Des multiples facettes des graphes circulants

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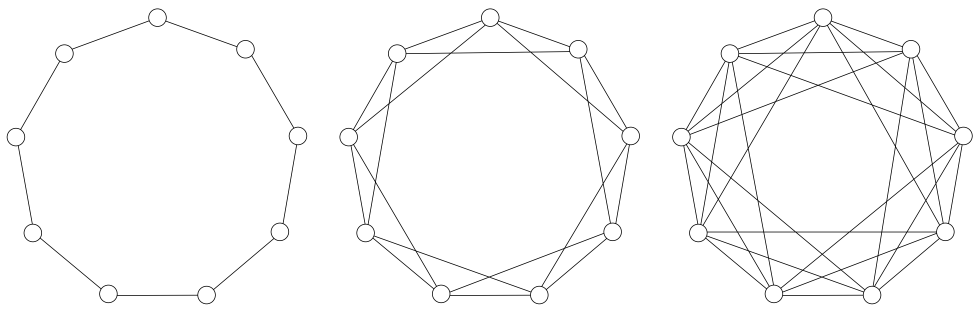
Avant-propos

Ce document présente une vue synthétique de mes travaux de recherche menés ces cinq dernières années, au sein du LaBRI. Les activités de recherche d'un enseignant-chercheur ne s'inscrivent pas souvent dans un plan de recherche soigneusement pensé. Elles évoluent en fonction de multiples impondérables, dont notamment les rencontres avec d'autres chercheurs ou encore les opportunités "stratégiques" de financement. De ce fait, il n'est pas toujours facile de dégager un fil conducteur qui permette de regrouper un ensemble des résultats obtenus "au fil de l'eau" sans avoir recours à des raccourcis un peu "artificiels".

Lorsque je me suis efforcé de dégager un point commun à mes travaux, je me suis aperçu que des objets mathématiques bien particuliers n'étaient jamais très loin de mes activités : les groupes cycliques finis. En creusant un peu plus cette perception, il m'est apparu que mes travaux accordent une place considérable à des graphes élémentaires associés aux groupes cycliques, dits *graphes circulants* ou encore *webs*, dont voici un aperçu :



Ce document est donc consacré à la mise en valeur des multiples facettes de ces graphes. "Facettes" est ici à double sens, puisqu'une partie conséquente de mes résultats est précisément dédiée à la détermination des facettes de certains polytopes associés aux graphes !

Sur la forme, les preuves ont été omises afin d'alléger le texte, à l'exception de quelques preuves sélectionnées pour leur brièveté et pour la pertinence du résultat qu'elles procurent. Des hyperliens pointent vers la version anglaise des preuves manquantes, telles qu'elles figurent dans le recueil d'articles en annexe. Pour faciliter également la lecture, l'index à la fin de l'ouvrage redonne toutes les principales définitions.

Sur le fond, ce document est structuré de la manière suivante :

- le premier chapitre est consacré aux principaux résultats connus sur les graphes parfaits. Ceci permet de définir les objets mathématiques utilisés par la suite, et de rappeler l'extraordinaire richesse conceptuelle des graphes parfaits ;
- dans le second chapitre, nous abordons un raffinement de la coloration usuelles des graphes, appelé "coloration circulaire". Cette coloration est à l'origine d'une généralisation récente des graphes parfaits : les "graphes circulaires-parfaits". Nous étudions la possibilité d'une caractérisation analogue à celles des graphes parfaits, que ce soit par sous-graphes exclus ou bien polyédrale ;
- dans le troisième chapitre, nous nous intéressons à une généralisation naturelle des webs : "les graphes quasi-adjoints". Il s'agit d'une sous-famille des graphes sans griffe, et à ce titre, l'étude de leur polytope des stables est de première importance ;
- dans le quatrième chapitre, nous menons des investigations directes sur le polytope des stables des graphes sans griffe ;

– la conclusion est donnée dans le dernier et cinquième chapitre, qui contient également une brève présentation de quelques résultats préliminaires quant au calcul en temps polynomial du nombre circulaire-chromatique des graphes circulaires-parfaits et au calcul du nombre de stabilité des graphes quasi-adjoints. Tout repose sur l'introduction d'un nouveau polytope construit à partir des webs ...

Chapitre 1

Introduction : de la diversité des graphes parfaits

Sommaire

Dans ce premier chapitre, nous passons en revue les principales caractérisations des graphes parfaits (section 1.1), et à partir de ce socle, nous définissons les familles de graphes, qui sont étudiées dans les chapitres suivants (section 1.2). Ces familles sont issues de trois approches visant à affaiblir les contraintes de la perfection d'un graphe : ainsi les graphes circulaires-parfaits sont basés sur la coloration circulaire (sous-section 1.2.1), les graphes partitionnables satisfont certaines propriétés des graphes minimaux imparfaits (sous-section 1.2.2) , et les graphes rang-parfaits sont issus de la caractérisation des graphes parfaits via le polytope des stables (sous-section 1.2.3). Ce polytope est l'objet central de ce document, et plus précisément les inégalités des familles de cliques que nous introduisons dans le paragraphe 1.2.3.4. Enfin, nous évoquons quelques-unes des pistes similaires qui ont été explorées par d'autres auteurs (section 1.3).

1.1 Graphes parfaits, les bien nommés

Les graphes étudiés dans ce document sont non-orientés, simples, finis et sans boucles. Par abus de langage, nous désignons donc par graphe G un couple (V, E) où V est un ensemble fini et E un ensemble de paires d'éléments de V. Les éléments de V sont appelés *sommets* de G et les éléments de E sont appelés *arêtes* de G. Si e est un élément $\{v, v'\}$ de E, nous utilisons fréquemment la notation simplifiée vv' pour désigner e. Deux sommets v et v' d'un graphe G sont dits adjacents si vv' est une arête de G. Le complémentaire $\overline{G}$ d'un graphe G est le graphe $(V, \overline{E})$ où $\overline{E}$ est l'ensemble des paires d'éléments de V qui n'appartiennent pas à E. De fait, la terminologie utilisée dans ce document suit les conventions usuelles de la théorie des graphes et tous les objets "classiques" ne seront pas nécessairement définis : les définitions manquantes peuvent être obtenues dans le texte fondateur de Berge [3] par exemple. La plupart des définitions de l'optimisation combinatoire ont été omises : le livre "Geometric Algorithms and Combinatorial Optimization " de Grötschel, Lovász et Schrijver [56] est une excellente référence, d'autant plus que son neuvième chapitre est entièrement consacré au polytope des stables d'un graphe.

Soit donc $G = (V, E)$ un graphe ; une k-*coloration* de G est une application $f : V \to \{1, \ldots, k\}$ telle que $f(u) \neq f(v)$ pour tout $uv \in E$, i.e., des sommets adjacents de G reçoivent des couleurs différentes. Le plus petit

k pour lequel G possède une k-coloration est appelé le *nombre chromatique* $\chi(G)$; calculer $\chi(G)$ est un problème NP-difficile en général. Un ensemble de k sommets deux-à-deux adjacents est appelé une *clique* ; les k sommets doivent recevoir des couleurs différentes. Ainsi la taille d'une clique maximum de G, le *nombre de clique* $\omega(G)$, est une minoration triviale de $\chi(G)$; cette minoration peut être très mauvaise [71] et est également difficile à déterminer. Le nombre de stabilité de G est noté $\alpha(G)$ et est égal à la plus grande taille d'un stable de G, i.e. d'un ensemble de sommets deux-à-deux non-adjacents.

Claude Berge a introduit les graphes parfaits au début des années 60 [4], pour étudier la capacité de Shannon des canaux sans mémoire [100]. Pour tout graphe $G = (V, E)$ et tout entier $n \geq 1$, notons G_n le graphe dont l'ensemble des sommets est V^n et tel que deux sommets $(u_1, \ldots, u_n)$ et $(v_1, \ldots, v_n)$ sont adjacents si et seulement si pour tout indice $1 \leq i \leq n$, nous avons $u_i = v_i$ ou $u_i v_i \in E$. La capacité de Shannon $c(G)$ est égale à $\lim_{k\to\infty} \omega(G_k)^{\frac{1}{k}}$. Ce paramètre est motivé par la théorie de l'information : en effet, si V est l'ensemble des symboles qu'un canal peut transmettre à chaque utilisation, et que deux symboles sont adjacents s'ils ne peuvent pas être confondus (à la réception), alors $\omega(G_k)$ est le nombre maximum de messages pouvant être transmis en k utilisations du canal, sans possibilité de confusion.

Nous avons toujours $\omega(G)^k \leq \omega(G^k) \leq \chi(G^k) \leq \chi(G)^k$. Ainsi si G est un graphe coloriable en $\omega(G)$ couleurs, alors $c(G) = \omega(G) = \chi(G)$. Ceci a conduit Claude Berge à introduire les *graphes parfaits* :

Définition 1.1 (graphe parfait - Berge (1961) [4])**.**

Un graphe G est parfait si et seulement si

$$\text{tout sous-graphe induit } G' \text{ de } G \text{ est coloriable en } \omega(G') \text{ couleurs.} \tag{1.1}$$

Berge a observé que tous les cycles impairs sans corde C_{2k+1} ($k \geq 2$), appelés *trous impairs*, et leurs complémentaires, les antitrous impairs $\overline{C_{2k+1}}$ ($k \geq 2$), ne sont pas parfaits.

Cette observation est la base de la célèbre Conjecture Forte des Graphes Parfaits : un graphe est parfait si et seulement s'il ne contient pas comme sous-graphe induit un trou impair ou un antitrou impair.

Cette conjecture stipule en particulier qu'un graphe est parfait si et seulement si son complémentaire l'est. Cette conséquence était connue sous le nom de la Conjecture des Graphes Parfaits . En développant la théorie des polyèdres antibloquants, Fulkerson [42] a fortement ébranlé cette conjecture, juste avant que Lovász ne la démontre de deux manières différentes [66] [67].

En particulier, Lovász a démontré cette caractérisation :

Théorème 1.2 (Lovász (1972) [66] [67])**.**

Un graphe G est parfait si et seulement si

$$\omega(G')\alpha(G') \geq |G'| \text{ pour tout sous-graphe induit } G' \text{ de } G. \tag{1.2}$$

Gasparian a donné une preuve combinatoire courte et élégante en 1996 [46]. La conjecture forte, quant à elle, a fait l'objet de très nombreux travaux pendant plus de quarante ans. Finalement, Chudnovsky, Robertson, Seymour et Thomas ont réussi à la démontrer en explorant la structure des graphes sans trous impairs et sans antitrous impairs [15] :

Théorème 1.3 (Chudnovsky, Robertson, Seymour et Thomas (2004) [15])**.**

Un graphe G est parfait si et seulement si

$$G \text{ ne contient pas de trous impairs induits et d'antitrous impairs induits.} \tag{1.3}$$

La reconnaissance en temps polynomiale des graphes parfaits est tombée peu de temps après la conjecture forte. Elle ne résulte pas directement du théorème fort des graphes parfaits :

Théorème 1.4 (Chudnovsky, Cornuéjols, Liu, Seymour et Vušković (2005) [13])**.** *Un graphe parfait est reconnaissable en temps polynomial.*

La diversité des attaques menées contre la conjecture forte a révélé des propriétés fascinantes des graphes parfaits (voir le livre [94] pour un état de l'art récent) ainsi que de multiples connexions avec d'autres domaines mathématiques.

En particulier, pour tout graphe parfait G, la taille d'une clique maximum $\omega(G)$ et la taille d'un stable maximum $\alpha(G)$ peuvent être déterminés en temps polynomial [56].

Ce résultat repose en partie sur l'étude des facettes du *polytope des stables* :

Définition 1.5 (polytope des stables STAB(G) d'un graphe G)**.** *Enveloppe convexe de tous les stables du graphe G :*

$$\mathit{STAB}(G) \quad = \quad \mathit{conv}\,\{\chi_S : S \text{ est un stable de } G\}$$

Exemple 1.6. *La figure 1.1 donne une illustration du polytope des stables d'un graphe a 3 sommets s_1, s_2 et s_3. Celui-ci possède 5 facettes : les contraintes de positivité $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$ et les contraintes $s_1 + s_3 \leq 1$, $s_2 + s_3 \leq 1$ données par les deux cliques maximales $\{s_1, s_3\}$ et $\{s_2, s_3\}$.*

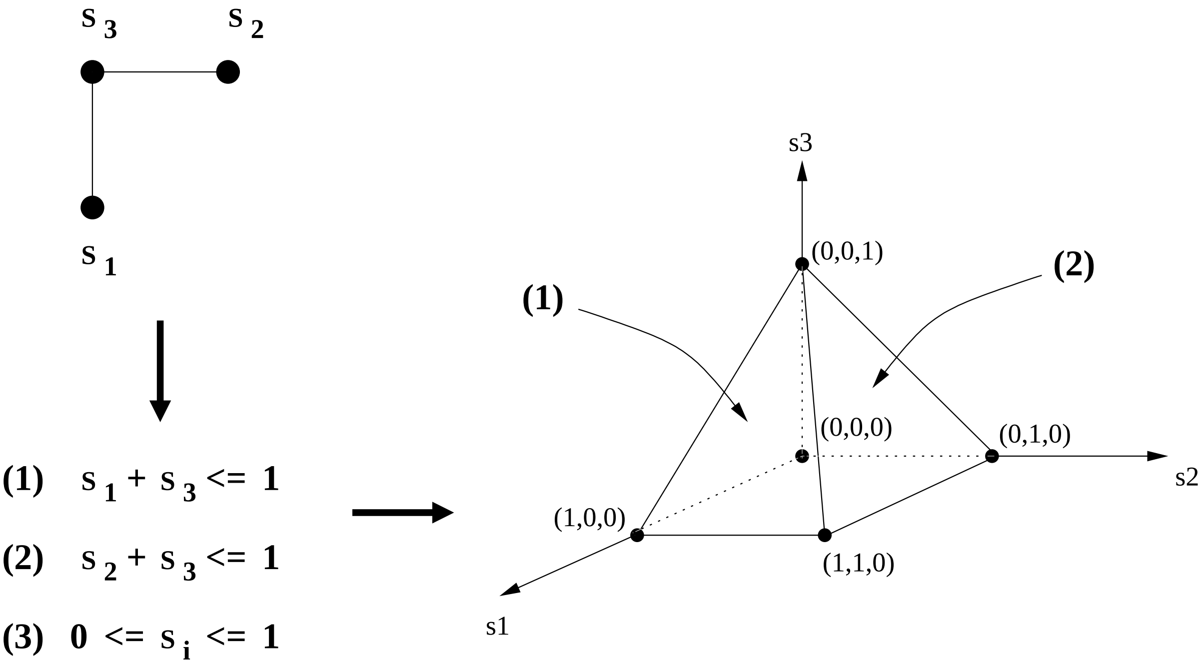


FIG. 1.1 – Exemple de polytope des stables

Comme tout stable a au plus un sommet dans une clique, le polytope des stables d'un graphe $G = (V, E)$, STAB(G) peut être également décrit sous cette forme :

$$\text{STAB}(G) = \text{conv}\left\{x \in \{0,1\}^V : x(Q) = \sum_{i \in Q} x_i \leq 1, Q \text{ clique de } G\right\}$$

Ainsi, le polytope STAB(G) est-il contenu dans sa relaxation linéaire, obtenue en supprimant les contraintes d'intégralité :

Définition 1.7 (polytope des contraintes des cliques QSTAB(G) d'un graphe G)**.**

$$\mathit{QSTAB}(G) = \left\{x \in \mathbb{R^+}^V : \sum_{i \in Q} x_i \leq 1, Q \textit{ clique de } G\right\}$$

Si STAB(G) $\subseteq$ QSTAB(G) est vérifié pour tous les graphes G, l'égalité n'est vraie que pour les graphes parfaits seulement :

Théorème 1.8 (Chvátal (1975) [19], Fulkerson (1972) [42] et Padberg (1974) [74]).

$$\textit{Un graphe } G \textit{ est parfait si et seulement si } STAB(G) = QSTAB(G) \tag{1.4}$$

Cette caractérisation polyédrale des graphes parfaits est fondamentale, et ouvre la voie vers le calcul en temps polynomial de la taille pondérée maximale d'un stable. Etant donné un vecteur de poids $\mathbf{c}$ et un graphe G quelconque, la taille pondérée maximale d'un stable $\alpha(G, \mathbf{c})$ relative à $\mathbf{c}$ vérifie successivement :

$$\begin{aligned}
\alpha(G, \mathbf{c}) &= \max \left\{ \sum_{i \in S} c_i : S \subseteq G \text{ stable} \right\} \\
&= \max \left\{ \mathbf{c}^T \mathbf{x} : \mathbf{x} \in \text{STAB}(G) \right\} \\
&= \max \left\{ \mathbf{c}^T \mathbf{x} : x(Q) \leq 1, \text{ pour toute clique } Q, \, \mathbf{x} \in \{0,1\}^V \right\} \\
&\leq \max \left\{ \mathbf{c}^T \mathbf{x} : \mathbf{x} \in \text{QSTAB}(G) \right\}
\end{aligned}$$

Lorsque le graphe G est parfait, la dernière inégalité est une égalité. Cependant, calculer l'optimum par un programme linéaire ne marche pas directement car les contraintes des cliques ne sont pas séparables en temps polynomial [56]. Pour contourner cette difficulté, Lovász [68] a défini un convexe intermédiaire TH(G) entre STAB(G) et QSTAB(G) en introduisant un nouveau système de contraintes valides pour le polytope des stables, basé sur les représentations orthonormales d'un graphe :

Définition 1.9 (représentation orthonormale d'un graphe - Lovász (1979) [68]). *Soit $G = (V, E)$ un graphe. Une représentation orthonormale de G dans l'espace vectoriel $\mathbb{R}^d$ est la donnée pour tout sommet v de G d'un vecteur u_v de $\mathbb{R}^d$ tel que*
- $||u_v|| = 1$ *pour tout sommet* v *;*
- $u_v^T u_{v'} = 0$ *pour toute arête* vv'.

Naturellement, tout graphe possède au moins une représentation orthonormale, ne serait-ce que dans $\mathbb{R}^{|V|}$ en prenant une base orthonormale.

Définition 1.10 (contrainte d'une représentation orthonormale - Lovász (1979)[68]). *Étant donnés une représentation orthonormale $(u_v)_{v \in V}$ d'un graphe $G = (V, E)$ dans $\mathbb{R}^d$ et un vecteur unitaire c de $\mathbb{R}^d$, la contrainte de la représentation orthonormale $(u_v)_{v \in V}$ est*

$$\sum_{v \in V} (c^T u_v)^2 x_v \quad \leq \quad 1$$

L'ensemble des points à coordonnées positives vérifiant les contraintes des représentations orthonormales d'un graphe G forment un convexe :

Définition 1.11 (convexe TH(G) d'un graphe G - Lovász (1979) [68]).

$$TH(G) \quad = \quad conv \left\{ x \in \mathbb{R^+}^V : \sum_{v \in V} (c^T u_v)^2 x_v \leq 1, \, (u_v)_{v \in V} \textit{ représentation orthonormale de } G, \, ||c|| = 1 \right\}$$

Si S est un stable, $(u_v)_{v \in V}$ est une représentation orthonormale et c un vecteur unitaire, nous avons $\sum_{s \in S} (c^T u_s)^2 \leq 1$: les vecteurs d'incidence des stables d'un graphe satisfont les contraintes des représentations orthonormales. Comme de plus, toute contrainte de clique peut s'écrire sous la forme d'une contrainte de représentation orthonormale, nous avons donc pour tout graphe G :

$$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G)$$

En particulier si G est parfait alors le convexe TH(G) est un polytope. Lovász a démontré que la réciproque est vraie également :

Théorème 1.12 (Lovász (1979) [68]).

$$\textit{Un graphe } G \textit{ est parfait si et seulement si le convexe } TH(G) \textit{ est un polytope.} \quad (1.5)$$

Le plus remarquable est qu'il est possible d'optimiser en temps polynomial sur ce convexe : pour n'importe quel graphe $G = (V, E)$ et n'importe quel vecteur de cout $\mathbf{c} \in \mathbb{R}^{+V}$, la solution du problème d'optimisation (avec généralement un nombre infini de contraintes linéaires), $\max\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in \mathrm{TH}(G)\}$ est calculable en temps polynomial :

Théorème 1.13 (Grötschel, Lovász et Schrijver (1984) [54, 55, 56]). *Le calcul du poids maximal d'un stable pondéré se fait en temps polynomial pour un graphe parfait.*

La résolution repose sur la méthode de l'ellipsoïde, dont on ne connaît pas d'implémentation exploitable : procurer un algorithme polynomial combinatoire pour calculer le poids pondéré maximal d'un stable d'un graphe parfait est le problème ouvert restant le plus fondamental concernant les graphes parfaits.

1.2 Rompre la perfection

Ainsi les graphes parfaits possèdent une richesse structurelle exceptionnelle et n'usurpent vraiment pas leur nom. Cependant, la plupart des graphes ne sont pas parfaits et ne possèdent pas d'aussi belles propriétés. Il est donc important de déterminer parmi les graphes imparfaits, lesquels sont les plus proches des graphes parfaits, et dans quelle mesure ils partagent les mêmes propriétés. Ceci revient à étendre "naturellement" la famille des graphes parfaits.

Fort heureusement, la diversité des caractérisations des graphes parfaits ouvre un nombre considérable de possibilités pour étendre cette classe. Dans cette section, nous allons passer en revue les pistes explorées dans les prochains chapitres.

1.2.1 Colorations circulaires, un raffinement de la coloration usuelle

La manière la plus naturelle de généraliser les graphes parfaits est de revenir à la définition, i.e. la caractérisation 1.1 en termes de propriétés de coloration. Le concept de coloration est particulièrement riche, et de nombreuses variantes ont été définies, dont notamment la coloration fractionnaire, la coloration circulaire [106], la coloration par liste [107, 38] ...

Dans ce document, nous utiliserons les colorations fractionnaire et circulaire :

Définition 1.14 ((k, d)-coloration circulaire). *Étant donnés deux entiers k et d tels que $k \geq 2d$, une (k, d)-coloration circulaire d'un graphe $G = (V, E)$ est une application $c : V \mapsto \{0, \ldots, k-1\}$ telle que $d \leq |c(x) - c(y)| \leq k - d$ pour toute arête $xy \in E$.*

Définition 1.15 ((k, d)-coloration fractionnaire). *Étant donnés deux entiers k et d tels que $k \geq 2d$, une (k, d)-coloration fractionnaire d'un graphe $G = (V, E)$ est une application $c : V \mapsto \mathcal{P}_d^k$ telle que $c(x) \cap c(y) = \emptyset$ pour toute arête $xy \in E$, où $\mathcal{P}_d^k$ désigne l'ensemble des parties à d éléments de $\{0, \ldots, k-1\}$*

Les nombres fractionnaire et circulaire chromatique sont alors définis ainsi à partir des colorations circulaire et fractionnaire :

Définition 1.16 (nombre circulaire chromatique $\chi_c(G)$ - Vince (1988) [106] et fractionnaire $\chi_f(G)$).

$$\begin{aligned}
\chi_c(G) &= \inf\{k/d : G \textit{ admet une } (k,d) - \textit{coloration circulaire}\} \\
\chi_f(G) &= \inf\{k/d : G \textit{ admet une } (k,d) - \textit{coloration fractionnaire}\}
\end{aligned}$$

Comme une $(k,d)-$coloration circulaire induit une $(k,d)-$coloration fractionnaire et qu'une $(k,1)-$coloration circulaire est une k-coloration, nous avons pour tout graphe G

$$\chi_f(G) \leq \chi_c(G) \leq \chi(G)$$

Le nombre circulaire chromatique a été étudié en premier par Vince [106] qui a établi que l'infimum est atteint, via des arguments d'analyse mathématique :

Théorème 1.17 (Vince (1988) [106])**.**

$$\chi_c(G) = \min\{k/d : \textit{ G admet une } (k,d) - \textit{coloration circulaire}\}$$

Pour étendre les graphes parfaits, la coloration circulaire possède un sérieux atout : connaissant le nombre circulaire chromatique $\chi_c(G)$ d'un graphe G, nous connaissons également son nombre chromatique usuel $\chi(G)$ car $\chi(G) = \lceil \chi_c(G) \rceil$. Ceci signifie que le nombre circulaire chromatique est un raffinement du nombre chromatique. Le nombre circulaire chromatique a fait l'objet de très nombreux travaux, voir [113] pour un état de l'art par Zhu qui fait référence et [116] pour un état de l'art très récent du même auteur.

Zhu a proposé en 2000 une extension des graphes parfaits, définie élégamment à partir du nombre circulaire chromatique. Pour ce faire, il a introduit la notion de *clique circulaire* :

Définition 1.18 (clique circulaire $K_{p/q}$)**.** *Étant donné deux entiers naturels p et q non-nuls, tels que $p \geq 2q$, la clique circulaire $K_{p/q}$ est le graphe à p sommets $\{0, \ldots, p-1\}$ et arêtes ij telles que $q \leq |i-j| \leq p-q$.*

Remarquons que les cliques circulaires contiennent notamment les cliques ($q = 1$), les trous impairs ($p = 2q+1$) et les antitrous impairs (p impair et $q = 2$). La figure 1.2 liste les cliques circulaires à 9 sommets.

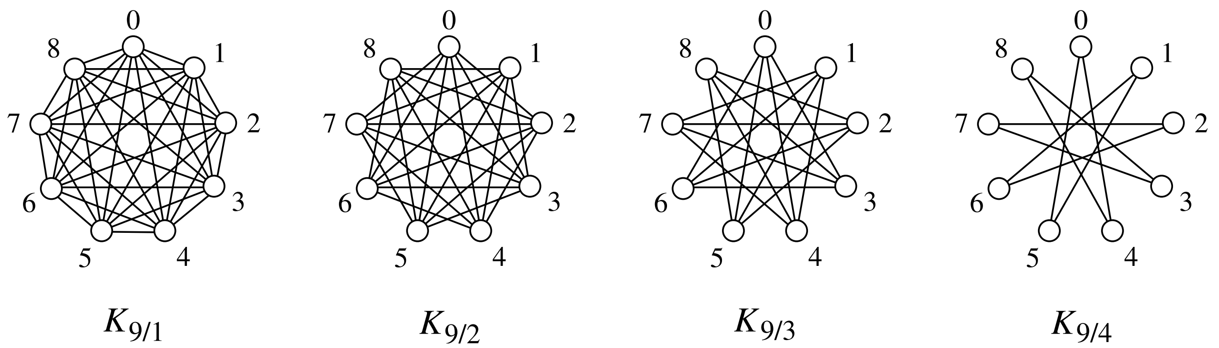


FIG. 1.2 – Les cliques circulaires à 9 sommets

La notion d'homomorphisme permet de définir élégamment les colorations circulaires (ainsi que la plupart des concepts de colorations) :

Définition 1.19 (homomorphisme de graphes)**.** *Étant donnés deux graphes $G = (V,E)$ et $G' = (V', E')$, un homomorphisme h de G dans G' est une application de V dans V' qui préserve l'adjacence, i.e. si uv est une arête de G alors $f(u)f(v)$ est une arête de G'.*

Ainsi un graphe admet une $(k,d)-$coloration circulaire si et seulement s'il est homomorphe à la clique circulaire $K_{k/d}$. Bondy et Hell ont proposé une preuve combinatoire du théorème 1.17 remarquablement courte et élégante. Elle résulte de ce résultat :

Théorème 1.20 (Bondy et Hell (1990) [7])**.** *Si $K_{p/q}$ et $K_{p'/q'}$ sont deux cliques circulaires telles que $p/q > p'/q'$ alors $K_{p/q}$ n'est pas homomorphe à $K_{p'/q'}$.*

Le complémentaire d'une clique circulaire est appelé *web* (voir la figure 1.3 pour quelques exemples) :

Définition 1.21 (web W_p^q - Sebö (1996) [99]). *Complémentaire de la clique circulaire $K_{p/q}$, i.e. graphe à p sommets $\{0,\ldots,p-1\}$ et arêtes ij telles que* $\min\{|i-j|,|p-(i-j)|\} < q$

Les cliques circulaires et leurs complémentaires, les webs, sont les acteurs principaux de ce document. La terminologie est assez fluctuante dans la littérature : ils sont parfois appelés *antiwebs* [58] ou simplement *circulants* [64, 73, 31].

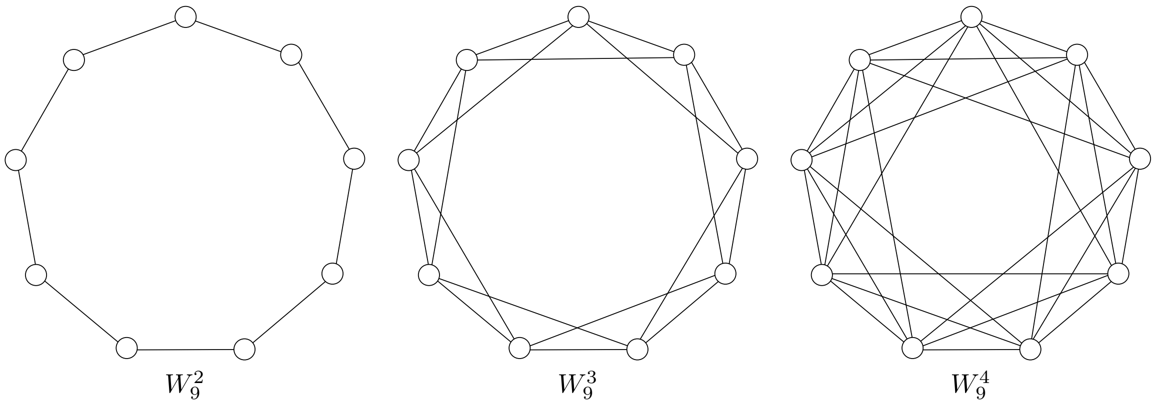


FIG. 1.3 – Quelques webs à 9 sommets

Définition 1.22 (nombre de clique circulaire $\omega_c(G)$ d'un graphe G - Zhu (2005) [115]). *Plus grand ratio p/q tel que la clique circulaire $K_{p/q}$ (p et q premiers entre eux) soit induite dans G.*

D'après le théorème 1.20, nous avons $\omega_c(G) \leq \chi_c(G)$. Il en résulte la chaîne d'inégalité fondamentale

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) \tag{1.6}$$

qui a conduit Zhu à proposer la définition suivante [115] :

Définition 1.23 (graphe circulaire-parfait - Zhu (2001) [115]).

Un graphe G est circulaire-parfait si et seulement si

$$\textit{pour tout sous-graphe induit } G' \textit{ de } G, \omega_c(G') = \chi_c(G'). \tag{1.7}$$

Un graphe parfait est bien évidemment circulaire-parfait, en raison de la chaîne d'inégalités 1.6. Les cliques circulaires sont toutes circulaires-parfaites (Zhu [115]) ainsi que les graphes convexes-ronds (Bang-Jensen et Huang [2]). Des conditions suffisantes pour qu'un graphe soit circulaire-parfait ont été obtenues par Zhu [114, 115].

Une propriété élémentaire mais néanmoins remarquable des graphes circulaires-parfaits est qu'ils sont presque "parfaitement" coloriables :

Lemme 1.24 (Zhu (2005) [115]). *Si G est un graphe circulaire-parfait alors G est coloriable en $\omega(G)+1$ couleurs.*

Naturellement cette propriété est partagée également par tous les sous-graphes induits : en ce sens, les graphes circulaires-parfaits sont très proches des graphes parfaits et il est naturel de se demander quelles propriétés des graphes parfaits s'étendent aux graphes circulaires-parfaits (cf chapitre 2).

1.2.2 Des plus petits graphes imparfaits aux graphes partitionnables

Une autre possibilité pour étendre les graphes parfaits est d'ajouter à cette famille un ensemble de graphes qui sont "proches" des graphes parfaits.

Définition 1.25 (graphe minimal imparfait)**.** *Un graphe minimal imparfait est un graphe qui n'est pas parfait, mais tel qu'en enlevant n'importe quel sommet, on obtienne un graphe parfait.*

Ce sont donc les plus petits graphes imparfaits d'un point de vue ensembliste (pour les sommets). Le théorème fort des graphes parfaits (caractérisation 1.2) ne stipule rien d'autre que les graphes minimaux imparfaits sont exactement les trous impairs et les antitrous impairs.

Nous avons observé que les trous impairs et les antitrous impairs sont des cliques circulaires et par conséquent circulaires-parfaits. Ainsi l'ensemble des graphes parfaits et minimaux imparfaits sont inclus dans les graphes circulaires-parfaits. Une des voies explorées pour tenter de résoudre la conjecture forte des graphes parfaits a consisté à déterminer les propriétés des graphes minimaux imparfaits. Ainsi le théorème des graphes parfaits implique que le complémentaire d'un graphe minimal imparfait est également minimal imparfait. Les travaux de Lovász [67] et Padberg [74] ont établi que les cliques et stables maximaux des graphes minimaux imparfaits possèdent des symétries remarquables :

Théorème 1.26 (Lovász (1972) [67] - Padberg (1974) [74])**.** *Un graphe minimal imparfait G avec un stable maximum de taille α et une clique maximum de taille ω est tel que :*
- *il possède $n = \alpha\omega + 1$ sommets ;*
- *pour tout sommet v, le graphe obtenu en enlevant v a une partition en α cliques maximum et ω stables maximums ;*
- *il a exactement n cliques maximums et n stables maximums ;*
- *chaque sommet appartient a exactement ω cliques maximums et α stables maximums ;*
- *pour toute clique maximum, il existe un unique stable maximum ne la rencontrant pas.*

Ce résultat a amené Bland, Huang et Trotter a définir la famille des graphes partiionnables :

Définition 1.27 (graphe partitionnable - Bland, Huang et Trotter (1979) [6])**.** *Un graphe est dit partitionnable s'il vérifie l'ensemble des propriétés énoncées dans le théorème 1.26.*

La figure 1.4 donne un exemple de graphe partitionnable à 10 sommets.

