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Rachid Lebbar

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THESE

présentée à

l'Université Scientifique et Médicale de Grenoble

et à

l'Institut National Polytechnique de Grenoble

pour obtenir le grade de

DOCTEUR DE 3ème CYCLE

Mathématiques Appliquées

(Analyse Numérique)

par

Rachid LEBBAR

SUR LES PROPRIETES DE SUPERCONVERGENCE DES SOLUTIONS APPROCHEES DE CERTAINES EQUATIONS INTEGRALES ET DIFFERENCIELLES.

Thèse soutenue le 29 septembre 1981 devant la Commission d'Examen :

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à mes parents

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INTRODUCTION

Dans sa conférence sur "L'avenir des Mathématiques", au Congrès de Rome en 1908, POINCARE a remarqué que l'on devait pouvoir appliquer la théorie des équations intégrales linéaires à la théorie des équations différentielles ordinaires non linéaires. Un premier pas pour réaliser l'idée de Poincaré a été fait par Fredholm dans une Note dans les Comptes Rendus 23 aôut 1920. FREDHOLM arrive à une équation intégrale linéaire mais il constate en même temps que l'état actuel de la théorie des équations intégrales ne paraît cependant pas permettre une étude suffisamment approfondie de l'équation obtenue...

A Stockolm, le 18 mars 1932

TORSTEN CARLEMAN

INTRODUCTION

Cette thèse se compose de deux parties : une partie traitant de problèmes intégraux et une autre de problèmes différentiels.

A) T est un opérateur intégral linéaire compact

Il s'agit de trouver une approximation numérique du problème :

(1) $Tx = zx + f$, $x \in X$ (X : espace de Banach)

par des méthodes de projections.

On supposera que $0 \neq z$ appartient à l'ensemble résolvant de T ce qui assurera l'existence et l'unicité de x solution du problème . L'approximation x_n est solution de :

(2) $\pi_n (Tx_n - z\, x_n - f) = 0$, $x_n \in X_n$ (X_n : sous espace de dimension finie de X);

où π_n est une projection de X dans X_n , ponctuellement convergent

T étant compact (2) admet une solution unique pour n assez grand.

En 1976 I.H.SLOAN a considéré l'itéré du point fixe $\tilde{x}_n$ de l'équation $x = \frac{1}{z} Tx + \frac{1}{z} f$ à partir de l'approximation x_n :

$$\tilde{x}_n = \frac{1}{z} Tx_n + \frac{1}{z} f \text{ , et a démontré que}$$

$||\tilde{x}_n - x|| \leq \xi_n ||x_n - x||$ où $\xi_n \to 0$ quand $n \to \infty$.

En 1978 F.CHATELIN a introduit la suite des itérés des éléments propres $\phi_n^{(k)}$ et $\lambda_n^{(k)}$ associés à une valeur propre simple, dont elle a démontré la convergence.

I.G.GRAHAM en collaboration avec I.H.SLOAN a donné en 1979 une caractérisation de la compacité des opérateurs intégraux à noyau ; puis en 1980 il a soutenu une thèse sur la superconvergence de $\tilde{x}_n$, associé à un opérateur intégral dont le noyau est un noyau de convolution faiblement singulier.

Cependant en 1979 G.A.CHANDLER (sous la direction d'Anderson) démontre dans sa thèse que si $X_n = P_{r,\Delta}$ où Δ est une partition quasiuniforme et si π_n est la projection orthogonale définie de X dans X_n alors :

$$||x-x_n|| = O(h^{r+1})$$

$$||x-\tilde{x}_n|| = O(h^{2r+2}) \text{ Si le noyau est lisse.}$$

$$|(x-\tilde{x}_n)(t_i)| = O(h^{2r+2}) \text{ Si le noyau est de Green.}$$

(h étant le plus grand pas de la maille).

B) T est un opérateur différentiel d'ordre p

Il s'agit de chercher une approximation numérique du problème :

$$(3)\quad \begin{cases} (Ty)(t) := y^{(p)}(t) - \sum_{i=0}^{p-1} a_i(t)\, y^{(i)}(t) = f(t),\ t\in[0,1] \ ; \\ \text{sous p conditions aux limites linéairement indépendantes} \\ \text{sur l'espace des polynômes de degré p-1.} \end{cases}$$

Sous les hypothèses que :

i) le problème (3) admet une solution unique

ii) les a_i sont de classe $C^{r+1}_{[0,1]}$

(mais cependant sans aucune hypothèse sur la partition $\Delta := (ti)_{i=1}^{n}$) ;

C. de BOOR et B.SWARTZ ont démontré en 1973 que la solution y_n du problème (4) vérifie :

$$||y-y_n|| = O(h^{r+1+p})$$

$$|(y-y_n)(t_i)| = O(h^{2r+2}) \qquad 1\le i\le n$$

(4) étant :

$$(4)\quad \begin{cases} (Ty_n)(\tau_i^{\,j}) = f(\tau_i^{\,j}) \quad 1\le i\le n \quad , 1\le j\le r+1 \\ y_n \text{ vérifiant les p conditions aux limites, et appartenant à} \\ \text{l'espace } P^{p-1}_{p+r,\Delta} \text{ des polynomes par morceaux de degré} \le p+r \\ \text{qui se raccordent aux noeuds de la partition } \Delta \text{ jusqu'à l'ordre p-1.} \end{cases}$$

où les τ_i^j $1 \leq j \leq r+1$, sont les (r+1) points de Gauss du sous intervalle $[t_{i-1}, t_i]$ de $[0,1]$, $1 \leq i \leq n$.

Ils ont démontré ensuite en 1979 que sous les mêmes hypothèses qu'en 1973 et toujours en interpolant aux points de Gauss, la moyenne arithmétique $\hat{\lambda}_n$, des m valeurs propres qui approximent la valeur propre λ de multiplicité algébrique m finie, est elle aussi en $O(h^{2r+2})$ cad :

$$|\lambda - \hat{\lambda}_n| = O(h^{2r+2})$$

Enfin en 1980, ils ont démontré que leurs résultats de 1973 et 1979 ne sont pas valables uniquement pour l'interpolation aux points de Gauss : collocation aux points de Gauss, mais pour toute une classe de méthodes dérivant de projections qu'ils ont appelées les "superprojecteurs".

Les "superprojecteurs" sont au fait des projections "mixtes" cad elles sont de la projection orthogonale sur une partie de la partiti et de la collocation sur l'autre. Ainsi, ils semblent avoir appliqué la méthode de collocation à la projection orthogonale Il en va tout autrement dans cette thèse, surtout dans le 2eme article où le lien entre l'interpolation aux points de Gauss et la projection orthogonale est clairement établi.

Le tableau qui suit est un résumé des parties A) et B) de l'introduction. Nous y avons adopté la notation :

$\boxed{\alpha} \rightarrow \boxed{\beta}$ si les idées de $\boxed{\beta}$ dérivent ou s'inspirent de celles de

De plus celui-ci présente également les quatre articles qui constituent cette thèse, en précisant les différentes interactions pouvant exister entre chacun des articles ou des thèses jusqu'ici présentées. Ces quatre articles sont rédigés indépendamment des uns des autres , chacun possédant une introduction propre et une bibliographie propre.

Cette thèse enfin se termine par une expérimentation numérique de quelques exemples de superconvergence.

T operateur intégral à noyau k

Sloan 1976

H_1 T compact

introduction de $\tilde{x}_n$

$\tilde{x}_n := \frac{1}{z} Tx_n + f$

$||\tilde{x}_n - x|| \leq \varepsilon_n ||x_n - n||$

$\varepsilon_n \to 0$ qd $n \to \infty$

Chatelin 1978

$H_2 \begin{cases} T \text{ compact, la valeur propre} \\ \lambda \text{ est simple} \end{cases}$

introduction de la suite d'itérés d'éléments propres $\phi_n^{(k)}$, $\lambda_n^{(k)}$ qui converge vers ϕ, λ.

Graham - Sloan 1979

Conditions nécessaires et suffisantes de compacité d'un opérateur intégral à noyau

Chandler 1979

$H_2 \begin{cases} T \text{ compact} \\ \Delta \text{ quasi uniforme} \\ \pi_n \text{ projection orthogonale} \end{cases}$

$||x_n - x|| = O(h^{r+1})$

$||\tilde{x}_n - x|| = O(h^{2r+2})$ si k est lisse

$|(\tilde{x}_n - x)(ti)| = O(h^{2r+2})$ si k est de Gre

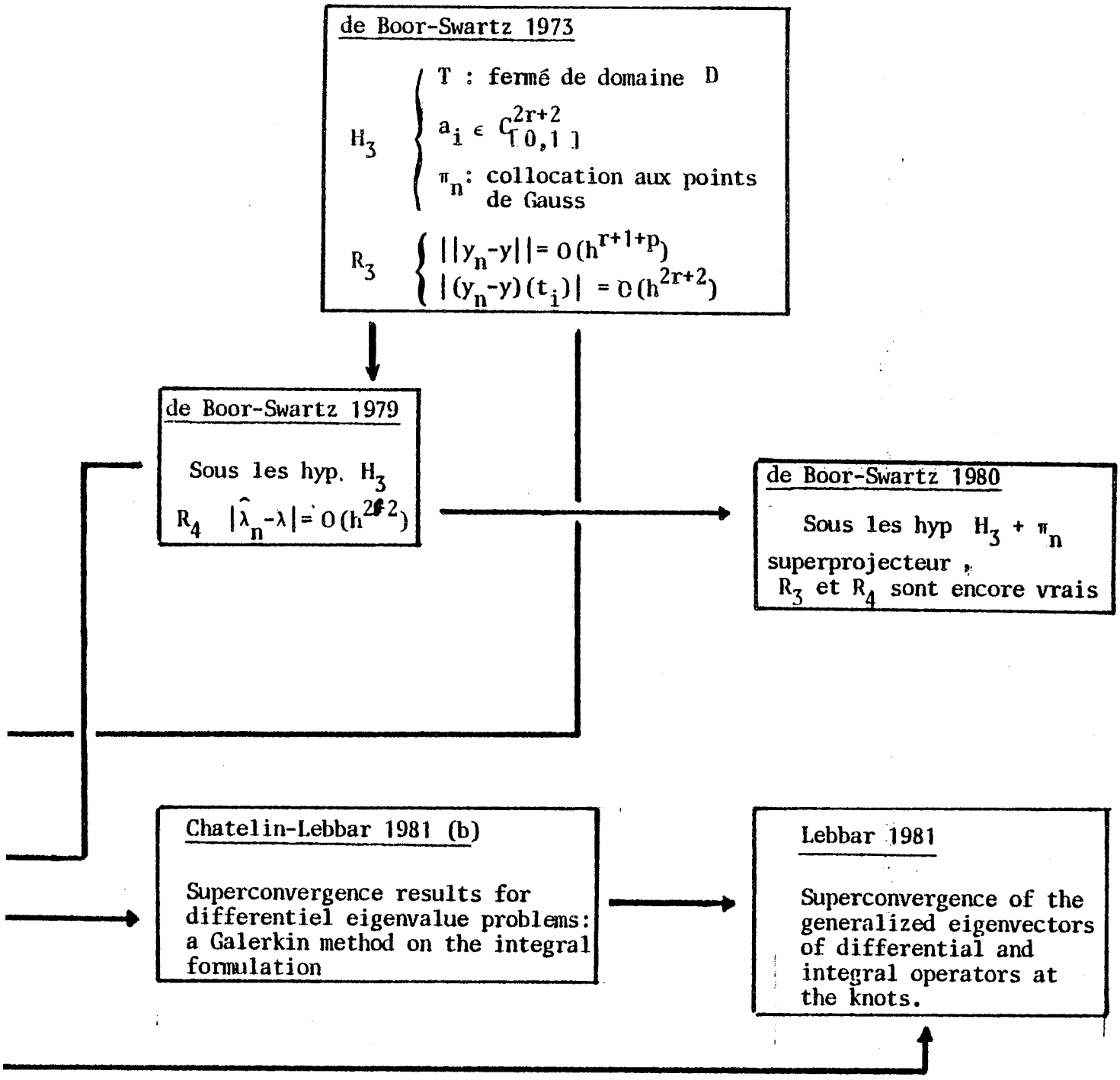
Chatelin - Lebbar 1980

The iterated projection solution for the Fredholm integral equation of second kind

Chatelin - Lebbar 1981(a)

Superconvergence results for the iterated projection method applied to a second kind Fredholm integral equations and eigenvalue problem.

T opérateur differentiel d'ordre p



CHAPITRE 1

THE ITERATED PROJECTION SOLUTION FOR THE FREDHOLM INTEGRAL EQUATION OF SECOND KIND

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THE ITERATED PROJECTION SOLUTION FOR THE FREDHOLM INTEGRAL EQUATION OF SECOND KIND

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Abstract

We are concerned with the solution of the second kind Fredholm equation (and eigenvalue problem) by a projection method, where the projection is either an orthogonal projection on a set of piecewise polynomials or an interpolatory projection at the Gauss points of subintervals.

We study these cases of superconvergence of the Sloan iterated solution: global superconvergence for a smooth kernel, and superconvergence at the partition points for a kernel of "Green's function" type. The mathematical analysis applies for the solution of the inhomogeneous equation as well as for an eigenvector.

1. Introduction

We consider some projection methods for the solution of second kind integral equations of the form

$$(Tx)(s) - zx(s) = f(s), \qquad 0 < s < 1, \tag{1}$$

where T is the operator defined by

$$x(s) \mapsto \int_0^1 k(s, t)x(t)\, dt, \qquad 0 < s < 1.$$

Along with (1), we consider the eigenvalue problem

$$(T\phi)(s) = \lambda\phi(s), \qquad 0 < s < 1, \phi \neq 0. \tag{2}$$

(1) and (2) are regarded as equations in an appropriate subspace X of the complex Banach space $L^\infty(0, 1)$ with the norm $\|\cdot\|_\infty$. T is supposed to be

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compact and $z \in \rho(T)$, the resolvent set of T, so that $(T - z)^{-1}$ is bounded with domain X. Let X_n be a finite dimensional subspace of X and let Π_n be a projection onto X_n. Then the projection method consists in approximating (1) and (2) respectively by

$$(\Pi_n T - z)x_n = \Pi_n f, \qquad x_n \in X_n, \tag{3}$$

$$\Pi_n T\phi_n = \lambda_n \phi_n, \qquad 0 \neq \phi_n \in X_n, \tag{4}$$

where x_n (resp. ϕ_n) is the projection solution (resp. eigenvector), corresponding to the approximation $T_n^P = \Pi_n T$ of T (P for projection).

Given a projection $\Delta = \{t_i\}_0^n$ of $[0, 1]$, $t_0 = 0$, $t_n = 1$, let X_n be a space S_Δ of piecewise polynomials of degree $\leqslant r$ on each subinterval $\Delta_i = [t_{i-1}, t_i]$, $i = 1, \ldots, n$. We set $h = \max_{1 \leqslant i \leqslant n}(t_i - t_{i-1})$. We shall consider two types of projection methods:

(a) Π_n is the orthogonal projection (in $L^2(0, 1)$) on S_Δ,

(b) Π_n is an interpolatory projection defined so that $\Pi_n x$ is the piecewise polynomial of degree $\leqslant r$ which interpolates x at $r + 1$ points $\{\tau_j^i\}_{j=1}^{r+1}$, on each Δ_i, $i = 1, \ldots, n$.

Case (a) corresponds to a *Galerkin method*, and case (b) to a *collocation method* at the collocation points $\{\tau_j^i\}$.

If $z \neq 0$ (resp. $\lambda_n \neq 0$) we consider the iterated projection solution $\tilde{x}_n$ (resp. eigenvector $\tilde{\phi}_n$) introduced by Sloan [13], [14] and given by the formulae:

$$\tilde{x}_n = \frac{1}{z}(Tx_n - f), \qquad \tilde{\phi}_n = \frac{1}{\lambda_n} T\phi_n,$$

where $\tilde{x}_n$ and $\tilde{\phi}_n$ are solutions of the equations

$$(T\Pi_n - z)\tilde{x}_n = f, \tag{5}$$

and

$$T\Pi_n\tilde{\phi}_n = \lambda_n\tilde{\phi}_n, \tag{6}$$

corresponding to the approximation $T_n^S = T\Pi_n$ of T (S for Sloan). Now $\Pi_n\tilde{x}_n = x_n$ and $\Pi_n\tilde{\phi}_n = \phi_n$, so that in case (b), the iterated solutions and the solutions themselves agree at the collocation points.

If k and f are smooth enough, it is known that $\|x_n - x\|_\infty = O(h^{r+1})$, while $\|\tilde{x}_n - x\|_\infty = O(h^{2r+2})$ for case (b) for example, provided that the $\{\tau_j^i\}$ are the $r + 1$ Gauss points on Δ_i, $i = 1, \ldots, n$. The optimal rate of convergence, relative to S_Δ, which is $\inf_{y \in S_\Delta}\|x - y\|_\infty = O(h^{r+1})$, is then overshot by $\tilde{x}_n \notin S_\Delta$, when k and f are smooth. Such fast convergence is often called *superconvergence*.

When k is the Green's function of an ordinary differential equation (o.d.e.) of order p with smooth coefficients, $\tilde{x}_n$ is still superconvergent at the partition points $\{t_i\}_0^n$, but *not* globally: the global rate of convergence is now $O(h^{r+1+p})$.

Similar results hold for $\tilde{\phi}_n$. This problem is studied for the equation (1) and the Galerkin method in Chandler's thesis [6]. The collocation method for a non linear o.d.e. has been looked at by de Boor-Swartz (see [1] for the solution of (1), and [2], [3] for the linear eigenvalue problem (2)), where T is the associated differential operator. In de Boor-Swartz [4] the "essential" least squares method (or local moment method) for an o.d.e. is also studied.

We present in this paper an analysis of the convergence rates which is a blend of the techniques of Chandler and of de Boor-Swartz. It applies for the iterated solution $\tilde{x}_n$ as well as for the iterated eigenvector $\tilde{\phi}_n$ (the result seems to be new for the *eigenvector* in the most general case). It is . ased on a study of the error at the point t of [0, 1] in terms of the scalar product $\langle l_t, (1 - \Pi_n)\tilde{x}_n\rangle$ (resp. $\langle l_t', (1 - \Pi_n)\tilde{\phi}_n\rangle$) where l_t (resp. l_t') is a function having the same smoothness properties as $k_t(\cdot) := k(t, \cdot)$, and where $\langle f, g\rangle = \int_0^1 f\bar{g}$.

In case (a), we use the orthogonality of Π_n:

$$\langle l_t, (1 - \Pi_n)\tilde{x}_n\rangle = \langle (1 - \Pi_n)l_t, (1 - \Pi_n)\tilde{x}_n\rangle.$$

In case (b) we use firstly that the function $(1 - \Pi_n)\tilde{x}_n$ vanishes at the collocation points τ_j^i, and secondly that the $\{\tau_j^i\}$ being the $r + 1$ Gauss points in Δ_i, then $\int_{\Delta_i} p(s)\Pi_{j=1}^{r+1}(s - \tau_j^i)\, ds = 0$ for all polynomials p of degree $< r$.

The superconvergence in case (a) is proved under the assumption that Δ is quasi-uniform. In case (b), Δ is arbitrary but more smoothness properties are required for k and f.

2. The setting of the problem

2.1. Piecewise continuous functions

Let be given $\Delta = \{t_i\}_0^n$, a strict partition of [0, 1], $0 = t_0 < t_1 < \cdots < t_n = 1$. It is quasi-uniform if there exists $\sigma > 0$: $\max(t_i - t_{i-1})/\min(t_i - t_{i-1}) < \sigma$ for $n = 1, 2, \ldots$. Then $nh < \sigma$. $\Delta_i := [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. We define $C_\Delta := \Pi_{i=1}^n C_{(\Delta_i)}$: $f \in C_\Delta$ consists of n components $f_i \in C_{(\Delta_i)}$, f is a piecewise continuous function having (possibly) different left and right values at the partition points t_i. With the norm $\|\cdot\|_\Delta$ defined by $\|f\|_\Delta = \max_{i=1,\ldots,n}\|f_i\|_\infty$, C_Δ is a Banach space. $C_\Delta \subset L^\infty(0, 1)$ by $\|f\|_\Delta < \|f\|_\infty$ and if f is continuous on [0, 1], then $\|f\|_\infty = \|f\|_\Delta$. We define, more generally, C_Δ^l for positive integer l by $C_\Delta^l = \Pi_{i=1}^n C_{(\Delta_i)}^l$ where $f_i \in C_{(\Delta_i)}^l$ iff its lth derivative $f_i^{(l)}$ is continuous on Δ_i. Clearly $S_\Delta \subset C_\Delta$ and the projection Π_n is defined $C_\Delta \to S_\Delta$ with $f = (f_1, \ldots, f_n) \mapsto \Pi_n f = (\Pi f_1, \ldots, \Pi f_n)$, where Πf_i is the projection of $f_i \in C_{(\Delta_i)}$ on the polynomials of degree $< r$ on Δ_i.

2.2. Spectral definitions

T is supposed to be compact in the complex Banach space $X = C_\Delta$. $\mathcal{L}(X)$ is the algebra of bounded operators on X. The *resolvent set* of T is $\rho(T) = \{z \in \mathbf{C};\ (T - z)^{-1} \in \mathcal{L}(X)\}$ where z stands for $z1$. For z in $\rho(T)$, $R(z) = (T - z)^{-1}$ is the *resolvent* of T and $TR(z) = R(z)T$. The unique solution of (1) is then $x = R(z)f$.

Let $\lambda \neq 0$ be an *isolated eigenvalue* of T with algebraic (resp. geometric) multiplicity m (resp. g), and ascent μ, $1 \leqslant \mu \leqslant m$, $1 \leqslant g \leqslant m$. The associated eigenspace is $E = \ker(T - \lambda)$, the null space of $T - \lambda$ so $\dim E = g$; the invariant subspace is

$$M = \ker(T - \lambda)^m, \quad \dim M = m, \quad \text{and} \quad \ker(T - \lambda)^\mu \equiv \ker(T - \lambda)^m.$$

Let Γ be a Jordan curve in $\rho(T)$, around λ, which contains neither 0 nor any other eigenvalue of T. $P := -1/2i\pi \int_\Gamma R(z)\, dz$ is the *spectral projection* associated with λ, $M = PX$. Let T_n be a sequence of operators in $\mathcal{L}(X)$ such that T_n converge to T pointwise. T_n will be either $T_n^P = \Pi_n T$ or $T_n^S = T\Pi_n$. If $\Gamma \subset \rho(T_n)$, we may define for T_n the resolvent $R_n(z)$ for $z \in \Gamma$ and the spectral projection $P_n := -1/2i\pi \int_\Gamma R_n(z)\, dz$. If T_n is strongly stable inside Γ (Chatelin [8], [9]), there are, for n large enough, exactly m eigenvalues $\{\lambda_{in}\}_{i=1}^m$ of T_n inside Γ (counting their algebraic multiplicities), $\hat{\lambda}_n$ is their arithmetic mean, and λ_n is any one of them.

For the projections Π_n under consideration, both T_n^P and T_n^S are strongly stable around any non-zero eigenvalue of T (Chatelin [7], [9]). The solution x_n of (3) is such that $x_n = R_n^P(z)\Pi_n f$, and $\tilde{x}_n = R_n^S(z)f$. Similarly ϕ_n is an eigenvector of T_n^P and $\tilde{\phi}_n$ of T_n^S, associated with the same eigenvalue λ_n.

2.3. The errors $x_n - x$, $\phi_n - P\phi_n$, $\tilde{x}_n - x$, $\tilde{\phi}_n - P\tilde{\phi}_n$ and $\lambda - \hat{\lambda}_n$

C is a generic constant, which may depend on r and σ, but is otherwise independent of Δ.

2.3.1. The projection method

We recall the following equality:

$$x_n - x = zR_n^P(z)(1 - \Pi_n)x, \quad \text{then } \|x - x_n\|_\infty \leqslant C\|(1 - \Pi_n)x\|_\infty.$$

As for the resolvents,

$$(R_n^P(z) - R(z))\phi_n = R(z)(T - T_n^P)R_n^P(z)\phi_n = \frac{R(z)}{\lambda_n - z}(1 - \Pi_n)T\phi_n,$$

because $R_n^P(z)\phi_n = \phi_n/\lambda_n - z$. To integrate on Γ, we distinguish whether $\lambda_n = \lambda$ or not. If $\lambda_n = \lambda$, then $-1/2i\pi\int_\Gamma(R(z)/\lambda - z)dz = S = \lim_{z\to\lambda} R(z)(1-P)$; S is the *reduced resolvent* with respect to λ. If $\lambda_n \neq \lambda$, $R(z) - R(\lambda_n) = (z-\lambda_n)R(\lambda_n)R(z)$, and

$$\frac{-1}{2i\pi}\int_\Gamma \frac{R(z)}{\lambda_n - z}\,dz = R(\lambda_n)\left[\frac{-1}{2i\pi}\int_\Gamma \frac{dz}{\lambda_n - z} + \frac{1}{2i\pi}\int_\Gamma R(z)\,dz\right] = R(\lambda_n)(1-P).$$

λ is the only pole of $R(z)$ inside Γ, $R(\lambda_n)(1-P)$ is well defined and when $n\to\infty$, $\lambda_n\to\lambda$, $R(\lambda_n)(1-P)\to S$. $R(\lambda_n)(1-P)$ is then uniformly bounded in n, for n large enough. To have a unique formula for the cases $\lambda_n = \lambda$ and $\lambda_n \neq \lambda$, we set $R(\lambda)(1-P) = S$.

By integration in z on Γ, we get $\phi_n - P\phi_n = R(\lambda_n)(1-P)(1-\Pi_n)T\phi_n$, and

$$\operatorname{dist}(\phi_n, M) = \inf_{\phi\in M}\|\phi_n - \phi\|_\infty \leq \|\phi_n - P\phi_n\|_\infty \leq C\|(1-\Pi_n)T\phi_n\|_\infty.$$

2.3.2. The Sloan method

1) $\tilde{x}_n - x = (R_n^S(z) - R(z))f = R(z)(T - T_n^S)R_n^S f = R(z)T(1-\Pi_n)\tilde{x}_n = TR(z)(1-\Pi_n)\tilde{x}_n$.

Then for any fixed t in [0, 1], and any fixed z in $\rho(T)$,

$$\begin{aligned}(\tilde{x}_n - x)(t) &= \int_0^1 k(t,s)[R(z)(1-\Pi_n)\tilde{x}_n](s)\,ds\\ &= \langle k_t, R(z)(1-\Pi_n)\tilde{x}_n\rangle\\ &= \langle (R(z))^* k_t, (1-\Pi_n)\tilde{x}_n\rangle = \langle l_t, (1-\Pi_n)\tilde{x}_n\rangle.\end{aligned}$$

Because $R^*(z) := (R(z))^* = (T^* - \bar{z})^{-1}$, $l_t := R^*(z)k_t$ is the solution of $(T^* - \bar{z})l_t = k_t$; the solution l_t (which depends on z) is unique since $z \in \rho(T) \Leftrightarrow \bar{z} \in \rho(T^*)$.

2) Similarly

$$(R_n^S(z) - R(z))\tilde{\phi}_n = R(z)(T - T_n^S)R_n^S(z)\tilde{\phi}_n = \frac{R(z)}{\lambda_n - z}T(1-\Pi_n)\tilde{\phi}_n.$$

By integration on Γ, we get for any fixed t on [0, 1]

$$\begin{aligned}\tilde{\phi}_n(t) - (P\tilde{\phi}_n)(t) &= \left[T\left(\frac{-1}{2i\pi}\int_\Gamma \frac{R(z)}{\lambda_n - z}\,dz\right)(1-\Pi_n)\tilde{\phi}_n\right](t)\\ &= [TR(\lambda_n)(1-P)(1-\Pi_n)\tilde{\phi}_n](t).\end{aligned}$$

We define $l_t' := R^*(\lambda_n)(1-P^*)k_t$, that is l_t' is the unique solution of $(T^* - \bar{\lambda}_n)l_t' = (1-P^*)k_t$. We define accordingly $R^*(\lambda)(1-P^*) := S^*$. Then $\tilde{\phi}_n(t) - (P\tilde{\phi}_n)(t) = \langle l_t', (1-\Pi_n)\tilde{\phi}_n\rangle$. We have just proved that the error $(x - \tilde{x}_n)(t)$ (resp. $(\tilde{\phi}_n - P\tilde{\phi}_n)(t)$) at $t \in [0, 1]$ can be expressed in terms of the scalar product $\langle l_t, (1-\Pi_n)\tilde{x}_n\rangle$ (resp. $\langle l_t', (1-\Pi_n)\tilde{\phi}_n\rangle$).

