^2\alpha^3 \\
& + 153600\mu^2\delta^3 - 15360\sqrt{5}\alpha\eta^2 + 184320\sqrt{5}\alpha^5 + 138240\sqrt{5}\alpha^3 - 430080\alpha\mu^4 \\
& - 30720\eta^2\delta^3 - 61440\alpha\eta^4 + 215040\delta^2\alpha^3 + 15360\alpha\mu^2 + 30720\delta\sqrt{5}\alpha^2 \\
& - 307200\alpha\nu^2\mu^2 + 61440\alpha\eta^2\nu^2 + 61440\alpha\eta^2\mu^2 - 215040\alpha\eta^2\beta^2 \\
& + 522240\alpha\beta^2\mu^2 - 215040\eta^2\mu^2\delta - 307200\alpha\nu^2\delta^2 + 61440\alpha\eta^2\delta^2 \\
& - 307200\alpha\mu^2\delta^2 + 30720\mu^2\alpha^3, 101376000\mu^2\alpha^2 + 38937600\mu^2\delta^2 - 1105920\beta^6 \\
& - 81408000\alpha^2\nu^4 - 7603200\alpha^4\delta^2 + 66355200\eta^2\nu^4 + 1814400 + 24883200\nu^2\beta^2\delta^2 \\
& - 6359040\eta^2\beta^2\nu^2 - 2488320\delta^2\beta^4 - 7257600\eta^2\alpha^2 - 1175040\eta^2\beta^2 \\
& - 6220800\eta^2\delta^2 + 6912000\nu^2\delta^2 + 27417600\mu^2 - 144000\sqrt{5} + 6912000\eta^2\mu^2 \\
& + 172800\delta^2\sqrt{5} + 1382400\nu^2\beta^2 + 921600\beta^4\sqrt{5} + 2073600\eta^4\sqrt{5}\beta^2 \\
& + 9676800\nu^2\alpha^2 - 1382400\delta^2\alpha^2 - 4147200\alpha^4 - 70348800\nu^4\beta^2 + 12026880\beta^4\alpha^2 \\
& + 1209600\delta^2 + 518400\alpha^2 + 1036800\nu^2 - 1036800\eta^2 + 10368000\delta^2\alpha^2\beta^2 \\
& + 10091520\eta^2\alpha^2\beta^2 - 10368000\eta^2\alpha^2\delta^2 - 5114880\delta^2\eta^2\beta^2 + 1866240\beta^4 \\
& - 460800\nu^4 + 483840\eta^4 + 7948800\alpha^2\beta^2 - 532480\nu^6 - 3456000\eta^2\delta^4 \\
& - 2073600\sqrt{5}\nu^2 + 109670400\mu^2\alpha^2\beta^2 + 91699200\mu^2\delta^2\alpha^2 - 257126400\mu^2\nu^2\alpha^2 \\
& - 117964800\mu^2\nu^2\beta^2 - 13824000\mu^2\nu^2\delta^2 - 70410240\mu^2\eta^2\beta^2 - 22036480\mu^6
\end{aligned}$$

$$\begin{aligned}
& + 2246400 \beta^2 + 2073600 \beta^2 \delta^2 + 202752000 \mu^2 \eta^2 \nu^2 + 16588800 \beta^2 \mu^2 \delta^2 \\
& + 10368000 \eta^4 \beta^2 + 5114880 \eta^4 \alpha^2 + 1036800 \delta^4 + 2764800 \eta^2 \nu^2 - 6220800 \alpha^6 \\
& - 108134400 \nu^2 \mu^2 + 27648000 \alpha^2 \nu^2 \delta^2 + 6912000 \eta^2 \alpha^2 \nu^2 + 10506240 \beta^4 \nu^2 \\
& - 42024960 \eta^4 \nu^2 + 42393600 \mu^2 \beta^2 \sqrt{5} \nu^2 - 15667200 \mu^2 \alpha^2 \sqrt{5} \beta^2 \\
& + 10137600 \mu^2 \sqrt{5} \nu^2 \alpha^2 + 8755200 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 1382400 \mu^2 \eta^2 \sqrt{5} \delta^2 \\
& + 15206400 \eta^2 \alpha^2 \mu^2 - 90624000 \mu^4 + 8601600 \sqrt{5} \nu^2 \mu^2 - 7065600 \beta^2 \sqrt{5} \nu^4 \\
& + 16896000 \sqrt{5} \nu^4 \alpha^2 + 32870400 \mu^4 \beta^2 \sqrt{5} - 15974400 \mu^4 \sqrt{5} \alpha^2 - 6451200 \mu^2 \eta^4 \sqrt{5} \\
& + 1843200 \mu^4 \eta^2 \sqrt{5} - 11059200 \mu^2 \eta^2 \sqrt{5} \nu^2 - 16588800 \mu^2 \eta^2 \sqrt{5} \beta^2 \\
& - 12902400 \eta^2 \sqrt{5} \nu^4 + 3686400 \eta^4 \sqrt{5} \nu^2 + 1843200 \eta^4 \sqrt{5} \delta^2 - 230400 \eta^4 \sqrt{5} \alpha^2 \\
& - 576000 \sqrt{5} \mu^2 - 19814400 \beta^4 \sqrt{5} \mu^2 + 6451200 \beta^4 \sqrt{5} \nu^2 + 518400 \beta^2 \sqrt{5} \\
& + 460800 \eta^2 \sqrt{5} \alpha^2 \nu^2 + 7372800 \eta^2 \sqrt{5} \delta^2 \nu^2 + 230400 \eta^2 \sqrt{5} \delta^2 \beta^2 + 1036800 \alpha^4 \sqrt{5} \\
& + 518400 \alpha^2 \sqrt{5} - 11980800 \beta^2 \sqrt{5} \mu^2 \delta^2 + 9262080 \eta^6 + 8755200 \eta^2 \sqrt{5} \alpha^2 \mu^2 \\
& + 5990400 \eta^2 \sqrt{5} \nu^2 \beta^2 - 1152000 \eta^2 \sqrt{5} \alpha^2 \beta^2 - 9420800 \sqrt{5} \mu^4 \nu^2 \\
& - 14950400 \sqrt{5} \mu^2 \nu^4 - 9216000 \mu^4 \delta^2 \sqrt{5} + 3916800 \sqrt{5} \mu^2 \alpha^2 - 10368000 \eta^2 \alpha^4 \\
& + 230400 \beta^2 \sqrt{5} \nu^2 + 7833600 \delta^2 \sqrt{5} \mu^2 - 3456000 \delta^2 \sqrt{5} \nu^2 + 6912000 \alpha^4 \beta^2 \\
& + 31795200 \alpha^4 \nu^2 + 230400 \eta^4 \sqrt{5} + 2995200 \eta^2 \sqrt{5} \nu^2 + 7142400 \eta^2 \sqrt{5} \mu^2 \\
& + 691200 \alpha^2 \sqrt{5} \delta^2 - 4838400 \sqrt{5} \nu^2 \alpha^2 - 2073600 \eta^2 \sqrt{5} \alpha^4 - 691200 \delta^4 \sqrt{5} \eta^2 \\
& - 9216000 \beta^2 \sqrt{5} \mu^2 - 691200 \eta^2 \sqrt{5} \delta^2 - 576000 \eta^2 \sqrt{5} \beta^2 + 2419200 \alpha^2 \sqrt{5} \beta^2 \\
& + 61056000 \beta^2 \mu^2 + 230400 \alpha^6 \sqrt{5} - 4608000 \mu^2 \delta^2 \sqrt{5} \nu^2 - 2073600 \eta^2 \sqrt{5} \alpha^2 \delta^2 \\
& + 4423680 \eta^4 \delta^2 + 5990400 \mu^2 \alpha^4 \sqrt{5} + 460800 \beta^4 \sqrt{5} \delta^2 + 1152000 \alpha^2 \sqrt{5} \beta^4 \\
& - 9676800 \alpha^4 \sqrt{5} \nu^2 + 691200 \alpha^4 \sqrt{5} \delta^2 - 4147200 \mu^2 \delta^4 - 27648000 \mu^4 \delta^2 \\
& + 1382400 \alpha^2 \nu^2 \beta^2 + 4147200 \mu^2 \delta^4 \sqrt{5} + 4608000 \nu^4 \delta^2 \sqrt{5} + 691200 \beta^2 \sqrt{5} \delta^2 \\
& - 1728000 \eta^2 \sqrt{5} \alpha^2 - 5068800 \alpha^2 \sqrt{5} \beta^2 \nu^2 + 13824000 \nu^4 \delta^2 - 13824000 \alpha^2 \sqrt{5} \delta^2 \nu^2 \\
& - 3686400 \beta^2 \sqrt{5} \nu^2 \delta^2 + 2073600 \eta^6 \sqrt{5} + 136396800 \eta^2 \mu^4 - 101928960 \eta^4 \mu^2 \\
& - 460800 \sqrt{5} \delta^6 + 5324800 \sqrt{5} \mu^6 - 1433600 \sqrt{5} \nu^6 + 1382400 \delta^6 + 4608000 \sqrt{5} \nu^4 \\
& - 5683200 \sqrt{5} \mu^4 + 2073600 \alpha^2 \sqrt{5} \delta^2 \beta^2 + 36771840 \mu^2 \beta^4 - 345600 \sqrt{5} \eta^2 \\
& + 1382400 \alpha^4 \sqrt{5} \beta^2 - 75110400 \eta^2 \mu^2 \delta^2 - 27340800 \mu^4 \beta^2 - 54476800 \mu^4 \nu^2 \\
& - 63692800 \mu^2 \nu^4 + 113817600 \mu^2 \alpha^4 - 166502400 \mu^4 \alpha^2 - 11059200 \delta^2 \eta^2 \nu^2, \\
& - 53680320 \mu^2 \alpha^2 - 3576960 \mu^2 \delta^2 + 5702400 \alpha^2 \nu^4 - 20090880 \nu^6 \alpha^2 \\
& - 186624000 \alpha^2 \mu^2 \nu^2 \delta^2 + 154828800 \alpha^2 \mu^2 \nu^2 \beta^2 - 481075200 \alpha^2 \mu^2 \eta^2 \nu^2 \\
& - 130752000 \alpha^4 \mu^2 \beta^2 + 404352000 \alpha^4 \mu^2 \nu^2 - 8514720 \mu^2 - 20736000 \beta^2 \mu^2 \nu^2 \delta^2 \\
& - 186624000 \alpha^2 \mu^4 \delta^2 - 259200 \eta^2 \mu^2 - 53452800 \beta^2 \mu^2 \eta^2 \nu^2 - 481075200 \alpha^2 \eta^2 \mu^4 \\
& - 32832000 \alpha^4 \nu^2 \beta^2 + 224870400 \alpha^2 \mu^2 \nu^4 + 1518796800 \alpha^2 \mu^4 \nu^2 \\
& - 615475200 \alpha^2 \mu^4 \beta^2 + 2649600 \beta^4 \mu^2 \alpha^2 + 12211200 \beta^4 \mu^2 \nu^2 - 19008000 \beta^4 \alpha^2 \nu^2 \\
& - 20736000 \beta^2 \mu^4 \delta^2 + 24985600 \beta^2 \mu^2 \nu^4 + 168755200 \beta^2 \mu^4 \nu^2 - 53452800 \beta^2 \eta^2 \mu^4 \\
& + 2841600 \nu^4 \beta^4 + 5702400 \nu^4 \beta^2 - 1728000 \beta^6 \nu^2 - 4451328 \nu^6 \\
& - 100961280 \mu^2 \alpha^2 \beta^2 - 80766720 \mu^2 \delta^2 \alpha^2 + 157386240 \mu^2 \nu^2 \alpha^2 \\
& + 111936000 \mu^2 \nu^2 \beta^2 - 8812800 \mu^2 \nu^2 \delta^2 + 7153920 \mu^2 \eta^2 \beta^2 + 152497152 \mu^6 \\
& - 22083840 \mu^2 \eta^2 \nu^2 - 7660800 \beta^2 \mu^2 \delta^2 + 1728000 \beta^6 \mu^2 + 1267200 \beta^6 \mu^2 \sqrt{5} \\
& - 1267200 \beta^6 \nu^2 \sqrt{5} + 12908160 \nu^2 \mu^2 - 147225600 \alpha^4 \mu^2 \delta^2 + 147225600 \alpha^2 \eta^4 \mu^2 \\
& - 3265920 \beta^4 \nu^2 - 16358400 \beta^2 \mu^2 \delta^2 \alpha^2 + 16358400 \beta^2 \eta^2 \mu^2 \delta^2 \\
& + 5253120 \mu^2 \beta^2 \sqrt{5} \nu^2 - 33454080 \mu^2 \alpha^2 \sqrt{5} \beta^2 + 67184640 \mu^2 \sqrt{5} \nu^2 \alpha^2 \\
& - 33592320 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 6635520 \mu^2 \eta^2 \sqrt{5} \delta^2 - 518400 \eta^2 \alpha^2 \mu^2 - 35786880 \mu^4 \\
& + 16358400 \beta^2 \eta^4 \mu^2 + 16358400 \beta^4 \mu^2 \eta^2 + 4821120 \sqrt{5} \nu^2 \mu^2 + 1762560 \beta^2 \sqrt{5} \nu^4
\end{aligned}$$

$$\begin{aligned}
& \mu^2 + 1762560 \beta^2 \sqrt{5} \nu^4 \\
& + 1555200 \sqrt{5} \nu^4 \alpha^2 + 7015680 \mu^4 \beta^2 \sqrt{5} + 68739840 \mu^4 \sqrt{5} \alpha^2 + 4302720 \mu^2 \eta^4 \sqrt{5} \\
& - 8605440 \mu^4 \eta^2 \sqrt{5} - 8605440 \mu^2 \eta^2 \sqrt{5} \nu^2 + 2488320 \mu^2 \eta^2 \sqrt{5} \beta^2 - 1205280 \sqrt{5} \mu^2 \\
& - 1992960 \beta^4 \sqrt{5} \mu^2 - 2643840 \beta^4 \sqrt{5} \nu^2 - 2937600 \beta^2 \sqrt{5} \mu^2 \delta^2 - 518400 \eta^2 \sqrt{5} \alpha^2 \mu^2 \\
& - 52876800 \alpha^6 \mu^2 \sqrt{5} + 111974400 \alpha^4 \mu^4 \sqrt{5} + 844800 \beta^4 \sqrt{5} \nu^4 - 1689600 \beta^4 \mu^4 \sqrt{5} \\
& - 4665600 \mu^4 \delta^2 \sqrt{5} + 6220800 \alpha^4 \sqrt{5} \nu^4 + 147225600 \alpha^2 \mu^2 \eta^2 \beta^2 \\
& - 19206720 \sqrt{5} \mu^2 \alpha^2 + 147225600 \alpha^2 \eta^2 \mu^2 \delta^2 - 1244160 \delta^2 \sqrt{5} \mu^2 - 3265920 \alpha^4 \nu^2 \\
& - 259200 \eta^2 \sqrt{5} \mu^2 - 3723840 \beta^2 \sqrt{5} \mu^2 + 6912000 \beta^4 \mu^2 \eta^2 \sqrt{5} \\
& + 8294400 \alpha^2 \beta^2 \sqrt{5} \nu^4 - 2764800 \alpha^2 \mu^4 \beta^2 \sqrt{5} - 124416000 \alpha^2 \mu^4 \eta^2 \sqrt{5} \\
& - 62208000 \alpha^4 \mu^2 \sqrt{5} \delta^2 + 105753600 \alpha^4 \mu^2 \sqrt{5} \nu^2 - 13708800 \beta^4 \alpha^2 \sqrt{5} \nu^2 \\
& - 13824000 \beta^2 \mu^4 \eta^2 \sqrt{5} - 11597760 \beta^2 \mu^2 - 2534400 \beta^4 \mu^2 \sqrt{5} \nu^2 \\
& - 4665600 \mu^2 \delta^2 \sqrt{5} \nu^2 - 60030720 \mu^2 \alpha^4 \sqrt{5} - 15552000 \alpha^6 \nu^2 - 2332800 \alpha^4 \sqrt{5} \nu^2 \\
& + 777600 \mu^2 \delta^4 - 8812800 \mu^4 \delta^2 - 6531840 \alpha^2 \nu^2 \beta^2 + 2332800 \mu^2 \delta^4 \sqrt{5} \\
& + 25574400 \alpha^4 \nu^4 - 4976640 \alpha^2 \sqrt{5} \beta^2 \nu^2 - 9331200 \sqrt{5} \alpha^6 \nu^2 + 6912000 \beta^2 \mu^2 \eta^4 \sqrt{5} \\
& + 6796800 \beta^4 \mu^2 \alpha^2 \sqrt{5} - 47347200 \alpha^4 \mu^2 \sqrt{5} \beta^2 - 21772800 \alpha^4 \sqrt{5} \beta^2 \nu^2 \\
& + 62208000 \alpha^2 \mu^2 \eta^4 \sqrt{5} - 22083840 \eta^2 \mu^4 + 7413120 \eta^4 \mu^2 + 4821120 \sqrt{5} \mu^4 \\
& + 28416000 \nu^4 \alpha^2 \beta^2 - 16058880 \mu^2 \beta^4 + 8190720 \eta^2 \mu^2 \delta^2 - 110227200 \mu^4 \beta^2 \\
& + 199618560 \mu^4 \nu^2 - 42946560 \mu^2 \nu^4 - 13824000 \beta^2 \mu^2 \eta^2 \sqrt{5} \nu^2 \\
& - 11059200 \alpha^2 \mu^2 \beta^2 \sqrt{5} \nu^2 + 62208000 \alpha^2 \mu^2 \eta^2 \sqrt{5} \beta^2 - 150336000 \mu^2 \alpha^4 \\
& - 124416000 \alpha^2 \mu^2 \eta^2 \sqrt{5} \nu^2 - 64819200 \beta^4 \mu^4 + 96686080 \beta^2 \mu^6 - 2232320 \beta^2 \nu^6 \\
& - 206703360 \mu^4 \alpha^2 - 288921600 \alpha^4 \mu^4 - 131673600 \alpha^6 \mu^2 + 870174720 \alpha^2 \mu^6 \\
& + 6912000 \beta^2 \mu^2 \eta^2 \sqrt{5} \delta^2 - 6912000 \beta^2 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 62208000 \alpha^2 \mu^2 \eta^2 \sqrt{5} \delta^2, \\
& 7040 \alpha \mu \eta^3 \delta + 7040 \alpha \mu \eta \delta^3 - 14080 \alpha \mu^3 \eta \delta - 11520 \mu \sqrt{5} \eta \nu^2 + 6240 \mu \sqrt{5} \eta \beta^2 \\
& + 5760 \mu \sqrt{5} \eta \delta^2 + 11520 \mu \sqrt{5} \eta \alpha^2 + 3520 \alpha \delta \eta \mu - 19200 \mu^3 \sqrt{5} \eta \nu^2 \\
& - 1280 \mu \sqrt{5} \eta \nu^4 - 800 \eta \mu \beta^2 + 7040 \alpha^3 \mu \eta \delta - 20480 \mu^5 \sqrt{5} \eta - 6400 \eta \mu \nu^2 \beta^2 \\
& - 9600 \eta \mu \nu^2 \alpha^2 - 6720 \mu^3 \sqrt{5} \eta + 4320 \mu \sqrt{5} \eta - 13440 \mu^3 \sqrt{5} \eta \alpha^2 \\
& + 7680 \mu^3 \sqrt{5} \eta \beta^2 - 5760 \mu \sqrt{5} \eta^3 \delta^2 - 10560 \mu \sqrt{5} \eta^3 \beta^2 - 20480 \eta \mu^5 \\
& + 23040 \mu^3 \sqrt{5} \eta^3 - 8640 \mu \sqrt{5} \eta^5 + 1920 \beta \nu \alpha^4 + 4800 \eta \mu \delta^4 + 23040 \mu \sqrt{5} \eta^3 \nu^2 \\
& + 12800 \eta \mu^3 \beta^2 - 19200 \eta \mu^3 \nu^2 - 1280 \eta \mu \nu^4 + 1280 \eta \mu \beta^4 + 2880 \eta \mu \alpha^4 \\
& - 1600 \eta^3 \mu \beta^2 + 9600 \eta \mu^3 \alpha^2 + 4800 \eta \mu^3 + 2880 \mu \sqrt{5} \eta \delta^4 + 14400 \mu \sqrt{5} \eta \alpha^4 \\
& + 3840 \mu \sqrt{5} \eta \beta^4 + 1440 \eta \mu - 14080 \delta \eta \mu \nu^2 \alpha + 384 \beta^5 \nu + 3264 \eta^5 \mu \\
& + 4160 \beta^2 \eta \mu \alpha^2 + 18240 \beta^2 \sqrt{5} \eta \mu \alpha^2 - 1600 \beta^2 \delta^2 \eta \mu + 960 \beta^2 \sqrt{5} \delta^2 \eta \mu \\
& - 32640 \mu \sqrt{5} \eta \nu^2 \alpha^2 + 11520 \mu \sqrt{5} \eta \delta^2 \alpha^2 - 11520 \mu \sqrt{5} \eta \nu^2 \beta^2 + 1920 \eta^3 \mu \delta^2 \\
& - 3840 \beta^3 \nu \alpha^2 + 7040 \alpha \mu \eta \delta \beta^2, -627264 - 5207040 \mu^2 \alpha^2 + 6935040 \mu^2 \delta^2 \\
& + 1036800 \beta^6 + 64757760 \alpha^2 \nu^4 - 1658880 \delta \eta^2 \alpha + 4631040 \delta \nu^2 \alpha \\
& + 14515200 \alpha^4 \delta^2 - 37324800 \eta^2 \nu^4 + 10368000 \beta^2 \nu^2 \alpha - 1157760 \delta - 829440 \nu^2 \alpha \\
& + 1420416 \alpha - 8294400 \nu^2 \beta^2 \delta^2 - 36910080 \eta^2 \beta^2 \nu^2 - 829440 \delta^2 \beta^4 - 414720 \eta^2 \alpha^2 \\
& + 7257600 \eta^2 \beta^2 + 6428160 \eta^2 \delta^2 + 6428160 \nu^2 \delta^2 - 3317760 \delta \eta^2 \alpha^2 \\
& - 2626560 \delta \eta^2 \nu^2 + 4008960 \delta \eta^2 \beta^2 + 18984960 \mu^2 + 254016 \sqrt{5} - 3732480 \eta^2 \mu^2 \\
& + 1036800 \eta^2 \alpha - 4631040 \delta \alpha^2 + 414720 \delta^2 \sqrt{5} - 13271040 \nu^4 \alpha - 4769280 \nu^2 \beta^2 \\
& - 1036800 \beta^4 \sqrt{5} + 7050240 \eta^4 \sqrt{5} \beta^2 + 1036800 \nu^2 \alpha^2 + 414720 \delta^2 \alpha^2 + 9538560 \alpha^4 \\
& - 3317760 \delta \eta^2 \alpha^3 + 9262080 \alpha^3 \delta \mu^2 + 9262080 \alpha^3 \delta \nu^2 - 1935360 \delta \alpha^3 \beta^2 \\
& + 116736000 \nu^4 \beta^2 + 2695680 \delta \beta^4 - 8294400 \beta^4 \alpha^2 - 3421440 \delta^2 - 1140480 \alpha^2
\end{aligned}$$



