

# Mathematical modelling of tsunami waves

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PhD Thesis Defense  
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# Contents

- 1 Tsunami generation
  - Linear theory
  - Dislocations dynamics
- 2 Visco-potential free surface flows
  - Physical considerations
  - Systematic study
  - Long wave approximation
- 3 Water wave impacts and two phase flows
  - Physical context
  - Mathematical model
  - Finite volumes scheme
- 4 Perspectives

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# Several characteristic values

for a typical tsunami application in Indian Ocean

<i>physical parameter</i>	<i>typical value</i>
wave amplitude, $a$	0.5 m
water depth, $h_0$	4 km
wavelength, $\ell$	100 km

We can construct three dimensionless combinations :

Nonlinearity :  $\varepsilon := \frac{a}{h_0} \approx 10^{-4}$

Dispersion :  $\mu^2 := \left(\frac{h_0}{\ell}\right)^2 \approx 10^{-4}$

Stokes-Ursell number :  $S := \frac{\varepsilon}{\mu^2} \approx 1$

- Propagation stage is **nondispersive**  $\Rightarrow$  NSWE codes
- Flow is almost **linear**
  - $\Rightarrow$  Equations can be linearised
  - $\Rightarrow$  analytical solutions for simple geometries

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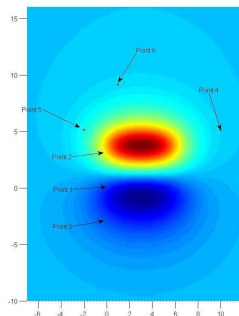
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  - $\Rightarrow$  analytical solutions for simple geometries

# Comparison between linear and nonlinear models

First minutes of tsunami propagation

## Main objective :

- To check the importance of nonlinear effects
  - Frequency dispersion
- 
- Complete water wave problem
    - BIEM accelerated by FMM (C. Fochesato)
  - Nonlinear shallow-water equations
    - VFFC scheme
  - Linearized water wave problem
    - Analytical solution for separable geometries



# Comparison results - I

Weakly Dispersive and weakly nonlinear waves

$$\varepsilon := \frac{a_0}{h_0} \cong 5 \times 10^{-4} \quad \mu^2 := \left(\frac{h_0}{\ell}\right)^2 \cong 10^{-4} \quad S := \frac{\varepsilon}{\mu^2} = 5$$

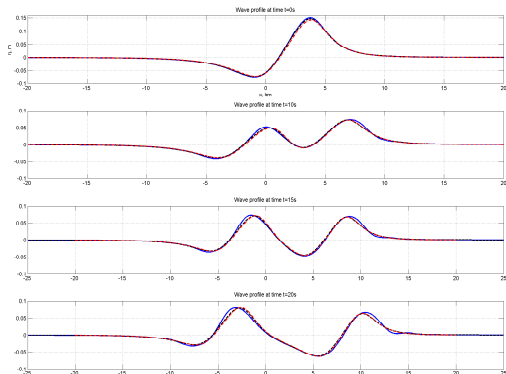


FIG.: - · - · - Linearized solution, — Fully nonlinear, - - - NSWE

# Comparison results - II

Dispersive and weakly nonlinear waves

$$\varepsilon := \frac{a_0}{h_0} \cong 0.0045 \quad \mu^2 := \left(\frac{h_0}{\ell}\right)^2 \cong 0.02 \quad S := \frac{\varepsilon}{\mu^2} = 0.225$$

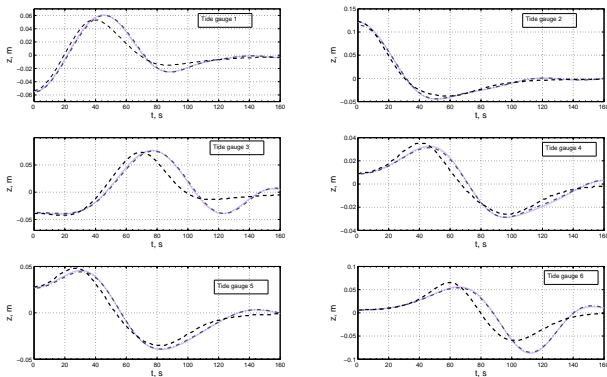


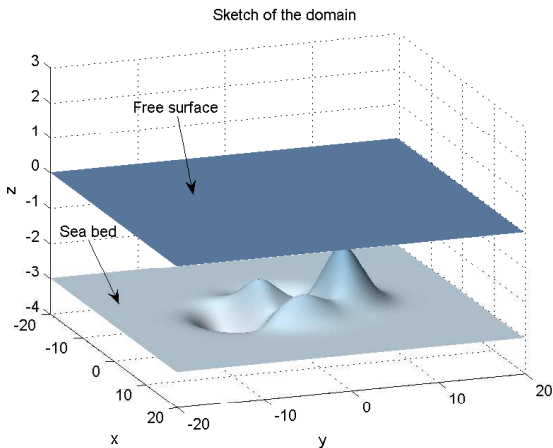
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# Traditional approach