REMARK. Another way to bound

$$(\tilde{\phi}_n - P\tilde{\phi}_n)(t) = \left[T(-1/2i\pi \textstyle\int_\Gamma (R(z)/\lambda_n - z)\, dz)(1 - \Pi_n)\tilde{\phi}_n\right](t)$$

is the following (Lebbar [10]). Let Γ' be the circle centered at λ_n, with radius r, containing λ and contained in Γ (for n large enough, there exists such a circle). We set $z = \lambda_n + re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, for $z \in \Gamma'$.

$$\frac{-1}{2i\pi}\int_\Gamma \frac{R(z)}{\lambda_n - z}\, dz = \frac{-1}{2i\pi}\int_{\Gamma'} \frac{R(z)}{\lambda_n - z}\, dz$$

$$= \frac{-1}{2i\pi}\int_0^{2\pi} \frac{R(\lambda_n + re^{i\theta})}{-re^{i\theta}}\, rie^{i\theta}\, d\theta.$$

Then

$$|(\tilde{\phi}_n - P\tilde{\phi}_n)(t)| = \frac{1}{2\pi}\left|\int_0^{2\pi}\left[TR(\lambda_n + re^{i\theta})(1 - \Pi_n)\tilde{\phi}_n\right](t)\, d\theta\right|$$

$$\leqslant \sup_{0<\theta<2\pi} \left|\left[TR(\lambda_n + re^{i\theta})(1 - \Pi_n)\tilde{\phi}_n\right](t)\right|$$

$$= \sup_{z\in\Gamma'} \left|\left[TR(z)(1 - \Pi_n)\tilde{\phi}_n\right](t)\right|.$$

For $z \in \Gamma'$, we define $l_t(z) := R^*(z)k_t$. Then

$$|(\tilde{\phi}_n - P\tilde{\phi}_n)(t)| \leqslant \sup_{z\in\Gamma'} |\langle l_t(z), (1 - \Pi_n)\tilde{\phi}_n\rangle|.$$

As for the global bounds on [0, 1], they are easy to get:

$$r\tilde{x}_n - x = R(z)T(1 - \Pi_n)\tilde{x}_n$$

implies $\|\tilde{x}_n - x\|_\infty \leqslant C\|T(1 - \Pi_n)\tilde{x}_n\|_\infty$, and

$$\|T(1 - \Pi_n)\tilde{x}_n\|_\infty = \sup_{t\in[0,1]} |\langle k_t, (1 - \Pi_n)\tilde{x}_n\rangle|.$$

$\tilde{\phi}_n - P\tilde{\phi}_n = R(\lambda_n)(1 - P)T(1 - \Pi_n)\tilde{\phi}_n$ implies

$$\operatorname{dist}(\tilde{\phi}_n, M) := \inf_{\phi\in M} \|\tilde{\phi}_n - \phi\|_\infty \leqslant \|\tilde{\phi}_n - P\tilde{\phi}_n\|_\infty \leqslant C\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty,$$

and

$$\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty = \sup_{t\in[0,1]} |\langle k_t, (1 - \Pi_n)\tilde{\phi}_n\rangle|.$$

3) Now we set $M_n := P_nX$. For n large enough, $P_{\restriction M_n}$ has a bounded inverse and $m(\lambda - \hat{\lambda}_n) = \Sigma_{i=1}^m \langle x_i^*, (1 - \Pi_n)T(P_{\restriction M_n})^{-1}x_i\rangle$ where $\{x_i\}_1^m$ (resp. $\{x_i^*\}_1^m$) is a basis of M (resp. the adjoint basis of M^*) (see de Boor-Swartz [2]). The error $\lambda - \hat{\lambda}_n$ is then of the same type as the errors $(\tilde{x}_n - x)(t)$ and $(\tilde{\phi}_n - P\tilde{\phi}_n)(t)$.

4) Let Q be the eigenprojection on $E = \ker(T - \lambda)$, along a supplementary subspace F.

$$\tilde{\phi}_n - Q\tilde{\phi}_n = [(T-\lambda)(1-Q)]^{-1}(T-\lambda)(1-Q)\tilde{\phi}_n,$$

and

$$(T-\lambda)(1-Q)\tilde{\phi}_n = (T-\lambda)\tilde{\phi}_n = T(1-\Pi_n)\tilde{\phi}_n + (\lambda_n - \lambda)\tilde{\phi}_n.$$

Therefore

$$\begin{aligned}\operatorname{dist}(\tilde{\phi}_n, E) &= \inf_{\phi \in E} \|\tilde{\phi}_n - \phi\|_\infty < \|\tilde{\phi}_n - Q\tilde{\phi}_n\|_\infty \\ &< C(\|T(1-\Pi_n)\tilde{\phi}_n\|_\infty + |\lambda_n - \lambda|).\end{aligned}$$

Note that this method does not provide a pointwise estimate for $\tilde{\phi}_n - Q\tilde{\phi}_n$. This is due to the fact that, unlike the spectral projection P, the eigenprojection Q has no expression in terms of the resolvent.

3. Two basic results

We shall be concerned with two types of continuous kernels k that we define now.

i) k is *smooth* (of order $l > 0$) if $k \in C^l_\Delta([0, 1] \times [0, 1])$, that is $k_{ij} \in C^l_{(\Delta_i \times \Delta_j)}$, for $1 < i, j < n$, and k is continuous on $[0, 1] \times [0, 1]$.

ii) k is a *Green's* kernel (of order $l > 1$, and continuity δ, $0 < \delta < l$) if

$$k(t, s) = \begin{cases} k_1(t, s) & \text{for } t > s, \\ k_2(t, s) & \text{for } t < s, \end{cases}$$

is such that

$$\begin{aligned} k_1 &\in C^l(\{0 < s < t < 1\}), \\ k_2 &\in C^l(\{0 < t < s < 1\}), \\ k &\in C^\delta([0, 1] \times [0, 1]). \end{aligned}$$

An obvious example of case ii) is the Green's function of an o.d.e. of order $\delta + 2$.

For any z in $\rho(T)$, we consider the solution $x = R(z)f$ of (1), along with $\tilde{x}_n$ and $\tilde{\phi}_n$, solutions of (5) and (6).

LEMMA 1. *Let T be an integral operator with a kernel k of order l, of type* i) *or* ii). *If $f \in C^l_\Delta$ then, in both cases, x, $\tilde{x}_n$ and $\tilde{\phi}_n$ are in C^l_Δ.*

Now with $k_t(\cdot) := k(t, \cdot)$ for t fixed in $[0, 1]$, we consider the equation $(T^* - \bar{z})l_t = k_t$, for $z \in \rho(T)$.

LEMMA 2. *When k is a smooth kernel of order l, then $l_t \in C^l_\Delta(0, 1)$ for any t in $[0, 1]$. When k is a Green's kernel of order l and continuity δ, then $l_{t_i} \in C^l_\Delta(0, 1)$ for $t_i \in \Delta$, $i = 0, \ldots, n$, and $l_t \in C^\delta(0, 1)$ for $t \notin \Delta$.*

It is left to the reader to check the two lemmas (see Lebbar [10]). Note that when k is a Green's kernel, l_t is defined by the functions $l_{1t} \in C^l(0, t)$, $l_{2t} \in C^l(t, 1)$.

Lemma 2 shows that l_t has the same smoothness properties as k_t. The same is true for l_t'.

We define $\alpha := \min(l, r + 1)$ and $\alpha^* := \min(l, r + 1, \delta + 2)$.

3.1. Π_n is an orthogonal projection

THEOREM 3. *Let Δ be quasi-uniform. With the above definitions, then for $f \in C^l_\Delta$, and z in $\rho(T)$:*

i) *if k is a smooth kernel of order l, then for $t \in [0, 1]$: $|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{2\alpha}\|l_t^{(\alpha)}\|_\Delta$, and globally $\|T(1 - \Pi_n)f\|_\infty \leqslant Ch^{2\alpha}$.*

ii) *if k is a Green's kernel of order l and continuity δ, $0 \leqslant \delta < l$, then for $t_i \in \Delta$,*

$$|\langle l_{t_i}, (1 - \Pi_n)f\rangle| \leqslant Ch^{2\alpha}\|l_{t_i}^{(\alpha)}\|_\Delta, \qquad i = 0, \ldots, n,$$

for $t \notin \Delta$,

$$|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{\alpha+\alpha^*} \max\left(\|l_{1t}^{(\delta+1)}\|_\infty, \|l_{2t}^{(\delta+1)}\|_\infty\right),$$

and globally, $\|T(1 - \Pi_n)f\|_\infty \leqslant Ch^{\alpha+\alpha^*}$.

PROOF. It is adapted from Chandler [6]. Since Π_n is an orthogonal projection: $\langle l_t, (1 - \Pi_n)f\rangle = \langle (1 - \Pi_n)l_t, (1 - \Pi_n)f\rangle$. And

$$\int_0^1 (1 - \Pi_n)l_t(s)(1 - \Pi_n)f(s)\, ds = \sum_{i=1}^n \int_{\Delta_i} (1 - \Pi)l_{it}(s)(1 - \Pi)f_i(s)\, ds.$$

Given $f_i \in C_{(\Delta_i)}$, Πf_i is the orthogonal projection of f_i on the set of polynomials of degree $\leqslant r$ on Δ_i. When f, $l_t \in C^l_\Delta(0, 1)$, f_i, $l_{it} \in C^l_{(\Delta_i)}$, and

$$\|(1 - \Pi)f_i\|_1 \leqslant Ch^{\alpha+1}\|f_i^{(\alpha)}\|_\infty, \ \|(1 - \Pi)l_{it}\|_\infty \leqslant Ch^\alpha\|l_{it}^{(\alpha)}\|_\infty.$$

When $l_t \in C^\delta(0, 1)$, with $t_{i-1} < t < t_i$, then on Δ_i, if $\delta \leqslant r$,

$$\|(1 - \Pi)l_{it}\|_\infty \leqslant Ch^{\delta+1} \max\left(\|l_{it}^{(\delta+1)}\|_\infty, \|l_{2t}^{(\delta+1)}\|_\infty\right).$$

The result follows by summing over i, and using $nh \leqslant \sigma$.

3.2. Π_n is an interpolatory projection

Let f be a function of $C^{l+1}(a, b)$, such that $f(w_j) = 0, j = 1, \ldots, r + 1$, where the $\{w_i\}_1^{r+1}$ are $r + 1$ distinct points in (a, b). The $(r + 1)$th divided difference

of f on the points $w_1, \ldots, w_{r+1}$ is denoted by $\delta[w_1, w_2, \ldots, w_{r+1}, \cdot]f$. Then

$$f(s) = (s - w_1) \cdots (s - w_{r+1})\delta[w_1, w_2, \ldots, w_{r+1}, s]f, \quad \text{for } s \notin \{w_j\}_1^{r+1}.$$

We set, for $s \in [a, b]$,

$$g^*(s) = \begin{cases} f(s)/v(s) & \text{if } s \notin \{w_j\}_1^{r+1}, \\ \lim\limits_{t \to w_j} \left(\dfrac{f(s)}{v(s)} \right) & \text{if } t = w_j, j = 1, \ldots, r+1, \end{cases}$$

where $v(s) = (s - w_1) \cdots (s - w_{r+1})$.

LEMMA 4. *If $f \in C^{l+1}(a, b)$, then $g^* \in C^l(a, b)$.*

There is only a need to prove that g^* is C^l in the neighborhood of any w_j, $j = 1, \ldots, r+1$ (see Lebbar [10]). If $f \in C^{l+1}(a, b)$, the divided difference $\delta[w_1, \ldots, w_{r+1}, \cdot]f$ may therefore be prolongated by continuity on $[a, b]$, up to the order l.

We shall apply this lemma on each Δ_i, with the $\{w_j\}_1^{r+1}$ being the Gauss points $\{\tau_j^i\}_{j=1}^{r+1}$. For $f \in C_\Delta^{l+1}$, Πf_i is the polynomial of degree $\leqslant r$ on Δ_i which interpolates f_i at the Gauss points $\{\tau_j^i\}_{j=1}^{r+1}$. Hence $(1 - \Pi)f_i(\tau_j^i) = 0$ for $j = 1, \ldots, r+1$. We consider the divided difference $\delta[\tau_1^i, \ldots, \tau_{r+1}^i, \cdot](1 - \Pi)f_i$, and set $q_{it} := l_{it}\delta[\tau_1^i, \ldots, \tau_{r+1}^i, \cdot](1 - \Pi)f_i$. $k_{it} \in C_{(\Delta_i)}^{l+1}$ (resp. $C_{(\Delta_i)}^{\delta}$) implies that $l_{it} \in C_{(\Delta_i)}^{l+1}$ (resp. $C_{(\Delta_i)}^{\delta}$) and $q_{it} \in C_{(\Delta_i)}^{l}$ (resp. $C_{(\Delta_i)}^{\delta}$, for $\delta \leqslant l$).

THEOREM 5. *With the above definitions, then for $f \in C_\Delta^{l+1}$ and z in $\rho(T)$*

i) *if k is a smooth kernel of order l, then for $t \in [0, 1]$, $|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha}\|q_t^{(\alpha)}\|_\Delta$ and globally $\|T(1 - \Pi_n)f\|_\infty \leqslant M_\Delta h^{r+1+\alpha}$,*

ii) *if k is a Green's kernel, of order l and continuity δ, $0 \leqslant \delta < l$, then for $t_i \in \Delta$,*

$$|\langle l_{t_i}, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha}\|q_{t_i}^{(\alpha)}\|_\Delta, \qquad i = 0, \ldots, n,$$

for $t \notin \Delta$,

$$|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha^*} \max(\|q_{1t}^{(\delta+1)}\|_\infty, \|q_{2t}^{(\delta+1)}\|_\infty),$$

and globally $\|T(1 - \Pi_n)f\|_\infty \leqslant M_\Delta h^{r+1+\alpha^*}$.
If $f \in C_\Delta^{l+r+1}$, then $M_\Delta \leqslant C$.

PROOF. It is adapted from de Boor-Swartz [1].

$$\int_0^1 l_t(s)(1 - \Pi_n)f(s)\,ds = \sum_{i=1}^n \int_{\Delta_i} l_{it}(s)(1 - \Pi)f_i(s)\,ds$$

$$= \sum_{i=1}^n \int_{\Delta_i} \underbrace{l_{it}(s)\delta[\tau_1^i, \ldots, \tau_{r+1}^i, s](1 - \Pi)f_i(s)}_{q_{it}(s)} \underbrace{(s - \tau_1^i) \cdots (s - \tau_{r+1}^i)}_{v(s)}\,ds.$$

When $q_{it} \in C^l_{(\Delta_i)}$, $q_{it}(s) = q_{it}(t_{i-1}) + \cdots + ((s - t_{i-1})^l/l!)q_{it}^{(l)}(\theta_s)$, $t_{i-1} < \theta_s < s$. Making use of $\int_{\Delta_i} v(s)p(s)\,ds = 0$ for all polynomial p of degree $\leqslant r$ on Δ_i, we get $|\int_{\Delta_i} q_{it}(s)v(s)\,ds| \leqslant Ch^{r+2+\alpha}\|q_{it}^{(\alpha)}\|_\infty$, which gives, for $l_t \in C^l_\Delta$,

$$|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha}\|q_t^{(\alpha)}\|_\Delta.$$

When $l_{it} \in C^\delta_{(\Delta_i)}$,

$$\left|\int_{\Delta_i} l_{it}(s)(1 - \Pi)f_i(s)\,ds\right| \leqslant Ch^{r+2+\min(r+1,\delta+2)}$$

and $|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha^*}$, by summing over i.

Theorems 3 and 5 play a central role to derive the convergence rates, as we shall see in the next section.

4. Convergence rates

We recall that $\alpha = \min(l, r + 1)$ and $\alpha^* = \min(l, r + 1, \delta + 2)$. In practice $\delta + 2 \leqslant r + 1 \leqslant l$, so that $\alpha = r + 1$ and $\alpha^* = \delta + 2$. We assume throughout this section that the kernel k is of order l for the Galerkin method ($f \in C^l_\Delta \Rightarrow \tilde{x}_n \in C^l_\Delta$, $\tilde{\phi}_n \in C^l_\Delta$) and of order $l + r + 1$ for the collocation method ($f \in C^{l+r+1}_\Delta \Rightarrow \tilde{x}_n \in C^{l+r+1}_\Delta$ and $\tilde{\phi}_n \in C^{l+r+1}_\Delta$).

4.1. Convergence rate for the eigenvalues

The definitions are those of Section 2.2.

THEOREM 6. *For both types of kernel k*

$$\lambda - \hat{\lambda}_n = O(\varepsilon_n), \quad \max_i |\lambda - \lambda_{in}| = O(\varepsilon_n^{1/\mu}), \quad \min_i |\lambda - \lambda_{in}| = O(\varepsilon_n^{g/m})$$

where (a) $\varepsilon_n = h^{2\alpha}$ *for the Galerkin method, and* (b) $\varepsilon_n = h^{r+1+\alpha}$ *for the collocation method.*

PROOF. It is adapted from de Boor-Swartz [2] where it is noticed that λ (resp. λ_{in}) are the eigenvalues of two $m \times m$ matrices such that the (i, j)th coefficient of the difference is $\langle x_i^*, (1 - \Pi_n)T(P_{\restriction M_n})^{-1}x_j\rangle$. Theorem 3 applies where l_t is replaced by $x_i^* \in C^l_\Delta$ and $T(P_{\restriction M_n})^{-1}x_j \in C^l_\Delta$, if the kernel is of order l. Similarly, Theorem 5 applies if k is of order $l + r + 1$. And the results follow from classical theorems in *matrix* theory (see Wilkinson [18], pp. 80–81).

4.2. Convergence rate for the solutions and the eigenvectors

(a) **The Galerkin method.** We suppose that $f \in C_\Delta^l$ and k is of order l. Then $\tilde{x}_n$, $\tilde{\phi}_n \in C_\Delta^l$. Δ is quasi-uniform.

THEOREM 7. *With a smooth kernel,* $\|x - \tilde{x}_n\|_\infty$ *and* $\operatorname{dist}(\tilde{\phi}_n, M)$ *are of the order* $h^{2\alpha}$, $\operatorname{dist}(\tilde{\phi}_n, E) = O(h^{2\alpha/\mu})$. *With a Green's kernel, then: at* $t_i \in \Delta$, $|x(t_i) - \tilde{x}_n(t_i)|$ *and* $|\tilde{\phi}_n(t_i) - (P\tilde{\phi}_n)(t_i)|$ *are of the order* $h^{2\alpha}$, $i = 0, \ldots, n$, *whereas globally* $\|x - \tilde{x}_n\|_\infty$ *and* $\operatorname{dist}(\tilde{\phi}_n, M)$ *are of the order* $h^{\alpha+\alpha^*}$, $\operatorname{dist}(\tilde{\phi}_n, E) = O(h^{2\alpha/\mu})$, *for* $\mu > 1$.

PROOF. We apply Theorem 3 to $(\tilde{x}_n - x)(t) = \langle l_t, (1 - \Pi_n)\tilde{x}_n\rangle$, $(\tilde{\phi}_n - P\tilde{\phi}_n)(t) = \langle l_t', (1 - \Pi_n)\tilde{\phi}_n\rangle$, and Theorem 6 to

$$\operatorname{dist}(\tilde{\phi}_n, E) < C(\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty + |\lambda_n - \lambda|).$$

(b) **The collocation method.** We suppose that $f \in C_\Delta^{l+r+1}$ and k is of order $l + r + 1$. Then $\tilde{x}_n$, $\tilde{\phi}_n \in C_\Delta^{l+r+1}$. We get, as Theorem 8, the analog of Theorem 7, where $h^{2\alpha}$ (resp. $h^{\alpha+\alpha^*}$) is replaced by $h^{r+1+\alpha}$ (resp. $h^{r+1+\alpha^*}$). The convergence rates in Theorems 7 and 8 are the best we could hope from the known results. It should be noticed that the computation of $\tilde{x}_n$ (resp. $\tilde{\phi}_n$) from x_n (resp. λ_n, ϕ_n) does not require much extra work: let $\dim X_n = n$ (say), let $\{e_i^n\}_1^n$ be a basis of X_n: if $x_n = \sum_{i=1}^n \xi_i^n e_i^n$, then $Tx_n = \sum_{i=1}^n \xi_i^n Te_i^n$ where the $\{Te_i^n\}_1^n$ have already been computed to get the coefficients of the matrix associated with the projection method.

5. Numerical Example

We end this paper with a numerical example illustrating the behavior of the iterated collocation solution for the Fredholm equation

$$\int_0^1 k(t, s)x(s)\,ds - \frac{1}{4}x(t) = -\cosh(1), \qquad 0 < t < 1,$$

with

$$k(t, s) = \begin{cases} -t(1 - s) & \text{if } s > t, \\ -s(1 - t) & \text{if } s < t. \end{cases}$$

The exact solution is $x(t) = \cosh(2t - 1)$.

We choose the partition $\Delta = \{i/5\}_0^5$, $h = \frac{1}{5}$, and on each interval Δ_i, the $r + 1 = 4$ Gauss points. We display in Table 1 the values of $x - x_n$ and $x - \tilde{x}_n$

at the partition points t_i, $i = 1, 2, 3, 4$. The kernel k is of "Green's function" type with $\delta = 0$.

TABLE 1

Error values at the partition points

i	$(x - x_n)(t_i^-)$	$(x - x_n)(t_i^+)$	$(x - \tilde{x}_n)(t_i)$
1	8.10^{-1}	7.10^{-5}	-5.10^{-12}
2	6.10^{-5}	6.10^{-5}	-7.10^{-12}
3	6.10^{-5}	6.10^{-5}	-7.10^{-12}
4	7.10^{-5}	8.10^{-5}	-5.10^{-12}

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CHAPITRE 2

SUPERCONVERGENCE RESULTS FOR THE ITERATED PROJECTION METHOD APPLIED TO A SECOND KIND FREDHOLM INTEGRAL EQUATION AND EIGENVALUE PROBLEM

SUPERCONVERGENCE RESULTS FOR THE ITERATED PROJECTION METHOD APPLIED TO A SECOND KIND FREDHOLM INTEGRAL EQUATION AND EIGENVALUE PROBLEM

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Abstract.

We consider a Fredholm integral equation of second kind and the corresponding eigenvalue problem. The projection method uses projections on piecewise polynomials and amounts either to an orthogonal-Galerkin method in L^2 or to a collocation method at gaussian points in C. The aim of the paper is threefold :

- give a new proof of superconvergence results for the orthogonal-Galerkin method which does not assume that the partition is quasi-uniform,
- treat the eigenvectors in much the same manner as the solutions,
- do a parallel study of the two projection methods cited above, so that many proofs can be factorized.

1. INTRODUCTION

We consider some projection methods for the solution of second kind integral equations of the form :

$$(Tx)(s) - z\,x(s) = f(s) \quad , \quad 0 \leq s \leq 1 \qquad (1)$$

where T is the operator defined by :

$$x(s) \mapsto \int_0^1 k(s,t)x(t)dt \quad , \quad 0 \leq s \leq 1 \qquad (2)$$

and where (1) is supposed to be well posed : $(T-z1)^{-1}$ exists and is bounded on the whole space.
Along with (1), we consider the eigenvalue problem :

$$(T\varphi)(s) = \lambda\varphi(s) \quad , \quad 0 \leq s \leq 1 \ , \quad \varphi \neq 0 \qquad (3)$$

where λ is an isolated eigenvalue of T
(1) and (3) are regarded as equations in the complex Banach space X (specialized to be L^∞ or L^2 later on).
T is supposed to be *compact* in X and $z \in \rho(T)$, the resolvant set of T. Let X_n be a finite dimensional subspace of X and let π_n be a projection onto X_n . Then the projection method consists in approximating (1) and (3) respectively by :

$$(\pi_n T - z)x_n = \pi_n f \quad , \quad x_n \in X_n \ , \qquad (4)$$

$$\pi_n T \varphi_n = \lambda_n \varphi_n \ , \ 0 \neq \varphi_n \in X_n \quad , \quad \|\varphi_n\| = 1 \ . \qquad (5)$$

x_n (resp. φ_n) is the projection solution (resp. eigenvector), corresponding to the approximation $T_n^P = \pi_n T$ of T (P for projection).
Given a partition $\Delta = \{t_i\}_0^n$ of $[0,1]$, $t_0 = 0$, $t_n = 1$,

let X_n be the set $\mathbb{P}_{r,\Delta}$ of piecewise polynomials of degree $\leq r$ on each subinterval $[t_{i-1},t_i[$, $i=1,\dots,n$, the value at 1 being defined by continuity.
We shall consider two types of projection methods :

(a) in case (a), $X = L^2(0,1)$ and π_n^1 is the orthogonal projection in $L^2(0,1)$ on $\mathbb{P}_{r,\Delta}$;

(b) in case (b), $X = L^\infty(0,1)$ and π_n^2 is the interpolatory projection defined for x in $C_\Delta(0,1) \subset L^\infty(0,1)$, so that $\pi_n^2 x$ interpolates x at the $r+1$ Gauss points $\{\tau_j^i\}_{j=1}^{r+1}$, on each subinterval , $i=1,\dots,n$.

Case (a) corresponds to an *orthogonal-Galerkin* method, (abbreviated 1-Galerkin) , and case (b) to a *collocation method* at the gaussian points $\{\tau_j^i\}$. Let π_n represent any one of the two projections π_n^1 or π_n^2 .
If $z \neq 0$ (resp. $\lambda_n \neq 0$) we consider the iterated projection solution $\tilde{x}_n$ (resp. eigenvector $\tilde{\varphi}_n$) introduced by Sloan [14], cf. also Brakhage [6], and given by the formulae :

$$\tilde{x}_n = \frac{1}{z}\,(Tx_n - f) \quad , \quad \tilde{\varphi}_n = \frac{1}{\lambda_n}\,T\varphi_n \ .$$

Since T is compact , z in $\rho(T)$ is non zero, and $\lambda_n \neq 0$ is not a restriction either because the isolated eigenvalues of T are non zero.

$\tilde{x}_n$ and $\tilde{\varphi}_n$ are solutions of the equations :

$$(T\,\pi_n - z)\tilde{x}_n = f \tag{6}$$

$$T\,\pi_n\,\tilde{\varphi}_n = \lambda_n\,\tilde{\varphi}_n \quad , \tag{7}$$

corresponding to the approximation $T_n^S = T\,\pi_n$ of T (S for Sloan). $\pi_n\,\tilde{x}_n = x_n$, $\pi_n\,\tilde{\varphi}_n = \varphi_n$ so that in case (b) the iterated solutions and the solutions themselves agree at the collocation points.

If k and f are smooth enough, it is known that $\|x_n - x\|_\nu = O(h^{r+1})$ while $\|\tilde{x}_n - x\|_\nu = O(h^{2r+2})$ for both cases (a) and (b). Similar results hold for $\tilde{\varphi}_n$.
Here, as in the rest of the text, $\|\cdot\|_\nu$ is taken to be $\|\cdot\|_2$ in case (a) and $\|\cdot\|_\infty$ in case (b).
Such fast convergence for $\tilde{x}_n$ and $\tilde{\varphi}_n$, often called *super-convergence*, is still valid at the partition points $\{t_i\}_0^n$ when k is the Green's function of an ordinary differential equation (o.d.e.).
These questions of high order convergence have already been studied in Chandler [7], de Boor-Swartz [2-4], and Chatelin-Lebbar [9].
[7] and [9] deal with the $\perp$-Galerkin method where the partition Δ is assumed to be *quasi-uniform*. Δ may be quite arbitrary in the present paper.
For the $\perp$-Galerkin method, the problem is to find either pointwise or global L^∞-estimate for $x-\tilde{x}_n$ (resp. $\varphi-\tilde{\varphi}_n$).
This is easily done if $\|\pi_n^1\|_\infty < \infty$ (cf. [7] for $x-\tilde{x}_n$ and [9] for $\varphi-\tilde{\varphi}_n$). There are two cases where $\|\pi_n^1\|_\infty < \infty$:
Δ is quasi-uniform, or π_n^1 is a projection on $\mathbb{P}_{r,\Delta} \cap C(0,1)$, cf
The analysis is then possible in L^∞ or C, using L^∞-and L^1-norms (cf. [7],[9]).
In this paper we wish to get a quite general result for case (a), and the proof is in four steps :

1. prove the convergence in L^2,
2. bound $x-\tilde{x}_n$ and $\varphi-\tilde{\varphi}_n$ in L^2,
3. bound $x-\tilde{x}_n$ and $\varphi-\tilde{\varphi}_n$ pointwise,
4. deduce global bounds in L^∞.