$$\begin{aligned}
& -1935360 \delta \alpha^2 \beta^2 - 5391360 \delta \alpha \beta^2 \nu^2 + 7257600 \nu^2 - 881280 \eta^2 \\
& -14100480 \delta^2 \alpha^2 \beta^2 - 13685760 \eta^2 \alpha^2 \beta^2 - 3732480 \eta^2 \alpha^2 \delta^2 + 14515200 \delta^2 \eta^2 \beta^2 \\
& -1451520 \beta^4 + 1866240 \alpha^3 - 3110400 \delta^3 - 24744960 \nu^4 + 3421440 \eta^4 \\
& + 414720 \alpha \beta^2 - 5806080 \alpha^2 \beta^2 + 1036800 \beta^4 \alpha + 1105920 \alpha \delta^3 \beta^2 + 16130048 \nu^6 \\
& + 4631040 \alpha \delta \mu^2 + 1520640 \alpha \eta^4 \delta + 5806080 \eta^2 \delta^4 - 4354560 \sqrt{5} \nu^2 \\
& - 691200 \alpha \eta^2 \delta^3 + 4631040 \delta \nu^2 + 6681600 \mu^2 \alpha^2 \beta^2 + 17602560 \mu^2 \delta^2 \alpha^2 \\
& - 41195520 \mu^2 \nu^2 \alpha^2 + 97689600 \mu^2 \nu^2 \beta^2 + 8294400 \mu^2 \nu^2 \delta^2 - 37923840 \mu^2 \eta^2 \beta^2 \\
& - 96059392 \mu^6 - 3991680 \beta^2 - 3939840 \beta^2 \delta^2 + 18616320 \mu^2 \eta^2 \nu^2 \\
& - 2626560 \alpha \delta \eta^2 \nu^2 - 12441600 \beta^2 \mu^2 \delta^2 - 967680 \delta \alpha \beta^2 + 4008960 \alpha \delta \eta^2 \beta^2 \\
& - 6220800 \delta^3 \alpha^2 - 4631040 \delta \alpha^4 + 3179520 \delta^3 \nu^2 + 1105920 \delta^3 \beta^2 - 155520 \delta \sqrt{5} \\
& - 2073600 \delta^4 \beta^2 + 8709120 \eta^4 \beta^2 - 12441600 \eta^4 \alpha^2 - 5391360 \alpha \beta^2 \delta \mu^2 \\
& + 9262080 \alpha^2 \delta \nu^2 - 967680 \delta \beta^2 + 2695680 \beta^4 \alpha \delta + 9262080 \alpha^2 \delta \mu^2 - 933120 \delta^4 \\
& - 4631040 \delta \alpha^3 - 1157760 \delta \alpha - 3110400 \alpha \delta^3 - 5806080 \eta^2 \nu^2 + 17252352 \alpha^6 \\
& - 44267520 \nu^2 \mu^2 - 2626560 \alpha \eta^2 \mu^2 \delta + 16588800 \alpha^2 \nu^2 \delta^2 + 36080640 \eta^2 \alpha^2 \nu^2 \\
& - 5391360 \beta^2 \delta \mu^2 - 31656960 \beta^4 \nu^2 + 1244160 \delta \sqrt{5} \eta^2 \beta^2 - 1244160 \delta \sqrt{5} \eta^2 \nu^2 \\
& - 1244160 \delta \sqrt{5} \eta^2 \mu^2 - 2488320 \sqrt{5} \alpha \mu^2 \delta^2 - 4147200 \sqrt{5} \alpha \nu^2 \mu^2 \\
& + 5391360 \sqrt{5} \alpha \beta^2 \mu^2 - 2488320 \sqrt{5} \alpha \eta^2 \beta^2 + 1658880 \sqrt{5} \alpha \eta^2 \mu^2 \\
& + 1658880 \sqrt{5} \alpha \eta^2 \nu^2 - 2488320 \sqrt{5} \alpha \nu^2 \delta^2 - 5391360 \delta \beta^2 \nu^2 - 2073600 \delta^2 \alpha \beta^2 \\
& + 414720 \alpha \delta^2 + 311040 \delta^4 \sqrt{5} + 2903040 \eta^4 \nu^2 + 829440 \alpha^2 \delta^4 + 829440 \delta \sqrt{5} \eta^4 \\
& - 414720 \delta^3 \sqrt{5} \eta^2 - 73359360 \mu^2 \beta^2 \sqrt{5} \nu^2 - 28984320 \mu^2 \alpha^2 \sqrt{5} \beta^2 \\
& + 182753280 \mu^2 \sqrt{5} \nu^2 \alpha^2 - 51978240 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 33315840 \mu^2 \eta^2 \sqrt{5} \delta^2 \\
& - 1244160 \sqrt{5} \alpha \eta^4 + 6635520 \sqrt{5} \alpha \nu^4 + 207360 \sqrt{5} \alpha \beta^4 - 5806080 \sqrt{5} \alpha \mu^4 \\
& + 40227840 \eta^2 \alpha^2 \mu^2 - 261120 \mu^4 + 1244160 \delta^3 \sqrt{5} \mu^2 + 1244160 \delta^3 \sqrt{5} \nu^2 \\
& + 3179520 \alpha \delta^3 \nu^2 + 76861440 \sqrt{5} \nu^2 \mu^2 - 14254080 \beta^2 \sqrt{5} \nu^4 + 53637120 \sqrt{5} \nu^4 \alpha^2 \\
& - 37847040 \mu^4 \beta^2 \sqrt{5} + 111974400 \mu^4 \sqrt{5} \alpha^2 + 29998080 \mu^2 \eta^4 \sqrt{5} \\
& - 66631680 \mu^4 \eta^2 \sqrt{5} - 87367680 \mu^2 \eta^2 \sqrt{5} \nu^2 + 30827520 \mu^2 \eta^2 \sqrt{5} \beta^2 \\
& - 20736000 \eta^2 \sqrt{5} \nu^4 + 7050240 \eta^4 \sqrt{5} \nu^2 + 3179520 \alpha \delta^3 \mu^2 + 622080 \eta^4 \sqrt{5} \delta^2 \\
& - 4976640 \eta^4 \sqrt{5} \alpha^2 - 16312320 \sqrt{5} \mu^2 - 622080 \delta^3 \sqrt{5} + 23823360 \beta^4 \sqrt{5} \mu^2 \\
& - 3916800 \beta^4 \sqrt{5} \nu^2 - 259200 \beta^2 \sqrt{5} - 6220800 \eta^2 \sqrt{5} \alpha^2 \nu^2 + 10368000 \eta^2 \sqrt{5} \delta^2 \nu^2 \\
& + 3732480 \eta^2 \sqrt{5} \delta^2 \beta^2 + 6635520 \alpha^4 \sqrt{5} + 2488320 \alpha^2 \sqrt{5} + 7464960 \beta^2 \sqrt{5} \mu^2 \delta^2 \\
& + 5598720 \eta^2 \sqrt{5} \beta^4 - 1658880 \eta^6 - 1658880 \delta \eta^2 - 2073600 \eta^2 \sqrt{5} \alpha^2 \mu^2 \\
& + 12026880 \eta^2 \sqrt{5} \nu^2 \beta^2 - 5391360 \eta^2 \sqrt{5} \alpha^2 \beta^2 + 4147200 \mu^4 \delta^2 \sqrt{5} \\
& - 68567040 \sqrt{5} \mu^2 \alpha^2 + 11404800 \eta^2 \beta^4 + 2695680 \eta^2 \alpha^4 + 622080 \delta \sqrt{5} \mu^2 \\
& + 622080 \delta \sqrt{5} \nu^2 + 15966720 \beta^2 \sqrt{5} \nu^2 - 17902080 \delta^2 \sqrt{5} \mu^2 - 8501760 \delta^2 \sqrt{5} \nu^2 \\
& + 7257600 \alpha^4 \beta^2 - 43683840 \alpha^4 \nu^2 - 1140480 \eta^4 \sqrt{5} + 2073600 \eta^2 \sqrt{5} \nu^2 \\
& + 4147200 \eta^2 \sqrt{5} \mu^2 + 3732480 \alpha^2 \sqrt{5} \delta^2 - 15552000 \sqrt{5} \nu^2 \alpha^2 - 1451520 \eta^2 \sqrt{5} \alpha^4 \\
& - 829440 \delta^4 \sqrt{5} \eta^2 - 414720 \delta^4 \sqrt{5} \beta^2 - 2488320 \delta^4 \sqrt{5} \nu^2 - 16565760 \beta^2 \sqrt{5} \mu^2 \\
& - 2280960 \eta^2 \sqrt{5} \delta^2 - 622080 \eta^2 \sqrt{5} \beta^2 + 829440 \alpha^2 \sqrt{5} \beta^2 - 18777600 \beta^2 \mu^2 \\
& + 6469632 \alpha^6 \sqrt{5} - 622080 \sqrt{5} \alpha \delta^3 + 8294400 \mu^2 \delta^2 \sqrt{5} \nu^2 + 2073600 \alpha^3 \eta^2 \\
& - 12856320 \sqrt{5} \alpha^3 \nu^2 + 2488320 \sqrt{5} \alpha^3 \delta^2 - 414720 \sqrt{5} \alpha^3 \mu^2 - 3317760 \sqrt{5} \alpha \nu^2 \\
& - 207360 \sqrt{5} \alpha \mu^2 + 1244160 \sqrt{5} \alpha \delta^2 - 1244160 \delta^3 \sqrt{5} \alpha^2 - 7050240 \eta^2 \sqrt{5} \alpha^2 \delta^2 \\
& + 829440 \sqrt{5} \alpha \beta^2 + 2488320 \sqrt{5} \alpha^3 \beta^2 + 4354560 \eta^4 \delta^2 + 829440 \sqrt{5} \alpha \delta^4 \\
& + 1244160 \alpha \delta^3 \sqrt{5} \mu^2 + 622080 \sqrt{5} \alpha \delta \nu^2 - 1244160 \sqrt{5} \alpha^3 \delta^3 - 580608 \sqrt{5} \alpha \delta^5 \\
& - 71884800 \mu^2 \alpha^4 \sqrt{5} + 622080 \sqrt{5} \alpha \delta \beta^4 - 155520 \sqrt{5} \alpha \delta + 4147200 \beta^2 \alpha^3
\end{aligned}$$

$$\begin{aligned}
& + 1520640 \eta^4 \delta + 3773952 \alpha^5 + 4631040 \delta \mu^2 - 414720 \sqrt{5} \alpha \eta^2 \delta^3 \\
& - 414720 \beta^4 \sqrt{5} \delta^2 - 2488320 \alpha^2 \sqrt{5} \beta^4 - 32348160 \alpha^4 \sqrt{5} \nu^2 + 5806080 \alpha^4 \sqrt{5} \delta^2 \\
& - 4147200 \mu^2 \delta^4 + 622080 \delta \sqrt{5} \beta^4 - 1548288 \delta^5 - 622080 \delta \sqrt{5} \alpha^4 + 53913600 \mu^4 \delta^2 \\
& - 19768320 \alpha^2 \nu^2 \beta^2 + 829440 \alpha^2 \delta^4 \sqrt{5} - 2488320 \mu^2 \delta^4 \sqrt{5} + 4147200 \nu^4 \delta^2 \sqrt{5} \\
& - 7050240 \sqrt{5} \alpha \beta^2 \nu^2 + 902016 \sqrt{5} \alpha - 414720 \sqrt{5} \alpha \delta^2 \eta^2 - 1451520 \beta^2 \sqrt{5} \delta^2 \\
& - 2073600 \eta^2 \sqrt{5} \alpha^2 - 17187840 \alpha^2 \sqrt{5} \beta^2 \nu^2 - 45619200 \nu^4 \delta^2 \\
& - 1244160 \sqrt{5} \alpha \beta^2 \delta \mu^2 - 33177600 \alpha^2 \sqrt{5} \delta^2 \nu^2 - 622080 \sqrt{5} \alpha^3 \delta - 622080 \sqrt{5} \alpha^5 \delta \\
& + 11612160 \beta^2 \sqrt{5} \nu^2 \delta^2 - 1244160 \delta \sqrt{5} \beta^2 \mu^2 - 1244160 \delta \sqrt{5} \beta^2 \nu^2 \\
& + 1244160 \delta \sqrt{5} \alpha^2 \mu^2 + 1244160 \delta \sqrt{5} \alpha^2 \nu^2 + 414720 \sqrt{5} \alpha^3 \eta^2 - 580608 \sqrt{5} \delta^5 \\
& + 1658880 \eta^6 \sqrt{5} + 207360 \beta^6 \sqrt{5} - 414720 \beta^2 \sqrt{5} \alpha \delta^2 + 829440 \alpha \delta^4 \\
& + 55941120 \eta^2 \mu^4 + 4561920 \nu^2 \alpha^3 + 3179520 \mu^2 \delta^3 + 207360 \sqrt{5} \alpha \eta^2 \\
& - 2257920 \eta^4 \mu^2 + 207360 \sqrt{5} \delta^6 - 207360 \delta^6 + 4561920 \sqrt{5} \nu^4 + 52392960 \sqrt{5} \mu^4 \\
& + 2944512 \sqrt{5} \alpha^5 + 2280960 \sqrt{5} \alpha^3 - 4147200 \alpha^2 \sqrt{5} \delta^2 \beta^2 - 829440 \alpha \mu^4 \\
& + 1244160 \alpha \delta^3 \sqrt{5} \nu^2 - 691200 \eta^2 \delta^3 + 829440 \sqrt{5} \alpha \eta^4 \delta + 622080 \sqrt{5} \alpha \delta \mu^2 \\
& + 1244160 \sqrt{5} \alpha^3 \delta \mu^2 + 1244160 \sqrt{5} \alpha^3 \delta \nu^2 + 2073600 \alpha \eta^4 + 829440 \delta^2 \alpha^3 \\
& - 4631040 \delta \alpha^5 - 3939840 \alpha \mu^2 - 31380480 \mu^2 \beta^4 - 6220800 \delta^3 \alpha^3 - 673920 \sqrt{5} \eta^2 \\
& - 1548288 \alpha \delta^5 + 3110400 \alpha^4 \sqrt{5} \beta^2 - 622080 \delta \sqrt{5} \alpha^2 - 22164480 \eta^2 \mu^2 \delta^2 \\
& + 100761600 \mu^4 \beta^2 + 4147200 \alpha \nu^2 \mu^2 + 1658880 \alpha \eta^2 \nu^2 + 1658880 \alpha \eta^2 \mu^2 \\
& + 35471360 \mu^4 \nu^2 - 829440 \alpha \eta^2 \beta^2 + 127262720 \mu^2 \nu^4 - 2073600 \alpha \beta^2 \mu^2 \\
& - 2626560 \eta^2 \mu^2 \delta - 17510400 \mu^2 \alpha^4 + 4147200 \alpha \nu^2 \delta^2 + 2903040 \alpha \eta^2 \delta^2 \\
& + 4147200 \alpha \mu^2 \delta^2 + 4147200 \delta^4 \nu^2 - 1244160 \sqrt{5} \alpha \delta \eta^2 \nu^2 + 1244160 \sqrt{5} \alpha \delta \eta^2 \beta^2 \\
& - 1244160 \sqrt{5} \alpha \delta \beta^2 \nu^2 - 1244160 \sqrt{5} \alpha \eta^2 \mu^2 \delta - 7879680 \mu^2 \alpha^3 + 17725440 \mu^4 \alpha^2 \\
& - 8709120 \delta^2 \eta^2 \nu^2, -1900800 + 265512960 \mu^2 \alpha^2 + 156810240 \mu^2 \delta^2 - 4561920 \beta^6 \\
& + 63897600 \alpha^2 \nu^4 - 30274560 \alpha^4 \delta^2 - 11059200 \eta^2 \nu^4 + 44236800 \nu^2 \beta^2 \delta^2 \\
& + 40089600 \eta^2 \beta^2 \nu^2 - 3456000 \delta^2 \beta^4 - 27786240 \eta^2 \alpha^2 - 12994560 \eta^2 \beta^2 \\
& - 50181120 \eta^2 \delta^2 - 414720 \nu^2 \delta^2 + 51793920 \mu^2 - 4717440 \sqrt{5} + 32071680 \eta^2 \mu^2 \\
& - 5702400 \delta^2 \sqrt{5} - 4008960 \nu^2 \beta^2 + 207360 \beta^4 \sqrt{5} - 414720 \eta^4 \sqrt{5} \beta^2 + 829440 \nu^2 \alpha^2 \\
& - 414720 \delta^2 \alpha^2 - 25297920 \alpha^4 + 19660800 \nu^4 \beta^2 + 47554560 \beta^4 \alpha^2 + 7361280 \delta^2 \\
& + 7050240 \alpha^2 - 14515200 \nu^2 - 3525120 \eta^2 + 53913600 \delta^2 \alpha^2 \beta^2 + 33730560 \eta^2 \alpha^2 \beta^2 \\
& - 29859840 \eta^2 \alpha^2 \delta^2 - 26818560 \delta^2 \eta^2 \beta^2 + 1382400 \beta^4 - 1105920 \nu^4 - 12303360 \eta^4 \\
& + 19906560 \alpha^2 \beta^2 + 25133056 \nu^6 - 20736000 \eta^2 \delta^4 + 9331200 \sqrt{5} \nu^2 \\
& + 142755840 \mu^2 \alpha^2 \beta^2 + 315279360 \mu^2 \delta^2 \alpha^2 - 383754240 \mu^2 \nu^2 \alpha^2 \\
& - 64512000 \mu^2 \nu^2 \beta^2 + 5529600 \mu^2 \nu^2 \delta^2 - 274360320 \mu^2 \eta^2 \beta^2 + 84606976 \mu^6 \\
& + 13789440 \beta^2 + 13271040 \beta^2 \delta^2 + 722903040 \mu^2 \eta^2 \nu^2 + 44236800 \beta^2 \mu^2 \delta^2 \\
& + 414720 \delta^4 \beta^2 + 29445120 \eta^4 \beta^2 + 26818560 \eta^4 \alpha^2 - 9538560 \delta^4 + 32071680 \eta^2 \nu^2 \\
& - 30965760 \alpha^6 - 318935040 \nu^2 \mu^2 + 829440 \alpha^2 \nu^2 \delta^2 + 1105920 \eta^2 \alpha^2 \nu^2 \\
& + 30136320 \beta^4 \nu^2 - 3525120 \delta^4 \sqrt{5} - 98979840 \eta^4 \nu^2 + 323297280 \mu^2 \beta^2 \sqrt{5} \nu^2 \\
& - 120453120 \mu^2 \alpha^2 \sqrt{5} \beta^2 - 7741440 \mu^2 \sqrt{5} \nu^2 \alpha^2 + 12994560 \mu^2 \alpha^2 \sqrt{5} \delta^2 \\
& + 26818560 \mu^2 \eta^2 \sqrt{5} \delta^2 + 1105920 \eta^2 \alpha^2 \mu^2 - 304189440 \mu^4 + 12718080 \sqrt{5} \nu^2 \mu^2 \\
& + 63651840 \beta^2 \sqrt{5} \nu^4 + 44236800 \sqrt{5} \nu^4 \alpha^2 + 187883520 \mu^4 \beta^2 \sqrt{5} \\
& - 29859840 \mu^4 \sqrt{5} \alpha^2 + 1105920 \mu^2 \eta^4 \sqrt{5} - 25436160 \mu^4 \eta^2 \sqrt{5} \\
& - 91791360 \mu^2 \eta^2 \sqrt{5} \nu^2 - 32901120 \mu^2 \eta^2 \sqrt{5} \beta^2 - 66355200 \eta^2 \sqrt{5} \nu^4 \\
& + 21565440 \eta^4 \sqrt{5} \nu^2 + 2903040 \eta^4 \sqrt{5} \delta^2 + 9123840 \eta^4 \sqrt{5} \alpha^2 + 10298880 \sqrt{5} \mu^2 \\
& - 109393920 \beta^4 \sqrt{5} \mu^2 + 6543360 \beta^4 \sqrt{5} \nu^2 - 3421440 \beta^2 \sqrt{5} + 9953280 \eta^2 \sqrt{5} \alpha^2 \nu^2
\end{aligned}$$

$$\begin{aligned}
& + 38983680 \eta^2 \sqrt{5} \delta^2 \nu^2 - 9123840 \eta^2 \sqrt{5} \delta^2 \beta^2 - 25090560 \alpha^4 \sqrt{5} - 13893120 \alpha^2 \sqrt{5} \\
& - 49766400 \beta^2 \sqrt{5} \mu^2 \delta^2 - 7464960 \eta^2 \sqrt{5} \beta^4 + 37324800 \eta^6 + 9953280 \eta^2 \sqrt{5} \alpha^2 \mu^2 \\
& - 4147200 \eta^2 \sqrt{5} \nu^2 \beta^2 + 11612160 \eta^2 \sqrt{5} \alpha^2 \beta^2 - 16588800 \mu^4 \delta^2 \sqrt{5} \\
& + 40780800 \sqrt{5} \mu^2 \alpha^2 - 12441600 \eta^2 \beta^4 - 41472000 \eta^2 \alpha^4 - 29859840 \beta^2 \sqrt{5} \nu^2 \\
& + 10229760 \delta^2 \sqrt{5} \mu^2 - 4147200 \delta^2 \sqrt{5} \nu^2 + 21150720 \alpha^4 \beta^2 + 105062400 \alpha^4 \nu^2 \\
& + 9745920 \eta^4 \sqrt{5} - 8709120 \eta^2 \sqrt{5} \nu^2 - 8709120 \eta^2 \sqrt{5} \mu^2 - 16588800 \alpha^2 \sqrt{5} \delta^2 \\
& + 28615680 \sqrt{5} \nu^2 \alpha^2 - 6635520 \eta^2 \sqrt{5} \alpha^4 - 2488320 \delta^4 \sqrt{5} \eta^2 + 414720 \delta^4 \sqrt{5} \beta^2 \\
& - 829440 \delta^4 \sqrt{5} \nu^2 - 67276800 \beta^2 \sqrt{5} \mu^2 + 1658880 \eta^2 \sqrt{5} \delta^2 + 12026880 \eta^2 \sqrt{5} \beta^2 \\
& - 12026880 \alpha^2 \sqrt{5} \beta^2 + 136627200 \beta^2 \mu^2 - 9953280 \alpha^6 \sqrt{5} - 16588800 \mu^2 \delta^2 \sqrt{5} \nu^2 \\
& + 17280000 \eta^4 \delta^2 + 40366080 \mu^2 \alpha^4 \sqrt{5} - 1244160 \beta^4 \sqrt{5} \delta^2 + 13271040 \alpha^2 \sqrt{5} \beta^4 \\
& - 13271040 \alpha^4 \sqrt{5} \nu^2 - 10368000 \alpha^4 \sqrt{5} \delta^2 + 7464960 \mu^2 \delta^4 + 60825600 \mu^4 \delta^2 \\
& - 40089600 \alpha^2 \nu^2 \beta^2 + 7464960 \mu^2 \delta^4 \sqrt{5} + 4561920 \beta^2 \sqrt{5} \delta^2 - 4147200 \eta^2 \sqrt{5} \alpha^2 \\
& - 58152960 \alpha^2 \sqrt{5} \beta^2 \nu^2 - 55296000 \nu^4 \delta^2 - 15759360 \alpha^2 \sqrt{5} \delta^2 \nu^2 \\
& - 49766400 \beta^2 \sqrt{5} \nu^2 \delta^2 + 5806080 \eta^6 \sqrt{5} - 1244160 \beta^6 \sqrt{5} + 733962240 \eta^2 \mu^4 \\
& - 405135360 \eta^4 \mu^2 + 414720 \sqrt{5} \delta^6 - 691200 \delta^6 + 12718080 \sqrt{5} \nu^4 + 8847360 \sqrt{5} \mu^4 \\
& + 14100480 \alpha^2 \sqrt{5} \delta^2 \beta^2 + 30689280 \mu^2 \beta^4 - 414720 \sqrt{5} \eta^2 + 4561920 \alpha^4 \sqrt{5} \beta^2 \\
& - 284313600 \eta^2 \mu^2 \delta^2 - 143155200 \mu^4 \beta^2 + 206110720 \mu^4 \nu^2 + 157450240 \mu^2 \nu^4 \\
& + 287354880 \mu^2 \alpha^4 - 829440 \delta^4 \nu^2 - 595107840 \mu^4 \alpha^2 + 13547520 \delta^2 \eta^2 \nu^2, 192 \\
& + 5760 \beta^2 \nu^2 \alpha - 1440 \delta - 4800 \nu^2 \alpha + 1440 \alpha - 3840 \delta \eta^2 \alpha^2 - 3840 \delta \eta^2 \nu^2 \\
& + 2880 \delta \eta^2 \beta^2 - 960 \eta^2 \alpha - 5760 \delta \alpha^2 - 1280 \nu^4 \alpha + 960 \delta \beta^4 - 4800 \delta \alpha^2 \beta^2 \\
& - 2880 \delta^3 + 960 \alpha \beta^2 + 6720 \beta^4 \alpha + 5760 \delta \nu^2 - 5760 \delta^3 \alpha^2 - 5760 \delta \alpha^4 + 960 \delta^3 \beta^2 \\
& + 11520 \alpha^2 \delta \nu^2 - 2400 \delta \beta^2 + 11520 \alpha^2 \delta \mu^2 - 1920 \beta^2 \delta \mu^2 - 960 \delta \sqrt{5} \eta^2 \beta^2 \\
& + 3840 \delta \sqrt{5} \eta^2 \nu^2 + 3840 \delta \sqrt{5} \eta^2 \mu^2 - 9600 \sqrt{5} \alpha \mu^2 \delta^2 + 76800 \sqrt{5} \alpha \nu^2 \mu^2 \\
& - 24960 \sqrt{5} \alpha \beta^2 \mu^2 - 11520 \sqrt{5} \alpha \eta^2 \beta^2 + 9600 \sqrt{5} \alpha \eta^2 \mu^2 + 9600 \sqrt{5} \alpha \eta^2 \nu^2 \\
& - 9600 \sqrt{5} \alpha \nu^2 \delta^2 - 1920 \delta \beta^2 \nu^2 + 1920 \delta^2 \alpha \beta^2 - 1920 \delta \sqrt{5} \eta^4 - 1920 \delta^3 \sqrt{5} \eta^2 \\
& - 6720 \sqrt{5} \alpha \eta^4 - 1280 \sqrt{5} \alpha \nu^4 + 960 \sqrt{5} \alpha \beta^4 + 37120 \sqrt{5} \alpha \mu^4 - 1920 \delta \eta^2 \\
& - 960 \delta \sqrt{5} \eta^2 + 480 \delta \sqrt{5} \beta^2 - 1920 \alpha^3 \eta^2 - 13440 \sqrt{5} \alpha^3 \nu^2 + 11520 \sqrt{5} \alpha^3 \delta^2 \\
& - 51840 \sqrt{5} \alpha^3 \mu^2 - 16320 \sqrt{5} \alpha \nu^2 - 25920 \sqrt{5} \alpha \mu^2 + 5760 \sqrt{5} \alpha \delta^2 + 4800 \sqrt{5} \alpha \beta^2 \\
& + 15360 \sqrt{5} \alpha^3 \beta^2 + 960 \delta^3 \sqrt{5} \beta^2 + 2880 \sqrt{5} \alpha \delta^4 + 3840 \beta^2 \alpha^3 + 3840 \eta^4 \delta + 3264 \alpha^5 \\
& + 5760 \delta \mu^2 + 960 \delta \sqrt{5} \beta^4 + 384 \delta^5 + 13440 \sqrt{5} \alpha \beta^2 \nu^2 + 4320 \sqrt{5} \alpha \\
& - 1920 \delta \sqrt{5} \eta^2 \alpha^2 - 3840 \sqrt{5} \alpha \delta^2 \eta^2 - 1920 \delta \sqrt{5} \beta^2 \mu^2 - 1920 \delta \sqrt{5} \beta^2 \nu^2 \\
& + 1920 \sqrt{5} \alpha^3 \eta^2 + 960 \delta \sqrt{5} \alpha^2 \beta^2 - 1920 \beta^2 \sqrt{5} \alpha \delta^2 + 2880 \alpha \delta^4 + 9600 \nu^2 \alpha^3 \\
& + 960 \sqrt{5} \alpha \eta^2 + 14400 \sqrt{5} \alpha^5 + 11520 \sqrt{5} \alpha^3 + 37120 \alpha \mu^4 - 1920 \eta^2 \delta^3 + 960 \alpha \eta^4 \\
& - 14400 \alpha \mu^2 + 76800 \alpha \nu^2 \mu^2 - 5760 \alpha \eta^2 \nu^2 - 5760 \alpha \eta^2 \mu^2 - 32640 \alpha \beta^2 \mu^2 \\
& - 3840 \eta^2 \mu^2 \delta - 9600 \alpha \nu^2 \delta^2 + 3840 \alpha \eta^2 \delta^2 - 9600 \alpha \mu^2 \delta^2 - 28800 \mu^2 \alpha^3, -10752 \\
& + 107520 \mu^2 \alpha^2 + 76800 \mu^2 \delta^2 + 721920 \beta^2 \nu^2 \alpha + 798720 \beta^4 \nu^2 \alpha + 16640 \delta \\
& - 161280 \nu^2 \alpha + 23040 \alpha + 15360 \eta^2 \beta^2 + 76800 \nu^2 \delta^2 + 194560 \delta \eta^2 \alpha^2 \\
& - 61440 \delta \eta^2 \nu^2 + 87040 \delta \eta^2 \beta^2 + 53760 \mu^2 + 11520 \sqrt{5} + 46080 \eta^2 \mu^2 + 99840 \delta \alpha^2 \\
& + 38400 \delta^2 \sqrt{5} + 51200 \nu^4 \alpha + 76800 \nu^2 \beta^2 - 23040 \beta^4 \sqrt{5} + 107520 \nu^2 \alpha^2 \\
& - 15360 \delta^2 \alpha^2 - 23040 \alpha^4 - 983040 \nu^4 \alpha \beta^2 + 61440 \beta^6 \alpha - 215040 \beta^6 \delta + 56320 \delta \beta^4 \\
& - 7680 \delta^2 - 21504 \alpha^2 + 460800 \delta \alpha^2 \beta^2 + 53760 \nu^2 - 7680 \beta^4 + 76800 \alpha^3 + 33280 \delta^3 \\
& - 133120 \nu^4 + 7680 \eta^4 - 92160 \alpha \beta^2 - 30720 \alpha^2 \beta^2 - 38400 \beta^4 \alpha + 430080 \beta^2 \delta \eta^2 \nu^2 \\
& + 174080 \beta^2 \delta \eta^2 \alpha^2 - 655360 \beta^2 \alpha^2 \delta \nu^2 - 23040 \sqrt{5} \nu^2 - 655360 \beta^2 \alpha^2 \delta \mu^2 \\
& - 66560 \delta \nu^2 - 13824 \beta^2 - 122880 \alpha^2 \delta \eta^2 \nu^2 + 133120 \delta^3 \alpha^2 + 199680 \delta \alpha^4
\end{aligned}$$

