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Université Pierre et Marie Curie-Paris 6

Thèse de doctorat

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Représentations décomposables et sous-variétés lagrangiennes des espaces de modules associés aux groupes de surfaces
-
Decomposable representations and Lagrangian submanifolds of moduli spaces associated to surface groups

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L'écriture, mon travail noir, ne renvoyait à rien, et du coup, se prenait elle-même pour fin. J'écrivais pour écrire. Je ne regrette rien. Eussé-je été lu, je tentais de plaire, je redevenais merveilleux. Clandestin, je fus vrai.

Jean-Paul Sartre, dans *Les mots*.

Mais c'est pas une raison pour plus vous laver les joues !

Jacques, dans *La classe américaine*.

Comment des brouillons illisibles sont devenus une thèse

Paris, septembre 2005.

L'air de rien, cette partie est l'une des plus difficiles à écrire. Pour commencer, il va falloir dire *je* tout le temps, ce qui est un exercice assez désagréable, dans la lignée des classiques rédactions de collège où l'on est censé raconter ses vacances. On ne se rend pas compte à quel point c'est dur pour un enfant de raconter ses vacances, alors qu'il s'est ennuyé pendant deux mois à attendre de retrouver l'école et les copains. Du coup, il a été insupportable avec ses parents, mais il ne peut quand même pas raconter ça. Ses parents mériteraient pourtant d'être les premiers remerciés, pour l'avoir supporté tout ce temps-là, ainsi, le cas échéant, que son petit frère, qui lui a passé un bon été, comme tous les étés, et comme le reste de l'année d'ailleurs. De toute façon, maintenant que la rentrée est là et bien là, on se dit que finalement les vacances avaient du bon.

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Chapter 1

Introduction and motivation

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1.1 French version **9**
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1.1 French version

Le but de cette thèse est de donner un exemple de sous-variété lagrangienne de l'espace des modules

$$\mathcal{M}_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$$

où $\pi := \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ est le groupe fondamental de la sphère privé de l points ($l \geq 1$) et où U est un groupe de Lie compact connexe quelconque. Nous préciserons sous peu les notations utilisées ci-dessus et nous verrons par la suite qu'il nous faudra par moments supposer que le groupe compact connexe U est de plus simplement connexe. Nous reviendrons sur ces questions au fil de ce travail et plus particulièrement dans la conclusion. Pour l'heure, nous souhaiterions donner un bref aperçu du domaine d'étude dans lequel s'inscrit cette thèse et dresser un panorama succint des principaux thèmes qui fondent la géométrie symplectique des espaces de modules.

On désigne communément par *modules* des coordonnées sur l'espace des orbites associé à une action de groupe. Si l'on considère par exemple l'action par conjugaison du groupe unitaire sur lui-même, l'espace des orbites est l'ensemble $\mathrm{Conj}(U(n))$ des classes de conjugaison de $U(n)$, et chacune de ces classes est entièrement déterminée par l'ensemble des valeurs propres de l'un quelconque de ses représentants, comptées avec leurs multiplicités respectives :

$$\mathrm{Conj}\big(U(n)\big) \simeq \mathbb{T}^n/\mathfrak{S}_n$$

Les modules de cette action sont alors par définition les éléments de $\mathbb{T}^n/\mathfrak{S}_n$. De manière plus élémentaire encore, les modules de l'action par rotations de $S^1 \simeq SO(2)$ sur l'ensemble des droites du plan euclidien sont les nombres réels appartenant à l'intervalle $[0, \pi[$, généralement appelés *angles orientés*. Les espaces de modules auxquels nous nous intéresserons dans cette thèse sont les espaces (des classes d'équivalence) de représentations du groupe fondamental $\pi_{g.l} := \pi_1(\Sigma_{g,l})$ d'une surface de Riemann $\Sigma_{g,l} := \Sigma_g \backslash \{s_1, \ldots, s_l\}$ où Σ_g est une surface de Riemann compacte de genre $g \geq 0$, où l est un entier naturel $l \geq 0$ (en convenant que $\Sigma_{g,0} := \Sigma_g$) et où $s_1, \ldots, s_l$ sont l points distincts de Σ_g. Ces *variétés de représentations* sont un objet d'étude important depuis plusieurs décennies maintenant, et se situent à l'intersection de diverses branches des mathématiques, toutes très riches, qui apportent chacune un éclairage différent sur ces espaces. Ainsi l'espace

$$\mathrm{Rep}(\pi_{g,l}, U) := \mathrm{Hom}(\pi_{g,l}, U)/U$$

des classes d'équivalence de représentations de $\pi_{g,l}$ dans un groupe de Lie U apparaît-il de manière naturelle en géométrie algébrique complexe car il s'identifie à l'espace des classes d'équivalence de fibrés vectoriels holomorphes sur $\Sigma_{g,l}$, comme l'ont montré Narasimhan et Seshadri dans les années 60 (voir [NS65]). Au début des années 80, Atiyah et Bott ont donné une nouvelle impulsion (voir [AB83]) à l'étude de ces espaces en les identifiant aux modules des connexions plates sur les fibrés principaux de groupe U sur $\Sigma_{g,l}$, révélant ainsi l'importance des variétés de représentations en théorie de jauge. Ces espaces apparaissent également en théorie de Galois différentielle et en théorie des algèbres d'opérateurs. Enfin, il est possible d'utiliser ces espaces pour construire des déformations de sous-groupes discrets de groupes de Lie (voir par exemple [MG88]). La diversité des théories auxquelles sont reliées ces variétés de représentations justifie l'étude de leur structure géométrique, dont la description est susceptible d'interprétation dans chacun des domaines ci-dessus. On trouvera une introduction à l'étude de ces structures par exemple dans [Gol88]. En ce qui nous concerne, nous privilégierons l'étude de la structure *symplectique* de certains de ces espaces de représentations. Cette structure symplectique peut être obtenue et décrite de diverses manières (voir par exemple [GHJW97, AM95, AMM98, MW99]), qui présentent toutes des avantages. Une description particulièrement bien adaptée à l'étude des représentations de $\pi_{g,l}$ est celle donnée par Alekseev, Malkin et Meinrenken dans [AMM98]. Elle repose sur la notion d'*espace quasi-hamiltonien*, qui permet notamment d'éviter le recours à des variétés de dimension infinie tout en se limitant à des objets relativement simples pour construire une forme symplectique sur les variétés de représentations. Nous verrons en détail les étapes de cette construction dans le chapitre 4 et nous poursuivrons l'étude de la géométrie symplectique de ces espaces de modules dans le chapitre 7. Pour ce qui est d'exhiber une sous-variété lagrangienne, nous nous limiterons dans la suite au groupe fondamental de la sphère épointée mais nous pensons que cet exemple et les méthodes utilisées dans cette thèse peuvent servir de point de départ pour la recherche de sous-variétés lagrangiennes dans l'espace des modules associé à un groupe de surface quelconque. En particulier, nous verrons que les résultats généraux sur les espaces quasi-hamiltoniens obtenus ici (chapitres 7 et 8) s'appliquent indépendamment du groupe de surface considéré. Dans la suite, nous étudierons exclusivement les représentations de $\pi = \pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ où l'on a fixé la classe de conjugaison de chacun des générateurs. Afin de pouvoir être plus précis, rappelons que le groupe $\pi = \pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ admet la présentation finie par générateurs et relations suivante :

$$\pi =< \gamma_1, \dots, \gamma_l \mid \gamma_1 \dots \gamma_l = 1 >$$

On suppose donné un système de représentants $\gamma_1, \dots, \gamma_l$ des générateurs de π, et on se donne par ailleurs l classes de conjugaison $\mathcal{C}_1, \dots, \mathcal{C}_l$ de U. On étudie alors l'ensemble

$$\mathrm{Hom}_{\mathcal{C}}(\pi, U) := \{\rho : \pi \to U \mid \forall j \in \{1, \dots, l\}, \rho(\gamma_j) \in \mathcal{C}_j\}$$

qui est un sous-ensemble (éventuellement vide) de l'ensemble $\mathrm{Hom}(\pi, U)$ des morphismes de groupes de π dans U. Les éléments de $\mathrm{Hom}(\pi, U)$ sont également appelés *représentations de π dans U*. Remarquons que grâce au choix des générateurs $\gamma_1, \dots, \gamma_l$ de π, on a :

$$\mathrm{Hom}(\pi, U) \simeq \left\{(u_1, \dots, u_l) \in U \times \cdots \times U \mid u_1 \dots u_l = 1\right\}$$

et

$$\mathrm{Hom}_{\mathcal{C}}(\pi, U) \simeq \left\{(u_1, \dots u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \dots u_l = 1\right\}$$

Dans toute la suite, nous supposerons que les classes de conjugaison $\mathcal{C}_1, \dots \mathcal{C}_l$ de U sont choisies de manière à ce que $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. Dans le cas où $U = SU(n)$, un ensemble de conditions nécessaires et suffisantes portant sur les $\mathcal{C}_j$ pour que ceci soit vrai a été donné par exemple par Agnihotri et Woodward dans [AW98] (voir aussi [Bis98, Bis99, Bel01, JW92, Gal97, KM99]). Il s'agit d'inégalités linéaires portant sur les arguments des valeurs propres qui définissent les $\mathcal{C}_j$. La forme générale de ces inégalités fait appel à des outils sophistiqués mais nous verrons qu'il est possible, dans le cas particulier où $U = U(2)$ et $l = 3$, de les obtenir par des méthodes géométriques élémentaires (voir corollaire 5.4.12).

Deux représentations $\rho, \rho' \in \mathrm{Hom}(\pi, U)$ de π dans U sont dites *équivalentes* s'il existe un élément $\varphi \in U$ tel que $\varphi\rho(\gamma_j)\varphi^{-1} = \rho'(\gamma_j)$ pour tout $j \in \{1, \dots, l\}$ (en particulier, si $U \subset Gl(V)$ est un groupe de

transformations linéaires d'un espace vectoriel V, cette notion coïncide bien avec la notion d'équivalence pour les représentations linéaires : il existe un automorphisme φ de V tel que $\forall\gamma \in \pi$ on ait $\varphi(\rho(\gamma).v) = \rho'(\gamma).\varphi(v)$ pour tout $v \in V$). Cette relation d'équivalence laisse stable la classe de conjugaison de chacun des $\rho(\gamma_j)$ et induit donc une relation d'équivalence sur $\mathrm{Hom}_{\mathcal{C}}(\pi, U)$. Remarquons que si l'on utilise la description de $\mathrm{Hom}(\pi, U)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(\pi, U)$) donnée plus haut, alors $\rho = (u_1, \ldots, u_l)$ est équivalent à $\rho' = (u'_1, \ldots, u'_l)$ si et seulement si $(u_1, \ldots, u_l)$ et $(u'_1, \ldots, u'_l)$ sont dans une même orbite de l'action diagonale de U sur $U \times \cdots \times U$ (resp. $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$) donnée par :

$$\varphi.(u_1, \ldots, u_l) := (\varphi u_1 \varphi^{-1}, \ldots, \varphi u_l \varphi^{-1})$$

L'espace des classes d'équivalence pour cette relation est appelé *l'espace des modules* des représentations de π dans U et est noté $\mathcal{M}$ (resp. $\mathcal{M}_{\mathcal{C}}$) :

$$\mathcal{M} := \mathrm{Hom}(\pi, U)/U = \{(u_1, \ldots, u_l) \in U \times \cdots \times U \mid u_1 \ldots u_l = 1\}/U$$

$$\mathcal{M}_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U = \{(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \ldots u_l = 1\}/U$$

Ces espaces ne sont pas des variétés lisses en général, mais ils possèdent une structure stratifiée que nous évoquerons au chapitre 4. Dans la suite, nous étudierons exclusivement l'espace $\mathcal{M}_{\mathcal{C}}$, dont l'une des propriétés les plus remarquables est de porter une structure symplectique stratifiée. Pour nous, cela signifiera simplement que $\mathcal{M}_{\mathcal{C}}$ est réunion disjointe de variétés lisses (de dimensions différentes) appelées strates portant chacune une structure symplectique, et nous appellerons sous-variété lagrangienne de $\mathcal{M}_{\mathcal{C}}$ un sous-ensemble de $\mathcal{M}_{\mathcal{C}}$ dont l'intersection avec chaque strate est une sous-varité lagrangienne de la strate considérée. Nous appellerons *représentation de* π aussi bien les éléments de $\mathrm{Hom}_C(\pi, U)$ que les éléments de $\mathcal{M}_{\mathcal{C}}$, sauf si le contexte ne permet pas de dire duquel de ces deux ensembles il est question.

La démarche suivie pour trouver une sous-variété lagrangienne de $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$ consiste à :

1. introduire une notion de représentation décomposable.
2. caractériser ces représentations comme les éléments du lieu des points fixes d'une involution définie sur $\mathcal{M}_{\mathcal{C}}$.
3. montrer que l'involution considérée est antisymplectique et que l'ensemble de ses points fixes est non vide (formant ainsi une sous-variété lagrangienne de $\mathcal{M}_{\mathcal{C}}$).

Comme nous le verrons dans le chapitre 5, l'idée d'introduire une notion de représentation décomposable a une origine géométrique très simple. Cette origine géométrique nous conduira à étudier les configurations de sous-espaces lagrangiens de $\mathbb{C}^n$ et à définir une notion d'angle entre deux tels sous-espaces. Le fait de vouloir ensuite caractériser l'ensemble des représentations décomposables comme le lieu des points fixes d'une involution découlera alors d'une tentative de formulation d'une version infinitésimale de ce problème de configurations. L'autre mérite de cette version infinitésimale sera de montrer pourquoi l'on doit s'attendre à ce que l'involution considérée sur $\mathcal{M}_{\mathcal{C}}$ soit antisymplectique.

Le cadre de ce travail sera celui de la géométrie quasi-hamiltonienne, que nous utiliserons à la fois pour décrire la structure symplectique de $\mathcal{M}_{\mathcal{C}}$ et pour étudier la notion de représentation décomposable (caractérisation et existence). La structure symplectique de $\mathcal{M}_{\mathcal{C}}$ sera en effet obtenue par *réduction symplectique* à partir de l'espace quasi-hamiltonien $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ et l'involution permettant de caractériser les représentations décomposables sera induite par une involution sur l'espace total $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. Nous donnerons en particulier des conditions suffisantes pour qu'une involution construite sur un espace quasi-hamiltonien induise une involution antisymplectique sur le quotient symplectique associé. Le point de notre étude le plus difficile techniquement sera de montrer l'existence des représentations décomposables (c'est-à-dire de montrer que l'involution construite sur $\mathcal{M}_{\mathcal{C}}$ admet effectivement des points fixes). Comme nous le verrons, cela découle d'un théorème de convexité, dit théorème de convexité *réel*, pour les applications moment à valeurs dans un groupe de Lie, dont la démonstration fait l'objet du chapitre 8, et pour lequel nous supposerons de plus que le groupe compact connexe U est simplement connexe.

Nous pouvons désormais résumer la discussion ci-dessus de la manière suivante, qui sera considérablement détaillée par la suite. On se donne un groupe de Lie compact connexe quelconque U, muni d'un automorphisme involutif τ et d'un produit scalaire Ad-invariant $(.\,|\,.)$ sur $\mathfrak{u} = \mathrm{Lie}(U)$. On note τ^- l'involution $\tau^- : u \in U \mapsto \tau(u^{-1})$. Un élément $w \in U$ vérifiant $\tau^-(w) = w$ (soit $\tau(w) = w^{-1}$) est dit *symétrique*. On supposera de plus que le lieu des points fixes $Fix(\tau^-)$ de l'involution τ^- est un ensemble connexe. Cette hypothèse est par exemple vérifiée par le groupe $U(n)$ muni de l'involution $\tau(u) := \overline{u}$ (voir les remarques 5.2.3 et 7.4.2 pour des commentaires sur cette hypothèse) . On se donne enfin l classes de conjugaison $\mathcal{C}_1, \ldots, \mathcal{C}_l$ de U, choisies de manière à ce que $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. L'espace $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ est un espace quasi-hamiltonien lorsqu'on le munit de l'action diagonale de U, d'une certaine 2-forme notée ω, et de l'application

$$\begin{array}{rccc} \mu : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \ldots, u_l) & \longmapsto & u_1 \ldots u_l \end{array}$$

appelée *application moment*. L'ensemble des représentations de π est alors la fibre du moment au-dessus de 1 :

$$\mathrm{Hom}_{\mathcal{C}}(\pi, U) = \mu^{-1}(\{1\}) = \{(u_1, \ldots .u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \,|\, u_1 \ldots u_l = 1\}$$

et l'espace des modules $\mathcal{M}_{\mathcal{C}}$ est le *quotient symplectique* associé à l'espace quasi-hamiltonien $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\mathcal{M}_{\mathcal{C}} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \,//\, U := \mu^{-1}(\{1\})/U$$

Donnons maintenant la définition d'une représentation décomposable. Nous verrons dans le chapitre 5 comment arriver à cette définition de nature algébrique par des considérations géométriques.

Définition (Représentation décomposable). Soit (U, τ) un groupe de Lie muni d'un automorphisme involutif τ. Une représentation $(u_1, \ldots, u_l) \in \mu^{-1}(\{1\})$ de π dans U est dite *décomposable* s'il existe l éléments $w_1, \ldots, w_l$ de U vérifiant :

(i) $\tau(w_j) = w_j^{-1}$ pour tout j (chaque w_j est un élément *symétrique* de U au sens de l'involution τ).

(ii) $u_1 = w_1 w_2^{-1}$, $u_2 = w_2 w_3^{-1}$, $\ldots$, et $u_l = w_l w_1^{-1}$.

Une représentation est dite σ_0-*décomposable* si elle est décomposable avec $w_1 = 1$.

On définit alors l'involution suivante sur $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\begin{array}{rccc} \beta : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ldots, u_l) & \longmapsto & \big(\tau^-(u_l) \ldots \tau^-(u_2)\tau^-(u_1)\tau(u_2) \ldots \tau(u_l), \ldots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

Nous verrons au chapitre 6 comment cette involution est obtenue et nous démontrerons alors le théorème suivant :

Théorème 1 (Caractérisation des représentations décomposables). *Une représentation $u = (u_1, \ldots, u_l) \in \mu^{-1}(\{1\})$ est σ_0-décomposable si et seulement si $\beta(u) = u$. Elle est décomposable si et seulement si $\beta(u) \sim u$ en tant que représentations de π.*

Nous verrons par ailleurs que l'involution β vérifie $Fix(\beta) \neq \emptyset$, $\beta(\varphi.u) = \tau(\varphi).\beta(u)$ pour tout $u \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ et tout $\varphi \in U$, et $\mu \circ \beta = \tau^- \circ \mu$. Ceci montre que β induit une involution

$$\hat{\beta} : \; [u] \in \mathcal{M}_{\mathcal{C}} \mapsto [\beta(u)]$$

sur l'espace $\mathcal{M}_{\mathcal{C}}$ des classes d'équivalence de représentations de π dans U. On remarque que si u est décomposable alors $\varphi.u$ est décomposable pour tout $\varphi \in U$, et on a alors immédiatement :

Corollaire 2. *$[u] \in \mathcal{M}_{\mathcal{C}}$ est décomposable si et seulement si $\hat{\beta}([u]) = [u]$.*

De plus, nous verrons au chapitre 7 que l'on a :

Proposition 3. *$\beta^*\omega = -\omega$ sur $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, de sorte que $\hat{\beta}$ est antisymplectique sur $\mathcal{M}_{\mathcal{C}}$.*

Il reste à montrer que $\hat{\beta}$ a effectivement des points fixes, ou, de manière équivalente, que $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$. Ceci est un corollaire du théorème suivant :

Théorème 4 (Un théorème de convexité réel pour les applications moment à valeurs dans un groupe de Lie). *Soit $(U, (.\,|\,.), \tau)$ un groupe de Lie compact connexe et simplement connexe muni d'un automorphisme involutif τ tel que l'involution $\tau^- : u \mapsto \tau(u^{-1})$ laisse un tore maximal T de U fixe point par point, et soit $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$ une alcôve de Weyl fermée. Soit $(M, \omega, \mu : M \to U)$ un U-espace quasi-hamiltonien connexe tel que l'application moment $\mu : M \to U$ soit propre et soit $\beta : M \to M$ une involution sur M vérifiant :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *pour tout* $x \in M$ *et tout* $u \in U$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Alors :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \mu(M) \cap \exp(\overline{\mathcal{W}})$$

En particulier, $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ est un sous-polytope convexe de $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$, égal au polytope moment $\mu(M) \cap \exp(\overline{\mathcal{W}})$ tout entier.

Corollaire 5 (Existence de points fixes pour $\hat{\beta}$). *Si $\mu^{-1}(\{1\}) \neq \emptyset$ alors $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$.*

L'existence des représentations décomposables est donc garantie dès lors que $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. On peut alors conclure de la manière suivante :

Théorème 6 (Une sous-variété lagrangienne de $\mathcal{M}_{\mathcal{C}}$). *L'ensemble des classes d'équivalence de représentations décomposables du groupe $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ est une sous-variété lagrangienne de l'espace symplectique stratifié $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$, égale au lieu des points fixes d'une involution anti-symplectique $\hat{\beta}$ définie sur $\mathcal{M}_{\mathcal{C}}$.*

Les chapitres 2 et 3 de cette thèse rappellent quelques notions et résultats sur les groupes de Lie qui seront utiles par la suite et qui sont exposés en détail par exemple dans [Hel01] et [Loo69b]. Le chapitre 4 donne la définition et les principaux exemples d'espaces quasi-hamiltoniens, ainsi que les propriétés dont nous aurons besoin ultérieurement. Il suit [AMM98] de très près. Les chapitres 5 à 9 constituent quant à eux le coeur de cette thèse. Les résultats énoncés ci-dessus y sont démontrés et l'on tente d'y exposer au mieux les motivations et les idées qui ont permis de les obtenir. Outre les résultats principaux mentionnés ci-dessus, on trouvera aussi dans cette thèse une formule, obtenue en collaboration avec Elisha Falbel et Jean-Pierre Marco, pour calculer l'indice d'inertie d'un triplet de sous-espaces lagrangiens de $\mathbb{C}^n$ à partir des angles mesurés deux à deux entre les lagrangiens considérés (voir proposition 5.5.10).

L'un des intérêts de ce travail de thèse me semble être de donner, en plus d'un exemple explicite de sous-variéte lagrangienne d'un espace de modules (voir aussi [Gol88] et [Ho04] [1]), une façon d'en chercher d'autres lorsque l'on change le groupe de surface considéré initialement. Ainsi, pour peu que l'on sache construire une involution β vérifiant certaines propriétés sur l'espace quasi-hamiltonien dont la réduction symplectique donne l'espace de modules qui nous intéresse, les résultats obtenus ici garantissent l'existence

[1] Dans cet article, Ho construit une involution antisymplectique sur l'espace

$$\mathcal{M}_{g,0} := \{(a_1, b_1, \ldots, a_g, b_g) \in SU(n) \times \cdots \times SU(n) \mid \prod_{i=1}^{n} [a_i, b_i] = 1\}/SU(n)$$

des représentations du groupe fondamental d'une surface de Riemann *compacte* de genre $g \geq 1$.

de points fixes pour l'involution $\hat{\beta}$ induite sur le quotient, et montrent que l'ensemble de ces points fixes forme une sous-variété lagrangienne de cet espace de modules. Il reste à obtenir effectivement de telles involutions β dans le cas d'un groupe de surface π quelconque, et l'on peut penser que cela passe par la définition d'une notion appropriée de représentation décomposable.

Les principales références ayant servi de point de départ à ce travail de thèse sont d'une part l'article d'Alekseev, Meinrenken et Woodward ([AMW01]) sur la conjecture de Thompson et l'article d'Alekseev, Malkin et Meinrenken ([AMM98]) sur la notion d'espace quasi-hamiltonien, et d'autre part l'article de Hilgert, Neeb et Plank ([HNP94]) sur les propriétés de convexité du moment pour les espaces hamiltoniens usuels et l'article de O'Shea et Sjamaar ([OS00]) donnant une version réelle de ces résultats dont nous démontrons ici un analogue dans le cadre quasi-hamiltonien.

Une partie des résultats nouveaux contenus dans cette thèse a déjà fait l'objet de publications. Ainsi le chapitre 5 reprend-il les résultats obtenus en collaboration avec Elisha Falbel et Jean-Pierre Marco dans [FMS04], tandis que les chapitres 6 et 9 contiennent les résultats publiés dans [Sch06].

La notion de représentation décomposable a été introduite dans le cas du groupe unitaire $U = U(n)$ par Elisha Falbel et Richard Wentworth, avec qui j'ai eu la chance de pouvoir beaucoup discuter de ces questions, et que je remercie beaucoup pour l'aide qu'ils m'ont apportée lors de ces discussions. Le fait que l'ensemble des représentations décomposables constitue une sous-variété lagrangienne de l'espace des modules dans le cas particulier $U = U(n)$ a été obtenu simultanément et par des méthodes différentes dans [FW] et dans [Sch06], ce dernier article utilisant le résultat principal de [FW] pour prouver la non-vacuité de l'ensemble des représentations décomposables de $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ dans $U(n)$. Dans cette thèse, la notion de représentation décomposable est étendue à un groupe de Lie quelconque U muni d'une involution τ, et la caractérisation de ces représentations décomposables obtenue dans le chapitre 6 est valable pour tout groupe de Lie compact connexe (sous l'hypothèse que $Fix(\tau^-)$ est connexe). L'existence des représentations décomposables est quant à elle obtenue pour les groupes de Lie compacts connexes et simplement connexes, ce qui ne permet donc pas de retrouver le résultat obtenu par Falbel et Wentworth dans [FW] (voir à ce propos la section 9.3). Je souhaiterais également remercier Alan Weinstein pour m'avoir encouragé dans l'idée d'aborder l'étude des représentations décomposables à l'aide de la notion d'application moment et pour m'avoir suggéré l'approche infinitésimale développée dans la section 6.1, ainsi que Johannes Huebschmann pour m'avoir aiguillé vers la notion d'espace quasi-hamiltonien. Enfin, je remercie vivement Sam Evens et Jiang Hua Lu pour la discussion qui m'a conduit à écrire les sections 6.3 et 6.4, étapes cruciales vers la caractérisation des représentations décomposables obtenue dans le chapitre 6.

1.2 English version

The purpose of this thesis is to give an example of a Lagrangian submanifold of the moduli space

$$\mathcal{M}_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$$

where $\pi := \pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ is the fundamental group of an l-punctured sphere ($l \geq 1$), and where U is an arbitrary compact connected Lie group. We will specify the above notations shortly, and see that it will sometimes be necessary to suppose that the compact connected group U is in addition simply connected. We shall come back to these considerations later on in this work, notably in the last chapter. For now, we would like to give an outline of the field which this thesis is attached to, and an overview of the foundational material for the study of symplectic geometry of moduli spaces.

It is customary to call *modules* coordinates on the orbit space associated to a group action. For instance, if we consider the conjugacy action of the unitary group on itself, then the orbit space is the set $\mathrm{Conj}(U(n))$ of conjugacy classes of $U(n)$, and each of these conjugacy classes is completely determined

by the eigenvalues of any of its representatives, counted with their respective multiplicities :

$$\mathrm{Conj}\big(U(n)\big) \simeq \mathbb{T}^n/\mathfrak{S}_n$$

Then by definition the modules of this action are the elements of $\mathbb{T}^n/\mathfrak{S}_n$. In an even more elementary way, the modules of the rotation action of $S^1 \simeq SO(2)$ on the set of lines of the Euclidean plane are the real numbers in the interval $[0, \pi[$, usually called *oriented angles*. In this thesis, the moduli spaces that we shall be interested in are the spaces of (equivalence classes of) representations of the fundamental group $\pi_{g,l} := \pi_1(\Sigma_{g,l})$ of a Riemann surface $\Sigma_{g,l} := \Sigma_g \backslash \{s_1, \ldots, s_l\}$ where Σ_g is a compact Riemann surface of genus $g \geq 0$, where l is an integer $l \geq 0$ (with the convention that $\Sigma_{g,0} := \Sigma_g$) and where $s_1, \ldots, s_l$ are l pairwise distinct points of Σ_g. These *representations varieties* have been an important object of study for several decades now, and are located at the intersection of various areas of mathematics, each of which is very rich and sheds interesting lightning on these spaces. Thus, the space

$$\mathrm{Rep}(\pi_{g,l}, U) := \mathrm{Hom}(\pi_{g,l}, U)/U$$

of equivalence classes of representations of $\pi_{g,l}$ in a Lie group U arises naturally in complex algebraic geometry as it can be identified to the space of equivalence classes of holomorphic vector bundles on $\Sigma_{g,l}$, as it was shown by Narasimhan and Seshadri in the 1960s (see [NS65]). At the beginning of the 1980s, Atiyah and Bott gave a new impulsion (see [AB83]) to the subject by identifying these spaces as the moduli spaces of flat connections on principal bundles of group U on $\Sigma_{g,l}$, thereby revealing the importance of the representation varieties in gauge theory. These spaces also arise in differential Galois theory and in operator algebra theory. Finally, it is possible to use these spaces to construct deformations of discrete subgroups of Lie groups (see for instance [MG88]). The diversity of the fields which these representation spaces are attached to justifies the fact that they are such an important object of study and that their geometry should be investigated. One may for instance find an introduction to the study of geometric structures of moduli spaces in [Gol88]. As for us, we shall focus our attention on studying the *symplectic* structure of some of these representation spaces. This symplectic structure can be obtained and described in a wide variety of ways (see for instance [GHJW97, AM95, AMM98, MW99]), each of which has its own advantages. The description given by Alekseev, Malkin and Meinrenken in [AMM98] is in our sense particularly well-suited for studying representations of $\pi_{g,l}$. This description rests on the notion of *quasi-Hamiltonian space*, which enables one to avoid infinite-dimensional manifolds while limiting oneself to relatively simple objects to construct a symplectic form on representation varieties. We will get into the details of each step of this construction in chapter 4 and carry on studying symplectic geometry of moduli spaces in chapter 7. As for giving examples of Lagrangian submanifolds, we shall restrict ourselves to the case of the fundamental group of a punctured sphere, but we think that this example and the methods used in this thesis can be used as a starting point to find Lagrangian submanifolds in the moduli space associated to an arbitrary surface group. In particular, we will see that the general results on quasi-Hamiltonian spaces obtained here (chapters 7 and 8) can be applied regardless of the considered surface group. In the following, we will study representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ whose generators lie in a prescribed conjugacy class exclusively. To be able to be more precise in our statements, let us recall that the group $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ admits the following finite presentation by generators and relations :

$$\pi =< \gamma_1, \ldots, \gamma_l \mid \gamma_1 \ldots \gamma_l = 1 >$$

We start with a system of representatives $\gamma_1, \ldots, \gamma_l$ of generators of π, and l conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l$ of U. We then study the set

$$\mathrm{Hom}_{\mathcal{C}}(\pi, U) := \{\rho : \pi \to U \mid \forall j \in \{1, \ldots, l\}, \rho(\gamma_j) \in \mathcal{C}_j\}$$

which is a (possibly empty) subset of the set $\mathrm{Hom}(\pi, U)$ of group morphisms of π into U. Elements of $\mathrm{Hom}(\pi, U)$ are also called *representations of π into U*. Thanks to the choice of generators $\gamma_1, \ldots, \gamma_l$ of π, one has :

$$\mathrm{Hom}(\pi, U) \simeq \big\{(u_1, \ldots, u_l) \in U \times \cdots \times U \mid u_1 \ldots u_l = 1\big\}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(\pi, U) \simeq \left\{(u_1, \dots u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \dots u_l = 1\right\}$$

In all of the following, we will assume that the conjugacy classes $\mathcal{C}_1, \dots \mathcal{C}_l$ of U satisfy the condition $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. In the case where $U = SU(n)$, a set of necessary and sufficient conditions lying on the $\mathcal{C}_j$ for this to be true has been given for instance by Agnihotri and Woodward in [AW98] (see also [Bis98, Bis99, Bel01, JW92, Gal97, KM99]). These conditions are linear inequalities satisfied by the arguments of the eigenvalues defining the $\mathcal{C}_j$. The general form of these inequalities calls for sophisticated tools, but we will see that in the case where $U = U(2)$ and $l = 3$, it is possible to obtain them using only elementary geometric methods (see corollary 5.4.12).

Two representations $\rho, \rho' \in \mathrm{Hom}(\pi, U)$ of π into U are said to be *equivalent* if there exists an element $\varphi \in U$ such that $\varphi\rho(\gamma_j)\varphi^{-1} = \rho'(\gamma_j)$ for all $j \in \{1, \dots, l\}$ (in particular, if $U \subset Gl(V)$ is a group of linear transformations of a vector space V, this is indeed the same notion as equivalent linear representations : there exists an automorphism φ of V such that $\forall\gamma \in \pi$ one has $\varphi(\rho(\gamma).v) = \rho'(\gamma).\varphi(v)$ for all $v \in V$). This equivalence relation preserves the conjugacy class of each of the $\rho(\gamma_j)$, so that it induces an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(\pi, U)$. Observe that if we use the above-given description of $\mathrm{Hom}(\pi, U)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(\pi, U)$), then $\rho = (u_1, \dots, u_l)$ is equivalent to $\rho' = (u'_1, \dots, u'_l)$ if and only if $(u_1, \dots, u_l)$ and $(u'_1, \dots, u'_l)$ lie in a same orbit of the diagonal action of U on $U \times \cdots \times U$ (resp. $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$) given by :

$$\varphi.(u_1, \dots, u_l) := (\varphi u_1 \varphi^{-1}, \dots, \varphi u_l \varphi^{-1})$$

The space of equivalence classes for this relation is called the *moduli space* of representations of π into U and denoted by $\mathcal{M}$ (resp. $\mathcal{M}_{\mathcal{C}}$) :

$$\mathcal{M} := \mathrm{Hom}(\pi, U)/U = \{(u_1, \dots, u_l) \in U \times \cdots \times U \mid u_1 \dots u_l = 1\}/U$$

$$\mathcal{M}_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U = \{(u_1, \dots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \dots u_l = 1\}/U$$

These spaces are generally not smooth manifolds but they carry a stratified structure that we shall evoke in chapter 4. In the following, we will focus our attention on the space $\mathcal{M}_{\mathcal{C}}$, whose main remarkable property is that it carries a stratified symplectic structure. To us, this will simply mean that $\mathcal{M}_{\mathcal{C}}$ is a disjoint union of smooth manifolds (of different dimensions) called strata, each of which carries a symplectic structure. And we will call Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}}$ a subset of $\mathcal{M}_{\mathcal{C}}$ whose intersection with each stratum is a Lagrangian submanifold of the considered stratum. Elements of $\mathrm{Hom}_C(\pi, U)$ will be called *representations of* π. We shall also call elements of $\mathcal{M}_{\mathcal{C}}$ representations of π, unless it is not clear from the context which of these two sets we are precisely alluding to.

The path we shall follow to find a Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$ consists in :

1. introducing a notion of decomposable representation.
2. characterizing these representations as the elements of the fixed-point set of an involution defined on $\mathcal{M}_{\mathcal{C}}$.
3. showing that this involution is anti-symplectic and that its fixed-point set is non-empty (being therefore a Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}}$).

As we shall see in chapter 5, the idea of introducing a notion of decomposable representation has a very simple geometric origin, which will lead us to studying configurations of Lagrangian subspaces of $\mathbb{C}^n$ and to defining a notion of angle between two such subspaces. The idea of trying to characterize the set of decomposable representations as the fixed-point set of an in involution will then follow from an attempt at giving an infinitesimal formulation of this configuration problem. Another upshot of this infinitesimal formulation is to show why one should expect the involution at stake to be anti-symplectic on $\mathcal{M}_{\mathcal{C}}$.

The setting of this thesis will be quasi-Hamiltonian geometry. We shall encounter this geometry both to describe the symplectic structure of $\mathcal{M}_{\mathcal{C}}$ and to study the notion of decomposable representation

(characterization and existence). Indeed, the symplectic structure on $\mathcal{M}_{\mathcal{C}}$ will be obtained by *symplectic reduction* from the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, and the involution that shall enable us to characterize decomposable representations will be induced by an involution on the total space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. In particular, we will give sufficient conditions on an involution defined on a quasi-Hamiltonian space for it to induce an anti-symplectic involution on the associated symplectic quotient. The most technically difficult point of our study will be to prove the existence of decompoable representations (that is, to prove that the fixed-point set of the involution constructed on $\mathcal{M}_{\mathcal{C}}$ is non-empty). As we shall see later on, this will follow from a convexity theorem, more precisely from what is usually called a *real* convexity theorem, for group-valued momentum maps, whose proof, for which we will suppose that the compact connected Lie group U is in addition simply connected, will be presented in chapter 8.

We may now summarize the above discussion in a way that shall be considerably detailed in the following. We start with a compact connected Lie group U, endowed with an involutive automorphism τ and whose Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$ is equipped with an Ad-invariant scalar product $(.\,|\,.)$. We denote by τ^- the involution $\tau^- : u \in U \mapsto \tau(u^{-1})$. An element $w \in U$ satisfying $\tau^-(w) = w$ (or equivalently, $\tau(w) = w^{-1}$) is said to be *symmetric*. We shall suppose additionally that the fixed-point set $Fix(\tau^-)$ of the involution τ^- is a connected set. This assumption is for instance satisfied by the unitary group $U(n)$ endowed with the involution $\tau(u) := \overline{u}$ (see remarks 5.2.3 and 7.4.2 for comments on this assumption). Finally, we suppose given l conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l$ of U, picked in a way that $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. The space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ is a quasi-Hamiltonian space when it is endowed with the diagonal action of U, a certain 2-form ω, and the map

$$\begin{array}{rccc} \mu : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \ldots, u_l) & \longmapsto & u_1 \ldots u_l \end{array}$$

called the *momentum map*. Then the set of representations of π is the fibre above 1 of the momentum map :

$$\mathrm{Hom}_{\mathcal{C}}(\pi, U) = \mu^{-1}(\{1\}) = \{(u_1, \ldots .u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \,|\, u_1 \ldots u_l = 1\}$$

and the moduli space $\mathcal{M}_{\mathcal{C}}$ is the *symplectic quotient* associated to the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\mathcal{M}_{\mathcal{C}} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \,//\, U := \mu^{-1}(\{1\})/U$$

Let us now give the definition of a decomposable representation. We will see in chapter 5 how to reach this definition using geometric considerations.

Definition (Decomposable representations). Let (U, τ) be a Lie group endowed with an involutive automorphism τ. A representation $(u_1, \ldots, u_l)$ of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into U is called *decomposable* if there exist l elements $w_1, \ldots, w_l \in U$ satisfying :

(i) $\tau(w_j) = w_j^{-1}$ for all j (each w_j is a *symmetric* element of U with respect to τ).

(ii) $u_1 = w_1 w_2^{-1}$, $u_2 = w_2 w_3^{-1}$, $\ldots$, $u_l = w_l w_1^{-1}$.

A representation will be called σ_0-*decomposable* if it is decomposable with $w_1 = 1$.

We then define the following involution on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\begin{array}{rccc} \beta : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ldots, u_l) & \longmapsto & \big(\tau^-(u_l) \ldots \tau^-(u_2)\tau^-(u_1)\tau(u_2) \ldots \tau(u_l), \ldots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

We shall see in chapter 6 how this involution is obtained and we will then show the following result :

Theorem 1 (Characterization of decomposable representations). *A representation $u = (u_1, \ldots, u_l) \in \mu^{-1}(\{1\})$ is σ_0-decomposable if and only if $\beta(u) = u$. It is decomposable if and only if $\beta(u) \sim u$ as representations of π.*

We shall also see that the involution β satisfies $Fix(\beta) \neq \emptyset$, $\beta(\varphi.u) = \tau(\varphi).\beta(u)$ for all $u \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ and all $\varphi \in U$, and $\mu \circ \beta = \tau^- \circ \mu$. This shows that β induces an involution

$$\hat{\beta} : \ [u] \in \mathcal{M}_{\mathcal{C}} \mapsto [\beta(u)]$$

on the space $\mathcal{M}_{\mathcal{C}}$ of equivalence classes of representations of π in U. One then observes that if u is decomposable then so is $\varphi.u$ for all $\varphi \in U$, therefore one immediately has :

Corollary 2. *$[u] \in \mathcal{M}_{\mathcal{C}}$ is decomposable if and only if $\hat{\beta}([u]) = [u]$.*

Additionally, we shall see in chapter 7 that one has :

Proposition 3. *$\beta^*\omega = -\omega$ on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, so that $\hat{\beta}$ is anti-symplectic on $\mathcal{M}_{\mathcal{C}}$.*

It remains to show that $\hat{\beta}$ indeed has fixed points, or equivalently, that $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$. This is a corollary of the following theorem :

Theorem 4 (A real convexity theorem for group-valued momentum maps). *Let $(U, (.\,|\,.), \tau)$ be a compact connected simply connected Lie group endowed with an involutive automorphism τ such that the involution $\tau^- : u \mapsto \tau(u^{-1})$ leaves a maximal torus T of U pointwise fixed and let $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$ be a closed Weyl alcove. Let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian U-space with proper momentum map $\mu : M \to U$ and let $\beta : M \to M$ be an involution on M such that :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $x \in M$ *and all* $u \in U$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Then :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \mu(M) \cap \exp(\overline{\mathcal{W}})$$

In particular, $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ is a convex subpolytope of $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$, equal to the whole momentum polytope $\mu(M) \cap \exp(\overline{\mathcal{W}})$.

Corollary 5 (Existence of fixed points for $\hat{\beta}$). *If $\mu^{-1}(\{1\}) \neq \emptyset$ then $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$.*

Thus, the existence of decomposable representations is guaranteed as soon as $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. One may then conclude in the following way :

Theorem 6 (A Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}}$). *The set of equivalence classes of decomposable representations of the group $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ is a Lagrangian submanifold of the stratified symplectic space $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$, equal to the fixed-point set of an anti-symplectic involution $\hat{\beta}$ defined on $\mathcal{M}_{\mathcal{C}}$.*

Chapters 2 and 3 of this thesis are devoted to recalling a few notions and results on Lie groupe that are exposed in detail for instance in [Hel01] and [Loo69b]. In chapter 4, we give the definition and main examples of quasi-Hamiltonian spaces, as well as the properties we shall need in the following. It follows [AMM98] closely. Chapters 5 to 9 constitute the heart of this thesis. The results announced above are proved there, and we try to explain the motivation and ideas that led to them. In addition to the main results mentioned above, this thesis contains a formula, obtained in collaboration with Elisha Falbel and Jean-Pierre Marco, that enables ones to compute the inertia index of a Lagrangian triple of $\mathbb{C}^n$ from the angles between them (see proposition 5.5.10).

In addition to giving an explicit example of a Lagrangian submanifold in a moduli space (see also [Gol88] and [Ho04] [2]), one of the interests of this thesis work is, in my opinion, to give a way of finding others when one changes the surface group considered initially. As a matter of fact, if one is able to obtain an involution β satisfying certain properties on the quasi-Hamiltonian space whose symplectic reduction is the moduli space one is interested in, the results contained in this work ensure that the involution $\hat{\beta}$ induced on the associated quotient indeed has fixed points and that the set of such fixed points is a Lagrangian submanifold of the moduli space at hand. And we would think that obtaining such involutions in the case of an arbitrary surface group π is a matter of defining an appropriate notion of decomposable representation.

The main references serving for starting point for this thesis work are on the one hand the article of Alekseev, Meinrenken and Woodward on the Thompson conjecture ([AMW01]) and the article of Alekseev, Malkin and Meinrenken on quasi-Hamiltonian spaces ([AMM98]), and on the other hand the article of Higert, Neeb an Plank on convexity properties of momentum maps in the usual Hamiltonian setting and the article of O'shea and Sjamaar giving a real version of these results, which we prove here a quasi-Hamiltonian analogue of.

Some of the results contained in this thesis has already been accepted for publication. Thus, chapter 5 is an expanded version of results obtained in collaboration with Elisha Falbel and Jean-Pierre Marco in [FMS04], whereas chapters 6 and 9 contain the results published in [Sch06].

[2] In this paper, Ho constructs an antisymplectic involution on the space

$$\mathcal{M}_{g,0} := \big\{(a_1, b_1, \ldots, a_g, b_g) \in SU(n) \times \cdots \times SU(n) \mid \prod_{i=1}^{n} [a_i, b_i] = 1\big\}/SU(n)$$

of representations of the fundamental group of a *compact* Riemann surface of genus $g \geq 1$.

Chapter 2

Generalities on actions of compact connected Lie groups

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2.2 The conjugacy action of a compact connected simply connected Lie group on itself 23

In this chapter, we recall some properties of actions of compact Lie groups. In particular, we study the conjugacy action of a compact connected simply connected Lie group on itself.

The sole purpose of this chapter is to serve as a reference for results that we shall need in the remainder of this thesis. The properties we remind here are all standard, and proofs might be found in the books indicated below. Whenever we need one of these results in the forthcoming chapters, a precise reference will be made, so that one may for now skip the present chapter and come back to it when necessary.

2.1 The action of a compact Lie group on an arbitrary manifold

In this section, we recall a few facts about compact group actions that we will need in the forthcoming chapters. We freely quote results from [GS84c], [DK00], [Bre72] and [Bou82], and begin with the notion of *manifold of symmetry* :

Proposition 2.1.1. *Let U be a compact Lie group acting on a manifold M and let $K \subset U$ be a closed subgroup of U. Let M_K denote the set of points of M whose stabilizer is exactly K :*

$$M_K := \{x \in M \mid U_x = K\}$$

Then M_K is a submanifold of M, called the manifold of symmetry K*, whose tangent space at any $x \in M_K$ consists of K-fixed vectors of T_xM :*

$$T_xM_K = \{v \in T_xM \mid \text{for all } k \in K, k.v = v\}$$

We refer to [GS84c] (p.203) for a proof of this result. Observe that for $x \in M_K$, the group K indeed acts on T_xM : since x is fixed by any $k \in K$, the tangent map to the diffeomorphism $y \in M \mapsto k.y$ sends T_xM to itself. Also observe that the subgroup K is assumed to be closed because a stabilizer always is.

When studying the action of a Lie group U on a manifold M, it is often useful to understand what the manifold M looks like in the neighbourhood of an orbit $U.x$ of this action. One key notion in this context is that of a *slice through $x \in M$* (see for instance [DK00], p.98) :

Definition 2.1.2. Let U be a Lie group acting on a manifold M and let $x \in M$ be a point of M. A submanifold $\mathcal{S} \subset M$ is called a *slice through* x if :

(i) $x \in \mathcal{S}$

(ii) $\mathcal{S}$ is U_x-stable

(iii) if $u.\mathcal{S} \cap \mathcal{S} \neq \emptyset$ then $u \in U_x$

(iv) $T_xM = T_x(U.x) \oplus T_x\mathcal{S}$ and for all $y \in \mathcal{S}$, $T_yM = T_y(U.y) + T_y\mathcal{S}$

In particular, the set $U.\mathcal{S} = \{u.y \ : \ y \in \mathcal{S}, u \in U\}$ is an open neighbourhood of the orbit $U.x$, and $\mathcal{S}$ is closed in $U.\mathcal{S}$.

We then have :

Proposition 2.1.3. *If U is a compact Lie group acting on a manifold M, then for every $x \in M$ there exists a slice through x. Furthermore, we can choose coordinates on $\mathcal{S}$ so that $\mathcal{S}$ is an open ball in a vector space upon which U_x acts linearly.*

We refer to [GS84c] (p.201) for the proof. See also [DK00] for the case of proper actions of arbitrary Lie groups. One very interesting consequence of the existence of slices is the possibility of establishing a local normal form for compact group actions (see for instance [DK00], p.102) :

Proposition 2.1.4. *Let U be a compact Lie group acting on a manifold M. Then any $x \in M$ possesses a U-stable open neighbourhood $\mathcal{V}_x$ such that :*

$$M \supset \mathcal{V}_x \simeq U \times_{U_x} \Big(T_xM/T_x(U.x)\Big)$$

where $U \times_{U_x} (T_xM/T_x(U.x)) =: U \times_{U_x} V_x$ is the quotient of the manifold $U \times V_x$ by the free action of the compact group U_x given by $u.(g,v) := (gu^{-1}, u.v)$. Furthermore, if we denote by $[g,v]$ the U_x-orbit of (g,v) in $U \times V_x$ for this action, then the manifold $U \times_{U_x} V_x$ inherits a U-action given by $u.[g,v] = [ug,v]$ and the above diffeomorphism between $\mathcal{V}_x$ and $U \times_{U_x} V_x$ is equivariant.

This has the following consequence :

Corollary 2.1.5. *Every $x \in M$ possesses a U-stable open neighbourhood $\mathcal{V}_x$ such that the stabilizer of any $y \in \mathcal{V}_x$ is conjugate to a subgroup of U_x. In particular,* $\dim(U.y) \geq \dim(U.x)$ *for any $y \in \mathcal{V}_x$.*

Proof. Take $y = [g,v] \in U \times_{U_x} V_x \simeq \mathcal{V}_x$. Then $u.y = y$ if and only if $[ug,v] = [g,v]$, that is, if and only if there exists $k \in U_x$ such that $ugk^{-1} = g$ and $k.v = v$. In particular, $u = gkg^{-1} \in gU_xg^{-1}$. □

I would like to thank Pierre Sleewaegen for discussions on these topics, and refer to [Slea] for further properties of compact group actions and a comprehensive study of convexity properties of momentum maps for torus actions on symplectic manifolds.

Finally, we quote one last result on actions of compact Lie groups, for the proof of which we refer to [DK00] (p.118, see also [Bou82], pp.95-99).

Proposition 2.1.6. *Let U be a compact Lie group acting on a manifold M and let $N \subset M$ be a connected subset of M such that N is the union of open connected pieces of U-orbits. Set :*

$$q := \max\{\dim(U.x) \ : \ x \in N\}$$

and :

$$N_q := \{x \in N \mid \dim(U.x) = q\}$$

Then N_q is an open, connected, and dense subset of N.

2.2 The conjugacy action of a compact connected simply connected Lie group on itself

In this section, we study the conjugacy action of a compact connected simply connected Lie group on itself. More precisely, we investigate the geometry of a fundamental domain for this action. We freely quote results from [Bou82], [Hum78], [BtD95], [Ada69] and [Loo69b]. The material presented here will be useful to us in chapter 8.

Let U be a compact connected Lie group and let $T \subset U$ be a maximal torus of U. We denote by $\mathcal{N}(T)$ the normalizer of T in U and by $W(T) := \mathcal{N}(T)/T$ (or simply W) the associated Weyl group, which is a finite group operating on T. The compact connected group U acts on itself by conjugation, and we denote the orbit space by $U/\mathrm{Int}(U)$. We then have :

$$U/\mathrm{Int}(U) \quad \simeq \quad T/W \tag{2.1}$$

(see for instance [BtD95], p.166). If we additionally assume U to be simply connected, we can obtain a fundamental domain $\mathcal{D} := \exp(\overline{\mathcal{W}}) \subset U$ for the conjugacy action as the exponential of a convex polyhedron $\overline{\mathcal{W}} \subset \mathfrak{t} := Lie(T) \subset \mathfrak{u} := Lie(U)$ called a *closed Weyl alcove* on which the exponential map is injective. We will give a description of such a closed Weyl alcove in terms of roots of (U, T). Let us first recall the following definition :

Definition 2.2.1 (Fundamental domain). A subset $\mathcal{D} \subset X$ of a U-space X is called a *fundamental domain* for the action of U if it intersects each U-orbit in exactly one point. Consequently, the map $\mathcal{D} \subset X \to X/U$ is a bijection from the fundamental domain $\mathcal{D}$ onto the orbit space X/U.

We now consider the adjoint action of the maximal torus $T \subset U$ on the complexification $\mathfrak{u}^{\mathbb{C}}$ of $\mathfrak{u} = Lie(U)$, and we denote by R the corresponding root system (see for instance [Loo69b], ch. 5) :

$$R := \left\{\alpha \in \mathfrak{t}^* \mid \mathfrak{u}^{\mathbb{C}}_\alpha \neq \{0\}\right\}$$

where for any linear form $\alpha : \mathfrak{t} \to \mathbb{R}$ we set :

$$\mathfrak{u}^{\mathbb{C}}_\alpha := \{Y \in \mathfrak{u}^{\mathbb{C}} \mid \text{for all } X \in \mathfrak{t}, [X, Y] = 2i\pi\alpha(X)\, Y\}$$

In the following, we will always assume that $\mathfrak{u} = Lie(U)$ is equipped with an Ad-invariant *positive definite* scalar product $(.\,|\,.)$. In particular, we may identify the co-adjoint action of U on $\mathfrak{u}^*$ with the adjoint action of U on $\mathfrak{u}$. Since U is compact connected and simply connected, it is in particular semisimple, and we may for instance take $(.\,|\,.)$ to be minus the Killing form $\kappa(X, Y) = tr(ad\, X\ ad\, Y)$. To every root $\alpha \in R$ we associate the hyperplane

$$\mathcal{H}_\alpha = \{X \in \mathfrak{t} \mid \alpha(X) = 0\} = \ker \alpha \subset \mathfrak{t}$$

Definition 2.2.2 (Weyl chamber). A connected component of $\mathfrak{t} \setminus \cup_{\alpha \in R} \mathcal{H}_\alpha$ is called a *Weyl chamber* of the root system R. By definition, it is an open cone of $\mathfrak{t}$. In particular, it is convex. A Weyl chamber is commonly denoted by $\mathfrak{t}^*_+ \subset \mathfrak{t}^* \simeq \mathfrak{t}$. We will denote its closure by $\overline{\mathfrak{t}^*_+}$ and call it a *closed Weyl chamber*. It is a closed convex subset of $\mathfrak{t}$.

We now choose a Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}$ and denote by $R_+(\mathfrak{t}^*_+)$ (or simply R_+) the set of associated positive roots :

$$R_+(\mathfrak{t}^*_+) := \{\alpha \in R \mid \alpha(X) > 0 \text{ for one and therefore all } X \in \mathfrak{t}^*_+\}$$

A positive root $\alpha \in R_+$ is then said to be *decomposable* if it can be written as a sum $\alpha = \sum_{\beta \in R_+} n_\beta . \beta$ where $n_\beta \geq 0$ are integers. Otherwise it is called indecomposable, or *simple*. We denote by $\Delta(\mathfrak{t}^*_+)$ (or simply Δ) the set of simple roots in R_+, also called a basis of R, since it can be shown that every root $\alpha \in R$ is of the form $\sum_{\beta \in \Delta} n_\beta . \beta$ where $n_\beta \in \mathbb{Z}$. As a matter of fact, bases of R and Weyl chambers are in one-to-one correspondence (see for instance [BtD95], p.204). One of the interests of the notion of a Weyl chamber is that it provides a fundamental domain for the (co-)adjoint action of U on $\mathfrak{u} \simeq \mathfrak{u}^*$.

Proposition 2.2.3. *The inclusion maps* $\overline{\mathfrak{t}_+^*} \hookrightarrow \mathfrak{t} \hookrightarrow \mathfrak{u}$ *induce homeomorphisms :*

$$\overline{\mathfrak{t}_+^*} \xrightarrow{\simeq} \mathfrak{t}/W(T) \xrightarrow{\simeq} \mathfrak{u}/Ad\,U$$

We refer to [Bou82] (p.46) for a proof of this result.

The set of simple roots Δ associated to the choice of a Weyl chamber $\mathfrak{t}_+^*$ enables one to give a very nice description of the polyhedral structure of the closure $\overline{\mathfrak{t}_+^*}$ of the Weyl chamber, which we will recall shortly. This description is key to the proof of the momentum convexity theorem presented in [HNP94]. By describing in a similar way the polyhedral structure of a fundamental domain for the conjugacy action of U on itself, we lay the ground for the proof of the momentum convexity theorem that we will give in chapter 8. Generalizing the definition of the hyperplanes $\mathcal{H}_\alpha$, we set, for all $\alpha \in R$ and all $n \in \mathbb{Z}$:

$$\mathcal{H}_{\alpha,n} := \{X \in \mathfrak{t} \mid \alpha(X) = n\} \subset \mathfrak{t}$$

The set

$$D := \bigcup_{\alpha \in R,\ n \in \mathbb{Z}} \mathcal{H}_{\alpha,n}$$

is called the *(Stiefel) diagram* of $\mathfrak{t}$. It is a family of affine hyperplanes of $\mathfrak{t}$.

Definition 2.2.4 (Weyl alcove). A connected component $\mathcal{W}$ of $\mathfrak{t}\backslash D$ is called a *Weyl alcove* of the root system R. By definition, it is an open bounded convex polyhedron. For each choice of a Weyl chamber $\mathfrak{t}_+^* \subset \mathfrak{t}$ (with associated set of positive roots R_+ and set of simple roots Δ), there exists a unique Weyl alcove $\mathcal{W}$ whose closure contains $0 \in \mathfrak{t}$:

$$\mathcal{W} = \{X \in \mathfrak{t} \mid \forall \alpha \in \Delta, \alpha(X) > 0 \text{ and } \forall \alpha \in R_+\backslash\Delta, \alpha(X) < 1\}$$

We call it the *fundamental alcove* associated to the Weyl chamber $\mathfrak{t}_+^*$. Its closure $\overline{\mathcal{W}}$, called the *closed fundamental alcove*, is a convex polytope of $\mathfrak{t}$.

We then have :

Proposition 2.2.5. *Let U be a compact connected simply connected Lie group and let $\mathcal{W} \subset \mathfrak{u} = Lie(U)$ be a Weyl alcove for U. Then the set* $\exp(\overline{\mathcal{W}}) \subset U$ *is a fundamental domain for the conjugacy action of U on itself. Moreover, the exponential map* $\exp : \mathfrak{u} \to U$ *induces a one-to-one map from the compact convex polytope $\overline{\mathcal{W}}$ onto the closed set* $\exp(\overline{\mathcal{W}}) \subset U$*. Consequently, we have homeomorphisms :*

$$\overline{\mathcal{W}} \xrightarrow[\exp]{\simeq} \exp(\overline{\mathcal{W}}) \xrightarrow{\simeq} U/\mathrm{Int}(U)$$

We refer to [Bou82] (p.45) or to [Loo69b] (p.37) for a proof of this result. We now wish to describe the polyhedral structure of the convex polytope $\overline{\mathcal{W}} \subset \mathfrak{t}$. We begin with the polyhedral structure of the closed Weyl chamber $\mathfrak{t}_+^*$ (see [HNP94]). By definition of $\Delta \subset R_+$, we have :

$$\mathfrak{t}_+^* = \{X \in \mathfrak{t} \mid \forall \alpha \in \Delta, \alpha(X) > 0\}$$

and :

$$\overline{\mathfrak{t}_+^*} = \{X \in \mathfrak{t} \mid \forall \alpha \in \Delta, \alpha(X) \geq 0\}$$

For each subset $S \subset \Delta$, we set :

$$F_S := \{X \in \mathfrak{t} \mid \forall \alpha \in S, \alpha(X) = 0 \text{ and } \forall \alpha \in \Delta\backslash S, \alpha(X) > 0\} \subset \overline{\mathfrak{t}_+^*}$$

And we then have :

$$\mathfrak{t}_+^* = F_\emptyset \quad \text{and} \quad \overline{\mathfrak{t}_+^*} = \bigsqcup_{S \subset \Delta} F_S$$

One remarkable feature of the sets $(F_S)_{S\subset\Delta}$, which we will call the *cells* of $\overline{\mathfrak{t}_+^*}$, is that two elements X, Y lying in a same F_S have the same stabilizer $U_X = U_Y$ for the (co-)adjoint action of U on $\mathfrak{u} \simeq \mathfrak{u}^*$ (see lemma 6.3 in [HNP94]). We will establish an analogous property for the closed fundamental alcove $\overline{\mathcal{W}} \subset \overline{\mathfrak{t}_+^*}$ (see proposition 2.2.8). We first observe that $\overline{\mathcal{W}}$ is also a union of cells. Instead of corresponding to subsets $S \subset \Delta$ of the set of simple roots, these cells correspond to subset $S \subset R_+$ of the whole set of positive roots. More precisely, each subset $S \subset R_+$ can be uniquely written $S = S_1 \cup S_2$ where $S_1 \subset \Delta$ and $S_2 \subset R_+\backslash\Delta$, and for such an $S = S_1 \cup S_2$, we set :

$$\mathcal{W}_S := \left\{ X \in \mathfrak{t} \mid \left\{ \begin{array}{rl} \forall\alpha \in S_1, & \alpha(X) = 0 \\ \forall\alpha \in \Delta\backslash S_1, & \alpha(X) > 0 \end{array} \right. \text{ and } \left\{ \begin{array}{rl} \forall\alpha \in S_2, & \alpha(X) = 1 \\ \forall\alpha \in (R_+\backslash\Delta)\backslash S_2, & \alpha(X) < 1 \end{array} \right. \right\}$$

In particular :

$$\mathcal{W} = \{X \in \mathfrak{t} \mid \forall\alpha \in \Delta, \alpha(X) > 0 \text{ and } \forall\alpha \in R_+\backslash\Delta, \alpha(X) < 1\} = \mathcal{W}_\emptyset$$

and :

$$\overline{\mathcal{W}} = \{X \in \mathfrak{t} \mid \forall\alpha \in \Delta, \alpha(X) \geq 0 \text{ and } \forall\alpha \in R_+\backslash\Delta, \alpha(X) \leq 1\} = \bigsqcup_{S\subset R_+} \mathcal{W}_S$$

As a matter of fact, by using the notion of *highest root* (see [Bou68], p.165), this description can be simplified : there exists a unique positive root $\alpha_0 \in R_+\backslash\Delta$, called the highest root, such that for all $X \in \mathfrak{t}_+^*$, $\alpha_0(X) > \alpha(X)$. In particular, if $\alpha_0(X) < 1$ then necessarily $\alpha(X) < 1$. Then :

$$\mathcal{W} = \{X \in \mathfrak{t} \mid \forall\alpha \in \Delta, \alpha(X) > 0 \text{ and } \alpha_0(X) < 1\}$$

And in fact :

$$\overline{\mathcal{W}} = \{X \in \mathfrak{t} \mid \forall\alpha \in \Delta, \alpha(X) \geq 0 \text{ and } \alpha_0(X) \leq 1\}$$

and the cells of $\overline{\mathcal{W}}$ correspond to subsets $S \subset \Delta \cup \{\alpha_0\}$:

$$\overline{\mathcal{W}} = \bigsqcup_{S\subset\Delta\cup\{\alpha_0\}} \mathcal{W}_S$$

Definition 2.2.6. The sets $(\mathcal{W}_S)_{S\subset\Delta\cup\{\alpha_0\}}$ are called the *cells* of the closed fundamental alcove $\overline{\mathcal{W}}$.

We now write this down explicitly in the case where $U = SU(4)$ (see [Loo69b] pp.16-18 for the case $U = SU(3)$). We do so because in the $SU(3)$ case there is only one positive root which is not simple, so that it is automatically the highest root. In contrast, in the $SU(4)$ case, there are three positive roots which are not simple (see below) and we will see that it is enough for our purposes to consider the highest one. Additionally, since $SU(4)$ is of rank 3, it is still possible to draw the Weyl alcove explicitly, which helps developing intuition on this object. This will be useful in chapter 8. We choose the following *Ad*-invariant positive definite scalar product on $\mathfrak{su}(4)$:

$$(X \,|\, Y) = -tr(XY) = -\frac{1}{8}\, \kappa_{\mathfrak{su}(4)}(X, Y)$$

The 3-dimensional torus

$$T = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & t_4 \end{pmatrix} \;:\; t_j \in S^1,\ t_1t_2t_3t_4 = 1 \right\} \subset SU(4)$$

is a maximal torus of $SU(4)$, with Lie algebra

$$\mathfrak{t} = \left\{ \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & x_4 \end{pmatrix} \;:\; x_j \in i\mathbb{R},\ x_1 + x_2 + x_3 + x_4 = 0 \right\} \subset \mathfrak{su}(4)$$

The associated set of roots is

$$R = \{\pm\alpha_{12},\ \pm\alpha_{23},\ \pm\alpha_{34},\ \pm\alpha_{13},\ \pm\alpha_{24},\ \pm\alpha_{14}\}$$

where

$$\alpha_{jk}(X) = \frac{x_j - x_k}{2i\pi} \in \mathbb{R} \quad \text{for any } X = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & x_4 \end{pmatrix} \in \mathfrak{t}$$

A set of positive roots is

$$R_+ = \{\alpha_{12},\ \alpha_{23},\ \alpha_{34},\ \alpha_{13},\ \alpha_{24},\ \alpha_{14}\}$$

The corresponding set of simple roots is

$$\Delta = \{\alpha_{12},\ \alpha_{23},\ \alpha_{34}\}$$

and

$$\alpha_{14} = \alpha_{12} + \alpha_{23} + \alpha_{34} \in R_+ \backslash \Delta$$

is the highest root. As in [Loo69b], we define the *inverse roots* $\{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{34}^\vee\}$ to be the following elements of $\mathfrak{t}$:

$$\alpha_{jk}^\vee := \frac{2\overrightarrow{\alpha_{jk}}}{(\overrightarrow{\alpha_{jk}} \,|\, \overrightarrow{\alpha_{jk}})}$$

where $\overrightarrow{\alpha_{jk}}$ satisfies $(\overrightarrow{\alpha_{jk}} \,|\, X) = \alpha(X)$ for all $X \in \mathfrak{t}$. Explicitly here :

$$\alpha_{12}^\vee = 2i\pi \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},\ \alpha_{23}^\vee = 2i\pi \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix},\ \text{and } \alpha_{34}^\vee = 2i\pi \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

Then $\|\alpha_{jk}^\vee\| = 2\pi\sqrt{2}$ and the angles between the inverse roots are :

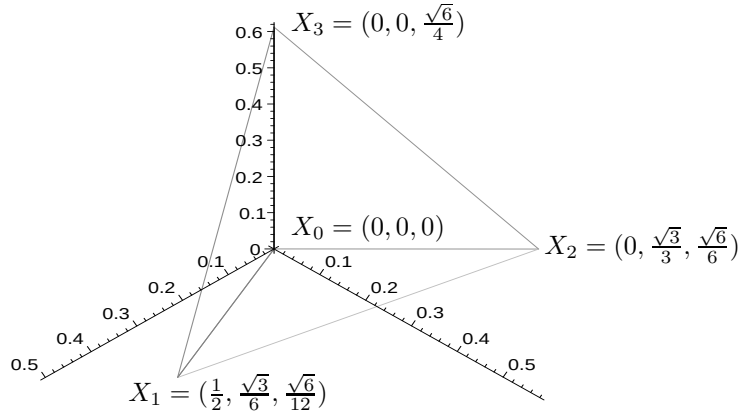
$$(\alpha_{12}^\vee, \alpha_{23}^\vee) = \frac{2\pi}{3},\ (\alpha_{23}^\vee, \alpha_{34}^\vee) = \frac{2\pi}{3},\ \text{and } (\alpha_{12}^\vee, \alpha_{34}^\vee) = \frac{\pi}{2}$$

As a basis for the central lattice $\exp^{-1}(\mathcal{Z}(SU(4)))$, we obtain, by inverting the Cartan matrix (see for instance [Ada69]) :

$$\begin{aligned} X_1 &:= \frac{1}{4}(3\alpha_{12}^\vee + 2\alpha_{23}^\vee + \alpha_{34}^\vee) \\ X_2 &:= \frac{1}{4}(2\alpha_{12}^\vee + 4\alpha_{23}^\vee + 2\alpha_{34}^\vee) \\ X_3 &:= \frac{1}{4}(\alpha_{12}^\vee + 2\alpha_{23}^\vee + 3\alpha_{34}^\vee) \end{aligned}$$

and the tetrahedron of $\mathfrak{t}$ whose vertices are $(X_0 := 0, X_1, X_2, X_3)$ is a closed fundamental alcove for $SU(4)$. In particular, $\{\exp(X_j)\}_{0\leq j\leq 3} = \{1, -i, -1, i\} = \mathcal{Z}(SU(4))$. As all the $\overrightarrow{\alpha_{jk}}$ have same norm, we have $\alpha_{14}^\vee = \alpha_{12}^\vee + \alpha_{23}^\vee + \alpha_{34}^\vee$, and we can then represent the closed fundamental alcove of $(SU(4), \Delta)$ as shown in figure 2.1. The cells of the alcove $\overline{\mathcal{W}}$ in the sense of definition 2.2.6 are the cells of the tetrahedron represented in figure 2.1. In particular, two elements $X, Y \in \overline{\mathcal{W}}$ lying in the same cell have the same number of distinct eigenvalues with the same respective multiplicities, so that the conjugacy classes of $\exp(X)$ and $\exp(Y)$ have the same dimension (see proposition 2.2.8 for the general case).

We now go back to our general study of the conjugacy action of a compact connected simply connected Lie group U on itself. First, we will show in all generality that the stabilizer, for the conjugacy action, of an element $\exp(X) \in T$ for some $X \in \overline{\mathcal{W}}$ only depends on the cell of $\overline{\mathcal{W}}$ containing X.



$$\overline{\mathcal{W}} = \left\{(x, y, z) \in \mathbb{R}^3 \mid \alpha_{12}(x, y, z) \geq 0,\ \alpha_{23}(x, y, z) \geq 0,\ \alpha_{34}(x, y, z) \geq 0. \text{ and } \alpha_{14}(x, y, z) \leq 1\right\}$$

Figure 2.1: The closed fundamental alcove of $(SU(4), \Delta)$

Lemma 2.2.7. *Let U be a compact connected simply connected Lie group. Then for any $u \in U$, the centralizer $U_u = \{v \in U \mid vuv^{-1} = u\}$ is connected.*

We refer to [Bou82] (p.48) for a proof of this result.

Proposition 2.2.8. *If $u, v \in \exp \overline{\mathcal{W}} = \sqcup_{S \subset \Delta \cup \{\alpha_0\}} \exp(\mathcal{W}_S)$ lie in a same $\exp(\mathcal{W}_S)$, then the centralizers U_u and U_v are equal.*

Proof. Since U is compact connected and simply connected, lemma 2.2.7 shows that U_u and U_v are compact connected subgroups of U. Therefore $U_u = U_v$ if and only if their Lie algebras are equal. We then know from [Loo69b] (p.7) that the Lie algebra of U_u is :

$$Lie(U_u) \quad = \quad \mathfrak{t} \quad \oplus \sum_{\alpha \ \mid \ \exp\left(i2\pi\alpha(X)\right)=1} \mathfrak{u} \cap \mathfrak{u}_\alpha^{\mathbb{C}}$$

where $X \in \mathfrak{t}$ satisfies $\exp(X) = u$. But for $X \in \overline{\mathcal{W}}$, the set

$$\left\{\alpha \in \Delta \cup \{\alpha_0\} \mid \exp\left(i2\pi\alpha(X)\right) = 1\right\}$$

is equal to

$$\left\{\alpha \in \Delta \cup \{\alpha_0\} \mid \alpha(X) = 0 \text{ or } \alpha(X) = 1\right\}$$

so that it only depends on the cell $\mathcal{W}_S \subset \overline{\mathcal{W}}$ in which X lies, which prove the proposition. □

Definition 2.2.9. For any subset $S \subset \Delta \cup \{\alpha_0\}$, we denote by U_S the stabilizer of any element $u \in \exp(\mathcal{W}_S)$.

Finally, if we consider, for any integer j, the set

$$\Sigma_j := \{u \in U \mid \dim U.u = j\}$$

of points of U whose conjugacy class is of dimension j, we have :

Proposition 2.2.10. *The intersection of Σ_j with $\exp(\overline{\mathcal{W}})$ is :*

$$\Sigma_j \cap \exp(\overline{\mathcal{W}}) \quad = \quad \bigsqcup_{S \ \mid \ \dim U_S = \dim U - j} \exp(\overline{\mathcal{W}}_S)$$

In addition to that, Σ_j *is a submanifold of* U *and so is every* $\exp(\mathcal{W}_S)$*. For any* $u \in \exp(\mathcal{W}_S)$*, one has :*

$$T_u\Sigma_j = T_u(U.u) \oplus T_u \exp(\mathcal{W}_S)$$

Proof. This result is a consequence of the slice theorem (proposition 2.1.3) : for any slice $\mathcal{S}$ through $u \in U$, the set of elements whose conjugacy class has the same dimension as $U.u$ consists of elements which are fixed by U_u (since the stabilizer of such an element $y \in \mathcal{S}$ is a subgroup of U_u by (iii) in definition 2.1.2), and for a linear action the set of such fixed points is a subspace, so that Σ_j is a submanifold of U, and it is U-invariant. Now for any $S \subset \Delta \cup \{\alpha_0\}$, $\exp(\mathcal{W}_S)$ is a submanifold of U (the chart is given by the exponential map) and it follows from the fact that $\exp(\overline{\mathcal{W}}) = \sqcup \exp(\mathcal{W}_S)$ is a fundamental domain for the conjugacy action and from proposition 2.2.8 that

$$\Sigma_j \cap \exp(\overline{\mathcal{W}}) \quad = \bigsqcup_{S \ | \ \dim U_S = \dim U - j} \exp(\overline{\mathcal{W}}_S)$$

and that $\exp(\mathcal{W}_S)$ is a slice through *any* $u \in \exp(\mathcal{W}_S)$, which concludes the proof. □

Chapter 3

A few facts about compact connected Lie groups as compact symmetric spaces

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In this chapter, we recall a few facts about symmetric spaces and symmetric pairs that will be useful to us in the rest of this work. The standard references for these questions are [Hel01] and [Loo69a, Loo69b] (see also appendix B in [OS00] for a summary of the theory of symmetric pairs with a view towards symplectic geometry).

The sole purpose of this chapter is to serve as a reference for results that we shall need in the remainder of this thesis. The properties we remind here are all standard, and proofs might be found in the books indicated below. Whenever we need one of these results in the forthcoming chapters, a precise reference will be made, so that one may for now skip the present chapter and come back to it when necessary.

3.1 Symmetric spaces and symmetric pairs

Here we briefly recall the definition and main examples of symmetric spaces, following [Loo69a, Loo69b]. Roughly speaking, a *symmetric space* is a manifold M on which there is a notion of symmetry (or reflection) around each point $x \in M$, the correct definition of a symmetry being that it is an involutive transformation having x as an isolated fixed point and satisfying the composition rule depicted in figure 3.1.

Definition 3.1.1 (Symmetric space). A *symmetric space* $(M, (s_x)_{x\in M})$ is a manifold M endowed with an application

$$\begin{array}{rccc} s : & M & \longrightarrow & \mathrm{Diff}(M) \\ & x & \longmapsto & s_x \end{array}$$

from M to the group of its diffeomorphisms, satisfying, for all $x \in M$:

(i) $s_x(x) = x$

(ii) $s_x(s_x(y)) = y$ for all $y \in M$

(iii) $s_x \circ s_y(z) = s_{s_x(y)} \circ s_x(z)$ for all $y, z \in M$ (see figure 3.1)

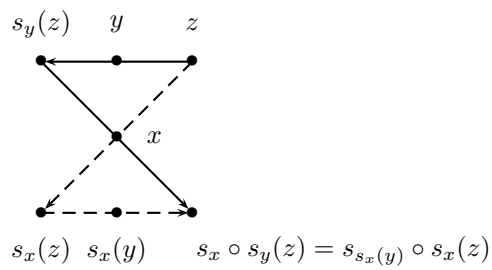

Figure 3.1: Composition rules for symmetries on M

(iv) x possesses a neighbourhood $\mathcal{V}_x$ such that if $y \in \mathcal{V}_x$ satisfies $s_x(y) = y$ then $y = x$.

All the symmetric spaces $(M, (s_x)_{x \in M})$ that we shall consider henceforth will be supposed to be connected (meaning that the underlying manifold M is connected). Basic examples of symmetric spaces are the vector spaces $\mathbb{R}^n$, with central symmetry s_x at each $x \in \mathbb{R}^n$:

$$s_x(y) = -(y - x) + x = 2x - y$$

(in particular $s_0(y) = -y$). More generally, any Lie group U becomes a symmetric space by setting :

$$s_u(v) = u(u^{-1}v)^{-1} = uv^{-1}u$$

(in particular $s_1(v) = v^{-1}$). Another example is the sphere of radius r in $\mathbb{R}^n$: $S_r = \{x \in \mathbb{R}^n \mid (x \,|\, x) = r^2\}$ with symmetry at x the transformation :

$$s_x(y) = 2\frac{(x \,|\, y)}{(x \,|\, x)}x - y$$

For any symmetric space $(M, (s_x))$, the subgroup G of $\mathrm{Diff}(M)$ generated by all transformations of the form $s_x s_y$ $(x, y \in M)$ is called the *group of displacements of* M. It is a normal subgroup of $\mathrm{Diff}(M)$ (since $\varphi s_x \varphi^{-1} = s_{\varphi(x)}$) and it is actually a (finite-dimensional) Lie group acting transitively on M. By choosing a base point x_0 in M, one obtains the following homogeneous description of M :

$$M \simeq G/G_{x_0}$$

where G_{x_0} is the stabilizer of the base-point x_0. The map

$$\begin{array}{rccc} \sigma \ : & G & \longrightarrow & G \\ & g & \longmapsto & s_{x_0} g s_{x_0} \end{array}$$

is an involutive automorphism of G and G_{x_0} lies between the group G^σ of fixed-points of σ and its neutral component :

$$(G^\sigma)^0 \subset G_{x_0} \subset G^\sigma$$

(see [Loo69a], p.91). Conversely, if G is a Lie group and σ is an involutive automorphism of G, then any subgroup $K \subset G$ satisfying $(G^\sigma)^0 \subset K \subset G^\sigma$ is necessarily closed and the coset space G/K endowed with the transformations

$$s_{[x]}\big([y]\big) := \big[x\sigma(x^{-1})\sigma(y)\big]$$

(in particular, $s_{[1]}([y]) = [\sigma(y)]$ for all $y \in G$) is a symmetric space. As an example, when the symmetric space M is a compact connected Lie group U, one may consider the group $G = U \times U$ and the automorphism $\sigma(u_1, u_2) = (u_2, u_1)$. Then

$$G^\sigma = \big\{(u,u) \,:\, u \in U\big\} =: U_\Delta$$

and the map

$$\begin{array}{rccc} \Phi \,:\, & U \times U & \longrightarrow & U \\ & (u_1, u_2) & \longmapsto & u_1 u_2^{-1} \end{array}$$

induces an isomorphism of symmetric spaces

$$\overline{\varphi} \,: U \times U/U_\Delta \xrightarrow{\simeq} U$$

(meaning that $\overline{\varphi}(s_{[x]}([y])) = s_{\overline{\varphi}([x])}(\overline{\varphi}([y])))$.

In the following, the only example of symmetric space that will be really useful to us is the space $M = U/U^\tau$, where U is a compact connected Lie group and τ is an involutive automorphism of U. In this case, the symmetric space U/U^τ may in fact be thought of as a subspace of U :

Proposition 3.1.2. *If U is a compact connected Lie group and τ is an involutive automorphism of U, the map*

$$\begin{array}{rccc} q \,:\, & U & \longrightarrow & U \\ & u & \longmapsto & u\tau(u^{-1}) \end{array}$$

induces a homeomorphism

$$U/U^\tau \simeq \big\{u\tau(u^{-1}) \,:\, u \in U\big\} \subset U$$

If one denotes by τ^- the map

$$\begin{array}{rccc} \tau^- \,:\, & U & \longrightarrow & U \\ & u & \longmapsto & \tau(u^{-1}) \end{array}$$

then the set $\{u\tau(u^{-1}) \,:\, u \in U\}$ is the connected component of 1 in $Fix(\tau^-)$. In particular, if $Fix(\tau^-)$ is connected then any $w \in Fix(\tau^-)$ may be written $w = u\tau(u^{-1})$ for some $u \in U$.

We refer to [Loo69a] (pp.73 and 182) for a proof of this result. We will call such a pair (U, τ) a *symmetric pair*. This definition is less general than that appearing in [Hel01] and [OS00] but it will be sufficient for us.

Definition 3.1.3 (Symmetric pair). A compact connected Lie group (U, τ) endowed with an involutive automorphism τ will be called a *symmetric pair*.