Note that in [5] the $\perp$-Galerkin method is also treated with a quite general Δ. But more smoothness assumptions are required there on k and f than we do. We shall go back to this question in section 5. Also in [5], the high order convergence properties of π_n^1 are presented as special cases of those of π_n^2.

In this paper, we present an analysis of both projections which goes the other way round. Properties of the collocation method are deduced in section 6 from those of the $\perp$-Galerkin method. By doing so, we stress the intimate relationship between the two methods, and we factorize many of the proofs. There is only a small amount of work which is left to carry superconvergence results from (a) to (b). Namely there is only to bound, uniformly in n, all the derivatives of $\tilde{x}_n$ (resp. $\tilde{\varphi}_n$) up to a certain order. This part uses the technique for the collocation method developed in [2]. Moreover this part is even not necessary when the kernel is smooth.
As mentioned in [9], the error on the eigenvector $\tilde{\varphi}_n$ is treated much like the error $x-\tilde{x}_n$ on the solution, and the result is new in the most general case (ascent of the eigenvalue superior to one).

2. DEFINITION OF THE PROBLEM

c will be a generic constant throughout the text.

2.1. PROPERTIES OF THE INTEGRAL OPERATOR T

$\alpha \geq 0$ and $\gamma \geq -1$ are integers. We are concerned with the following class of kernels :

DEFINITION : The kernel k is of *class* $\mathcal{G}(\alpha,\gamma)$, with $\alpha \geq 0$ and $\alpha \geq \gamma \geq -1$, iff

$$k(t,s) = \begin{cases} k_1(t,s) & \text{for } 1 \geq t > s \geq 0 , \\ k_2(t,s) & \text{for } 0 \leq t \leq s \leq 1 , \end{cases} \tag{8}$$

with $k_1 \in C^\alpha(\{0 \leq s \leq t \leq 1\})$, $k_2 \in C^\alpha(\{0 \leq t \leq s \leq 1\})$
and $k \in C^\gamma([0,1] \times [0,1])$.
γ is the *continuity* of k and α is its *order*.

The obvious examples of such kernels are Green's functions of o.d.e. and kernels of Volterra integral operators. Note that this definition includes smooth kernels of order α , if $\gamma \equiv \alpha$.

We first state the following theorem proved by means of the characterization of compact operators given in Graham-Sloan [12] .

THEOREM 1 :

Let the kernel of T *be of class* $\mathcal{G}(\alpha,\gamma)$. *Then :*

i) T *is compact from* L^2 *into* C ,

ii) *if* $\gamma \geq 1$, *the range of* T *is in* C^{γ} ,

iii) *for any* p , $0 \leq p \leq \alpha - 1$, T *is bounded from* C^p *into* C^{p+1} .

2.2. ABTRACT SETTING AND CONVERGENCE RESULTS

T is supposed to be compact in the complex Banach space X , $\mathcal{L}(X)$ is the algebra of bounded operators on X . The resolvent set of T is $\rho(T) := \{z \in \mathbb{C} \; ; \; (T-z)^{-1} \in \mathcal{L}(X)\}$ where z stands for $z1$. For z in $\rho(T)$, $R(z) := (T-z)^{-1}$ is the resolvent of T and $TR(z) = R(z)T$. The unique solution of (1) is then $x = R(z)f$.

Let $\lambda \neq 0$ be an isolated eigenvalue of T with algebraic (resp. geometric) multiplicity m (resp. g) , and ascent ℓ , $1 \leq \ell \leq m$, $1 \leq g \leq m$. The associated eigenspace is $E := \mathrm{Ker}(T-\lambda)$ the null space of $T-\lambda$, $\dim E = g$; the invariant subspace is $M := \mathrm{Ker}\,(T-\lambda)^m$, $\dim M = m$, and $\mathrm{Ker}\,(T-\lambda)^{\ell} \equiv \mathrm{Ker}\,(T-\lambda)^m$.

Let Γ be a Jordan curve in $\rho(T)$, around λ , which contains neither 0 nor any other eigenvalue of T .

$P := \frac{-1}{2i\pi}\int_\Gamma R(z)dz$ is the *spectral projection* associated with λ, $M = PX$. The operator $T-z$ is invertible on $(1-P)X$ for any z inside Γ and $S(z) := (T-z)^{-1}(1-P)$ is called the *reduced resolvent* of T with respect to λ, at the point z.

Let T_n be a sequence of operators in $\mathcal{L}(X)$ such that T_n converge to T pointwise. T_n will be either $T_n^P = \pi_n T$ or $T_n^S = T\pi_n$, where π_n is a projection onto X_n, a subspace of X.

If $\Gamma \subset \rho(T_n)$, we may define for T_n the resolvent $R_n(z)$ for $z \in \Gamma$, and the spectral projection $P_n := \frac{-1}{2i\pi}\int_\Gamma R_n(z)dz$.

If T_n is strongly stable inside Γ (Chatelin [8]), there are exactly m eigenvalues $\{\lambda_{in}\}_{i=1}^{m}$ of T_n inside Γ (counting their algebraic multiplicities), $\hat{\lambda}_n$ is their arithmetic mean, and λ_n is any one of them.

Let $h := \max_{1\le i\le n} (t_i - t_{i-1})$ and $X_n = \mathbb{P}_{r,\Delta}$. If $h \to 0$, then in case (
(resp. (b)) for any x in L^2 (resp. in C), $||\pi_n^1 x - x||_2 \to 0$ in L^2 (resp. $||\pi_n^2 x - x||_\infty \to 0$ in L^∞). T is compact in L^2 (resp. L^∞) with range in C, then $||(\pi_n^1 - 1)T||_2 \to 0$, $T\pi_n^1 \xrightarrow{c.c.} T$ in L^2 (resp. $||(\pi_n^2 - 1)T||_\infty \to 0$ in L^∞ and $T\pi_n^2 \xrightarrow{c.c.} T$ in C).

It is then standard (cf [8] for example) to deduce the strong stability of T_n^P and T_n^S and the following convergences :

- $||x - x_n||_\nu \to 0$, $||x - \tilde{x}_n||_\nu \to 0$, if $f \in C$,
- $|\lambda - \lambda_n| \to 0$, $|\lambda - \hat{\lambda}_n| \to 0$,
- $||(1-P)\varphi_n||_\nu \to 0$, $||(1-P)\tilde{\varphi}_n||_\nu \to 0$.

We also deduce the uniform bounds in n, for n large enough :

$$||R_n(z)||_\nu < \infty, \quad ||S(\lambda_n)||_\nu < \infty$$

both for $T_n^P = \pi_n T$ and $T_n^S = T\pi_n$

2.3 THE ERRORS $x_n - x$, $\varphi_n - P\varphi_n$, $\tilde{x}_n - x$, $\tilde{\varphi}_n - P\tilde{\varphi}_n$ AND $\hat{\lambda}_n - \lambda$.

c is a generic constant which may depend on r but is otherwise independant of Δ .

$M_n := P_n^P X$ is the direct sum of the invariant subspaces of $T_n^P = \pi_n T$ associated with all its distinct eigenvalues inside Γ. Similarly $\tilde{M}_n := P_n^S X$ corresponds to $T_n^S = T\pi_n$. It is easy to check that $\tilde{M}_n = TM_n$, we say that $\tilde{M}_n$ is the iterated invariant subspace.

We recall the definition of the gap between two closed subspace M and N :

$$\Theta(M,N) := \max\left(\sup_{\substack{x\in M\\ ||x||=1}} \operatorname{dist}(x,N),\ \sup_{\substack{x\in N\\ ||x||=1}} \operatorname{dist}(x,M)\right)$$

where $\operatorname{dist}(x,M) = \min_{y\in M}||x-y||$. dist_ν is the distance in the L^ν-norm.

We cite the general resultproved in [8] .

PROPOSITION 2 :

If T_n is a strongly stable approximation of T inside Γ, then for n large enough

$$\Theta(M,M_n) \le c\ ||(T-T_n)P||.$$

The proof is based on the decomposition

$(R(z)-R_n(z))P=R_n(z)(T_n-T)R(z)P=R_n(z)(T_n-T)P\ R(z)$.

For $T_n^P = \pi_n T$, $\Theta(M,M_n) \le c||(1-\pi_n)P||$, whereas

for $T_n^S = T\pi_n$, $\Theta(M,\tilde{M}_n) \le c||T(1-\pi_n)P||$.

Clearly for any Ψ_n in M_n ,$||(1-P)\Psi_n||\le||1-P||\operatorname{dist}(\Psi_n,M)\le c\ \Theta(M,M_n)$

and similary, for any $\tilde{\Psi}_n$ in $\tilde{M}_n$,$||(1-P)\ \tilde{\Psi}_n||\le c\ \Theta(M,\tilde{M}_n)$.

The following bounds may then be derived (cf. [9]):

$$\left.\begin{array}{l} \cdot\ ||x-x_n||_\nu \le c\ ||(1-\pi_n)x||_\nu\ , \\ \cdot\ ||(1-P)\varphi_n||_\nu \le c\ ||(1-\pi_n)P||_\nu\ ,\ \text{for}\ \ \varphi_n \in \operatorname{Ker}(T_n^P - \lambda_n) \\ \cdot\ ||(1-P)\ \Psi_n||_\nu \le c||(1-\pi_n)P||_\nu\ ,\ \text{for}\ \ \Psi_n \in M_n\ . \end{array}\right\} \quad (9)$$

Let l_t (resp. l_t^1) be the unique solution of

$(T^* - \bar{z})l_t = k_t$ (resp. $(T^* - \bar{\lambda}_n)(1-P^*)\, l_t^1 = (1-P^*)k_t$).

We proved in [9] the pointwise equalities for any t in [0,1] :

$$\left.\begin{aligned} &(\tilde{x}_n - x)(t) = \langle l_t , (1-\pi_n)\tilde{x}_n \rangle , \\ &[(1-P)\tilde{\varphi}_n](t) = \langle l_t^1 , (1-\pi_n)\, \tilde{\varphi}_n \rangle , \end{aligned}\right\} \qquad (10)$$

where $\langle\ ,\ \rangle$ is the L^2 - scalar product.

We shall also need the following global bounds :

$$\left.\begin{aligned} &.\|\tilde{x}_n - x\|_\nu \le c\|T(1-\pi_n)x\|_\nu , \\ &.\ \|(1-P)\tilde{\varphi}_n\|_\nu \le c\|T(1-\pi_n)P\|_\nu , \text{ for } \tilde{\varphi}_n \in \mathrm{Ker}(T_n^S - \lambda_n) \\ &.\|(1-P)\tilde{\Psi}_n\|_\nu \le c\|T(1-\pi_n)P\|_\nu \text{ for } \tilde{\Psi}_n \in \tilde{M}_n . \end{aligned}\right\} \qquad (11)$$

Let $P_{\restriction M_n} : M_n \to M$ be the operator P restricted to $M_n := P_n X$. Then

$$m(\lambda - \hat{\lambda}_n) = \sum_{i=1}^{m} \langle x_i^* , (1-\pi_n)\ T\ (P_{\restriction M_n})^{-1}\ x_i \rangle \qquad (12)$$

where $\{x_i\}_1^m$ (resp. $\{x_i^*\}_1^m$) is a basis of M (resp. the adjoint basis of M^*), as proved in [3] .

For $x \in M$, $(P_{\restriction M_n})^{-1}\ x - x = (1-P)(P_{\restriction M_n})^{-1}x$. Therefore

$$\|(1-P)(P_{\restriction M_n})^{-1}x\|_\nu \le \|1-P\|_\nu \ \ \mathrm{dist}_\nu((P_{\restriction M_n})x^{-1}, M) \le c\ \ \Theta_\nu(M,M_n)$$

where Θ_ν is the gap in the L^ν-norm between M and M_n .

We conclude that for any x in M

$$\|(P_{\restriction M_n})^{-1}x - x\|_\nu \le c\|(1-\pi_n)P\|_\nu \qquad (13)$$

The eigenspace $E = \mathrm{Ker}(T-\lambda)$ is finite dimensional, and has therefore a supplementary subspace F : $X = E \oplus F$. Let Q be the projection onto E along F. Then, as proved in [9] ,

$$\begin{aligned} \mathrm{dist}_\nu\ (\tilde{\varphi}_n\ , E) &\le \|\tilde{\varphi}_n - Q\,\tilde{\varphi}_n\|_\nu \\ &\le c\ [\|T(1-\pi_n)\tilde{\varphi}_n\|_\nu + |\lambda_n - \lambda|]. \end{aligned}$$

3. SMOOTHNESS PROPERTIES OF THE SOLUTIONS

3.1 PIECEWISE CONTINUOUS FUNCTIONS

Let be given $\Delta = \{t_i\}_0^n$, a strict partition of $[0,1]$, $0 = t_0 < t_1 < \ldots < t_n = 1$. We set $h_i := t_i - t_{i-1}$, $h := \max_{1 \le i \le n} h_i$, $\Delta_i := [t_{i-1}, t_i]$, $i = 1,\ldots,n$.

We define $C_\Delta := \prod_{i=1}^{n} C_{(\Delta_i)}$: $f \in C_\Delta$ consists of n components $f_i \in C_{(\Delta_i)}$, f is a piecewise continuous function having (possibly) different left and right values at the partition points t_i . With the norm $||.||_\Delta$ defined by $||f||_\Delta = \max_i ||f_i||$, C_Δ is a Banach space and $C_\Delta \subset L^\infty(0,1)$ since $||f||_\Delta = ||f||_\infty$ for $f \in C_\Delta$.

We define more generally, C_Δ^α for a positive integer α by $C_\Delta^\alpha := \prod_{i=1}^{n} C^\alpha_{(\Delta_i)}$: $f_i \in C^\alpha_{(\Delta_i)}$ iff its αth derivative $f_i^{(\alpha)}$ is continuous on Δ_i. For $f \in \mathbb{P}_{r,\Delta}$, if the value at t_i^- is defined by continuity, then $\mathbb{P}_{r,\Delta} \subset C_\Delta$ and the projection π_n is defined from C_Δ on to $\mathbb{P}_{r,\Delta}$ with $f = (f_1,\ldots,f_n) \mapsto \pi_n f = (\pi f_1,\ldots,\pi f_n)$, where πf_i is the projection of $f_i \in C_{(\Delta_i)}$ on the polynomials of degree $\le r$ on Δ_i .

We denote $||f_i||_{p,\Delta_i}$ the L^p-norm of f_i on Δ_i , $1 \le p \le \infty$.

3.2 PROPERTIES OF $x, \varphi, \tilde{x}_n$ AND $\tilde{\varphi}_n$

THEOREM 3 :

Let the kernel of T be of class $\mathcal{G}(\alpha,\gamma)$. Then

i) if $f \in C^\alpha$, the solution x of (1) (resp. $\tilde{x}_n$ of (6)) is such that $x \in C^\alpha$ (resp. $\tilde{x}_n \in C_\Delta^\alpha$).

ii) the eigenvector φ of (3) (resp. $\tilde{\varphi}_n$ of (7)) is such that $\varphi \in C^\alpha$ (resp. $\tilde{\varphi}_n \in C_\Delta^\alpha$). Moreover $M \subset C^\alpha$.

Proof

i) $x = \frac{1}{z}Tx - \frac{1}{z}f$, and by induction

$$x = \frac{1}{z^\alpha}T^\alpha x - [\frac{1}{z^\alpha} T^{\alpha-1} f + \frac{1}{z^{\alpha-1}} T^{\alpha-2} f + \ldots + \frac{1}{z} f]$$

where the two terms in x belong to C^α .

Now $\tilde{x}_n = \frac{1}{z} T x_n + f$, $x_n \in \mathbb{P}_{r,\Delta} \subset C^{\infty}_{\Delta}$, then by theorem 1, $Tx_n \in C^{\alpha}_{\Delta}$.

ii) $\lambda\varphi = T\varphi = \frac{1}{\lambda} T^2\varphi = \frac{1}{\lambda^{\alpha-1}} T^{\alpha}\varphi$ implies $\varphi \in C^{\alpha}$.

Similarly $\varphi_n \in \mathbb{P}_{r,\Delta}$ implies $\varphi_n = \frac{1}{\lambda_n} T\varphi_n \in C^{\alpha}_{\Delta}$.

We now consider $M = \{\varphi \in L^{\infty}(0,1) \; ; \; (T-\lambda)^m \varphi = 0\}$. By induction it can be shown that $M \subset C^{\alpha}$. Similarly $M^* \subset C^{\alpha}$. $\square$

For $x \in C^{\alpha}$, we define $||x||_{\alpha,p} := \sum_{i=0}^{\alpha} ||x^{(i)}||_p$, for $1 \le p \le \infty$.

PROPOSITION 4 :

Let the kernel of T be of class $\mathcal{G}(\alpha,\gamma)$, and f belong to C^{α} . Then the following bounds hold for n large enough

i) $||x||_{\alpha,\infty} \le c||f||_{\alpha,\infty}$

ii) $||Py||_{\alpha,\infty} \le c||y||_1$ *for any y in* $L^1(0,1)$

Proof

i) $x = R(z)f$ proves $||x||_{\infty} \le c||f||_{\infty}$. From the proof of theorem 3, we get $||x^{(\alpha)}||_{\infty} \le c(||f||_{\alpha,\infty} + ||x||_{\infty})$.

Therefore $||x||_{\alpha,\infty} \le c||f||_{\alpha,\infty}$.

ii) Let $\{x_i\}_1^m$ (resp. $\{x_i^*\}_1^m$) be a basis of M (resp. the adjoint basis of M^*). P is a finite rank integral operator defined by $P = \sum_{i=1}^{m} < x_i^* , \cdot > x_i$. Therefore

$$(Px)(s) = \sum_{i=1}^{m} [\int_0^1 x_i^*(t)\, x(t)\, dt]\, x_i(s)$$

and P is defined by the degenerate kernel $p(s,t) := \sum_{i=1}^{m} x_i(s)\, x_i^*($

where p is a smooth kernel of continuity α .We deduce that for any y in $L^1(0,1)$

$$||Py||_{\alpha,\infty} \le (\sum_{i=1}^{m} ||x_i||_{\alpha,\infty} ||x_i^*||_{\alpha,\infty})\, ||y||_1 \quad . \quad \square$$

3.3 PROPERTIES OF k_t, l_t AND l_t^1

LEMMA 5 :

Let the kernel of T be of class $\mathcal{G}(\alpha,\gamma)$. Then the functions k_t, ℓ_t and ℓ_t^1 belong to C^{α}_{Δ} for any t of the partition Δ and to C^{γ} for $t \notin \Delta$, if $\gamma \ge 0$

Proof

By (8), $k_{t_i} \in C^{\alpha}_{\Delta}$ for $t_i \in \Delta$ and $k_t \in C^{\gamma}$ for $t \notin \Delta$, $\gamma \geq 0$. The kernels of $T^* - \bar{z}$ and $(T^* - \bar{\lambda}_n)(1-P^*)$ have the same smoothness properties as k. We may then apply theorem 3 to the equations $(T^* - \bar{z})1_t = k_t$ and $(T^* - \bar{\lambda}_n)(1-P^*)1^1_t = (1-P^*)k_t$

3.4 APPROXIMATION PROPERTIES OF $\mathbb{P}_{r,\Delta}$

Let $[a,b]$ be a given bounded interval of $\mathbb{R}$. For $f \in C^{\alpha}(a,b)$, we define $\beta := \min(\alpha, r+1)$ and $[\mathcal{T}_{\beta}(f)](t) = \sum_{j=0}^{\beta-1} \frac{f^{(j)}(a)}{j!}(t-a)^j$, the Taylor series expansion of order $\beta-1$ for f at the point a, for $t \in [a,b]$.

THEOREM 6 :

For $f \in C^{\alpha}(a,b)$, then for $1 \leq p \leq \infty$

$$||f^{(j)} - \mathcal{T}^{(j)}_{\beta}(f))||_p \leq c(b-a)^{\beta-j}||f^{(\beta)}||_p \quad , \; 0 \leq j \leq \beta .$$

Proof

The proof is by induction on j : $||f^{(\beta)} - \mathcal{T}^{(\beta)}_{\beta}(f)||_p = ||f^{(\beta)}||_p$

For $\beta-1$,

$[f^{(\beta-1)} - \mathcal{T}^{(\beta-1)}_{\beta}(f)](t) = \int_a^t [f^{(\beta)} - \mathcal{T}^{(\beta)}_{\beta}(f)](s)ds$. Then, by the Hölder inequality we get :

$||f^{(\beta-1)} - \mathcal{T}^{(\beta-1)}_{\beta}(f)||_{\infty} \leq (b-a)^{1/q} ||f^{(\beta)} - \mathcal{T}^{(\beta)}_{\beta}(f)||_p$ with $1/p + 1/q = 1$, and

$$||f^{(\beta-1)} - \mathcal{T}^{(\beta-1)}_{\beta}(f)||_p \leq (b-a)^{1/p} ||f^{(\beta-1)} - \mathcal{T}^{(\beta-1)}_{\beta}(f)||_{\infty}$$
$$\leq (b-a)||f^{(\beta)}||_p .$$

We suppose that the inequality holds at the step j+1 :

$||f^{(j+1)} - \mathcal{T}^{(j+1)}_{\beta}(f)||_p \leq (b-a)^{\beta-j-1} ||f^{(\beta)}||_p$. Then

$(f^{(j)} - \mathcal{T}^{(j)}_{\beta}(f))(t) = \int_a^t [f^{(j+1)} - \mathcal{T}^{(j+1)}_{\beta}(f)](s)ds$,

$||f^{(j)} - \mathcal{T}^{(j)}_{\beta}(f)||_{\infty} \leq (b-a)^{1/q} ||f^{(j+1)} - \mathcal{T}^{(j+1)}_{\beta}(f)||_p$,

$||f^{(j)} - \mathcal{T}^{(j)}_{\beta}(f)||_p \leq (b-a)^{1/p} ||f^{(j)} - \mathcal{T}^{(j)}_{\beta}(f)||_{\infty}$.

The last two inequalities imply

$$||f^{(j)} - \mathcal{T}_\beta^{(j)}(f)||_p \le (b-a)^{\beta-j}||f^{(\beta)}||_p \ . \square$$

COROLLARY 7 :

Let π_n be a projection from C_Δ on-to $\mathbb{P}_{n,\Delta}$, such that, uniformly in n, $||\pi_n||_p < \infty$ for $1 \le p \le \infty$. Then for any f in C_Δ^α

$$||(1-\pi_n)f||_p \le c\ h^\beta||f^{(\beta)}||_p \ .$$

Proof

On Δ_i, by theorem 6,

$$||(1-\pi)f_i||_{p,\Delta_i} = ||(1-\pi)(f_i - \mathcal{T}_\beta(f_i)||_{p,\Delta_i} \le c||f_i - \mathcal{T}_\beta(f_i)||_{p,\Delta_i}$$

$$\le c\ h_i^\beta\ ||f_i^{(\beta)}||_{p,\Delta_i}$$

We first suppose that $1 \le p < \infty$.

$$(||(1-\pi_n)f||_p)^p = \sum_{i=1}^n (||(1-\pi)f_i||_{p,\Delta_i})^p \ .$$

Therefore

$$\sum_{i=1}^n (||(1-\pi)f_i||_{p,\Delta_i})^p \le c \sum_{i=1}^n h_i^{p\beta}(||f_i^{(\beta)}||p,\Delta_i)^p$$

$$\le c(\max_i h_i)^{p\beta} \sum_{i=1}^n (||f_i^{(\beta)}||_{p,\Delta_i})^p \ .$$

For $p = \infty$, $||(1-\pi_n)f||_\infty = \max_i ||(1-\pi)f_i||_{\infty,\Delta_i}$,

and $||(1-\pi)f_i||_{\infty,\Delta_i} \le c\ h_i^\beta\ ||f_i^{(\beta)}||_{\infty,\Delta_i}$

prove that $||(1-\pi_n)f||_\infty \le c\ h^\beta||f^{(\beta)}||_\infty$. $\square$

We define $\beta_1 := \min(\beta,\gamma+1) \ge 0$,

COROLLARY 8 :

Let the kernel k be of class $\mathcal{G}(\alpha,\gamma)$. Then

for $t \in \Delta$, $||(1-\pi_n)k_t||_p = O(h^\beta)$, for $1 \le p \le \infty$,

for $t \notin \Delta$, $||(1-\pi_n)k_t||_p = O(h^{\beta_1})$.

Proof

For $t \in \Delta$, $k_t \in C^{\alpha}_{\Delta}$ by lemma 5. And the proof of corollary 7 applies : we get for $1 \le p < \infty$

$$||(1-\pi_n)k_t||_p \le c\, h^{\beta}[(||k^{(\beta)}_{1t}||_{p,[0,t]})^p+(||k^{(\beta)}_{2t}||_{p,[t,1]})^p]^{1/p}$$

and

$$||(1-\pi_n)k_t||_{\infty} \le c\, h^{\beta} \max\,(||k^{(\beta)}_{1t}||_{\infty,[0,t]},\ ||k^{(\beta)}_{2t}||_{\infty,[t,1]})\ .$$

For $t \notin \Delta$, we suppose that $t \in\ ]t_{i-1}\ ,\ t_i[$.

For $s < t$ or $s > t$, k_t has continuous derivatives at s up to the order α, while for $s = t$, $(k_t)^{(\gamma)}$ is continuous for $\gamma \ge 0$. We then write for $1 \le p < \infty$.

$$(||(1-\pi_n)k_t||_p)^p = \underbrace{\sum_{j\neq i}(||(1-\pi)(k_t)_j||_{p,\Delta_j})^p}_{A}+(||(1-\pi)(k_t)_i||_{p,\Delta_i})$$

The sum $A^{1/p}$ can easily be bounded in h^{β}, as above.

On Δ_i, we have

$$(||(1-\pi)(k_t)_i||_p)^p \le (||k_t-\mathcal{T}_{\gamma+1}(k_{1t})||_{p,[t_{i-1},t]})^p+(||k_t-\mathcal{T}_{\gamma+1}(k_{2t})||_{p,[t,t_i]})$$

Clearly $(k_t-\mathcal{T}_{\gamma+1}(k_{1t}))(s) = \frac{1}{(\gamma+1)!}\, k_1^{(\gamma+1)}(t,\theta_s)(s-t)^{\gamma+1}$ for some θ_s such that $t_{i-1} < \theta_s < t$. Together with the similar equality on $[t,t_i]$ this proves that $||(1-\pi)(k_t)_i||_{p,\Delta_i}$ can be bounded in $h_i^{\gamma+1}$. The resulting global order is $\beta_1 = \min\,(\beta,\gamma+1)$. If $\gamma = -1$, $\beta_1 = 0$ and the result is clear. For $p = \infty$, the proof is similar. □

We remark that corollary 8 also applies to l_t and l_t^1 . For this latter function, which depends on n through λ_n, we have to check that the derivative can be bounded uniformly in n for n large enough. This is easily done, using theorem 3, by noting that since $\lambda_n \to \lambda \neq 0$, $|\lambda_n| > \lambda-\varepsilon$ for a given $\varepsilon > 0$ and n large enough.

4. SUPERCONVERGENCE OF $\tilde{x}_n$ AND $\tilde{\varphi}_n$ FOR THE 1-GALERKIN METHOD

In this section $X = L^2(0,1)$, π_n^1 is the orthogonal projection on $\mathbb{P}_{r,\Delta}$, and $\beta^* := \min(\beta, \gamma+2)$.

We begin by the

LEMMA 9 :

If k is of class $\mathcal{G}(\alpha,\gamma)$ and if $x \in C^\alpha$, then for $t \notin \Delta$, $|< k_t , (1 - \pi_n^1)x > | = O(h^{\beta+\beta^*})$.