$$\begin{aligned}
& + 240640 \delta^3 \beta^2 + 614400 \beta^2 \alpha \mu^2 \delta^2 - 122880 \beta^2 \alpha \eta^2 \delta^2 + 614400 \beta^2 \alpha \nu^2 \delta^2 \\
& + 430080 \beta^2 \eta^2 \mu^2 \delta - 122880 \beta^2 \alpha \eta^2 \mu^2 - 122880 \beta^2 \alpha \eta^2 \nu^2 + 614400 \beta^2 \alpha \nu^2 \mu^2 \\
& + 3840 \delta \sqrt{5} - 266240 \alpha^2 \delta \nu^2 + 115200 \delta \beta^2 - 266240 \alpha^2 \delta \mu^2 - 7680 \delta^4 \\
& + 46080 \eta^2 \nu^2 - 307200 \nu^2 \mu^2 - 327680 \beta^2 \delta \mu^2 - 122880 \alpha^2 \eta^2 \mu^2 \delta \\
& - 46080 \delta \sqrt{5} \eta^2 \beta^2 + 61440 \delta \sqrt{5} \eta^2 \nu^2 + 61440 \delta \sqrt{5} \eta^2 \mu^2 - 138240 \sqrt{5} \alpha \mu^2 \delta^2 \\
& + 1966080 \sqrt{5} \alpha \nu^2 \mu^2 - 875520 \sqrt{5} \alpha \beta^2 \mu^2 + 261120 \sqrt{5} \alpha \eta^2 \beta^2 \\
& - 599040 \sqrt{5} \alpha \eta^2 \mu^2 - 599040 \sqrt{5} \alpha \eta^2 \nu^2 - 138240 \sqrt{5} \alpha \nu^2 \delta^2 - 327680 \delta \beta^2 \nu^2 \\
& - 215040 \delta^2 \alpha \beta^2 + 23040 \alpha \delta^2 + 7680 \delta^4 \sqrt{5} - 30720 \delta \sqrt{5} \eta^4 - 30720 \delta^3 \sqrt{5} \eta^2 \\
& + 430080 \delta \beta^4 \nu^2 + 207360 \sqrt{5} \alpha \eta^4 + 51200 \sqrt{5} \alpha \nu^4 - 84480 \sqrt{5} \alpha \beta^4 \\
& + 1095680 \sqrt{5} \alpha \mu^4 - 133120 \mu^4 - 266240 \alpha^4 \delta \mu^2 - 266240 \alpha^4 \delta \nu^2 + 194560 \alpha^4 \delta \eta^2 \\
& - 307200 \sqrt{5} \nu^2 \mu^2 - 307200 \alpha^3 \nu^2 \delta^2 - 122880 \alpha^3 \eta^2 \beta^2 - 184320 \alpha^3 \eta^2 \mu^2 \\
& - 184320 \alpha^3 \eta^2 \nu^2 + 2211840 \alpha^3 \nu^2 \mu^2 + 184320 \alpha^2 \eta^2 \delta^3 - 23040 \sqrt{5} \mu^2 + 7680 \delta^3 \sqrt{5} \\
& - 307200 \alpha^3 \mu^2 \delta^2 + 23040 \beta^2 \sqrt{5} + 53760 \alpha^4 \sqrt{5} + 61440 \alpha^2 \sqrt{5} + 48640 \delta \eta^2 \\
& - 307200 \beta^2 \mu^2 \delta^3 - 46080 \sqrt{5} \mu^2 \alpha^2 - 7680 \delta \sqrt{5} \eta^2 - 7680 \delta \sqrt{5} \beta^2 \\
& + 1187840 \alpha^3 \sqrt{5} \mu^4 - 15360 \delta \sqrt{5} \mu^2 - 15360 \delta \sqrt{5} \nu^2 + 204800 \alpha^3 \sqrt{5} \nu^4 \\
& + 675840 \alpha^5 \sqrt{5} \nu^2 - 307200 \alpha^5 \sqrt{5} \delta^2 - 307200 \alpha^5 \sqrt{5} \mu^2 + 30720 \alpha^4 \sqrt{5} \delta^3 \\
& + 107520 \beta^2 \sqrt{5} \nu^2 + 46080 \delta^2 \sqrt{5} \mu^2 + 46080 \delta^2 \sqrt{5} \nu^2 - 69120 \eta^4 \sqrt{5} \\
& - 122880 \beta^6 \sqrt{5} \alpha + 199680 \eta^2 \sqrt{5} \nu^2 + 199680 \eta^2 \sqrt{5} \mu^2 - 491520 \beta^2 \sqrt{5} \alpha^5 \\
& + 76800 \alpha^2 \sqrt{5} \delta^2 - 46080 \sqrt{5} \nu^2 \alpha^2 + 107520 \beta^2 \sqrt{5} \mu^2 - 61440 \eta^2 \sqrt{5} \delta^2 \\
& - 76800 \eta^2 \sqrt{5} \beta^2 + 30720 \alpha^2 \sqrt{5} \beta^2 + 76800 \beta^2 \mu^2 - 61440 \beta^2 \delta^3 \sqrt{5} \nu^2 \\
& - 61440 \beta^2 \delta^3 \sqrt{5} \mu^2 + 61440 \beta^2 \delta^3 \sqrt{5} \eta^2 - 552960 \beta^4 \sqrt{5} \alpha \mu^2 + 1290240 \beta^4 \sqrt{5} \alpha \nu^2 \\
& - 61440 \alpha^2 \delta^3 \sqrt{5} \eta^2 + 860160 \beta^2 \sqrt{5} \alpha \mu^4 - 983040 \beta^2 \sqrt{5} \alpha \nu^4 \\
& + 1843200 \beta^2 \sqrt{5} \alpha^3 \nu^2 - 184320 \alpha^3 \sqrt{5} \mu^2 \delta^2 + 30720 \alpha^6 \sqrt{5} \delta - 798720 \alpha^3 \sqrt{5} \eta^2 \mu^2 \\
& + 2211840 \alpha^3 \sqrt{5} \nu^2 \mu^2 - 798720 \alpha^3 \sqrt{5} \eta^2 \nu^2 - 61440 \alpha^2 \delta \sqrt{5} \eta^4 \\
& - 184320 \alpha^3 \sqrt{5} \nu^2 \delta^2 + 752640 \sqrt{5} \alpha^3 \nu^2 - 384000 \sqrt{5} \alpha^3 \delta^2 - 537600 \sqrt{5} \alpha^3 \mu^2 \\
& + 69120 \sqrt{5} \alpha \nu^2 - 192000 \sqrt{5} \alpha \mu^2 - 115200 \sqrt{5} \alpha \delta^2 + 30720 \delta^3 \sqrt{5} \alpha^2 \\
& - 184320 \sqrt{5} \alpha \beta^2 - 460800 \sqrt{5} \alpha^3 \beta^2 - 15360 \delta^3 \sqrt{5} \beta^2 - 23040 \sqrt{5} \alpha \delta^4 \\
& - 92160 \beta^2 \alpha^3 + 124416 \alpha^5 - 66560 \delta \mu^2 - 15360 \delta \sqrt{5} \beta^4 - 6144 \delta^5 + 46080 \delta \sqrt{5} \alpha^4 \\
& - 30720 \alpha^2 \sqrt{5} \delta \beta^4 + 204800 \alpha^3 \nu^4 + 629760 \sqrt{5} \alpha \beta^2 \nu^2 + 1187840 \alpha^3 \mu^4 \\
& + 61440 \alpha^5 \nu^2 - 12288 \alpha^2 \delta^5 + 133120 \alpha^6 \delta + 133120 \alpha^4 \delta^3 - 34560 \sqrt{5} \alpha \\
& - 921600 \alpha^5 \mu^2 + 61440 \alpha^5 \delta^2 - 30720 \alpha^3 \eta^4 + 276480 \alpha^3 \sqrt{5} \eta^4 - 30720 \delta \sqrt{5} \eta^2 \alpha^2 \\
& + 184320 \sqrt{5} \alpha \delta^2 \eta^2 + 30720 \beta^6 \delta \sqrt{5} - 399360 \beta^4 \sqrt{5} \alpha^3 + 61440 \beta^4 \delta^3 \sqrt{5} \\
& + 61440 \delta \sqrt{5} \beta^2 \mu^2 + 61440 \delta \sqrt{5} \beta^2 \nu^2 - 61440 \delta \sqrt{5} \alpha^2 \mu^2 - 61440 \delta \sqrt{5} \alpha^2 \nu^2 \\
& - 30720 \beta^2 \delta \sqrt{5} \alpha^4 - 30720 \beta^2 \delta^3 \sqrt{5} \alpha^2 + 307200 \beta^4 \sqrt{5} \alpha \eta^2 - 61440 \beta^4 \delta \sqrt{5} \nu^2 \\
& - 61440 \beta^4 \delta \sqrt{5} \mu^2 + 30720 \beta^2 \delta \sqrt{5} \eta^4 + 61440 \beta^4 \delta \sqrt{5} \eta^2 - 30720 \delta \sqrt{5} \alpha^2 \beta^2 \\
& + 245760 \alpha^3 \sqrt{5} \eta^2 \delta^2 + 368640 \alpha^3 \sqrt{5} \eta^2 \beta^2 - 30720 \alpha^4 \delta \sqrt{5} \eta^2 \\
& - 983040 \beta^2 \sqrt{5} \alpha^3 \mu^2 - 307200 \beta^2 \sqrt{5} \alpha^3 \delta^2 + 245760 \beta^2 \sqrt{5} \alpha \eta^4 \\
& - 153600 \beta^2 \sqrt{5} \alpha \delta^2 + 23040 \alpha \delta^4 - 15360 \nu^2 \alpha^3 + 30720 \alpha^3 \delta^4 - 30720 \alpha^3 \delta^4 \sqrt{5} \\
& - 133120 \sqrt{5} \nu^4 - 133120 \sqrt{5} \mu^4 - 407040 \sqrt{5} \alpha^5 - 230400 \sqrt{5} \alpha^3 + 1095680 \alpha \mu^4 \\
& + 30720 \beta^2 \sqrt{5} \delta^5 + 92160 \eta^2 \delta^3 + 79872 \alpha^7 - 23040 \alpha \eta^4 + 76800 \delta^2 \alpha^3 \\
& - 422400 \alpha \mu^2 - 61440 \alpha^4 \sqrt{5} \delta \mu^2 - 61440 \alpha^4 \sqrt{5} \delta \nu^2 - 61440 \beta^4 \delta^3 + 23040 \delta \sqrt{5} \alpha^2 \\
& + 1966080 \alpha \nu^2 \mu^2 - 138240 \alpha \eta^2 \nu^2 - 138240 \alpha \eta^2 \mu^2 - 76800 \alpha \eta^2 \beta^2 \\
& - 783360 \alpha \beta^2 \mu^2 - 61440 \eta^2 \mu^2 \delta - 230400 \alpha \nu^2 \delta^2 - 153600 \beta^4 \alpha^3 - 230400 \alpha \mu^2 \delta^2
\end{aligned}$$

$$\begin{aligned}
& -135168\beta^2\alpha^5 + 141312\beta^2\delta^5 + 122880\alpha^2\delta\sqrt{5}\eta^2\mu^2 + 122880\alpha^2\delta\sqrt{5}\eta^2\nu^2 \\
& -61440\beta^2\delta\sqrt{5}\eta^2\nu^2 - 61440\beta^2\delta\sqrt{5}\eta^2\mu^2 - 368640\beta^2\sqrt{5}\alpha\eta^2\mu^2 \\
& -368640\beta^2\sqrt{5}\alpha\eta^2\nu^2 + 614400\beta^2\sqrt{5}\alpha\nu^2\mu^2 + 112640\beta^4\delta\alpha^2 - 430080\beta^4\delta\eta^2 \\
& + 460800\beta^2\delta\alpha^4 + 481280\beta^2\delta^3\alpha^2 + 430080\beta^4\delta\mu^2 - 276480\beta^2\eta^4\delta \\
& -245760\beta^2\alpha\delta^4 + 122880\beta^2\alpha\eta^4 + 61440\beta^2\eta^2\delta^3 + 860160\beta^2\alpha\mu^4 \\
& + 1966080\beta^2\nu^2\alpha^3 - 430080\beta^2\delta^2\alpha^3 + 430080\beta^4\alpha\eta^2 - 860160\beta^2\mu^2\alpha^3 \\
& -1044480\beta^4\alpha\mu^2 - 1305600\mu^2\alpha^3 + 61440\beta^4\alpha\delta^2 - 307200\delta^3\beta^2\nu^2 \\
& -61440\beta^4\alpha\delta^2\sqrt{5} - 122880\beta^2\alpha\delta^4\sqrt{5} - 215040\alpha^7\sqrt{5} - 92160\beta^2\delta\sqrt{5}\eta^2\alpha^2 \\
& + 122880\beta^2\delta\sqrt{5}\alpha^2\mu^2 + 122880\beta^2\delta\sqrt{5}\alpha^2\nu^2 + 122880\beta^2\alpha\eta^2\delta^2\sqrt{5} \\
& + 368640\beta^2\alpha\mu^2\delta^2\sqrt{5} + 368640\beta^2\sqrt{5}\alpha\delta^2\nu^2, -768 + 23040\beta^2\nu^2\alpha + 4160\delta \\
& -26880\nu^2\alpha + 3840\alpha + 24320\delta\eta^2\alpha^2 - 15360\delta\eta^2\nu^2 - 7680\delta\eta^2\beta^2 + 16640\delta\alpha^2 \\
& + 25600\nu^4\alpha - 15360\delta\beta^4 + 1280\delta\alpha^2\beta^2 + 7680\alpha^3 + 8320\delta^3 + 3840\alpha\beta^2 \\
& -3840\beta^4\alpha - 16640\delta\nu^2 + 16640\delta^3\alpha^2 + 16640\delta\alpha^4 - 15360\delta^3\beta^2 + 960\delta\sqrt{5} \\
& -33280\alpha^2\delta\nu^2 + 640\delta\beta^2 - 33280\alpha^2\delta\mu^2 + 30720\beta^2\delta\mu^2 - 7680\delta\sqrt{5}\eta^2\beta^2 \\
& + 15360\delta\sqrt{5}\eta^2\nu^2 + 15360\delta\sqrt{5}\eta^2\mu^2 - 23040\sqrt{5}\alpha\mu^2\delta^2 + 276480\sqrt{5}\alpha\nu^2\mu^2 \\
& -115200\sqrt{5}\alpha\beta^2\mu^2 + 38400\sqrt{5}\alpha\eta^2\beta^2 - 99840\sqrt{5}\alpha\eta^2\mu^2 - 99840\sqrt{5}\alpha\eta^2\nu^2 \\
& -23040\sqrt{5}\alpha\nu^2\delta^2 + 30720\delta\beta^2\nu^2 + 3840\alpha\delta^2 - 7680\delta\sqrt{5}\eta^4 - 7680\delta^3\sqrt{5}\eta^2 \\
& + 34560\sqrt{5}\alpha\eta^4 + 25600\sqrt{5}\alpha\nu^4 + 11520\sqrt{5}\alpha\beta^4 + 148480\sqrt{5}\alpha\mu^4 + 1920\delta^3\sqrt{5} \\
& + 12160\delta\eta^2 - 1920\delta\sqrt{5}\eta^2 + 1920\delta\sqrt{5}\beta^2 - 3840\delta\sqrt{5}\mu^2 - 3840\delta\sqrt{5}\nu^2 \\
& + 84480\sqrt{5}\alpha^3\nu^2 - 38400\sqrt{5}\alpha^3\delta^2 - 38400\sqrt{5}\alpha^3\mu^2 + 11520\sqrt{5}\alpha\nu^2 \\
& -19200\sqrt{5}\alpha\mu^2 - 19200\sqrt{5}\alpha\delta^2 + 3840\delta^3\sqrt{5}\alpha^2 - 11520\sqrt{5}\alpha\beta^2 \\
& -15360\sqrt{5}\alpha^3\beta^2 - 3840\sqrt{5}\alpha\delta^4 + 30720\beta^2\alpha^3 + 9984\alpha^5 - 16640\delta\mu^2 - 1536\delta^5 \\
& + 3840\delta\sqrt{5}\alpha^4 + 7680\sqrt{5}\alpha\beta^2\nu^2 - 5760\sqrt{5}\alpha - 3840\delta\sqrt{5}\eta^2\alpha^2 \\
& + 30720\sqrt{5}\alpha\delta^2\eta^2 - 7680\delta\sqrt{5}\alpha^2\mu^2 - 7680\delta\sqrt{5}\alpha^2\nu^2 + 3840\delta\sqrt{5}\alpha^2\beta^2 \\
& + 3840\alpha\delta^4 + 7680\nu^2\alpha^3 - 26880\sqrt{5}\alpha^5 - 30720\sqrt{5}\alpha^3 + 148480\alpha\mu^4 + 23040\eta^2\delta^3 \\
& -3840\alpha\eta^4 + 7680\delta^2\alpha^3 - 57600\alpha\mu^2 + 3840\delta\sqrt{5}\alpha^2 + 276480\alpha\nu^2\mu^2 \\
& -23040\alpha\eta^2\nu^2 - 23040\alpha\eta^2\mu^2 - 7680\alpha\eta^2\beta^2 - 99840\alpha\beta^2\mu^2 - 15360\eta^2\mu^2\delta \\
& -38400\alpha\nu^2\delta^2 - 38400\alpha\mu^2\delta^2 - 115200\mu^2\alpha^3, 15360\eta^3\mu + 23040\delta^2\eta\mu \\
& + 28160\alpha\mu\eta^3\delta + 28160\alpha\mu\eta\delta^3 - 56320\alpha\mu^3\eta\delta - 23040\mu\sqrt{5}\eta\nu^2 \\
& -7680\mu\sqrt{5}\eta\beta^2 - 7680\mu\sqrt{5}\eta\delta^2 - 15360\mu\sqrt{5}\eta\alpha^2 + 14080\alpha\delta\eta\mu \\
& + 307200\mu^3\sqrt{5}\eta\nu^2 + 51200\mu\sqrt{5}\eta\nu^4 - 84480\eta\mu\nu^2 + 17920\eta\mu\beta^2 \\
& + 28160\alpha^3\mu\eta\delta + 51200\mu^5\sqrt{5}\eta + 97280\eta\mu\nu^2\beta^2 - 76800\eta\mu\nu^2\delta^2 \\
& + 15360\eta\mu\delta^2\alpha^2 + 46080\eta\mu\nu^2\alpha^2 - 23040\mu^3\sqrt{5}\eta - 3840\mu\sqrt{5}\eta \\
& -46080\mu^3\sqrt{5}\eta\alpha^2 - 76800\mu^3\sqrt{5}\eta\beta^2 + 15360\mu\sqrt{5}\eta^3\delta^2 + 15360\mu\sqrt{5}\eta^3\beta^2 \\
& + 76800\eta\mu^3\delta^2 + 51200\eta\mu^5 + 46080\mu^3\sqrt{5}\eta^3 + 15360\mu\sqrt{5}\eta^5 + 15360\beta\nu\alpha^4 \\
& + 15360\eta\mu\delta^4 - 107520\mu\sqrt{5}\eta^3\nu^2 - 56320\eta\mu^3\beta^2 + 307200\eta\mu^3\nu^2 \\
& -46080\eta^3\mu\nu^2 + 51200\eta\mu\nu^4 - 10240\eta\mu\beta^4 + 15360\eta\mu\alpha^4 - 25600\eta^3\mu\beta^2 \\
& -107520\eta\mu^3\alpha^2 - 7680\sqrt{5}\alpha\beta^2\delta\eta\mu + 15360\alpha\mu\eta\sqrt{5}\delta\nu^2 - 84480\eta\mu^3 \\
& -15360\mu\sqrt{5}\eta\alpha^4 + 11520\eta\mu - 56320\delta\eta\mu\nu^2\alpha + 107520\eta^3\mu^3 + 3072\beta^5\nu \\
& -12288\eta^5\mu + 76800\mu^3\sqrt{5}\eta\delta^2 + 30720\eta\mu\alpha^2 - 3840\sqrt{5}\alpha\delta\eta\mu \\
& + 5120\beta^2\eta\mu\alpha^2 - 15360\beta^2\sqrt{5}\eta\mu\alpha^2 - 10240\beta^2\delta^2\eta\mu + 107520\mu\sqrt{5}\eta\nu^2\alpha^2 \\
& -15360\mu\sqrt{5}\eta\delta^2\alpha^2 + 76800\mu\sqrt{5}\eta\nu^2\beta^2 - 7680\alpha^3\mu\eta\sqrt{5}\delta - 7680\alpha\mu\eta\sqrt{5}\delta^3 \\
& -46080\eta^3\mu\delta^2 - 30720\beta^3\nu\alpha^2 - 7680\sqrt{5}\alpha\delta\eta^3\mu - 76800\mu\sqrt{5}\eta\nu^2\delta^2
\end{aligned}$$

$$\begin{aligned}
& + 28160 \alpha \mu \eta \delta \beta^2 + 15360 \sqrt{5} \alpha \eta \mu^3 \delta, 737280 \mu^2 \alpha^2 + 368640 \mu^2 \delta^2 \\
& - 307200 \alpha^2 \nu^4 + 46080 \mu^2 - 307200 \nu^4 \beta^2 - 16384 \nu^6 + 184320 \mu^2 \alpha^2 \beta^2 \\
& + 737280 \mu^2 \delta^2 \alpha^2 - 552960 \mu^2 \nu^2 \alpha^2 + 921600 \mu^2 \nu^2 \beta^2 - 737280 \mu^2 \eta^2 \beta^2 \\
& + 1335296 \mu^6 + 1474560 \mu^2 \eta^2 \nu^2 - 721920 \nu^2 \mu^2 + 276480 \beta^4 \nu^2 \\
& + 138240 \mu^2 \beta^2 \sqrt{5} \nu^2 + 184320 \mu^2 \alpha^2 \sqrt{5} \beta^2 - 506880 \mu^2 \sqrt{5} \nu^2 \alpha^2 \\
& + 276480 \mu^2 \alpha^2 \sqrt{5} \delta^2 - 276480 \mu^2 \eta^2 \sqrt{5} \delta^2 - 1044480 \mu^4 - 276480 \sqrt{5} \nu^2 \mu^2 \\
& - 46080 \beta^2 \sqrt{5} \nu^4 - 15360 \sqrt{5} \nu^4 \alpha^2 + 92160 \mu^4 \beta^2 \sqrt{5} - 522240 \mu^4 \sqrt{5} \alpha^2 \\
& - 276480 \mu^2 \eta^4 \sqrt{5} + 552960 \mu^4 \eta^2 \sqrt{5} + 552960 \mu^2 \eta^2 \sqrt{5} \nu^2 - 276480 \mu^2 \eta^2 \sqrt{5} \beta^2 \\
& + 69120 \sqrt{5} \mu^2 - 69120 \beta^4 \sqrt{5} \mu^2 + 69120 \beta^4 \sqrt{5} \nu^2 + 264960 \sqrt{5} \mu^2 \alpha^2 \\
& + 138240 \delta^2 \sqrt{5} \mu^2 + 276480 \alpha^4 \nu^2 + 103680 \beta^2 \sqrt{5} \mu^2 + 368640 \beta^2 \mu^2 \\
& + 253440 \mu^2 \alpha^4 \sqrt{5} + 23040 \alpha^4 \sqrt{5} \nu^2 + 552960 \alpha^2 \nu^2 \beta^2 + 92160 \alpha^2 \sqrt{5} \beta^2 \nu^2 \\
& + 1474560 \eta^2 \mu^4 - 737280 \eta^4 \mu^2 - 276480 \sqrt{5} \mu^4 - 276480 \mu^2 \beta^4 - 737280 \eta^2 \mu^2 \delta^2 \\
& - 614400 \mu^4 \beta^2 + 1187840 \mu^4 \nu^2 + 81920 \mu^2 \nu^4 + 460800 \mu^2 \alpha^4 - 2088960 \mu^4 \alpha^2, \\
& 613416960 \beta^6 \alpha \delta - 56038440960 \mu^2 \alpha^2 - 26123304960 \mu^2 \delta^2 - 22118400 \beta^6 \\
& + 22316974080 \alpha^2 \nu^4 + 37355520 \delta \eta^2 \alpha - 67584000 \delta \nu^2 \alpha - 540426240 \alpha^4 \delta^2 \\
& + 309657600 \eta^2 \nu^4 + 4432896 + 629145600 \nu^6 \alpha^2 - 1843200000 \beta^2 \nu^2 \alpha \\
& - 8729395200 \beta^4 \nu^2 \alpha - 16957440 \delta + 119439360 \nu^2 \alpha - 131235840 \alpha \\
& + 11737497600 \alpha^2 \mu^2 \nu^2 \beta^2 - 132710400 \eta^2 \beta^2 \nu^2 - 69835161600 \alpha^2 \mu^2 \eta^2 \nu^2 \\
& - 463994880 \alpha^5 \delta \nu^2 - 1022361600 \alpha^3 \delta^3 \beta^2 - 11059200 \delta^2 \beta^4 - 463994880 \alpha^5 \delta \mu^2 \\
& - 82575360 \eta^2 \alpha^2 - 18063360 \eta^2 \beta^2 - 41656320 \eta^2 \delta^2 + 164413440 \nu^2 \delta^2 \\
& - 235929600 \alpha^3 \eta^4 \delta - 231997440 \delta \eta^2 \alpha^2 - 28140503040 \alpha^4 \mu^2 \beta^2 \\
& + 444334080 \delta \eta^2 \nu^2 - 516096000 \delta \eta^2 \beta^2 + 34021048320 \alpha^4 \mu^2 \nu^2 - 5100871680 \mu^2 \\
& - 26542080000 \beta^2 \mu^2 \nu^2 \delta^2 - 1391616 \sqrt{5} - 629145600 \alpha \mu^2 \nu^4 \\
& - 9122611200 \alpha \mu^4 \nu^2 - 234455040 \eta^2 \mu^2 - 68419584000 \beta^2 \mu^2 \eta^2 \nu^2 \\
& + 18874368000 \alpha \eta^4 \mu^2 - 69835161600 \alpha^2 \eta^2 \mu^4 - 37748736000 \alpha \eta^2 \mu^4 \\
& + 2211840 \eta^2 \alpha - 29508894720 \alpha^4 \nu^2 \beta^2 - 3145728000 \alpha^2 \mu^2 \nu^4 \\
& - 165150720 \alpha^3 \eta^2 \delta^3 - 45613056000 \alpha^2 \mu^4 \nu^2 - 25236602880 \alpha^2 \mu^4 \beta^2 \\
& + 849346560 \alpha^4 \eta^2 \beta^2 - 943718400 \alpha^4 \eta^2 \mu^2 - 943718400 \alpha^4 \eta^2 \nu^2 \\
& - 117964800 \alpha^6 \eta^2 + 48168960 \delta \alpha^2 - 707788800 \beta^4 \mu^2 \alpha^2 - 584908800 \nu^4 \alpha \delta \\
& + 15630336000 \beta^4 \mu^2 \nu^2 - 15482880 \delta^2 \sqrt{5} + 2091909120 \nu^4 \alpha + 668712960 \nu^2 \beta^2 \\
& + 5529600 \beta^4 \sqrt{5} + 28753920 \eta^4 \sqrt{5} \beta^2 - 18343526400 \beta^4 \alpha^2 \nu^2 \\
& - 26542080000 \beta^2 \mu^4 \delta^2 + 679034880 \nu^2 \alpha^2 + 31981568000 \beta^2 \mu^2 \nu^4 \\
& + 216006656000 \beta^2 \mu^4 \nu^2 - 68419584000 \beta^2 \eta^2 \mu^4 + 251658240 \alpha \beta^4 \delta^3 \\
& - 152248320 \delta^2 \alpha^2 - 599654400 \alpha \mu^4 \delta - 559595520 \alpha^4 + 66846720 \delta \eta^2 \alpha^3 \\
& - 715653120 \alpha^3 \delta \eta^2 \beta^2 + 3869245440 \nu^4 \alpha \beta^2 - 359792640 \alpha^3 \delta \mu^2 \\
& - 352419840 \alpha^3 \delta \nu^2 + 306708480 \alpha^4 \beta^2 \delta^2 + 471859200 \alpha^2 \beta^2 \eta^4 \\
& - 569180160 \delta \alpha^3 \beta^2 - 37748736000 \alpha \mu^2 \eta^2 \nu^2 + 3637248000 \nu^4 \beta^4 \\
& - 424673280 \alpha^4 \nu^2 \delta^2 + 20358758400 \nu^4 \beta^2 - 188743680 \alpha^2 \beta^4 \eta^2 \\
& - 212336640 \alpha^3 \delta^3 \nu^2 + 471859200 \beta^6 \alpha - 2211840000 \beta^6 \nu^2 + 1226833920 \beta^6 \delta \\
& - 53084160 \delta \beta^4 + 41287680 \beta^4 \alpha^2 - 212336640 \alpha^3 \mu^2 \delta^3 + 11796480 \delta^2 \\
& - 88694784 \alpha^2 - 1269104640 \delta \alpha^2 \beta^2 + 1328578560 \delta \alpha \beta^2 \nu^2 + 235929600 \beta^6 \alpha^2 \\
& + 34283520 \nu^2 - 4608000 \eta^2 - 23592960 \alpha^4 \eta^2 \delta^2 + 207912960 \delta^2 \alpha^2 \beta^2 \\
& + 598671360 \eta^2 \alpha^2 \beta^2 + 250675200 \eta^2 \alpha^2 \delta^2 + 88473600 \delta^2 \eta^2 \beta^2 + 7004160 \beta^4 \\
& - 929267712 \alpha^3 - 29245440 \delta^3 - 574832640 \nu^4 - 31703040 \eta^4 + 365985792 \alpha \beta^2
\end{aligned}$$