As a matter of fact, we will now restrain ourselves even further by considering symmetric pairs of maximal rank (see definition 3.2.1).

3.2 Symmetric pairs of maximal rank

As we did not go into enough detail in the general theory of symmetric spaces, we do not have a notion of *rank* of a symmetric space, and therefore cannot define a symmetric space M of maximal rank the way it should be, as a symmetric space whose rank is equal to the rank of its group of displacements. We refer to [Loo69b] (pp.49-86) for this matter, and concentrate on the case where $M = U/U^\tau$, where (U, τ) is a symmetric pair. Following [Loo69b] (p.78), we may then set :

Definition 3.2.1 (Symmetric pair of maximal rank). A symmetric pair (U, τ) is said to be *of maximal rank* if one has :

$$\dim U/U^\tau = \frac{1}{2}(\dim U + \mathrm{rk}\, U)$$

(where rk U is the dimension of any maximal torus $T \subset U$ of U).

As noted in [Loo69b] (p.80), compact semisimple symmetric spaces of maximal rank correspond to *normal* (or *split*) real forms of complex semisimple Lie algebras. In particular :

Proposition 3.2.2. *If U is a compact connected simply connected Lie group, there exists an involutive automorphism τ of U such that the symmetric pair (U,τ) is of maximal rank. Additionally, two such automorphisms τ and τ' are conjugate by an inner automorphism of U.*

We refer to [Loo69b] (pp.78-81) for a proof of this result. To us, the most important feature of symmetric pairs of maximal rank will be the existence of a *maximal* torus T of U such that $\tau(t) = t^{-1}$ for all $t \in T$.

Proposition 3.2.3. *If (U,τ) is a symmetric pair of maximal rank, then there exists a maximal torus T of U such that $\tau(t) = t^{-1}$ for all $t \in T$. Equivalently, denoting by τ^- the involution $\tau^-(u) := \tau(u^{-1})$ on U, one has $\tau^-(t) = t$ for all $t \in T$.*

We refer to [Loo69b] (pp.78-81) for a proof of this result. In the rest of this chapter, we shall assume that the fixed-point set

$$Fix(\tau^-) = \big\{w \in U \mid \tau(w^{-1}) = w\big\}$$

is connected, in which case proposition 3.1.2 shows that $Fix(\tau^-) = \{u\tau^-(u) \,:\, u \in U\}$. This assumption is in particular satisfied for the Lie group $SU(n)$ equipped with the involutive automorphism $\tau(u) := \overline{u} \in SU(n)$ (see proposition 5.1.3), which is of maximal rank since one has :

$$\dim\, SU(n)/SO(n) = (n^2-1) - \frac{n(n-1)}{2} = \frac{1}{2}(n^2+n-2)$$

and :

$$\frac{1}{2}(\dim\, SU(n) + \mathrm{rk}\, SU(n)) = \frac{1}{2}\big((n^2-1)+(n-1)\big) = \frac{1}{2}(n^2+n-2)$$

Observe that the same is true for the symmetric pair $(U(n), \tau(u) := \overline{u})$. These are the examples that we will keep in mind throughout this work, as they motivated and inspired most of our results. We refer to remarks 5.2.3 and 7.4.2 for additional comments on the assumption that $Fix(\tau^-)$ is connected.

We now quote a series of results that will be useful to us in the forthcoming chapters.

Lemma 3.2.4. *Let (U,τ) be a symmetric pair of maximal rank and let $T \subset U$ be a maximal torus of U fixed pointwise by τ^-. Then any element of the associated Weyl group $W(T) := \mathcal{N}(T)/T$ can be represented by an element in the neutral component K^0 of the group $K := U^\tau$.*

Proof. We need to prove that for any $n \in \mathcal{N}(T)$, there exists $k \in K^0$ such that for all $t \in T$, $ntn^{-1} = ktk^{-1}$. For all $n \in \mathcal{N}(T)$, we have, for all $t \in T$, $ntn^{-1} \in T \subset Fix(\tau^-)$, so that $\tau(n)t\tau(n^{-1}) = ntn^{-1}$, hence $(\tau^-(n)n)t(\tau^-(n)n)^{-1} = t$ for all $t \in T$. Therefore $\tau^-(n)n \in \mathcal{Z}(T) = T$ since a maximal torus is its own centralizer (see for instance [Loo69b], p.4). Write $\tau^-(n)n = \exp(X)$ for some $X \in \mathfrak{t} = Lie(T)$ and set $w = \exp(\frac{X}{2})$ (so that $w \in T \subset Fix(\tau^-)$ and $w^2 = \tau^-(n)n$). Set now $k := nw^{-1}$. Then :

$$\begin{aligned}
\tau(k)k^{-1} &= \tau(n)\ \underbrace{\tau(w^{-1})w}\ n^{-1} \\
&= \tau(n)\qquad w^2\qquad n^{-1} \\
&= \tau(n)\big(\tau^-(n)n\big)n^{-1} \\
&= 1
\end{aligned}$$

so that $k \in Fix(\tau)$. Since $w^{-1} \in T$ acts trivially on T, one has, for all $t \in T$:

$$ktk^{-1} = nw^{-1}twn^{-1} = ntn^{-1}$$

To prove that we may even choose k to lie in K^0, we refer to lemma B.1 in the appendix of [OS00] : $K = U^\tau = K^0.T[2]$, where $T[2] := \{t \in T \,:\, t^2 = 1\} = T^\tau$, so that our k above writes $k = k_0 a$ with $k_0 \in K^0$ and $a \in T[2]$. Since $a \in T[2] \subset T$, it acts trivially on T and $ktk^{-1} = k_0 t k_0^{-1}$ for all $t \in T$. $\square$

Proposition 3.2.5. *Let (U,τ) be a symmetric pair of maximal rank and let $T \subset U$ be a maximal torus of U fixed pointwise by τ^-. Recall that we assume $Fix(\tau^-)$ to be connected. Then for all $w \in Fix(\tau^-)$, there exists $k \in K^0$ (where $K = U^\tau$) such that $kwk^{-1} \in T$.*

We refer to [Loo69b] (p.56) for a proof of this result (recall that we assumed $Fix(\tau^-)$ to be connected and see also proposition 5.1.3 for a proof of this result in the case where $(U,\tau) = (U(n), \tau(u) = \overline{u})$).

Corollary 3.2.6. *Assume U to be simply connected. If $\exp(\overline{\mathcal{W}}) \subset T$ is a fundamental domain for the conjugacy action of U on itself (see proposition 2.2.5) and if $w \in Fix(\tau^-)$, then there exists $k \in K^0$ such that $kwk^{-1} \in \exp(\overline{\mathcal{W}})$.*

Proof. By proposition 3.2.5, there exists $k \in K^0$ such that $kwk^{-1} \in T$. Recall that $\exp(\overline{\mathcal{W}}) \simeq T/W(T)$ (see proposition 2.2.5 and relation (2.1) in section 2.2), so that by conjugating by an appropriate Weyl group element, which may be taken in K^0 according to lemma 3.2.4, we obtain $k'w(k')^{-1} \in \exp(\overline{\mathcal{W}})$ for some $k' \in K^0$. □

Finally :

Proposition 3.2.7 ($K \times K$-orbits in U). *Let (U,τ) be a symmetric pair of maximal rank and let $T \subset U$ be a maximal torus of U fixed pointwise by τ^-. Take $u, v \in U$. Then there exists $(k_1, k_2) \in K \times K$ such that $v = k_1 u k_2^{-1}$ if and only if $\tau^-(v)v$ and $\tau^-(u)u$ lie in a same conjugacy class of U.*

Proof. The first implication is obvious. Conversely, suppose that $\Delta_v := \tau^-(v)v$ is conjugate to $\Delta_u := \tau^-(u)u$ in U. Then, by proposition 3.2.5, there exists $k_1, k_2 \in U^\tau$ such that $k_1\Delta_u k_1^{-1} = k_2\Delta_v k_2^{-1} \in T = \exp(\mathfrak{t})$, where $\mathfrak{t} = Lie(T)$. Write $k_1\Delta_u k_1^{-1} = k_2\Delta_v k_2^{-1} = \exp(X)$ for some $X \in \mathfrak{t}$ and set $w := \exp(\frac{X}{2})$, $\delta_u := k_1^{-1} w k_1$ and $\delta_v := k_2^{-1} w k_2$. Set now $k_u := u\delta_u^{-1}$ and $k_v := v\delta_v^{-1}$. Then :

$$\begin{aligned}
\tau(k_u)k_u^{-1} &= \tau(u)\tau(\delta_u^{-1})\delta_u u^{-1} \\
&= \tau(u)\delta_u^2 u^{-1} \\
&= \tau(u)\tau^-(u)uu^{-1} \\
&= 1
\end{aligned}$$

so that $\tau(k_u) = k_u$. Likewise $\tau(k_v) = k_v$. And we then have :

$$\begin{aligned}
v &= k_v\delta_v \\
&= k_v k_2^{-1} w k_2 \\
&= k_v k_2^{-1} k_1 \delta_u k_1^{-1} k_2 \\
&= \underbrace{(k_v k_2^{-1} k_1 k_u^{-1})}_{\in K}\ u\ \underbrace{(k_1^{-1} k_2)}_{\in K}
\end{aligned}$$

□

Chapter 4

Quasi-Hamiltonian spaces

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In this chapter, we give the definition of a *quasi-Hamiltonian space*, as well as the main examples of manifolds carrying such a structure. The purpose of doing so is to obtain a fairly easy and convenient description of the symplectic structure of moduli spaces of representations of surface groups (see section 4.6).

The notion of quasi-Hamiltonian space was derived from the definition of a usual Hamiltonian space in [AMM98]. A related construction appeared before that in [GHJW97], where it was noticed that the Cartan 3-form χ of a Lie group whose Lie algebra is equipped with an Ad-invariant non-degenerate product, which is always closed, is exact when restricted to a conjugacy class. The resulting 2-form ω such that $d\omega = \chi$ is, up to a sign, the form used in [AMM98] to show that a conjugacy class of a Lie group is a quasi-Hamiltonian space. A larger notion, the one of *quasi-Poisson manifold*, was later on identified in [AKS00] and investigated in [AKSM02]. Loosely speaking, a quasi-Poisson manifold is a manifold endowed with an action of a Lie group G (whose Lie algebra is supposed to be equipped with an Ad-invariant non-degenerate symmetric bilinear form) and an invariant bivector field satisfying a compatibility condition with this group action. The basic example of a quasi-Poisson manifold is the group G itself, endowed with the conjugacy action, in analogy with the dual of a Lie algebra being a basic example of Poisson manifold (one should notice, though, that in the quasi-Poisson setting, the quasi-Poisson structure is always defined *with respect to a given group action*, for instance the action of G on itself by conjugation). Of particular interest are the *Hamiltonian* quasi-Poisson manifolds (that is, those admitting a (group-valued) *momentum map*). Again, the basic example of a Hamiltonian quasi-Poisson manifold is the group G itself, with momentum map the identity $Id : G \to G$. A large part of the

theory of quasi-Poisson manifolds can then be derived from the analogy with usual Hamiltonian Poisson manifolds :

- Hamiltonian quasi-Poisson manifolds are foliated by *non-degenerate* quasi-Poisson manifolds, corresponding to symplectic leaves of Poisson manifolds. When the quasi-Poisson manifold at hand is a Lie group, its non-degenerate leaves are the conjugacy classes, in analogy with the co-adjoint orbits being the symplectic leaves of the dual of a Lie algebra.

- non-degenerate quasi-Poisson manifolds correspond to quasi-Hamiltonian spaces in the sense of [AMM98].

- homogeneous non-degenerate Hamiltonian quasi-Poisson manifolds are coverings of conjugacy classes, just like homogeneous Hamiltonian symplectic manifolds are coverings of co-adjoint orbits.

All these properties are discussed in detail in [AKSM02], along with many other very nice features of quasi-Poisson manifolds, like reduction, products and cohomology. The framework of quasi-Poisson geometry was made use of in [Tre02] to study the symplectic geometry of the space of polygons on the sphere $S^3 \simeq SU(2)$, where a symplectic structure was obtained on the moduli space of polygons with fixed sidelengths by reduction from the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ where $\mathcal{C}_j$ is a conjugacy class in $SU(2)$.

Here, we will likewise obtain, given a surface group π and a Lie group U, a symplectic structure on the moduli space $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$ (see section 4.6 for a precise definition) by reduction from a quasi-Hamiltonian space, which is why we will present this notion only, without entering the broader and richer notion of quasi-Poisson manifold. We will follow [AMM98] very closely, except for the fact that we do not assume the Lie group entering the definition of quasi-Hamiltonian space to be compact. Indeed, the results of this part of the theory hold for non-compact groups as well, as we shall see in the course of the proofs. So we start with a Lie group U, and we assume the existence of an Ad-invariant non-degenerate symmetric bilinear form on its Lie algebra $\mathfrak{u}$ (which could for instance be a Euclidean scalar product obtained by averaging in the case a compact group, or the Killing form of a semi-simple group, compact or not). We point this out now to stress the fact that there is probably little to be done to generalize the results contained in this thesis to the case where U belongs to a large class of non-compact groups (although more serious problems regarding this generalization will appear for example in chapter 8, where we shall prove a convexity theorem for momentum maps defined on a quasi-Hamiltonian space).

4.1 From Hamiltonian to quasi-Hamiltonian spaces

In this section, we will show how to derive the notion of quasi-Hamiltonian space from the notion of a usual Hamiltonian space, following the process of [AMM98], in which the aim was to develop a theory of Lie-group valued momentum maps. Previous examples of such theories include the notion of momentum maps for Poisson Lie groups (taking value in the Poisson dual of a Poisson Lie group acting on a given *symplectic* manifold, see [Lu91, LW90, Vai94]) and S^1-valued momentum maps for actions of S^1 on *symplectic* manifolds considered in [McD88, Wei93] (so that this time the target space for the momentum map is the acting group itself). The Poisson Lie group setting provided a whole new series of examples of symplectic manifolds, that were later related to Kostant's famous nonlinear convexity theorems (see [Kos74, LR91, FR96]) and to matrix spectral problems (see for instance the work on the Thompson conjecture in [AMW01, EL05]). A possible proof for both these applications is to show that the Poisson Lie situation is *equivalent* to the usual Hamiltonian one (meaning that there exists another symplectic form on the given manifold for which the group action admits a Lie-algebra valued momentum map, see [Ale97, AMW01]). On the contrary, the examples of symplectic manifolds laid forward in [McD88] did not reduce to usual Hamiltonian manifolds. The starting point of [AMM98] is to study this same situation in the case of non-abelian groups. The first consequence of this fact is that the manifolds at hand are no longer symplectic : the 2-form defining the quasi-Hamiltonian structure is neither closed

nor non-degenerate, except when the acting group is abelian, in which case one recovers the situation of [McD88].

Let us now proceed to defining quasi-Hamiltonian spaces. We will follow [AMM98] closely (another presentation of the notion of a quasi-Hamiltonian space, emphasizing the comparison with Hamiltonian spaces, can be found in [Rac03]). Throughout this section, we shall designate by U a Lie group whose Lie algebra $\mathfrak{u} = Lie(U) = T_1U$ is equipped with an Ad-invariant non-degenerate symmetric bilinear form (for instance an invariant Euclidean product in the compact case, or the Killing form in the (non-necessarily compact) semi-simple case) denoted by $(.\,|\,.)$. We will call such a form an Ad-invariant *scalar product* (or simply product). To fix notation right away, let χ be (half) the Cartan 3-form of U, that is, the left-invariant 3-form on U defined on $\mathfrak{u} = T_1U$ by :

$$\chi_1(X,Y,Z) := \frac{1}{2}(X\,|\,[Y,Z]) = \frac{1}{2}([X,Y]\,|\,Z)$$

where the last equality follows by derivation from the Ad-invariance property of $(.\,|\,.)$. It is then a consequence of the Jacobi identity that $d\chi = 0$. Observe that, since $(.\,|\,.)$ is Ad-invariant, χ is also right-invariant. Further, let us denote by θ^L and θ^R the respectively left-invariant and right-invariant Maurer-Cartan 1-forms on U : they take value in $\mathfrak{u}$ and are the identity on $\mathfrak{u}$, meaning that for any $u \in U$ and any $\xi \in T_uU$,

$$\theta^L_u(\xi) = u^{-1}.\xi \quad \text{and} \quad \theta^R_u(\xi) = \xi.u^{-1}$$

(where we denote by a point . the effect of translations on tangent vectors). In particular, by definition of χ, one has, for all $\in U$ and all $\xi_1, \xi_2, \xi_3 \in T_uU$:

$$\chi_u(\xi_1,\xi_2,\xi_3) \quad = \quad \frac{1}{2}\big(\theta^L_u(\xi_1)\,|\,[\theta^L_u(\xi_2),\theta^L_u(\xi_3)]\big) \tag{4.1}$$

As earlier, it then follows from the Ad-invariance of $(.\,|\,.)$ and from the fact that

$$Ad\,u.(\theta^L_u(\xi)) = u.(u^{-1}.\xi).u^{-1} = \xi.u^{-1} = \theta^R_u(\xi)$$

that one has :

$$\chi_u(\xi_1,\xi_2,\xi_3) = \frac{1}{2}\big(\theta^R_u(\xi_1)\,|\,[\theta^R_u(\xi_2),\theta^R_u(\xi_3)]\big)$$

Finally, we denote by M a manifold on which the group U acts, and by $X^\sharp$ the fundamental vector field on M defined, for any $X \in \mathfrak{u}$, by the action of U in the following way :

$$X^\sharp_x := \tfrac{d}{dt}|_{t=0}\,\big(exp(tX).x\big) \tag{4.2}$$

for any $x \in M$. In particular, we will denote by $X^\dagger$ the fundamental vector field on U associated to $X \in \mathfrak{u}$ by the conjugacy action of U on itself :

$$X^\dagger_u \quad = \quad \frac{d}{dt}|_{t=0}\,\big(\exp(tX)u\exp(-tX)\big) \quad = \quad X.u - u.X \tag{4.3}$$

One then has :

$$\theta^L_u(X^\dagger_u) = Ad\,u^{-1}.X - X \quad \text{and} \quad \theta^R_u(X^\dagger_u) = X - Ad\,u.X$$

The map $X \in \mathfrak{u} \mapsto X^\# \in \Gamma(TM)$ is an *anti-homomorphism* of Lie algebras from $\mathfrak{u}$ to the Lie algebra $\Gamma(TM)$ of vector fields on M, meaning that it is linear and that it satisfies :

$$[X,Y]^\# = -[X^\#, Y^\#]$$

Indeed, observe that, for any $Z \in \mathfrak{u}$:

$$Z^\#_x = \frac{d}{dt}|_{t=0}\,\big(\exp(tZ).x\big) = T_1\Phi_x.Z$$

where Φ_x is the map :

$$\begin{array}{rccc} \Phi_x : & U & \longrightarrow & M \\ & u & \longmapsto & u.x \end{array}$$

so that $(X+Y)^\#_x = X^\#_x + Y^\#_x$ and $(\lambda X)^\#_x = \lambda X^\#_x$ for all $x \in M$. Further, the bracket of two vector fields satisfies :

$$[X^\#, Y^\#]_x = -\frac{d}{dt}|_{t=0} \left((\psi_t^{X^\#})_* Y^\# \right)_x$$

where $\psi_t^{X^\#} : x \in M \mapsto \exp(tX).x$ is the local flow of $X^\#$ (see for instance [Spi99], p.150). Hence, for a given $x \in M$:

$$\begin{aligned} [X^\#, Y^\#]_x &= -\frac{d}{dt}|_{t=0} \left(T_{\exp(-tX).x}\psi_t^{X^\#}.Y^\#_{\exp(-tX).x} \right) \\ &= -\frac{d}{dt}|_{t=0} \left(\frac{d}{ds}|_{s=0} \left(\psi_t^{X^\#} \left(\exp(sY)\exp(-tX).x \right) \right) \right) \\ &= -\frac{d}{dt}|_{t=0} \left(\frac{d}{ds}|_{s=0} \left(\underbrace{\exp(tX)\exp(sY)\exp(-tX).x}_{=\Phi_x\left(\exp(tX)\exp(sY)\exp(-tX)\right)} \right) \right) \\ &= -\frac{d}{dt}|_{t=0} \left(T_1\Phi_x \circ \left(\underbrace{Ad\ \exp(tX)}_{=e^{ad(tX)}} \right).Y \right) \\ &= -T_1\Phi_x.\left(\frac{d}{dt}|_{t=0} \left(e^{t(ad\,X)}.Y \right) \right) \quad \text{since } T_1\Phi_x \text{ is a linear map} \\ &= -T_1\Phi_x.(ad\,X.Y) = -T_1\Phi_x.[X,Y] \end{aligned}$$

Hence :

$$[X^\#, Y^\#]_x = -T_1\Phi_x.[X,Y] = -[X,Y]^\#_x$$

Sometimes, a minus sign is introduced in the definition of fundamental vector fields

$$X^\#_x := \frac{d}{dt}|_{t=0} \left(\exp(-tX).x \right)$$

in order to make the map $X \mapsto X^\#$ a homomorphism of Lie algebras. Indeed, if $\varphi : \mathfrak{u} \to \mathfrak{g}$ is an anti-homomorphism of Lie algebras, then the map $\widetilde{\varphi}(X) := \varphi(-X)$ satisfies :

$$\begin{aligned} \widetilde{\varphi}([X,Y]) &= \varphi(-[X,Y]) \\ &= \varphi([Y,X]) \\ &= -[\varphi(Y), \varphi(X)] \\ &= [\varphi(X), \varphi(Y)] \\ &= [-\widetilde{\varphi}(X), -\widetilde{\varphi}(Y)] \\ &= [\widetilde{\varphi}(X), \widetilde{\varphi}(Y)] \end{aligned}$$

We shall nevertheless not use this definition and continue with the one introduced in (4.2). We will follow the conventions in [Spi99] to compute exterior products and exterior differentials of differential forms.

We start by recalling the definition of a Hamiltonian space in the usual sense. Let (M, ω) be a symplectic manifold (that is to say, a manifold M equipped with a closed non-degenerate 2-form ω). Since ω is non-degenerate, one can associate to each function $f : M \to \mathbb{R}$ an unique vector field V_f on M, called the *Hamiltonian vector field associated to* f, satisfying $\iota_{V_f}\omega = df$, where ι designates the interior product between a vector field and a differential form. If now the group U acts on M leaving ω invariant (that is, for all $u \in U$, the corresponding diffeomorphism φ_u of M satisfies $\varphi_u^*\omega = \omega$) then, since ω is

closed, one sees from the Cartan homotopy formula (see for instance [Mor01], p.74) that for all $X \in \mathfrak{u}$, the 1-form $\iota_{X^\sharp}\omega$ is closed :

$$d(\iota_{X^\sharp}\omega) = \mathcal{L}_{X^\sharp}\omega - \iota_{X^\sharp}(dw) = 0 - 0 = 0$$

When this closed 1-form is exact, that is, when every fundamental vector field is a Hamiltonian vector field, the action is called *Hamiltonian*. Actually, one usually asks for the following stronger definition (see for instance [LM87] for a discussion on this).

Definition 4.1.1 (Hamiltonian space in the usual sense). A symplectic action of a Lie group U on a symplectic manifold (M, ω) is said to be *Hamiltonian* if there exists an U*-equivariant* map (with respect to the co-adjoint action of U on $\mathfrak{u}^*$) $\mu : M \to \mathfrak{u}^*$, called the *momentum map*, satisfying for all $X \in \mathfrak{u}$ the relation :

$$\iota_{X^\sharp}\omega \quad = \quad d < \mu, X > \tag{4.4}$$

where $<\,,\,>$ denotes the duality bracket between $\mathfrak{u}$ and $\mathfrak{u}^*$ and $< \mu, X >$ is the function on M defined by $x \in M \mapsto < \mu(x)\,,\, X >$.

When $\mathfrak{u}$ is endowed with an Ad-invariant scalar product $(.\,|\,.)$, we can identify equivariantly $\mathfrak{u}^*$ with $\mathfrak{u}$ and write $(\mu\,|\,X)$ instead of $< \mu\,,\, X >$, where $\mu : M \to \mathfrak{u}$ is equivariant with respect to the adjoint action of U on $\mathfrak{u}$ and $(\mu\,|\,X)$ is the function on M defined by $x \in M \mapsto (\mu(x)\,|\,X)$. Now $(.\,|\,X)$ is a (linear) function on $\mathfrak{u}$ and $(\mu\,|\,X)$ is simply the pull-back of this function by the map μ, so that one has, for all $X \in \mathfrak{u}$:

$$\begin{aligned} \iota_{X^\sharp}\omega &= d(\mu\,|\,X) \\ &= d\big(\mu^*(.\,|\,X)\big) \\ &= \mu^*\big(d(.\,|\,X)\big) \end{aligned}$$

Since $(.\,|\,X)$ is linear, the 1-form $d(.\,|\,X)$ on $\mathfrak{u}$ is :

$$\begin{array}{rccl} d(.\,|\,X): & \mathfrak{u} & \longrightarrow & T^*\mathfrak{u} \\ & Y & \longmapsto & \big(\underbrace{Y+H}_{\in T_Y\mathfrak{u}=Y+\mathfrak{u}} \mapsto (H\,|\,X)\big) \end{array}$$

If one identifies $T_Y\mathfrak{u}$ with $\mathfrak{u}$ by translation from vector Y in the vector space $\mathfrak{u}$, one is led to introduce the $\mathfrak{u}$-valued 1-form θ on $\mathfrak{u}$ defined by :

$$\begin{array}{rccl} \theta: & \mathfrak{u} & \longrightarrow & T^*\mathfrak{u}\otimes\mathfrak{u} \\ & Y & \longmapsto & (\underbrace{Y+H}_{\in T_Y\mathfrak{u}} \mapsto \underbrace{H}_{\in\mathfrak{u}}) \end{array}$$

and the $\mathbb{R}$-valued 1-form $(\theta\,|\,X)$ on $\mathfrak{u}$ defined by :

$$\begin{array}{rccl} (\theta\,|\,X): & \mathfrak{u} & \longrightarrow & T^*\mathfrak{u} \\ & Y & \longmapsto & \big(Y+H \mapsto (H\,|\,X)\big) \end{array}$$

so that $d(.\,|\,X) = (\theta\,|\,X)$, and we can therefore rewrite the momentum condition (4.4) under the form :

$$\iota_{X^\sharp}\omega \quad = \quad \mu^*(\theta\,|\,X) \tag{4.5}$$

As a consequence of this point of view, one understands that to change the space where μ takes its values and still obtain a momentum condition similar to (4.5), one has to change the 1-form θ, which translates vectors in $T_Y\mathfrak{u} = Y + \mathfrak{u}$ to vectors in $\mathfrak{u} = T_0\mathfrak{u}$ (translation of vector Y). In the Lie group U, one has a choice as how to translate tangent vectors from T_uU to T_1U for a given $u \in U$. Namely, one can use

either θ^L or θ^R. If the group U is abelian, then $\theta^L = \theta^R$, and one can define the a priori momentum map condition to be, for all $X \in \mathfrak{u}$:

$$\iota_{X^\sharp}\omega = \mu^*(\theta^L \,|\, X) = \mu^*(\theta^R \,|\, X)$$

Of course, up to this point, we have not verified that this will indeed lead to a theory of group-valued momentum maps, but is a first step. To complete this first step, let us consider the non-abelian case. In this case, if one wants the relation $\omega(X^\sharp, Y^\sharp) = -\omega(Y^\sharp, X^\sharp)$ (which is simply the antisymmetry of ω expressed on fundamental vector fields) to hold for all $X, Y \in \mathfrak{u}$ when one tries to define a momentum map condition, one sees that the only possibility is to set :

$$\iota_{X^\sharp}\omega \quad = \quad \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X) \tag{4.6}$$

Indeed, if one asks $\mu : M \to U$ to be equivariant with respect to the conjugacy action of U on itself (which is the analogue of the (co-) adjoint action in the usual Lie-algebra setting), one has, using equation (4.5) with $\theta = \theta^L$, for all $x \in M$:

$$\begin{aligned}
\omega_x(X^\sharp_x, Y^\sharp_x) &= (\iota_{X^\sharp}\omega)_x(Y^\sharp_x) \\
&= \big(\mu^*(\theta^L \,|\, X)\big)_x(Y^\sharp)_x \\
&= \big(\theta^L_{\mu(x)}(T_x\mu.Y^\sharp_x) \,|\, X\big)
\end{aligned}$$

Since μ is equivariant, $T_x\mu$ sends the vector $Y^\#_x$ to the value at $\mu(x)$ of the fundamental vector field $Y^\dagger$ (see (4.3)) on U. Indeed, for all $x \in M$:

$$\begin{aligned}
T_x\mu.Y^\#_x &= \frac{d}{dt}|_{t=0}\ \big(\mu(\exp(tY).x)\big) \\
&= \frac{d}{dt}|_{t=0}\ \big(\exp(tY)\mu(x)\exp(-tY)\big) \\
&= Y.\mu(x) - \mu(x).Y \\
&= Y^\dagger_{\mu(x)}
\end{aligned}$$

One therefore has, for all $x \in M$ (setting $u = \mu(x) \in U$) :

$$\begin{aligned}
\omega_x(X^\sharp_x, Y^\sharp_x) &= \big(u^{-1}.(Y.u - u.Y) \,|\, X\big) \\
&= (Ad\, u^{-1}.Y - Y \,|\, X) \\
&= (Ad\, u^{-1}.Y \,|\, X) - (Y \,|\, X) \\
&= (Y \,|\, Ad\, u.X) - (Y \,|\, X) \\
&= -\big(Y \,|\, (X.u - u.X).u^{-1}\big) \\
&= -\big(Y \,|\, \theta^R_u(T_u\mu.X^\sharp_u)\big) \\
&= -\big(\mu^*(\theta^R \,|\, Y)\big)_u(X^\sharp_u)
\end{aligned}$$

which would be equal to $-\omega_x(X^\sharp_x, Y^\sharp_x)$ if one had $\theta^R = \theta^L$. Making the same computation with $\iota_{X^\sharp}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)$ gives indeed $\omega(X^\sharp, Y^\sharp) = -\omega(Y^\sharp, X^\sharp)$.

Let us now try and understand the consequences of the momentum condition (4.6). If we still want ω to be invariant by the action of U, one has $\mathcal{L}_{X^\sharp}\omega = 0$ for all $X \in \mathfrak{u}$, so that by the Cartan homotopy formula, one has $d(\iota_{X^\sharp}\omega) + \iota_{X^\sharp}(d\omega) = 0$, which is then equivalent, using (4.6), to :

$$d\big(\frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)\big) + \iota_{X^\sharp}(d\omega) = 0$$

that is :

$$\mu^*(\frac{1}{2}d(\theta^L + \theta^R \,|\, X)\big) + \iota_{X^\sharp}(d\omega) = 0 \tag{4.7}$$

Let us compute $d(\theta^L + \theta^R \,|\, X) = (d(\theta^L + \theta^R) \,|\, X) = (d\theta^L + d\theta^R \,|\, X)$ on fundamental vector fields on U. The structure equations for Lie groups (see for instance [Spi99], ch. 10 p.404) say that :

$$d\theta^L = -\frac{1}{2}[\theta^L \wedge \theta^L] \text{ and } d\theta^R = \frac{1}{2}[\theta^R \wedge \theta^R]$$

where $\frac{1}{2}[\theta^j \wedge \theta^j]$ is the $\mathfrak{u}$-valued 2-form defined, for all $u \in U$ and all $\xi_1, \xi_2 \in T_uU$, by :

$$\frac{1}{2}[\theta^j \wedge \theta^j]_u(\xi_1, \xi_2) := \frac{1}{2}\big([\theta^j_u(\xi_1), \theta^j_u(\xi_2)] - [\theta^j_u(\xi_2), \theta^j_u(\xi_1)]\big) = [\theta^j_u(\xi_1), \theta^j_u(\xi_2)]$$

This, together with the Ad-invariance of $(.\,|\,.)$, shows that for all $X, Y, Z \in \mathfrak{u}$, one has :

$$\begin{aligned}
\frac{1}{2}\big(d(\theta^L + \theta^R \,|\, X)\big)_u(Y^\dagger_u, Z^\dagger_u) &= \frac{1}{2}\Big(- [\theta^L_u(Y^\dagger_u), \theta^L(Z^\dagger_u)] + [\theta^R_u(Y^\dagger_u), \theta^R(Z^\dagger_u)] \,|\, X\Big) \\
&= \frac{1}{2}\Big(\big([Ad\,u^{-1}.Y - Y, Ad\,u^{-1}.Z - Z] \,|\, X\big) \\
&\qquad + \big([Y - Ad\,u.Y, Z - Ad\,u.Z] \,|\, X\big)\Big) \\
&= \frac{1}{2}\Big(\big(- Ad\,u^{-1}.[Y - Ad\,u.Y, Z - Ad\,u.Z] \,|\, X\big) \\
&\qquad + \big([Y - Ad\,u.Y, Z - Ad\,u.Z] \,|\, X\big)\Big) \\
&= \frac{1}{2}\big([Y - Ad\,u.Y, Z - Ad\,u.Z] \,|\, X - Ad\,u.X\big) \\
&= \frac{1}{2}\big(\theta^L_u(X^\dagger_u) \,|\, [\theta^L_u(Y^\dagger_u), \theta^L_u(Z^\dagger_u)]\big) \\
&= \big(\iota_{X^\dagger}(\chi)\big)_u\big(Y^\dagger_u, Z^\dagger_u\big)
\end{aligned}$$

(where χ is the Cartan 3-form, see (4.1)), whence we obtain :

$$\frac{1}{2}d(\theta^L + \theta^R \,|\, X) = \iota_{X^\dagger}(\chi)$$

Since μ is equivariant, this yields :

$$\mu^*\big(\frac{1}{2}d(\theta^L + \theta^R \,|\, X)\big) = \mu^*(\iota_{X^\dagger}(\chi)) = \iota_{X^\sharp}(\mu^*\chi)$$

so that by re-injecting in (4.7), we obtain :

$$\iota_{X^\sharp}(\mu^*\chi + d\omega) = 0, \text{ for all } X \in \mathfrak{u}$$

The easiest way to ensure this is to ask that $d\omega = -\mu^*\chi$. In particular, one will *not* ask the 2-form ω to be closed.

We shall now see that the momentum condition (4.6) actually also implies that ω is degenerate. Indeed, let us compute $\omega_x(X^\sharp_x, v)$ for $x \in M$, $X \in \mathfrak{u}$ and $v \in T_xM$ (setting $u = \mu(x)$ and $\xi = T_x\mu.v$) :

$$\begin{aligned}
\omega_x(X^\sharp_x, v) &= \frac{1}{2}\big(\mu^*(\theta^L + \theta^R \,|\, X)\big)_x.v \\
&= \frac{1}{2}\big(\theta^L_{\mu(x)}(T_x\mu.v) + \theta^R_{\mu(x)}(T_x\mu.v) \,|\, X\big) \\
&= \frac{1}{2}(u^{-1}.\xi + \xi.u^{-1} \,|\, X) \\
&= \frac{1}{2}(\xi.u^{-1} \,|\, Ad\,u.X + X)
\end{aligned}$$

whence we see that if $(Ad\,\mu(x) + Id).X = 0$ then $\omega_x(X_x^\sharp, v) = 0$ for all $v \in T_xM$, that is :

$$\{X_x^\sharp \,:\, X \in \ker(Ad\,\mu(x) + Id)\} \subset \ker \omega_x \text{ for all } x \in M$$

One then asks this necessary degeneracy condition to be minimal in the sense that the above inclusion is required to be an equality. We can now sum up the above discussion and give the definition of a quasi-Hamiltonian space. What is truly remarkable is that although the last two conditions might seem somewhat arbitrary (albeit derived from a quite reasonable momentum condition), we will see in the following section that there are very natural examples of spaces satisfying the axioms appearing in the following definition, and that these spaces are moreover very nice analogues of known Hamiltonian spaces.

Definition 4.1.2 (Quasi-Hamiltonian space). Let (M, ω) be a manifold endowed with a 2-form ω and an action of the Lie group $(U, (.\,|\,.))$ leaving the 2-form ω invariant. Recall that $(.\,|\,.)$ is an Ad-invariant non-degenerate symmetric bilinear form on $\mathfrak{u} = Lie(U)$, that χ is the Cartan 3-form of U, and that θ^L and θ^R are the Maurer-Cartan 1-forms of U. Let $\mu : M \to U$ be a U-equivariant map (for the conjugacy action of U on itself).
Then $(M, \omega, \mu : M \to U)$ is said to be a *quasi-Hamiltonian space* with respect to the action of U if the map $\mu : M \to U$ satisfies the following three conditions :

(i) $d\omega = -\mu^*\chi$

(ii) for all $x \in M$, $\ker \omega_x = \{X_x^\sharp \;:\; X \in \mathfrak{u}\,|\,(Ad\,\mu(x) + Id).X = 0\}$

(iii) for all $X \in \mathfrak{u}$, $\iota_{X^\sharp}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R\,|\,X)$

where $(\theta^L + \theta^R\,|\,X)$ is the real-valued 1-form defined on U for any $X \in \mathfrak{u}$ by $(\theta^L + \theta^R\,|\,X)_u(\xi) := (\theta_u^L(\xi) + \theta_u^R(\xi)\,|\,X)$ (where $u \in U$ and $\xi \in T_uU$).
In analogy with the usual Hamiltonian case, the map μ is called the *momentum map*.

4.2 Fundamental examples of quasi-Hamiltonian spaces

There are two fundamental examples of quasi-Hamiltonian spaces : a conjugacy class of the Lie group U and the manifold $U \times U$ endowed with a particular action of the product group $U \times U$. Only the first example will be really fundamental to us in this work, since the quasi-Hamiltonian space we are mainly interested in is a product of conjugacy classes. Before entering considerations on products of quasi-Hamiltonian spaces, let us describe these two examples explicitly.

4.2.1 Conjugacy classes of a Lie group

In the usual Hamiltonian setting, orbits $\mathcal{O} \subset \mathfrak{u}^*$ of the co-adjoint action of U on $\mathfrak{u}^*$ are basic examples of Hamiltonian spaces, with momentum map the inclusion $i : \mathcal{O} \hookrightarrow \mathfrak{u}^*$. Here, it is natural to study the orbits of the conjugacy action of U on itself : the conjugacy classes of U. Let us consider a conjugacy class $\mathcal{C} \subset U$ and the inclusion map

$$\mu \;: \mathcal{C} \hookrightarrow U$$

and let us compute $\frac{1}{2}\mu^*(\theta^L + \theta^R\,|\,X)$ for all $X \in \mathfrak{u}$. First, let us observe that, for each $u \in \mathcal{C}$, one can describe $\mathcal{C}$ as a homogeneous space under U in the following way :

$$\mathcal{C} = \{gug^{-1} \,:\, g \in U\}$$

Consequently :

$$T_u\mathcal{C} = \{\frac{d}{dt}|_{t=0}\,(g_t u g_t^{-1}) \mid g_0 = 1 \in U\} = \{X.u - u.X \,:\, X = \frac{d}{dt}|_{t=0}\, g_t \in \mathfrak{u} = T_1U\}$$

that is, a tangent vector to $\mathcal{C}$ at $u \in \mathcal{C}$ is always the value of a fundamental vector field for the conjugacy action of U on $\mathcal{C}$:

$$X.u - u.X = \frac{d}{dt}|_{t=0}\big(\exp(tX)u\exp(-tX)\big) = X_u^\dagger \text{ for all } X \in \mathfrak{u} \quad \text{(see (4.3))}$$

This is just a consequence of the fact that $\mathcal{C}$ is a homogeneous space under U. It will also be useful to write :

$$X.u - u.X = u.(Ad\,u^{-1}.X - X) = (X - Ad\,u.X).u$$

In particular :

$$\theta_u^L(X.u - u.X) = Ad\,u^{-1}.X - X \quad \text{and} \quad \theta_u^R(X.u - u.X) = X - Ad\,u.X$$

Finally, another way to say the above is to say that the map

$$\begin{array}{rccl} \Psi_u : & \mathfrak{u} & \longrightarrow & T_u\mathcal{C} \\ & X & \longmapsto & X_u^\dagger = X.u - u.X \end{array}$$

is surjective with kernel $\ker\Psi_u = \{X \in \mathfrak{u} \mid Ad\,u.X = X\}$. We may now compute $\frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)$ using fundamental vector fields :

$$\begin{array}{rcl} \text{for all } Y \in \mathfrak{u},\ \frac{1}{2}\big(\mu^*(\theta^L + \theta^R \,|\, X)\big)_u\big(Y_u^\dagger\big) & = & \frac{1}{2}\big(\theta_u^L(Y_u^\dagger) + \theta^R(Y_u^\dagger) \,|\, X\big) \\ & = & \frac{1}{2}(Ad\,u^{-1}.Y - Y + Y - Ad\,u.Y \,|\, X) \\ & = & \frac{1}{2}\big((Y \,|\, Ad\,u.X) - (Ad\,u.Y \,|\, X)\big) \end{array}$$

This expression defines a 2-form on $T_u\mathcal{C}$. Indeed, if $Y, Y' \in \mathfrak{u}$ satisfy $Y.u - u.Y = Y'.u - u.Y'$ (that is , $(Y - Y') \in \ker\Psi_u$), then $Ad\,u.Y - Y = Ad\,u.Y' - Y'$ and one has :

$$\begin{array}{rcl} (Y \,|\, Ad\,u.X) - (Ad\,u.Y \,|\, X) & = & (Ad\,u.Y - Ad\,u.Y' + Y' \,|\, Ad\,u.X) - (Y + Ad\,u.Y' - Y' \,|\, X) \\ & = & (Y' \,|\, Ad\,u.X) - (Ad\,u.Y' \,|\, X) \end{array}$$

whence we see that the expression

$$\omega_u(X_u^\dagger, Y_u^\dagger) = \frac{1}{2}\big((Ad\,u.X \,|\, Y) - (Ad\,u.Y \,|\, X)\big)$$

gives a well-defined 2-form ω on $\mathcal{C}$. The U-invariance of ω follows from the fact that $\mathcal{C}$ is a homogeneous space. By construction, the 2-form ω satisfies :

$$\iota_{X^\dagger}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X) \quad \text{for all } X \in \mathfrak{u}$$

It actually follows from the construction that such a 2-form is unique. Let us now consider $X_u^\dagger \in \ker\omega_u$. Then, for all $Y \in \mathfrak{u}$, $(Ad\,u.X - Ad\,u^{-1}.X \,|\, Y) = 0$, so that $(Ad\,u.X - Ad\,u^{-1}.X) = 0$ since $(.\,|\,.)$ is non-degenerate. Equivalently, $X \in F_u := \ker((Ad\,u)^2 - Id)$. But $(Ad\,u)|_{F_u}$ is diagonalizable (since the polynom $P = (X-1)(X+1) = X^2 - 1$ satisfies $P((Ad\,u)|_{F_u}) = 0$ on $F_u = \ker((Ad\,u)^2 - Id)$), and therefore we have :

$$\ker\big((Ad\,u)^2 - Id\big) = \ker(Ad\,u - Id) \oplus \ker(Ad\,u + Id)$$

But for $X_2 \in \ker(Ad\,u - Id)$, one has $(X_2)_u^\dagger = 0$, so that $X_u^\dagger = (X_1)_u^\dagger$, with $X_1 \in \ker(Ad\,u + Id)$. Hence :

$$\ker\omega_u = \{X_u^\dagger \ :\ X \in \ker(Ad\,u + Id)\}$$

Finally, let us compute dw. One has, for all $X, Y, Z \in \mathfrak{u}$:

$$\begin{aligned}\big((d\omega)(X^\dagger, Y^\dagger, Z^\dagger)\big) &= \mathcal{L}_{X^\dagger}\big(\omega(Y^\dagger, Z^\dagger)\big) - \mathcal{L}_{Y^\dagger}\big(\omega(X^\dagger, Z^\dagger)\big) + \mathcal{L}_{Z^\dagger}\big(\omega(X^\dagger, Y^\dagger)\big) \\ &\quad -\omega\big([X^\dagger, Y^\dagger], Z^\dagger\big) + \omega\big([X^\dagger, Z^\dagger], Y^\dagger\big) - \omega\big([Y^\dagger, Z^\dagger], X^\dagger\big)\end{aligned}$$

where, for all $u \in U$:

$$\begin{aligned}\mathcal{L}_{X^\dagger}\big(\omega(Y^\dagger, Z^\dagger)\big)(u) &= \frac{d}{dt}|_{t=0}\ \big(\omega(Y^\dagger, Z^\dagger)\big)\big(\exp(tX)u\exp(-tX)\big) \\ &= \frac{1}{2}\Big(\big([X, Ad\,u.Y]\,|\,Z\big) - \big(Ad\,u.[X,Y]\,|\,Z\big) \\ &\qquad -\big([X, Ad\,u.Z]\,|\,Y\big) + \big(Ad\,u.[X,Z]\,|\,Y\big)\Big)\end{aligned}$$

and for all $u \in U$:

$$\begin{aligned}\Big(\omega\big([X^\dagger, Y^\dagger], Z^\dagger\big)\Big)(u) &= \omega_u\big(-[X,Y]^\dagger_u, Z^\dagger_u\big) \\ &= -\frac{1}{2}\Big(\big(Ad\,u.[X,Y]\,|\,Z\big) - \big(Ad\,u.Z\,|\,[X,Y]\big)\Big)\end{aligned}$$

so that, all computations made, one obtains :

$$\begin{aligned}(d\omega)_u(X^\dagger_u, Y^\dagger_u, Z^\dagger_u) &= \frac{1}{2}\Big(\big([X, Ad\,u.Y]\,|\,Z\big) - \big([Y, Ad\,u.X]\,|\,Z\big) \\ &\quad +\big(Ad\,u.[Y,X]\,|\,Z\big) + \big([Y, Ad\,u.Z]\,|\,X\big) \\ &\quad -\big(Ad\,u.[Z,X]\,|\,Y\big) + \big(Ad\,u.[Z,Y]\,|\,X\big)\Big)\end{aligned}$$

On the other hand, μ being the inclusion map $\mathcal{C} \hookrightarrow U$:

$$\begin{aligned}(\mu^*\chi)_u(X^\dagger_u, Y^\dagger_u, Z^\dagger_u) &= \frac{1}{2}\big(u^{-1}.X^\dagger_u\,|\,[u^{-1}.Y^\dagger_u, u^{-1}.Z^\dagger_u]\big) \\ &= \frac{1}{2}\big(Ad\,u^{-1}.X - X\,|\,[Ad\,u^{-1}.Y - Y, Ad\,u^{-1}.Z - Z]\big)\end{aligned}$$

Expanding this expression and comparing it with the above using the Ad-invariance of $(.\,|\,.)$ shows that :

$$d\omega = -\mu^*\chi$$

Other proofs of this fact can be found for instance in [AMM98] and [GHJW97]. We can then sum up the above in the following proposition :

Proposition 4.2.1 ([AMM98]). *Let $\mathcal{C} \subset U$ be a conjugacy class of a Lie group $(U, (.\,|\,.))$. The tangent space to $\mathcal{C}$ at $u \in \mathcal{C}$ is $T_u\mathcal{C} = \{X.u - u.X \ :\ X \in \mathfrak{u}\}$. The 2-form ω on $\mathcal{C}$ given at $u \in \mathcal{C}$ by*

$$\omega_u(X.u - u.X, Y.u - u.Y) = \frac{1}{2}\big((Ad\,u.X\,|\,Y) - (Ad\,u.Y\,|\,X)\big)$$

is well-defined and makes $\mathcal{C}$ a quasi-Hamiltonian space for the conjugacy action with momentum map the inclusion $\mu : \mathcal{C} \hookrightarrow U$. Such a 2-form is actually unique.

4.2.2 The double of a Lie group

The second example of quasi-Hamiltonian space associated to a given Lie group $(U, (.\,|\,.))$ is the manifold $D(U) := U \times U$ *equipped with the action of the product group* $U \times U$ defined for all $(a, b) \in D(U)$ and all $(u_1, u_2) \in U \times U$ by :

$$(u_1, u_2).(a, b) := (u_1 a u_2^{-1}, u_2 b u_1^{-1})$$

Let us explain how this is obtained. This example is actually a first step in the construction of another example of a quasi-Hamiltonian U-space, which we will denote later by $\mathfrak{D}(U)$ and which is obtained from $D(U)$ by *internal fusion* (see section 4.4 and particularly proposition 4.4.4, and observe that as a manifold, $\mathfrak{D}(U)$ is just $U \times U$ as well). This space $\mathfrak{D}(U)$ comes into play when one wants to obtain symplectic structures on spaces of representations of fundamental groups of surfaces of genus $g \geq 1$, which shows its importance. To this end, the manifold $\mathfrak{D}(U) = U \times U$ is equipped with the diagonal U-action and the quasi-Hamiltonian structure is defined with respect to the momentum map :

$$\begin{array}{rrcl} \mu : & \mathfrak{D}(U) = U \times U & \longrightarrow & U \\ & (a,b) & \longmapsto & aba^{-1}b^{-1} \end{array}$$

The relevance of this map when it comes to describing symplectic structures on moduli spaces associated to surface groups will become clearer later on (see section 4.6). We shall also see that the momentum map μ above is obtained from the momentum map $\mu_D : D(U) \to U \times U$, defining the quasi-Hamiltonian structure on the $U \times U$-space $D(U)$, by multiplication of its two components. This is a consequence of the notion of fusion that we will study later (see section 4.4) and this explains that one is led to define :

$$\begin{array}{rrcl} \mu_D : & D(U) = U \times U & \longrightarrow & U \times U \\ & (a,b) & \longmapsto & (ab, a^{-1}b^{-1}) \end{array}$$

Then in order to obtain an equivariant map μ_D (where the target space $U \times U$ acts on itself by conjugation), one has to set :

$$(u_1, u_2).(a.b) = (u_1 a u_2^{-1}, u_2 b u_1^{-1})$$

So that indeed :

$$\begin{array}{rcl} \mu_D\big((u_1,u_2).(a,b)\big) & = & (u_1 a u_2^{-1} u_2 b u_1^{-1}, u_2 a^{-1} u_1^{-1} u_1 b^{-1} u_2^{-1}) \\ & = & (u_1 a b u_1^{-1}, u_2 a^{-1} b^{-1} u_2^{-1}) \\ & = & (u_1,u_2)\mu_D(a,b)(u_1,u_2)^{-1} \end{array}$$

We now want to determine an expression for a 2-form ω^D on $D(U)$ satisfying, for all $(X,Y) \in \mathfrak{u} \times \mathfrak{u} = Lie(U \times U)$:

$$\iota_{(X,Y)^\#}\omega^D = \frac{1}{2}\mu_D^*\big(\theta^L_{U\times U} + \theta^R_{U\times U} \,|\, (X,Y)\big)_{\mathfrak{u}\times\mathfrak{u}} \tag{4.8}$$

where the Lie algebra $\mathfrak{u} \times \mathfrak{u} = Lie(U \times U)$ is equipped with the Ad-invariant scalar product

$$\big((X,Y) \,|\, (X',Y')\big)_{\mathfrak{u}\times\mathfrak{u}} := (X \,|\, X') + (Y \,|\, Y')$$

the scalar product $(.\,|\,.)$ being the given invariant product on $\mathfrak{u}$. Observe that, for all $(X,Y) \in \mathfrak{u} \times \mathfrak{u}$, one has :

$$(X,Y)^\#_{(a,b)} = (X.a - a.Y, Y.b - b.X) \in T_{(a.b)}D(U) = T_aU \times T_bU$$

And that for all $(u_1,u_2) \in U \times U$ and all $(\xi_1,\xi_2) \in T_{u_1}U \times T_{u_2}U$:

$$(\theta^L_{U\times U})_{(u_1,u_2)} = (u_1^{-1}.\xi_1, u_2^{-1}.\xi_2) \quad \text{and} \quad (\theta^R_{U\times U})_{(u_1,u_2)}(\xi_1,\xi_2) = (\xi_1.u_1^{-1}, \xi_2.u_2^{-1})$$

Further, since $\mu_D(a,b) = (ab, a^{-1}b^{-1})$, one has, for all $(v,w) \in T_{(a,b)}D(U) = T_aU \times T_bU$:

$$T_{(a,b)}\mu.(v,w) = \big(v.b + a.w, -(a^{-1}.v).a^{-1}b^{-1} - a^{-1}b^{-1}.(w.b^{-1})\big)$$

Therefore, by computing

$$\frac{1}{2}\Big(\mu_D^*\big(\theta^L_{U\times U} + \theta^R_{U\times U} \,|\, (X,Y)\big)_{\mathfrak{u}\times\mathfrak{u}}\Big)_{(a,b)}.\big(v,w\big)$$

we can rewrite equation (4.8) and obtain the expression of ω^D on tangent vectors which are values of fundamental vector fields. Explicitly, we have :

$$\begin{aligned}
\big(\iota_{(X,Y)\#}\omega^D\big)_{(a,b)}.(v,w) &= \frac{1}{2}\Big(\mu_D^*\big(\theta^L_{U\times U}+\theta^R_{U\times U}\,|\,(X,Y)\big)_{\mathfrak{u}\times\mathfrak{u}}\Big)_{(a,b)}.\big(v,w\big)\\
&= \frac{1}{2}\Big(\big(\theta^L\big)_{(ab,a^{-1}b^{-1})}\big(v.b+a.w,-(a^{-1}.v).a^{-1}b^{-1}-a^{-1}b^{-1}.(w.b^{-1})\big)\\
&\quad+\big(\theta^R\big)_{(ab,a^{-1}b^{-1})}\big(v.b+a.w,-(a^{-1}.v).a^{-1}b^{-1}-a^{-1}b^{-1}.(w.b^{-1})\big)\\
&\quad|\,\big(X,Y\big)\Big)_{\mathfrak{u}\times\mathfrak{u}}\\
&= \frac{1}{2}\Big(\big(Ad\,b^{-1}.(a^{-1}.v)+b^{-1}.w,-Ad\,b.(v.a^{-1})-w.b^{-1}\big)\\
&\quad+\big(v.a^{-1}+Ad\,a.(w.b^{-1}),-a^{-1}.v-Ad\,a^{-1}.(b^{-1}.w)\big)\,|\,\big(X,Y\big)\Big)_{\mathfrak{u}\times\mathfrak{u}}\\
&= \frac{1}{2}\big((a^{-1}.v\,|\,Ad\,b.X)+(b^{-1}.w\,|\,X)-(v.a^{-1}\,|\,Ad\,b^{-1}.Y)-(w.b^{-1}\,|\,Y)\\
&\quad+(v.a^{-1}\,|\,X)+(w.b^{-1}\,|\,Ad\,a^{-1}.X)-(a^{-1}.v\,|\,Y)-(b^{-1}.w\,|\,Ad\,Y)\big)\\
&= \frac{1}{2}\big((a^{-1}.v\,|\,Ad\,b.X-Y)+(b^{-1}.w\,|\,X-Ad\,a.Y)\\
&\quad-(v.a^{-1}\,|\,Ad\,b^{-1}.Y-X)-(w.b^{-1}\,|\,Y-Ad\,a^{-1}.X)\big)\\
&= \frac{1}{2}\Big(\big(a^{-1}.(X.a-a.Y)\,|\,w.b^{-1}\big)-\big(a^{-1}.v\,|\,(Y.b-b.X).b^{-1}\big)\Big)\\
&\quad+\frac{1}{2}\Big(\big((X.a-a.Y).a^{-1}\,|\,b^{-1}.w\big)-\big(v.a^{-1}\,|\,b^{-1}.(Y.b-b.X)\big)\Big)
\end{aligned}$$

Therefrom, one can guess the expression of the 2-form ω^D on tangent vectors which are not necessarily values of fundamental vector fields and define, for all $(a,b)\in D(U)=U\times U$ and all $(v_i,w_i)\in T_{(a,b)}D(U)=T_aU\times T_bU$:

$$\begin{aligned}
\omega^D_{(a,b)}\big((v_1,w_1),(v_2,w_2)\big) &:= \frac{1}{2}\big((a^{-1}.v_1\,|\,w_2.b^{-1})-(a^{-1}.v_2\,|\,w_1.b^{-1})\big)\\
&\quad+\frac{1}{2}\big((v_1.a^{-1}\,|\,b^{-1}.w_2)-(v_2.a^{-1}\,|\,b^{-1}.w_1)\big)
\end{aligned}$$

or more concisely :

$$\omega^D := \frac{1}{2}(\alpha^*\theta^L\wedge\beta^*\theta^R)+\frac{1}{2}(\alpha^*\theta^R\wedge\beta^*\theta^L)$$

where :

$$\begin{array}{rccc}\alpha: & D(U) & \longrightarrow & U\\ & (a,b) & \longmapsto & a\end{array}\qquad\text{and}\qquad\begin{array}{rccc}\beta: & D(U) & \longrightarrow & U\\ & (a,b) & \longmapsto & b\end{array}$$

We then refer to [AMM98] for the proof of the following result :

Proposition 4.2.2 ([AMM98]). *The manifold $D(U)=U\times U$, equipped with the $U\times U$-action defined by*

$$(u_1,u_2).(a,b)=(u_1au_2^{-1},u_2bu_1^{-1})$$

the $U\times U$ invariant 2-form

$$\omega^D=\frac{1}{2}(\alpha^*\theta^L\wedge\beta^*\theta^R)+\frac{1}{2}(\alpha^*\theta^R\wedge\beta^*\theta^L)$$

and the equivariant momentum map

$$\begin{array}{rccc}\mu_D: & D(U)=U\times U & \longrightarrow & U\times U\\ & (a,b) & \longmapsto & (ab,a^{-1}b^{-1})\end{array}$$

is a quasi-Hamiltonian space called the double of U.

The term *double* alludes to the theory of quantum groups (see [AMM98] for an explanation).

4.3 Properties of quasi-Hamiltonian spaces

We now give a few properties of quasi-Hamiltonian spaces that we shall need in the following, especially when considering the reduction theory of quasi-Hamiltonian spaces (see section 4.5) and convexity properties of Lie-group valued momentum maps (see chapter 8). The results below are quasi-Hamiltonian analogues of classical lemmas entering the reduction theory and momentum convexity properties for usual Hamiltonian spaces (see for instance [GS84c] or [MS98]). For other results, specific to quasi-Hamiltonian spaces, we refer to [AMM98].

Proposition 4.3.1 ([AMM98]). *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space and let $x \in M$. Then :*

(i) The map

$$\begin{array}{rccc} \Lambda_x : & \ker(Ad\,\mu(x) + Id) & \longrightarrow & \ker \omega_x \\ & X & \longmapsto & X_x^{\#} = \frac{d}{dt}|_{t=0} \left(\exp(tX).x\right) \end{array}$$

is an isomorphism.

(ii) $\ker T_x\mu \cap \ker \omega_x = \{0\}$

(iii) The left translation

$$\begin{array}{ccc} U & \longrightarrow & U \\ u & \longmapsto & \left(\mu(x)\right)^{-1} u \end{array}$$

induces an isomorphism

$$\mathrm{Im}\ T_x\mu \simeq \mathfrak{u}_x^{\perp}$$

where $\mathfrak{u}_x = \{X \in \mathfrak{u} \mid X_x^{\#} = 0\}$ is the Lie algebra of the stabilizer U_x of x and $\mathfrak{u}_x^{\perp}$ denotes its orthogonal with respect to $(.\,|\,.)$. Equivalently, $\mathrm{Im}\ (\mu^*\theta^L)_x = \mathfrak{u}_x^{\perp}$ *(and likewise,* $\mathrm{Im}\ (\mu^*\theta^R)_x = \mathfrak{u}_x^{\perp}$*).*

(iv) $(\ker T_x\mu)^{\perp_\omega} = \{X_x^{\#} : X \in \mathfrak{u}\}$*, where* $(\ker T_x\mu)^{\perp_\omega} \subset T_xM$ *denotes the subspace of T_xM orthogonal to* $\ker T_x\mu$ *with respect to ω_x.*

(v) $\ker T_x\mu \subset (T_x(U.x))^{\perp_\omega}$ *where $U.x$ denotes the orbit of x in M under U.*

We will need the following lemma, coming from the general theory of bilinear forms on vector spaces, in the course of the proof.

Lemma 4.3.2. *For every subspace $F \subset E := T_xM$, one has* $\dim E = \dim F + \dim F^{\perp_{\omega_x}} - \dim(F \cap \ker \omega_x)$.

Proof of lemma 4.3.2. The bilinear form $\omega_x : E \times E \to \mathbb{R}$ induces a non-degenerate bilinear form $\overline{\omega_x} : E/\ker\omega_x \times E/\ker\omega_x \to \mathbb{R}$ and the map $F^{\perp_{\omega_x}} \hookrightarrow E \to E/\ker\omega_x$ sends $F^{\perp_{\omega_x}}$ onto

$$\left(\left(F + \ker\omega_x\right)/\ker\omega_x\right)^{\perp_{\overline{\omega_x}}}$$

and its kernel is $F^{\perp_{\omega_x}} \cap \ker\omega_x = \ker\omega_x$ (since $\ker\omega_x = E^{\perp_{\omega_x}} \subset F^{\perp_{\omega_x}}$). Hence :

$$F^{\perp_{\omega_x}}/\ker\omega_x \simeq \left(\left(F + \ker\omega_x\right)/\ker\omega_x\right)^{\perp_{\overline{\omega_x}}}$$

Since $\overline{\omega_x}$ is non-degenerate, this yields :

$$\begin{aligned} \dim F^{\perp_{\omega_x}} - \dim \ker\omega_x &= \dim E/\ker\omega_x - \dim(F + \ker\omega_x)/\ker\omega_x \\ &= \dim E - \dim \ker\omega_x - \dim(F + \ker\omega_x) + \dim \ker\omega_x \end{aligned}$$

so that

$$\dim\, F^{\perp_{\omega_x}} = \dim\, E - \dim\, F - \dim\, \ker \omega_x + \dim\,(F \cap \ker \omega_x) + \dim\, \ker \omega_x$$

□

Proof of proposition 4.3.1. (i) It follows from the definition of a quasi-Hamiltonian space that the map Λ_x is surjective. Now take $X \in \ker(Ad\,\mu(x) + Id)$ such that $X_x^{\#} = 0$. Then $T_x\mu.X_x^{\#} = 0$ and, since μ is equivariant, this yields $X^{\dagger}_{\mu(x)} = T_x\mu.X_x^{\#} = 0$, that is $X.\mu(x) - \mu(x).X = 0$, or equivalently $Ad\,\mu(x).X = X$. But $Ad\,\mu(x).X = -X$ by assumption, so that $X = 0$ and Λ_x is injective.

(ii) Consider $v \in \ker T_x\mu \cap \ker \omega_x$. It follows from the definition of a quasi-Hamiltonian space that we can write $v = X_x^{\#}$ for some $X \in \ker(Ad\,\mu(x) + Id)$. Then $T_x\mu.v = 0$ implies $X^{\dagger}_{\mu(x)} = 0$ as above. Hence, we have again : $X \in \ker(Ad\,\mu(x) - Id) \cap \ker(Ad\,\mu(x) + Id) = \{0\}$, so that $v = 0$.

(iii) Here we follow [Rac03]. Let us first show that Im $(\mu^*\theta^L)_x \subset \mathfrak{u}_x^{\perp}$. Take $v \in T_xM$ and $X \in \mathfrak{u}_x$. Then $X_x^{\#} = 0$ and as above one has $X_x^{\dagger} = 0$, that is, $Ad\,\mu(x).X = X$. Therefore :

$$\begin{aligned}\big((\mu^*\theta^L)_x.v \,|\, X\big) &= \Big(Ad\,\mu(x).\big((\mu^*\theta^L)_x.v\big) \,|\, Ad\,\mu(x).X\Big)\\ &= \big((\mu^*\theta^R)_x.v \,|\, X\big)\end{aligned}$$

hence :

$$\begin{aligned}\big((\mu^*\theta^L)_x.v \,|\, X\big) &= \frac{1}{2}\big((\mu^*\theta^L + \mu^*\theta^R)_x.v) \,|\, X\big)\\ &= \omega_x(X_x^{\#}, v)\\ &= 0 \quad \text{since } X_x^{\#} = 0\end{aligned}$$

so that $(\mu^*\theta^L)_x.v \in \mathfrak{u}_x^{\perp}$.
Let us now consider $X \in \mathfrak{u}_x^{\perp}$ and show that there exists a $v \in T_xM$ such that $X = (\mu^*\theta^L)_x.v$. If this is true then for all $Y \in \mathfrak{u}$ one has :

$$\begin{aligned}\omega_x(Y_x^{\#}, v) &= \frac{1}{2}\big((\mu^*\theta^L + \mu^*\theta^R)_x.v \,|\, Y\big)\\ &= \frac{1}{2}\Big(\big(Id + Ad\,\mu(x)\big).\big((\mu^*\theta^L)_x.v\big) \,|\, Y\Big)\\ &= \frac{1}{2}\Big(\big(Id + Ad\,\mu(x)\big).X \,|\, Y\Big)\end{aligned}$$

So in order to show the existence of v, we will show that the map

$$\alpha_x : Y_x^{\#} \longmapsto \frac{1}{2}\Big(\big(Id + Ad\,\mu(x)\big).X \,|\, Y\Big)$$

is a well-defined linear map on $V_x := \{Y_x^{\#} \,:\, Y \in \mathfrak{u}\} \subset T_xM$ and that if one extends it to the whole of T_xM (for instance by choosing a complement W_x to V_x in T_xM and deciding that $\alpha_x|_{W_x} = 0$) then this extended α_x can be represented by some $v \in T_xM$. To this end, it is enough to show that α_x is well-defined on V_x and that $\ker w_x$ (which is included in V_x by (i)) satisfies $\alpha_x|_{\ker \omega_x} = 0$. First, if $Y_x^{\#} = 0$ (that is, $Y \in \mathfrak{u}_x$), then as usual $Ad\,\mu(x).Y = Y$, therefore :

$$\begin{aligned}\frac{1}{2}\Big(\big(Id + Ad\,\mu(x)\big).X \,|\, Y\Big) &= \frac{1}{2}\Big(\big(Ad\,\mu(x).X \,|\, Y\big) + \big(X \,|\, Y\big)\Big)\\ &= \frac{1}{2}\Big(\big(X \,|\, \underbrace{Ad\,\big(\mu(x)\big)^{-1}.Y}_{=Y}\big) + \big(X \,|\, Y\big)\Big)\\ &= (X \,|\, Y)\\ &= 0 \quad \text{since } Y \in \mathfrak{u}_x \text{ and } X \in \mathfrak{u}_x^{\perp}\end{aligned}$$

Thus, α_x is well-defined on V_x and can be extended linearly to T_xM. Second, if $w \in \ker \omega_x$, then by (i), $w = Z_x^{\#}$ with $Ad\,\mu(x).Z = -Z$. Hence :

$$\begin{aligned}\alpha_x(w) &= \frac{1}{2}\Big(\big(Ad\,\mu(x) + Id\big).X \mid Z\Big) \\ &= \frac{1}{2}\Big(X \mid \underbrace{\big(Ad\,\big(\mu(x)\big)^{-1} + Id\big).Z}_{=0}\Big) \\ &= 0\end{aligned}$$

Therefore, there exists some $v \in T_xM$ such that $\alpha_x = \omega_x(.,v)$, that is, for all $Y \in \mathfrak{u}$, $\alpha_x(Y_x^{\#}) = \omega_x(Y_x^{\#}, v)$. Hence, for all $Y \in \mathfrak{u}$:

$$\frac{1}{2}\Big(\big(Ad\,\mu(x) + Id\big).X \mid Y\Big) = \frac{1}{2}\Big(\big(Ad\,\mu(x) + Id\big)\big((\mu^*\theta^L)_x.v\big) \mid Y\Big)$$

Since $(.\mid.)$ is non-degenerate, this yields :

$$\big(Ad\,\mu(x) + Id\big).X = \big(Ad\,\mu(x) + Id\big)\big((\mu^*\theta^L)_x.v\big)$$

Therefore :

$$X' := X - (\mu^*\theta^L)_x.v \in \ker\big(Ad\,\mu(x) + Id\big)$$

But $\ker(Ad\,\mu(x) + Id) \subset \mathrm{Im}\ (\mu^*\theta^L)_x$. Indeed, if $Y \in \mathfrak{u}$ satisfies $Ad\,\mu(x).Y = -Y$ then :

$$\begin{aligned}(\mu^*\theta^L)_x.(-\frac{1}{2}Y)_x^{\#} &= -\frac{1}{2}\theta^L_{\mu(x)}.(\underbrace{T_x\mu.Y_x^{\#}}_{=Y_\mu^\dagger(x)}) \\ &= -\frac{1}{2}\Big(\underbrace{Ad\,\big(\mu(x)\big)^{-1}.Y}_{=-Y} - Y\Big) \\ &= Y\end{aligned}$$

so that $Y \in \mathrm{Im}\ (\mu^*\theta^L)_x$. and therefore $X = X' + (\mu^*\theta^L)_x.v \in \mathrm{Im}\ (\mu^*\theta^L)_x$, which proves that $\mathfrak{u}_x^\perp \subset \mathrm{Im}\ (\mu^*\theta^L)_x$. Thus we have proved that $\mathrm{Im}\ (\mu^*\theta^L)_x = \mathfrak{u}_x^\perp$. The proof that $\mathrm{Im}\ (\mu^*\theta^R)_x = \mathfrak{u}_x^\perp$ is obtained similarly by writing $(\mu^*\theta^L + \mu^*\theta^R)_x.v = (Ad\,(\mu(x))^{-1} + Id).((\mu^*\theta^R)_x.v)$ in the above proof and proceeding accordingly.

(iv) Take $X \in \mathfrak{u}$ and $v \in \ker T_x\mu$. Then :

$$\omega_x(X_x^{\#}, v) = \frac{1}{2}\big((\theta^L + \theta^R)_{\mu(x)}.(\underbrace{T_x\mu.v}_{=0}) \mid X\big) = 0$$

that is :

$$\{X_x^{\#} \,:\, X \in \mathfrak{u}\} \subset (\ker T_x\mu)^{\perp_{\omega_x}} \tag{4.9}$$

Further, $\{X_x^{\#} \,:\, X \in \mathfrak{u}\} \simeq \mathfrak{u}/\mathfrak{u}_x$, therefore its dimension is :

$$\dim\ \mathfrak{u} - \dim\ \mathfrak{u}_x = \dim\ \mathfrak{u}_x^\perp = \dim\ \mathrm{Im}\ T_x\mu$$

since $\mathrm{Im}\ T_x\mu \simeq \mathfrak{u}_x^\perp$ by (iii). Therefore :

$$\dim\{X_x^{\#} \,:\, X \in \mathfrak{u}\} = \dim\ T_xM - \dim\ \ker T_x\mu$$

But $\ker T_x\mu \cap \ker \omega_x = \{0\}$ by (ii), so that by lemma 4.3.2 :

$$\dim\,(\ker T_x\mu)^{\perp_{\omega_x}} = \dim\ T_xM - \dim\ \ker T_x\mu$$

and the inclusion (4.9) above is therefore an equality by dimension count.

(v) Observe first that :

$$T_x(U.x) = \{\frac{d}{dt}|_{t=0}\ (u_t.x)\ :\ u_t \in U, u_0 = 1\} = \{X^{\#}_x\ :\ X \in \mathfrak{u}\}$$

so that (iv) means that $(\ker T_x\mu)^{\perp_{\omega_x}} = T_x(U.x)$. Therefore, $((\ker T_x\mu)^{\perp_{\omega_x}})^{\perp_{\omega_x}} = (T_x(U.x))^{\perp_{\omega_x}}$. But $\ker T_x\mu \subset ((\ker T_x\mu)^{\perp_{\omega_x}})^{\perp_{\omega_x}}$, hence :

$$\ker T_x\mu \subset \big(T_x(U.x)\big)^{\perp_{\omega_x}}$$

In fact, one can show, using lemma 4.3.2, that $((\ker T_x\mu)^{\perp_{\omega_x}})^{\perp_{\omega_x}} = \ker T_x\mu + \ker \omega_x$, and this last sum is actually a direct sum since $\ker T_x\mu \cap \ker \omega_x = \{0\}$ by (ii).

□

4.4 Products of quasi-Hamiltonian spaces

For now, given a Lie group $(U, (.\,|\,.))$, we only have two examples of associated quasi-Hamiltonian spaces at our disposal : a conjugacy class $\mathcal{C} \subset U$ and the $U \times U$-space $D(U) = U \times U$ (see subsection 4.2). In particular, only one of these two examples is a U-space. We now wish to construct new examples of quasi-Hamiltonian U-spaces. Drawing from the usual Hamiltonian setting, one can for instance look for a quasi-Hamiltonian structure on a product $M_1 \times M_2$ of two quasi-Hamiltonian U-spaces $(M_1, \omega_1, \mu_1 : M_1 \to U)$ and $(M_2, \omega_2, \mu_2 : M_2 \to U)$. In the usual Hamiltonian framework, the product manifold is endowed with the diagonal action of U, the direct symplectic form $\omega_1 \oplus \omega_2$ and the direct sum momentum map :

$$\begin{array}{rccl} \mu_1 \oplus \mu_2 : & M_1 \times M_2 & \longrightarrow & \mathfrak{u}^* \\ & (x_1, x_2) & \longmapsto & \mu_1(x_1) + \mu_2(x_2) \end{array}$$

Of particular interest in the Hamiltonian setting is the case of a product of two co-adjoint orbits (see for instance [Knu00] for the relation of this with the Weyl-Horn problem). Here, it is therefore natural to consider the diagonal action of U on a product of conjugacy classes. Further, since we are in a Lie group setting, it seems reasonable to expect the map $\mu_1 \oplus \mu_2$ to be replaced by the map :

$$\begin{array}{rccl} \mu_1 \cdot \mu_2 : & M_1 \times M_2 & \longrightarrow & U \\ & (x_1, x_2) & \longmapsto & \mu_1(x_1)\mu_2(x_2) \end{array}$$

Observe that this map is U-equivariant. The question then is : which 2-form is appropriate on $M_1 \times M_2$ to obtain a quasi-Hamiltonian structure on this product manifold when endowed with the diagonal action of U and the U-equivariant map $\mu_1 \cdot \mu_2$? We will see shortly that it is *not* the direct sum 2-form $\omega_1 \oplus \omega_2$ but rather this 2-form plus a residual term ω_{res}. To obtain an expression for the correct 2-form ω on $M_1 \times M_2$, one can try and guess it from the momentum map condition :

$$\iota_{X^{\#}}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)$$

If one computes $\iota_{X^{\#}}(\omega_1 \oplus \omega_2) - \frac{1}{2}(\mu_1 \cdot \mu_2)^*(\theta^L + \theta^R \,|\, X)$, one obtains a non-zero term which turns out to be of the form $\iota_{X^{\#}}\omega_{res}$, where ω_{res} is a 2-form on $M_1 \times M_2$. As a matter of fact, to obtain an expression for ω_{res}, it is even enough to compute this in the simple explicit case where $M_1 \times M_2 = \mathcal{C}_1 \times \mathcal{C}_2$ is a product of two conjugacy classes. To do so, take $X \in \mathfrak{u}$, $(u_1, u_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ and $(\xi_1, \xi_2) \in T_{u_1}\mathcal{C}_1 \times T_{u_2}\mathcal{C}_2$. Then there exist $Y_1, Y_2 \in \mathfrak{u}$ such that $\xi_1 = Y_1.u_1 - u_1.Y_1 = Y^{\dagger}_{u_1}$ and $\xi_2 = Y_2.u_2 - u_2.Y_2 = Y^{\dagger}_{u_2}$ (see proposition 4.2.1). We denote by ω_i the 2-form defining the quasi-Hamiltonian structure on $\mathcal{C}_i$ (see proposition 4.2.1). Since U acts diagonally on $\mathcal{C}_1 \times \mathcal{C}_2$, one has :

$$X^{\#}_{(u_1,u_2)} = (X^{\dagger}_{u_1}, X^{\dagger}_{u_2})$$

Consequently, one has, on the one hand :

$$\begin{aligned}
\big(\iota_{X^\#}(\omega_1\oplus\omega_2)\big)_{(u_1,u_2)}(\xi_1,\xi_2) &= (\omega_1\oplus\omega_2)_{(u.1,u_2)}\Big(\big(X^\dagger_{u_1},X^\dagger_{u_2}\big),\big((Y_1)^\dagger_{u_1},(Y_2)^\dagger_{u_2}\big)\Big)\\
&= (\omega_1)_{u_1}\big(X^\dagger_{u_1},(Y_1)^\dagger_{u_1}\big)+(\omega_2)_{u_2}\big(X^\dagger_{u_2},(Y_2)^\dagger_{u_2}\big)\\
&= \frac{1}{2}\big((Ad\,u_1.X\,|\,Y_1)-(Ad\,u_1.Y_1\,|\,X)\big)\\
&\quad+\frac{1}{2}\big((Ad\,u_2.X\,|\,Y_2)-(Ad\,u_2.Y_2\,|\,X)\big)
\end{aligned}$$

and on the other hand :

$$\begin{aligned}
T_{(u_1,u_2)}(\mu_1\cdot\mu_2).(\xi_1,\xi_2) &= (T_{u_1}\mu_1.\xi_1).\mu_2(u_2)+\mu_1(u_1).(T_{u_2}\mu_2.\xi_2)\\
&= \xi_1.u_2+u_1.\xi_2 \quad \text{since } \mu_i:\mathcal{C}_i\hookrightarrow U\\
&= (Y_1)^\dagger_{u_1}.u_2+u_1.(Y_2)^\dagger_{u_2}
\end{aligned}$$

and therefore :

$$\begin{aligned}
\Big(\frac{1}{2}(\mu_1\cdot\mu_2)^*(\theta^L+\theta^R)\Big)_{(u_1,u_2)}.(\xi_1,\xi_2) &= \frac{1}{2}(\theta^L+\theta^R)_{u_1u_2}\big((Y_1)^\dagger_{u_1}.u_2+u_1.(Y_2)^\dagger_{u_2}\big)\\
&= \frac{1}{2}\Big(\big(u_1u_2)^{-1}.(Y_1)^\dagger_{u_1}.u_2+u_2^{-1}.(Y_2)^\dagger_{u_2}\\
&\qquad+(Y_1)^\dagger_{u_1}.u_1^{-1}+u_1.(Y_2)^\dagger_{u_2}.(u_1u_2)^{-1}\,|\,X\Big)\\
&= \frac{1}{2}\Big(\big(u_1^{-1}.(Y_1)^\dagger_{u_1}\,|\,Ad\,u_2.X\big)+\big(u_2^{-1}.(Y_2)^\dagger_{u_2}\,|\,X\big)\\
&\qquad+\big((Y_1)^\dagger_{u_1}.u_1^{-1}\,|\,X\big)+\big((Y_2)^\dagger_{u_2}.u_2^{-1}\,|\,Ad\,u_1^{-1}.X\big)\Big)\\
&= \frac{1}{2}\Big((Ad\,u_1^{-1}.Y_1-Y_1\,|\,Ad\,u_2.X)+(Ad\,u_2^{-1}.Y_2-Y_2\,|\,X)\\
&\qquad+(Y_1-Ad\,u_1.Y_1\,|\,X)+(Y_2-Ad\,u_2.Y_2\,|\,Ad\,u_1^{-1}.X)\Big)
\end{aligned}$$

so that :

$$\begin{aligned}
&\Big(\iota_{X^\#}(\omega_1\oplus\omega_2)-\frac{1}{2}(\mu_1\cdot\mu_2)^*(\theta^L+\theta^R\,|\,X)\Big)_{(u_1,u_2)}.(\xi_1,\xi_2)\\
=\;&\frac{1}{2}\Big((Ad\,u_1.X\,|\,Y)-(Ad\,u_2.Y_2\,|\,X)-(Ad\,u_1^{-1}.Y_1-Y_1\,|\,Ad\,u_2.X)\\
&\quad+(Y_2\,|\,X)-(Y_1\,|\,X)-(Y_2-Ad\,u_2.Y_2\,|\,X)\Big)\\
=\;&\frac{1}{2}\Big(\big(X\,|\,Ad\,u_1^{-1}.Y_1\big)-\big(Ad\,u_2.Y_2\,|\,X\big)-\big(u_1^{-1}.(Y_1)^\dagger_{u_1}\,|\,Ad\,u_2.X\big)\\
&\quad\big(Y_1\,|\,X\big)-\big((Y_2)^\dagger_{u_2}.u_2^{-1}\,|\,Ad\,u_1^{-1}.X\big)\Big)\\
=\;&\frac{1}{2}\Big(\big(X\,|\,u_1^{-1}.(Y_1)^\dagger_{u_1}\big)+\big((Y_2)^\dagger_{u_2}.u_2^{-1}\,|\,X\big)\\
&\quad-\big(u_1^{-1}.(Y_1)^\dagger_{u_1}\,|\,Ad\,u_2.X\big)-\big((Y_2)^\dagger_{u_2}\,|\,Ad\,u_1^{-1}.X\big)\Big)\\
=\;&\frac{1}{2}\Big(\big(X-Ad\,u_2.X\,|\,u_1^{-1}.(Y_1)^\dagger_{u_1}\big)-\big((Y_2)^\dagger_{u_2}.u_2^{-1}\,|\,Ad\,u_1^{-1}.X-X\big)\Big)\\
=\;&\frac{1}{2}\Big(\big(X^\dagger_{u_2}.u_2^{-1}\,|\,u_1^{-1}.(Y_1)^\dagger_{u_1}\big)-\big((Y_2)^\dagger_{u_2}.u_2^{-1}\,|\,u_1^{-1}.X^\dagger_{u_1}\big)\Big)
\end{aligned}$$

This last quantity can be rewritten under the form :

$$-\frac{1}{2}\Big(\big(\theta^L_{u_1}(X^\dagger_{u_1})\,|\,\theta^R_{u_2}\big((Y_2)^\dagger_{u_2}\big)\big)-\big(\theta^L_{u_1}\big((Y_1)^\dagger_{u_1}\big)\,|\,\theta^R_{u_2}(X^\dagger_{u_2})\big)\Big) \tag{4.10}$$

This expression defines a 2-form on $\mathcal{C}_1 \times \mathcal{C}_2$. To guess the expression of the appropriate 2-form on any product $M_1 \times M_2$, denote by $\widetilde{\mu_i}$ the map :

$$\begin{array}{rccc} \widetilde{\mu_i}: & M_1 \times M_2 & \longrightarrow & U \\ & (x_1, x_2) & \longmapsto & \mu_i(x_i) \end{array}$$

Then, by the equivariance of μ_i, one has $T_{(x_1,x_2)}\widetilde{\mu_i}.X^{\#}_{(x_1,x_2)} = X^{\dagger}_{\mu_i(x_i)}$, so that expression (4.10) transforms into :

$$\begin{aligned} (4.10) &= -\frac{1}{2}\Big(\big((\widetilde{\mu_1}^*\theta^L)_{x_1}.X^{\#}_{x_1} \,|\, (\widetilde{\mu_2}^*\theta^R)_{x_2}.v_2\big) - \big((\widetilde{\mu_1}^*\theta^L)_{x_1}.v_1 \,|\, (\widetilde{\mu_2}^*\theta^R).X^{\#}_{x_2}\big)\Big) \\ &= -\frac{1}{2}\big(\widetilde{\mu_1}^*\theta^L \wedge \widetilde{\mu_2}^*\theta^R\big)_{(x_1,x_2}\big(X^{\sharp}_{(x_1,x_2)}, (v_1, v_2)\big) \\ &= -(\iota_{X^{\#}}\omega_{res})_{(x_1,x_2)}.(v_1, v_2) \end{aligned}$$

where $(v_1, v_2) \in T_{x_1}M_1 \times T_{x_2}M_2$ is supposed to satisfy $T_{(x_1,x_2)}\widetilde{\mu_i}.(v_1, v_2) = \xi_i \in T_{\mu_i(x_i)}U$ and where $\omega_{res} = \frac{1}{2}(\widetilde{\mu_1}^*\theta^L \wedge \widetilde{\mu_2}^*\theta^R)$. In the following, we shall use the conventions of [AMM98] and write simply :

$$\omega_{res} = \frac{1}{2}(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$$

That is :

$$\begin{aligned} & (\mu_1^*\theta^L \wedge \mu_2^*\theta^R)_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big) \\ := \quad & \big(\theta^L_{\mu_1(x_1)}(T_{x_1}\mu_1.v_1) \,|\, \theta^R_{\mu_2(x_2)}(T_{x_2}\mu_2.w_2)\big) - \big(\theta^L_{\mu_1(x_1)}(T_{x_1}\mu_1.w_1) \,|\, \theta^R_{\mu_2(x_2)}(T_{x_2}\mu_2.v_2)\big) \end{aligned}$$

Observe that this expression coincides indeed with (4.10) when $M_1 \times M_2 = \mathcal{C}_1 \times \mathcal{C}_2$. The above calculations then show that for all $X \in \mathfrak{u}$:

$$\iota_{X\ \#}\big((\omega_1 \oplus \omega_2) + \omega_{res}\big) = \frac{1}{2}(\mu_1 \cdot \mu_2)^*(\theta^L + \theta^R \,|\, X)$$

The product space $M_1 \times M_2$ endowed with the 2-form $\omega := (\omega_1 \oplus \omega_2) + \frac{1}{2}(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$ is called the *fusion product* of M_1 and M_2 (it is denoted $M_1 \circledast M_2$ in [AMM98]). The term *fusion product* alludes to the theory of quantum groups (see [AMM98] for an explanation and references).