Ainsi la famille des graphes partitionnables contient tous les graphes minimaux imparfaits, et un angle d'attaque de la conjecture forte des graphes parfaits a été de rechercher les graphes minimaux imparfaits parmi les graphes partitionnables. Cependant, on ne sait toujours pas construire tous les graphes partitionnables, bien que plusieurs constructions partielles aient été proposées (Chvátal et al [21], Boros et al [8]).

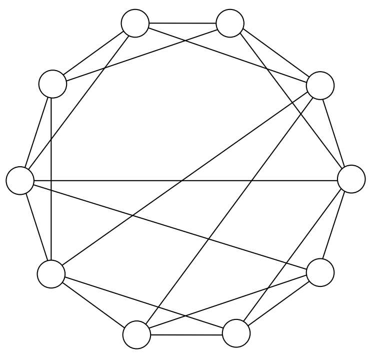


FIG. 1.4 – Un exemple de graphe partitionnable à 10 sommets.

Définition 1.28 (graphe de Cayley [5])**.** *Soit X un groupe fini et S un sous-ensemble symétrique de X, le graphe de Cayley $G(X, S)$ est le graphe dont les sommets sont les éléments de X, et $\{x, y\}$ est une arête si et seulement si $xy^{-1} \in S$.*

Même l'ensemble des graphes de Cayley partitionnables n'est pas connu [79] [80], y compris pour les groupes cycliques, bien que Grinstead ait proposé une conjecture pour ceux-ci en 1984 [53] [1].

Remarque (cliques circulaires et graphes de Cayley). *Les graphes de Cayley définis sur les groupes cycliques sont habituellement appelés des graphes circulants. Toutes les cliques circulaires sont donc des graphes circulants, et ceci est la justification du titre de ce document.*

Tout graphe partitionnable G est naturellement coloriable en $\omega(G)+1$ couleurs (il suffit d'enlever un sommet pour pouvoir partitionner en ω stables). Ceci est une propriété partagée avec les graphes circulaires-parfaits : dans le second chapitre, nous étudions plus finement les colorations circulaires et la perfection circulaire des graphes partitionnables.

1.2.3 Relaxations polyédrales

Rappelons qu'en raison de la caractérisation polyédrale (1.4) des graphes parfaits, nous avons $\mathrm{QSTAB}(G) \subsetneq \mathrm{STAB}(G)$ pour tous les graphes imparfaits G. Nous pouvons donc utiliser la différence entre ces deux polytopes pour mesurer le "degré" d'imperfection d'un graphe.

1.2.3.1 Le ratio d'imperfection

Ainsi Gerke et McDiarmid ont introduit le ratio d'imperfection pour évaluer l'écart entre ces deux polytopes.

Définition 1.29 (ratio d'imperfection - Gerke et McDiarmid (2001) [47, 48]). *Le ratio d'imperfection $imp(G)$ d'un graphe G est le plus petit réel positif t, tel que $QSTAB(G) \subseteq t.STAB(G)$.*

Naturellement un graphe est parfait si et seulement son ratio d'imperfection est égal à 1.

1.2.3.2 Graphes rang-parfaits

Une autre conséquence de la caractérisation polyédrale est que pour un graphe imparfait, les contraintes des cliques maximales ne suffisent pas pour décrire le polytope des stables. Si les graphes minimaux imparfaits sont par définition les plus proches des graphes parfaits d'un point de vue ensembliste, Padberg a été le premier à démontrer que c'était également vrai d'un point de vue polyédral :

Théorème 1.30 (Padberg (1974) [74, 75]). *Un graphe $G = (V, E)$ est minimal imparfait si et seulement si $QSTAB(G)$ possède un seul point extrême qui ne soit pas un sommet de $STAB(G)$, le point $1/\omega(G)\mathbf{1}$, et*

$$STAB(G) \quad = \quad QSTAB(G) \cap \left\{ \mathbf{x} \in \mathbb{R} : x(V) = \sum_{v \in V} x_v \leq \alpha(G) \right\} \tag{1.8}$$

Ainsi une seule facette supplémentaire seulement est requise pour décrire le polytope des stables des graphes minimaux imparfaits. En se basant sur ce résultat, Shepherd [101] a introduit la famille des graphes *proche-parfaits* comme étant les graphes G dont le polytope $\mathrm{STAB}(G)$ vérifie l'équation 1.8.

Les contraintes de rang permettent de généraliser élégamment ce concept :

Définition 1.31 (contrainte de rang). *une contrainte de rang associée à un sous-graphe induit $G' = (V', E')$ d'un graphe $G = (V, E)$ est l'inéquation $x(V') = \sum_{v \in V'} x_v \leq \alpha(G')$.*

Lorsque $G' = G$, la contrainte de rang est dite *totale*. Remarquons que les contraintes de rang sont valides pour le polytope des stables de n'importe quel graphe, sont à coefficients 0/1 seulement et qu'elles contiennent les contraintes des cliques. A partir de cette observation, Wagler a introduit les graphes *rang-parfaits* :

Définition 1.32 (graphe rang-parfait - Wagler (2002) [108]). *Un graphe est rang-parfait si toutes les facettes non-triviales de son polytope des stables sont des contraintes de rang.*

Le principal intérêt des graphes rang-parfaits est qu'ils définissent une extension des graphes parfaits, pour laquelle les polytopes des stables sont encore "simples" (coefficients 0/1 associés à des graphes induits). Plusieurs familles de graphes sont définies en restreignant l'ensemble des facettes à des contraintes de rang associées à des familles particulières de sous-graphes induits : outre les graphes parfaits (contraintes des cliques), nous avons notamment les graphes t-parfaits [19] (contraintes des arêtes, des triangles et des trous impairs) et les graphes h-parfaits [56] (contraintes des cliques et des trous impairs). Les cliques circulaires sont également rang-parfaites [109].

Dans le chapitre 3, nous introduisons une famille de graphes rang-parfaits généralisant les graphes h-parfaits : les graphes a-parfaits, dont les facettes sont associées aux cliques circulaires, puis nous étudions leur ratio d'imperfection. De plus, nous montrons que la plupart des webs (de taille de clique maximum donnée au moins 4) ne sont pas rang-parfaits.

1.2.3.3 Le rang de Chvátal

Chvátal [18] et, de manière implicite, Gomory [51] ont introduit un procédé par approximations successives pour déterminer l'enveloppe convexe entière d'un polytope quelconque. Si P est un polytope, dénotons par P_I l'*enveloppe convexe entière* de ce dernier, i.e. le plus grand convexe contenant tous les points entiers de P. Si $\sum a_i x_i \leq b$ est une inégalité valide à coefficients a_i entiers pour P alors $\sum a_i x_i \leq \lfloor b \rfloor$ est évidemment une inégalité valide pour P_I : ces nouvelles inégalités sont appelées des *coupes de Chvátal-Gomory* :

Définition 1.33 (coupe de Chvátal-Gomory)**.** *Inégalité obtenue à partir d'une inégalité à coefficients gauches entiers en prenant la partie entière du membre de droite.*

Soit P' l'ensemble des points vérifiant les coupes de Chvátal-Gomory de P. Posant $P^0 = P$ et $P^{t+1} = P^{t'}$ pour tout indice $t \geq 0$, nous obtenons

$$P_I \subseteq P^t \subseteq P$$

Chvátal a démontré que pour tout P, il existe un t fini tel que $P^t = P_I$: le plus petit indice t vérifiant $P^t = P_I$ est appelé le *rang de Chvátal* de P :

Définition 1.34 (rang de Chvátal d'un polytope P)**.** *Nombre minimal d'approximations par applications de coupes de Chvátal-Gomory nécessaire pour obtenir l'enveloppe convexe entière de P.*

En appliquant les coupes de Chvátal-Gomory au polytope des cliques QSTAB(G), nous obtenons une autre manière naturelle de généraliser les graphes parfaits : une famille de graphes $\mathcal{G}$ quelconque a pour rang de Chvátal au plus t, si pour tout graphe G de la famille, nous avons QSTAB$(G)^t$ = STAB(G). Ainsi les graphes parfaits forment la classe des graphes de rang de Chvátal au plus 0. Les graphes minimaux imparfaits, les graphes h-parfaits ou t-parfaits, les graphes adjoints ont un rang de Chvátal au plus 1, tandis que le rang de Chvátal des graphes rang-parfaits ne peut pas être borné [20]. Dans le chapitre 3, nous établissons une majoration du rang de Chvátal pour les inégalités des familles de cliques, décrites ci-après.

1.2.3.4 Des inégalités des ensembles de taille impaire aux inégalités des familles de cliques

Dans ce paragraphe, nous allons introduire des inégalités valides pour le polytope des stables qui sont associées à des familles de cliques quelconques. Ces inégalités ont été découvertes lors de l'étude des graphes sans griffe et sont au cœur de ce document. Leur origine est liée aux travaux d'Edmonds [35] sur le polytope des couplages d'un graphe :

Définition 1.35 (couplage et polytope des couplages)**.** *Un couplage est un ensemble d'arêtes deux-à-deux disjointes d'un graphe G et le polytope des couplages $M(G)$ est l'enveloppe convexe des vecteurs d'incidence des couplages.*

Edmonds a introduit les inégalités des ensembles de taille impaire pour décrire les facettes de ce polytope [34] :

Théorème 1.36 (Edmonds (1965) [34]). *Les facettes du polytope des couplages $M(G)$ d'un graphe $G = (V, E)$ sont*

- *des contraintes de non-négativité $x_e \geq 0$ pour toute arête $e \in E$;*
- *des contraintes, dites contraintes des étoiles, $\sum_{v \in e} x_e \leq 1$ pour tout sommet $v \in V$;*
- *des contraintes d'ensembles de taille impaire $\sum_{e \in E(H)} x_e \leq (|H| - 1)/2$ pour tout sous-graphe H dont le nombre de sommets est impair et au moins 3 (où $E(H)$ désigne l'ensemble des arêtes de H).*

Remarquons que les contraintes des étoiles stipulent simplement que dans un couplage au plus une arête est incidente à un sommet et que les contraintes d'ensemble de taille impaire correspondent à l'observation que dans un couplage, le nombre d'arêtes contenues dans un sous-graphe H est naturellement au plus la moitié du nombre de sommets dans H.

Le polytope des couplages fractionnaire est le polytope dont les facettes sont données par les contraintes de non-négativité et les contraintes des étoiles. Chvátal a observé que les contraintes d'ensembles de taille impaire sont obtenues à partir du polytope des couplages fractionnaire avec une seule application de la coupe de Chvátal-Gomory. Autrement dit, le polytope des couplages a un rang de Chvátal inférieur ou égal à un.

Toutes les contraintes d'ensemble impair ne sont pas forcément des facettes : Edmonds et Pulleyblank [36] ont caractérisé ensuite celles qui induisent des facettes, ce sont celles données par les sous-graphes H 2-connexe et sous-couplable (i.e. pour tout sommet v, $H - v$ est connexe et admet un couplage couvrant tous ses sommets)

Définition 1.37 (graphe adjoint (ou graphe représentatif des arêtes) $L(G)$ d'un graphe G). *le graphe adjoint d'un graphe $G = (V, E)$ est le graphe avec pour ensemble de sommets E, et tel que deux sommets e et e' sont adjacents si et seulement si les deux arêtes e et e' de G sont incidentes.*

Remarque. *Nous utilisons la dénomination "graphe adjoint" de préférence à "graphe représentatif des arêtes" dans ce document, car elle est plus concise et va de pair avec celle des graphes "quasi-adjoints" définis ci-après.*

Comme les couplages d'un graphe G sont en bijection avec les stables de son graphe adjoint, la description d'Edmonds possède la reformulation suivante sous forme de description des facettes du polytope des stables des graphes adjoints :

Corollaire 1.38 (Edmonds (1965) [34]). *Les facettes du polytope des stables du graphe adjoint $L(G)$ d'un graphe G sont*

(i) les contraintes de non-négativité $x_v \geq 0$ pour tout sommet v ;
(ii) les contraintes des cliques maximales de $L(G)$;
(iii) les conraintes de rang $\sum_{v \in V(L(H))} x_v \leq (|H| - 1)/2$ pour tout sous-graphe H 2-connexe, sous-couplable, de G .

En particulier, les graphes adjoints ont donc un rang de Chvátal au plus un.

Pour pouvoir introduire les inégalités de familles de cliques et expliquer pourquoi elles généralisent les contraintes de rang (iii), il est utile de faire la constatation suivante : considérons un sous-graphe H avec un nombre impair de sommets d'un graphe G ; pour chaque sommet v de H, l'ensemble des arête incidentes à v induit une clique dans le graphe adjoint $L(G)$ et nous avons donc une famille $\mathcal{F}$ de $|H|$ cliques de $L(G)$, telle que deux cliques s'intersectent s'il y a une arête entre les deux sommets correspondants de G (voir figure 1.5).

En notant par $V(\mathcal{F}, 2)$ l'ensemble des sommets de G qui sont couvertes deux fois par des cliques de $\mathcal{F}$, nous pouvons donc reformuler la contrainte de rang (ii) associée à $L(H)$ par

$$\sum_{v \in V(\mathcal{F}, 2)} x_v \leq (|\mathcal{F}| - 1)/2$$

Définition 1.39 (graphe quasi-adjoint). *Un graphe est quasi-adjoint si le voisinage de tout sommet peut être couvert par 2 cliques seulement.*

Naturellement, un graphe adjoint est quasi-adjoint. Le corollaire 1.38 stipule que ces contraintes de rang sont les seules facettes non-triviales, avec les contraintes de cliques, requises pour décrire le polytope des stables des graphes adjoints.

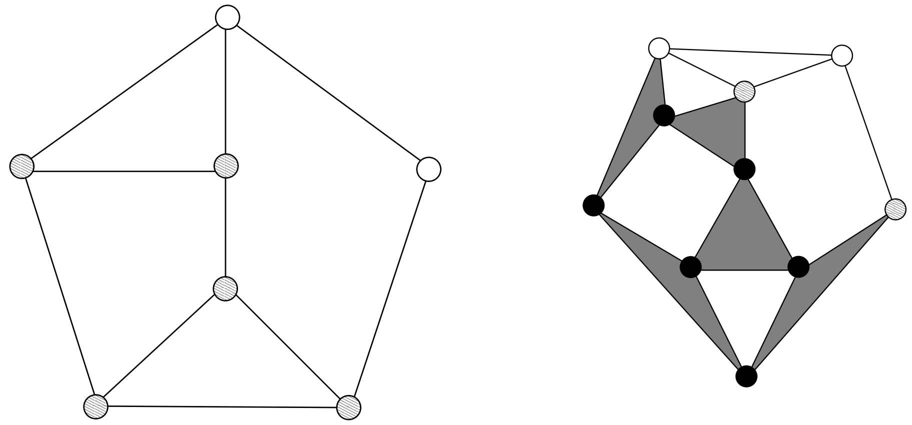


FIG. 1.5 – Des inégalités d'ensembles impairs aux inégalités de familles de cliques

Définition 1.40 (graphe circulaire d'intervalle - Chudnovsky et Seymour (2005) [17]). *Soit Σ un cercle et soient $F_1, F_2, \ldots, F_k$ des intervalles de Σ. Soit V un ensemble fini de points Σ, et soit G le graphe dont l'ensemble des sommets est V et tel que deux sommets x et y sont adjacents si et seulement si il existe un indice i tel que $x, y \in F_i$. Un tel graphe est appelé un graphe circulaire d'intervalle.*

Définition 1.41 (graphe circulaire d'intervalle flou - Chudnovsky et Seymour (2005) [17]). *Soit G un graphe circulaire d'intervalle. Une arête xy de G est dite* maximale *si pour tout intervalle F_i dans la représentation par intervalle de G contenant x et y alors x et y soient les extrémités de F_i. Soit M un couplage d'arêtes maximales : pour toute arête xy de M, remplaçons x par une clique A et y par une clique B de telle sorte que les sommets de A (resp. B) aient les mêmes voisins que x (resp. y) dans $V \setminus \{x, y\}$; les arêtes entre A et B étant arbitraires. Le graphe obtenu est appelé graphe circulaire d'intervalle flou.*

Définition 1.42 (graphe semi-adjoint - Chudnovsky et Seymour (2005) [17]). *Graphe quasi-adjoint qui n'est pas un graphe circulaire d'intervalle flou*

Chudnovsky et Seymour ont établi la liste des facettes des polytopes de stables des graphes semi-adjoints :

Théorème 1.43 (Chudnovsky et Seymour (2004) [16]). *Un graphe connexe quasi-adjoint G est un graphe circulaire d'intervalle flou ou un graphe dont le polytope des stables est donné par les inégalités triviales, les contraintes de cliques et les inégalités de famille de cliques $(\mathcal{Q}, 2)$*

$$\sum_{i \in V_{\geq 2}(\mathcal{Q},2)} x_i \leq \tfrac{|\mathcal{Q}|-1}{2}$$

associées avec des familles de cliques $\mathcal{Q}$ de taille impaire.

Cependant, les contraintes de rang ne sont pas toujours suffisantes pour décrire le polytope des graphes quasi-adjoints : en fait, dans le chapitre 3, nous établissons que la plupart des webs ne sont pas rang-parfaits ! Ben Rebea a proposé de généraliser les contraintes (ii) ainsi :

Définition 1.44 (inégalité d'une famille de cliques - Ben Rebea (1981) [95]). *Soient $G = (V, E)$ un graphe, $\mathcal{F}$ une famille d'au moins 3 cliques non-nécessairement distinctes, $p \leq |\mathcal{F}|$ un entier, et soient les deux ensembles*

$$\begin{aligned} V_{\geq p} &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\} \\ V_{p-1} &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p - 1\} \end{aligned}$$

Alors l'inégalité $(\mathcal{F}, p)$ de la famille de cliques $\mathcal{F}$ est définie par

$$(p-r) \sum_{v \in V_{\geq p}})x_v + (p-r-1) \sum_{v \in V_{p-1}} x_v \leq (p-r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor$$

où $r = |\mathcal{F}| \pmod p$.

Le fait remarquable est que ces inégalités sont valides pour le polytope des stables de n'importe quel graphe (Oriolo [73]). Dans le chapitre 3, nous bornons en particulier le rang de Chvátal d'une telle inégalité par $\min\{r, p-r\}$. Ben Rebea a affirmé dans sa thèse [95] qu'elles sont suffisantes pour décrire les polytopes des stables des graphes quasi-adjoints. Malheureusement Ben Rebea est décédé peu de temps après sa thèse, et quelques preuves se sont révélées inexactes. Vingt ans plus tard, Oriolo a repris et diffusé une partie des travaux de Ben Rebea [73]. Conjointement avec Eisenbrand, Stauffer et Ventura, il a établi la validité de la conjecture de Ben Rebea, en se basant sur les travaux de Chudnovsky et Seymour :

Théorème 1.45 (Eiseibrand, Oriolo, Stauffer et Ventura (2005) [37] & Chudnovsky et Seymour (2004) [16] - conjecture de Ben Rebea (1981) [95])**.** *Les facettes du polytope des stables des graphes quasi-adjoints sont :*
- *les contraintes triviales de non-négativité ;*
- *les contraintes des cliques maximales ;*
- *des inégalités de familles de cliques.*

Ainsi ce théorème généralise la description d'Edmonds du polytope des stables des graphes adjoints aux graphes quasi-adjoints. La preuve de ce théorème repose fortement sur la récente décomposition de Chudnovsky et Seymour des graphes sans griffe, dans son volet sur les graphes quasi-adjoints.

Pour les graphes sans griffe, Edmonds a conjecturé que leur rang de Chvátal est au plus un.

Conjecture 1.46 (Edmonds (cf Giles et Trotter (1981) [49]))**.** *Le rang de Chvátal d'un graphe sans griffe est au plus un.*

Si cette conjecture est vraie pour les graphes adjoints, elle est fausse en général [49, 20], même lorsque l'on se restreint aux graphes quasi-adjoints [73]. Dans le chapitre 3, nous établissons sa validité pour la famille intermédiaire des graphes semi-adjoints.

1.3 D'autres pistes

Le théorème fort des graphes parfaits ne marque pas l'achèvement de la théorie des graphes parfaits. De fait, les résultats fondamentaux des graphes parfaits (le calcul du nombre de clique pondéré et la reconnaissance en temps polynomiale) ne sont pas des conséquences de ce théorème. Les différentes caractérisations des graphes parfaits ont eu un impact considérable et inattendu sur d'autres branches mathématiques. La caractérisation (1.5) via les représentations orthonormales en est le plus bel exemple puisqu'elle est à l'origine de la théorie de la programmation quadratique en recherche opérationnelle !

Nous avons passé en revue dans ce chapitre quelques unes des caractérisations des graphes parfaits, qui nous ont permis de définir des familles de graphes contenant les graphes parfaits, et qui sont étudiées dans les chapitres suivants.

D'autres pistes ne sont pas explorées dans ce document. Mentionnons en particulier
- la caractérisation des graphes parfaits basée sur l'entropie d'un graphe : soient $\mathbf{p} \in \mathbb{R}^n_+$ une distribution de probabilités et G un graphe, l'entropie du polytope des stables $H(G, \mathbf{p})$, introduite par Körner (1973) [61], et l'entropie de $\mathbf{p}$ sont définies par

$$\begin{aligned} H(G,\mathbf{p}) &= \min\left\{\sum_{i\leq n} p_i \log_2 \frac{1}{x_i} : \mathbf{x} \in \text{STAB}(G)\right\} \\ H(\mathbf{p}) &= \sum_{i\leq n} p_i \log_2 \frac{1}{p_i} \end{aligned}$$

L'entropie d'un graphe est sous-additive : pour toute distribution de probabilités $\mathbf{p}$, nous avons

$$H(\mathbf{p}) \leq H(G,\mathbf{p}) + H(\overline{G}, \mathbf{p}$$

Cziszár et al. [29] ont démontré que les graphes parfaits sont exactement les graphes réalisant l'égalité :

Théorème 1.47 (Czsiszár et al. (1990) [29])**.**

$$\text{Un graphe } G \text{ est parfait si et seulement si } H(\mathbf{p}) = H(G, \mathbf{p}) + H(\overline{G}, \mathbf{p}),\ \forall \mathbf{p} \text{ distribution de probabilités} \tag{1.9}$$

– les graphes normaux : un graphe est dit normal s'il possède une couverture par des cliques $\mathcal{Q}$ et une couverture par des stables $\mathcal{S}$ telles que toute clique Q de $\mathcal{Q}$ intersecte tout stable S de $\mathcal{S}$. Les graphes parfaits sont naturellement tous normaux. De nombreuses propriétés des graphes parfaits s'étendent aux graphes normaux. En particulier, on peut également les caractériser via l'entropie d'un graphe !
Théorème 1.48 (Körner et Marton (1988) [62])**.**

Un graphe G est normal si et seulement si $\exists \mathbf{p} > 0$ distribution de probabilités , $H(\mathbf{p}) = H(G, \mathbf{p}) + H(\overline{G}, \mathbf{p})$,

De Simone et Körner ont posé une conjecture très intéressante, introduisant une condition suffisante pour les graphes normaux :
Conjecture 1.49 (De Simone et Körner (1999) [103])**.** *Un graphe sans C_5, C_7 et $\overline{C_7}$ induits est normal.*

Chapitre 2

Colorations circulaires

Sommaire

L'ensemble de ce chapitre est dédié à la présentation de nos résultats concernant les colorations circulaires et les graphes circulaires-parfaits. En bref, nous avons démontré qu'une caractérisation par sous-graphes exclus similaire à celle du théorème fort des graphes parfaits (cf caractérisation 1.3) est très improbable, car nous avons exhibé trois familles de graphes minimaux circulaires-imparfaits aux propriétés structurelles très distinctes [89]. Nous avons également exhibé une famille de graphes circulaires-parfaits dont les polytopes des stables ont des facettes relativement compliquées [24]. Cela signifie qu'une caractérisation polyédrale analogue à celle des graphes parfaits (cf caractérisation 1.4) est également peu vraisemblable.

Une des faiblesses des graphes circulaires-parfaits est que le complémentaire d'un graphe circulaire-parfait n'est pas forcément circulaire-parfait : le théorème des graphes parfaits ne s'étend pas aux graphes circulaires-parfaits. Ainsi nous avons décidé d'introduire la classe des *graphes fortement circulaires-parfaits* : un graphe fortement circulaire-parfait est un graphe circulaire-parfait dont le complémentaire est également circulaire-parfait. Cette famille inclut naturellement les graphes parfaits. Une caractérisation par sous-graphes exclus des graphes fortement circulaires-parfaits est peut être possible : rappelons qu'un graphe parfait sans triangle est exactement un graphe sans triangle ni trous impairs ; nous avons réussi à donner une caractérisation analogue pour les graphes fortement circulaires-parfaits sans triangle [25, 28].

2.1 Colorations circulaires des graphes partitionnables

Dans l'état de l'art de Xuding Zhu [113], il est demandé de déterminer les graphes tels qu'en enlevant n'importe quel sommet, le nombre circulaire chromatique diminue de 1.

Les graphes partitionnables dont le nombre circulaire chromatique est égal au nombre chromatique vérifient cette propriété : en effet, pour tout sommet x, nous avons $\chi(G-x) = \omega(G-x) = \chi_c(G-x) = \chi(G) - 1$.

Avec Xuding Zhu, nous avons exploré la famille des graphes circulants partitionnables [92] et exhibé la première famille infinie de graphes ayant la propriété mentionnée ci-dessus (paragraphe 2.1.1). Nous avons également montré que la quasi-totalité des graphes partitionnables n'étaient pas circulaires-parfaits (paragraphes 2.1.2 et 2.2.1) !

2.1.1 Colorations circulaires des graphes circulants partitionnables

La famille des graphes circulants partitionnables a été introduite par Chvátal, Graham, Perold et Whitesides [21] : si X, Y sont deux ensembles d'entiers, nous notons $X + Y$ l'ensemble $\{x + y : x \in X,\ y \in Y\}$.

Définition 2.1 (graphe circulant partitionnable - Chvátal, Graham, Perold et Whitesides [21])**.** *Soient des entiers $m_i \geq 2$ ($i = 1, 2, \cdots, 2r$) : les paramètres μ_i (pour $i = 0, 1, \ldots, 2r$), les ensembles M_i (pour $i = 1, 2, \ldots, 2r$) et les ensembles C et S sont définis de la manière suivante :*

$$\begin{aligned} \mu_i &= m_1 m_2 \cdots m_i \ (\mu_0 = 1), \\ M_i &= \{0, \mu_{i-1}, 2\mu_{i-1}, \cdots, (m_i - 1)\mu_{i-1}\}, \\ C &= M_1 + M_3 + \cdots + M_{2r-1}, \\ S &= M_2 + M_4 + \cdots + M_{2r}. \end{aligned}$$

Soit $n = m_1 m_2 \cdots m_{2r} + 1$. Nous notons $C[m_1, m_2, \ldots, m_{2r}]$ le graphe circulant avec pour ensemble de sommets $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, tel que xy est une arête si et seulement si $x \neq y$ et $(x - y)$ modulo n est égal à la différence de deux éléments de C (autrement dit $C[m_1, m_2, \ldots, m_{2r}]$ est le graphe de Cayley sur le groupe cyclique $\mathbb{Z}_n$ avec pour ensemble connectant $(C - C) \setminus \{0\}$). Cette construction des ensembles C et S avait été introduite auparavant par De Bruijn dans un autre contexte [32].

Chvátal, Graham, Perold et Whiteside ont démontré que le graphe $C[m_1, m_2, \ldots, m_{2r}]$ est toujours partitionnable [21]. Remarquons que lorsque $r = 1$, le graphe $C[m_1, m_2]$ est un web.

Exemple 2.2 (Graphe circulant partitionnable)**.** *Prenons le graphe $C[2, 2, 2, 2]$ pour exemple. Nous avons*

$$\begin{aligned} \mu_i &= 2^i,\ \ i = 0, 1, 2, 3, 4; \\ M_1 &= \{0, 1\},\ M_2 = \{0, 2\},\ M_3 = \{0, 4\},\ M_4 = \{0, 8\}; \\ C &= \{0, 1, 4, 5\}; \\ S &= \{0, 2, 8, 10\}. \end{aligned}$$

L'ensemble des sommets de $C[2, 2, 2, 2]$ est $\mathbb{Z}_{17}$, et ij est une arête si et seulement si $|i - j| \in \{1, 3, 4, 5\}$. Le complémentaire de ce graphe est décrit dans la figure 2.1.

Nous avons démontré qu'un nombre infini de ces graphes possèdent un nombre circulaire chromatique égal à leur nombre chromatique :

Théorème 2.3 (P., Zhu (2006) [92] B.4.2)**.** *Soient $x_1, \ldots, x_p$ ($p \geq 2$) des entiers tels que $x_i \geq 2$ pour chaque indice $1 \leq i \leq p$. Soit $\delta = \max\{x_i\}$ et $G = C[x_1, 2, \ldots, x_p, 2]$. Si $p = 2$ ou $x_1 x_2 \ldots x_p \geq 2^{p+1}\delta$ alors $\chi_c(G) = \chi(G)$.*

Un certains nombre de travaux concernent le nombre circulaire chromatique des graphes circulants. Ainsi, il est connu que pour tout graphe G, le nombre fractionnaire chromatique $\chi_f(G)$ est majoré par le nombre circulaire chromatique (cf chapitre 1), et un graphe est dit *étoile-extrémal* si ces deux paramètres sont égaux. Les graphes étoile-extrémaux *circulants* sont explorés dans les articles [45, 65] notamment. Notre théorème introduit donc une famille de graphes circulants présentant une autre sorte d'extrémalité.

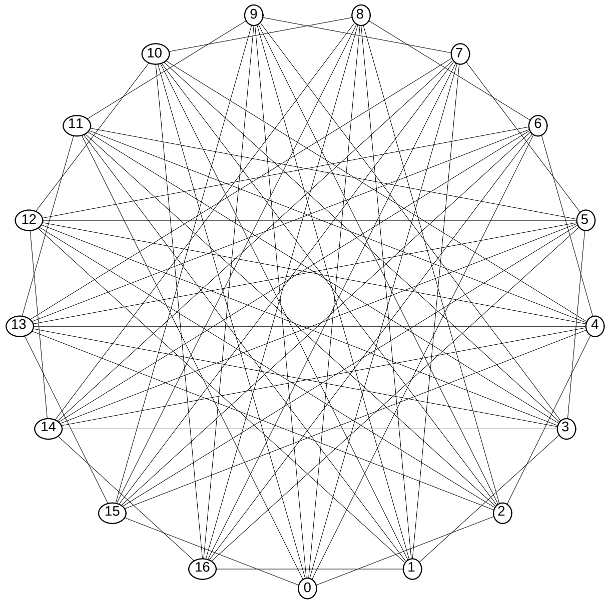


FIG. 2.1 – Graphe complémentaire de $C[2,2,2,2]$.

2.1.2 Imperfection circulaire des graphes partitionnables

Comme les graphes partitionnables sont toujours $\omega + 1$ coloriables, tout comme les graphes parfaits, et qu'ils sont proches des graphes parfaits, par définition, ils devraient constituer un vivier "idéal" pour débusquer des graphes circulaires-parfaits. En fait, il n'en est rien !

Lemme 2.4 (P., Wagler, Zhu (2005) [91] [89])**.** *Les seuls graphes partitionnables circulaires-parfaits sont des cliques circulaires.*

Démonstration. Soit G un graphe partitionnable. Si $\omega_c(G) = \omega(G)$ alors $\chi_c(G) > \omega(G) = \omega_c(G)$, puisque $\chi(G) = \omega(G) + 1$, et G est donc circulaire-imparfait.

Si $\omega_c(G) = p/q > \omega$ alors soit $\{0, \ldots, p-1\}$ les sommets d'une clique circulaire induite $K_{p/q}$ (selon l'étiquetage usuel). Pour tout indice $0 \leq i < \omega$, soit Q_i la clique maximum $\{jq | 0 \leq j \leq i\} \cup \{jq+1 | i < j < \omega\}$. Les cliques $Q_0, \ldots, Q_{\omega-1}$ sont ω cliques maximums distinctes de G contenant toutes le sommet 0. Si $p > \omega q + 1$ alors l'ensemble $(Q_0 \setminus \{(\omega-1)q+1\}) \cup \{(\omega-1)q+2\}$ est une autre clique maximum de G contenant 0, ce qui est impossible car un sommet appartient à exactement ω cliques maximum [6]. Ainsi $p = \omega q + 1$ et donc G contient la clique circulaire partitionnable $K_{(\omega q+1)/q}$ comme sous-graphe induit, ce qui implique que G est la clique circulaire $K_{(\omega q+1)/q}$. ■

2.2 Graphes circulaires-parfaits

Nous avons vu dans le premier chapitre que les graphes circulaires-parfaits ont été introduits par Xuding Zhu, comme le pendant "circulaire" des graphes parfaits. Compte tenu du théorème fort des graphes parfaits et de l'importance des applications des travaux que la conjecture forte elle-même a suscité, il est naturel de rechercher une caractérisation par sous-graphes exclus des graphes circulaires-parfaits.

Nous avons exhibé trois familles de graphes minimaux circulaires-imparfaits [89] [91] aux propriétés structurelles assez distinctes, si bien qu'une caractérisation attractive des graphes minimaux circulaires-imparfaits est improbable.

2.2.1 Quelques familles de graphes minimaux circulaires-imparfaits

2.2.1.1 Une famille de graphes minimaux circulaires-imparfaits ... circulants !

Le concept des graphes normalisés à été étudié lors de l'approche de la conjecture forte des graphes parfaits via les graphes partitionnables :

Définition 2.5 (graphe normalisé)**.** *Le graphe normalisé d'un graphe G est le graphe obtenu en enlevant les arêtes qui ne sont pas dans des cliques maximum. Un graphe est dit normalisé s'il est isomorphe à son normalisé, i.e. si toute arête est contenue dans une clique maximum.*

Naturellement, le normalisé d'un graphe partitionnable est lui-même partitionnable, et la quasi totalité des preuves démontrant que tels graphes partitionnables ne sont pas minimaux imparfaits, reposent uniquement sur la structure en cliques maximums et stables maximums [1, 99, 98]. Fort de ces observations, les constructions de familles de graphes partitionnables qui ont été élaborées sont des constructions de familles de graphes partitionnables *normalisés* [21, 8, 79, 80].