$$\begin{aligned}
& + 195747840 \alpha^2 \beta^2 + 216023040 \beta^4 \alpha - 602112000 \alpha \delta^3 \beta^2 - 1195376640 \beta^2 \delta \eta^2 \nu^2 \\
& - 1431306240 \beta^2 \delta \eta^2 \alpha^2 + 1533542400 \alpha^3 \beta^2 \delta \mu^2 + 3067084800 \beta^2 \alpha^2 \delta \nu^2 \\
& - 1164902400 \nu^2 \alpha \delta \mu^2 + 929300480 \nu^6 - 63897600 \alpha \delta \mu^2 - 140328960 \alpha \eta^4 \delta \\
& + 22118400 \eta^2 \delta^4 + 1533542400 \alpha^3 \delta \beta^2 \nu^2 - 39444480 \sqrt{5} \nu^2 - 37355520 \alpha \eta^2 \delta^3 \\
& + 3067084800 \beta^2 \alpha^2 \delta \mu^2 + 141557760 \delta \nu^2 - 53149040640 \mu^2 \alpha^2 \beta^2 \\
& + 302284800 \nu \mu \eta \beta^3 - 70831964160 \mu^2 \delta^2 \alpha^2 + 302284800 \nu \mu \eta^3 \beta \\
& + 90015989760 \mu^2 \nu^2 \alpha^2 - 604569600 \nu^3 \mu \eta \beta - 604569600 \nu \mu^3 \eta \beta \\
& - 47795404800 \mu^2 \nu^2 \beta^2 + 4866048000 \mu^2 \nu^2 \delta^2 + 52442726400 \mu^2 \eta^2 \beta^2 \\
& - 85420933120 \mu^6 + 35241984 \beta^2 - 7372800 \beta^2 \delta^2 + 151142400 \nu \mu \eta \beta \\
& - 98338406400 \mu^2 \eta^2 \nu^2 + 990904320 \alpha^2 \delta \eta^2 \nu^2 + 560824320 \alpha \delta \eta^2 \nu^2 \\
& + 495452160 \alpha^3 \eta^2 \mu^2 \delta + 723517440 \alpha \delta^3 \beta^2 \nu^2 + 723517440 \alpha \delta^3 \beta^2 \mu^2 \\
& - 9584640000 \beta^2 \mu^2 \delta^2 - 150896640 \delta \alpha \beta^2 - 500367360 \alpha \delta \eta^2 \beta^2 \\
& + 279674880 \delta^3 \alpha^2 + 396165120 \delta \alpha^4 + 128778240 \delta^3 \nu^2 - 1018920960 \delta^3 \beta^2 \\
& + 1887436800 \beta^2 \alpha \mu^2 \delta^2 + 1321205760 \beta^2 \alpha \eta^2 \delta^2 + 1887436800 \beta^2 \alpha \nu^2 \delta^2 \\
& - 1195376640 \beta^2 \eta^2 \mu^2 \delta + 19629342720 \beta^2 \alpha \eta^2 \mu^2 + 754974720 \beta^2 \alpha \eta^2 \nu^2 \\
& - 27839692800 \beta^2 \alpha \nu^2 \mu^2 + 912261120 \alpha \beta^4 \delta \eta^2 - 1351680 \delta \sqrt{5} \\
& - 1226833920 \alpha \beta^4 \delta \mu^2 + 11059200 \delta^4 \beta^2 + 77414400 \eta^4 \beta^2 + 238878720 \eta^4 \alpha^2 \\
& + 1321205760 \alpha \beta^2 \delta \mu^2 + 2211840000 \beta^6 \mu^2 - 328335360 \alpha^2 \delta \nu^2 - 367165440 \delta \beta^2 \\
& - 491520000 \mu^4 \delta - 206438400 \beta^4 \alpha \delta - 33423360 \alpha^2 \delta \mu^2 + 1622016000 \beta^6 \mu^2 \sqrt{5} \\
& - 1622016000 \beta^6 \nu^2 \sqrt{5} - 9953280 \delta^4 + 269352960 \delta \alpha^3 + 52838400 \delta \alpha \\
& + 151633920 \alpha \delta^3 - 157286400 \alpha \beta^2 \eta^2 \delta^3 + 207912960 \eta^2 \nu^2 \\
& + 943718400 \alpha^2 \beta^2 \delta^2 \nu^2 - 587464704 \alpha^8 - 895057920 \alpha^6 + 54929817600 \nu^2 \mu^2 \\
& - 35342254080 \alpha^4 \mu^2 \delta^2 + 34917580800 \alpha^2 \eta^4 \mu^2 + 582942720 \alpha \eta^2 \mu^2 \delta \\
& + 660602880 \alpha^2 \beta^2 \eta^2 \delta^2 - 797736960 \alpha^2 \nu^2 \delta^2 - 1251901440 \eta^2 \alpha^2 \nu^2 \\
& + 377487360 \alpha^2 \beta^2 \eta^2 \nu^2 + 1627914240 \beta^2 \delta \mu^2 - 17827430400 \beta^4 \nu^2 \\
& - 19995033600 \beta^2 \mu^2 \delta^2 \alpha^2 + 990904320 \alpha^2 \eta^2 \mu^2 \delta - 25559040 \delta \sqrt{5} \eta^2 \beta^2 \\
& + 88473600 \delta \sqrt{5} \eta^2 \nu^2 + 88473600 \delta \sqrt{5} \eta^2 \mu^2 - 908328960 \sqrt{5} \alpha \mu^2 \delta^2 \\
& + 20938752000 \beta^2 \eta^2 \mu^2 \delta^2 - 996802560 \sqrt{5} \alpha \nu^2 \mu^2 + 477757440 \sqrt{5} \alpha \beta^2 \mu^2 \\
& - 141557760 \alpha^4 \delta^4 + 92897280 \sqrt{5} \alpha \eta^2 \beta^2 + 194641920 \sqrt{5} \alpha \eta^2 \mu^2 \\
& + 194641920 \sqrt{5} \alpha \eta^2 \nu^2 + 153354240 \sqrt{5} \alpha \nu^2 \delta^2 + 1333002240 \delta \beta^2 \nu^2 \\
& + 184320000 \delta^2 \alpha \beta^2 - 273530880 \alpha \delta^2 - 10690560 \delta^4 \sqrt{5} - 154828800 \eta^4 \nu^2 \\
& + 11796480 \alpha^2 \delta^4 - 19906560 \delta \sqrt{5} \eta^4 - 98795520 \delta^3 \sqrt{5} \eta^2 - 2453667840 \delta \beta^4 \nu^2 \\
& - 2093875200 \mu^2 \beta^2 \sqrt{5} \nu^2 - 25433210880 \mu^2 \alpha^2 \sqrt{5} \beta^2 + 47772794880 \mu^2 \sqrt{5} \nu^2 \alpha^2 \\
& - 25161891840 \mu^2 \alpha^2 \sqrt{5} \delta^2 + 19509903360 \mu^2 \eta^2 \sqrt{5} \delta^2 + 14745600 \sqrt{5} \alpha \eta^4 \\
& + 298844160 \sqrt{5} \alpha \nu^4 + 388546560 \sqrt{5} \alpha \beta^4 - 196608000 \delta \nu^4 + 186777600 \sqrt{5} \alpha \mu^4 \\
& - 2136637440 \eta^2 \alpha^2 \mu^2 + 72853340160 \mu^4 + 20938752000 \beta^2 \eta^4 \mu^2 \\
& + 20938752000 \beta^4 \mu^2 \eta^2 + 11796480 \delta^3 \sqrt{5} \mu^2 + 11796480 \delta^3 \sqrt{5} \nu^2 \\
& + 103710720 \alpha \delta^3 \nu^2 - 471859200 \alpha^2 \eta^4 \delta - 927989760 \alpha^4 \delta \mu^2 - 927989760 \alpha^4 \delta \nu^2 \\
& - 424673280 \alpha^2 \delta^3 \nu^2 - 31457280 \alpha^4 \delta \eta^2 + 20031897600 \sqrt{5} \nu^2 \mu^2 \\
& + 3204710400 \beta^2 \sqrt{5} \nu^4 + 910295040 \sqrt{5} \nu^4 \alpha^2 + 1110835200 \mu^4 \beta^2 \sqrt{5} \\
& + 48860037120 \mu^4 \sqrt{5} \alpha^2 + 19275448320 \mu^2 \eta^4 \sqrt{5} - 38603980800 \mu^4 \eta^2 \sqrt{5} \\
& - 38825164800 \mu^2 \eta^2 \sqrt{5} \nu^2 + 19934576640 \mu^2 \eta^2 \sqrt{5} \beta^2 - 849346560 \alpha^3 \nu^2 \delta^2 \\
& + 1698693120 \alpha^3 \eta^2 \beta^2 - 1887436800 \alpha^3 \eta^2 \mu^2 - 1887436800 \alpha^3 \eta^2 \nu^2 \\
& + 29821501440 \alpha^3 \nu^2 \mu^2 - 330301440 \alpha^2 \eta^2 \delta^3 - 221184000 \eta^2 \sqrt{5} \nu^4
\end{aligned}$$

$$\begin{aligned}
& - 424673280 \alpha^2 \mu^2 \delta^3 + 84049920 \eta^4 \sqrt{5} \nu^2 + 125829120 \alpha \delta^3 \mu^2 + 8847360 \eta^4 \sqrt{5} \delta^2 \\
& + 91422720 \eta^4 \sqrt{5} \alpha^2 - 4931297280 \sqrt{5} \mu^2 - 5652480 \delta^3 \sqrt{5} + 1875640320 \beta^4 \sqrt{5} \mu^2 \\
& - 4789370880 \beta^4 \sqrt{5} \nu^2 - 19723714560 \alpha^3 \mu^2 \delta^2 - 47185920 \alpha^3 \eta^2 \delta^2 \\
& + 4276224 \beta^2 \sqrt{5} - 45711360 \eta^2 \sqrt{5} \alpha^2 \nu^2 + 97320960 \eta^2 \sqrt{5} \delta^2 \nu^2 \\
& + 17694720 \eta^2 \sqrt{5} \delta^2 \beta^2 - 231505920 \alpha^4 \sqrt{5} - 66945024 \alpha^2 \sqrt{5} \\
& - 3574333440 \beta^2 \sqrt{5} \mu^2 \delta^2 + 6635520 \eta^2 \sqrt{5} \beta^4 + 22118400 \eta^6 - 108134400 \delta \eta^2 \\
& - 930447360 \eta^2 \sqrt{5} \alpha^2 \mu^2 - 9142272000 \alpha^6 \mu^2 \sqrt{5} + 19472056320 \alpha^4 \mu^4 \sqrt{5} \\
& + 79626240 \eta^2 \sqrt{5} \nu^2 \beta^2 + 1081344000 \beta^4 \sqrt{5} \nu^4 - 2162688000 \beta^4 \mu^4 \sqrt{5} \\
& + 1447034880 \beta^2 \mu^2 \delta^3 + 197591040 \eta^2 \sqrt{5} \alpha^2 \beta^2 - 398131200 \mu^4 \delta^2 \sqrt{5} \\
& + 55050240 \alpha^4 \sqrt{5} \nu^4 + 35295068160 \alpha^2 \mu^2 \eta^2 \beta^2 - 21669396480 \sqrt{5} \mu^2 \alpha^2 \\
& + 34917580800 \alpha^2 \eta^2 \mu^2 \delta^2 + 33177600 \eta^2 \beta^4 - 205701120 \eta^2 \alpha^4 \\
& - 24821760 \delta \sqrt{5} \eta^2 - 64143360 \delta \sqrt{5} \beta^2 + 2846883840 \alpha^3 \sqrt{5} \mu^4 + 5406720 \delta \sqrt{5} \mu^2 \\
& + 5406720 \delta \sqrt{5} \nu^2 - 951582720 \alpha^3 \sqrt{5} \nu^4 + 908328960 \alpha^5 \sqrt{5} \nu^2 \\
& - 188743680 \alpha^5 \sqrt{5} \delta^2 - 766771200 \alpha^5 \sqrt{5} \mu^2 + 110100480 \alpha^4 \sqrt{5} \delta^3 \\
& - 387072000 \beta^2 \sqrt{5} \nu^2 - 10029957120 \delta^2 \sqrt{5} \mu^2 - 102481920 \delta^2 \sqrt{5} \nu^2 \\
& - 241827840 \alpha^4 \beta^2 - 18135613440 \alpha^4 \nu^2 + 24330240 \eta^4 \sqrt{5} + 94371840 \beta^6 \sqrt{5} \alpha \\
& - 92897280 \eta^2 \sqrt{5} \nu^2 - 535265280 \eta^2 \sqrt{5} \mu^2 + 868220928 \beta^2 \sqrt{5} \alpha^5 \\
& - 105062400 \alpha^2 \sqrt{5} \delta^2 - 152616960 \sqrt{5} \nu^2 \alpha^2 - 70041600 \eta^2 \sqrt{5} \alpha^4 \\
& - 4423680 \delta^4 \sqrt{5} \eta^2 + 6635520 \delta^4 \sqrt{5} \beta^2 - 22118400 \delta^4 \sqrt{5} \nu^2 - 9392947200 \beta^2 \sqrt{5} \mu^2 \\
& + 4792320 \eta^2 \sqrt{5} \delta^2 + 32071680 \eta^2 \sqrt{5} \beta^2 + 8847360000 \beta^4 \mu^2 \eta^2 \sqrt{5} \\
& + 4164157440 \alpha^2 \beta^2 \sqrt{5} \nu^4 + 11065098240 \alpha^2 \mu^4 \beta^2 \sqrt{5} + 131604480 \alpha^2 \sqrt{5} \beta^2 \\
& - 21233664000 \alpha^2 \mu^4 \eta^2 \sqrt{5} - 10286530560 \alpha^4 \mu^2 \sqrt{5} \delta^2 + 18001428480 \alpha^4 \mu^2 \sqrt{5} \nu^2 \\
& - 7207649280 \beta^4 \alpha^2 \sqrt{5} \nu^2 - 17694720000 \beta^2 \mu^4 \eta^2 \sqrt{5} - 32350371840 \beta^2 \mu^2 \\
& - 3244032000 \beta^4 \mu^2 \sqrt{5} \nu^2 - 286654464 \alpha^6 \sqrt{5} + 33669120 \sqrt{5} \alpha \delta^3 \\
& - 353894400 \mu^2 \delta^2 \sqrt{5} \nu^2 - 113541120 \alpha^3 \eta^2 + 566231040 \beta^2 \delta^3 \sqrt{5} \nu^2 \\
& + 566231040 \beta^2 \delta^3 \sqrt{5} \mu^2 - 188743680 \beta^2 \delta^3 \sqrt{5} \eta^2 - 94371840 \sqrt{5} \alpha^6 \delta^2 \\
& + 2984509440 \beta^4 \sqrt{5} \alpha \mu^2 + 302284800 \alpha^2 \mu \eta \beta \nu - 3739484160 \beta^4 \sqrt{5} \alpha \nu^2 \\
& + 47185920 \sqrt{5} \alpha^2 \beta^6 - 47185920 \alpha^2 \delta^3 \sqrt{5} \eta^2 - 3350200320 \beta^2 \sqrt{5} \alpha \mu^4 \\
& - 3686400 \nu \mu \eta \beta \sqrt{5} + 3373793280 \beta^2 \sqrt{5} \alpha \nu^4 - 5945425920 \beta^2 \sqrt{5} \alpha^3 \nu^2 \\
& - 1462763520 \alpha^3 \sqrt{5} \mu^2 \delta^2 + 39321600 \alpha^6 \sqrt{5} \delta + 377487360 \alpha^3 \sqrt{5} \eta^2 \mu^2 \\
& + 967311360 \alpha^3 \sqrt{5} \nu^2 \mu^2 + 377487360 \alpha^3 \sqrt{5} \eta^2 \nu^2 - 141557760 \alpha^2 \delta^3 \sqrt{5} \mu^2 \\
& - 141557760 \alpha^2 \delta^3 \sqrt{5} \nu^2 - 47185920 \alpha^2 \delta \sqrt{5} \eta^4 + 37748736 \sqrt{5} \alpha^3 \delta^5 \\
& + 660602880 \alpha^3 \sqrt{5} \nu^2 \delta^2 - 265420800 \sqrt{5} \alpha^3 \nu^2 - 92897280 \sqrt{5} \alpha^3 \delta^2 \\
& - 41287680 \sqrt{5} \alpha^3 \mu^2 + 143032320 \sqrt{5} \alpha \nu^2 + 171048960 \sqrt{5} \alpha \mu^2 + 737280 \sqrt{5} \alpha \delta^2 \\
& + 43745280 \delta^3 \sqrt{5} \alpha^2 + 61931520 \eta^2 \sqrt{5} \alpha^2 \delta^2 - 94371840 \alpha^2 \beta^2 \eta^2 \delta^2 \sqrt{5} \\
& - 566231040 \alpha^2 \beta^2 \delta^2 \nu^2 \sqrt{5} + 382205952 \sqrt{5} \alpha \beta^2 + 850821120 \sqrt{5} \alpha^3 \beta^2 \\
& + 44236800 \eta^4 \delta^2 - 270336000 \delta^3 \sqrt{5} \beta^2 + 35389440 \sqrt{5} \alpha \delta^4 - 58982400 \alpha \delta^3 \sqrt{5} \mu^2 \\
& - 41533440 \sqrt{5} \alpha \delta \nu^2 - 136642560 \sqrt{5} \alpha \delta^3 \beta^2 + 94863360 \sqrt{5} \alpha^3 \delta^3 \\
& + 29638656 \sqrt{5} \alpha \delta^5 - 28184739840 \mu^2 \alpha^4 \sqrt{5} - 24576000 \sqrt{5} \alpha \delta \beta^4 \\
& - 11065098240 \alpha^6 \nu^2 + 9461760 \sqrt{5} \alpha \delta - 48660480 \beta^2 \alpha^3 - 128532480 \eta^4 \delta \\
& - 1357037568 \alpha^5 + 215285760 \delta \mu^2 - 51609600 \sqrt{5} \alpha \eta^2 \delta^3 - 2211840 \beta^4 \sqrt{5} \delta^2 \\
& + 193167360 \alpha^2 \sqrt{5} \beta^4 - 765296640 \alpha^4 \sqrt{5} \nu^2 - 195379200 \alpha^4 \sqrt{5} \delta^2 \\
& + 141557760 \sqrt{5} \alpha \beta^6 \delta - 1393459200 \mu^2 \delta^4 + 18677760 \delta \sqrt{5} \beta^4 + 13320192 \delta^5
\end{aligned}$$



$$\begin{aligned}
& + 33914880 \delta \sqrt{5} \alpha^4 - 4246732800 \alpha \mu^4 \eta^2 \sqrt{5} + 4644864000 \mu^4 \delta^2 \\
& - 47185920 \alpha^5 \eta^2 \sqrt{5} - 37340282880 \alpha^2 \nu^2 \beta^2 - 39321600 \alpha^2 \sqrt{5} \delta \beta^4 \\
& + 302284800 \delta^2 \beta \nu \eta \mu + 14745600 \sqrt{5} \alpha \nu^4 \delta - 11796480 \alpha^2 \delta^4 \sqrt{5} \\
& - 19660800 \sqrt{5} \alpha^3 \delta \beta^4 + 199065600 \mu^2 \delta^4 \sqrt{5} + 44236800 \nu^4 \delta^2 \sqrt{5} \\
& - 47185920 \alpha^4 \delta^4 \sqrt{5} + 330301440 \sqrt{5} \alpha^4 \nu^2 \delta^2 - 23592960 \sqrt{5} \alpha^3 \eta^2 \delta^3 \\
& - 23592960 \alpha^6 \eta^2 \sqrt{5} + 5945425920 \alpha^3 \nu^4 - 2376990720 \sqrt{5} \alpha \beta^2 \nu^2 \\
& + 42372956160 \alpha^3 \mu^4 - 3019898880 \alpha^5 \nu^2 + 217055232 \alpha^2 \delta^5 + 463994880 \alpha^6 \delta \\
& + 676331520 \alpha^4 \delta^3 + 13589544960 \alpha^4 \nu^4 - 7299072 \sqrt{5} \alpha - 13967032320 \alpha^5 \mu^2 \\
& - 754974720 \alpha^5 \delta^2 + 235929600 \alpha^3 \eta^4 + 47185920 \alpha^3 \sqrt{5} \eta^4 - 65372160 \delta \sqrt{5} \eta^2 \alpha^2 \\
& + 50135040 \sqrt{5} \alpha \delta^2 \eta^2 - 737280 \beta^2 \sqrt{5} \delta^2 - 31703040 \eta^2 \sqrt{5} \alpha^2 \\
& + 283115520 \beta^6 \delta \sqrt{5} + 1061683200 \beta^4 \sqrt{5} \alpha^3 - 6996787200 \alpha^2 \sqrt{5} \beta^2 \nu^2 \\
& + 221184000 \nu^4 \delta^2 + 190709760 \sqrt{5} \alpha \beta^2 \delta \mu^2 + 1474560 \alpha^2 \sqrt{5} \delta^2 \nu^2 \\
& + 42762240 \sqrt{5} \alpha^3 \delta + 57507840 \sqrt{5} \alpha^5 \delta - 35389440 \beta^2 \sqrt{5} \nu^2 \delta^2 \\
& + 245760000 \delta \sqrt{5} \beta^2 \mu^2 + 245760000 \delta \sqrt{5} \beta^2 \nu^2 - 28508160 \delta \sqrt{5} \alpha^2 \mu^2 \\
& - 28508160 \delta \sqrt{5} \alpha^2 \nu^2 - 42762240 \sqrt{5} \alpha^3 \eta^2 - 342097920 \sqrt{5} \alpha^6 \nu^2 \\
& + 19660800 \sqrt{5} \alpha^7 \delta + 8847360000 \beta^2 \mu^2 \eta^4 \sqrt{5} - 2017198080 \beta^4 \mu^2 \alpha^2 \sqrt{5} \\
& - 14338621440 \alpha^4 \mu^2 \sqrt{5} \beta^2 - 15728640 \alpha^5 \delta \eta^2 - 153354240 \alpha^3 \delta \beta^4 \\
& - 534773760 \alpha^5 \delta \beta^2 - 1032192 \sqrt{5} \delta^5 - 7484866560 \alpha^4 \sqrt{5} \beta^2 \nu^2 \\
& + 10616832000 \alpha^2 \mu^2 \eta^4 \sqrt{5} + 23592960 \sqrt{5} \alpha^4 \eta^4 + 434110464 \sqrt{5} \alpha^6 \beta^2 \\
& - 283115520 \beta^2 \delta \sqrt{5} \alpha^4 - 534773760 \beta^2 \delta^3 \sqrt{5} \alpha^2 - 1132462080 \beta^4 \sqrt{5} \alpha \eta^2 \\
& - 566231040 \beta^4 \delta \sqrt{5} \nu^2 + 13271040 \eta^6 \sqrt{5} - 566231040 \beta^4 \delta \sqrt{5} \mu^2 \\
& + 377487360 \beta^2 \delta \sqrt{5} \eta^4 + 566231040 \beta^4 \delta \sqrt{5} \eta^2 - 269844480 \delta \sqrt{5} \alpha^2 \beta^2 \\
& - 8847360 \beta^6 \sqrt{5} - 47185920 \alpha^3 \sqrt{5} \eta^2 \delta^2 + 377487360 \alpha^3 \sqrt{5} \eta^2 \beta^2 \\
& - 31457280 \alpha^4 \delta \sqrt{5} \eta^2 - 896532480 \beta^2 \sqrt{5} \alpha^3 \mu^2 + 1179648000 \beta^2 \sqrt{5} \alpha^3 \delta^2 \\
& - 566231040 \beta^2 \sqrt{5} \alpha \eta^4 + 585400320 \beta^2 \sqrt{5} \alpha \delta^2 + 11796480 \alpha \delta^4 \\
& - 162791424 \sqrt{5} \alpha^8 - 98648064000 \eta^2 \mu^4 + 91422720 \nu^2 \alpha^3 + 128778240 \mu^2 \delta^3 \\
& - 9584640 \sqrt{5} \alpha \eta^2 + 50208768000 \eta^4 \mu^2 - 283115520 \alpha^3 \delta^4 - 235929600 \alpha^5 \eta^2 \\
& - 94371840 \alpha^3 \delta^4 \sqrt{5} + 397393920 \sqrt{5} \nu^4 + 19964805120 \sqrt{5} \mu^4 - 238436352 \sqrt{5} \alpha^5 \\
& - 143327232 \sqrt{5} \alpha^3 + 75497472 \alpha^2 \sqrt{5} \delta^5 + 334725120 \alpha^2 \sqrt{5} \delta^2 \beta^2 \\
& + 22086942720 \alpha \mu^4 - 283115520 \sqrt{5} \alpha \beta^2 \delta \eta^2 \nu^2 - 66355200 \alpha \delta^3 \sqrt{5} \nu^2 \\
& - 264241152 \beta^2 \sqrt{5} \delta^5 - 7372800 \delta^2 \sqrt{5} \beta \nu \eta \mu - 178913280 \eta^2 \delta^3 \\
& - 78888960 \sqrt{5} \alpha \eta^4 \delta - 37847040 \sqrt{5} \alpha \delta \mu^2 - 20889600 \sqrt{5} \alpha \eta^2 \delta \\
& - 95354880 \sqrt{5} \alpha^3 \delta \mu^2 - 102727680 \sqrt{5} \alpha^3 \delta \nu^2 - 49643520 \sqrt{5} \alpha^3 \eta^2 \delta \\
& + 16188702720 \nu^4 \alpha^2 \beta^2 - 1174929408 \alpha^7 + 563281920 \alpha \eta^4 \\
& - 283115520 \sqrt{5} \alpha \beta^4 \delta \mu^2 - 267386880 \sqrt{5} \alpha^3 \beta^2 \delta^3 - 141557760 \sqrt{5} \alpha^5 \beta^2 \delta \\
& - 28753920 \sqrt{5} \alpha \beta^2 \delta - 283115520 \sqrt{5} \alpha^2 \beta^2 \eta^4 + 589824000 \sqrt{5} \alpha^4 \beta^2 \delta^2 \\
& - 132120576 \sqrt{5} \alpha \beta^2 \delta^5 - 128286720 \sqrt{5} \alpha^3 \beta^2 \delta - 566231040 \sqrt{5} \alpha^2 \beta^4 \eta^2 \\
& - 924549120 \delta^2 \alpha^3 + 443351040 \delta \alpha^5 + 2123366400 \alpha \mu^2 \eta^4 \sqrt{5} \\
& - 23592960 \sqrt{5} \alpha^4 \eta^2 \delta^2 - 3227811840 \alpha \mu^2 + 8994816000 \mu^2 \beta^4 + 472350720 \delta^3 \alpha^3 \\
& - 78643200 \alpha^4 \sqrt{5} \delta \mu^2 - 78643200 \alpha^4 \sqrt{5} \delta \nu^2 + 377487360 \sqrt{5} \alpha^2 \beta^2 \eta^2 \nu^2 \\
& + 188743680 \sqrt{5} \alpha^4 \eta^2 \nu^2 + 188743680 \sqrt{5} \alpha^4 \eta^2 \mu^2 + 188743680 \sqrt{5} \alpha^4 \eta^2 \beta^2 \\
& - 39321600 \sqrt{5} \alpha^5 \delta \mu^2 - 39321600 \sqrt{5} \alpha^5 \delta \nu^2 - 15728640 \sqrt{5} \alpha^5 \eta^2 \delta \\
& - 23592960 \sqrt{5} \alpha^3 \eta^4 \delta - 7372800 \nu \mu \eta \beta^3 \sqrt{5} + 503316480 \beta^4 \delta^3 - 1290240 \sqrt{5} \eta^2
\end{aligned}$$