## Approaches to generation

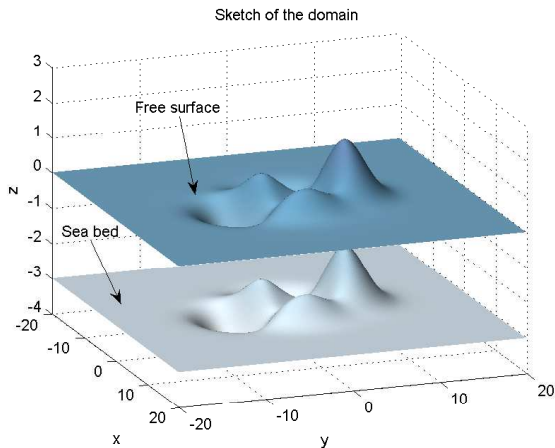
Put coseismic displacements directly on the free surface and let it propagate :



# Traditional approach

## Approaches to generation

Put coseismic displacements directly on the free surface and let it propagate :



# Analytical solution on flat bottom

## Cauchy-Poisson system

- General solution by Fourier-Laplace transform :

$$\eta(\vec{x}, t) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \frac{e^{i\mathbf{k}\cdot\vec{x}}}{\cosh(|\mathbf{k}|h)} \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{s^2 \bar{\zeta}(\mathbf{k}, s)}{s^2 + \omega^2} e^{st} ds d\mathbf{k}$$

$$\omega^2 = g|\mathbf{k}| \tanh(|\mathbf{k}|h)$$

- $\zeta(\vec{x}, t)$  : is the unknown **dynamic** sea-bed displacement
  - There is no available analytical solution for fault dynamics

### Issue :

Use static solution and make assumptions about the dynamics

# Dynamic sea-bed deformation

## Main ingredients

Variables separation :  $\zeta(\vec{x}, t) = D(\vec{x})T(t)$

- Celebrated Okada solution provides  $D(\vec{x})$
- Assumptions about time evolution  $T(t)$  (dynamic scenarios) :

<i>Instantaneous</i>	$T_i(t) = \mathcal{H}(t)$
<i>Exponential</i>	$T_e(t) = (1 - e^{-\alpha t})\mathcal{H}(t)$
<i>Trigonometric</i>	$T_c(t) = \mathcal{H}(t - t_0) + \frac{1}{2}[1 - \cos(\frac{\pi t}{t_0})]\mathcal{H}(t_0 - t)$
<i>Linear</i>	$T_l(t) = (\mathcal{H}(t - t_0) + \frac{t}{t_0}\mathcal{H}(t_0 - t))\mathcal{H}(t)$

- Application to linear waves :

$$\eta(\vec{x}, t) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \frac{\hat{D}(\mathbf{k}) e^{i\mathbf{k} \cdot \vec{x}}}{\cosh(|\mathbf{k}|h)} \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{s^2 \hat{T}(s)}{s^2 + \omega^2} e^{st} ds d\mathbf{k}$$

# Application to tsunami generation problems

How large is error in translating sea-bed deformation onto free surface ?

Passive : deformation translated on free surface

$$\eta(\vec{x}, t) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \widehat{D}(\mathbf{k}) e^{i\mathbf{k}\cdot\vec{x}} \cos \omega t \, d\mathbf{k}$$

Active : instantaneous scenario

$$\eta_i(\vec{x}, t) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} \frac{\widehat{D}(\mathbf{k}) e^{i\mathbf{k}\cdot\vec{x}}}{\cosh(|\mathbf{k}|h)} \cos \omega t \, d\mathbf{k}$$

Drawbacks :

- initial velocity is neglected
- dynamic character of the rupture
- wave amplitude is always slightly exceeded
- water has effect of low-pass filter

# Towards more realistic dynamic source model

## Modelling of fault dynamics

- Earth crust is a linear viscoelastic material (Kelvin-Voigt model)
- Isotropic homogeneous or heterogeneous medium
- Fault modeled as a Volterra dislocation
  - Displacement field is increased by the amount of the Burgers vector along any loop enclosing the dislocation

$$\oint_C d\vec{u} = \vec{b}$$

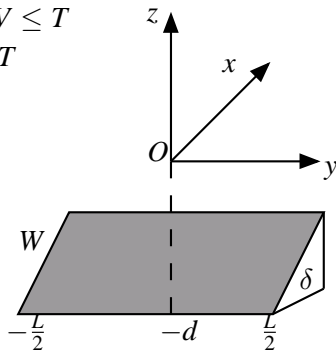
- Simplified situation with respect to fracture mechanics : location and displacement jump are known

# Haskell's model (1969)

## Rupture propagation and rise time

$$\vec{b}(\vec{x}, t) = \begin{cases} 0 & t - \zeta/V < 0 \\ (\vec{b}_0/T)(t - \zeta/V) & 0 \leq t - \zeta/V \leq T \\ \vec{b}_0 & t - \zeta/V > T \end{cases}$$

- $T$  is the rise time,  $V$  the rupture velocity
- $\zeta$  is a coordinate along the fault
- Front propagates unilaterally along the  $y$ -axis