La conjecture de Grinstead précise la liste des graphes normalisés de Cayley partitionnables sur les groupes cycliques :

Conjecture 2.6 (Grinstead (1984) [53])**.** *Tout graphe de Cayley sur un groupe cyclique, partitionnable, normalisé est isomorphe à un graphe circulant partitionnable (définition 2.1).*

Cette conjecture a été vérifiée pour les graphes avec un nombre de cliques au plus 5 [1] ou avec moins de 123 sommets [79]. Il est difficile d'exhiber un graphe de Cayley (sur un groupe quelconque) partitionnable normalisé qui ne soit pas un graphe circulant partitionnable : de fait, cela revient à étudier les quasi-factorisations des groupes finis [78, 9] et le plus petit tel graphe a 50 sommets. Il s'agit du seul connu à ce jour [80] !

Ce concept propose également un intérêt vis-à-vis des graphes circulaires-parfaits. En effet, si G est un graphe circulaire-parfait, en prenant son graphe normalisé, le nombre de clique circulaire diminue. Si le nombre circulaire chromatique ne varie pas, nous obtenons alors un graphe circulaire-imparfait. C'est le point de départ du théorème 2.7 ci-après, qui présente notre première famille de graphes minimaux circulaires-imparfaits :

Théorème 2.7 (P., Wagler (2008) [89] B.6.2)**.** *Soit $K_{p/q}$ une clique circulaire première. Alors son graphe normalisé est*

1. *circulaire-imparfait si et seulement si $p \not\equiv -1 \pmod q$ et $\lfloor p/q \rfloor \geq 3$;*
2. *minimal circulaire-imparfait si et seulement si $p = 3q + 1$ et $q \geq 3$.*

Une conséquence inattendue du théorème 2.7 est que le concept de perfection circulaire permet de discriminer les graphes minimaux imparfaits des autres graphes partitionnables normalisés !

Corollaire 2.8. *Un graphe partitionnable normalisé est circulaire-parfait si et seulement s'il est un trou impair ou un antitrou impair.*

Démonstration. Soit G un graphe partitionnable normalisé circulaire-parfait : par le lemme 2.4, nous savons que G est une clique circulaire $K_{p/q}$. Si $\omega(G) \geq 3$, alors $p = 1 \pmod q$ (puisque G est partitionnable) et $p = -1 \pmod q$ (par le théorème 2.7). Par conséquent $q = 2$ et G est un antitrou impair. Si $\omega(G) = 2$ alors G est un trou impair. ∎

2.2.1.2 Une famille de graphes minimaux circulaires-imparfaits planaires

Nous avons découvert par ordinateur des exemples de graphes minimaux circulaires-imparfaits *planaires*. Citons par exemple la roue impaire à 5 sommets. D'autres exemples sont donnés dans la figure 2.2.

Ceci nous a conduit à étudier les graphes circulaires-parfaits planaires. Un *graphe planaire extérieur* est un graphe admettant un plongement dans le plan où tous ses sommets sont sur la face externe. Rappelons que tout

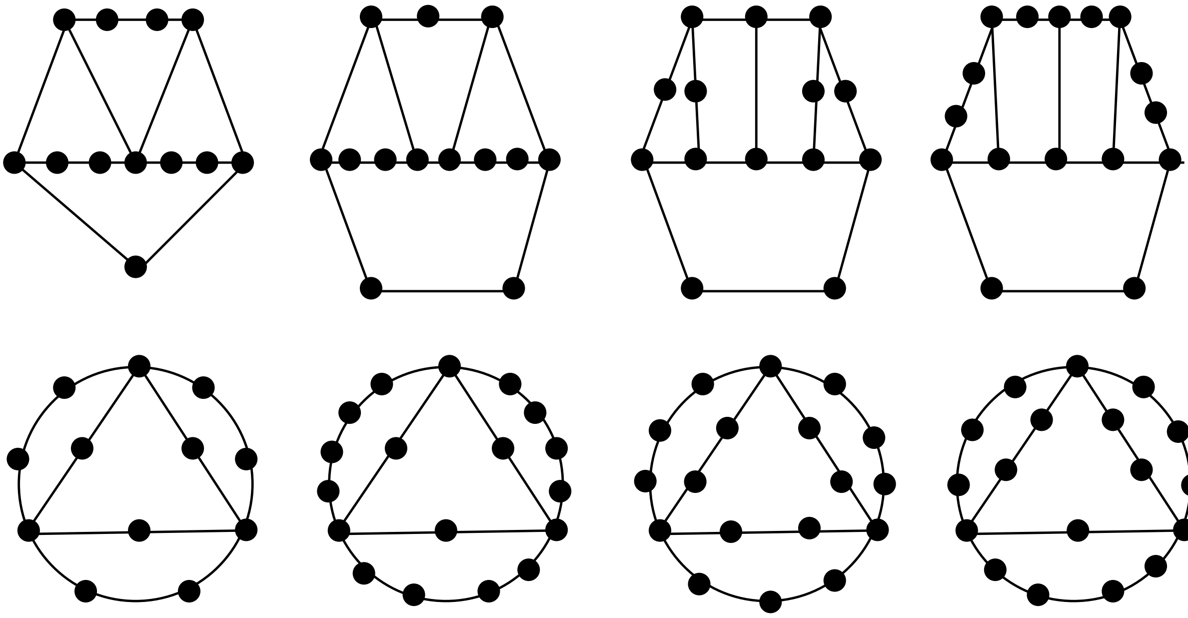


FIG. 2.2 – Quelques exemples de graphes minimaux circulaires-imparfaits planaires

graphe planaire peut être recouvert par deux graphes planaires extérieurs (conjecture de Chartrand, Geller et Hedetniemi [10] posée en 1971, prouvée récemment par Gonçalves [52]).

Les graphes planaires extérieurs forment une classe élémentaire de graphes circulaires-parfaits :

Théorème 2.9 (P., Wagler (2008) [89] B.6.6)**.** *Les graphes planaires extérieurs sont circulaires-parfaits.*

Ainsi, d'après ce théorème, le nombre circulaire chromatique d'un graphe planaire extérieur est égal à son nombre de clique circulaire, soit 2 si tous les cycles sont de taille paire, $2 + \frac{1}{d}$ où $2d + 1$ est la taille du plus petit cycle impair, sinon. Ce qui nous redonne un résultat récent de Kemnitz et Wellmann [59].

A partir de la perfection circulaire des graphes planaires extérieurs, nous pouvons construire une classe simple de graphes minimaux circulaires-imparfaits, appelés *colliers de trous impairs* :

Définition 2.10 (collier de trous impairs)**.** *Pour tous entiers positifs k et l tels que $(k, l) \neq (1, 1)$, soit $T_{k,l}$ le graphe planaire possédant $2l+1$ faces internes $F_1, F_2, \ldots F_{2l+1}$ de taille $2k+1$ arrangées circulairement autour d'un sommet central, où tous les autres sommets sont sur la face externe, comme décrit dans la figure 2.3.*

Il est facile de voir que ces graphes sont circulaires-imparfaits, leur minimalité résultant du théorème 2.9, puisqu'en enlevant n'importe quel sommet, on obtient un graphe planaire extérieur.

D'autres familles de graphes minimaux circulaires-imparfaits planaires ont été obtenues très récemment par Kuhpfahl, Wagler et Wagner [63].

2.2.1.3 Les graphes minimaux circulaires-imparfaits avec un sommet universel sont les roues et les antiroues impaires

Définition 2.11 (jointure complète de deux graphes)**.** *La jointure complète de deux graphes G_1 et G_2 est le graphe formé de la réunion disjointe de ces deux graphes, munie de l'ensemble des arêtes entre G_1 et G_2.*

Notre troisième famille repose sur l'étude de la perfection circulaire de la *jointure complète* de deux graphes :

Théorème 2.12 (P., Wagler (2008) [89] B.6.7)**.** *La jointure complète $G * G'$ de deux graphes G et G' est*

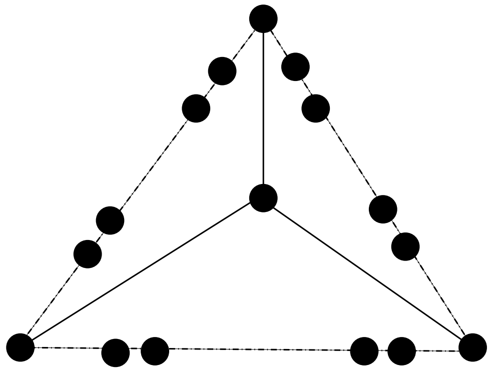


FIG. 2.3 – Exemple d'un collier de trous impairs ($T_{k,1}$)

(i) *circulaire-parfaite si et seulement si G et G' sont tous les deux parfaits ;*

(ii) *minimale circulaire-imparfaite si et seulement si G est un trou impair ou un antitrou impair et G' un singleton (ou vice versa), i.e. si et seulement si elle est une roue impaire ou une antiroue impaire.*

Ainsi notre troisième famille est-elle formée des *roues impaires* et des *antiroues impaires*. Remarquons que les roues impaires sont les colliers de roues impaires $T_{1,l}$, déjà mentionnés, et que les antiroues impaires sont des exemples de graphes minimaux circulaires-imparfaits avec une taille de clique maximum arbitrairement grande.

2.2.2 Le cas sans griffe

Définition 2.13 (graphe sans griffe)**.** *Un graphe est dit sans griffe si tout sommet ne possède pas un stable de taille 3 dans son voisinage, autrement dit s'il ne contient pas le graphe $K_{1,3}$ comme sous-graphe induit.*

De très nombreuses publications sont consacrées aux graphes sans griffe (voir [40], pour un état de l'art). Ils sont l'objet des chapitres suivants de ce document. Ce paragraphe est dédié aux graphes sans griffe circulaires-parfaits.

Chudnovsky et Seymour ont très récemment proposé [17, 16] une décomposition structurelle des graphes sans griffe. Nous avons utilisé celle-ci pour la sous-classe des graphes quasi-adjoints pour démontrer une propriété partagée par tous les graphes minimaux circulaires-imparfaits sans griffe : leur taille de clique maximum ou de stable maximum est au plus 3.

Avant que la conjecture forte des graphes parfaits ne devienne un théorème, celle-ci avait été vérifiée pour de nombreuses familles de graphes (la preuve du théorème repose d'ailleurs fortement sur le lemme "merveilleux" de Roussel et Rubio [96], qui a été découvert en essayant de vérifier la conjecture pour la famille des graphes sans C_4 induit), dont les graphes sans griffe [50, 77, 102]. En étudiant les graphes circulaires-parfaits sans griffe, nous avons obtenu une preuve alternative du théorème fort des graphes parfaits pour les graphes sans griffe.

Le point de départ est une propriété structurelle très forte des graphes sans griffe, établie par Fouquet :

Théorème 2.14 (Lemme de Ben Rebea renforcé - Fouquet (1993) [41])**.** *Si G est un graphe sans griffe avec un stable maximum de taille au moins 3 alors le voisinage de tout sommet contient un C_5 induit ou bien peut être couvert à l'aide de deux cliques seulement.*

Si G est de plus circulaire-parfait alors le voisinage d'un sommet ne peut contenir un C_5 induit, car la roue impaire à 5 sommets est circulaire-imparfaite. Par conséquent, nous avons la propriété fondamentale suivante :

Lemme 2.15. *Si G est un graphe sans griffe circulaire-parfait avec un nombre de stabilité supérieur ou égal à 3 alors G est quasi-adjoint.*

Nous avons tout d'abord montré que les graphes sans griffe circulaires-parfaits contenant un antitrou impair de taille au moins 7 ont une structure très simple :

Théorème 2.16 (P., Zhu (2007) [93] D.4.4)**.** *Si G est un graphe connexe sans griffe circulaire-parfait avec un antitrou impair H de taille au moins 7 alors $G \setminus H$ est une clique. De plus $\alpha(G) = 2$.*

Il est bien connu que l'identification de deux graphes parfaits en une clique K est parfaite. Pour les graphes circulaires-parfaits, nous avons observé que cela est également vrai si la clique K est de taille au plus 2 [91], bien que cela ne soit pas vrai en général. Ceci implique que tout graphe minimal circulaire-imparfait est 2-connexe. Par conséquent, il résulte du théorème précédent :

Corollaire 2.17. *Un graphe sans griffe minimal circulaire-imparfait contenant un antitrou impair induit de taille au moins 7 a une taille de stable maximum au plus 3.*

Il reste à étudier le cas des graphes circulaires-parfaits sans griffe, qui ne contiennent pas d'antitrous impairs de taille au moins 7. D'après le théorème 2.16, c'est notamment le cas des graphes circulaires-parfaits sans griffe G, ayant une taille de stable maximum au moins 3. Nous avons vu que dans ce cas, G doit être quasi-adjoint (lemme 2.15). Nous avons démontré que si G a une taille de clique au moins 4 alors G possède un stable qui intersecte toute clique maximum.

Ceci résulte de l'énoncé suivant :

Théorème 2.18 (P., Zhu (2007) [93] D.4.8)**.** *Si G est un graphe quasi-adjoint avec une taille de clique k au moins égale à 4 et tel que pour tout sommet x, le graphe $G - x$ a une coloration en k couleurs, alors G est le complément d'une clique circulaire (i.e. G est un web), ou G a un stable qui intersecte toutes les cliques maximums de G.*

Une conséquence de ce théorème est qu'un graphe G sans griffe avec $\omega(G) = k \geq 4$ et $\alpha(G) \geq 4$ ne peut pas être minimal circulaire-imparfait. En effet, sinon G ne peut pas être le complémentaire d'une clique circulaire [25] (cf lemme 2.22 dans ce chapitre), est quasi-adjoint puisqu'il ne peut contenir la roue impaire W_5 (qui est elle-même minimale circulaire-imparfaite). Ainsi G possède un stable I intersectant toutes les cliques maximums. Comme $G - I$ est circulaire-parfait, nous avons $\omega_c(G - I) = \chi_c(G - I)$. D'après le corollaire 2.17, G et donc $G - I$ ne peut pas contenir un antitrou impair de taille au moins 7. Ceci signifie en particulier que $\omega_c(G - I) = \omega(G - I) = k - 1$, et donc $\chi(G - I) = \omega(G) = k - 1$. Mais alors $\chi(G) = \omega(G) = k$, i.e. G est circulaire-parfait, une contradiction.

En résumé, nous avons :

Corollaire 2.19. *Un graphe minimal circulaire-imparfait sans griffe a une taille de clique maximum ou de stable maximum au plus égal à 3.*

2.2.3 Prouver partiellement le théorème fort des graphes parfaits

Le théorème fort des graphes parfaits est équivalent au fait que les graphes minimaux imparfaits ont une taille de clique maximum ou de stable maximum égal à 2. Le lemme 2.20 établit une connexion remarquable entre les graphes minimaux circulaires-imparfaits et cette reformulation, sans utiliser naturellement le théorème fort des graphes parfaits :

Lemme 2.20. *Soit $\mathcal{F}$ une famille de graphes telle que tout graphe minimal circulaire-imparfait de celle-ci a une taille de clique maximum ou de stable maximum au plus 3, alors tout graphe minimal imparfait de $\mathcal{F}$ est un trou impair ou un antitrou impair.*

Démonstration. Soit G un graphe minimal imparfait dans $\mathcal{F}$. En particulier, G est partitionnable. Si G est une clique circulaire alors G est un trou impair ou un antitrou impair [1]. Si G ou son complémentaire a une taille de clique maximum au plus 3 alors G doit être un trou impair ou un antitrou impair [105]. Nous pouvons donc supposer que G n'est pas une clique circulaire et que G, ou son complémentaire $\overline{G}$, a une clique de taille au moins 4. Par conséquent, G est circulaire-imparfait d'après le Lemme 2.4. Comme $G \in \mathcal{F}$, ceci implique que G contient un sous-graphe induit propre circulaire-imparfait, et donc un sous-graphe induit propre minimal imparfait, une contradiction. ■

Ainsi l'étude des graphes minimaux circulaires-imparfaits d'une famille de graphes $\mathcal{F}$ donnée peut amener à démontrer la restriction du théorème fort des graphes parfaits à celle-ci. Observons que tous les graphes minimaux circulaires-imparfaits présentés dans ce chapitre satisfont les hypothèses du lemme 2.20. En particulier, nous avons vu que c'est le cas pour la famille des graphes sans griffe. Ceci donne donc une nouvelle preuve que les graphes minimaux-imparfaits sans griffe sont les trous et les antitrous impairs [22][50][77]. Il est alors naturel de demander s'il est possible de dériver le théorème fort des graphes parfaits du lemme 2.20 [91], autrement dit si tout graphe minimal circulaire-imparfait a une taille de clique maximum ou de stable maximum au plus 3. Selon, Pan et Zhu [76], la réponse est non, car il existe des graphes minimaux circulaires-imparfaits avec une taille de clique maximum *et* de stable maximum arbitrairement grands.

2.2.4 Un "théorème fort des graphes circulaires-parfaits" bien improbable

Pour conclure cette partie sur les graphes circulaires-parfaits, les différentes familles de graphes minimaux circulaires-imparfaits que nous avons présentées rendent illusoire l'existence d'une caractérisation séduisante des graphes circulaires-parfaits par sous-graphes induits exclus. D'autant plus que d'autres familles de graphes minimaux circulaires-imparfaits ont été données par Pan et Zhu [76], ainsi que Baogang Xu [111, 112]. En outre, ce dernier [111] a démontré que le graphe adjoint $L(G)$ d'un graphe cubique G est circulaire-parfait si et seulement si G est d'indice chromatique au plus 3. Par conséquent, une telle caractérisation des graphes circulaires-parfaits, même restreinte au cas sans griffe, donnerait une caractérisation des graphes cubiques critiques d'indice chromatique 4, ce qui est un problème NP-complet [57].

2.3 Graphes fortement circulaires-parfaits

Le théorème des graphes parfaits stipule qu'un graphe est parfait si et seulement si son complémentaire l'est aussi : autrement dit, la perfection est stable par complémentarité. Ceci n'est pas vrai pour la perfection circulaire : le complémentaire de la clique circulaire $K_{10/3}$ par exemple n'est pas circulaire-parfait. Comme le théorème des graphes parfaits est crucial pour établir de nombreuses propriétés des graphes parfaits, nous avons décidé d'introduire la sous-classe des graphes circulaires-parfaits stables par complémentarité :

Définition 2.21 (Graphe fortement circulaire-parfait [25])**.** *Graphe circulaire-parfait dont le complémentaire est également circulaire-parfait.*

Pour étudier les graphes fortement circulaires-parfaits, il est naturellement nécessaire de déterminer au préalable quelles cliques circulaires sont fortement circulaires-parfaites. Cette liste est facile à établir et est assez restreinte :

Lemme 2.22 (Coulonges, P., Wagler (2005) [25] C.1.3)**.** *Une clique circulaire est fortement circulaire-parfaite si et seulement si c'est une clique, un stable, un trou impair, un antitrou impair ou une clique circulaire $K_{3q/3}$ avec $q \geq 3$.*

Ainsi le nombre circulaire chromatique d'un graphe fortement circulaire-parfait est toujours de la forme k, $k + 1/2$ ou $2 + 1/k$ pour un entier k.

2.3.1 Le cas sans triangle

On ne connaît pas de caractérisation des graphes circulaires-parfaits sans triangle, ni des graphes minimaux circulaires imparfaits sans triangle d'ailleurs. Bien que la famille des colliers de trous impairs $T_{k,l}$ (paragraphe 2.2.1.2) soit composée de graphes sans triangle (pour $l \geq 2$), il semble très difficile de donner la liste des graphes minimaux circulaires imparfaits sans triangle ...

Le cas des graphes fortement circulaires-parfaits est bien plus simple : en effet, il est possible de montrer que les graphes fortement circulaires-parfaits sans triangle sont formés des graphes bipartis et d'une famille élémentaire, que nous appelons les *trous impairs entrelacés*. De cette caractérisation, nous dérivons ensuite un algorithme de reconnaissance en temps polynomial et nous en déduisons que la détermination d'un stable maximum pondéré est également polynomiale. Enfin, il en résulte que les graphes fortement minimaux circulaires-imparfaits sans triangle sont des trous impairs avec au plus 2 sommets supplémentaires !

Avant de donner la définition précise des trous impairs entrelacés, observons tout d'abord que tout repose sur une propriété basique, facile à démontrer :

Lemme 2.23 (Coulonges, P., Wagler (2005) [25]). *Si G est un graphe fortement circulaire-parfait sans triangle contenant un plus petit trou impair $\mathcal{O}$ alors toute arête a au moins une de ses extrémités dans $\mathcal{O}$.*

Démonstration. Supposons qu'il existe une arête xy qui n'a aucune de ses extrémités dans $\mathcal{O}$, et soit $2p+1$ la taille de $\mathcal{O}$. Alors le sous-graphe H induit par $\mathcal{O}$ et les sommets x et y est un graphe fortement circulaire-parfait, avec un nombre de stabilité au plus $p+1$.

Par le lemme 2.22, ceci signifie que le nombre de clique circulaire de $\overline{H}$ est au plus $p+1$. Comme $\overline{H}$ est circulaire-parfait, nous avons donc $\chi_c(\overline{H}) \leq p+1$. Puisque $\chi(\overline{H})$ est la partie entière par excès de $\chi_c(\overline{H})$, le graphe $\overline{H}$ a une coloration en $(p+1)$-couleurs. Ainsi H a une couverture avec au plus $p+1$ cliques $Q_1, \ldots Q_{p+1}$. Soit Q_x (resp. Q_y) la clique contenant x (resp. y). Alors au moins une des deux cliques Q_x et Q_y intersecte $\mathcal{O}$ en deux sommets consécutifs, et a donc au moins 3 sommets, en contradiction avec l'absence de triangles dans G. ■

Autrement dit, dans un graphe fortement circulaire-parfait sans triangle non biparti, les arêtes sont toutes incidentes à un trou impair donné (Fig. 2.4). Ceci nous a conduit à une caractérisation précise :

Définition 2.24 (trou impair entrelacé [25]). *Un graphe G est appelé trou impair entrelacé si et seulement si son ensemble des sommets admet une partition $((A_i)_{1\leq i\leq 2p+1}, (B_i)_{1\leq i\leq 2p+1})$ en $2p+1$ (où $p \geq 2$) ensembles non-vides $A_1, \ldots, A_{2p+1}$ et $2p+1$ ensembles $B_1, \ldots, B_{2p+1}$ (éventuellement vides) tels que*

1. *$\forall 1 \leq i \leq 2p+1$, $|A_i| > 1$ implique $|A_{i-1}| = |A_{i+1}| = 1$, (les indices sont modulo $2p+1$),*
2. *$\forall 1 \leq i \leq 2p+1$, $B_i \neq \emptyset$ implique $|A_i| = 1$,*

et l'ensemble des arêtes de G est égal à $\cup_{i=1,\ldots,2p+1}(E_i \cup E'_i)$, où E_i (resp. E'_i) est formé de l'ensemble des arêtes entre A_i et A_{i+1} (resp. entre A_i et B_i) ;

Voir la figure 2.4 pour un exemple (les ensembles de sommets B_i sont en gris).

Théorème 2.25 (Coulonges, P., Wagler (2005) [25] C.1.15). *Un graphe G sans triangle est fortement circulaire-parfait si et seulement si G est biparti (i.e. parfait) ou un trou impair entrelacé.*

Ainsi reconnaître un graphe sans triangle fortement circulaire-parfait revient à reconnaître un graphe biparti ou un trou impair entrelacé. Ceci peut être fait en temps polynomial, via l'algorithme 2.1.

Justifions brièvement que cet algorithme est correct et polynomial :

1-3 La reconnaissance d'un graphe biparti en temps polynomial est un exercice standard.

4 Le graphe n'est pas biparti. S'il est sans triangle et sans trou impair, alors il est parfait et donc biparti, une contradiction. Par conséquent il possède soit un triangle soit un plus petit trou impair. Dans les deux cas, il existe un plus petit cycle impair $\mathcal{O}$ que l'on peut exhiber en temps polynomial [70].

5-7 Si ce plus petit cycle impair est de taille 3 alors le graphe n'est pas sans triangle.

8-11 Le graphe est sans triangle. Pour tout sommet o_i du plus petit trou impair $\mathcal{O}$, nous définissons l'ensemble B_i comme l'ensemble des voisins de o_i de degré 1, et A_i comme la réunion de o_i et des sommets de degré 2 qui son des voisins de o_{i-1} et o_{i+1}.

12- Si le graphe est un trou impair entrelacé, alors les ensembles A_i et B_i doivent être conformes à la définition. Ceci est testé dans cette dernière partie de l'algortihme.

Les graphes minimaux imparfaits sans triangle sont exactement les trous impairs, puisque les graphes bipartis sont les graphes sans cycles impairs. Nous avons obtenu un résultat similaire pour la perfection circulaire.

Définition 2.26 (trou impair étendu [25]). *Un trou impair étendu est la réunion d'un trou impair $\mathcal{O} = \{o_1, \ldots, o_{2p+1}\}$ et de deux sommets x et y de telle sorte que les sommets x et y aient pour ensemble de voisins l'un des 6 cas suivants :*

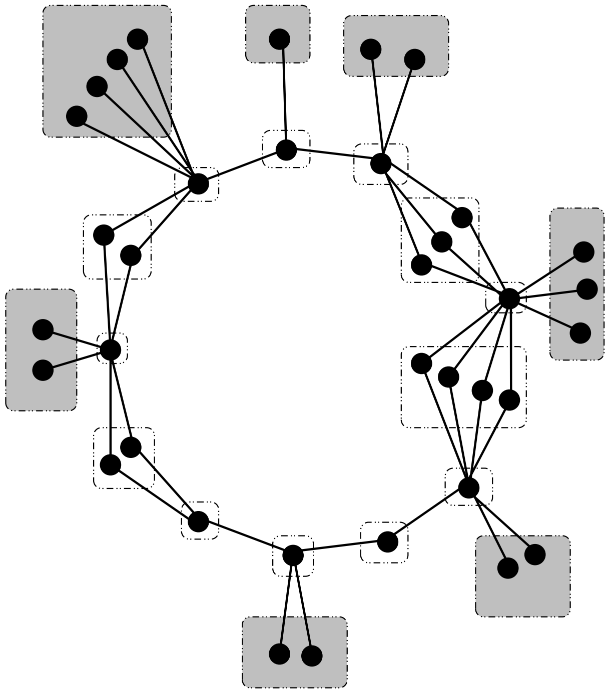


FIG. 2.4 – Exemple de trou impair entrelacé.

Input: un graphe G

Output: un booléen vrai si et seulement si G est fortement circulaire-parfait sans triangle.

Si G est biparti **Alors**

  retourner VRAI

déterminer un plus petit trou impair $\mathcal{O} = (o_1, \ldots, o_{2p+1})$. (Trouver un plus petit trou impair est facile pour un graphe sans triangle)

Dans la suite, les indices sont modulo $2p + 1$.

Si p=1 **Alors**

  retourner FAUX

Pour $i \in 1 \ldots 2p + 1$ **:**

  $B_i := \{v|\text{deg}(v) = 1, |vo_i \in E(G)\}$

  $A_i := \{v|vo_{i-1} \in E(G),\, vo_{i+1} \in E(G)\} \cup \{o_i\}$

Pour $i \in 1 \ldots 2p + 1$ **:**

  Si ($|A_i| > 1$ et ($|A_{i+1}| > 1$ ou $|A_{i-1}| > 1$)) ou ($B_i \neq \emptyset$ et $|A_i| > 1$) **Alors**

    retourner FAUX

$V := \emptyset$; $E := \emptyset$

Pour $i \in 1 \ldots 2p + 1$ **:**

  $V := V \cup A_i \cup B_i$

  $E_i := A_i \times A_{i+1}$; $E'_i := A_i \times B_i$; $E := E \cup E_i \cup E'_i$

Si $V \neq V(G)$ ou $E \neq E(G)$ **Alors**

  retourner FAUX

retourner VRAI

Algorithme 2.1: Un algorithme polynomial de reconnaissance des graphes sans triangle fortement circulaires-parfaits [25]

(a) $\{o_1x, xy, o_4y\}$

(b) $\{o_1x, xy, o_2y\}$

(c) $\{o_1x, o_3x, xy, o_4y\}$
(d) $\{o_1x, o_3x, xy, o_2y\}$
(e) $\{o_1x, o_3x, xy, o_2y, o_4y\}$
(f) $\{o_1x, o_3x, o_2y, o_4y\}$

Théorème 2.27 (Coulonges, P., Wagler (2005) [25] C.1.22)**.** *Un graphe sans triangle G est minimal fortement circulaire-imparfait si et seulement si G est la réunion disjointe d'un trou impair et d'un singleton, ou bien un trou impair étendu.*

2.4 Du polytope des stables des graphes (fortement) circulaires-parfaits

Dans le chapitre 3, nous étudions les graphes a-parfaits, i.e. les graphes dont les polytopes des stables sont donnés par les contraintes de non-négativité et les contraintes de rang associées aux cliques circulaires premières. Par analogie à la caractérisation polyédrale des graphes parfaits, on pourrait s'attendre à ce que les graphes circulaires-parfaits soient exactement les graphes a-parfaits, puisque les cliques circulaires sont sensées jouer le même rôle pour la coloration circulaire, que les cliques pour la coloration usuelle. Pour le moment, peu de choses sont connues à propos du polytope des stables des graphes circulaires-parfaits, bien que nous ayons remarqué que les graphes planaires extérieurs et les graphes convexes circulaires sont a-parfaits [91].

Pour clore ce chapitre, nous étudions le polytope des stables des graphes circulaires-parfaits ou fortement circulaires-parfaits et établissons que ceux-ci ne sont pas toujours rang-parfaits, i.e. leurs polytopes des stables peuvent avoir des facettes avec des coefficients différents de 0 ou 1.

Commençons par observer que les graphes fortement circulaires-parfaits ne sont pas tous rang-parfaits : étant donnés deux paramètres $1 \leq \delta \leq n$, soit $G_{n,\delta}$ le graphe à $n+1$ sommets dont les sommets sont étiquetés $\{0, \dots, n\}$, tel que le sous-graphe induit $G - \{0\}$ soit le trou à n sommets, et le sommet 0 soit adjacent aux sommets $1, \dots, \delta$ exactement (cf figure 2.5).

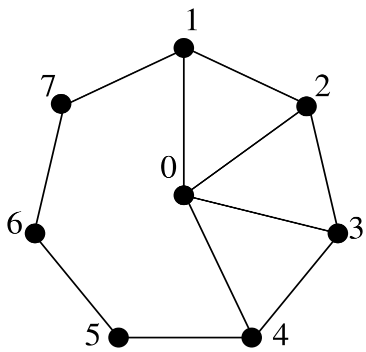


FIG. 2.5 – Le graphe $G_{7,4}$

Théorème 2.28 (Coulonges, P., Wagler (2005) [24] D.1.4)**.** *Soit $p \geq 2$ et $1 \leq \delta < 2p+1$. Alors le graphe $G_{2p+1,\delta}$ est fortement circulaire-parfait. De plus, si $\delta \geq 5$ alors $G_{2p+1,\delta}$ n'est pas rang-parfait car*

$$\left\lfloor \frac{\delta - 1}{2} \right\rfloor x_0 + \sum_{i=1}^{2p+1} x_i \quad \leq \quad p$$

est une facette de STAB $(G_{2p+1,\delta})$.

La famille des graphes $G_{2p+1,\delta}$ est très restrictive : en élargissant notre étude aux graphes circulaires-parfaits, nous avons exhibé une famille plus conséquente de graphes non rang-parfaits :

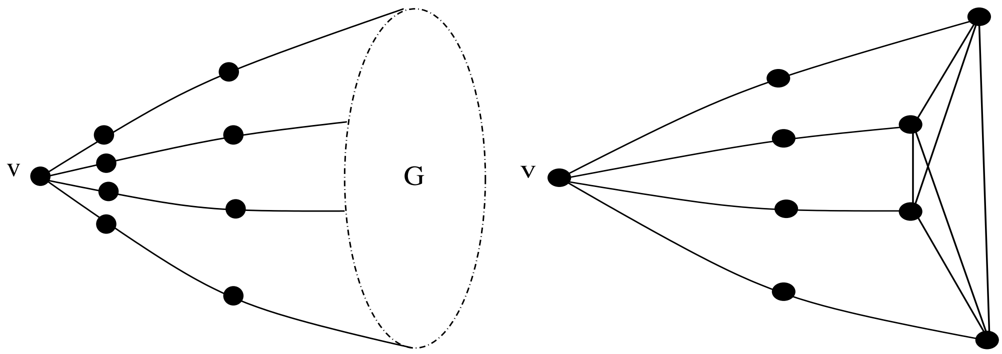
Théorème 2.29 (Coulonges, P., Wagler (2005) [24] D.1.1)**.** *Soit G un graphe, $k \geq 0$ un entier et $H(G, k)$ le graphe obtenu en ajoutant un sommet universel v à G et en subdivisant k fois chaque arête incidente à v (voir la figure 2.6 (a)). Nous avons :*

- *si $k = 1$ alors pour toute clique Q de G, la contrainte*

$$(|Q| - 1)x_v + \sum_{u \in Q} x_u + \sum_{u \in N(v) \cap N(Q)} x_u \quad \leq \quad |Q|$$

  induit une facette de STAB$(H(G, 1))$;
- *is G est une clique alors $H(G, k)$ est circulaire-parfait (figure 2.6 (b)).*

En particulier, $H(K_i, 1)$ est un graphe circulaire-parfait non rang-parfait pour tout $i \geq 3$.



(a) : graphe $H(G, 2)$ (b) : graphe $H(K_4, 1)$

FIG. 2.6 – Construction de graphes circulaires-parfaits non rang-parfaits

En résumé, l'existence d'une caractérisation élégante des graphes circulaires-parfaits (ou fortement circulaires-parfaits) via leurs polytopes des stables est assez improbable : en ce sens, les graphes circulaires-parfaits sont très différents des graphes parfaits.