Proposition 4.4.1 (Fusion product of quasi-Hamiltonian spaces, [AMM98]). *Let (M_1, ω_1, μ_1) and (M_2, ω_2, μ_2) be two quasi-Hamiltonian U-spaces. Endow $M_1 \times M_2$ with the diagonal action of U. Then the 2-form*

$$\omega := (\omega_1 \oplus \omega_2) + \frac{1}{2}(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$$

makes $M_1 \times M_2$ a quasi-Hamiltonian space with momentum map :

$$\begin{array}{rccc} \mu_1 \cdot \mu_2: & M_1 \times M_2 & \longrightarrow & U \\ & (x_1, x_2) & \longmapsto & \mu_1(x_1)\mu_2(x_2) \end{array}$$

We refer to [AMM98] for a complete proof of this result (one still has to verify that ω is U-invariant and to compute $d\omega$ and $\ker \omega_{(x_1,x_2)}$). As a consequence of this result, every finite product of quasi-Hamiltonian U-spaces is a quasi-Hamiltonian U-space for the diagonal action, with momentum map the product map

$$\begin{array}{rccc} \mu_1 \cdot \ldots \cdot \mu_l: & M_1 \times \cdots \times M_l & \longrightarrow & U \\ & (x_1, \ldots, x_l) & \longmapsto & \mu_1(x_1)\ldots\mu_l(x_l) \end{array}$$

In particular :

Corollary 4.4.2. *The product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ of l conjugacy classes of U is a quasi-Hamiltonian space for the diagonal action of U, with momentum map the product $\mu(u_1, \ldots, u_l) = u_1 \ldots u_l$.*

As pointed out in [AMM98], the fusion product defined above is associative; meaning that the fusion product form on $(M_1 \times M_2) \times M_3$ obtained by taking the product $M_1 \times M_2$ first and then the product $(M_1 \times M_2) \times M_3$ is equal to the fusion product form obtained similarly on $M_1 \times (M_2 \times M_3)$:

$$\begin{aligned} & \Big(\big((\omega_1 \oplus \omega_2) + \tfrac{1}{2}(\mu_1^* \theta^L \wedge \mu_2^* \theta^R) \big) \oplus \omega_3 \Big) + \tfrac{1}{2} \Big((\mu_1 \cdot \mu_2)^* \theta^L \wedge \mu_3^* \theta^R \Big) \\ = \; & \Big(\omega_1 \oplus \big((\omega_2 \oplus \omega_3) + \tfrac{1}{2}(\mu_2^* \theta^L \wedge \mu_3^* \theta^R) \big) \Big) + \tfrac{1}{2} \Big(\mu_1^* \theta^L \wedge (\mu_2 \cdot \mu_3)^* \theta^R \Big) \end{aligned}$$

To obtain an expression for the 2-form defining a quasi-Hamiltonian structure on a product $M_1 \times \cdots \times M_l$ of l quasi-Hamiltonian U-spaces, we re-arrange the above expressions and proceed by induction using the Ad-invariance of $(. \,|\, .)$:

- on $M_1 \times M_2$, the structural 2-form is :

$$\omega = (\omega_1 \oplus \omega_2) + \frac{1}{2}(\mu_1^* \theta^L \wedge \mu_2^* \theta^R) = (\omega_1 \oplus \omega_2) + \frac{1}{2}\big(\theta^R \wedge (\mu_1^* Ad).(\mu_2^* \theta^R)\big)$$

- on $M_1 \times M_2 \times M_3$, the structural 2-form is :

$$\begin{aligned} \omega &= \big(\omega_1 \oplus \omega_2 \oplus \omega_3\big) + \frac{1}{2}\big(\theta^R \wedge (\mu_1^* Ad).\mu_2^* \theta^R\big) + \frac{1}{2}\big(((\mu_1 \cdot \mu_2)^* Ad).\theta^L \wedge \mu_3^* \theta^R\big) \\ &= \big(\omega_1 \oplus \omega_2 \oplus \omega_3\big) + \frac{1}{2}\big(\theta^R \wedge (\mu_1^* Ad).\mu_2^* \theta^R\big) + \frac{1}{2}\big(\theta^R \wedge ((\mu_1 \cdot \mu_2)^* Ad).\mu_3^* \theta^R\big) \end{aligned}$$

- on $M_1 \times \cdots \times M_l$, the structural 2-form is :

$$\omega = (\omega_1 \oplus \ldots \oplus \omega_l) + \frac{1}{2} \sum_{i=1}^{l-1} \Big(\theta^R \wedge \big((\mu_1 \cdot \ldots \cdot \mu_i)^* Ad \big).\mu_{i+1}^* \theta^R \Big)$$

As one can see, these expressions might make computations somewhat long and intricate. In chapter 7, we will need to compute the pull-back of the 2-form ω on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ by a certain map β. By using certain properties of the map β, we will avoid the explicit computation of $\beta^* \omega$ (see lemma 7.3.3). See also [Tre02] for expressions of fusion product forms on products of conjugacy classes.

Another interesting problem is to consider a product of two quasi-Hamiltonian spaces $(M_1, \omega_1, \mu_1 : M_1 \to U_1)$ and $(M_2, \omega_2, \mu_2 : M_2 \to U_2)$ acted on by different groups U_1 and U_2. One then has the following result, which is verified immediately from the definition of a quasi-Hamiltonian space (see definition 4.1.2) :

Proposition 4.4.3 ([AMM98]). *Let $(M_1, \omega_1, \mu_1 : M_1 \to U_1)$ and $(M_2, \omega_2, \mu_2 : M_2 \to U_2)$ be two quasi-Hamiltonian spaces and endow $M_1 \times M_2$ with the $U_1 \times U_2$-action defined by :*

$$(u_1, u_2).(x_1, x_2) = (u_1.x_1, u_2.x_2)$$

the U-invariant 2-form $\omega^P = \omega_1 \oplus \omega_2$ and the equivariant map

$$\begin{array}{rccc} \mu^P : & M_1 \times M_2 & \longrightarrow & U_1 \times U_2 \\ & (x_1, x_2) & \longmapsto & \big(\mu_1(x_1), \mu_2(x_2)\big) \end{array}$$

Then the triple $(M_1 \times M_2, \omega^P, \mu^P)$ is a quasi-Hamiltonian $U_1 \times U_2$-space.

If now $U_1 = U_2 = U$, then one can embed U diagonally in $U_1 \times U_2$ and consider the induced U-action on $M_1 \times M_2$, which is just the diagonal action. We can then recover the fusion product 2-form on $M_1 \times M_2$ obtained in proposition 4.4.1 by using the following lemma, the upshot being that this lemma applies to $U \times U$-spaces which are not necessarily product manifolds.

Lemma 4.4.4 (Internal fusion of quasi-Hamiltonian $U \times U$-spaces). [AMM98] *Let $(M, \omega^D, \mu_D = (\mu_1, \mu_2) : M \to U \times U)$ be a quasi-Hamiltonian $U \times U$-space. Consider the action of U on M defined by $u.x := (u, u).x$ and the map*

$$\begin{array}{rccc} \mu : & M & \longrightarrow & U \\ & x & \longmapsto & \mu_1(x)\mu_2(x) \end{array}$$

Then, the 2-form

$$\omega := \omega_D + \frac{1}{2}(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$$

makes M a quasi-Hamiltonian U-space with momentum map the map μ.

When $(M, \omega^D, \mu_D) = (M_1 \times M_2, \omega^P, \mu_P : M_1 \times M_2 \to U \times U)$, applying lemma 4.4.4 to this $U \times U$-space, we indeed obtain the fusion product space of proposition 4.4.1. But a more interesting feature of the above lemma is the construction of a new quasi-Hamiltonian U-space associated to any given Lie group $(U, (.\,|\,.))$ called the *internally fused double* of U : it is obtained by applying internal fusion to the double $D(U) = U \times U$ constructed in proposition 4.2.2.

Proposition 4.4.5 ([AMM98]). *The manifold $\mathfrak{D}(U) := U \times U$ equipped with the diagonal conjugacy action of U, the U-invariant 2-form*

$$\omega = \frac{1}{2}(\alpha^*\theta^L \wedge \beta^*\theta^R) + \frac{1}{2}(\alpha^*\theta^R \wedge \beta^*\theta^L) + \frac{1}{2}\big((\alpha \cdot \beta)^*\theta^L \wedge (\alpha^{-1} \cdot \beta^{-1})^*\theta^R\big)$$

and the equivariant momentum map

$$\begin{array}{rccc} \mu : & \mathfrak{D}(U) = U \times U & \longrightarrow & U \\ & (a, b) & \longmapsto & aba^{-1}b^{-1} \end{array}$$

(where α and β are the projections respectively on the first and second factors of $\mathfrak{D}(U)$) is a quasi-Hamiltonian U-space, called the internally fused double *of U.*

This space plays a very important role in the description of symplectic structures on representation spaces of fundamental groups of Riemann surfaces whose genus is greater or equal to 1 (see [AMM98] and section 4.6 below). Before concluding this section, we would like to point out the fact that, in [AMM98], all the above results concerning products of quasi-Hamiltonian spaces, including internal fusion, are presented in a unified way. We chose to be more analytic here. Alekseev, Malkin and Meinrenken also prove in [AMM98] that the fusion product is commutative on isomorphism classes of quasi-Hamiltonian spaces and relate fusion to reduction to reproduce the "shifting trick" for symplectic reduction in the usual Hamiltonian setting. We refer to this paper for details.

4.5 Reduction theory of quasi-Hamiltonian spaces

In this section, we will show, mainly following [AMM98], how to obtain a symplectic manifold from a quasi-Hamiltonian space by a reduction procedure, that is to say, by taking the quotient of a fiber $\mu^{-1}(\{u\})$ of the momentum map by the action of the stabilizer group U_u, which preserves the fiber $\mu^{-1}(\{u\})$ since μ is equivariant. We refer to [MW01] for a historical account on the idea of reduction (which consists, to try a physical picture, in diminishing the number of degrees of liberty of a Hamiltonian system, that is, the dimension of a symplectic manifold called the phase space, by consideration of symmetry, that, is by taking the quotient by a group action to obtain a smaller phase space). The modern formulation of reduction that we will be dealing with in the following is due to Meyer and Marsden and Weinstein (see respectively [Mey73] and [MW74]).

Let us first recall how to obtain differential forms on an orbit space N/G where N is a manifold acted on by a Lie group G. We will assume that the action is proper and free, so that N/G is a manifold (and the submersion $p : N \to N/G$ is a locally trivial principal fibration with structural group G, see for

instance [DK00], pp.53-55). Let $[x]$ denote the G-orbit of $x \in N$. Then the tangent space $T_{[x]}(N/G)$ is isomorphic to $T_xN/\ker T_xp$ (since p is surjective). And $\ker T_xp$ consists exactly of the vectors tangent to N at x which are actually tangent to the G-orbit of x in N. Those are exactly the values at x of fundamental vector fields :

$$\ker T_xp = T_x(G.x) = \{X_x^{\#} \ : \ X \in \mathfrak{g} = Lie(G)\}$$

Let then α be a differential form on N (say, a 2-form). Under what conditions does α define a 2-form $\overline{\alpha}$ on N/G verifying $p^*\overline{\alpha} = \alpha$? This last condition amounts to saying that $\overline{\alpha}_{[x]}([v],[w]) = \alpha_x(v,w)$ for all $x \in N$ and all $v, w \in T_xN$. One then checks that the left-hand side term of this equation is well-defined by this relation if and only if the 2-form α is G-invariant. Further, since $X_x^{\#}$ is sent to 0 in $T_{[x]}(N/G)$ by the map T_xp, the relation $p^*\overline{\alpha} = \alpha$ implies that $\iota_{X^{\#}}\alpha = 0$ for all $X \in \mathfrak{g}$. These two conditions turn out to be enough :

Lemma 4.5.1. *Let $p : N \to B = N/G$ be a locally trivial principal fibration with structural group G and let α be a differential form on N. If α satisfies*

$$g^*\alpha = \alpha \quad \text{for all } g \in G \quad (G\text{−invariance})$$

and

$$\iota_{X^{\#}}\alpha = 0 \quad \text{for all } X \in \mathfrak{g} = Lie(G)$$

then there exists a unique differential form $\overline{\alpha}$ on B satisfying $p^*\overline{\alpha} = \alpha$. In such a case, the 2-form α on N is said to be basic.

Observe that if G is compact and connected (so that the exponential map is surjective), the condition $g^*\alpha = \alpha$ for all $g \in G$ may be replaced by $\mathcal{L}_X\alpha = 0$ for all $X \in \mathfrak{g}$ (which is always implied by the G-invariance).

We can now use this result to construct differential forms on orbit spaces associated to level manifolds of the momentum map. Let us start by considering the usual Hamiltonian case. Let (M, ω) be a symplectic manifold endowed with a Hamiltonian action of a Lie group U with momentum map $\mu : M \to \mathfrak{u}^*$, and take $N := \mu^{-1}(\{\zeta\})$ where $\zeta \in \mathfrak{u}^*$. Because of the equivariance of μ, the stabilizer $G := U_\zeta$ of ζ for the co-adjoint action of U on $\mathfrak{u}^*$ acts on $N = \mu^{-1}(\{\zeta\})$. Assuming that ζ is a regular value of μ and that U_ζ acts freely and properly on $\mu^{-1}(\{\zeta\})$, we then have a principal fibre bundle $p : \mu^{-1}(\{\zeta\}) \to \mu^{-1}(\{\zeta\})/U_\zeta$ and the following diagram :

$$\begin{array}{ccc} \mu^{-1}(\{\zeta\}) & \overset{i}{\hookrightarrow} & M \\ \Big\downarrow p & & \\ \mu^{-1}(\{\zeta\})/U_\zeta & & \end{array}$$

where $i : \mu^{-1}(\{\zeta\}) \hookrightarrow M$ is the inclusion map. The 2-form ω on M induces a 2-form $i^*\omega$ on $\mu^{-1}(\{\zeta\})$, which turns out to be basic (see the proof of proposition 4.5.2 for similar reasoning). Therefore, by lemma 4.5.1, there exists a unique 2-form ω^{red} on $\mu^{-1}(\{\zeta\})/U_\zeta$ such that $\pi^*\omega^{red} = i^*\omega$. Since ω is closed, so is ω^{red} (we may first check that if α is basic then $d\alpha$ is basic, as follows from the Cartan homotopy formula). And one may then notice that a vector $v \in T_xN = \ker T_x\mu$ is sent by T_xp to a vector in $\ker \omega^{red}_{[x]}$ if and only if v is contained in $(T_xN)^{\perp_\omega} = (\ker T_x\mu)^{\perp_\omega} = \{X_x^{\#} \ : \ X \in \mathfrak{u}\}$ as well. But then $v = X_x^{\#} \in \ker T_x\mu \cap (\ker T_x\mu)^{\perp_\omega}$, so that by the equivariance of μ, one has, denoting by $X^\dagger$ the fundamental vector field on $\mathfrak{u}^*$ associated to X by the co-adjoint action of U : $X_\zeta^\dagger = X_{\mu(x)}^\dagger = T_x\mu.X_x^{\#} = 0$, so that $X \in \mathfrak{u}_\zeta = Lie(U_\zeta)$. We have thus proved that $T_xp.v \in \ker \omega^{red}_{[x]}$ if and only if $v \in \{X_x^{\#} \ : \ X \in \mathfrak{u}_\zeta\}$. Consequently, for such a v, one has $T_xp.v = 0$, so that ω^{red} is non-degenerate and $\mu^{-1}(\{\zeta\})/U_\zeta$ is a symplectic manifold. When $\zeta = 0 \in \mathfrak{u}^*$, $U_\zeta = U$ and one usually denotes $\mu^{-1}(\{0\})/U$ by $M//U$. This manifold is called the *symplectic quotient* of M by U. Observe that in this case $\mu^{-1}(\{0\})$ is a

co-isotropic submanifold of M, since, if $\mu(x) = 0$, then for all $X \in \mathfrak{u}$, $T_x\mu.X_x^{\#} = X_0^{\dagger} = 0$, so that $(\ker T_x\mu)^{\perp_\omega} \subset \ker T_x\mu$. And the 2-form ω^{red} is then symplectic because the leaves of the null-foliation of $\omega|_N$ (that is, the foliation corresponding to the distribution $x \mapsto \ker(\omega|_N)_x = (T_xN)^{\perp_\omega} = (\ker T_x\mu)^{\perp_\omega}$) are precisely the U-orbits. One may also define the reduced space at ζ to be $\mu^{-1}(O_\zeta)/U$, where O_ζ is the co-adjoint orbit of ζ. We refer to [Mey73] and [MW74] for further details in that direction (in particular for the shifting trick, that reduces the study of $\mu^{-1}(O_\zeta)$ to the study of $(M \times O_{-\zeta})//U$).

In [LS91] and in [BL97], the authors study the case where the regularity assumptions (0 is a regular value of μ and the action of U on $\mu^{-1}(\{0\})$ is free) are dropped. In the rest of this section, we will restrict ourselves to the case where U is compact, so that the action is automatically proper. This is the situation studied in [LS91]. We refer to [BL97] for proper actions of non-compact Lie groups and references concerning singular symplectic quotients (notably the survey paper [AGJ90]). In [LS91], Lerman and Sjamaar showed that when the above regularity assumptions are dropped, the reduced space $M/\!\!/U$ is a union of symplectic manifolds which are the strata of a stratified space. Their proof relies on a normal form theorem for the momentum map obtained by Marle in [Mar86] and by Guillemin and Sternberg in [GS84b]. See subsection 4.5.2 for further comments.

4.5.1 The smooth case

Let us now come back to the quasi-Hamiltonian setting. In [AMM98], Alekseev, Malkin and Meinrenken showed how to construct new quasi-Hamiltonian spaces from a given quasi-Hamiltonian U-space $(M, \omega, \mu : M \to U)$ by a reduction procedure, assuming that U is a product group $U = U_1 \times U_2$ (so that μ has two components $\mu = (\mu_1, \mu_2)$). Their result says that the reduced space $\mu_1^{-1}(\{u\})/U_u$ is a quasi-Hamiltonian U_2-space. In particular, when $U_2 = \{1\}$, they obtain a *symplectic manifold*. Since this is the case we are interested in, we will state their result in this way and give a proof that is valid in this particular situation. We refer to [AMM98] for the general case. It is quite remarkable that one can obtain symplectic manifolds from quasi-Hamiltonian spaces by a reduction procedure. As a matter of fact, this is one of the nicest features of the notion of quasi-Hamiltonian spaces : it enables one to obtain symplectic structures on quotient spaces (typically, moduli spaces) using simple finite dimensional objects as a total space. The most important example in that respect is the moduli space of flat connections on a Riemann surface Σ, first obtained (in the case of a compact surface) by Atiyah and Bott in [AB83] by symplectic reduction of an infinite-dimensional symplectic manifold. We refer to [AMM98] and to section 4.6 below to see how one can recover these symplectic structures using quasi-Hamiltonian spaces. Let us now state and prove the result we are interested in.

Proposition 4.5.2 (Symplectic reduction of quasi-Hamiltonian spaces, the smooth case, [AMM98]). *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space. Assume that 1 is a regular value of μ and that U acts freely on $\mu^{-1}(\{1\})$. Let $i : \mu^{-1}(\{1\}) \hookrightarrow M$ be the inclusion of the level manifold $\mu^{-1}(\{1\})$ in M and let $p : \mu^{-1}(\{1\}) \to \mu^{-1}(\{1\})/U$ be the projection on the orbit space. Then there exists a unique symplectic form ω^{red} on the reduced manifold $M^{red} := \mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ on $\mu^{-1}(\{1\})$.*

Proof. The proof consists in showing that $i^*\omega$ is basic with respect to the principal fibration p and then verifying that the unique 2-form ω^{red} on $\mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ is indeed symplectic.
Let us first show that $i^*\omega$ is basic :

$$u^*(i^*\omega) = i^*\omega \quad \text{for all } u \in U$$

and

$$\iota_{X^\#} i^*\omega = 0 \qquad \text{for all } X \in \mathfrak{u}$$

The first condition is obvious since ω is U-invariant. Consider now $X \in \mathfrak{u}$. Then :

$$\begin{aligned}
\iota_{X^\#}(i^*\omega) &= i^*(\iota_{X^\#}\omega) \\
&= i^*\big(\frac{1}{2}\mu^*(\theta^L+\theta^R\,|\,X)\big) \\
&= \frac{1}{2}\big(i^*\circ\mu^*(\theta^L+\theta^R\,|\,X)\big) \\
&= \frac{1}{2}(\mu\circ i)^*(\theta^L+\theta^R\,|\,X) \\
&= 0
\end{aligned}$$

since $\mu\circ i$ is constant on $\mu^{-1}(\{1\})$ and therefore $T(\mu\circ i)=0$, hence $(\mu\circ i)^*=0$. Then there exists, by lemma 4.5.1, a unique 2-form ω^{red} on $\mu^{-1}(\{1\})/U$ such that $p^*\omega^{red}=i^*\omega$.
Let us now prove that ω^{red} is a symplectic form. First :

$$\begin{aligned}
p^*(d\omega^{red}) &= d(p^*\omega^{red}) \\
&= d(i^*\omega) \\
&= i^*(d\omega) \\
&= i^*(-\mu^*\chi) \\
&= -\underbrace{(\mu\circ i)^*}_{=0}\,\chi \\
&= 0
\end{aligned}$$

so that $d\omega^{red}=0$. Second, take $[x]\in\mu^{-1}(\{1\})/U$, where $x\in\mu^{-1}(\{1\})$, and $[v]\in\ker\omega^{red}_{[x]}$, where $v\in T_x\mu^{-1}(\{1\})=\ker T_x\mu$. Then, for all $w\in T_x\mu^{-1}(\{1\})=\ker T_x\mu$, one has :

$$(i^*\omega)_x(v,w)=(p^*\omega^{red})_x(v,w)=\omega^{red}_{[x]}([v],[w])=0$$

since $[v]\in\ker\omega^{red}_{[x]}$. Hence :

$$\begin{aligned}
v\in\ker(i^*\omega)_x &= \{s\in\ker T_x\mu\mid\forall\,w\in\ker T_x\mu,\ \omega_x(s,w)=0\} \\
&= \ker T_x\mu\cap(\ker T_x\mu)^{\perp_\omega}\subset T_xM
\end{aligned}$$

But, by proposition 4.3.1, $(\ker T_x\mu)^\perp=\{X^\#_x\,:\,X\in\mathfrak{u}\}$, so $v=X^\#_x$ for some $X\in\mathfrak{u}$. Hence :

$$[v]=T_xp.v=T_xp.X^\#_x=0$$

so that ω^{red} is non-degenerate. □

4.5.2 The stratified case

What happens if we now drop the regularity assumptions of proposition 4.5.2 ? Following the techniques used in [LS91] and [BL97] for usual Hamiltonian spaces, we will show that if we do not assume 1 to be a regular value of $\mu:M\to U$, nor that U acts freely on $\mu^{-1}(\{1\})$, then the orbit space $\mu^{-1}(\{1\})/U$ is a disjoint union, over subgroups $K\subset U$, of symplectic manifolds M^{red}_K :

$$\mu^{-1}(\{1\})/U=\bigsqcup_{K\subset U}M^{red}_K$$

each M^{red}_K being obtained by applying proposition 4.5.2 to a quasi-Hamiltonian space $(M_K,\omega_K,\widehat{\mu_K}:M_K\to L_K)$. Actually, the study conducted in [LS91] (and in [BL97] for the case of proper actions of non-compact groups) is far more precise and ensures that the reduced space $M^{red}:=\mu^{-1}(\{1\})/U$ is a *stratified space* (in particular, there is a notion of *smooth function* on M^{red}, and the set $\mathcal{C}^\infty(M^{red})$ of smooth functions is an algebra over the field $\mathbb{R}$, see [LS91] for a precise definition), with strata $(S_K)_{K\subset U}$, such that :

- each stratum S_K is a symplectic manifold.
- $\mathcal{C}^\infty(M^{red})$ is a Poisson algebra.
- the restriction maps $\mathcal{C}^\infty(M^{red}) \to \mathcal{C}^\infty(S_K)$ are Poisson maps.

A stratified space satisfying these additional three conditions is called a *stratified symplectic space*. In [LS91], to show that M^{red} is always a stratified symplectic space, Lerman and Sjamaar actually obtain this space as a disjoint union of symplectic manifolds in two differents ways. The first one enhances the stratified structure of M^{red} (the stratification being induced by the partition of M according to orbit types for the action of U), and relies on a normal form theorem for the momentum map obtained by Marle in [Mar86] and by Guillemin and Sternberg in [GS84b]. It also shows that each stratum carries a symplectic structure. The second description of M^{red} as a disjoint union of symplectic manifolds then aims at relating this reduction to the regular Marsden-Meyer-Weinstein procedure (see for instance [dS01]) : the symplectic structure on each stratum is obtained by symplectic reduction from a submanifold of M.

Here, we shall not be dealing with the notion of stratified space and we will content ourselves with a description of $\mu^{-1}(\{1\})/U$ as a disjoint union of symplectic manifolds obtained by reduction from a quasi-Hamiltonian space $M_K \subset M$. We will nonetheless call the case at hand the stratified case.

We start with a quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ and use the partition of M given by what we may call the *isotropy type* :

$$M = \bigsqcup_{K \subset U} M_K$$

where $K \subset U$ is a closed subgroup of U and M_K is the set of points of M whose stabilizer is exactly K :

$$M_K = \{x \in M \mid U_x = K\}$$

Observe that if one wants K to be the stabilizer of some $x \in M$, one has to assume that K is closed, since a stabilizer always is. If M_K is non-empty, it is a submanifold of M (see proposition 2.1.1), called the *manifold of symmetry* K in [LS91], whose tangent space at some point $x \in M_K$ consists of all vectors in T_xM which are fixed by K :

$$T_xM_K = \{v \in T_xM \mid \text{for all } k \in K, k.v = v\}$$

where $k \in K$ acts on T_xM as the tangent map of the diffeomorphism $y \in M \mapsto k.y$ which sends x to itself by definition. The action of U does not preserve M_K but M_K is globally stable under the action of elements $n \in \mathcal{N}(K) \subset U$, where $\mathcal{N}(K)$ denotes the normalizer of K in U :

$$\mathcal{N}(K) := \{u \in U \mid \text{for all } k \in K, uku^{-1} \in K\}$$

It is actually the largest subgroup of U leaving M_K invariant, since the stabilizer of $u.x$ for some $x \in M_K$ and some $u \in U$ is still U_x if and only if $uU_xu^{-1} = U_x$, that is, $uKu^{-1} = K$. Observe that we have :

$$Lie\big(\mathcal{N}(K)\big) = \{X \in \mathfrak{u} \mid \text{for all } Y \in \mathfrak{k}, [X, Y] \in \mathfrak{k}\}$$

That is, the Lie algebra of the normalizer of K in U is the normalizer of $n(\mathfrak{k})$ of the Lie algebra $\mathfrak{k} := Lie(K)$ in $\mathfrak{u} = Lie(U)$. The subgroup K is normal in $\mathcal{N}(K)$ and acts trivially on M_K by definition of the manifold of symmetry K, so that M_K inherits an action of the quotient group $\mathcal{N}(K)/K$. It actually follows from the definition of M_K that this induced action is free : if $n \in \mathcal{N}(K)$ stabilizes some x in M_K, then $n \in K$ and so is the identity in $\mathcal{N}(K)/K$. We now wish to show that M_K is a quasi-Hamiltonian space with respect to this action. We need to find a momentum map $\mu_K : M_K \to \mathcal{N}(K)/K$ and a 2-form ω_K satisfying the axioms of definition 4.1.2. The natural candidates are $\mu_K := \mu|_{M_K}$ and $\omega_K := \omega|_{M_K}$, but the problem is then that μ_K does not take its values in $\mathcal{N}(K)/K$. We will now show that $\mu(M_K) \subset \mathcal{N}(K)$ and that we can therefore consider the composed map $\widehat{\mu_K} := p_K \circ \mu_K : M_K \to \mathcal{N}(K)/K$, where p_K is

the projection map $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K$. Denote then by L_K the group $L_K := \mathcal{N}(K)/K$. As K is closed in U, so is $\mathcal{N}(K)$, and since U is compact, $\mathcal{N}(K)$ is compact. Therefore $L_K = \mathcal{N}(K)/K$ is a compact Lie group. We will then show that $(M_K, \omega|_{M_K}, \widehat{\mu_K})$ is a quasi-Hamiltonian space. Moreover, we will show that $1 \in L_K$ is a regular value of $\widehat{\mu_K}$ and that L_K acts freely on $\widehat{\mu_K}^{-1}(\{-1\})$, so that, by proposition 4.5.2, the reduced space $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ is a symplectic manifold.

To do so, we start by studying $\mu(M_K)$. This whole analysis adapts the ideas of [LS91] to the quasi-Hamiltonian setting. Let us denote $\omega_K := \omega|_{M_K}$ and $\mu_K := \mu|_{M_K}$. First, since K acts trivially on M_K, we have, for all $x \in M_K$ and all $k \in K$:

$$\mu_K(x) = \mu_K(k.x) = k\mu_K(x)k^{-1}$$

so that $\mu(x)$ belongs to the subgroup U^K of points of U whose centralizer contains K :

$$U^K := \{u \in U \mid \text{for all } k \in K, kuk^{-1} = u\} \tag{4.11}$$

Thus : $\mu(M_K) \subset U^K$, and therefore, for all $x \in M_K$:

$$\text{Im } T_x\mu_K \hookrightarrow Lie(U^K) = \{X \in \mathfrak{u} \mid \text{for all } Y \in \mathfrak{k}, [X, Y] = 0\}$$

(this is not strictly speaking an inclusion since Im $T_x\mu_K \subset T_{\mu_K(x)}U$, but it is true up to a translation : Im $(\mu_K^*\theta^L)_x \subset Lie(U^K)$, as in proposition 4.3.1). Observe that the Lie algebra of U^K is the subalgebra $\mathfrak{u}^{\mathfrak{k}}$ of elements of $\mathfrak{u}$ whose centralizer in $\mathfrak{u}$ contains $\mathfrak{k}$:

$$\mathfrak{u}^{\mathfrak{k}} := \{X \in \mathfrak{u} \mid \text{for all } Y \in \mathfrak{k}, [X, Y] = 0\}$$

Second, for all $X \in \mathfrak{k}$, we have :

$$\iota_{X^\#}\omega_K = \tfrac{1}{2}\mu_K^*(\theta^L + \theta^R \,|\, X) \tag{4.12}$$

(where θ^L and θ^R denote as usual the Maurer-Cartan 1-forms of U, so that the above relationship simply follows from the fact that $(M, \omega, \mu : M \to U)$ is a quasi-Hamiltonian space). Hence, by proposition 4.3.1, we have, for all $x \in M_K$:

$$\text{Im } T_x\mu_K \subset \text{Im } T_x\mu \simeq \mathfrak{u}_x^\perp = \mathfrak{k}^\perp \tag{4.13}$$

It follows from (4.11) and (4.13) that for all $x \in M_K$:

$$\text{Im } T_x\mu_K \hookrightarrow \mathfrak{u}^{\mathfrak{k}} \cap \mathfrak{k}^\perp$$

(again, this could be written : Im $(\mu_K^*\theta^L)_x \subset \mathfrak{u}^{\mathfrak{k}} \cap \mathfrak{k}^\perp$). We then have :

Lemma 4.5.3. *The Lie subalgebra $\mathfrak{u}^{\mathfrak{k}} \cap \mathfrak{k}^\perp \subset \mathfrak{u}$ is equal to the orthogonal of $\mathfrak{k}$ in $n(\mathfrak{k})$:*

$$\mathfrak{u}^{\mathfrak{k}} \cap \mathfrak{k}^\perp = \mathfrak{k}^{\perp_{n(\mathfrak{k})}}$$

Proof. To prove the first inclusion, it suffices to show that any $X \in \mathfrak{u}^{\mathfrak{k}} \cap \mathfrak{k}^\perp$ belongs to $n(\mathfrak{k})$. Since $X \in \mathfrak{u}^{\mathfrak{k}}$, we have, for all $Y \in \mathfrak{k}$, $[X, Y] = 0 \in \mathfrak{k}$, so that $X \in n(\mathfrak{k})$.
To prove the converse inclusion, it is enough to show that any $X \in \mathfrak{k}^{\perp_{n(\mathfrak{k})}}$ belongs to $\mathfrak{u}^{\mathfrak{k}}$, that is, to show that for all $Y \in \mathfrak{k}$, $[X, Y] = 0$. Take $Y \in \mathfrak{k}$. Then for all $Z \in n(\mathfrak{k})$:

$$([X, Y] \,|\, Z) = (X \,|\, [Y, Z]) = 0$$

since $[Y, Z] \in \mathfrak{k}$ (because $Z \in n(\mathfrak{k})$) and $X \in \mathfrak{k}^{\perp_{n(\mathfrak{k})}}$. Since the restriction $(. \mid .)_{n(\mathfrak{k})}$ is non-degenerate, this implies that $[X, Y] = 0$. □

Observe now that $\mathfrak{k}$ is an ideal in $n(\mathfrak{k})$ and that $\mathfrak{k}^{\perp_{n(\mathfrak{k})}}$ is therefore a Lie subalgebra of $n(\mathfrak{k})$ (hence a Lie subalgebra of $\mathfrak{u}$) which is isomorphic to $n(\mathfrak{k})/\mathfrak{k}$. Moreover, we have just seen that, for all $x \in M_K$:

$$\text{Im } T_x\mu_K \hookrightarrow \mathfrak{k}^{\perp_{n(\mathfrak{k})}} \tag{4.14}$$

(again, this could be written : Im $(\mu_K^*\theta^L)_x \subset \mathfrak{k}^{\perp_{n(\mathfrak{k})}}$). In particular, for all $x \in M_K$, Im $T_x\mu_K \hookrightarrow n(\mathfrak{k})$, so that $\mu_K(M_K) \subset \mathcal{N}(K)$. We can therefore consider the map $\widehat{\mu_K} := p_K \circ \mu_K : M_K \to L_K = \mathcal{N}(K)/K$, where $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K$. Furthermore, we may identify the Lie algebra of L_K to $n(\mathfrak{k})/\mathfrak{k}$. Under this identification, the Maurer-Cartan 1-forms $\theta^L_{L_K}$ and $\theta^R_{L_K}$ of L_K are obtained by restricting those of U to $\mathcal{N}(K)$ (which gives $n(\mathfrak{k})$-valued 1-forms) and composing by the projection $n(\mathfrak{k}) \to n(\mathfrak{k})/\mathfrak{k}$. It is then immediate from relation (4.12), that for all $X \in Lie(L_K)$, one has :

$$\iota_{X\#}\omega_K = \frac{1}{2}\widehat{\mu_K}^*(\theta^L_{L_K} + \theta^R_{L_K} \mid X)$$

Likewise, the Cartan 3-form χ_{L_K} of L_K is obtained by restricting that of U to $\mathcal{N}(K)$ and composing the $n(\mathfrak{k})$-valued 3-form thus obtained by the projection $n(\mathfrak{k}) \to n(\mathfrak{k})/\mathfrak{k}$. Then, it follows from the fact that $d\omega = -\mu^*\chi$ that we have :

$$d\omega_K = -\mu_K^*\chi|_{\mathcal{N}(K)} = -\widehat{\mu_K}^*\chi_{L_K}$$

Thus, we have almost proved that $(M_K, \omega_K, \widehat{\mu_K})$ is a quasi-Hamiltonian L_K-space. In order to compute $\ker(\omega_K)_x$ for all $x \in M_K$, we observe the following two facts, the first of which is classical in symplectic geometry and the second of which is a quasi-Hamiltonian analogue :

Lemma 4.5.4. *Let (V, ω) be a symplectic vector space and let K be a compact group acting linearly on V preserving ω. Then the subspace*

$$V_K := \{v \in V \mid \text{for all } k \in K, k.v = v\}$$

of K-fixed vectors in V is a symplectic subspace of V.

Proof. Since K is compact, there exists a K-invariant positive definite scalar product on V, that we shall denote by $(.\,|\,)$. Since ω is non-degenerate, there exists, for any $v \in V$, a unique vector $Av \in V$ satisfying

$$(v\,|\,w) = \omega(Av, w)$$

for all $w \in V$, and the map $A : V \to V$ thus defined is an automorphism of V. Moreover, it satisfies $A(V_K) \subset V_K$. Indeed, if $v \in V_K$, then for all $k \in K$, one has, for all $w \in V$:

$$\begin{aligned}
\omega(k.Av, w) &= \omega(Av, k^{-1}.w) \\
&= (v\,|\,k^{-1}.w) \\
&= (k.v\,|\,w) \\
&= \omega(A(k.v), w) \\
&= \omega(Av, w)
\end{aligned}$$

and therefore $k.Av = Av$ for all $k \in K$ (incidentally, if one forgets the last equality, which used the fact that $k.v = v$, this also proves that $Ak = kA$ for all $k \in K$), hence $Av \in V_K$. If now $v \in V_K$ satisfies $\omega(v, w) = 0$ for all $w \in V_K$, then in particular for $w = Av$, one obtains $\omega(v, Av) = 0$, that is, $(v\,|\,v) = 0$, hence $v = 0$, since $(.\,|\,)$ is positive definite. □

Lemma 4.5.5. *Let (V, ω) be a vector space endowed with a possibly degenerate antisymmetric bilinear form and let K be a compact group acting linearly on V preserving w. Then the 2-form $w_K := \omega|_{V_K}$ defined on the subspace*

$$V_K := \{v \in V \mid \text{for all } k \in K, k.v = v\}$$

of K-fixed vectors of V has kernel :

$$\ker \omega_K = \ker \omega \cap V_K$$

Proof. If ω is non-degenerate then this is simply lemma 4.5.4. Assume now that $\ker\omega \neq \{0\}$. Observe that $\ker\omega_K = V_K^{\perp_\omega} \cap V_K \supset \ker\omega \cap V_K$. We now consider the reduced vector space $V^{red} := V/\ker\omega$. The 2-form ω induces a 2-form ω^{red} on V^{red}, which is non-degenerate by construction. The map $V_K \hookrightarrow V \to V/\ker\omega$ induces an inclusion $V_K/(\ker\omega \cap V_K) \hookrightarrow V/\ker\omega$. Further, the action of K on V induces an action $k.[v] := [k.v]$ on V^{red} : this action is well-defined because K preserves ω and therefore if $r \in \ker\omega$ then $k.r \in \ker\omega$. The subspace $(V^{red})_K$ of K-fixed vectors for this action can be identified with $V_K/(\ker\omega \cap V_K)$. Indeed, if $[v] \in V^{red}$ satisfies, for all $k \in K$, $[k.v] = [v]$, then set :

$$w := \int_{k\in K} (k.v)d\lambda(k)$$

where λ is the Haar measure on the compact Lie group K (such that $\lambda(K) = 1$, see for instance [BtD95], p.46). Then for all $k' \in K$:

$$\begin{aligned} k'.w &= k'.\Big(\int_{k\in K} (k.v)d\lambda(k)\Big) \\ &= \int_{k\in K} (k'k.v)d\lambda(k) \\ &= \int_{h\in K} (h.v)d\lambda(h) \end{aligned}$$

since the Haar measure on K is invariant by translation. Thus $w \in V_K$ and we have :

$$\begin{aligned} [w] &= \Big[\int_{k\in K} (k.v)d\lambda(k)\Big] \\ &= \int_{k\in K} \underbrace{[k.v]}_{=[v]} d\lambda(k) \\ &= [v] \times \int_{k\in K} d\lambda(k) \\ &= [v] \end{aligned}$$

Thus $[v] \in V_K/(\ker\omega \cap V_K) \subset V^{red}$, which proves that $(V^{red})_K \subset V_K/(\ker\omega \cap V_K)$, and therefore :

$$(V^{red})_K = V_K/(\ker\omega \cap V_K)$$

(the converse inclusion being obvious). Consequently, since V^{red} is a symplectic space, lemma 4.5.4 applies and we obtain :

$$\ker\omega^{red}|_{(V^{red})_K} = \{0\}$$

Now $\omega_K = \omega|_{V_K}$ induces a 2-form $(\omega_K)^{red}$ on $V_K/(\ker\omega \cap V_K) = (V^{red})_K$, whose kernel is, by definition :

$$\ker(\omega_K)^{red} = \ker\omega_K/(\ker\omega \cap V_K)$$

But, again by definition, $(\omega_K)^{red} = \omega^{red}|_{(V^{red})_K}$, so that $\ker(w_K)^{red} = \{0\}$, hence $\ker\omega_K = \ker\omega \cap V_K$, which proves the lemma. □

We then obtain a new class of examples of quasi-Hamiltonian spaces :

Proposition 4.5.6. *For each closed subgroup $K \subset U$, the compact Lie group $L_K := \mathcal{N}(K)/K$ acts freely on the manifold of symmetry*

$$M_K = \{x \in M \mid U_x = K\}$$

In addition to that, $\mu(M_K) \subset \mathcal{N}_K$ and $(M_K, \omega_K := \omega|_{M_K}, \widehat{\mu_K} := p_K \circ \mu|_{M_K})$, where p_K is the projection map $p_K : \mathcal{N}(K) \to \mathcal{N}(K)/K = L_K$, is a quasi-Hamiltonian space.

Proof. The only thing left to prove is that, for all $x \in M_K$:

$$\ker(\omega_K)_x = \{X_x^{\#} \,:\, X \in n(\mathfrak{k})/\mathfrak{k} \mid Ad\,\widehat{\mu_K}(x).X = -X\}$$

Since Im $T_x\mu_K \hookrightarrow \mathfrak{k}^{\perp_{n(\mathfrak{k})}}$ (see (4.14)), this is equivalent to proving that :

$$\ker(\omega_K)_x = \{X_x^{\#} \,:\, X \in n(\mathfrak{k}) \mid Ad\,\mu_K(x).X = -X\}$$

Further, for $X \in \mathfrak{u}$, one has $X_x^{\#} \in T_xM_K$ if and only if $X \in n(\mathfrak{k})$, so that what we really need to prove is that :

$$\begin{aligned} \ker(\omega_K)_x &= \{X_x^{\#} \,:\, X \in \mathfrak{u} \mid Ad\,\mu(x).X = -X\} \cap T_xM_K \\ &= \ker\omega_x \cap T_xM_K \end{aligned}$$

But $T_xM_K = \{v \in T_xM \mid \text{for all } k \in K, k.v = v\}$, so that what we want follows from lemma 4.5.5. □

And we then observe that :

Corollary 4.5.7. *$1 \in L_K$ is a regular value of $\widehat{\mu_K}$ and the reduced space $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ is a symplectic manifold.*

Proof. By proposition 4.3.1, we have, for all $x \in M_K$, Im $T_x\widehat{\mu_K} \simeq \mathfrak{l}_x^{\perp}$, where $\mathfrak{l}_x$ is the Lie algebra of the stabilizer of x in L_K. Since the action of L_K on M_K is free, we have $\mathfrak{l}_x = 0$, and therefore $\widehat{\mu_K} : M_K \to L_K$ is a submersion. In particular, $1 \in L_K$ is a regular value of $\widehat{\mu_K}$. The fact that $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ is a symplectic manifold then follows from proposition 4.5.2. □

Observe that for a given $K \subset U$ closed, either M_K or $\widehat{\mu_K}^{-1}(\{1\})$ may very well be empty. We are obviously only interested in closed subgroups of U such that this is not the case, and we then have the following description of the orbit space $M^{red} := \mu^{-1}(\{1\})/U$ as a disjoint union of symplectic manifolds :

Proposition 4.5.8 (Symplectic reduction of quasi-Hamiltonian spaces, the stratified case). *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian U-space. Then the orbit space $M^{red} := \mu^{-1}(\{1\})/U$ is the disjoint union, over closed subgroups $K \subset U$, of the symplectic manifolds $M_K^{red} := \widehat{\mu_K}^{-1}(\{1\})/L_K$ introduced in proposition 4.5.6 and corollary 4.5.7 :*

$$\mu^{-1}(\{1\})/U = \bigsqcup_{K \subset U} \left(\widehat{\mu_K}^{-1}(\{1\})\right)/L_K$$

Proof. This is purely set-theoretic. Let us write $K \in (U_x)$ to say that K is conjugate to (U_x) in U. Then :

$$\begin{aligned} \mu^{-1}(\{1\})/U &= \bigsqcup_{x \in \mu^{-1}(\{1\})} U.x \\ &= \bigcup_{x \in \mu^{-1}(\{1\})} \bigcup_{K \in (U_x)} \bigcup_{y \in M_K \cap \mu^{-1}(\{1\})} L_K.y \\ &= \bigcup_{K \subset U} \bigsqcup_{y \in \widehat{\mu_K}^{-1}(\{1\})} L_K.y \\ &= \bigcup_{K \subset U} \left(\widehat{\mu_K}^{-1}(\{1\})\right)/L_K \end{aligned}$$

To show that this union is disjoint, consider K, K' such that there exists $y \in \widehat{\mu_K}^{-1}(\{1\})$ satisfying $(L_K.y) \in \widehat{\mu_{K'}}^{-1}(\{1\})/L_{K'}$. Then there exists $y' \in \widehat{\mu_{K'}}^{-1}(\{1\})$ such that $L_K.y = L_{K'}.y'$. In particular, $y' \in L_K.y$, hence $U_{y'} = nU_yn^{-1}$ for some $n \in L_K$. Since $U_y = K$ and L_K normalizes K, one has $K' = U_{y'} = U_y = K$. Therefore $\widehat{\mu_{K'}} = \widehat{\mu_K}$, $L_{K'} = L_K$ and $\mu_{K'}^{-1}(\{1\})/L_{K'} = \mu_K^{-1}(\{1\})/L_K$. □

Thus, for every quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$, the reduced space $M^{red} := \mu^{-1}(\{1\})/U$ is a disjoint union of symplectic manifolds. We may then denote it by $M//U$, as in the usual Hamiltonian case :

Definition 4.5.9 (Quasi-Hamiltonian quotient). The reduced space $M//U := \mu^{-1}(\{1\})/U$ associated, by means of propositions 4.5.2 and 4.5.8, to a given quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ is called the *quasi-Hamiltonian quotient* associated to M.

Remark 4.5.10. Observe that when the action of U on M is free, then $1 \in U$ is necessarily a regular value of μ (see the proof of corollary 4.5.7) and the only subgroup $K \subset U$ such that the manifold of symmetry M_K is non-empty is $K = \{1\}$, so that the results of propositions 4.5.2 and 4.5.8 do coincide in this case, which is the nicest case one can hope for.

As we shall see in section 4.6, representation spaces naturally arise as quasi-Hamiltonian quotients. Since in this case it is known that representation spaces are stratified symplectic spaces in the sense of [LS91] (see for instance [Hue95a]), it should be possible to obtain this stratified symplectic structure in the quasi-Hamiltonian framework. Following [LS91], the first step to do so should be a normal form for momentum maps defined on quasi-Hamiltonian spaces.

4.6 Symplectic structure of the moduli space of representations of a surface group : the quasi-Hamiltonian description

In this section, we wish to explain how the notion of quasi-Hamiltonian space provides a proof of the fact that, for any Lie group $(U, (.\,|\,))$ endowed with an Ad-invariant non-degenerate product and any collection $\mathcal{C} = \{\mathcal{C}_j\}_{1 \leq j \leq l}$ of l conjugacy classes of U, there exists a symplectic structure on the representation spaces

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$$

(see 4.6.1 below for a precise definition). Here, $\pi_{g,l}$ denotes the fundamental group of the surface $\Sigma_{g,l} := \Sigma_g \backslash \{s_1, \ldots, s_l\}$, Σ_g being a compact Riemann surface of genus $g \geq 0$, l being an integer $l \geq 1$ and $s_1, \ldots, s_l$ being l pairwise distinct points of Σ_g. When $l = 0$, we set $\mathcal{C} := \emptyset$ and $\Sigma_{g,0} := \Sigma_g$. Everything we will say is valid for any $g \geq 0$ and any $l \geq 0$ but we will not always distinguish between the cases $l = 0$ and $l \geq 1$, to lighten the presentation.

Before entering the description of the symplectic structure of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$, we would like to say that we will keep this description naive and elementary : first, we will not enter considerations about the stratified structure of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$ (meaning that we choose to forget that these spaces are *not* smooth manifolds, even if U is assumed to be compact, see for instance [Wei95, Hue95a, Hue95b, Hue01b]) and second, we will not compare the symplectic structure we obtain with known symplectic structures on representation spaces (see [AB83, Gol84, Kar92, Wei95, Hue95a, Hue95b, Jef94, Jef95, HJ94, GHJW97, AM95, MW99, FR93, FR97, BF99]), which are in fact one and the same. As briefly mentioned in the introduction to this work, the description of symplectic structures on moduli spaces has a quite large history and we refer to [AMM98] for a comparison of the quasi-Hamiltonian description and the original gauge-theoretic description of [AB83]. The point of keeping this description naive and elementary is to hopefully make it easier to grasp the nice features of quasi-Hamiltonian spaces when it comes to obtaining symplectic structures on representation spaces. The upshot of the quasi-Hamiltonian description of these symplectic structures is, first, that it is fairly easy (insofar as it does not call for infinite-dimensional manifolds nor tools of group cohomology) and second, that it seems very natural (insofar as the total space M of the symplectic quotient $M//U$ entering the description is a simple object appearing naturally in the set-theoretic description of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$). Of course, all descriptions of the symplectic structure of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$ are interesting in their own right and they all have their own advantages, depending on the motivation for studying these representation spaces. Here, we felt that the quasi-Hamiltonian description was the most suited for our problem and we simply wish to explain why.

Recall that the fundamental group of the surface $\Sigma_{g,l} = \Sigma_g \backslash \{s_1, \ldots, s_l\}$ has the following finite presentation :

$$\pi_{g,l} =< A_1, \ldots, A_g, B_1, \ldots, B_g \gamma_1, \ldots, \gamma_l \mid \prod_{i=1}^{g} [A_i, B_i] \prod_{j=1}^{l} \gamma_j = 1 >$$

each γ_j being the homotopy class of a loop around the puncture s_j. In particular, if $l \geq 1$, it is a free group on $(2g+l-1)$ generators. As a consequence of this presentation, we see that giving a *representation of* $\pi_{g,l}$ *in the group* U (that is, a group morphism from $\pi_{g,l}$ to U) amounts to giving $(2g+l)$ elements $(a_i, b_i, u_j)_{1 \leq i \leq g, 1 \leq j \leq l}$ of U satisfying :

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{l} u_j = 1$$

Two representations $(a_i, b_i, u_j)_{i,j}$ and $(a'_i, b'_i, u'_j)_{i,j}$ are then called *equivalent* if there exists an element $u \in U$ such that $a'_i = u a_i u^{-1}$, $b'_i = u b_i u^{-1}$, $u'_j = u u_j u^{-1}$ for all i, j. The original approach to describing symplectic structures on spaces of representations (see [NS65, MS80, AB83]) shows that, in order to obtain *symplectic* structures, one has to prescribe the conjugacy class of each u_j, $1 \leq j \leq l$. Otherwise, one may obtain Poisson structures, but we shall not enter these considerations and refer to [Hue01a] and [AKSM02] instead. We are then led to studying the space $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ of representations of $\pi_{g,l}$ in U with prescribed conjugacy classes for the $(u_j)_{1 \leq j \leq l}$:

Definition 4.6.1. We define the space $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ to be the following set of group morphisms :

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U) = \{\rho : \pi_{g,l} \to U \mid \rho(\gamma_j) \in \mathcal{C}_j \text{ for all } j \in \{1, \ldots, l\}\}$$

Observe that this space may very well be empty, depending on the choice of the conjugacy classes $(\mathcal{C}_j)_{1 \leq j \leq l}$. As a matter of fact, conditions on the $(\mathcal{C}_j)$ for this set to be non-empty are quite difficult to obtain (see for instance [AW98] for the case $g = 0$ and $U = SU(n)$, and [TW03] for the case $g = 0$ and U compact). As earlier, giving such a morphism $\rho \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ amounts to giving appropriate elements of U :

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U) \simeq \{(a_1, \ldots, a_g, b_1, \ldots, b_g, u_1, \ldots u_l) \in \underbrace{U \times \cdots \times U}_{2g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{l} u_j = 1\}$$

In particular, two representations $(a_i, b_i, u_j)_{i,j}$ and $(a'_i, b'_i, u'_j)_{i,j}$ are equivalent if and only if they are in a same orbit of the diagonal action of U on $U \times \cdots \times U \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. The *representation space* $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ is then defined to be the quotient space for this action :

$$\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) := \mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)/U$$

Following for instance [Hue95a], the idea to obtain a symplectic structure on the representation space, or moduli space, $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ is then to see this quotient as a *symplectic quotient*, meaning that one wishes to identify $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,l}, U)$ with the fibre of a momentum map defined on an *extended moduli space* (the expression comes from [Jef94, Hue95a]). The notion of quasi-Hamiltonian space then arises naturally from the choice of

$$\underbrace{U \times \cdots \times U}_{2g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$$

as an extended moduli space, and of the map

$$\mu_{g,l}(a_1, \ldots, a_g, b_1, \ldots, b_g, u_1, \ldots, u_l) = [a_1, b_1] \ldots [a_g, b_g] u_1 \ldots u_l$$

as U-valued momentum map, so that :

$$\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) = \mu_{g,l}^{-1}(\{1\})/U$$

Actually, because of the occurence of the commutators $[a_i, b_i]$, it is more appropriate to re-arrange the arguments of the map $\mu_{g,l}$ in the following way :

$$\mu_{g,l}(a_1, b_1, \ldots, a_g, b_g, , u_1, \ldots, u_l) = [a_1, b_1]\ldots[a_g, b_g]u_1\ldots u_l = 1$$

and to write the extended moduli space :

$$\underbrace{(U \times U)\cdots\times(U \times U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$$

In the case where $g = 0$, one simply has :

$$\begin{array}{rccc} \mu_{0,l}: & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \ldots, u_l) & \longmapsto & u_1\ldots u_l \end{array}$$

When $g = 1$ and $l = 0$, one has :

$$\begin{array}{rccc} \mu_{1,0}: & U \times U & \longrightarrow & U \\ & (a, b) & \longmapsto & aba^{-1}b^{-1} \end{array}$$

These two particular cases correspond to the examples we have studied in propositions 4.4.2 and 4.4.5 and motivate the notion of quasi-Hamiltonian space. Thus, in general, the extended moduli space is the following quasi-Hamiltonian space :

$$\mathcal{M}_{g,l} := \underbrace{\mathfrak{D}(U) \times \cdots \times \mathfrak{D}(U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$$

(where $\mathfrak{D}(U)$ is the internally fused double of U of proposition 4.4.5) equipped with the diagonal U-action and the momentum map

$$\begin{array}{rccc} \mu_{g,l}: & \mathfrak{D}(U) \times \cdots \times \mathfrak{D}(U) \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (a_1, b_1, \ldots, a_g, b_g, u_1, \ldots, u_l) & \longmapsto & [a_1, b_1]\ldots[a_g, b_g]u_1\ldots u_l \end{array}$$

The representation space $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U)$ is then the associated quasi-Hamiltonian quotient (see definition 4.5.9) :

$$\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) = \mathcal{M}_{g,l}//U = (\underbrace{\mathfrak{D}(U) \times \cdots \times \mathfrak{D}(U)}_{g \text{ times}} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_l)//U$$

In particular, in the case of an l-punctured sphere ($g = 0$), which is the one we are mainly interested in in this work, we have :

$$\mathrm{Hom}_{\mathcal{C}}\big(\pi_1(S^2\backslash\{s_1, \ldots, s_l\}), U\big)/U = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_l)//U$$

For the record, we also spell out the case of torus :

$$\mathrm{Hom}\big(\pi_1(\mathbb{T}^2), U\big)/U = \mathfrak{D}(U)//U$$

(there are no conjugacy classes necessary here, as the surface $\mathbb{T}^2$ is closed) and of the punctured torus :

$$\mathrm{Hom}_{\mathcal{C}}\big(\pi_1(\mathbb{T}^2\backslash\{s\}), U\big)/U = (\mathfrak{D}(U) \times \mathcal{C})//U$$

We then know from propositions 4.5.2 and 4.5.8 that these representation spaces $\mathrm{Rep}_{\mathcal{C}}(\pi_{g,l}, U) = \mathcal{M}_{g,l}//U$ carry a symplectic structure, obtained by reduction from the quasi-Hamiltonian space $\mathcal{M}_{g,l}$. Observe that one essential ingredient to obtain this symplectic structure was the fact that $\pi_{g,l}$ admits a finite presentation with a single relation, which was used as a momentum relation. We refer to [Hue95a] for further comments on representations of groups which are not necessarily surface groups.

One sees that the choice of the quasi-Hamiltonian description of the symplectic structure of the representation space $\mathrm{Rep}_{\mathcal{C}}(\pi_1(S^2\backslash\{s_1, \ldots, s_l\}), U)$ was dictated by the very form of our problem : as we were interested (see chapter 1) in characterizing decomposable representations $(u_1, \ldots, u_l)$ of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$, which are elements of $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ satisfying $u_1 \ldots u_l = 1$, it seemed appropriate to favour a description using the space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ as an extended moduli space.

We now have all the theoretical prerequisites to prove the results announced in the introduction. In the following two chapters, we shall define and study decomposable representations of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$. Chapter 5 is elementary in nature and provides nice applications of the notion of decomposable representation. In chapter 6, we use a Lie-theoretic point of view to obtain a characterization of decomposable representations by reducing the problem to a fundamental class called σ_0-decomposable representations. These particular decomposable representations are then characterized as the elements of the fixed-point set of an involution β defined on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. As a consequence, we shall return to the general theory of quasi-Hamiltonian spaces in chapter 7. There, we will show how to obtain an anti-symplectic involution $\hat{\beta}$ on a quasi-Hamiltonian quotient $M//U$ starting from an involution β on the given quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$. In particular, we will study the case where the manifold M is a product space $M = M_1 \times M_2$, in order to apply the results obtained there to the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$.

Chapter 5

Decomposable representations of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ and configurations of Lagrangian subspaces of $\mathbb{C}^n$

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In this chapter, we define and study decomposable representations of the fundamental group of an l-punctured sphere ($l \geq 1$) :

$$\pi := \pi_1(S^2\backslash\{s_1, \ldots, s_l\})$$

This is the only surface group we will be dealing with in the rest of this work. Recall from section 4.6 that we are interested in representations of π with prescribed conjugacy classes of generators, that is, in elements of $\mathrm{Hom}_{\mathcal{C}}(\pi, U)$ (see definition 4.6.1), where $(U, (.\,|\,.))$ is an arbitrary Lie group equipped with an Ad-invariant non-degenerate product.

First, we shall describe a geometric approach to the notion of decomposable representation, using the unitary group $U = U(n)$ as a prototype. This approach relies on the notion of *Lagrangian involution* σ_L associated to a Lagrangian subspace L of $\mathbb{C}^n$ (see definition 5.1.1)

Second, we shall investigate the notion of Lagrangian involution so as to obtain an algebraic definition of decomposable representations, which will be easier to work with in chapter 6, and which is valid for an arbitrary Lie group (U, τ) equipped with an involutive automorphism.

Third, we will see that this geometric viewpoint naturally leads to studying *angles* of Lagrangian subspaces of $\mathbb{C}^n$: we will show that the orbit of a pair (L_1, L_2) of Lagrangian subspaces of $\mathbb{C}^n$ under the diagonal action of the unitary group $U(n)$ is characterized by the spectrum of the transformation $\sigma_{L_1} \circ \sigma_{L_2}$, where σ_{L_j} is the Lagrangian involution associated to L_j (see proposition 5.3.10). We shall also verify that this classification result for pairs of Lagrangian subspaces is equivalent to known results on this matter (see [Nic91] and propositions 5.3.4, 5.3.6 and 5.3.10). As they enter the definition of a decomposable representation, it is important to acquire geometric intuition on Lagrangian involutions. Indeed, in chapter 6, we will use this geometric intuition to obtain a characterization of decomposable representations.

Fourth, we shall conduct an elementary study of decomposable representations in the case where $U = U(2)$ and $l = 3$. Our main concern will be to prove the existence of such decomposable representations. This difficulty arises from the fact that we have fixed the conjugacy classes of the generators. Geometrically, this amounts to asking if there exists a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^n$ with prescribed angles (L_1, L_2), (L_2, L_3) and (L_3, L_1) (see section 5.3). To handle the case of arbitrary n and arbitrary $l \geq 1$, we will have to wait until chapter 8, where we shall prove the existence of decomposable representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ for an *arbitrary* compact connected Lie group $(U, (.\,|\,))$ and for any choice of $l \geq 1$ conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l \subset U$ such that $\mathrm{Hom}_{\mathcal{C}}(\pi, U) \neq \emptyset$. There, the existence result will be a consequence of a general real convexity theorem for Lie-group valued momentum maps (see theorems 8.3.9 and 8.3.14).

Finally, we shall give a nice application of the notion of angle between Lagrangian subspaces by showing how it provides a way of computing the inertia index of a triple of Lagrangian subspaces of $\mathbb{C}^n$. The results stated in sections 5.4 and 5.5 have been obtained in collaboration with Elisha Falbel and Jean-Pierre Marco and have been published in [FMS04].

5.1 A geometric approach to the notion of decomposable representation

Recall that a decomposition of the fundamental group $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into a given Lie group U consists of l elements $u_1, \ldots, u_l$ of U satisfying the relation $u_1 \ldots u_l = 1$. In this section, we shall use the unitary group $U = U(n)$ as a prototype to acquire geometric intuition on representations. In particular, when $n = 1$, eah $u_j = e^{i2\theta_j}$, $\theta_j \in [0, \pi[$ is a rotation of the complex plane $\mathbb{C} \simeq \mathbb{R}^2$. Consequently, it can be decomposed as a product of two orhtogonal symmetries $u = \sigma_1 \sigma_2$ with respect to real lines of $\mathbb{C} \simeq \mathbb{R}^2$ and the direct angle between these two lines is θ_j. How does this situation extend to a unitary matrix $u \in U(n)$? The appropriate orthogonal symmetries (also called reflections) to consider turn out to be what we will call *Lagrangian involutions* in the following : they are orthogonal symmetries with respect to a Lagrangian subspace of $\mathbb{C}^n$. Let us now write this down with further details.

Recall that $\mathbb{C}^n$ is endowed with the symplectic form $\omega := -\mathrm{Im}\ h$ where h is the canonical Hermitian product $h := \sum_{k=1}^n dz_k \otimes d\overline{z}_k$, for which it is symplectomorphic to $\mathbb{R}^{2n}$ endowed with the canonical symplectic form $\omega = \sum_{k=1}^n dx_k \wedge dy_k$. Multiplication by $i \in \mathbb{C}$ in $\mathbb{C}^n$ corresponds to an $\mathbb{R}$-endomorphism J of $\mathbb{R}^{2n}$ satisfying $J^2 = -Id$. Denoting by $g := \mathrm{Re}\ h = \sum_{k=1}^n (dx_k \otimes dx_k + dy_k \otimes dy_k)$ the canonical Euclidean product on $\mathbb{R}^{2n}$, we have $g = \omega(., J.)$ (J is called a *complex structure* and is said to be *compatible*

with ω). A real subspace L of $\mathbb{C}^n$ is said to be Lagrangian if $\omega|_{L\times L} = 0$ and if $\dim_{\mathbb{R}} L = n$ (that is, L is maximal isotropic with respect to ω). One may then check that L is Lagrangian if and only if its g-orthognal complement is $L^{\perp_g} = JL$. We then have the following definition :

Definition 5.1.1 (Lagrangian involution). For any Lagrangian subspace L of $\mathbb{C}^n$, the $\mathbb{R}$-linear map

$$\begin{array}{rccc} \sigma_L : & \mathbb{C}^n = L \oplus JL & \longrightarrow & \mathbb{C}^n \\ & x + Jy & \longmapsto & x - Jy \end{array}$$

is called the *Lagrangian involution associated to* L.

Observe that σ_L is *anti-holomorphic* : $\sigma_L \circ J = -J \circ \sigma_L$. In the following, we denote by $\mathcal{L}(n)$ the set of all Lagrangian subspaces of $\mathbb{C}^n$ (the Lagrangian Grassmannian of $\mathbb{C}^n$). Finally, recall that, under the identification $(\mathbb{C}^n, h) \simeq (\mathbb{R}^{2n}, J, \omega)$, we have $U(n) = O(2n) \cap Sp(n)$. As Lagrangian involutions are orthogonal symmetries which are anti-holomorphic, they are elements of $O(2n)$ which are *not* contained in $U(n)$. Furthermore, the action of $U(n)$ on $\mathcal{L}(n)$ is transitive and the stabilizer of the horizontal Lagrangian $L_0 := \mathbb{R}^n \subset \mathbb{C}^n$ is the orthogonal group $O(n) \subset U(n)$, giving the usual homogeneous description $\mathcal{L}(n) = U(n)/O(n)$ (see for instance [MS98], p.51). Observe that $O(n) = Fix(\tau)$ where $\tau : u \mapsto \overline{u}$ is complex conjugation on $U(n)$, so that $\mathcal{L}(n)$ is a compact symmetric space.

Proposition 5.1.2. *Let $L \in \mathcal{L}(n)$ be a Lagrangian subspace of $\mathbb{C}^n$. Then :*

(i) There exists a unique anti-holomorphic map σ_L whose fixed point set is exactly L.

(ii) If L' is a Lagrangian subspace such that $\sigma_L = \sigma_{L'}$, then $L = L'$: there is a one-to-one correspondence between Lagrangian subspaces and Lagrangian involutions.

(iii) σ_L is anti-unitary *: for all $z, z' \in \mathbb{C}^n$, $h(\sigma_L(z), \sigma_L(z')) = \overline{h(z, z')}$.*

(iv) For any $\varphi \in U(n)$, $\sigma_{\varphi(L)} = \varphi\sigma_L\varphi^{-1}$.

Proof. (i) Since L is Lagrangian, $\mathbb{C}^n = L \oplus JL$. Let σ be an anti-holomorphic map leaving L pointwise fixed. Let $z = x + Jy \in \mathbb{C}^n$, where $x, y \in L$. Then $\sigma(z) = \sigma(x) - J\sigma(y) = x - Jy$, so that σ_L is uniquely defined. If $z = x + Jy$ satisfies $\sigma_L(z) = z$ then $2Jy = z - \sigma_L(z) = 0$, hence $y = 0$ and $z \in L$. That is, the fixed-point set of σ_L is exactly L.

(ii) If now $\sigma_{L'} = \sigma_L$ then one has, for all $x \in L$, $\sigma_{L'}(x) = \sigma_L(x) = x$, therefore $x \in L'$ by (i) above, so that $L \subset L'$. Likewise, $L' \subset L$, hence $L = L'$.

(iii) For any $x, y, x', y' \in L$, we have

$$\begin{aligned} h(x - Jy, x' - Jy') &= h(x, x') + h(y, y') - h(x, Jy') - h(Jy, x') \\ &= g(x, x') + g(y, y') + i(g(x, y') - g(x', y)) \\ &= \overline{h(x + Jy, x' + Jy')} \end{aligned}$$

(iv) $\varphi\sigma_L\varphi^{-1}$ is anti-holomorphic and leaves $\varphi(L)$ pointwise fixed. By unicity of such a map, we then have $\varphi\sigma_L\varphi^{-1} = \sigma_{\varphi(L)}$.

□

We now recall an alternative description of the Lagrangian Grassmannian $\mathcal{L}(n)$ of $\mathbb{C}^n$. The underlying idea is that the elements of the compact symmetric space $\mathcal{L}(n) = U(n)/O(n)$ can be identified with the *symmetric* elements of $U(n)$ (that is, elements of $U(n)$ satisfying $\tau(u) = u^{-1}$, see chapter 3), all of them being of the form $\varphi^t\varphi$, where $\varphi \in U(n)$ and φ^t denotes the transpose of φ (so that the symmetric elements of $U(n)$ are indeed symmetric unitary matrices).

Proposition 5.1.3. *Let $W(n) := \{w \in U(n) \mid w^t = w\}$ be the set of symmetric unitary matrices.*

(i) Let $u \in U(n)$. Then $u \in W(n)$ if and only if there exists $k \in O(n)$ such that kuk^{-1} is diagonal. Moreover, any $w \in W(n)$ may be written $w = \exp(iS)$ where S is a real symmetric matrix, which shows in particular that $W(n)$ is connected.

(ii) If $w \in W(n)$, then there exists $\varphi \in W(n)$ such that $\varphi^2 = w$.

(iii) For any $w \in W(n)$, define $L_w := \{z \in \mathbb{C}^n \mid z - w\overline{z} = 0\}$. Then, if φ is any element in $W(n)$ such that $\varphi^2 = w$, we have $\varphi(L_0) = L_w$. Consequently, L_w is a Lagrangian subspace of $\mathbb{C}^n$. Furthermore, $\sigma_{L_w}\sigma_{L_0} = w$.

(iv) The map $w \in W(n) \mapsto L_w \in \mathcal{L}(n)$ is a diffeomorphism whose inverse is the well-defined map

$$\begin{array}{rcl} \mathcal{L}(n) = U(n)/O(n) & \longrightarrow & W(n) \\ L = u(L_0) & \longmapsto & uu^t \end{array}$$

(v) For any $L \in \mathcal{L}(n)$, we have $\sigma_{L_0}\sigma_L = v^t v$, where v is any unitary map such that $v(L) = L_0$.

Proof. (i) Observe that, alternatively, $W(n) = \{w \in U(n) \mid w^{-1} = \overline{w}\}$. Now take $w \in W(n)$ and write $w = x + iy$ where x, y are real matrices. Then $w^t = w$ implies $x^t = x$ and $y^t = y$, and $w\overline{w} = Id$ implies $x^2 + y^2 = Id$ and $xy - yx = 0$. Thus x and y are commuting real symmetric matrices, so there exists $k \in O(n)$ such that $d_x := kxk^{-1}$ and $d_y := kyk^{-1}$ are both diagonal. Therefore, $kwk^{-1} = d_x + id_y$ is diagonal. The converse is obvious. Since $d_x^2 + d_y^2 = k(x^2+y^2)k^{-1} = Id$, one has $d_x + id_y = \exp(iS)$ where S is a real symmetric (diagonal) matrix. Consequently, $w = \exp(ik^{-1}Sk)$ with $k^{-1}Sk$ real and symmetric. In particular, $W(n)$ is the continuous image of a vector space, and therefore is connected.

(ii) is an immediate consequence of (i).

(iii) Take $\varphi \in W(n) \mid \varphi^2 = w$. Then $z - w\overline{z} = 0$ iff $z - \varphi^2\overline{z} = 0$, that is, $\varphi^{-1}z - \varphi\overline{z} = 0$. But $\varphi^{-1} = \overline{\varphi}$ so that $z \in L_w$ is equivalent to $\varphi^{-1}z = \overline{\varphi^{-1}z}$ hence to $\varphi^{-1}z \in L_0$, hence to $z \in \varphi(L_0)$, which shows that $L_w = \varphi(L_0)$ is a Lagrangian subspace of $\mathbb{C}^n$. Furthermore, $\sigma_{L_w}\sigma_{L_0} = \varphi\sigma_{L_0}\varphi^{-1}\sigma_{L_0}$. But since σ_{L_0} is complex conjugation in $\mathbb{C}^n$ and since φ is both symmetric and unitary, we have $\varphi^{-1}\sigma_{L_0} = \overline{\varphi^t}\sigma_{L_0} = (\sigma_{L_0}\varphi^t\sigma_{L_0})\sigma_{L_0} = \sigma_{L_0}\varphi$, therefore $\sigma_{L_w}\sigma_{L_0} = \varphi\sigma_{L_0}^2\varphi = \varphi^2 = w$.

(iv) Observe that if u, v are two unitary maps sending L_0 to $L \in \mathcal{L}(n)$ then $v^{-1}u \in Stab(L_0) = O(n)$ so that $uu^t = vv^t$. Then, if $L = u(L_0) \in \mathcal{L}(n)$, one has $L_{uu^t} = \{z - uu^t\overline{z} = 0\}$. But $z - uu^t\overline{z} = 0$ iff $u^{-1}z = \overline{u}^{-1}\overline{z}$, that is, $u^{-1}z \in L_0$ so $L_{uu^t} = u(L_0)$. Conversely, we know that $L_w = \varphi(L_0)$ where $\varphi \in W(n) \mid \varphi^2 = w$ so that indeed $\varphi\varphi^t = \varphi^2 = w$.

(v) For a given $L \in \mathcal{L}(n)$, take $v \in U(n)$ such that $v(L) = L_0$. Then $L = v^{-1}(L_0)$ and so we know from (iii) and (iv) that $L = \{z - (v^{-1})(v^{-1})^t\overline{z} = 0\}$ and that $\sigma_L\sigma_{L_0} = v^{-1}(v^{-1})^t$. Hence $\sigma_{L_0}\sigma_L = (\sigma_L\sigma_{L_0})^{-1} = v^t v$.

□

Statement (v) may seem a bit useless at this point as it is just a way of rephrasing (ii), but it will prove useful to us when formulating the *centered Lagrangian problem* (see section 6.2). We now obtain from proposition 5.1.3 the following result, which when $n = 1$, boils down to saying that a rotation in the complex plane is a product of two reflections.

Proposition 5.1.4. *For any unitary matrix $u \in U(n)$, there exist two Lagrangian subspaces $L_1, L_2 \in \mathcal{L}(n)$ such that $u = \sigma_{L_1}\sigma_{L_2}$.*

Proof. Let $d = \operatorname{diag}(\alpha_1, \ldots, \alpha_l) \in U(n)$ be a diagonal matrix such that $u = \varphi d^2\varphi^{-1}$ for some $\varphi \in U(n)$, and set $L := d(L_0)$. Then we know from statement (iii) of proposition 5.1.3 that $\sigma_L\sigma_{L_0} = d^2$, hence $u = \varphi\sigma_L\sigma_{L_0}\varphi^{-1} = \sigma_{\varphi(L)}\sigma_{\varphi(L_0)}$. □

Thus, any unitary matrix $u \in U(n)$ is a product $u = \sigma_1\sigma_2$ of two orthogonal symmetries with respect to Lagrangian subspaces of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. These symmetries are no longer unitary transformations (they are anti-holomorphic) but are elements of order 2 in the orthogonal group $O(2n)$. Consequently, *one may notice* that if $u_1 = \sigma_1\sigma_2$, $u_2 = \sigma_2\sigma_3$, ... and $u_l = \sigma_l\sigma_1$, then one automatically has :

$$\begin{aligned} u_1 \ldots u_l &= (\sigma_1\sigma_2)(\sigma_2\sigma_3)\ldots(\sigma_l\sigma_1) \\ &= \sigma_1(\sigma_2\sigma_2)\ldots(\sigma_3\sigma_3)\sigma_1 \\ &= 1 \end{aligned}$$

That is : giving l Lagrangian subspaces $L_1, \ldots, L_l$ of $\mathbb{C}^n$ automatically furnishes a representation of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$. We call such a representation a *Lagrangian representation*. The natural question to ask is then the following one : when is a given representation a Lagrangian one ? We will give a precise answer to this in this work (see corollary 6.6.5). Of particular interest will be the Lagrangian representations $(u_1 = \sigma_1\sigma_2, \ldots, u_l = \sigma_l\sigma_1)$ where $\sigma_1 = \sigma_0$ is the Lagrangian involution with respect to the horizontal Lagrangian $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$. We will call this particular class of Lagrangian representations the class of σ_0-Lagrangian representations. Recall finally that we are interested in representations of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ with prescribed conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l \subset U(n)$ of generators, that is, in elements of

$$\mathrm{Hom}_{\mathcal{C}}\big(\pi, U(n)\big) = \{(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \ldots u_l = 1\}$$

(see definition 4.6.1). Since Lagrangian representations are those for which generators decompose in a good way as products of Lagrangian involutions, we shall also call them decomposable representations.