# Coupled computation

Seismology/hydrodynamics coupling

## Viscoelastodynamics

- Space derivatives are discretized by FEM
- Implicit time stepping

## Hydrodynamics

- Governing equations are NSWE :

$$\eta_t + \nabla \cdot ((h + \eta)\vec{v}) = -\partial_t h,$$
$$\vec{v}_t + \frac{1}{2}\nabla|\vec{v}|^2 + g\nabla\eta = 0.$$

- Solved by VFFC scheme

Coupling with FEM computation is done through the bathymetry  $h = h(x, y, t)$



# Comparison of two approaches

## Remarks about simulation

### Active generation :

- We simulate only first  $10s$  of the Earthquake and couple it with hydrodynamic solver
- For  $t > 10s$  we assume that bottom remains in its latest configuration

### Passive generation :

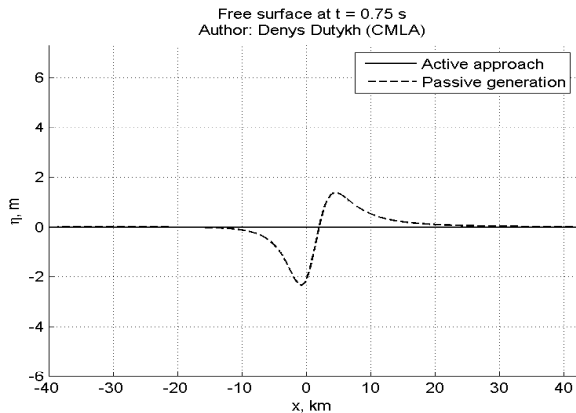
- Translate static dislocation solution onto free surface as initial condition

### Multiscale nature :

Two different scales : *elastic waves* and *water gravity waves*

# Results of numerical computation

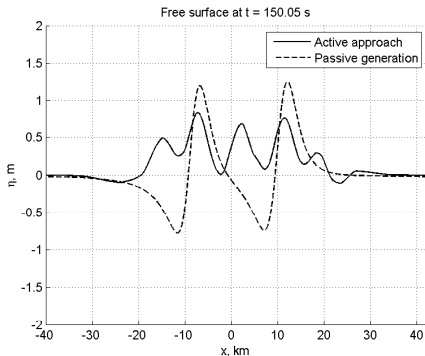
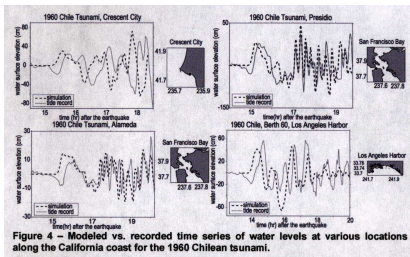
## Comparison between passive and active generation



# Discrepancy with tide gauges records

Chilean 1960 event

Reference : J.C. Borrero, B. Uslu, V. Titov, C.E. Synolakis (2006). *Modeling tsunamis for California ports and harbors*. Proceedings of the thirtieth International Conference on Coastal Engineering, ASCE



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# Importance of viscous effects

## Experimental evidences

- 1 Boussinesq (1895), Lamb (1932) formula

$$\frac{d\alpha}{dt} = -2\nu k^2 \alpha(t)$$

- 2 J. Bona, W. Pritchard & L. Scott, *An Evaluation of a Model Equation for Water Waves*. Phil. Trans. R. Soc. Lond. A, 1981, 302, 457-510

In « **Resumé** » section :

*[...] it was found that the inclusion of a dissipative term was much more important than the inclusion of the nonlinear term, although the inclusion of the nonlinear term was undoubtedly beneficial in describing the observations [...]*

# Mechanisms of dissipation

## 1 Wave breaking

- The main effect of wave breaking is the dissipation of energy. This can be modelled by adding dissipative terms in coastal regions where the wave becomes steeper

## 2 Turbulence

- For tsunami wave  $Re \geq 10^6$ , so the flow is turbulent
- $\Rightarrow$  energy extraction from waves in upper ocean

## 3 Boundary layers

- Regions where the viscosity is the most important
  - 1 free surface boundary layer
  - 2 bottom boundary layer

## 4 Molecular viscosity

- The least important factor for long waves

## Energy balance in a fluid flow

- We assume that flow is governed by incompressible Navier-Stokes equations :

$$\begin{aligned}\nabla \cdot \vec{u} &= 0 \\ \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} &= \vec{g} - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \tau\end{aligned}$$

- We multiply the second equation by  $\vec{u}$  and integrate on control volume  $\Omega$  :

$$\begin{aligned}\frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} (\rho |\vec{u}|^2) d\Omega + \frac{1}{2} \int_{\partial\Omega} \rho |\vec{u}|^2 \vec{u} \cdot \vec{n} d\sigma &= \\ = \int_{\partial\Omega} (-p\mathbb{I} + \tau) \vec{n} \cdot \vec{u} d\sigma + \int_{\Omega} \rho \vec{g} \cdot \vec{u} d\Omega - \underbrace{\frac{1}{2\mu} \int_{\Omega} \tau : \tau d\Omega}_{\mathcal{T}}\end{aligned}$$