Chapitre 3

De quelques graphes circulants aux graphes quasi-adjoints

Sommaire

Dans le second chapitre, nous avons mis en évidence que les cliques circulaires sont essentielles pour les colorations circulaires. Ceci est parfaitement naturel compte tenu de la définition même des colorations circulaires. De manière un peu plus surprenante, il se trouve qu'elles jouent un rôle de tout premier plan pour une famille de graphes majeure : les graphes sans griffe. Dans ce contexte, leurs complémentaires sont connus sous le nom de *webs*. Rappelons leur définition :

Définition 3.1 (web)**.** *Étant donné deux entiers k et n, le web à n sommets et taille de clique k, noté W_n^k, est le complémentaire $\overline{K_{n/k}}$ de la clique circulaire $K_{n/k}$.*

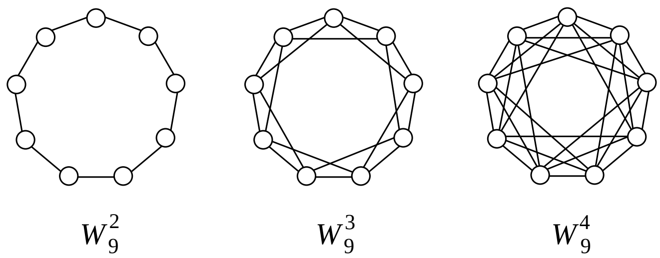


FIG. 3.1 – Des webs à 9 sommets

Autrement dit, l'ensemble des sommets de W_n^k sont les éléments de $\mathbb{Z}_n$ et deux sommets i et j sont adjacents si et seulement si i et j sont distants d'au plus $k-1$ dans $\mathbb{Z}_n$. Ainsi, les webs sont-ils des graphes circulants, i.e. des graphes de Cayley sur des groupes cycliques. En ce sens, ils forment une famille de graphes circulants élémentaires.

La caractérisation des facettes de rang du polytope des stables des graphes sans griffe est un résultat remarquable de Gallucio et Sassano, qui illustre le rôle des webs dans le cadre des graphes sans griffe :

Théorème 3.2 (Galluccio & Sassano (1997) [44]). *Soit G un graphe sans griffe produisant une facette de rang minimal (tout sous-graphe induit propre ne produit pas une facette de rang). Alors une des trois propriétés suivantes est vraie :*
- *G est une clique ;*
- *G est le graphe adjoint d'un graphe 2-connexe sous-couplable ;*
- *G est le web $W^{\omega(G)}_{\omega(G)\alpha(G)+1}$.*

Ainsi les webs apparaissent-ils comme incontournables à la fois pour comprendre le polytope des stables des graphes sans griffe, mais aussi pour la sous-classe des graphes quasi-adjoints. La première partie de ce chapitre est consacrée au polytope des stables des webs. Les inégalités de famille de cliques sont l'objet du second chapitre : ces inégalités contiennent toutes les facettes du polytope des stables des graphes quasi-adjoints. Enfin, la troisième partie est dédiée à l'étude du ratio d'imperfection des graphes a-parfaits (i.e. les graphes dont les facettes non-triviales de leurs polytopes des stables sont les contraintes de rangs associées aux cliques circulaires). De cette étude résulte que le ratio d'imperfection des graphes quasi-adjoints est borné par 3/2.

3.1 Polytope des stables des webs

3.1.1 Les premiers travaux

Trotter a donné une condition nécessaire et suffisante pour qu'un web produise une facette de rang :

Théorème 3.3 (Trotter (1975) [58]). *Soit $G = W^\omega_n$ un web. Alors*

$$\sum_{v \in V(G)} x_v \leq \alpha(G)$$

induit une facette de STAB(G) si et seulement si ω est premier avec n.

En étudiant le problème de la recherche d'un arbre couvrant de profondeur au plus 2 de poids minimum (dont les applications sont essentiellement en télécommunications), Geir Dahl a établi une équivalence entre les solutions de ce problème et celles du problème du stable pondéré de poids minimum pour les webs avec un nombre de clique égal à 3 [30]. Ceci l'a naturellement amené à étudier le polytope des stables de ces webs et à fournir une description complète des facettes de celui-ci :

Théorème 3.4 (Dahl (1998) [31]). *Pour tout entier $n \geq 3$, les facettes du polytope des stables du web W^3_n sont :*
- *les contraintes triviales de positivité : $x_v \geq 0$, $\forall v \in V$;*
- *les inégalités des cliques maximales : $\sum_{v\in Q} x_v \leq 1$ pour toute clique maximale Q ;*
- *l'inégalité de rang pleine : $\sum_{v\in V} x_v \leq \alpha(W^3_n)$ si $3 \nmid n$;*
- *les inégalités de rang$\sum_{v\in T} x_v \leq \alpha(G[T])$ pour tout ensemble de sommets T tel que T soit l'union disjointe de t intervalles $I_i = [l_i, r_i]$ (de $[0, n-1]$) tels que*
  - *t soit impair ;*
  - *pour tout $i = 1 \dots t$, $|I_i| = 1 \pmod 3$;*
  - *pour tout $i = 1 \dots t-1$, $l_{i+1} = r_i + 2$.*

En particulier, ces facettes sont toutes des facettes de rang. Pour le polytope des stables des webs avec un nombre de clique supérieur, Kind a exhibé des exemples de facettes qui ne sont pas de rang, à l'aide du logiciel Porta [60, 12]. Nous établissons ci-après que ces exemples sont loin d'être isolés pour les webs de taille de clique au moins 4.

3.1.2 La plupart des webs ne sont pas rang-parfaits

Dans ce paragraphe, nous introduisons un procédé constructif qui permet, à partir d'une facette du polytope des stables d'un web W^ω_n, d'exhiber une facette "similaire" du polytope des stables du web $W^\omega_{n+\omega}$. Ainsi, en considérant un web W_n^ω qui n'est pas rang-parfait, nous pouvons établir que la famille des webs $W_{n+\omega}^\omega$, $W_{n+2\omega}^k$, $W_{n+3\omega}^k, \ldots$ est une famille de webs qui ne sont pas rang-parfaits et qui possèdent le même nombre de clique.

Pour énoncer formellement le résultat qui nous permet de faire cette construction, nous avons besoin d'introduire la notion de *facette presque de rang* :

Définition 3.5 (facette presque de rang - Wagler [108]). *Soit G un graphe, une facette $\mathbf{a}^T\mathbf{x} \leq c\alpha(G')$ de STAB(G) est une facette presque de rang associée à un sous-graphe induit G' de G si $a_i = c$ pour tout sommet i de G' et si G' produit une facette de rang (i.e. $\sum_{i\in V(G')} x_i \leq \alpha(G')$ est une facette de STAB(G')).*

Théorème 3.6 (P. et Wagler (2006) [84] B.2.11). *Soit $\mathbf{a}^T\mathbf{x} \leq c\alpha'$ une facette presque de rang associée à un sous-graphe G' d'un web W_n^ω. Alors, si G' n'induit pas une clique de W_n^ω, STAB$(W_{n+\omega}^\omega)$ a pour facette l'inégalité*

$$\sum_{1\leq i\leq n} a_i x_i + \sum_{n<i\leq n+\omega} c x_i \quad \leq \quad c(\alpha'+1) \tag{3.1}$$

Ce procédé d' “injection d'une clique” pour obtenir une facette “similaire” a été inspiré par la preuve de Dahl pour le cas $\omega = 3$ [31].

Exemple 3.7. *Prenons pour exemple le plus petit web (en nombre de sommets) avec une facette qui ne soit pas de rang, i.e. W_{25}^6 : son polytope des stables admet pour facette l'inégalité suivante :*

$$(2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2)^T\mathbf{x} \leq 6$$

Soit V_1 l'ensemble des sommets correspondant aux coefficients de valeur 2 (i.e., les sommets noirs de la figure 3.2(a)). Remarquons que le graphe $G[V_1]$ est isomorphe au web partitionnable W_{10}^3, qui produit donc une facette de rang. Ainsi nous pouvons appliquer le théorème 3.6, avec $c = 2$, $G' = G[V_1]$ et $\alpha(G') = 3$: l'inégalité

$$(2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,2,2,2,2,2)^T\mathbf{x} \leq 8$$

est une facette presque de rang de STAB(W_{31}^6) (les sommets avec coefficient 2 sont les sommets noirs de la figure 3.2(b)). En itérant indéfiniment ce raisonnement, nous obtenons ainsi que les webs W_{25}^6, W_{31}^6, W_{37}^6, ... ne sont pas rang-parfaits.

Une conséquence utile du théorème 3.6 est le corollaire suivant :

Corollaire 3.8 (P. et Wagler (2006) [84]). *S'il existe ω webs $W_{n_0}^\omega, \ldots, W_{n_{\omega-1}}^\omega$ tels que*
- *STAB$(W_{n_i}^\omega)$ a une facette presque de rang qui n'est pas associée à une clique et qui n'est pas de rang ;*
- $n_i = i \pmod{\omega}$

alors tous les webs W_n^ω tels que $n \geq \max\{n_0, \ldots, n_{\omega-1}\} - (\omega - 1)$ ne sont pas rang-parfaits.

Cela signifie en particulier que si nous sommes capables, pour ω fixé, d'exhiber un ensemble ω de webs satisfaisant les hypothèses du corollaire 3.8, alors il n'existe qu'un nombre fini de webs rang-parfaits avec un nombre de clique égal à ω. C'est ce que nous avons réussi à faire dans un premier article pour $\omega = 4$ [86], puis dans un second article pour $\omega \geq 5$ [83] :

Théorème 3.9 (P. et Wagler (2006) [86, 83] B.3.11). *Soit ω un entier. Si $\omega \leq 3$ alors tout web avec un nombre de clique égal à ω est rang-parfait. Si $\omega \geq 4$, il n'existe qu'un nombre fini de webs rang-parfaits avec un nombre de clique égal à ω et donc presque tous ne sont pas rang-parfaits.*

Dans le premier chapitre, nous avons énoncé le théorème 1.45 d'Eisenbrand, Oriolo, Stauffer et Ventura affirmant la validité de la conjecture de Ben Rebea : les facettes du polytope des stables des graphes quasi-adjoints non-triviales sont des inégalités de familles de cliques. Cependant, ce résultat ne précise pas quelles inégalités de familles de cliques sont essentielles, i.e. quelles sont celles induisant des facettes.

Pour la sous-classe des webs, nous avons mis en évidence que les sous-webs induits jouent un rôle esssentiel :

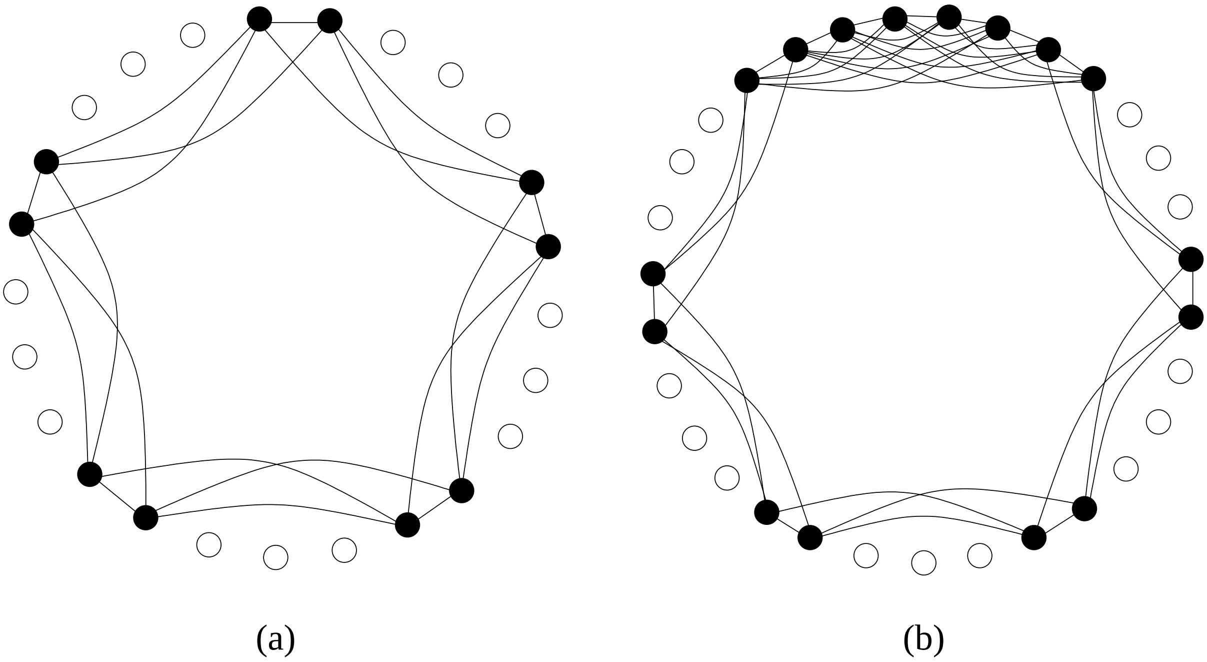


FIG. 3.2 – Illustration du théorème 3.6

Définition 3.10 (inégalité d'une famille de cliques associée à un sous-web)**.** *Étant donné un web W_n^ω et un sous-web induit $W_{n'}^{\omega'}$, une inégalité de famille de cliques $(\mathcal{Q}, p)$ est dite associée au sous-web $W_{n'}^{\omega'}$ si $p = \omega'$ et $\mathcal{Q} = \{Q_i : i \in W_{n'}^{\omega'}\}$ où pout tout sommet i de $W_{n'}^{\omega'}$, la clique Q_i est la clique maximum $\{i, \ldots, i+\omega-1\}$ de W_n^ω.*

Nous avons alors conjecturé et Stauffer a établi dans sa thèse que les inégalités de familles de cliques associées à certains sous-webs sont suffisantes pour le polytope des stables des webs :

Théorème 3.11 (Stauffer (2005) [104] - conjecture de P. et Wagler (2006) [83])**.** *Toute facette du polytope des stables d'un web W_n^ω est soit*
- *une contrainte de non-négativité ;*
- *une contrainte de clique maximale ;*
- *la contrainte de rang pleine ;*
- *une inégalité de familles de cliques associée à un sous-web $W_{n'}^{\omega'}$ tel que $\omega' \not| \, n'$ et $\alpha(W_{n'}^{\omega'}) < \alpha(W_n^\omega)$.*

3.2 Inégalités de familles de cliques

Nous avons mentionné dans le chapitre d'introduction que les inégalités des familles de cliques sont des inégalités valides pour le polytope des stables de n'importe quel graphe (Oriolo [73], Ben Rebea [95]), bien qu'elles aient été conçues pour l'étude du polytope des stables des graphes quasi-adjoints. Elles sont toutes de la forme $a\sum_{v\in W} x_v + (a-1)\sum_{v\in W'} x_v \leq a\delta$ (avec a entier positif non-nul).

Dans cette section, nous allons voir tout d'abord que leur rang de Chvátal est majoré par a (i.e. le plus grand coefficient gauche), ce qui donne une autre preuve de leur validité, en ne faisant appel qu'à des techniques d'arrondi classiques. Ensuite nous améliorons cette borne et en tirons quelques conséquences pour les graphes quasi-adjoints.

3.2.1 Majoration de leur rang de Chvátal

Commençons par rappeler la définition d'une inégalité de familles de cliques, pour pouvoir ensuite établir une propriété assez fondamentale : soit donc $\mathcal{F}$ une famille de cliques (non nécessairement distinctes) d'un graphe G, $p \leq |\mathcal{F}|$ un entier, et les deux ensembles

$$\begin{aligned} V_{\geq p} &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\} \\ V_{p-1} &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p-1\} \end{aligned}$$

L'inégalité de la famille de cliques $I(\mathcal{F}, p)$ est définie par

$$(p-r) \sum_{v \in V_{\geq p}})x_v + (p-r-1) \sum_{v \in V_{p-1}} x_v \quad \leq \quad (p-r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \tag{3.2}$$

où $r = |\mathcal{F}| \pmod p$.

L'observation suivante est cruciale pour le calcul de notre majorant de Chvátal de ces inégalités : en sommant les inégalités de cliques correspondant aux cliques de $\mathcal{F}$, avec éventuellement des contraintes de non-négativité $-x_v \leq 0$ pour les sommets $v \in V_{\geq p}$ qui sont contenus dans plus de p cliques de $\mathcal{F}$, l'inégalité

$$p \sum_{v \in V_p} x_v + (p-1) \sum_{v \in V_{p-1}} x_v \quad \leq \quad p \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor + r \tag{3.3}$$

est valide pour QSTAB(G).

La preuve pour montrer que le rang de Chvátal de l'inégalite 3.2 est majoré par $p-r$ se fait par une simple récurrence :

Théorème 3.12 (P. , Wagler (2007) [88] B.5.1)**.** *Soit $I(\mathcal{F}, p)$ une inégalité de familles de cliques et soit $r = |\mathcal{F}| \pmod p$. Pour tout indice $1 \leq i \leq p-r$, l'inégalité $h(i)$*

$$i \sum_{v \in V_{\geq p}} x_v + (i-1) \sum_{v \in V_{p-1}} x_v \leq i \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor$$

a un rang de Chvátal au plus i. En particulier, $I(\mathcal{F}, p)$ a un rang de Chvátal au plus $p-r$.

Démonstration. Pour tout indice $1 \leq i \leq p-r$, soit $H(i)$ l'assertion : "L'inégalité $h(i)$ a un rang de Chvátal au plus i." La preuve est par récurrence sur i :

$H(1)$ est vraie car l'inégalité (3.3) implique que $\sum_{v \in V_p} x_v \leq \frac{|\mathcal{F}|}{p}$ est valide pour QSTAB(G), et donc $\sum_{v \in V_p} x_v \leq \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor$ a un rang de Chvátal 1.

Supposons maintenant que $H(i)$ est vraie pour tout indice $i < p-r$. Pour montrer la validité de $H(i+1)$, nous allons obtenir l'inégalité $h(i+1)$ en appliquant une seule fois une coupe de Chvátal-Gomory à $h(i)$ et l'inégalité (3.3). Ainsi nous devons exhiber, et c'est là toute la difficulté de la preuve, une paire de solutions rationnelles (λ, μ) au système :

$$\begin{aligned} \lambda i + \mu p &= i+1 \\ \lambda(i-1) + \mu(p-1) &= i \\ \left\lfloor \lambda i \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor + r \right) \right\rfloor &= (i+1) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \end{aligned}$$

En choisissant $\lambda = \frac{p-i-1}{p-i}$ et $\mu = \frac{1}{p-i}$ nous obtenons une telle paire de solutions, car $\left\lfloor \lambda i \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor + r \right) \right\rfloor = \left\lfloor (\lambda i + \mu p) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor + \frac{r}{p-i} \right\rfloor = (i+1) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor$, puisque $0 \leq r/(p-i) < 1$. ■

Ainsi, les inégalité des familles de clique *de rang* ont un rang de Chvátal égal à un. Ceci n'est pas vrai en général pour les inégalités de rang [20] et pour les inégalités de familles de cliques [73].

Naturellement, la borne du théorème 3.12 devient moins bonne lorsque r diminue. En se basant sur une technique de preuve similaire, nous l'avons améliorée lorsque $r < p/2$:

Théorème 3.13 (P. , Wagler (2007) [88] B.5.2)**.** *Chaque inégalité de famille de clique $I(\mathcal{F}, p)$ avec $r = |\mathcal{F}| \pmod p$ a un rang de Chvátal au plus $\min\{r, p - r\}$ où $r = |\mathcal{F}| \pmod p$. En particulier, le rang de Chvátal est toujour inférieur ou égal à $\lfloor \frac{p}{2} \rfloor$.*

3.2.2 Application aux graphes quasi-adjoints

Maintenant que nous avons établi un majorant du rang de Chvátal des inégalités de familles de cliques, utilisons celui-ci pour évaluer le rang de Chvátal de certains graphes quasi-adjoints. Ainsi puisque Chudnovsky et Seymour ont montré que les graphes semi-adjoints (les graphes quasi-adjoints qui ne sont pas des graphes circulaires d'intervalles flous) sont rang-parfaits [16], le rang de Chvátal de ces derniers est au plus un :

Corollaire 3.14. *Les graphes semi-adjoints vérifient la conjecture 1.46 d'Edmonds.*

Par conséquent, les seuls graphes quasi-adjoints susceptibles d'avoir une structure polyédrale complexe sont les graphes circulaires d'intervalles flous. Bien que les webs soient des graphes quasi-adjoints des plus “élémentaires”, ceci signifie que ceux-ci font partie des graphes quasi-adjoints dont le polytope des stables est le plus difficile à caractériser !

Passons maintenant en revue quelques-unes des facettes les plus remarquables du polytope des stables des graphes quasi-adjoints : Oriolo [73] a démontré qu'une facette découverte par Giles et Trotter [49] avait un rang de Chvátal *au moins* égal à 2. Comme cette facette est une inégalité de famille de cliques $(\mathcal{F}, 8)$ où $\mathcal{F}$ possède 37 éléments, notre théorème 3.12 stipule que celle-ci a un rang de Chvátal au plus 3, puisque $r = 5$ et donc $p - r = 3$.

Giles et Trotter [49] ont également introduit une famille de graphes $(G_k)_{k \geq 1}$ circulaires d'intervalles flous dont les polytopes des stables possèdent des facettes avec des coefficients gauches égaux à k et $k + 1$: pour tout $k \geq 1$, le graphe G_k a une inégalité de famille de cliques $(\mathcal{F}, k+2)$ induisant une facette, où $|\mathcal{F}| = 2k(k+2)+1$. En leur appliquant le théorème 3.13, nous avons donc que leur rang de Chvátal est au plus 1, i.e. ce ne sont pas des contre-exemples à la conjecture 1.46.

Liebling, Oriolo, Spille et Stauffer [64] ont également démontré que le polytope des stables des webs pouvait également avoir de grands coefficients gauches

Théorème 3.15 (Liebling, Oriolo, Spille et Stauffer (2004) [64])**.** *Pour tout entier $a \geq 1$, le polytope des stables du web $W_{(2a+3)^2}^{2(a+2)}$ a une facette dont les coefficients gauches sont a et $a + 1$. De plus cette facette est une inégalité de familles de cliques $(\mathcal{F}, a + 2)$ où la famille $\mathcal{F}$ possède $(a + 2)(2a + 3)$ éléments.*

Comme $(a + 2)(2a + 3) = 1 \pmod{a + 2}$, pour tout a, le théorème 3.13 montre encore que ces facettes ont un rang de Chvátal au plus 1.

En appliquant le théorème 3.11 à notre borne du rang de Chvátal énoncée dans le théorème 3.13, nous obtenons donc que le rang de Chvátal d'un web W_n^ω est au plus $\omega/2$. Ainsi, même pour une famille aussi restreinte que celle des webs, notre théorème ne stipule pas que le rang de Chvátal soit borné. De fait, il n'y a pas de telle borne pour les graphes sans griffe :

Théorème 3.16 (Chvátal, Cool et Hartmann (1989) [20])**.** *Pour tout M, il existe un graphe G tel que $\alpha(G) = 2$, de rang de Chvátal au moins $1/3 \ln n$ et $n \geq M$ où n désigne le nombre de sommets de G.*

3.3 Ratio d'imperfection des graphes quasi-adjoints

Puisque les cliques circulaires sont censées jouer, selon Xuding Zhu [115], le même rôle pour les graphes circulaires-parfaits que les cliques pour les graphes parfaits, nous nous sommes intéressés à leurs propriétés “polyédrales” :

Définition 3.17 (graphe a-parfait - Coulonges, P., Wagler (2005) [26]). *Un graphe est a-parfait si et seulement si les facettes de son polytope des stables sont les contraintes de non-négativité et les contraintes de rang associées aux cliques circulaires.*

Le a de a-parfait provient du terme *antiweb*, qui est également utilisé pour désigner les cliques circulaires. Aussitôt quelques questions émergent naturellement :

Question 3.18. *Avons-nous une caractérisation polyédrale des graphes circulaires-parfaits analogue à celle des graphes parfaits, i.e. est-ce qu'un graphe est circulaire-parfait si et seulement s'il est a-parfait ?*

Question 3.19. *Est-ce que les graphes a-parfaits sont proches des graphes parfaits ? Qu'en est-il de leur ratio d'imperfection ?*

La réponse à la première question est négative, tel en témoigne le théorème 2.29 du chapitre 2 qui exhibe des graphes circulaires-parfaits qui ne sont pas rang-parfaits, et a fortiori, qui ne sont pas a-parfaits.

L'étude de la seconde question s'est révélée beaucoup plus fructueuse : non seulement il est possible de borner le ratio d'imperfection des graphes a-parfaits par 3/2, mais de plus, il est possible d'appliquer cette borne au ratio d'imperfection des graphes quasi-adjoints ! C'est l'objet de cette section.

Dans leur article fondateur sur le ratio d'imperfection, Gerke et McDiarmid ont étudié les graphes adjoints et h-parfaits :

Théorème 3.20 (Gerke et McDiarmid (2001) [47]). *Soit G un graphe adjoint ou h-parfait. Alors le ratio d'imperfection de G est 1 si G est parfait, égal à $g/(g-1)$ si g est la taille d'un plus petit trou impair de G sinon. En particulier le ratio d'imperfection de G est majoré par 5/4.*

Nous avons étendu ces résultats de deux manières. Tout d'abord, nous avons obtenu la même borne pour les graphes semi-adjoints, qui contiennent tous les graphes adjoints.

Dans un second temps, nous avons montré que le ratio d'imperfection des graphes a-parfaits est déterminé par les cliques circulaires $K_{p/q}$ induites : ainsi le fait que le ratio d'imperfection des graphes h-parfaits ne dépende que des trous impairs induits peut être vu comme un cas particulier des graphes a-parfaits, puisque les trous impairs sont des cliques circulaires :

Théorème 3.21 (Coulonges, P. et Wagler (2005) [26, 27] C.2.2). *Pour tout graphe a-parfait G, le ratio d'imperfection ratio est donné par*

$$\operatorname{imp}(G) \quad = \quad \max\left\{\frac{p}{\alpha\omega} : K_{p/\alpha} \subseteq G \ \gcd(p,\alpha) = 1\right\} \ \textit{où } \omega = \lfloor p/\alpha \rfloor$$

Outre le cas des graphes h-parfaits, ce résultat permet donc de déterminer le ration d'imperfection des familles de graphes pour lesquelles la a-perfection a été établie, comme les cliques circulaires [109] ou plus généralement les complémentaires des graphes d'intervalles circulaires flous [110].

A partir de cette formule du ratio d'imperfection des graphes a-parfaits, il est aisé de tirer un majorant :

Corollaire 3.22. *Pour tout graphe a-parfait G, nous avons* $\operatorname{imp}(G) < \frac{3}{2}$.

Bien que les facettes du polytope des stables d'un graphe a-parfait soient des contraintes de rang associées aux cliques circulaires induites par définition, on ne peut pas calculer en général le nombre de stabilité pondéré en temps polynomial via un programme linéaire à partir de ce polytope, puisque les contraintes des cliques circulaires ne sont pas séparables en temps polynomial [11]. De plus, nous n'avons pas réussi à déterminer précisément quelles cliques circulaires premières sont essentielles aussi bien pour calculer le ratio d'imperfection de ces graphes que pour connaître explicitement les facettes de leur polytope des stables.

Le complémentaire d'un graphe quasi-adjoint est appelé un *graphe proche-biparti*. Les graphes proche-bipartis ont été introduits par Shepherd en 1995 [102], qui a notamment établi la liste des facettes de leurs polytope des stables :

Théorème 3.23 (Shepherd (1995) [102]). *Pour tout graphe proche-biparti G, les facettes non-triviales de son polytope des stables sont des inégalités*

$$\sum_{i\leq k} \frac{1}{\alpha_i} x(K_{n_i/\alpha_i}) \leq 1$$

associées à des jointures complètes de cliques circulaires premières $K_{n_1/\alpha_1}, \ldots, K_{n_k/\alpha_k}$.

Du théorème 3.21, nous pouvons donc en déduire la formule suivante pour leur ratio d'imperfection :

Corollaire 3.24. *Pour tout graphe proche-biparti ou quasi-adjoint G,*

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(K_{n'/\alpha'}) : K_{n'/\alpha'} \subseteq G \text{ prime}\}$$

et majoré strictement par $\frac{3}{2}$.

Il n'est pas possible d'étendre ce résultat aux graphes sans griffe : en effet, bien que ceux-ci contiennent les graphes quasi-adjoints, ils contiennent tous les graphes avec un nombre de stabilité égal à 2 et peuvent donc avoir un ratio d'imperfection arbitrairement grand [47].

Chudnovsky et Ovetsky ont donné très récemment un majorant du nombre chromatique des graphes quasi-adjoints :

Théorème 3.25 (Chudnovsky, Ovetsky (2007) [14]). *Si G est un graphe quasi-adjoint alors $\chi(G) \leq 3/2\omega(G)$.*

Ce résultat redonne pratiquement notre borne du corollaire 3.24. Pour expliquer ceci, nous avons besoin d'une caractérisation du ratio d'imperfection due à Gerke et McDiarmid [47] basée sur les graphes dupliqués : étant donné un vecteur x de $\mathbb{N}^{V(G)}$, le graphe dupliqué G_x relativement à x est obtenu en remplaçant chaque sommet v du graphe G par une clique Q_v de taille x_v et tel que pour toute arête ij de G, les cliques Q_i et Q_j sont complètement jointes, et il n'y a pas d'autres arêtes. Nous avons alors :

Théorème 3.26 (Gerke, McDiarmid (2001) [47]). *Pour tout graphe G,*

$$\textit{imp}(G) \quad = \quad \sup\left\{\frac{\chi_f(G_x)}{\omega(G_x)} : 0 \neq x \in \mathbb{N}^V\right\} \tag{3.4}$$

Comme la famille $\mathcal{G}$ des graphes quasi-adjoint est close par duplication d'un sommet et en appliquant le théorème 3.25, nous avons donc la chaîne d'inégalités suivante

$$\begin{aligned}
\max\{\mathrm{imp(G)} : \mathrm{G} \in \mathcal{G}\} &= \sup\left\{\frac{\chi_f(G_x)}{\omega(G_x)} : 0 \neq x \in \mathbb{N}^V, G \in \mathcal{G}\right\} \\
&= \sup\left\{\frac{\chi_f(G)}{\omega(G)} : G \in \mathcal{G}\right\} \\
&\leq \sup\left\{\frac{\chi(G)}{\omega(G)} : G \in \mathcal{G}\right\} \\
&\leq 3/2
\end{aligned}$$

Pour la sous-classe des graphes semi-adjoints, nous avons obtenu une meilleure borne :

Lemme 3.27. *C.2.8 Pour tout graphe semi-adjoint G, nous avons* $\mathrm{imp}(G) \leq \frac{5}{4}$.

La figure 3.3 synthétise les différentes bornes que nous avons évoquées pour le ratio d'imperfection.

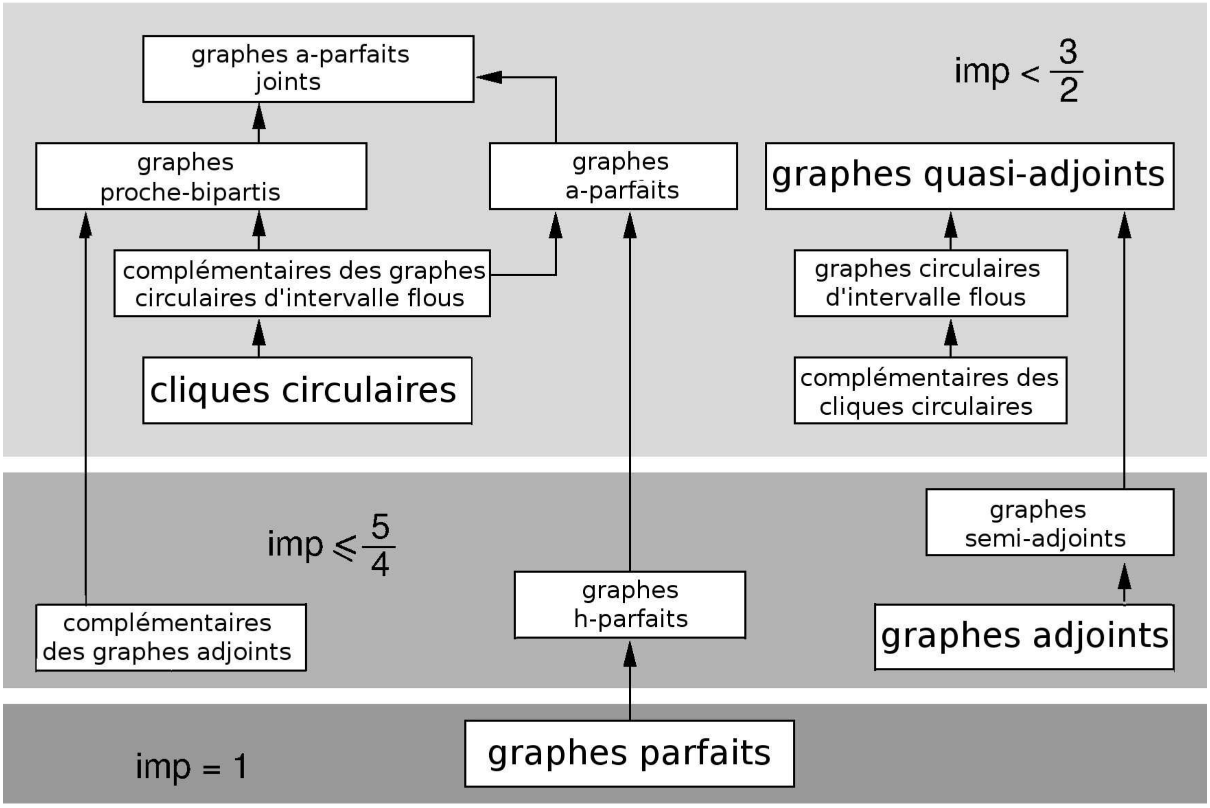


FIG. 3.3 – Ratio d'imperfection de quelques familles de graphes

Chapitre 4

Des graphes quasi-adjoints aux graphes sans griffe

Sommaire

Dans ce chapitre, nous élargissons notre étude du polytope des stables aux graphes sans griffe. En s'inspirant de l'approche novatrice d'Edmonds pour déterminer la taille d'un couplage maximum d'un graphe [34], Sbihi a proposé un algorithme polynomial pour déterminer la taille d'un stable de cardinalité maximum en 1980 [97]. La même année, Minty a donné un algorithme traitant du cas pondéré [69]. Nakamura et Tamura ont établi que ce dernier algorithme était incorrect dans certain cas et proposé un correctif [72]. Du fait de la correspondance entre la complexité de la séparabilité des facettes du polytope des stables et du calcul du nombre de stabilité pondéré, ceci suggère que les facettes du polytope des stables devraient être “simples” à déterminer. Ainsi Grötschel, Lovász et Schrijver ont mentionné ce problème ouvert comme étant fondamental, dès 1988 [56].