Definition 5.1.5 (Decomposable representations of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ into $U = U(n)$). A given unitary representation $(u_1, \ldots, u_l) \in \mathrm{Hom}_{\mathcal{C}}(\pi, U(n))$ is said to be *Lagrangian* (or *decomposable*) if there exist l Lagrangian subspaces $L_1, \ldots, L_l$ of $\mathbb{C}^n$ such that $u_j = \sigma_j\sigma_{j+1}$ for all $j \in \{1, \ldots, l\}$, where σ_j is the Lagrangian involution associated to L_j and where $\sigma_{l+1} = \sigma_1$.
A Lagrangian representation is said to be σ_0-*Lagrangian* (or σ_0-*decomposable*) if in addition $\sigma_1 = \sigma_0$ (the Lagrangian involution associated to the horizontal Lagrangian $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$).

Observe that, as we have shown that any unitary matrix $u \in U(n)$ admits a decomposition $u = \sigma_1\sigma_2$ as a product of two reflections, it is also natural to ask, in analogy with the $n = 1$ case, if this defines a notion of angle between the Lagrangian subspaces L_1 and L_2. We postpone work on this question until section 5.3. For now, we wish to work out additional properties of Lagrangian involutions in order to be able to generalize the notion of decomposable representation to Lie groups other than $U(n)$.

5.2 An algebraic definition of decomposable representations

Denote by $\mathcal{LI}nv(n) := \{\sigma_L \ : \ L \in \mathcal{L}(n)\}$ the subset of $O(2n)$ consisting of Lagrangian involutions. Observe that it is not a subgroup, as it is not stable by composition of maps. Statement (iv) of proposition 5.1.2 then shows that the subgroup $\widehat{U(n)} :=< U(n) \cup \mathcal{LI}nv(n) > \subset O(2n)$ generated by Lagrangian involutions and unitary transformations is in fact generated by $U(n)$ and σ_{L_0} : $\widehat{U(n)} =< U(n) \cup \{\sigma_{L_0}\} >$. As a word in $< U(n) \cup \{\sigma_{L_0}\} >$ contains either an even or an odd number of occurrences of σ_{L_0} (depending only on whether it represents a holomorphic or an anti-holomorphic transformation of $(\mathbb{R}^{2n}, J) \simeq \mathbb{C}^n$), it can be written uniquely under the reduced form $u\varepsilon$ where $u \in U(n)$ and $\varepsilon = 1$ or $\varepsilon = \sigma_{L_0}$. Consequently, we have $< U(n) \cup \{\sigma_{L_0}\} >= U(n) \sqcup U(n)\sigma_{L_0}$, so that $U(n)$ is a subgroup of index 2 of $\widehat{U(n)}$. Further, if we write $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma_{L_0}\}$ and consider the action of this group on $U(n)$ given by $\sigma_{L_0}.u := \sigma_{L_0}u\sigma_{L_0} = \overline{u} = \tau(u)$, then the map

$$\begin{array}{ccc} U(n) \rtimes \mathbb{Z}/2\mathbb{Z} & \longrightarrow & U(n) \sqcup U(n)\sigma_{L_0} \\ (u, \varepsilon) & \longmapsto & u\varepsilon \end{array}$$

(where $\varepsilon = 1$ or $\varepsilon = \sigma_{L_0}$) is a group isomorphism. Finite subgroups of $U(2) \rtimes \mathbb{Z}/2\mathbb{Z}$ generated by Lagrangian involutions are studied in [Fal01] and [FP04]. If one uses the description of $\widehat{U(n)}$ as a

semi-direct product given above, the notion of Lagrangian representation makes sense even when the compact connected Lie group U at hand is not necessarily the unitary group $U(n)$. Indeed, endow such a Lie group U with an involutive automorphism τ and define an action of $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma_0\}$ on U by $\sigma_0.u := \tau(u)$. We then consider the semi-direct product $U \rtimes \mathbb{Z}/2\mathbb{Z}$ for this action. When $U = U(n)$ and $\tau(u) = \overline{u}$ then we have indeed $U(n) \rtimes \mathbb{Z}/2\mathbb{Z} = \widehat{U(n)} = U(n) \sqcup U(n)\sigma_{L_0}$. Under this identification, $\sigma_0.u = \tau(u) = \overline{u} = \sigma_{L_0} u \sigma_{L_0}$ and the Lagrangian involutions are the elements $\sigma_L = \sigma_{\varphi(L_0)} = \varphi\sigma_{L_0}\varphi^{-1} = (\varphi\sigma_{L_0}\varphi^{-1}\sigma_{L_0})\sigma_{L_0} = (\varphi\overline{\varphi}^{-1})\sigma_{L_0} = (\varphi\varphi^t)\sigma_{L_0} \leftrightarrow (\varphi\varphi^t, \sigma_{L_0}) \in U(n) \rtimes \mathbb{Z}/2\mathbb{Z}$. Observe that the element $\varphi\varphi^t$ does not depend on the choice of $\varphi \in U(n)$ such that $L = \varphi(L_0)$, as was shown in proposition 5.1.3. If $Fix(\tau^-)$ is connected (see remarks 5.2.3 and 7.4.2 for comments on this assumption), any *symmetric* element of U (that is, an element w of U satisfying $\tau(w) = w^{-1}$) can be written $u\tau(u^{-1})$ for some $u \in U$ (see proposition 3.1.2), and the elements of order 2 that we are interested in are the elements $(w, \sigma_0) \in U \rtimes \mathbb{Z}/2\mathbb{Z}$ where $w \in U$ satisfies $\tau(w) = w^{-1}$ (in particular, we see again that these elements of order 2 are in one-to-one correspondence with the symmetric elements of U). The product of two such elements is then of the form $(w_1, \sigma_0).(w_2, \sigma_0) = (w_1(\sigma_0.w_2), \sigma_0^2) = (w_1\tau(w_2), 1) \in U \subset U \rtimes \mathbb{Z}/2\mathbb{Z}$ (observe that when $w_2 = w_1$, we indeed obtain 1 because $\tau(w_1) = w_1^{-1}$). One can then say that a U-representation $(u_1, \ldots, u_l)$ of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ is *decomposable* (or *Lagrangian*) if there exist $w_1, \ldots, w_l \in U$ such that $\tau(w_j) = w_j^{-1}$ for all j and $u_1 = (w_1, \sigma_0).(w_2, \sigma_0), u_2 = (w_2, \sigma_0).(w_3, \sigma_0), \ldots, u_l = (w_l, \sigma_0).(w_1, \sigma_0)$. Observe that we then have indeed $u_1 \ldots u_l = 1$, for $u_1 \ldots u_l = (w_1\tau(w_2), 1).(w_2\tau(w_3), 1) \ldots (w_l\tau(w_1), 1) = (w_1\tau(w_2)w_2\tau(w_3) \ldots w_l\tau(w_1), 1) = 1$ since $\tau(w_j) = w_j^{-1}$. A representation will be called *σ_0-decomposable* if it is decomposable with $w_1 = Id$. Observe further that by definition of the semi-direct product we have in fact : $(w_j, \sigma_0).(w_{j+1}, \sigma_0) = w_j\sigma_0 w_{j+1}\sigma_0 = w_j\tau(w_{j+1}) = w_j w_{j+1}$ in U. We can now define decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into an arbitrary Lie group (U, τ) equipped with an involution.

Definition 5.2.1 (Decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$). Let (U, τ) be a Lie group endowed with an involutive automorphism τ. A representation $(u_1, \ldots, u_l)$ of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into U is called *decomposable* if there exist l elements $w_1, \ldots, w_l \in U$ satisfying :

(i) $\tau(w_j) = w_j^{-1}$ for all j (each w_j is a *symmetric* element of U with respect to τ).

(ii) $u_1 = w_1 w_2^{-1}$, $u_2 = w_2 w_3^{-1}$, $\ldots$, $u_l = w_l w_1^{-1}$.

A representation will be called *σ_0-decomposable* if it is decomposable with $w_1 = 1$.

In particular, for σ_0-decomposable representations, the elements u_1 and u_l are themselves symmetric. We then observe the following fact :

Proposition 5.2.2. *Let $(u_1, \ldots, u_l)$ be a representation of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into U. Then $(u_1, \ldots, u_l)$ is decomposable if and only if the representation $\varphi.(u_1, \ldots, u_l) := (\varphi u_1 \varphi^{-1}, \ldots, \varphi u_l \varphi^{-1})$ is decomposable for any $\varphi \in U$.*

Proof. If $(u_1, \ldots, u_l)$ is decomposable, then there exist $w_1, \ldots, w_l \in Fix(\tau^-)$ satisfying $u_j = w_j w_{j+1}^{-1}$ for all j (with $w_{l+1} = w_1$). Therefore :

$$\begin{aligned} \varphi u_j \varphi^{-1} &= \varphi w_j w_{j+1}^{-1} \varphi^{-1} \\ &= \varphi w_j \tau^-(\varphi)\tau(\varphi) w_{j+1}^{-1} \varphi^{-1} \\ &= \big(\varphi w_j \tau^-(\varphi)\big)\big(\varphi w_{j+1}\tau^-(\varphi)\big)^{-1} \end{aligned}$$

and each $\varphi w_j \tau^-(\varphi)$ lies in $Fix(\tau^-)$, so that the representation $\varphi.(u_1, \ldots, u_l)$ is indeed decomposable. The converse implication is obvious. □

Observe that this result is clear in the case where $U = U(n)$ since one has, by proposition 5.1.2 :

$$\varphi\sigma_{L_j}\sigma_{L_{j+1}}\varphi^{-1} = \varphi\sigma_{L_j}\varphi\varphi^{-1}\sigma_{L_{j+1}}\varphi^{-1} = \sigma_{\varphi(L_j)}\sigma_{\varphi(L_{j+1})}$$

Remark 5.2.3. The above definition rests on the fact that $Fix(\tau^-)$ is assumed to be connected. If this is not the case, condition (i) is to be replaced by the condition $w_j \in \{\tau^-(u)u \,:\, u \in U\}$, which is the connected component of 1 in $Fix(\tau^-)$ (see proposition 3.1.2). We refer to remark 7.4.2 for further comments on this assumption.

5.3 Pairs of Lagrangian subspaces

As mentioned at the end of section 5.1, introducing orthogonal symmetries with respect to Lagrangian subspaces of $\mathbb{C}^n$ leads to studying angles between two such subspaces. When $n = 1$ and $e^{i2\theta} = \sigma_{L_1}\sigma_{L_2}$, the direct angle between the lines L_1 and L_2 is $\theta \in [0, \pi[$. This means that given two pairs (L_1, L_2) and (L'_1, L'_2) of real lines of $\mathbb{C}$, there exists a unitary map φ sending L_1 to L'_1 and L_2 to L'_2 if and only if $\sigma_{L_1}\sigma_{L_2} = \sigma_{L'_1}\sigma_{L'_2}$. We will now see how this situation extends to the case of an arbitrary n by studying the diagonal action of $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n)$.

Recall that the unitary group $U(n)$ acts transitively on the Lagrangian Grassmannian $\mathcal{L}(n)$. Fixing a Lagrangian L in $\mathcal{L}(n)$, its stabilizer can be identified to $O(n)$, and $\mathcal{L}(n)$ is therefore a compact homogenous space diffeomorphic to $U(n)/O(n)$. We shall here be concerned with the diagonal action of $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n)$. Observe that requiring $\psi(L)$ to be Lagrangian when L is Lagrangian and $\psi \in O(2n)$ is equivalent to requiring that ψ be unitary (since $L \oplus JL = \mathbb{C}^n$, a g-orthogonal basis B of L over $\mathbb{R}$ is a unitary basis of $\mathbb{C}^n$ over $\mathbb{C}$, and if L is Lagrangian and ψ orthogonal with $\psi(L)$ Lagrangian, then $\psi(B)$ is also a unitary basis, so that ψ is a unitary map). Equivalently, the orbit of a pair (L_1, L_2) of Lagrangian subspaces under the diagonal action of $U(n)$ is the intersection with $\mathcal{L}(n) \times \mathcal{L}(n)$ of the orbit of (L_1, L_2) under the diagonal action of $O(2n)$. The orbit $[L_1, L_2]$ of the pair (L_1, L_2) under the diagonal action of $U(n)$ may therefore be called the *Lagrangian angle* formed by L_1 and L_2. In the following, we shall simply speak of the angle (L_1, L_2) to designate the orbit $[L_1, L_2]$. We now wish to find complete numerical invariants for this action : to each angle (L_1, L_2) we shall associate a *measure*, denoted by $\mathrm{meas}(L_1, L_2)$, in a way that two pairs (L_1, L_2) and (L'_1, L'_2) lie in a same orbit of the action of $U(n)$ if and only if $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$. This can be done in three equivalent ways, which we shall describe and compare (see propositions 5.3.4, 5.3.6 and 5.3.10).

5.3.1 Projective properties of Lagrangian subspaces of $\mathbb{C}^n$

A real subspace W of $\mathbb{C}^n$ is said to be *totally real* if $h(u, v) \in \mathbb{R}$ for all $u, v \in W$. Therefore, a real subspace L of V is Lagrangian if and only if it is totally real and of maximal dimension with respect to this property. Let p be the projection $p : \mathbb{C}^n\backslash\{0\} \to \mathbb{C}\mathrm{P}^{n-1}$ on the $(n-1)$-dimensional complex projective space, and for any real subspace W of $\mathbb{C}^n$, let $p(W)$ be the image of $W\backslash\{0\}$. When L is a Lagrangian subspace of $\mathbb{C}^n$, recall that we denote by σ_L the only anti-holomorphic involution of $\mathbb{C}^n$ leaving L pointwise fixed (called the Lagrangian involution associated to L, see definition 5.1.1). The map σ_L being anti-holomorphic, it induces a map

$$\begin{array}{rccc} \widehat{\sigma_L} : & \mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}^{n-1} \\ & [z] & \longmapsto & [\sigma_L(z)] \end{array}$$

Further, σ_L is anti-unitary so that, if we endow $\mathbb{C}\mathrm{P}^{n-1}$ with the Fubiny-Study metric (see for instance [ABK+94] p.40 or [Kli82] pp.106-108), $\widehat{\sigma_L}$ is an isometry, and $p(L)$ is the fixed point set of this isometry. Therefore, for any Lagrangian L of $\mathbb{C}^n$, the subspace $l := p(L)$ of $\mathbb{C}\mathrm{P}^{n-1}$, called a *projective Lagrangian*, is a totally geodesic embedded submanifold of $\mathbb{C}\mathrm{P}^{n-1}$ (see for instance [Kli82] p.94). More generally, every totally real subspace W of $\mathbb{C}^n$ is sent by p to a closed embedded submanifold of $\mathbb{C}\mathrm{P}^{n-1}$ which is diffeomorphic to $\mathbb{R}\mathrm{P}(W)$ (see [Nic91], p.73). These projective properties can be used to prove the first diagonalization lemma (proposition 5.3.3), as shown in [Nic91]. They will also be important to us in the study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$.

5.3.2 First diagonalization lemma and unitary classification of Lagrangian pairs

We state here the results obtained by Nicas in [Nic91]. Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^n$ and let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be orthonormal bases of L_1 and L_2 respectively. Let A be the $n \times n$ complex matrix with coefficients $A_{ij} = h(v_j, u_i)$. Observe that A is the matrix of a unitary transformation sending L_1 to L_2.

Definition 5.3.1 (Souriau matrix, [Nic91]). The matrix AA^t, where A^t is the transpose of A, is called the *Souriau matrix* of the pair (L_1, L_2) with respect to the bases B_1 and B_2.

The matrix AA^t is both unitary and symmetric. If $B_1' = (u_1', \ldots, u_n')$ and $B_2' = (v_1', \ldots, v_n')$ are other orthonormal bases of L_1 and L_2 respectively, and $A'A'^t$ is the corresponding Souriau matrix where $A'_{ij} = h(v_j', u_i')$, let P and Q be the matrices with coefficients $P_{ij} = h(u_i, u_j')$ and $Q_{ij} = h(v_i, v_j')$. Since L_1 and L_2 are Lagrangian, P and Q are real orthogonal matrices. Furthermore $A = PA'Q$, hence $AA^t = PA'QQ^t(A')^tP^t = P(A'A'^t)P^t$. Thus AA^t is conjugate to $A'A'^t$. It follows that the characteristic polynomial of a Souriau matrix of the pair (L_1, L_2) is independent of the choice of the orthonormal bases B_1 and B_2.

Definition 5.3.2 ([Nic91]). The *characteristic polynomial* of the pair (L_1, L_2), denoted by $P(L_1, L_2)$, is by definition the characteristic polynomial of any Souriau matrix of the pair (L_1, L_2).

In particular, $P(L_1, L_2)$ is a monic complex polynomial of degree n, and since a Souriau matrix is unitary, the roots of $P(L_1, L_2)$ lie in the unit circle of $\mathbb{C}$.

Proposition 5.3.3 (First diagonalization lemma, [Nic91]). *Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^n$. Then there exists an orthonormal basis $(u_1, \ldots, u_n)$ of L_1 and unit complex numbers $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ such that $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ is an orthonormal basis of L_2. Furthermore, the squares $e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}$ of these numbers are the roots of the characteristic polynomial of the pair (L_1, L_2), counted with their multiplicities.*

We refer to [Nic91] for a proof of this result exploiting the positivity of the sectional curvature of the complex projective space endowed with the Fubiny-Study metric. The name given to this result is justified by the fact that the Souriau matrix of the pair (L_1, L_2) with respect to the bases provided by the lemma is the diagonal matrix $\mathrm{diag}(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$.

Proposition 5.3.4 (Unitary classification of Lagrangian pairs of $\mathbb{C}^n$, [Nic91]). *Let (L_1, L_2) and (L_1', L_2') be two pairs of Lagrangian subspaces of $\mathbb{C}^n$. Then, there exists a unitary map $\psi \in U(n)$ such that $\psi(L_1) = L_1'$ and $\psi(L_2) = L_2'$ if and only if the characteristic polynomials $P(L_1, L_2)$ and $P(L_1', L_2')$ are equal.*

Proof. If such a ψ exists, let $(u_1, \ldots, u_n)$ be any orthonormal basis of L_1 and let $(v_1, \ldots, v_n)$ be any orthonormal basis of L_2. Since the map ψ is unitary, $(\psi(u_1), \ldots, \psi(u_n))$ is an orthonormal basis of L_1' and $(\psi(v_1), \ldots, \psi(v_n))$ is an orthonormal basis of L_2', and we have $h(\psi(v_j), \psi(u_j)) = h(u_j, v_j)$. Therefore, the Souriau matrices of (L_1, L_2) and (L_1', L_2') in the above bases are equal, so that $P(L_1, L_2)$ and $P(L_1', L_2')$ have the same roots with the same multiplicities. Since both these polynomials are monic, we then have $P(L_1, L_2) = P(L_1', L_2')$.
Conversely, suppose that $P(L_1, L_2) = P(L_1', L_2')$ and let $\alpha_1^2, \ldots, \alpha_n^2$ be the roots of this polynomial counted with their multiplicities. By the first diagonalization lemma 5.3.3, there exists an orthonormal basis $(u_1, \ldots, u_n)$ of L_1 and an orthonormal basis $(u_1', \ldots, u_n')$ of L_1' such that $(\alpha_1 u_1, \ldots, \alpha_n u_n)$ is an orthonormal basis of L_2 and $(\alpha_1 u_1', \ldots, \alpha_n u_n')$ is an orthonormal basis of L_2'. L_1 and L_1' being Lagrangian, $(u_1, \ldots, u_n)$ and $(u_1', \ldots, u_n')$ are unitary bases of $\mathbb{C}^n$ over $\mathbb{C}$. Let ψ be the $\mathbb{C}$-linear map defined by sending u_k to v_k. Then ψ is unitary and sends $\alpha_k u_k$ to $\alpha_k u_k'$. Therefore $\psi(L_1) = L_1'$ and $\psi(L_2) = L_2'$. $\square$

5.3.3 Second diagonalization lemma

It is possible to express the result of the first diagonalization lemma in terms of unitary maps sending L_1 to L_2, in a way that generalizes the situation of real lines in $\mathbb{C}$.

Proposition 5.3.5 (Second diagonalization lemma). *Given two Lagrangian subspaces of L_1 and L_2 of $\mathbb{C}^n$, there exists a unique unitary map $\varphi_{12} \in U(n)$ sending L_1 to L_2 and verifying the following diagonalization conditions :*

(i) the eigenvalues of φ_{12} are unit complex numbers $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ satisfying $\pi > \lambda_1 \geq \ldots \geq \lambda_n \geq 0$

(ii) there exists an orthonormal basis $(u_1, \ldots, u_n)$ of L_1 such that u_k is an eigenvector of φ_{12} (with eigenvalue $e^{i\lambda_k}$) .

Proof. The existence is a direct consequence of the first diagonalization lemma. As for unicity, observe that two such unitary maps have the same eigenspaces and the same corresponding eigenvalues (see subsection 5.3.6 to fully understand this : the eigenspaces of φ_{12} as a map from L_1 to L_2 are exactly the W_k of proposition 5.3.12), and are therefore equal. □

It is also possible to give a direct proof of this result, which then proves the first diagonalization lemma 5.3.3 without making use of projective geometry (see [Arn67] or [LMS03]). Observe that condition (i) is essential for the unicity part: for any Lagrangian L, the two maps J and $-J$ are both unitary, they both send L to $JL = (-J)L$ and satisfy condition (ii) for any orthonormal basis of L, but J is the only one of these two maps whose eigenvalues are located in the upper half of the unit circle of $\mathbb{C}$.

Observe that the Souriau matrix of the pair (L_1, L_2) with respect to the bases $(u_1, \ldots, u_n)$ and $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ is the diagonal matrix $\mathrm{diag}(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$. Therefore, the roots of the characteristic polynomial $P(L_1, L_2)$ are the squares of the eigenvalues of φ_{12}.

Finally, observe that if (L_1, L_2) and (L_1', L_2') lie in a same orbit of the diagonal action of $U(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n)$, then the two associated unitary maps φ_{12} and φ_{12}' are conjugate in $U(n)$. Indeed, if $\psi(L_1) = L_1'$ and $\psi(L_2) = L_2'$ with $\psi \in U(n)$, then $\psi \circ \varphi_{12} \circ \psi^{-1}$ sends L_1' to L_2' and satisfies the conditions of the second diagonalization lemma 5.3.5, hence by unicity of such a map : $\psi \circ \varphi_{12} \circ \psi^{-1} = \varphi_{12}'$. The unitary maps φ_{12} will be very useful in the study of the diagonal action of $U(2)$ on triples of Lagrangian subspaces of $\mathbb{C}^2$ (see section 5.4). For now, we can already use them to reformulate the classification result for pairs of Lagrangian subspaces (proposition 5.3.4) :

Proposition 5.3.6. *Let (L_1, L_2) and (L_1', L_2') be two pairs of Lagrangian subspaces of $\mathbb{C}^n$, and let φ_{12} (resp. φ_{12}') be the only unitary map sending L_1 (resp. L_1') to L_2 (resp. L_2') and satisfying the conditions of the second diagonalization lemma 5.3.5. Let $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$, where $\pi > \lambda_1 \geq \ldots \geq \lambda_n \geq 0$, be the eigenvalues of φ_{12} counted with their multiplicities, and let $e^{i\lambda_1'}, \ldots, e^{i\lambda_n'}$, where $\pi > \lambda_1' \geq \ldots \geq \lambda_n' \geq 0$, be the eigenvalues of φ_{12}' counted with their multiplicities.*
Then, there exists a unitary map $\psi \in U(n)$ such that $\psi(L_1) = L_1'$ and $\psi(L_2) = L_2'$ if and only if $\lambda_k = \lambda_k'$ for $k = 1, \ldots, n$.

Proof. Let $(u_1, \ldots, u_n)$ (resp. $(u_1', \ldots, u_n')$) be an orthonormal basis of L_1 (resp. L_1') formed by eigenvectors of φ_{12} (resp. φ_{12}'). Then $e^{i\lambda_k}u_k = \varphi_{12}(u_k) \in L_2$ and $e^{i\lambda_k'}u_k' = \varphi_{12}'(u'k) \in L_2'$. The maps φ_{12} and φ_{12}' being unitary, $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ is an orthonormal basis of L_2 and $(e^{i\lambda_1'}u_1', \ldots, e^{i\lambda_n'}u_n')$ is an orthonormal basis of L_2'. The Souriau matrix of the pair (L_1, L_2) (resp. (L_1', L_2')) in the bases $(u_1, \ldots, u_n)$ and $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ (resp. $(u_1', \ldots, u_n')$ and $(e^{i\lambda_1'}u_1', \ldots, e^{i\lambda_n'}u_n')$) is $\mathrm{diag}(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$ (resp. $\mathrm{diag}(e^{i2\lambda_1'}, \ldots, e^{i2\lambda_n'})$). Therefore, by proposition 5.3.4, (L_1, L_2) and (L_1', L_2') lie in a same orbit of the action of $U(n)$ if and only if the eigenvalues $e^{i2\lambda_k}$ and $e^{i2\lambda_k'}$ are the same up to permutation, and since we forced $\pi > \lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $\pi > \lambda_1' \geq \ldots \geq \lambda_n' \geq 0$, this is equivalent to $e^{i2\lambda_k} = e^{i2\lambda_k'}$ for all k. Since $\lambda_k, \lambda_k' \in [0, \pi[$, this last condition is equivalent to $\lambda_k = \lambda_k'$ for all k. □

Observe that this point of view indeed generalizes the classification result for pairs of real lines in $\mathbb{C} \simeq \mathbb{R}^2$ under the diagonal action of the unitary group $U(1) \simeq SO(2)$, λ_1 being in that case the measure of the oriented Euclidean angle between the two real lines L_1 and L_2.

5.3.4 Lagrangian involutions and angles of Lagrangian subspaces

The following result establishes a relation between Lagrangian involutions and angles of Lagrangian subspaces.

Proposition 5.3.7. *Let L_1 and L_2 be two Lagrangian subspaces of $\mathbb{C}^n$. The eigenvalues of the unitary map $\sigma_{L_2} \circ \sigma_{L_1}$ are the roots of the characteristic polynomial $P(L_1, L_2)$ of the pair (L_1, L_2), with the same multiplicity. Equivalently, since $P(L_1, L_2)$ is monic, it is the characteristic polynomial of the map $\sigma_{L_2} \circ \sigma_{L_1}$.*

Proof. By the first diagonalization lemma 5.3.3, there exists an orthonormal basis $(u_1, \ldots, u_n)$ of L_1 and unit complex numbers $\alpha_1, \ldots, \alpha_n \in S^1$ such that $(\alpha_1 u_1, \ldots, \alpha_n u_n)$ is an orthonormal basis of L_2 and $\alpha_1^2, \ldots, \alpha_n^2$ are the roots of $P(L_1, L_2)$, counted with their multiplicities. Let ψ be the unitary map sending u_k to $\alpha_k u_k$ for $k = 1, \ldots, n$. Then ψ sends L_1 to L_2 and $\alpha_1^2, \ldots, \alpha_n^2$ are the eigenvalues of ψ^2, counted with their multiplicities, and it is therefore sufficient to prove that $\sigma_{L_2} \circ \sigma_{L_1} = \psi^2$. The map $\psi \circ \sigma_{L_1} \circ \psi^{-1}$ is anti-holomorphic and leaves L_2 pointwise fixed, hence $\sigma_{L_2} = \psi \circ \sigma_{L_1} \circ \psi^{-1}$. Furthermore, for all $j = 1, \ldots, n$, we have $\sigma_{L_1} \circ \psi^{-1}(u_j) = \sigma_{L_1}(\frac{1}{\alpha_j} u_j) = \alpha_j \sigma_{L_1}(u_j) = \alpha_j u_j = \psi(u_j) = \psi \circ \sigma_{L_1}(u_j)$, so that $\sigma_{L_2} = \psi \circ \sigma_{L_1} \circ \psi^{-1} = \sigma_{L_1} \circ (\psi^{-1})^2$, hence $\sigma_{L_2} \circ \sigma_{L_1} = \psi^2$, which is what we needed. □

In particular, setting $\psi = \varphi_{12}$ in the above proof, we obtain the following corollary :

Corollary 5.3.8. *Let φ_{12} be the only unitary map sending L_1 to L_2 and satisfying the conditions of proposition 5.3.5. Then $\varphi_{12}^2 = \sigma_{L_2} \circ \sigma_{L_1}$.*

5.3.5 Measure of a Lagrangian angle

In order to reformulate one more time the classification result of propositions 5.3.4 and 5.3.6, we introduce the following notion :

Definition 5.3.9 (Measure of a Lagrangian angle). Let L_1 and L_2 be two Lagrangians of $\mathbb{C}^n$ and let $e^{i\theta_1}, \ldots, e^{i\theta_n}$ be the eigenvalues of the unitary map $\sigma_{L_2} \circ \sigma_{L_1}$, counted with their multiplicities. The symmetric group $\mathfrak{S}_n$ acts on $S^1 \times \cdots \times S^1$ by permuting the elements of the n-tuples of unit complex numbers, and we denote by $[e^{i\theta_1}, \ldots, e^{i\theta_n}]$ the equivalence class of $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in S^1 \times \cdots \times S^1$, and call it the *measure* of the angle formed by L_1 and L_2 :

$$\mathrm{meas}(L_1, L_2) = [e^{i\theta_1}, \ldots, e^{i\theta_n}] \in (S^1 \times \cdots \times S^1)/\mathfrak{S}_n$$

As $\sigma_{\psi(L)} = \psi \circ \sigma_L \circ \psi^{-1}$ for any unitary map $\psi \in U(n)$, we have $\mathrm{meas}(\psi(L_1), \psi(L_2)) = \mathrm{meas}(L_1, L_2)$, so this notion is well-defined. This definition of a measure does not extend the usual one (in the case $n = 1$, we obtain $e^{i2\lambda}$, where $\lambda \in [0, \pi[$ is the usual measure). It will nonetheless prove relevant.

Observe that, since $\sigma_{L_1} \circ \sigma_{L_2} = (\sigma_{L_2} \circ \sigma_{L_1})^{-1}$, it would be equivalent to define $\mathrm{meas}(L_1, L_2)$ to be the eigenvalues of the unitary map $\sigma_{L_1} \circ \sigma_{L_2}$, counted with their multiplicities. As a consequence, if $\mathrm{meas}(L_1, L_2) = [e^{i\theta_1}, \ldots, e^{i\theta_n}]$ then $\mathrm{meas}(L_2, L_1) = [e^{-i\theta_1}, \ldots, e^{-i\theta_n}]$. In particular, $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L_1', L_2')$ if and only if $\mathrm{meas}(L_2, L_1) = \mathrm{meas}(L_2', L_1')$.

In the following, we shall identify $S^1 \times \cdots \times S^1$ with the n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, to which it is homeomorphic. The measure of the angle (L_1, L_2) will be denoted by $\mathrm{meas}(L_1, L_2) = [e^{i\theta_1}, \ldots, e^{i\theta_n}] \in \mathbb{T}^n/\mathfrak{S}_n$. In view of proposition 5.3.7 above, we may now reformulate the classification result for Lagrangian pairs of proposition 5.3.4 in the following way :

Proposition 5.3.10. *Given two pairs of Lagrangian subspaces (L_1, L_2) and (L_1', L_2') of Lagrangian subspaces of $\mathbb{C}^n$, there exists a unitary map $\psi \in U(n)$ such that $\psi(L_1) = L_1'$ and $\psi(L_2) = L_2'$ if and only if $\sigma_{L_1} \circ \sigma_{L_2}$ is conjugate to $\sigma_{L_1'} \circ \sigma_{L_2'}$. Equivalently, the map*

$$\begin{array}{rccc} \chi: & (\mathcal{L}(n) \times \mathcal{L}(n))/U(n) & \longrightarrow & \mathbb{T}^n/\mathfrak{S}_n \\ & [L_1, L_2] & \longmapsto & \mathrm{meas}(L_1, L_2) \end{array}$$

is one-to-one.

The map χ is in fact a bijection : given $[e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}] \in \mathbb{T}^n/\mathfrak{S}_n$, consider any Lagrangian $L_1 \in \mathcal{L}(n)$, $(u_1, \ldots, u_n)$ an orthonormal basis of L_1 and let L_2 be the real subspace of $\mathbb{C}^n$ generated by $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$. Since $(e^{i\lambda_1}u_1, \ldots, e^{i\lambda_n}u_n)$ is a unitary basis of $\mathbb{C}^n$ over $\mathbb{C}$, L_2 is Lagrangian and $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_1}, \ldots, e^{i2\lambda_n}]$.

Corollary 5.3.11. *The angle space $(\mathcal{L}(n) \times \mathcal{L}(n))/U(n)$, endowed with the quotient topology, is homeomorphic to the quotient space $\mathbb{T}^n/\mathfrak{S}_n$, both being Hausdorff and compact.*

As a final remark, observe that the corresponding symplectic problem admits a simple answer : a necessary and sufficient condition for the existence of a symplectic map $\psi \in Sp(n)$ such that $\psi(L_1) = L_1'$ and $\psi(L_2) = L_2'$ is that $\dim(L_1 \cap L_2) = \dim(L_1' \cap L_2')$ that is, the measure of the symplectic angle formed by two Lagrangian subspaces of $\mathbb{C}^n$ simply is the dimension of their intersection (see for instance [Vai87]).

5.3.6 Orthogonal decomposition of L_1 associated to meas(L_1, L_2)

The presentation given here follows that of [Nic91]. The notion of orthogonal decomposition will enable us to classify triples of Lagrangian subspaces of $\mathbb{C}^2$ (proposition 5.4.1).

Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^n$, and let $(\alpha_1^2, \ldots, \alpha_n^2)$ be a representative of $\mathrm{meas}(L_1, L_2) \in \mathbb{T}^n/\mathfrak{S}_n$. By proposition 5.3.7, the unit complex numbers $\alpha_1^2, \ldots, \alpha_n^2$ then are the roots of the characteristic polynomial $P(L_1, L_2)$ of the pair (L_1, L_2). Let $\alpha_{j_1}^2, \ldots, \alpha_{j_m}^2$ be the distinct roots of $P(L_1, L_2)$. For $k = 1, \ldots, m$, define the real subspace W_k of L_1 by $W_k = \{u \in L_1 \mid \alpha_{j_k} u \in L_2\}$ Observe that W_k is independent of the choice of the square root of $\alpha_{j_k}^2$, and that $W_1 \oplus \cdots \oplus W_m$ is independent, up to permutation of the subspaces, of the choice of the representative $(\alpha_1^2, \ldots, \alpha_n^2)$ of $\mathrm{meas}(L_1, L_2) \in \mathbb{T}^n/\mathfrak{S}_n$.

Proposition 5.3.12 ([Nic91]). *L_1 decomposes as an orthogonal direct sum : $L_1 = W_1 \oplus \cdots \oplus W_m$, the dimension of W_k being the multiplicity of $\alpha_{j_k}^2$ as a root of $P(L_1, L_2)$.*

Proof. By the first diagonalization lemma 5.3.3, there exists an orthonormal basis $(u_1, \ldots, u_n)$ of L_1 such that $(\alpha_1 u_1, \ldots, \alpha_n u_n)$ is an orthonormal basis of L_2, so that u_i belongs to W_k if and only if $\alpha_i = \alpha_{j_k}$. Thus, $\{u_i \mid \alpha_i = \alpha_{j_k}\}$ is a basis of W_k, which proves the proposition. □

Observe that L_2 then also decomposes as an orthogonal direct sum : $L_2 = \alpha_{j_1} W_1 \oplus \cdots \oplus \alpha_{j_m} W_m$. Furthermore, by considering the representative $(e^{i2\lambda_1}, \ldots, e^{i2\lambda_n})$ of $\mathrm{meas}(L_1, L_2)$, where $e^{i\lambda_1}, \ldots, e^{i\lambda_n}$ are the eigenvalues of the unitary map φ_{12}, we see that the subspace W_k of L_1 is the intersection with L_1 of the eigenspace of φ_{12} with respect to the eigenvalue $e^{i\lambda_{j_k}}$. Given a Lagrangian triple (L_1, L_2, L_3), the unitary maps φ_{12} and φ_{13} therefore have the same eigenspaces if and only if the orthogonal decompositions of L_1 associated to $\mathrm{meas}(L_1, L_2)$ and $\mathrm{meas}(L_1, L_3)$ are the same (see definition 5.5.6).

5.4 The case where $U = U(2)$ and $l = 3$

In this section, we study decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into $U(2)$. Geometrically, this amounts to studying configurations of triples of Lagrangian subspaces (L_1, L_2, L_3) of $\mathbb{C}^2$, or more

precisely, to studying the diagonal action of $U(2)$ on $\mathcal{L}(2)\times\mathcal{L}(2)\times\mathcal{L}(2)$. In particular, we will completely describe the image of the map

$$\begin{array}{rccc} \hat{\kappa}: & \mathcal{L}(2)\times\mathcal{L}(2)\times\mathcal{L}(2) & \longrightarrow & \mathbb{T}^2/\mathfrak{S}_2\times\mathbb{T}^2/\mathfrak{S}_2\times\mathbb{T}^2/\mathfrak{S}_2 \\ & (L_1,L_2,L_3) & \longmapsto & (\mathrm{meas}(L_1,L_2),\mathrm{meas}(L_2,L_3),\mathrm{meas}(L_3,L_1)) \end{array}$$

and prove that it induces a homeomorphism κ from the orbit space $(\mathcal{L}(2)\times\mathcal{L}(2)\times\mathcal{L}(2))/U(2)$ onto a closed subset of $\mathbb{T}^2/\mathfrak{S}_2\times\mathbb{T}^2/\mathfrak{S}_2\times\mathbb{T}^2/\mathfrak{S}_2$ (see proposition 5.4.7). As a consequence, we will obtain necessary and sufficient conditions on three conjugacy classes $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3\subset U(2)$ for the representation space $\mathrm{Hom}_{\mathcal{C}}(\pi_1(S^2\backslash\{s_1,s_2,s_3\}),U(2))$ to be non-empty. This provides an alternative elementary description of these conditions that were already known to Jeffrey and Weitsman (see [JW92]), to Gallitzer (see [Gal97]) and to Biswas (see [Bis98]), among others. For a description of these conditions for arbitrary dimension n and arbitrary number of punctures l, we refer for instance to the work of Agnihotri and Woodward in [AW98], Biswas in [Bis99], Kapovich and Millson in [KM99] and Belkale in [Bel01] (see also Teleman and Woodward in [TW03]).

5.4.1 A first classification result for triples of Lagrangian subspaces of $\mathbb{C}^2$

The following remark is valid for any n. If (L_1,L_2,L_3) and (L_1',L_2',L_3') are two triples of Lagrangian subspaces of $\mathbb{C}^n$ which lie in a same orbit of the diagonal action of $U(n)$ on $\mathcal{L}(n)\times\mathcal{L}(n)\times\mathcal{L}(n)$, it follows from section 5.3 that we have in particular $\mathrm{meas}(L_1,L_2)=\mathrm{meas}(L_1',L_2')$ and $\mathrm{meas}(L_1,L_3)=\mathrm{meas}(L_1',L_3')$. Let $L_1=W_1\oplus\cdots\oplus W_m$ be the orthogonal decomposition of L_1 associated to $\mathrm{meas}(L_1,L_2)$ (see proposition 5.3.12) and let $L_1=Z_1\oplus\cdots\oplus Z_p$ be the orthogonal decomposition of L_1 associated to $\mathrm{meas}(L_1,L_3)$. Define $L_1'=W_1'\oplus\cdots\oplus W_m'$ and $L_1'=Z_1'\oplus\cdots\oplus Z_p'$ similarly. Since $\mathrm{meas}(L_1,L_2)=\mathrm{meas}(L_1',L_2')$ and $\mathrm{meas}(L_1,L_3)=\mathrm{meas}(L_1',L_3')$, the respective numbers of factors m and p in the above decompositions are indeed pairwise the same. Furthermore, $\dim W_k=\dim W_k'$ for $k=1,\ldots,m$ and $\dim Z_l=\dim Z_l'$ for $l=1,\ldots,p$. More specifically, if the unitary map $\psi\in U(n)$ sends L_j to L_j' for $j=1,2,3$, then $\psi(W_k)=W_k'$ for $k=1,\ldots,m$ and $\psi(Z_l)=Z_l'$ for $l=1,\ldots,p$, as follows from the definition of W_k and Z_l. Since ψ is unitary, we even have $\psi(W_k\oplus JW_k)=W_k'\oplus JW_k'$ for all k and $\psi(Z_l\oplus JZ_l)=Z_l'\oplus JZ_l'$ for all l.

When $n=2$, the above remark admits an easy converse, which gives a first classification result for triples of Lagrangians of $\mathbb{C}^2$. We shall use the following notations : given two triples (L_1,L_2,L_3) and (L_1',L_2',L_3') of Lagrangian subspaces of $\mathbb{C}^2$, let φ_{12} be the only unitary map sending L_1 to L_2 and satisfying the conditions of the second diagonalization lemma (proposition 5.3.5), and let $(e^{i\lambda_{12}},e^{i\mu_{12}})$ be its eigenvalues, where $\pi>\lambda_{12}\geq\mu_{12}\geq 0$, and define $\varphi_{13},\varphi_{12}',\varphi_{13}'$ and $(e^{i\lambda_{13}},e^{i\mu_{13}}),(e^{i\lambda_{12}'},e^{i\mu_{12}'}),(e^{i\lambda_{13}'},e^{i\mu_{13}'})$ similarly. As a preliminary remark to the statement of the classification result, observe that when both φ_{12} and φ_{13} have two distinct eigenvalues, respectively denoted by $(e^{i\lambda_{12}},e^{i\mu_{12}})$ and by $(e^{i\lambda_{13}},e^{i\mu_{13}})$, where $\pi>\lambda_{12}>\mu_{12}\geq 0$ and $\pi>\lambda_{13}>\mu_{13}\geq 0$, then $W_1=\{u\in L_1\mid e^{i\lambda_{12}}u\in L_2\}$ and $Z_1=\{u\in L_1\mid e^{i\lambda_{13}}u\in L_3\}$ are one-dimensional real subspaces of the Euclidean space L_1, and therefore form a (non-oriented) angle measured by a real number $\theta\in[0,\frac{\pi}{2}]$, that will be denoted by $\mathrm{meas}(W_1,Z_1)$. A real number θ' may be defined similarly in L_1', since W_1' are Z_1' are also one-dimensional.

Proposition 5.4.1 (Unitary classification of Lagrangian triples of $\mathbb{C}^2$, first version). *Given two triples (L_1,L_2,L_3) and (L_1',L_2',L_3') of Lagrangian subspaces of $\mathbb{C}^2$, there exists a unitary map $\psi\in U(n)$ such that $\psi(L_1)=L_1',\psi(L_2)=L_2'$ and $\psi(L_3)=L_3'$ if and only if one either has :*

$$\text{(A)}\quad \lambda_{12}\neq\mu_{12},\ \lambda_{13}\neq\mu_{13}\ \text{and}\ \left\{\begin{array}{rcl}(\lambda_{12},\mu_{12}) & = & (\lambda_{12}',\mu_{12}')\\ (\lambda_{13},\mu_{13}) & = & (\lambda_{13}',\mu_{13}')\\ \theta & = & \theta'\end{array}\right.$$

where $\theta=\mathrm{meas}(W_1,Z_1)\in[0,\frac{\pi}{2}]$ and $\theta'=\mathrm{meas}(W_1',Z_1')\in[0,\frac{\pi}{2}]$ are defined as above, or :

$$\text{(B)}\quad \lambda_{12}=\mu_{12}\ \text{or}\ \lambda_{13}=\mu_{13},\ \text{and}\left\{\begin{array}{rcl}(\lambda_{12},\mu_{12}) & = & (\lambda_{12}',\mu_{12}')\\ (\lambda_{13},\mu_{13}) & = & (\lambda_{13}',\mu_{13}')\end{array}\right.$$

Observe that, in each case, the condition $(\lambda_{jk}, \mu_{jk}) = (\lambda'_{jk}, \mu'_{jk})$ is equivalent to the condition $\mathrm{meas}(L_j, L_k) = \mathrm{meas}(L'_j, L'_k)$.

Proof. Suppose that such a $\psi \in U(2)$ exists. Then, as we have seen earlier, $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$. Furthermore, $\psi(W_1) = W'_1$ and $\psi(Z_1) = Z'_1$, so that if φ_{12} and φ_{13} both have distinct eigenvalues (that is, we are in the situation (A) above), we have $\theta = \theta'$ since $\psi|_{L_1} : L_1 \to L'_1$ is an orthogonal map.
Conversely, suppose first that conditions (A) are fulfilled. Let $w_1 \in L_1$ be a generator of W_1 and let $w'_1 \in L'_1$ be a generator of W'_1. By choosing w_2 in L_1 orthogonal to w_1 and w'_2 in L'_1 orthogonal to w'_1, we may define an orthogonal map $\nu : L_1 \to L'_1$ sending W_1 to W'_1 (and therefore $W_2 = W_1^\perp$ to $W'_2 = (W'_1)^\perp$). Then the measure of the angle $(W'_1, \nu(Z_1)) = (\nu(W_1), \nu(Z_1))$ is $\theta = \theta'$, so that there exists an orthogonal map $\xi \in O(L'_1)$ satisfying $\xi \circ \nu(W_1) = W'_1$ and $\xi \circ \nu(Z_1) = Z'_1$. The subspace L_1 being Lagrangian, the orthogonal map $\xi \circ \nu$ can be extended $\mathbb{C}$-linearly to a unitary transformation $\psi \in U(2)$ of $\mathbb{C}^2 = L_1 \oplus JL_1$ sending L_1 to L'_1 by construction. But $L_2 = e^{i\lambda_{12}} W_1 \oplus e^{i\mu_{12}} W_2$ and $L_3 = e^{i\lambda_{13}} Z_1 \oplus e^{i\mu_{13}} Z_2$ (see proposition 5.3.12), hence $\psi(L_2) = e^{i\lambda_{12}} W'_1 \oplus e^{i\mu_{12}} W'_2 = L'_2$ and $\psi(L_3) = e^{i\lambda_{13}} Z'_1 \oplus e^{i\mu_{13}} Z'_2 = L'_3$.
If now the conditions (B) are fulfilled, then for instance $L_2 = e^{i\lambda} L_1$ and the result is a consequence of the classification of pairs. □

Observe that, given real numbers $(\lambda_{12}, \mu_{12}, \lambda_{13}, \mu_{13}, \theta)$ as in (A), it is always possible to find a triple (L_1, L_2, L_3) such that $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$, $\mathrm{meas}(L_1, L_3) = [e^{i2\lambda_{13}}, e^{i2\mu_{13}}]$ and $\mathrm{meas}(W_1, Z_1) = \theta$. Indeed, let L_1 be any Lagrangian of $\mathbb{C}^2$ and let (u_1, u_2) be an orthonormal basis of L_1, let $d_1 = \mathbb{R}u_1$, $d_2 = \mathbb{R}u_2$, and let d be the image of d_1 by the rotation of the Euclidean space L_1 with matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

in the basis (u_1, u_2), and set $L_2 = e^{i\lambda_{12}} d_1 \oplus e^{i\mu_{12}} d_2$ and $L_3 = e^{i\lambda_{13}} d \oplus e^{i\mu_{13}} d^\perp$. Given numbers $(\lambda_{12} = \mu_{12} = \lambda, \lambda_{13}, \mu_{13})$ as in (B), we only need to set $L_2 = e^{i\lambda} L_1$ and $L_3 = e^{i\lambda_{13}} d_1 \oplus e^{i\mu_{13}} d_2$.

Thus, the orbits of the diagonal action of $U(2)$ on $\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2)$ are generically characterized by the five invariants $\lambda_{12}, \mu_{12}, \lambda_{13}, \mu_{13}$ and θ.

5.4.2 Geometric study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

The aim of this section is to study the space $(\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2)$ of the orbits of the diagonal action of $U(2)$ on triples of Lagrangians subspaces of $\mathbb{C}^2$, and more specifically to describe it in terms of the map

$$\begin{array}{rccl} \kappa : & \big(\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2)\big) & \longrightarrow & \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \\ & [L_1, L_2, L_3] & \longmapsto & \big(\mathrm{meas}(L_1, L_2), \mathrm{meas}(L_2, L_3), \mathrm{meas}(L_3, L_1)\big) \end{array}$$

which will enable us to obtain another classification result for Lagrangian triples of $\mathbb{C}^2$, and to state it in a way (proposition 5.4.7) that is similar to the corresponding result for Lagrangian pairs (proposition 5.3.10). We shall see in subsection 5.4.3 that this way of doing things is equivalent to our previous approach which consisted in considering orthogonal decompositions of one of the three subspaces (see subsection 5.4.1). We are first going to describe the image of the map κ and then prove that it is one-to-one. This will also give a topological description of the orbit space. Our main tool to characterize the image of κ will be the study of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$.

5.4.2.1 Configurations of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

In the following, we shall constantly identify the complex projective line $\mathbb{C}\mathrm{P}^1$, endowed with the Fubini-Study metric (see for instance [ABK+94] p.40 or [Kli82] pp.106-108), with the Euclidean sphere $S^2 \subset \mathbb{R}^3$

endowed with its usual structure of oriented Riemannian manifold. We will denote by p the projection :

$$\begin{array}{rrcl} p: & \mathbb{C}^2\backslash\{0\} & \longrightarrow & \mathbb{C}\mathrm{P}^1 \\ & z=(z_1,z_2) & \longmapsto & p(z)=[z]=[z_1,z_2] \end{array}$$

As seen in 5.3.1, the image of a Lagrangian subspace of $\mathbb{C}^2$ is a totally geodesic submanifold of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ that is diffeomorphic to $\mathbb{R}\mathrm{P}^1 \simeq S^1$. Therefore $l = p(L)$ is a great circle of S^2, and the isometry $\widehat{\sigma_L}$ of $\mathbb{C}\mathrm{P}^1$, induced by the Lagrangian involution σ_L, acts on S^2 as the reflexion with respect to the plane of $\mathbb{R}^3$ containing the great circle $l = p(L)$. Recall that the unitary group $U(2)$ acts transitively on the Lagrangian Grassmannian $\mathcal{L}(2)$. The action of $U(2)$ on $\mathbb{C}\mathrm{P}^1$ is the same as the action of the special unitary group $SU(2)$, which acts on S^2 by the 2-sheeted universal covering map $h : SU(2) \to SO(3) = SU(2)/\{\pm 1\}$. The map $L \in \mathcal{L}(2) \mapsto l = p(L) \subset \mathbb{C}\mathrm{P}^1 \simeq S^2$ is equivariant for these actions. For any $\varphi \in GL(2,\mathbb{C})$, we shall denote by $\widehat{\varphi}$ the induced map of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ into itself : $\widehat{\varphi}.[z] = [\varphi(z)]$. If $\varphi \in U(2)$ then $\widehat{\varphi}$ acts on S^2 as an element of $SO(3)$: indeed $\varphi = e^{i\frac{\delta}{2}}\psi$, where $e^{i\delta} = \det\varphi$ and $\psi \in SU(2)$, and then $\widehat{\varphi} = \widehat{\psi}$ in $\mathrm{Aut}(\mathbb{C}\mathrm{P}^1)$, the action on S^2 being obtained by considering $h(\psi)$, which we shall from now on simply denote by $\widehat{\psi}$.

In the following, let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$ and let (l_1, l_2, l_3) be the triple of corresponding great circles of S^2 : $l_j = p(L_j)$ for $j = 1, 2, 3$. As above, we denote by φ_{jk} the only unitary map sending L_j to L_k and satisfying the conditions of the second diagonalization lemma (proposition 5.3.5). Let $(e^{i\lambda_{jk}}, e^{i\mu_{jk}})$ be its eigenvalues, where $\pi > \lambda_{jk} \geq \mu_{jk} \geq 0$, and let (u_{jk}, v_{jk}) be an orthonormal basis of L_j formed by eigenvectors of φ_{jk} : $\varphi_{jk}(u_{jk}) = e^{i\lambda_{jk}}u_{jk}$ and $\varphi(v_{jk}) = e^{i\mu_{jk}}v_{jk}$. Recall that $(e^{i2\lambda_{jk}}, e^{i2\mu_{jk}})$ is then a representative of $\mathrm{meas}(L_j, L_k) \in \mathbb{T}^2/\mathfrak{S}_2$. We denote by L_0 the Lagrangian subspace $L_0 = \{(x, y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}$ of $\mathbb{C}^2$. We denote its projection on $\mathbb{C}\mathrm{P}^1$ by $l_0 = p(L_0)$.

We are now going to relate the angles of projective Lagrangians of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ with the Lagrangian angles defined in section 5.3. Furthermore, in order to study configurations of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$, we are going to define a notion of sign of a projective Lagrangian triple. To do so, we shall first define such a notion in a generic case and then extend it to the remaining cases. Finally, we shall see that there is also a notion of sign for Lagrangian triples of $\mathbb{C}^n$ and that in the case $n = 2$, the triples (L_1, L_2, L_3) and (l_1, l_2, l_3) have same sign.

Proposition 5.4.2 (Projection of a Lagrangian pair). *Let (L_1, L_2) be a pair of Lagrangian subspaces of $\mathbb{C}^2$ and let $(e^{i\lambda_{12}}, e^{i\mu_{12}})$ be the eigenvalues of φ_{12}. Then $l_1 = l_2$ if and only if $\lambda_{12} = \mu_{12}$. Furthermore, if $\lambda_{12} \neq \mu_{12}$, then l_2 is the image of l_1 by the (direct) rotation of angle $\alpha_{12} = \lambda_{12} - \mu_{12} \in]0, \pi[$ around the point $[v_{12}] \in \mathbb{C}\mathrm{P}^1 \simeq S^2 \subset \mathbb{R}^3$, $[v_{12}] = \mathbb{C}v_{12}$ being the complex eigenline of φ_{12} associated to the eigenvalue $e^{i\mu_{12}}$ of lowest argument.*

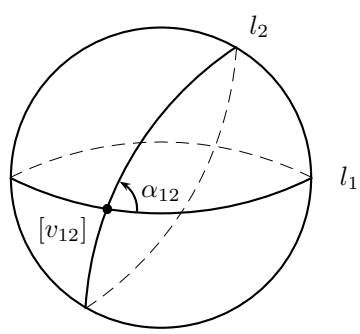


Figure 5.1: Two projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

Proof. If $\lambda_{12} = \mu_{12} = \lambda$ then $L_2 = e^{i\lambda}L_1$ and therefore $l_2 = l_1$ in $\mathbb{C}\mathrm{P}^1$.
If now $\lambda_{12} \neq \mu_{12}$, suppose first that $L_1 = L_0$ and that (u_{12}, v_{12}) is the standard basis of $\mathbb{C}^2$. Then L_2 is

the image of L_1 by the unitary map whose matrix in the standard basis of $\mathbb{C}^2$ is :

$$\begin{pmatrix} e^{i\lambda_{12}} & 0 \\ 0 & e^{i\mu_{12}} \end{pmatrix}$$

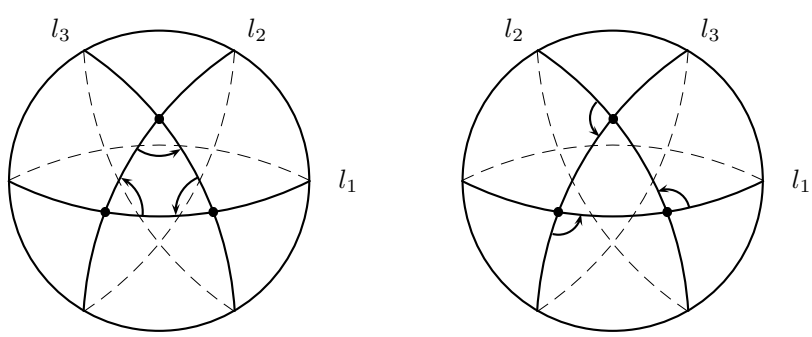
so that $L_2 = \{(e^{i\lambda_{12}}x, e^{i\mu_{12}}y) \,:\, x, y \in \mathbb{R}\}$ and $l_2 = p(L_2) = \{[e^{i\lambda_{12}}x, e^{i\mu_{12}}y] \,:\, x, y \in \mathbb{R}\}$. Therefore, in the chart $[z_1, z_2] \mapsto \frac{z_1}{z_2}$ of $\mathbb{C}\mathrm{P}^1$ containing $[v_{12}] = [0, 1]$, l_2 is sent diffeomorphically onto the real line $\{e^{i(\lambda_{12}-\mu_{12})}\frac{x}{y} \,:\, x, y \in \mathbb{R},\ y \neq 0\} = e^{i(\lambda_{12}-\mu_{12})}d_0$ of the plane $\mathbb{C} \simeq \mathbb{R}^2$, where d_0 is the image of $l_0 = l_1$ in this same chart. Thus, l_2 and l_1 intersect at $a_{12} = [u_{12}]$ and $b_{12} = [v_{12}]$, and l_2 is the image of l_1 by the rotation of angle $\alpha_{12} = \lambda_{12} - \mu_{12} \in]0, \pi[$ around the point $b_{12} = v_{12}$, which means that the oriented angle formed by l_1 and l_2 at b_{12} has measure $\alpha_{12} = \lambda_{12} - \mu_{12}$. Note that the oriented angle at a_{12} has measure $\pi - \alpha_{12} \in]0, \pi[$, since in the chart $[z_1, z_2] \mapsto \frac{z_2}{z_1}$, l_2 is diffeomorphic to the real line $e^{i(\mu_{12}-\lambda_{12})}d_0 = e^{i(\pi-(\lambda_{12}-\mu_{12}))}d_0$.
If now (u_{12}, v_{12}) is not the standard basis of $\mathbb{C}^2$, consider the unitary map $\psi \in U(2)$ sending the standard basis (e, f) of $\mathbb{C}^2$ to (u_{12}, v_{12}). Then $L_0 = \psi^{-1}(L_1)$, and let $L = \psi^{-1}(L_2)$. Then $[v_{12}] = \widehat{\psi}.[f]$, $l_2 = \widehat{\psi}(l)$ and $l_1 = \widehat{\psi}(l_0)$. Then, since $\mathrm{meas}(L_0, L) = \mathrm{meas}(L_1, L_2)$, we deduce from the above paragraph that $l = p(L)$ is the image of l_0 by the rotation of angle α_{12} around the point $[f]$. Hence, since $\widehat{\psi} \in SO(3)$, the oriented angle between l_1 and l_2 at $b_{12} = [v_{12}] \in l_1 \cap l_2$ also has measure α_{12}. □

Observe that this proof also provides an elementary way of seeing why L_0, and therefore every Lagrangian subspace of $\mathbb{C}^2$, projects to a great circle of $S^2 \simeq \mathbb{C}\mathrm{P}^1$. We shall state a converse to the above result later (see proposition 5.4.5).

Note that the preceding result gives a complete description of the relative position of the projective Lagrangians l_1 and l_2 only by means of the unitary map φ_{12}. In particular, the rotation described above is no other than the map $\widehat{\varphi_{12}}$ of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ into itself : $l_2 = \widehat{\varphi_{12}}(l_1)$. The axis of this rotation is the real line of $\mathbb{R}^3$ generated by any of the antipodal points $a_{12} = [u_{12}]$ and $b_{12} = [v_{12}]$ of S^2, (u_{12}, v_{12}) being a unitary basis of $\mathbb{C}^2$ into which the matrix of φ_{12} is diagonal.

We are now going to describe all possible configurations of the projective Lagrangians l_1, l_2 and l_3 of $\mathbb{C}\mathrm{P}^1 \simeq S^2$ satisfying the following condition for $(j, k) = (1, 2), (2, 3), (3, 1)$: if $l_j \neq l_k$ then l_k is the image of l_j by the direct rotation $\widehat{\varphi_{jk}}$ of $S^2 \subset \mathbb{R}^3$ of angle $\alpha_{jk} \in]0, \pi[$ around a specified point $b_{jk} \in l_j \cap l_k$.

First case : l_1, l_2 and l_3 are pairwise distinct.
(a) Suppose first that the three points b_{12}, b_{23}, b_{31} are linearly independent in $\mathbb{R}^3$ (that is, l_1, l_2, l_3 do not have a common diameter). We may then consider the spherical triangle (b_{12}, b_{23}, b_{31}), whose sides $[b_{12}, b_{23}]$, $[b_{23}, b_{31}]$, $[b_{31}, b_{12}]$ are respectively contained in the geodesics l_2, l_3, l_1. Since l_k is the image of l_j by a direct rotation around b_{jk}, the only possible configurations are the ones shown in figure 5.2.



A negative triple A positive triple

Figure 5.2: Triples of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$ in general position

On each sphere, we represent the angles α_{jk} around the point b_{jk} and we shall continue to do so in the following. We call the first triangle *negative* and the second triangle *positive*. Let us explain this terminology and prove that these cases are indeed the only possible ones when the b_{jk} are pairwise distinct.
Let $\varphi = \varphi_{31} \circ \varphi_{23} \circ \varphi_{12} \in U(2)$. Then $\varphi(L_1) = L_1$ and therefore $\widehat{\varphi}(l_1) = l_1$. There are only two possible cases : either $\widehat{\varphi}$ preserves a given orientation on l_1, or it reverses that orientation. But $\widehat{\varphi} = \widehat{\varphi_{31}} \circ \widehat{\varphi_{23}} \circ \widehat{\varphi_{12}}$ is the map obtained by composing the three rotations φ_{jk} around the b_{jk}. When $\widehat{\varphi}$ reverses the orientation of l_1, which we will call the *negative case*, then $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ are the angles of the spherical triangle (b_{12}, b_{23}, b_{31}). When $\widehat{\varphi}$ preserves the orientation of l_1, which we will call the *positive case*, the angles of the triangle (b_{12}, b_{23}, b_{31}) are β_{jk}, where $\beta_{jk} = \pi - \alpha_{jk} \in]0, \pi[$. Observe that this gives a series of necessary conditions for the existence of a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ projecting onto a triple (l_1, l_2, l_3) of great circles of S^2 that do not have a common diameter. Indeed, assume for instance that the triangle (b_{12}, b_{23}, b_{31}) has angles α_{jk} (negative case), then we necessarily have the following conditions on these angles :

$$(\Delta) \quad \alpha_{12}, \alpha_{23}, \alpha_{31} \in]0, \pi[\quad \text{and} \quad \begin{cases} \alpha_{12} + \alpha_{23} + \alpha_{31} & > & \pi \\ \alpha_{12} + \pi & > & \alpha_{23} + \alpha_{31} \\ \alpha_{23} + \pi & > & \alpha_{31} + \alpha_{12} \\ \alpha_{31} + \pi & > & \alpha_{12} + \alpha_{23} \end{cases}$$

since $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ are the angles of a spherical triangle (see for instance [Ber], pp.396 sqq). In the positive case, the same conditions apply to $(\beta_{12}, \beta_{23}, \beta_{31})$. In the following we shall write $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \Delta$ to say that $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ satisfy this set of conditions. Δ is an open subset of $\mathbb{R}^3$ and its closure $\overline{\Delta}$ in $\mathbb{R}^3$ is a tetrahedron (see figure 5.3). In the following, we will relate the set of equations and inequations describing the faces, edges and vertices of this tetrahedron to the possible configurations of Lagrangian subspaces of $\mathbb{C}^n$.

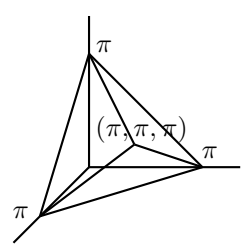


Figure 5.3: The tetrahedron $(\overline{\Delta})$

(b) Suppose now that b_{12}, b_{23} and b_{31} are not linearly independent. Then l_1, l_2, l_3 have a common diameter and we either have $b_{12} = b_{23} = b_{31}$ or, for instance, $b_{12} = b_{23}$ and $b_{31} \neq b_{12}$. Since l_1, l_2, l_3 are still supposed to be pairwise distinct and satisfying $l_k = \widehat{\varphi_{jk}}(l_j)$, the only possible configurations are the ones shown in figure 5.4 (we indicate in each case the sign of the triple (l_1, l_2, l_3)).

These 4 cases correspond to degenerate spherical triangles, so that we respectively have the following

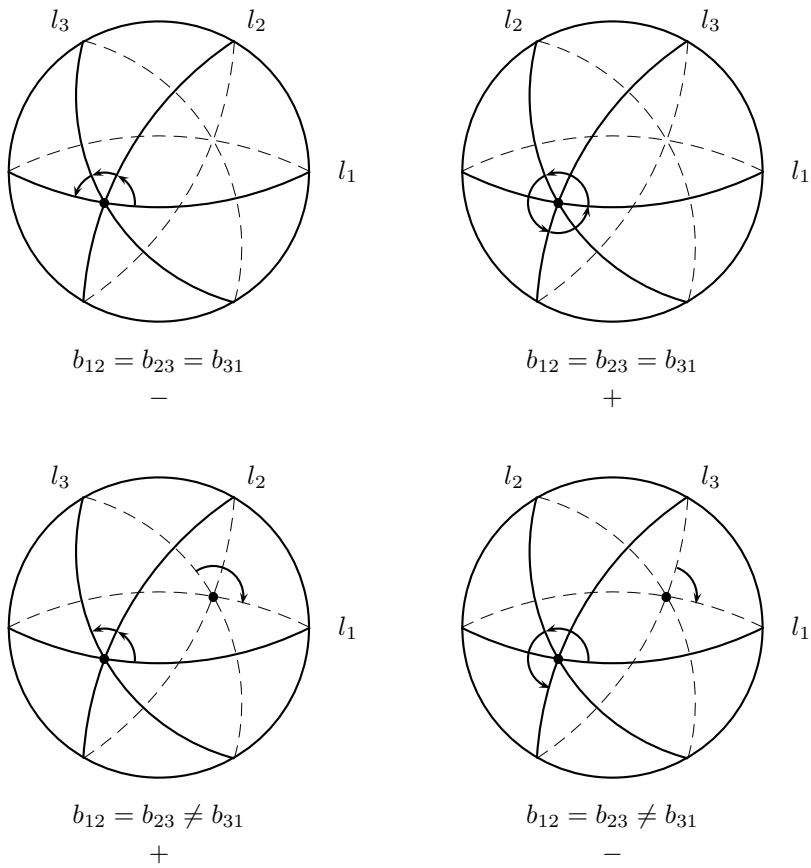


Figure 5.4: Exceptional triples of pairwise distinct projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

necessary conditions :

$\alpha_{12}, \alpha_{23}, \alpha_{31}$	$\in$	$]0, \pi[$	$\beta_{12}, \beta_{23}, \beta_{31}$	$\in$	$]0, \pi[$
$\alpha_{12} + \alpha_{23} + \alpha_{31}$	$=$	π	$\beta_{12} + \beta_{23} + \beta_{31}$	$=$	π
$\alpha_{12} + \pi$	$>$	$\alpha_{23} + \alpha_{31}$	$\beta_{12} + \pi$	$>$	$\beta_{23} + \beta_{31}$
$\alpha_{23} + \pi$	$>$	$\alpha_{31} + \alpha_{12}$	$\beta_{23} + \pi$	$>$	$\beta_{31} + \beta_{12}$
$\alpha_{31} + \pi$	$>$	$\alpha_{12} + \alpha_{23}$	$\beta_{31} + \pi$	$>$	$\beta_{12} + \beta_{23}$
$\beta_{12}, \beta_{23}, \beta_{31}$	$\in$	$]0, \pi[$	$\alpha_{12}, \alpha_{23}, \alpha_{31}$	$\in$	$]0, \pi[$
$\beta_{12} + \beta_{23} + \beta_{31}$	$>$	π	$\alpha_{12} + \alpha_{23} + \alpha_{31}$	$>$	π
$\beta_{12} + \pi$	$>$	$\beta_{23} + \beta_{31}$	$\alpha_{12} + \pi$	$>$	$\alpha_{23} + \alpha_{31}$
$\beta_{23} + \pi$	$>$	$\beta_{31} + \beta_{12}$	$\alpha_{23} + \pi$	$>$	$\alpha_{31} + \alpha_{12}$
$\beta_{31} + \pi$	$=$	$\beta_{12} + \beta_{23}$	$\alpha_{31} + \pi$	$=$	$\alpha_{12} + \alpha_{23}$

This means that either the α_{jk} or the β_{jk}, depending on the negativity or positivity of the triple (l_1, l_2, l_3), are located in an open face of the tetrahedron $\overline{\Delta}$ (see figure 5.3). The remaining faces are obtained when $b_{23} = b_{31}$ and $b_{12} \neq b_{23}$, and when $b_{31} = b_{12}$ and $b_{23} \neq b_{31}$.

Second case : l_1, l_2 and l_3 are not pairwise distinct.
(a) Suppose first, for instance, that $l_1 = l_2$ and $l_3 \neq l_1$. Since $l_1 = l_2$, we may consider either that $\alpha_{12} = 0$ or that $\alpha_{12} = \pi$ and that it is the angle of a direct rotation around $b_{23} \in l_2 \cap l_3 = l_1 \cap l_3$, *so that the*

notion of negative and positive triples is still valid. Then the only possible configurations of l_1, l_2, l_3 are the ones shown in figure 5.5.

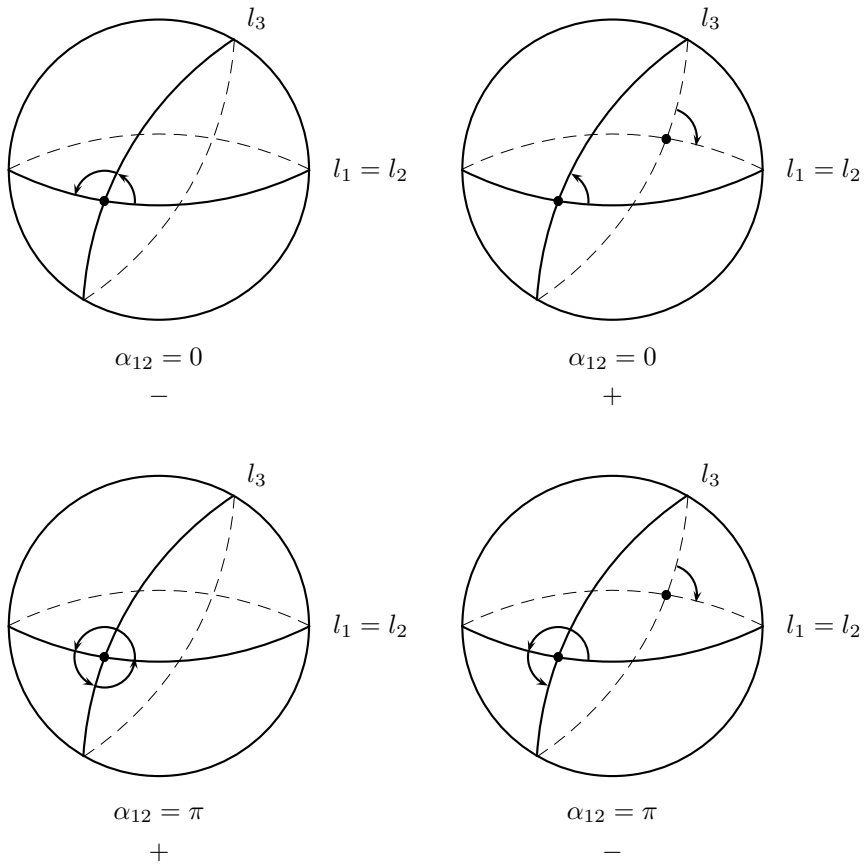


Figure 5.5: Triples of non pairwise distinct projective Lagrangians of $\mathbb{C}\mathrm{P}^1$

Those configurations correspond to open edges of $\overline{\Delta}$ (see figure 5.3) :

$\alpha_{12} = 0$		$\alpha_{23}, \alpha_{31} \in]0, \pi[$	$\beta_{12} = \pi$		$\beta_{23}, \beta_{31} \in]0, \pi[$
$\alpha_{12} + \alpha_{23} + \alpha_{31}$	$=$	π	$\beta_{12} + \beta_{23} + \beta_{31}$	$>$	π
$\alpha_{12} + \pi$	$=$	$\alpha_{23} + \alpha_{31}$	$\beta_{12} + \pi$	$>$	$\beta_{23} + \beta_{31}$
$\alpha_{23} + \pi$	$>$	$\alpha_{31} + \alpha_{12}$	$\beta_{23} + \pi$	$=$	$\beta_{31} + \beta_{12}$
$\alpha_{31} + \pi$	$>$	$\alpha_{12} + \alpha_{23}$	$\beta_{31} + \pi$	$=$	$\beta_{12} + \beta_{23}$
$\beta_{12} = 0$		$\beta_{23}, \beta_{31} \in]0, \pi[$	$\alpha_{12} = \pi$		$\alpha_{23}, \alpha_{31} \in]0, \pi[$
$\beta_{12} + \beta_{23} + \beta_{31}$	$=$	π	$\alpha_{12} + \alpha_{23} + \alpha_{31}$	$>$	π
$\beta_{12} + \pi$	$=$	$\beta_{23} + \beta_{31}$	$\alpha_{12} + \pi$	$>$	$\alpha_{23} + \alpha_{31}$
$\beta_{23} + \pi$	$>$	$\beta_{31} + \beta_{12}$	$\alpha_{23} + \pi$	$=$	$\alpha_{31} + \alpha_{12}$
$\beta_{31} + \pi$	$>$	$\beta_{12} + \beta_{23}$	$\alpha_{31} + \pi$	$=$	$\alpha_{12} + \alpha_{23}$

The remaining edges are obtained when $l_2 = l_3$ and $l_1 \neq l_2$ and when $l_3 = l_1$ and $l_2 \neq l_3$.
(b) Finally, suppose that $l_1 = l_2 = l_3$. The notion of negative and positive triples remains valid by considering either that $\alpha_{jk} = 0$ or that $\alpha_{jk} = \pi$, and that the b_{jk} all are a same b chosen arbitrarily in $l_1 = l_2 = l_3$. Then the possible configurations on S^2 correspond to vertices of $\overline{\Delta}$, that is, in the negative

case, $(\alpha_{12}, \alpha_{23}, \alpha_{31}) = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ or (π, π, π), and in the positive case : $(\beta_{12}, \beta_{23}, \beta_{31}) = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ or (π, π, π). Observe that in the cases where the three rotations $\widehat{\varphi_{jk}}$ occur around a same point b_{jk} or around two diametrically opposed points, then the negative case corresponds to $(\alpha_{12} + \alpha_{23} + \alpha_{31}) \equiv \pi \pmod{2\pi}$, and the positive case corresponds to $(\beta_{12} + \beta_{23} + \beta_{31}) \equiv \pi \pmod{2\pi}$, that is to $(\alpha_{12} + \alpha_{23} + \alpha_{31}) \equiv 0 \pmod{2\pi}$. Also note that if l_1, l_2, l_3 are pairwise distinct great circles of S^2 that do not have a common diameter, the pairwise intersections $l_j \cap l_k$ determine 6 points on S^2, which in turn give rise to 8 spherical triangles, four of which are negative, the other four being positive. Two triangles with a common edge have opposite sign, whereas two triangles with only a common vertex have same sign.

From the study above, we deduce that a Lagrangian triple (L_1, L_2, L_3) projects on a triple (l_1, l_2, l_3) of great circles of S^2, that is either positive with $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \overline{\Delta}$ or negative with $(\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta}$. In particular, these conditions are necessary conditions for $([e^{i2\lambda_{12}}, e^{i2\mu_{12}}], [e^{i2\lambda_{23}}, e^{i2\mu_{23}}], [e^{i2\lambda_{31}}, e^{i2\mu_{31}}])$ to be the triple of measures of a Lagrangian triple. Before showing that these conditions are sufficient, we shall give another way of determining if a triple (l_1, l_2, l_3) is negative or positive.

Proposition 5.4.3. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$, and set $\varphi = \varphi_{31} \circ \varphi_{23} \circ \varphi_{12}$. Write $\det \varphi = e^{i\delta}$, where $\delta = (\lambda_{12} + \mu_{12}) + (\lambda_{23} + \mu_{23}) + (\lambda_{31} + \mu_{31})$. Then $\delta \equiv 0 \pmod{\pi}$, and (l_1, l_2, l_3) is negative if $\delta \equiv \pi \pmod{2\pi}$ and positive if $\delta \equiv 0 \pmod{2\pi}$.*

Proof. Suppose first that $L_1 = L_0$ and that (u_{12}, v_{12}) is the standard basis of $\mathbb{C}^2$. Write

$$\varphi_{jk} = e^{i\frac{\lambda_{jk} + \mu_{jk}}{2}} \psi_{jk}$$

where $\psi_{jk} \in SU(2)$ and $e^{i(\lambda_{jk} + \mu_{jk})} = \det \varphi_{jk}$. Set $\psi = \psi_{31} \circ \psi_{23} \circ \psi_{12}$, so that $\varphi = e^{i\frac{\delta}{2}} \psi$, where $\delta = \sum_{j,k} (\lambda_{jk} + \mu_{jk})$. Note that $\widehat{\varphi_{jk}} = \widehat{\psi_{jk}}$ and $\widehat{\varphi} = \widehat{\psi}$. In particular, $\widehat{\psi}(l_0) = l_0$. But the matrix of ψ in the standard basis of $\mathbb{C}^2$ is of the form

$$A = \begin{pmatrix} s & -\overline{t} \\ t & \overline{s} \end{pmatrix}$$

where $s, t \in \mathbb{C}$ and satisfy $|s|^2 + |t|^2 = 1$. Since $l_0 = \{[x, y] \in \mathbb{C}\mathrm{P}^1 : x, y \in \mathbb{R}\}$, $\widehat{\psi}(l_0) = l_0$ if and only if

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ or } A = \begin{pmatrix} ia & ib \\ ib & -ia \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and satisfy $a^2 + b^2 = 1$.
In the first case $\psi(L_0) = L_0$, so that $L_0 = \varphi(L_0) = e^{i\frac{\delta}{2}}.L_0$, and since L_0 is totally real we have $\frac{\delta}{2} \equiv 0 \pmod{\pi}$, that is $\delta \equiv 0 \pmod{2\pi}$. In the second case $\psi(L_0) = i.L_0$, so that $L_0 = \varphi(L_0) = e^{i\frac{\delta}{2}} i.L_0$ and therefore $\frac{\delta}{2} \equiv \frac{\pi}{2} \pmod{\pi}$, that is $\delta \equiv \pi \pmod{2\pi}$. Now recall that $\widehat{\varphi}(l_0) = (l_0)$. When

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$\widehat{\psi}$ preserves a given orientation on l_0 (since, in the chart $[z_1, z_2] \mapsto \frac{z_1}{z_2}$, the map $x \in \mathbb{R} \mapsto \frac{ax-b}{bx+a}$ is increasing), so that the triple (l_0, l_2, l_3) is positive. When

$$A = \begin{pmatrix} ia & ib \\ ib & -ia \end{pmatrix}$$

$\widehat{\psi}$ reverses a given orientation on l_0 (since, in the chart $[z_1, z_2] \mapsto \frac{z_1}{z_2}$, the map $x \in \mathbb{R} \mapsto \frac{ax+b}{bx-a}$ is decreasing), so that the triple (l_0, l_2, l_3) is negative.
Suppose now that (u_{12}, v_{12}) is not the standard basis of $\mathbb{C}^2$, and define the unitary map $\nu \in U(2)$ sending the standard basis to (u_{12}, v_{12}). Let $L_2' = \nu^{-1}(L_2)$, $L_3' = \nu^{-1}(L_3)$, $l_2' = p(L_2')$ and $l_3' = p(L_3)$. Then the map $\nu^{-1} \circ \varphi \circ \nu$ sends L_0 to L_0 and $\det(\nu^{-1} \circ \varphi \circ \nu) = \det \varphi = e^{i\delta}$. From the study above, the triple (l_0, l_2', l_3') is positive if and only if $\delta \equiv 0 \pmod{2\pi}$, and negative if and only if $\delta \equiv \pi \pmod{2\pi}$. But since $l_1 = \widehat{\nu}(l_0)$, $l_2 = \widehat{\nu}(l_2')$ and $l_3 = \widehat{\nu}(l_3')$ with $\widehat{\nu} \in SO(3)$, the triples (l_1, l_2, l_3) and (l_0', l_2', l_3') have same sign. □

Remark 5.4.4. In particular, we have shown that we always have $\delta \equiv 0 \pmod{\pi}$. Observe that when $\delta \equiv 0 \pmod{2\pi}$, we have $\det \varphi = 1$, so that we might also say that the triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ is positive. Similarly, when $\delta \equiv \pi \pmod{2\pi}$, $\det \varphi = -1$ and (L_1, L_2, L_3) will be said to be negative. The above proposition then says that the triples (L_1, L_2, L_3) and (l_1, l_2, l_3) have same sign. Note that the notion of sign of a Lagrangian triple (L_1, L_2, L_3) is also valid for Lagrangian subspaces of $\mathbb{C}^n$. Indeed, by corollary 5.3.8, we have $\varphi_{jk}^2 = \sigma_{L_k} \circ \sigma_{L_j}$. But $(\sigma_{L_1}\sigma_{L_3})(\sigma_{L_3}\sigma_{L_2})(\sigma_{L_2}\sigma_{L_1}) = 1$, hence $(\det \varphi)^2 = 1$ (where $\varphi = \varphi_{31} \circ \varphi_{23} \circ \varphi_{12}$), so that $e^{i2\delta} = 1$. Consequently, $2\delta \equiv 0 \pmod{2\pi}$ and therefore $\delta \equiv 0 \pmod{\pi}$. When $\delta \equiv 0 \pmod{2\pi}$, the triple (L_1, L_2, L_3) is said to be positive and when $\delta \equiv \pi \pmod{2\pi}$ it is said to be negative.