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# Anatomy of dissipation

## Estimation of viscous dissipation rate

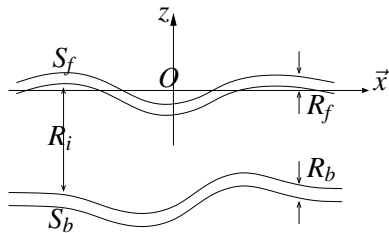


FIG.: Flow regions

- Interior region :

$$\mathcal{T}_{R_i} \sim \frac{1}{\mu} \left( \mu \frac{a}{t_0 \ell} \right)^2 \cdot \ell^3 \sim \mu$$

- Free surface boundary layer :

$$\mathcal{T}_{R_f} \sim \frac{1}{\mu} \left( \mu \frac{a}{t_0 \delta} \right)^2 \cdot \delta \ell^2 \sim \mu^{\frac{3}{2}}$$

- Bottom boundary layer :

$$\mathcal{T}_{R_b} \sim \frac{1}{\mu} \left( \mu \frac{a}{t_0 \delta} \right)^2 \cdot \delta \ell^2 \sim \mu^{\frac{1}{2}}$$

The previous scalings suggest us the following diagram :

$$\underbrace{\mathcal{O}(\mu^{\frac{1}{2}})}_{R_b} \longleftrightarrow \underbrace{\mathcal{O}(\mu)}_{R_i \cup S_f} \longleftrightarrow \underbrace{\mathcal{O}(\mu^{\frac{3}{2}})}_{R_f} \longleftrightarrow \underbrace{\mathcal{O}(\mu^2)}_{S_f} \longleftrightarrow \dots$$

# Visco-potential flows

How to add dissipation in potential flows ?

- 1 Consider Navier-Stokes equations
- 2 Write Helmholtz-Leray decomposition  $\vec{u} = \nabla\phi + \nabla \times \vec{\psi}$
- 3 Express vortical components  $\vec{\psi}$  of velocity field in terms of potential  $\phi$  and  $\eta$  using (**pseudo**) differential (**fractional**) operators

Kinematic condition :

$$\eta_t = \phi_z + \psi_{2x} - \psi_{1y} \quad \Rightarrow \quad \eta_t = \phi_z + 2\nu\nabla^2\eta$$

Dynamic condition :

$$\phi_t + g\eta + 2\nu\phi_{zz} + \mathcal{O}(\nu^{\frac{3}{2}}) = 0$$

# Boundary layer correction

## Bottom boundary condition

### Ideas of derivation :

- 1 Consider semi-infinite domain :  $z > -h$
- 2 Use pure Leray decomposition :  $\vec{v} = \nabla\phi + \vec{u}$ ,  $\nabla \cdot \vec{u} = 0$
- 3 Introduce boundary layer coordinate :  $\zeta = \frac{(z+h)}{\delta}$ , where  $\delta = \sqrt{\nu}$
- 4 Asymptotic expansion :  $\phi = \phi_0 + \delta\phi_1 + \dots$

### Bottom condition :

$$\left. \frac{\partial\phi}{\partial z} \right|_{z=-h} = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{zz}|_{z=-h}}{\sqrt{t-\tau}} d\tau = -\sqrt{\nu\mathcal{I}[\phi_{zz}]}$$

# Nonlocal visco-potential formulation

Resulting governing equations (cf. Liu & Orfila, JFM, 2004)

- Continuity equation

$$\Delta\phi = 0, \quad (x, y, z) \in \Omega,$$

- Kinematic free surface condition

$$\frac{\partial\eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial\phi}{\partial z} + 2\nu\nabla^2\eta, \quad z = \eta(x, y, t),$$

- Dynamic free surface condition

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 2\nu\nabla^2\phi, \quad z = \eta(x, y, t).$$

- Kinematic bottom condition

$$\frac{\partial\phi}{\partial z} + \nabla\phi \cdot \nabla h = -\sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\phi_{zz}}{\sqrt{t-\tau}} d\tau, \quad z = -h(x, y),$$



# Long wave approximation

## 1 Nonlocal Boussinesq equations :

- Mass conservation :

$$\eta_t + \nabla \cdot ((h + \eta)\vec{u}) + A_\theta h^3 \nabla^2 (\nabla \cdot \vec{u}) = 2\nu \Delta \eta + \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\nabla \cdot \vec{u}}{\sqrt{t - \tau}} d\tau$$

- Horizontal momentum :

$$\vec{u}_t + \frac{1}{2} \nabla |\vec{u}|^2 + g \nabla \eta - B_\theta h^2 \nabla (\nabla \cdot \vec{u}_t) = 2\nu \Delta \vec{u}$$

## 2 Nonlocal KdV equation :

$$\eta_t + \sqrt{\frac{g}{h}} \left( (h + \frac{3}{2}\eta)\eta_x + \frac{1}{6}h^3\eta_{xxx} - \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\eta_x}{\sqrt{t - \tau}} d\tau \right) = 2\nu\eta_{xx}$$

# Solitary wave attenuation

Effect of nonlocal term on the amplitude

## Numerical solution to nonlocal Boussinesq equations

- Solitary wave initial condition
- Fourier-type spectral method
- Comparison between :
  - 1 Classical Boussinesq equations
  - 2 Local dissipative terms
  - 3 Local + nonlocal dissipation