La première partie dresse un état de l'art des principales facettes connues du polytope des stables de ces graphes. Si les inégalités de familles de cliques couvrent toutes les facettes du polytope des stables des graphes quasi-adjoints, c'est loin d'être le cas pour les graphes sans griffe en général. En effet, le plus petit graphe sans griffe qui n'est pas quasi-adjoint, la roue impaire à 5 sommets (figure 4.1(a)) possède une facette qui n'est pas une inégalité de famille de cliques. Les roues impaires sont des graphes avec un nombre de stabilité égal à 2. Cook [101] a démontré que toutes les facettes de ces graphes sont des contraintes de voisinage de clique. Stauffer a proposé dans sa thèse [104] une conjecture basée sur celles-ci pour les facettes des graphes sans griffe G avec un nombre de stabilité au moins 4. Galluccio, Gentile et Ventura ont récemment découvert un contre-exemple et ont proposé une nouvelle conjecture, plus technique [43]. Pour les graphes sans griffe avec un nombre de stabilité exactement égal à 3, la situation est beaucoup plus ouverte, et de fait, toutes les facettes les plus “originales” ou variées connues concernent ce cas. Par exemple, Giles et Trotter [49] puis Liebling et ses co-auteurs [64] ont présenté des facettes qui ont jusqu'à 5 coefficients gauches distincts (2 exemples sont donnés dans la figure 4.1(b),(c)).

Dans la seconde partie, nous présentons une reformulation mathématique, permettant d'apporter un éclairage commun à ces facettes. Cette approche nous a permis de découvrir de nouvelles facettes, qui peuvent contenir un nombre arbitrairement grand de coefficients gauches (paragraphe 4.2.2.1) et de répondre par la négative à une question de Liebling, Oriolo, Spille et Stauffer [64] : il existe un graphe sans griffe dont le polytope des stables a une facette dont les coefficients ne sont pas tous consécutifs (paragraphe 4.2.2.2).

Dans la dernière partie, nous généralisons le concept d'inégalités de familles de cliques et proposons [82] une "extension" de la conjecture de Ben Rebea, i.e. que ces "inégalités étendues" suffisent à décrire le polytope des stables des graphes sans griffe.

4.1 Panorama des facettes du polytope des stables des graphes sans griffe

4.1.1 Avec un nombre de stabilité égal à 2

Les graphes ayant un nombre de stabilité égal à 2 sont naturellement sans griffe. Leurs polytopes des stables possèdent une structure faciale très simple. En effet, leurs facettes sont des *contraintes de voisinage de clique* :

Définition 4.1 (contrainte de voisinage de clique - Cook (1990) [23, 101])**.** *Etant donnée une clique Q, la contrainte de voisinage $F(Q)$ de la clique Q est*

$$2x(Q) + 1x(N'(Q)) \leq 2 \tag{4.1}$$

où $Q \subseteq G$ est une clique et $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$

Nous avons alors :

Théorème 4.2 (Cook (1990) [23, 101])**.** *Si G est un graphe tel que $\alpha(G) = 2$ alors les facettes non-triviales de son polytope des stables sont les contraintes de voisinage de clique*

$$2x(Q) + 1x(N'(Q)) \leq 2 \tag{4.2}$$

où $Q \subseteq G$ est une clique, $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$ et le sous graphe induit $\overline{G}[N'(Q)]$ du complémentaire $\overline{G}$ ne possède pas de composante connexe bipartie.

Remarquons qu'une contrainte de voisinage de clique $F(Q)$ est
- la contrainte de clique donnée par Q si et seulement si Q est maximale (et donc $N'(Q) = \emptyset$) ;
- la contrainte de rang pleine associée à G si et seulement si $Q = \emptyset$ (et donc $N'(Q) = G$) ;
- une contrainte qui n'est pas de rang sinon.

Les inégalités des familles des cliques contiennent-elles les contraintes de voisinage de clique ? Assez curieusement, la réponse est négative, pour beaucoup de contraintes de voisinage de clique :

Lemme 4.3 (P., Wagler (2006) [87])**.** *Soit G un graphe tel que $\alpha(G) = 2$. Si Q est une clique non vide et n'est pas maximale alors la contrainte de voisinage $F(Q)$ n'est pas une inégalité de famille de cliques.*

Démonstration. Supposons au contraire que la contrainte $F(Q)$ soit une inégalité de famille de cliques $(\mathcal{Q}, p)$, alors nous avons $V_{\geq p} = \{Q\}$, $V_{p-1} = N'(Q)$, $p - r = 2$ et $(p-r)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 2$. Par conséquent, nous avons $\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 1$.

Comme le sous graphe induit $\overline{G}[N'(Q)]$ du complémentaire $\overline{G}$ ne possède pas de composante connexe bipartie, il contient un antitrou impair $\overline{C}_{2k+1}$. En comptant le nombre de fois qu'un sommet de $\overline{C}_{2k+1}$ est couvert par les cliques de $\mathcal{Q}$, nous avons :

$$(2k+1)(p-1) \;=\; \sum_{Q \in \mathcal{Q}} |Q \cap \overline{C}_{2k+1}| \leq\; |\mathcal{Q}| \cdot k$$

Nous en déduisons la contradiction suivante :

$$2 < 2 + \frac{1}{k} = \frac{2k+1}{k} \leq \frac{p\left\lfloor\frac{|Q|}{p}\right\rfloor + r}{p-1} = \frac{p+r}{p-1} = \frac{2p-2}{p-1} = 2$$

■

En particulier, les inégalités des roues impaires $x(\overline{C}_{2k+1}) + 2x_v \leq 2$ ne sont pas inégalités des familles de cliques. Le rôle des inégalités des familles de cliques semble donc assez limité dès lors que l'on considère le polytope des stables des graphes sans griffe qui ne sont pas quasi-adjoints. Dans la section 4.3, nous aborderons la problématique de leur généralisation et de leur renforcement pour l'expression de *toutes* les facettes du polytope des stables des graphes sans griffe.

4.1.2 Avec un nombre de stabilité égal à 3

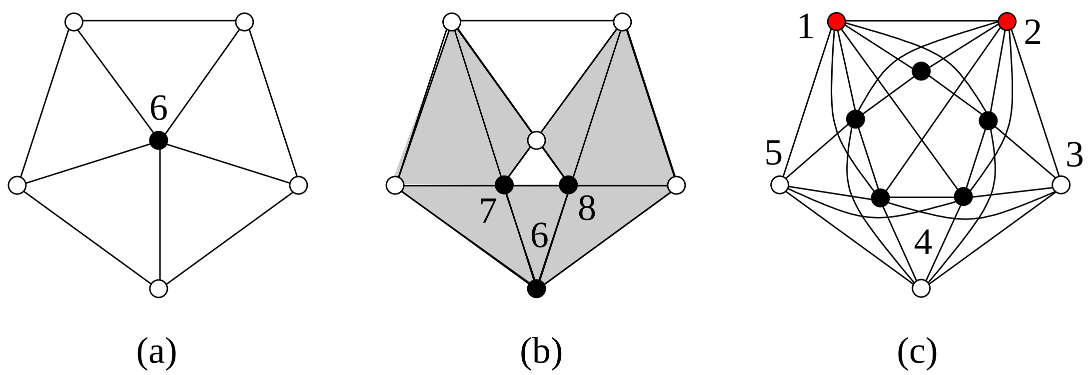


FIG. 4.1 – Des graphes avec des facettes qui ne sont pas des contraintes de familles de cliques

Plusieurs auteurs ont étudié le polytope des stables des graphes sans griffe avec un nombre de stabilité 3, et exhibé des facettes qui ne sont pas de rang. Nous allons voir que celles-ci ne sont pas des inégalités de familles de cliques. Ainsi Giles et Trotter ont introduit une famille de tels graphes, appelés *wedges* en anglais, et ont montré que leurs polytopes des stables ont des facettes de la forme

$$1x(A) + 2x(B) \leq 3$$

pour des ensembles de sommets A et B bien choisis (exemple : figure 4.1(b)).

Dans une inégalité d'une famille de cliques, le plus grand coefficient gauche doit être un diviseur du membre de droite. Ainsi ces facettes $1x(A) + 2x(B) \leq 3$ ne sont évidemment pas des inégalités de familles de cliques.

Dans le même article, Giles et Trotter ont donné un autre exemple de facette (figure 4.1(c))

$$3(x_1 + x_2) + 1(x_3 + x_4 + x_5) + 2(x_6 + \ldots + x_{10}) \leq 4$$

et bien évidemment, les deux coefficients gauches non-nuls d'une inégalité de famille de clique ne peuvent convenir pour exprimer cette inéquation. Liebling et al. [64] ont découvert récemment d'autres facettes similaires avec jusqu'à 5 coefficients consécutifs gauches, qui ne peuvent pas être des inégalités de familles de cliques, par le même argument.

4.1.3 Avec un nombre de stabilité au moins 4

Rappelons que Fouquet a démontré [41] qu'un graphe sans griffe connexe avec un nombre de stabilité au moins 3 est quasi-adjoint ou bien contient une roue impaire à 5 sommets. Stauffer a mis en évidence dans sa thèse [104] que les roues impaires à 5 sommets sont au premier plan pour décrire les facettes des polytopes des stables des graphes sans griffe G tels que $\alpha(G) \geq 4$, car les contraintes des roues impaires à 5 sommets peuvent être élevées aux inégalités plus générales $1x(A) + 2x(B) \leq 2$. Ceci l'a amené à conjecturer :

Conjecture 4.4 (Stauffer (2005) [104])**.** *Les facettes du polytope des stables d'un graphe sans griffe G qui n'est pas quasi-adjoint et tel que $\alpha(G) \geq 4$ sont les contraintes de non-négativité, des contraintes de rang et des contraintes de voisinage de clique.*

En combinaison avec le théorème de Cook, cette conjecture met en valeur la singularité des facettes lorsque le nombre de stabilité vaut 3. Pour comprendre ce qui se passe dans ce cas précis, nous avons défini une nouvelle approche qui fait l'objet de la seconde partie de ce chapitre. Galluccio, Gentile et Ventura ont récemment démontré que cette conjecture est fausse, en proposant une nouvelle construction d'inégalités valides pour le polytope de stables, non-limitée aux graphes sans griffe [43]. Ils ont conjecturé que ces inégalités sont suffisantes pour décrire le polytope des stables des graphes de la conjecture 4.4.

4.2 De nouvelles facettes du polytope des stables des graphes sans griffe

Nous venons de voir que toutes les facettes "complexes" connues du polytope des stables des graphes sans griffe G concernent principalement le cas $\alpha(G) = 3$. Dans cette partie, nous donnons une description du polytope des stables des graphes G tels que $\alpha(G) = 3$ en général, puis en discutons les conséquences pour les graphes sans griffe.

4.2.1 Notre approche via la programmation linéaire

Notre approche est basée sur une reformulation combinatoire (en termes de graphes) de la notion de facette, que nous traduisons ensuite sous forme d'un programme linéaire. L'implémentation de ce dernier nous a alors permis de découvrir de nouvelles facettes.

4.2.1.1 Une reformulation combinatoire

Le polytope des stables de la jointure complète G de deux graphes G_1 et G_2 a été décrit par Chvátal en 1975 [19] : d'une manière informelle, la liste des facettes de STAB(G) est la concaténation de celles de STAB(G_1) et de STAB(G_2).

Soit maintenant un graphe G à n sommets tel que $\alpha(G) = 3$ et soit $a^T x \leq b$ une facette quelconque de son polytope des stables STAB(G). Par définition d'une facette, il existe n ensembles stables $S_1, \ldots, S_n$ linéairement indépendants tels que $a^T \chi^{S_i} = b$ pour tout $1 \leq i \leq n$. Ce sont des *racines* de la facette $a^T x \leq b$.

Puisque $\alpha(G) = 3$, toute racine contient au plus 3 sommets. Si une des racines ne contient qu'un seul sommet s alors ce sommet doit avoir dans son voisinage tous les autres sommets v tel que $a_v > 0$: le graphe G est donc une jointure complète de deux sous-graphes G_1 et G_2 et la facette $a^T x \leq b$ est en fait une facette de STAB(G_1) ou de STAB(G_2).

Intéressons nous au cas où G n'est pas une jointure complète de deux graphes. Toutes les racines de $a^T x \leq b$ sont donc de taille 2 ou 3, et correspondent dans le complémentaire à des arêtes ou des triangles. Le graphe donné par ces arêtes ou triangles possède une structure très simple : pour l'expliciter nous avons besoin de quelques définitions.

Définition 4.5 (1-forêt couvrante - P., Wagler (2006) [87])**.** *Une 1-forêt couvrante est un triplet $(T,\ w,\ B)$ où T est un graphe muni de la pondération w sur les sommets et B un entier tels que :*

- *la taille d'une clique maximum est 3 ;*
- *si k est le nombre de triangles de T, le graphe obtenu en enlevant les arêtes de ces k triangles vérifie :*
  (i) chaque composante connexe est soit un arbre, soit un arbre avec une arête additionnelle ;
  (ii) le nombre de composantes connexes qui sont des arbres est k.
- *pour toute clique maximale Q, la somme $w(x), x \in Q$ est égale à B.*

Définition 4.6 (contrainte de 1-forêt couvrante - P., Wagler (2006) [87])**.** *La contrainte de 1-forêt couvrante associée à (T, w, B) est $\sum_{x \in T} w(x) \leq B$.*

Naturellement une 1-forêt couvrante T possède exactement n cliques maximales (où n est le nombre de sommets de T) et celle-ci définit une facette $\sum_{i \in V} w(i)x_i \leq b$ du polytope des stables de $\overline{T}$ dès lors que la matrice d'incidence de ces n cliques maximales est inversible.

Définition 4.7 (co-contrainte de 1-forêt couvrante - P., Wagler (2006) [87])**.** *Facette $a^T x \leq b$ du polytope des stables d'un graphe G telle qu'il existe un ensemble de n racines $S_1, \dots, S_n$ linérairement indépendantes telles que $(H_{\mathcal{S}}, a, b)$ soit une contrainte de 1-forêt couvrante où $H_{\mathcal{S}}$ désigne le sous-graphe couvrant partiel de $\overline{G}$ dont les cliques maximales de H sont exactement les racines S_i pour $1 \leq i \leq n$ (ij est une arête si et seulement s'il existe une racine S_k contenant i et j).*

Le fait remarquable est alors qu'il est possible de choisir un ensemble de n racines de manière à former une 1-forêt couvrante :

Théorème 4.8 (P., Wagler (2006) [87] D.2.8)**.** *Soit G un graphe avec $\alpha(G) \leq 3$ qui n'est pas la jointure complète de deux graphes. Alors toute facette non-triviale de son polytope des stables est soit une inégalité de clique, soit une co-contrainte de 1-forêt couvrante.*

Ainsi les 1-forêts couvrantes sont en fait une reformulation combinatoire des facettes du polytope des stables des graphes avec un nombre de stabilité égal à 3. Dans l'exemple 4.9 ci-après, nous exprimons à l'aide de 1-forêts couvrantes quelques-unes des facettes du polytope des stables de graphes sans griffe mises en avant dans la littérature.

Exemple 4.9 (contraintes de 1-forêts couvrantes)**.** *Liebling et al. [64] ont démontré que le graphe sans griffe "poisson dans un filet" a un polytope des stables dont une des facettes a pour inéquation :*

$$1x(\circ) + 2x(\bullet) + 3x(\square) + 4x(\otimes) \leq 5$$

Dans son complémentaire, des racines induisent une 1-forêt couvrante formée de deux arbres et deux triangles (voir figure 4.2(a)). Nous avons étendu cette structure pour exhiber une suite de graphes sans griffe dont les polytopes des stables possèdent des facettes avec un nombre de coefficients gauches arbitrairement et un membre de droite grands (cf prochaîne partie). Deux graphes de cette suite sont illustrés dans les figures 4.2(b),(c) et les inéquations des facettes correspondantes sont

$$1x(\circ) + 2x(\bullet) + 3x(\square) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) \leq 7$$

et

$$1x(\circ) + 2x(\bullet) + 3x(\square) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) + 7x(\diamond) + 8x(\oplus) \leq 9.$$

Giles et Trotter ont exhibé un graphe sans griffe dans [49] dont le polytope des stables a une des facettes ayant pour inéquation :

$$1x(\circ) + 2x(\bullet) + 3x(\square) \leq 4$$

Dans son complémentaire, des racines induisent une 1-forêt couvrante formée d'un chemin, d'un triangle et d'un 1-arbre correspondant à un C_5 (voir figure 4.3(a)).

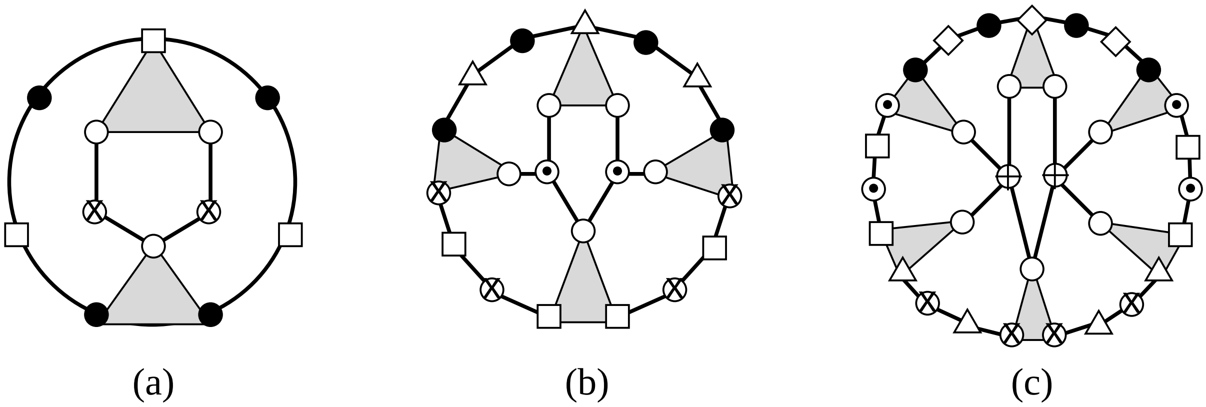


FIG. 4.2 – Complémentaire du graphe "poisson dans un filet" et deux généralisations

Le plus petit coefficient gauche n'est pas forcément égal à 1 : ainsi le graphe sans griffe, dont le complémentaire est illustré dans la figure 4.3(b), a une facette de son polytope des stables d'inéquation

$$2x(\bullet) + 3x(\square) + 4x(\otimes) \leq 6$$

Dans son complémentaire, des racines induisent une 1-forêt couvrante formée d'un arbre, d'un triangle et d'un C_5.

Dernier exemple, Liebling et al. [64] ont montré que le graphe sans griffe "poisson dans un filet avec une bulle", dont le complémentaire est illustré dans la figure 4.3(c), a une facette de son polytope des stables d'inéquation

$$2x(\bullet) + 3x(\square) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) \leq 8$$

Dans son complémentaire, des racines induisent une 1-forêt couvrante formée de deux chemins, de deux triangles et d'un C_5.

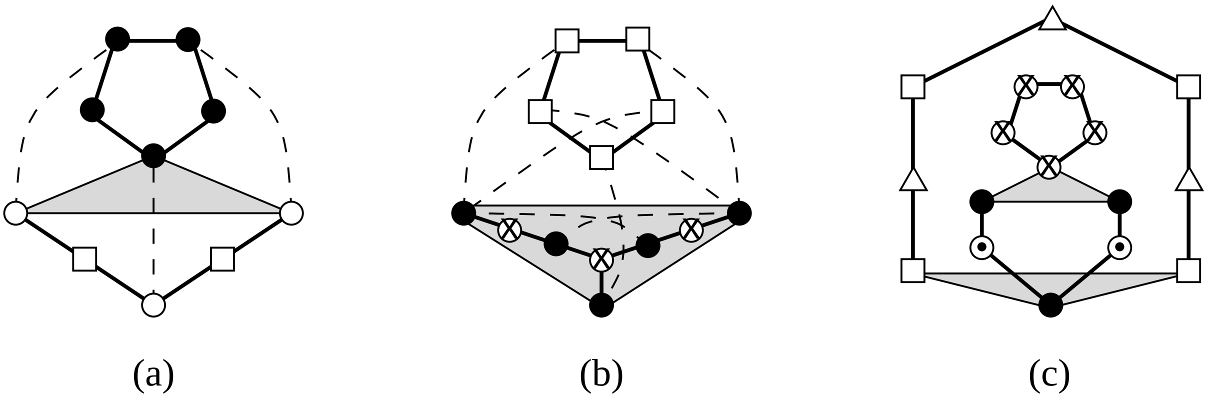


FIG. 4.3 – Quelques exemples de 1-forêts couvrantes

4.2.1.2 Spécialisation aux graphes sans griffe via un programme linéaire

Soit donc $(T = (V, E), w, B)$ (où V est l'ensemble des sommets et E l'ensemble des arêtes de T) une contrainte de forêt couvrante telle que la matrice d'incidence de ses cliques maximales soit inversible. Si $H =$

(V, F) est un graphe dont T est un sous-graphe tel que :

C$_1$ pour tout $ij \in F$, $w_i + w_j \leq B$,

C$_2$ pour tout $i, j, k \in V$ tels que $ij, jk, ik \in F$, $w_i + w_j + w_k \leq B$,

C$_3$ le graphe H ne contient pas de clique de taille 4,

C$_4$ le graphe complémentaire de H soit sans griffe, cela signifie que si i, j, k et l sont quatre sommets de V tels que ij, ik et il sont des arêtes de H alors au moins une des arêtes jk, kl et jl appartient également à H,

alors le complémentaire de H est un graphe sans griffe, dont les stables maximums sont de taille 3, et tel que $\sum_{i\in V} w(i)x_i \leq B$ est une facette de son polytope des stables.

Afin de résoudre le problème de l'existence de H, on considère le graphe complet K_n sur $n = |V|$ sommets. Pour chaque arête ij de V_n on définit une variable x_{ij} valant 1 si l'arête ij appartient à la solution du problème et 0 sinon. Le problème considéré est alors équivalent au programme linéaire en nombres entiers suivant :

$$\text{Minimiser} \sum_{ij\in V_n} x_{ij} \tag{4.3}$$

$$x_{ij} = 1 \qquad \text{pour tout } ij \in E, \tag{4.4}$$

$$x_{ij} = 0 \qquad \text{pour tout } i, j \in V \text{ tels que } w_i + w_j > B, \tag{4.5}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \qquad \text{pour tout } i, j, k \in V \text{ tels que } w_i + w_j + w_k > B, \tag{4.6}$$

$$x_{ij} + x_{ik} + x_{il} + x_{jk} + x_{jl} + x_{kl} \leq 5 \qquad \text{pour tout } i, j, k, l \in V, \tag{4.7}$$

$$x_{jk} + x_{jl} + x_{kl} - x_{ij} - x_{ik} - x_{il} \leq 2 \qquad \text{pour tout } i, j, k, l \in V, \tag{4.8}$$

$$x_{ij} \in \{0, 1\} \qquad \text{pour tout } i, j \in V_n. \tag{4.9}$$

Les contraintes (4.4) imposent que le graphe recherché contienne le graphe G comme sous-graphe. Respectivement, les contraintes (4.5), (4.6), (4.7) et (4.8) imposent que la solution vérifie les condition C$_1$, C$_2$, C$_3$ et C$_4$. La fonction de coût (4.3) minimise le nombre d'arêtes à ajouter pour parvenir à une solution, lorsqu'elle existe.

4.2.2 Mise en pratique : de nouvelles facettes

La caractérisation des facettes des graphes quasi-adjoints établit en particulier que pour ceux-ci, une facette du polytope des stables a tout au plus deux coefficients gauches et ils sont de plus consécutifs. Nous avons vu que pour un graphe sans griffe en général, on peut avoir plus de deux coefficients gauches. Il est naturel de se demander si le nombre de coefficients gauches est borné et si la consécutivité des coefficients est préservée :

Question 4.10. *Existe-t'il une constante M tel que toute facette du polytope des stables des graphes sans griffe a au plus M coefficients gauches distincts ?*

Question 4.11 (Liebling et al. [64])**.** *Les coefficients d'une facette du polytope des stables d'un graphe sans griffe sont-ils toujours consécutifs ?*

En menant des investigations à l'aide du programme linéaire que nous venons d'introduire, nous avons réussi à apporter une réponse, malheureusement négative, à ces deux questions.

4.2.2.1 Avec un nombre arbitraire de coefficients différents

Nous allons exhiber, pour tout entier $b \geq 4$, un graphe sans griffe à $5b - 13$ sommets dont le polytope des stables possède une facette avec b coefficients gauches distincts (et consécutifs).

Théorème 4.12 (P., Pesneau, Wagler (2006) [81, 90] C.3.1)**.** *Soit un entier $b \geq 5$. Alors il existe un graphe sans griffe G_b tel que $\alpha(G_b) = 3$ et dont l'ensemble des sommets admet une partition en $b-1$ ensembles non vides $V_1, V_2, \ldots, V_{b-1}$ tels que*

$$\sum_{v \in V_1} x_v + 2 \sum_{v \in V_2} x_v + \ldots + (b-1) \sum_{v \in V_{b-1}} x_v \quad \leq \quad b \tag{4.10}$$

soit une facette de son polytope des stables.

Notre construction du graphe G_b repose sur 1-forêt qui ne comporte que des arbres. Nous procédons en deux temps :

– dans le premier temps, nous construisons une contrainte 1-forêt couvrante (H_b, c, b) ;
– dans le second temps, nous déterminons un ensemble d'arêtes à ajouter pour obtenir un graphe H'_b pour que le complémentaire G_b du graphe soit sans griffe, et la contrainte de 1-forêt couvrante (H_b, w, b) soit une facette de son polytope des stables.

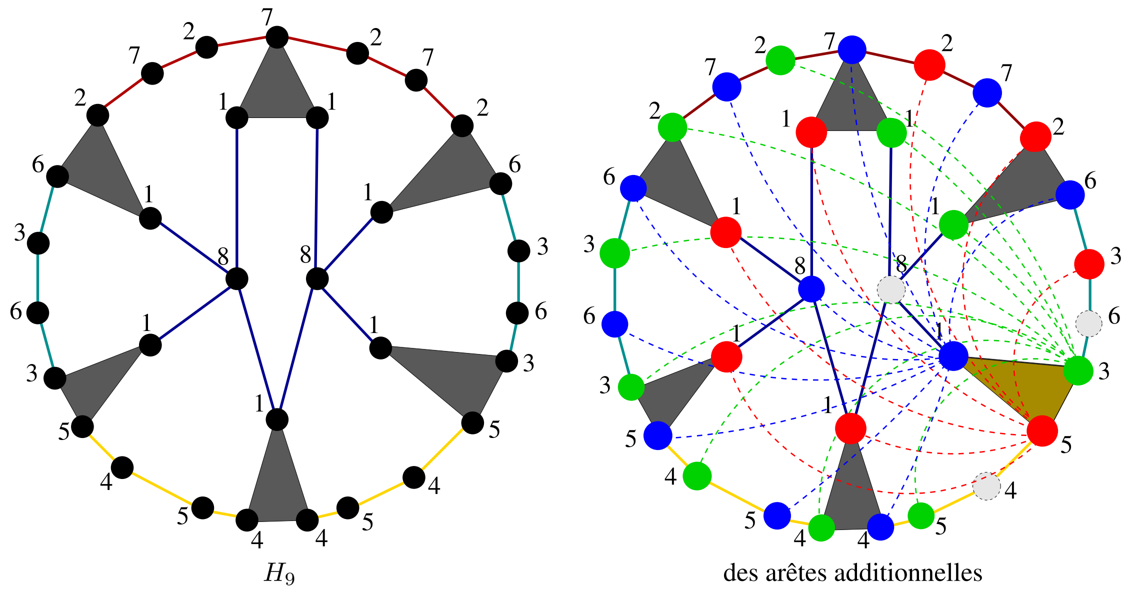


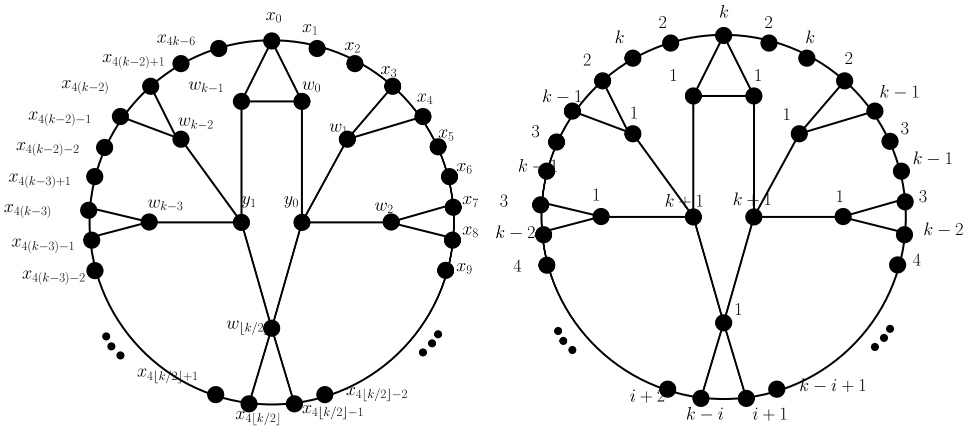
FIG. 4.4 – Le graphe H_9 et des arêtes à ajouter pour rendre le complémentaire sans griffe.

Construction de la contrainte 1-forêt couvrante (H_b, c, b)**.** Soit donc $k = b-2$ et $n = 5k-3$. Dénotons par H_b le graphe (voir la figure 4.4 pour une illustration de H_9) dont l'ensemble des sommets V est formé des n sommets $\{x_0, x_1, \ldots, x_{4k-6}\} \cup \{w_0, w_1, \ldots, w_{k-1}\} \cup \{y_0, y_1\}$, et H_b possède exactement les n cliques maximales suivantes :

– $k-1$ triangles $U_1, U_2, \ldots, U_{k-1}$ où $U_j = \{w_j, x_{4j-1}, x_{4j}\}$ pour tout indice $1 \leq j \leq k-2$ et $U_{k-1} = \{w_0, w_{k-1}, x_0\}$;
– les arêtes de la forêt couvrante formée des $k-1$ arbres $T_1, T_2, \ldots, T_{k-1}$ où pour tout indice $1 \leq i \leq k-3$, l'arbre T_i est la chaîne de taille 4 $\{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, l'arbre T_{k-2} est la chaîne de taille 7 $\{x_{4(k-2)}, x_{4(k-2)+1}, x_{4k-6}, x_0, x_1, x_2, x_3\}$, et le dernier arbre T_{k-1} a pour sommets $\{w_0, w_1, \ldots, w_{k-1}\} \cup \{y_0, y_1\}$ et arêtes $\{y_0w_0, y_0w_1, \ldots, y_0w_{\lfloor k/2 \rfloor}\} \cup \{y_1w_{\lfloor k/2 \rfloor}, y_1w_{\lfloor k/2 \rfloor+1}, \ldots, y_1w_{k-1}\}$.

La figure 4.5 décrit de manière informelle les coefficients des sommets de cette forêt couvrante.

Détermination de l'ensemble d'arêtes à ajouter. Nous devons maintenant déterminer un ensemble d'arêtes à ajouter à H_b pour obtenir un graphe H'_b tel que :



graphe H_b coefficients $c(v)$

FIG. 4.5 – La forêt couvrante H_b et les coefficients c de la facette associée

- le complémentaire de H'_b soit sans griffe, autrement dit, dans H'_b tout triangle est dominant (un ensemble est dit dominant s'il contient tous les autres sommets dans son voisinage) ;
- la taille d'une clique maximum de H'_b reste égale à 3 ;
- la contrainte (4.10) de forêt couvrante (H_b, c, b) induit une facette du polytope des stables du complémentaire de H'_b (autrement dit, l'ajout des arêtes ne compromet pas sa validité).

Nous appelons *arête additionnelle* une arête de H'_b qui n'est pas dans H_b. La description de l'ensemble des arêtes additionnelles est assez fastidieuse et la preuve pour démontrer qu'il vérifie les critères énoncés en début de paragraphe est technique. La clef de cette preuve est de montrer tout d'abord qu'en ajoutant ces arêtes additionnelles, on ne crée pas de nouveaux triangles, si bien que pour vérifier que le complémentaire est sans griffe il suffit de s'assurer que chaque triangle de H_b est dominant, et pour s'assurer que la contrainte est toujours valide, il suffit de vérifier qu'aucune des nouvelles arêtes ne viole celle-ci. La figure 4.6 donne une illustration de cet ensemble d'arêtes additionnelles.

4.2.2.2 Avec des coefficients non-consécutifs

Liebling, Oriolo, Spille et Stauffer ont demandé si pour tout graphe sans griffe, les coefficients d'une facette de son polytope des stables sont toujours consécutifs, y compris en tenant compte des coefficients nuls.

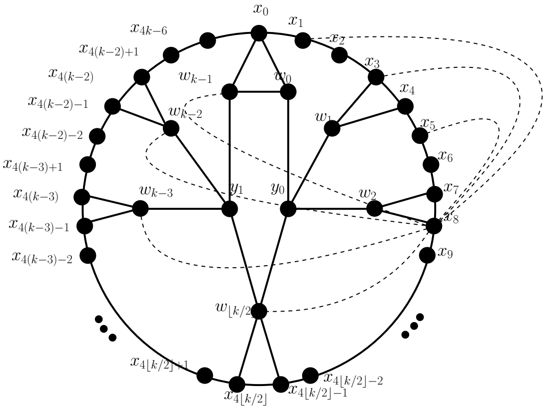
Nous avons découvert un graphe pour lequel ce n'est pas vrai, illustré dans la figure 4.7.