Proof

$<k_t, (1-\pi_n^1)x> = <(1-\pi_n^1)k_t , (1-\pi_n^1)x>$ by the orthogonality of π_n^1. We first notice that if we use corollary 8, we get only the order $\beta+ \beta_1$ by writing

$$|<k_t,(1-\pi_n^1)x >| \le ||(1-\pi_n^1)k_t||_2 \; ||(1-\pi_n^1)x||_2 \le c\, h^{\beta_1+\beta} .$$

The optimal order $\beta + \beta^*$ is recovered by a careful use of the following bound for $x \in C_\Delta$, $i = 1,\dots,n$:

$$||x_i||_{2,\Delta_i} \le h_i^{1/2} \; ||x_i||_{\infty,\Delta_i} \le h_i^{1/2} \; ||x||_\infty .$$

$$|<k_t,(1-\pi_n^1)x >| \le \sum_{i=1}^n |< (k_t)_i,(1-\pi^1)x_i> |$$

$$\le \sum_{i=1}^n [\; ||(1-\pi^1)(k_t)_i||_{2,\Delta_i} ||(1-\pi^1)x_i||_{2,\Delta_i}]$$

$$||(1-\pi^1)x_i||_{2,\Delta_i} \le c\, h_i^\beta ||x_i^{(\beta)}||_{2,\Delta_i} \le c\, h_i^{\beta+1/2} ||x^{(\beta)}||_\infty, \text{for } i=1,\dots$$

For $t \notin \Delta$, let $t \in]t_{i-1},t_i[$, then $(k_t)_j \in C^\alpha_{(\Delta_j)}$ for $j \ne i$ and $(k_t)_i \in C^\gamma_{(\Delta_i)}$ for $\gamma \ge 0$. Then

$$||(1-\pi^1)(k_t)_j||_{2,\Delta_j} \le c\, h_j^\beta \max(||(k_{1t})_j^{(\beta)}||_{2,\Delta_j}, ||(k_{2t})_j^{(\beta)}||_{2,\Delta_j})$$

$$\le c\, h_j^{\beta+1/2} \max(||k_{1t}||_{\beta,\infty}, ||k_{2t}||_{\beta,\infty}) .$$

$$||(1-\pi^1)(k_t)_i||_{2,\Delta_i} \le c\, h_i^{\beta_1} [(||k_{1t}^{(\beta_1)}||_{2,[t_{i-1},t]})^2+(||k_{2t}^{(\beta_1)}||_{2,[t,t_i]})^2]^{1/2}$$

$$\le c\, h_i^{\beta_1+1/2} [(||k_{1t}||_{\beta_1,\infty})^2+(||k_{2t}||_{\beta_1,\infty})^2]^{1/2} .$$

The remark $\min(2\beta, \beta+\beta_1+1) = \beta+\beta^*$ ends the proof. □

We are now ready for the main

THEOREM 10 :

If the kernel of T is of class $\mathcal{G}(\alpha,\gamma)$ and if $f \in C^\alpha$, then for n large enough :

i) $||x-x_n||_2 = O(h^\beta)$, $||(1-P)\varphi_n||_2 = O(h^\beta)$

ii) $||x-\tilde{x}_n||_2 = O(h^{\beta+\beta*})$, $||(1-P)\tilde{\varphi}_n||_2 = O(h^{\beta+\beta*})$

iii) $|(x-\tilde{x}_n)(t_i)| = O(h^{2\beta})$, $|[(1-P)\tilde{\varphi}_n](t_i)| = O(h^{2\beta})$

iv) $||x-\tilde{x}_n||_\infty = O(h^{\beta+\beta*})$, $||(1-P)\tilde{\varphi}_n||_\infty = O(h^{\beta+\beta*})$

where $\beta = \min(\alpha, r+1)$ *and* $\beta* = \min(\beta, \gamma+2)$.

Proof

i) Since $x \in C^\alpha$ and $M \subset C^\alpha$, then by corollary 7

$||(1-\pi_n^1)x||_2 \le ch^\beta ||x^{(\beta)}||_2$ and

$||(1-\pi_n^1)P||_2 \le c\, h^\beta \max_{\Psi\in M}(||\Psi^{(\beta)}||_2, ||\Psi||_2 = 1)$

The result follows by (9). In full generality $\Theta_2(M,M_n) = O(h^\beta)$.

ii) $||T(1-\pi_n^1)x||_2 \le ||T(1-\pi_n^1)x||_\infty = \sup_{t\in[0,1]} |\langle k_t, (1-\pi_n^1)x\rangle|$

Application of lemma 9 proves that

$||T(1-\pi_n^1)x||_\infty \le c\, h^{\beta+\beta*} ||x^{(\beta)}||_\infty$.

$||T(1-\pi_n^1)P||_\infty$ can be bounded accordingly. The desired results follow by (11). And $\Theta_2(M,\tilde{M}_n) = O(h^{\beta+\beta*})$.

To prove iii) and iv) we write for a fixed t in [0,1]

$\langle 1_t, (1-\pi_n^1)\tilde{x}_n\rangle = \langle (1-\pi_n^1)1_t, (1-\pi_n^1)[(\tilde{x}_n - x) + x]\rangle$

$\langle 1_t^1, (1-\pi_n^1)\tilde{\varphi}_n\rangle = \langle (1-\pi_n^1)1_t^1, (1-\pi_n^1)[(1-P)\tilde{\varphi}_n + P\tilde{\varphi}_n]\rangle$

For $t \in \Delta$ $||(1-\pi_n^1)1_t||_2$ and $||(1-\pi_n^1)1_t^1||_2$ are both of order β by corollary 8, as well as the quantities

$||\tilde{x}_n - x||_2$, $||(1-\pi_n^1)x||_2$, $||(1-P)\tilde{\varphi}_n||_2$ and $||(1-\pi_n^1)P\tilde{\varphi}_n||_2$

which are of order β at least by i) and ii). This proves iii). Now for $t \notin \Delta$, $<1_t, (1-\pi_n^1)x>$ and $<1_t^1, (1-\pi_n^1)P\tilde{\varphi}_n>$ are of the order $\beta + \beta^*$ by making use of lemma 9 and of the uniform bound $||(P\tilde{\varphi}_n)^{(\beta)}||_\infty \le c||\tilde{\varphi}_n||_\infty \le c$. As for the remaining scalar products, we deduce from corollary 8 that

$\max\ (||(1-\pi_n^1)1_t||_2 , ||(1-\pi_n^1)1_t^1||_2) \le c\ h^{\beta_1}$

Therefore, since $\beta_1 \ge 0$,

$|<(1-\pi_n^1)1_t , \tilde{x}_n - x>| \le c\ h^{\beta_1}||\tilde{x}_n - x||_2 \le c\ h^{\beta+\beta*}$,

$|<(1-\pi_n^1)1_t, (1-P)\tilde{\varphi}_n>| \le c\ h^{\beta_1}||(1-P)\tilde{\varphi}_n||_2 \le c\ h^{\beta+\beta*}$. □

5. CONNECTION BETWEEN THE ⊥-GALERKIN AND COLLOCATION METHODS

We introduce the following definitions and notation .
Let be given the r+1 Gauss points $\{\tau_i\}_1^{r+1}$ of the interval [a,b], for $g \in C(a,b)$, let $\pi^2 g$ be the Lagrange interpolation polynomial of degree $\le r$ at the r+1 points $\{\tau_i\}$. $\pi^1 g$ is the least squares polynomial of degree $\le r$ on [a,b].
We define the polynomial $v : s \in [a,b] \mapsto \prod_{i=1}^{r+1} (s-\tau_i)$. Given a function f vanishing at $\{\tau_i\}$, its $(r+1)^{th}$ divided difference at the points $\{\tau_i\}$ is denoted $\delta[\tau_1,\ldots,\tau_{r+1},.]\ f$, and is abbreviated $\delta^{(r+1)}f$. We have

$f(s) = v(s)\ \delta[\tau_1,\ldots,\tau_{r+1},s]\ f$ for $s \notin \{\tau_i\}_1^{r+1}$, the value for $s \in \{\tau_i\}$ being defined by continuity.

LEMMA 11 :

For f and g in $C(a,b)$, then
$<f,(1-\pi^2)g> = <(1-\pi^1)\ f\ \delta^{(r+1)}g\ ,\ v>$

Proof

By definition $(1-\pi^2)g$ vanishes at the $\{\tau_i\}$, $\pi^2 g \in \mathbb{P}_r$ and $\delta^{(r+1)}\pi^2 g = 0$. Then

$(1-\pi^2)g = v\,\delta^{(r+1)}((1-\pi^2)g) = v\,\delta^{(r+1)}g$.

Therefore

$\langle f,(1-\pi^2)g\rangle = \langle f\delta^{(r+1)}g,\ v\rangle$.

Because the $\{\tau_i\}_1^{r+1}$ are the *r+1 Gausspoints* on [a,b] , then $\int_a^b v(s)\,p(s)ds = 0$ for any polynomial p of degree $\leq r$.

v is orthogonal to $\mathbb{P}_r$, that is $\pi^1 v = 0$, or

$\langle f,(1-\pi^2)g\rangle = \langle f\delta^{(r+1)}g,\ (1-\pi^1)v\rangle = \langle(1-\pi^1)f\,\delta^{(r+1)}g,v\rangle$. □

COROLLARY 12 :

Let η *be a nonnegative integer. If* $f \in C^{\eta}(a,b)$ *and* $g \in C^{\eta+r+1}(a,b)$ *then*

$$|\langle f,(1-\pi^2)g\rangle| \leq c(b-a)^{r+1+\beta'+1}\|f\|_{\beta',\infty}\|g\|_{\beta'+r+1,\infty}\ ,$$

where $\beta' := \min(\eta,\ r+1)$.

Proof By lemma 11

$$|\langle f,(1-\pi^2)g\rangle| \leq \|(1-\pi^1)v\|_2\ \|(1-\pi^1)f\,\delta^{(r+1)}g\|_2$$

Since $v \in \mathbb{P}_{r+1}$

$$\|(1-\pi^1)v\|_2 \leq (b-a)^{r+1}\|v^{(r+1)}\|_2 \leq (b-a)^{r+1+1/2}\ \|v^{(r+1)}\|_\infty$$

Similarly

$$\|(1-\pi^1)f\delta^{(r+1)}g\|_2 \leq (b-a)^{\beta'}\|(f\delta^{(r+1)}g)^{(\beta')}\|_2$$
$$\leq (b-a)^{\beta'+1/2}\|f\|_{\beta',\infty}\|g\|_{\beta'+r+1,\infty}\ .\ \square$$

Lemma 11 expresses the relationship between the $\perp$-Galerkin method and the collocation method at Gaussian points. This may be related to the fact that in the engineering literature, collocation at Gaussian points is very adequately called *orthogonal* collocation, cf. Prenter [13] .

Lemma 11 explains the asymmetry of the roles played by f and g, by the presence of the divided difference $\delta^{(r+1)}g$ .One consequence is expressed by corollary 12 where more smoothness is required

from g. The asymmetry is stressed if we write the corresponding bound for π^1 which is symmetric in f and g :

if $f,g \in C^{\alpha}(a,b)$, $|\langle f,(1-\pi^1)g\rangle| \le c(b-a)^{2\beta+1} \|f^{(\beta)}\|_{\infty}\|g^{(\beta)}\|_{\infty}$,

where $\beta = \min(\alpha, r+1)$, (cf. corollary 7 and lemma 9).

As another consequence, higher order convergence results on $\tilde{x}_n$ and $\tilde{\varphi}_n$ for collocation at Gaussian points will be obtained only if $\alpha \ge r+1$, in opposition with what happens in the $\perp$-Galerkin method. It is fair to say that this fact has no practical importance since r is chosen such that $r+1 \le \alpha$ to take full advantage of the accuracy of functions in $\mathbb{P}_{r,\Delta}$. Nevertheless the *double* accuracy h^{2r+2} will be obtained only when $\alpha \ge 2r+2$ in case (b), when it is obtained already for $\alpha \ge r+1$ in case (a).

Our analysis of higher order convergence results obtained by methods (a) and (b) can be compared to that given in [5] by de Boor-Swartz in the following way. We start from (a) to explain (b), when they start from (b) to study (a). As a result, we would need more smoothness requirement on k and f to deduce the superconvergence results given in theorem 10, if we applied results of [5] to an integral equation.

6. SUPERCONVERGENCE OF $\tilde{x}_n$ AND $\tilde{\varphi}_n$ FOR THE COLLOCATION METHOD AT GAUSSIAN POINTS

We consider now the collocation method (b) defined in C(0,1). The collocation points $\{\tau_j^i\}$ are chosen such that the $\{\tau_j^i\}_{j=1}^{r+1}$ are the r+1 Gauss points on each Δ_i , $i = 1,\dots,n$.

π_n^1 (resp. π_n^2) is the orthogonal (resp. interpolatory) projection on $\mathbb{P}_{r,\Delta}$.

When we deal with the L^2-scalar product and adjoint operators, the continuous functions under consideration are taken as elements of L^2.

The following result is standard.

PROPOSITION 13 :

If the kernel of T is of class $\mathcal{G}(\alpha,\gamma)$ and if $f \in C^{\alpha}$ then $||x-x_n||_{\infty}$, $||(1-P)\varphi_n||_{\infty}$ and $\Theta_{\infty}(M,M_n)$ are of the order of $\beta = min(\alpha,\ r+1)$.

If we suppose that $\alpha \geq r+1$, then $\beta = r+1$ and we can prove the superconvergence results for $\tilde{x}_n$ and $\tilde{\varphi}_n$ given in theorem 15, by using corollary 12 on the Δ_i , $i = 1,\ldots,n$.
Before proving this main result we need some preparation.
We recall the Markov inequality [2] for polynomials of degree $\leq$ r written on Δ_i :

$$||q||_{r,\infty,\Delta_i} \leq c\ h_i^{-r}||q||_{\infty,\Delta_i} \quad \text{for any } q \in \mathbb{P}_r(\Delta_i)$$

LEMMA 14 :

Under the hypothesis of proposition 13, if $\alpha \geq r+1$ then

$$||x_n-x||_{r,\infty,\Delta_i} \leq c(h/h_i)^r\ h\ ||x||_{r+1,\infty}\ ,\ \text{for } i = 1,\ldots,n.$$

$$||(1-P)\varphi_n||_{r,\infty,\Delta_i} \leq c(h/h_i)^r\ h||P\varphi_n||_{r+1,\infty}\ ,\ \text{for } i = 1,\ldots,$$

Since there is no ambiguity we have omitted the index i for the functions defined on Δ_i .

Proof

Since $\alpha \geq r+1$, x has a continuous r^{th} derivative at least. We define on Δ_i the Taylor expansion of x of order r, $\mathcal{T}_{r+1}(x) \in \mathbb{P}_r(\Delta$

$$||x-x_n||_{r,\infty,\Delta_i} \leq ||x-\mathcal{T}_{r+1}(x)||_{r,\infty,\Delta_i}+||\mathcal{T}_{r+1}(x)-x_n||_{r,\infty,\Delta_i}$$

By application of the Markov inequality to $\mathcal{T}_{r+1}(x)-x_n$ we get

$$||\mathcal{T}_{r+1}(x)-x_n||_{r,\infty,\Delta_i} \leq c\ h_i^{-r}||\mathcal{T}_{r+1}(x)-x_n||_{\infty,\Delta_i}$$

$$||\mathcal{T}_{r+1}(x)-x_n||_{\infty,\Delta_i} \leq ||\mathcal{T}_{r+1}(x)-x||_{\infty,\Delta_i}+||x-x_n||_{\infty,\Delta_i}$$

By proposition 13, $||x-x_n||_{\infty,\Delta_i} \leq c\ h^{r+1}||x||_{r+1,\infty}$

By theorem 6,

$$||x-\mathcal{T}_{r+1}(x)||_{r,\infty,\Delta_i} \leq ||x-\mathcal{T}_{r+1}(x)||_{\infty,\Delta_i} \leq c\ h^{r+1}||x||_{r+1,\infty}\ ,$$

and the desired bound follows for $||x-x_n||_{r,\infty,\Delta_i}$.

The proof for $\varphi_n - P\varphi_n$ is quite similar by writing

$\varphi_n - P\varphi_n = \varphi_n - \mathcal{C}_{r+1}(P\varphi_n) + \mathcal{C}_{r+1}(P\varphi_n) - P\varphi_n$

and using $||P\varphi_n||_{r+1,\infty} < c$ since $\alpha \geq r+1$. □

α being $\geq r+1$, we write $\alpha = \eta + r + 1$ where η is a nonnegative integer. And we define

$\beta' := \min(\eta, r+1), \beta'_1 := \min(\beta', \gamma+1), \beta'^* := \min(\beta', \gamma+2)$.

<u>THEOREM 15</u> :

Let the kernel of T be of class $\mathcal{G}(\alpha,\gamma)$ with $\alpha \geq r+1$, $\alpha \geq \gamma \geq -1$. If $f \in C^\alpha$ then for n large enough,

i) $||x-\tilde{x}_n||_\infty = O(h^{r+1+\beta'^*})$, $||(1-P)\tilde{\varphi}_n||_\infty = O(h^{r+1+\beta'^*})$

ii) $|(x-\tilde{x}_n)(t_i)| = O(h^{r+1+\beta'})$, $|[(1-P)\tilde{\varphi}_n](t_i)| = O(h^{r+1+\beta'})$

<u>Proof</u>

i) By (11)

$||x-\tilde{x}_n||_\infty \leq c||T(1-\pi_n^2)x||_\infty$, $||(1-P)\tilde{\varphi}_n||_\infty \leq c||T(1-\pi_n^2)P||_\infty$,

with $||T(1-\pi_n^2)x||_\infty = \sup_{t\in[0,1]} |\langle k_t, (1-\pi_n^2)x\rangle|$.

For $t = t_i \in \Delta$, $k_t \in C^\alpha_\Delta$, therefore by corollary 12 on Δ_j , $j = 1,\dots,n$ we get

$|\langle k_t, (1-\pi^2)x_j\rangle_{\Delta_j}| \leq c\, h_j^{r+1+\beta'+1} ||(k_t)_j||_{\beta',\infty,\Delta_j} ||x_j||_{\beta'+r+1,\infty,\Delta_j}$

By summing over j

$|\langle k_t, (1-\pi_n^2)x\rangle| \leq c\, h^{r+1+\beta'} ||x||_{\beta'+r+1,\infty}$.

For $t \notin \Delta$, we suppose that $t \in]t_{i-1}, t_i[$. The above bounds are valid on Δ_j for all $j \neq i$.

On Δ_i, $(k_t)_i \in C^{\gamma+1}$ for $s \in [t_{i-1}, t]$ and $[t, t_{i+1}]$

$|\langle k_t, (1-\pi^2)\, x\rangle_{\Delta_i}| \leq c\, h_i^{r+1+\beta'_1+1} ||x_i||_{\beta'_1+r+1,\infty,\Delta_i}$

By summing over j

$|\langle k_t, (1-\pi_n^2)\, x\rangle| \leq c\, h^{r+1+\min(\beta',\beta'_1+1)}$.

The order $r + 1 + \beta'^*$ results from $\min(\beta', \beta_1' + 1) = \beta'^*$.

$||T(1-\pi_n^2)P||_\infty$ is treated in the same way. We deduce in general that $\Theta_\infty(M, \tilde{M}_n) = 0(h^{\beta+\beta'*})$.

ii) We first study $x-\tilde{x}_n$. For $t = t_i \in \Delta$ we write by (10)

$$(x-\tilde{x}_n)(t) = < 1_t , (1-\pi_n^2)\tilde{x}_n >$$

$$= <1_t, (1-\pi_n^2)x> + <1_t, (1-\pi_n^2)(\tilde{x}_n-x)>$$

$$= <1_t, (1-\pi_n^2)x> + \sum_{j=1}^{n} <1_t, (1-\pi^2)(\tilde{x}_n-x)>_{\Delta_j}$$

The first scalar product is of the same order as $< k_t, (1-\pi_n^2)x >$ that is $r+1+\beta'$. As for the second

$$|<(1_t)_j, (1-\pi^2)(\tilde{x}_n-x)_j>| \leq c\, h_j^{r+1+\beta'+1} ||(1_t)_j||_{\beta',\infty,\Delta_j} ||(\tilde{x}_n-x)_j||_{\beta'+r+1,\infty,\Delta_j} .$$

It remains to bound $||\tilde{x}_n-x||_{\beta'+r+1,\infty}$ uniformly in n, with $\beta'+r+1 \leq \alpha$.

First we note that $\tilde{x}_n-x = \frac{1}{z} T(x_n-x)$, then

$$||x_n-x||_{\beta'+r+1,\infty,\Delta_j} \leq c[\, ||x-x_n||_\infty + ||x_n-x||_{\beta'+r,\infty,\Delta_j} \,] .$$

$||x-x_n||_\infty = 0(h^{r+1})$ is bounded, and x_n on Δ_j is a polynomial of degree $\leq r$, therefore

$$||x-x_n||_{\beta'+r,\infty,\Delta_j} \leq ||x||_{\beta'+r,\infty,\Delta_j} + ||x_n-x||_{r,\infty,\Delta_j} .$$

By application of lemma 14 we may write for $j = 1,\ldots,n$

$(h/h_j)^r\, h \leq (h/h_j)^{r+1}$ and

$$|< 1_t, (1-\pi^2)(\tilde{x}_n-x)>_{\Delta_j}| \leq c\, h_j^{r+1+\beta'+1} (h/h_j)^{r+1} ||x||_{\beta'+r+1,\infty,\Delta_j}$$

By summing over j

$$|< 1_t, (1-\pi_n^2)(\tilde{x}_n-x)>| \leq c\, h^{r+1+\beta'} ||x||_{\beta'+r+1,\infty} .$$

The analysis of $(1-P)\tilde{\varphi}_n$ is similar to that of $x-\tilde{x}_n$:

$$[(1-P)\tilde{\varphi}_n](t) = < 1_t^1, (1-\pi_n^2)P\tilde{\varphi}_n> + < 1_t^1, (1-\pi_n^2)(1-P)\tilde{\varphi}_n > .$$

For the first scalar product, $||P\tilde{\varphi}_n||_{\beta'+r+1,\infty}$ is uniformly bounded in n by proposition 4, since $\beta' + r + 1 \leq \alpha$.

For the second scalar product $(1-P)\tilde{\varphi}_n = \frac{1}{\lambda_n} T(1-P)\varphi_n$ and $||((1-P)\varphi_n)_j||_{\beta'+r,\infty,\Delta_j}$ can be bounded by means of lemma 14, much like this was done for the solution. This ends the proof. □

The only nontrivial part of the proof of theorem 15 is to bound $||\tilde{x}_n - x||_{\alpha,\infty}$ (resp. $||(1-P)\tilde{\varphi}_n||_{\alpha,\infty}$) uniformly in n. It uses the technique developed by de Boor-Swartz in [2] . It should be noticed that in case of a smooth kernel, when $\gamma=\alpha$, $||\tilde{x}_n||_{\alpha,\infty}$ (resp. $||\tilde{\varphi}_n||_{\alpha,\infty}$) is bounded uniformly by the lemma 16 given below. Then the proof of theorem 15 becomes straight-forward.

LEMMA 16 :

If the kernel of T is smooth of order α, then the following bounds hold for n large enough

$$||\tilde{x}_n||_{\alpha,\infty} \le c||f||_{\alpha,\infty} \quad , \quad ||\tilde{\varphi}_n||_{\alpha,\infty} \le c||\varphi_n||_\infty \ .$$

Proof

$\tilde{x}_n = \frac{1}{z} T x_n + f$, then

$||\tilde{x}_n||_{\alpha,\infty} \le c(||x_n||_\infty + ||f||_{\alpha,\infty}) \le c\ ||f||_{\alpha,\infty}$

Similarly $\tilde{\varphi}_n = \frac{1}{\lambda_n} T\varphi_n$,

$||\tilde{\varphi}_n||_{\alpha,\infty} \le \frac{c}{|\lambda_n|}||\varphi_n||_\infty$ and $||\varphi_n||_\infty = 1$, $|\lambda_n| > |\lambda| - \varepsilon$ for n large enough. □

7. BOUNDS ON $\lambda - \hat{\lambda}_n$, $\lambda - \lambda_n$ AND $\mathrm{dist}_\infty(\tilde{\varphi}_n, E)$

For the eigenvalues and eigenvectors, the following orders of convergence can be established (assuming that $\alpha \ge r+1$ for (b)) :

i) $\lambda - \hat{\lambda}_n = O(\varepsilon_n)$, $\max_i|\lambda-\lambda_{in}| = O(\varepsilon_n^{1/\ell})$, $\min_i|\lambda-\lambda_{in}| = O(\varepsilon_n^{g/m})$

where in case (a) $\varepsilon_n = h^{2\beta}$,

(b) $\varepsilon_n = h^{r+1+\beta'}$.

ii) for $\ell = 1$, $\mathrm{dist}_\infty(\tilde{\varphi}_n, E) \equiv \mathrm{dist}_\infty(\tilde{\varphi}_n, M) = \begin{cases} O(h^{\beta+\beta^*}) & \text{for (a)} \\ O(h^{r+1+\beta'^*}) & \text{for (b)} \end{cases}$,

iii) for $\ell > 1$, $\mathrm{dist}_\infty(\tilde{\varphi}_n, E) = O(\varepsilon_n^{1/\ell})$, where ε_n is as defined in i).

Everything relies on the bound for

$$\lambda - \hat{\lambda}_n = \frac{1}{m} \sum_{i=1}^{m} \langle x_i^*, (1-\pi_n)T(P_{\wedge M_n})^{-1} x_i \rangle \qquad (12)$$

Let us set $y_{in} := (P_{\wedge M_n})^{-1} x_i \in \mathbb{P}_{r,\Delta}$, $i = 1,\dots,m$. The question is to give the order of the scalar product

$\langle x_i^*, (1-\pi_n)T\, y_{in} \rangle = \langle x_i^*, (1-\pi_n)T[(y_{in} - x_i) + x_i] \rangle$ with $x_i^* \in C^\alpha$, $y_{in} \in M_n$.

By (13), $||y_{in} - x_i||_\nu \le c\, ||(1-\pi_n)P||_\nu$,

which shows that $||y_{in} - x_i||_\nu$ is of the order of $\beta = \min(\alpha, r+1)$.

The order of $\langle x_i^*, (1-\pi_n)T\, x_i \rangle$ is clear by application of corolla 7 with $p = 2$ for the case (a), and of theorem 15 for the case (b).

It remains to deal with $\langle x_i^*, (1-\pi_n)T(y_{in} - x_i) \rangle$.

(a) The order 2β follows readily from

$$|\langle x_i^*, (1-\pi_n^1)T\,(y_{in} - x_i) \rangle| \le c\, ||(1-\pi_n^1)x_i^*||_2 \; ||y_{in} - x_i||_2 .$$

(b) It remains to bound uniformly in n all the derivatives of $(1-P)y_{in} = y_{in} - x_i$ up to the order $\beta' + r$. This can be done by means of lemma 14, since $||y_{in} - x_i||_\infty = O(h^{r+1})$ when $\alpha \ge r+1$. The proof of theorem 15 applies and the order $r+1+\beta'$ follows.

The bounds on $\mathrm{dist}_\infty(\tilde{\varphi}_n, E)$ are derived from the order of $|\lambda - \lambda_n|$ where λ_n is the eigenvalue of T_n such that $\tilde{\varphi}_n = \frac{1}{\lambda_n} T \varphi_n$.

CONCLUSION

The aim of this paper has been threefold :

- give a new proof of superconvergence results for the $\perp$-Galerkin method which does not use the assumption that Δ is quasi-uniform
- treat the eigenvectors in much the same manner as the solutions.
- do a parallel study of the $\perp$-Galerkin method and collocation at Gaussian points.

One may notice that in our treatment of the $\perp$-Galerkin method, it is not necessary to bound uniformly in n some derivative of $\tilde{x}_n$ or $\tilde{\varphi}_n$ even to prove the double accuracy at the partition points. This is only required for the collocation method.

In a forthcoming paper [10] , the same analysis is applied to the solution of an o.d.e. and to the generalized (differential and integro-differential) eigenvalue problem.

Also in [11] , superconvergence at the partition points for the iterated subspace TM_n is established. This cannot be done with the above analysis because the identity (10) depends crucially on the fact that $\tilde{\varphi}_n$ is an eigenvector.