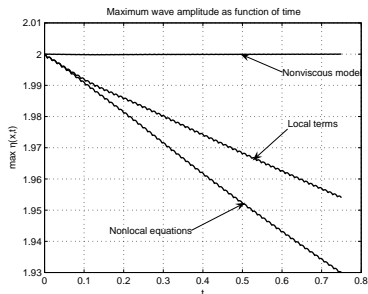


FIG.: Soliton amplitude

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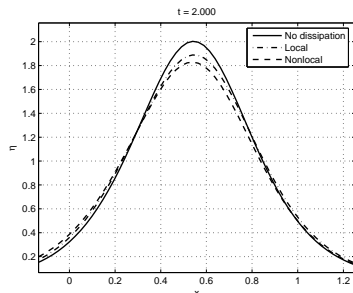


FIG.: Zoom on soliton crest

# Linear progressive waves attenuation

Generalization of Boussinesq/Lamb formula

- Consider nonlocal dissipative Airy equation

$$\eta_t + \sqrt{\frac{g}{h}} \left( h\eta_x + \frac{1}{6}h^3\eta_{xxx} - \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\eta_x}{\sqrt{t-\tau}} d\tau \right) = 2\nu\eta_{xx}$$

- Special form of solutions

$$\eta(x, t) = \mathcal{A}(t)e^{ik\xi}, \quad \xi = x - \sqrt{ght}, \quad \mathcal{A}(t) \in \mathbb{C}$$

Integro-differential equation :

$$\frac{d|\mathcal{A}|^2}{dt} + 4\nu k^2 |\mathcal{A}(t)|^2 + ik\sqrt{\frac{g\nu}{\pi h}} \int_0^t \frac{\bar{\mathcal{A}}(t)\mathcal{A}(\tau) - \mathcal{A}(t)\bar{\mathcal{A}}(\tau)}{\sqrt{t-\tau}} d\tau = 0$$

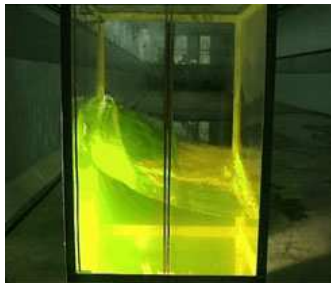
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# Physical phenomena

Two applications which motivated this study

- Wave sloshing in Liquefied Natural Gas (LNG) carriers



- Wave impacts on coastal structures



FIG.: GWK, Hannover

# Wave impacts on a wall

Ref : Bullock, Obhrai, Peregrine, Bredmose, 2007

## Impacts classification :

- **low-aeration** : the water adjacent to the wall contains typically 5% of air
- **high-aeration** : higher level of entrained air with clear evidence of entrapment



# Main results of the experimental study

Ref : Bullock, Obhrai, Peregrine, Bredmose, 2007

- Low-aeration impact
  - temporary and spatially localised pressure impulse
- High-aeration impact
  - less localised pressure spike with a longer rise time, fall time and duration
  - peak values of the pressure are lower

## Conclusion :

« Even when the pressures during a high-aeration impact are lower, the fact that the impact is less spatially localised and lasts longer may well lead to a higher total impulse »



# Influence of aeration

## Ideas for mathematical modelling

For low-aeration water wave impact ( $\alpha_g \approx 0.05$ ) :

- Sound speed drops down to  $\approx 54 \frac{m}{s}$
- Compressible effects are very important
  - Mach number is not tiny anymore
- CFL condition is not so severe
  - Explicit in time scheme

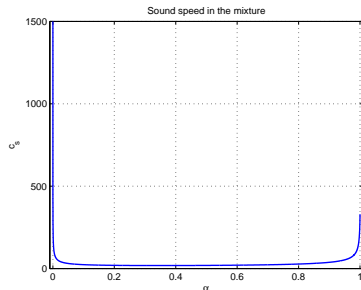


FIG.: Sound speed in the air/water mixture

# Two-phase homogenous model - I

## Governing equations

Mass conservation for each phase :

$$\partial_t(\alpha^\pm \rho^\pm) + \nabla \cdot (\alpha^\pm \rho^\pm \vec{u}) = 0,$$

Momentum equation :

$$\partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u} + p \mathbb{I}) = \rho \vec{g},$$

Energy conservation :

$$\partial_t(\rho E) + \nabla \cdot (\rho H \vec{u}) = \rho \vec{g} \cdot \vec{u},$$

$$\alpha^+ + \alpha^- = 1, \rho := \alpha^+ \rho^+ + \alpha^- \rho^-, H := E + \frac{p}{\rho}, E := e + \frac{1}{2} |\vec{u}|^2.$$