4.3 Généraliser les inégalités de familles de cliques pour les graphes sans griffe

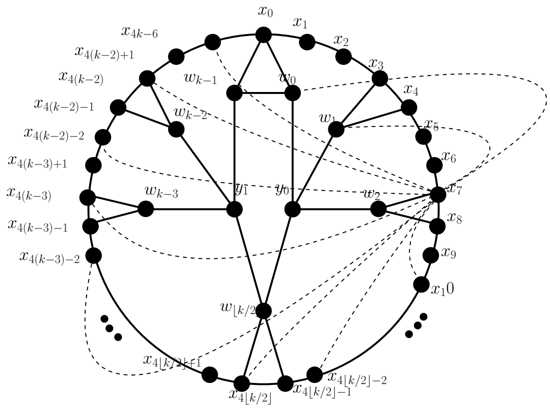
Nous avons vu à plusieurs reprises dans ce chapitre que les inégalités de familles de cliques ne suffisent pas à exprimer toutes les facettes des polytopes des graphes sans griffe. Pourtant, ce sont des généralisation élégantes des inégalités d'Edmonds pour les graphes adjoints, et elles suffisent pour la classe intermédiaire des graphes quasi-adjoints.

Nous avons donc essayé d'étendre cette famille en jouant sur plusieurs paramètres :
- en autorisant plus de deux coefficients à gauche ;
- en autorisant plus de flexibilité sur les coefficients gauches ;
- en renforçant le membre de droite.

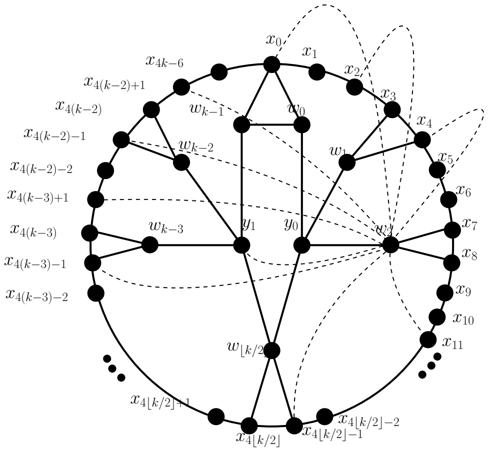
Cette dernière partie est consacrée à cette extension des inégalités de familles de cliques.



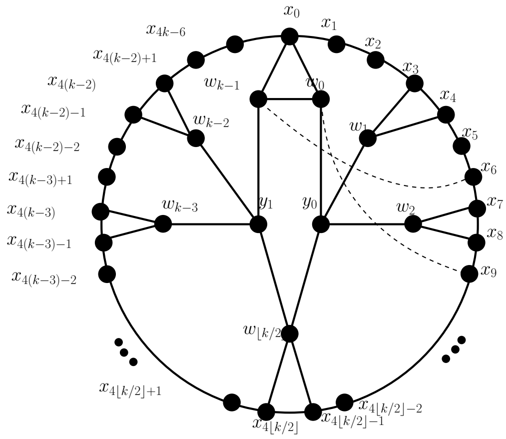
Arêtes additionnelles incidentes à x_{4i}



Arêtes additionnelles incidentes à x_{4i-1}



Arêtes additionnelles incidentes à w_i



Arêtes additionnelles incidentes à x_{4i-2} ou x_{4i+1}

FIG. 4.6 – Arêtes additionnelles pour $i = 2$ (en pointillé)

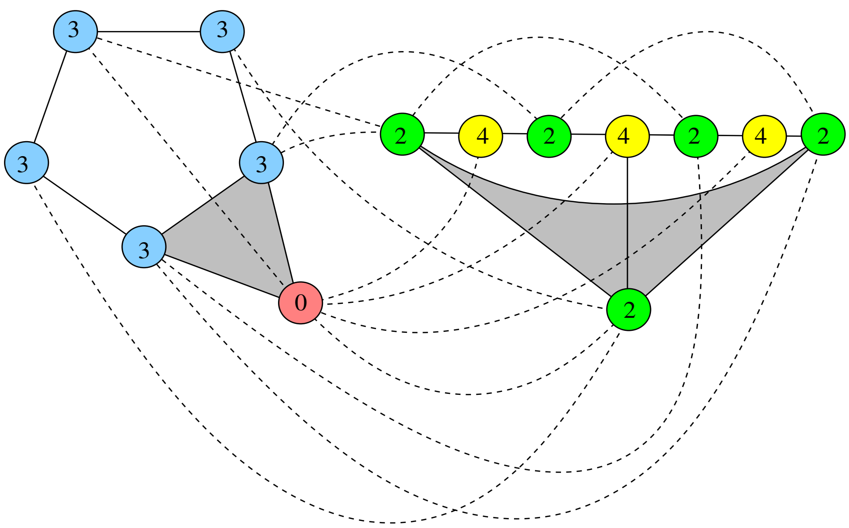


FIG. 4.7 – Exemple de facette avec des coefficients non-consécutifs

4.3.1 Inégalités étendues de familles de cliques

Définition 4.13 (Inégalité étendue d'une famille de cliques - P., Wagler (2006) [85]). *Soit $G = (V, E)$ un graphe et une famille de cliques $\mathcal{Q}$ de G. Soient des entiers $p \leq |\mathcal{Q}|$, r avec $0 \leq r \leq R = |\mathcal{Q}| \bmod p$, et J avec $0 \leq J \leq p - r$. Définissons*

$$\begin{aligned} V_p = &\quad \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| \geq p\}, \\ V_{p-j} = &\quad \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| = p - j \end{aligned}$$

pour tout $1 \leq j \leq J$

L'inégalité étendue $(\mathcal{Q}, p, r, J, b)$ de la famille de cliques $\mathcal{Q}$ est

$$\sum_{0 \leq j \leq J} (p - r - j)\, x(V_{p-j}) \;\leq\; b$$

où b est un entier permettant d'assurer la validité de cette inégalité pour le polytope des stables.

Ces inégalités sont si générales qu'elles permettent d'exprimer n'importe quelle facette du polytope des stables d'un graphe. Il s'agit donc ni plus ni moins de l'introduction d'un nouveau formalisme. Le but est d'exprimer b sous forme d'une fonction des autres paramètres et de déterminer la famille des graphes dont le polytope correspond à ces inégalités.

Avec ce formalisme, il est facile de voir que la preuve originelle de la validité des inégalités des familles de cliques, telle que donnée par Oriolo [73] peut être facilement adaptée dans le contexte d'un nombre arbitraire de coefficients gauches.

Lemme 4.14 (P., Wagler (2006) [85]). *Soit $(\mathcal{Q}, p)$ une famille de n cliques d'un graphe G et entier p, $R = n(\mathrm{mod} p)$ et $0 \leq J \leq p - R$. Toute inégalité étendue $(\mathcal{Q}, p, R, J, b)$ de la famille de cliques $\mathcal{Q}$ où $b \geq (p - R)\lfloor n/p \rfloor$ est valide.*

La preuve est très courte :

Démonstration. Soit S un stable quelconque. Pour tout $0 \leq j \leq J$, notons s_j le nombre de sommets de S dans V_{p-j}. Soit $s = \sum_{0 \leq j \leq J} s_j$.

Si $s \leq \lfloor n/p \rfloor$ alors $\sum_{0 \leq j \leq J}(p - R - j)s_j \leq (p - R)s \leq (p - R)\lfloor n/p \rfloor$.

Si $s \geq \lfloor n/p \rfloor + 1$ alors, comme $\sum_{0 \leq j \leq J}(p - j)s_j \leq n = p\lfloor n/p \rfloor + R$, nous avons $\sum_{0 \leq j \leq J}(p - R - j)s_j \leq p\lfloor n/p \rfloor + R - Rs \leq (p - R)\lfloor n/p \rfloor$.

Ainsi nous avons dans les deux cas $\sum_{0 \leq j \leq J}(p - R - j)s_j \leq (p - R)\lfloor n/p \rfloor \leq b$. ∎

Naturellement le lemme 4.14 pour $J = 2$ redonne la validité des inégalités des familles de cliques “classiques”. Cependant, contrairement aux inégalités des familles de cliques pour les graphes quasi-adjoints, le membre de droite n'est pas assez fort pour obtenir par ce biais toutes les facettes des graphes sans griffe. Considérons par exemple le graphe de la figure 4.1 (c) : le membre de droite est 4, et ce n'est pas un multiple du plus grand coefficient gauche. Par conséquent, pour pouvoir obtenir cette facette, nous devons renforcer le membre de droite un peu plus.

Malheureusement, il n'est pas possible d'améliorer le lemme 4.14 sans hypothèse additionnelle sur le graphe considéré, car la borne du membre de droite est atteinte pour de nombreux graphes. Le lemme suivant se base sur une utilisation un peu plus fine des arguments de la preuve du lemme 4.14.

Lemme 4.15 (P., Wagler (2006) [85] D.3.2). *Soit $(\mathcal{Q}, p)$ une famille de n cliques d'un graphe G, p un entier, $R = n(\mathrm{mod} p)$ et $0 \leq J \leq p - R$. Soit $0 \leq \delta \leq \min\{R, p - R\}$.*

Si pour tout stable avec $\lfloor n/p \rfloor$ ou $\lfloor n/p \rfloor + 1$ sommets dans $V(\mathcal{Q}) = \cup_{Q \in \mathcal{Q}} Q$ l'inégalité suivante est vérifiée

$$(p - R)x(V_p) + \ldots + (p - R - J)x(V_{p-J}) \quad \leq \quad (p - R)\lfloor n/p \rfloor - \delta \tag{4.11}$$

alors l'inégalité étendue $(\mathcal{Q}, p, R, J, (p - R)\lfloor n/p \rfloor - \delta)$ est valide pour STAB(G).

Le graphe de la figure 4.1 (c) à nouveau est un exemple de graphe dont la facette peut être obtenue en appliquant le lemme 4.15, en utilisant une famille de 12 cliques. Dans le théorème 4.18, nous exprimerons cette même facette à l'aide d'une inégalité étendue d'une famille de 9 cliques seulement.

4.3.2 Les inégalités étendues et les graphes sans griffe

Nous avons défini les inégalités étendues des familles de cliques dans un cadre très général. Abordons maintenant leur potentiel pour les graphes sans griffe.

Pour les graphes ayant un nombre de stabilité 2, les inégalités de Cook s'expriment sous forme d'inégalités étendues de la manière suivante :

Théorème 4.16 (P., Wagler (2006) [85] D.3.9)**.** *Soit G un graphe tel que $\alpha(G) = 2$ et soit $2k+1$ la taille d'un plus petit antitrou impair. Alors*

- *la contrainte de rang pleine $x(G) \leq 2$ est une inégalité étendue $(\mathcal{Q}, k, r, 1, 2)$ pour un $0 \leq r \leq R$;*
- *si Q est une clique non-maximale alors la contrainte de voisinage de clique $F(Q)$ est une inégalité étendue $(\mathcal{Q}, k+1, k-1, 1, 2)$.*

De plus, dans les deux cas, la famille de cliques $\mathcal{Q}$ est de taille $2k+1$.

Qu'en est-il pour les autres graphes sans griffe, en particulier, pour ceux avec un nombre de stabilité égal à 3 ? Nous avons quelques éléments de réponse pour les contraintes de forêts couvrantes les plus élémentaires, dont celles données par les "wedges" :

Théorème 4.17 (P., Wagler (2006) [85] D.3.11)**.** *Pour tout graphe sans griffe G tel que $\alpha(G) = 3$, toute contrainte de forêt couvrante formée d'un seul arbre*

$$1x(\circ) + 2x(\bullet) \leq 3$$

est une inégalité etendue $(\mathcal{Q}, p, R, p-2, p)$ où $|\mathcal{Q}| = 7$ et $p = 3$.

Nous avons obtenu un résultat similaire pour des facettes à coefficients gauches 1, 2 et 3, et un membre de droite égal à 4 :

Théorème 4.18 (P., Wagler (2006) [85] D.3.12)**.** *Si G est un graphe sans griffe avec exactement un seul stable de taille 3, alors toute contrainte de forêt couvrante de la forme*

$$1x(\circ) + 2x(\bullet) + 3x(\square) \leq 4$$

est une inégalité étendue $(\mathcal{Q}, p, R, p-2, p)$ où $|\mathcal{Q}| = 9$ et $p = 4$.

Comme annoncé précédemment, ce résultat s'applique en particulier à la facette du graphe de la figure 4.1 (c).

Bien que très préliminaires, ces premiers résultats sur l'expression des facettes des graphes sans griffe sous forme d'inégalités étendues de familles de cliques semblent indiquer des valeurs très simples de paramètres, à savoir qu'il ne suffit de considérer que des familles à $2p+1$ cliques et un membre de gauche égal à p. Plus précisément :

Question 4.19 (P., Wagler (2006) [85])**.** *Est-ce que pour toute facette non-triviale du polytope des stables des graphes sans griffe, soit cette facette est une inégalité d'une famille de cliques, soit il existe une famille de $2p+1$ cliques $\mathcal{Q}$ telle que cette facette soit l'inégalité étendue $(\mathcal{Q}, p, R, p-2, p)$ (en particulier $b = p$) ?*

En cas de réponse positive, la version polyédrale du problème du stable de poids maximum pour les graphes sans griffe serait une extension de la description du polytope des couplages, tout comme la version combinatoire, car les inégalités étendues de familles de cliques sont elles-mêmes une extension des inégalités des ensembles impairs du polytope des couplages.

Chapitre 5

Conclusion : de la diversité des cliques circulaires

Sommaire

Un résultat majeur de la théorie des graphes est que le nombre chromatique d'un graphe parfait [4] est calculable en temps polynomial (Grötschel, Lovász et Schrijver [54, 56]). Le nombre circulaire chromatique est un raffinement du nombre chromatique usuel d'un graphe. Xuding Zhu a mis en évidence que les cliques circulaires sont le pendant naturel des cliques pour les colorations circulaires. Ceci lui a permis de définir élégamment les graphes circulaires-parfaits [115], une famille de graphes contenant strictement les graphes parfaits. La complexité du calcul du nombre circulaire chromatique d'un graphe circulaire-parfait est inconnue.

Dans ce document, nous avons dressé un panorama de l'apport des cliques circulaires et de leurs complémentaires, les webs, dans des domaines variés et, à priori, bien distincts de la théorie des graphes, tels que les colorations de graphes et l'optimisation combinatoire.

Nous avons étudié les graphes circulaires-parfaits et établi que :

- la perfection circulaire permet de séparer les graphes minimaux imparfaits des autres graphes partitionnables (partie 2.1) ;
- les graphes minimaux circulaires-imparfaits sont très divers si bien qu'une caractérisation "simple" par sous-graphes induits exclus des graphes circulaires-parfaits est improbable (partie 2.2) ;
- les graphes circulaires-parfaits sans triangle dont le complémentaire est également circulaire-parfait conservent les bonnes propriétés des graphes parfaits : caractérisation par quelques sous-graphes induits exclus basiques, reconnaissance en temps polynomial, calcul du nombre de stabilité en temps polynomial, énumération facile des facettes du polytope des stables, etc (partie 2.3)...
- le polytope des stables n'est pas adapté à une caractérisation polyédrale des graphes circulaires-parfaits (partie 2.4).

La détermination des facettes du polytope des stables des graphes sans griffe est un problème ouvert de premier plan en programmation mathématique depuis le début des années 1980. La sous-famille des graphes quasi-adjoints est la seule pour laquelle une conjecture séduisante, due à Ben Rebea, était proposée. Le rôle des cliques circulaires, ou plus exactement de leurs complémentaires, les webs, est plus inattendu dans ce contexte. En abordant l'étude du polytope des stables des webs, nous pensions commencer par un cas très simple : en fait, il n'en est rien ! Ainsi, nous avons montré que presque tous ne sont pas rang-parfaits, i.e. on ne peut exprimer toutes leurs facettes avec les coefficients 0 et 1 seulement (partie 3.1).

Ben Rebea a introduit les inégalités de famille de cliques, des inégalités valides pour le polytope des stables de tout graphe, et qui généralisent les inégalités des ensembles de taille impaire d'Edmonds. Nous avons donné un majorant de leur rang de Chvátal, ce qui a permis entre autre d'évaluer la force des inégalités proposées dans la littérature pour le polytope des stables des graphes quasi-adjoints (partie 3.2).

A l'aide des webs, nous avons défini les graphes a-parfaits, qui contiennent tous les graphes parfaits. En étudiant leur ratio d'imperfection, nous avons établi que le ratio d'imperfection des graphes quasi-adjoints est inférieur à 3/2. En ce ce sens, les graphes quasi-adjoints sont proches des graphes parfaits d'un point de vue polyédral, ce qui n'est pas le cas des graphes sans griffe en général, puisque leur ratio d'imperfection est non-borné (partie 3.3).

Les facettes des polytopes des stables des graphes sans griffe avec un nombre de stabilité égal à trois sont particulièrement mal comprises : nous avons proposé une nouvelle approche qui nous a permis, en nous appuyant sur les solveurs de la programmation mathématique, de découvrir des facettes bien plus "exotiques" encore (partie 4.2) ! Il semble donc que l'on soit bien loin de proposer une description complète des facettes du polytope des stables des graphes sans griffe. Nous avons alors proposé une extension des inégalités des familles de cliques à même d'unifier toutes les facettes connues (partie 4.3). Néanmoins, cela ne fait qu'amplifier les inconnues du cas quasi-adjoint : en effet, si Eisenbrand, Oriolo, Stauffer et Ventura, en s'appuyant sur des travaux de Chudnovsky et Seymour, ont réussi à prouver la conjecture de Ben Rebea, à savoir que toutes les facettes du polytope des stables des graphes quasi-adjoints sont des inégalités de familles de cliques, l'ensemble des inégalités de familles de cliques pour un graphe donné est infini (puisque les cliques d'une famille ne sont pas nécessairement distinctes). Ainsi tout l'enjeu reste de déterminer lesquelles de ces inégalités sont essentielles ...

Pour clore ce document, nous terminons sur une note plus positive en présentant quelques résultats préliminaires (avec leurs preuves) quant au calcul en temps polynomial du nombre circulaire-chromatique des graphes circulaires-parfaits et au calcul du nombre de stabilité de graphes quasi-adjoints via la voie polyédrale. Tout repose sur l'introduction d'un nouveau polytope construit à partir des cliques circulaires ...

Ainsi dans la première partie, nous introduisons le polytope des cliques circulaires, un polytope dont les points extrémaux sont en correspondance avec les cliques circulaires induites d'un graphe, et nous montrons à l'aide de celui-ci et le résultat de Grötschel, Lovász et Schrijver déjà mentionné, que le nombre circulaire-chromatique d'un graphe fortement circulaire-parfait est calculable en temps polynomial.

La seconde partie détaille une autre application, plus inattendue, de ce polytope : en le combinant avec la caractérisation des facettes du polytope des stables d'un graphe proche-biparti de Shepherd [102], nous avons pu établir que le nombre de stabilité d'un graphe circulaire d'intervalles flou est calculable en temps polynomial. Il s'agit de la première preuve de ce résultat de nature polyédrale, mais elle est limitée au cas non-pondéré. Conjointement avec la caractérisation des facettes du polytope des stables des graphes semi-adjoints par Chudnovsky et Seymour, ceci couvre donc tous les graphes quasi-adjoints. Cependant, pour la version pondérée, une compréhension plus fine du polytope des stables reste nécessaire.

5.1 Perspectives : le polytope des cliques circulaires

X. Zhu a exhibé la propriété suivante des graphes circulaires-parfaits, qui est également une conséquence directe du théorème fort des graphes parfaits :

Théorème 5.1 (Zhu (2005) [115])**.** *Si G est un graphe circulaire-parfait, alors pour tout sommet v, le graphe induit par son voisinage est parfait.*

Puisque toute clique est nécessairement contenue dans le voisinage d'un des sommets du graphe, nous avons naturellement :

Corollaire 5.2. *Si G est un graphe circulaire-parfait alors pour toute pondération rationnelle $w : V \to \mathbb{Q}_+^V$ de ses sommets, son nombre de clique pondéré $\omega(G, w)$ est calculable en temps polynomial.*

Ainsi comme un graphe circulaire-parfait G est toujours $\omega(G) + 1$ coloriable, pour calculer $\chi(G)$ en temps polynomial, il suffirait de tester si $\chi(G) = \omega(G)$ en temps polynomial, ce qui est un problème ouvert pour les graphes circulaires-parfaits.

L'objet de cette section est de prouver que le nombre circulaire-chromatique est calculable en temps polynomial pour les graphes fortement circulaires-parfaits. La preuve est basée sur une approche polyédrale, dont l'intuition repose sur un résultat remarquable de Padberg :

Théorème 5.3 (Padberg (1974) [74]). *Un graphe G est minimal imparfait si et seulement si son polytope des contraintes des cliques QSTAB(G) a un unique point extrême non-entier, le point $1/\omega(G)\mathbf{1}$.*

Pour présenter les résultats de ce chapitre, il est convénient d'utiliser le polytope des stables fractionnaire, i.e. le polytope des contraintes des cliques du complémentaire :

Définition 5.4 (Polytope des stables fractionnaire SCLI(G) d'un graphe G).

$$\begin{aligned} \mathit{SCLI}(G) &= \left\{ x \in \mathbb{R}_+^V : \sum_{i \in S} x_i \leq 1, S \text{ stable de } G \right\} \\ &= \mathit{QSTAB}(\overline{G}) \end{aligned}$$

De manière analogue, nous définissons le polytope des cliques d'un graphe comme le polytope des stables de son complémentaire :

Définition 5.5 (polytope des cliques CLI(G) d'un graphe G). *Enveloppe convexe de tous les vecteurs d'incidence des cliques du graphe G :*

$$\begin{aligned} \mathit{CLI}(G) &= \mathit{conv}\,\{\chi_H : H \text{ est une clique de } G\} \\ &= \mathit{STAB}(\overline{G}) \end{aligned}$$

En combinant le théorème 5.3 avec le théorème des graphes parfaits de Lovász, nous avons donc :

Corollaire 5.6. *Un graphe G est minimal imparfait si et seulement si son polytope des stables fractionnaire SCLI(G) a un unique point extrême non-entier, le point $1/\alpha(G)\mathbf{1}$.*

Pour un trou impair ou un antitrou impair, les cliques circulaires induites sont les cliques et le graphe tout entier. Ainsi le corollaire 5.6 implique que le polytope des stables fractionnaire d'un trou impair ou un antitrou impair G est l'enveloppe convexe des vecteurs $1/\alpha(H)\chi_H$, pour toutes les cliques circulaires induites H de G. C'est cette observation qui nous a amenés à introduire le polytope des cliques circulaires :

Définition 5.7 (polytope des cliques circulaires CLI$_c(G)$ d'un graphe G). *Enveloppe convexe de tous les vecteurs d'incidence des cliques circulaires multipliés par l'inverse de leurs nombres de stabilité :*

$$\mathit{CLI}_c(G) = \mathit{conv}\,\{1/\alpha(H)\chi_H : H \text{ est une clique circulaire première induite de } G\}$$

Pour tout graphe G, il résulte de ces définitions, la chaîne d'inclusion

$$\mathrm{CLI}(G) \subseteq \mathrm{CLI}_c(G) \subseteq \mathrm{SCLI}(G) \tag{5.1}$$

A partir du polytope des cliques circulaires, il est aisé de définir le nombre pondéré de clique circulaire :

Définition 5.8 (nombre de clique circulaire pondéré $\omega_c(G, w)$ du graphe G pour une pondération w). *Etant donné un graphe $G = (V, E)$ et une pondération $w : V \rightarrow \mathbb{R}_+^V$ de ses sommets, le nombre de clique pondéré $\omega(G, w)$ est défini par :*

$$\omega(G, w) = \max\{\mathbf{w}.\mathbf{x} : \mathbf{x} \in \mathit{CLI}(G)\}$$

et le nombre de clique circulaire pondéré $\omega_c(G, w)$ est donné par :

$$\omega_c(G, w) = \max\{\mathbf{w}.\mathbf{x} : \mathbf{x} \in \mathit{CLI}_c(G)\}$$

Dans le premier chapitre, nous avons donné une définition du nombre fractionnaire chromatique $\chi_f(G)$ d'un graphe G. Il est bien connu que ce paramètre peut être calculé via un programme linéaire. En appliquant le théorème de dualité forte, nous avons alors une expression du nombre fractionnaire chromatique naturelle basée sur le polytope des stables fractionnaire. Dans ce contexte, le nombre fractionnaire chromatique d'un graphe G est le plus souvent appelé nombre fractionnaire de clique et noté $\omega_f(G)$:

Définition 5.9 (nombre de clique fractionnaire pondéré $\omega_f(G,w)$ du graphe G pour une pondération w / nombre chromatique fractionnaire pondéré $\omega_f(G,w)$ du graphe G pour une pondération w **).** *Etant donné un graphe $G=(V,E)$ et une pondération $w : V \to \mathbb{R}_+^V$ de ses sommets, le nombre de clique fractionnaire pondéré $\omega_f(G,w)$ est défini par :*

$$\begin{aligned}\omega_f(G,w) &= \max\{\mathbf{w}.\mathbf{x} : \mathbf{x} \in \mathit{SCLI}(G)\}\\ &= \min\left\{\sum_{S\in\mathcal{S}} \Delta(S) : \left(\Delta : \mathcal{S} \to \{0,1\} \textit{ telle que } \forall v \in V, \sum_{S\in\mathcal{S}, v\in S} \Delta(S) \geq w(v)\right)\right\}\\ &= \chi_f(G,w)\end{aligned}$$

Lemme 5.10. *Pour tout graphe $G=(V,E)$ et toute pondération $w : V \to \mathbb{R}_+^V$ de ses sommets, nous avons la chaîne d'inégalités*

$$\omega(G,w) \leq \omega_c(G,w) \leq \omega_f(G,w) = \chi_f(G,w) \tag{5.2}$$

En outre, nous avons

$$\chi_f(G) = \omega_f(G,1) \leq \chi_c(G) \leq \chi(G) \tag{5.3}$$

Démonstration. Cela résulte directement des définitions et de l'inégalité 5.1. ■

Observation : des versions pondérées du nombre circulaire chromatique et du nombre chromatique ont été étudiées dans la littérature. En particulier, Deuber et Zhu ont observé que $\omega_f(G,w) \leq \chi_c(G,w) \leq \chi(G,w)$ pour tout graphe G et toute pondération positive w [33], de sorte que nous avons en fait la chaîne d'inégalité :

$$\omega(G,w) \leq \omega_c(G,w) \leq \omega_f(G,w) = \chi_f(G,w) \leq \chi_c(G,w) \leq \chi(G,w)$$

Cependant, la preuve n'est pas donnée explicitement dans l'article de Deuber et Zhu et la version plus faible donnée dans le lemme 5.10 suffit à notre propos.

Rappelons brièvement le théorème d'équivalence de la complexité de la séparabilité des facettes de Grötschel, Lovász et Schrijver :

Théorème 5.11 (Grötschel, Lovász et Schrijver (1988) [56]**).** *Soit $\mathbf{Ax} \leq \mathbf{b}$ un système de contraintes linéaires. Si $\mathbf{v}$ est un vecteur, $<\mathbf{v}>$ désigne le nombre de bits utilisés pour stocker $\mathbf{v}$ en mémoire. Alors le problème d'optimisation $\max\{\mathbf{c}.\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}\}$ est résoluble en temps polynomial en n (où n est égal au minimum des $<\mathbf{a_i}> + <\mathbf{b_i}> + <\mathbf{c}>$ pris sur les lignes du système) si et seulement si, le problème de séparation, i.e. déterminer pour tout vecteur $\mathbf{y}$ si $\mathbf{Ay} \leq \mathbf{b}$ et dans le cas négatif exhiber une ligne $\mathbf{a_i}$ de $\mathbf{A}$ telle que $\mathbf{a_i}.\mathbf{x} > b_i$, est résoluble en temps polynomial en n' (où n' est égal au minimum des $<\mathbf{a_i}> + <\mathbf{b_i}> + <\mathbf{y}>$ pris sur les lignes du système).*

Nous pouvons maintenant énoncer le principal résultat de cette partie :

Théorème 5.12. *Si $G=(V,E)$ est un graphe tel que pour tout sommet v, le graphe induit $G-N(v)$ est parfait, alors pour toute pondération rationnelle $w : V \to \mathbb{Q}_+^V$ de ses sommets, le nombre fractionnaire de clique pondéré $\omega_f(G,w)$ de G est calculable en temps polynomial.*

Démonstration. Nous avons $\omega_f(G,w) = \max\{\mathbf{w}.\mathbf{x} : \mathbf{x} \in \mathrm{SCLI}(G)\}$ par définition. D'après le théorème 5.11, $\omega_f(G,w)$ est calculable en temps polynomial si et seulement si les facettes de SCLI(G) sont séparables en temps polynomial. Soit donc $\mathbf{y}$ un vecteur de $\mathbb{Q}_+^V$. Pour tout sommet v, soit $\alpha_{v,y}$ le nombre de stabilité de $G-N(v)$ pour la pondération de ses sommets donnée par y : $\alpha_{v,y}$ est calculable en temps polynomial puisque $G-N(v)$ est parfait. S'il existe un sommet v tel que $\alpha_{v,y} > 1$ alors $\mathbf{y} \notin$ SCLI(G) et on peut exhiber en temps polynomial un stable maximal S de $G-N(v)$ tel que $\sum_S y_i > 1$. Si pour tout sommet v, $\alpha_{v,y} \leq 1$ alors $\mathbf{y} \in$ SCLI(G) puisque tout stable maximal de G est contenu dans le non-voisinage d'un des sommets de G. ■

Ainsi si G est un graphe fortement circulaire-parfait, alors pour tout sommet v, le graphe induit $G - N(v)$ est parfait (Théorème 5.1), et comme $\chi_c(G) = \chi_f(G) = \omega_f(G, \mathbf{1})$, nous obtenons :

Théorème 5.13. *Si G est un graphe fortement circulaire-parfait alors son nombre chromatique $\chi(G)$ et son nombre circulaire-chromatique $\chi_c(G)$ sont calculables en temps polynomial.*

W. Deuber et X. Zhu ont observé qu'un graphe G est parfait si et seulement si pour tout graphe $G = (V, E)$ et toute pondération $w : V \to \mathbb{R}_+^V$ de ses sommets, nous avons $\omega(G, w) = \omega_f(G, w)$ [33]. Compte tenu de la chaîne d'inégalités 5.2, cela amène la question suivante :

Question 5.14. *Un graphe G est-il circulaire-parfait si et seulement si pour tout graphe $G = (V, E)$ et toute pondération $w : V \to \mathbb{R}_+^V$ de ses sommets, nous avons $\omega_c(G, w) = \omega_f(G, w)$? Autrement dit, un graphe G est-il circulaire-parfait si et seulement si CLI$_c(G)$ = SCLI(G) ?*

Il est facile de voir que malheureusement la réponse est négative :

Lemme 5.15. *Pour tout graphe G, CLI$_c(G)$ = SCLI(G) si et seulement si G est a-parfait.*

Démonstration. Si Q est un polytope de $\mathbb{R}_+^n$ tel que pour tout $x \in Q$, $\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}$ implique $y \in Q$, nous notons Q^* son antibloqueur :

$$Q^* \quad = \quad \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}.\mathbf{y} \leq 1, \forall \mathbf{y} \in Q\}$$

Un résultat classique (voir [56] par exemple) est que cet opérateur est une involution et est donc injectif

$$Q^{**} \quad = \quad Q$$

Notons ASTAB(G) le polytope délimité par les contraintes des cliques circulaires induites premières d'un graphe G :

$$\text{ASTAB}(G) \quad = \quad \left\{ \mathbf{x} \in \mathbb{R}_+^V : \sum_W x_v \leq \alpha(W),\, W \text{ est une clique circulaire induite première de } G \right\}$$

Il est bien connu que pour tout graphe G, STAB$(G)^*$ = SCLI(G), soit encore STAB(G) = SCLI$(G)^*$. Par définition, CLI$_c(G)^*$ = ASTAB(G), et donc G est a-parfait si et seulement si CLI$_c(G)$ = SCLI(G) .

■

En particulier, il existe des graphes fortement circulaires-parfaits G tels que CLI$_c(G) \subsetneq$ SCLI(G) (cf chapitre 2). Ceci signifie notamment qu'il existe des graphes fortement circulaires-parfaits et des pondérations w tels que $\omega_c(G, w) < \omega_f(G, w)$.

5.2 Une conséquence pour les graphes quasi-adjoints

Rappelons qu'un graphe proche-biparti est le complémentaire d'un graphe quasi-adjoint. Le polytope des stables des graphes proche-biparti a été étudié par Shepherd (cf théorème 3.23).

Dans le cas des graphes circulaires d'intervalle flous, Wagler a renforcé la caractérisation des facettes due à Shepherd :

Théorème 5.16 (Wagler (2005) [110]). *Tout complémentaire d'un graphe circulaire d'intervalles flou est a-parfait.*

Si G est un graphe quasi-adjoint alors le voisinage de tout sommet est parfait, puisque ce voisinage est le complémentaire d'un graphe biparti. Il résulte de cette observation que le théorème 5.12 s'applique aux graphes proche-bipartis. En combinant cette observation avec le théorème 5.16, nous obtenons donc le résultat suivant :

Théorème 5.17. *Si G est un graphe circulaire d'intervalle flou alors STAB$_c(G)$ = QSTAB(G), et $\alpha_c(G)$, $\alpha(G)$ sont calculables en temps polynomial.*

Ainsi pour un graphe circulaire d'intervalle flou, les points extrêmes de QSTAB(G) sont donnés par les webs induits, et cette connaissance permet de calculer effectivement le nombre de stabilité de G : en effet, si le théorème 5.12 ne fournit pas d'algorithme combinatoire explicite pour calculer le nombre de clique fractionnaire puisqu'il fait appel aux graphes parfaits en général, dans le cas des graphes proche-biparti, la preuve revient à résoudre n problèmes de stables de poids maximal dans des graphes bipartis, ce qui se fait en $O(n^4)$ [39] (n étant naturellement le nombre de sommets).