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CHAPITRE 3

SUPERCONVERGENCE RESULTS FOR DIFFERENTIAL EIGENVALUE PROBLEMS : A GALERKIN METHOD ON THE INTEGRAL FORMULATION

SUPERCONVERGENCE RESULTS FOR DIFFERENTIAL EIGENVALUE PROBLEMS : A GALERKIN METHOD ON THE INTEGRAL FORMULATION

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Abstract

We consider the simple and generalized differential eigenvalue problem associated with a 2-point linear boundary value problem of order p. The approximation method is the method of moments and the collocation method at Gaussian points, defined on piewewise polynomial functions of degree $\leq p+r$, where the partition is quite arbitrary. h is the mesh size, optimal orders of convergence are established, h^{2r+2} at the partition points, h^{r+1+p} globally, $p<r+1$, which are the same than for the solution. More smoothness is required on the coefficients $\{a_i\}_0^{p-1}$ for the collocation method $(a_i \in C^{2r+2})$ than for the method of moments $(a_i \in C^{r+1})$.

1. INTRODUCTION

We consider some projection methods of moment- and collocation- type for the differential equation

$$\underline{L}u = f \tag{1}$$

and the eigenvalue problems

$$\underline{L}\Psi = \lambda\Psi \quad , \text{ and} \tag{2}$$

$$\underline{L}\Psi = \lambda\underline{N}\Psi \quad , \tag{3}$$

where $\underline{L}$ and $\underline{N}$ are respectively the linear differential forms :

$$\underline{L}u := u^{(p)} - \sum_{i=0}^{p-1} a_i(.)\, u^{(i)} = u^{(p)} - \underline{M}u \quad ,$$

$$\underline{N}u := \sum_{i=0}^{p-1} b_i(.)\, u^{(i)} \quad .$$

The variable runs over $[0,1]$, and the coefficients $\{a_i, b_i\}_0^{p-1}$ are continuous real or complex functions.

u and Ψ are subjected to the p homogeneous boundary conditions :

$$\ell_k(u) := \sum_{i=0}^{p-1} [\alpha_{ik}\, u^{(i)}(0) + \beta_{ik}\, u^{(i)}(1)] = 0, \; k = 1,\dots,p. \tag{4}$$

Let T be the differential operator in $X = L^2(0,1)$ or $L^\infty(0,1)$ defined for $u \in \underline{D}$ by $Tu = \underline{L}u$, where

$$\underline{D} := \{u \in X \; ; \; u^{(p)} \in X \text{ and } \ell_k(u) = 0 \; , \; k = 1,\dots,p \} \; .$$

The equations can be written respectively

$$Tu = f \quad , \quad f \in C(0,1) \tag{1'}$$

$$T\Psi = \lambda\Psi \tag{2'}$$

$$T\Psi = \lambda\underline{N}\Psi \tag{3'}$$

We assume that (1') has a unique solution, then T^{-1} is bounded and $\lambda \neq 0$ for (2'). We also assume that $\lambda \neq 0$ for (3') . We will see later that λ is an isolated eigenvalue with finite multiplicity.

The principle of the projection method is to compute an approximate solution u_n (resp. eigenvector Ψ_n) in a subspace X_n of dimension N(say) by imposing N linear conditions on the residual $Tu_n - f$ (resp. $T\Psi_n - \lambda_n \Psi_n$ or $T\Psi_n - \lambda_n \underline{N} \Psi_n$) :

(a) in the method of moments, the residual should be orthogonal to some given subspace Y_n of same dimension N,

(b) in the collocation method, the residual should vanish at N prescribed points of $[0,1]$.

Various methods are obtained from various choices of the subspaces X_n and Y_n .

We describe now the choice we have made.

Let be given $\Delta = \{t_i\}_0^n$ a strict partition of $[0,1]$, $t_0 = 0$, $t_n = 1$. Y_n is the set $\mathbb{P}_{r,\Delta}$ of piecewise polynomials of degree $\leq r$ on each subinterval $[t_{i-1},t_i[$, $i = 1,\dots,n$, the value at 1 being defined by continuity. Let $\mathbb{P}^{p-1}_{p+r,\Delta}$ be the set of functions in $C^{p-1}(0,1)$ which are polynomials of degree $\leq p+r$ on $[t_{i-1},t_i[$, $i = 1,\dots,n$. We define $X_n := C_n$ the set of functions in $\mathbb{P}^{p-1}_{p+r,\Delta}$ which satisfy the boundary conditions (4). X_n and Y_n have the same dimension $N=n(r+1)$. C_Δ is the space of piecewise continuous functions on each subinterval, $C_\Delta \subset L^\infty(0,1)$.

The approximate solution $u_n \in C_n$ is determined so that

(a) $\underline{L}u_n - f$ is orthogonal to $\mathbb{P}_{r,\Delta}$ in $L^2(0,1)$,

(b) $\underline{L}u_n - f$ vanishes at the r+1 Gauss points $\{\tau_i^j\}_{j=1}^{r+1}$ defined in each subinterval $[t_{i-1},t_i]$, $i=1,\dots,n$.

Let π_n represent any one of the two projections :

$\pi_n^1 : L^2(0,1) \rightarrow \mathbb{P}_{r,\Delta}$, the orthogonal projection on Y_n ,

$\pi_n^2 : C_\Delta \rightarrow \mathbb{P}_{r,\Delta}$: for $x \in C_\Delta$, $\pi_n^2 x$ is its piecewise Lagrange interpolati of degree $\leq r$ at the points $\{\tau_i^j\}$.

The condition to determine u_n may be written under the single form

$$\pi_n[\underline{L}u_n - f] = 0 \ , \ u_n \in C_n \ . \qquad (5)$$

Similarly λ_n, Ψ_n are determined by

$$\pi_n [\underline{L}\ \Psi_n - \lambda_n\ \Psi_n] = 0 \ , \quad \Psi_n \in C_n \ , \qquad (6)$$

$$\pi_n [\underline{L}\ \Psi_n - \lambda_n\ \underline{N}\ \Psi_n] = 0 \ , \quad \Psi_n \in C_n \ . \qquad (7) \ .$$

The aim of the paper is to prove that the same superconvergence properties hold for the eigenvector Ψ_n at the partition points, than for the solution u_n . The proof is based on the interpretation of the projection method on the differential equation as a Galerkin method on the integral formulation. The analysis given in Chatelin-Lebbar [6] can then be applied here to prove very easily the optimal orders. This paper should be seen as a companion paper for [6] .

This approach differs significantly from the one taken by de Boor-Swartz [1,2,3,4] . We feel that the present alternative point of view has its own interest, since once the integral problem is treated, the result for the differential problem follows from the same theory : the key is again to write the error formulae as appropriate inner products.

All the remarks given in [6] about the differences between our approach and de Boor-Swartz's remain valid.

2. INTEGRAL FORMULATION

X is again $L^2(0,1)$ or $L^\infty(0,1)$. We suppose throughout that the problem $\begin{cases} u^{(p)}=0 \\ (4) \end{cases}$ has u = 0 for unique solution. Then there exists a unique Green's function g(t,s) such that the solution u of $\begin{cases} u^{(p)}=x \\ (4) \end{cases}$ is given by $u(t) = \int_0^1 g(t,s)x(s)ds$:

this defines the integral operator G by u = Gx . G is the inverse of the differential operator D^P : $u \in \underline{D} \to u^{(p)}$.

$$(1') \iff \begin{cases} u=Gx \\ x = Kx + f \end{cases} \qquad (8)$$

$$(2') \iff \begin{cases} \Psi=G\phi \\ (1-K)\phi = \lambda G\phi \end{cases} \qquad (9)$$

$$(3') \iff \begin{cases} \Psi=G\phi \\ (1-K)\phi = \lambda \underline{N}G\phi = \lambda L\phi \end{cases} \qquad (10)$$

where the integral operator $K := \underline{M}G$ is defined by

$$Kx = \int_0^1 k(.,s)x(s)ds \ ,$$
$$k(t,s) = \sum_{i=0}^{p-1} a_i(t) \frac{\partial^i g(t,s)}{\partial t^i} \ , \quad 0 \le t \ , \ s \le 1 \ .$$

We remark that k has a first kind discontinuity on $s = t$. K is compact as an operator from $L^p(0,1)$ into $C(0,1)$, $2 \le p \le \infty$, by the characterization of compact operators given in Graham-Sloan [8] . K is also compact from L^p to L^p . The same property is true for the integral operator $L := \underline{N}G$ whose kernel is

$$\sum_{i=0}^{p-1} b_i(t) \frac{\partial^i g(t,s)}{\partial t^i} \ .$$

By assumption $(1-K)^{-1}$ is bounded, so that (8) has the unique solution $x = (1-K)^{-1}f$, $u = Gx$.

(9) is a *generalized* eigenvalue problem. It has the following equivalent form of a simple eigenvalue problem :

$$(9) \iff \phi = \lambda \ (1-K)^{-1}G\phi = \lambda U\phi \qquad (11)$$

with $U := (1-K)^{-1}G = G^{-1} \ G(1-K)^{-1}G = G^{-1} \ T^{-1} \ G$.

Let Γ be a closed Jordan curve around $\lambda \neq 0$ which does not enclose 0. $P = \frac{-1}{2i\pi} \int_\Gamma (T-z)^{-1} dz$ is the spectral projection for T, associated with λ . z stands for z1 , where 1 is the identity operator on X . Then

$$Q = \frac{-1}{2i\pi} \int_\Gamma (1-K-zG)^{-1}Gdz \underset{t=1/z}{=} \frac{-1}{2i\pi} \int_{\Gamma(1/\lambda)} (U-t)^{-1}dt = G^{-1}PG$$

is the spectral projection for U associated with $1/\lambda$. $\Gamma(1/\lambda)$ is a Jordan curve around $1/\lambda$.

Similarly

$$(10) \iff \phi = \lambda \ (1-K)^{-1}\underline{N}G\phi = \lambda(1-K)^{-1}L\phi = \lambda V\phi \qquad (12)$$

with $L := \underline{N}G$, $V := (1-K)^{-1} \ L = G^{-1}(T^{-1} \ \underline{N})G$.

Now the spectral projection for T, associated with λ is

$$P = \frac{-1}{2i\pi} \int_\Gamma (T-z\underline{N})^{-1} \underline{N} \, dz ,$$

and the spectral projection for V, associated with $1/\lambda$ is

$$Q = \frac{-1}{2i\pi} \int_\Gamma (1-K-z\underline{N}G)^{-1} \underline{N}G \, dz = G^{-1}PG .$$

The main difference between (11) and (12) is that $G = 1_\bullet G$ is invertible, whereas $L = \underline{N}G$ is not invertible.

We are then lead to introduce a second equivalent form to (10) and (12) :

$$(10) \Longleftrightarrow (12) \Longleftrightarrow \begin{cases} \phi = (1-K)^{-1}\xi \\ \xi = \lambda \underline{N}T^{-1} \xi \end{cases} \qquad (13)$$

with $\underline{N}\, T^{-1} = L(1-K)^{-1} = (1-K)V(1-K)^{-1}$. The corresponding spectral projection is denoted R, and the following relationships hold

$$R = (1-K)\, Q(1-K)^{-1} = TPT^{-1}$$

Note that if $\underline{N} = 1$, $L=G$, $\xi=\phi$ and $R=P$ since (13) $\Longleftrightarrow$ $\phi = \lambda T^{-1}\phi$.

The approximation equations (5), (6), (7) have now the respective equivalent forms :

$$(5) \Longleftrightarrow \begin{cases} u_n = Gx_n \\ (1-\pi_n K)x_n = \pi_n f , \quad x_n \in \mathbb{P}_{r,\Delta} \end{cases} \qquad (14)$$

$$(6) \Longleftrightarrow \begin{cases} \Psi_n = G\,\phi_n \\ (1-\pi_n K)\,\phi_n = \lambda_n\, \pi_n\, G\,\phi_n , \quad \phi_n \in \mathbb{P}_{r,\Delta} \end{cases} \qquad (15)$$

$$(7) \Longleftrightarrow \begin{cases} \Psi_n = G\,\phi_n \\ (1-\pi_n K)\phi_n = \lambda_n \pi_n L\,\phi_n , \quad \phi_n \in \mathbb{P}_{r,\Delta} \end{cases} \qquad (16)$$

The projection method on the differential problem is then equivalent to a Galerkin method on the integral formulation, defined by the subspace $\mathbb{P}_{r,\Delta}$. We remark that , even though (2') is a simple eigenvalu problem, its approximation (6) leads to a generalized eigenvalue problem (12). This fact will then force us to some preparation before being able to apply the results of [6] to (2'). This is the subject of the next section.

3. CONVERGENCE

Since $u-u_n = G(x-x_n)$ and $\Psi - \Psi_n = G(\phi-\phi_n)$, the convergences $u_n \to u$ and $\Psi_n \to \Psi$ will follow easily from $x_n \to x$ and $\phi_n \to \phi$. Hence we have to study the equations (14), (15) and (16).

The reference space X is taken to be $L^2(0,1)$ in case (a) (resp. $C(0,1)$ in case (b)). The norm in L^2 (resp. L^∞ or C)) is denoted $||.||_2$ (resp. $||.||_\infty$). Throughout the text, $||.||_\nu$ is taken to be $||.||_2$ in case (a) (resp. $||.||_\infty$ in case (b)).

It is well known that if $h := \max_i (t_i-t_{i-1}) \to 0$ then $||(\pi_n^1-1)x||_2 \to 0$ for any x in $L^2(0,1)$ (resp. $||(\pi_n^2-1)x||_\infty \to 0$ for any x in $C(0,1)$). We recall that π_n^2 defined from $C(0,1)$ onto $\mathbb{P}_{r,\Delta}$ is *not* a projection. Now K is compact in L^2 (resp. L^∞), as well as $G, L, U = (1-K)^{-1}G$ and $V = (1-K)^{-1} L$. Then $||(1-\pi_n)K||_\nu \to 0$ and the convergence $||x_n-x||_\nu \to 0$ follows at once. For n large enough $(1-\pi_n K)^{-1} = (1-K)^{-1} [1+(1-\pi_n)K(1-K)^{-1}]^{-1}$ exists and is uniformly bounded. Hence we may define for n large enough :

$$U_n := (1-\pi_n K)^{-1} \pi_n G \quad \text{and} \quad V_n := (1-\pi_n K)^{-1} \pi_n L ,$$

and we write

$$(15) \iff \phi_n = \lambda_n U_n \phi_n ,$$

$$(16) \iff \phi_n = \lambda_n V_n \phi_n .$$

We study the convergences $U_n \to U$ and $V_n \to V$.

$$\begin{aligned} U-U_n &= (1-K)^{-1}G - (1-\pi_n K)^{-1} \pi_n G = \\ &= (1-\pi_n K)^{-1}(1-\pi_n)G + [(1-K)^{-1}-(1-\pi_n K)^{-1}] G \\ &= (1-\pi_n K)^{-1}(1-\pi_n) [1+K(1-K)^{-1}]G \\ &= (1-\pi_n K)^{-1}(1-\pi_n)(1-K)^{-1}G = (1-\pi_n K)^{-1}(1-\pi_n)U . \end{aligned}$$

Similarly $V-V_n = (1-\pi_n K)^{-1} (1-\pi_n)V$. Therefore $||U-U_n||_\nu \to 0$ and $||V-V_n||_\nu \to 0$, as $n \to \infty$.

Let $\lambda \neq 0$ be an eigenvalue of (11) or (12). From the compactness of U and V, we deduce that λ is isolated with finite multiplicity m (say) It is standard (cf. [5] for example) to deduce the strong stability of the approximations U_n and V_n around λ, and the convergences

$$|\lambda-\lambda_n| \to 0 \text{ and } ||(1-Q)\phi_n||_V \to 0 \text{ , as } n \to \infty.$$

For n large enough, there are exactly m approximate eigenvalues (counting their multiplicities) converging to λ . Let $\hat{\lambda}_n$ be their arithmetic mean : $|\lambda-\hat{\lambda}_n| \to 0$. The convergence of the projection method for any one of the equations (1') (2') or (3') is then proved.

4. ERROR FORMULAE

As in [6] , the key point is to write the pointwise error $(u-u_n)(t)$ and $[(1-P)\Psi_n](t)$ for $t \in [0,1]$ under an appropriate form .

Lemma 1

The following identity holds

$$u-u_n = T^{-1}(1-\pi_n)\tilde{x}_n \text{ , } \textit{with } \tilde{x}_n = Kx_n + f \text{ .}$$

Proof :

$$\begin{aligned} x-x_n &= (1-K)^{-1} f - (1-\pi_n K)^{-1} \pi_n f \\ &= (1-K)^{-1}(1-\pi_n)K(1-\pi_n)K^{-1} \pi_n f + (1-K)^{-1}(1-\pi_n)f \\ &= (1-K)^{-1} (1-\pi_n)(Kx_n + f) = (1-K)^{-1} (1-\pi_n) \tilde{x}_n \text{ .} \end{aligned}$$

Then

$$u-u_n = G(1-K)^{-1} (1-\pi_n)\tilde{x}_n = T^{-1} (1-\pi_n)\tilde{x}_n \text{ . } \square$$

By definition T-z is invertible on (1-P)X for any z inside Γ and the operator $S(z) := (T-z)^{-1}(1-P)$ is the *reduced resolvent* of T with respect to λ , at the point z.

Lemma 2

The following error formula holds

$$(1-P)\Psi_n = S(\lambda_n)(\pi_n-1)\tilde{\phi}_n \textit{ with } \tilde{\phi}_n = K\phi_n + \lambda_n G\phi_n \text{ .}$$

Proof :

Let Q be the spectral projection for $(1-K)\phi = \lambda G\phi \Leftrightarrow \phi = \lambda\, U\, \phi$.

$$Q = \frac{-1}{2i\pi} \int_\Gamma (1-K-zG)^{-1}\, G\, dz, \text{ and } P = G\, Q\, G^{-1} .$$

Then

$$\begin{aligned}(1-P)\Psi_n &= G(1-Q)\phi_n \\ &= G(1-K-\lambda_n G)^{-1}\, G(1-Q)G^{-1}\, (1-K-\lambda_n\, G)\phi_n \\ &= (T-\lambda_n)^{-1}(1-P)(\phi_n - \tilde{\phi}_n) = S(\lambda_n)(\phi_n - \tilde{\phi}_n).\end{aligned}$$

The desired result follows from $\pi_n\, \tilde{\phi}_n = \phi_n$. □

Lemma 3

The following error formula holds

$$(1-P)\Psi_n = (T-\lambda_n\, \underline{N})^{-1}\, (1-R)\tilde{\phi}_n \text{ , with } \tilde{\phi}_n = K\phi_n + \lambda_n\, L\, \phi_n .$$

Proof :

$$(1-P)\Psi_n = G(1-Q)\phi_n ,$$

$$\begin{aligned}(1-Q)\phi_n &= (\lambda_n(1-K)^{-1}\, L-1)^{-1}\, (1-Q)(\lambda_n(1-K)^{-1}\, L-1)\phi_n \\ &= [\lambda_n\, L- (1-K)]^{-1}\, (1-K)(1-Q)(1-K)^{-1}\, [\lambda_n\, L\, \phi_n - (1-K)\phi_n] \\ &= (1-K)^{-1}\, (\lambda_n\, L(1-K)^{-1} -1)^{-1}(1-R)\, (\tilde{\phi}_n - \phi_n) .\end{aligned}$$

And $(\lambda_n\, \underline{N}\, T^{-1}-1\,)^{-1}\, (1-R) = \lambda_n\, S(\underline{N}\, T^{-1}\, , 1/\lambda_n)$ where S is the reduced resolvent of $\underline{N}T^{-1}$ with respect to $1/\lambda$ at the point $1/\lambda_n$.

Formally we write

$$\begin{aligned}G(1-K)^{-1}(\lambda_n\, \underline{N}T^{-1}-1)^{-1} &= [\lambda_n\, (\underline{N}G(1-K)^{-1} -1)(1-K)G^{-1}]^{-1} \\ &= (\lambda_n\, \underline{N}-T)^{-1} .\end{aligned}$$

We conclude that

$$\begin{aligned}(1-P)\Psi_n &= (T-\lambda_n\, \underline{N}\,)^{-1}\, (1-R)(\phi_n - \tilde{\phi}_n) \\ &= (T-\lambda_n\, \underline{N})^{-1}\, (1-R)(\pi_n-1)\, \tilde{\phi}_n)\end{aligned}$$

where $(T-\lambda_n \underline{N})^{-1}(1-R) = \lambda_n T^{-1} S(\underline{N}T^{-1}, 1/\lambda_n)$ is well defined as the product of T^{-1} by the reduced resolvent S. The formula reduces to that of lemma 2 if $\underline{N} = 1$. □

These lemmas are fundamental, which express the connection between the error $u-u_n$ (resp. $(1-P)\Psi_n$) and the iterate $\tilde{x}_n = Kx_n + f$ (resp. $\tilde{\phi}_n = K\phi_n + \lambda_n G\,\phi_n$ or $\tilde{\phi}_n = K\phi_n + \lambda_n L\,\phi_n$) relative to the Galerkin method applied on the integral formulation. This will be used in section 5 to establish the optimal orders of convergence.

Before, we need to establish some properties of the iterates $\tilde{x}_n$ and $\tilde{\phi}_n$. $\tilde{x}_n$ is the solution of

$$(1-K\pi_n)\tilde{x}_n = f$$

The collectively compact convergence (abbreviated c.c.) $K\pi_n \xrightarrow{c.c.} K$ is easily proved in L^2 (resp. C). Hence $||\tilde{x}_n - x||_\nu \to 0$ as $n \to \infty$, and $(1-K\pi_n)^{-1}$ exists and is uniformly bounded for n large enough. Now for the eigenvalue problems, $\tilde{\phi}_n$ is solution of

$$\tilde{\phi}_n = \lambda_n(1-K\pi_n)^{-1}\, G\,\pi_n\,\tilde{\phi}_n = \lambda_n\,\tilde{U}_n\,\phi_n ,$$

or

$$\tilde{\phi}_n = \lambda_n(1-K\pi_n)^{-1}\, L\,\pi_n\,\tilde{\phi}_n = \lambda_n\,\tilde{V}_n\,\phi_n .$$

We have

$$\begin{aligned} U-\tilde{U}_n &= (1-K)^{-1}\, G - (1-K\pi_n)^{-1}\, G\,\pi_n \\ &= (1-K\pi_n)^{-1}\, G(1-\pi_n) + [(1-K)^{-1} - (1-K\pi_n)^{-1}]\, G \\ &= (1-K\pi_n)^{-1}\, [G(1-\pi_n) + K(1-\pi_n)U] , \end{aligned}$$

and similarly

$$V-\tilde{V}_n = (1-K\pi_n)^{-1}\, [L(1-\pi_n) + K(1-\pi_n)V] .$$

Hence $\tilde{U}_n \xrightarrow{c.c.} U$, $\tilde{V}_n \xrightarrow{c.c.} V$ and $||(1-Q)\,\tilde{\phi}_n||_\nu \to 0$.

5. ORDERS OF CONVERGENCE

5.1. Properties of the kernels of G, K and T^{-1}

We recall the definition given in [6] of the class $\mathcal{G}(\alpha,\gamma)$ of kernels, where $\alpha\geq 0$ and $\gamma\geq -1$ are integers.

Definition : The kernel k is of *class* $\mathcal{G}(\alpha,\gamma)$ with $\alpha\geq 0$, $\alpha\geq\gamma\geq -1$ iff

$$k(t,s) = \begin{cases} k_1(t,s) & \text{for } 0\leq s<t\leq 1 , \\ k_2(t,s) & \text{for } 0\leq t\leq s\leq 1 , \end{cases}$$

with $k_1\in C^{\alpha}(\{0\leq s\leq t\leq 1\})$, $k_2\in C^{\alpha}(\{0\leq t\leq s\leq 1\})$ and $k \in C^{\gamma}([0,1]\times[0,1])$

discontinuity if $\gamma \geq 0$. $\gamma = -1$ allows k to have a first kind on $s=t$.

γ is the *continuity* of k and α is its *order*.

Another notation is

$$k_t : s \mapsto k_t(s) = k(t,s) \text{ for any } t,s\in[0,1] .$$

Lemma 4

If the coefficients a_i *belong to* C^{α}, $i = 0,\dots,p-1$ *then*

i) the kernel of K *is of class* $\mathcal{G}(\alpha, -1)$,

ii) the kernel of T^{-1} *is of class* $\mathcal{G}(\alpha+1, p-2)$.

Proof :

Let g,k,v be the respective kernels of G, K and T^{-1} . It is well known that g is of class $\mathcal{G}(\infty, p-2)$. The kernel k is such that

$$k(t,s) = \sum_{i=0}^{p-1} a_i(t) \frac{\partial^i}{\partial t^i} g(t,s) ,$$

and i) follows easily.

As for the kernel v, it is related to g in the following way :

$T^{-1}(1-K)f = Gf$ for any f in X implies

$$\int_0^1 v(t,s) \quad [(1-K)f\ ](s)ds = \int_0^1 g(t,s)f(s)ds \text{ for any } t \text{ in } [0,1] .$$

Therefore, v_t is solution of

$$(1-K^*)v_t = g_t .$$

Hence $v_t \in C^{p-2}$ and the ℓth partial derivative $(v_t)^{(\ell)}$ is such that

$$(v_t)^{(\ell)} = (K^* v_t)^{(\ell)} + (g_t)^{(\ell)} \text{ , with}$$

$$(K^* v_t)(\tau) = \int_0^1 \sum_{i=0}^{p-1} \overline{a_i(s)} \frac{\partial^i}{\partial s^i} \overline{g(s,\tau)} \; v_t(s)ds , \qquad 0 \le \tau \le 1 .$$

For $\ell = 1$,

$$\frac{\partial}{\partial \tau} (K^* v_t)(\tau) = \int_0^1 \sum_{i=0}^{p-1} \overline{a_i(s)} \frac{\partial^i}{\partial s^i} \frac{\partial}{\partial \tau} \overline{g(s,\tau)} \; v_t(s)ds$$

$$+ \overline{a_{p-1}(\tau)} \frac{\partial^{p-1}}{\partial s^{p-1}} [\overline{g_1(\tau,\tau)} - \overline{g_2(\tau,\tau)}] \, v_t(\tau) .$$

Similarly the second partial derivative will depend on $\overline{a^{(1)}_{p-1}(\tau)}$, $\overline{a_{p-1}(\tau)}$ and $\overline{a_{p-2}(\tau)}$. This proves by induction that the kernel v is of class $\mathcal{G}(\alpha+1, p-2)$. □

To establish the orders of convergence, the partition Δ is fixed. The reference space X is $L^2(0,1)$ for case (a) and C_Δ for case (b). Then $\pi_n^2 : C_\Delta \to \mathbb{P}_{r,\Delta}$ is a projection. In the error bounds to follow, the constants will be independant of Δ, hence of n .

5.2. <u>The equation (1')</u>

<u>Theorem 5</u>

If $a_i \in C^\alpha$, $i = 0,\dots,p-1$, *if* $f \in C^\alpha$ *and if* $\alpha \ge r+1$, *then for* n *large enough, the following rates of convergence hold for the method of moments (resp. collocation method)*

i) $|(u-u_n)(t_i)| = 0\,(h^{2(r+1)})$, (resp. $h^{r+1+\beta'}$), $i = 0,1,\dots,n$,

ii) $||u-u_n||_\infty = 0(h^{r+1+\beta^*})$ (resp. $h^{r+1+\beta'^*}$).

where $\beta^* = \min(r+1,p)$

$\beta' = \min(\alpha-r-1,r+1)$, $\beta'^* = \min(\beta',p)$.

Proof :

The kernel v of T^{-1} is of class $\mathcal{G}(\alpha+1,p-2)$. And for any t in $[0,1]$:

$$(u-u_n)(t) = (v_t\ ,\ (1-\pi_n)\overline{(\tilde{x}_n-x)})+(v_t,(1-\pi_n)\bar{x}),$$

by lemma 1, where $\bar{x}$ denotes the function $s \mapsto \overline{x(s)}$, for $0\le s\le 1$, and $(.,.)$ denotes the inner product of L^2.

We note that $\beta := \min(\alpha,r+1) = r+1$ by assumption.

1) Case (a) : k has a kernel of class $\mathcal{G}(\alpha,-1)$, hence $||\tilde{x}_n-x||_2$ is of the order r+2 by theorem 10 of [6] .