$$\begin{aligned}
& + 91176960 \alpha \delta^5 + 302284800 \alpha^4 \sqrt{5} \beta^2 - 283115520 \sqrt{5} \alpha \beta^2 \eta^2 \mu^2 \delta \\
& - 94371840 \sqrt{5} \alpha \beta^2 \eta^2 \delta^3 + 4423680 \delta \sqrt{5} \alpha^2 + 48815308800 \eta^2 \mu^2 \delta^2 \\
& + 20024524800 \mu^4 \beta^2 + 20938752000 \alpha \nu^2 \mu^2 - 1958215680 \alpha \eta^2 \nu^2 \\
& - 1958215680 \alpha \eta^2 \mu^2 - 884736000 \nu^2 \delta \mu^2 - 87693721600 \mu^4 \nu^2 \\
& + 1018920960 \alpha \eta^2 \beta^2 - 15656550400 \mu^2 \nu^4 - 5795020800 \alpha \beta^2 \mu^2 \\
& - 17694720000 \beta^2 \mu^2 \eta^2 \sqrt{5} \nu^2 + 8788377600 \alpha^2 \mu^2 \beta^2 \sqrt{5} \nu^2 + 444334080 \eta^2 \mu^2 \delta \\
& + 10994319360 \alpha^2 \mu^2 \eta^2 \sqrt{5} \beta^2 - 69419335680 \mu^2 \alpha^4 - 855244800 \alpha \nu^2 \delta^2 \\
& + 1722286080 \beta^4 \alpha^3 + 575078400 \alpha \eta^2 \delta^2 - 10292428800 \alpha \mu^2 \delta^2 \\
& - 21233664000 \alpha^2 \mu^2 \eta^2 \sqrt{5} \nu^2 + 301989888 \beta^2 \alpha^5 - 704643072 \beta^2 \delta^5 \\
& + 141557760 \alpha^2 \delta \sqrt{5} \eta^2 \mu^2 + 141557760 \alpha^2 \delta \sqrt{5} \eta^2 \nu^2 - 66355200 \delta^4 \nu^2 \\
& + 530841600 \sqrt{5} \alpha^4 \beta^4 - 566231040 \beta^2 \delta \sqrt{5} \eta^2 \nu^2 - 566231040 \beta^2 \delta \sqrt{5} \eta^2 \mu^2 \\
& + 2878341120 \beta^2 \sqrt{5} \alpha \eta^2 \mu^2 + 754974720 \beta^2 \sqrt{5} \alpha \eta^2 \nu^2 - 2949120000 \beta^2 \sqrt{5} \alpha \nu^2 \mu^2 \\
& - 82968576000 \beta^4 \mu^4 + 123758182400 \beta^2 \mu^6 - 2857369600 \beta^2 \nu^6 \\
& - 306708480 \beta^4 \delta \alpha^2 + 1824522240 \beta^4 \delta \eta^2 - 1069547520 \beta^2 \delta \alpha^4 \\
& - 2044723200 \beta^2 \delta^3 \alpha^2 + 151879680 \sqrt{5} \alpha \delta \eta^2 \nu^2 - 7372800 \alpha^2 \mu \eta \sqrt{5} \beta \nu \\
& - 377487360 \alpha^6 \delta^2 - 104202240 \sqrt{5} \alpha \delta \eta^2 \beta^2 - 2453667840 \beta^4 \delta \mu^2 \\
& + 117964800 \alpha^4 \eta^4 + 692060160 \beta^2 \eta^4 \delta + 183336960 \sqrt{5} \alpha \delta \beta^2 \nu^2 \\
& + 159252480 \sqrt{5} \alpha \eta^2 \mu^2 \delta + 377487360 \beta^2 \alpha \delta^4 + 943718400 \beta^2 \alpha \eta^4 \\
& - 314572800 \beta^2 \eta^2 \delta^3 - 10758389760 \beta^2 \alpha \mu^4 - 16373514240 \beta^2 \nu^2 \alpha^3 \\
& + 613416960 \beta^2 \delta^2 \alpha^3 - 377487360 \beta^4 \alpha \eta^2 - 6086983680 \beta^2 \mu^2 \alpha^3 \\
& + 12504268800 \beta^4 \alpha \mu^2 - 16624189440 \mu^2 \alpha^3 + 55050240 \sqrt{5} \alpha^5 \delta^3 \\
& - 283115520 \sqrt{5} \alpha \beta^4 \delta \nu^2 + 283115520 \sqrt{5} \alpha \beta^4 \delta \eta^2 + 322437120 \sqrt{5} \alpha^3 \beta^2 \delta \mu^2 \\
& + 108527616 \alpha^3 \delta^5 - 7372800 \nu \mu \eta^3 \beta \sqrt{5} + 150994944 \alpha^6 \beta^2 \\
& + 14745600 \nu^3 \mu \eta \beta \sqrt{5} + 14745600 \nu \mu^3 \eta \beta \sqrt{5} + 188743680 \sqrt{5} \alpha \beta^2 \eta^4 \delta \\
& - 597688320 \alpha \beta^2 \delta \eta^2 \nu^2 + 861143040 \alpha^4 \beta^4 + 338165760 \alpha^5 \delta^3 + 231997440 \alpha^7 \delta \\
& - 471859200 \alpha^2 \beta^4 \delta^2 + 188743680 \alpha^2 \beta^2 \delta^4 + 70778880 \sqrt{5} \alpha^3 \delta \eta^2 \nu^2 \\
& - 55050240 \sqrt{5} \alpha^3 \delta \eta^2 \beta^2 - 943718400 \beta^4 \alpha \delta^2 + 192175472640 \mu^4 \alpha^2 \\
& + 1447034880 \delta^3 \beta^2 \nu^2 + 322437120 \sqrt{5} \alpha^3 \delta \beta^2 \nu^2 - 221184000 \delta^2 \eta^2 \nu^2 \\
& + 283115520 \alpha \beta^2 \delta^3 \sqrt{5} \nu^2 + 283115520 \alpha \beta^2 \delta^3 \sqrt{5} \mu^2 + 70778880 \sqrt{5} \alpha^3 \eta^2 \mu^2 \delta \\
& - 70778880 \sqrt{5} \alpha^3 \delta^3 \nu^2 - 70778880 \sqrt{5} \alpha^3 \delta^3 \mu^2 - 1226833920 \alpha \beta^4 \delta \nu^2 \\
& - 188743680 \beta^4 \alpha \delta^2 \sqrt{5} + 377487360 \beta^2 \alpha \delta^4 \sqrt{5} + 93380935680 \alpha^4 \mu^4 \\
& + 2123366400 \alpha \mu^2 \eta^2 \sqrt{5} \delta^2 - 4246732800 \alpha \mu^2 \eta^2 \sqrt{5} \nu^2 - 22908764160 \alpha^6 \mu^2 \\
& + 495452160 \alpha^3 \delta \eta^2 \nu^2 + 14745600 \sqrt{5} \alpha \nu^2 \delta \mu^2 - 51275366400 \alpha^2 \mu^6 \\
& + 346030080 \alpha \beta^2 \eta^4 \delta - 352321536 \alpha \beta^2 \delta^5 - 325582848 \alpha^7 \sqrt{5} \\
& - 94371840 \alpha^2 \beta^4 \delta^2 \sqrt{5} + 188743680 \alpha^2 \beta^2 \delta^4 \sqrt{5} - 597688320 \alpha \beta^2 \eta^2 \mu^2 \delta \\
& + 8847360000 \beta^2 \mu^2 \eta^2 \sqrt{5} \delta^2 - 9413591040 \beta^2 \mu^2 \alpha^2 \sqrt{5} \delta^2 \\
& + 10616832000 \alpha^2 \mu^2 \eta^2 \sqrt{5} \delta^2 - 10255073280 \alpha \mu^6 + 125829120 \alpha \nu^6 \\
& + 18874368000 \alpha \eta^2 \mu^2 \delta^2 - 110100480 \beta^2 \delta \sqrt{5} \eta^2 \alpha^2 + 644874240 \beta^2 \delta \sqrt{5} \alpha^2 \mu^2 \\
& + 644874240 \beta^2 \delta \sqrt{5} \alpha^2 \nu^2 - 188743680 \beta^2 \alpha \eta^2 \delta^2 \sqrt{5} - 1132462080 \beta^2 \alpha \mu^2 \delta^2 \sqrt{5} \\
& - 1132462080 \beta^2 \sqrt{5} \alpha \delta^2 \nu^2]
\end{aligned}$$

```
> R := convert(map(L->reduce(L, [radic^2-5, 72*nu^4-60*nu^2+7,
6*mu^2-5+6*nu^2, 12*beta^2-12*eta^2+15*nu^2-13,
12*alpha^2+12*eta^2-3*nu^2+1, 6*delta^2+6*eta^2-6*nu^2-1],
termorder(poly_algebra(delta,alpha,beta,eta,mu,nu,radic,alg_relations=
{radic^2-5
}),plex(delta,alpha,beta,eta,mu,nu,radic))),subs(sqrt(5)=radic,Cond_6
)),set);
```

$$\begin{aligned}
R := \{ & -1, 0, 1, 27648 \nu \beta \eta^4 \delta - 3456 \beta \nu \alpha \delta \eta^2 + 2016 \beta \nu^3 \alpha \delta + 10728 \beta \nu \alpha - 120 \beta \nu \alpha \delta \\
& - 14112 \beta \nu^3 \alpha - 34560 \eta^4 \nu \beta + 2520 \eta \mu \nu^2 + 27648 \beta \nu \alpha \eta^4 \delta - 5616 \nu \beta \eta^2 \\
& - 20736 \beta \nu^3 \alpha \delta \eta^2 - 1440 \mu \eta^3 + 55296 \eta^4 \nu^3 \beta - 27648 \eta^6 \nu \beta + 2016 \nu^3 \beta \delta \\
& - 8640 \eta^3 \mu \nu^2 + 27648 \nu \beta \alpha \eta^4 - 3456 \nu \beta \delta \eta^2 - 120 \nu \beta \delta - 48384 \nu^3 \beta \alpha \eta^2 \\
& - 20736 \nu^3 \beta \delta \eta^2 + 39168 \eta^2 \alpha \nu \beta - 150 \eta \mu + 1087 \nu \beta - 378 \nu^3 \beta + 9216 \nu^3 \beta \eta^2 \\
& + 6912 \eta^5 \mu, 1152 \eta^4 \nu \beta - 1440 \nu^3 \beta \eta^2 + 1056 \nu \beta \eta^2 - 180 \nu^3 \beta + 117 \nu \beta \\
& + 1152 \eta^5 \mu - 1440 \eta^3 \mu \nu^2 - 240 \mu \eta^3 + 420 \eta \mu \nu^2 - 25 \eta \mu, 2772 \delta - 388476 \alpha \\
& + 1542528 \eta^4 - 25776 \delta \nu^2 + 14981 \text{radic} - 1976832 \alpha \eta^4 - 663552 \alpha \eta^6 \delta \\
& - 6336 \delta \eta^2 + 1701516 + 1562112 \eta^6 \text{radic} + 425088 \alpha \text{radic} \eta^2 \nu^2 + 66960 \alpha \nu^2 \\
& + 38016 \delta \text{radic} \eta^2 \nu^2 + 860544 \alpha \eta^2 \nu^2 + 331776 \alpha \eta^4 \delta \nu^2 - 2087424 \eta^4 \nu^2 \\
& + 1990656 \eta^4 \alpha \nu^2 - 50688 \alpha \text{radic} \delta \nu^2 - 732672 \alpha \eta^4 \delta + 663552 \eta^4 \delta \nu^2 \\
& + 92736 \alpha \delta \eta^2 - 68832 \alpha \delta \nu^2 + 1692 \text{radic} \delta + 1893024 \eta^2 + 69120 \eta^4 \delta \\
& - 1589760 \nu^2 \eta^4 \text{radic} - 1728000 \alpha \text{radic} \eta^4 + 1810944 \eta^6 - 663552 \alpha \eta^6 \text{radic} \delta \\
& - 483840 \alpha \text{radic} \eta^4 \delta + 61632 \alpha \text{radic} \delta \eta^2 + 317952 \delta \text{radic} \eta^4 \\
& + 1990656 \eta^4 \text{radic} \alpha \nu^2 + 501120 \alpha \text{radic} \delta \eta^2 \nu^2 - 1327104 \eta^6 \text{radic} \delta \\
& - 1327104 \eta^6 \text{radic} \alpha + 3636 \alpha \delta \text{radic} + 663552 \eta^8 + 687744 \alpha \delta \eta^2 \nu^2 \\
& - 37440 \delta \text{radic} \eta^2 - 7632 \text{radic} \delta \nu^2 + 1169280 \eta^4 \text{radic} - 335232 \alpha \text{radic} \eta^2 \\
& - 60048 \alpha \text{radic} \nu^2 - 400464 \eta^2 \text{radic} \nu^2 + 225456 \eta^2 \text{radic} - 25395 \text{radic} \nu^2 \\
& + 20076 \alpha \text{radic} - 421713 \nu^2 - 1161216 \eta^6 \text{radic} \nu^2 - 561168 \eta^2 \nu^2 - 687744 \alpha \eta^2 \\
& + 224640 \delta \eta^2 \nu^2 + 4716 \alpha \delta + 663552 \delta \text{radic} \eta^4 \nu^2 - 1327104 \eta^6 \delta \\
& + 331776 \alpha \text{radic} \eta^4 \delta \nu^2 - 1327104 \eta^6 \alpha - 1161216 \eta^6 \nu^2 + 663552 \eta^8 \text{radic}, 134045 \\
& + 360 \delta - 32184 \alpha + 124416 \eta^4 - 6048 \delta \nu^2 - 1155 \text{radic} - 82944 \alpha \eta^4 + 10368 \delta \eta^2 \\
& + 82944 \eta^6 \text{radic} + 145152 \alpha \text{radic} \eta^2 \nu^2 + 42336 \alpha \nu^2 + 62208 \delta \text{radic} \eta^2 \nu^2 \\
& + 145152 \alpha \eta^2 \nu^2 - 165888 \eta^4 \nu^2 - 6048 \alpha \text{radic} \delta \nu^2 - 82944 \alpha \eta^4 \delta + 10368 \alpha \delta \eta^2 \\
& - 6048 \alpha \delta \nu^2 + 360 \text{radic} \delta + 35856 \eta^2 - 82944 \eta^4 \delta - 165888 \nu^2 \eta^4 \text{radic} \\
& - 82944 \alpha \text{radic} \eta^4 + 82944 \eta^6 - 82944 \alpha \text{radic} \eta^4 \delta + 10368 \alpha \text{radic} \delta \eta^2 \\
& - 82944 \delta \text{radic} \eta^4 + 62208 \alpha \text{radic} \delta \eta^2 \nu^2 + 360 \alpha \delta \text{radic} + 62208 \alpha \delta \eta^2 \nu^2 \\
& + 10368 \delta \text{radic} \eta^2 - 6048 \text{radic} \delta \nu^2 + 124416 \eta^4 \text{radic} - 117504 \alpha \text{radic} \eta^2 \\
& + 42336 \alpha \text{radic} \nu^2 - 53568 \eta^2 \text{radic} \nu^2 + 35856 \eta^2 \text{radic} - 2106 \text{radic} \nu^2 \\
& - 32184 \alpha \text{radic} - 2106 \nu^2 - 53568 \eta^2 \nu^2 - 117504 \alpha \eta^2 + 62208 \delta \eta^2 \nu^2 + 360 \alpha \delta, \\
& 1152 \eta^4 \delta - 864 \delta \eta^2 \nu^2 - 144 \delta \eta^2 + 84 \delta \nu^2 - 5 \delta + 1152 \alpha \eta^4 - 2016 \alpha \eta^2 \nu^2 \\
& + 1632 \alpha \eta^2 - 588 \alpha \nu^2 + 411 \alpha + 36 \}
\end{aligned}$$

```
> R := convert(map(L->reduce(L, [72*nu^4-60*nu^2+7, 6*mu^2-5+6*nu^2,
12*beta^2-12*eta^2+12*nu^2-11, 12*alpha^2+12*eta^2-1,
6*delta^2+6*eta^2-6*nu^2-1],
termorder(poly_algebra(delta,alpha,beta,eta,mu,nu,radic,alg_relations=
{radic^2-5
}),plex(delta,alpha,beta,eta,mu,nu,radic))),subs(sqrt(5)=radic,Cond_6
)),set);
```

$$\begin{aligned}
R := & \{-1, 0, 1, -1152 \beta \nu \eta^4 + 864 \beta \nu^3 \eta^2 - 672 \beta \nu \eta^2 + 12 \beta \nu^3 - \beta \nu - 1152 \eta^5 \mu \\
& + 1440 \eta^3 \mu \nu^2 + 240 \eta^3 \mu - 420 \eta \mu \nu^2 + 25 \eta \mu, 240 \eta^2 + 30 \nu^2 - 107, 1152 \eta^4 \delta \\
& - 864 \delta \eta^2 \nu^2 - 144 \delta \eta^2 + 84 \delta \nu^2 - 5 \delta + 1152 \alpha \eta^4 - 1440 \alpha \eta^2 \nu^2 + 1248 \eta^2 \alpha \\
& - 300 \nu^2 \alpha + 213 \alpha + 36, -3 \eta^2 + 3 \nu^2 + 1, -1152 \eta^4 \delta + 864 \delta \eta^2 \nu^2 + 144 \delta \eta^2 \\
& - 84 \delta \nu^2 + 5 \delta - 1152 \alpha \eta^4 + 1440 \alpha \eta^2 \nu^2 - 1248 \eta^2 \alpha + 300 \nu^2 \alpha - 213 \alpha - 36, \\
& 1152 \beta \nu \eta^4 - 864 \beta \nu^3 \eta^2 + 672 \beta \nu \eta^2 - 12 \beta \nu^3 + \beta \nu + 1152 \eta^5 \mu - 1440 \eta^3 \mu \nu^2 \\
& - 240 \eta^3 \mu + 420 \eta \mu \nu^2 - 25 \eta \mu, -5 \text{ radic} + 3, 40685 - 1728 \delta \eta^2 \alpha + 1008 \delta \nu^2 \alpha \\
& - 60 \delta - 3600 \nu^2 \alpha + 2988 \alpha - 10368 \delta \eta^2 \nu^2 + 14976 \eta^2 \alpha - 1308 \text{ radic} \eta^2 \\
& - 13824 \eta^4 \text{ radic} - 300 \nu^2 - 1308 \eta^2 + 645 \text{ radic} - 300 \text{ radic} \nu^2 - 13824 \eta^4 \\
& + 13824 \alpha \eta^4 \delta + 1008 \delta \nu^2 - 10368 \alpha \delta \eta^2 \nu^2 + 17280 \eta^4 \text{ radic} \nu^2 \\
& + 14976 \text{ radic} \alpha \eta^2 - 3600 \text{ radic} \alpha \nu^2 + 13824 \text{ radic} \alpha \eta^4 + 13824 \delta \text{ radic} \eta^4 \\
& - 60 \delta \text{ radic} - 60 \text{ radic} \alpha \delta - 60 \delta \alpha + 2160 \eta^2 \nu^2 - 10368 \delta \text{ radic} \eta^2 \nu^2 \\
& - 17280 \text{ radic} \alpha \eta^2 \nu^2 + 1008 \text{ radic} \alpha \delta \nu^2 + 13824 \text{ radic} \alpha \eta^4 \delta + 17280 \eta^4 \nu^2 \\
& - 13824 \eta^6 - 1728 \delta \eta^2 - 1728 \text{ radic} \alpha \eta^2 \delta + 13824 \eta^4 \delta + 2160 \eta^2 \text{ radic} \nu^2 \\
& + 1008 \delta \text{ radic} \nu^2 - 1728 \delta \text{ radic} \eta^2 + 13824 \alpha \eta^4 - 17280 \alpha \eta^2 \nu^2 \\
& - 10368 \text{ radic} \alpha \delta \eta^2 \nu^2 + 2988 \text{ radic} \alpha - 13824 \eta^6 \text{ radic}, 3456 \eta^4 \nu^2 \delta - 5184 \eta^4 \delta \\
& + 1296 \delta \eta^2 \nu^2 + 900 \delta \eta^2 - 183 \delta \nu^2 - 2 \delta + 3456 \eta^4 \alpha \nu^2 - 5184 \alpha \eta^4 + 6624 \alpha \eta^2 \nu^2 \\
& - 5196 \eta^2 \alpha + 1239 \nu^2 \alpha - 871 \alpha + 108 \nu^2 - 162, -487296 \delta \eta^2 \alpha + 16560 \delta \nu^2 \alpha \\
& + 11172 \delta - 1500048 \nu^2 \alpha + 2561676 \alpha - 300672 \delta \eta^2 \nu^2 + 6757632 \eta^2 \alpha \\
& - 43956 \text{ radic} \eta^2 - 2395584 \eta^4 \text{ radic} + 2599716 \nu^2 - 10316412 \eta^2 + 231667 \text{ radic} \\
& - 213684 \text{ radic} \nu^2 - 2146752 \eta^4 + 2032128 \alpha \eta^4 \delta + 145008 \delta \nu^2 + 425088 \alpha \delta \eta^2 \nu^2 \\
& + 3362688 \eta^4 \text{ radic} \nu^2 + 7027200 \text{ radic} \alpha \eta^2 - 1564848 \text{ radic} \alpha \nu^2 \\
& + 10658304 \text{ radic} \alpha \eta^4 + 5847552 \delta \text{ radic} \eta^4 + 10092 \delta \text{ radic} + 7836 \text{ radic} \alpha \delta \\
& + 11631457 + 8916 \delta \alpha + 218736 \eta^2 \nu^2 - 487296 \delta \text{ radic} \eta^2 \nu^2 \\
& - 9189504 \text{ radic} \alpha \eta^2 \nu^2 - 1990656 \eta^8 \text{ radic} + 34704 \text{ radic} \alpha \delta \nu^2 \\
& + 2280960 \text{ radic} \alpha \eta^4 \delta + 3051648 \eta^4 \nu^2 - 4271616 \eta^6 - 1990656 \eta^8 - 1166400 \delta \eta^2 \\
& + 3981312 \delta \eta^6 \text{ radic} + 3981312 \delta \eta^6 + 5142528 \eta^6 \nu^2 - 518400 \text{ radic} \alpha \eta^2 \delta \\
& - 8294400 \eta^4 \nu^2 \delta \text{ radic} + 1990656 \eta^6 \alpha \delta \text{ radic} + 1990656 \eta^6 \alpha \delta \\
& - 10285056 \eta^4 \alpha \nu^2 + 3981312 \alpha \eta^6 + 5598720 \eta^4 \delta - 4147200 \eta^4 \alpha \nu^2 \delta \\
& - 4147200 \eta^4 \alpha \nu^2 \delta \text{ radic} + 5142528 \eta^6 \nu^2 \text{ radic} + 257616 \eta^2 \text{ radic} \nu^2 \\
& + 163152 \delta \text{ radic} \nu^2 - 1197504 \delta \text{ radic} \eta^2 + 10409472 \alpha \eta^4 + 3981312 \alpha \eta^6 \text{ radic} \\
& - 8878464 \alpha \eta^2 \nu^2 + 238464 \text{ radic} \alpha \delta \eta^2 \nu^2 - 10285056 \alpha \eta^4 \nu^2 \text{ radic} \\
& - 8294400 \eta^4 \nu^2 \delta + 1117860 \text{ radic} \alpha - 4520448 \eta^6 \text{ radic}\}
\end{aligned}$$

> Pas de plan de force 6.