Ceci constitue donc une approche polyédrale remarquablement simple pour calculer le nombre de stabilité d'un graphe circulaire d'intervalle flou, i.e. la famille des graphes quasi-adjoints la plus retorse pour la conjecture de Ben Rebea ! Il n'est pas possible par ce biais de calculer le nombre de stabilité d'un graphe quasi-adjoint en général car il est facile de voir que le nombre de clique fractionnaire peut être arbitrairement plus grand que le nombre de clique pour un graphe proche-biparti.

Concluons sur une dernière question ouverte : est-il possible de donner une autre preuve de la conjecture de Ben Rebea basée sur la connaissance des points extrêmes de QSTAB(G) d'un graphe circulaire d'intervalle flou G ?

Remerciements

Rétrospectivement, mes travaux post-doctoraux sont en quelque-sorte un retour aux sources : en effet, lorsque j'ai commencé ma thèse sous la direction d'Henri Thuillier à Orléans, j'étais néophyte en théorie des graphes, et Henri m'a rapidement mis au fait de la théorie des graphes parfaits. C'est à cette occasion que j'ai découvert les travaux de Grötschel, Lovász et Schrijver. Mes premières tentatives sur le polytope des stables ont donc eu lieu pendant la première année de ma thèse ; à l'issue de cette première année, j'ai du me résoudre à reconnaître que ce n'était pas chose aisée, que de poursuivre le travail de ces trois "géants", et je me suis réorienté vers une branche plus "ouverte" : une approche algébrique (par pavages des groupes finis) des graphes partitionnables.

J'ai ensuite enchainé par deux postdocs courts : le premier a eu lieu au IASI, à Rome, auprès de Anna Galluccio et Antonio Sassano. Bien qu'ils soient les auteurs d'un résultat de premier ordre sur le polytope des stables des graphes sans griffe, nous nous sommes en fait intéressés à autre chose : les colorations circulaires. Il faut reconnaître que cela n'a pas été très fructueux à cette époque ... A la fin de mon postdoc, j'ai rencontré Gianpaolo Oriolo, et le polytope des stables s'est à nouveau invité dans mes recherches ...

Mon second postdoc a eu lieu au ZIB, à Berlin, auprès d'Annegret Wagler et de Martin Grötschel. Comme je connaissais bien les récents travaux de Gianpaolo sur le polytope des stables des graphes quasi-adjoints, nous avons regardé de plus près le cas des webs, qui m'étaient familiers en tant que graphes 'cibles' pour les homomorphismes des colorations circulaires. Nous pensions que ceux-ci devaient être une des familles les plus simples des graphes quasi-adjoints : une intuition qui s'est révélée on ne peut plus fausse, comme je l'ai expliqué dans ce document ! Nous avons alors entamé une collaboration continue qui s'est révélée très productive.

Le fil de mes recherches s'est ensuite déroulé sans heurts : après la prise de mes fonctions, j'ai rencontré Xuding Zhu, invité par André Raspaud au LaBRI, et nous avons naturellement étudié ensemble les colorations circulaires et les graphes circulaires-parfaits. C'est un honneur et un grand plaisir pour moi de pouvoir collaborer avec un chercheur aussi fort et accueillant !

En conclusion, *tout* ce que je sais des graphes parfaits m'a été enseigné par Henri Thuillier, et je ne l'en remercierai jamais assez. Mes deux postdocs ont été deux périodes de ma vie professionnelle réellement merveilleux, et constituent le fondement de tous les travaux que j'ai présentés ici : qu'Anna Galluccio et Annegret Wagler en soient tous particulièrement remerciées ! A l'issue de ces postsdocs, j'ai eu la chance d'obtenir un poste de maître de conférence dans un laboratoire très dynamique, le LaBRI. Ceci est donc l'occasion de remercier tous mes collègues du LaBRI et aussi de la jeune équipe INRIA RealOpt. Je n'oublie pas bien sûr ma petite famille, Séverine, Mattéo et Alexian, mes parents, qui ont tous contribué à leur manière à ce document. Et pour conclure, je remercie chaleureusement tous les membres de mon jury ; je suis très honoré que Robert Cori, Frédéric Maffray et Ridha Mahjoub m'aient fait la joie d'accepter d'évaluer ce manuscript, tant ce sont des références dans les domaines abordés par ce mémoire. Et je termine par une mention spéciale à mon "mentor", André Raspaud, exemplaire à tout point de vue, qui n'a pas ménagé sa peine pour m'embarquer dans ses tribulations diverses et variées !

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Bibliographie

[1] G. Bacsó, E. Boros, V. Gurvich, F. Maffray, and M. Preissmann. On minimally imperfect graphs with circular symmetry. *J. of Graph Theory*, 29 :209–225, 1998.

[2] J. Bang-Jensen and J. Huang. Convex-round graphs are circular-perfect. *J. of Graph Theory*, 40 :182–184, 2002.

[3] C. Berge. *Théorie des graphes et ses applications*. Dunod, 1958.

[4] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[5] N. Biggs. *Algebraic Graph Theory*. Syndics of the Cambridge University Press, 1974.

[6] R.G. Bland, H.C. Huang, and L.E. Trotter. Graphical properties related to minimal imperfection. *Discrete Mathematics*, 27 :11–22, 1979.

[7] J.A. Bondy and P. Hell. A note on the star chromatic number. *J. of Graph Theory*, 14 :479–482, 1990.

[8] E. Boros, V. Gurvich, and S. Hougardy. Recursive generation of partitionable graphs. *J. of Graph Theory*, 41(4) :259–285, 2002.

[9] D. De Caen, D.A. Gregory, I.G. Hughes, and D.L. Kreher. Near-factors of finite groups. *Ars Combinatoria*, 29 :53–63, 1990.

[10] G. Chartrand, D. Geller, and S. Hedetniemi. Graphs with forbidden subgraphs. *J. Combinatorial Theory Ser. B*, 10 :12–41, 1971.

[11] E. Cheng and S. de Vries. Antiweb inequalities : Strength and intractability. *Congr. Numeration*, 152 :5–19, 2001.

[12] T. Christof and A. Löbel. Porta software. http ://www.zib.de.

[13] M. Chudnovsky, G. Cornuéjols, X. Liu, P.D. Seymour, and K. Vušković. Recognizing berge graphs. *Combinatorica*, 25 :143–186, 2005.

[14] M. Chudnovsky and A. Ovetsky. Coloring quasi-line graphs. *J. Graph Theory*, 54(1) :41–50, 2007.

[15] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1) :51–229, 2006.

[16] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[17] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In B.S. Webb, editor, *Surveys in combinatorics*, volume 327 of *London. Math. Soc. Lecture Notes*, pages 153–171, 2005.

[18] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4 :305–337, 1973.

[19] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18 :138–154, 1975.

[20] V. Chvátal, W. Cook, and M. Hartmann. On cutting-plane proofs in combinatorial optimization. *Linear Algebra and its Applications*, 114/115 :455–499, 1989.

[21] V. Chvátal, R.L. Graham, A.F. Perold, and S.H. Whitesides. Combinatorial designs related to the perfect graph conjecture. *Applied Discrete Mathematics*, 21 :197–206, 1984.

[22] V. Chvátal and N. Sbihi. Recognizing claw-free perfect graphs. *J. Combin. Theory Ser. B*, 44(2) :154–176, 1988.

[23] W. Cook. private communication. 1990.

[24] S. Coulonges, A. Pêcher, and A. Wagler. On stable set polytopes of circular-perfect graphs. In *5th ALIO/EURO conference on combinatorial optimization*, oct 2005.

[25] S. Coulonges, A. Pêcher, and A. Wagler. On strongly circular-perfectness. In *VIIth International Conference on Graph Theory*, volume 22 of *Electonic Notes in Discrete Mathematics*, pages 369–374. Elsevier, oct 2005.

[26] S. Coulonges, A. Pêcher, and A. Wagler. On the imperfection ratio of a-perfect graphs and some consequences. In *CTW05 - Cologne-Twente Workshop on Graphs and Combinatorial Optimization*, mai 2005.

[27] S. Coulonges, A. Pêcher, and A. Wagler. Characterizing and bounding the imperfection ratio for some classes of graphs. *Mathematical Programming A*, in press.

[28] S. Coulonges, A. Pêcher, and A. Wagler. Triangle-free strongly circular-perfect graphs. *Discrete Mathematics*, in press.

[29] I. Cziszár, J. Körner, L. Lovász, K. Marton, and G. Simonyi. Entropy splitting for antiblocking corners and perfect graphs. *Combinatorica*, 10 :27–40, 1990.

[30] G. Dahl. The 2-hop spanning tree problem. *Operations Research Letters*, 23 :21–26, 1998.

[31] G. Dahl. Stable set polytopes for a class of circulant graphs. *SIAM Journal on Optimization*, 9(2) :493–503, 1998.

[32] N.G. DeBruijn. On number systems. *Nieuw Arch. Wisk.*, 3(4) :15–17, 1956.

[33] W. Deuber and X. Zhu. Circular coloring of weighted graphs. *J. Graph Theory*, 23 :365–376, 1996.

[34] J.E. Edmonds. Maximum matching and a polyhedron with (0,1) vertices. *J. Res. Nat. Bur. Standards*, 69(B) :125–130, 1965.

[35] J.E. Edmonds. Paths, trees, and flowers. *Canad. J. Math.*, 17 :449–467, 1965.

[36] J.E. Edmonds and W.R. Pulleyblank. Facets of 1-matching polyhedra. In C. Berge and D.R. Chuadhuri, editors, *Hypergraph Seminar*, pages 214–242. Springer, 1974.

[37] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. Circular one matrices and the stable set polytope of quasi-line graph. *Lecture Notes in Computer Science*, 3509 :291–305, 2005.

[38] P. Erdös, A.L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, Congr. Num. 26*, pages 125–157, 1979.

[39] Ulrich Faigle and Gereon Frahling. A combinatorial algorithm for weighted stable sets in bipartite graphs. *Discrete Applied Mathematics*, 154(9) :1380–1391, 2006.

[40] R. Faudree, E. Flandrin, and Z. Ryjacek. Claw-free graphs – A survey. *Discrete Mathematics*, 164 :87–147, 1997.

[41] J. L. Fouquet. A strengthening of Ben Rebea's lemma. *J. Comb. Theory Ser. B*, 59(1) :35–40, 1993.

[42] D.R. Fulkerson. Anti-blocking polyhedra. *J. of Combinatorial Theory B*, 12 :50–71, 1972.

[43] A. Galluccio, C. Gentile, and P. Ventura. Gear composition of stable set polytopes and g-perfection. *Operations Research Letters*, to appear.

[44] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1) :1–38, 1997.

[45] Guogang Gao and Xuding Zhu. Star-extremal graphs and the lexicographic product. *Discrete Math.*, 152(1-3) :147–156, 1996.

[46] G.S. Gasparian. Minimal imperfect graphs : a simple approach. *Combinatorica*, 16(2) :209–212, 1996.

[47] S. Gerke and C. McDiarmid. Graph imperfection. *J. of Combinatorial Theory B*, 83 :58–78, 2001.

[48] S. Gerke and C. McDiarmid. Graph imperfection ii. *J. of Combinatorial Theory B*, 83 :59–101, 2001.

[49] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31 :313–326, 1981.

[50] R. Giles, L.E. Trotter jr, and A. Tucker. The strong perfect graph theorem for a class of partitionable graphs. *Annals of Discrete Mathematics*, 21 :161–167, 1984.

[51] R.E. Gomory. Outline of an algorithm for integer solutions to linear programs. *Bul. of the American Math. Soc.*, 64 :275–278, 1958.

[52] D. Gonçalves. Edge partition of planar graphs into two outerplanar graphs. In *Proceedings of STOC '05*, pages 504–512, 2005.

[53] C.M. Grinstead. On circular critical graphs. *Discrete Mathematics*, 51 :11–24, 1984.

[54] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2) :169–197, 1981.

[55] M. Grötschel, L. Lovász, and A. Schrijver. *Topics on perfect graphs*, volume 88, chapter Polynomial algorithms for perfect graphs, pages 325–356. North-Holland Math. Stud., 1984.

[56] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[57] I. Holyer. The NP-completeness of edge-colouring. *SIAM J. Comput.*, 10(4) :718–720, 1981.

[58] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12 :373–388, 1975.

[59] A. Kemnitz and P. Wellmann. Circular chromatic number of certain planar graphs and small graphs. *Congressus Numerantium*, 169 :199–209, 2004.

[60] J. Kind. *Mobilitätsmodelle für zellulare Mobilfunknetze : Produktformen und Blockierung*. PhD thesis, RWTH Aachen, 2000.

[61] J. Körner. An extension of the class of perfect graphs. *Studia Math. Hung.*, 8 :405–409, 1973.

[62] J. Körner and K. Marton. Graphs that split entropies. *SIAM J. Discrete Math.*, 1 :71–79, 1988.

[63] J. Kuhpfahl, A. Wagler, and C. Wagner. Circular-imperfection of triangle free graphs. manuscript, 2007.

[64] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1) :25–35, 2004.

[65] K.-W. Lih, D. Liu, and X. Zhu. Star extremal circulant graphs. *SIAM J. Discrete Math.*, 12(4) :491–499 (electronic), 1999.

[66] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13 :95–98, 1972.

[67] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2 :253–267, 1972.

[68] L. Lovász. On the shannon capacity of a graph. *Trans. Inform. Theory*, 25(1) :1–7, January 1979.

[69] G.J. Minty. On maximal independent sets of vertices in claw-free graphs. *J. Comb. Theory Series B*, 28 :284–304, 1980.

[70] B. Monien. The complexity of determining a shortest cycle of even length. *Comput.*, 31 :355–369, 1983.

[71] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3 :161–162, 1955.

[72] D. Nakamura and A. Tamura. A revision of Minty's algorithm for finding a maximum weighted stable set of a claw-free graph. *Journal of the OR Society of Japan*, 44 :194–204, 2001.

[73] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[74] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6 :180–196, 1974.

[75] M.W. Padberg. Almost integral polyhedra related to certain combinatorial optimization problems. *Linear Algebra and its Applications*, 15 :69–88, 1976.

[76] Z. Pan and X. Zhu. Minimal circular-imperfect graphs of large clique number and large independence number. manuscript, 2006.

[77] K.R. Parthasarathy and G. Ravindra. The strong perfect graph conjecture is true for $k_{1,3}$-free graphs. *J. Combin. Theory Ser. B*, 21 :212–223, 1976.

[78] A. Pêcher. *Graphes de Cayley partitionnables*. PhD thesis, Université d'Orléans, 2000.

[79] A. Pêcher. Partitionable graphs arising from near-factorizations of finite groups. *Discrete Mathematics*, 269(1-3) :191–218, 2003.

[80] A. Pêcher. Cayley partitionable graphs and near-factorizations of finite groups. *Discrete Mathematics*, 276(1-3) :295–311, 2004.

[81] A. Pêcher, P. Pesneau, and A. Wagler. On facets of stable set polytope of claw-free graphs with maximum stable set size three. In *CS06 - Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications*, jul 2006.

[82] A. Pêcher and A. Wagler. A conjecture on the stable set polytope for claw-free graphs. Preprint 11-2006, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik.

[83] A. Pêcher and A. Wagler. Almost all webs are not rank-perfect. *Mathematical Programming B*, 105(2-3) :311–328, 2006.

[84] A. Pêcher and A. Wagler. A construction for non-rank facets of stable set polytopes of webs. *European Journal of Combinatorics*, 27(7) :1172–1185, 2006.

[85] A. Pêcher and A. Wagler. Generalized clique family inequalities for claw-free graphs. In Elsevier, editor, *CTW06 - Cologne-Twente Workshop on Graphs and Combinatorial Optimization*, volume 25, pages 117–121. Electronic Notes in Discrete Mathematics, jun 2006.

[86] A. Pêcher and A. Wagler. On non-rank facets in stable set polytopes of webs with clique number four. *Discrete Applied Mathematics*, 154(9) :1408–1415, jun 2006.

[87] A. Pêcher and A. Wagler. Results and conjectures on the stable set polytope of claw-free graphs. *19th International Symposium on Mathematical Programming, Rio de Janeiro, Brazil*, jul 2006.

[88] A. Pêcher and A. Wagler. A note on the Chvátal rank of clique family inequalities. *RAIRO - Operations Research*, 41 :289–294, 2007.

[89] A. Pêcher and A. Wagler. On classes of minimal circular-imperfect graphs. *Discrete Applied Mathematics*, 156 :998–1010, 2008.

[90] A. Pêcher and A. Wagler. On facets of stable set polytope of claw-free graphs with maximum stable set size three. *Disc. Appl. Math.*, to appear.

[91] A. Pêcher, A. Wagler, and X. Zhu. Three classes of minimal circular imperfect graphs. In *Proceedings of GRACO05*, volume 19 of *Electonic Notes in Discrete Mathematics*, pages 9–15. Elsevier, avril 2005.

[92] A. Pêcher and X. Zhu. On the circular chromatic number of circular partitionable graphs. *Journal of Graph Theory*, 52(4) :294–306, aug 2006.

[93] A. Pêcher and X. Zhu. Claw-free circular-perfect graphs. In *Eurocomb07 - European Conference on Combinatorics, Graph Theory and Applications*, Electonic Notes in Discrete Mathematics. Elsevier, sep 2007.

[94] J. Ramirez-Alfonsin and B. Reed, editors. *Perfect Graphs*. Wiley, 2001.

[95] A. Ben Rebea. *Etude des stables dans les graphes quasi-adjoints*. PhD thesis, Grenoble, 1981.

[96] F. Roussel and P. Rubio. About skew partitions in minimal imperfect graphs. *J. Comb. Theory, Ser. B*, 83(2) :171–190, 2001.

[97] N. Sbihi. Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile. *Disc. Math.*, 29 :53–76, 1980.

[98] A. Sebő. The connectivity of minimal imperfect graphs. *J. of Graph Theory*, 23(1) :77–85, 1996.

[99] A. Sebő. On critical edges in minimal imperfect graphs. *J. of Comb. Theory*, 67 :62–85, 1996.

[100] C. E. Shannon. The zero-error capacity of a noisy channel. *IRE Trans. Inform. Theory*, 2 :8–19, 1956.

[101] F.B. Shepherd. Near-perfect matrices. *Mathematical Programming*, 64 :295–323, 1994.

[102] F.B. Shepherd. Applying Lehman's theorem to packing problems. *Mathematical Programming*, 71 :353–367, 1995.

[103] C. De Simone and J. Körner. On the odd cycles of normal graphs. *Discrete App. Math.*, 94 :161–169, 1999.

[104] G. Stauffer. *On the Stable Set Polytope of Claw-free Graphs*. PhD thesis, EPF Lausanne, 2005.

[105] Alan Tucker. Critical perfect graphs and perfect 3-chromatic graphs. *J. Combinatorial Theory Ser. B*, 23(1) :143–149, 1977.

[106] A. Vince. Star chromatic number. *J. of Graph Theory*, 12 :551–559, 1988.

[107] V.G. Vizing. Vertex colorings with given colors (in russian). *Metody Diskret. Analiz.*, 29 :3–10, 1976.

[108] A. Wagler. Rank-perfect and weakly rank-perfect graphs. *Mathematical Methods of Operations Research*, 95 :127–149, 2002.

[109] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2 :149–152, 2004.

[110] A. Wagler. On rank-perfect subclasses of near-bipartite graphs. *4OR*, pages 329–336, 2005.

[111] B. Xu. On minimally circular-imperfect graphs. manuscript.

[112] B. Xu. Some results on circular perfect graphs and perfect graphs. *J. Syst. Sci. Complex.*, 18 :167–173, 2005.

[113] X. Zhu. Circular chromatic number : a survey. *Discrete Mathematics*, 229 :371–410, 2001.

[114] X. Zhu. Perfect graphs for generalized colouring—circular perfect graphs. In *Graphs, morphisms and statistical physics*, volume 63 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 177–193. Amer. Math. Soc., Providence, RI, 2004.

[115] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3) :186–209, 2005.

[116] X. Zhu. Recent developments in circular colouring of graphs. *Topics in Discrete Mathematics*, pages 497–550, 2006.

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HABILITATION À DIRIGER DES RECHERCHES

22 octobre 2008

au titre de l'École doctorale de Mathématiques et Informatique de Bordeaux

par Monsieur Arnaud PÊCHER

Des multiples facettes des graphes circulants
Annexes

– Curriculum Vitæ
– Articles parus
– Articles acceptés
– Manuscripts

devant la commission d'examen formée de :

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Annexe A

Curriculum Vitæ

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Juin 2002 - Septembre 2002	Postdoc au Z.I.B., Berlin, Allemagne
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- Pêcher, *Cayley partitionable graphs*, *Discrete Mathematics* (269), pages 191-218, 2003
- Pêcher, *Partitionable graphs arising from near-factorizations of finite groups*, *Discrete Mathematics* (276), pages 295-311, 2004
- Pêcher, Wagler, *Almost all webs are not rank-perfect*, *Mathematical Programming* (105), pages 311-328, 2006
- Pêcher, Wagler, *A construction for non-rank facets of stable set polytopes of webs*, *European Journal of Combinatorics* (27), no 7, pages 1172-1185, 2006
- Pêcher, Zhu, *On the Circular Chromatic Number of Circular Partitionable Graphs*, *Journal of Graph Theory* (52), no 4, pages 294-306, 2006

Responsabilités administratives

- directeur des études de la MIAGE (2006-2008) : prime de responsabilités pedagogiques
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Domaine de recherche

J'ai un poste de Maître de Conférences au LaBRI (Bordeaux), au sein de l'équipe Graphes et Applications, depuis septembre 2002. Mes activités de recherche s'inscrivent dans deux axes principaux :

– axe *colorations circulaires de graphes* : le nombre circulaire chromatique d'un graphe est un raffinement du nombre chromatique usuel, qui possède des applications pour les problèmes d'allocation de ressources, d'ordonnancement de processus ... En 2001, Xuding Zhu a introduit une extension de la célèbre classe des graphes parfaits : la classe des graphes circulaires-parfaits. Une partie conséquente de mes travaux est consacrée à ces derniers [7,9,12,14,18]. Nous avons notamment démontré qu'une caractérisation par sous-graphes exclus, analogue à celle pour les graphes parfaits, est très improbable, en exhibant trois familles de graphes minimaux circulaires-imparfaits aux propriétés structurelles très différentes [9,17]. Nous avons également introduit la sous-classe des graphes fortement circulaires-parfaits : pour ces graphes, les problèmes classiques de reconnaissance en temps polynomial, de calcul des paramètres usuels (stable maximal pondéré etc...) sont ouverts. Nous avons caractérisé ceux qui n'ont pas de triangles, et conçu pour ces derniers des algorithmes polynomiaux pour les problème précités [7,12].
– axe *optimisation combinatoire* : étude des facettes d'un polytope fondamental en théorie des graphes : le polytope (des vecteurs d'incidence) des stables [3,4,5,8,10,13,14,16,17,19,20,21]. Établir la liste des facettes des polytopes des stables des graphes sans griffe est un problème ouvert depuis une vingtaine d'années : seule la famille des facettes à coefficients 0/1, dites *facettes de rang*, est bien comprise. Les webs forment une famille de graphes sans griffe, considérée comme élémentaire. Nous avons démontré, que même pour ceux-ci, les *facettes de rang* ne sont pas suffisantes [3,4,5]. Mes derniers travaux ont été consacrés au polytope des stables des graphes sans griffe, en général. Ainsi, nous avons proposé de nouvelles approches [18,19], conduisant à la première conjecture quant à la nature de toutes les facettes [21].

Perspectives : le projet INRIA RealOpt

La modélisation par des graphes de Feketete et Shepers (1997) du problème du placement de parallélépipèdes dans un container a permis de faire un saut quantitatif quant à la taille des problèmes traités. En effet, en représentant les solutions par des graphes, de nombreuses symétries (i.e. des solutions identiques) sont éliminées et les bonnes propriétés algorithmiques des graphes mis en jeu, des graphes d'intervalle, sont la clef pour la conception de l'algorithme de Fekete/Shepers.

Nous avons créé récemment en mars 2007 une équipe-projet INRIA RealOpt formée de quelques chercheurs du LaBRI (pour les aspects graphes) et du MAB (pour les aspects programmation mathématique) dont un des objectifs est de reprendre cette approche en proposant des reformulations par des graphes de problèmes d'optimisation combinatoire, de manière à restreindre l'espace de recherche des solveurs "classiques" de la programmation mathématique.

De part mon positionnement, mêlant graphes et approches polyédrales, je suis au cœur de ce projet. Nous avons déjà obtenu des résultats directement liés à cette approche pluridisciplinaire [11,20].

Site web du projet : `http ://realopt.math.cnrs.fr`

Publications

Revues d'audience internationale avec comité de rédaction

1. Pêcher, *Cayley partitionable graphs*, *Discrete Mathematics* (269), pages 191-218, 2003
2. Pêcher, *Partitionable graphs arising from near-factorizations of finite groups*, *Discrete Mathematics* (276), pages 295-311, 2004
3. Pêcher, Wagler, *Almost all webs are not rank-perfect*, *Mathematical Programming* (105), pages 311-328, 2006
4. Pêcher, Wagler, *On Non-Rank Facets in Stable Set Polytopes of Webs with Clique Number Four*, *Discrete Applied Mathematics* (154), no 9, pages 1408-1415, 2006

5. Pêcher, Wagler, *A construction for non-rank facets of stable set polytopes of webs*, *European Journal of Combinatorics* (27), no 7, pages 1172-1185, 2006
6. Pêcher, Wagler, *A note on the Chvátal rank of clique family inequalities*, *RAIRO - Operations Research* (41) pages 289-294, 2007
7. Pêcher, Wagler, *On classes of minimal circular-imperfect graphs*, *Discrete Applied Mathematics* (156) pages 998-1010, 2008
8. Pêcher, Zhu, *On the Circular Chromatic Number of Circular Partitionable Graphs*, *Journal of Graph Theory* (52), no 4, pages 294-306, 2006

Articles à paraître dans des revues d'audience internationale avec comité de rédaction

9. Coulonges, Pêcher, Wagler, *Triangle-free strongly circular-perfect graphs*, à paraître dans *Discrete Mathematics*
10. Coulonges, Pêcher, Wagler, *Characterizing and bounding the imperfection ratio for some classes of graphs*, à paraître dans *Mathematical Programming A*
11. Pêcher, Wagler, *On facets of stable set polytopes of claw-free graphs with stability number three*, à paraître dans *Discrete Applied Mathematics*

Conférences internationales avec comité de sélection et actes

12. Coulonges, Pêcher, Wagler, *On strongly circular-perfectness*, 7th International Colloquium on Graph Theory, Electronic Notes in Discrete Mathematics (22), pages 369-374, 2005.
13. Pêcher, *Cayley partitionable graphs*, 6th International Conference on Graph Theory, Electronic Notes in Discrete Mathematics (5), pages 256-259, 2000.
14. Pêcher, Wagler, *A construction for non-rank facets of stable set polytopes of webs*, Eurocomb, Prague, Tchéquie - ITI Series 2003-145, pages 301-305, 2003
15. Pêcher, Wagler, *On Non-Rank Facets in Stable Set Polytopes of Webs with Clique Number Four*, 2nd Cologne Twente Workshop (CTW) on Graphs and Combinatorial Optimization, Twente, Pays-Bas - Electronic Notes in Discrete Mathematics (13), pages 96-99, 2003
16. Pêcher, Wagler, Zhu, *Three classes of minimal circular imperfect graphs*, Proceedings of GRACO2005, Electronic Notes Discrete Mathematics (19), Rio de Janeiro, Brésil, pages 9-15, 2005
17. Pêcher, Wagler, *Generalized clique family inequalities for claw-free graphs*, CTW'06 - Cologne-Twente Workshop on Graphs and Combinatorial Optimization, Electronic Notes in Discrete Mathematics (25), pages 117-121, 2006
18. Pêcher, Pesneau, Wagler, *On facets of stable set polytope of claw-free graphs with maximum stable set size three*, CS'06 - Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications, Electronic Notes in Discrete Mathematics (28), pages 185-190, 2006
19. Pêcher, Zhu, *Claw-free circular-perfect graphs*, Proceedings of Eurocomb'07, Electronic Notes in Discrete Mathematics (29), Seville, Espagne, pages 451-455, 2007

Conférences internationales avec comité de sélection

20. Coulonges, Pêcher, Wagler, *On the imperfection ratio of a-perfect graphs and some consequences*, CTW'05 - Cologne-Twente Workshop on Graphs and Combinatorial Optimization, mai 2005.
21. Coulonges, Pêcher, Wagler, *On stable set polytopes of circular-perfect graphs*, ALIO/EURO V conference on combinatorial optimization, octobre 2005.
22. Pêcher, Wagler, *Results and Conjectures on the Stable Set Polytope of Claw-Free Graphs*, ISMP06 - 19th International Symposium on Mathematical Programming, Rio de Janeiro, Brésil, juillet 2006.

Conférences invité

23. Pêcher, *Stable set polytopes of claw-free graphs*, Coloring workshop, Kaohsiung, Taiwan, 2008

Rayonnement scientifique

Séjours invité

– janvier 2008 (2 semaines), National Sun Yat-sen University, Kaohsiung, Taiwan
– février 2007 (2 semaines), National Sun Yat-sen University, Kaohsiung, Taiwan
– août 2005 (1 mois), IMO, Magdeburg, Allemagne

Organisations de rencontres scientifiques

– co-organisateur de la conférence internationale Eurocomb 2009
– co-organisateur de la conférence nationale JPOC6
– organisateur de l'Ecole Jeunes Chercheurs en calcul formel 2006
– organisateur des journées graphes 2005 (GDR ALP)
– co-organisateur de la rencontre internationale CombStru'04,

Arbitre pour les revues ou conférences internationales

– revues : *European Journal of Combinatorics*, *Journal of Combinatorial Optimization*, *Journal of Discrete Applied Mathematics*, *Journal of Discrete Mathematics*, *Journal of Discrete Optimization*, *Journal of Graph Theory*, *RAIRO*, *SIAM Journal of Discrete Mathematics*
– conférences : I.C.G.T. 2005, S.T.A.C.S. 2003, S.T.A.C.S. 2004, S.T.A.C.S. 2006, Eurocomb 2007, Lagos 2007

Participations aux projets de recherche

– membre permanent et représentant du projet INRIA RealOpt (en cours de création, 2007-)
– webmestre du projet RealOpt (2007-)
`http ://realopt.math.cnrs.fr`
– membre du projet ANR "IDEA" (2009-)
– membre du projet européen "Donet" (2002), de l'Action Spécifique "Grands Graphes" (2002-2003), du projet européen "CombStru" (2003-2006)
– membre du GDR ALP (2002-2006), du GDR IM (2006-)
– partenaire du projet "Stable sets" du ZIB, Allemagne (2003)

Fonctions électives

– membre élu de la commission de spécialistes du LaBRI (2005-2008), du conseil d'administration de l'UFR Mathématiques-Informatique (2004-2008)
– représentant du conseil d'administration au conseil de la Licence (2005-2008)

Mobilité thématique et/ou géographique

– géographique : deux post-docs en Italie et Allemagne (2002), un séjour invité à l'université de Magdeburg, Allemagne (août 2005), deux séjours invité à l'université de Kaohsiung, Taiwan (février 2007, janvier 2008)
– thématique : thèse de combinatoire (pavages) et algébrique (théorie des groupes finis) ; recherche post-doctorale : théorie des graphes (colorations) et optimisation combinatoire, à la frontière de la recherche opérationnelle.

Thèses

- Septembre 2004 - Décembre 2007, (90%) avec Prof. Raspaud (10%) : Sylvain Coulonges - thème "*Etude polyédrale des généralisations des graphes parfaits*" - mention très honorable
- Septembre 2007 - , (40%) avec Prof. Vanderbeck (40%) et P. Pesneau (20%) : Cédric Joncour - thème "*Graphes pour les problèmes de placement*"

Stages de DEA ou master Recherche

- Janvier 2003 - Juin 2003 (100%) : Jonathan Limouzineau (DEA) - thème "*Réseaux neuronaux et grands graphes*". Dans son stage, Jonathan a proposé un nouveau modèle "uniquement graphe" pour les réseaux neuronaux, et mis au point un algorithme d'apprentissage.
- Janvier 2003 - Juin 2003, (50%) avec Prof. Métivier : Reda Noushi (DEA) - thème "*Informatique distribuée asynchrone et homologie*". Reda a exploré des développements récents et plus simples de l'utilisation de la topologie algébrique (théorème d'Herlihy et Shavit) pour l'informatique distribuée.
- Janvier 2003 - Juin 2003, (50%) avec Prof. Raspaud : Charles-Henri Collin (DEA) - thème initial "*Colorations circulaires des graphes planaires*". Charles-Henri s'est intéressé aux colorations circulaires des graphes de Petersen généralisés.
- Janvier 2004 - Juin 2004 (100%) : Céline Cordier (master Recherche) - thème "*Graphes circulaires parfaits*". Contrairement aux graphes parfaits, peu de choses sont connues sur les graphes circulaires parfaits, du fait de leur jeunesse. Le travail demandé a consisté à découvrir de nouvelles classes de graphes circulaires parfaits.
- Janvier 2004 - Juin 2004 (100%) : Sylvain Coulonges (master Recherche) - thème "*Capacité de Shannon des trous impairs*". La capacité de Shannon permet de mesurer la capacité de transmission d'une liaison avec pertes. La capacité de Shannon du trou impair à 7 sommets est inconnue. Le but du stage était d'affiner le meilleur encadrement connu par des techniques simples de pavages.
- Janvier 2006 - Juin 2006 (100%) : Paul Cadroas (master Recherche) - thème "*Etude du ratio d'imperfection*". Le ratio d'imperfection mesure le degré d'imperfection d'un graphe fini. Le travail demandé a notamment consisté à rechercher une généralisation pour les graphes infinis.