# Two-phase homogenous model - II

## Equation of state

- Ideal gas law for light fluid :

$$p^- = (\gamma - 1)\rho^- e^-,$$

$$e^- = c_v^- T^-,$$

- Tate's law for heavy fluid :

$$p^+ + \pi_0 = (\mathcal{N} - 1)\rho^+ e^+,$$

$$e^+ = c_v^+ T^+ + \frac{\pi_0}{\mathcal{N}\rho^+},$$

where  $\gamma$ ,  $c_v^\pm$ ,  $\pi_0$ ,  $\mathcal{N}$  are constants

**Additional assumption :** Two phases are in thermodynamic equilibrium :

$$p := p^+ = p^-, \quad T := T^+ = T^-$$

# Motivation for the choice of this model

Trade-off between model complexity and accuracy of the results

## Main reasons

- This model is hyperbolic
  - We have only four equations in 1D
  - Equations do not contain nonconservative products
  - Eigenvalues and eigenvectors can be computed analytically
- ⇒ computation is not expensive

We believe that this model gives qualitatively correct results for the flow and right impact pressure

# System of balance laws

## General ideas

Rewrite governing equations :

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathcal{F}(w) = \mathcal{S}(\vec{x}, t, w),$$

Integrate them over control volume :

$$\frac{d}{dt} \int_K w \, d\Omega + \int_{\partial K} \mathcal{F}(w) \cdot \vec{n}_{KL} \, d\sigma = \int_K \mathcal{S}(w) \, d\Omega$$

Introduce cell averages :

$$w_K(t) := \frac{1}{\text{vol}(K)} \int_K w(\vec{x}, t) \, d\Omega$$

How to express  $(\mathcal{F} \cdot \vec{n})|_{\partial K}$  in terms of  $\{w_K\}_{K \in \Omega}$  ?

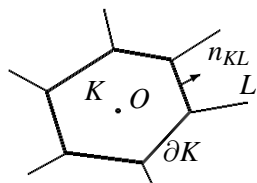


FIG.: Control volume

# Finite volumes scheme

## Volumes Finis à Flux Caractéristique (VFFC)

Use numerical flux of VFFC scheme to discretize advection operator :

$$\Phi(w_K, w_L; \vec{n}_{KL}) = \frac{\mathcal{F}_n(w_K) + \mathcal{F}_n(w_L)}{2} - U(\mu; \vec{n}_{KL}) \frac{\mathcal{F}_n(w_L) - \mathcal{F}_n(w_K)}{2}$$

where  $\mu$  is a mean state

$$\mu := \frac{\text{vol}(K)w_K + \text{vol}(L)w_L}{\text{vol}(K) + \text{vol}(L)}$$

and  $U(\mu; \vec{n}_{KL})$  is the sign matrix

$$U := \text{sign}(\mathbb{A}_n) \equiv \mathbf{R} \text{sign}(\Lambda) \mathbf{R}^{-1}, \quad \mathbb{A}_n := \frac{\partial(\mathcal{F} \cdot \vec{n})(w)}{\partial w}$$

**Remark :** Since, the advection operator is relatively simple,  $U$  can be computed analytically.

# Second order extension

## Monotone Upstream-centered Schemes for Conservation Laws (MUSCL)

We find our solution in class of affine by cell functions :

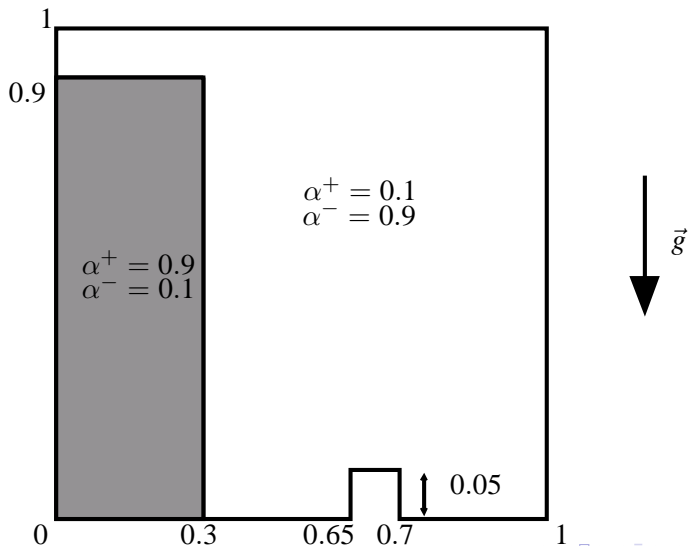
$$w_K(\vec{x}, t) := \bar{w}_K + (\nabla w)_K \cdot (\vec{x} - \vec{x}_0)$$

Conservation requirement :  $\frac{1}{\text{vol}(K)} \int_K w_K(\vec{x}, t) d\vec{x} \equiv \bar{w}_K$

- Gradient reconstruction procedure
  - Least squares method
  - Green-Gauss procedure
- Slope limiter
  - Barth - Jespersen (1989)
- Time stepping methods
  - classical Runge-Kutta schemes
  - SSP-RK (3,4) with  $CFL = 2$