> Famille infinie de plans de force 4 -> par exemple si eta=0, il y a deux dispositifs différents.

```

> unassign('alpha','beta','nu','delta','eta','mu');
> assign({eta = 0, mu = 1/6*(15-3*11^(1/2))^(1/2), delta =
(-eta^2+7/12+1/12*11^(1/2))^(1/2), alpha =
1/2*(-4*eta^2+1/12+1/12*11^(1/2))^(1/2), beta =
1/2*(4*eta^2+9/4-5/12*11^(1/2))^(1/2), nu =
-1/6*(15+3*11^(1/2))^(1/2)});

```

> alpha;beta;nu;delta;eta;mu;

$$\frac{\sqrt{\frac{1}{12} + \frac{\sqrt{11}}{12}}}{2}$$

$$\frac{\sqrt{\frac{9}{4} - \frac{5\sqrt{11}}{12}}}{2}$$

$$-\frac{\sqrt{15 + 3\sqrt{11}}}{6}$$

$$\frac{\sqrt{\frac{7}{12} + \frac{\sqrt{11}}{12}}}{0}$$

$$\frac{\sqrt{15 - 3\sqrt{11}}}{6}$$

> N25:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN25) union  
OrbPG(Vector([delta,eta,mu]),GrN25) union  
OrbPG(Vector([1,0,0]),GrN25)));

$$N25 :=$$

$$\left[ \%1, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%2 \right]$$

$$\left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right.$$

$$+ \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96}$$

$$\left. - \frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} - \frac{\sqrt{81-15\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%3 \right]$$

$$\left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right.$$

$$\left. \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%3 \right]$$

$$\left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right.$$

$$\left. \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%4 \right]$$

$$\begin{aligned}
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
& \quad - \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \\
& \quad \left. + \frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} - \frac{\sqrt{81-15\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%3 \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right. \\
& \quad \left. -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{81-15\sqrt{11}}}{48}, \%4 \right] \\
& \left[ \frac{\sqrt{3+3\sqrt{11}}}{12}, -\frac{\sqrt{81-15\sqrt{11}}}{12}, \%4 \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
& \quad - \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \\
& \quad \left. - \frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} + \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%4 \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right. \\
& \quad \left. -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%3 \right] \\
& \left[ \frac{\sqrt{3+3\sqrt{11}}}{12}, \frac{\sqrt{81-15\sqrt{11}}}{12}, \%3 \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
& \quad + \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \\
& \quad \left. + \frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} + \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%4 \right] \\
& \left[ \frac{\sqrt{21+3\sqrt{11}}}{6}, 0, \%2 \right] \\
& \left[ \%5, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%6 \right] \\
& \left[ \frac{\sqrt{21+3\sqrt{11}}}{6}, 0, \%6 \right]
\end{aligned}$$

$$\begin{aligned}
 & \left[ -\frac{1}{4} + \frac{\sqrt{5}}{4}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, 0 \right] \\
 & \left[ \%1, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%6 \right] \\
 & \left[ -\frac{1}{4} - \frac{\sqrt{5}}{4}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}, 0 \right] \\
 & \left[ \%5, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%2 \right] \\
 & \left[ \%5, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%6 \right] \\
 & [1, 0, 0] \\
 & \left[ -\frac{1}{4} - \frac{\sqrt{5}}{4}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}, 0 \right] \\
 & \left[ \%1, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%6 \right] \\
 & \left[ -\frac{1}{4} + \frac{\sqrt{5}}{4}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, 0 \right] \\
 & \left[ \%5, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%2 \right] \\
 & \left[ \%1, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%2 \right] \\
 & \%1 := \frac{(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{24} \\
 & \%2 := \frac{\sqrt{15-3\sqrt{11}}}{6} \\
 & \%3 := -\frac{\sqrt{15+3\sqrt{11}}}{6} \\
 & \%4 := \frac{\sqrt{15+3\sqrt{11}}}{6} \\
 & \%5 := -\frac{(1+\sqrt{5})\sqrt{21+3\sqrt{11}}}{24} \\
 & \%6 := -\frac{\sqrt{15-3\sqrt{11}}}{6}
 \end{aligned}$$

> UpdatePlan(N25,"N25");

“Update complete”

> pointplot3d((N25),symbol=box,scaling=constrained,shading=Z,orientation=[32,79]);

> Graph:=convhull(op(map(convert,[evalf(op(convert(N25,listlist)))]),'rational',5)));

Graph := POLYTOPE(3, 3, 25, 40)

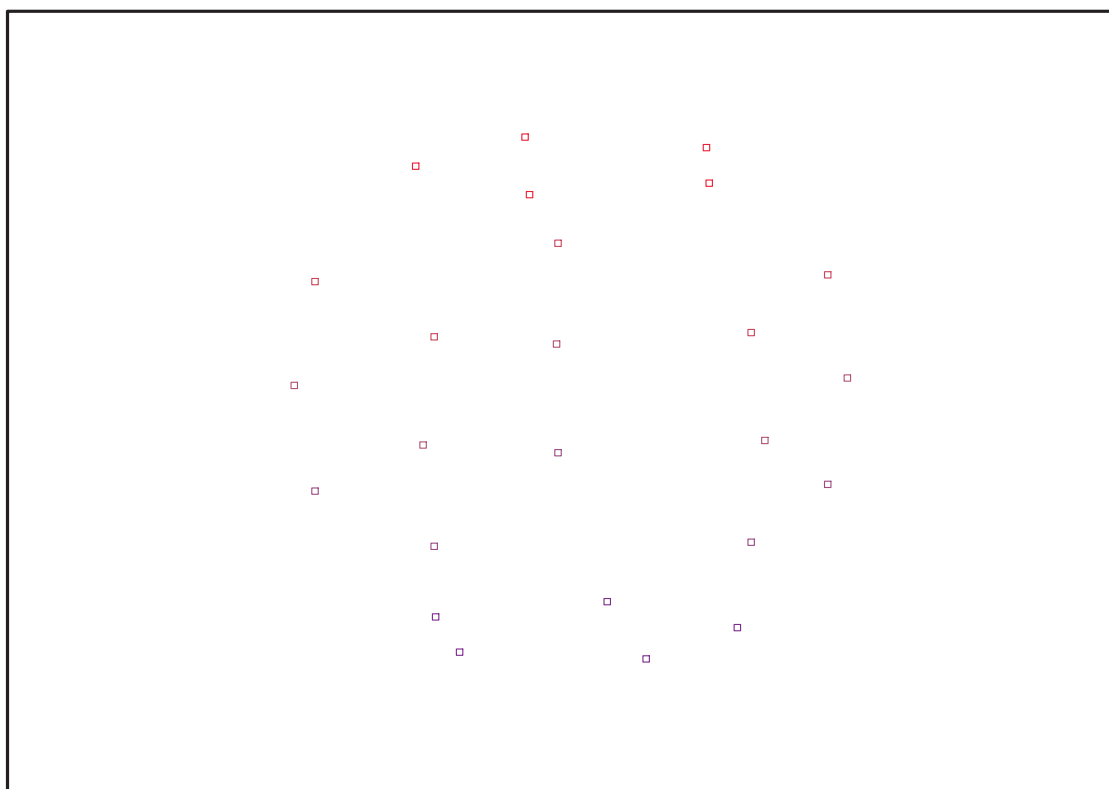


FIG. 11.80 – Graphique : Support du plan N25.

```
> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z,orientation=[32,79]);
```

```
> unassign('alpha','beta','nu','delta','eta','mu');
> assign({eta = 0, mu = 1/6*(15-3*11^(1/2))^(1/2), delta =
-(-eta^2+7/12+1/12*11^(1/2))^(1/2), alpha =
1/2*(-4*eta^2+1/12+1/12*11^(1/2))^(1/2), beta =
1/2*(4*eta^2+9/4-5/12*11^(1/2))^(1/2), nu =
-1/6*(15+3*11^(1/2))^(1/2)});
```

```
> alpha;beta;nu;delta;eta;mu;
```

$$\frac{\sqrt{\frac{1}{12} + \frac{\sqrt{11}}{12}}}{2}$$

$$\frac{\sqrt{\frac{9}{4} - \frac{5\sqrt{11}}{12}}}{2}$$

$$-\frac{\sqrt{15 + 3\sqrt{11}}}{6}$$

$$-\sqrt{\frac{7}{12} + \frac{\sqrt{11}}{12}}$$

$$0$$

$$\frac{\sqrt{15 - 3\sqrt{11}}}{6}$$



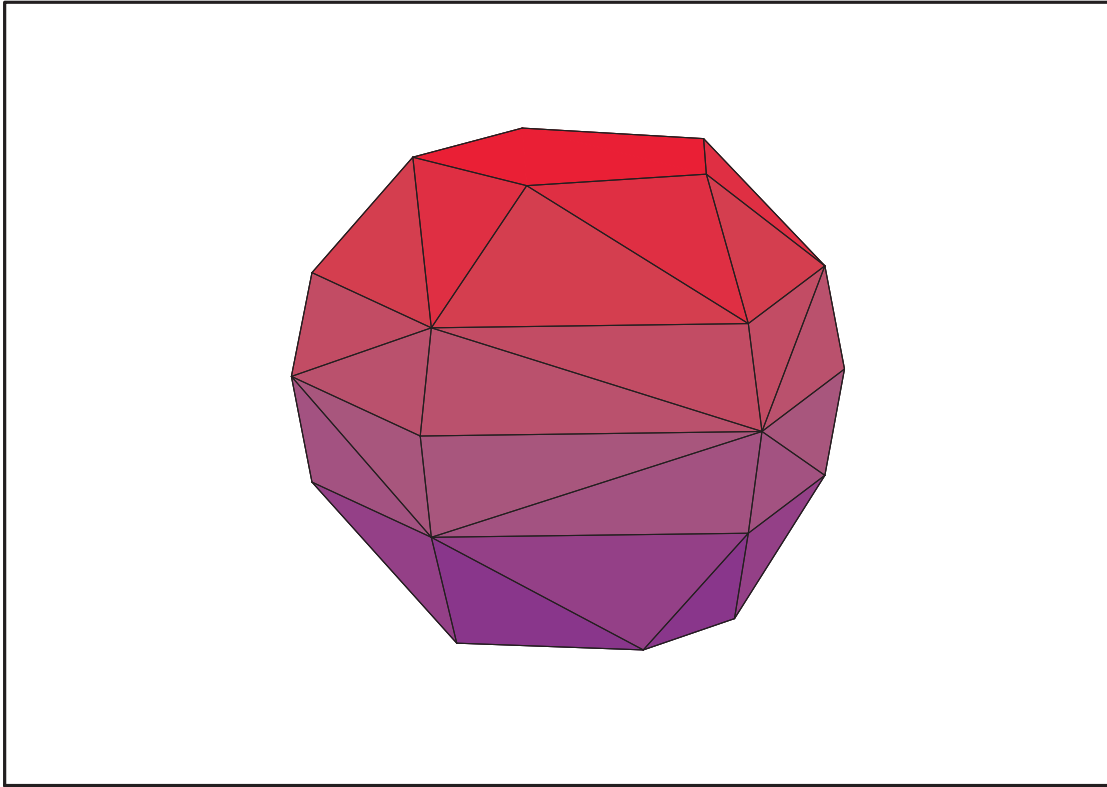


FIG. 11.81 – Graphique : Enveloppe convexe du support du plan N25.

```
> N25bis:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN25)
union OrbPG(Vector([delta,eta,mu]),GrN25) union
OrbPG(Vector([1,0,0]),GrN25)));
```

$$\begin{aligned}
 N25bis := & \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right. \\
 & \left. -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%1 \right] \\
 & \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right. \\
 & \left. \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%2 \right] \\
 & \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
 & \left. - \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \right. \\
 & \left. + \frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} - \frac{\sqrt{81-15\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%1 \right]
 \end{aligned}$$

$$\begin{aligned}
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right. \\
& \left. -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{81-15\sqrt{11}}}{48}, \%2 \right] \\
& \left[ \frac{\sqrt{3+3\sqrt{11}}}{12}, -\frac{\sqrt{81-15\sqrt{11}}}{12}, \%2 \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
& -\frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \\
& -\frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} + \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%2 \left. \right] \\
& \left[ \%3, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%4 \right] \\
& \left[ \%5, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%6 \right] \\
& \left[ \frac{\sqrt{3+3\sqrt{11}}}{12}, \frac{\sqrt{81-15\sqrt{11}}}{12}, \%1 \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
& + \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \\
& + \frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} + \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%2 \left. \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{96} \right. \\
& + \frac{\sqrt{81-15\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{96} \\
& -\frac{\sqrt{3+3\sqrt{11}}\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{96} - \frac{\sqrt{81-15\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%1 \left. \right] \\
& \left[ -\frac{\sqrt{3+3\sqrt{11}}}{48} + \frac{\sqrt{3+3\sqrt{11}}\sqrt{5}}{48} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{81-15\sqrt{11}}}{48}, \right. \\
& \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{3+3\sqrt{11}}}{48} - \frac{\sqrt{81-15\sqrt{11}}}{48} + \frac{\sqrt{81-15\sqrt{11}}\sqrt{5}}{48}, \%1 \left. \right] \\
& \left[ \%3, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%4 \right] \\
& \left[ \%3, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%6 \right] \\
& \left[ \%5, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%6 \right] \\
& \left[ \%5, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%4 \right]
\end{aligned}$$

$$\begin{aligned}
 & \left[ \%3, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48}, \%6 \right] \\
 & \left[ \%5, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24}, \%4 \right] \\
 & \left[ -\frac{\sqrt{21+3\sqrt{11}}}{6}, 0, \%4 \right] \\
 & \left[ -\frac{\sqrt{21+3\sqrt{11}}}{6}, 0, \%6 \right] \\
 & \left[ -\frac{1}{4} + \frac{\sqrt{5}}{4}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, 0 \right] \\
 & \left[ -\frac{1}{4} - \frac{\sqrt{5}}{4}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}, 0 \right] \\
 & [1, 0, 0] \\
 & \left[ -\frac{1}{4} + \frac{\sqrt{5}}{4}, \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, 0 \right] \\
 & \left[ -\frac{1}{4} - \frac{\sqrt{5}}{4}, -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8}, 0 \right] \\
 & \%1 := -\frac{\sqrt{15+3\sqrt{11}}}{6} \\
 & \%2 := \frac{\sqrt{15+3\sqrt{11}}}{6} \\
 & \%3 := \frac{(1+\sqrt{5})\sqrt{21+3\sqrt{11}}}{24} \\
 & \%4 := -\frac{\sqrt{15-3\sqrt{11}}}{6} \\
 & \%5 := -\frac{(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{24} \\
 & \%6 := \frac{\sqrt{15-3\sqrt{11}}}{6}
 \end{aligned}$$

> UpdatePlan(N25bis,"N25bis");

“Update complete”

> pointplot3d((N25bis),symbol=box,scaling=constrained,shading=Z,orientation=[32,79]);

> Graph:=convhull(op(map(convert,[evalf(op(convert(N25bis,listlist))), 'rational',5)]));

*Graph* := POLYTOPE(3, 3, 25, 42)

> plotdes:=draw(Graph):

> display(plotdes,scaling=constrained,shading=Z,orientation=[32,79]);

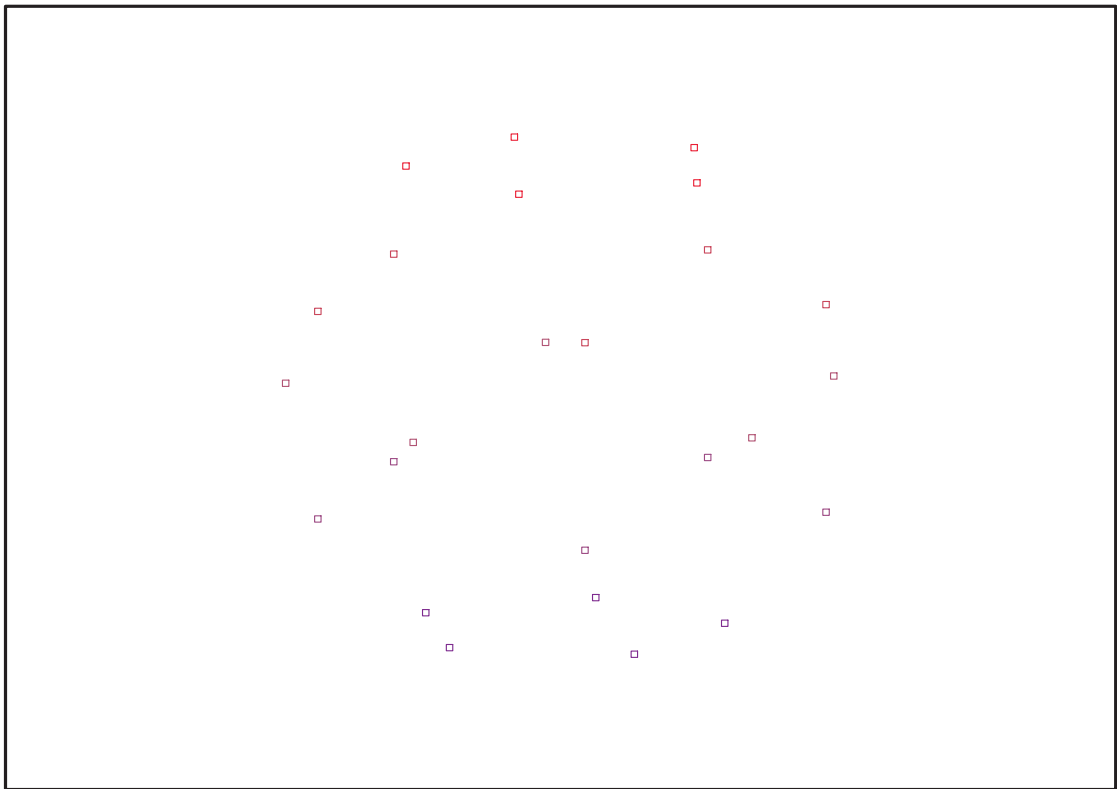


FIG. 11.82 – Graphique : Support du plan N25bis.

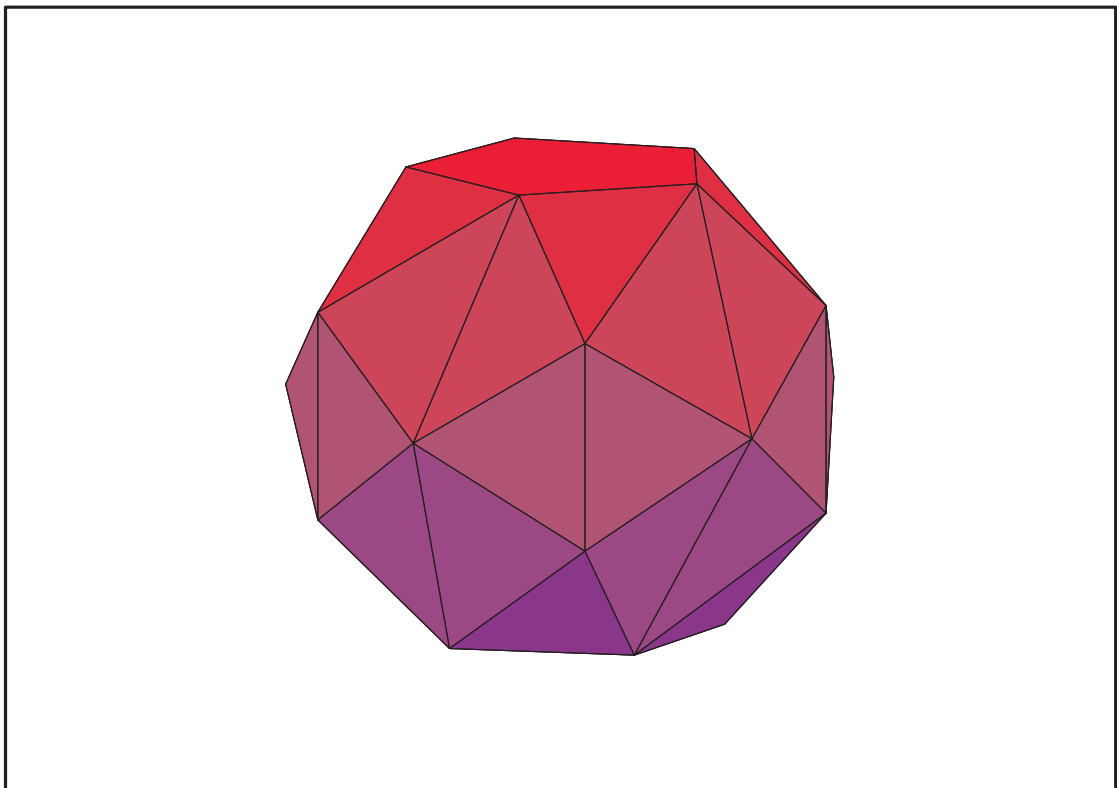


FIG. 11.83 – Graphique : Enveloppe convexe du support du plan N25bis.

### 11.30.1. Variance

#### Modèle de Degré 1

```
> ReadPlan("N25");Plan:N25:=Plan:
> Var:=Var_Norm_Blue_D_Centre(N25,1,0);
 Var := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
> ReadPlan("N25bis");Plan:N25bis:=Plan:
> Var:=Var_Norm_Blue_D_Centre(N25,1,0);
 Var := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

#### Modèle de Degré 2

```
> Var:=Var_Norm_Blue_D_Centre(N25,2,0);
 "M_D is not full rank"
> Var:=expand(Var_Norm_Blue_D_Centre(N25,2,1));
 Var := 25 - 47 var32 - 47 var22 - 47 var12 + 62 var22 var32 + 62 var12 var32 + 62 var12 var22
 + 31 var34 + 31 var24 + 31 var14
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 25 + 31 IndetDecompInKfi12 - 47 IndetDecompInKfi1,
 IndetDecompInKfi1 = var12 + var22 + var32
> Var:=Var_Norm_Blue_D_Centre(N25bis,2,0);
 "M_D is not full rank"
> Var:=expand(Var_Norm_Blue_D_Centre(N25bis,2,1));
 Var := 25 - 47 var32 - 47 var22 - 47 var12 + 62 var22 var32 + 62 var12 var32 + 62 var12 var22
 + 31 var34 + 31 var24 + 31 var14
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 25 + 31 IndetDecompInKfi12 - 47 IndetDecompInKfi1,
 IndetDecompInKfi1 = var12 + var22 + var32
> Plan de force 4 -> il suffit beta=eta=0 alpha -> -alpha - + +
```

```
> unassign('alpha','beta','nu','delta','eta','mu');
```

```
> nu := 1/6*(15+3*11^(1/2))^(1/2); beta := 0; eta:=0; alpha :=
-(-beta^2-nu^2+1)^(1/2); mu := (-nu^2+5/6)^(1/2); delta :=
(-eta^2+1/6+nu^2)^(1/2);
```

$$\nu := \frac{\sqrt{15 + 3\sqrt{11}}}{6}$$

$$\beta := 0$$

$$\eta := 0$$

$$\alpha := -\sqrt{\frac{7}{12} - \frac{\sqrt{11}}{12}}$$

$$\mu := \sqrt{\frac{5}{12} - \frac{\sqrt{11}}{12}}$$

$$\delta := \sqrt{\frac{7}{12} + \frac{\sqrt{11}}{12}}$$

```
> N25ter:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN25)
union OrbPG(Vector([delta,eta,mu]),GrN25) union
OrbPG(Vector([1,0,0]),GrN25)));
```

$$N25ter := \left[ \begin{array}{ccc} \%1 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 - 3 \sqrt{11}}}{24} & \%2 \\ \%3 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 - 3 \sqrt{11}}}{48} & \%4 \\ \%1 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 - 3 \sqrt{11}}}{24} & \%4 \\ -\frac{\sqrt{21 - 3 \sqrt{11}}}{6} & 0 & \%4 \\ \%3 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 - 3 \sqrt{11}}}{48} & \%2 \\ \%1 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 - 3 \sqrt{11}}}{24} & \%4 \\ \%3 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 - 3 \sqrt{11}}}{48} & \%4 \\ \%3 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 - 3 \sqrt{11}}}{48} & \%2 \\ \%1 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 - 3 \sqrt{11}}}{24} & \%2 \\ \%5 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 + 3 \sqrt{11}}}{48} & \%6 \\ \%7 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 + 3 \sqrt{11}}}{24} & \%8 \\ -\frac{\sqrt{21 - 3 \sqrt{11}}}{6} & 0 & \%2 \\ \%7 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 + 3 \sqrt{11}}}{24} & \%6 \\ \frac{\sqrt{21 + 3 \sqrt{11}}}{6} & 0 & \%6 \\ \%7 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 + 3 \sqrt{11}}}{24} & \%6 \\ \%5 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 + 3 \sqrt{11}}}{48} & \%8 \\ \%7 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{21 + 3 \sqrt{11}}}{24} & \%8 \\ \%5 & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 + 3 \sqrt{11}}}{48} & \%8 \\ \%5 & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} (\sqrt{5} - 1) \sqrt{21 + 3 \sqrt{11}}}{48} & \%6 \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}}}{4} & 0 \\ \frac{\sqrt{21 + 3 \sqrt{11}}}{6} & 0 & \%8 \\ 1 & 0 & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}}}{8} - \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{5}}{8} & 0 \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2} \sqrt{5 + \sqrt{5}}}{4} & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & -\frac{\sqrt{2} \sqrt{5 + \sqrt{5}}}{8} + \frac{\sqrt{2} \sqrt{5 + \sqrt{5}} \sqrt{5}}{8} & 0 \end{array} \right]$$

$$\begin{aligned} \%1 &:= -\frac{(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{24} \\ \%2 &:= \frac{\sqrt{15+3\sqrt{11}}}{6} \\ \%3 &:= \frac{(1+\sqrt{5})\sqrt{21-3\sqrt{11}}}{24} \\ \%4 &:= -\frac{\sqrt{15+3\sqrt{11}}}{6} \\ \%5 &:= -\frac{(1+\sqrt{5})\sqrt{21+3\sqrt{11}}}{24} \\ \%6 &:= -\frac{\sqrt{15-3\sqrt{11}}}{6} \\ \%7 &:= \frac{(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{24} \\ \%8 &:= \frac{\sqrt{15-3\sqrt{11}}}{6} \end{aligned}$$

```
> UpdatePlan(N25ter,"N25ter");
```

“Update complete”

```
> radnormal(eval(Cond_4));
```

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

```
> pointplot3d(N25ter,symbol=box,scaling=constrained,shading=Z,orientation=[32,79]);
```

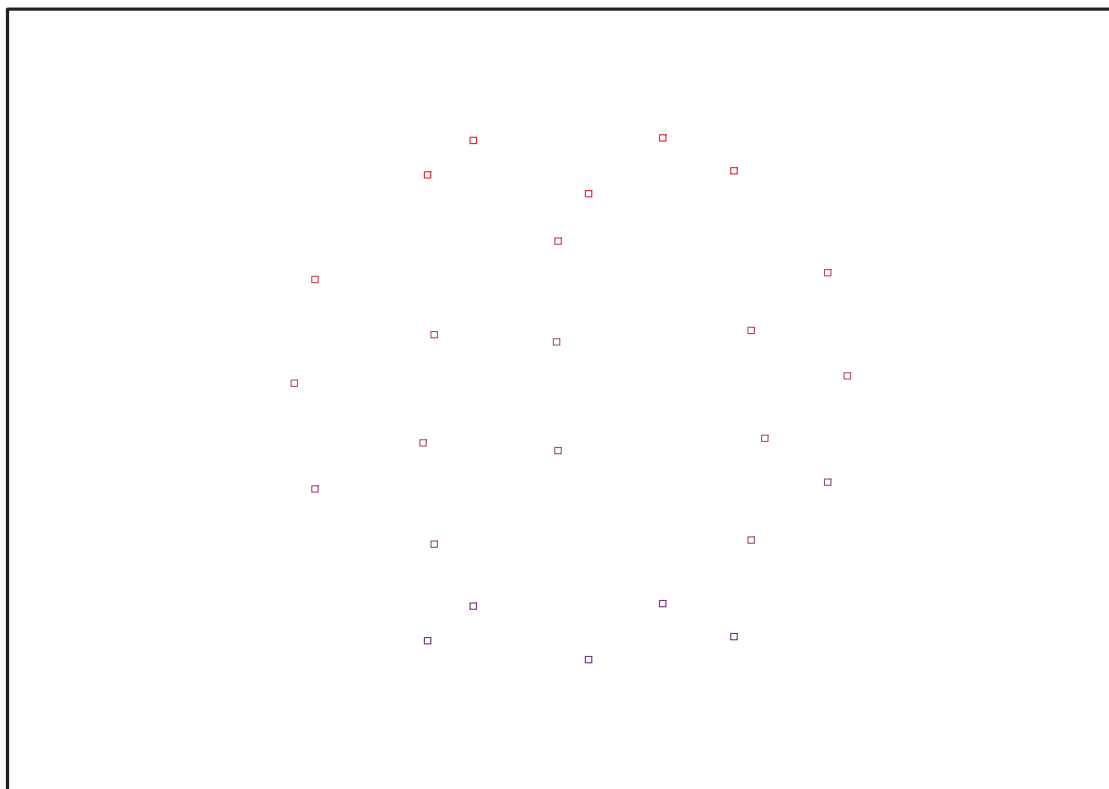


FIG. 11.84 – Graphique : Support du plan N25ter.