Then $|((1-\pi_n^1)v_t\ ,\ \overline{\tilde{x}_n-x})|$ is of the order

. $(r+1)+(r+2)$ for $t \in \Delta$,

. $\min\ (r+1,p-1)+r+2$ for $t \notin \Delta$, by corollary 8 of [6] .

And $|((1-\pi_n^1)v_t\ ,\ (1-\pi_n^1)\bar{x})|$ is of the order

. $2(r+1)$ for $t \in \Delta$,

. $\min(r+1,p)+r+1$ for $t \notin \Delta$, by lemma 9 of [6] .

The desired orders follow.

2) Case (b) : by application of corollary 12 and theorem 15 of [6] , $|(v_t,(1-\pi_n^2)\bar{x})|$ is of the order

. $r+1+\beta'$ for $t \in \Delta$,

. $r+1+\beta'^*$ for $t \notin \Delta$.

As for the inner product $|(v_t,(1-\pi_n^2)\overline{(\tilde{x}_n-x)})|$, the quantity $||\tilde{x}_n-x||_{\beta'+r+1,\infty}$ for $t \in \Delta$ (resp. $||\tilde{x}_n-x||_{\min(\beta',p-1)+r+1,\infty}$ for $t \notin \Delta$), can be bounded uniformly in n in the way it is done in the proof of theorem 15 of [6] , where T has to be replaced by K, and z by 1. And the kernel of K is of class $\mathcal{G}(\alpha,-1)$. □

The orders of convergence are 2r+2 at the partition points and r+1+p globally, for the method of moments if $\alpha\ge r+1>p$ and for the collocation method if $\alpha\ge 2r+2$ and $p<r+1$. The optimal orders, namely h^{2r+2} and h^{r+1+p} , are obtained by the method of moments if the coefficients a_i are in C^{r+1} ; but by the method of collocation they are obtained if $a_i \in C^{2r+2}$, $i = 0,\dots,p-1$.

De Boor-Swartz [1] deal with the collocation method on a non linear differential equation. Putting their work into our framework on (1'), we can say that write the error $u-u_n$ under the form

$$u-u_n = T^{-1}T(u-u_n) \quad \text{and} \quad (u-u_n)(t) = (v_t,(1-\pi_n^2)(\bar{f}-T\bar{u}_n)) ,$$

then bound uniformly the derivatives of $T\bar{u}_n$.
It is worth noticing again that the superconvergence results for the method of moments are proved here for an arbitrary partition , without the use of any kind of Markov inequality. This could not be done by using results in de Boor-Swartz [4], cf. also Graham [7] for a related result.

5.3. The equation (2')

Lemma 6

If $a_i \in C^\alpha$, $i=0,\dots,p-1$, P (resp. Q) *has a degenerate kernel of continuity* $\alpha+1$ (resp. α).

Proof : Cf. the proof of proposition 4 in [6].
There is only to prove that $N = QX \subset C^\alpha$ (resp. $N^* \subset Q^*X^* \subset C^\alpha$). We recall that $(1-K)^{-1}M = N$, where $M = PX \subset C^{\alpha+1}$ (theorem 3 in [6] applied on T^{-1}) . For $f \in M$, $g \in N$ is the solution of $(1-K)g = f$ and $g \in C^\alpha$. □

Lemma 7

For n *large enough, the following bounds hold*

$$\|(1-Q)\phi_n\|_\nu \leq c\|(1-\pi_n)Q\|_\nu ,$$
$$\|(1-Q)\tilde{\phi}_n\|_\nu \leq c[\|K(1-\pi_n)Q\|_\nu + \|G(1-\pi_n)Q\|_\nu] .$$

Proof : It is standard (cf. [5]) that

$$\|(1-Q)\phi_n\|_\nu \leq c \|(U-U_n)Q\|_\nu ,$$
$$\|(1-Q)\tilde{\phi}_n\|_\nu \leq c\|(U-\tilde{U}_n)Q\|_\nu .$$

We recall that the spectral projection Q commute with U.
Then

$$(U-U_n)Q = (1-\pi_n K)^{-1}(1-\pi_n)UQ = (1-\pi_n K)^{-1}(1-\pi_n)QU .$$
$$(U-\tilde{U}_n)Q = (1-K\pi_n)^{-1}[K(1-\pi_n)QU + G(1-\pi_n)Q] . \quad □$$

Theorem 8

If $a_i \in C^{\alpha}$, $\alpha \geq r+1$, *then for* n *large enough, the orders of convergence given in theorem* 5 *hold for*

$|[(1-P)\Psi_n](t_i)|$, $i=0,1,\ldots,n$, and $\|(1-P)\Psi_n\|_\infty$ *respectively.*

Proof :

Let v^1 be the kernel of the reduced resolvent $S(\lambda_n) = (T-\lambda_n)^{-1}(1-P)$. v^1 is, like v , of class $\mathcal{G}(\alpha+1,p-2)$.

$$[(1-P)\Psi_n](t) = (v^1_t, (\pi_n-1)\overline{\tilde{\phi}_n})$$

$$= (v^1_t, (\pi_n-1)Q\overline{\tilde{\phi}_n}) + (v^1_t, (\pi_n-1)(1-Q)\overline{\tilde{\phi}_n}) .$$

(a) The method of moments. Q is of continuity α and $\|K(1-\pi^1_n)Q\|_2$ is clearly of the order r+2 by lemma 9 in [6]. We deduce from lemma 7 that $\|(1-Q)\tilde{\phi}_n\|_2 \leq c\ h^{r+2}$. And the proof of theorem 5 applies.

(b) The collocation method. The proof is adapted from the proof of theorem 15 in [6]. We need that $\|(1-Q)\phi_n\|_\infty \leq c\ h^{r+1}$ which follows from lemma 7. □

Let Γ be a Jordan curve which encloses λ but not zero. We define $N:=QX$ the invariant subspace for U associated with $1/\lambda$. $N_n:=Q_nX$ is the direct sum of the invariant subspaces for U_n associated with all the eigenvalues $1/\lambda_n$, where λ_n lies inside Γ . Then for n large enough, $Q_{(n)} := Q_{\restriction N_n}$ $:N_n \to N$ is a bijection from N_n onto N (cf. [5]).

Lemma 9

$\lambda-\tilde{\lambda}_n$ *is of the order of*

$$\max_{z\in\Gamma} \left| \sum_{i=1}^{m} ((U-U_n)Q^{-1}_{(n)}\phi_{in}(z), \phi^*_i(z)) \right|$$

where $\phi_{in}(z)$ (resp. $\phi^*_i(z)$) *belongs to* N (resp. $N^* = Q^*\ X^*$) *for* $i=1,\ldots,m$.

Proof : This is standard [5], and $\phi_{in}(z)$ is uniformly bounded in n. □

Theorem 10 :

If $a_i \in C^\alpha$, $\alpha \geq r+1$, *then for* n *large enough* $|\lambda-\hat{\lambda}_n| = o(h^{2(r+1)})$, (resp. $h^{r+1+\beta'}$) *for the method of moments (resp. collocation method).*

Proof : By lemma 9, we have to find the order of

$$\max_{z\in\Gamma} \left| \sum_{i=1}^{m} ((U-U_n)Q_{(n)}^{-1}\phi_{in}(z),\phi_i^*(z)) \right|$$

$$= \max_{z\in\Gamma} \left| \sum_{i=1}^{m} ((1-\pi_n)(G+K\ U_n)Q_{(n)}^{-1}\phi_{in}(z),(1-K^*)^{-1}\phi_i^*(z)) \right| ,$$

where we have used the identity

$$\begin{aligned} U-U_n &= (1-K)^{-1}G - (1-\pi_n K)^{-1}G \\ &= (1-K)^{-1}(1-\pi_n)G + [(1-K)^{-1}-(1-\pi_n K)^{-1}]\pi_n G \\ &= (1-K)^{-1}(1-\pi_n)[G+KU_n] . \end{aligned}$$

$\phi_{in}(z) \in N$ is uniformly bounded in n , $Q_{(n)}^{-1}\phi_{in}(z) \in N_n$ and $U_n\ Q_{(n)}^{-1}\phi_{in}(z) \in N_n$; $(1-K^*)^{-1}\phi_i^*(z) \in C^\alpha$. Then the proof given in [6], section 7, applies where T is replaced by K . □

5.4. The equation (3')

To study the kernel of $(T-\lambda_n\underline{N})^{-1}(1-R)$, we prove the

Lemma 11

If $a_i, b_i \in C^\alpha$, $i=0,\dots,p-1$, *then*

i) $(T-z\ \underline{N})^{-1}$ *has a kernel of class* $\mathcal{G}(\alpha+1,p-2)$, *for* z *such that* $(T-z\ \underline{N})^{-1}$ *is bounded,*

ii) L *has a kernel of class* $\mathcal{G}(\alpha,-1)$,

iii) R *and* Q *have a degenerate kernel of continuity* α .

Proof :

i) $(T-z\,\underline{N})u = \underline{L}u - z\,\underline{N}u = u^{(p)} - \sum_{i=0}^{p-1} (a_i - z\,b_i)u^{(i)}$.

If $(T-z\,\underline{N})^{-1}$ is bounded, its kernel is then of the same class than the kernel of T^{-1} given by lemma 4.

ii) $L = \underline{N}\,G$ and $K = \underline{M}\,G$ both have a kernel of class $\mathcal{G}(\alpha,-1)$.

iii) We first deal with R, the sepctral projection associated with $\underline{N}\,T^{-1}$. For any integer $q \geq 1$, let the space $X^{(q)}$ be defined by

$$X^{(q)} := \{f \in C^{q-1}(0,1)\ ;\ f^{(q-1)} \text{ is absolutly continuous and } f^{(q)} \in X\}\ ,\quad X \text{ being } L^2 \text{ or } C\ .$$

For $q \leq \alpha$, T^{-1} is an operator : $X^{(q)} \to X^{(q+p)}$ and $\underline{N}T^{-1} : X^{(q)} \to X^{(q+1)}$ for $q < \alpha$. Therefore, any invariant function ξ for $\underline{N}T^{-1}$, which is solution of $(\underline{N}T^{-1}-\lambda)^m\,\xi = 0$, belongs to C^{α} . Now an invariant function ϕ for V is such that $\phi = (1-K)^{-1}\,\xi$: this proves that $\phi \in C^{\alpha}$. □

We conclude that $(T-\lambda_n\,\underline{N})^{-1}(1-R)$ has a kernel of class $\mathcal{G}(\alpha,p-2)$. In the previous two cases the kernels of T^{-1} and of $(T-\lambda_n)^{-1}(1-P)$ were of class $\mathcal{G}(\alpha+1,p-2)$. This change does not affect the resulting orders since with the assumption $\alpha \geq r+1$, we get in all cases $\min(\alpha,r+1) = \min(\alpha+1,r+1) = r+1$.

It is now straightforward to prove that, when $a_i, b_i \in C^{\alpha}$, $\alpha \geq r+1$, $i=0,\dots,p-1$, then

i) the orders of convergence for $[(1-P)\Psi_n](t_i)$ and $\|(1-P)\Psi_n\|_\infty$ respectively are those given in theorem 8,

ii) the orders of convergence of $\lambda-\hat{\lambda}_n$ are those given in theorem 10 .

CONCLUSION

We have proved that the error $u-u_n$ for the solution and $(1-P)\Psi_n$ for the eigenvector are of the order h^{2r+2} at the partition points and h^{r+1+p} globally if $p < r + 1$, both for the method of moments if the coefficients a_i are in C^{r+1}, and for the collocation method if the a_i are in C^{2r+2}.

From these orders, the derivation of the orders of $|\lambda-\lambda_n|$ and $\mathrm{dist}_\infty(\Psi_n, \mathrm{Ker}(T-\lambda))$ or $\mathrm{dist}_\infty(\Psi_n, \mathrm{Ker}(T-\lambda_n \underline{N}))$ is standard [6].

Superconvergence results for the gap between the invariant subspaces are given in Lebbar [9]. In Lebbar [10], it is proved that the orders are preserved in presence of a finite number of weak singularities for the coefficients and the second hand side, provided the partition is of type "graded mesh", cf. also Graham [7].

Superconvergence results for another type of moment method, the least squares method, are given in Locker-Prenter [1].

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CHAPITRE 4

SUPERCONVERGENCE OF THE GENERALIZED EIGENVECTORS OF DIFFERENTIAL AND INTEGRAL OPERATORS AT THE KNOTS

SUPERCONVERGENCE OF THE GENERALIZED EIGENVECTORS OF DIFFERENTIAL AND INTEGRAL OPERATORS AT THE KNOTS

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Abstract

Superconvergence at the partition points for the iterated subspace TM_n (resp invariant subspace M_n) associated to an integral (resp differential) eigenvalue problem is established for the orthogonal Galerkin-method and the collocation method at Gaussian points.

1. INTRODUCTION

Let X be a complex Banach space and let T be a linear closed operator defined in X. We are concerned with the numerical solution of the problem :

(1) $T\phi = \lambda\phi$, $0 \neq \phi \in X$ where λ is an isolated eigenvalue of T .

Two main examples of operator T are considered

i) T will be the integral operator defined by

$(Tx)(t) = \int_0^1 k(t,s)x(s)ds$ for $0 \leq t \leq 1$

with emphasis in particular when k is a Green's kernel.

ii) T will be the differential operator with domain D defined by :

$$(Tx)(t) = x^{(p)}(t) - \sum_{i=0}^{p-1} a_i(t)\, x^{(i)}(t) \text{ for } 0 \leq t \leq 1 .$$

Throughout this paper X will be either $L^2[0,1]$ or $L^\infty[0,1]$ and D will be the space of all functions f defined in $C^{p-1}[0,1]$ such that $f^{(p)}$ belongs to X and f satisfies the p homogeneous boundary conditions :

$$(2) \quad l_k(f) = \sum_{i=0}^{p-1} [\alpha_{ik}\, f^{(i)}(0) + \beta_{ik}\, f^{(i)}(1)] = 0 \quad ; \quad 1 \leq k \leq p$$

The coefficients $\{a_i\}_{i=0}^{p-1}$ are continuous real or complex functions, the constant coefficients α_{ik} , β_{ik} are real or complex.

The aim of this paper is to show that the superconvergence results discribed in Chatelin-Lebbar [8-9] hold not only for the eigenvectors but also for the *generalized eigenvectors*. For this, new error formulae for generalized eigenvectors are given under the very general hypothesis that the approximation T_n of T is strongly stable in a neighborhood of λ .

We recall that the superconvergence results shown in [8-9] are based on the interpretation of the projection method on the differential equation as a *Galerkin method* on the integral formulation. However we take adventage of this paper to give a rather different approach from the one taken in [9]. Thus in section 6 the projection method on the differential equation will be interpreted as a *Petrov method* on the integral formulation.

Another reason to introduce the Petrov method is that it seems to be a real application of the integral theory. Furthermore it allows us to give the superconvergence results for the gap between the invariant subspaces which may not be given easily from the Galerkin method. And at last it clarifies why we iterate in the integral case to obtain the superconvergence and not in the differential one.

2. PROJECTION METHOD

2.1. THE GALERKIN METHOD AND THE PETROV METHOD

Let X_n and Y_n be finite dimensional subspaces of X such that $\dim X_n = \dim Y_n$; and let π_n be a projection on Y_n, then the projection method consists in approximating (1) by :

(3) $\pi_n [(T-\lambda_n)\phi_n] = 0 \quad , \; 0 \neq \phi_n \in X_n$.

X_n is called the right subspace, and Y_n is the left subspace.

If $X_n = Y_n$ (resp $X_n \neq Y_n$) the projection method is known as the Galerkin method (resp the Petrov method).

In the Galerkin method since π_n is a projection on $Y_n = X_n$ (3) may be rewritten as

(4) $\pi_n T \phi_n = \lambda_n \phi_n \quad 0 \neq \phi_n \in X_n$.

Thus in the Galerkin method λ_n(resp ϕ_n) is the eigenvalue (resp eigenvector) corresponding to the approximation $\pi_n T$ of T ; while in the Petrov method, since $Y_n \ni \pi_n \phi_n \neq \phi_n \in X_n$, λ_n (resp ϕ_n) it is not.

However if the following condition is satisfied

(5) $\begin{cases} \text{for } n \text{ large enough } \pi_n\restriction_{X_n} : X_n \to Y_n \text{ is invertible and} \\ (\pi_n\restriction_{X_n})^{-1} \text{ is uniformly bounded.} \end{cases}$

Then the Petrov method (3) is equivalent to the Galerkin method

(6) $\bar{\pi}_n T \phi_n = \lambda_n \phi_n \quad , \; 0 \neq \phi_n \in X_n$

where the projection $\bar{\pi}_n$ on X_n is defined by (cf Krasnosel'skii [14]) :

$$\bar{\pi}_n := (\pi_n\restriction_{X_n})^{-1} \pi_n .$$

Thus in the Petrov method λ_n(resp ϕ_n) is the eigenvalue (resp eigenvector) corresponding to the approximation $\bar{\pi}_n T$ of T when $\bar{\pi}_n$ exists.

Remark

Petrov method which will play an important role in the differentia case will be somewhat omited in the first part of this paper. In fact,when condition (5) is satisfied the Petrov method is theoretically equivalent to the Galerkin method, so our omission is not actually relevant.

2.2. THE ITERATED GALERKIN METHOD

If λ_n , ϕ_n are the eigenelements obtained by the Galerkin method (4), then we may define, if $\lambda_n \neq 0$, the iterated eigenvector $\tilde{\phi}_n$ from λ_n , ϕ_n by :

(7) $\tilde{\phi}_n = \frac{1}{\lambda_n} T \phi_n$.

$\tilde{\phi}_n$ can be regarded as the solution of the iterated Galerkin method :

(8) $(T\pi_n - \lambda_n)\tilde{\phi}_n = 0$, $0 \neq \tilde{\phi}_n \in X$;

corresponding to the approximation $T\pi_n$ of T .

Remarks

1) Applying π_n to each side of (7) and comparing with (4) it follows that $\tilde{\phi}_n$ and ϕ_n are related by $\pi_n \tilde{\phi}_n = \phi_n$.

2) The interesting case to define iterated eigenvectors is when the linear operator T is compact (cf Chatelin-Lebbar [8]). Furthermore under this assumption the condition "λ_n is non zero" is not a restriction since any isolated eigenvalue of T compact is non zero so that if $\lambda_n \to \lambda$ as $n \to \infty$ then λ_n is non zero for n large enough.

3. SETTING OF THE PROBLEM

Let be given $\Delta = \{t_i\}_0^n$ a strict partition of $[0,1]$, $t_o = 0$, $t_n = 1$, Y_n is the set $P_{r,\Delta}$ of piecewise polynomials of degree $\leq r$ on each subinterval $[t_{i-1}, t_i[$, $i = 1,\ldots,n$, the value at 1 being defined by continuity. The choice of the subspace X_n will then specify the method of projection.

In this paper we are concerned with :

Case (i) :

T is defined by (i) and X_n will be $P_{r,\Delta}$. Thus the method of projection applied in this first case is a Galerkin method with $Y_n = X_n = P_{r,\Delta}$. The eigenvector compucted by the Galerkin method (resp iterated Galerkin method)(4)(resp(8)) will be denoted by ϕ_n(resp $\tilde{\phi}_n$).

Case (ii) :

T is defined by (ii) and X_n will be the set $P^{p-1}_{p+r,\Delta} := (P_{r+p,\Delta} \cap D)$. Thus the method of projection considered in this second case is a Petrov method with $P^{p-1}_{p+r,\Delta} = X_n \neq Y_n = P_{r,\Delta}$. The eigenvector computed by the Petrov method (3) will be denoted by Ψ_n .

Note that in both cases X_n and Y_n have the same dimension $N = n(r+1)$.

π_n will, throughout the text, represent any one of the following projections :

$\pi^1_n : L^2[0,1] \to P_{r,\Delta}$ the orthogonal projection on Y_n

$\pi^2_n : C_\Delta \to P_{r,\Delta}$ the collocation method on Y_n

i.e. for $x \in C_\Delta$, $\pi^2_n x$ is its piecewise Lagrange interpolation of degree $\leq r$ at the r+1 Gauss points on each subinterval $\Delta_i := [t_{i-1}, t_i]$ $1 \leq i \leq n$.

We recall that C_Δ denotes the space $\prod_{i=1}^{n} C(\Delta_i)$.

In other words $f \in C_\Delta$ consists on n components $f_i \in C(\Delta_i)$. f is a piecewise continuous function having (possibly) different left and right values at the partition points t_i.

With the norm $||.||_\Delta$ defined by :

$$||f||_\Delta = \max_{1 \leq i \leq n} ||f_i||_\infty \quad , C_\Delta \text{ is a Banach space}$$

and $C_\Delta \subset L^\infty[0,1]$ since $||f||_\Delta = ||f||_\infty$ for $f \in C_\Delta$.

Again we recall that for the two projections under consideration, when the operator T is defined by (i) (resp (ii)) with smooth coefficients, it is known (cf. Chatelin-Lebbar [8] resp [9]) that the eigenvector $\tilde{\phi}_n$ (resp Ψ_n) is superconvergent at the partition points $(t_i)_{i=0}^{n}$; and this, without any assumption on the partition

4. BASIC CONCEPTS AND ERROR FORMULAE

4.1. BASIC CONCEPTS

Let X be a Banach space on $\mathbb{C}$, and let T be a closed linear operator in X, with domain D .

Let us denote by L(X) the algebra of bounded operators on X and by $\rho(T) := \{z \in \mathbb{C}$ such that $R(z) := (T-z)^{-1} \in L(x)\}$ the resolvent set of T.

If λ is an isolated eigenvalue of T, of finite algebraic multiplicity m then there exists a closed Jordan curve Γ in $\rho(T)$ arround λ and isolating λ , that is such that the domain ∇ enclosed by Γ contains no points of the spectrum $\sigma(t) := \mathbb{C} \setminus \rho(T)$ other than λ . And hence

(9) $P := \frac{-1}{2i\pi} \int_\Gamma R(z)dz$ is the spectral projection associated with λ , $P \in L(x)$.

(10) $M := P\,X$ is the invariant subspace

$$m = \dim P\,X .$$

(11) $S(z) := \frac{-1}{2i\pi} \int_\Gamma \frac{R(z')}{z-z'} dz' = R(z)(1-P)$ is the reduced resolvent of T with respect to λ at the point z in ∇ ; $S(z) \in L(x)$.

Let T_n $n = 1,2,\ldots$ be a sequence of closed operators with domain D and assume furthermore that the sequence $\{T_n\}_N$ is a strongly stable approximation of T in ∇ (cf Chatelin [7]) then for n large enough Γ lies in $\rho(T_n)$ and hence we may define in L(x) for the approximation T_n :

(12) $R_n(z) := (T_n - z)^{-1}$ for any $z \in \Gamma$, the resolvent of T_n

(13) $P_n := \frac{-1}{2i\pi} \int_\Gamma R_n(z)dz$, the spectral projection of T_n.

The reduced resolvent of T_n with respect to its eigenvalues inside Γ at any point z in ∇ :

(14) $S_n(z) := \frac{-1}{2i\pi} \int_\Gamma \frac{P_n(z')}{z-z'} d\, z' = R_n(z)(1-P_n)$.

Furthermore the invariant subspace $M_n := P_n X$ is of dimension m. In other words there are exactly m eigenvalues $\{\lambda_{ni}\}_{i=1}^{m}$ of T_n inside Γ (counting their algebraic multiplicity m_{ni})

so that $m = \sum_{i=1}^{\mu} m_{ni}$, where μ is the number of different λ_{ni}. Defining $M_{ni} = \ker(T-\lambda_{ni})^{m_{ni}}$, we have $M_n = \underset{i=1}{\overset{\mu}{+}} M_{ni}$; i.e. M_n is the direct sum of the invariant subspaces of T_n associated with all its distinct eigenvalues inside Γ .

4.2. ERROR FORMULAE

Let us assume that n is large enough

LEMME 1

We have the identities

$(1-P) = S(z)(T-z)$ *for any* z *inside* Γ

$(1-P_n) = S_n(z)(T_n-z)$ " " "

Proof

since $S(z) = \frac{-1}{2i\pi} \int_\Gamma \frac{R(z')}{z-z'} dz'$ we have

$S(z)(T-z) = \frac{-1}{2i\pi} \int_\Gamma \frac{R(z')(T-z'+z'-z)}{z-z'} dz' = (1-P)$

For the 2nd identity the proof is similar

THEOREM 2

We have the two identities

$(1-P_n)\phi = S_n(\lambda)(T_n-T)\phi$ *for any* ϕ *in* $\ker(T-\lambda)$

$(1-P)\phi_{ni} = S(\lambda_{ni})(T-T_n)\phi_{ni}$ *for any* ϕ_{ni} *in* $\ker(T_n - \lambda_{ni})$

Proof

We start from the identity

$R(z)-R_n(z) = R_n(z)(T_n-T)R(z) = R(z)(T_n-T)R_n(z)$

For the eigenvectors ϕ , $R(z)\phi = \frac{1}{\lambda - z}\phi$ and

$(1-P_n)\phi = (\frac{-1}{2i\pi}\int_\Gamma \frac{R_n(z)}{\lambda - z}dz)(T_n - T)\phi = S_n(\lambda)\ (T_n - T)\phi$

Similarly if $\phi_{ni} \in \ker(T_n - \lambda_{ni})$ then $R_n(z)\phi_{ni} = \frac{\phi_{ni}}{\lambda_{ni} - z}$

and hence $(1-P)\phi_{ni} = S(\lambda_{ni})(T - T_n)\phi_{ni}$.

Theorem 2 and its proof were already given in Chatelin-Lebbar [8] . This proof depends heavely on the equality $R(z)\phi = \frac{1}{\lambda - z}\phi$ valid for the eigenvectors.

Since there is no known equivalent equality for the generalized eigenvectors it seems that there is no extention of theorem 2. However an application of lemma (1) for $z = \lambda$ (resp $z = \lambda_{ni}$), shows that $(1-P)\phi = S_n(\lambda)(T_n - \lambda)\phi$ (resp $(1-P)\phi_{ni} = S(\lambda_{ni})(T - \lambda_{ni})(\phi_{ni})$. Then noting that $\lambda\phi = T\phi$ (resp $\lambda_{ni}\ \phi_{ni} = T_n\ \phi_{ni}$), we end a second proof for theorem 2.

A more important application of lemma (1) is the following theorem.

THEOREM 3

A) Let λ_{ni} of index l_{ni} be one of the distinct eigenvalues of T_n inside Γ ; M_{ni} is the associated invariant subspace. Then for any ϕ_{ni} in M_{ni} we have :

$$(1-P)\phi_{ni} = \sum_{j=1}^{l_{ni}} S^j(\lambda_{ni})(T - T_n)(T_n - \lambda_{ni})^{j-1}\ \phi_{ni} .$$

B) If l is the index of λ , then for any ϕ in M we have :

$$(1-P_n)\phi = \sum_{j=1}^{l} S_n^j(\lambda)(T_n - T)(T - \lambda)^{j-1}\ \phi .$$

Proof

The proof of A and B are similar. Let us give B's proof which notation is simpler.

$(T-\lambda)^l\ \phi = 0$ for any ϕ in M . And we write

$(T_n - \lambda)\phi = (T - \lambda)\phi + (T_n - T)\phi$

$$
\begin{aligned}
(1-P)\phi &= S_n(\lambda)(T_n-\lambda)\phi \\
&= S_n(\lambda)(T_n-T)\phi + S_n(\lambda)(T-\lambda)\phi \\
&= S_n(\lambda)(T_n-T)\phi + S_n(\lambda)(1-P_n)(T-\lambda)\phi \\
&= S_n(\lambda)(T_n-T)\phi + S_n^2(\lambda)(T_n-\lambda)(T-\lambda)\phi \\
&= S_n(\lambda)(T_n-T)\phi + S_n^2(\lambda)(T_n-T)(T-\lambda)\phi + S_n^2(\lambda)(T-\lambda)^2\phi
\end{aligned}
$$

... the recursion goes on until $(T-\lambda)^1 \phi = 0$ is reached.

We recall now the definition of the gap between two closed subspaces M and N

$\theta(M,N) := \max(\delta(M,N), \delta(N,M))$ where

$\delta(M,N) := \sup\limits_{\substack{x\in M \\ ||x||=1}} \mathrm{dis}(x,N)$ and

$\mathrm{dis}(x,N) := \min\limits_{y\in N} ||x-y||$.

dis_δ is the distance in the L^δ - norm.

We may hence cite the well - known result.