5.4.2.2 A second classification result for triples of Lagrangian subspaces of $\mathbb{C}^2$

As a converse to proposition 5.4.2, it is possible, given two distinct great circles $l_1 \neq l_2$ of $S^2 \simeq \mathbb{C}\mathrm{P}^1$, to describe the measure of the angle (L_1, L_2) between two Lagrangians of $\mathbb{C}^2$ that project respectively to l_1 and l_2. Recall that two distinct great circles $l_1 \neq l_2$ intersect along two antipodal points a, b, and that $\alpha \in]0, \pi[$ is said to be the measure of the oriented angle between l_1 and l_2 at $b \in l_1 \cap l_2$ if l_2 is the image of l_1 by the (direct) rotation of angle α around b.

Proposition 5.4.5 (Lifting lemma). *Let $l_1 \neq l_2$ be two distinct projective Lagrangians of $\mathbb{C}\mathrm{P}^1 \simeq S^2$, let $b \in l_1 \cap l_2$ and let $\alpha \in]0, \pi[$ be the measure of the oriented angle (l_1, l_2) at b. Then, given λ and μ such that $\pi > \lambda > \mu \geq 0$, and given a Lagrangian subspace $L_1 \in p^{-1}(l_1)$, there exists a unique Lagrangian subspace $L_2 \in p^{-1}(l_2)$ such that* $\mathrm{meas}(L_1, L_2) = [e^{i2\lambda}, e^{i2\mu}]$.

Proof. Let $v \in L_1$ such that $p(v) = b$. We may choose v such that $\|v\| = h(v, v) = 1$. Then, take $u \in L_1$ such that (u, v) is an orthonormal basis of L_1. Since L_1 is Lagrangian, (u, v) is a unitary basis of $\mathbb{C}^2$. Let ψ be the unitary transformation of $\mathbb{C}^2$ whose matrix in the basis (u, v) is

$$\begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{i\mu} \end{pmatrix}$$

and let $L = \psi(L_1)$. Then L is Lagrangian and $\mathrm{meas}(L_1, L) = [e^{i2\lambda}, e^{i2\mu}]$. Therefore, by proposition 5.4.2, $l = p(L)$ is a great circle of S^2, distinct of l_1 since $\lambda \neq \mu$, that intersects l_1 at $p(v) = b$ and the measure of the oriented angle between l_1 and l at b is $\lambda - \mu = \alpha$, so that $l = l_2$.
As for unicity, if $L' \in p^{-1}(l_2)$, then, again by proposition 5.4.2, we know that $L' = e^{i\theta}.L$, where $\theta \in]0, \pi[$. The unitary map $e^{i\theta}.\psi$ then sends L_1 to L', and its matrix in the unitary basis (u, v), which is an orthonormal basis of L_1 is

$$\begin{pmatrix} e^{i(\theta+\lambda)} & 0 \\ 0 & e^{i(\theta+\mu)} \end{pmatrix}$$

so that $\mathrm{meas}(L_1, L') = [e^{i2((\theta+\lambda) \mod \pi)}, e^{i((\theta+\mu) \mod \pi)}]$, with $\pi > (\theta + \lambda) \mod \pi > (\theta + \mu) \mod \pi \geq 0$. Since $\mathrm{meas}(L_1, L) = \mathrm{meas}(L_1, L')$, we have in particular $(\theta + \lambda) \mod \pi = \lambda$, hence $\theta \mod \pi = 0$ (and so $\theta = 0$) and $L' = e^{i\theta}.L = L$. □

The next proposition completely describes the image of the map κ and lays the ground for the second classification result for triples of Lagrangian subspaces of $\mathbb{C}^2$.

Proposition 5.4.6 (Possible triples of measures for triples of Lagrangian subspaces of $\mathbb{C}^2$). *Given a triple of measures*

$$([e^{i2\lambda_{12}}, e^{i2\mu_{12}}], [e^{i2\lambda_{23}}, e^{i2\mu_{23}}], [e^{i2\lambda_{31}}, e^{i2\mu_{31}}])$$

satisfying the conditions $\pi \geq \lambda_{jk} \geq \mu_{jk} \geq 0$, set $\alpha_{jk} = \lambda_{jk} - \mu_{jk} \in [0, \pi]$, $\beta_{jk} = \pi - \alpha_{jk} \in [0, \pi]$ and

$$\delta = (\lambda_{12} + \mu_{12}) + (\lambda_{23} + \mu_{23}) + (\lambda_{31} + \mu_{31})$$

Then, there exists a triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^2$ such that

$$\begin{cases} \text{meas}(L_1, L_2) &= [e^{i2\lambda_{12}}, e^{i2\mu_{12}}] \\ \text{meas}(L_2, L_3) &= [e^{i2\lambda_{23}}, e^{i2\mu_{23}}] \\ \text{meas}(L_3, L_1) &= [e^{i2\lambda_{31}}, e^{i2\mu_{31}}] \end{cases}$$

if and only if :

$$\delta \equiv \pi \pmod{2\pi} \quad \text{and} \quad (\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \overline{\Delta} \text{ (negative case)}$$

or

$$\delta \equiv 0 \pmod{2\pi} \quad \text{and} \quad (\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta} \text{ (positive case)}$$

(Here we allow $\lambda_{jk} = \pi$ so that we may have $\alpha_{jk} = \pi$ and $\beta_{jk} = 0$).

Proof. The study made in 5.4.2.1 shows that these conditions are necessary.
Conversely, suppose first that $\delta \equiv \pi \pmod{2\pi}$ and that $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ lie in the open set Δ. Then there exists a negative triple (l_1, l_2, l_3) of pairwise distinct great circles of S^2 such that l_k is the image of l_j by the direct rotation of angle α_{jk} around a certain point $b_{jk} \in l_j \cap l_k$ for $(j,k) = (1,2), (2,3), (3,1)$, and we may suppose that $l_1 = l_0$. Let $L_1 = L_0$. Then, by proposition 5.4.5, there exists a unique Lagrangian $L_2 \in p^{-1}(l_2)$ such that $\text{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$. Again by proposition 5.4.5, there exists a unique Lagrangian $L_3 \in p^{-1}(l_3)$ such that $\text{meas}(L_2, L_3) = [e^{i2\lambda_{23}}, e^{i2\mu_{23}}]$, and a unique Lagrangian $L_4 \in p^{-1}(l_1)$ such that $\text{meas}(L_3, L_4) = [e^{i2\lambda_{31}}, e^{i2\mu_{31}}]$. Let φ_{34} be the unique unitary map sending L_3 to L_4 and satisfying the conditions of the second diagonalization lemma 5.3.5, and let $\varphi = \varphi_{34} \circ \varphi_{23} \circ \varphi_{12}$. Then $\varphi(L_1) = L_4$ and $\det \varphi = e^{i\delta}$. Write $\varphi = e^{i\frac{\delta}{2}}\psi$, where $\psi \in SU(2)$. Then $\widehat{\psi}(l_1) = l_1$, and since (l_1, l_2, l_3) is negative, we deduce from the study made in 5.4.2.1 that $\psi(L_1) = i.L_1$, hence, as $\delta \equiv \pi \pmod{2\pi}$, we have $L_4 = \varphi(L_1) = e^{i\frac{\delta}{2}} i.L_1 = L_1$.
Suppose now that $(\alpha_{12}, \alpha_{23}, \alpha_{31}) \in \partial\Delta$. If $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ lay in an open face of $\overline{\Delta}$, there exists a negative triple (l_1, l_2, l_3) of pairwise distinct great circles of S^2 such that l_k is the image of l_j by the direct rotation of angle α_{jk} around a certain point $b_{jk} \in l_j \cap l_k$ for $(j,k) = (1,2), (2,3), (3,1)$, and we can therefore conclude as earlier. If now $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ lay in an open edge of $\overline{\Delta}$, there exists a negative triple, for instance of the form $(l_1, l_2 = l_1, l_3 \neq l_2)$, such that l_3 is the image of l_2 by the rotation of angle α_{23} around $b_{23} \in l_2 \cap l_3$ and such that l_1 is the image of l_3 by the rotation of angle α_{31} around $b_{31} \in l_3 \cap l_1$. Since $l_2 = l_1$, α_{12} is either 0 or π, and by setting $b_{12} = b_{23}$ (or $b_{12} = b_{31}$), we have that l_2 is the image of l_1 by the rotation of angle α_{12} around $b_{12} \in l_1 \cap l_2$ (see figure 5.5). Let $L_1 = L_0$. If $\alpha_{12} = 0$, then $\lambda_{12} = \mu_{12}$ and we set $L_2 = e^{i\lambda_{12}}.L_1$. If $\alpha_{12} = \pi$, then $\lambda_{12} = \pi$ and $\mu_{12} = 0$, and we set $L_2 = L_1$. In both cases $L_2 \in p^{-1}(l_2) = p^{-1}(l_1)$ and $\text{meas}(L_1, L_2) = [e^{i2\lambda_{12}}, e^{i2\mu_{12}}]$. Since $l_1 = l_2 \neq l_3$, there exists, by proposition 5.4.5, a unique Lagrangian $L_3 \in p^{-1}(l_3)$ such that $\text{meas}(L_2, L_3) = [e^{i2\lambda_{23}}, e^{i2\mu_{23}}]$, and a unique Lagrangian $L_4 \in p^{-1}(l_1)$ such that $\text{meas}(L_3, L_4) = [e^{i2\lambda_{31}}, e^{i2\mu_{31}}]$. As earlier, since the triple (l_1, l_2, l_3) is negative, we have $L_4 = e^{i\frac{\delta}{2}} i.L_1 = L_1$.
Finally, if $(\alpha_{12}, \alpha_{23}, \alpha_{31})$ is a vertex of $\overline{\Delta}$, that is, if $(\alpha_{12}, \alpha_{23}, \alpha_{31}) = (\pi, 0, 0), (0, \pi, 0), (0, 0, \pi)$ or (π, π, π), then $L_1 = L_2 = L_3 = L_0$ meet the required conditions.
If now, $\delta \equiv 0 \pmod{2\pi}$, the condition $(\beta_{12}, \beta_{23}, \beta_{31}) \in \overline{\Delta}$ implies the existence of a positive triple (l_1, l_2, l_3) of pairwise distinct great circles of S^2, with angles α_{jk} as required. Reasoning the same way, we find 4 Lagrangians L_1, L_2, L_3 and L_4 with prescribed angles $[e^{i2\lambda_{jk}}, e^{i2\mu_{jk}}]$, and since (l_1, l_2, l_3) is positive we have: $L_4 = \varphi(L_1) = e^{i\frac{\delta}{2}}.L_1$, and therefore, as $\delta \equiv 0 \pmod{2\pi}$, $L_4 = L_1$.
The other cases are treated identically. □

We now obtain the following classification result for triples of Lagrangian subspaces of $\mathbb{C}^2$.

Proposition 5.4.7 (Unitary classification of Lagrangian triples of $\mathbb{C}^2$, second version). *Given two triples (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) of Lagrangian subspaces of $\mathbb{C}^2$, there exists a unitary map $\varphi \in U(2)$ such that $\varphi(L_1) = L'_1$, $\varphi(L_2) = L'_2$ and $\varphi(L_3) = L'_3$ if and only if :*

$$\begin{cases} \text{meas}(L_1, L_2) &= \text{meas}(L'_1, L'_2) \\ \text{meas}(L_2, L_3) &= \text{meas}(L'_2, L'_3) \\ \text{meas}(L_3, L_1) &= \text{meas}(L'_3, L'_1) \end{cases}$$

Equivalently, the map

$$\kappa : \big(\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2)\big) \longrightarrow \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2$$

is one-to-one and is therefore a homeomorphism from the orbit space $(\mathcal{L}(2) \times \mathcal{L}(2) \times \mathcal{L}(2))/U(2)$ *onto a closed subset of the measure space* $\mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2 \times \mathbb{T}^2/\mathfrak{S}_2$.

Proof. It only remains to prove that the above conditions are sufficient. Let (L_1, L_2, L_3) and (L'_1, L'_2, L'_3) be two Lagrangian triples such that $\mathrm{meas}(L_j, L_k) = \mathrm{meas}(L'_j, L'_k)$ for all j, k. Then, the (generalized) triangles (b_{12}, b_{23}, b_{31}) and $(b'_{12}, b'_{23}, b'_{31})$ have the same angles, so that there exists a map $\psi \in SU(2)$ such that $\widehat{\psi}(b_{jk}) = b'_{jk}$ for all j, k. Since moreover $\delta = \delta'$, the triples (l_1, l_2, l_3) and (l'_1, l'_2, l'_3) have same sign and we therefore even have $\widehat{\psi}(l_j) = l'_j$ for all j. Equivalently, $p(\psi(L_j)) = \widehat{\psi}(p(L_j)) = \widehat{\psi}(l_j) = l'_j = p(L'_j)$. In particular, by proposition 5.4.2, we have $L'_1 = e^{i\theta}.\psi(L_1)$ for some $\theta \in [0, \pi[$. Set $\varphi = e^{i\theta}.\psi \in U(2)$. Then $\varphi(L_1) = L'_1$ and $p(\varphi(L_2)) = \widehat{\varphi}(p(L_2)) = \widehat{\psi}(l_2) = l'_2$ and $\mathrm{meas}(L'_1, \varphi(L_2)) = \mathrm{meas}(\varphi(L_1), \varphi(L_2)) = \mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$, hence, by unicity in proposition 5.4.5, $\varphi(L_2) = L'_2$. Likewise, $p(\varphi(L_3)) = l'_3$ and $\mathrm{meas}(L'_2, \varphi(L_3)) = \mathrm{meas}(L'_2, L'_3)$, therefore $\varphi(L_3) = L'_3$. □

5.4.3 Equivalence of the two classification results

We now wish to explain why the two classification results that we have obtained (propositions 5.4.1 and 5.4.7) are equivalent.

Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$. If one of the unitary maps φ_{jk} is of the form $e^{i\lambda} Id$ (for instance if $L_2 = e^{i\lambda}.L_1$), and if (L'_1, L'_2, L'_3) is a triple of Lagrangian subspaces such that $\mathrm{meas}(L_1, L_2) = \mathrm{meas}(L'_1, L'_2)$ and $\mathrm{meas}(L_1, L_3) = \mathrm{meas}(L'_1, L'_3)$ (or equivalently $\mathrm{meas}(L_3, L_1) = \mathrm{meas}(L'_3, L'_1)$), we necessarily have $\mathrm{meas}(L_2, L_3) = \mathrm{meas}(L'_2, L'_3)$. Indeed, since

$$\mathrm{meas}(L'_1, L'_2) = \mathrm{meas}(L_1, L_2) = [e^{i2\lambda}, e^{i2\lambda}]$$

one has $L'_2 = e^{i\lambda}.L'_1$. Therefore :

$$\sigma_{L'_3} \circ \sigma_{L'_2} = \sigma_{L'_3} \circ \sigma_{e^{i\lambda} L'_1} = \sigma_{L'_3} \circ (e^{i\lambda} \sigma_{L'_1} e^{-i\lambda}) = e^{-i\lambda} (\sigma_{L'_3} \circ \sigma_{L'_1}) e^{-i\lambda}$$

But $\mathrm{meas}(L_3, L_1) = \mathrm{meas}(L'_3, L'_1)$, so that, by proposition 5.3.10, $\sigma_{L_3} \circ \sigma_{L_1}$ is conjugate to $\sigma_{L'_3} \circ \sigma_{L'_1}$. Since we also have $L_2 = e^{i\lambda}.L_1$, the above computation shows that $\sigma_{L_3} \circ \sigma_{L_2}$ is conjugate to $\sigma_{L'_3} \circ \sigma_{L'_2}$ and this means that $\mathrm{meas}(L_2, L_3) = \mathrm{meas}(L'_2, L'_3)$, which proves that in this case the two classification results are indeed the same.

If now each unitary map φ_{jk} has two distinct eigenvalues $e^{i\lambda_{jk}}$ and $e^{i\mu_{jk}}$, where $\pi > \lambda_{jk} > \mu_{jk} \geq 0$, set $d_{12} = \mathbb{R}u_{12} \subset L_1$ and $d_{13} = \mathbb{R}u_{13} \subset L_1$ (where u_{12} and u_{13} are defined as earlier by means of φ_{12} and φ_{13}), and let $\theta = \mathrm{meas}(d_{12}, d_{13}) \in [0, \frac{\pi}{2}]$ be the measure of the non-oriented angle formed by the real lines d_{12} and d_{13} in the Euclidean space L_1. Recall that $L_1 = d_{12} \oplus d_{12}^{\perp} = d_{13} \oplus d_{13}^{\perp}$, where $d_{12}^{\perp} = \mathbb{R}v_{12}$ and $d_{13}^{\perp} = \mathbb{R}v_{13}$, and observe that θ is also the measure of the angle $(d_{12}^{\perp}, d_{13}^{\perp})$. As earlier, define $b_{jk} = [v_{jk}] \in l_j \cap l_k \subset \mathbb{C}\mathrm{P}^1 \simeq S^2$. We then formulate the following remark :

Lemma 5.4.8. *The measure of the non-oriented angle formed by the two vectors* b_{12} *and* b_{13} *of* $S^2 \subset \mathbb{R}^3$ *is* $2\theta \in [0, \pi]$. *In particular, two orthogonal vectors of* L_1 *project onto antipodal points of* S^2.

Proof. Suppose first that $L_1 = L_0$ and that (u_{12}, v_{12}) is the standard basis of $\mathbb{C}^2$. Since $\mathrm{meas}(\mathbb{R}v_{12}, \mathbb{R}v_{13}) = \theta$ and since v_{13} has norm 1, we either have $v_{13} = (\sin\theta, \cos\theta)$ or $v_{13} = (-\sin\theta, \cos\theta)$ in L_0. We may assume $\theta \neq 0$, otherwise $v_{13} = v_{12}$, hence $b_{13} = b_{12}$ and $\mathrm{meas}(b_{12}, b_{13}) = 0$. If for instance $v_{13} = (\sin\theta, \cos\theta)$ with $\theta \in]0; \frac{\pi}{2}]$, then $[v_{13}] = [\sin\theta, \cos\theta] \in \mathbb{C}\mathrm{P}^1$ is sent by the diffeomorphism $\mathbb{C}\mathrm{P}^1 \xrightarrow{\simeq} S^2$ to $(-\sin 2\theta, 0, -\cos 2\theta)$. Since $[v_{12}] = [0; 1]$ is sent to $(0, 0, -1)$, the measure of the non-oriented angle between these two vectors of $\mathbb{R}^3$ is indeed 2θ. The case $v_{13} = (-\sin\theta, \cos\theta)$ is similar ($[v_{13}]$ is then sent

to $(\sin 2\theta, 0, -\cos 2\theta))$.
If now (u_{12}, v_{12}) is not the standard basis (e, f) of $\mathbb{C}^2$, define the unitary map ψ sending (e, f) to (u_{12}, v_{12}). Since $\psi|_{L_0} : L_0 \to L_1$ is an orthogonal map, $\text{meas}(f, \psi^{-1}(v_{13})) = \text{meas}(v_{12}, v_{13}) = \theta$ and since $\widehat{\psi} \in SO(3)$, we deduce from the preceding case that $\text{meas}(b_{12}, b_{13}) = \text{meas}(\widehat{\psi}([f]), \widehat{\psi}([\psi^{-1}(v_{13})])) = \text{meas}([f], [\psi^{-1}(v_{13})]) = 2\theta$. □

Observe now that $b_{31} \in l_1 \cap l_3$ is one of the two antipodal points a_{13} or $b_{13} \in l_1 \cap l_3$. Therefore, we have :

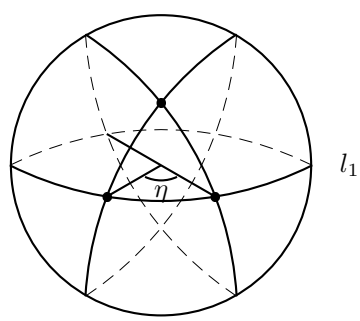
Lemma 5.4.9. *If $\mu_{13} = 0$ then $b_{31} = b_{13}$ and therefore* $\text{meas}(b_{12}, b_{31}) = 2\theta$. *If $\mu_{13} \neq 0$ then $b_{31} = a_{13}$ and therefore* $\text{meas}(b_{12}, b_{31}) = \pi - 2\theta$.

Proof. Recall that we have supposed that $\lambda_{13} \neq \mu_{13}$, and that we have : $\varphi_{13}(u_{13}) = e^{i\lambda_{13}} u_{13}$, $\varphi_{13}(v_{13}) = e^{i\mu_{13}} v_{13}$, and $\pi > \lambda_{13} > \mu_{13} \geq 0$. Similarly : $\varphi_{31}(u_{31}) = e^{i\lambda_{31}} u_{31}$, $\varphi_{31}(v_{31}) = e^{i\mu_{31}} v_{31}$, and, since $\lambda_{13} \neq \mu_{13}$, $\pi > \lambda_{31} > \mu_{31} \geq 0$. Set $w_{31} = e^{i\lambda_{13}} u_{13} \in L_3$. Then $e^{i(\pi - \lambda_{13})} w_{31} = -u_{13} \in L_3$, with $\pi - \lambda_{13} \in]0; \pi[$. Therefore, w_{31} is an eigenvector of φ_{31}, so $w_{31} \in \mathbb{R}u_{31}$ or $w_{31} \in \mathbb{R}v_{31}$. If $\mu_{13} \neq 0$, we have $\pi > \pi - \mu_{13} > \pi - \lambda_{13} > 0$, so that the eigenvalues of φ_{31} are $\pi - \lambda_{13}$ and $\pi - \mu_{13}$, hence $\mu_{31} = \pi - \lambda_{13}$ and $w_{31} \in \mathbb{R}v_{31}$. Consequently, $[u_{13}] = [w_{31}] = [v_{31}]$ in $\mathbb{C}\text{P}^1$, that is : $a_{13} = b_{31}$. By lemma 5.4.8, $\text{meas}(b_{12}, b_{31}) = \pi - 2\theta$.
If now $\mu_{13} = 0$, then $v_{13} \in L_1 \cap L_3$, hence $\mu_{31} = 0$ and $v_{31} = v_{13}$, so that $b_{31} = [v_{31}] = [v_{13}] = b_{13}$, and therefore $\text{meas}(b_{12}, b_{31}) = 2\theta$. □

Let now $(\gamma_{12}, \gamma_{23}, \gamma_{31})$ be the measures of the angles of the spherical triangle (b_{12}, b_{23}, b_{31}) (from the study of projective Lagrangians of $\mathbb{C}\text{P}^1$ conducted in 5.4.2.1, we know that either $(\gamma_{12}, \gamma_{23}, \gamma_{31}) = (\alpha_{12}, \alpha_{23}, \alpha_{31})$ or $(\gamma_{12}, \gamma_{23}, \gamma_{31}) = (\beta_{12}, \beta_{23}, \beta_{31})$, where $\alpha_{jk} = \lambda_{jk} - \mu_{jk}$ and $\beta_{jk} = \pi - \alpha_{jk}$). Let $\eta \in [0, \pi]$ be the measure of the non-oriented angle (b_{12}, b_{31}) (from the study above, we know that either $\eta = 2\theta$ or $\eta = \pi - 2\theta$). Then we know from spherical trigonometry that :

$$\cos \gamma_{23} = \sin \gamma_{12} \sin \gamma_{31} \cos \eta - \cos \gamma_{12} \cos \gamma_{31}$$

(see for instance [Ber], pp.396 sqq). The next proposition completes the explanation why our two classification results are indeed equivalent.



$\eta = 2\theta$ or $\pi - 2\theta$

Figure 5.6: Relation between the two classification results

Proposition 5.4.10 (Equivalence of the two classification results). *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$ such that $\varphi_{12}, \varphi_{23}$ and φ_{31} have distinct eigenvalues. Let (L'_1, L'_2, L'_3) be a triple of Lagrangian subspaces of $\mathbb{C}^2$ such that* $\text{meas}(L_1, L_2) = \text{meas}(L'_1, L'_2)$ *and* $\text{meas}(L_1, L_3) = \text{meas}(L'_1, L'_3)$, *this last condition being equivalent to* $\text{meas}(L_3, L_1) = \text{meas}(L'_3, L'_1)$. *Define* $\theta = \text{meas}(d_{12}, d_{13}) \in [0, \frac{\pi}{2}]$ *to be the measure of the non-oriented angle (d_{12}, d_{13}) in L_1 and define* $\theta' = \text{meas}(d'_{12}, d'_{13}) \in [0, \frac{\pi}{2}]$ *in L'_1 similarly. Then* $\text{meas}(L_2, L_3) = \text{meas}(L'_2, L'_3)$ *if and only if $\theta = \theta'$.*

Proof. Assume first that $\text{meas}(L_2, L_3) = \text{meas}(L'_2, L'_3)$. Since we also have $\text{meas}(L_1, L_2) = \text{meas}(L'_1, L'_2)$ and $\text{meas}(L_1, L_3) = \text{meas}(L'_1, L'_3)$, we get $\delta = \delta'$: the triples (l_1, l_2, l_3) and (l'_1, l'_2, l'_3) have same sign. As a consequence, the spherical triangles (b_{12}, b_{23}, b_{31}) and $(b'_{12}, b'_{23}, b'_{31})$ have the same angles : $\gamma_{jk} = \gamma'_{jk} \in]0, \pi[$ for all j, k. Since $\text{meas}(L_1, L_3) = \text{meas}(L'_1, L'_3)$ we have $\mu_{13} = \mu'_{13}$, therefore, by lemma 5.4.9, either $b_{31} = b_{13}$ and $b'_{31} = b'_{13}$ (when μ_{13} and μ'_{13} equal zero) or $b_{31} = a_{13}$ and $b'_{31} = a'_{13}$ (when $\mu_{13} = \mu'_{13} \neq 0$), so that either $\eta = 2\theta$ and $\eta' = 2\theta'$ or $\eta = \pi - 2\theta$ and $\eta' = \pi - 2\theta'$. But then from the relation from spherical trigonometry recalled above, since $\sin \gamma_{jk} \neq 0$ for all j, k, we have $\cos \eta = \cos \eta'$, and since $\eta, \eta' \in [0, \pi]$ we get $\eta = \eta'$, therefore $\theta = \theta'$.
Assume now that $\theta = \theta'$. Then, as in proposition 5.4.1, there exists a unitary map $\psi \in U(2)$ such that $\psi(L_j) = L'_j$ for $j = 1, 2, 3$, so that $\text{meas}(L'_2, L'_3) = \text{meas}(L_2, L_3)$. □

5.4.4 Two-dimensional unitary representations of $\pi_1(S^2 \backslash \{s_1, s_2, s_3\})$

The purpose of this subsection is twofold :

- to show that when $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset U(2)$ are three conjugacy classes of the unitary group $U(2)$ such that $Hom_{\mathcal{C}}(\pi_1(S^2 \backslash \{s_1, s_2, s_3\}), U(2)) \neq \emptyset$ (that is, when there exist two-dimensional unitary representations of $\pi_1(S^2 \backslash \{s_1, s_2, s_3\})$ with generators lying in the prescribed conjugacy classes) then there exist two-dimensional Lagrangian representations for the same conjugacy classes (that is, there exist three Lagrangian subspaces of $\mathbb{C}^2$ with prescribed angles $(L_j, L_{j+1}) \sim \mathcal{C}_j$).

- to determine, using Lagrangian representations, explicit necessary and sufficient conditions on such $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset U(2)$ for the representation space $\text{Hom}_{\mathcal{C}}(\pi_1(S^2 \backslash \{s_1, s_2, s_3\}), U(2))$ to be non-empty.

The point of doing so is first to show that, in the very particular case where $n = 2$ and $l = 3$, we can prove the existence of Lagrangian representations by elementary methods, whereas for the general case of a compact Lie group $(U, (.\,|\,))$ and arbitrary $l \geq 1$, we will need to prove a real convexity theorem for group-valued momentum maps (see chapter 8). Second, the analysis of this particular situation will give a simple and geometric interpretation of the conditions on $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset U(2)$ for $\text{Hom}_{\mathcal{C}}(\pi_1(S^2 \backslash \{s_1, s_2, s_3\}), U(2))$ to be non-empty, using spherical geometry and the results of subsection 5.4.2. This result is to be compared with other answers to this question given for instance in [JW92], [Gal97] and [Bis98]. In particular, in [Gal97], Gallitzer deals with the case of an arbitrary l using spherical polygons.

Before entering the detailed statements and proofs, we would like to add one more comment. As it turns out, we will prove that when $n = 2$ and $l = 3$, *every* (two-dimensional) unitary representation (of $\pi_1(S^2 \backslash \{s_1, s_2, s_3\})$) is in fact Lagrangian. This is a very special case, as we will see in chapter 9, where we will prove that the set of Lagrangian representations is actually a Lagrangian submanifold of the representation space $\text{Hom}_{\mathcal{C}}(\pi, U)/U$ (in particular its dimension is half the dimension of the representation space). To see why there is no contradiction in this, one has to notice that when $n = 2$ and $l = 3$, this representation space is zero-dimensional and actually reduces to a point as it is connected. This was for instance proved in [FW], where dimensions of representation spaces for $U = U(n)$ and arbitrary choice of conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l \subset U(n)$ were computed, and where the Lagrangian nature of decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ in $U = U(n)$ was also proved (see also [HL03] and[HL04] for a study of connected components of representation varieties). We now have the following result :

Proposition 5.4.11 (Every two-dimensional unitary representation of $\pi_1(S^2 \backslash \{s_1, s_2, s_3\})$ is decomposable). *Let $u_1, u_2, u_3 \in U(2)$ be three unitary 2×2 matrices satisfying $u_1 u_2 u_3 = 1$. Then there exist three Lagrangian subspaces L_1, L_2, L_3 of $\mathbb{C}^2$ such that $u_1 = \sigma_1 \sigma_2$, $u_2 = \sigma_2 \sigma_3$ and $u_3 = \sigma_3 \sigma_1$, where σ_j is the Lagrangian involution associated to L_j.*

Proof. Denote by $(e^{i2\lambda_j}, e^{i2\mu_j})$ the eigenvalues of u_j (where $\pi > \lambda_j \geq \mu_j \geq 0$) and let (v_j, w_j) be a basis of eigenvectors for u_j : $u_j(v_j) = e^{i2\lambda_j} v_j$ and $u_j(w_j) = e^{i2\mu_j} w_j$. Let l_1 be a great circle of $\mathbb{C}\text{P}^1 \simeq S^2$ containing both $[w_1]$ and $[w_3]$ (such a great circle always exists). Define l_3 to be unique great circle of S^2 such that l_1 is the image of l_3 by the rotation of angle $\lambda_3 - \mu_3 \in [0, \pi]$ around $[w_3]$ and choose a

Lagrangian subspace of $L_1 \in p^{-1}(l_1)$ arbitrarily. Set $L := \mathbb{R}v_3 \oplus \mathbb{R}w_3$. Since (v_3, w_3) is a unitary basis of $\mathbb{C}^2$, we have that L is Lagrangian. Set $l := p(L)$. Then $[w_3] \in l$ and consequently l_1 is the image of l by a rotation $\widehat{\psi} \in SO(3)$ around $[w_3]$ for some $\psi \in SU(2)$. Set $L' = \psi(L)$. Then L' is a Lagrangian subspace of $\mathbb{C}^2$ and $p(L') = \widehat{\psi}(l) = l_1$. Therefore, by proposition 5.4.2, we have $L_1 = e^{i\theta}.L'$ for some $\theta \in [0, \pi[$. Set $v_3' := e^{i\theta}v_3 \in L_1$ and $w_3' := e^{i\theta}w_3 \in L_1$. Then (v_3', w_3') is an orthonormal basis of L_1. Set $L_3 := \mathbb{R}e^{i\lambda_3}v_3' \oplus \mathbb{R}e^{i\mu_3}w_3'$. Then L_3 is Lagrangian (and one may check that, by proposition 5.4.2, $p(L_3)$ is the image of $p(L) = l_1$ by the rotation of angle $(\lambda_3 - \mu_3)$ around $[w_3'] = [w_3]$, so that $p(L_3) = l_3$). Finally, denote by σ_j the Lagrangian involution associated to L_j for $j = 1$ and $j = 3$. Then :

$$\begin{aligned}
\sigma_3 \circ \sigma_1(v_3') &= \sigma_3(v_3') \\
&= \sigma_3(e^{-i\lambda_3}e^{i\lambda_3}v_3') \\
&= e^{i\lambda_3}\sigma_3(e^{i\lambda_3}v_3') \\
&= e^{i2\lambda_3}v_3'
\end{aligned}$$

so that $\sigma_3\sigma_1(e^{i\theta}v_3) = e^{i2\lambda_3}e^{i\theta}v_3$ and therefore $\sigma_3\sigma_1(v_3) = e^{i2\lambda_3}v_3 = u_3(v_3)$. Likewise, $\sigma_3\sigma_1(w_3) = u_3(w_3)$, so that $\sigma_3\sigma_1 = u_3$.
Now, set $L_0 := \mathbb{R}v_1 \oplus \mathbb{R}w_1$. As earlier, L_0 is Lagrangian and we set $l_0 := p(L_0)$. Then $[w_1] \in l_0$, so that l_1 is the image of l_0 by a rotation $\widehat{\psi_0} \in SO(3)$ around $[w_1]$, where $\psi_0 \in SU(2)$ is a special unitary map having v_1 and w_1 as eigenvectors (since $\widehat{\psi_0}$ is a rotation around $[w_1] = -[v_1] \in \mathbb{R}^3$) : $\psi_0(v_1) = \alpha v_1$ and $\psi_0(w_1) = \beta w_1$ for some $\alpha, \beta \in S^1$. Set $L_0' := \mathbb{R}\alpha v_1 \oplus \mathbb{R}\beta w_1 = \psi_0(L_0)$. Then L_0' is a Lagrangian subspace of $\mathbb{C}^2$ and $p(L_0') = \widehat{\psi_0}(l_0) = l_1$. But $L_1 \in p^{-1}(l_1)$ so that, by proposition 5.4.2, $L_1 = e^{i\theta'}L_0'$ for some $\theta' \in [0, \pi[$. Set $v_1' := e^{i\theta'}\alpha v_1$ and $w_1' := e^{i\theta'}\beta w_1$. Then (v_1', w_1') is an orthonormal basis of L_1. Set $L_2 := \mathbb{R}e^{-i\lambda_1}v_1' \oplus \mathbb{R}e^{-i\mu_1}w_1'$. Then L_2 is Lagrangian (and one may check that $p(L_2)$ is the image of l_1 by the rotation of angle $\lambda_1 - \mu_1$ around $[w_1'] = [w_1]$). Finally, denote by σ_2 the Lagrangian involution associated to L_2. Then one has :

$$\begin{aligned}
\sigma_1 \circ \sigma_2(v_1') &= \sigma_1\sigma_2(e^{i\lambda_1}e^{-i\lambda_1}v_1') \\
&= \sigma_1(e^{-i\lambda_1}\sigma_2(e^{-i\lambda_1}v_1')) \\
&= \sigma_1(e^{-i2\lambda_1}v_1') \\
&= e^{i2\lambda_1}\sigma_1(v_1') \\
&= e^{i2\lambda_1}v_1'
\end{aligned}$$

so that $\sigma_1\sigma_2(v_1) = \sigma_1\sigma_2(e^{-i\theta'}\alpha^{-1}v_1') = e^{-i\theta'}\alpha^{-1}\sigma_1\sigma_2(v_1') = e^{-i\theta'}\alpha^{-1}e^{i2\lambda_1}v_1' = e^{i2\lambda_1}v_1 = u_1(v_1)$. Likewise, $\sigma_1\sigma_2(w_1) = u_1(w_1)$, so that $\sigma_1\sigma_2 = u_1$. Therefore : $\sigma_2\sigma_3 = (\sigma_2\sigma_1)(\sigma_1\sigma_3) = u_1^{-1}u_3^{-1} = u_2$ since $u_1u_2u_3 = 1$. □

Corollary 5.4.12 (Inequalities for $n = 2$ and $l = 3$). *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset U(2)$ be three conjugacy classes of the unitary group $U(2)$ given by the respective eigenvalues $(e^{i2\lambda_1}, e^{i2\mu_1})$, $(e^{i2\lambda_2}, e^{i2\mu_2})$ and $(e^{i2\lambda_3}, e^{i2\mu_3})$, where $\pi \geq \lambda_j \geq \mu_j \geq 0$. Set*

$$\delta := (\lambda_1 + \mu_1) + (\lambda_2 + \mu_2) + (\lambda_3 + \mu_3)$$

as well as

$$\alpha_j := \lambda_j - \mu_j \in [0, \pi] \quad \textit{and} \quad \beta_j := \pi - \alpha_j \in [0, \pi]$$

Then, there exist three unitary matrices $u_1, u_2, u_3 \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ satisfying $u_1u_2u_3 = 1$ if and only if :

$$\begin{aligned}
&\delta \equiv \pi \pmod{2\pi} \quad \textit{and} \quad (\alpha_1, \alpha_2, \alpha_3) \in \overline{\Delta} \\
\textit{or} \\
&\delta \equiv 0 \pmod{2\pi} \quad \textit{and} \quad (\beta_1, \beta_2, \beta_3) \in \overline{\Delta}
\end{aligned}$$

where $\overline{\Delta}$ is the tetrahedron defined in 5.4.2.1 (see figure 5.3).

Proof. By proposition 5.4.11, there exist three 2×2 unitary matrices $u_1, u_2, u_3 \in U(2)$ satisfying $u_j \in \mathcal{C}_j$ and $u_1 u_2 u_3 = 1$ if and only if there exist three Lagrangian subspaces L_1, L_2, L_3 of $\mathbb{C}^2$ satisfying $\sigma_j \sigma_{j+1} \in \mathcal{C}_j$, where σ_j is the Lagrangian involution associated to L_j for $j = 1, 2, 3$ and where $\sigma_4 := \sigma_1$. By proposition 5.4.6, such a triple exists if and only if :

$$\begin{array}{ll} & \delta \equiv \pi \pmod{2\pi} \quad \text{and} \quad (\alpha_1, \alpha_2, \alpha_3) \in \overline{\Delta} \\ \textit{or} & \\ & \delta \equiv 0 \pmod{2\pi} \quad \textit{and} \quad (\beta_1, \beta_2, \beta_3) \in \overline{\Delta} \end{array}$$

□

This result provides a set of necessary and sufficient conditions for $\mathrm{Hom}_{\mathcal{C}}(\pi_1(S^2 \backslash \{s_1, s_2, s_3\}), U(2))$ to be non-empty. These conditions are linear inequalities to be satisfied by the arguments of the eigenvalues defining the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subset U(2)$. They had already been obtained by Jeffrey and Weitsman in [JW92], by Gallitzer in [Gal97] and by Biswas in [Bis98], among others. For the case of arbitrary l and n, we refer for instance to [AW98, Bel01].

5.5 Angles of Lagrangian subspaces and computation of the inertia index of a Lagrangian triple

In this section, we give a formula to compute the inertia index $\tau(L_1, L_2, L_3)$ of a triple (L_1, L_2, L_3) of Lagrangian subspaces of a Hermitian vector space (see proposition 5.5.10). This formula relates the index to the measures of the angles (L_1, L_2), (L_2, L_3) and (L_3, L_1), as defined in section 5.3. The point of doing this is to show how the additional structure provided in this case by a compatible complex structure gives a new description of the symplectic invariant $\tau(L_1, L_2, L_3)$.

5.5.1 Basic properties of the inertia index

In contrast with the corresponding situation for pairs of Lagrangian subspaces, the orbit of a triple (L_1, L_2, L_3) of Lagrangian subspaces of a $2n$-dimensional symplectic vector space (V, ω) under the diagonal action of the symplectic group $Sp(V)$ is *not* characterized by the integers $n_{12} = \dim(L_1 \cap L_2)$, $n_{23} = \dim(L_2 \cap L_3)$, $n_{31} = \dim(L_3 \cap L_1)$ and $n_0 = \dim(L_1 \cap L_2 \cap L_3)$, which are invariants of this action. To classify the orbits, one introduces the notion of *inertia index* (sometimes called *Maslov index*, or simply index, or signature) of a Lagrangian triple (L_1, L_2, L_3). For the following definition and properties of the inertia index, we refer to [KS90], pp.486 sqq (see also [LM87], [LV80] and [Sou76]).

Definition 5.5.1 (Inertia index). The *inertia index* of the Lagrangian triple (L_1, L_2, L_3), denoted by $\tau(L_1, L_2, L_3)$, is the signature of the quadratic form q defined on the $3n$-dimensional vector space $L_1 \times L_2 \times L_3$ by : $q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$.

In a suitable basis of $L_1 \times L_2 \times L_3$, one can represent q by a diagonal matrix whose entries consist of r terms equal to $+1$, s terms equal to -1 and $3n - r - s$ terms equal to 0, the integers r and s being independent from the choice of the basis. What is called *signature* of q here, and denoted by $\mathrm{sgn}(q)$, is the integer $\mathrm{sgn}(q) := r - s$. From the definition, we see that for any symplectic map $\psi \in Sp(n)$, we have $\tau(\psi(L_1), \psi(L_2), \psi(L_3)) = \tau(L_1, L_2, L_3)$. We summarize here some of the properties of the inertia index that we will need in the following.

Proposition 5.5.2 ([KS90]). *The inertia index has the following properties :*

(i) $\tau(L_1, L_2, L_3) \equiv n - (n_{12} + n_{23} + n_{31}) \pmod{2\mathbb{Z}}$

(ii) $|\tau(L_1, L_2, L_3)| \leq n + 2n_0 - (n_{12} + n_{23} + n_{31})$

We may now state the theorem of symplectic classification of triples of Lagrangian subspaces of (V,ω), which is due to Kashiwara. For $d=(n_0,n_{12},n_{23},n_{31},\tau)\in\mathbb{N}^4\times\mathbb{Z}$, we set :

$$O_d=\left\{(L_1,L_2,L_3)\in\mathcal{L}(V)\times\mathcal{L}(V)\times\mathcal{L}(V)\ \left|\ \begin{array}{l}\dim(L_1\cap L_2\cap L_3)=n_0,\\ \dim(L_1\cap L_2)=n_{12},\\ \dim(L_2\cap L_3)=n_{23},\\ \dim(L_3\cap L_1)=n_{31},\\ \tau(L_1,L_2,L_3)=\tau\end{array}\right.\right\}$$

Proposition 5.5.3 (Symplectic classification of Lagrangian triples, [KS90], p.493)). *O_d is non-empty if and only if $d=(n_0,n_{12},n_{23},n_{31},\tau)$ satisfies the conditions:*

$$\begin{array}{ll}(i) & 0\le n_0\le n_1,n_2,n_3\le n\\ (ii) & n_{12}+n_{23}+n_{31}\le n+2n_0\\ (iii) & |\tau|\le n+2n_0-(n_{12}+n_{23}+n_{31})\\ (iv) & \tau\equiv n-(n_{12}+n_{23}+n_{31})\mod 2\mathbb{Z}\end{array}$$

If (L_1,L_2,L_3) and (L_1',L_2',L_3') are two triples of Lagrangian subspaces of V, there exists a symplectic map $\psi\in Sp(V)$ such that $\psi(L_1)=L_1'$, $\psi(L_2)=L_2'$ and $\psi(L_3)=L_3'$ if and only if $n_0=n_0'$, $n_{12}=n_{12}'$, $n_{23}=n_{23}'$, $n_{31}=n_{31}'$ and $\tau=\tau'$.

Thus, the diagonal action of $Sp(V)$ on $\mathcal{L}(V)\times\mathcal{L}(V)\times\mathcal{L}(V)$ has only finitely many orbits and these orbits are the O_d, where d satisfies conditions (i) to (iv) of proposition 5.5.3. We now specialize to the case where $V=\mathbb{C}^n$, that is, we endow the symplectic vector space (V,ω) with a compatible complex structure J and choose a unitary basis of (V,ω,J), and we show how in this context it is possible to compute the inertia index of a triple (L_1,L_2,L_3) from the angles (L_1,L_2), (L_2,L_3) and (L_3,L_1).

5.5.2 From angles to inertia index

We saw earlier (proposition 5.4.3) that the quantity $\delta=(\lambda_{12}+\mu_{12})+(\lambda_{23}+\mu_{23})+(\lambda_{31}+\mu_{31})$ defined for triples of Lagrangian subspaces of $\mathbb{C}^2$, satisfies $\delta\equiv 0\pmod\pi$ and contains information about the triple (L_1,L_2,L_3). Namely, if $\delta\equiv 0\pmod{2\pi}$ the triple (L_1,L_2,L_3) is positive (that is, setting $\varphi=\varphi_{31}\circ\varphi_{23}\circ\varphi_{12}$, we have $\det\varphi=e^{i\delta}=1>0$), and if $\delta\equiv\pi\pmod{2\pi}$ the triple is negative (that is $\det\varphi=e^{i\delta}=-1<0$). The interest of that notion was that the triple (l_1,l_2,l_3) of projective Lagrangians of $\mathbb{C}\mathrm{P}^1$ had same sign as (L_1,L_2,L_3) : if $\delta\equiv 0\pmod{2\pi}$ the transformation $\widehat{\varphi}=\widehat{\varphi_{31}}\circ\widehat{\varphi_{23}}\circ\widehat{\varphi_{12}}$ of $\mathbb{C}\mathrm{P}^1$ preserves a given orientation on l_1 (the triple (l_1,l_2,l_3) is then said to be positive), and if $\delta\equiv\pi\pmod{2\pi}$ then $\widehat{\varphi}$ reverses a given orientation (the triple (l_1,l_2,l_3) is said to be negative), and this enabled us to distinguish between positive and negative spherical triangles, which was essential in order to determine the image of the map $\kappa:(\mathcal{L}(2)\times\mathcal{L}(2)\times\mathcal{L}(2))/U(2)\to\mathbb{T}^2/\mathfrak{S}_2\times\mathbb{T}^2/\mathfrak{S}_2\times\mathbb{T}^2/\mathfrak{S}_2$. But δ can actually be defined for a triple of Lagrangian subspaces of $\mathbb{C}^n$ for any integer n. Since $\sigma_{L_j}^2=Id$, we have, for such a triple (L_1,L_2,L_3), the following relation : $(\sigma_{L_1}\circ\sigma_{L_3})\circ(\sigma_{L_3}\circ\sigma_{L_2})\circ(\sigma_{L_2}\circ\sigma_{L_1})=Id$, and the determinant of this unitary map is therefore of the form $e^{i2\delta}$ with $\delta\equiv 0\pmod\pi$. When $n=2$, the eigenvalues of the unitary map $\sigma_{L_k}\circ\sigma_{L_j}$ are $e^{i2\lambda_{jk}}$ and $e^{i2\mu_{jk}}$, so that we have indeed $\delta=(\lambda_{12}+\mu_{12})+(\lambda_{23}+\mu_{23})+(\lambda_{31}+\mu_{31})$. In the following, we shall consider a triple (L_1,L_2,L_3) of Lagrangian subspaces of $\mathbb{C}^n$, for arbitrary n. We shall denote the measures of the angles (L_1,L_2), (L_2,L_3) and (L_3,L_1) by $\mathrm{meas}(L_1,L_2)=[e^{i2\alpha_1},\ldots,e^{i2\alpha_n}]$, $\mathrm{meas}(L_2,L_3)=[e^{i2\beta_1},\ldots,e^{i2\beta_n}]$ and $\mathrm{meas}(L_3,L_1)=[e^{i2\gamma_1},\ldots,e^{i2\gamma_n}]$, where $\pi>\alpha_1\ge\ldots\ge\alpha_n\ge 0$, $\pi>\beta_1\ge\ldots\ge\beta_n\ge 0$ and $\pi>\gamma_1\ge\ldots\ge\gamma_n\ge 0$. We then have $\delta=\sum_{j=1}^n(\alpha_j+\beta_j+\gamma_j)$, where $e^{i2\delta}=1$ is the determinant of the unitary map $(\sigma_{L_1}\circ\sigma_{L_3})\circ(\sigma_{L_3}\circ\sigma_{L_2})\circ(\sigma_{L_2}\circ\sigma_{L_1})=Id$, so that $\delta\equiv 0\pmod\pi$. Since δ, which we shall also denote $\delta(L_1,L_2,L_3)$ to avoid confusion, is defined by means of the measures of the angles (L_j,L_k) (that is, up to permutation, the eigenvalues of the unitary maps $\sigma_{L_k}\circ\sigma_{L_j}$), δ is invariant under the diagonal action of the unitary group $U(n)$ on $\mathcal{L}(n)\times\mathcal{L}(n)\times\mathcal{L}(n)$: if $\varphi\in U(n)$, then $\delta(\varphi(L_1),\varphi(L_2),\varphi(L_3))=\delta(L_1,L_2,L_3)$. We now study properties of δ and show that it is in fact a *symplectic* invariant of (L_1,L_2,L_3)

Lemma 5.5.4. *If $c : t \in [0,1] \mapsto (L_1(t), L_2(t), L_3(t)) \in \mathcal{L}(n) \times \mathcal{L}(n) \times \mathcal{L}(n)$ is a continuous map such that the dimensions $n_{jk}(t) = \dim(L_j(t) \cap L_k(t))$ of the pairwise intersections are constant functions of t, then the map $\delta : t \longmapsto \delta(L_1(t), L_2(t), L_3(t))$ is a constant map.*

Observe that this result is also true for the inertia index (see [KS90], pp.487-488).

Proof of lemma 5.5.4. Since the n_{jk}'s remain constant along the deformation, the non-zero $\alpha_j(t), \beta_j(t)$ and $\gamma_j(t)$ vary continuously. Therefore, $\delta(L_1, L_2, L_3)$ varies continuously. As $\delta(t) \equiv 0 \pmod{\pi}$, δ is a constant map. □

Proposition 5.5.5. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^n$ and let $\psi \in Sp(n)$ be a symplectic map. Then $\delta(\psi(L_1), \psi(L_2), \psi(L_3)) = \delta(L_1, L_2, L_3)$, that is : δ is a symplectic invariant.*

Proof. Since the symplectic group is connected, there exists a continuous path $t \in [0,1] \mapsto \psi_t \in Sp(n)$ such that $\psi_0 = Id$ and $\psi_1 = \psi$. For all $t \in [0,1]$, set $L_j(t) = \psi_t(L_j)$ for $j = 1, 2, 3$. As ψ_t is invertible, $n_{12}(t), n_{23}(t)$ and $n_{31}(t)$ are constant, and, by lemma 5.5.4, so is $\delta(t)$, so that $\delta(\psi(L_1), \psi(L_2), \psi(L_3)) = \delta(1) = \delta(0) = \delta(L_1, L_2, L_3)$. □

Since we now know that δ is a symplectic invariant of the triple (L_1, L_2, L_3), it is natural to try and relate it to the inertia index τ. To be able to do so, we need to learn to compute δ for a particular class of Lagrangian triples called *exceptional triples* :

Definition 5.5.6. A triple (L_1, L_2, L_3) of Lagrangian subspaces of $\mathbb{C}^n$ is said to be an *exceptional triple* if the unitary maps φ_{12} and φ_{13} (defined as in proposition 5.3.5) have the same eigenspaces.

As can be seen from the case $n = 2$, a triple (L_1, L_2, L_3) is generically not exceptional (since being exceptional in this case requires either that φ_{12} and φ_{13} have non-distinct eigenvalues or, in the notation of proposition 5.4.1, that $\theta \equiv 0 \pmod{\frac{\pi}{2}}$), which justifies the terminology. Exceptional Lagrangian triples of $\mathbb{C}^2$ project to triples (l_1, l_2, l_3) of great circles of S^2 which have at least one common diameter (the angles between any two of them may have measure 0, see figures 5.4 and 5.5).
The interest of the notion of exceptional Lagrangian triple is first that we know how to compute δ for those triples (and we shall soon see that the inertia index is computed very similarly for such triples, see lemmas 5.5.7 and 5.5.8), and second that every Lagrangian triple is symplectically equivalent to an exceptional triple (which then has same δ, see proposition 5.5.9).

Lemma 5.5.7. *Let (L_1, L_2, L_3) be an exceptional triple of Lagrangian subspaces of $\mathbb{C}^n$. Denote by $(u_1, \ldots, u_n)$ an orthonormal basis of L_1 formed of eigenvectors of φ_{12} : $\varphi_{12}(u_k) = e^{i\alpha_k} u_k$, where $[e^{i2\alpha_1}, \ldots, e^{i2\alpha_n}] = \mathrm{meas}(L_1, L_2)$. For all k, set $\mathbb{C}^{(k)} := \mathbb{C}u_k$, $d_1^k := L_1 \cap \mathbb{C}^{(k)}$, $d_2^k := L_2 \cap \mathbb{C}^{(k)}$ and $d_3^k := L_3 \cap \mathbb{C}^{(k)}$. Then d_1^k, d_2^k and d_3^k are real lines of $\mathbb{C}^{(k)}$ and, if one denotes by $\mathrm{meas}(d_1^k, d_2^k)$, $\mathrm{meas}(d_k^2, d_k^3)$ and $\mathrm{meas}(d_3^k, d_1^k) \in [0, \pi[$ the measures of the oriented angles (d_1^k, d_2^k), (d_2^k, d_3^k) and (d_3^k, d_1^k) in $\mathbb{C}^{(k)}$, one has :*

$$\delta(L_1, L_2, L_3) = \sum_{k=1}^{n} \left(\mathrm{meas}(d_1^k, d_2^k) + \mathrm{meas}(d_k^2, d_k^3) + \mathrm{meas}(d_3^k, d_1^k) \right)$$

Proof. Set $\mathrm{meas}(L_1, L_3) = [e^{i2\varepsilon_1}, \ldots, e^{i2\varepsilon_n}]$. Observe first that L_1 intersects the complex line $\mathbb{C}^{(k)} = \mathbb{C}u_k$ because $u_k \in L_1$. Since $(u_1, \ldots, u_n)$ is a basis of L_1 formed of eigenvectors of φ_{12}, and since φ_{12} and φ_{13} have the same eigenspaces, there exists a permutation $g \in \mathfrak{S}_n$ such that, for all $k \in \{1, \ldots, n\}$, $\varphi_{13}(u_k) = e^{i\varepsilon_{g(k)}} u_k \in L_3$. Therefore, we have $e^{i\alpha_k} u_k \in L_2$ and $e^{i\varepsilon_k} u_k \in L_3$, so that $\mathbb{C}^{(k)}$ also intersects both L_2 and L_3. But if $u \in \mathbb{C}^n \backslash \{0\}$ is contained in a Lagrangian subspace L of $\mathbb{C}^n$ then $L \cap \mathbb{C}u = \mathbb{R}u$. Indeed, if $v \in L \cap \mathbb{C}u$ then $v = \lambda u + \mu J u$ with $\lambda, \mu \in \mathbb{R}$, and since L is Lagrangian $\omega(u, v) = 0$. But $\omega(u, v) = \lambda\omega(u, u) + \mu\omega(u, Ju) = \mu g(u, u)$ with $g(u, u) \neq 0$, therefore $v = \lambda u \in \mathbb{R}u$. Therefore, since $\varphi_{12}(u_k) = e^{i\alpha_k} u_k \in L_2$, we have $d_1^k = L_1 \cap \mathbb{C}u_k = \mathbb{R}u_k$ and $d_k^2 = L_2 \cap \mathbb{C}u_k = \mathbb{R}(e^{i\alpha_k} u_k) = e^{i\alpha_k} d_1^k$, hence $\mathrm{meas}(d_1^k, d_2^k) = \alpha_k \in [0, \pi[$. Likewise, since $e^{i\varepsilon_{g(k)}} u_k \in L_3$, we have $d_3^k = e^{i\varepsilon_{g(k)}} d_1^k$, so that $\mathrm{meas}(d_1^k, d_3^k) = \varepsilon_{g(k)}$, hence, setting $\xi_k = \pi - \varepsilon_{g(k)} \mod \pi$, $\mathrm{meas}(d_3^k, d_1^k) = \xi_k \in [0, \pi[$. Setting $w_k = e^{i\varepsilon_{g(k)}} u_k \in L_3$, we have $e^{i\xi_k} w_k = \pm e^{i(\pi - \varepsilon_{g(k)})} w_k = \pm u_k \in L_1$. The $(e^{i\xi_k})$ therefore are the roots of the characteristic

polynomial $P(L_3, L_1)$ of the pair (L_3, L_1), hence $[e^{i2xi_1}, \ldots, e^{i2\xi_n}] = \text{meas}(L_3, L_1) = [e^{i2\gamma_1}, \ldots, e^{i2\gamma_n}]$, and since $\xi_k, \gamma_k \in [0, \pi[$, there exists a permutation $g_3 \in \mathfrak{S}_n$ such that, for all k, $\xi_k = \gamma_{g_3(k)}$. Similarly, setting $v_k = e^{i\alpha_k} u_k \in L_2$ and $\zeta_k = (\varepsilon_{g(k)} - \alpha_k) \mod \pi$, we have $e^{i\zeta_k} v_k = \pm e^{i\varepsilon_{g(k)}} u_k \in L_1$, hence $[e^{i2\zeta_1}, \ldots, e^{i2\zeta_n}] = \text{meas}(L_2, L_3) = [e^{i2\beta_1}, \ldots, e^{i2\beta_n}]$, and since $\zeta_k, \beta_k \in [0, \pi[$, there exists $g_2 \in \mathfrak{S}_n$ such that, for all k, $\zeta_k = \beta_{g_2(k)}$. Furthermore, since $d_2^k = \mathbb{R}v_k$ and $d_3^k = \mathbb{R}e^{i\varepsilon_{g(k)}} u_k = \mathbb{R}e^{i\zeta_k} v_k$, we have $\text{meas}(d_k^2, d_3^k) = \zeta_k$. Hence :

$$\begin{aligned}
\sum_{k=1}^{n} (\text{meas}(d_1^k, d_2^k) + \text{meas}(d_2^k, d_3^k) + \text{meas}(d_3^k, d_1^k)) &= \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \zeta_k + \sum_{k=1}^{n} \xi_k \\
&= \sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \beta_{g_2(k)} + \sum_{k=1}^{n} \gamma_{g_3(k)} \\
&= \sum_{k=1}^{n} (\alpha_k + \beta_k + \gamma_k) \\
&= \delta(L_1, L_2, L_3)
\end{aligned}$$

□

Lemma 5.5.8. *Let (L_1, L_2, L_3) be an exceptional triple of Lagrangian subspaces of $\mathbb{C}^n$. Define d_1^k, d_2^k, d_3^k as in lemma 5.5.7 for $k = 1, \ldots, n$. Let τ_k be the inertia index of the Lagrangian triple (d_1^k, d_2^k, d_3^k) in the complex line $\mathbb{C}^{(k)}$. Then $\tau(L_1, L_2, L_3) = \sum_{k=1}^{n} \tau_k$.*

Proof. Since, with the notations introduced in lemma 5.5.7, we have $L_1 = d_1^1 \oplus \cdots \oplus d_n^1$, $L_2 = d_1^2 \oplus \cdots \oplus d_n^2$ and $L_3 = d_1^3 \oplus \cdots \oplus d_n^3$, and since $\mathbb{C}^n = \mathbb{C}^{(1)} \oplus \cdots \oplus \mathbb{C}^{(n)}$ is the symplectic direct sum of the $\mathbb{C}^{(k)}$, d_1^k, d_2^k, d_3^k being Lagrangian in the symplectic space $\mathbb{C}^{(k)}$, we have, by definition of the index, $\tau(L_1, L_2, L_3) = \sum_{k=1}^{n} \tau(d_1^k, d_2^k, d_3^k) = \sum_{k=1}^{n} \tau_k$. □

Proposition 5.5.9. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^n$. Then there exists an exceptional triple (L_1', L_2', L_3') and a symplectic map $\psi \in Sp(n)$ such that $L_j' = \psi(L_j)$ for $j = 1, 2, 3$. In particular, by proposition 5.5.5, $\delta(L_1, L_2, L_3) = \delta(L_1', L_2', L_3')$.*

This means that each orbit O_d of the diagonal action of the symplectic group $Sp(n)$ on $\mathcal{L}(n) \times \mathcal{L}(n) \times \mathcal{L}(n)$ contains at least one exceptional triple.

Proof of proposition 5.5.9. Set $\tau = \tau(L_1, L_2, L_3)$. We are now going to construct an exceptional triple (L_1', L_2', L_3') such that $\tau(L_1', L_2', L_3') = \tau$, and such that $\dim(L_1' \cap L_2' \cap L_3') = \dim(L_1 \cap L_2 \cap L_3)$ and $\dim(L_j' \cap L_k') = \dim(L_j \cap L_k)$ for all j, k. By theorem 5.5.3, there will then exist a symplectic map $\psi \in Sp(n)$ such that $\psi(L_j) = L_j'$ for $j = 1, 2, 3$.
As earlier, set $n_0 = \dim(L_1 \cap L_2 \cap L_3)$, $n_{jk} = \dim(L_j \cap L_k)$. Recall that $n_{12} + n_{23} + n_{31} \leq n + 2n_0$. Let $(u_1, \ldots, u_n)$ be the standard basis of $\mathbb{C}^n$ over $\mathbb{C}$ and let L_1' be the Lagrangian subspace $L_1' = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_n$.

- for $k \in \{1, \ldots, n_0\}$, set $v_k = w_k = u_k$
- for $k \in \{n_0 + 1, \ldots, n_{12}\}$, set $v_k = u_k$ and $w_k = e^{i\frac{\pi}{4}} u_k$
- for $k \in \{n_{12} + 1, \ldots, n_{12} + n_{23} - n_0\}$, set $v_k = w_k = e^{i\frac{\pi}{2}} u_k$
- for $k \in \{n_{12} + n_{23} - n_0 + 1, \ldots, n_{12} + n_{23} + n_{31} + 2n_0\}$, set $w_k = u_k$ and $v_k = e^{i\frac{\pi}{2}} u_k$

Since $|\tau| \leq n + 2n_0 - (n_{12} + n_{23} + n_{31})$ and $\tau \equiv n - (n_{12} + n_{23} + n_{31}) \mod 2\mathbb{Z}$, τ can be written as a sum

$$\tau = \sum_{k=n_{12}+n_{23}+n_{31}-2n_0+1}^{n} \tau_k$$

of $n + 2n_0 - (n_{12} + n_{23} + n_{31})$ summands $\tau_k = \pm 1$. One then has :

- for k such that $\tau_k = -1$, set $v_k = e^{i\frac{\pi}{2}}u_k$ and $w_k = e^{i\frac{\pi}{4}}u_k$
- for k such that $\tau_k = 1$, set $v_k = e^{i\frac{\pi}{2}}u_k$ and $w_k = e^{i\frac{3\pi}{4}}u_k$

Now set, for all k, $d_1^k = \mathbb{R}u_k$, $d_2^k = \mathbb{R}v_k$, and $d_3^k = \mathbb{R}w_k$. Then $L_1' = d_1^1 \oplus \cdots \oplus d_1^n$ and we set $L_2' = d_2^1 \oplus \cdots \oplus d_2^n$ and $L_3' = d_3^1 \oplus \cdots \oplus d_3^n$. Since $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are unitary bases of $\mathbb{C}^n$ over $\mathbb{C}$, L_2' and L_3' are Lagrangian. By construction, $\dim(L_1' \cap L_2' \cap L_3') = n_0$ and $\dim(L_j' \cap L_k') = n_{jk}$ for all j, k. For all k, set $\tau_k' = \tau(d_1^k, d_2^k, d_3^k)$. For $k \in \{1, \ldots, n_{12} + n_{23} + n_{31} - 2n_0\}$ there are always two non-distinct Lagrangians among the d_j^k, so that $\tau_k' = 0$. For $k \in \{n_{12} + n_{23} + n_{31} - 2n_0 + 1, \ldots, n\}$, we have by construction $\tau_k' = \tau_k$, with τ_k defined as above. Since, for all k, there exist $\alpha_k \in [0, \pi[$ and $\varepsilon_k \in [0, \pi[$ such that $e^{i\alpha_k}u_k \in L_2'$ and $e^{i\varepsilon_k}u_k \in L_3'$, each u_k is an eigenvector of both φ_{12} and φ_{13} and since $(u_1, \ldots, u_n)$ is a basis of L_1, these unitary maps have the same eigenspaces up to permutation, so that (L_1', L_2', L_3') is an exceptional triple. Therefore, by lemma 5.5.8 :

$$\tau(L_1', L_2', L_3') = \sum_{k=1}^{n} \tau_k' = \sum_{k=n_{12}+n_{23}+n_{31}-2n_0+1}^{n} \tau_k = \tau$$

This completes the proof, as indicated above. □

Incidentally, we have proved that O_d is non-empty when d satisfies the conditions of proposition 5.5.3.

We now have all the material that we need to relate δ to τ and show that the inertia index can be computed from the measures of the Lagrangian angles (L_1, L_2), (L_2, L_3) and (L_3, L_1), that is, from the eigenvalues of the unitary maps $\sigma_{L_k} \circ \sigma_{L_j}$, where σ_{L_j} is the Lagrangian involution associated to L_j.

Proposition 5.5.10. *Let (L_1, L_2, L_3) be a triple of Lagrangian subspaces of $\mathbb{C}^n$, and set $n_{jk} = \dim(L_j \cap L_k)$, $\tau = \tau(L_1, L_2, L_3)$ and $\delta = \delta(L_1, L_2, L_3)$. Then :*

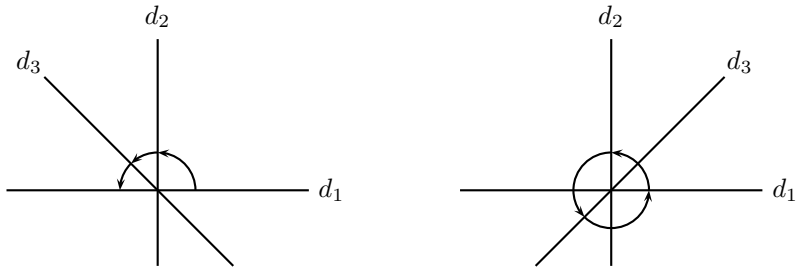
$$\tau = 3n - \frac{2\delta}{\pi} - (n_{12} + n_{23} + n_{31})$$

In particular, when $L_j \cap L_k = \{0\}$ for all j, k, one has :

$$\tau = 3n - \frac{2\delta}{\pi}$$

Proof. By proposition 5.5.9, there exists a symplectic map $\psi \in Sp(n)$ such that $(\psi(L_1), \psi(L_2), \psi(L_3))$ is an exceptional triple. Since such a transformation leaves τ, δ and the n_{jk} invariant, we may assume that (L_1, L_2, L_3) is itself exceptional. Let us recall the notations $\mathrm{meas}(L_1, L_2) = [e^{i2\alpha_1}, \ldots, e^{i2\alpha_n}]$ and $\mathrm{meas}(L_1, L_3) = [e^{i2\varepsilon_1}, \ldots, e^{i2\varepsilon_n}]$, where $\pi > \alpha_1 \geq \ldots \geq \alpha_n \geq 0$ and $\pi > \varepsilon_1 \geq \ldots \geq \varepsilon_n \geq 0$. Then, since (L_1, L_2, L_3) is exceptional, there exists an orthonormal basis $(u_1, \ldots, u_n)$ of L_1 and a permutation $g \in \mathfrak{S}_n$ such that $(e^{i\alpha_1}u_1, \ldots, e^{i\alpha_n}u_n)$ is an orthonormal basis of L_2 and $(e^{i\varepsilon_1}u_{g(1)}, \ldots, e^{i\varepsilon_n}u_{g(n)})$ is an orthonormal basis of L_3. By abandoning the condition $\pi > \varepsilon_1 \geq \ldots \geq \varepsilon_n \geq 0$, we may suppose that $g = Id$. Set $d_1^k = \mathbb{R}u_k$, $d_2^k = e^{i\alpha_k}d_1^k$, $d_3^k = e^{i\varepsilon_k}d_1^k$, and $\tau_k = \tau(d_1^k, d_2^k, d_3^k)$ in the symplectic space $\mathbb{C}u_k$. Set $\delta_k = \mathrm{meas}(d_1^k, d_2^k) + \mathrm{meas}(d_2^k, d_3^k) + \mathrm{meas}(d_3^k, d_1^k)$ and set, as in lemma 5.5.7, $\zeta_k = (\varepsilon_k - \alpha_k) \mod \pi$ and $\xi_k = (\pi - \varepsilon_k) \mod \pi$, so that $\delta_k = \alpha_k + \zeta_k + \xi_k$. Observe that $\delta_k = \delta(d_1^k, d_2^k, d_3^k)$ in the symplectic space $\mathbb{C}u_k$. In particular, this implies that $\delta_k \equiv 0 \mod \pi$. If $d_1^k = d_2^k = d_3^k$, which happens n_0 times, then $\tau_k = 0$ and $\delta_k = 0$. If either $d_1^k = d_2^k \neq d_3^k$ or $d_2^k = d_3^k \neq d_1^k$ or $d_3^k = d_1^k \neq d_2^k$, which happens $(n_{12} - n_0) + (n_{23} - n_0) + (n_{31} - n_0)$ times, then $\tau_k = 0$ and $0 < \delta_k = \alpha_k + \zeta_k + \xi_k < 2\pi$ (since one of these numbers is 0 and since all of them are $< \pi$ and two of them are non-zero), but $\delta_k \equiv 0 \mod \pi$ so $\delta_k = \pi$. If $d_1^k \neq d_2^k \neq d_3^k \neq d_1^k$, which happens $n + 2n_0 - (n_{12} + n_{23} + n_{31})$ times, then either $\tau_k = 1$ and $\delta_k = \pi$ or $\tau_k = -1$ and $\delta_k = 2\pi$, so that $\tau_k = 3 - \frac{2\delta_k}{\pi}$ (see figure 5.7). To sum up :

number of occurences	δ	τ
n_0	0	0
$(n_{12} - n_0) + (n_{23} - n_0) + (n_{31} - n_0)$	$\delta_k = \pi$	0
$n + 2n_0 - (n_{12} + n_{23} + n_{31})$	$\delta_k = \pm 1$	$3 - \frac{2\delta_k}{\pi}$



$\tau = 1$ and $\delta = \pi$ $\qquad$ $\tau = -1$ and $\delta = 2\pi$

Figure 5.7: Relation between δ and τ for exceptional triples of Lagrangians

Since (L_1, L_2, L_3) is an exceptional triple, we have, by proposition 5.5.7, $\delta = \sum_{k=1}^{n} \delta_k$. Likewise, $\tau = \sum_{k=1}^{n} \tau_k$, so that we have :

$$\begin{aligned}
\tau &= \sum_{k=1}^{n+2n_0-(n_{12}+n_{23}+n_{31})} (3 - \frac{2\delta_k}{\pi}) \\
&= 3(n + 2n_0 - (n_{12} + n_{23} + n_{31})) - \frac{2}{\pi} \sum_{k=1}^{n+2n_0-(n_{12}+n_{23}+n_{31})} \delta_k \\
&= 3(n + 2n_0 - (n_{12} + n_{23} + n_{31})) - \frac{2}{\pi}(\delta - \pi((n_{12} - n_0) + (n_{23} - n_0) + (n_{31} - n_0))) \\
&= 3n - \frac{2\delta}{\pi} - (n_{12} + n_{23} + n_{31})
\end{aligned}$$

$\square$

Chapter 6

Decomposable representations of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ as fixed-point set of an involution

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In this chapter, we obtain a characterization of decomposable representations of the fundamental group $\pi = \pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ in terms of the fixed-point set of an involution β defined on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. More precisely, we show that the σ_0-decomposable representations $(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ introduced in definition 5.2.1 are exactly the elements of the fixed-point set of an involution β. This enables us to characterize all decomposable representations in terms of β : they are the elements $u = (u_1, \ldots, u_l) \in \mathrm{Hom}_{\mathcal{C}}(\pi, U)$ satisfying $\beta(u) \sim u$ as representations of π.

To understand why an involution comes into play here, we use the geometric intuition on Lagrangian involutions acquired in chapter 5. Indeed, by continuing to use the Lie group $U = U(n)$ as a prototype, we formulate an infinitesimal version of the problem of knowing whether or not a given representation is decomposable (or in this case, Lagrangian, see definition 5.1.5). This will give us insight on why decomposable representations should be characterized using involutions, and even on why such an involution should induce an anti-symplectic involution on the moduli space $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi, U)/U$, as we shall see eventually in chapters 7 and 9.

The first five sections explain how these results (in particular the involution β) were obtained, but may be skipped if one wants to go straight to the characterization results of propositions 6.6.2 and 6.6.5, whose proofs may be read without knowledge of the previous sections.

Despite its relative short length, this chapter is *really* the core of this thesis work, around which everything else revolves, the key step being to find the involution β. The results obtained here and in chapter 9 were accepted for publication in [Sch06].

6.1 The infinitesimal picture and the momentum map approach

Let us recall our problem : given l unitary matrices $u_1, \ldots, u_l \in U(n)$ satisfying $u_j \in \mathcal{C}_j$ and $u_1 \ldots u_j = 1$, do there exist l Lagrangian subspaces $L_1, \ldots, L_l$ of $\mathbb{C}^n$ such that $\sigma_j \sigma_{j+1} = u_j$ (where σ_j is the Lagrangian involution associated with L_j and $\sigma_{l+1} = \sigma_1$) ? As was shown in proposition 5.3.10, the condition $\sigma_j \sigma_{j+1} \in \mathcal{C}_j$, which lies on the spectrum of the unitary map $\sigma_j \sigma_{j+1}$, can be interpreted geometrically as the measure of an angle between Lagrangian subspaces. The *Lagrangian problem* above can therefore be thought of as a configuration problem in the Lagrangian Grassmannian $\mathcal{L}(n)$ of $\mathbb{C}^n$: given eigenvalues $\exp(i\lambda_j)$, $\lambda_j \in \mathbb{R}^n$, do there exist l Lagrangian subspaces $L_1, \ldots, L_l$ such that measure$(L_j, L_{j+1}) = \exp(i\lambda_j)$? Under this geometrical form, the Lagrangian problem is slightly more general than our original representation theory problem. It is very much linked to the unitary problem studied for instance in [JW92, Gal97, Bis98, AW98, KM99, Bis99, Bel01], which is the following : given $\lambda_j \in \mathbb{R}^n$, do there exist l unitary matrices $u_1, \ldots, u_l$ satisfying $\operatorname{Spec} u_j = \exp(i\lambda_j)$ and $u_1 \ldots u_l = 1$? In fact, a solution $(L_1, \ldots, L_l)$ to the Lagrangian problem (second version) provides a solution $u_j = \sigma_j \sigma_{j+1}$ to the unitary problem.

The fact that the unitary problem admits a symplectic description (see for instance [AW98]) was our first motivation to study the Lagrangian problem from a symplectic point of view. The second motivation is derived from the above-given geometrical formulation of the problem. To better understand this, let us try and formulate an infinitesimal version of the Lagrangian problem. Take three Lagrangian subspaces L_1, L_2, L_3 close enough so that we can think of these points in $\mathcal{L}(n)$ as tangent vectors to $\mathcal{L}(n)$ at some point L_0 representing the center of mass of L_1, L_2, L_3. Tangent vectors to the Lagrangian Grassmannian are identified with real symmetric matrices S_1, S_2, S_3 and the center of mass condition then turns into $S_1 + S_2 + S_3 = 0$. It seems reasonable in this context to translate the angle condition measure$(L_j, L_{j+1}) = \exp(i\lambda_j)$ (that is, $\operatorname{Spec} \sigma_j \sigma_{j+1} = \exp(i\lambda_j)$) into the spectral condition $\operatorname{Spec} S_j = \lambda_j \in \mathbb{R}^n$. We then recognize a real version (replacing complex Hermitian matrices with real symmetric ones) of a famous problem in mathematics (see [Ful98] for a review of this problem and those related to it) : given $\lambda_j \in \mathbb{R}^n$, do there exist Hermitian matrices H_1, H_2, H_3 such that $\operatorname{Spec} H_j = \lambda_j$ and $H_1 + H_2 + H_3 = 0$? In fact, these last two problems are equivalent (meaning that, for given $(\lambda_j)_j$, one of them has a solution if and only if the other one does) and this can be shown in a purely symplectic framework (see [AMW01] and section 6.5) using momentum maps to translate the condition $H_1 + H_2 + H_3 = 0$ into $(H_1, H_2, H_3) \in \mu^{-1}(\{0\})$. Therefrom, it seems promising to try to think of the Lagrangian problem as a real version, in a sense that will be made precise in section 6.5, of the unitary problem (since a solution to the Lagrangian problem provides an obvious solution to the unitary problem). We shall come back to the infinitesimal picture later on in this work (see section 9.2), and formalize further the analogy between the Lagrangian problem and the symmetric problem (that is, the real version of the Hermitian problem above).

6.2 The centered Lagrangian problem

As a consequence of the above infinitesimal picture, we replace our Lagrangian problem with a *centered problem*, meaning that instead of measuring the angles (L_j, L_{j+1}), we measure the angles (L_0, L_j) where L_0 is the horizontal Lagrangian $L_0 = \mathbb{R}^n \subset \mathbb{C}^n$ (playing the role of an origin in $\mathcal{L}(n)$). Recall from chapter 5 (proposition 5.3.10) that this angle is measured by the spectrum of $\sigma_0 \sigma_j = u_j^t u_j$, where u_j is any unitary map sending L_j to L_0 (see proposition 5.1.3 for this last statement). We then ask the following question, which we will call the *centered Lagrangian problem* :

> given l conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l \subset U(n)$, does there exist l unitary matrices $u_1, \ldots, u_l$ such that $u_j^t u_j \in \mathcal{C}_j$ and $u_1 \ldots u_l = 1$?