```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N25ter,listlist))),
'rational',5))));
 Graph := POLYTOPE(3, 3, 25, 40)

> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z,orientation=[32,79]);
```

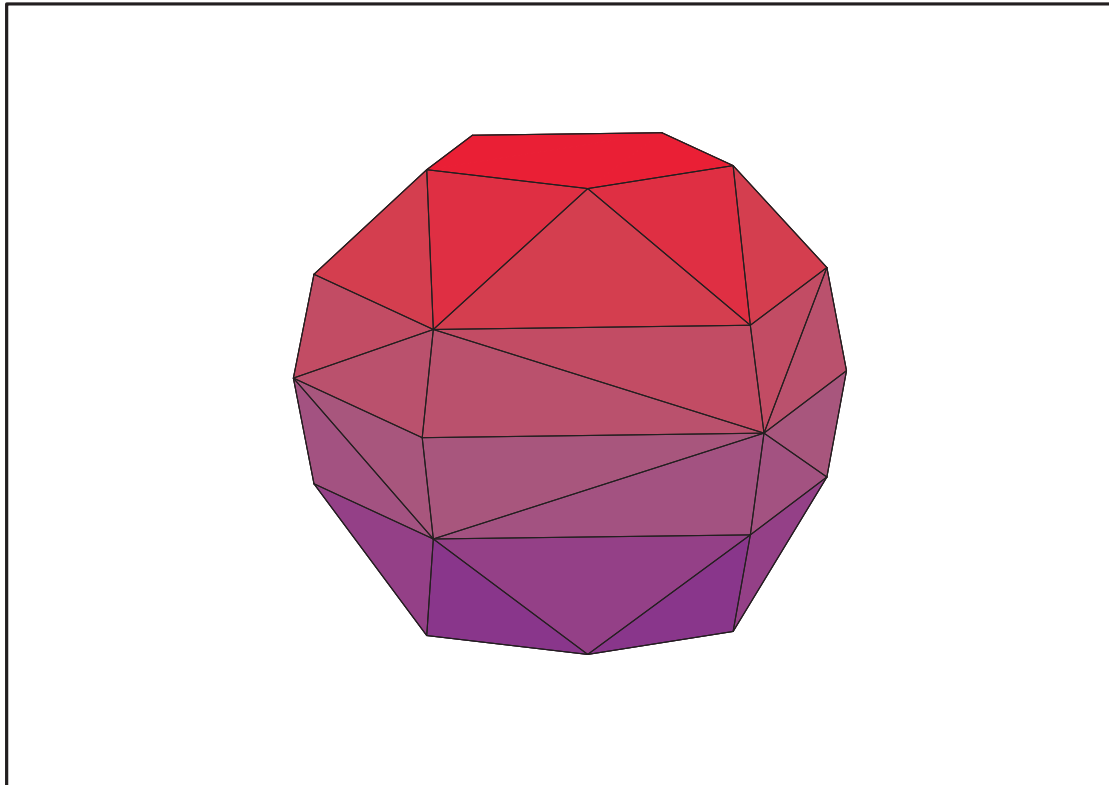


FIG. 11.85 – Graphique : Enveloppe convexe du support du plan N25ter.

## 11.30.2. Variance

### Modèle de Degré 1

```
> ReadPlan("N25ter");Plan:N25ter:=Plan:
> Var:=Var_Norm_Blue_D_Centre(N25ter,1,0);
 Var := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

## Modèle de Degré 2

```

> Var:=Var_Norm_Blue_D_Centre(N25ter,2,0);
 "M_D is not full rank"

> Var:=expand(Var_Norm_Blue_D_Centre(N25ter,2,1));

Var := 25 - 47 var32 - 47 var22 - 47 var12 + 62 var22 var32 + 62 var12 var32 + 62 var12 var22
+ 31 var34 + 31 var24 + 31 var14

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));

25 + 31 IndetDecompInKfi12 - 47 IndetDecompInKfi1,
IndetDecompInKfi1 = var12 + var22 + var32

> Plan de force 4 -> il suffit beta=eta=0 1 -> -1 + + -

> unassign('alpha','beta','nu','delta','eta','mu');

> nu := 1/6*(15+3*11^(1/2))^(1/2); beta := 0; eta:=0; alpha :=
(-beta^2-nu^2+1)^(1/2); mu := (-nu^2+5/6)^(1/2); delta :=
(-eta^2+1/6+nu^2)^(1/2);

$$\nu := \frac{\sqrt{15+3\sqrt{11}}}{6}$$

$$\beta := 0$$

$$\eta := 0$$

$$\alpha := \sqrt{\frac{7}{12} - \frac{\sqrt{11}}{12}}$$

$$\mu := \sqrt{\frac{5}{12} - \frac{\sqrt{11}}{12}}$$

$$\delta := \sqrt{\frac{7}{12} + \frac{\sqrt{11}}{12}}$$

> N25qua:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN25)
union OrbPG(Vector([delta,eta,mu]),GrN25) union
OrbPG(Vector([-1,0,0]),GrN25)));

```

$$N25qua := \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -1 & 0 & 0 \\ \%1 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%2 \\ \%3 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%4 \\ \%3 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%2 \\ \frac{\sqrt{21-3\sqrt{11}}}{6} & 0 & \%2 \\ \%1 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%4 \\ \%3 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%4 \\ \%3 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%2 \\ \%1 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%4 \\ \frac{\sqrt{21-3\sqrt{11}}}{6} & 0 & \%4 \\ \%1 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%2 \\ \%5 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%6 \\ \frac{\sqrt{21+3\sqrt{11}}}{6} & 0 & \%7 \\ \%8 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%7 \\ \%5 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%7 \\ \%8 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%6 \\ \%5 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%7 \\ \%8 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%7 \\ \%8 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%6 \\ \%5 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%6 \\ \frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -\frac{\sqrt{5}}{4} + \frac{1}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ \frac{\sqrt{21+3\sqrt{11}}}{6} & 0 & \%6 \\ -\frac{\sqrt{5}}{4} + \frac{1}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \end{bmatrix}$$

$$\begin{aligned}\%1 &:= -\frac{(1+\sqrt{5})\sqrt{21-3\sqrt{11}}}{24} \\ \%2 &:= -\frac{\sqrt{15+3\sqrt{11}}}{6} \\ \%3 &:= \frac{(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{24} \\ \%4 &:= \frac{\sqrt{15+3\sqrt{11}}}{6} \\ \%5 &:= \frac{(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{24} \\ \%6 &:= \frac{\sqrt{15-3\sqrt{11}}}{6} \\ \%7 &:= -\frac{\sqrt{15-3\sqrt{11}}}{6} \\ \%8 &:= -\frac{(1+\sqrt{5})\sqrt{21+3\sqrt{11}}}{24}\end{aligned}$$

```
> UpdatePlan(N25qua,"N25qua");
```

“Update complete”

```
> radnormal(eval(Cond_4));
```

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

```
> pointplot3d((N25qua),symbol=box,scaling=constrained,shading=Z,orientation=[32,79]);
```

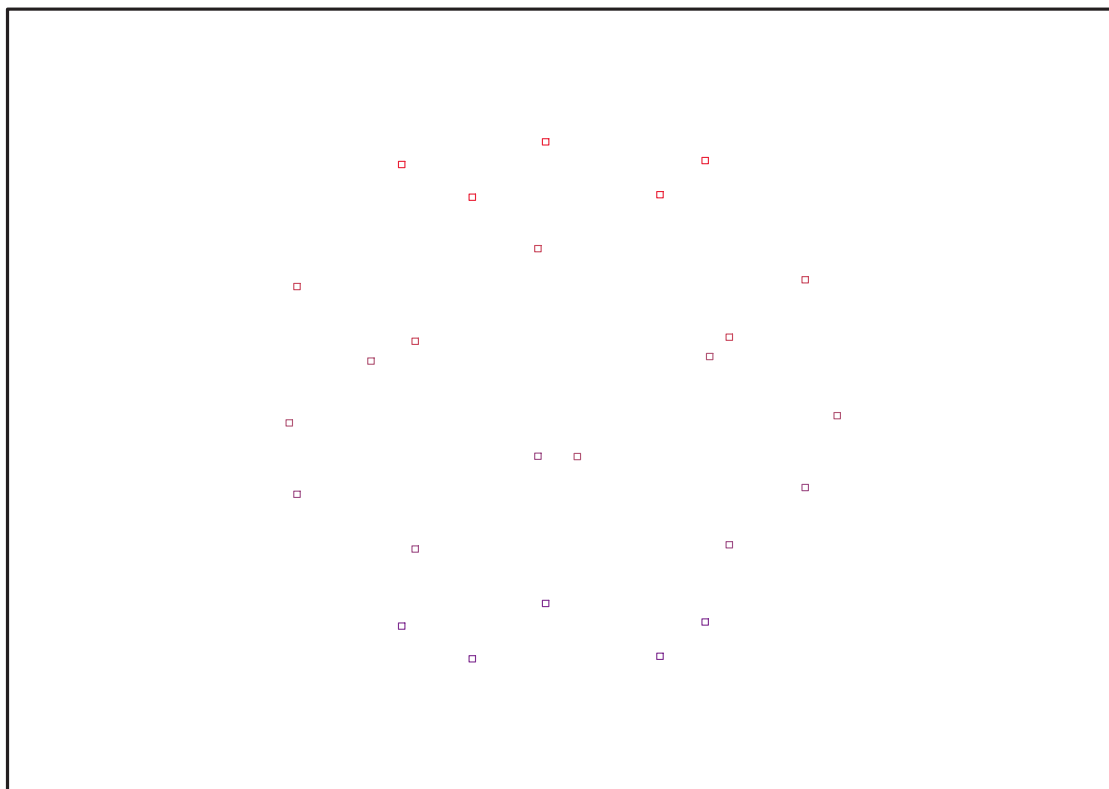


FIG. 11.86 – Graphique : Support du plan N25qua.

```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N25qua,listlist))),
'rational',5))));
 Graph := POLYTOPE(3, 3, 25, 42)

> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z,orientation=[32,79]);
```

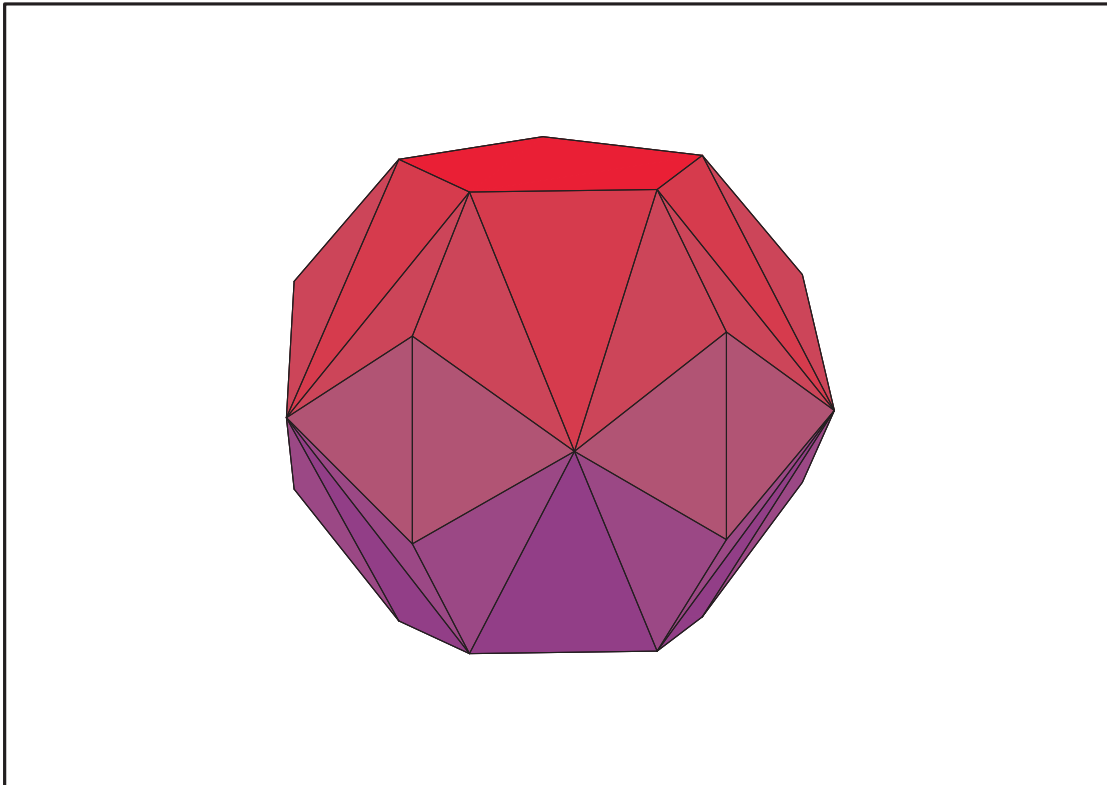


FIG. 11.87 – Graphique : Enveloppe convexe du support du plan N25qua.

### 11.30.3. Variance

#### Modèle de Degré 1

```
> ReadPlan("N25qua");Plan:N25qua:=Plan:
> Var:=Var_Norm_Blue_D_Centre(N25ter,1,0);
 Var := 1 + 3 var32 + 3 var22 + 3 var12
> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32
> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

## Modèle de Degré 2

```

> Var:=Var_Norm_Blue_D_Centre(N25qua,2,0);
 "M_D is not full rank"

> Var:=expand(Var_Norm_Blue_D_Centre(N25qua,2,1));

Var := 25 - 47 var32 - 47 var22 - 47 var12 + 62 var22 var32 + 62 var12 var32 + 62 var12 var22
+ 31 var34 + 31 var24 + 31 var14

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));

25 + 31 IndetDecompInKfi12 - 47 IndetDecompInKfi1,
IndetDecompInKfi1 = var12 + var22 + var32

> Plan de force 4 -> il suffit beta=eta=0 et delta -> -delta + -+

> nu := 1/6*(15+3*11^(1/2))^(1/2); beta := 0; eta:=0; alpha :=
(-beta^2-nu^2+1)^(1/2); mu := (-nu^2+5/6)^(1/2); delta :=
-(-eta^2+1/6+nu^2)^(1/2);

ν := $\frac{\sqrt{15+3\sqrt{11}}}{6}$
β := 0
η := 0
α := $\sqrt{\frac{7}{12} - \frac{\sqrt{11}}{12}}$
μ := $\sqrt{\frac{5}{12} - \frac{\sqrt{11}}{12}}$
δ := $-\sqrt{\frac{7}{12} + \frac{\sqrt{11}}{12}}$

> N25qui:=simplify(OrbToMatrix(OrbPG(Vector([alpha,beta,nu]),GrN25)
union OrbPG(Vector([delta,eta,mu]),GrN25) union
OrbPG(Vector([1,0,0]),GrN25)));

```

$$N25qui := \left[ \begin{array}{ccc} \%1 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%2 \\ -\frac{\sqrt{21+3\sqrt{11}}}{6} & 0 & \%2 \\ \%3 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%2 \\ \%1 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%4 \\ \frac{\sqrt{21-3\sqrt{11}}}{6} & 0 & \%5 \\ \%6 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%7 \\ \%6 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%5 \\ \%8 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%5 \\ \%8 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%7 \\ \%6 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%5 \\ \%8 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%7 \\ \frac{\sqrt{21-3\sqrt{11}}}{6} & 0 & \%7 \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} + \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ 1 & 0 & 0 \\ -\frac{1}{4} - \frac{\sqrt{5}}{4} & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{8} - \frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{5}}{8} & 0 \\ -\frac{1}{4} + \frac{\sqrt{5}}{4} & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4} & 0 \\ -\frac{\sqrt{21+3\sqrt{11}}}{6} & 0 & \%4 \\ \%3 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%2 \\ \%3 & \frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%4 \\ \%1 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%4 \\ \%1 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21+3\sqrt{11}}}{24} & \%2 \\ \%3 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{48} & \%4 \\ \%8 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}\sqrt{21-3\sqrt{11}}}{24} & \%5 \\ \%6 & -\frac{\sqrt{2}\sqrt{5+\sqrt{5}}(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{48} & \%7 \end{array} \right]$$

$$\begin{aligned} \%1 &:= -\frac{(\sqrt{5}-1)\sqrt{21+3\sqrt{11}}}{24} \\ \%2 &:= -\frac{\sqrt{15-3\sqrt{11}}}{6} \\ \%3 &:= \frac{(1+\sqrt{5})\sqrt{21+3\sqrt{11}}}{24} \\ \%4 &:= \frac{\sqrt{15-3\sqrt{11}}}{6} \\ \%5 &:= \frac{\sqrt{15+3\sqrt{11}}}{6} \\ \%6 &:= -\frac{(1+\sqrt{5})\sqrt{21-3\sqrt{11}}}{24} \\ \%7 &:= -\frac{\sqrt{15+3\sqrt{11}}}{6} \\ \%8 &:= \frac{(\sqrt{5}-1)\sqrt{21-3\sqrt{11}}}{24} \end{aligned}$$

```
> UpdatePlan(N25qui,"N25qui");
```

“Update complete”

```
> radnormal(eval(Cond_4));
```

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

```
> pointplot3d((N25qui),symbol=box,scaling=constrained,shading=Z,orientation=[32,79]);
```

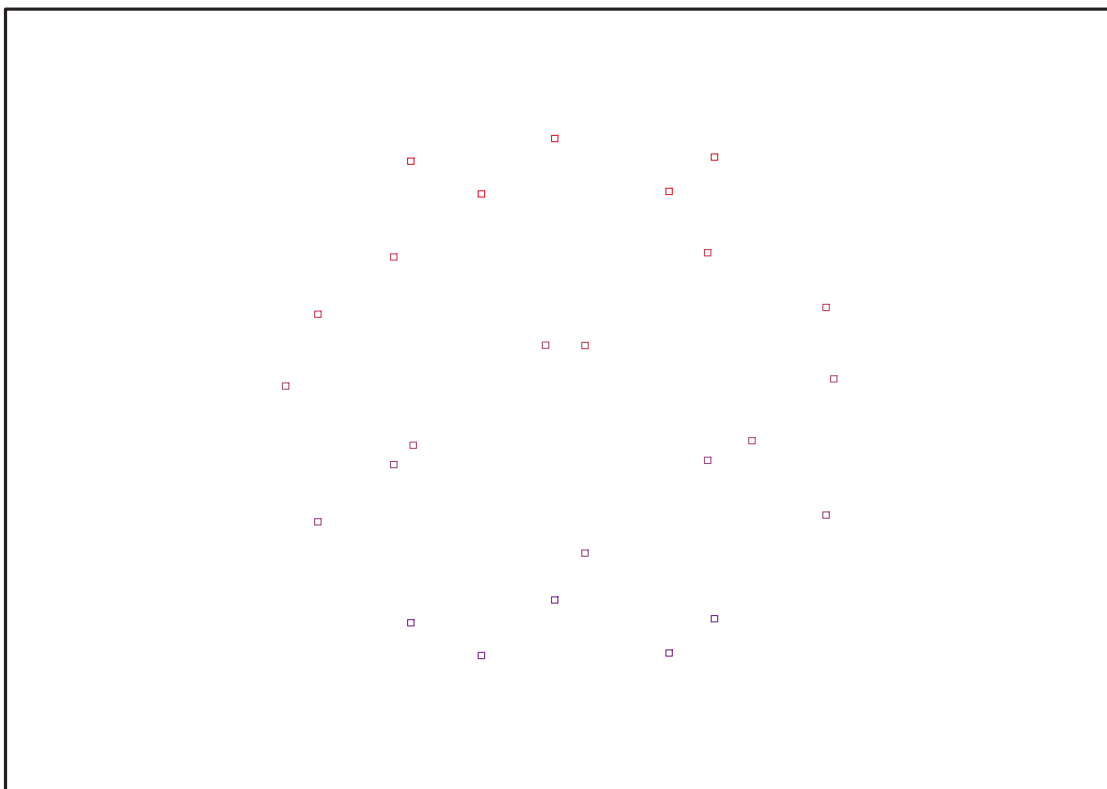


FIG. 11.88 – Graphique : Support du plan N25qui.