# Water column test case - I

Geometry and description of the test case



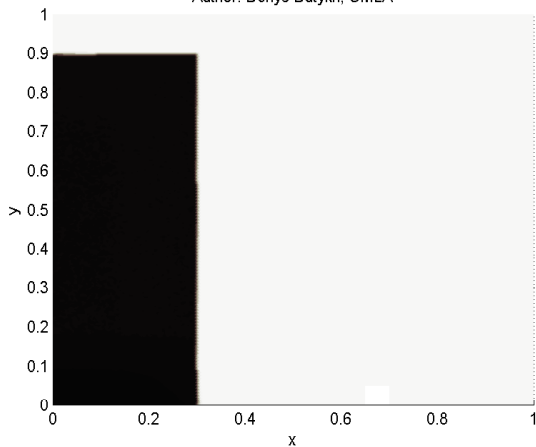


# Water column test case - II

Gravity acceleration  $g = 100m/s^2$ , in heavy fluid  $\alpha^+ = 0.9$ , in light fluid  $\alpha^+ = 0.1$

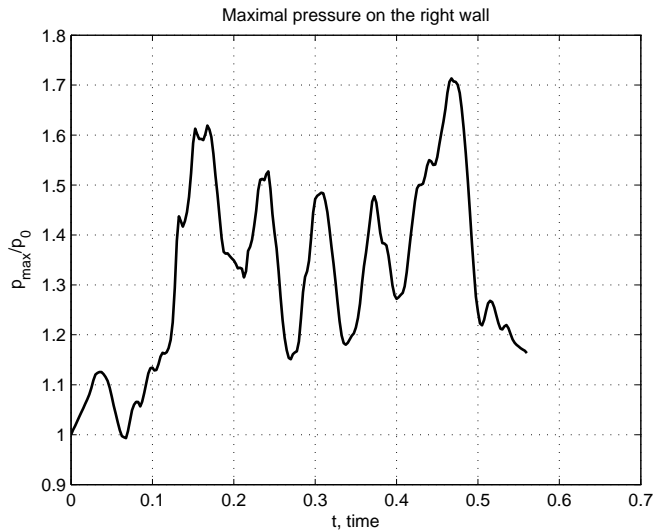
Fully compressible homogeneous two phase solver. Mixture density at  $t = 0.005$