PROPOSITION 4

If T_n is strongly stable approximation of T inside Γ then for n large enough :

i) $\theta(M,M_n) \le \max(||(1-P)P_n||, ||(1-P_n)P||)$

ii) $\max(||(1-P)P_n||, ||(1-P_n)P||) \le C\,\theta(M,M_n)$

iii) $\theta(M,M_n) \le C \min(||(T-T_n)P||\ ,\ ||(T-T_n)P_n||)$.

Sketchy proof

For a detailed proof cf Kato [13] and Chatelin [7] .

i) Since $\delta(M,M_n) \le ||(1-P_n)P||$ and $\delta(M_n,M) \le ||(1-P)P_n||$

(i) is obvious from the definition of the gap $\theta(M,M$

ii) Let us consider respectively $||(1-P)P_n||$ and $||(1-P_n)P||$.

$||(1-P)P_n|| = \sup\limits_{\substack{x\in M_n \\ ||x||=1}} ||(1-P)x|| \le ||1-P|| \sup\limits_{\substack{x\in M_n \\ ||x||=1}} \mathrm{dis}(x,M) \le ||1-P||\ \delta(M_n$

$||(1-P_n)P|| \le ||1-P_n||\ \delta(M,M_n)$ similarly.

iii) From the identity :

for $z \in \rho(T)$: $R(z) - R_n(z) = R_n(z)(T-T_n)R(z) = R(z)(T_n-T)R_n(z)$,

We deduce easily that $||(1-P_n)P|| \le C||(T-T_n)P||$ and $||(1-P)P_n|| \le C||(T-T_n)P_n||$. Then using the fact that $\delta(M,N) \le \frac{\delta(N,M)}{1-\delta(N,M)}$ when dim M = dim N $<\infty$ we get the desired result.

Remark

Proposition (4) proves that there is no need to look for error formulaeof the type given in theorem (3) if we are just interested in global error bounds. However local error bounds may be established from theorem (3) to prove the desired local superconvergence of generalized eigenvectors.

5. SUPERCONVERGENCE RESULTS FOR THE INTEGRAL EIGENVALUE PROBLEM

5.1. PROPERTIES OF THE INTEGRAL OPERATOR T

C is a generic constant throughout the text $\alpha \ge 0$ and $\gamma \ge -1$ are integers. We are concerned with the following class of kernels.

DEFINITION 5

The kernel k *is of class* $\mathcal{G}(\alpha,\gamma)$ *with* $\alpha \ge 0$ *and* $\alpha \ge \gamma \ge -1$ *iff* :

$$k(t,s) = \begin{cases} k_1(t,s) & \text{for } 1 \ge t \ge s \ge 0 \\ k_2(t,s) & \text{for } 1 \ge s > t \ge 0 \end{cases}$$

with $k_1 \in C^\alpha(\{0 \le s \le t \le 1\})$, $k_2 \in C^\alpha(\{0 \le t \le s \le 1\})$ *and* $k \in C^\gamma([0,1] \times [0,1])$.

γ *is the continuity of* k *and* α *its order.*

(when $\gamma = -1$ *,* k *is allowed to be discontinuous at* t = s*)*

A characterization of compact operators given in Graham-Sloan [12 applied to an integral operator T with kernel k of class $\mathcal{G}(\alpha,\gamma)$ proves that T is compact from $L^2[0,1]$ into $C[0,1]$. Thus T is compact on $L^2[0,1]$ and on $L^\infty[0,1]$.

5.2. CONVERGENCE RESULTS

As said in section 3 the approximation method used in this firs case is the Galerkin method with $X_n = Y_n = P_{r,\Delta}$.

Let $h := \max_{1\le i\le n} (t_i - t_{i-1})$. If $h \to 0$ then for any x in $L^2[0,1]$ (resp $C[0,1]$) $||\pi_n^1 x - x||_2 \to 0$ in $L^2[0,1]$ (resp $||\pi_n^2 x - x||_\infty \to 0$ in $L^\infty[0,1]$).

T is compact in $L^2[0,1]$ (resp $L^\infty[0,1]$) with range in $C[0,1]$ then $||(\pi_n^1 - 1)T||_2 \to 0$ and $T\pi_n^1 \xrightarrow{c.c.} T$ in $L^2[0,1]$(resp $||(\pi_n^2 - 1)T||_\infty \to$ and $T\pi_n^2 \xrightarrow{c.c.} T$ in $C[0,1]$).

($\xrightarrow{c.c.}$ is the notation of the collectively compact convergence).

The strong stability of $T_n^P := \pi_n T$ and $T_n^S := T\pi_n$ inside Γ follows then easily, as well as the convergence of the approximat eigenelements. (cf Chatelin [7]).

Let then P_n^P (resp P_n^S) be the spectral projection associated to the approximation T_n^P (resp T_n^S) of T.

$M_n^P := P_n^P X$ (resp $M_n^S := P_n^S X$) is the direct sum of the invarian subspaces of T_n^P (resp T_n^S) associated with all its distinct eigenvalues inside Γ.

PROPOSITION 6

M_n^S and M_n^P are related as follows :

A) $M_n^S = \mathop{+}_{i=1}^{\mu} M_{ni}^S = \mathop{+}_{i=1}^{\mu} T M_{ni}^P = T M_n^P$

B) $M_n^P = \mathop{+}_{i=1}^{\mu} M_{ni}^P = \mathop{+}_{i=1}^{\mu} \pi_n M_{ni}^S = \pi_n M_n^S$

We say that M_n^S is the <u>iterated</u> invariant subspace.

Proof

A (resp B) follows straightforward from the identity :

$(T\pi_n - \lambda_{ni})^{m_{ni}} T = T(\pi_n T - \lambda_{ni})^{m_{ni}}$

(resp $(\pi_n T - \lambda_n)^{m_{ni}} \pi_n = \pi_n (T\pi_n - \lambda_{ni})^{m_{ni}}$)

where m_{ni} is the algebraic multiplicity of λ_{ni} , $\sum_{i=1}^{\mu} m_{ni} = m$;

and the fact that $\dim M_{ni}^P = \dim M_{ni}^S = m_{ni}$.

5.3. ERROR FORMULAE

A straightforward application of proposition (4) to T_n^P and T_n^S yields the bounds :

(15) $\max (\theta_\delta(M, M_n^P) , ||(1-P_n)P||_\delta) \le C \, ||(1-\pi_n)P||_\delta$

(16) $\max (\theta_\delta(M, M_n^S) , ||(1-\tilde{P}_n)P||_\delta) \le C \, ||T(1-\pi_n)P||_\delta$

when $\delta = 2$ (resp $\delta = \infty$) if $\pi_n = \pi_n^1$ (resp $\pi_n = \pi_n^2$).

However an application of theorem (3) to T_n^S yields the pointwise following equality :

for any $t \in [0,1]$ and any $\tilde{\phi}_n \in M_{ni}^S$ we have

$$[(1-P)\tilde{\phi}_{ni}](t) = \sum_{j=1}^{l_{ni}} [S^j(\lambda_{ni})T(1-\pi_n)(T\pi_n-\lambda_{ni})^{j-1}\tilde{\phi}_{ni}](t)$$

$$= \sum_{j=1}^{l_{ni}} [T\, S^j(\lambda_{ni})(1-\pi_n)(T\pi_n-\lambda_{ni})^{j-1}\tilde{\phi}_{ni}](t)$$

$$= \sum_{j=1}^{l_{ni}} \int_0^1 k(t,s)\, [S^j(\lambda_{ni})(T\pi_n)(T\pi_n-\lambda_{ni})^{j-1}\phi_{ni}](s)d$$

$$= \sum_{j=1}^{l_{ni}} \langle k_t \, ; S^j(\lambda_{ni})(1-\pi_n)(T\pi_n - \lambda_{ni})^{j-1}\tilde{\phi}_{ni}\rangle$$

where $\langle , \rangle$ is the L^2-inner product and k_t is defined by

$k_t : s \to k_t(s) \equiv k(t,s)$ for $s \in [0,1]$.

Let l_t be the unique solution of $(T^* - \bar{\lambda}_{ni})^j (1-P^*) l_t = (1-P^*)\, k_t$ where the adjoint operators are defined in $L^2[0,1]$. The above equalities may be rewritten :

(17) $[(1-P)\tilde{\phi}_{ni}](t) = \sum_{j=1}^{l_{ni}} \langle l_t \; ; \; (1-\pi_n)(T\pi_n - \lambda_{ni})^{j-1} \; \tilde{\phi}_{ni} \rangle$

(Note that since $\tilde{\phi}_{ni}$ belongs to M_{ni}^S , $(T\pi_n - \lambda_{ni})^{j-1} \tilde{\phi}_{ni}$ also does).

Remark

Let $\hat{\lambda}_n$ be the arithmetic mean of the m approximated eigenvalues inside Γ , and $P_{\restriction M_n^P} : M_n^P \to M$ be the spectral projection P restricted to M_n^P , then it is known (cf de Boor-Shwartz [3]) that

(18) $m(\lambda - \hat{\lambda}_n) = \sum_{i=1}^{m} \langle x_i^* \; , \; (1-\pi_n)\; T\; (P_{\restriction M_n^P})^{-1}\; x_i \rangle$

where $\{x_i\}_{i=1}^m$ (resp $\{x_i^*\}_{i=1}^m$) is a basis of M (resp the adjoint basis of M^*).

$(P_{\restriction M_n^P})^{-1}\, x_i$ is an element of M_n^P , thus $T\,(P_{\restriction M_n^P})^{-1}\, x_i$ is (by proposition (6)) an element of M_n^S . Then, comparing (17) and (18) we can say that the error formulae of $\lambda - \hat{\lambda}_n$ and $[(1-P)\tilde{\phi}_n]\,(t)$ are of the same type.

5.4. SMOOTHNESS PROPERTIES OF THE INVARIANT SUBSPACES

Since the notation C_Δ is already known we define more generally C_Δ^{α} for a positive integer α by $C_\Delta^{\alpha} := \prod_{i=1}^{n} C^{\alpha}(\Delta i)$.
We state the following theorems :

THEOREM 7

Let the kernel of T be of class $\mathcal{G}(\alpha,\gamma)$ *then*

i) *T is compact from* L^2 *into* C

ii) *if* $\gamma \geq 1$ *the range of T is in* C^{γ}

iii) for any p, $0 \le p \le \alpha-1$, T *is bounded from* C^p *into* C^{p+1}

iv) " " T *is* " " C^p_Δ *into* C^{p+1}_Δ

Proof

i) ii) and iii) are shown in Chandler [6] . The proof of iv) is similar to that of (iii).

THEOREM 8

If the kernel of T *is of class* $\mathcal{G}(\alpha,\gamma)$ *then*

i) $M \subset C^\alpha$, $M^* \subset C^\alpha$

ii) $M_n^P \subset P_{r,\Delta}$, $M_n^S \subset C^\alpha_\Delta$

Proof

i) Since $M = \{\phi \in L^2, (T-\lambda)^m \phi = 0\}$, ϕ may be written as $\lambda^m \phi = \sum_{i=1}^{m} \alpha_i T^i \phi$. Thus applying theorem 7 inductively we show that $\phi \in C^\alpha$. $M^* \in C^\alpha$ similarly.

ii) $M_n^P \subset P_{r,\Delta}$ by definition of the Galerkin method and $M_n^S \subset C^\alpha$ by application of theorem 7 and proposition 6.

5.5. SUPERCONVERGENCE RESULTS

We are ready now for the main theorem of this section.

THEOREM 9

If the kernel of T *is of class* $\mathcal{G}(\alpha,\gamma)$ (resp $\mathcal{G}(\alpha+r+1,\gamma)$) *and if* $\pi_n = \pi_n^1$ (resp $\pi_n = \pi_n^2$) *then for* n *large enough*

$$\begin{cases} \text{i) } \max(\theta_2(M,M^P_n),\ \|(1-P)P^P_n\|_2) = O(h^\beta) \\ \text{ii) } \max(\theta_\infty(M,M^S_n),\ \|(1-P)\tilde{P}_n\|_\infty) = O(h^{\beta+\beta^*}) \\ \text{iii) } \max\limits_{1\le i\le n}(|\lambda-\hat{\lambda}_n|,\ |[(1-P)\tilde{\phi}_n](t_i)|) = O(h^{2\beta}) \\ \tilde{\phi}_n \in M_n^S,\ \|\tilde{\phi}_n\| = 1 \end{cases}$$

respectively

$$\begin{cases} \text{i) } \max(\theta_\infty(M, M_n^P), \|(1-P)P_n^P\|_\infty) = O(h^{r+1}) \\ \text{ii) } \max(\theta_\infty(M, M_n^S), \|(1-P)\tilde{P}_n\|_\infty) = O(h^{r+1+\beta^*}) \\ \text{iii) } \max\limits_{1\le i\le n}(|\lambda - \hat{\lambda}_n|, |((1-P)\tilde{\phi}_n)(t_i)|) = O(h^{r+1+\beta}) \\ \quad \tilde{\phi}_n \in M_n^S, \|\tilde{\phi}_n\| = 1 \end{cases}$$

where $\beta = \min(\alpha, r+1)$ and

$\beta^* = \min(\beta, \gamma+2)$.

Proof

Using the smoothness properties of the invariant subspaces given in theorem 8, the proof of theorem 9 for $\pi_n = \pi_n^1$ (resp $\pi_n = \pi_n^2$) is then similar to the proof of theorem 10 (resp 19) given in Chatelin-Lebbar [8] . It may be remarked however that more smoothness is assumed for the collocation method than for the orthogonal projection method. The explanation of this phenomenon is based on the relation between π_n^1 and π_n^2 given in [8] .

CONCLUSION

In summary the most relevant point we remark in the integral case is that in the superconvergence eigenvalue problem the most important concept is not the iterated eigenvector but the iterated invariant subspace since the superconvergence phenomenon does not give any priority to the approximated eigenvectors versus the approximated generalized eigenvectors.

6. SUPERCONVERGENCE RESULTS FOR DIFFERENTIAL EIGENVALUE PROBLEMS

6.1. THE APPROXIMATED DIFFERENTIAL EIGENVALUE PROBLEM

We are now concerned with the eigenvalue problem (19) $T\Psi = \lambda\Psi$ $0 \neq \Psi \in D$ where T is the differential operator defined by (ii). We assume throughout that $0 \in \rho(T)$ and that the problem $(u^{(P)} = 0 \ , \ u \in D)$ has $u = 0$ for unique solution. Then there exists a Green's function $g(t,s)$ of class $\mathcal{G}(\infty, p-2)$ such that for any $x \in X$, the unique solution u of the problem $(u^{(P)} = x \ , \ u \in D)$ is given by $u(t) = \int_0^1 g(t,s)\, x(s)ds$. This defines the integral operator G by $u = Gx$.

G is the inverse of the differential operator $D^P : u \in D \to u^{(P)}$. Then the operator T may be written as (20) $T := (1-K)G^{-1}$ where the integral operator K is defined by :

$$Kx = \int_0^1 k(.,s)x(s)ds \text{ where}$$

$$k(t,s) = \sum_{i=0}^{P-1} ai(t) \ \frac{\delta^i g(t,s)}{\delta t^i} \qquad 0 \leq t,s \leq 1$$

We remark that if $a_{p-1} \neq 0$ then k has a first kind discontinuity on $s = t$. However by the characterization of compact operators given in Graham-Sloan [12] , K is compact from $L^2[0,1]$ into $C[0,1]$ and hence on $L^2[0,1]$ and on $L^\infty[0,1]$.

As said in section 3 the approximation method which will be used in this case is the Petrov method with $Y_n = P_{r,\Delta}$ and $X_n = GY_n = P^{p-1}_{p+r,\Delta}$. The approximated differential eigenvalue problem is then

(21) $\quad \pi_n T \Psi_n = \lambda_n \pi_n \Psi_n \qquad 0 \neq \Psi_n \in X_n$.

Applying (20), (21) may be written again as

(22) $\pi_n (1-K)G^{-1} \Psi_n = \lambda_n \pi_n \Psi_n \quad 0 \neq \Psi_n \in X_n$,

then noting that $G^{-1} \Psi_n \in Y_n$, (22) is equivalent to

(23) $(1- \pi_n K)G^{-1} \Psi_n = \lambda_n \pi_n \Psi_n \quad 0 \neq \Psi_n \in X_n$.

Now since $0 \in \rho(T)$ and in view of the compactness of the operator K it is clear that $1 \in \rho(K) \cap \rho(\pi_n K)$ for the two projection under consideration (for the details cf. section 5.) and hence that (23) is equivalent to

(24) $\Psi_n = \lambda_n G(1-\pi_n K)^{-1} \pi_n \Psi_n \quad 0 \neq \Psi_n \in X_n$.

We denote by P (resp P_n) the spectral projection associated with the operator T (resp $G(1-\pi_n K)^{-1} \pi_n$) and the eigenvalue λ (resp $1/\lambda$) . M and M_n denote respectively the invariant subspaces PX and $P_n X$.

6.2. TWO LEMMAS

Before giving the integral formulation of problem (19) some lemmas are needed.

LEMMA 10

Let $\lambda \neq 0$ be an isolated eigenvalue of T of finite multiplicity. If $0 \in \rho(T)$, $1/\lambda$ is an eigenvalues of $T^{-1} = A$, the associated eigenvectors and invariant subspaces are identical.

Proof

$1/\lambda$ is an isolated point of $\sigma(A)$. From the identity :

$T-\lambda = \lambda T(\frac{1}{\lambda} - A)$, $\lambda \neq 0$, follows that $1/\lambda$ is an isolated eigenvalue of A and $\ker(T-\lambda) = \ker(A-1/\lambda)$.

Let $\Gamma(\lambda)$ be a Jordan curve enclosing λ . Let $P(T,\lambda)$ be the spectra projection :

$$\frac{-1}{2i\pi} \int_{\Gamma(\lambda)} (T-z)^{-1}dz = \frac{-1}{2i\pi} \int_{\Gamma(\lambda)} \frac{1}{z} A(\frac{1}{z} - A)^{-1}dz$$

$= \frac{-1}{2i\pi} \int_{\Gamma(1/\lambda)} A(A-t)^{-1} \frac{dt}{t} = P(A,1/\lambda)$ if the curve $\Gamma(1/\lambda)$ around $1/\lambda$

$t = \frac{1}{z}$

does not enclose 0 . Hence $P(T,\lambda)X = P(A,1/\lambda)X$.

LEMMA 11

X_n and Y_n are two finite dimensional subspaces on Y such that. $\dim X_n = \dim Y_n$.

Let K be a compact operator on X and π_n a pointwise convergent projection on Y_n (abreviated $\pi_n \xrightarrow{p.} 1$ on Y_n). Assume furthermor that $1 \in \rho(K)$ and that $X_n = (1-K)Y_n$.

Then the projection $\overline{\pi_n}$ defined on X_n by $\overline{\pi_n} := (\pi_{n\restriction X_n})^{-1}\ \pi_n$ exists, is uniformly bounded and satisfies the following properties :

a) $\overline{\pi_n} \xrightarrow{p.} 1$ on X_n

b) $\overline{\pi_n} = (1-K)(1-\pi_n K)^{-1}\ \pi_n$

c) $1-\overline{\pi_n} = (1-\pi_n)(1-K\pi_n)^{-1}$

Proof

a) To prove that $\overline{\pi_n} \xrightarrow{p} 1$ we apply the Galerkin method defined by Y_n to the equation $(1-K)u = g$ for an arbitrary g in X .

It yields the equation $\begin{cases} \pi_n(1-K)u_n = \pi_n g \\ u_n \in Y_n \end{cases}$

which has for n large enough, a unique solution u_n converging to $u = (1-K)^{-1}g$ as $n \to \infty$; since K is compact and $\pi_n \xrightarrow{p.} 1$.

Setting $v_n := (1-K)u_n$ the equation can be rewritten

$$\pi_n v_n = \pi_n g .$$

For any g in X, the equation has a unique solution $v_n \in X_n$: this proves that $(\pi_{n\restriction X_n})$ is invertible on X_n :

$$v_n = (\pi_{n\ X_n})^{-1}\ \pi_n g = \overline{\pi_n}\ g .$$

Now $v_n \to g$ can be rewritten $\overline{\pi_n}\ g \to g$ for any g in X .

By the Banach-Steinhauss theorem $||\overline{\pi_n}|| \le C$.

b) The desired identity follows from

$$(1-\overline{\pi}_n)g = (1-K)(u-u_n) = (1-K)(1-\pi_n K)^{-1}(1-\pi_n)(1-K)^{-1}g$$

and hence
$$\begin{aligned} 1-\overline{\pi}_n &= (1-K)(1-\pi_n K)^{-1}\,(1-\pi_n K - \pi_n(1-K))(1-K)^{-1} \\ &= (1-K)(1-\,(1-\pi_n K)^{-1}\,\pi_n(1-K))(1-K)^{-1} \\ &= 1 - (1-K)(1-\pi_n K)^{-1}\,\pi_n\,. \end{aligned}$$

c) Since $\overline{\pi}_n = (1-K)(1-\pi_n K)^{-1}\,\pi_n$ we have

$$\begin{aligned} 1-\overline{\pi}_n &= (1-K)\,[(1-K)^{-1} - (1-\pi_n K)^{-1}\,\pi_n] \\ &= (1-K)(1-K)^{-1}(1-\pi_n)(1-K\,\pi_n)^{-1} \\ &= (1-\pi_n)(1-K\,\pi_n)^{-1}\,. \end{aligned}$$

6.3. THE INTEGRAL FORMULATION OF (19)

Since $0 \in \rho(T)$, and $A = T^{-1}$, $T\Psi = \lambda\Psi$ is equivalent to

$$(25)\quad \begin{cases} O = \lambda A\,O \\ \Psi = A\,O \end{cases}$$

Let denote by R the spectral projection associated with the integral operator A and its isolated eigenvalue $1/\lambda$. Then by Lemma (10) P and R are identical (ie $P \equiv R$) .

Let apply now to (25) the Petrov method defined by

$X_n = T\,(P^{p-1}_{p+r,\Delta})$ and $Y_n = P_{r,\Delta}$.

Since $P^{p-1}_{p+r,\Delta} = G\,P_{r,\Delta}$ and $T = (1-K)G^{-1}$, X_n and Y_n are related by :

(26) $X_n = (1-K)Y_n$, and hence the approximated eigenvalue problem

$$(27)\quad \begin{cases} \pi_n\,O_n = \lambda_n\,\pi_n\,A\,O_n \qquad 0 \neq O_n \in X_n \\ \Psi_n = A\,O_n \end{cases}$$

is by lemma (19) equivalent to the problem

$$(28)\quad \begin{cases} O_n = \lambda_n\,\overline{\pi}_n\,A\,O_n \qquad 0 \neq O_n \in X_n \\ \Psi_n = A\,O_n \end{cases}$$

which corresponds to an application of a Galerkin method defined by $\overline{\pi_n}$ and X_n .

We recall that $\overline{\pi_n}$ is the projection on X_n defined by

$$\overline{\pi_n} := (\pi_n \restriction_{(1-K)P_{r,\Delta}})^{-1} \pi_n .$$

Let us denote by R_n the spectral projection associated to the approximation $\overline{\pi_n}$ A of A , then

PROPOSITION 12

$$R_n = T P_n T^{-1}$$

Proof

Since $A = G(1-K)^{-1}$ and $\overline{\pi_n} = (1-K)(1-\pi_n K)^{-1} \pi_n$

we have for $z \in \rho(\overline{\pi_n} A)$:

$$\begin{aligned}(\overline{\pi_n} A-z)^{-1} &= ((1-K)(1-\pi_n K)^{-1}\pi_n G(1-K)^{-1}-z)^{-1} \\ &= (1-K)G^{-1} (G(1-\pi_n K)^{-1} \pi_n -z)^{-1} G(1-K)^{-1} \\ &= T (G(1-\pi_n K)^{-1} \pi_n-z)^{-1} A .\end{aligned}$$

Then by integration over $\Gamma(1/\lambda)$ we obtain the desired result.

As in the integral case let consider now the iterated eigenvector $\tilde{O}_n$ solution of the equation

(29) $\tilde{O}_n = \lambda_n A \overline{\pi_n} \tilde{O}_n \quad 0 \neq \tilde{O}_n \in X$ and let us denote by $\tilde{R}_n$ the spectral projection associated to the approximation $A\overline{\pi_n}$ of A.

Then if $\lambda_n \neq 0$

PROPOSITION 13

$$P_n = \tilde{R}_n \textit{ and } \Psi_n = \frac{1}{\lambda_n} \tilde{O}_n$$

Proof

As in the proof of proposition (12), if $z \in \rho(A\overline{\pi_n})$

$$\begin{aligned}\text{then } (A\pi_n -z)^{-1} &= (G(1-K)^{-1}(1-K)(1-\pi_n K)^{-1}\pi_n - z)^{-1} \\ &= (G(1-\pi_n K)^{-1} \pi_n - z)^{-1}\end{aligned}$$

and hence the desired equality is obtained by integration over the Jordan curve $\Gamma(1/\lambda)$.

Remark

Proposition (13) proves that the approximated invariant subspace of the differential operator T is nothing but the approximated iterated invariant subspace of the integral operator $A = T^{-1}$. This explains why we iterate in the integral case and not in the differential one.

6.4. CONVERGENCE RESULTS

Since $P = R$ and $P_n = \tilde{R}_n$, the convergence of the eigenelements of the differential problem (21) is equivalent to the convergence of the eigenelements of the integral problem (29) which follows straight forward from the convergence results given in section 5.

6.5. ERROR FORMULAE

Since $P = R$ and $P_n = \tilde{R}_n$ the following error formulae are just obtained from those given in section 5.

$$(30)\quad \begin{cases} \max(\ \Theta_\delta(M,M_n), ||(1-P)P_n||_\delta) \le C||A(1-\overline{\pi}_n)P_n||_\delta \\ \text{where } \delta = 2 \text{ (resp } \delta=\infty) \text{ if } \overline{\pi}_n \text{ is derived form } \pi_n^{\,1} \text{(resp } \pi_n^{\,2}). \end{cases}$$

$$(31)\quad \begin{cases} \text{for any } t \in [0,1] \text{ and any } \Psi_{ni} \in M_{ni} \text{ we have} \\ [(1-P)\Psi_{ni}]\ (t) = \sum_{j=1}^{l_{ni}} < l_t\ ,\ (1-\overline{\pi}_n)(A\ \overline{\pi}_n - 1/\lambda_{ni})^{j-1}\ \ \Psi_{ni}> \\ \text{where the function } l_t \text{ is the unique solution of the problem} \\ (A^* - 1/\overline{\lambda}_{ni})(1-P^*)l_t = (1-P^*)k_t\ . \end{cases}$$

We remark that, as in the integral case, $|\lambda - \hat{\lambda}_n|$ and $((1-P)\Psi_n)(t)$ are of the same type since (by application of theorem 6.1 of [7]) $\lambda - \hat{\lambda}_n$ is of the order of :

$$(32)\quad \max_{z\in\Gamma} \sum_{i=1}^{m} < \Phi_i^{*}(z)\ ;\ (1-\overline{\pi}_n)A(P_{R_nX})^{-1}\ \Phi_{in}(z) >$$

where $O_{in}(z)$ (resp $O_i^*(z)$) belongs to M(resp M^*)

Again we note that (30),(31) and (32) may be given in terms of $(1-\pi_n)$ instead of $(1-\overline{\pi_n})$ since $(1-\overline{\pi_n}) = (1-\pi)(1-K\pi_n)^{-1}$.

6.6. SMOOTHNESS PROPERTIES OF THE INVARIANT SUBSPACES

PROPOSITION 14

If the coefficients a_i *belong to* C^α $i = 0,\dots,p-1$

then i) the kernel of K *is of class* $\mathcal{G}(\alpha,-1)$

ii) the kernel of $A=T^{-1}$ *is of class* $\mathcal{G}(\alpha+1,p-2)$

Proof

cf. Chatelin-Lebbar [9] .

Thus by application of theorem 8 we show.

THEOREM 15

If the coefficients a_i *belong to* C^α $i = 0,\dots,p-1$

then : $M \in C^{\alpha+1}$, $M^* \subset C^{\alpha+1}$

and $M_n \subset P^{p-1}_{p+r,\Delta}$

6.7. SUPERCONVERGENCE RESULTS

We are ready now for the main theorem of this section.