```
> Graph:=convhull(op(map(convert,[evalf(op(convert(N25qui,listlist))),
'rational',5)]));
 Graph := POLYTOPE(3, 3, 25, 42)

> plotdes:=draw(Graph):
> display(plotdes,scaling=constrained,shading=Z,orientation=[32,79]);
```

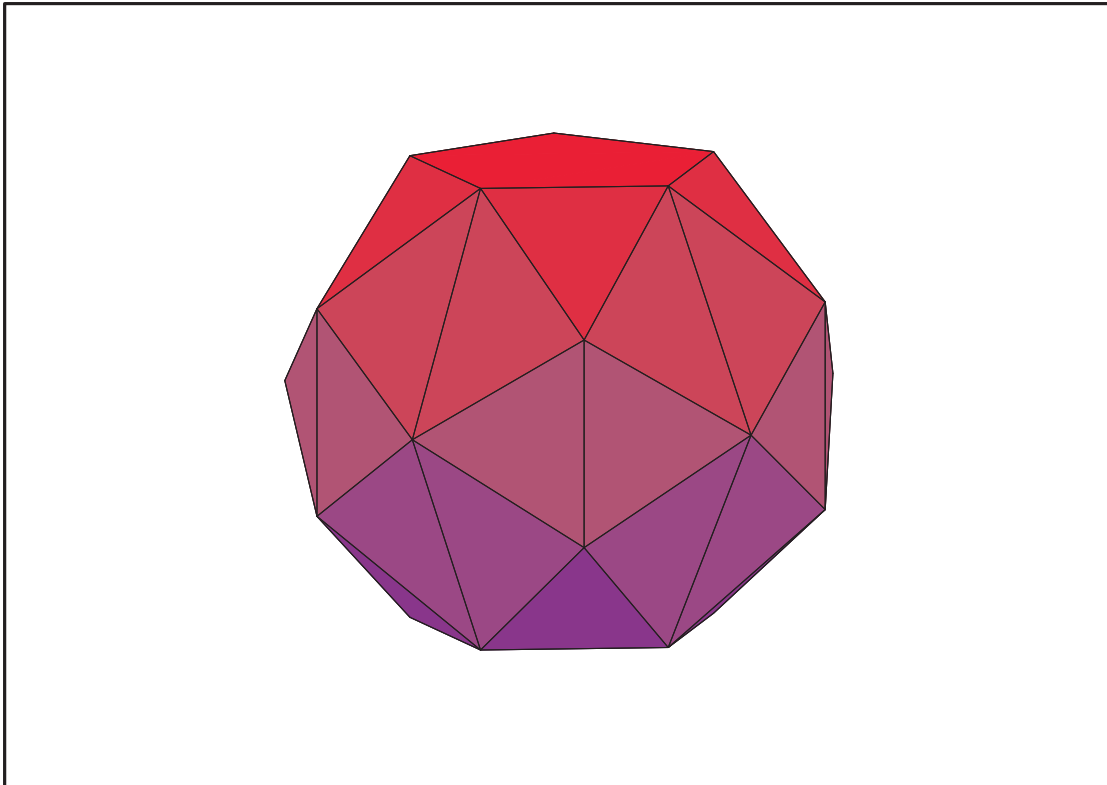


FIG. 11.89 – Graphique : Enveloppe convexe du support du plan N25qui.

#### 11.30.4. Variance

##### Modèle de Degré 1

```
> ReadPlan("N25qui");Plan:N25qui:=Plan:
> Var:=Var_Norm_Blue_D_Centre(N25qua,1,0);
 Var := 1 + 3 var32 + 3 var22 + 3 var12

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));
 1 + 3 IndetDecompInKfi1, IndetDecompInKfi1 = var12 + var22 + var32

> animate(plot3d,[Var,var[1]=-2..2,var[2]=-2..2],var[3]=-2..2):
```

**Modèle de Degré 2**

```
> Var:=Var_Norm_Blue_D_Centre(N25qui,2,0);
 "M_D is not full rank"

> Var:=expand(Var_Norm_Blue_D_Centre(N25qui,2,1));

 Var := 25 - 47 var32 - 47 var22 - 47 var12 + 62 var22 var32 + 62 var12 var32 + 62 var12 var22
 + 31 var34 + 31 var24 + 31 var14

> DecompInKfiInfo({add(y^2,y=seq(var[i],i=1..3))
},Var,poly_algebra(seq(var[i],i=1..3)));

 25 + 31 IndetDecompInKfi12 - 47 IndetDecompInKfi1,
 IndetDecompInKfi1 = var12 + var22 + var32
```

## 11.31. Conclusion

L'étude des dispositifs précédents a comme première conséquence les théorèmes suivants sur l'existence de plans isovariants pour un modèle de degré 2.

**Théorème 11.31.1.** *Il existe pour  $N \in \{13, 15\} \cup [17, +\infty[$  des dispositifs expérimentaux de cardinal  $N$  qui, analysés avec un modèle polynomial complet de degré 2, sont isovariants. De plus ces dispositifs présentent la particularité suivante : le support de chacun des plans est l'union d'un plan sphérique et d'un point au centre du domaine expérimental. Dans tous les cas, ils peuvent ne comporter aucune répétition dans leur partie sphérique. En fonction du cardinal du plan recherché, ils peuvent comporter un nombre partiel ou total de répétitions.*

*Pour chacune des valeurs de  $N$  suivantes 13, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 31, 33, 37, 39, 49, 55, 57, 61, 73, 81, les coordonnées exactes de ces plans sont connues et ces plans ne sont pas construits comme l'union de deux plans de cardinal inférieur.*

**Démonstration 11.31.1.** L'existence et le calcul des coordonnées exactes des plans sphériques de force supérieure ou égale à 4 lorsque  $N$  est égal à 12, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30, 32, 36, 38, 48, 54, 56, 60, 72, 80 découlent de l'étude de cas précédente. L'ajout d'un point au centre à ces dispositifs les rend donc isovariants lorsqu'ils sont analysés par un modèle de degré 2 et nous obtenons les valeurs de cardinaux mentionnées dans le théorème ci-dessus. Ce qui prouve la seconde partie du théorème.

Rappelons que l'union de deux plans sphériques de force  $t$  est un plan sphérique de force  $t$ . Ainsi pour tout cardinal  $N$  pouvant s'écrire comme  $1 + N_1 + N_2 + \dots + N_k$  où  $N_1, N_2, \dots, N_k$  sont  $k$  entiers de la liste 12, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30, 32, 36, 38, 48, 54, 56, 60, 72, 80 sont des plans isovariants pour un modèle de degré 2. En particulier si nous disposons d'une suite de 12 valeurs consécutives  $C_1, \dots, C_{12}$  pour lesquelles il existe un tel dispositif alors en ajoutant le nombre convenable de fois un plan sphérique de cardinal 12, l'icosaèdre, nous pourrions obtenir des plans isovariants pour un modèle d'ordre pour tous les cardinaux supérieurs ou égaux à  $C - 1$ .

Considérons la suite de valeurs 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28. Il manque la valeur 27. Or  $27=1+12+14$ , ainsi il existe un plan isovariant de cardinal 27 pour un modèle d'ordre 2. Nous le construisons comme l'union d'un plan sphérique de cardinal 12 de force supérieure à 4, d'un plan sphérique de cardinal 14 de force supérieure à 4 et d'un point au centre. Nous obtenons alors 12 valeurs consécutives de 17 à 28, ce qui donne l'existence pour  $N \in \{13, 15\} \cup [17, +\infty[$ . Ces plans peuvent comporter des répétitions. Nous raffinons la construction précédente de la manière suivante : l'image d'un plan sphérique par une transformation orthogonale est un plan sphérique. Nous éviterons ainsi les répétitions en transformant les dispositifs dont nous prenons l'union par une rotation. Reste à trouver une famille infinie de rotations dont les coefficients de la matrice sont algébriques. Ce problème se réduit à savoir s'il existe une famille infinie  $\Theta$  de valeurs de  $\theta$  pour lesquelles  $\cos(\theta)$  est algébrique. Nous pouvons aborder ce problème à l'aide de la théorie des polygones réguliers à  $n$  côtés constructibles à la règle et au compas. En effet leurs coordonnées dans un repère s'expriment alors comme  $(\cos(2\pi * k/n), \sin(2\pi * k/n))$ . Si un polygone régulier à  $n$  côtés est constructible à la règle et au compas alors  $\cos(2\pi * k/n)$  et  $\sin(2\pi * k/n)$  sont algébriques sur  $\mathbb{Q}$  et leurs degrés sont une puissance de 2. Il existe

une famille infinie de tels polygones, ceux pour lesquels nous avons  $n = 2^i F_1^{j_1} F_2^{j_2} \dots F_k^{j_k}$  où  $i$  est un entier naturel supérieur à 1 et pour  $p \in [1, k]$ ,  $j_p = 0$  ou  $j_p = 1$  et  $F_j$  sont les nombres premiers de Fermat c'est-à-dire les nombres premiers de la forme  $2^{2^j} + 1$ , voir par exemple [Car01]. Puisqu'il existe des nombres de Fermat premiers, par exemple 3, 5, 17, 257 et 65537, il existe bien une infinité de polygones réguliers dont les coordonnées sont algébriques.

**Conjecture 11.31.1.** *La valeur, conjecturée comme égale à 3 par Hardin et Sloane, de  $\tau(13)$  et  $\tau(15)$  permet de formuler la conjecture suivante : il n'existe pas de dispositif, union d'un plan sphérique et d'un point au centre et de cardinal 14 et 16, qui soit isovariant s'il est analysé à l'aide d'un modèle polynomial de degré 2.*

**Théorème 11.31.2.** *Pour  $N \in [13, +\infty[$ , il existe des dispositifs expérimentaux de cardinal  $N$  qui, analysés avec un modèle polynomial de degré 2, sont isovariants.*

*Précisons la nature de chacun de ces dispositifs :*

- *Le support de chacun d'entre eux est l'union d'un plan sphérique et d'au moins un point au centre. Il est possible de spécifier le nombre total d'essais  $N$  et le nombre  $k \geq 1$  de répétitions au centre du domaine arbitrairement dès que  $N \geq k + 16$ ,  $N = k + 14$  ou  $N = k + 12$ . Les valeurs de  $N$  et de  $k$  pour lesquelles nous avons  $N = k + 15$  ou  $N = k + 13$  ne sont pas accessibles, il suffit de prendre à la place de la valeur  $k$  la valeur  $k' = k + 1$  ou la valeur  $k'' = k - 1$ .*
- *Pour toutes les valeurs de  $N$  possibles, il est possible de construire des dispositifs ne comportant aucune répétition hormis à l'origine du repère.*
- *En fonction de la valeur de  $N$ , il est possible de construire des dispositifs possédant un nombre partiel ou total de répétitions dans la partie sphérique du dispositif.*

**Démonstration 11.31.2.** Puisque nous autorisons plus d'une répétition au centre du domaine, les cardinaux de 14 et 16 deviennent accessibles :  $14 = 12 + 2$ ,  $16 = 12 + 4 = 14 + 2$  ce qui prouve le premier point du théorème.

Si  $N \geq k + 16$ ,  $N = k + 14$  ou  $N = k + 12$ , le théorème 11.31.1 précédent indique qu'il existe un plan isovariant pour un modèle de degré construit comme l'union d'un plan sphérique de cardinal  $N - k$  et d'un point au centre. L'ajout d'essais au centre ne modifiant pas le caractère isovariant d'un dispositif, un dispositif formé d'un plan sphérique de cardinal  $N - k$  et de  $k$  points au centre est isovariant lorsqu'il est analysé à l'aide d'un modèle d'ordre 2. Si  $N = 13$  ou  $N = 15$ , le changement de  $k$  en  $k'$  ou en  $k''$  découle des mêmes considérations.

Signalons enfin, que les plans construits ici, héritent, pour leur partie sphérique, des propriétés des plans construits au théorème 11.31.1, en particulier la possibilité de les obtenir sans répétitions dans leur partie sphérique.

La situation se complique lorsque nous considérons des plans analysés par des modèles de degré supérieur ou égaux à trois en raison des confusions d'effets qui sont plus complexes à éliminer. En effet il faut alors ajouter d'autres points que des points au centre du domaine ce qui modifiera la fonction génératrice des moments du plan et affectera la propriété d'isovariance.

Il existe plusieurs manières de traiter cette difficulté. L'approche par les  $t$ -mesures euclidiennes est une possibilité pour obtenir des dispositifs isovariants. Une autre manière de remédier à cette situation est de considérer des modèles qui sont naturellement adaptés

au contexte sphérique.

Nous introduisons pour cela les fonctions polynomiales définies sur la sphère unité, pour plus de détails voir le chapitre 8 sur les notions algébriques.

**Proposition 11.31.1.** *Les fonctions polynomiales définies sur la sphère unité sont isomorphes à l'anneau quotient  $M_S = \mathbb{R}[\mathbf{x}]/(X^2 + Y^2 + Z^2 - 1)$ , nous notons  $M_{S,k} = \mathbb{R}_k[\mathbf{x}]/(X^2 + Y^2 + Z^2 - 1)$ . Une base de  $M_{S,k}$  est constituée par la somme directe orthogonale de  $l$  sous-espaces vectoriels chacun engendré par une base orthonormale de polynômes harmoniques de degré  $l$ , pour  $0 \leq l \leq k$ .*

**Définition 11.31.1.** *Nous appelons modèle polynomial sphérique de degré  $d$ , un modèle pour lequel  $\text{Vect}(\mathbf{f}(\mathcal{S})) = M_{S,d}$ .*

**Théorème 11.31.3.** *Si, à la place du modèle polynomial classique d'ordre  $k$ , où  $k$  est un entier supérieur ou égal à 1, nous considérons le modèle polynomial sphérique pour analyser un dispositif expérimental sphérique, alors il n'y a plus aucune confusion d'effets systématique liée à la nature du domaine expérimental considéré : quelque soit le sous-ensemble de fonctions polynomiales linéairement indépendantes sur la sphère, il existe un plan sphérique pour lequel tous les termes de ce modèle sont estimables.*

**Démonstration 11.31.3.** Soit  $d$  le degré maximal des termes du modèle. Le théorème 4.2.2 du chapitre 4, non constructif, de Seymour et Slavasky sur l'existence des plans sphériques de force arbitraire et donc en particulier de force  $2d$ . Or la preuve de l'inégalité sur le nombre minimal de points d'un plan sphérique de force  $2d$  implique qu'un tel plan rend estimable tous les paramètres associés au modèle sphérique d'ordre  $k$ .

**Théorème 11.31.4.** *Les dispositifs de cardinal  $N=4,6$  à 100 sont des dispositifs orthogonaux lorsque qu'ils sont analysés avec un modèle polynomial sphérique d'ordre 1. Ils sont optimaux pour un modèle formé par toutes les des harmoniques sphériques d'ordre inférieur ou égal à 1 pour tous les critères d'optimalité  $\Phi_p$  introduits par Kiefer et repris dans l'exemple 2.2.1 du chapitre 2.*

*Les dispositifs de cardinal  $N=12,14,16$  à 100 sont des dispositifs orthogonaux lorsque qu'ils sont analysés avec un modèle polynomial sphérique d'ordre 2. Ils sont optimaux pour un modèle formé par toutes les des harmoniques sphériques d'ordre inférieur ou égal à 2 pour tous les critères d'optimalité  $\Phi_p$  introduits par Kiefer et repris dans l'exemple 2.2.1 du chapitre 2.*

*Les dispositifs de cardinal  $N=24,26,28$  à 100 sont des dispositifs orthogonaux lorsque qu'ils sont analysés avec un modèle polynomial sphérique d'ordre 3. Ils sont optimaux pour un modèle formé par toutes les des harmoniques sphériques d'ordre inférieur ou égal à 3 pour tous les critères d'optimalité  $\Phi_p$  introduits par Kiefer et repris dans l'exemple 2.2.1 du chapitre 2.*

*Les dispositifs de cardinal  $N=36,40,42,44$  à 100 sont des dispositifs orthogonaux lorsque qu'ils sont analysés avec un modèle polynomial sphérique d'ordre 4. Ils sont optimaux pour un modèle formé par toutes les des harmoniques sphériques d'ordre inférieur ou égal à 4 pour tous les critères d'optimalité  $\Phi_p$  introduits par Kiefer et repris dans l'exemple 2.2.1 du chapitre 2.*

*Les dispositifs de cardinal  $N=60,62,64$  à 100 sont des dispositifs orthogonaux lorsque qu'ils*

| Degré | Cardinal du plan $N$                                                                                                           |
|-------|--------------------------------------------------------------------------------------------------------------------------------|
| 1     | 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30, 32, 36, 38, 48, 54, 56, 60, 72, 80. |
| 2     | 12, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30, 32, 36, 38, 48, 54, 56, 59, 72, 80.                                |
| 3     | 24, 26, 27, 28, 30, 32, 36, 38, 48, 54, 56, 59, 72, 80.                                                                        |
| 4     | 36, 48, 54, 56, 59, 72, 80.                                                                                                    |
| 5     | 60, 72, 80.                                                                                                                    |

TAB. 11.2 – Plans sphériques orthogonaux exacts

sont analysés avec un modèle polynomial sphérique d'ordre 5. Ils sont optimaux pour un modèle formé par toutes les des harmoniques sphériques d'ordre inférieur ou égal à 5 pour tous les critères d'optimalité  $\Phi_p$  introduits par Kiefer et repris dans l'exemple 2.2.1 du chapitre 2.

Les dispositifs de cardinal  $N=84, 86$  à 100 sont des dispositifs orthogonaux lorsque qu'ils sont analysés avec un modèle polynomial sphérique d'ordre 6. Ils sont optimaux pour un modèle formé par toutes les des harmoniques sphériques d'ordre inférieur ou égal à 6 pour tous les critères d'optimalité  $\Phi_p$  introduits par Kiefer et repris dans l'exemple 2.2.1 du chapitre 2.

Le tableau 7.1.5 indique les plans dont les coordonnées exactes sont connues ainsi que le degré jusqu'auquel les propriétés d'orthogonalité et de  $\Phi_p$ -optimalité sont vérifiées.

**Démonstration 11.31.4.** Il s'agit d'une conséquence directe de la caractérisation des plans sphériques à l'aide des polynômes harmoniques et du résultat de Dette, Melas et Pepelyshev, cf. [DMP05].

## 11.32. Cas de la dimension 4

### 11.32.1. Les résultats de Hardin et Sloane

Dans ce chapitre nous nous intéressons à une série de dispositifs ayant été construits ou étudiés par Sloane, Hardin et Cara dans leur rapport préliminaire [SHC03].

L'objectif des trois auteurs a été similaire à celui que nous avons exposé pour le cas de la dimension 3 à la section 7.1.1 : pour un cardinal donné, construction du plan sphérique dont la force est maximale et les symétries les plus importantes.

Les résultats obtenus par Sloane, Hardin et Cara forment le tableau 7.3 :

- La première colonne  $N$  correspond au cardinal des plans sphériques considérés.
- La deuxième colonne  $\tau_4(N)$  correspond à la force maximale qu'il a été possible d'obtenir avec un plan sphérique de cardinal  $N$ .
- La troisième colonne « Group Order » est égal à l'ordre du plus grand groupe de symétries du dispositif.
- La quatrième colonne « Remarks » précise parfois succinctement la structure du plan.

Rappelons qu'un  $t$ -plan sphérique de  $\mathbb{R}^4$  muni de sa structure euclidienne canonique est un ensemble de  $N$  points  $\mathcal{P} = \{\mathbf{P}_1, \dots, \mathbf{P}_N\}$  de la sphère unité  $\mathcal{S}^4 = \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x}\| = 1\}$  tels que l'égalité

$$\int_{\mathcal{S}^4} f(\mathbf{x}) d\mu(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^N f(\mathbf{P}_i), \quad (11.32.1)$$

où  $\mu$  est la mesure de probabilité uniforme sur  $\mathcal{S}^4$ , soit vérifiée pour tout polynôme de  $\mathbb{R}[\mathbf{x}]$  de degré inférieur ou égal à  $t$ .

Le processus de recherche des plans sphériques est identique à celui utilisé dans le cas de la dimension 3 et décrit à la section 7.1.1.

TAB. 11.3: « Conjectured values of  $\tau_4(N)$ , the largest  $t$  for which an  $N$ -point configuration on the sphere in 4 dimensions forms a spherical  $t$ -designs. »

| $N$ | $\tau_4(N)$ | Group Order | Remarks                       |
|-----|-------------|-------------|-------------------------------|
| 1   | 0           | $\infty$    | single point                  |
| 2   | 1           | $\infty$    | two antipodal points          |
| 3   | 1           | $\infty$    | equilateral triangle          |
| 4   | 1           | 48          | regular tetrahedron           |
| 5   | 2           | 120         | regular simplex               |
| 6   | 2           | 72          | join of two triangles         |
| 7   | 2           | 14          | $\{(\omega^j, \omega^{2j})\}$ |
| 8   | 3           | 384         | orthoplex                     |
| 9   | 2           | 72          | $\{(\omega^j, \omega^k)\}$    |
| 10  | 3           | 240         | diplo-simplex                 |
| 11  | 3           | 22          |                               |
| 12  | 3           | 288         | join of two hexagons          |
| 13  | 3           | 52          | $\{(\omega^j, \omega^{5j})\}$ |
| 14  | 3           | 392         |                               |
| 15  | 3           | 60          |                               |
| 16  | 3           | 512         | join of two octagons          |
| 17  | 3           | 68          |                               |
| 18  | 3           | 648         |                               |
| 19  | 3           | 48          |                               |
| 20  | 4           | 20          | [HS92]                        |
| 21  | 4           | 14          |                               |
| 22  | 4           | 6           |                               |
| 23  | 4           | 6           |                               |
| 24  | 5           | 1152        | 24-cell                       |
| 25  | 4           | 20          |                               |
| 26  | 4           | 52          |                               |
| 27  | 4           | 18          |                               |

### 11.32 Cas de la dimension 4

---

| $N$ | $\tau_4(N)$ | Group Order | Remarks      |
|-----|-------------|-------------|--------------|
| 28  | 5           | 28          |              |
| 29  | 4           | 8           |              |
| 30  | 5           | 40          |              |
| 31  | 4           | 12          |              |
| 32  | 5           | 192         |              |
| 33  | 5           | 10          |              |
| 34  | 5           | 12          |              |
| 35  | 5           | 10          |              |
| 36  | 5           | 144         |              |
| 37  | 5           | 12          |              |
| 38  | 5           | 48          |              |
| 39  | 5           | 26          |              |
| 40  | 5           | 240         |              |
| 41  | 5           | 12          |              |
| 42  | 6           | 42          |              |
| 43  | 5           | 12          |              |
| 44  | 5           | 96          |              |
| 45  | 5           | 30          |              |
| 46  | 5           | 96          |              |
| 47  | 5           | 12          |              |
| 48  | 7           | 2304        | two 24-cells |
| 49  | 5           | 4           |              |
| 50  | 6           | 100         |              |
| 51  | 6           | 4           |              |
| 52  | 6           | 52          |              |
| 53  | 6           | 6           |              |
| 54  | 7           | 54          |              |
| 55  | 6           | 20          |              |
| 56  | 7           | 16          |              |
| 57  | 6           | 9           |              |
| 58  | 6           | 116         |              |
| 59  | 6           | 12          |              |
| 60  | 7           | 240         |              |
| 61  | 6           | 8           |              |
| 62  | 7           | 8           |              |
| 63  | 6           | 18          |              |
| 64  | 7           | 256         |              |
| 65  | 6           | 52          |              |
| 66  | 7           | 66          |              |
| 67  | 6           | 12          |              |
| 68  | 7           | 136         |              |
| 69  | 7           | 3           |              |
| 70  | 7           | 40          |              |
| 71  | 7           | 6           |              |



| $N$      | $\tau_4(N)$ | Group Order | Remarks          |
|----------|-------------|-------------|------------------|
| 72       | 7           | 576         |                  |
| 73       | 7           | 6           |                  |
| 74       | 7           | 16          |                  |
| 75       | 7           | 6           |                  |
| 76       | 7           | 38          |                  |
| 77       | 7           | 22          |                  |
| 78       | 7           | 52          |                  |
| 79       | 7           | 8           |                  |
| 80       | 7           | 256         |                  |
| 81       | 7           | 36          |                  |
| 82       | 7           | 164         |                  |
| 83       | 7           | 12          |                  |
| 84       | 7           | 144         |                  |
| 85       | 7           | 34          |                  |
| 86       | 7           | 24          |                  |
| 87       | 7           | 18          |                  |
| 88       | 7           | 128         |                  |
| 89       | 7           | 12          |                  |
| 90       | 7           | 180         |                  |
| 91       | 7           | 52          |                  |
| 92       | 7           | 48          |                  |
| 93       | 7           | 36          |                  |
| 94       | 7           | 96          |                  |
| 95       | 7           | 38          |                  |
| 96       | 8           | 6           |                  |
| 97       | 8           | 1           |                  |
| 98       | 8           | 4           |                  |
| 99       | 8           | 11          |                  |
| 100      | 8           | 20          |                  |
| $\vdots$ | $\vdots$    | $\vdots$    | $\vdots$         |
| 120      | 11          | 14400       | 600-cell [DGS77] |

Nous résumons ici les informations concernant la nature de la preuve de la force des dispositifs de la liste précédente.

- Pour des plans de cardinal  $N$  de 1 à 20, 24, 48, et 120 les résultats d’existence se trouvent dans [Mim90], [Baj92], [Baj98], [HS92] et [DGS77].
- L’existence des plans de cardinaux  $N = 21, 22, 23, 25, 26, 27, 31$  avait été conjecturée dans [HS92].
- Sloane, Hardin et Cara affirment avoir montré l’existence des plans de cardinaux  $N = 21, 26, 28, 32, 36, 39, 40, 42, 44, 52, 54, 60, 65, 66, 72, 78, 84, 90$ . Toutefois aucune indication précise sur la structure, comme union d’orbites de groupes finis, de ces plans n’est indiquée dans [SHC03].

### 11.32.2. Utilisation des informations de Hardin et Sloane

Rappelons que nous avons caractérisé la nature radiale de la variance de l'estimateur de la réponse moyenne à l'aide de la fonction génératrice des moments. Un plan est un  $2t$ —plan sphérique si et seulement si la partie polynomiale limitée au degré  $2t$  de la fonction génératrice des moments est radiale. Tous les plans du tableau 11.3 dont la force est supérieure à  $2t$  sont donc des plans pour lesquels la partie polynomiale limitée au degré  $2t$  de la fonction génératrice des moments est radiale.

**Théorème 11.32.1.** *Les dispositifs de cardinal  $N=5$  à 100 et 120 sont des dispositifs isovariants lorsqu'ils sont analysés avec un modèle polynomial d'ordre 1.*

*Les dispositifs de cardinal  $N=20$  à 100 et 120, augmentés d'au moins un point au centre, sont des dispositifs fortement isovariants lorsqu'ils sont analysés avec un modèle polynomial d'ordre 2.*

**Démonstration 11.32.1.** En premier, les dispositifs dont le support est un plan sphérique de force supérieure ou égale à 2 et analysés à l'aide d'un modèle d'ordre 1 sont isovariants.

Considérons en second lieu un dispositif expérimental analysé à l'aide d'un modèle d'ordre 2. Tous les plans sphériques pour lesquels  $\tau_4(N) \geq 2$ , augmentés d'au moins un point au centre, sont des dispositifs isovariants.

Enfin précisons que certains des dispositifs proposés par Sloane, Hardin et Cara sont de force impaire  $2t + 1$  ce qui n'est pas utile pour les considérations d'isovariance. Considérer un dispositif de force paire  $2t$  de même cardinal permettra l'ajout d'une recherche d'optimalité alphabétique.

## 11.33. Conclusion

Cette étude de cas a permis de montrer que la méthodologie introduite dans le chapitre 5 permet d'aboutir à la construction de plans expérimentaux dont les coordonnées sont connues de manière exacte. Nous avons alors montré l'existence et indiqué comment obtenir de tels dispositifs pour tous les cardinaux pour lesquels il existe des plans isovariants pour des modèles polynomiaux complets de degré 2 en dimension 3. Nous en avons également tiré l'existence de dispositifs orthogonaux et  $\Phi_p$ —optimaux pour des problèmes de régression sphérique dont le modèle est spécifié en termes d'harmoniques sphériques d'ordre inférieur 1, 2, 3, 4, 5 ou 6 dans le cas de la dimension 3 et 4. Rappelons que ces dispositifs ont une application particulièrement intéressante dans les problèmes de reconnaissance de formes tridimensionnelles, cf. [DMP05].



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**Résumé** - Ce travail comporte deux parties, l'une théorique et l'autre pratique, et porte sur l'utilisation combinée d'outils combinatoires et algébriques pour la construction et l'analyse de plans d'expérience. Nous nous intéressons en particulier à des caractérisations polynomiales des propriétés d'invariance faible d'un plan expérimental et proposons une définition ainsi qu'un cadre de résolution d'un problème de construction de type polynomial à l'aide de la géométrie algébrique réelle et du lien entre l'optimisation semi-définie positive et le théorème des zéros réels. Nous nous intéresserons ici également à la méthodologie des surfaces de réponse et plus particulièrement à la propriété d'isovariance statistique, ce qui nous amène à étudier plus particulièrement des plans dont le support est inclus dans une sphère. Les principaux avantages de l'approche développée dans ce travail sont sa grande généralité, son automatisation et l'obtention des coordonnées exactes des points support du plan ce qui permet une détermination complète des confusions d'effets contrairement à la construction numérique de plans d'expérience euclidiens qui ne permet pas l'analyse exacte des confusions d'effets qui apparaissent nécessairement lorsque nous nous intéressons à des plans euclidiens de petite taille. Or une connaissance précise des confusions d'effets est nécessaire pour rendre possible l'utilisation de modèles polynomiaux qui ne seront plus limités au degré 2 comme c'est trop souvent le cas dans la théorie et dans la pratique. De nombreux exemples de construction de plans isovariants, l'étude de leurs caractéristiques ainsi que les programmes ayant permis d'obtenir ces résultats sont également présentés.

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