The main observation here is then to see that the condition $\operatorname{Spec} u^t u = \exp(i\lambda)$, for some $\lambda \in \mathbb{R}^n$ (that is, $u^t u$ lies in some fixed conjugacy class of $U(n)$) means that u belongs to a fixed orbit of the action of $O(n) \times O(n)$ on $U(n)$ given by $(k_1, k_2).u = k_1 u k_2^{-1}$, as is shown by the following result :

Lemma 6.2.1. *For any $u, v \in U(n)$, $\operatorname{Spec} u^t u = \operatorname{Spec} v^t v$ if and only if there exists $(k_1, k_2) \in O(n) \times O(n)$ such that $v = k_1 u k_2^{-1}$.*

Proof. Take $u, v \in U(n)$ and suppose that $\operatorname{Spec} u^t u = \operatorname{Spec} v^t v$ and set $\Delta_u := u^t u$ and $\Delta_v = v^t v$. Then, Δ_u and Δ_v are symmetric unitary matrices with the same spectrum so that, by proposition 5.1.3 (and the fact that the entries of a diagonal matrix can be permuted by conjugating by an appropriate $SO(n)$ element, see lemma 3.2.4), there exist orthogonal matrices $k_1, k_2 \in O(n)$ such that $k_1 \Delta_u k_1^{-1} = k_2 \Delta_v k_2^{-1}$ is a *diagonal* unitary matrix. Now denote by $\sqrt{\Delta_u}$ (resp. $\sqrt{\Delta_v}$) any symmetric unitary matrix satisfying $(\sqrt{\Delta_u})^2 = \Delta_u$ (resp. $(\sqrt{\Delta_v})^2 = \Delta_v$). Such matrices always exist since $k_1 \Delta_u k_1^{-1}$ is diagonal and is therefore the exponential of i times some real symmetric matrix : for instance $k_1 \Delta_u k_1^{-1} = \exp(iS)$ where $S = diag(\lambda_1, \ldots, \lambda_n)$ is a real diagonal matrix, and we set $\sqrt{\Delta_u} := k_1^{-1} \exp(i\frac{S}{2}) k_1$, which is unitary and symmetric. Since $k_1 \Delta_u k_1^{-1} = k_2 \Delta_v k_2^{-1}$, we then have $\sqrt{\Delta_v} = k \sqrt{\Delta_u} k^{-1}$ for $k := k_2^{-1} k_1 \in O(n)$. Set now $k_u := u(\sqrt{\Delta_u})^{-1}$ and $k_v := u(\sqrt{\Delta_v})^{-1}$. Then it is immediate that $k_u^t k_u = 1$ and $k_v^t k_v = 1$, and we then have :

$$v = k_v \sqrt{\Delta_v} = k_v k \sqrt{\Delta_u} k^{-1} = (k_v k k_u^{-1}) k_u \sqrt{\Delta_u} k^{-1} = \underbrace{(k_v k k_u^{-1})}_{\in O(n)} \, u \, \underbrace{k^{-1}}_{\in O(n)}$$

The converse implication is obvious. □

Remark 6.2.2 (Another proof of lemma 6.2.1, using Lagrangian subspaces). The proof of lemma 6.2.1 is directly modelled on the proof of proposition 3.2.7. However, we can take advantage of the fact that we are working with the unitary group to write a proof which is different in spirit. To prove the non-obvious implication, set, as in proposition 5.1.3, $L = u^{-1}(L_0)$ and $L' = v^{-1}(L_0)$. Then one has $u^t u = \sigma_{L_0} \sigma_L$ and $v^t v = \sigma_{L_0} \sigma_{L'}$. Assuming that $\operatorname{Spec} u^t u = \operatorname{Spec} v^t v$, proposition 5.3.10 then shows that there exists $\psi \in U(n)$ such that $\psi(L_0) = L_0$ and $\psi(L) = L'$. Since $\psi(L_0) = L_0$, one actually has $\psi \in Stab(L_0) = O(n)$ and we set $k_2 := \psi$. Then $L_0 = v(L') = v k_2(L) = v k_2 u^{-1}(L_0)$, and therefore $k_1 := v k_2 u^{-1} \in Stab(L_0) = O(n)$, hence $v = k_1 u k_2^{-1}$ with $k_1, k_2 \in O(n)$.

Since we think of the above problem as a real version of some complex problem, we now wish to find this complex version, which is done by abstracting a bit our situation to put it in the appropriate framework, which turns out to be adopting a Lie-theoretic viewpoint.

6.3 Complexification of the centered Lagrangian problem

Let us formulate the centered Lagrangian problem in greater generality. For everything regarding the theory of Lie groups and symmetric spaces, especially regarding real forms and duality, we refer to [Hel01] (see also [Loo69b] and chapter 3).

We start with a real Lie group H. Let $G = H^{\mathbb{C}}$ be its complexification and let τ be the Cartan involution on G associated to H, that is to say, the involutive automorphism of G such that $Fix(\tau) = H$. Let U be a compact connected real form of G such that the associated Cartan involution θ satisfies $\theta\tau = \tau\theta$. Such a compact group always exists and is stable under τ. The group H is then stable under θ and U and H are said to be dual to each other (when H is non-compact, they indeed define dual symmetric spaces $U/(U \cap H)$ and $H/(U \cap H)$). Moreover, because of the fact that τ is the Cartan involution associated to the non-compact dual H of U, the compact connected group U contains a maximal torus T such that $\tau(t) = t^{-1}$ for all $t \in T$ ((U, τ) is said to be *of maximal rank*, see proposition 3.2.3). Let $K := U \cap H$. Then $K = Fix(\tau|_U) \subset U$ and $K = Fix(\theta|_H) \subset H$. We consider the action of $K \times K$ on U given by $(k_1, k_2).u = k_1 u k_2^{-1}$. Notice that if H is compact to start with, then $K = U = H$ and the above action defines congruence in U. As for us though, we are interested in the case where H is non-compact. For $H = Gl(n, \mathbb{R})$, we have $U = U(n)$ and $K = O(n)$, and we are then led to asking the following question, which is the abstract formulation of our centered Lagrangian problem : given l orbits $\mathcal{D}_1, \ldots, \mathcal{D}_l$ of the action of $K \times K$ on U, do there exist $u_1, \ldots, u_l \in U$ such that $u_j \in \mathcal{D}_j$ and $u_1 \ldots u_l = 1$? This is a typical Lie theory problem (see for instance [EL05]), the goal being to find necessary and sufficient conditions

on the $\mathcal{D}_j$ for this question to admit a positive answer. Observe that, as a generalization of lemma 6.2.1, these orbits are in one-to-one correspondence with the conjugacy classes in U of elements of the form $\tau^-(u)u$, where u is any element in a given orbit $\mathcal{D}$ and $\tau^-(u) = \tau(u^{-1})$. Indeed, this was recalled in proposition 3.2.7 : given two elements $u, v \in U$, there exists $(k_1, k_2) \in K \times K$ such that $v = k_1 u k_2^{-1}$ if and only if $\tau^-(v)v$ and $\tau^-(u)u$ lie in a same conjugacy class of U.

Now, to find the complex version of our problem, we apply the same construction to the complex Lie group $G = H^{\mathbb{C}}$ viewed as a real Lie group. Then $G^{\mathbb{C}} = G \times G$ is the complexification of G and $\widetilde{U} = U \times U \subset G \times G = G^{\mathbb{C}}$ is a compact real form of $G^{\mathbb{C}}$. Its non-compact dual (which needs to be a subgroup of $G^{\mathbb{C}} = G \times G$) is then $\widetilde{H} = \{(g, \theta(g)) \,:\, g \in G\} \simeq G$ where θ is the Cartan involution associated to U. The Cartan involution associated to $\widetilde{U}$ is $\widetilde{\theta} : (g_1, g_2) \in \widetilde{G} \mapsto (\theta(g_1), \theta(g_2))$ and the Cartan involution associated to $\widetilde{H}$ is $\widetilde{\tau} : (g_1, g_2) \mapsto (\theta(g_2), \theta(g_1))$. Indeed, $Fix(\widetilde{\theta}) = \widetilde{U}$, $Fix(\widetilde{\tau}) = \widetilde{H}$ and $\widetilde{\theta}\widetilde{\tau} = \widetilde{\tau}\widetilde{\theta}$. We then define :

$$\begin{aligned}
\widetilde{K} &:= \widetilde{U} \cap \widetilde{H} \\
&= \Big\{ \big(g, \theta(g)\big) \mid \widetilde{\theta}\big(g, \theta(g)\big) = \big(g, \theta(g)\big) \Big\} \\
&= \Big\{ \big(g, \theta(g)\big) \mid \theta(g) = g \Big\} \\
&= \big\{ (u, u) \,:\, u \in U \big\}
\end{aligned}$$

We will also use the notation $U_\Delta := \{(u, u) \,:\, u \in U\}$ instead of $\widetilde{K}$. We now consider the action of $\widetilde{K} \times \widetilde{K} = U_\Delta \times U_\Delta$ on $\widetilde{U} = U \times U$ defined by :

$$\big((u_1, u_1), (u_2, u_2)\big).(u, v) = (u_1 u u_2^{-1}, u_1 v u_2^{-1})$$

Our problem then states : given l orbits $\widetilde{\mathcal{D}}_1, \ldots, \widetilde{\mathcal{D}}_l$ of the above action, do there exist l pairs $(u_1, v_1), \ldots, (u_l, v_l) \in \widetilde{U} = U \times U$ such that $(u_j, v_j) \in \widetilde{\mathcal{D}}_j$ and $(u_1, v_1). \ldots .(u_l, v_l) = 1$, that is, $u_1 \ldots u_l = 1$ and $v_1 \ldots v_l = 1$? We will call this problem the *complexification of the centered Lagrangian problem.*

Before passing on to the next section, we wish to point out that if we consider the action of $K \times K$ not on U but rather on its dual H, then the orbits of this action are characterized by the singular values ($\mathrm{Sing}\, h := \mathrm{Spec}\,(\theta^-(h)h)$ where $h \in H$ and $\theta^-(h) = \theta(h^{-1})$) of any of their elements. As a consequence, our (centered) Lagrangian problem appears as a compact version of the (real) Thompson problem, replacing θ with τ in the latter to formulate the former (see [AMW01] and [EL05] for a proof of the Thompson conjecture in the real case). To understand this better, we consider the simple original case where $H = Gl(n, \mathbb{R})$. Then $U = U(n)$ and $K = O(n)$. The $O(n) \times O(n)$-orbits in $Gl(n, \mathbb{R})$ are the sets of matrices with fixed singular values :

$$\begin{aligned}
O(n) \times O(n) \,.\, h_0 &= \{h \in Gl(n, \mathbb{R}) \mid \mathrm{Sing}\,(h) = \mathrm{Sing}\,(h_0)\} \\
&= \{h \in Gl(n, \mathbb{R}) \mid \mathrm{Spec}\,(h^t h) = \mathrm{Spec}\,(h_0^t h_0)\}
\end{aligned}$$

Observe that $h_0^t h_0$ is a positive definite real symmetric matrix, so that it is conjugate to a diagonal matrix $d = \exp(\lambda)$ with $\lambda \in \mathbb{R}^n$. The (real) Thompson problem then asks : given $\lambda_1, \ldots, \lambda_l \in \mathbb{R}^n$, do there exist l invertible matrices $h_1, \ldots, h_l \in Gl(n, \mathbb{R})$ such that $\mathrm{Spec}\,(h_j^t h_j) = \exp(\lambda_j)$ and $h_1 \ldots h_l = 1$? Our centered Lagrangian problem, as defined in section 6.2, asked : given $\lambda_1, \ldots, \lambda_l \in \mathbb{R}^n$, do there exist l invertible matrices $u_1, \ldots, u_l \in U(n)$ such that $\mathrm{Spec}\,(u_j^t u_j) = \exp(i\lambda_j)$ and $u_1 \ldots u_l = 1$? We see here that these two questions are formally the same, replacing the non-compact group $H = Gl(n, \mathbb{R})$ by its compact dual $U = U(n)$, which is why we think of our problem as a compact version of the Thompson problem. We shall come back upon this analogy in propositions 6.4.3 and 6.5.7. Further, if we consider $\widetilde{K} \times \widetilde{K}$-orbits in $\widetilde{H}$ instead of in $\widetilde{U}$, we obtain :

$$\begin{aligned}
\widetilde{K} \times \widetilde{K} \,.\, \big(g, \theta(g)\big) &= \big\{ \big(u_1, u_1\big)\big(g, \theta(g)\big)\big(u_2^{-1}, u_2^{-1}\big) \;:\; u_1, u_2 \in U \big\} \\
&\simeq \{u_1 g u_2^{-1} \,:\, u_1, u_2 \in U\}
\end{aligned}$$

since $\theta(u_j) = u_j$. And once again these orbits are characterized by the singular values of g, $\mathrm{Sing}\,(g) := \mathrm{Spec}\,(\theta^-(g)g)$. In other words, the map $g \in \widetilde{H} \mapsto \theta^-(g)g$ sends a $\widetilde{K} \times \widetilde{K}$-orbit in $\widetilde{H}$ onto a conjugacy class in G. If $H = Gl(n, \mathbb{R})$, then $G = Gl(n, \mathbb{C})$, $\mathrm{Sing}\,(g) := \mathrm{Spec}\,(g^*g)$, and we have :

$$g = u_1 g_0 u_2^{-1} \text{ for some } (u_1, u_2) \in U(n) \times U(n) \text{ if and only if } \mathrm{Spec}\,(g^*g) = \mathrm{Spec}\,(g_0^* g_0)$$

As $g_0^* g_0$ is a positive definite Hermitian matrix, it is conjugate to some diagonal matrix $\exp(\lambda)$ with $\lambda \in \mathbb{R}^n$. The Thompson problem then asks : given $\lambda_1, \ldots, \lambda_l \in \mathbb{R}^n$, do there exist l invertible matrices $g_1, \ldots, g_l \in Gl(n, \mathbb{C})$ such that $\mathrm{Spec}\,(g_j^* g_j) = \exp(\lambda_j)$ and $g_1 \ldots g_l = 1$? We then have a Thompson problem for both $Gl(n, \mathbb{C})$ and $Gl(n, \mathbb{R})$, the latter being in fact a *real version* of the former in a sense that will be made precise in section 6.5. For an explanation of this fact and further reference on the Thompson problem and the Thompson conjecture, we refer to [Kly00], [AMW01] and [EL05], and we recall the following result, first proved in [Kly00] and then proved using symplectic geometry in [AMW01] :

Proposition 6.3.1. [Kly00, AMW01] *Consider $\lambda_1, \ldots, \lambda_l \in \mathbb{R}^n$. Then the following statements are equivalent :*

(i) There exist l invertible complex matrices $g_1, \ldots, g_l \in Gl(n, \mathbb{C})$ such that :

$$\mathrm{Spec}\,(g_j^* g_j) = \exp(\lambda_j) \quad \textit{and} \quad g_1 \ldots g_l = 1$$

(ii) There exist l complex Hermitian matrices $H_1, \ldots H_l \in \mathcal{H}(n)$ such that :

$$\mathrm{Spec}\, H_j = \lambda_j \quad \textit{and} \quad H_1 + \cdots + H_l = 0$$

(iii) There exist l invertible real matrices $h_1, \ldots, h_l \in Gl(n, \mathbb{R})$ such that :

$$\mathrm{Spec}\,(h_j^t h_j) = \exp(\lambda_j) \quad \textit{and} \quad h_1 \ldots h_l = 1$$

(iv) There exist l real symmetric matrices $S_1, \ldots S_l \in \mathcal{S}(n)$ such that :

$$\mathrm{Spec}\, S_j = \lambda_j \quad \textit{and} \quad S_1 + \cdots + S_l = 0$$

The equivalence of (i) and (ii) is the Thompson conjecture, and the equivalence of (iii) and (iv) is the real Thompson conjecture. We refer to [AMW01] to see how the equivalence of (ii) and (iv) relies on a *real* convexity result on Lie-algebra-valued momentum maps. Statements (iii) and (iv) are real versions of statements (i) and (ii) respectively, and we refer to section 6.5 for further comments on that terminology. Observe in particular that statement (iv) corresponds to the infinitesimal version of our Lagrangian problem (see section 6.1, and subsection 9.2). Finally, we refer to [Kly98] for necessary and sufficient conditions on the λ_j for problem (ii) (and therefore any of the other three) to have a solution. These conditions are linear inequalities satisfied by the λ_j.

We now wish to relate the complexification of our centered Lagrangian problem to another know problem.

6.4 Equivalence between the complexification of the centered Lagrangian problem and the unitary problem

From now on, the initial data is a compact connected Lie group U. For such a group, we can formulate :

- the centered Lagrangian problem (concerning $K \times K$-orbits in U, where $K = U \cap H$ with H the non-compact dual of U).
- a complex version of this (concerning $U_\Delta \times U_\Delta$-orbits in $U \times U$).

- the unitary problem (concerning conjugacy classes in U).

We now claim that these last two problems are in fact equivalent, meaning that the first one has a solution if and only if the second one does. To show that this is indeed the case, the main observation to make is the following one :

Lemma 6.4.1. *The map*

$$\begin{array}{rccc} \eta : & U \times U & \longrightarrow & U \\ & (u,v) & \longmapsto & u^{-1}v \end{array}$$

sends a $U_\Delta \times U_\Delta$-orbit $\widetilde{\mathcal{D}}$ in $U \times U$ onto a conjugacy class $\mathcal{C}$ in U, and if $\eta(u',v')$ is conjugate to $\eta(u,v)$ in U then (u',v') and (u,v) lie in a same $U_\Delta \times U_\Delta$-orbit $\widetilde{\mathcal{D}}$ in $U \times U$.

Proof. If $(u,v) = (u_1u_0u_2^{-1}, u_1v_0u_2^{-1})$ then $u^{-1}v = u_2(u_0^{-1}v_0)u_2^{-1}$ so that $\eta(\widetilde{\mathcal{D}}) \subset \mathcal{C}$ where $\mathcal{C}$ is a well-defined conjugacy class in U. Further, let $(u,v) \in \widetilde{\mathcal{D}}$ and take any $w \in \mathcal{C}$. Then there exists $u_2 \in U$ such that $w = u_2u^{-1}vu_2^{-1}$ so that :

$$(1,w) = (1, u_2u^{-1}vu_2^{-1}) = \underbrace{\big((u_2u^{-1}, u_2u^{-1}), (u_2,u_2)\big)}_{\in U_\Delta \times U_\Delta} . \underbrace{(u,v)}_{\in \widetilde{\mathcal{D}}}$$

hence $(1,w) \in \widetilde{\mathcal{D}}$, therefore $w = \eta(1,w) \in \eta(\widetilde{\mathcal{D}})$. Likewise, if $\eta(u',v') = u_0\eta(u,v)u_0^{-1}$ for some $u_0 \in U$, then :

$$(u',v') \underset{U_\Delta \times U_\Delta}{\sim} \big(1, (u')^{-1}v\big) = \big(1, u_0(u^{-1}v)u_0^{-1}\big) \underset{U_\Delta \times U_\Delta}{\sim} (u,v)$$

□

We now have the following result, which says that the complexification of the centered Lagrangian problem has a solution if and only if the unitary problem has a solution (that is, these two problems are equivalent) :

Proposition 6.4.2. *Let $\widetilde{\mathcal{D}}_1, \dots, \widetilde{\mathcal{D}}_l$ be l orbits of $U_\Delta \times U_\Delta$ in $U \times U$ and let $\mathcal{C}_1, \dots, \mathcal{C}_l \subset U$ be the corresponding conjugacy classes : $\mathcal{C}_j = \eta(\widetilde{\mathcal{D}}_j)$. Then there exists $\big((u_1,v_1), \dots, (u_l,v_l)\big) \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ such that $u_1 \dots u_l = 1$ and $v_1 \dots v_l = 1$ if and only if there exists $(w_1, \dots, w_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $w_1 \dots w_l = 1$.*

Proof. Setting $(u_j, v_j) := (1, w_j)$ for every j, we see that the second condition implies the first one. Conversely, assume that $((u_1,v_1), \dots, (u_l,v_l)) \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ satisfy $u_1 \dots u_l = 1$ and $v_1 \dots v_l = 1$. Then $(u_1 \dots u_l)^{-1}v_1 \dots v_l = 1$, hence $u_l^{-1} \dots u_2^{-1}(u_1^{-1}v_1)v_2 \dots v_l = 1$, with $u_1^{-1}v_1 \in \mathcal{C}_1$. Hence :

$$\underbrace{u_l^{-1} \dots u_2^{-1}(u_1^{-1}v_1)u_2 \dots u_l}_{\in \mathcal{C}_1} \underbrace{u_l^{-1} \dots u_3^{-1}(u_2^{-1}v_2)u_3 \dots u_l}_{\in \mathcal{C}_2} \dots \underbrace{(u_l^{-1}v_l)}_{\in \mathcal{C}_l} = 1$$

Setting $w_1 = u_l^{-1} \dots u_2^{-1}(u_1^{-1}v_1)u_2 \dots u_l$, $w_2 = u_l^{-1} \dots u_3^{-1}(u_2^{-1}v_2)u_3 \dots u_l$, $\dots$, and $w_l = u_l^{-1}v_l$ then gives a solution $(w_1, \dots, w_l)$ to the unitary problem. □

In analogy with a result on double cosets of $U(n)$ in $Gl(n,\mathbb{C})$ (which are characterized by the singular values $\operatorname{Sing} g = \operatorname{Spec}(\theta^-(g)g)$ of any of their elements) and dressing orbits of $U(n)$ in $(U(n))^* = \{b \in Gl(n,\mathbb{C}) \mid b$ is upper triangular and $diag(b) \in (\mathbb{R}^{*+})^n\}$ appearing in [AMW01], the above proposition can be formulated more precisely in the following way. Consider the action of U^l on $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ given by

$$(\varphi_1, \dots, \varphi_l).\big((u_1,v_1), \dots, (u_l,v_l)\big) = \big(\underbrace{\varphi_1.(u_1,v_1).\varphi_2^{-1}}_{=(\varphi_1u_1\varphi_2^{-1}, \varphi_1v_1\varphi_2^{-1})}, \varphi_2.(u_2,v_2).\varphi_3^{-1}, \dots, \varphi_l.(u_l,v_l).\varphi_1^{-1}\big)$$

and the diagonal action of U on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$: $\varphi.(w_1, \ldots, w_l) = (\varphi w_1 \varphi^{-1}, \ldots, \varphi w_l \varphi^{-1})$. These actions respectively preserve the relations $u_1 \ldots u_l = v_1 \ldots v_l = 1$ and $\omega_1 \ldots \omega_l = 1$. We may then define the orbit spaces :

$$\mathcal{M}_{\widetilde{\mathcal{D}}} := \Big\{ \big((u_j, v_j)\big)_j \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \mid u_1 \ldots u_l = v_1 \ldots v_l = 1 \Big\} \Big/ U^l$$

and :

$$\mathcal{M}_{\mathcal{C}} = \big\{ (w_j)_j \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid w_1 \ldots w_l = 1 \big\} \Big/ U$$

And we then have :

Proposition 6.4.3. *The map*

$$\begin{array}{rccl} \eta^{(l)} : & \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & \big((u_1, v_1), \ldots, (u_l, v_l)\big) & \longmapsto & \big(u_l^{-1} \ldots u_2^{-1}(u_1^{-1} v_1) u_2 \ldots u_l, \ldots, u_l^{-1}(u_{l-1}^{-1} v_{l-1}) u_l, u_l^{-1} v_l\big) \end{array}$$

induces a homeomorphism $\mathcal{M}_{\widetilde{\mathcal{D}}} \simeq \mathcal{M}_{\mathcal{C}}$.

Proof. First, observe that if $u_1 \ldots u_l = v_1 \ldots v_l = 1$ and $w_j = u_l^{-1} \ldots u_{j+1}^{-1}(u_j^{-1} v_j) u_{j+1} \ldots u_l$, then $w_1 \ldots w_l = (u_1 \ldots u_l)^{-1} v_1 \ldots v_l = 1$. Second, the map $\eta^{(l)}$ sends a U^l-orbit in $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ into an orbit of the diagonal action of U on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\eta^{(l)}\Big((\varphi_1, \ldots, \varphi_l).\big((u_1, v_1), \ldots, (u_l, v_l)\big)\Big) = \varphi_1.\eta^{(l)}\big((u_1, v_1), \ldots, (u_l, v_l)\big)$$

so that we indeed have a map $\mathcal{M}_{\widetilde{\mathcal{D}}} \to \mathcal{M}_{\mathcal{C}}$, that we shall still denote by $\eta^{(l)}$. Take now $((u_j, v_j))_{1 \leq j \leq l}$ and $((u'_j, v'_j))_{1 \leq j \leq l}$ in $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ such that $u_1 \ldots u_l = v_1 \ldots v_l = 1$ and $u'_1 \ldots u'_l = v'_1 \ldots v'_l = 1$, and suppose that there exists $\varphi \in U$ such that :

$$\eta^{(l)}\big((u'_j, v'_j)\big) = \varphi.\eta^{(l)}\big((u_j, v_j)\big)$$

Then for all $j \in \{1, \ldots l\}$:

$$(u'_l)^{-1} \ldots (u'_{j+1})^{-1}\big((u'_j)^{-1} v'_j\big) u'_{j+1} \ldots u'_l = \varphi\big(u_l^{-1} \ldots u_{j+1}^{-1}(u_j^{-1} v_j) u_{j+1} \ldots u_l\big)\varphi^{-1} \tag{6.1}$$

Hence, using (6.1), we have for all j :

$$\begin{aligned} (u'_j, v'_j) &= \big((u'_j, u'_j), (1,1)\big).\big(1, (u'_j)^{-1} v'_j\big) \\ &= \big((u'_j, u'_j), (1,1)\big).\big(1, (u'_{j+1} \ldots u'_l \varphi u_l^{-1} \ldots u_{j+1}^{-1})(u_j^{-1} v_j)(u'_{j+1} \ldots u'_l \varphi u_l^{-1} \ldots u_{j+1}^{-1})^{-1}\big) \\ &= \Big(\big((u'_j \ldots u'_l)\varphi(u_j \ldots u_l)^{-1}, \ \ldots\big), \big((u'_{j+1} \ldots u'_l)\varphi(u_{j+1} \ldots u_l)^{-1}, \ \ldots\big)\Big).\big(u_j, v_j\big) \end{aligned}$$

So that, if we set $\varphi_j := (u'_j \ldots u'_l)\varphi(u_j \ldots u_l)^{-1}$, we have :

$$\big(\varphi_1, \ldots, \varphi_l\big).\big((u_1, v_1), \ldots, (u_l, v_l)\big) = \big((u'_1, v'_1), \ldots, (u'_l, v'_l)\big)$$

which shows that the induced map $\eta^{(l)} : \mathcal{M}_{\widetilde{\mathcal{D}}} \to \mathcal{M}_{\mathcal{C}}$ is injective. Let us now show that it is surjective : take $(w_1, \ldots, w_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ satisfying $w_1 \ldots w_l = 1$, and set $u_j := 1$ and $v_j := w_j$. Then lemma 6.4.1 shows that $(u_j, v_j) \in \widetilde{\mathcal{D}}_j$ and we have indeed $u_1 \ldots u_l = v_1 \ldots v_l = 1$ and $\eta^{(l)}((u_j, v_j)) = (w_1, \ldots, w_l)$, which concludes the proof. □

We point out the fact that this result reinforces the analogy between our problem and the Thompson problem. We now wish to explain in what precise sense the Lagrangian problem is a real version of these two equivalent problems.

6.5 Seeing the Lagrangian problem as a real version of the unitary problem

The important idea of thinking of possible solutions to a real problem as the fixed point set of an involution defined on the set of possible solutions to a corresponding complex problem is well-established in symplectic geometry and is due to Michael Atiyah and Alan Weinstein (see [Ati82, Dui83] and [LR91]). In fact, the idea is that the set of possible solutions to a complex problem carries a symplectic structure and that the corresponding real problem is formulated for elements of the fixed point set of an anti-symplectic involution defined on this symplectic manifold. Examples of results obtained using this idea include the (linear and non-linear) real Kostant convexity theorems (see [Dui83, LR91]) and the real Thompson conjecture (see [AMW01, EL05]). Although we will have to replace symplectic manifolds with quasi-Hamiltonian spaces for technical considerations, the above idea plays a key role in our approach. Keeping this in mind, we will eventually define an involution $\beta^{(l)}$ on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$. But, to explain how this involution is obtained, we will first work on the product $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ of l $U_\Delta \times U_\Delta$-orbits in $U \times U$.
The key here is to try and see the $K \times K$-orbit of $w \in U$ as a subset of some $U_\Delta \times U_\Delta$-orbit $\widetilde{\mathcal{D}} \subset U \times U$. This is done by observing the following fact :

Lemma 6.5.1. *The condition ($w \in \mathcal{D}$) is equivalent to ($\tau^-(w)w \in \mathcal{C}$), where $\mathcal{C}$ is defined as the conjugacy class of $\tau^-(w_0)w_0$ for any $w_0 \in \mathcal{D}$, which in turn is equivalent to $(\tau(w), w) \in \widetilde{\mathcal{D}}$, where $\widetilde{\mathcal{D}}$ is the $U_\Delta \times U_\Delta$-orbit of $(\tau(w), w)$.*

Proof. The first equivalence is proposition 3.2.7. Then we know from lemma 6.4.1 that $\mathcal{C} = \eta(\widetilde{\mathcal{D}})$ where $\widetilde{\mathcal{D}}$ is the $U_\Delta \times U_\Delta$-orbit of $(1, \tau^-(w)w) \sim (\tau(w), w)$, which gives the second equivalence. □

In order to obtain elements of the form $(\tau(w), w)$ as fixed points of an involution, we set :

$$\begin{array}{rccc} \alpha : & U \times U & \longrightarrow & U \times U \\ & (u, v) & \longmapsto & \big(\tau(v), \tau(u)\big) \end{array}$$

Then $\alpha^2 = Id$ and $Fix(\alpha) = \{(\tau(v), v) \mid v \in U\} \simeq U$. In particular, $Fix(\alpha)$ is always non-empty. Moreover, we have :

Lemma 6.5.2. *$\alpha(\widetilde{\mathcal{D}}) = \widetilde{\mathcal{D}}$, so that α defines an involution on $\widetilde{\mathcal{D}}$, whose fixed point set is isomorphic to $\mathcal{D}$ and therefore non-empty.*

Proof. If $(u, v) \in \widetilde{\mathcal{D}}$, we have $\eta(\alpha(u, v)) = \tau^-(v)\tau(u) = \tau(v^{-1}u) = \tau^-(u^{-1}v)$. But if $w \in U$, then $\tau^-(w)$ is conjugate to w. Indeed, there exists a maximal torus of U which is pointwise fixed by τ^-, and w is conjugate to an element in such a torus : $w = \varphi t \varphi^{-1}$ with $\tau^-(t) = t$ so that $\tau^-(w) = \tau(\varphi)t\tau(\varphi^{-1}) = \tau(\varphi)\varphi^{-1}w\varphi\tau(\varphi^{-1})$ (observe that when $U = U(n)$ then $\tau^-(w) = w^t$ and all of this becomes clear). Thus, $\eta(\alpha(u, v)) = \tau(u^{-1}v)$ and $u^{-1}v = \eta(u, v)$ lie in the same conjugacy class $\mathcal{C} = \eta(\widetilde{\mathcal{D}})$, so, by lemma 6.4.1, we have indeed $\alpha(u, v) \in \widetilde{\mathcal{D}}$. Further, by lemma 6.5.1 :

$$Fix(\alpha|_{\widetilde{\mathcal{D}}}) = \Big\{ \big(\tau(v), v\big) \mid \big(\tau^-(v)v\big) \in \mathcal{C} \Big\}$$

hence $Fix(\alpha|_{\widetilde{\mathcal{D}}}) \simeq \mathcal{D} \neq \emptyset$. □

On the product $\widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$ of l $U_\Delta \times U_\Delta$-orbits in $U \times U$, we can therefore define the involution :

$$\begin{array}{rccc} \alpha^{(l)} : & \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l & \longrightarrow & \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \\ & \big((u_1, v_1), \ldots, (u_l, v_l)\big) & \longmapsto & \Big(\big(\tau(v_1), \tau(u_1)\big), \ldots, \big(\tau(v_l), \tau(u_l)\big)\Big) \end{array}$$

Observe that its fixed point set satisfies $Fix(\alpha^{(l)}) \simeq \mathcal{D}_1 \times \cdots \times \mathcal{D}_l$ and is therefore non-empty. We then have the following result, which says that the centered Lagrangian problem has a solution if and only if there exists a solution of the complexified problem which is fixed by $\alpha^{(l)}$:

Proposition 6.5.3. *Let $\mathcal{D}_1, \dots, \mathcal{D}_l$ be l $K \times K$-orbits in U. For every $j \in \{1, \dots, l\}$, let $\mathcal{C}_j$ be the conjugacy class of $\tau^-(w)w$ where w is any element in $\mathcal{D}_j$, and let $\widetilde{\mathcal{D}}_j$ be the corresponding $U_\Delta \times U_\Delta$-orbit in $U \times U$ (i.e., such that $\eta(\widetilde{\mathcal{D}}_j) = \mathcal{C}_j$, where $\eta(u, v) = u^{-1}v$). Then there exists*

$$(w_1, \dots, w_l) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_l$$

such that $w_1 \dots w_l = 1$ if and only there exists

$$\big((u_1, v_1), \dots, (u_l, v_l)\big) \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l$$

such that

$$u_1 \dots u_l = 1, \ v_1 \dots v_l = 1 \quad \textit{and} \quad u_j = \tau(v_j)$$

for all $j \in \{1, \dots, l\}$, that is, $((u_1, v_1), \dots, (u_l, v_l)) \in Fix(\alpha^{(l)})$.

Proof. For a given $(w_1, \dots, w_l) \in \mathcal{D}_1 \times \cdots \times \mathcal{D}_l \mid w_1 \dots w_l = 1$, set $(u_j, v_j) := (\tau(w_j), w_j)$. By lemma 6.4.1, (u_j, v_j) then belongs to $\widetilde{\mathcal{D}}_j$ and we have indeed $u_1 \dots u_l = v_1 \dots v_l = 1$. Conversely, for $((u_j, v_j))_j \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \mid u_1 \dots u_l = v_1 \dots v_l = 1$ and such that $u_j = \tau(v_j)$ for all j, set $w_j := v_j$. Then $w_1 \dots w_l = 1$ and $\tau^-(w_j)w_j = u_j^{-1}v_j \in \mathcal{C}_j$, so that, by proposition 3.2.7, $w_j \in \mathcal{D}_j$. □

This type of result is exactly why some given problem (A) is called a real version of another problem (B) : if $\mathcal{S}_\mathbb{C}$ denotes the set of solutions to problem (B) (we assume that $\mathcal{S}_\mathbb{C} \neq \emptyset$) and $\mathcal{S}_\mathbb{R}$ the set of solutions to problem (A), then there exists an involution α on some space $M \supset \mathcal{S}_\mathbb{C}$, whose fixed point set is non-empty, such that $\mathcal{S}_\mathbb{R} \neq \emptyset$ if and only if $\mathcal{S}_\mathbb{C} \cap Fix(\alpha) \neq \emptyset$. See for instance [Fot] for a more systematic treatment of these questions.

The question then is : what is the real version of the unitary problem ? Given what we have done so far, we see that giving an answer to this question amounts to defining an involution $\beta^{(l)}$ on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $\beta^{(l)} \circ \eta^{(l)} = \eta^{(l)} \circ \alpha^{(l)}$, where $\eta^{(l)} : \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ is defined as in proposition 6.4.3, so that $\eta^{(l)}(Fix(\alpha^{(l)})) \subset Fix(\beta^{(l)})$, which in particular implies that $Fix(\beta^{(l)}) \neq \emptyset$. The only possibility is then to set, for any $(w_1, \dots, w_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\beta^{(l)}(w_1, \dots, w_l) \ = \ \big(\tau^-(w_l) \dots \tau^-(w_2)\tau^-(w_1)\tau(w_2) \dots \tau(w_l), \dots, \tau^-(w_l)\tau^-(w_{l-1})\tau(w_l), \tau^-(w_l)\big)$$

The fact that this map β is well-defined is a consequence of the following lemma :

Lemma 6.5.4. *The involution $\tau^- : U \to U$ sends a conjugacy class $\mathcal{C} \subset U$ into itself.*

Proof. This follows from the fact that there exists a maximal torus T of U which is pointwise fixed by τ^- (see proposition 3.2.3). Indeed, if we take $u \in \mathcal{C}$, then $u = vdv^{-1}$ for some $v \in U$ and some $d \in T$. Then $\tau^-(u) = \tau(v)\tau^-(d)\tau^-(v) = \tau(v)d\tau^-(v)$ is conjugate to d and therefore to u. □

We now set :

Definition 6.5.5 (Definition of the involution β). Let $\mathcal{C}_1, \dots, \mathcal{C}_l$ denote l conjugacy classes of U. Let then β denote the map :

$$\begin{array}{rccl} \beta \ : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \dots, u_l) & \longmapsto & \big(\tau^-(u_l) \dots \tau^-(u_2)\tau^-(u_1)\tau(u_2) \dots \tau(u_l), \dots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

Proposition 6.5.6. *β is a well-defined involution on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, and satisfies $Fix(\beta) \neq \emptyset$. Additionally, β is compatible with the diagonal action of U on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ and with the momentum map $\mu : (u_1, \dots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mapsto u_1 \dots u_l \in U$ in the sense that :*

$$\beta\big(u.(u_1, \dots, u_l)\big) = \tau(u).\beta(u_1, \dots, u_l) \quad \text{and} \quad \mu \circ \beta = \tau^- \circ \mu$$

Proof. Lemma 6.5.4 shows that β is well-defined. Let us compute $\beta \circ \beta(u_1, \ldots, u_l)$. The jth element in $\beta(u_1, \ldots, u_l)$ is :

$$\tau^-(u_l)\ldots\tau^-(u_{j+1})\tau^-(u_j)\tau(u_{j+1})\ldots\tau(u_l)$$

So the jth element of $\beta \circ \beta(u_1, \ldots, u_l)$ is :

$$\begin{array}{rcl} & & \tau^-\big(\tau^-(u_l)\big)\ldots\tau^-\big(\tau^-(u_l)\ldots\tau^-(u_{j+2})\tau^-(u_{j+1})\tau(u_{j+2})\ldots\tau(u_l)\big) \\ & \times & \tau^-\big(\tau^-(u_l)\ldots\tau^-(u_{j+1})\tau^-(u_j)\tau(u_{j+1})\ldots\tau(u_l)\big) \\ & \times & \tau\big(\tau^-(u_l)\ldots\tau^-(u_{j+2})\tau^-(u_{j+1})\tau(u_{j+2})\ldots\tau(u_l)\big)\ldots\tau\big(\tau^-(u_l)\big) \\ = & & u_l\ldots u_l^{-1}\ldots u_{j+2}^{-1}u_{j+1}u_{j+2}\ldots u_l \\ & \times & u_l^{-1}\ldots u_{j+1}^{-1}u_ju_{j+1}\ldots u_l \\ & \times & u_l^{-1}\ldots u_{j+2}^{-1}u_{j+1}^{-1}u_{j+2}\ldots u_l\ldots u_l^{-1} \\ = & & u_j \end{array}$$

so that $\beta \circ \beta = Id$. Now, since there exists a maximal torus of T which is fixed pointwise by τ^-, each conjugacy class $\mathcal{C}_j$ contains an element $d_j \in Fix(\tau^-) \cap T$. We then have :

$$\beta(d_1, \ldots, d_l) = (d_l\ldots d_2d_1d_2^{-1}\ldots d_l^{-1}, \ldots, d_ld_{l-1}d_l^{-1}, d_l) = (d_1, \ldots, d_l)$$

as T is abelian. Hence $Fix(\beta) \neq \emptyset$. Compatibility with the action of U and the momentum map μ is a simple verification, along the same lines (see section 7.2). □

We then have the following result (proposition 6.5.7), along the lines of proposition 6.4.3. As earlier, we see that the group K^l acts on $Fix(\alpha^{(l)})$ and preserves the relations $u_1\ldots u_l = v_1\ldots v_l = 1$. Likewise, K acts diagonally on $Fix(\beta^{(l)})$, preserving the relation $w_1\ldots w_l = 1$. We may therefore define :

$$\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} := \Big\{\big((u_j, v_j)\big)_j \in \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \mid u_1\ldots u_l = v_1\ldots v_l = 1 \text{ and } \big((u_j, v_j)\big)_j \in Fix(\alpha^{(l)})\Big\}\Big/K^l$$

and

$$\mathcal{M}^{\beta}_{\mathcal{C}} = \big\{(w_j)_j \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \,|\, w_1\ldots w_l = 1 \text{ and } (w_j)_j \in Fix(\beta^{(l)})\big\}\Big/K$$

We then have :

Proposition 6.5.7. *The map $\eta^{(l)} : \widetilde{\mathcal{D}}_1 \times \cdots \times \widetilde{\mathcal{D}}_l \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ induces a homeomorphism $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \simeq \mathcal{M}^{\beta}_{\mathcal{C}}$.*

Proof. Since $\eta^{(l)}(Fix(\alpha^{(l)})) \subset Fix(\beta^{(l)})$, the map $\eta^{(l)}$ induces a map $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \to \mathcal{M}^{\beta}_{\mathcal{C}}$, that we still denote by $\eta^{(l)}$. Take now $((u_j, v_j))_j$ and $((u'_j, v'_j))$ in $Fix(\alpha^{(l)})$ (in particular $u_j = \tau(v_j)$ and $u'_j = \tau(v'_j)$) satisfying $u_1\ldots u_l = v_1\ldots v_l = 1$ and $u'_1\ldots u'_l = v'_1\ldots v'_l = 1$, and suppose that there exists $k \in K$ such that :

$$\eta^{(l)}\big((u'_j, v'_j)\big) = k.\eta^{(l)}\big((u_j, v_j)\big)$$

Then for all $j \in \{1, \ldots, l\}$:

$$\tau^-(v'_l)\ldots\tau^-(v'_{j+1})\big(\tau^-(v'_j)v'_j\big)\tau(v'_{j+1})\ldots\tau(v'_l) \quad = \quad k\big(\tau^-(v_l)\ldots\tau^-(v_{j+1})\big(\tau^-(v_j)v_j\big)\tau(v_{j+1})\ldots\tau(v_l)\big)k^{-1}$$

that is :

$$\tau(v'_j\ldots v'_l)k\tau^-(v_j\ldots v_l) \quad = \quad v'_j\tau(v'_{j+1}\ldots v'_l)k\tau^-(v_{j+1}\ldots v_l)v_j^{-1} \tag{6.2}$$

And we have, as in the proof of proposition 6.4.3 :

$$\big((\tau(v'_j), v'_j)\big) = \big(\varphi_1, \ldots, \varphi_l\big).\big((\tau(v_j), v_j)\big)$$

with $\varphi_j = \tau(v'_j \ldots v'_l) k \tau^-(v_j \ldots v_l)$, so that for all $j < l$:

$$\begin{aligned}
\tau(\varphi_j) &= \tau\big(\tau(v'_j \ldots v'_l) k \tau^-(v_j \ldots v_l)\big) \\
&= \tau(v'_j \tau(v'_{j+1} \ldots v'_l) k \tau^-(v_{j+1} \ldots v_l)) \quad \text{(using (6.2))} \\
&= \tau(v'_j)\tau(\varphi_{j+1})\tau^-(v_j)
\end{aligned}$$

Additionally :

$$\begin{aligned}
\tau(\varphi_l) &= \tau\big(\tau(v'_l) k \tau^-(v_l)\big) \\
&= \tau(v'_l k v_l^{-1}) \quad \text{(using(6.2))} \\
&= \tau(v'_l)\tau(k)\tau^-(v_l) \\
&= \tau(v'_l) k \tau^-(v_l) \quad \text{since } \tau(k) = k \\
&= \varphi_l
\end{aligned}$$

so by induction (using (6.2) again), one has $\tau(\varphi_j) = \varphi_j$ for all j, which shows that $(\varphi_1, \ldots, \varphi_l) \in K^l$, thus that the induced map $\eta^{(l)} : \mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \to \mathcal{M}^{\beta}_{\mathcal{C}}$ is injective.
Let us now show that it is surjective. Consider $(w_1, \ldots, w_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $w_1 \ldots w_l = 1$ and $\beta(w_1, \ldots, w_l) = (w_1, \ldots, w_l)$. Then in particular $\tau^-(w_l) = w_l$. As $Fix(\tau^-)$ is connected, $w_l = \tau^-(v_l) v_l$ for some $v_l \in U$, and lemma 6.5.1 shows that $(\tau(v_l), v_l) \in \widetilde{\mathcal{D}}_l$. Further, since $(w_j)_j \in Fix(\beta^{(l)})$, we also have :

$$\tau^-(w_l)\tau^-(w_{l-1})\tau(w_l) = w_{l-1}$$

that is :

$$\tau^-(v_l) v_l \tau^-(w_{l-1}) v_l^{-1} \tau(v_l) = w_{l-1}$$

hence :

$$\tau^-\big(\tau(v_l) w_{l-1} \tau^-(v_l)\big) = \tau(v_l) w_{l-1} \tau^-(v_l)$$

so that $\tau(v_l) w_{l-1} \tau^-(v_l) = \tau^-(v_{l-1}) v_{l-1}$ for some $v_{l-1} \in U$, and as earlier $(\tau(v_{l-1}, v_{l-1})) \in \widetilde{\mathcal{D}}_{l-1}$. Continuing like this, we obtain for all $j \in \{2, \ldots, l\}$:

$$\tau^-(v_j) v_j = \tau(v_{j+1} \ldots v_l) w_j \tau^-(v_{j+1} \ldots v_l) \tag{6.3}$$

In particular, $(\tau(v_j), v_j) \in \widetilde{\mathcal{D}}_j$ by lemma 6.5.1. We then set $v_1 := (v_2 \ldots v_l)^{-1}$. Then :

$$\begin{aligned}
\tau^-(v_2 \ldots v_l)\tau^-(v_1) v_1 \tau(v_2 \ldots v_l) &= \tau^-(v_2 \ldots v_l)\tau(v_2 \ldots v_l)(v_2 \ldots v_l)^{-1}\tau(v_2 \ldots v_l) \\
&= \big(\tau^-(v_2 \ldots v_l) v_2 \ldots v_l\big)^{-1} \\
&= (w_2 \ldots w_l)^{-1} \quad \text{(using (6.3))} \\
&= w_1
\end{aligned}$$

Thus :

$$\tau^-(v_2 \ldots v_l)\tau^-(v_1) v_1 \tau(v_2 \ldots v_l) = w_1 \tag{6.4}$$

and $\tau^-(v_1) v_1$ is conjugate to $w_1 \in \mathcal{C}_1$, so that $(\tau(v_1), v_1) \in \widetilde{\mathcal{D}}_1$ by lemma 6.5.1. Further, relations (6.3) and (6.4) show that we have, for all j :

$$\tau^-(v_{j+1} \ldots v_l)\tau^-(v_j) v_j \tau(v_{j+1} \ldots v_l) = w_j$$

hence :

$$\eta^{(l)}\big((\tau(v_j), v_j)_j\big) = (w_1, \ldots, w_l)$$

Since $(\tau(v_j), v_j)_j \in Fix(\alpha^{(l)}$ by definition of $\alpha^{(l)}$, we have have shown that the map $\eta^{(l)} : \mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \to \mathcal{M}^{\beta}_{\mathcal{C}}$ is indeed surjective. □

Again, this is an analogue of a result in [AMW01], which justifies that we may consider our Lagrangian problem a compact version of the Thompson problem. We may now move on the main results of this chapter.

6.6 The set of σ_0-decomposable representations

Let $\mathcal{C}_1, \dots, \mathcal{C}_l$ be l conjugacy classes in U such that there exist $(u_1, \dots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ satisfying $u_1 \dots u_l = 1$. We denote by μ the map

$$\begin{array}{rccc} \mu : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \dots, u_l) & \longmapsto & u_1 \dots u_l \end{array}$$

Recall that a representation $(u_1, \dots, u_l)$ of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ is said to be *decomposable* if there exist $w_1, \dots w_l \in Fix(\tau^-)$ such that $u_j = w_j w_{j+1}^{-1}$ (see definition 5.2.1). It is said to be σ_0-*decomposable* if it is decomposable with $w_1 = 1$. Also recall that two representations $(u_1, \dots, u_l)$ and $(v_1, \dots, v_l)$ of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ are called equivalent if there exists an element $\varphi \in U$ such that $\varphi u_j \varphi^{-1} = v_j$ for all $j \in \{1, \dots, l\}$. Recall finally that $Fix(\tau^-)$ is assumed to be connected (see remark 5.2.3). We then make the following observation :

Lemma 6.6.1. *A representation $(u_1, \dots, u_l)$ of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ is decomposable if and only if it is equivalent to a σ_0-decomposable representation.*

Proof. Assume first that $(u_1, \dots, u_l)$ is decomposable. Then there exist $w_1, \dots, w_l \in Fix(\tau^-)$ such that $u_1 = w_1 w_2^{-1}, \dots, u_l = w_l w_1^{-1}$. Then, since $Fix(\tau^-)$ is connected, there exists, by proposition 3.1.2, an element $\varphi \in U$ such that $w_1 = \tau^-(\varphi)\varphi$. In particular, $\tau(\varphi) w_1 \varphi^{-1} = 1$. If we set $w'_j := \tau(\varphi) w_j \varphi^{-1}$, we have $\omega'_j \in Fix(\tau^-)$, as $\tau^-(w'_j) = \tau^-(\varphi^{-1})\tau^-(w_j)\tau^-(\tau(\varphi)) = \tau(\varphi) w_j \varphi^{-1} = w'_j$, as well as :

$$w'_j (w'_{j+1})^{-1} = \tau(\varphi) w_j \varphi^{-1} \varphi w_{j+1}^{-1} \tau^-(\varphi) = \tau(\varphi) w_j \omega_{j+1}^{-1} \tau^-(\varphi) = \tau(\varphi) u_j \tau^-(\varphi)$$

and $w'_1 = 1$ by definition, so that the representations $\tau(\varphi).(u_1, \dots, u_l)$ is σ_0-decomposable.
Conversely, if there exists $u \in U$ such that $u.(u_1, \dots, u_l)$ is σ_0-decomposable, then there exist $w_1, \dots, w_l \in Fix(\tau^-)$ such that $w_1 = 1$ and $u u_j u^{-1} = w_j w_{j+1}^{-1}$. Set $w'_j := u^{-1} w_j \tau^-(u^{-1})$. Then $w'_j \in Fix(\tau^-)$, since $\tau^-(w'_j) = \tau^-(\tau^-(u^{-1}))\tau^-(w_j)\tau^-(u^{-1}) = u^{-1} w_j \tau^-(u^{-1}) = w'_j$, and :

$$\begin{array}{rcl} w'_j (w'_{j+1})^{-1} & = & u^{-1} w_j \tau^-(u^{-1}) \big(u^{-1} w_{j+1} \tau^-(u^{-1})\big)^{-1} \\ & = & u^{-1} w_j \tau^-(u^{-1}) \tau(u^{-1}) w_{j+1}^{-1} u \\ & = & u^{-1} w_j w_{j+1}^{-1} u \\ & = & u_j \end{array}$$

so that the representation $(u_1, \dots, u_l)$ is decomposable. □

Recall now that we have an involution β on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ defined by :

$$\begin{array}{rccc} \beta : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \dots, u_l) & \longmapsto & \big(\tau^-(u_l) \dots \tau^-(u_2)\tau^-(u_1)\tau(u_2) \dots \tau(u_l), \dots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

(see definition 6.5.5 and proposition 6.5.6). We may then state and prove the following characterization of σ_0-decomposable representations :

Theorem 6.6.2 (Characterization of σ_0-decomposable representations). *Consider $(u_1, \dots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $u_1 \dots u_l = 1$. Then, the representation of $\pi_1(S^2 \backslash \{s_1, \dots, s_l\})$ corresponding to $(u_1, \dots, u_l)$ is σ_0-decomposable if and only if $\beta(u_1, \dots, u_l) = (u_1, \dots, u_l)$.*

Remark 6.6.3. We could as well have defined β on $U \times \cdots \times U$ and obtained a similar result but we deliberately stated our result this way, as it will be more appropriate to work with the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ in the following.

We will give two proofs of theorem 6.6.2, the first of which is valid in the special case where $U = U(n)$ and emphasizes the geometric point of view that we have adopted to guide us in the earlier sections and chapters. In this case, $\tau(u) = \overline{u}$, and we have :

Proof of theorem 6.6.2 in the case where $U = U(n)$. Let us start with $(u_1, \dots, u_l) \in Fix(\beta)$, that is :

$$\begin{aligned}
\overline{u}_l^{-1} \dots \overline{u}_2^{-1} u_1^t \overline{u}_2 \dots \overline{u}_l &= u_1 \\
\overline{u}_l^{-1} \dots \overline{u}_3^{-1} u_2^t \overline{u}_3 \dots \overline{u}_l &= u_2 \\
&\vdots \\
\overline{u}_l^{-1} \dots \overline{u}_{j+1}^{-1} u_j^t \overline{u}_{j+1} \dots \overline{u}_l &= u_j \\
&\vdots \\
\overline{u}_l^{-1} u_{l-1}^t \overline{u}_l &= u_{l-1} \\
u_l^t &= u_l
\end{aligned}$$

Then we have $u_l^t = u_l$ (so that $\overline{u}_l = u_l^{-1}$), $(u_{l-1}u_l)^t = (\overline{u}_l^{-1} u_{l-1}^t \overline{u}_l u_l)^t = (u_l^t u_{l-1}^t)^t = u_{l-1}u_l$, $\dots$, $(u_j \dots u_l)^t = (\overline{u}_l^{-1} \dots \overline{u}_{j+1}^{-1} u_j^t \overline{u}_{j+1} \dots \overline{u}_l \dots \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l u_l)^t = (u_l^t u_{l-1}^t \dots u_{j+1}^t u_j^t)^t = u_j \dots u_l$, $\dots$, and $(u_1 \dots u_l)^t = (\overline{u}_l^{-1} \dots \overline{u}_2^{-1} u_1^t \overline{u}_2 \dots \overline{u}_l \overline{u}_l^{-1} u_{l-1}^t \overline{u}_l u_l)^t = (u_l^t u_{l-1}^t \dots u_2^t u_1^t)^t = u_1 \dots u_l$. To these l symmetric unitary matrices we can associate, by proposition 5.1.3, l Lagrangian subspaces :

$$\begin{aligned}
L_1 &:= \{z \in \mathbb{C}^n \mid z - (u_1 \dots u_l)\overline{z} = 0\} \\
L_2 &:= \{z \in \mathbb{C}^n \mid z - (u_2 \dots u_l)\overline{z} = 0\} \\
&\vdots \\
L_j &:= \{z \in \mathbb{C}^n \mid z - (u_j \dots u_l)\overline{z} = 0\} \\
&\vdots \\
L_{l-1} &:= \{z \in \mathbb{C}^n \mid z - (u_{l-1}u_l)\overline{z} = 0\} \\
L_l &:= \{z \in \mathbb{C}^n \mid z - u_l\overline{z} = 0\}
\end{aligned}$$

and denote by σ_j the Lagrangian involution associated to L_j. Let us now assume that $(u_1, \dots, u_l)$ satisfy the full hypotheses of the theorem, that is, that we have $u_1 \dots u_l = 1$. Then $L_1 = L_0$. Therefore, by proposition 5.1.3, since $L_l = \{z - u_l\overline{z} = 0\}$, we have $\sigma_l\sigma_0 = u_l$, that is, $\sigma_l\sigma_1 = u_l$. Further, since $L_2 = \{z - (u_2 \dots u_l)\overline{z} = 0\}$, we have $\sigma_2\sigma_0 = u_2 \dots u_l = u_1^{-1}$ hence $u_1 = \sigma_1\sigma_2$. Finally, for all $j \in \{2, \dots, l-1\}$, since $(u_j \dots u_l)^t = u_j \dots u_l$, there exists, by proposition 5.1.3, a unitary map $\varphi_j \in U(n) \mid \varphi_j^t = \varphi_j$ and $\varphi_j^2 = u_j \dots u_l$, and we then have $\varphi_j(L_0) = L_j$. Set $L'_j = \varphi_2^{-1}(L_j) = L_0$ and $L'_{j+1} = \varphi_j^{-1}(L_{j+1})$, and denote by σ'_j and σ'_{j+1} the associated involutions. Then :

$$\begin{aligned}
L'_{j+1} &= \{z \mid \varphi_j(z) \in L_{j+1}\} \\
&= \{z \mid \varphi_j(z) - u_{j+1} \dots u_l \overline{\varphi_j(z)} = 0\} \\
&= \{z \mid \varphi_j(z) - u_{j+1} \dots u_l \underbrace{\overline{\varphi_j}}_{=\varphi_j^{-1}}(\overline{z}) = 0\} \\
&= \{z \mid z - (\varphi_j^{-1} u_{j+1} \dots u_l \varphi_j^{-1})\overline{z} = 0\}
\end{aligned}$$

but $(\varphi_j^{-1} u_{j+1} \dots u_l \varphi_j^{-1})^t = \varphi_j^{-1} u_{j+1} \varphi_j^{-1}$ since $(\varphi_j^{-1})^t = (\varphi_j^t)^{-1} = \varphi_j^{-1}$ and $(u_{j+1} \dots u_l)^t = u_{j+1} \dots u_l$. Therefore, by proposition 5.1.3, we have $\sigma'_{j+1}\sigma'_j = \varphi_j^{-1} u_{j+1} \dots u_l \varphi_j^{-1}$. Since $\varphi_j^2 = u_j \dots u_l$, we then have $\varphi_j^{-1} u_{j+1} \dots u_l \varphi_j^{-1} = \varphi_j^{-1}(u_j^{-1}\varphi_j^2)\varphi_j^{-1} = \varphi_j^{-1} u_j^{-1} \varphi_j$, therefore $u_j^{-1} = \varphi_j \sigma'_{j+1} \sigma'_j \varphi_j^{-1} = \sigma_{j+1}\sigma_j$ since $L_j = \varphi_j(L'_j)$, $L_{j+1} = \varphi_j(L'_{j+1})$ and $\sigma_{\varphi(L)} = \varphi\sigma_L\varphi_{-1}$. Hence $u_j = \sigma_j\sigma_{j+1}$ and the representation of π corresponding to $(u_1, \dots, u_l)$ is σ_0-Lagrangian.
Conversely, assume that a given representation $(u_1, \dots, u_l)$ is σ_0-Lagrangian. Then $u_l = \sigma_l\sigma_0$. Now observe that for any unitary map u, one has $\overline{u} = \sigma_0 u \sigma_0$, therefore here $u_l^t = \overline{u}_l^{-1} = \sigma_0 u_l^{-1} \sigma_0 =$

$\sigma_0(\sigma_l\sigma_0)^{-1}\sigma_0 = \sigma_0(\sigma_0\sigma_l)\sigma_0 = \sigma_l\sigma_0 = u_l$. Likewise :

$$\begin{aligned}
\overline{u_l}^{-1}u_{l-1}^t\overline{u}_l &= (\sigma_0 u_l^{-1}\sigma_0)(\sigma_0 u_{l-1}^{-1}\sigma_0)(\sigma_0 u_l\sigma_0)\\
&= \sigma_0(u_l^{-1}u_{l-1}^{-1}u_l)\sigma_0\\
&= \sigma_0(\sigma_0\sigma_l)(\sigma_l\sigma_{l-1})(\sigma_l\sigma_0)\sigma_0\\
&= \sigma_{l-1}\sigma_l\\
&= u_{l-1}
\end{aligned}$$

and so on, until :

$$\begin{aligned}
\overline{u_l}^{-1}\dots\overline{u_2}^{-1}u_1^t\overline{u}_2\dots\overline{u}_l &= \sigma_0(\sigma_0\sigma_l)\dots(\sigma_3\sigma_2)(\sigma_2\sigma_1)(\sigma_2\sigma_3)\dots(\sigma_3\sigma_0)\sigma_0\\
&= \sigma_1\sigma_2\\
&= u_1
\end{aligned}$$

so that $\beta(u_1, \dots, u_l) = (u_1, \dots, u_l)$. □

Proof of theorem 6.6.2 in the general case. Let us start with $(u_1, \dots, u_l) \in Fix(\beta)$, that is :

$$\begin{aligned}
\tau^-(u_2\dots u_l)\tau^-(u_1)\tau(u_2\dots u_l) &= u_1\\
&\vdots\\
\tau^-(u_l)\tau^-(u_{l-1})\tau(u_l) &= u_{l-1}\\
\tau^-(u_l) &= u_l
\end{aligned}$$

Then we have $\tau^-(u_l) = u_l$, $\tau^-(u_{l-1}u_l) = u_{l-1}\tau^-(u_l) = u_{l-1}u_l$, ... , until $\tau^-(u_1\dots u_l) = u_1\dots u_l$. Set $w_j := u_j\dots u_l$. In particular, $w_1 = u_1\dots u_l = 1$. Then $\tau^-(w_j) = w_j$ for all $j \in \{1, \dots, l\}$ and one has :

$$\begin{aligned}
w_j w_{j+1}^{-1} &= (u_j\dots u_l)(u_{j+1}\dots u_l)^{-1}\\
&= u_j
\end{aligned}$$

so that the representation $(u_1, \dots, u_l)$ is σ_0-decomposable. Observe that the algebraic definition of σ_0-decomposable representations indeed enabled us to write a simple proof of our characterization result. Conversely, if $u_j = w_j w_{j+1}^{-1}$ with $w_j \in Fix(\tau^-)$ for all j and $w_1 = 1$, then :

$$\begin{aligned}
\tau^-(u_{j+1}\dots u_l)\tau^-(u_j)\tau(u_{j+1}\dots u_l) &= \tau^-(w_{j+1}w_1^{-1})\tau^-(w_j w_{j+1}^{-1})\tau(w_{j+1}w_1^{-1})\\
&= \tau^-(w_{j+1})\tau(w_{j+1})\tau^-(w_j)\tau(w_{j+1})\\
&= w_j w_{j+1}^{-1}\\
&= u_j
\end{aligned}$$

so that $(u_1, \dots, u_l) \in Fix(\beta)$. □

Remark 6.6.4. Should we choose to work with $U \times \dots \times U$ instead of $\mathcal{C}_1 \times \dots \times \mathcal{C}_l$, there is an interesting "non-homogeneous" description of the involution β. The map

$$\begin{array}{rccl}
\Phi : & U \times \dots \times U & \longrightarrow & U \times \dots \times U\\
& (u_1, \dots, u_l) & \longmapsto & (u_1\dots u_l, u_2\dots u_l, \dots, u_{l-1}u_l, u_l) =: (v_1, \dots, v_l)
\end{array}$$

is a diffeomorphism from $U \times \dots \times U$ on itself, whose inverse is the map

$$\Phi^{-1}(v_1, \dots, v_l) = (v_1 v_2^{-1}, \dots, v_{l-1}v_l^{-1}, v_l)$$

These maps are U-equivariant with respect to the diagonal action of U on $U \times \cdots \times U$. Using these "non-homogeneous" coordinates on $U \times \cdots \times U$, the involution β takes the simpler form :

$$\begin{array}{rccc} \Phi \circ \beta \circ \Phi^{-1} : & U \times \cdots \times U & \longrightarrow & U \times \cdots \times U \\ & \big(v_1, \ldots, v_l\big) & \longmapsto & \big(\tau^-(v_1), \ldots, \tau^-(v_l)\big) \end{array}$$

This is to be related to the work of Treloar on the moduli space of polygons in S^3 (see [Tre02]), where the map Φ is used to give a Lie group description of this moduli space.

We can then characterize those among representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ which are Lagrangian in the following way :

Corollary 6.6.5 (Characterization of decomposable representations). *Consider $(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ such that $u_1 \ldots u_l = 1$. Then the representation of $\pi := \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ corresponding to $(u_1, \ldots, u_l)$ is decomposable if and only if $\beta(u_1, \ldots, u_l) \sim (u_1, \ldots, u_l)$ as representations of π, that is, if and only if there exists $u \in U$ such that :*

$$\beta(u_1, \ldots, u_l) = u.(u_1, \ldots, u_l)$$

Proof. We know from lemma 6.6.1 that if $(u_1, \ldots, u_l)$ is decomposable, then there exists $\varphi \in U$ such that $\tau(\varphi).(u_1, \ldots, u_l)$ is σ_0-decomposable, so that, by theorem 6.6.2, one has :

$$\beta\big(\tau(\varphi).(u_1, \ldots, u_l)\big) = \tau(\varphi).(u_1, \ldots, u_l)$$

Hence, from the compatibility of β and τ :

$$\varphi.\beta(u_1, \ldots, u_l) = \tau(\varphi).(u_1, \ldots, u_l)$$

That is :

$$\beta(u_1, \ldots, u_l) = \big(\varphi^{-1}\tau(\varphi)\big).\big(u_1, \ldots, u_l\big)$$

Conversely, assume that $\beta(u_1, \ldots, u_l) = u.(u_1, \ldots, u_l)$ for some $u \in U$. Observe then that, because of the compatibility of β with τ and μ, the involution β induces an involution $\hat{\beta}$ on $\mu^{-1}(\{1\})/U$:

$$\begin{array}{rccc} \hat{\beta} : & \mu^{-1}(\{1\})/U & \longrightarrow & \mu^{-1}(\{1\})/U \\ & \big[(u_1, \ldots, u_l)\big] & \longmapsto & \big[\beta(u_1, \ldots, u_l)\big] \end{array}$$

And the condition $\beta(u_1, \ldots, u_l) = u.(u_1, \ldots, u_l)$ means that $\hat{\beta}(\,[(u_1, \ldots, u_l]\,) = [(u_1, \ldots, u_l)]$. We then anticipate on a result from the general theory of quasi-Hamiltonian spaces that we will prove in chapter 7 : as $Fix(\tau^-)$ is connected, proposition 7.4.5 shows that the map

$$p_\beta : Fix(\beta) \cap \mu^{-1}(\{1\}) \to Fix(\hat{\beta}) \subset \mu^{-1}(\{1\})/U$$

is surjective. Consequently, there exists $(v_1, \ldots, v_l) \in Fix(\beta) \cap \mu^{-1}(\{1\})$ such that $[(v_1, \ldots, v_l)] = [(u_1, \ldots, u_l)]$ that is, $(u_1, \ldots, u_l)$ is equivalent to a σ_0-decomposable representation so that, by lemma 6.6.1, $(u_1, \ldots, u_l)$ is decomposable. □

We shall now move on to the next chapter, where we will see that the involution β that we have just used to characterize decomposable representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ induces an anti-symplectic involution of the moduli space $\mathcal{M}_\mathcal{C} = \mathrm{Hom}_\mathcal{C}(\pi, U)/U$. This brings us back to studying general properties of symplectic quotients associated to quasi-Hamiltonian spaces.

Chapter 7

Anti-symplectic involutions on quasi-Hamiltonian quotients

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In this chapter, we introduce the tools that we shall need in chapter 9 to show that the decomposable representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ which we have characterized in chapter 6 project to a Lagrangian submanifold of the moduli space $\mathcal{M}_{\mathcal{C}} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_l//U$.

7.1 Motivation

Recall that the representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ are the elements of $\mu^{-1}(\{1\})$, where μ is the map :

$$\begin{array}{rccc} \mu : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \ldots, u_l) & \longmapsto & u_1 \ldots u_l \end{array}$$

As we have seen in the previous chapter, the σ_0-decomposable representations of π are the elements $u = (u_1, \ldots, u_l)$ of $\mu^{-1}(\{1\})$ satisfying $\beta(u) = u$. Furthermore, the heuristic approach that we have been following so far suggests that the set of equivalence classes of decomposable representations should be obtained as the fixed-point set of an *antisymplectic* involution defined on the moduli space $\mathcal{M}_{\mathcal{C}}$ (see section 6.1). If we can prove that the involution β, defined on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, induces such an antisymplectic involution $\hat{\beta}$ on $\mathcal{M}_{\mathcal{C}} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_l//U$ which fixes equivalence classes of decomposable representations, then the result we hope for (the Lagrangian nature of decomposable representations) will be a consequence of the following result, which is classical in symplectic geometry :

Lemma 7.1.1. *Let (N, ω) be a symplectic manifold and let σ be an antisymplectic involution on N (meaning that $\sigma^*\omega = -\omega$ and $\sigma^2 = Id_N$). Denote by $N^\sigma := Fix(\sigma)$ the fixed-point set of σ. Then : if $N^\sigma \neq \emptyset$, it is a Lagrangian submanifold of N.*

Proof. There always exists a Riemannian metric g_0 on M such that σ is an isometry for g_0 (to obtain it, simply average any Riemannian metric over the group $\{1, \sigma\}$). We can then associate to the metric g_0 an almost complex structure $J = r(g_0)$ (see [MS98], proposition 2.50, (ii), p.63, for a definition of the

map r) such that J is compatible with ω (that is, such that $g(.,.) := \omega(.,J.)$ is a Riemannian metric on N). It then follows from the construction of $J = r(g_0)$ (see [MS98], p.64), that the g_0-isometry σ is anti-holomorphic with respect to J (that is, $T\sigma \circ J = -J \circ T\sigma$). Consequently, σ is an isometry for the Riemannian metric g :

$$\begin{aligned} g_x(T_x\sigma.v, T_x\sigma.w) &= \omega_x(T_x\sigma.v, JT_x\sigma.w) \\ &= -\omega_x(T_x\sigma.v, T_x\sigma.Jw) \\ &= -(\sigma^* w)_x(v, Jw) \\ &= \omega_x(v, Jw) \\ &= g_x(v, w) \end{aligned}$$

and $N^\sigma = Fix(\sigma)$ is therefore a totally geodesic submanifold of N whenever it is non-empty (see for instance [Kli82] p.95).
Let us now show that for all $x \in N^\sigma$, the subspace $T_xN^\sigma \subset T_xN$ is a Lagrangian subspace of T_xN. For all $v \in T_xN^\sigma$, $T_x\sigma.v = \frac{d}{dt}|_{t=0}\ (\sigma(x_t))$, where $x_t \in N^\sigma$ for all t (that is, $\sigma(x_t) = x_t$ for all t), $x_0 = x$ and $\frac{d}{dt}|_{t=0}\ x_t = v$, so that $T_x\sigma.v = v$ and $T_xN^\sigma = \ker(T_x\sigma - Id)$ (the inclusion $\supset$ being a consequence of the fact that N^σ is totally geodesic). Further, since $T_x\sigma$ is an involution on T_xN, one has :

$$T_xN = \ker(T_x\sigma - Id) \oplus \ker(T_x\sigma + Id)$$

But $\ker(T_x\sigma + Id) = J(T_xN^\sigma)$. Indeed, if $v \in T_xN^\sigma$, $T_x\sigma.(Jv) = -J(T_x\sigma.v) = -Jv$, so that $Jv \in \ker(T_x\sigma + Id)$ and conversely, if $T_x\sigma.w = -w$, then $T_x\sigma.(Jw) = -J(T_x\sigma.w) = Jw$, so that $w = J(-Jw) \in J(T_xN^\sigma)$. Therefore :

$$T_xN = T_xN^\sigma \oplus J(T_xN^\sigma)$$

and consequently :

$$\dim\ T_xN^\sigma = \frac{1}{2} \dim\ T_xN$$

Finally, since $T_x\sigma$ is antisymplectic, T_xN^σ is isotropic. Indeed, for all $v, w \in T_xN^\sigma$, one has :

$$\omega_x(v, w) = -(\sigma^*\omega)_x(v, w) = -w_{\sigma(x)}(T_x\sigma.v, T_x\sigma.w) = -\omega_x(v, w)$$

so that $\omega_x(v, w) = 0$. □

Remark 7.1.2. Observe that an anti-symplectic involution does not necessarily have fixed points. For instance, the map $(-Id_{\mathbb{R}^3})|_{S^2}\ : (x, y, z) \in S^2 \mapsto -(x, y, z)$ reverses orientation on S^2 (so that it is anti-symplectic with respect to the volume form $x\,dy \wedge dz - y\,dx \wedge dz - z\,dx \wedge dy$ on S^2), and has no fixed points on S^2.

As a matter of fact, to apply the preceding lemma, what we really need to prove is that β induces an involution $\hat{\beta}$ on $\mathcal{M}_{\mathcal{C}}$, which is antisymplectic, *and which satisfies* $Fix(\hat{\beta}) \neq \emptyset$. In the remaining part of this chapter, we will give general sufficient conditions on an involution β defined on a quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ for it to induce an antisymplectic involution on the associated quasi-Hamiltonian quotient $M//U$ (see proposition 7.2.2). Giving such conditions mainly consists in carrying over a standard procedure for usual symplectic quotients (appearing for instance in [OS00]) to the quasi-Hamiltonian setting. We will then show that the map β constructed in chapter 6 satisfies these conditions. To do this, we will actually give a general way of obtaining such involutions on product spaces, and apply this to $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ to prove the result for β (although the general result we shall state, namely lemma 7.3.3, was of course really inspired by the form of β itself). As for the existence of fixed points for $\hat{\beta}$, we postpone work on this question until the next chapter, as it is technically more difficult and calls for notions and methods which are very different from the ones we have encountered so far in this work. It is true, though, that if β satisfies the hypotheses of proposition 7.2.2 then $\hat{\beta}$ necessarily has fixed points (see corollary 8.3.11).

7.2 Lagrangian submanifolds of a quasi-Hamiltonian quotient

The purpose of this section is to give a way of finding Lagrangian submanifolds in a symplectic manifold $M//U$ obtained by reduction from a quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ (see proposition 4.5.2). As a matter of fact, we would like to apply lemma 7.1.1 to an antisymplectic involution σ defined on the symplectic manifold $N = M//U$. The idea is to obtain such a σ from an involution β defined on the whole quasi-Hamiltonian space M. We would then have to give sufficient conditions on β for it to induce indeed an antisymplectic involution $\sigma = \hat{\beta}$ on $M//U = \mu^{-1}(\{1\})/U$. To obtain these conditions, we draw from the corresponding situation in the usual Hamiltonian setting, which is studied in [OS00]. If we want β to induce a map on $\mu^{-1}(\{1\})/U$, it has to let $\mu^{-1}(\{1\})$ stable, and to map U-orbits to U-orbits. In the usual Hamiltonian case, the appropriate conditions, given in [OS00], are the following ones. Let U be a Lie group acting on a symplectic manifold (M, ω) in a Hamiltonian fashion with momentum map $\Phi : M \to \mathfrak{u}^*$. Let τ denote an involutive automorphism of U and still denote by τ the involution :

$$\begin{array}{rccc} (T_1\,\tau)^* : & \mathfrak{u}^* & \longrightarrow & \mathfrak{u}^* \\ & \lambda & \longmapsto & \lambda \circ T_1\,\tau \end{array}$$

that it induces on the dual $\mathfrak{u}^*$ of the Lie algebra $\mathfrak{u} = T_1\,U$ of U. Let β be an anti-symplectic involution on M (that is, such that $\beta^*\omega = -\omega$ and $\beta^2 = Id_M$). In the above notations, β is said to be *compatible with the action* of U if for all $u \in U$, for all $x \in M$, $\beta(u.x) = \tau(u).\beta(x)$ and β is said to be *compatible with the momentum map* $\Phi : M \to \mathfrak{u}^*$ if for all $x \in M$, $\Phi \circ \beta(x) = -\tau \circ \Phi(x)$. Since in the quasi-Hamiltonian case the momentum map takes value in a group instead of a vector space, we are led to formulate the following compatibility conditions :

Definition 7.2.1 (Compatible involutions). Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian space and let τ be an involutive automorphism of U. Denote by τ^- the involution on U defined by $\tau^-(u) = \tau(u^{-1})$. An involution β on M is said to be *compatible with the action of* U if $\beta(u.x) = \tau(u).\beta(x)$ for all $x \in M$ and all $u \in U$, and it is said to be *compatible with the momentum map* μ if $\mu \circ \beta = \tau^- \circ \mu$.

Let us mention here that when $U = T$ is a torus and τ is the involutive automorphism $\tau(t) = t^{-1}$ of the abelian group T, compatibility with the momentum map amounts to saying that $\mu \circ \beta = \mu$. This condition, that one may recognize from the work of Duistermaat in [Dui83], will play an important role in chapter 8, where we will study the image, under the momentum map μ of the fixed-point set M^β of an involution β defined on M and satisfying the compatibility conditions of definition 7.2.1 above and the additional condition $\beta^*\omega = -\omega$.

Proposition 7.2.2. *Let $(M, \omega, \mu : M \to U)$ be a quasi-Hamiltonian space and let τ be an involutive automorphism of U. Denote by τ^- the involution on U defined by $\tau^-(u) = \tau(u^{-1})$ and let β be an involution on M such that :*

(i) $\forall u \in U$, $\forall x \in M$, $\beta(u.x) = \tau(u).\beta(x)$

(ii) $\forall x \in M$, $\mu \circ \beta(x) = \tau^- \circ \mu(x)$

(iii) $\beta^*\omega = -\omega$

then β induces an anti-symplectic involution $\hat{\beta}$ on the reduced space $M^{red} := \mu^{-1}(\{1\})/U$. If $\hat{\beta}$ has fixed points, then $Fix(\hat{\beta})$ is a Lagrangian submanifold of M^{red}.