THEOREM 16

If $a_i \in C^\alpha$ (resp $C^{\alpha+r+1}$) $i = 0,\dots,p-1$ *and if*

$\pi_n = \pi_n^1$ (resp $\pi_n = \pi_n^2$) *then for* n *large enough :*

$$\begin{cases} \text{i) } \max\left(\|((1-P)\Psi_n)^{(P)}\|_2 ; \|((1-P_n)\Psi)^{(P)}\|_2\right) = O(h^\beta) \\ \quad \textit{for any } \Psi_n \in M_n \quad \|\Psi_n\| = 1 \textit{ and any } \Psi \in M \quad \|\Psi\| = 1 . \\ \text{ii) } \max\left(\theta_\infty(M,M_n), \|(1-P)P_n\|_\infty\right) = O(h^{\beta+\beta^*}) \\ \text{iii) } \max_{1\le i\le n}\left(|\lambda-\hat{\lambda}_n|, |((1-P)\Psi_n)(t_i)|\right) = O(h^{2\beta}) \\ \quad \Psi_n \in M_n , \|\Psi_n\| = 1 \end{cases}$$

respectively

$$\begin{cases} \text{i) } \max(||((1-P)\Psi_n)^{(P)}||_\infty \ , \ ||((1-P_n)\Psi)^{(P)}||_\infty) = O(h^{r+1}) \\ \quad \textit{for any } \Psi_n \in M_n \ ||\Psi_n|| = 1 \textit{ and any } \Psi \in M \ ||\Psi|| = 1 \ . \\ \text{ii) } \max\ (\ \Theta_\infty(M,M_n), \ ||(1-P)P_n||_\infty\ ) = O(h^{r+1+\beta^*}) \\ \text{iii) } \max\limits_{1\le i\le n}\ ((\lambda - \hat{\lambda}_n)\ , \ |((1-P)\Psi_n)(t_i)|) = O(h^{r+1+\beta}) \\ \quad \Psi_n \in M_n \ , \ ||\Psi_n|| = 1 \end{cases}$$

where $\beta = \min(\alpha, r+1)$ *and* $\beta^* = \min(\beta, \gamma+2)$.

<u>Proof</u>

To begin let us consider the spectral projection

$Q = G^{-1} P G = (1-K)^{-1} P(1-K)$, $N := QX$

$Q_n = G^{-1}P_n G$, $N_n := Q_nX$

and $\tilde{Q}_n = (1-K\pi_n)^{-1} P_n(1-K\pi_n)$, $\tilde{N}_n = \tilde{Q}_nX$

associated to the operators U, U_n, $\tilde{U}_n$ and the eigenvalues $1/\lambda$, $1/\lambda_n$, $1/\lambda_n$; where

$$U = (1-K)^{-1}G$$

$$U_n = (1-\pi_nK)^{-1} \pi_nG$$

$$\tilde{U}_n = (1-K\pi_n)^{-1}G\pi_n \ .$$

It is obvious to see that U is compact since G is compact and $(1-K)^{-1}$ is bounded. We also know from [9] that U_n and $\tilde{U}_n$ are strongly stable approximations of U, thus applying proposition (4) we obtain, in particular, that

$\max(||(1-Q)Q_n||,||(1-Q_n)Q||) \le C||(U-U_n)Q|| \le C|(1-\pi_n)\ UQ|$

which may be rewritten in terms of P and P_n as :

$\max\ (||G^{-1}(1-P)P_nG||,||G^{-1}(1-P_n)P\ G||) \le C\ ||(1-\pi_n)(1-K)^{-1}P\ G||$.

Thus the smoothness (resp (Spline)) properties of M (resp (π_n)) allows us to show the first part (i).

To prove ii) and iii) it is sufficient to remark that in the error formulae (30), (31) and (32), $1-\overline{\pi}_n$ was always applied to a functio

of the invariant subspace $\mathcal{M}_n$ and hence in writing $1-\overline{\pi}_n = (1-\pi_n)(1-K\,\pi_n)^{-1}$ and remarking that $(1-K\pi_n)^{-1}\,M_n = \tilde{N}_n$, The desired results are obtained by (30), (31), (32) and following the proof of theorem (8) in [9] .

CONCLUSION

The eigenvectors and the generalized eigenvectors have the same smoothness and convergence properties. This enables us to work with the gap between the invariant subspaces and to give the superconvergence results expected for the two examples under consideration. As it may be seen more smoothness is requiered for the collocation method than for the orthogonal projection : this is just a consequence of the relation (given in[8]) between the two projections π_n^1 and π_n^2 . Moreover the use of the Petrov interpretation in proposition (13) explains why we iterate in the integral case and not in the differential one.

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For a kernel which is the Green's function of an o.d.e. some interesting references of superconvergence results for iterated solution are Chandler [5] for the orthogonal Galerkin method, de Boor-Swartz [2,3,4] for the collocation method at Gaussian points, Chatelin-Lebbar [8,9] for both of them, and Telias-Ahues [19] for a least-square formulation of the advection-diffusion equation. For weakly singular kernels we refere to Graham [10,11] for the orghogonal-Galerkin method, to Schneider [17] , Chandler [6] and Bachlors [1] for the product integration method. Confere also Lebbar [16] where for some singular equations, superconvergence results are obtained for the orthogonal Galerkin method and the collocation method at Gaussian points.

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CHAPITRE 5

NUMERICAL ILLUSTRATIONS

NUMERICAL ILLUSTRATIONS

. We are concerned with the numerical solution of Fredholm's integral equation of the second kind :

(1) $\int_0^1 k(t,s)\, x(s)ds - z\, x(t) = f(t) \qquad 0 \le t \le 1$;

by some projection methods.

. In fact two types of projection methods are treated in the following examples :

i) the orthogonal Galerkin method in L^2, and

ii) the collocation method at Gaussian points in C.

The approximating subspaces being $P_{r,\Delta}$ of piecewise polynomials of degree $\le r$ on each subinterval $\Delta_i := [t_{i-1}, t_i]$, $i=1,\dots,n$.

. To obtain reliable estimates of the orders of convergence for our numerical calculations, we choose to solve equations which have known solutions. Thus in each of our examples k and f are chosen so that x has a particularly simple closed form.

Example 1

(2) $$k(t,s) = \begin{cases} -t(1-s) & \text{if } 0 \le t \le s \le 1 \\ -s(1-t) & \text{if } 0 \le s < t \le 1 \end{cases}$$

the right side f is chosen so that $x(t) = \cosh(2t-1)$.

We choose the uniform partition $\Delta = \{\frac{i}{5}\}_0^5$, and in each subinterval $\Delta_i = [\frac{i-1}{5}, \frac{i}{5}]$, the $r+1 = 4$ Gauss points.

We display in table (1) the values of $x-x_n$ and $x-\tilde{x}_n$ at the partition points t_i , $i = 1,2,3,4$. x_n (resp $\tilde{x}_n$) being the collocation (resp the iterated collocation) approximation of x.

We recall that since $x_n \in P_{r,\Delta}$, x_n may possess a first kind discontinuity at the partition points. Table (1) illustrates this discontinuity.

i	$(x-x_n)(t_i^-)$	$(x-x_n)(t_i^+)$	$(x-x_n)(t_i)$
1	8.10^{-5}	7.10^{-5}	-5.10^{-12}
2	6.10^{-5}	6.10^{-5}	-7.10^{-12}
3	6.10^{-5}	6.10^{-5}	-7.10^{-12}
4	7.10^{-5}	8.10^{-5}	-5.10^{-12}

Table (1) : Error values at the partition points

We display in figure (2), the graphs of the error functions :

$t \in \Delta_3$ $\quad (x-x_n)(t)$: the collocation error

$t \in \Delta_3$ $\quad (x-\tilde{x}_n)(t)$: the iterated collocation error

when $\Delta_3 = [\frac{2}{5}, \frac{3}{5}]$.

We have only *sketched* the two graphs because of the great difference in magnitude between the various values.

Example 2

We consider the same kernel as in example (1), but this time we choose f so that $x(t) = (t-\frac{1}{2})^4$. The partition Δ used in this second example is uniform with mesh size $h = \frac{1}{n}$.

In the tables below we list :

i) the dimension, under the heading dim, of the space in which the approximation of x is determined,

ii) the quantities : $\alpha_n = ||x-x_n||_\infty$, $\beta_n = ||x-x_n||_\infty$ and

$$\gamma_n = \max_{1 \le i \le n} |(x-\tilde{x}_n)(t_i)| .$$

iii) further, next to each column of errors $\{e_n\}$ we list in rew n, the number :

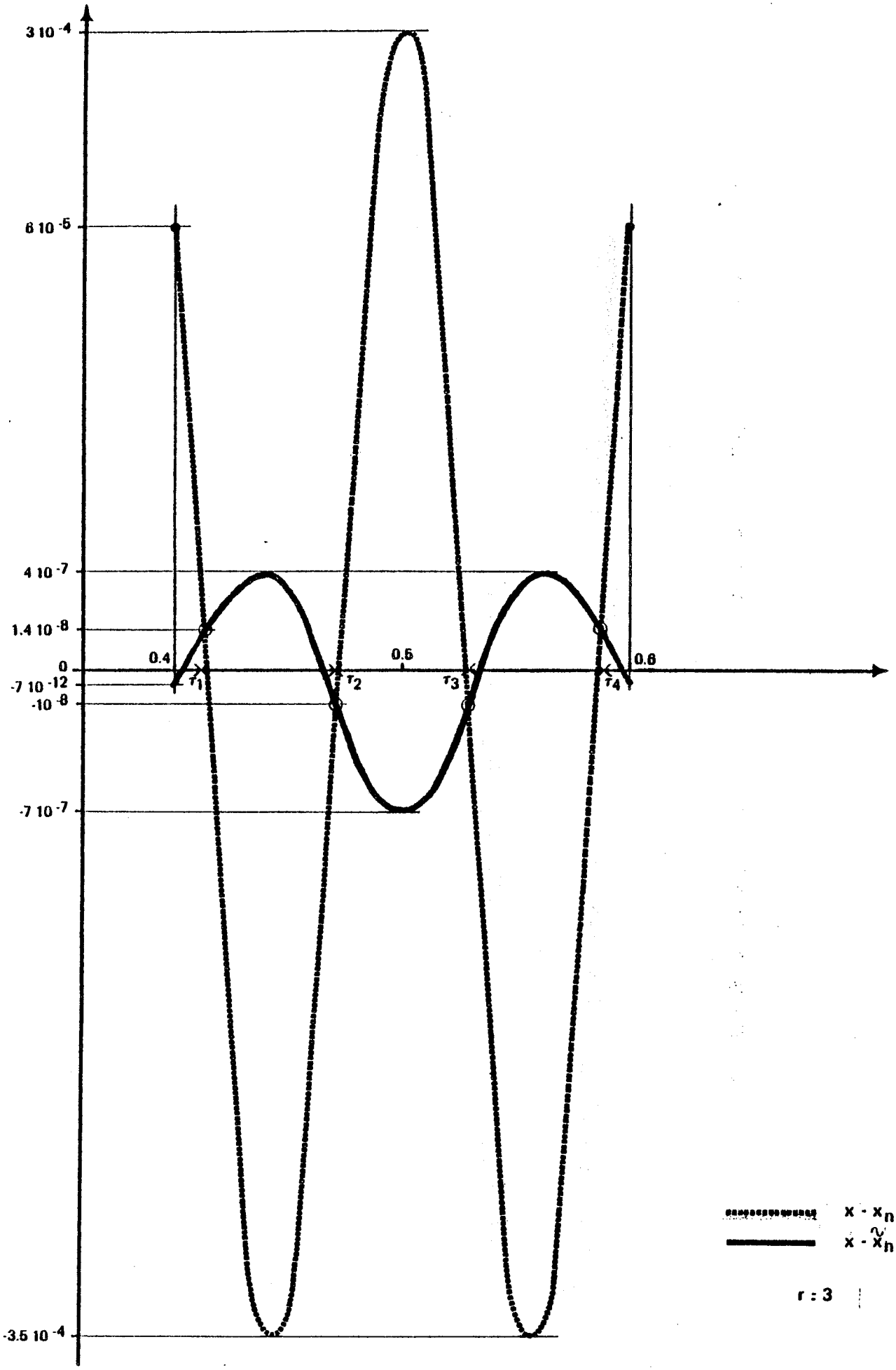

Figure .2 . Graphs of the errors on $\Delta_3 = [0.4, 0.6]$.

$$\lambda_n = \log(e_n/e_{2n}) / \log 2 ;$$

relating the decrease in e_n to the decrease in $h = \frac{1}{n}$. If the decrease of e_n to zero is of exact order h^λ , then $\lambda_n \to \lambda$. (the errors $\{e_n\}$ being $\{\alpha_n\}$, $\{\beta_n\}$ or $\{\gamma_n\}$).

. We recall that if the right side of belongs to C^α(resp $C^{\alpha+r+1}$) and if the projection method used for calculating x_n and $\tilde{x}_n$ is the orthogonal Galerkin method (resp the collocation method at Gaussian points) then, an application of theorem (**10**) (resp theorem (**15**)) of [1] to the Green's kernel (2) shows that the rates of convergence of x_n and $\tilde{x}_n$ are :

1) of exact order μ (resp (r+1)) globally for x_n
2) of an order greater or equal to $\mu + \mu^*$(resp $r+1+\mu$) globally for $\tilde{x}_n$
3) of an order greater or equal to 2μ (resp u+r+1) for $\tilde{x}_n$ at the partition points.

μ and μ^* being the numbers : $\mu := \min(\alpha, r+1)$

and $\mu^* := \min(\mu, 2)$.

n	dim	Theory predicts $\lambda=2$ α_n	λ_n	Theory predicts $\lambda\geq4$ β_n	λ_n	Theory predicts $\lambda\geq4$ γ_n	λ_n
2	4	$7,52 \times 10^{-1}$		$3,96 \times 10^{-3}$		$2,39 \times 10^{-4}$	
4	8	$3,04 \times 10^{-1}$	1,30	$4,26 \times 10^{-4}$	3,21	$1,05 \times 10^{-5}$	4,50
8	16	$9,52 \times 10^{-2}$	1,67	$4,36 \times 10^{-5}$	3,28	$5,91 \times 10^{-7}$	4,15
16	32	$2,64 \times 10^{-2}$	1,85	$3,13 \times 10^{-6}$	3,80	$3,59 \times 10^{-8}$	4,04

Table (3) : The orthogonal Galerkin method with r = 1

n	dim	Theory predicts $\lambda=2$ α_n	λ_n	Theory predicts $\lambda\geq 4$ β_n	λ_n	Theory predicts $\lambda\geq 4$ γ_n	λ_n
2	4	$8,81 \times 10^{-1}$		$4,40 \times 10^{-3}$		$3,41 \times 10^{-3}$	
4	8	$3,28 \times 10^{-1}$	1,43	$3,59 \times 10^{-4}$	3,61	$2,13 \times 10^{-4}$	4,00
8	16	$9,86 \times 10^{-2}$	1,73	$2,42 \times 10^{-5}$	3,89	$1,33 \times 10^{-5}$	4,00
16	32	$2,46 \times 10^{-2}$	2,00	$1,55 \times 10^{-6}$	3,96	$8,33 \times 10^{-7}$	4,00

Table (4) : The collocation Method at the r+1=2 Gauss points

n	dim	Theory predicts $\lambda=3$ α_n	λ_n	Theory predicts $\lambda\geq 5$ β_n	λ_n	Theory predicts $\lambda\geq 6$ γ_n	λ_n
2	6	$2,41 \times 10^{-1}$		$1,67 \times 10^{-4}$		$1,28 \times 10^{-6}$	
4	12	$3,68 \times 10^{-2}$	2,53	$9,64 \times 10^{-6}$	4,11	$1,98 \times 10^{-7}$	6,01
8	24	$5,23 \times 10^{-3}$	2,81	$4,72 \times 10^{-7}$	4,35	$3,08 \times 10^{-9}$	6,00
16	48	$6,93 \times 10^{-4}$	2,91	$1,65 \times 10^{-8}$	4,83	$4,81 \times 10^{-11}$	6,00

Table (5) : The orthogonal Galerkin Method with r=2

n	dim	Theory predicts $\lambda=3$ α_n	λ_n	Theory predicts $\lambda\geq 5$ β_n	λ_n	Theory predicts $\lambda\geq 6$ γ_n	λ_n
2	6	$2,34 \times 10^{-1}$		$2,22 \times 10^{-4}$		$1,87 \times 10^{-7}$	
4	12	$3,80 \times 10^{-2}$	2,62	$9,16 \times 10^{-6}$	4,59	$7,35 \times 10^{-10}$	7,99
8	24	$5,31 \times 10^{-3}$	2,83	$3,20 \times 10^{-7}$	4,83	$2,88 \times 10^{-12}$	7,99
16	48	$6,98 \times 10^{-4}$	2,92	$1,05 \times 10^{-8}$	4,92	$1,14 \times 10^{-14}$	7,98

Table (6) : The collocation Method at the r+1=3 Gauss points

The estimated rates of convergence in tables 3,4,5 and 6 are in good agreement with those suggested by the theory.
Moreover, in comparing tables 5 and 6, it follows that the collocati method at Gaussian points has a convergence rate in n which is faster than for the orthogonal Galerkin method in the case of problems with smooth solutions. However, since we have shwon in [1] , that we need fewer smoothness properties in the solution, for the orthogonal Galerkin method than for the collocation method, we may ask the question : what happens in the non-smooth case ?
This is the scope of the following example.

Example 3

We consider again the same kernel as in example (2) but this time f is chosen so that :

$$x(t) = \left|t - \frac{1}{2}\right|^{\alpha/2} , \quad \alpha = 5 \text{ or } 7 .$$

Then, if $\alpha = 5$ (resp ($\alpha=7$)) and $r=1$ (resp ($r=2$)), the solution x is still smooth for the orthogonal Galerkin method, but it is not for the collocation method.
The following tables 7,8,9 and 10 illustrate the theory.

n	dim	Theory predicts α_n	$\lambda=2$ λ_n	Theory predicts β_n	$\lambda\geq4$ λ_n	Theory predicts γ_n	$\lambda\geq6$ λ_n
2	4	6.65×10^{-1}		4.50×10^{-3}		2.79×10^{-4}	
4	8	1.94×10^{-1}	1.77	3.59×10^{-4}	3.64	1.48×10^{-5}	4.23
8	16	5.15×10^{-2}	1.91	2.45×10^{-5}	3.87	8.75×10^{-7}	4.08
16	32	1.32×10^{-2}	1.96	1.59×10^{-6}	3.94	5.36×10^{-8}	4.02

Table (7) : The orthogonal Galerkin method with r=1 and $\alpha=5$

n	dim	Theory predicts $\lambda = 2$ α_n	λ_n	Theory predicts $\lambda \geq 2.5$ β_n	λ_n	Theory predicts $\lambda \geq 2.5$ γ_n	λ_n
2	4	7.06×10^{-1}		3.40×10^{-3}		1.66×10^{-3}	
4	8	1.98×10^{-1}	1.83	2.12×10^{-4}	4.00	1.35×10^{-4}	3.62
8	16	5.20×10^{-2}	1.92	1.41×10^{-5}	3.91	1.07×10^{-5}	3.65
16	32	1.33×10^{-2}	1.96	9.94×10^{-7}	3.82	8.44×10^{-7}	3.66

Table (8) : The collocation method with r=1 and $\alpha=5$

n	dim	Theory predicts $\lambda=3$ α_n	λ_n	Theory predicts $\lambda \geq 5$ β_n	λ_n	Theory predicts $\lambda \geq 6$ γ_n	λ_n
2	6	1.78×10^{-1}		1.42×10^{-4}		1.14×10^{-5}	
4	12	2.64×10^{-2}	2.75	6.96×10^{-6}	4.35	1.72×10^{-7}	6.05
8	24	3.54×10^{-3}	2.89	2.74×10^{-7}	4.66	2.66×10^{-9}	6.01
16	48	4.54×10^{-4}	2.96	9.22×10^{-9}	4.89	4.14×10^{-11}	6.00

Table (9) : The orthogonal Galerkin method with r=2 and $\alpha=7$

n	dim	Theory predicts $\lambda=3$ α_n	λ_n	Theory predicts $\lambda \geq 3.5$ β_n	λ_n	Theory predicts $\lambda \geq 3.5$ γ_n	λ_n
2	6	1.87×10^{-1}		1.78×10^{-4}		1.80×10^{-5}	
4	12	2.69×10^{-2}	2.79	6.47×10^{-6}	4.78	7.36×10^{-7}	4.61
8	24	3.55×10^{-3}	2.92	2.74×10^{-7}	4.56	2.81×10^{-8}	4.71
16	48	4.56×10^{-4}	2.96	1.19×10^{-8}	4.52	1.10×10^{-9}	4.67

Table (10) : Collocation method with r=2 and $\alpha=7$

It may be remarked in comparing tables 3, 5 with tables 7, 9 that the estimated rates of convergence are the same. This implies that the singularitie of the solution does not affect the orthogonal Galerkin method, while in comparing tables 4, 6 with tables 8, 10 it follows that it does for the collocation method.

We have shown in [1], that there is no need for the partition Δ to be uniform to get the superconvergence results. The question is then : what is the interest to choose a non uniform partition, and when ?

This is the scope of the following example.

Example 4

We consider the same equation than in example 4, but this time we take $\alpha = 1$. Then, the theoretical solution is : $x(t) = |t-\frac{1}{2}|^{1/2}$ which singularity at $t = \frac{1}{2}$ is too strong even for the orthogonal Galerkin method as it may be remarked through tables 11 and 12.

		Theory predicts $\lambda=0.5$		Theory predicts $\lambda\geq 2.5$		Theory predicts $\lambda\geq 2.5$	
n	dim	α_n	λ_n	β_n	λ_n	γ_n	λ_n
2	4	7.08×10^{-1}		1.80×10^{-3}		1.27×10^{-4}	
4	8	5.00×10^{-1}	0.50	3.19×10^{-4}	2.49	1.68×10^{-5}	2.91
8	16	3.53×10^{-1}	0.50	5.74×10^{-5}	2.47	1.85×10^{-6}	3.18
16	32	2.50×10^{-1}	0.49	1.02×10^{-5}	2.49	1.86×10^{-7}	3.31

Table (11) : The orthogonal Galerkin method with uniform mesh (r=1)

n	dim	Theory predicts $\lambda=0.5$ α_n	λ_n	Theory predicts $\lambda\geq 2.5$ β_n	λ_n	Theory predicts $\lambda\geq 3.5$ γ_n	λ_n
2	6	4.54×10^{-1}		8.07×10^{-5}		8.10×10^{-6}	
4	12	3.21×10^{-1}	0.50	1.38×10^{-5}	2.54	3.79×10^{-7}	4.41
8	24	2.27×10^{-1}	0.50	2.42×10^{-6}	2.51	1.72×10^{-8}	4.46
16	48	1.60×10^{-1}	0.50	4.27×10^{-7}	2.50	7.76×10^{-10}	4.47

Table (12) : The orthogonal Galerkin method with uniform mesh (r=2)

Tables 11 and 12 illustrate these poor rates of convergence which arise partly as a result of our rather naïve approach of using splines defined on uniform meshes.

In fact, better results may be obtained using meshes specially chosen to take into account the singularities in the solution x.

J.Rice in 1968 has introduced a classification of functions which includes piecewise smooth functions with singularities of certain type. He has also given the corresponding mesh which allows the approximation error of such functions to be optimal.

In a forthcoming paper [2], using Rice's idea, we give the rates of convergence of some integral and differential equations which includes Rice's singularities.

A direct application of [2], for the orthogonal Galerkin method, on the example under consideration yields the following orders of convergence :

$$\lambda(\alpha_n) = \min(q\alpha,\ r+1)$$

$$\lambda(\beta_n) = \min(q(\alpha+2), r+3)$$

$$\lambda(\gamma_n) = \min(q(\alpha+r+2)-1\ ;\ 2r+2)$$

where the real $q \geq 1$, is the number associated with the partition $\Delta := (t_i)_{i=0}$ by the relation :

$$t_0 = 0 \quad , \quad t_1 = \frac{1}{2}\left(\frac{2}{n}\right)^q$$

$$t_i = \frac{1}{2} - \left(\frac{n}{2} - i\right)^q t_1 \quad , \quad 0 \leq i \leq \frac{n}{2}$$

$$t_i = \frac{1}{2} + \left(\frac{n}{2} - i\right)^q t_1 \quad , \quad \frac{n}{2} \leq i \leq n$$

Note that $q = 1$ corresponds to a uniform partition and $q > 1$ to a non uniform mesh, which knots $\{t_i\}$ are "bunched up" near the point $t = \frac{1}{2}$ where x behaves badly, and "spread out" near the end points 0 and 1 where x is well-behaved.

The following tables illustrate the predicted rates of convergence.

n	dim	Theory predicts λ=0.71		Theory predicts $\lambda \geq 3.55$		Theory predicts $\lambda \geq 4$	
		α_n	λ_n	β_n	λ_n	γ_n	λ_n
2	4	7.08×10^{-1}		1.80×10^{-3}		1.27×10^{-4}	
4	8	4.31×10^{-1}	0.71	1.50×10^{-4}	3.58	9.23×10^{-6}	3.78
8	16	2.62×10^{-1}	0.71	1.27×10^{-5}	3.56	6.39×10^{-7}	3.85
16	32	1.60×10^{-1}	0.71	1.08×10^{-6}	3,55	4.22×10^{-8}	3.92

Table (13) : The orthogonal Galerkin method with r=1 and q=1,428571

n	dim	Theory predicts $\lambda=0.80$ α_n	λ_n	Theory predicts $\lambda\geq 4$ β_n	λ_n	Theory predicts $\lambda\geq 4$ γ_n	λ_n
2	4	7.08×10^{-1}		1.80×10^{-3}		1.27×10^{-4}	
4	8	4.06×10^{-1}	0.80	1.66×10^{-4}	3.43	8.93×10^{-6}	3.83
8	16	2.33×10^{-1}	0.80	1.07×10^{-5}	3.95	6.20×10^{-7}	3.84
16	32	1.33×10^{-1}	0.80	6.82×10^{-7}	3.97	4.05×10^{-8}	3.93

Table (14) : The orthogonal Galerkin method with r=1 and q=1.6

n	dim	Theory predicts $\lambda=2$ α_n	λ_n	Theory predicts $\lambda\geq 4$ β_n	λ_n	Theory predicts $\lambda\geq 4$ γ_n	λ_n
2	4	7.08×10^{-1}		1.80×10^{-3}		1.27×10^{-4}	
4	8	2.71×10^{-1}	1.38	1.08×10^{-3}	0.73	6.41×10^{-3}	1.00
8	16	6.76×10^{-2}	2.00	1.90×10^{-4}	2.50	6.09×10^{-6}	3.39
16	32	1.63×10^{-2}	2.05	1.60×10^{-5}	3.57	3.77×10^{-7}	4.01

Table (15) : The orthogonal Galerkin method with r=1 and q=4

n	dim	Theory predicts $\lambda=0.77$ α_n	λ_n	Theory predicts $\lambda\geq 3.88$ β_n	λ_n	Theory predicts $\lambda\geq 6$ γ_n	λ_n
2	6	4.54×10^{-1}		8.07×10^{-5}		8.10×10^{-6}	
4	12	2.65×10^{-1}	0.77	5.34×10^{-6}	3.91	1.59×10^{-7}	5.67
8	24	1.55×10^{-1}	0.77	3.54×10^{-7}	3.91	2.79×10^{-9}	5.83
16	48	9.02×10^{-2}	0.78	2.38×10^{-8}	3.89	4.61×10^{-11}	5.91

Table (16) : The orthogonal Galerkin method with r=2 and q=1,555

n	dim	Theory predicts $\lambda=1$ α_n	λ_n	Theory predicts $\lambda\geq 5$ β_n	λ_n	Theory predicts $\lambda\geq 6$ γ_n	λ_n
2	6	4.54×10^{-1}		8.07×10^{-5}		8.10×10^{-6}	
4	12	2.27×10^{-1}	1.00	2.71×10^{-6}	4.89	3.11×10^{-7}	4.70
8	24	1.13×10^{-1}	1.00	9.11×10^{-8}	4.89	5.61×10^{-9}	5.79
16	48	5.68×10^{-2}	0.99	2.85×10^{-9}	4.99	9.05×10^{-11}	5.95

Table (17) : The orthogonal Galerkin method with r=2 and q=2

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