Author: Denys Dutykh, CMLA



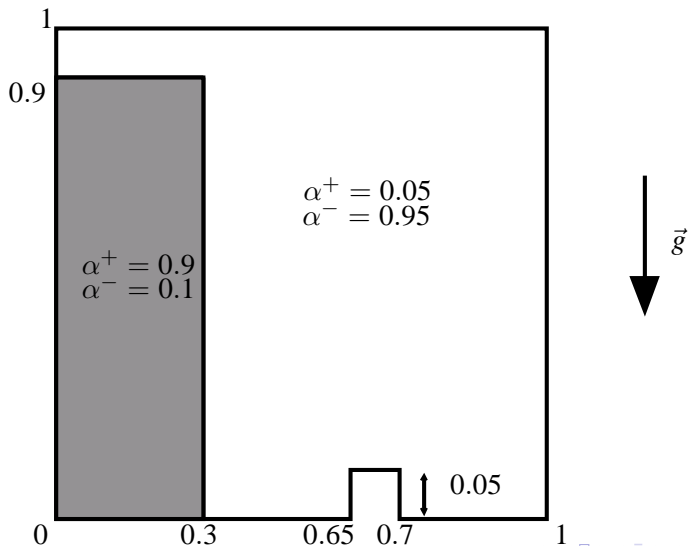
# Maximal pressure on the right wall

as a function of time  $t \mapsto \max_{(x,y) \in \Gamma \times [0,1]} p(x,y,t)$



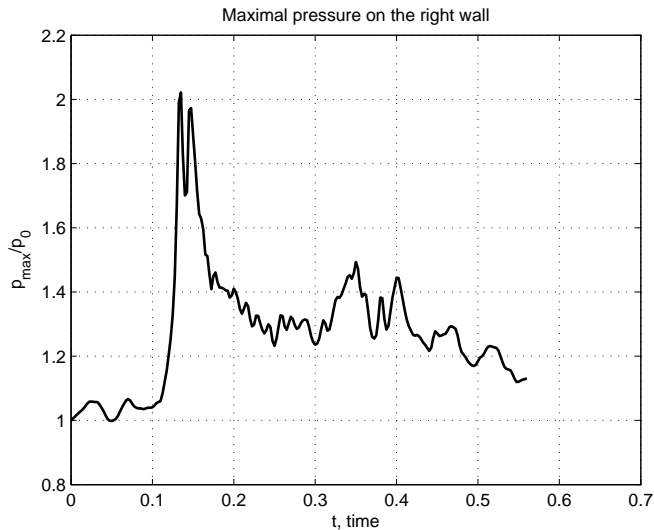
# Water column test case - III

Lighter gas case



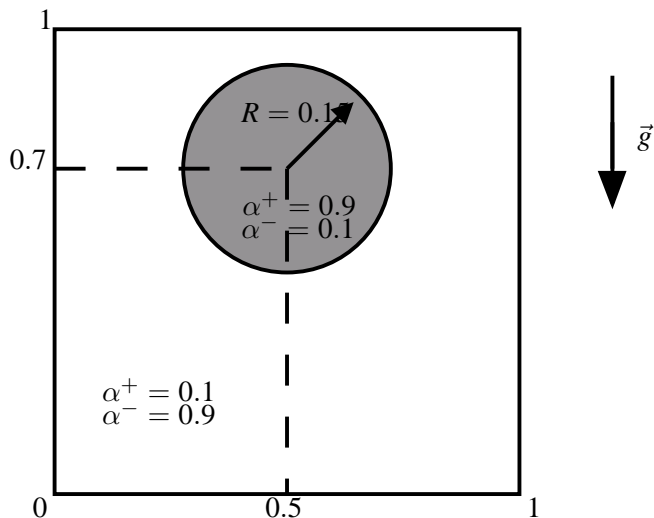
# Maximal pressure on the right wall

as a function of time  $t \mapsto \max_{(x,y) \in \Gamma \times [0,1]} p(x,y,t)$



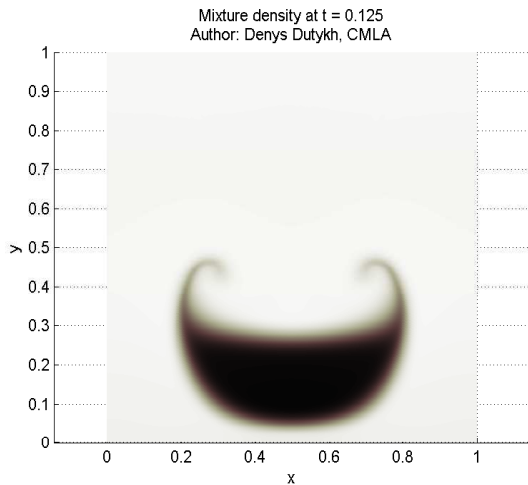
# Water drop test case - I

Geometry and description of the test case



# Water drop test case - II

Gravity acceleration  $g = 100m/s^2$



# Application to long wave propagation

## Viscous shallow water equations

- Governing equations :

$$\begin{aligned}\partial_t \eta + \nabla \cdot ((h + \eta) \vec{u}) &= -\partial_t h + \nu \nabla^2 \eta, \\ \partial_t \vec{u} + \nabla |\vec{u}|^2 + g \nabla \eta &= \nu \nabla^2 \vec{u}.\end{aligned}$$

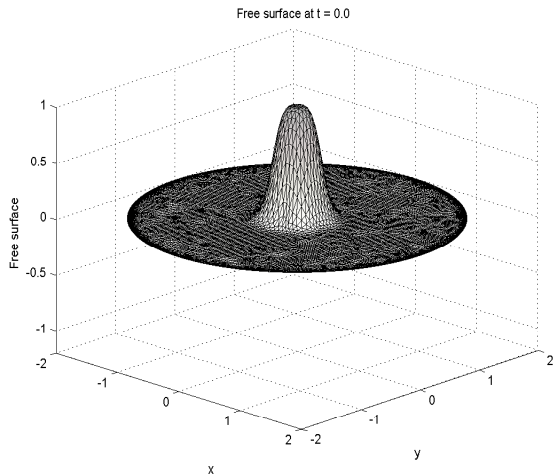
- System of balance laws :

$$\partial_t \mathbf{w} + \nabla \cdot \mathcal{F}(\mathbf{w}) = \nabla \cdot (\mathcal{D} \nabla \mathbf{w}) + \mathcal{S}(\mathbf{w})$$

Finite volumes scheme described above can be easily applied to these equations!

# Water drop in a basin - I

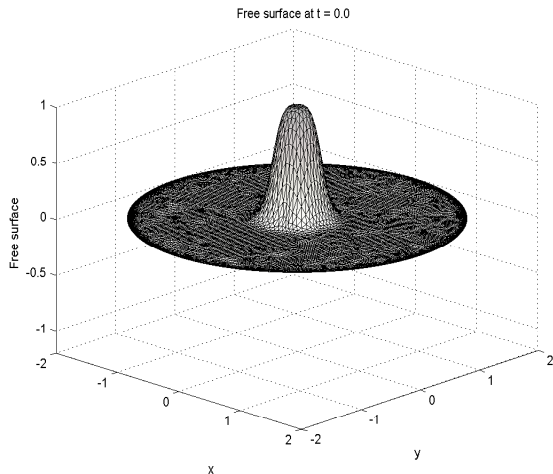
Nonviscous case :  $\nu_t = 0$





# Water drop in a basin - II

Viscous case :  $\nu_t = 0.015$



# Contents

- 1 Tsunami generation
  - Linear theory
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- 2 Visco-potential free surface flows
  - Physical considerations
  - Systematic study
  - Long wave approximation
- 3 Water wave impacts and two phase flows
  - Physical context
  - Mathematical model
  - Finite volumes scheme
- 4 Perspectives

# Directions for future research

- Tsunami generation
  - more realistic source models (rupture propagation, friction on the fault)
  - understand the role of Rayleigh waves in tsunami formation
- Visco-potential flows
  - rigorous justification of new formulation
  - relation to Navier-Stokes equations
- Two-phase compressible flows
  - Quantitative comparison with 6-equations model
  - Extension to *pure* phases
  - Test cases with air/water interface
    - Implicit solver because of CFL condition
    - $Ma \ll 1$

Thank you for your attention !

<http://www.cmla.ens-cachan.fr/~dutykh>