Proof. We give a proof under the following regularity assumptions : 1 is a regular value of μ and the compact group U acts freely on the level manifold $\mu^{-1}(\{1\})$. The proof in the stratified case works the same since, by proposition 4.5.8, the symplectic structure on each stratum of the reduced space $\mu^{-1}(\{1\})/U$ is obtained using the reduction procedure of the smooth case (see proposition 4.5.2). In particular, $Fix(\hat{\beta})$ is a disjoint union of Lagrangian submanifolds.
Compatibility with the momentum map (condition (ii)) shows that β maps $\mu^{-1}(\{1\})$ into $\mu^{-1}(\{1\})$ (since

$\tau^-(1) = 1$). Compatibility with the action (condition (i)) then shows that $\beta(u.x)$ and $\beta(x)$ lie in the same U-orbit, so that we have a map :

$$\begin{array}{rccc} \hat{\beta}: & \mu^{-1}(\{1\})/U & \longrightarrow & \mu^{-1}(\{1\})/U \\ & U.x & \longmapsto & U.\beta(x) \end{array}$$

We know from quasi-Hamiltonian reduction (see proposition 4.5.2) that there exists a unique symplectic form ω^{red} on $M^{red} = \mu^{-1}(\{1\})/U$ such that $p^*\omega^{red} = i^*\omega$ where $i : \mu^{-1}(\{1\}) \hookrightarrow M$ and $p : \mu^{-1}(\{1\}) \to M^{red}$. Observe that, by definition of $\hat{\beta}$, one has $p \circ \beta = \hat{\beta} \circ p$. To show that $\hat{\beta}^*\omega^{red} = -\omega^{red}$, we first prove that $i^*(\beta^*\omega)$ is basic with respect to the fibration p. Then there will exist a unique 2-form γ on M^{red} such that $p^*\gamma = i^*(\beta^*\omega)$. Since both $\gamma = -\omega^{red}$ and $\gamma = \hat{\beta}^*\omega^{red}$ satisfy this condition, they have to be equal. The last part of the proposition then follows from lemma 7.1.1. Let us now write this explicitly. Verifying that $i^*(\beta^*\omega)$ is basic is easy since $\beta^*\omega = -\omega$ and $i^*\omega$ is basic (see proposition 4.5.2) but it is actually true without this assumption so we prove it for β satisfying only conditions (i) and (ii) above. We have to show that $i^*(\beta^*\omega)$ is U-invariant and that for every $X \in \mathfrak{u} = Lie(U)$, we have $\iota_{X^\sharp}(i^*(\beta^*\omega)) = 0$, where $X^\sharp$ is the fundamental vector field $X^\sharp_x = \frac{d}{dt}|_{t=0}(\exp(tX).x)$ (for any $x \in M$) associated to $X \in \mathfrak{u}$ by the action of U on M. Let $u \in U$ and denote by φ_u the corresponding diffeomorphism of M. The map μ being equivariant φ_u sends $\mu^{-1}(\{1\})$ into itself, hence $i \circ \varphi_u = \varphi_u \circ i$ on $\mu^{-1}(\{1\})$. Furthermore, compatibility with the action yields $\beta \circ \varphi_u = \varphi_{\tau(u)} \circ \beta$. We then have, on $\mu^{-1}(\{1\})$:

$$\begin{aligned} \varphi_u^*(i^*(\beta^*\omega)) &= (\beta \circ i \circ \varphi_u)^*\omega \\ &= (\varphi_{\tau(u)} \circ \beta \circ i)^*\omega \\ &= i^*(\beta^*(\underbrace{\varphi_{\tau(u)}^*\omega}_{=\omega})) \end{aligned}$$

where the very last equality follows from the U-invariance of ω. Further, let $X \in \mathfrak{u}$. Since β is compatible with the action, one has $\beta(\exp(tX).x) = \tau(\exp(tX)).\beta(x) = \exp(t\tau(X)).\beta(x)$ (where we still denote by τ the involution $T_1\,\tau$ on $\mathfrak{u} = T_1\,U$), hence $T_x\beta.X^\sharp_x = (\tau(X))^\sharp_{\beta(x)}$, hence $\iota_{X^\sharp}(\beta^*\omega) = \beta^*(\iota_{(\tau(X))^\sharp}\omega)$. Since $\iota_{X^\sharp}(i^*(\beta^*\omega)) = i^*(\iota_{X^\sharp}(\beta^*\omega))$, we have, using the fact that β is compatible with μ :

$$\begin{aligned} \iota_{X^\sharp}(\beta^*\omega) &= \beta^*(\iota_{(\tau(X))^\sharp}\omega) \\ &= \beta^*\Big(\frac{1}{2}\,\mu^*\big(\theta^L + \theta^R \,|\, \tau(X)\big)\Big) \\ &= \frac{1}{2}\,(\mu \circ \beta)^*\big(\theta^L + \theta^R \,|\, \tau(X)\big) \\ &= \frac{1}{2}\,(\tau^- \circ \mu)^*\big(\theta^L + \theta^R \,|\, \tau(X)\big) \\ &= \frac{1}{2}\,\mu^*\Big((\tau^-)^*\big(\theta^L + \theta^R \,|\, \tau(X)\big)\Big) \end{aligned}$$

hence $i^*(\iota_{X^\sharp}(\beta^*\omega)) = \frac{1}{2}\,i^* \circ \mu^*\,(\ldots) = \frac{1}{2}\,(\mu \circ i)^*\,(\ldots)$. But $\mu \circ i : \mu^{-1}(\{1\}) \to U$ is a constant map, therefore $T(\mu \circ i)$, and consequently $(\mu \circ i)^*$, are zero, which completes the proof that $i^*(\beta^*\omega)$ is basic. Finally, let us show that $p^*(\hat{\beta}\omega^{red}) = i^*(\beta^*\omega) = p^*(-\omega^{red})$ (this is where we really use $\beta^*\omega = -\omega$). We have, on $\mu^{-1}(\{1\})$, $p^*(\hat{\beta}^*\omega^{red}) = (\hat{\beta} \circ p)^*\omega^{red} = (p \circ \beta)^*w^{red} = \beta^*(p^*\omega^{red}) = \beta^*(i^*\omega) = (i \circ \beta)^*\omega = (\beta \circ i)^*\omega = i^*(\beta^*\omega) = i^*(-\omega) = -i^*\omega = -p^*\omega^{red} = p^*(-\omega^{red})$. This completes the proof, as indicated above. □

Remark 7.2.3 (On the assumption that $Fix(\hat{\beta}) \neq \emptyset$). The assumption that $Fix(\hat{\beta}) \neq \emptyset$ in proposition 7.2.2 is in fact automatically satisfied by an involution β satisfying the compatibility relations of definition 7.2.1 and the additional condition $Fix(\beta) \neq \emptyset$, as we shall see in corollary 8.3.11.

We now would like to apply this result to the involution β on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ that we constructed in chapter 6. To that end, we must show that β satisfies the conditions of proposition 7.2.2. Recall from definition

6.5.5 that :

$$\begin{array}{rccl} \beta \ : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ \ldots, u_l) & \longmapsto & \big(\tau^-(u_l)\ldots\tau^-(u_2)\tau^-(u_1)\tau(u_2)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

Here, the involution τ on U is supposed to satisfy the following assumption, already used in chapter 6 : the involution $\tau^- : u \mapsto \tau(u^{-1})$ leaves a maximal torus of U pointwise fixed. In particular, this implies that τ^- leaves each conjugacy class $\mathcal{C} \subset U$ globally invariant, therefore the map β above is well-defined. Let us then show that this map β satisfies the compatibility conditions of definition 7.2.1 :

$$\begin{aligned} \beta\big(u.(u_1, \ \ldots, u_l)\big) &= \beta(uu_1u^{-1}, \ \ldots \ , uu_lu^{-1}) \\ &= \big(\tau^-(u^{-1})\tau(u_l)\tau^-(u)\ldots\tau^-(u^{-1})\tau^-(u_1)\tau^-(u)\ldots\tau(u)\tau(u_l)\tau(u^{-1}), \\ &\qquad \ldots, \tau^-(u^{-1})\tau^-(u_l)\tau^-(u)\big) \\ &= \tau(u).\big(\tau^-(u_l)\ldots\tau^-(u_1)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\big) \\ &= \tau(u).\beta(u_1, \ \ldots \ , u_l) \end{aligned}$$

and :

$$\begin{aligned} \mu \circ \beta(u_1, \ \ldots, u_l) &= \mu\big(\tau^-(u_l)\ldots\tau^-(u_2)\tau^-(u_1)\tau(u_2)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\big) \\ &= \tau^-(u_l)\ldots\tau^-(u_1) \\ &= \tau^-(u_1\ldots u_l) \\ &= \tau^- \circ \mu \ (u_1, \ \ldots, u_l) \end{aligned}$$

Thus, to apply proposition 7.2.2 to the involution β, it remains to show that $\beta^*\omega = -\omega$, where ω is the 2-form defining the quasi-Hamiltonian structure on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, and that $\hat{\beta}$ has fixed points. As we mentioned earlier, we postpone work on this last question to chapter 8, and we shall only prove for now that $\beta^*\omega = -\omega$ (see proposition 7.3.4). To do so, we will show that β is constructed by induction from the involution $\tau^- : u \in \mathcal{C} \to \tau(u^{-1})$ on a single conjugacy class (see lemma 7.3.2), and that this construction gives rise to form-reversing involutions on product spaces when starting from form-reversing involutions on each factor (see lemma 7.3.3).

7.3 Constructing form-reversing involutions on product spaces

From now on, we make the further assumption that the involutive automorphism τ of U is such that the involution $T_1\tau$ of $\mathfrak{u} = T_1U$ is an isometry for the Ad-invariant scalar product $(.\,|\,.)$ on $\mathfrak{u}$. In the following, we shall still denote by τ the map $T_1\tau$. Again, recall from definition 6.5.5, that :

$$\begin{array}{rccl} \beta \ : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ \ldots, u_l) & \longmapsto & \big(\tau^-(u_l)\ldots\tau^-(u_2)\tau^-(u_1)\tau(u_2)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

In the rest of this section, to avoid confusion, we shall denote by $\beta^{(l)}$ the involution β above defined on a product of l conjugacy classes. For instance, $\beta^{(2)}$ designates the involution β on a product of two conjugacy classes, regardless of whether this product space is $\mathcal{C}_1 \times \mathcal{C}_2$ or $\mathcal{C}_2 \times \mathcal{C}_3$. Likewise, we will denote by $\mu^{(l)}$ the momentum map $\mu : (u_1, \ldots, u_l) \mapsto u_1\ldots u_l$ defined on a product of l conjugacy classes, and by $\omega^{(l)}$ the 2-form defining the quasi-Hamiltonian structure on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ with respect to the momentum map $\mu^{(l)}$ (see corollary 4.4.2). Observe now that when $l = 1$, we simply have $\beta^{(1)}(u) = \tau^-(u)$ on a single conjugacy class $\mathcal{C}$. The first thing to notice is then the following result :

Lemma 7.3.1. *The involution $\beta^{(1)} = \tau^- : u \mapsto \tau(u^{-1})$, restricted to a single conjugacy class $\mathcal{C}$ of U, reverses the 2-form $\omega^{(1)}$ defining the quasi-Hamiltonian structure on $\mathcal{C}$, that is : $(\beta^{(1)})^*\,\omega^{(1)} = -\omega^{(1)}$.*

Proof. To lighten notations, we will denote $\omega^{(1)}$ by ω and $\beta^{(1)}$ by β in the course of this proof. First, recall from proposition 6.5.4 that the involution τ^- on U sends any conjugacy class $\mathcal{C} \subset U$ to itself. Second, recall from proposition 4.2.1 that we have, for any $X, Y \in \mathfrak{u}$ (denoting $[X]_u = X.u - u.X \in T_u\mathcal{C}$) :

$$\omega_u([X]_u, [Y]_u) = \frac{1}{2}\big((Adu.X \,|\, Y) - (Adu.Y \,|\, X)\big)$$

Further, $\beta(u) = \tau(u^{-1})$ and $T_u\beta.[X]_u = [\tau(X)]_{\tau(u^{-1})}$. Therefore :

$$\begin{aligned}
(\beta^*\omega)_u\big([X]_u, [Y]_u\big) &= \omega_{\beta(u)}\big(T_u\beta.[X]_u, T_u\beta.[Y]_u\big) \\
&= \frac{1}{2}\Big(\big(Ad\,\tau(u^{-1}).\tau(X) \,|\, \tau(Y)\big) - \big(Ad\,\tau(u^{-1}).\tau(Y) \,|\, \tau(X)\big)\Big) \\
&= \frac{1}{2}\Big(\big(\tau(Ad\,u^{-1}.X) \,|\, \tau(Y)\big) - \big(\tau(Ad\,u^{-1}.Y) \,|\, \tau(X)\big)\Big)
\end{aligned}$$

Since τ is an isometry for $(.\,|\,.)$, we then have :

$$\begin{aligned}
(\beta^*\omega)_u\big([X]_u, [Y]_u\big) &= \frac{1}{2}\big((Ad\,u^{-1}.X \,|\, Y) - (Ad\,u^{-1}.Y \,|\, X)\big) \\
&= \frac{1}{2}\big((X \,|\, Ad\,u.Y) - (Y \,|\, Ad\,u.X)\big) \\
&= -\omega_u\big([X]_u, [Y]_u\big)
\end{aligned}$$

□

When now $l = 2$, the involution β writes :

$$\begin{array}{rccc}
\beta^{(2)}: & \mathcal{C}_1 \times \mathcal{C}_2 & \longrightarrow & \mathcal{C}_1 \times \mathcal{C}_2 \\
& (u_1, u_2) & \longmapsto & \big(\tau^-(u_2)\tau^-(u_1)\tau(u_2)\ ,\ \tau^-(u_2)\big)
\end{array}$$

So that :

$$\beta^{(2)}(u_1, u_2) = \big(\tau^-(u_2).\tau^-(u_1)\ ,\ \tau^-(u_2)\big) \tag{7.1}$$

where the action denoted by a point . is the conjugacy action of U on itself. The fruitful observation to make is then to notice that :

$$\beta^{(2)}(u_1, u_2) = \Big(\big(\underbrace{\tau^- \circ \mu^{(1)}}_{=\mu^{(1)}\circ\beta^{(1)}}(u_2)\big).\beta^{(1)}(u_1)\ ,\ \beta^{(1)}(u_2)\Big)$$

When $\mu^{(1)} : \mathcal{C}_2 \hookrightarrow U$ is the inclusion map, we indeed obtain expression (7.1). Likewise :

$$\begin{aligned}
\beta^{(3)}(u_1, u_2, u_3) &= \big(\tau^-(u_3)\tau^-(u_2)\tau^-(u_1)\tau(u_2)\tau(u_3)\ ,\ \tau^-(u_3)\tau^-(u_2)\tau(u_3),\ \tau^-(u_3)\big) \\
&= \big(\tau^-(u_2u_3).\tau^-(u_1)\ ,\ \beta^{(2)}(u_2, u_3)\big) \\
&= \Big(\big(\underbrace{\tau^- \circ \mu^{(2)}}_{=\mu^{(2)}\circ\beta^{(2)}}(u_2, u_3)\big).\tau^-(u_1)\ ,\ \beta^{(2)}(u_2, u_3)\Big) \\
&= \Big(\big(\mu^{(2)} \circ \beta^{(2)}(u_2, u_3)\big).\beta^{(1)}(u_1)\ ,\ \beta^{(2)}(u_2, u_3)\Big)
\end{aligned}$$

In fact, the involution $\beta^{(l)}$ is obtained in the same way from the involution $\beta^{(l-1)}$ on $\mathcal{C}_2 \times \cdots \times \mathcal{C}_l$ and the involution $\beta^{(1)}$ on $\mathcal{C}_1$. More precisely, we can sum up the above discussion in the following way :

Lemma 7.3.2. *Consider an integer $l \geq 1$ and let $\mathcal{C}_1, \ldots, \mathcal{C}_l$ be l conjugacy classes in U. Let $\beta^{(1)}$ be the involution defined on $\mathcal{C}_1$ by*

$$\begin{array}{rccc}
\beta^{(1)}: & \mathcal{C}_1 & \longrightarrow & \mathcal{C}_1 \\
& u_1 & \longmapsto & \tau^-(u_1)
\end{array}$$

and let $\beta^{(l-1)}$ be the involution defined on the product $\mathcal{C}_2 \times \cdots \times \mathcal{C}_l$ of $(l-1)$ conjugacy classes by :

$$\begin{array}{rccc} \beta^{(l-1)}: & \mathcal{C}_2 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_2 \times \cdots \times \mathcal{C}_l \\ & (u_2, \ \ldots, u_l) & \longmapsto & \big(\tau^-(u_l)\ldots\tau^-(u_3)\tau^-(u_2)\tau(u_3)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\big) \end{array}$$

Let $\mu^{(l-1)}$ be the map from $\mathcal{C}_2 \times \mathcal{C}_l$ to U defined by :

$$\begin{array}{rccc} \mu^{(l-1)}: & \mathcal{C}_2 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_2 \times \cdots \times \mathcal{C}_l \\ & (u_2, \ \ldots, u_l) & \longmapsto & u_2 \ldots u_l \end{array}$$

Finally, let $\beta^{(l)}$ be the involution defined on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l = \mathcal{C}_1 \times (\mathcal{C}_2 \times \cdots \times \mathcal{C}_l)$ by :

$$\begin{array}{rccc} \beta^{(l)}: & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ \ldots, u_l) & \longmapsto & \big(\tau^-(u_l)\ldots\tau^-(u_2)\tau^-(u_1)\tau(u_2)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\big) \end{array}$$

Then we have :

$$\beta^{(l)}(u_1, \ \ldots, u_l) = \Big(\big(\mu^{(l-1)} \circ \beta^{(l-1)}(u_2, \ \ldots, u_l)\big).\beta^{(1)}(u_1) \ , \ \beta^{(l-1)}(u_2, \ \ldots, u_l)\Big)$$

which we will write :

$$\beta^{(l)} = \Big(\big(\mu^{(l-1)} \circ \beta^{(l-1)}\big).\beta^{(1)} \ , \ \beta^{(l-1)}\Big)$$

Now, to prove that $(\beta^{(l)})^* \omega^{(l)} = -\omega^{(l)}$, we will use the following lemma, which is general in nature and gives a way of constructing form-reversing involutions on products of quasi-Hamiltonian spaces starting from form-reversing involutions on each factor. It was inspired by the form of our involution β.

Lemma 7.3.3. *Let $(M_1, \omega_1, \mu_1 : M_1 \to U)$ and $(M_2, \omega_2, \mu_2 : M_2 \to U)$ be two quasi-Hamiltonian U-spaces. Let τ be an involutive automorphism of $(U, (.\,|\,.))$ and let β_i be an involution on M_i satisfying :*

(i) $\beta_i^* \omega_i = -\omega_i$

(ii) $\beta_i(u.x_i) = \tau(u).\beta_i(x_i)$ *for all* $u \in U$ *and all* $x_i \in M_i$

(iii) $\mu_i \circ \beta_i = \tau^- \circ \mu_i$

Consider the quasi-Hamiltonian U-space $(M := M_1 \times M_2, \omega := \omega_1 \oplus \omega_2 + \frac{1}{2}(\mu_1^* \theta^L \wedge \mu_2^* \theta^R), \mu := \mu_1 \cdot \mu_2)$ (with respect to the diagonal action of U) and the map :

$$\begin{array}{rccc} \beta := \big((\mu_2 \circ \beta_2).\beta_1, \beta_2\big): & M & \longrightarrow & M \\ & (x_1, x_2) & \longmapsto & \big((\mu_2 \circ \beta_2(x_2)).\beta_1(x_1), \beta_2(x_2)\big) \end{array}$$

Then β is an involution on M satisfying :

(i) $\beta^* \omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $u \in U$ *and all* $x \in M$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

Proof. First, we have :

$$\begin{aligned} \beta(\beta(x_1, x_2)) &= \Big(\Big(\mu_2 \circ \beta_2\big(\beta_2(x_2)\big)\Big).\beta_1\Big(\big(\mu_2 \circ \beta_2(x_2)\big).\beta_1(x_1)\Big), \beta_2\big(\beta_2(x_2)\big)\Big) \\ &= \Big(\big(\mu_2(x_2)\big).\Big(\tau\big(\underbrace{\mu_2 \circ \beta_2}_{=\tau^- \circ \mu_2}(x_2)\big).\beta_1\big(\beta_1(x_1)\big)\Big), x_2\Big) \\ &= \Big(\big(\mu_2(x_2)\big)\big(\mu_2(x_2)\big)^{-1}.x_1, x_2\Big) \\ &= (x_1, x_2) \end{aligned}$$

so that β is indeed an involution. Second :

$$\begin{aligned}
\beta(u.x_1, u.x_2) &= \big(\mu_2 \circ \beta_2(u.x_2).\beta_1(u.x_1), \beta_2(u.x_2)\big) \\
&= \Big(\underbrace{\mu_2\big(\tau(u).\beta_2(x_2)\big)}_{=\tau(u)\mu_2\left(\beta_2(x_2)\right)\tau(u)^{-1}} .\big(\tau(u).\beta_1(x_1)\big), \tau(u)\beta_2(x_2)\Big) \\
&= \Big(\tau(u).\Big(\big(\mu_2 \circ \beta_2(x_2)\big).\beta_1(x_1)\Big), \tau(u).\beta_2(x_2)\Big) \\
&= \tau(u).\beta(x_1, x_2)
\end{aligned}$$

and :

$$\begin{aligned}
\mu \circ \beta(x_1, x_2) &= \mu_1\Big(\big(\mu_2 \circ \beta_2(x_2)\big).\beta_1(x_1)\Big)\mu_2\big(\beta_2(x_2)\big) \\
&= \Big(\mu_2 \circ \beta_2(x_2)\mu_1 \circ \beta_1(x_1)\big(\mu_2 \circ \beta_2(x_2)\big)^{-1}\Big)\Big(\mu_2 \circ \beta_2(x_2)\Big) \\
&= \tau^- \circ \mu_2(x_2)\tau^- \circ \mu_1(x_1) \\
&= \tau^- \circ (\mu_1 \cdot \mu_2)(x_1, x_2) \\
&= \tau^- \circ \mu(x_1, x_2)
\end{aligned}$$

So the only thing left to prove is that $\beta^*\omega = -\omega$. Let us start by computing $T\beta$. For all $(x_1, x_2) \in M$, and all $(v_1, v_2) := \frac{d}{dt}|_{t=0}(x_1(t), x_2(t))$ (where $x_i(0) = x_i$), one has :

$$\begin{aligned}
T_{(x_1,x_2)}\beta.(v_1, v_2) &= \frac{d}{dt}|_{t=0}\Big(\big(\mu_2 \circ \beta_2(x_2)\big).\beta_1\big(x_1(t)\big), \beta_2\big(x_2(t)\big)\Big) \\
&= \Big(\big(\mu_2 \circ \beta_2(x_2)\big).\Big\{\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.v_1\Big\}, T_{x_2}\beta_2.v_2\Big)
\end{aligned}$$

Recall indeed that if a Lie group U acts on a manifold M then :

$$\frac{d}{dt}|_{t=0}\ (u_t.x_t) = u_0.X^\sharp_{x_0} + u_0.\Big(\frac{d}{dt}|_{t=0}\ x_t\Big)$$

where $X \in \mathfrak{u} = Lie(U)$ is such that $u_t = u_0 \exp(tX)$ for all t, that is :

$$X = u_0^{-1}.\Big(\frac{d}{dt}|_{t=0}\ u_t\Big) = \theta^L_{u_0}\Big(\frac{d}{dt}|_{t=0}\ u_t\Big)$$

Let us now compute $\beta^*(\omega_1 \oplus \omega_2)$. We obtain, for all $(x_1, x_2) \in M$ and all $(v_1, v_2), (w_1, w_2) \in T_{(x_1,x_2)}M$:

$$\begin{aligned}
&\big(\beta^*(\omega_1 \oplus \omega_2)\big)_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big) \\
&= (\omega_1)_{\left(\mu_2\circ\beta_2(x_2)\right).\beta_1(x_1)}\Bigg(\big(\mu_2 \circ \beta_2(x_2)\big).\Bigg\{\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)} + \underbrace{T_{x_1}\beta_1.v_1}_{(A)}\Bigg\}, \quad (7.2) \\
&\qquad \big(\mu_2 \circ \beta_2(x_2)\big).\Bigg\{\Big(\theta^L_{\mu_2\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).w_2\big)\Big)^\sharp_{\beta_1(x_1)} + \underbrace{T_{x_1}\beta_1.w_1}_{(A)}\Bigg\}\Bigg) \quad (7.3)
\end{aligned}$$

$$+ \underbrace{(\omega_2)_{\beta_2(x_2)}(T_{x_2}\beta_2.v_2, T_{x_2}\beta_2.w_2)}_{(B)}$$

Since ω_1 is U-invariant, we can drop the terms $\mu_2 \circ \beta_2(x_2) \in U$ appearing on lines (7.2) and (7.3). Further, since $\beta^*\omega_1 = -\omega_1$ and $\beta^*\omega_2 = -\omega_2$, we have, by the $l = 1$ case :

$$\begin{aligned} (A) + (B) &= -(w_1)_{x_1}(v_1, w_1) - (w_2)_{x_2}(v_2, w_2) && (7.4)\\ &= -(\omega_1 \oplus \omega_2)_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big) && (7.5)\end{aligned}$$

The remaining terms on lines (7.2) and (7.3) then are :

$$(w_1)_{\beta_1(x_1)}\Big(\Big(\theta^L_{\mu\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)}, T_{x_1}\beta_1.w_1\Big) \tag{7.6}$$

$$+ \quad (w_1)_{\beta_1(x_1)}\Big(T_{x_1}\beta_1.v_1\,, \Big(\theta^L_{\mu\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).w_2\big)\Big)^\sharp_{\beta_1(x_1)}\Big) \tag{7.7}$$

$$+ \quad (w_1)_{\beta_1(x_1)}\Big(\Big(\theta^L_{\mu\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).v_2\big)\Big)^\sharp_{\beta_1(x_1)}, \Big(\theta^L_{\mu\circ\beta_2(x_2)}\big(T_{x_2}(\mu_2 \circ \beta_2).w_2\big)\Big)^\sharp_{\beta_1(x_1)}\Big) \tag{7.8}$$

and we notice that each of these three terms is of the form $\iota_{X^\sharp}\omega_1 = \frac{1}{2}\mu_1^*(\theta^L + \theta^R \,|\, X)$ for some $X \in \mathfrak{u}$. To facilitate the computations, we set, for $i = 1, 2$:

$$\begin{aligned} g_i &:= \mu_i \circ \beta_i(x_i) \in U\\ \zeta_i &:= T_{x_i}(\mu_i \circ \beta_i).v_i \in T_{\mu_i\circ\beta_i(x_i)}U = T_{g_i}U\\ \eta_i &:= T_{x_i}(\mu_i \circ \beta_i).w_i \in T_{\mu_i\circ\beta_i(x_i)}U = T_{g_i}U\end{aligned}$$

We can then rewrite lines (7.6), (7.7) and (7.8) under the form :

$$\frac{1}{2}\big(\underbrace{\theta^L_{g_1}(\eta_1)}_{(1)} + \underbrace{\theta^R_{g_1}(\eta_1)}_{C} \,|\, \theta^L_{g_2}(\zeta_2)\big) \tag{7.9}$$

$$-\frac{1}{2}\big(\underbrace{\theta^L_{g_1}(\zeta_1)}_{(2)} + \underbrace{\theta^R_{g_1}(\zeta_1)}_{D} \,|\, \theta^L_{g_2}(\eta_2)\big) \tag{7.10}$$

$$+\frac{1}{2}\Big(\theta^L_{g_1}\big(\theta^L_{g_2}(\eta_2).g_1 - g_1.\theta^L_{g_2}(\eta_2)\big) + \theta^R_{g_1}\big(\theta^L_{g_2}(\eta_2).g_1 - g_1.\theta^L_{g_2}(\eta_2)\big) \,|\, \theta^L_{g_2}(\zeta_2)\Big) \tag{7.11}$$

where the expression for the last term follows from the equivariance of μ_1 :

$$T_{\beta_1(x_1)}\mu_1.\big(\theta^L_{\mu_2\circ\beta_2(x_2)}(T_{x_2}(\mu_2 \circ \beta_2).w_2)\big)^\sharp_{\beta_1(x_1)} = \big(\theta^L_{\mu_2\circ\beta_2(x_2)}(T_{x_2}(\mu_2 \circ \beta_2).w_2)\big)^\sim_{\mu_1\circ\beta_1(x_1)} = \big(\theta^L_{g_2}(\eta_2)\big)^\sim_{g_2}$$

(where $X^\sim_u = X.u - u.X$ is the value at u of the fundamental vector field associated to $X \in \mathfrak{u}$ by the action of U on itself by conjugation). We can simplify the expression in (7.11) further by using the definition of θ^L and θ^R and the Ad-invariance of $(.\,|\,.)$:

$$\begin{aligned} (7.11) &= \frac{1}{2}\big(Ad\, g_1^{-1}.\theta^L_{g_2}(\eta_2) - Ad\, g_1.\theta^L_{g_2}(\eta_2) \,|\, \theta^L_{g_2}(\zeta_2)\big) && (7.12)\\ &= \frac{1}{2}\big(\theta^L_{g_2}(\eta_2) \,|\, Ad\, g_1.\theta^L_{g_2}(\zeta_2)\big) - \frac{1}{2}\big(Ad\, g_1.\theta^L_{g_2}(\eta_2) \,|\, \theta^L_{g_2}(\zeta_2)\big) && (7.13)\end{aligned}$$

Let us now compute $\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$.

$$\begin{aligned}
&\big(\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)\big)_{(x_1,x_2)}\big((v_1,v_2),(w_1,w_2)\big) && (7.14)\\
=\;& (\mu_1^*\theta^L \wedge \mu_2^*\theta^R)_{\left(\mu_2\circ\beta_2(x_2).\beta_1(x_1),\beta_2(x_2)\right)}\big(T_{(x_1,x_2)}\beta.(v_1,v_2), T_{(x_1,x_2)}\beta.(w_1,w_2)\big) && (7.15)\\
=\;& \frac{1}{2}\Big(\theta^L_{\underbrace{\mu_1(g_2.\beta_1(x_1)}_{g_2\mu_1(\beta_1(x_1))g_2^{-1}}}\Big(T_{g_2.\beta_1(x_1)}\mu_1.\big(\mu_2\circ\beta_2(x_2)\big).\Big\{\big(\theta^L_{g_2}(\zeta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.v_1\Big\}\Big) \,|\, \theta^R_{g_2}(\eta_2)\Big) && (7.16)\\
&-\frac{1}{2}\Big(\theta^L_{\mu_1(g_2.\beta_1(x_1)}\Big(T_{g_2.\beta_1(x_1)}\mu_1.\big(\mu_2\circ\beta_2(x_2)\big).\Big\{\big(\theta^L_{g_2}(\eta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.w_1\Big\}\Big) \,|\, \theta^R_{g_2}(\zeta_2)\Big)
\end{aligned}$$

Since μ_1 is equivariant, we have, for any $v \in T_{\beta_1(x_1)}M_1$:

$$T_{g_2.\beta_1(x_1)}\mu_1.\big(\mu_2\circ\beta_2(x_2)\big).v = \big(\mu_2\circ\beta_2(x_2)\big).\Big(T_{\beta_1(x_1)}\mu_1.v\Big)$$

where the action in the right side term is conjugation. We then have :

$$\begin{aligned}
(7.16) \;=\;& \frac{1}{2}\Big(\theta^L_{g_2g_1g_2^{-1}}\Big(g_2.\Big(T_{\beta_1(x_1)}\mu_1.\Big(\big(\theta^L_{g_2}(\zeta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.v_1\Big)\Big).g_2^{-1}\Big) \,|\, \theta^R_{g_2}(\eta_2)\Big)\\
&-\frac{1}{2}\Big(\theta^L_{g_2g_1g_2^{-1}}\Big(g_2.\Big(T_{\beta_1(x_1)}\mu_1.\Big(\big(\theta^L_{g_2}(\eta_2)\big)^\sharp_{\beta_1(x_1)} + T_{x_1}\beta_1.w_1\Big)\Big).g_2^{-1}\Big) \,|\, \theta^R_{g_2}(\zeta_2)\Big)\\
=\;& \frac{1}{2}\Big(g_2g_1^{-1}g_2^{-1}g_2.\big(\theta^L_{g_2}(\zeta_2).g_1 - g_1.\theta^L_{g_2}(\zeta_2)\big).g_2^{-1} \,|\, \theta^R_{g_2}(\eta_2)\Big) + \frac{1}{2}\Big(g_2\underbrace{g_1^{-1}g_2^{-1}g_2.\zeta_1}_{=\theta^L_{g_1}(\zeta_1)}.g_2^{-1} \,|\, \theta^R_{g_2}(\eta_2)\Big)\\
&-\frac{1}{2}\Big(g_2g_1^{-1}g_2^{-1}g_2.\big(\theta^L_{g_2}(\eta_2).g_1 - g_1.\theta^L_{g_2}(\eta_2)\big).g_2^{-1} \,|\, \theta^R_{g_2}(\zeta_2)\Big) + \frac{1}{2}\Big(g_2\underbrace{g_1^{-1}g_2^{-1}g_2.\eta_1}_{=\theta^L_{g_1}(\eta_1)}.g_2^{-1} \,|\, \theta^R_{g_2}(\zeta_2)\Big)\\
=\;& \frac{1}{2}\Big(Ad\,g_2Ad\,g_1^{-1}.\theta^L_{g_2}(\zeta_2) \,|\, \theta^R_{g_2}(\eta_2)\Big) - \frac{1}{2}\Big(Ad\,g_2.\theta^L_{g_2}(\zeta_2) \,|\, \theta^R_{g_2}(\eta_2)\Big) + \frac{1}{2}\Big(Ad\,g_2.\theta^L_{g_1}(\zeta_1) \,|\, \theta^R_{g_2}(\eta_2)\Big)\\
&-\frac{1}{2}\Big(Ad\,g_2Ad\,g_1^{-1}.\theta^L_{g_2}(\eta_2) \,|\, \theta^R_{g_2}(\zeta_2)\Big) + \frac{1}{2}\Big(Ad\,g_2.\theta^L_{g_2}(\eta_2) \,|\, \theta^R_{g_2}(\zeta_2)\Big) - \frac{1}{2}\Big(Ad\,g_2.\theta^L_{g_1}(\eta_1) \,|\, \theta^R_{g_2}(\zeta_2)\Big)\\
=\;& \underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\zeta_2) \,|\, Ad\,g_1.\theta^L_{g_2}(\eta_2)\Big)}_{(3')} - \underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\zeta_2) \,|\, \theta^L_{g_2}(\eta_2)\Big)}_{(4)} + \underbrace{\frac{1}{2}\Big(\theta^L_{g_1}(\zeta_1) \,|\, \theta^L_{g_2}(\eta_2)\Big)}_{(2')} && (7.17)\\
&-\underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\eta_2) \,|\, Ad\,g_1.\theta^L_{g_2}(\zeta_2)\Big)}_{(3')} + \underbrace{\frac{1}{2}\Big(\theta^L_{g_2}(\eta_2) \,|\, \theta^L_{g_2}(\zeta_2)\Big)}_{(4')} - \underbrace{\frac{1}{2}\Big(\theta^L_{g_1}(\eta_1) \,|\, \theta^L_{g_2}(\zeta_2)\Big)}_{(1')}
\end{aligned}$$

(to obtain this last expression, one uses the Ad-invariance of $(.\,|\,.)$ and the fact that $Ad\,g_2^{-1}\circ\theta^R_{g_2} = \theta^L_{g_2}$). Observe that (4) and (4′) cancel in the above expression. Likewise, (1′), (2′) and (3′) in (7.17) cancel respectively with (1), (2) in (7.9) and (7.10) and with (7.13) when computing the sum $\beta^*(\omega_1 \oplus \omega_2) +$

$\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)$. The non-vanishing terms in this sum are therefore (A) and (B) from (7.4) and (C) and (D) from (7.9) and (7.10), so that :

$$\begin{aligned}
& (\beta^*\omega)_x(v,w) && (7.18)\\
= \; & \big(\beta^*(\omega_1 \oplus \omega_2)\big)_x(v,w) + \big(\beta^*(\mu_1^*\theta^L \wedge \mu_2^*\theta^R)\big)_x(v,w) && (7.19)\\
= \; & (A) + (B) + (C) + (D) && (7.20)\\
= \; & -(\omega_1 \oplus \omega_2)_x(v,w) - \frac{1}{2}\Big(\big(\theta^R_{g_1}(\zeta_1) \,|\, \theta^L_{g_2}(\eta_2)\big) - \big(\theta^R_{g_1}(\eta_1) \,|\, \theta^L_{g_2}(\zeta_2)\big)\Big) && (7.21)
\end{aligned}$$

But $\mu_i \circ \beta_i = \tau^- \circ \mu_i$, so that :

$$\begin{aligned}
\big(\theta^R_{g_1}(\zeta_1) \,|\, \theta^L_{g_2}(\eta_2)\big) &= \big(\theta^R_{\mu_1\circ\beta_1(x_1)}(T_{x_1}(\mu_1\circ\beta_1).v_1) \,|\, \theta^L_{\mu_2\circ\beta_2(x_2)}(T_{x_2}(\mu_2\circ\beta_2).w_2)\big) && (7.22)\\
&= \big(\theta^R_{\tau^-\circ\mu_1(x_1)}(T_{x_1}(\tau^-\circ\mu_1).v_1) \,|\, \theta^L_{\tau^-\circ\mu_2(x_2)}(T_{x_2}(\tau^-\circ\mu_2).w_2)\big) && (7.23)
\end{aligned}$$

and $\tau^- = Inv \circ \tau$, where $Inv : u \mapsto u^{-1}$ is inversion on U, so $T_u\tau^-.\xi = -\tau^-(u).(T_u\tau.\xi).\tau^-(u)$. Hence :

$$\begin{aligned}
\theta^R_{\tau^-(u)}(T_u\tau^-.\xi) &= \theta^R_{\tau^-(u)}\big(-\tau^-(u).(T_u\tau.\xi).\tau^-(u)\big)\\
&= -\tau^-(u).(T_u\tau.\xi)\\
&= -\theta^L_{\tau(u)}(T_u\tau.\xi)
\end{aligned}$$

(and likewise θ^L changes into θ^R). Since in addition to that τ is a group automorphism and an isometry for $(.\,|\,.)$, the expression (7.23) becomes :

$$\begin{aligned}
(7.23) &= \Big(\theta^L_{\tau\left(\mu_1(x_1)\right)}\big(T_{\mu_1(x_1)}\tau.(T_{x_1}\mu_1.v_1)\big) \,|\, \theta^R_{\tau\left(\mu_2(x_2)\right)}\big(T_{\mu_2(x_2)}\tau.(T_{x_2}\mu_2.w_2)\big)\Big)\\
&= \Big(T_1\tau.\big(\theta^L_{\mu_1(x_1)}(T_{x_1}\mu_1.v_1)\big) \,|\, T_1\tau.\big(\theta^R_{\mu_2(x_2)}(T_{x_2}\mu_2.w_2)\big)\Big)\\
&= \big(\theta^L_{\mu_1(x_1)}(T_{x_1}\mu_1.v_1) \,|\, \theta^R_{\mu_2(x_2)}(T_{x_2}\mu_2.w_2)\big)\\
&= \big((\mu_1^*\theta^L)_{x_1}(v_1) \,|\, (\mu_2^*\theta^R)_{x_2}(w_2)\big)
\end{aligned}$$

so that we have :

$$\begin{aligned}
(\beta^*\omega)_x(v,w) &= (7.21)\\
&= -(\omega_1 \oplus \omega_2)_x(v,w)\\
&\quad -\frac{1}{2}\Big(\big((\mu_1^*\theta^L)_{x_1}(v_1) \,|\, (\mu_2^*\theta^R)_{x_2}(w_2)\big) - \big((\mu_1^*\theta^L)_{x_1}(w_1) \,|\, (\mu_2^*\theta^R)_{x_2}(v_2)\big)\Big)\\
&= -(\omega_1 \oplus \omega_2)_x(v,w) - (\mu_1^*\theta^L \wedge \mu_2^*\theta^R)_x(v,w)\\
&= -\omega_x(v,w)
\end{aligned}$$

which completes the proof of lemma 7.3.3. □

Let us now conclude by showing that lemma 7.3.3 indeed guarantees that $(\beta^{(l)})^*\,\omega^{(l)} = -\omega^{(l)}$.

Proposition 7.3.4. *The involution $\beta = \beta^{(l)}$ (see definition 6.5.5) reverses the 2-form $\omega = \omega^{(l)}$ defining the quasi-Hamiltonian structure on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, that is : $\beta^*\omega = -\omega$.*

Proof. We proceed by induction. For $l = 1$, this is just lemma 7.3.1. Consider now $l \geq 2$ and assume that $\beta^{(l-1)}$ reverses the 2-form $\omega^{(l-1)}$ on the product $\mathcal{C}_2 \times \cdots \times \mathcal{C}_l$ of $(l-1)$ conjugacy classes. Then we know from lemma 7.3.2 that :

$$\beta^{(l)} = \Big(\big(\mu^{(l-1)} \circ \beta^{(l-1)}\big).\beta^{(1)}\ ,\ \beta^{(l-1)}\Big)$$

where $\mu^{(l-1)}(u_2, \ldots, u_l) = u_2 \ldots u_l$. Since, by the induction hypothesis, $(\beta^{(l-1)})^* \omega^{(l-1)} = -\omega^{(l-1)}$ on $\mathcal{C}_2 \times \cdots \times \mathcal{C}_l$, and since, by lemma 7.3.1, $(\beta^{(1)})^* \omega^{(1)} = -\omega^{(1)}$ on $\mathcal{C}_1$, lemma 7.3.3 applies (the other conditions of the lemma were verified at the end of section 7.2, and the fact that

$$\omega^{(l)} = (\omega^{(1)} \oplus \omega^{(l-1)}) + \frac{1}{2}\big((\mu^{(1)})^*\theta^L \wedge (\mu^{(l-1)})^*\theta^R\big)$$

follows from proposition 4.4.1, the map $\mu^{(1)}$ being the inclusion map $\mu^{(1)} : \mathcal{C}_1 \hookrightarrow U$), which shows that $(\beta^{(l)})^* \omega^{(l)} = -\omega^{(l)}$. □

Therefore, we have shown that the involution

$$\begin{array}{rccl} \beta : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ldots, u_l) & \longmapsto & \big(\tau^-(u_l)\ldots\tau^-(u_2)\tau^-(u_1)\tau(u_2)\ldots\tau(u_l), \ \ldots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

that we obtained in chapter 6 (see definition 6.5.5) is indeed an example of a map satisfying the conditions of proposition 7.2.2. Consequently, it induces an anti-symplectic involution $\hat{\beta}$ on the quasi-Hamiltonian quotient $M//U = \mu^{-1}(\{1\})/U$. We now wish to understand better the relation between $Fix(\hat{\beta})$ and the projection of $Fix(\beta) \cap \mu^{-1}(\{1\})$ under the map $p : \mu^{-1}(\{1\}) \to \mu^{-1}(\{1\})/U$.

7.4 Projection of the fixed-point set of a form-reversing involution

The purpose of this section is to study the image, under the projection map $p : \mu^{-1}(\{1\}) \to \mu^{-1}(\{1\})/U$, of the fixed-point set $Fix(\beta) \cap \mu^{-1}(\{1\})$ of the involution $\beta|_{\mu^{-1}(\{1\})}$, where β is a form-reversing involution on the quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$. We assume that U is endowed with an involutive automorphism τ and that β is compatible with the action of (U, τ) and the momentum map μ of this action, in the sense of definition 7.2.1. In this case, we have seen that β induces an anti-symplectic involution $\hat{\beta} : M//U \to M//U$ on the quasi-Hamiltonian quotient $M//U = \mu^{-1}(\{1\})$ (see proposition 7.2.2). By definition of $\hat{\beta}$, we see that if $x \in Fix(\beta) \cap \mu^{-1}(\{1\})$, then $p(x) \in Fix(\hat{\beta})$. Here, we shall give sufficient conditions for the projection map

$$p_\beta := p|_{Fix(\beta)\cap\mu^{-1}(\{1\})} : Fix(\beta) \cap \mu^{-1}(\{1\}) \longrightarrow Fix(\hat{\beta}) \subset M//U$$

to be surjective (here we implicitly assume that $Fix(\beta) \cap \mu^{-1}(\{1\})$ is non-empty but, as we shall see in chapter 8, this is always the case when β satisfies the assumptions proposition 7.2.1 and has a non-empty fixed-point set, see proposition 6.5.6). To prove the surjectivity of the map p_β, we adapt the ideas of [Fot] to the quasi-Hamiltonian setting (see also [GH04, Xu03]).

We begin with the case where the action of U on M is free, in which case we know from section 4.5 that 1 is a regular value of μ and that $M//U = \mu^{-1}(\{1\})$ is a symplectic manifold (see proposition 4.5.2 and remark 4.5.10). Recall that we denote by τ^- the involution $\tau^- : u \mapsto \tau(u^{-1})$ on U.

Lemma 7.4.1. *Assume that $Fix(\tau^-) \subset U$ is connected. Then, if U acts freely on M, the map*

$$p_\beta : Fix(\beta) \cap \mu^{-1}(\{1\}) \longrightarrow Fix(\hat{\beta})$$

is surjective.

Proof. Take $p(x) \in Fix(\hat{\beta})$ (where $x \in \mu^{-1}(\{1\})$). This means that $\beta(x) = u.x$ for some $u \in U$. Hence, by applying β we get :

$$x = \beta(u.x) = \tau(u).\beta(x) = \big(\tau(u)u\big).x$$

Since U acts freely, this yields $\tau(u)u = 1$ that is, $\tau^-(u) = u$. As $Fix(\tau^-)$ is assumed to be connected, proposition 3.1.2 shows that $u \in Fix(\tau^-)$ can be written $u = \tau^-(v)v$ for some $v \in U$. Therefore :

$$\beta(x) = u.x = \tau^-(v)v.x$$

Hence $\tau(v).\beta(x) = v.x$, that is :

$$\beta(v.x) = v.x$$

so that $v.x \in Fix(\beta) \cap \mu^{-1}(\{1\})$ and $p_\beta(v.x) = p(v.x) = p(x)$, which shows that p_β is surjective. □

Remark 7.4.2. We wish to make a few comments on the assumption that $Fix(\tau^-)$ is connected. On the one hand, it is not always the case, even if U is assumed to be simply connected, that $Fix(\tau^-)$ is connected, as is pointed out in [Loo69b], p.77. On the other hand, we know from proposition 5.1.3 that the set of symmetric elements of $(U(n), \tau(u) = \overline{u})$ that is, the set $W(n) = \{u \in U(n) \mid u^t = u\}$, is connected because each of its elements is of the form $\exp(iB)$, where B is a real symmetric matrix. The same is true for $SU(n)$. It is because of these two examples, which were ou main motivation and source of inspiration for this work, that we made the simplifying assumption that $Fix(\tau^-)$ is connected. One may observe that the result that we prove here (namely, proposition 7.4.5), which uses the above assumption on τ^-, is key to the proof of corollary 6.6.5.

If now the action of U on M is not free, let us recall from proposition 4.5.8 that we have :

$$M//U = \bigsqcup_{K \subset U} M_K//L_K$$

where the compact group $L_K = \mathcal{N}(K)/K$ (with K closed in U) acts freely on the quasi-Hamiltonian space $M_K = \{x \in M \mid U_x = K\}$. The only closed subgroups $K \subset U$ that we shall be interested in now are those for which $M_K \cap Fix(\hat{\beta}) \neq \emptyset$. For such a subgroup, we observe the following two facts :

Lemma 7.4.3. *If $K \subset U$ is a closed subgroup such that $M_K \cap Fix(\beta) \neq \emptyset$, then one has $\tau(K) \subset K$.*

Proof. Take $x \in M_K \cap Fix(\beta)$. Then if $k \in K$, one has $k.x = x$, so that $\beta(k.x) = \beta(x) = x$. Hence $\tau(k).\beta(x) = x$ that is, $\tau(k).x = x$, hence $\tau(k) \in U_x = K$. □

Lemma 7.4.4. *If $K \subset U$ is a closed subgroup such that $M_K \cap Fix(\beta) \neq \emptyset$, then one has $\beta(M_K) \subset M_K$.*

Proof. Take $y \in M_K$ and let us show that $U_{\beta(y)} = K$. A given $u \in U$ satisfies $u.\beta(y) = \beta(y)$ if and only if $\beta(\tau(u).y) = \beta(y)$ that is, by applying β, $\tau(u).y = y$. This means that $\tau(u) \in U_y = K$, hence $u \in \tau(K) = K$ by lemma 7.4.3. □

It is then immediate, from the definition of $(M_K, \omega_K = \omega|_{M_K}, \widehat{\mu_K} : M_K \to L_K)$ (see subsection 4.5.2) that the involution $\beta_K := \beta|_{M_K}$ is compatible with the action of L_K and the momentum map $\widehat{\mu_K}$ of this action, and that $\beta_K^* \omega_K = -\omega_K$. Since L_K acts freely on M_K, lemma 7.4.1 applies and one obtains :

$$Fix(\hat{\beta}) = \bigsqcup_{K \subset U} Fix(\hat{\beta_K}) = \bigsqcup_{K \subset U} p_{\beta_K}\big(Fix(\beta_K) \cap \widehat{\mu_K}^{-1}(\{1\})\big)$$

Summarizing, we have proved :

Proposition 7.4.5. *If $(M, \omega, \mu : M \to U)$ is a quasi-Hamiltonian (U, τ)-space and if $\beta : M \to M$ is involution on M satisfying $\beta^*\omega = -\omega$, $\beta(u.x) = \tau(u).\beta(x)$, and $\mu \circ \beta = \tau^- \circ \mu$, then one has : if $Fix(\tau^-)$ is connected, the map*

$$p_\beta : Fix(\beta) \cap \mu^{-1}(\{1\}) \longrightarrow Fix(\hat{\beta}) \subset \mu^{-1}(\{1\})/U$$

is surjective.

If $Fix(\tau^-)$ is not assumed to be connected, it is possible, following [Fot], to obtain a description of $Fix(\hat{\beta})$ as a disjoint union of quasi-Hamiltonian quotients indexed by the connected components of $Fix(\tau^-)$. When this set is connected, one then obtains proposition 7.4.5 above.

We shall now move on to the next chapter, where we will prove that the anti-symplectic involution $\hat{\beta}$ induced by β on the reduced space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l//U$ by means of proposition 7.2.2 always has fixed points.

Observe that, by corollary 6.6.5, this amounts to saying that there exists decomposable representations of $\pi = \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$. But we know from chapter 6 that this is equivalent to saying that there exists σ_0-decomposable representations, which are, by theorem 6.6.2, the elements u of $\mu^{-1}(\{1\})$ satisfying $\beta(u) = u$, that is, the elements u of $Fix(\beta) \cap \mu^{-1}(\{1\})$. Observe that we already know that $Fix(\beta) \neq \emptyset$ (see proposition 6.5.6). And we have now proved that :

$$Fix(\hat{\beta}) \neq \emptyset \quad \text{if and only if} \quad Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$$

This can also be written, denoting by M^β the fixed-point set $M^\beta := Fix(\beta)$ of β :

$$Fix(\hat{\beta}) \neq \emptyset \quad \text{if and only if} \quad 1 \in \mu(M^\beta)$$

This is why, in chapter 8, we will study the image under μ of the fixed-point set M^β of an involution β defined on the quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ and satisfying the conditions of proposition 7.2.2.

Chapter 8

Existence of decomposable representations : a real convexity theorem for group-valued momentum maps

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In this chapter, we study convexity properties of group-valued momentum maps. From a general point of view, this is motivated by the convexity properties of momentum maps in the usual Hamiltonian setting, which we will recall in section 8.1, and of which we will prove analogues in the remainder of this chapter. As for us though, the main motivation for this study is establishing the existence of decomposable representations, as explained at the end of chapter 7 and upon which we will come back in subsection 8.3.3.

First, we will be interested in understanding the Alekseev-Malkin-Meinrenken convexity theorem [1] (see [AMM98]), of which we will give a detailed proof, which will help us in the following. Our approach is based on adapting the ideas presented in [CDM88] and in [HNP94] to the quasi-Hamiltonian setting.

Second, we will give a *real version* of this convexity theorem, that is, a convexity result for the image $\mu(M^\beta)$ under the momentum map of the fixed-point set of a form-reversing involution β defined on a

[1] Actually, this theorem is due to Meinrenken and Woodward (see [MW98]), as pointed out to me by Anton Alekseev, whom I would like to thank. I apologize for this lack of care in the writing.

quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$. This will constitute a quasi-Hamiltonian analogue of the O'Shea-Sjamaar convexity theorem (see [OS00]). To prove this result, we shall again follow the approach of [HNP94], meaning that we shall not try to adapt the proof of [OS00] to the quasi-Hamiltonian setting. As a matter of fact, as the techniques in [HNP94] only cover the case of Hamiltonian *torus* actions when it comes to describing *real* convexity properties of momentum maps, the proof we shall give also suggests a different proof of the O'Shea-Sjamaar theorem.

In this whole chapter, the Lie group U that we shall consider will always be assumed to be compact connected and simply connected, unless stated otherwise explicitly. We shall explain this last topological assumption later on (see remark 8.2.2).

8.1 Convexity results for Lie-algebra-valued momentum maps

In this section, we recall the convexity properties of momentum maps in usual Hamiltonian geometry. The first two results we shall recall deal with Hamiltonian actions of *tori*. The original statements of these theorems are due to Atiyah and Guillemin-Sternberg for the first one (see [Ati82, GS82, GS84a]) and to Duistermaat for the second one (see [Dui83]). The Atiyah-Guillemin-Sternberg (AGS) theorem says that whenever a *compact connected* symplectic manifold (M, ω) is endowed with a Hamiltonian action of a *torus* T with momentum map $\mu : M \to \mathfrak{t}^* = (Lie(T))^*$, then $\mu(M)$ is a convex polytope, whose vertices are the images under μ of the fixed points of the action. This polytope is sometimes called the *momentum polytope*. The Duistermaat theorem then provides what is usually called a *real version* of this convexity result : if β is an *antisymplectic* involution on M which is compatible with the action of T on M and the momentum map μ of this action, then $\mu(M^\beta) = \mu(M)$, that is, the image under μ of the fixed-point set M^β of β (as a matter of fact, of any of its connected components) is a convex polytope, which in this case is equal to the full momentum polytope. As an application of these symplectic geometry results, one recovers known convexity results from linear algebra and Lie theory, namely the Schur-Horn theorem, that says that the diagonal of a Hermitian matrix H is a convex combination of permutations of the eigenvalues of H (see for instance [Knu00]), as well as the related Kostant convexity results for semi-simple Lie groups (see [Kos74, LR91, FR96]). The convexity results for momentum maps that we are about to quote are in fact improved versions of the AGS and Duistermaat theorems, in the sense that they say that convexity of $\mu(M)$ and $\mu(M^\beta)$ holds *even if M is not assumed to be compact*, provided the momentum map μ is a proper map (that is, the inverse image $\mu^{-1}(K)$ of any compact set K is itself compact, this implies that μ is closed). These improved results have been obtained by Hilgert, Neeb and Plank in [HNP94] following the approach of Condevaux, Dazord and Molino in [CDM88], and also by Sjamaar in [Sja98], following an algebro-geometric approach. We refer to [HNP94] for the proofs of these results as well as for prerequisites on proper maps and convex sets.

Theorem 8.1.1 (Momentum convexity for Hamiltonian torus actions). [HNP94] *Let (M, ω) be a connected symplectic manifold endowed with a Hamiltonian action of a torus T with proper momentum map $\mu : M \to \mathfrak{t}^* = (Lie(T))^*$. Then :*

(i) $\mu(M)$ is a closed locally polyhedral convex set.

(ii) $\mu : M \to \mu(M)$ is an open map.

(iii) the fibre $\mu^{-1}(\{v\})$ of μ above any $v \in \mathfrak{t}^*$ is a connected set.

(iv) if $\mu(x) \in \mu(M)$ is an extremal point of the convex set $\mu(M)$ then x is a fixed point of the action : for all $t \in T$, $t.x = x$.

In particular, if M is compact, then $\mu(M)$ is a convex polytope and it is the convex hull of the images under μ of the fixed points of the action.

Theorem 8.1.2 (A real convexity result for Hamiltonian torus actions). [HNP94] *Suppose additionally that $\beta : M \to M$ is an involution on M satisfying :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(t.x) = t^{-1}.x$ *for all* $x \in M$ *and all* $t \in T$

(iii) $\mu \circ \beta = \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Then, for every connected component $Q \subset M^\beta$ *of the fixed-point set of* β, $\mu(Q)$ *is a convex polytope and one has :* $\mu(Q) = \mu(M)$.

Remark 8.1.3. Observe that, as T is abelian, the involution $\tau : t \mapsto t^{-1}$ is a group automorphism of T, whose tangent map at $1 \in T$ is $-Id : \mathfrak{t} \to \mathfrak{t}$, so that the compatibility conditions (ii) and (iii) above rewrite : $\beta(t.x) = \tau(t).\beta(x)$ and $\mu \circ \beta = -(T\tau)^* \circ \mu$, as was recalled in the beginning of chapter 7.

The next two convexity results that we shall state deal with Hamiltonian actions of non-abelian compact groups and may be obtained by reduction to the abelian case. Indeed, if U is a compact connected Lie group acting in a Hamiltonian fashion on a compact connected symplectic manifold (M, ω), convexity results for the momentum map $\mu : M \to \mathfrak{u}^* = (Lie(U))^*$ of the U-action are proved by reduction to the action, which turns out to be Hamiltonian, of a maximal torus $T \subset U$ on a T-stable symplectic connected submanifold $N \subset M$ satisfying $\overline{U.N} = M$ and $\overline{\mu(N)} = \mu(M) \cap \overline{\mathfrak{t}_+^*}$, where $\overline{\mathfrak{t}_+^*} \subset \mathfrak{t}^*$ is a closed Weyl chamber (see definition 2.2.2). Such a manifold N is called a *symplectic cross-section*. We refer to [HNP94] for details on the proof of the following result, which was originally conjectured by Guillemin and Sternberg in [GS82, GS84a] and proved by Kirwan in [Kir84] for compact connected symplectic manifolds, then extended by Hilgert-Neeb-Plank and Sjamaar to the case of arbitrary connected symplectic manifolds with proper momentum map. In the following, we assume that there is a given Ad-invariant product $(.\,|\,.)$ on $\mathfrak{u} = Lie(U)$, so that we can identify $\mathfrak{u}^*$ and $\mathfrak{u}$ and, for any subalgebra $\mathfrak{t} \subset \mathfrak{u}$, think of $\mathfrak{t}^*$ as a subset of $\mathfrak{u}^*$.

Theorem 8.1.4 (Momentum convexity for Hamiltonian actions of compact groups). [HNP94] *Let* $(U, (.\,|\,.))$ *be a compact Lie group acting on a connected symplectic manifold* (M, ω) *in a Hamiltonian fashion, with proper momentum map* $\mu : M \to \mathfrak{u}^* = (Lie(U))^*$. *Then, for any choice of a Cartan subalgebra* $\mathfrak{t} \subset \mathfrak{u}$ *and any choice of a closed Weyl chamber* $\overline{\mathfrak{t}_+^*} \subset \mathfrak{t}^* \subset \mathfrak{u}^*$, *the set* $\mu(M) \cap \overline{\mathfrak{t}_+^*}$ *is a convex subset of* $\mathfrak{u}^*$.

In section 8.2, we will give a proof of a quasi-Hamiltonian analogue, due to Alekseev, Malkin and Meinrenken (see [AMM98]), of the above result. Now, just as theorem 8.1.2 is a real version of theorem 8.1.1, there exists a real version of theorem 8.1.4, which is due to O'Shea and Sjamaar (see [OS00]). The setting is as follows : U is a compact Lie group acting on a connected symplectic manifold (M, ω) with proper momentum map $\mu : M \to \mathfrak{u}^*$, and $\beta : M \to M$ is an involution on M satisfying $\beta^*\omega = -\omega$, $\beta(u.x) = \tau(u).\beta(x)$ and $\mu \circ \beta = -\tau \circ \mu$, where $\tau : U \to U$ is an involutive automorphism of U and where we still denote by τ the involution $(T\tau)^* : \mathfrak{u}^* \to \mathfrak{u}^*$ that it induces on the dual of the Lie algebra of U. Because of the compatibility relations between β and both the action of U and the momentum map μ, one has $\mu(M^\beta) \subset (\mathfrak{u}^*)^{\tau^-}$ where $\tau^- : \xi \in \mathfrak{u}^* \mapsto -\tau(\xi)$, that is, $\mu(M^\beta)$ consists of points which are fixed by τ^-. The result one hopes for is that $\mu(M^\beta) \cap \overline{\mathfrak{t}_+^*}$ is a convex polytope, for some closed Weyl chamber $\overline{\mathfrak{t}_+^*} \subset \mathfrak{t}^* \subset \mathfrak{u}^*$, and to describe it as a subpolytope of $\mu(M) \cap \overline{\mathfrak{t}_+^*}$. As explained in [OS00], this is only possible for an appropriate choice of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{u}$. Namely, one has to choose $\mathfrak{t}$ in a way that $\mathfrak{t}^* \cap (\mathfrak{u}^*)^{\tau^-}$ is of maximal possible dimension. One way to obtain such a Cartan subalgebra is to start with an abelian subalgebra $\mathfrak{a} \subset \mathfrak{u}$ such that $\mathfrak{a}^* \subset \mathfrak{u}^*$ consists of τ^--fixed vectors and of maximal dimension with respect to this property, and then to consider a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{u}$ containing $\mathfrak{a}$. We refer to [OS00] for a description of $\mathfrak{a}$ in terms of roots, in particular for the notion of Weyl chamber $\mathfrak{a}_+^*$ for the restricted root system corresponding to $\mathfrak{a}$ (the important thing to understand being that $\overline{\mathfrak{a}_+^*}$ is a fundamental domain for the action of the neutral component K^0 of $K := U^\tau$ on the vector space $(\mathfrak{u}^*)^{\tau^-}$). One then obtains the following result, for the proof of which we refer to [OS00] :

Theorem 8.1.5 (A real convexity result for Hamiltonian actions of compact groups). [OS00] *Let $(U, (.\,|\,.), \tau)$ be a compact Lie group endowed with an involutive automorphism τ acting in a Hamiltonian fashion on a connected symplectic manifold (M, ω) with proper momentum map $\mu : M \to \mathfrak{u}^* = (Lie(U))^*$. Denote by τ^- the involution $\tau^- := (-T\tau)^* : \mathfrak{u}^* \to \mathfrak{u}^*$ and let $\beta : M \to M$ be an involution on M satisfying :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $x \in M$ *and all* $u \in U$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Let $\mathfrak{t} \subset \mathfrak{u}$ be a Cartan subalgebra of $\mathfrak{u} \simeq \mathfrak{u}^*$ such that $\mathfrak{t}^* \cap (\mathfrak{u}^*)^{\tau^-}$ is of maximal possible dimension, and let $\overline{\mathfrak{t}^*_+} \subset \mathfrak{t}^*$ be any closed Weyl chamber. Then, the set $\mu(M^\beta) \cap \overline{\mathfrak{t}^*_+}$ is convex and one has :

$$\mu(M^\beta) \cap \overline{\mathfrak{t}^*_+} = \left(\mu(M) \cap \overline{\mathfrak{t}^*_+}\right) \cap (\mathfrak{u}^*)^{\tau^-}$$

that is, $\mu(M^\beta) \cap \overline{\mathfrak{t}^*_+}$ is the subpolytope of $\mu(M) \cap \overline{\mathfrak{t}^*_+}$ obtained by intersecting the latter with the vector space $(\mathfrak{u}^*)^{\tau^-} \subset \mathfrak{u}^*$.

Observe that if $\mathfrak{a} \subset \mathfrak{u}$ designates an abelian subalgebra of $\mathfrak{u} \simeq \mathfrak{u}^*$ consisting of τ^--fixed points and of maximal dimension with respect to this property, and if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}$ containing $\mathfrak{a}$, then $\overline{\mathfrak{t}^*_+} \cap (\mathfrak{u}^*)^{\tau^-} = \overline{\mathfrak{a}^*_+}$, where $\mathfrak{a}^*_+$ is the Weyl chamber defined by the restricted root system corresponding to $\mathfrak{a}$ (see [OS00] for details). Therefore, since $\mu(M^\beta) \subset (\mathfrak{u}^*)^{\tau^-}$ because of the compatibility of β and μ, one has :

$$\begin{aligned} \mu(M^\beta) \cap \overline{\mathfrak{t}^*_+} &= \mu(M^\beta) \cap \overline{\mathfrak{t}^*_+} \cap (\mathfrak{u})^{\tau^-} \\ &= \mu(M^\beta) \cap \overline{\mathfrak{a}^*_+} \end{aligned}$$

and theorem 8.1.5 above says that :

$$\mu(M^\beta) \cap \overline{\mathfrak{a}^*_+} = \mu(M) \cap \overline{\mathfrak{a}^*_+}$$

We shall come back to this in subsection 8.3.2.

In the remainder of this chapter, we will state and prove a quasi-Hamiltonian analogue of theorem 8.1.5. The one truly remarkable feature of the proof we shall give for convexity properties of group-valued momentum maps is that, just as in the usual Hamiltonian case, we will reduce the situation at hand to that of a Hamiltonian torus action on a *symplectic* manifold N sitting inside the given quasi-Hamiltonian space M. More precisely, we will prove the existence of a connected symplectic cross-section $N \subset M$ for every connected quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$, where U is a compact connected simply connected Lie group (see proposition 8.2.3).

Other results and possible approaches to convexity properties of momentum maps may be found in [Sja98, LMTW98, MW99, Wei01, Sleb, Ben02, Del88, Zun, MT03, AL92, HN98, Nee94, Nee95, Bri87, BS00, Fot05].

8.2 A convexity theorem for momentum maps with value in a compact connected simply connected Lie group

In this section, we give a proof of a convexity theorem for momentum maps defined on a quasi-Hamiltonian space which is due to Alekseev, Malkin and Meinrenken (see also [AKSM02]). More precisely, we consider a compact connected simply connected Lie group U and a quasi-Hamiltonian U-space $(M, \omega, \mu : M \to U)$, and we study convexity properties of $\mu(M)$.

8.2.1 Making convexity make sense in a Lie group

The first issue is for convexity to make sense in a Lie group that is not, as it is compact, homeomorphic to a vector space. In the usual Hamiltonian case, a convex set $\mu(M) \cap \overline{\mathfrak{t}_+^*}$ was obtained (see theorem 8.1.4) by intersecting the image of μ with the closure of a Weyl chamber $\mathfrak{t}_+^* \in \mathfrak{u}^*$ in the dual $\mathfrak{u}^*$ of the Lie algebra of U, that is, with a fundamental domain for the co-adjoint action of U on $\mathfrak{u}^*$ (see proposition 2.2.3, recall that a fundamental domain is by definition a subset $\mathfrak{D} \subset X$ of some U-space X intersecting each U-orbit in X in exactly one point). In the quasi-Hamiltonian case, the analogous approach consists in intersecting $\mu(M)$ with a fundamental domain for the conjugacy action of U on itself. Convexity then makes sense because, *when the compact connected Lie group U is in addition simply connected*, this fundamental domain may be identified with a convex subset of the vector space $\mathfrak{u} = Lie(U)$. Let us recall how.

Instead of intersecting $\mu(M)$ with a fundamental domain for the conjugacy action of U, we could as well consider the projection of $\mu(M) \subset U$ to the orbit space $U/\mathrm{Int}(U)$ for this action. This is indeed equivalent *because $\mu(M)$ is a union of U-orbits* (as μ is an equivariant map) : if $\mathfrak{D} \subset X$ is a fundamental domain for the action of U on some space X and $Y \subset X$ is a *U-stable* subset of X, then the projection $p : X \to X/U$ from X to the orbit space X/U induces a bijection from $Y \cap \mathfrak{D}$ to $p(Y) = Y/U$. In other words, $Y \cap \mathfrak{D}$ is a fundamental domain for the action of U on Y. Recall now that we have assumed the compact connected Lie group U to be simply connected. In this case, the space $U/\mathrm{Int}(U)$ of conjugacy classes of U is homeomorphic to the closure $\overline{\mathcal{W}}$ of a Weyl alcove $\mathcal{W} \subset \mathfrak{t} = Lie(T)$ for any fixed maximal torus $T \subset U$ (see proposition 2.2.5). More precisely, $\exp(\overline{\mathcal{W}})$ is a fundamental domain for the conjugacy action of U on itself and the homeomorphism between $\overline{\mathcal{W}}$ and $U/\mathrm{Int}(U)$ is induced by the exponential map in the sense that we have :

$$\overline{\mathcal{W}} \xrightarrow[\exp]{\simeq} \exp(\overline{\mathcal{W}}) \xrightarrow[p]{\simeq} U/\mathrm{Int}(U)$$

This means that, for a simply connected U, the space $U/\mathrm{Int}(U)$ may be identified, topologically, to a convex polyhedron $\overline{\mathcal{W}} \subset \mathfrak{t}$ of a (finite-dimensional) vector space, so that it makes sense to speak of a convex subset of $U/\mathrm{Int}(U)$:

Definition 8.2.1 (Convex subsets of $U/\mathrm{Int}(U)$). A subset $C \subset U/\mathrm{Int}(U)$ is called *convex* if it is mapped, under the identification $U/\mathrm{Int}(U) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$, to a convex subset of $\mathfrak{t} \subset \mathfrak{u}$.

Observe that if we use the same approach for a usual Lie-algebra valued momentum map $\mu : M \to \mathfrak{u}^*$, we are led to considering subsets of the orbit space $u^*/Ad(U)$ for the co-adjoint action, which is homeomorphic to the closure $\overline{\mathfrak{t}_+^*} \subset \mathfrak{t}^*$ of a Weyl chamber $\mathfrak{t}_+^*$, which is also a convex subset of a finite-dimensional vector space.

Remark 8.2.2. When the compact connected Lie group U is not simply connected, the identification $U/\mathrm{Int}(U) \simeq \overline{\mathcal{W}}$ does not hold and has to be replaced by an identification $U/\mathrm{Int}(U) \simeq \overline{\mathcal{W}}/\pi_1(U)$ (see for instance [Loo69b] for an explanation of the action of $\pi_1(U)$ on $\overline{\mathcal{W}}$). In particular, $U/\mathrm{Int}(U)$ is not necessarily simply connected anymore, and therefore cannot be homeomorphic to a convex subset of a vector space. We refer to [Zun] for additional comments on this situation.

8.2.2 Constructing a symplectic cross-section in a quasi-Hamiltonian space

The purpose of this subsection is to prove the following result :

Proposition 8.2.3 (Existence of a connected symplectic cross-section). *Let U be a compact connected simply connected Lie group and let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian U-space. We assume the momentum map μ to be proper.*
Let $T \subset U$ be a maximal torus in U, let $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$ be the closure of a Weyl alcove, and let $p : U \to U/\mathrm{Int}(U)$ be the projection from U to the set of its conjugacy classes. Recall that the exponential map $\exp : \mathfrak{t} \to T$ induces a homeomorphism $\overline{\mathcal{W}} \xrightarrow{\simeq} U/\mathrm{Int}(U)$.
Then, there exists a submanifold $N \subset M$ such that :

(i) N is connected.

(ii) N is T-stable.

(iii) $\omega|_N$ is a symplectic form.

(iv) the action of T on N is Hamiltonian with momentum map the map

$$\widetilde{\mu} := p \circ \mu|_N : N \longrightarrow U/\mathrm{Int}(U) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$$

(v) The set $U.N := \{u.x \,:\, x \in N, u \in U\}$ is dense in M, and the set $\widetilde{\mu}(N)$ is dense in $\tilde{\mu}(M)$.

The manifold $N \subset M$ whose existence is guaranteed by proposition 8.2.3 is called a *symplectic cross-section* because it is a symplectic manifold satisfying $\overline{U.N} = M$ (see [GS84c]).

Remark 8.2.4 (On the assumption of properness of the momentum map). The assumption that μ is proper is satisfied in the examples of quasi-Hamiltonian spaces that we are interested in. As a matter of fact, when U is a compact Lie group, all the examples of quasi-Hamiltonian spaces that are of interest to us (a conjugacy class of U, the double of U, and products of those, see chapter 4) are all compact, so that $\mu : M \to U$ is automatically proper. The point of not assuming M to be compact in proposition 8.2.3 is to hopefully be able to consider quasi-Hamiltonian spaces associated to non-compact Lie groups G in the future, for this should be done by reducing the action of G to that of a maximal compact subgroup $U \subset G$, as does Weinstein in [Wei01] for usual Hamiltonian spaces, in which case we should be in the exact situation of proposition 8.2.3.

To prove proposition 8.2.3, we will in fact construct a submanifold $N \subset M$ such that $\mu(N) \subset \exp(\overline{\mathcal{W}}) \subset U$, so that the map $\widetilde{\mu} : N \to \overline{\mathcal{W}}$ is no other than $\widetilde{\mu} = \exp^{-1} \circ \mu|_N : N \to \mathfrak{t}$ (recall that $\exp|_{\overline{\mathcal{W}}}$ is a homeomorphism from $\overline{\mathcal{W}}$ to $\exp(\overline{\mathcal{W}})$) and we will show that it is a smooth map from N to $\mathfrak{t}$.

We begin by describing points of $\mu(M)$ whose conjugacy class in U is of maximal possible dimension *among points of* $\mu(M)$. This is a natural thing to do, as the set of such points is dense in $\mu(M)$ (see proposition 2.1.6). Define q to be the maximal dimension of a conjugacy class of a point of $\mu(M)$ and Σ_q to be the set of points of U whose conjugacy class is of dimension q :

$$q := \max\,\{\dim U.\mu(x) \;:\; x \in M\}$$

$$\Sigma_q := \{u \in U \mid \dim U.u = q\}$$

Recall that Σ_q is a submanifold of U (see proposition 2.2.10). Define then M_q to be the set of points of M whose image under μ lies in Σ_q :

$$M_q := \{x \in M \mid \dim U.\mu(x) = q\} = \mu^{-1}(\Sigma_q)$$

so that $\mu(M_q)$ is exactly the set of points of $\mu(M)$ whose conjugacy class in U is of maximal possible dimension.

The first thing to observe is that M_q is an open, connected, and dense subset of M. Let us first show that it is open. For any $x \in M_q$, there exists an open neighbourhood $\mathcal{V}$ of $\mu(x)$ in U such that for all $u \in \mathcal{V}$, $\dim U.u \geq \dim U.\mu(x)$ (see corollary 2.1.5). Since μ is continuous, we have $\mu(y) \in \mathcal{V}$ for all y in some open set $\mathcal{U}$ of M containing x. Since $\dim U.\mu(x)$ is maximal, we necessarily have $\dim U.\mu(y) = q$ for all $y \in \mathcal{U}$, so that $\mathcal{U} \subset M_q$. We now want to prove that M_q is dense and connected. To that end, we introduce the set M_{reg} of points of M whose orbit under U is of maximal possible dimension :

$$r := \max\,\{\dim U.x \,:\, x \in M\}$$

$$M_{reg} = \{x \in M \mid \dim U.x = r\}$$

Then it follows from proposition 2.1.6 that M_{reg} is an open, connected, and dense subset of M. As M_q is open, the intersection $M_{reg,q} := M_{reg} \cap M_q$ is a non-empty open set of M. In addition to that, since $(M, \omega, \mu : M \to U)$ is a quasi-Hamiltonian space, M_{reg} enjoys the following remarkable property :

$$M_{reg} = \{x \in M \mid \text{rk } T_x\mu \text{ is maximal}\}$$

Indeed, it follows from point (iii) of proposition 4.3.1 that :

$$\begin{aligned}
\max_{x\in M} \{\dim \text{Im } T_x\mu\} &= \max_{x\in M} \{\dim \mathfrak{u}_x^{\perp}\} \\
&= \dim \mathfrak{u} - \min_{x\in M} \{\dim \mathfrak{u}_x\} \\
&= \dim U - \min_{x\in M} \{\dim U_x\} \\
&= \max_{x\in M} \{\dim U.x\} \\
&= r
\end{aligned}$$

In particular, μ is of constant rank on M_{reg}. Now, to show that M_q is dense and connected in M, since we have $M_{reg,q} \subset M_q \subset M$, it is enough to prove that $M_{reg,q}$ is dense and connected in M. First, we observe that this is locally true in the following sense :

Lemma 8.2.5. *For all $x \in M_{reg}$, there exists an open neighbourhood $\mathcal{V}_x$ of $x \in M_{reg}$ such that $M_{reg,q} \cap \mathcal{V}_x$ is a dense and connected subset of $\mathcal{V}_x$.*

Proof. Since μ is of constant rank on M_{reg}, there exists an open connected neighbourhood $\mathcal{V}_x$ of x in M_{reg} such that $\mu(\mathcal{V}_x)$ is a (connected) submanifold of U (of dimension equal to rk $T_x\mu$) and such that $\mu|_{\mathcal{V}_x} : \mathcal{V}_x \to \mu(\mathcal{V}_x)$ is a locally trivial submersion onto a connected manifold with connected fibres (the constant rank theorem says that μ is locally equivalent to the linear projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$, see for instance [Ave83], p.86). Since μ is equivariant and continuous, we have $u.\mu(y) \in \mu(\mathcal{V}_x)$ for $y \in \mathcal{V}_x$ and u sufficiently close to 1 in U. Therefore, $\mu(\mathcal{V}_x)$ is a union of connected open pieces of conjugacy classes of U. Since in addition to that $\mu(\mathcal{V}_x)$ is connected, we have, if we set

$$q_x := \max \{\dim U.z \ : \ z \in \mu(\mathcal{V}_x)\}$$

(observe that q_x has no reason to be equal to dim $U.\mu(x)$) and

$$\Omega_x := \{\mu(y) \in \mu(\mathcal{V}_x) \mid \dim U.\mu(y) = q_x\}$$

that Ω_x is an open, connected, and dense subset of $\mu(\mathcal{V}_x)$ (see proposition 2.1.6). Now, since Ω_x is an open dense and connected subset of $\mu(\mathcal{V}_x)$ and since $\mu|_{\mathcal{V}_x} : \mathcal{V}_x \to \mu(\mathcal{V}_x)$ is a locally trivial submersion with connected fibres over the connected manifold $\mu(\mathcal{V}_x)$, we have that $(\mu|_{\mathcal{V}_x})^{-1}(\Omega_x) = \mu^{-1}(\Omega_x) \cap \mathcal{V}_x$ is an open dense and connected subset of $\mathcal{V}_x$ (recall that the submersion $\mu|_{\mathcal{V}_x}$ is equivalent to $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$). Moreover, if $x, y \in M_{reg}$, we can join them by a path $c : [0,1] \to M_{reg}$. Denote by $\hat{c}$ the compact connected set $\hat{c} := c([0,1])$. For every $z \in \hat{c}$, there exists an open neighbourhood $\mathcal{V}_z$ of z in M_{reg} such that the set

$$R_z := \left\{w \in \mathcal{V}_z \mid \dim U.\mu(z) = q_z := \max_{u\in\mu(\mathcal{V}_z)} \{\dim U.x\}\right\}$$

is open, connected, and dense in $\mathcal{V}_z$. By compactness, we can cover $\hat{c}$ by a finite number of such $\mathcal{V}_z$:

$$\hat{c} = \mathcal{V}_{z_1} \cup \ \ldots \cup \mathcal{V}_{z_p}$$

with $z_1 = x$ and $z_p = y$. If $\mathcal{V}_{z_i} \cap \mathcal{V}_{z_j} \neq \emptyset$, then by density and openness, $R_{z_i} \cap R_{z_j} \neq \emptyset$. Therefore, for $w \in R_{z_i} \cap R_{z_j}$, the conjugacy class of $\mu(w)$ has dimension $q_{z_i} = q_{z_j}$, whence we get $q_x = q_y$, so that q_x is the same for all $x \in M_{reg}$. As $M_{reg} \cap M_q \neq \emptyset$, one necessarily has $q_x = q$ for all $x \in M_{reg}$. Therefore $\mu^{-1}(\Omega_x) \cap \mathcal{V}_x = M_{reg,q} \cap \mathcal{V}_x$, which proves the lemma. □

We now go back to our global study :

Lemma 8.2.6. *The subset $M_{reg,q} := M_{reg} \cap M_q$ is an open, connected, and dense subset of M. Consequently, so is M_q.*

Proof. $M_{reg,q}$ is open as it is the intersection of two open sets. Since M_{reg} is dense in M, it suffices, in order to prove that $M_{reg,q}$ is dense in M, to show that $M_{reg} \subset \overline{M_{reg,q}}$. For any $x \in M_{reg}$, there exists, by lemma 8.2.5, an open neighbourhood $\mathcal{V}_x$ of x in M_{reg} such that $M_{reg,q} \cap \mathcal{V}_x$ is dense in $\mathcal{V}_x$, so that x is the limit of a sequence of points of $M_{reg,q}$, which proves that $M_{reg} \subset \overline{M_{reg,q}}$.
Let us now prove that $M_{reg,q}$ is connected. Take $x, y \in M_{reg,q}$. As M_{reg} is connected, there exists a path $c : [0,1] \to M_{reg}$ joining x to y in M_{reg}, and we set $\hat{c} := c([0,1])$. Then, as in the proof of lemma 8.2.5, there exists a finite open cover

$$\hat{c} = \mathcal{V}_{z_1} \cup \ \dots \ \cup \mathcal{V}_{z_p}$$

with $z_1 = x$ and $z_p = y$. Then $\mathcal{V}_{z_1} \cap \mathcal{V}_{z_i} \neq \emptyset$ for some $i \geq 2$ and by density one has $M_{reg,q} \cap (\mathcal{V}_{z_1} \cap \mathcal{V}_{z_i}) \neq \emptyset$. By connectedness of $\mathcal{V}_{z_1} \cap M_{reg,q}$, any $z_i' \in M_{reg,q} \cap (\mathcal{V}_{z_1} \cap \mathcal{V}_{z_i})$ can be joined to z_1 by a path in $M_{reg,q}$. Repeating this, we obtain a path from $z_1 = x$ to $z_p = y$ in $M_{reg,q}$.
Finally, we have proved that $M_{reg,q}$ is connected and dense in M, and we have $M_{reg,q} \subset M_q \subset M = \overline{M_{reg,q}}$, which proves that M_q is connected and dense in M. □

Instead of describing all of $\mu(M_q)$, what we are really interested in is describing $\mu(M_q) \cap \exp(\overline{\mathcal{W}}) \subset \Sigma_q \cap \exp(\overline{\mathcal{W}})$. Recall that $\overline{\mathcal{W}}$ is a convex polyhedron of $\mathfrak{t}$, which can de described entirely in terms of roots of (U, T) (see section 2.2). Moreover, we know from proposition 2.2.10 that the intersection of Σ_j with $\exp(\overline{\mathcal{W}})$ is a finite disjoint union of submanifolds of U :

$$\begin{aligned} \Sigma_q \cap \exp(\overline{\mathcal{W}}) &= \bigsqcup_{S \ | \ \dim U - \dim U_S = j} \exp(\mathcal{W}_S) \\ &= \exp(\mathcal{W}_{S^{(1)}}) \sqcup \ \dots \ \sqcup \exp(\mathcal{W}_{S^{(m)}}) \end{aligned}$$

(where U_S is the stabilizer of any element in $\exp(\mathcal{W}_S)$, see definition 2.2.9), so that we have :

$$\mu(M_q) \cap \exp(\overline{\mathcal{W}}) \subset \exp(\mathcal{W}_{S^{(1)}}) \sqcup \ \dots \ \sqcup \exp(\mathcal{W}_{S^{(m)}})$$

and we now want to study points in each $\exp(\overline{\mathcal{W}}_{S^{(i)}})$ which lie in the image of μ. To that end, we set, for all $i \in \{1, \ \dots, m\}$:

$$M_{S^{(i)}} := \mu^{-1}\big(\exp(\mathcal{W}_{S^{(i)}})\big)$$

By definition we have $M_{S^{(i)}} \subset M_q$, and since M_q is U-stable, we have $u.x \in M_q$ for all $x \in M_{S^{(i)}}$ and all $u \in U$.

Lemma 8.2.7. *If $M_{S^{(i)}} \neq \emptyset$, it is a submanifold of M, and for every open set $\mathcal{O}$ of $M_{S^{(i)}}$, the set*

$$U.\mathcal{O} := \{u.x \ : \ u \in U, x \in \mathcal{O}\}$$

is open in M_q.

Proof. Recall that $\exp(\mathcal{W}_{S^{(i)}})$ is a submanifold of Σ_q and that for all $u \in \exp(\mathcal{W}_{S^{(i)}})$, one has (see proposition 2.2.10) :

$$T_u\Sigma_q = T_u\big(U.u\big) \oplus T_u\big(\exp(\mathcal{W}_{S^{(i)}})\big)$$

Moreover, $M_{S^{(i)}} = \mu^{-1}(\exp(\mathcal{W}_{S^{(i)}}))$, where μ is seen as a map $\mu : M_q \to \Sigma_q$. Hence for all $x \in M_{S^{(i)}}$:

$$T_{\mu(x)}\Sigma_q = T_{\mu(x)}\big(U.\mu(x)\big) \oplus T_{\mu(x)}\big(\exp(\mathcal{W}_{S^{(i)}})\big)$$

But :

$$T_{\mu(x)}\big(U.\mu(x)\big) = T_x\mu.\big(T_x(U.x)\big) \subset \mathrm{Im}\ T_x\mu$$

(where the equality follows from the equivariance of μ), so that, for all $x \in M_{S^{(i)}}$:

$$\mathrm{Im}\ T_x\mu + T_{\mu(x)}\big(\exp(\mathcal{W}_{S^{(i)}})\big) = T_{\mu(x)}\Sigma_q$$

which means that the map μ is transverse to the submanifold $\exp(\mathcal{W}_{S^{(i)}})$. By the transversality theorem (see for instance [GP74], p.28), $M_{S^{(i)}} = \mu^{-1}(\exp(\mathcal{W}_{S^{(i)}}))$ is therefore a submanifold of M_q, hence of M, and one has, for all $x \in M_{S^{(i)}}$:

$$T_xM_{S^{(i)}} = \big(T_x\mu\big)^{-1}\Big(T_{\mu(x)}\big(\exp(\mathcal{W}_{S^{(i)}})\big)\Big) \tag{8.1}$$

Moreover, since $\mu(M_q) \subset \Sigma_q$, one has, for such an $x \in M_{S^{(i)}} \subset M_q$:

$$\mathrm{Im}\ T_x\mu \subset T_{\mu(x)}\Sigma_q = T_{\mu(x)}\big(U.\mu(x)\big) \oplus T_{\mu(x)}\big(\exp(\mathcal{W}_{S^{(i)}})\big)$$

Consequently, for all $v \in T_xM = T_xM_q$, one has :

$$T_x\mu.v = \xi_1 + \xi_2$$

where :

$$\xi_1 \in T_{\mu(x)}\big(U.\mu(x)\big) = T_x\mu\big(T_x(U.x)\big) \quad \text{and} \quad \xi_2 \in T_{\mu(x)}\big(\exp(\mathcal{W}_{S^{(i)}})\big)$$

Hence $\xi_2 = T_x\mu.(v - v_1)$ for some $v_1 \in T_x(U.x)$. Set $v_2 := v - v_1$. Then $T_x\mu.v_2 = \xi_2 \in T_{\mu(x)}(\exp(\mathcal{W}_{S^{(i)}}))$, therefore :

$$v_2 \in \big(T_x\mu\big)^{-1}\Big(T_{\mu(x)}\big(\exp(\mathcal{W}_{S^{(i)}})\big)\Big) = T_xM_{S^{(i)}}$$

hence :

$$v = v_1 + v_2 \in T_x(U.x) + T_xM_{S^{(i)}}$$

and therefore :

$$T_xM = T_x(U.x) + T_xM_{S^{(i)}} \tag{8.2}$$

One may observe that this sum is generally not a direct sum, since if $X \in \mathfrak{u}$ satisfies $X^\dagger_{\mu(x)} = 0$, then $X^\#_x \in \ker T_x\mu \subset T_xM_{S^{(i)}}$, and $X^\#_x$ may be non-zero in $T_x(U.x)$. The equality 8.2 shows that if $\mathcal{O}$ is an open subset of $M_{S^{(i)}}$ containing x, then $U.\mathcal{O}$ contains an open subset $\mathcal{V}_x$ of M containing x. Then for all $u \in U$, $u.\mathcal{V}_x$ is an open subset of M containing $u.x$ and contained in $U.\mathcal{O}$, which shows that $U.\mathcal{O}$ is open in M. □

In particular, $U.M_{S^{(i)}}$ is open in M. And we have :

Lemma 8.2.8. $M_q = \bigsqcup_{i=1}^m U.M_{S^{(i)}}$

Proof. Let us prove that $M_q \subset \sqcup_{i=1}^m U.M_{S^{(i)}}$, the other inclusion following from the facts that $M_{S^{(i)}} \subset M_q$ and that M_q is U-invariant. Consider an element $x \in M_q$. Then $\mu(x) \in \Sigma_q$. But there exists $u \in U$ such that $u.\mu(x) \in \exp(\overline{\mathcal{W}})$, so that $\mu(u.x) = u.\mu(x) \in \Sigma_q \cap \exp(\overline{\mathcal{W}}) = \sqcup_{i=1}^m \exp(\mathcal{W}_{S^{(i)}})$ hence $x \in \sqcup_{i=1}^m U.M_{S^{(i)}}$. □

Lemmas 8.2.7 and 8.2.8 have the following remarkable consequence :

Lemma 8.2.9. *Exactly one of the $M_{S^{(i)}}$ is non-empty.*

Proof. $M_q = \sqcup_{i=1}^m U.M_{S^{(i)}}$ by lemma 8.2.8, and $U.M_{S^{(i)}}$ is open in M_q by lemma 8.2.7. But M_q is connected by lemma 8.2.6, so that only one of the $U.M_{S^{(i)}}$, and consequently only one of the $M_{S^{(i)}}$, can be non-empty. □

Corollary 8.2.10. *It follows from lemmas 8.2.8 and 8.2.9 that $M_q = U.M_{S^{(i_0)}}$ for a unique $i_0 \in \{1, \ldots, m\}$. It then follows from lemma 8.2.6 that $U.M_{S^{(i_0)}}$ is an open, connected, and dense subset of M.*

From now on, we simply denote $S^{(i_0)}$ by S. The submanifold $M_S := \mu^{-1}(\exp(\mathcal{W}_S))$ will end up being our symplectic cross-section. We first prove the following result :

Lemma 8.2.11. *If $x \in M_S$ and $u \in U$ are such that $u.x \in M_S$, then $u \in U_S$ (where U_S is the stabilizer of any element in $\exp(\mathcal{W}_S)$, see definition 2.2.9).*

Proof. If $x \in M_S$ and $u \in U$ are such that $u.x \in M_S$, then $\mu(x)$ and $u.\mu(x) = \mu(u.x)$ are both elements of $\exp(\mathcal{W}_S) \subset \exp(\overline{\mathcal{W}})$, hence $\mu(x) = u.\mu(x)$ that is, u stabilizes some element of $\exp(\mathcal{W}_S)$. Consequently, $u \in U_S$. □

Together with the fact that U_S is connected, being the centralizer of an element of a compact connected simply connected Lie group (see proposition 2.2.7), lemma 8.2.11 has the following important consequence :

Lemma 8.2.12. *The manifold M_S is connected.*

Proof. Assume that $M_S = M_S^{(1)} \sqcup M_S^{(2)}$ is the disjoint union of two open subsets of M_S. Then, by lemma 8.2.7, $U.M_S^{(i)}$ is open in M. If $(U.M_S^{(1)}) \cap (U.M_S^{(2)}) \neq \emptyset$, there exist $x_1 \in M_S^{(1)}$, $x_2 \in M_S^{(2)}$ and $u_1, u_2 \in U$ such that $u_1.x_1 = u_2.x_2$, hence $u_2^{-1}u_1.x_1 = x_2$. But then, by lemma 8.2.11, $u_2^{-1}u_1 \in U_S$, which is connected by proposition 2.2.7. Therefore, there is a path (u_t) joining 1 to $u_2^{-1}u_1$ in U_S, hence $u_t.x_1$ is a path joining x_1 to x_2 in M_S, which contradicts the fact that x_1 and x_2 lie in disjoint open subsets of M_S. Therefore, $(U.M_S^{(1)}) \cap (U.M_S^{(2)}) = \emptyset$ and :

$$U.M_S = (U.M_S^{(1)}) \sqcup (U.M_S^{(2)})$$

with $U.M_S^{(i)}$ open in M. But $U.M_S$ is open in M and connected by corollary 8.2.10, so that $U.M_S^{(i)} = \emptyset$ for $i = 1$ or $i = 2$. Therefore, one of the $M_S^{(i)}$ is empty, which proves the lemma. □

We now want to study precisely the relation between $\mu(M_S)$ and $\mu(M) \cap \exp(\overline{\mathcal{W}})$, which was our initial motivation. Recall that $\mu(M_S) \subset \exp(\mathcal{W}_S) \subset \exp(\overline{\mathcal{W}})$, the latter being closed in U.

Lemma 8.2.13. *If μ is a closed map (in particular, if μ is proper), one has :*

$$\mu(M) \cap \exp(\overline{\mathcal{W}}) = \overline{\mu(M_S)}$$

Proof. Take $\mu(x) \in \mu(M) \cap \exp(\overline{\mathcal{W}})$. Since $M_q = U.M_S$ is dense in M by corollary 8.2.10, there exist a sequence $(x_j)_{j\in\mathbb{N}}$ of elements of M_q and a sequence $(u_j)_{j\in\mathbb{N}}$ of elements of U such that $x = \lim x_j$ and $u_j.x_j \in M_S$. Since U is compact, we may assume that (u_j) is convergent and denote its limit by $u := \lim u_j$. Then :

$$u.\mu(x) = \mu(u.x) = \mu\big(\lim (u_j.x_j)\big) = \lim \mu(u_j.x_j) \in \overline{\mu(M_S)}$$

In particular, $u.\mu(x) \in \exp(\overline{\mathcal{W}})$, so that $u.\mu(x) = \mu(x)$, since $\exp(\overline{\mathcal{W}})$ is a fundamental domain. Hence $\mu(x) \in \overline{\mu(M_S)}$, so that $\mu(M) \cap \exp(\overline{\mathcal{W}}) \subset \overline{\mu(M_S)}$.
Conversely, since μ is a closed map, $\mu(M)$ is closed in U and so is $\mu(M) \cap \exp(\overline{\mathcal{W}})$. But $\mu(M_S) \subset \mu(M) \cap \exp(\overline{\mathcal{W}})$, hence $\overline{\mu(M_S)} \subset \mu(M) \cap \exp(\overline{\mathcal{W}})$. □

Observe that lemma 8.2.13 is a consequence of corollary 8.2.10 and of the fact that $\mu(M_S) \subset \exp(\overline{\mathcal{W}})$. This last point also means that under the identification $U/\mathrm{Int}(U) \simeq \overline{\mathcal{W}}$, the map

$$\widetilde{\mu} := p \circ \mu|_{M_S} : M_S \longrightarrow U/\mathrm{Int}(U) \simeq \overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$$

is simply $\widetilde{\mu} = \exp^{-1} \circ \mu|_{M_S}$. As a matter of fact, it follows from the definition of M_S that $\mu(M_S)$ actually lies in the submanifold $\exp(\mathcal{W}_S)$ of U, which is diffeomorphic to $\mathcal{W}_S$ under $\exp^{-1}$, so that $\widetilde{\mu} = \exp^{-1} \circ \mu|_{M_S}$ is a smooth map from M_S to $\mathfrak{t}$:

$$\begin{tikzcd} & \mathcal{W}_S \arrow[d, "\exp", "\simeq"'] \\ M_S \arrow[ur, "\widetilde{\mu}"] \arrow[r, "\mu|_{M_S}"'] & \exp(\mathcal{W}_S) \end{tikzcd}$$

We now compute the differential of $\widetilde{\mu}$, which is defined to be the composed map $d\widetilde{\mu} := pr \circ T\widetilde{\mu}$ of the tangent map $T\widetilde{\mu} : TM_S \to T\mathfrak{t} \simeq \mathfrak{t} \times \mathfrak{t}$ and the projection $pr : T\mathfrak{t} \simeq \mathfrak{t} \times \mathfrak{t} \to \mathfrak{t}$ onto the second factor.

Lemma 8.2.14. *The differential $d\widetilde{\mu}$ of $\widetilde{\mu}$ is equal to the $\mathfrak{t}$-valued 1-form $\mu^*\theta$ on M_S, where θ is the Maurer-Cartan 1-form on T, that is, the $\mathfrak{t}$-valued 1-form defined for $t \in T$ and $\xi \in T_tT$ by $\theta_t(\xi) = t^{-1}.\xi = \xi.t^{-1}$:*

$$d\widetilde{\mu} = \mu^*\theta$$

Proof. Recall that the tangent map to the exponential map $\exp : \mathfrak{u} \to U$ is given, for all $X \in \mathfrak{u}$ and all $\xi \in T_X\mathfrak{u} = X + \mathfrak{u}$, by :

$$T_X \exp .\xi = \exp(X).\Big(\frac{1 - e^{-ad\,X}}{ad\,X} \cdot (\xi - X)\Big)$$

(see for instance [Hel01], p.105), where $\frac{1-e^{-ad\,X}}{ad\,X}$ is the endomorphism of $\mathfrak{u}$ given, for all $\zeta \in \mathfrak{u}$, by :

$$\frac{1 - e^{-ad\,X}}{ad\,X} \cdot \zeta = \sum_{k=1}^{+\infty} \frac{(-ad\,X)^{k-1}}{k!} \cdot \zeta$$

and where $\exp(X).\zeta$ denotes the effect on tangent vectors $\zeta \in \mathfrak{u} = T_1U$ of the left translation of element $\exp(X)$ in U. In the present case, we have to consider $\exp : \mathfrak{t} \to T$ with T abelian, since for $x \in M_S$, we have $\mu(x) \in \exp(\mathcal{W}_S) \subset T$, so that :

$$\frac{1 - e^{-ad\,X}}{ad\,X}.\zeta = \zeta$$

as $(ad\,X)^{k-1}.\zeta = 0$ as soon as $k - 1 \geq 1$. Therefore, for all $X \in \mathfrak{t}$ and all $\xi \in T_X\mathfrak{t} = X + \mathfrak{t}$:

$$T_X \exp .\xi = \exp(X).(\xi - X)$$

Therefore, for all $x \in M_S$ and all $v \in T_xM_S$, we have :

$$\begin{aligned} T_x(\underbrace{\exp \circ \widetilde{\mu}}_{=\mu}).v &= T_{\widetilde{\mu}(x)} \exp \circ T_x\widetilde{\mu}.v \\ &= \exp\big(\widetilde{\mu}(x)\big).\big(\underbrace{T_x\widetilde{\mu}.v - \widetilde{\mu}(x)}_{=(d\widetilde{\mu})_x.v}\big) \end{aligned}$$

so that :

$$T_x\mu.v = \exp\big(\widetilde{\mu}(x)\big).\big((d\widetilde{\mu})_x.v\big)$$

hence :

$$\begin{aligned} (d\widetilde{\mu})_x.v &= \Big(\exp\big(\widetilde{\mu}(x)\big)\Big)^{-1}.(T_x\mu.v) \\ &= \theta_{\exp \circ \widetilde{\mu}(x)}(T_x\mu.v) \\ &= \theta_{\mu(x)}(T_x\mu.v) \\ &= (\mu^*\theta)_x.v \end{aligned}$$

□

We may now prove proposition 8.2.3 :

Proof of proposition 8.2.3 (Existence of a connected symplectic cross-section). Set $N := M_S$, where $M_S = \mu^{-1}(\exp(\mathcal{W}_S))$ is the submanifold of M constructed above.

(i) Lemma 8.2.12 shows that N is connected.

(ii) Since $N = M_S = \mu^{-1}(\exp(\mathcal{W}_S))$ with μ equivariant, and since the conjugacy action of T on $\exp(\mathcal{W}_S)$ is trivial (as T is abelian), we have that N is T-stable.

(iii) Let us show that $\omega|_{M_S}$ is a symplectic form. We denote by i the inclusion map $i : M_S \hookrightarrow M$, so that $i^*\omega = \omega|_{M_S}$. First, we have :

$$d(i^*\omega) = i^*(d\omega) = i^*(-\mu^*\chi) = -(\mu \circ i)^*\chi$$

But $\mu \circ i = \mu|_{M_S}$ is T-valued and $\chi|_T = 0$ as T is abelian. Therefore, $d(i^*\omega) = 0$. Second, let us show that $i^*\omega$ is non-degenerate. Take $x \in M_S$ and $v \in T_xM_S$ such that for all $w \in T_xM_S$, $\omega_x(v, w) = 0$. In particular, $v \in (T_xM_S)^{\perp_\omega} \subset T_xM$. But we know from lemma 8.2.7 that :

$$T_xM_S = \big(T_x\mu\big)^{-1}\big(T_{\mu(x)}\exp(\mathcal{W}_S)\big)$$

(see (8.1)) hence :

$$\ker T_x\mu = T_x\mu^{-1}(\{0\}) \subset T_xM_S$$

and therefore :

$$(T_xM_S)^{\perp_\omega} \subset (\ker T_x\mu)^{\perp_\omega}$$

And we then know from proposition 4.3.1 that :

$$(\ker T_x\mu)^{\perp_\omega} = \{X^\#_x \ : \ X \in \mathfrak{u}\} = T_x(U.x)$$

Take now $X \in \mathfrak{u}$ such that $v = X^\#_x$. Then, by the equivariance of μ :

$$T_x\mu.v = T_x\mu.X^\#.x = X^\dagger_{\mu(x)} \in T_{\mu(x)}\big(\exp(\mathcal{W}_S)\big) \cap T_{\mu(x)}\big(U.\mu(x)\big) = \{0\}$$

(where $X^\dagger$ denotes the fundamental vector field associated to $X \in \mathfrak{u}$ by the conjugacy action of U on itself, and where the last equality follows from proposition 2.2.10). Hence $v \in \ker T_x\mu$. But by proposition 4.3.1, one has $\ker T_x\mu \subset (T_x(U.x))^{\perp_\omega}$, therefore :

$$v \in \big(T_x(U.x)\big)^{\perp_\omega} \cap \big(T_xM_S\big)^{\perp_\omega}$$

And we know from proposition 8.2.7 that :

$$T_xM_S + T_x(U.x) = T_xM$$

(see (8.2)). Therefore $v \in (T_xM)^{\perp_\omega} = \ker\omega_x$. Then $v \in \ker T_x\mu \cap \ker\omega_x$, which is equal to $\{0\}$ by proposition 4.3.1.

(iv) Let us now show that the action of T on M_S is Hamiltonian with momentum map $\widetilde{\mu} = \exp^{-1} \circ \mu : M_S \to \mathfrak{t}$. Take $X \in \mathfrak{t}$. Since M is a quasi-Hamiltonian space, we have :

$$\iota_{X^\#}\omega = \frac{1}{2}\mu^*(\theta^L + \theta^R \,|\, X)$$

Therefore, for all $x \in M_S$ and all $v \in T_xM_S = (T_x\mu)^{-1}(T_{\mu(x)}\exp(\mathcal{W}_S))$:

$$(\iota_{X^\#}\omega)_x.v = \frac{1}{2}\big(\theta^L_{\mu(x)}(T_x\mu.v) + \theta^R_{\mu(x)}(T_x\mu.v) \,|\, X\big)$$

But $T_x\mu.v \in T_{\mu(x)}\exp(\mathcal{W}_S)$ with $\exp(\mathcal{W}_S) \subset T$, so that, since T is abelian :

$$\theta^L_{\mu(x)}(T_x\mu.v) = \theta^R_{\mu(x)}(T_x\mu.v) = \theta_{\mu(x)}(T_x\mu.v) = (\mu^*\theta)_x.v$$

where θ is the Maurer-Cartan 1-form of T. Hence :

$$(\iota_{X^\#}\omega)_x.v = \big((\mu^*\theta)_x.v \,|\, X\big) = \big((d\widetilde{\mu})_x.v \,|\, X\big)$$

where the last equality follows from lemma 8.2.14. Denote by $(\widetilde{\mu} \,|\, X)$ the function :

$$\begin{array}{rccc} (\widetilde{\mu} \,|\, X): & M_S & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (\widetilde{\mu}(x) \,|\, X) \end{array}$$

(where $\widetilde{\mu} = \exp^{-1} \circ \mu|_{M_S} : M_S \to \mathfrak{t}$). We then have :

$$\big(d(\widetilde{\mu} \,|\, X)\big)_x.v = \big((d\widetilde{\mu})_x.v \,|\, X\big)$$

Therefore, for all $X \in \mathfrak{t}$:

$$\iota_{X^\#}\omega = d(\widetilde{\mu} \,|\, X)$$

that is : the Hamiltonian vector field associated to the function $(\widetilde{\mu} \mid X)$ is the fundamental vector field $X^\#$, which shows that the action of T on M_S is Hamiltonian.

(v) Corollary 8.2.10 shows that $\overline{U.N} = M$. Since μ is a proper map, lemma 8.2.13 shows that $\overline{\mu(N)} = \mu(M) \cap \exp(\overline{\mathcal{W}})$, or equivalently : $\overline{\widetilde{\mu}(N)} = \widetilde{\mu}(M)$.

□

8.2.3 The convexity statement

We can now state and prove the following convexity result :

Theorem 8.2.15 (Momentum convexity for group-valued momentum maps). [AMM98] *Let $(U, (.\,|\,.))$ be a compact connected simply connected Lie group and let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian space with proper momentum map μ. Then, for any choice of a maximal torus $T \subset U$ and any choice of a closed Weyl alcove $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$, the set $\mu(M) \cap \exp(\overline{\mathcal{W}}) \subset \exp(\overline{\mathcal{W}})$ is a convex subpolytope of $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}}$, called the* momentum polytope.

To prove this result, we will use the symplectic cross-section $N \subset M$ whose existence is guaranteed by proposition 8.2.3 and follow the strategy of [HNP94]. To that end, we recall the following results from [HNP94].

Theorem 8.2.16 (Local convexity results for Hamiltonian torus actions). [HNP94] *Let (N, ω) be a symplectic manifold endowed with a Hamiltonian action of a torus T with momentum map $\mu : N \to \mathfrak{t}^*$. Then for every $x \in N$, there exist an open neighbourhood $\mathcal{V}_x$ of $x \in N$ and a polyhedral cone $C_{\mu(x)} \subset \mathfrak{t}^*$ with vertex $\mu(x)$ such that :*

(i) $\mu : \mathcal{V}_x \to C_{\mu(x)}$ is an open map. In particular, $\mu(\mathcal{V}_x)$ is an open neighbourhood of $\mu(x)$ in $C_{\mu(x)}$.

(ii) $\mu^{-1}(\{\mu(y)\})$ is connected for all $y \in \mathcal{V}_x$.

If in addition β is an antisymplectic involution on N satisfying $\beta(t.x) = t^{-1}.\beta(x)$ and $\mu \circ \beta = \mu$, then assertion (i) above remains true for the manifold $N^\beta := Fix(\beta)$ and the same cones $C_{\mu(x)}$, $x \in N^\beta$, that is :

(iii) $\mu : \mathcal{V}_x \cap N^\beta \to C_{\mu(x)}$ is an open map. In particular, $\mu(\mathcal{V}_x \cap N^\beta)$ is an open neighbourhood of $\mu(x)$ in $C_{\mu(x)}$.

For additional local properties, including a description of the cones $C_{\mu(x)}$ using the local normal form of the action, we refer to [HNP94]. Conditions (ii) and (iii) above play a special role when it comes to convexity considerations insofar as they make it possible to obtain a global result from a local one (see theorem 8.2.18), which justifies the following definition :

Definition 8.2.17 (Local convexity data). Let X be a *connected* Hausdorff space, and let V be a finite dimensional vector space. Consider a continuous map $\psi : X \to V$. We will say that ψ *gives rise to local convexity data* $(\mathcal{V}_x, C_{\psi(x)})_{x\in X}$ if for any $x \in X$ there exist an open neighbourhood $\mathcal{V}_x$ of x in X and a convex cone $C_{\psi(x)} \subset V$ with vertex $\psi(x)$ such that :

(O) $\psi : \mathcal{V}_x \to C_{\psi(x)}$ is an open map.

(LC) $\psi^{-1}(\{\psi(y)\}) \cap \mathcal{V}_x$ is connected for all $y \in \mathcal{V}_x$.

A map $\psi : X \to V$ satisying condition (LC) alone is said to be *locally fibre-connected*.

We then have :

Theorem 8.2.18 (Local-global principle). [HNP94] *Let $\psi : X \to V$ be a map giving rise to local convexity data $(\mathcal{V}_x, C_{\mu(x)})_{x\in X}$ and assume that ψ is a proper map. Then $\psi(X)$ is a closed locally polyhedral convex subset of V, the fibres $\psi^{-1}(\{v\})$ are connected for all $v \in V$, $\psi : X \to \psi(X)$ is an open map and $C_{\mu(x)} = \psi(x) + \overline{\mathbb{R}^+.(\psi(X)\backslash\{\psi(x)\})}$.*

In particular, we see that theorem 8.1.1 follows immediately from theorems 8.2.16 and 8.2.18. We also state the following corollary of the local-global principle, which we will use in the proof of proposition 8.3.5.

Corollary 8.2.19. [HNP94] *Let V be a finite dimensional vector space and let $P \subset V$ be a closed connected subset of V such that for all $x \in P$, there exists a neighbourhood $\mathcal{O}_v$ of v in V and a cone $C_v \subset V$ with vertex v such that $\mathcal{O}_v \cap P = O_v \cap C_v$. Then P is a convex subset of V and for all $v \in V$, $C_v = v + \overline{\mathbb{R}^+.(P\backslash\{v\})}$.*

Going back to our case, we see that we have a symplectic cross-section $(N, \omega|_N) \subset (M, \omega)$ endowed with a Hamiltonian torus action, and that $\overline{\mu(N)} = \mu(M) \cap \exp(\overline{\mathcal{W}})$ (see proposition 8.2.3). In particular, by theorem 8.2.16, the momentum map $\mu|_N$ gives rise to local convexity data. But we cannot conclude immediately that $\mu(N)$ is convex because $\mu|_N$ has no reason to be proper, as $N = \mu^{-1}(\exp(\mathcal{W}_S))$ is in general not closed in M. But we can use another result from [HNP94] :

Proposition 8.2.20. [HNP94] *Let $\psi : X \to V$ be a map giving rise to local convexity data $(\mathcal{V}_x, C_{\psi(x)})_{x\in X}$. Consider any closed locally polyhedral convex subset $D \subset V$ and set $Y := \psi^{-1}(D) \subset X$. Then $\psi|_Y : Y \to V$ gives rise to local convexity data $(\mathcal{V}_y, C_{\psi(y)} \cap \overline{\mathbb{R}^+.(D\backslash\{\psi(y)\})})_{y\in Y}$.*

We then have :

Lemma 8.2.21. *Let $M_S := \mu^{-1}(\exp(\mathcal{W}_S))$ be a connected symplectic cross-section for the connected quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$. Then the set $\mu(M_S) \subset \exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}}$ is a convex polytope.*

Proof. By proposition 8.2.16, the map $\mu|_{M_S}$ gives rise to local convexity data $(\mathcal{V}_x, C_{\mu(x)})_{x\in M_S}$. Write the convex set $\mathcal{W}_S$ as an increasing sequence of *closed* locally polyhedral convex subsets $(D_n)_{n\in\mathbb{N}}$. Then :

$$\exp(\mathcal{W}_S) = \bigcup_{n\in\mathbb{N}} \exp(D_n)$$

Therefore, proposition 8.2.20 applies to the closed sets $Y_n := \mu^{-1}(\exp(D_n))$ and $\mu|_{Y_n}$ gives rise to local convexity data $(\mathcal{V}_x, C_{\mu(x)} \cap \overline{\mathbb{R}^+.(\exp(D_n)\backslash\{\mu(x)\})})_{x\in Y_n}$. Additionally, since Y_n is closed in M_S, $\mu|_{Y_n}$ is a proper map. Since M_S is connected and is an increasing union of closed subsets $M_S = \cup_{n\in\mathbb{N}} Y_n$, we can find an ascending sequence $(Z_n)_{n\in\mathbb{N}}$ of connected components of the $(Y_n)_{n\in\mathbb{N}}$ such that $M_S = \cup_{n\in\mathbb{N}} Z_n$.

Each Z_n is closed in Y_n, so that $\mu|_{Z_n}$ is a proper map which gives rise to local convexity data $(\mathcal{V}_x, C_{\mu(x)} \cap \overline{\mathbb{R}^+.(\exp(D_n)\backslash\{\mu(x)\})})_{x\in Z_n}$. Therefore, by theorem 8.2.18, $\mu(Z_n)$ is a convex polytope. We then have that $\mu(M_S)$ is an increasing union $\mu(M_S) = \cup_{n\in\mathbb{N}}\mu(Z_n)$ of convex subpolytopes of $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}}$, which implies that it is a convex polytope. □

We can now prove theorem 8.2.15 :

Proof of the convexity theorem 8.2.15. We have $\mu(M)\cap\exp(\overline{\mathcal{W}}) = \overline{\mu(M_S)}$ by proposition 8.2.3 and $\mu(M_S)$ is a convex polytope by lemma 8.2.21, hence so is $\overline{\mu(M_S)}$. □

Remark 8.2.22 (Addendum to theorem 8.2.15). As a matter of fact, a more complete statement of convexity theorem 8.2.15 would be to say that, in addition to the conclusion that $\widetilde{\mu}(M)$ is a convex polytope, one also has :

- the map $\widetilde{\mu} : M \to \widetilde{\mu}(M)$ is an open map.
- the fibres of $\widetilde{\mu}$ are connected. In particular, $\mu^{-1}(\{1\}) = \widetilde{\mu}^{-1}(\{1\})$ is a connected subset of M.

To prove this, observe that these results are true for the symplectic cross-section M_S in virtue of theorem 8.1.1, whence one can deduce that they are also true for $M = \overline{U.M_S}$. We refer for instance [Ben02] for a proof of this in the usual Hamiltonian setting. As we will not need these results in the following, we do not reproduce the proof here.

A slightly different strategy may be applied to prove the convexity of $\mu(M_S)$, for which we refer to [Ben02]. We now move on to establishing a real version of this convexity result.

8.3 A real convexity theorem for momentum maps with value in a compact connected simply connected Lie group

In this section, we study the image, under the momentum map μ, of the fixed-point set M^β of a form-reversing involution β defined on the quasi-Hamiltonian space $(M, \omega, \mu : M \to U)$ and compatible with the action of (U, τ) and the momentum map μ of this action in the sense of definition 7.2.1. More precisely, we will study the convexity properties of $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ (or equivalently $\widetilde{\mu}(M^\beta)$) and show that when the symmetric pair (U, τ) is of maximal rank (see definition 3.2.1) then the set $\mu(M^\beta)\cap\exp(\overline{\mathcal{W}}) \simeq \widetilde{\mu}(M^\beta) \subset \overline{\mathcal{W}}$ is a convex polytope, which turns to be equal to the full polytope $\mu(M)\cap\exp(\overline{\mathcal{W}}) \simeq \widetilde{\mu}(M)$. As this result is sufficient to prove the existence of decomposable representations (see subsection 8.3.3), we will not prove any convexity result for $\mu(M^\beta)\cap\exp(\overline{\mathcal{W}})$ in the case where (U, τ) is not of maximal rank. We shall nonetheless say a few words on this situation in subsection 8.3.2.

8.3.1 The case where (U, τ) is of maximal rank

Recall that the symmetric pair (U, τ) is said to be of maximal rank if $\dim U/U^\tau = \frac{1}{2}(\dim U + \mathrm{rk}\ U)$, where U^τ is the subgroup of U consisting of elements of U fixed by τ. In this case, there exists a *maximal* torus $T \subset U$ which is fixed pointwise by the involution $\tau^- : u \mapsto \tau(u^{-1})$ (see proposition 3.2.3). If we choose such a maximal torus $T \subset Fix(\tau^-) \subset U$ (in particular $\tau(t) = t^{-1}$ for all $t \in T$), then the construction of the symplectic cross-section $N \subset M$ of proposition 8.2.3 immediately implies that N is β-stable. Indeed, recall that $N = M_S := \mu^{-1}(\exp(\mathcal{W}_S))$ with $\exp(\mathcal{W}_S) \subset T$ so that, if $x \in M_S$, then :

$$\mu \circ \beta(x) = \tau^- \circ \mu(x) = \mu(x) \in \exp(\mathcal{W}_S)$$

hence $\beta(x) \in M_S$. Therefore, β induces an involution $\beta_S := \beta|_{M_S}$ on the symplectic manifold $(M_S, \omega_S := \omega|_{M_S})$, and this involution is antisymplectic since $\beta^*\omega = -\omega$. Further, β_S is compatible with the action of (T, τ) on M_S and with the momentum map $\widetilde{\mu}$. More precicely, $\beta(t.x) = \tau(t).\beta(x) = t^{-1}.\beta(x)$ for all $x \in M_S$ and all $t \in T$, and $\widetilde{\mu}\circ\beta = \widetilde{\mu}$, whence we see that we are almost in the situation of the Duistermaat

theorem (theorem 8.1.2). We shall now refine this analysis, and in particular relate the fixed-point sets $M_S^{\beta_S}$ and M^β, as was done for M_S and M in statement (v) of proposition 8.2.3.

Lemma 8.3.1. *The involution β leaves the set $M_q = \{x \in M \mid \dim U.\mu(x) = q\} \subset M$ invariant.*

Proof. For any $x \in M_q$, consider the element $\mu \circ \beta(x) = \tau^- \circ \mu(x) \in U$. There exists an element $u \in U$ such that $t := u(\tau^- \circ \mu(x))u^{-1}$ is an element of the maximal torus T. Consequently, $\tau^-(t) = t$, which means that $\tau(u)\mu(x)\tau^-(u) = u(\tau^- \circ \mu(x))u^{-1}$, therefore that $\mu(x)$ is conjugate to $\tau^- \circ \mu(x) = \mu \circ \beta(x)$. In particular, $\dim U.\mu(\beta(x)) = \dim U.\mu(x) = q$, which proves the lemma. □

Observe then that the group $K = U^\tau = Fix(\tau) \subset U$ acts on M^β since $\beta(k.x) = \tau(k).\beta(x) = k.x$ for all $x \in M^\beta$ and all $k \in K$. We then have :

Lemma 8.3.2. *Assume that $M^\beta \neq \emptyset$. Then the set $M_q^\beta = \{x \in Fix(\beta) \mid \dim U.\mu(x) = q\}$ is non-empty, open, and dense in M^β.*

Proof. Set :

$$q' := \max \{\dim K.\mu(x) \,:\, x \in M^\beta\}$$

and :

$$M_{K,q'}^\beta := \{x \in Fix(\beta) \mid \dim K.\mu(x) = q'\}$$

Then $M_{K,q'}^\beta$ non-empty by definition and it is an open and dense subset of M^β (apply proposition 2.1.6 to every connected component of M^β). Take now $x \in M_{K,q'}^\beta$. Then $\mu(x) \in Fix(\tau^-)$ (since $\tau^-(\mu(x)) = \mu(\beta(x)) = \mu(x)$), so that there exists, by corollary 3.2.6, an element $k \in K$ such that $k\mu(x)k^{-1} \in \exp(\overline{\mathcal{W}}) \subset T$. But if $w \in Fix(\tau^-)$, then $\dim K.w$ is maximal if and only if $\dim U.w$ is maximal. Indeed, $\dim K.w = \dim K - \dim (U_{S'})^\tau$ where S' is the uniquely defined set such that $K.w \cap \exp(\mathcal{W}_{S'}) \neq \emptyset$, and then $\dim U.w = \dim U - \dim U_{S'}$ *for the same* S' since $U.w \supset K.w$. Therefore, here :

$$\dim K.\mu(x) = q' \quad \text{if and only if} \quad \dim U.\mu(x) = q$$

Hence $M_{K,q'}^\beta = Fix(\beta) \cap M_q = M_q^\beta$, which proves the lemma. □

Consequently :

Lemma 8.3.3. *If $M^\beta \neq \emptyset$ then $M_S^{\beta_S} \neq \emptyset$, and one has : $M_q^\beta = U^\tau.M_S^{\beta_S}$.*

Proof. Since $M^\beta \neq \emptyset$ by assumption, lemma 8.3.2 shows that $M_q^\beta \neq \emptyset$. Take now $x \in M_q^\beta$. Then $\mu(x) \in Fix(\tau^-)$ and therefore there exists, by corollary 3.2.6, some $k \in Fix(\tau) = U^\tau$ such that :

$$k\mu(x)k^{-1} \in \exp(\overline{\mathcal{W}}) \cap \mu(M_q) \subset \exp(\mathcal{W}_S)$$

where the last inclusion follows from lemma 8.2.9. Hence :

$$k.x \in \mu^{-1}(\exp(\mathcal{W}_S)) = M_S$$

. Moreover, $\beta(k.x) = \tau(k).\beta(x) = k.x$, hence $k.x \in M_S^{\beta_S}$, which is therefore non-empty, and we have indeed $M_q^\beta = U^\tau.M_S^{\beta_S}$. □

We can now prove an analogue of statement (v) of proposition 8.2.3 (or equivalently, of lemma 8.2.13) :

Lemma 8.3.4. *If μ is a closed map (in particular, if μ is proper), then the set $\mu(M_S^{\beta_S})$ is dense in $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$.*

Proof. Assume that $M^\beta \neq \emptyset$ (otherwise there is nothing to prove) and take $\mu(x) \in \mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$. Then, by lemma 8.3.2, $x = \lim x_j$ with $x_j \in M_q^\beta$ and, by lemma 8.3.3, there exists, for all j, an element $k_j \in U^\tau$ such that $k_j.x_j \in M_S^{\beta_S}$. Since U^τ is compact, we may assume that the sequence (k_j) converges to a certain $k \in U^\tau$. Then :

$$k.\mu(x) = \mu(k.x) = \lim \mu(k_j.x_j) \in \overline{\mu(M_S^{\beta_S})}$$

In particular, $k.\mu(x) \in \exp(\overline{\mathcal{W}})$, so that $k.\mu(x) = \mu(x)$. Hence $\mu(x) \in \overline{\mu(M_S^{\beta_S})}$, so that $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) \subset \overline{\mu(M_S^{\beta_S})}$.
Conversely, since μ is a closed map, $\mu(M^\beta)$ is closed in U and so is $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$. But $\mu(M_S^{\beta_S}) \subset \mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$, so that $\overline{\mu(M_S^{\beta_S})} \subset \mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$. □

Thus, $\mu(M_S^{\beta_S})$ is almost the whole of $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$. This is interesting because we may now relate $\mu(M_S^{\beta_S})$ to $\mu(M_S)$ (which is almost $\mu(M) \cap \exp(\overline{\mathcal{W}})$, by lemma 8.2.13) in the following way :

Proposition 8.3.5. *Assume that $M^\beta \neq \emptyset$ and that $\mu : M \to U$ is a proper map (in particular, it is a closed map). Then, in the above notations :*

$$\mu(M_S^{\beta_S}) = \mu(M_S)$$

Recall that $M_S \subset M$ is the symplectic cross-section from proposition 8.2.3, so that the above result is very similar to Duistermaat's theorem 8.1.2. However, as in the proof of lemma 8.2.21, we cannot apply theorem 8.1.2 directly to M_S, since $\mu|_{M_S}$ is in general not proper. But we may work with the ascending sequence $(Z_n)_{n\in\mathbb{N}}$ introduced in the proof of lemma 8.2.21 : $\mathcal{W}_S$ is an acending union of closed convex subsets $\mathcal{W} = \cup_{n\in\mathbb{N}} D_n$, and $M_S := \mu^{-1}(\exp(\mathcal{W}_S))$ is an ascending union $M_S = \cup_{n\in\mathbb{N}} Z_n$ of *closed connected* sets $Z_n \subset \mu^{-1}(\exp(D_n))$. The map $\widetilde{\mu}|_{Z_n}$ is a proper map which gives rise to local convexity data $(\mathcal{V}_x, C_{\widetilde{\mu}(x)})_{x\in Z_n}$ and, by proposition 8.2.20 and theorem 8.2.18, the set $\widetilde{\mu}(Z_n) \subset \mathfrak{t}$ is convex. We then observe the following fact :

Lemma 8.3.6. *Consider $n \in \mathbb{N}$ such that $Z_n \cap M^\beta \neq \emptyset$. Then for any connected component $Q \subset (Z_n \cap M^\beta)$, the set $\widetilde{\mu}(Q)$ is convex.*

Proof. First, observe that Q is closed in M, and since $\mu : M \to U$ is a closed map, $\widetilde{\mu}(Q)$ is a closed connected subset of $\mathfrak{t}$. Second, take $x \in Q$. It follows from point (iii) of the local convexity theorem theorem 8.2.16 that there exists a neighbourhood $\mathcal{V}_x$ of x in M and a neighbourhood $\mathcal{O}_{\widetilde{\mu}(x)}$ of $\widetilde{\mu}(x)$ in $\mathfrak{t}$ such that $\widetilde{\mu}(\mathcal{V}_x \cap Q) = C_{\widetilde{\mu}(x)} \cap \mathcal{O}_{\widetilde{\mu}(x)}$. Further, $\widetilde{\mu}(Q)$ lies in the convex set $\widetilde{\mu}(Z_n)$, hence is contained in

$$\widetilde{\mu}(x) + \overline{\mathbb{R}^+.\big(\widetilde{\mu}(Z_n)\backslash\{\widetilde{\mu}(x)\}\big)} = C_{\widetilde{\mu}(x)} \tag{8.3}$$

where the equality follows from theorem 8.2.18. Hence :

$$C_{\widetilde{\mu}(x)} \cap \mathcal{O}_{\widetilde{\mu}(x)} = \widetilde{\mu}(Q) \cap \mathcal{O}_{\widetilde{\mu}(x)}$$

so that $\widetilde{\mu}(Q)$ is convex by corollary 8.2.19. □

Remark 8.3.7. In fact, if we apply the full of corollary 8.2.19, we also obtain :

$$\text{for all } x \in Q,\ C_{\widetilde{\mu}(x)} = \widetilde{\mu}(x) + \overline{\mathbb{R}^+.\big(\widetilde{\mu}(Q)\backslash\{\widetilde{\mu}(x)\}\big)} \tag{8.4}$$

We then recall the following result from [HNP94] :

Lemma 8.3.8. [HNP94] *If $P_1 \subset P_2$ is an inclusion between two convex subsets of a finite-dimensional vector space satisfying, for all $v \in P_1$ the condition $v + \overline{\mathbb{R}^+.(P_1\backslash\{v\})} = v + \overline{\mathbb{R}^+.(P_2\backslash\{v\})}$, then $P_1 = P_2$.*

Proof. A convex set is the intersection of cones containing it, so that :

$$P_1 = \bigcap_{v\in P_1} \left(v + \overline{\mathbb{R}^+.(P_1\backslash\{v\})}\right) = \bigcap_{v\in P_1} \left(v + \overline{\mathbb{R}^+.(P_2\backslash\{v\})}\right) \supset \bigcap_{v\in P_2} \left(v + \overline{\mathbb{R}^+.(P_2\backslash\{v\})}\right) = P_2$$

hence $P_1 = P_2$. □

And we may now prove proposition 8.3.5 :

Proof of proposition 8.3.5. Consider an n such that $Z_n \cap M^\beta \neq \emptyset$ and let Q be a connected component of $Z_n \cap M^\beta$. Then we know from lemma 8.3.6 that $\widetilde{\mu}(Q) \subset \widetilde{\mu}(Z_n)$ is an inclusion between two convex sets of a finite-dimensional vector space. Additionally, by comparing (8.3) and (8.4), we obtain :

$$\widetilde{\mu}(x) + \overline{\mathbb{R}^+.\left(\widetilde{\mu}(Q)\backslash\{\widetilde{\mu}(x)\}\right)} = \widetilde{\mu}(x) + \overline{\mathbb{R}^+.\left(\widetilde{\mu}(Z_n)\backslash\{\widetilde{\mu}(x)\}\right)}$$

Therefore, lemma 8.3.8 applies and $\widetilde{\mu}(Q) = \widetilde{\mu}(Z_n)$, hence $\widetilde{\mu}(Z_n \cap M^\beta) = \widetilde{\mu}(Z_n)$. Since $M_S = \cup_{n\in\mathbb{N}} Z_n$, one has $M_S^{\beta_S} = \cup_{n\in\mathbb{N}}(Z_n \cap M^\beta)$ and therefore :

$$\widetilde{\mu}(M_S) = \bigcup_{n\in\mathbb{N}} \widetilde{\mu}(Z_n) = \bigcup_{n\in\mathbb{N}} \widetilde{\mu}(Z_n \cap M^\beta) = \widetilde{\mu}(M^\beta)$$

Since $\mu(M_S)$ is contained in $\exp(\mathcal{W}_S)$ and $\widetilde{\mu} = \exp^{-1} \circ \mu|_{M_S}$, the above equality is equivalent to $\mu(M_S) = \mu(M_S^{\beta_S})$. □

We can now state our real convexity result in the case where the symmetric pair (U, τ) is of maximal rank :

Theorem 8.3.9 (A real convexity result for group-valued momentum maps). *Let $(U, (.\,|\,.), \tau)$ be a compact connected simply connected Lie group endowed with an involutive automorphism τ such that the involution $\tau^- : u \mapsto \tau(u^{-1})$ leaves a maximal torus T of U pointwise fixed and let $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$ be a closed Weyl alcove. Let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian U-space with proper momentum map $\mu : M \to U$ and let $\beta : M \to M$ be an involution on M such that :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $x \in M$ *and all* $u \in U$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Then :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \mu(M) \cap \exp(\overline{\mathcal{W}})$$

In particular, $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ is a convex subpolytope of $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$, equal to the whole momentum polytope $\mu(M) \cap \exp(\overline{\mathcal{W}})$.

Proof. Since μ is a proper map, lemmas 8.2.13 and 8.3.4 apply, as well as proposition 8.3.5. Therefore :

$$\mu(M) \cap \exp(\overline{\mathcal{W}}) = \overline{\mu(M_S)} = \overline{\mu(M_S^{\beta_S})} = \mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$$

□

Corollary 8.3.10. *For all $t \in \exp(\overline{\mathcal{W}})$, one has :*

$$\mu^{-1}(\{t\}) \neq \emptyset \quad \text{if and only if} \quad \mu^{-1}(\{t\}) \cap M^\beta \neq \emptyset$$

Proof. Assume that $\mu^{-1}(\{t\}) \neq \emptyset$. Then $t \in \mu(M) \cap \exp(\overline{\mathcal{W}}) = \mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$, so that there exists $y \in M^\beta$ satisfying $\mu(y) = t$. The converse implication is obvious. □

In particular, $1 \in \mu(M)$ if and only if $1 \in \mu(M^\beta)$, which we shall later relate to the existence of decomposable representations (see subsection 8.3.3). We also point out the following consequence of theorem 8.3.9 :

Corollary 8.3.11. *If $\hat{\beta}$ denotes the involution induced by $\beta : M \to M$ on the quasi-Hamiltonian quotient $M//U := \mu^{-1}(\{1\})/U$ (assumed to be non-empty), and if $Fix(\beta) \neq \emptyset$, then $Fix(\hat{\beta}) \neq \emptyset$.*

Proof. The assumption $M//U \neq \emptyset$ means that $\mu^{-1}(\{1\}) \neq \emptyset$. As noted in corollary 8.3.10, we then have $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$, which is equivalent to $Fix(\hat{\beta}) \neq \emptyset$, as seen in chapter 7 (see in particular proposition 7.4.5). □

These last results will be enough for us to prove the existence of decomposable representations in subsection 8.3.3. Before going into this, we would like to say a few words on what should happen if one does not assume the symmetric pair (U, τ) to be of maximal rank. The next subsection may be skipped if one wants to go straight to the proof of existence of decomposable representations.

8.3.2 The case where (U, τ) is not of maximal rank

In this subsection we conjecture, based on the work of O'Shea and Sjamaar in [OS00], a description of the set $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ as a subpolytope of $\mu(M) \cap \exp(\overline{\mathcal{W}})$ in the case where the symmetric pair (U, τ) is not assumed to be of maximal rank. We hope to return to this question in a future work. For now, we would just like to stress the fact that the convexity result that we have obtained in subsection 8.3.1 (namely, theorem 8.3.9) is sufficient to guarantee the existence of decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$, as we shall see in theorem 8.3.14. This is so because the proof of existence of decomposable representations (as a matter of fact, of σ_0-decomposable representations) relies on the fact that they have been characterized as the elements of the fixed-point set of an involution, and that this characterization was obtained under the assumption that (U, τ) was of maximal rank (that is, that there existed a maximal torus T of U which was fixed pointwise by τ^-). See remark 8.3.17 for additional comments on this.

Recall that (U, τ) is a compact connected simply connected Lie group endowed with an involutive automorphism τ, and that we denote by τ^- the involution $\tau^-(u) := \tau(u^{-1})$ on U. In the previous subsection, we assumed that τ^- left a *maximal* torus T of U pointwise fixed, which is not always true. Nonetheless, there always exists a torus $T' \subset U$ such that $T' \subset Fix(\tau^-)$, and any maximal torus T containing T' is τ-stable (see for instance [Loo69b], pp.72-73). Consider such a torus T' of maximal possible dimension with respect to the property that $\tau^-|_{T'} = Id$, and a maximal torus T of U containing T'. Then there is a corresponding Weyl alcove $\mathcal{W}' \subset \mathcal{W} \subset \mathfrak{t} = Lie(T)$ (and $\mathcal{W}' \subset \mathfrak{t}' = Lie(T')$), such that $\exp(\overline{\mathcal{W}}')$ is a fundamental domain for the action of U^τ on $Fix(\tau^-)$. Following [Loo69b] and [OS00], it should be possible to give a description of $\overline{\mathcal{W}}'$ in terms of the roots of (U, τ). We then expect the following result to hold, in analogy to theorem 8.1.5 :

Conjecture 8.3.12. *Let $(U, (.\,|\,.), \tau)$ be a compact connected simply connected Lie group endowed with an involutive automorphism τ. Let T be a maximal torus of U such that $T \cap Fix(\tau^-)$ is of maximal possible dimension, and let $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$ be a closed Weyl alcove. Let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian U-space with proper momentum map $\mu : M \to U$ and let $\beta : M \to M$ be an involution on M such that :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $x \in M$ *and all* $u \in U$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Then :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \big(\mu(M) \cap \exp(\overline{\mathcal{W}})\big) \cap Fix(\tau^-)$$

In particular, $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ *is a convex subpolytope of* $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$ *obtained by intersecting the momentum polytope* $\mu(M) \cap \exp(\overline{\mathcal{W}})$ *with the vector space* $Fix(\tau^-|_{\mathfrak{t}})$.

Remark 8.3.13. Observe that on the one hand $\mu(M^\beta) \subset Fix(\tau^-)$ because of the compatibility of β with μ, and on the other hand $\exp(\overline{\mathcal{W}}) \cap Fix(\tau^-) = \exp(\overline{\mathcal{W}}')$, so that the above result rewrites :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}') = \mu(M) \cap \exp(\overline{\mathcal{W}}')$$

8.3.3 Relation to the existence of decomposable representations

In this subsection, we write down in detail why there always exist decomposable representations of the fundamental group $\pi := \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into an arbitrary compact connected simply connected Lie group (U, τ) endowed with an involutive automorphism τ such that the involution $\tau^- : u \mapsto \tau(u^{-1})$ leaves a maximal torus of U pointwise fixed (such an involution always exists, see proposition 3.2.2). Recall from chapter 5 that decomposable representations can be defined only in terms of τ (see definition 5.2.1) and we saw in chapter 6 that decomposable representations are the elements $u \in \mathrm{Hom}_{\mathcal{C}}(\pi, U) = \mu^{-1}(\{1\})$ satisfying $\beta(u) \sim u$ as representations of π, where β is a form-reversing involution defined on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ (each $\mathcal{C}_j$ being a conjugacy class in U), compatible with the diagonal action of U on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ and the momentum map $\mu(u_1, \ldots, u_l) = u_1 \ldots u_l$ of this action. As we announced in chapter 7, this is enough to guarantee that $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ (provided $\mu^{-1}(\{1\}) \neq \emptyset$). The fact that $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ means that there exist σ_0-decomposable representations, which is equivalent, by lemma 6.6.1, to the fact that there exist decomposable representations. And we then have :

Theorem 8.3.14 (Existence of decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$). *Let* $\mathcal{C}_1, \ldots, \mathcal{C}_l$ *be* l *conjugacy classes in a compact connected simply connected Lie group* (U, τ) *satisfying :*

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(S^2 \backslash \{s_1, \ldots, s_l\}), U) := \big\{(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \ldots u_l = 1\big\} \neq \emptyset$$

then there exist $w_1, \ldots, w_l \in U$ *such that* $\tau^-(w_j) = w_j$ *for all* $j \in \{1, \ldots, l\}$ *and such that* $w_j \tau(w_{j+1}) \in \mathcal{C}_j$.

Proof. The assumption of the theorem says that $\mu^{-1}(\{1\}) \neq \emptyset$, where μ is the momentum map

$$\begin{array}{rccc} \mu : & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ldots, u_l) & \longmapsto & u_1 \ldots u_l \end{array}$$

Furthermore, saying that there exist $w_1, \ldots, w_l$ satisfying the prescribed conditions amounts to saying that there exist decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$, in which case there also exist σ_0-decomposable representations (see lemma 6.6.1). In turn, this is equivalent, by theorem 6.6.2, to saying that there exist representations $u_0 \in \mathrm{Hom}_{\mathcal{C}}(\pi, U)$ satisfying $\beta(u_0) = u_0$, which is exactly saying that $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$. But this is guaranteed by corollary 8.3.10 once $\mu^{-1}(\{1\}) \neq \emptyset$. □

Observe that to be able to apply theorem 6.6.2 and corollary 8.3.10 in the above proof we have to assume that the symmetric pair (U, τ) is of maximal rank and that $Fix(\tau^-)$ is connected.

Remark 8.3.15 (The case where U=U(n)). As a matter of fact, theorem 8.3.14 remains true even when the group U at hand is the unitary group $U = U(n)$, as was shown by Falbel and Wentworth in [FW]. The strategy that we have adopted in this work to prove the existence of decomposable representations does not apply to $U(n)$ because the convexity result 8.3.9 does not hold, as $U(n)$ is not simply connected. Nonetheless, the fact that we still have $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$ when $U = U(n)$ pleads for a local result that would be enough to ensure this (as opposed to studying the whole of $\mu(M^\beta)$).

We also point out the following consequence of corollary 8.3.10 which, in terms of the notation introduced in sections 6.4 and 6.5, says that $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \neq \emptyset$ if and only if $\mathcal{M}_{\mathcal{C}} \neq \emptyset$. The result as we formulate it may seem a bit odd at first sight, but it is merely a reformulation of what we have obtained so far for the case where $U = U(n)$. It is a type of result analogous to that of proposition 6.3.1.

Proposition 8.3.16 (An application to a matrix problem). *Consider $\lambda_1, \ldots, \lambda_l \in \mathbb{R}^n$. Then the following statements are equivalent :*

(i) There exist l unitary matrices $u_1, \ldots, u_l \in U(n)$ such that :

$$\operatorname{Spec} u_j = \exp(i\lambda_j) \quad \textit{and} \quad u_1 \ldots u_l = 1$$

(ii) There exist l unitary matrices $A_1, \ldots A_l \in U(n)$ such that :

$$\operatorname{Spec}(A_j^t A_j) = \exp(i\lambda_j) \quad \textit{and} \quad A_1 \ldots A_l = 1$$

Proof. We refer to chapter 6 for notation. Condition (ii) says that $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \neq \emptyset$. By proposition 6.5.7, $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \simeq \mathcal{M}^{\beta}_{\mathcal{C}}$, so that $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \neq \emptyset$ if and only if $\mathcal{M}^{\beta}_{\mathcal{C}} \neq \emptyset$, which is by definition equivalent to saying that $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$. By corollary 8.3.10, this last point is *equivalent* to saying that $\mu^{-1}(\{1\}) \neq \emptyset$, which is equivalent to saying that $\mathcal{M}_{\mathcal{C}} \neq \emptyset$, which proves the result. □

Observe that we already knew from sections 6.4 and 6.5 that the condition $\mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \neq \emptyset$ implied $\mathcal{M}_{\mathcal{C}} \neq \emptyset$, as we had a map $\eta^{(l)} : \mathcal{M}^{\alpha}_{\widetilde{\mathcal{D}}} \subset \mathcal{M}_{\widetilde{\mathcal{D}}} \xrightarrow{\simeq} \mathcal{M}_{\mathcal{C}}$ (see proposition 6.4.3). The above result then says that *the centered Lagrangian problem* (which was, in our terminology, a real problem) *has a solution if and only if the unitary problem has a solution*, the non-trivial implication being (the unitary problem has a solution) $\Longrightarrow$ (the centered Lagrangian problem has a solution). Pictorially :

$$\begin{array}{ccc}
\text{Complex centered Lagrangian problem (section 6.3)} & \overset{\text{proposition 6.4.3}}{\Longleftrightarrow} & \text{Unitary problem} \\
\Uparrow_{\text{proposition 6.5.3}} & & \Updownarrow_{\text{theorem 8.3.14}} \\
\text{Centered Lagrangian problem (section 6.2)} & \overset{\text{proposition 6.5.7}}{\Longleftrightarrow} & \text{Lagrangian problem}
\end{array}$$

Remark 8.3.17. Observe that the definition of decomposable representations, as well as the characterization that we obtained in chapter 6 depended on the facts that the involution $\tau^- : u \mapsto \tau(u^{-1})$ had a connected fixed-point set and left a *maximal* torus of U pointwise fixed. We used these assumptions to guarantee that every symmetric element $w \in U$ (that is, an element $w \in U$ satisfying $\tau^-(w) = w$) could be written $\tau^-(u)u$ for some $u \in U$, that β indeed sent a conjugacy class $\mathcal{C}$ of U into itself, and that the map $Fix(\beta) \cap \mu^{-1}(\{1\}) \to Fix(\hat{\beta})$ was surjective. All these facts were used to prove the characterization of decomposable representations obtained in corollary 6.6.5.

Chapter 9

The Lagrangian nature of decomposable representations

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This chapter concludes this thesis work. We will carefully review the results announced in chapter 1 and see how the theorems that we have proved in the course of this work provide an answer to the problem of finding a Lagrangian submanifold of the moduli space $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi_1(S^2\backslash\{s_1, \ldots, s_l\}), U)/U$. We shall also come back upon the infinitesimal formulation of our problem and see how the approach that we adopted in section 6.1 is justified a posteriori. Finally, we shall try and give directions for future work on these questions.

9.1 Decomposable representations in the moduli space

As announced in the introduction, the purpose of this thesis was to give an example of a Lagrangian submanifold in the moduli space

$$\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi_1(S^2\backslash\{s_1, \ldots, s_l\}), U)/U$$

where U is an arbitrary compact connected Lie group. To do so, the path we followed consisted in :

1. introducing a notion of decomposable representation.
2. characterizing these representations as the elements of the fixed-point set of an involution defined on $\mathcal{M}_{\mathcal{C}}$.
3. showing that this involution is anti-symplectic and that its fixed-point set is non-empty (being therefore a Lagrangian submanifold of $\mathcal{M}_{\mathcal{C}}$).

The definition of a decomposable representation we chose to work with was the following one :

Definition (Decomposable representations of $\pi_1(S^2\backslash\{s_1, \ldots, s_l\})$). Let (U, τ) be a Lie group endowed with an involutive automorphism τ. A representation $(u_1, \ldots, u_l)$ of $\pi = \pi_1(S^2\backslash\{s_1, \ldots, s_l\})$ into U is called *decomposable* if there exist l elements $w_1, \ldots, w_l \in U$ satisfying :

(i) $\tau(w_j) = w_j^{-1}$ for all j (each w_j is a *symmetric* element of U with respect to τ).

(ii) $u_1 = w_1 w_2^{-1}$, $u_2 = w_2 w_3^{-1}$, $\ldots$, $u_l = w_l w_1^{-1}$.

A representation will be called σ_0-*decomposable* if it is decomposable with $w_1 = 1$.

We refer to chapter 5 to see how this definition was obtained. For this definition to make sense, we had to endow the compact connected Lie group U with an involutive automorphism τ. It is a consequence of the existence of real forms of $U^{\mathbb{C}}$ that such an automorphism always exists. For the sake of simplicity, we assumed that the fixed-point set of the involution τ^- (defined by $\tau^-(u) := \tau(u^{-1})$ for any $u \in U$) was connected. This assumption is for instance satisfied by the involution $\tau(u) = \overline{u}$ on $U = U(n)$ or $U = SU(n)$. If we drop this assumption, the correct definition of a decomposable representation should be to ask that the w_j lie in the set $\{\tau^-(u)u \,:\, u \in U\} \subset Fix(\tau^-)$, which coincides with $Fix(\tau^-)$ when the latter is connected (see proposition 3.1.2).

From then on, the idea that such decomposable representations should be characterized as elements of the fixed-point set of an anti-symplectic involution was suggested by the infinitesimal formulation of our problem, as explained in section 6.1. Chapter 6 was devoted to obtaining this involution. The choice of the space we worked with for that matter was dictated by the description of the representation space $\mathcal{M}_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\pi_1(S^2 \backslash \{s_1, \ldots, s_l\}), U)/U$ as a quasi-Hamiltonian quotient :

$$\mathcal{M}_{\mathcal{C}} = \mu^{-1}(\{1\})/U$$

where μ is the momentum map

$$\begin{array}{rccc} \mu \;:\; & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & U \\ & (u_1, \ldots, u_l) & \longmapsto & u_1 \ldots u_l \end{array}$$

defining the quasi-Hamiltonian structure on the product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ of l conjugacy classes of U. We then defined the following involution on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$:

$$\begin{array}{rccc} \beta \;:\; & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \\ & (u_1, \ldots, u_l) & \longmapsto & \big(\tau^-(u_l) \ldots \tau^-(u_2)\tau^-(u_1)\tau(u_2) \ldots \tau(u_l), \ \ldots, \tau^-(u_l)\tau^-(u_{l-1})\tau(u_l), \tau^-(u_l)\big) \end{array}$$

and we proved the following result (theorem 6.6.2 and corollary 6.6.5) :

Theorem 1 (Characterization of decomposable representations). *A representation $u = (u_1, \ldots, u_l) \in \mu^{-1}(\{1\})$ is σ_0-decomposable if and only if $\beta(u) = u$. It is decomposable if and only if $\beta(u) \sim u$ as representations of π.*

To prove this result, we had to make an additional assumption on the involution τ, namely that the associated involution τ^- left a *maximal* torus of the compact connected Lie group U pointwise fixed. This assumption is in particular satisfied by the involution $\tau(u) = \overline{u}$ on $U = U(n)$ and $U = SU(n)$, and we saw in proposition 3.2.2 that such an involution always exists if U is a compact connected simply connected Lie group (as a matter of fact, such an involution exists on any compact connected semisimple Lie group, not just those which are simply connected, as explained in [Loo69b] pp. 78-81, but the assumption of simple connectedness will be used again shortly, this time in a crucial way, so we make it right away).

At this point, we are still working on the quasi-Hamiltonian space $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, upon which β is defined, and not on the quasi-Hamiltonian quotient

$$\mathcal{M}_{\mathcal{C}} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_l // U := \mu^{-1}(\{1\})/U$$

But we saw in chapter 7 that the involution β was compatible with the diagonal action of U on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$ and with the momentum map μ in the sense of definition 7.2.1, which proved that it induced an involution $\hat{\beta}$ on $\mathcal{M}_{\mathcal{C}}$ defined by :

$$\begin{array}{rccc} \hat{\beta} \;:\; & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l // U & \longrightarrow & \mathcal{C}_1 \times \cdots \times \mathcal{C}_l // U \\ & \big[(u_1, \ldots, u_l)\big] & \longmapsto & \big[\beta(u_1, \ldots, u_l)\big] \end{array}$$

Furthermore, we observed in proposition 5.2.2 that a representation $(u_1, \ldots, u_l)$ of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ is decomposable if and only if the representation $\varphi.(u_1, \ldots, u_l)$ is decomposable for any $\varphi \in U$, so that we may call an equivalence class $[u] = [(u_1, \ldots, u_l)] \in \mathcal{M}_{\mathcal{C}}$ decomposable if any of its representatives is decomposable. We then obtain the following result as a corollary of theorem 1 :

Corollary 2. *$[u] \in \mathcal{M}_{\mathcal{C}}$ is decomposable if and only if $\hat{\beta}([u]) = [u]$.*

Proof. Assume first that $[u] \in \mathcal{M}_{\mathcal{C}} = \mu^{-1}(\{1\})$ is decomposable. Then by proposition 5.2.2, $u \in \mu^{-1}(\{1\})$ is decomposable, so that by theorem 1, one has $\beta(u) \sim u$, hence by definition of $\hat{\beta}$: $\hat{\beta}([u]) = [u]$. The argument reverses to prove the converse. □

We then notice the one truly remarkable feature of the involution β :

Proposition 3. *$\beta^*\omega = -\omega$ on $\mathcal{C}_1 \times \cdots \times \mathcal{C}_l$, so that $\hat{\beta}$ is anti-symplectic on $\mathcal{M}_{\mathcal{C}}$.*

Although this was expected from the infinitesimal formulation, the analysis drawn in chapter 7 shows that the fact that $\beta^*\omega = -\omega$ is not obvious (see section 7.3 and proposition 7.3.4). The fact that it implies that $\hat{\beta}$ is anti-symplectic on $\mathcal{M}_{\mathcal{C}}$ is then a consequence of the construction of the symplectic structure of $\mathcal{M}_{\mathcal{C}}$ (see proposition 7.2.2).

We are now almost in a position to apply lemma 7.1.1. To do so, we still have to prove that $Fix(\hat{\beta}) \neq \emptyset$, or equivalently that $Fix(\beta) \cap \mu^{-1}(\{1\}) \neq \emptyset$. To prove this, we chose to study the whole of $\mu(Fix(\beta))$. This choice was motivated by the existence of convexity results for fixed-point sets of anti-symplectic involutions defined on usual Hamiltonian spaces, and by the use of such convexity results to prove a result similar to ours in [AMW01]. For convexity to make sense in a Lie group, we had to assume that the compact connected Lie group U was in addition simply connected (see subsection 8.2.1). We then obtained the following result (theorem 8.3.9) :

Theorem 4 (A real convexity theorem for group-valued momentum maps). *Let $(U, (.\,|\,.), \tau)$ be a compact connected simply connected Lie group endowed with an involutive automorphism τ such that the involution $\tau^- : u \mapsto \tau(u^{-1})$ leaves a maximal torus T of U pointwise fixed and let $\overline{\mathcal{W}} \subset \mathfrak{t} = Lie(T)$ be a closed Weyl alcove. Let $(M, \omega, \mu : M \to U)$ be a connected quasi-Hamiltonian U-space with proper momentum map $\mu : M \to U$ and let $\beta : M \to M$ be an involution on M such that :*

(i) $\beta^*\omega = -\omega$

(ii) $\beta(u.x) = \tau(u).\beta(x)$ *for all* $x \in M$ *and all* $u \in U$

(iii) $\mu \circ \beta = \tau^- \circ \mu$

(iv) $M^\beta := Fix(\beta) \neq \emptyset$

Then :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \mu(M) \cap \exp(\overline{\mathcal{W}})$$

In particular, $\mu(M^\beta) \cap \exp(\overline{\mathcal{W}})$ is a convex subpolytope of $\exp(\overline{\mathcal{W}}) \simeq \overline{\mathcal{W}} \subset \mathfrak{t}$, equal to the whole momentum polytope $\mu(M) \cap \exp(\overline{\mathcal{W}})$.

Observe that the assumption that τ^- leaves a maximal torus of U pointwise fixed is one we already made to obtain theorem 1. We then have the following corollary :

Corollary 5 (Existence of fixed points for $\hat{\beta}$). *If $\mu^{-1}(\{1\}) \neq \emptyset$ then $\mu^{-1}(\{1\}) \cap Fix(\beta) \neq \emptyset$.*

As explained earlier (see theorem 8.3.14 for details), this proves the existence of decomposable representations for any compact connected simply connected Lie group U and any choice of l conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_l$ satisfying the assumption :

$$\left\{(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \ldots u_l = 1\right\} \neq \emptyset$$

We are now in a position to prove the following result :

Theorem 6. *Let U be a compact connected simply connected Lie group endowed with an involutive automorphism τ satisfying :*

(i) there exists a maximal torus of U fixed pointwise by the involution $\tau^-(u) := \tau(u^{-1})$.

(ii) the fixed-point set $Fix(\tau^-)$ of the involution τ^- is connected.

Then the set of equivalence classes of decomposable representations of the group $\pi := \pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into U is a Lagrangian submanifold of the stratified symplectic space $\mathcal{M}_\mathcal{C} := \mathrm{Hom}_\mathcal{C}(\pi, U)/U$ (in particular it is non-empty), equal to the fixed-point set of an anti-symplectic involution $\hat{\beta}$ defined on $\mathcal{M}_\mathcal{C}$.

Proof. Corollary 2 shows that the set of equivalence classes of decomposable representations of π into U is exactly $Fix(\hat{\beta})$, and we know from proposition 3 that $\hat{\beta}$ is anti-symplectic. Corollary 5 then shows that $Fix(\hat{\beta}) \neq \emptyset$. Consequently, lemma 7.1.1 applies, showing that the set of equivalence classes of decomposable representations of π into U is a Lagrangian submanifold of the moduli space $\mathcal{M}_\mathcal{C}$. □

Therefore, we have obtained a Lagrangian submanifold of the moduli space $\mathcal{M}_\mathcal{C}$ for a certain class of compact connected Lie groups, keeping in mind the example of the Lie group $U = SU(n)$ troughout this work. As we mentioned a few times earlier, theorem 6 remains true for the compact connected non simply connected Lie group $U = U(n)$ endowed with the involution $\tau(u) = \overline{u}$.

Remark 9.1.1 (The case where $U = U(n)$). As a matter of fact, theorem 1 is true for $U = U(n)$, as we have seen in chapter 6, and the only thing we cannot pove with our methods is the existence of decomposable representations of π into $U(n)$. But this was proved by Falbel and Wentworth in [FW], so that theorem 6 is indeed true for $U = U(n)$.

9.2 Back to the infinitesimal picture

In this section, we briefly indicate a way to see why the infinitesimal formulation of the Lagrangian problem that we gave in section 6.1 was indeed a good one, in the sense that *for sufficiently small initial data $(\lambda_j)_{1 \leq j \leq l}$ the Lagrangian problem admits a solution if and only if its infinitesimal counterpart has a solution.* Recall that we started off with l elements $\lambda_j \in \mathbb{R}^n$, $1 \leq j \leq l$. For such initial data (λ_j), we formulated the following two problems :

- the Lagrangian problem : do there exist l Lagrangian subspaces $L_1, \ldots, L_l$ of $\mathbb{C}^n$ satisfying the condition $\mathrm{Spec}\,(\sigma_{L_j}\sigma_{L_{j+1}}) = \exp(i\lambda_j)$ for all j, where σ_{L_j} is the Lagrangian involution associated to L_j and where $L_{l+1} = L_1$?
- the symmetric problem, which we reached heuristically by trying to find an infinitesimal version of the Lagrangian problem : do there exist l real symmetric matrices $S_1, \ldots, S_l$ satisfying $\mathrm{Spec}\, S_j = \lambda_j$ and $S_1 + \cdots + S_l = 0$?

We will now show that for sufficiently small λ_j the Lagrangian problem has a solution if and only if the symmetric problem has a solution. This result is already mentioned in [Kly00], but our symplectic approach rests on the following result proved by Jeffrey in [Jef94] (theorem 6.6) :

Theorem 9.2.1. [Jef94] *For any $\lambda \in \mathbb{R}^n$, denote by $\mathcal{C}_\lambda$ the conjugacy class of the unitary matrix $\exp(i\lambda) \in U(n)$, and denote by $\mathcal{H}_\lambda$ the (co-adjoint) orbit of the Hermitian matrix $\mathrm{diag}(\lambda) \in \mathcal{H}(n)$ under $U(n)$. Then there exists an open neighbourhood $\mathcal{V}$ of 0 in $\mathbb{R}^n$ such that if $\lambda_1, \ldots, \lambda_l \in \mathcal{V}$ then the moduli spaces*

$$\mathcal{M}_\mathcal{C} := \big\{(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \ldots u_l = 1\big\}/U(n)$$

and

$$\mathcal{M}_\mathcal{H} := \big\{(H_1, \ldots, H_l) \in \mathcal{H}_{\lambda_1} \times \cdots \times \mathcal{H}_{\lambda_l} \mid H_1 + \cdots + H_l = 0\big\}/U(n)$$

are symplectomorphic. In particular, one of them is non-empty if and only if the other one is.

Corollary 9.2.2. *If $\lambda_1, \ldots, \lambda_l \in \mathcal{V}$ then the following two conditions are equivalent :*

(i) there exist l Lagrangian subspaces $L_1, \ldots, L_l$ of $\mathbb{C}^n$ satisfying the condition $\sigma_{L_j}\sigma_{L_{j+1}} \in \mathcal{C}_j$ for all j (that is, $\operatorname{Spec}(\sigma_{L_j}\sigma_{L_{j+1}}) = \exp(i\lambda_j)$).

(ii) there exist l real symmetric matrices $S_1, \ldots, S_l$ such that $S_j \in \mathcal{H}_{\lambda_j}$ (that is, $\operatorname{Spec} S_j = \lambda_j$) and $S_1 + \cdots + S_l = 0$.

Proof of the corollary. Condition (i) is by definition equivalent to saying that there exist decomposable representations of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ into $U(n)$, which is equivalent, by theorem 8.3.14, to saying that the moduli space

$$\mathcal{M}_{\mathcal{C}} = \big\{(u_1, \ldots, u_l) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_l \mid u_1 \ldots u_l = \big\}/U(n)$$

is non-empty (see remark 8.3.15). In turn, by theorem 9.2.1, this is equivalent, as the λ_j are assumed to lie in $\mathcal{V}$, to saying that the moduli space

$$\mathcal{M}_{\mathcal{H}} = \big\{(H_1, \ldots, H_l) \in \mathcal{H}_{\lambda_1} \times \cdots \times \mathcal{H}_{\lambda_l} \mid H_1 + \cdots + H_l = 0\big\}/U(n)$$

is non-empty. Then by proposition 6.3.1, this is equivalent to saying that there exist l real symmetric matrices $S_1, \ldots, S_l$ such that $\operatorname{Spec} S_j = \lambda_j$ and $S_1 + \cdots + S_l = 0$, which proves the corollary. □

This result can be made more precise by showing that the symplectomorphism between the above two moduli spaces actually carries the Lagrangian submanifold

$$\big\{\text{equivalence classes of decomposable representations of } \pi_1(S^2 \backslash \{s_1, \ldots, s_l\}) \text{ into } U(n)\big\} \subset \mathcal{M}_{\mathcal{C}}$$

onto the Lagrangian submanifold

$$\big\{(S_1, \ldots, S_l) \in \mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_l} \mid S_1 + \cdots + S_l = 0\big\}/O(n) \subset \mathcal{M}_{\mathcal{H}}$$

9.3 Directions for future work

We would like to conclude this thesis by an attempt at giving some possible directions for future work on the matters we have been dealing with here. We begin with questions for which obtaining an answer should simply be a matter of time and care, and then move on to questions that we deem to be a bit harder :

- what happens if one drops the assumption that $Fix(\tau^-)$ is connected ? Then we already mentioned in remark 5.2.3 that the definition of a decomposable representation should be slightly modified. As a matter of fact, following [Fot], we see that there is an involution τ_s of U associated to each connected component of $Fix(\tau^-)$, hence a notion of decomposable representations for each of these connected components, and a corresponding involution β_s characterizing these representations. The rest needs further investigation (particularly what happens when one descends to the moduli space, where all the β_s should induce the same involution $\hat{\beta}$).

- how can one interpret the results contained in this thesis in terms of the various equivalent formulations of the unitary problem ? For instance, what does the notion of decomposable representation become when one considers polygons on S^3 ? This corresponds to the case where $U = SU(2)$ and it was suggested to us by Philip Foth that polygons fixed by β should be those lying in an equatorial $S^2 \subset S^3$ (see [FH]). Or again, how does the notion of decomposable representation carry over to the vector bundle setting ? etc.

- is it possible to adapt the proof of our real convexity result 8.3.9 to prove conjecture 8.3.12 ? In the case where the symmetric pair (U, τ) is not of maximal rank, we think (in analogy with the O'Shea-Sjamaar theorem in the usual Hamiltonian setting, see [OS00]) that one has :

$$\mu(M^\beta) \cap \exp(\overline{\mathcal{W}}) = \big(\mu(M) \cap \exp(\overline{\mathcal{W}})\big) \cap U^{\tau^-}$$

but we were unable to prove this result following our approach of reducing the action of U on M to that of a torus $T \subset U$ on a symplectic cross-section $N \subset M$.

- what can one say about the notion of decomposable representation of $\pi_1(S^2 \backslash \{s_1, \ldots, s_l\})$ if we consider non-compact groups like $U = SL(2, \mathbb{R})$ or $U = SU(2,1)$? Limiting ourselves to semisimple groups, we still have quasi-Hamiltonian structures on conjugacy classes of such groups (see chapter 4). For $U = SU(n,1)$ $(n \geq 1)$, the notion of decomposable representation carries over immediately, and so should the characterization in terms of β, even though one should pay attention to the fact that not all elements in $SU(n,1)$ are diagonalizable. A much more serious problem is proving the existence of decomposable representations in this case since, if we want to follow the approach adopted in this thesis, we need a (real) convexity result for certain non-compact group actions, along the lines of [Wei01].

- finally, what would be an appropriate notion of decomposable representation for instance for the fundamental group of the punctured torus ? What about other surfaces ? Work in this direction can be found in [Wil], where representations of $< a, b, c \mid [a, b]c = 1 >$ into $SU(2,1)$ are discussed.

We hope to come back to these questions in a not-so-distant future. Merci d'avoir lu ce travail.

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