Annexe B

Articles parus - travaux post-doctoraux

Sommaire

B.1 On Non-Rank Facets of Stable Set Polytopes of Webs with Clique Number Four

par A. Pêcher et A. Wagler
Discrete Applied Mathematics 154 (2006) [86]

Graphs with circular symmetry, called webs, are relevant for describing the stable set polytopes of two larger graph classes, quasi-line graphs [6, 11] and claw-free graphs [5, 6]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [7]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [3, 8] while there are examples with clique number > 4 having non-rank facets [9, 11, 10]. The aim of the present paper is to treat the remaining case with clique number $= 4$: we provide an infinite sequence of such webs whose stable set polytopes admit non-rank facets.

Introduction

A natural generalization of odd holes and odd antiholes are graphs with circular symmetry of their maximum cliques and stable sets, called webs: a web W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge iff i and j differ by at most k (modulo n) and $i \neq j$. These graphs belong to the classes of quasi-line graphs and claw-free graphs and are, besides line graphs, relevant for describing the stable set polytopes of those larger graph classes [5, 6, 11]. (The line graph of a graph H is obtained by taking the edges of H as nodes and connecting two nodes iff the corresponding edges of H are incident. A graph is quasi-line (resp. claw-free) if the neighborhood of any node can be partitioned into two cliques (resp. does not contain any stable set of size 3).) All facets of the stable set polytope of line graphs are known from matching theory [4]. In contrary, providing all facets of the stable set polytopes of claw-free graphs is a long-standing problem [7] but we are even still far from having a complete description for the stable set polytopes of webs (and, therefore, of quasi-line and claw-free graphs, too).

In particular, as shown by Giles & Trotter [6], the stable set polytopes of claw-free graphs contain facets with a much more complex structure than those defining the matching polytope. Oriolo [11] discussed which of them occur in quasi-linegraphs. In particular, these non-rank facets rely on certain combinations of joined webs.

Several further authors studied the stable set polytopes of webs. Obviously, webs with clique number 2 are either even or odd holes (their stable set polytopes are known due to [1, 12]). Dahl [3] studied webs with clique number 3 and showed that their stable set polytopes admit rank facets only. On the other hand, Kind [9] found (by means of the PORTA software[1]) examples of webs with clique number > 4 whose stable set polytopes have *non*-rank facets. Oriolo [11] and Liebling et al. [10] presented further examples of such webs. It is natural to ask whether the stable set polytopes of webs with clique number $= 4$ admit rank facets only.

The aim of the present paper is to answer that question by providing an infinite sequence of webs with clique number $= 4$ whose stable set polytopes have *non*-rank facets.

[1] By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

Results on Stable Set Polytopes

The *stable set polytope* $\text{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of the graph $G = (V, E)$ (a set $V' \subseteq V$ is a stable set if the nodes in V' are mutually non-adjacent). A linear inequality $a^T x \leq b$ is said to be *valid* for $\text{STAB}(G)$ if it holds for all $x \in \text{STAB}(G)$. We call a stable set S of G a *root* of $a^T x \leq b$ if its incidence vector χ^S satisfies $a^T \chi^S = b$. A valid inequality for $\text{STAB}(G)$ is a *facet* if and only if it has $|V|$ roots with affinely independent incidence vectors. (Note that the incidence vectors of the roots of $a^T x \leq b$ have to be *linearly* independent if $b > 0$.)

The aim is to find a system $Ax \leq b$ of valid inequalities s.t. $\text{STAB}(G) = \{x \in \mathbb{R}_+^{|G|} : Ax \leq b\}$ holds. Such a system is unknown for the most graphs and it is, therefore, of interest to study certain linear relaxations of $\text{STAB}(G)$ and to investigate for which graphs G these relaxations coincide with $\text{STAB}(G)$.

One relaxation of $\text{STAB}(G)$ is the *fractional stable set polytope* $\text{QSTAB}(G)$ given by all "trivial" facets, the *nonnegativity constraints*

$$x_i \geq 0 \tag{B.0}$$

for all nodes i of G and by the *clique constraints*

$$\sum_{i \in Q} x_i \leq 1 \tag{B.1}$$

for all cliques $Q \subseteq G$ (a set $V' \subseteq V$ is a clique if the nodes in V' are mutually adjacent). Obviously, a clique and a stable set have at most one node in common. Therefore, $\text{QSTAB}(G)$ contains all incidence vectors of stable sets of G and $\text{STAB}(G) \subseteq \text{QSTAB}(G)$ holds for all graphs G. The two polytopes coincide precisely for perfect graphs [1, 12].

A graph G is called *perfect* if, for each (node-induced) subgraph $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$. That is, for all $G' \subseteq G$, as many stable sets cover all nodes of G' as a maximum clique of G' has nodes (maximum cliques resp. maximum stable sets contain a maximal number of nodes).

In particular, for all imperfect graphs G follows $\text{STAB}(G) \subset \text{QSTAB}(G)$ and, therefore, further constraints are needed to describe their stable set polytopes. A natural way to generalize clique constraints is to investigate *rank constraints*

$$\sum_{i \in G'} x_i \leq \alpha(G') \tag{B.2}$$

associated with *arbitrary* (node-)induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the stability number of G', i.e., the cardinality of a maximum stable set in G' (note that $\alpha(G') = 1$ holds iff G' is a clique). For convenience, we often write (B.2) in the form $x(G') \leq \alpha(G')$.

Let $\text{RSTAB}(G)$ denote the *rank polytope* of G given by all nonnegativity constraints (B.0) and all rank constraints (B.2). A graph G is called *rank-perfect* [14] if $\text{STAB}(G)$ coincides with $\text{RSTAB}(G)$.

By construction, every perfect graph is rank-perfect. Some further graphs are rank-perfect by definition: *near-perfect* [13] (resp. *t-perfect* [1], *h-perfect* [7]) graphs, where rank constraints associated with cliques and the graph itself (resp. edges and odd cycles, cliques and odd cycles) are allowed. Moreover, the result of Edmonds and Pulleyblank [4] implies that line graphs are rank-perfect as well (see [15] for a list with more examples).

Recall that a web W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (i.e., if $|i-j| \leq k \bmod n$) and $i \neq j$. We assume $k \geq 1$ and $n \geq 2(k+1)$ in the sequel in order to exclude the degenerated cases when W_n^k is a stable set or a clique. W_n^1 is a hole and W_{2k+1}^{k-1} an odd antihole for $k \geq 2$. All webs W_9^k on nine nodes are depicted in Figure B.1. It is easy to see that $\omega(W_n^k) = k+1$ and $\alpha(W_n^k) = \lfloor \frac{n}{k+1} \rfloor$ holds. Note that webs are also called circulant graphs C_n^k [2]. Furthermore, similar graphs $W(n,k)$ were introduced in [8].

So far, the following is known about stable set polytopes of webs. The webs W_n^1 are holes, hence they are perfect if n is even and near-perfect if n is odd (recall that we suppose $n \geq 2(k+1)$). Dahl [3] showed that all webs W_n^2 with clique number 3 are rank-perfect. But there are several webs with clique number > 4 known to be *not* rank-perfect [9, 11, 10], e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9; these results are summarized in Table 1.

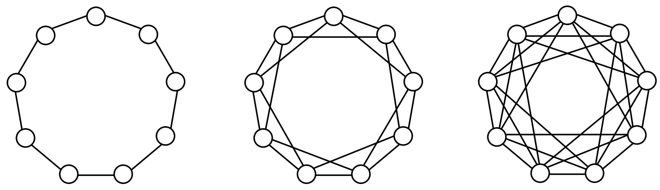

Figure B.1: Some webs

Table 1: Known results on rank-perfectness of webs

	$\omega = 2$	$\omega = 3$	$\omega = 4$	$\omega \geq 5$
All webs rank-perfect?	Yes	Yes	?	No
Infinitely many not rank-perfect webs?	No	No	?	?

A conjecture due to Ben Rebea (see [11]) claims that the stable set polytopes of quasi-line graphs admit only one type of facets besides nonnegativity constraints (B.0) and clique constraints (B.1), so-called clique family inequalities: Let $G = (V, E)$ be a graph, $\mathcal{F}$ be a family of (at least three inclusion-wise) maximal cliques of G, $p \leq |\mathcal{F}|$ be an integer, and define two sets as follows:

$$\begin{aligned} I(\mathcal{F}, p) = & \quad \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\} \\ O(\mathcal{F}, p) = & \quad \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p - 1\} \end{aligned}$$

The *clique family inequality* $(\mathcal{F}, p)$

$$(p - r) \sum_{i \in I(\mathcal{F},p)} x_i + (p - r - 1) \sum_{i \in O(\mathcal{F},p)} x_i \leq (p - r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \tag{B.3}$$

with $r = |\mathcal{F}| \bmod p$ and $r > 0$ is valid for the stable set polytope of *every* graph by Oriolo [11]. Since webs are quasi-line graphs in particular, the stable set polytopes of webs should admit, according to Ben Rebea's conjecture, facets coming from cliques and clique family inequalities only.

In order to answer the question whether the webs with clique number $= 4$ are rank-perfect or not, we introduce clique family inequalities associated with certain subwebs and prove the following: the clique family inequality associated with $W_{2l}^2 \subset W_{3l}^3$ induces a non-rank facet of STAB(W_{3l}^3) if $l \geq 11$ and $2 = l \bmod 3$ (Theorem B.1.6).

Non-Rank Facets

Consider a web W_n^k. We say that a clique family inequality $(\mathcal{F}, p)$ of STAB(W_n^k) is *associated* with a proper subweb $W_{n'}^{k'} \subset W_n^k$ if $\mathcal{F} = \{Q_i : i \in W_{n'}^{k'}\}$ is chosen as clique family, $p = k' + 1$, and $Q_i = \{i, \ldots, i + k\}$ denotes the maximum clique of W_n^k starting in node i. In order to explore the special structure of such inequalities, we need the following fact from Trotter [8].

Lemma B.1.1. [8] *$W_{n'}^{k'}$ is an induced subweb of W_n^k if and only if there is a subset $V' = \{i_1, \ldots, i_{n'}\} \subseteq V(W_n^k)$ s.t. $|V' \cap Q_{i_j}| = k' + 1$ for every $1 \leq j \leq n'$.*

We now prove the following.

Lemma B.1.2. *Let $W_{n'}^{k'} \subset W_n^k$ be any proper induced subweb. The clique family inequality $(\mathcal{F}, p)$ of STAB(W_n^k) associated with $W_{n'}^{k'}$ is*

$$(k' + 1 - r) \sum_{i \in I(\mathcal{F},p)} x_i + (k' - r) \sum_{i \in O(\mathcal{F},p)} x_i \leq (k' + 1 - r)\, \alpha(W_{n'}^{k'}) \tag{B.4}$$

with $p = k' + 1$, $r = n' \bmod (k'+1)$, $r > 0$*; we have* $W_{n'}^{k'} \subseteq I(\mathcal{F}, p)$ *and the union of* $I(\mathcal{F}, p)$ *and* $O(\mathcal{F}, p)$ *covers all nodes of* W_n^k.

Proof. Let $W_{n'}^{k'}$ be a proper subweb of W_n^k and choose $\mathcal{F} = \{Q_i : i \in W_{n'}^{k'}\}$, $p = k' + 1$. Obviously $|\mathcal{F}| = |W_{n'}^{k'}| = n'$ follows. Let $V' = \{i_1, \ldots, i_{n'}\}$ be the node set of $W_{n'}^{k'}$ in W_n^k. Observation B.1.1 implies that $Q_{i_j} = \{i_j, \ldots, i_j + k\}$ contains the nodes $i_j, \ldots, i_{j+k'}$ from V'. Obviously, the node $i_{j+k'}$ belongs exactly to the $(k'+1)$ cliques $Q_{i_j}, \ldots, Q_{i_{j+k'}}$ from $\mathcal{F}$. Since all indices are taken modulo n, every node in $W_{n'}^{k'}$ is covered precisely $(k'+1)$ times by $\mathcal{F}$ and $p = k'+1$ yields, therefore, $W_{n'}^{k'} \subseteq I(\mathcal{F}, p)$. Furthermore, $|\mathcal{F}| = n'$ and $p = \omega(W_{n'}^{k'})$ implies $\lfloor \frac{|\mathcal{F}|}{p} \rfloor = \alpha(W_{n'}^{k'})$. Hence the clique family inequality given by $(\mathcal{F}, p)$ is (B.4) which finishes the proof. ■

Let us turn to the clique family inequality associated with $W_{2l}^2 \subset W_{3l}^3$, i.e. n is divisible by 3 (for some $l \geq 3$ by $n \geq 2(k+1)$). Observation B.1.1 easily yields that every third node of W_{3l}^3 does not belong to the subweb W_{2l}^2 and that $W_{2l}^2 = I(\mathcal{F}, 3)$ holds if we choose $\mathcal{F} = \{Q_i : i \in W_{2l}^2\}$, see Figure B.1.

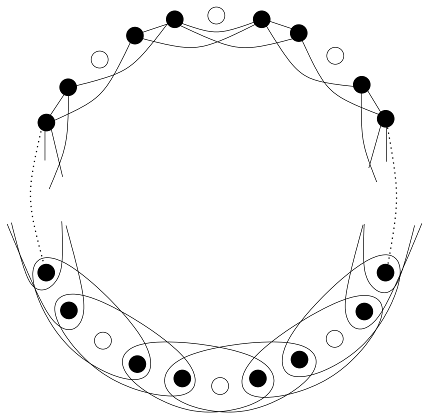

Figure B.2: The subweb $W_{2l}^2 \subset W_{3l}^3$

Furthermore, the nodes in $W_{3l}^3 - W_{2l}^2 = O(\mathcal{F}, 3)$ induce the hole W_{1l}^1. Thus, the clique family inequality $(\mathcal{F}, 3)$

$$(3-r)\, x(W_{2l}^2) + (2-r)\, x(W_{1l}^1) \leq (3-r)\, \alpha(W_{2l}^2)$$

associated with $W_{2l}^2 \subset W_{3l}^3$ is a *non*-rank constraint if $r = 1$ holds. The aim of this subsection is to prove that $(\mathcal{F}, 3)$ is a non-rank facet of STAB(W_{3l}^3) whenever $l \geq 11$ and $2 = l \bmod 3$ (note: $2 = l \bmod 3$ implies $r = 1 = 2l \bmod 3$).

For that, we have to present $3l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent. (Recall that a root of $(\mathcal{F}, 3)$ is a stable set of W_{3l}^3 satisfying $(\mathcal{F}, 3)$ at equality.)

It follows from [8] that a web W_n^k produces the full rank facet $x(W_n^k) \leq \alpha(W_n^k)$ iff $(k+1) \not| \, n$. Thus W_{2l}^2 is facet-producing if $2 = l \bmod 3$ and the maximum stable sets of W_{2l}^2 yield already $2l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent.

Let $V = V(W_{3l}^3)$ and $V' = V(W_{2l}^2)$. We need a set $\mathcal{S}$ of further l roots of $(\mathcal{F}, 3)$ which have a non-empty intersection with $V - V'$, called *mixed roots*, and are independent, in order to prove that $(\mathcal{F}, 3)$ is a facet of STAB(W_{3l}^3).

We show that there exists a set $\mathcal{S}$ of l mixed roots of $(\mathcal{F}, 3)$ whenever $l \geq 11$. Due to $2 = l \bmod 3$, we set $l = 2 + 3l'$ and obtain $|V| = 3l = 6 + 9l'$. Thus, V can be partitioned into 2 blocks D_1, D_2 with 3 nodes each and l' blocks $B_1, \ldots, B_{l'}$ with 9 nodes each s.t. every block ends with a node in $V - V'$ (this is possible since every third node of V belongs to $V - V'$ say $i \in V'$ if $3 \not| \, i$ and $i \in V - V'$ if $3|i$). Figure B.1 shows a block D_i and a block B_j (where circles represent nodes in V' and squares represent nodes in $V - V'$). For the studied mixed roots of $(\mathcal{F}, 3)$ we choose the black filled nodes in Figure B.1:

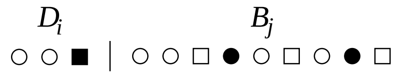

Figure B.3: A block D_i and a block B_j

Lemma B.1.3. *The set S containing the 3rd node of the blocks D_1, D_2 as well as the 4th and 8th node of any block B_j is a root of $(\mathcal{F}, 3)$ with $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$ for every ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ of the blocks s.t. D_1, D_2 are not neighbored.*

Proof. Consider a set S constructed that way. Since every block ends with a node in $V - V'$ by definition and every third node of V is in $V - V'$, we have that the last node of D_i and the 3rd, 6th, and 9th node of B_j belong to $V - V'$ while all other nodes are in V'. Thus, the two last nodes in D_1 and D_2 are the two studied nodes in $S \cap (V - V')$ and the 4th and 8th node in B_j for $1 \leq j \leq l'$ are the studied $2l'$ nodes in $S \cap V'$ (see Figure B.1).

S is a stable set provided the two blocks D_1 and D_2 are not neighbored: Obviously, there is no edge between the 4th and 8th node of any block B_j. Thus, we only have to discuss what happens between two consecutive blocks. Since the first 3 nodes of every block B_j do not belong to S, there is no problem with having any block before B_j, i.e., $B_k B_j$ or $D_i B_j$. For the remaining case $B_j D_i$, notice that the last node of B_j and the first two nodes of D_i do not belong to S and there cannot be an edge between two nodes of S in that case, too.

This shows that S is a stable set satisfying $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$. Due to $\alpha(W_{2l}^2) = \lfloor \frac{2(2+3l')}{3} \rfloor = 2l' + 1$, the set S is finally a root of $(\mathcal{F}, 3)$. ■

Lemma B.1.3 implies that there exist mixed roots S of $(\mathcal{F}, 3)$ with $|S| = 2 + 2l'$ if $l' \geq 2$. The next step is to show that there are l such roots if $l' \geq 3$ (resp. $l \geq 11$).

In the sequel, we denote by $S_{i,m}$ the stable set constructed as in Lemma B.1.3 when $D_1 = \{i-2, i-1, i\}$ and $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$. If there are more than $\lfloor \frac{l'}{2} \rfloor$ blocks between D_1 and D_2, there are less than $\lfloor \frac{l'}{2} \rfloor$ blocks between D_2 and D_1. Hence it suffices to consider $m \leq \lfloor \frac{l'}{2} \rfloor$.

By construction, $S_{i,m}$ contains a second node from $V - V'$, namely, the third node $i + 9m + 3$ of block D_2. If $2|l'$ and $m = \frac{l'}{2}$, then $(i+9m+3)+9m+3 = i+9l'+6 = i(\bmod n)$ and, therefore, $S_{i,m} = S_{i+9m+3,m}$ follows.

We are supposed to construct *distinct* mixed roots $S_{i,m}$ of $(\mathcal{F}, 3)$ with $2 + 2l'$ nodes, hence we choose orderings $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ and obtain easily:

Lemma B.1.4. *If $l' \geq 3$, then the stable sets $S_{i,m}$ for each $i \in V - V'$ obtained from any ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ yield $|V - V'| = l$ roots of $(\mathcal{F}, 3)$ with $2 + 2l'$ nodes each.*

Consequently, we can always choose a set of $3l$ roots of $(\mathcal{F}, 3)$ if $l' \geq 3$ resp. $l \geq 11$.

If $\mathcal{S}$ is a set of l distinct mixed roots, denote by $A_{\mathcal{S}}$ the square matrix containing the incidence vectors of the $2l$ maximum stable sets of W_{2l}^2 and the l mixed roots in $\mathcal{S}$. $A_{\mathcal{S}}$ can be arranged s.t. the first $2l$ and the last l columns correspond to the nodes in W_{2l}^2 and W_{1l}^1, respectively, and the first $2l$ rows contain the incidence vectors of the maximum stable sets of W_{2l}^2 where the last rows contain the incidence vectors of the l mixed roots in $\mathcal{S}$. (Note that the nodes corresponding to the last l columns of $A_{\mathcal{S}}$ are $3, 6, \ldots, 3l$.) Then $A_{\mathcal{S}}$ has the block structure

$$A_{\mathcal{S}} = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right)$$

where the $2l \times 2l$-matrix A_{11} is invertible (recall: W_{2l}^2 is facet-producing by [8] in the considered case with $1 = 2l \bmod 3$ resp. $2 = l \bmod 3$).

It is left to find a set $\mathcal{S}$ of l distinct mixed roots s.t. A_{22} is an invertible $l \times l$-matrix (then $A_{\mathcal{S}}$ is invertible due to its block structure).

$$
\begin{array}{c|ccccc|ccccccccccc|}
 & 1 & \dots & 5 & \dots & l\!-\!11 & l\!-\!10 & l\!-\!9 & l\!-\!8 & l\!-\!7 & l\!-\!6 & l\!-\!5 & l\!-\!4 & l\!-\!3 & l\!-\!2 & l\!-\!1 & l \\
\hline
1 & 1 & & 1 & & & & & & & & & & & & & \\
\vdots & & \ddots & & \ddots & & & & & & & & & & & & \\
\vdots & & & & \ddots & & & \ddots & & & & & & & & & \\
l\!-\!11 & & & & & 1 & & & & 1 & & & & & & & \\
\hline
l\!-\!10 & & & & & & 1 & & & & 1 & & & & & & \\
l\!-\!9 & & & & & & & 1 & & & & 1 & & & & & \\
l\!-\!8 & & & & & & & & 1 & & & & 1 & & & & \\
l\!-\!7 & & & & & & & & & 1 & & & & 1 & & & \\
l\!-\!6 & & & & & & & & & & 1 & & & & 1 & & \\
l\!-\!5 & & & & & & & & & & & 1 & & & & 1 & \\
l\!-\!4 & & & & & & & & & & & & 1 & & & & 1 \\
l\!-\!3 & & & & & & 1 & & & & & & & 1 & & & \\
l\!-\!2 & & & & & & & 1 & & & & & & & 1 & & \\
l\!-\!1 & & & & & & & & 1 & & & & & & & 1 & \\
l & & & & & & & & & 1 & & & & & & & 1 \\
\hline
\end{array}
$$

Fig. 4. The $l \times l$-matrix A_{22}

Lemma B.1.5. *For every $l \geq 11$, there is a set $\mathcal{S}$ of l mixed roots of $(\mathcal{F}, 3)$ containing 2 nodes from $V - V'$ s.t. the $l \times l$-submatrix A_{22} of $A_{\mathcal{S}}$ is invertible.*

Proof. Every root $S_{i,m}$ of $(\mathcal{F}, 3)$ corresponds to a row in $(A_{21}|A_{22})$ of $A_{\mathcal{S}}$ having precisely two 1-entries in the columns belonging to A_{22} (by $|S_{i,m} \cap (V - V')| = 2$ for all $i \in V - V'$). Lemma B.1.4 ensures that no such roots coincide if $1 \leq m < \frac{l'}{2}$ for all $i \in V - V'$.

The idea of finding cases when A_{22} is invertible goes as follows: Let $S_{3j,1}$ for $1 \leq j \leq l-4$ be the first $l-4$ roots in $\mathcal{S}$ with $S_{3j,1} \cap (V - V') = \{3j, 3(j+4)\}$. Choose as the remaining 4 roots in $\mathcal{S}$ the stable sets $S_{3j,2}$ for $l-10 \leq j \leq l-7$ with $S_{3j,2} \cap (V - V') = \{3j, 3(j+7)\}$. Then take their incidence vectors $\chi^{S_{3j,1}}$ for $1 \leq j \leq l-4$ as the first $l-4$ rows and $\chi^{S_{3j,2}}$ for $l-10 \leq j \leq l-7$ as the last 4 rows of $(A_{21}|A_{22})$. By construction, A_{22} is the $l \times l$-matrix in Figure 4 (1-entries are shown only, the column i corresponds to the node $3i$).

A_{22} has only 1-entries on the main diagonal (coming from the first nodes in $V - V'$ of $S_{3j,1}$ for $1 \leq j \leq l-4$ and from the second nodes in $V - V'$ of $S_{3j,2}$ for $l-10 \leq j \leq l-7$). The only non-zero entries of A_{22} below the main diagonal come from the first nodes in $V - V'$ of $S_{3j,2}$ for $l-10 \leq j \leq l-7$. Hence, A_{22} has the form

$$
A_{22} = \left(\begin{array}{c|c} A'_{22} & \\ \hline 0 & A''_{22} \end{array} \right)
$$

where both A'_{22} and A''_{22} are invertible due to the following reasons:

A'_{22} is an $(l-11) \times (l-11)$-matrix having 1-entries on the main diagonal and 0-entries below the main diagonal by construction. Hence A'_{22} is clearly invertible.

A''_{22} is an 11×11-matrix which has obviously the circular 1's property. In other words, A''_{22} is equivalent to the matrix $A(\overline{C}_{11})$ containing the incidence vectors of the maximum stable sets of the odd antihole $\overline{C}_{11}$ as rows. Since $A(\overline{C}_{11})$ is invertible due to Padberg [12], the matrix A''_{22} is invertible, too. (Note that $l = 11$ implies $A_{22} = A''_{22}$.)

This completes the proof that A_{22} is invertible for every $l \geq 11$ if we choose the set $\mathcal{S}$ of l roots of $(\mathcal{F}, 3)$ as constructed above. ∎

Finally, we have shown that, for every $l \geq 11$ with $2 = l \bmod 3$, there are $3l$ roots of $(\mathcal{F}, 3)$ whose incidence vectors are linearly independent:

Theorem B.1.6. *For any $W^2_{2l} \subset W^3_{3l}$ with $2 = l \bmod 3$ and $l \geq 11$, the clique family inequality*

$$2x(W^2_{2l}) + 1x(W^1_{1l}) \leq 2\alpha(W^2_{2l})$$

associated with W^2_{2l} is a non-rank facet of STAB(W^3_{3l}).

This gives us an infinite sequence of not rank-perfect webs W^3_{3l} with clique number 4, namely W^3_{33}, W^3_{42}, W^3_{51}, W^3_{60}, ... and answers the question whether the webs W^3_n with clique number 4 are rank-perfect negatively. Thus, we can update Table 1 as follows:

Table 2: Updated results on rank-perfectness of webs

	$\omega = 2$	$\omega = 3$	$\omega = 4$	$\omega \geq 5$
All webs rank-perfect?	Yes	Yes	No	No
Infinitely many not rank-perfect webs?	No	No	Yes	?

Concluding Remarks

It is open whether there exist, for each $\omega \geq 5$, infinitely many not rank-perfect webs, see Table 2. We beliefe that this is the case.

Assuming Ben Rebea's Conjecture as true, we conjecture further that all non-rank facets of STAB(W^k_n) are clique family inequalities $(\mathcal{F}, p)$

$$(k' + 1 - r) \sum_{i \in I(\mathcal{F},p)} x_i + (k' - r) \sum_{i \in O(\mathcal{F},p)} x_i \leq (k' + 1 - r)\, \alpha(W^{k'}_{n'})$$

associated with certain subwebs $W^{k'}_{n'} \subset W^k_n$. All non-rank facets would have, therefore, coefficients at most $k-1$ and $k-2$ (since $k' < k$ follows by $W^{k'}_{n'} \subset W^k_n$ and $(k'+1-r) \leq k'$ by $r > 0$). This would imply that the stable set polytopes of webs W^3_n could have non-rank facets with coefficients 2 and 1 only.

References

[1] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[2] V. Chvátal. On the strong perfect graph conjecture. *J. Comb. Theory Series B*, 20:139–141, 1976.

[3] G. Dahl. Stable set polytopes for a class of circulant graphs. *SIAM Journal on Optimization*, 9(2):493–503, 1998.

[4] J.E. Edmonds and W.R. Pulleyblank. Facets of 1-matching polyhedra. In C. Berge and D.R. Chuadhuri, editors, *Hypergraph Seminar*, pages 214–242. Springer, 1974.

[5] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1):1–38, 1997.

[6] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[7] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[8] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12:373–388, 1975.

[9] J. Kind. *Mobilitätsmodelle für zellulare Mobilfunknetze: Produktformen und Blockierung*. PhD thesis, RWTH Aachen, 2000.

[10] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[11] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[12] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[13] F.B. Shepherd. Near-perfect matrices. *Mathematical Programming*, 64:295–323, 1994.

[14] A. Wagler. Rank-perfect and weakly rank-perfect graphs. *Mathematical Methods of Operations Research*, 95:127–149, 2002.

[15] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2:149–152, 2004.

B.2 A construction for non-rank facets of stable set polytopes of webs

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Graphs with circular symmetry, called webs, are relevant w.r.t. describing the stable set polytopes of two larger graph classes, quasi-line graphs [9, 15] and claw-free graphs [8, 9]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [10]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [6, 11] while there are examples with clique number ≥ 4 having non-rank facets [12, 13, 15].

In this paper, we provide a construction for non-rank facets of stable set polytopes of webs. This construction is the main tool to obtain in a companion paper [17], for all fixed values of $\omega \geq 5$ that there are only finitely many webs with clique number ω whose stable set polytopes admit rank facets only.

Introduction

Graphs with circular symmetry of their maximum cliques and stable sets are called *webs*: a web W_n^k is a graph with vertices $1, \ldots, n$ where ij is an edge if i and j differ by at most k (mod n) and $i \neq j$. The webs W_9^k on nine vertices are depicted in Figure B.2. Notice that webs are also called circulant graphs C_n^k in [4] and that similar graphs $W(n, k)$ were introduced in [11].

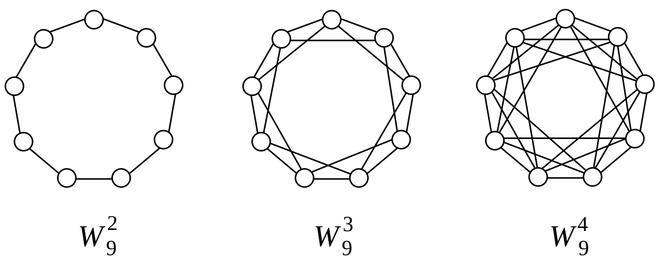


Figure B.4: Some webs

Webs and line graphs belong to the classes of quasi-line graphs and claw-free graphs and are relevant w.r.t. describing the stable set polytopes of those larger graph classes [8, 9, 15], as described in the sequel (all definitions are provided in the following section). The *stable set polytope* $\text{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of the graph G. In order to describe STAB(G) by means of facet-defining inequalities, the "trivial" facets $x_i \geq 0$ for all vertices i of G and the clique constraints

$$\sum_{i \in Q} x_i \leq 1$$

for all cliques $Q \subseteq G$ are necessary. These two types of facets are sufficient to describe STAB(G) for perfect graphs G only [3]. A natural way to generalize clique constraints is to investigate *rank constraints*, that are 0/1-constraints of the form

$$\sum_{i \in G'} x_i \leq \alpha(G')$$

associated with arbitrary induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G' (note $\alpha(G') = 1$ holds iff G' is a clique, as in stable sets all vertices are mutually non-adjacent but in cliques mutually adjacent). A graph is *rank-perfect* if all non-trivial facets of its stable set polytope are rank constraints. The class of rank-perfect graphs contains all perfect graphs [3], odd holes and odd antiholes [16], line graphs [7], and the complements of webs [19].

A characterization of the *rank* facets in stable set polytopes of claw-free graphs was given by Galluccio & Sassano [8]. They showed that all rank facets can be constructed by means of standard operations from rank constraints associated with cliques, partitionable webs, or line graphs of 2-connected, critical hypomatchable graphs. However, we are still far from having a complete description for the stable set polytopes of webs and, therefore, of quasi-line and claw-free graphs, too. Finding a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem (Grötschel, Lovász, and Schrijver [10]). Claw-free graphs are not rank-perfect: Giles & Trotter [9], Oriolo [15], and Liebling et al. [13] found non-rank facets which occur even in the stable set polytopes of quasi-line graphs. These non-rank facets rely on combinations of joined webs.

Several further authors studied the stable set polytopes of webs. The webs W_n^1 with clique number 2 are either perfect or odd holes and, therefore, rank-perfect due to [3, 16]. (Notice that the clique number, i.e. the size of a maximum clique, of a web W_n^k is $k+1$.) Dahl [6] showed that the webs W_n^2 with clique number 3 are rank-perfect as well. On the other hand, Kind [12] found (by means of the PORTA software[2]) examples of webs with clique number > 4 which are *not* rank-perfect, e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9. Oriolo [15], Liebling et al. [13], and Pêcher & Wagler [18] presented further examples of such webs.

The main contribution of this paper (Theorem B.2.1) is a construction that enables us to obtain, from certain non-rank-perfect webs W_n^k, an infinite sequence of non-rank-perfect webs $W_{n+(k+1)}^k, W_{n+2(k+1)}^k, W_{n+3(k+1)}^k, \dots$ with the same clique number. To be more precise, we introduce the notion of *proper weak non-rank facets*. A facet $\mathbf{a}^T\mathbf{x} \leq c\alpha(G')$ of STAB(G) is a *weak rank facet* w.r.t. $G' \subseteq G$, if $a_i = c$ for every vertex i of G' and if G' is rank facet-producing (i.e. $\sum_{i \in V(G')} x_i \leq \alpha(G')$ defines a facet of STAB(G')). A weak rank facet is *proper* if G' is not a clique and *non-rank* if it cannot be scaled to have 0/1-coefficients only (i.e., it is not a rank constraint).

Theorem B.2.1. *If STAB(W_n^k) has a proper weak non-rank facet then STAB(W_{n+k+1}^k) has a proper weak non-rank facet.*

Therefore, if W_n^k has a proper weak non-rank facet then all webs $W_{n+\lambda(k+1)}^k$ $(\lambda \geq 0)$ are not rank-perfect, too. Hence Theorem B.2.1 implies the following corollary:

Corollary B.2.2. *If there are $k+1$ webs $W_{n_0}^k, \dots, W_{n_k}^k$ such that*
- *STAB$(W_{n_i}^k)$ has a proper weak non-rank facet*
- $n_i = i \pmod{k+1}$

then all webs W_n^k with $n \geq \max\{n_0, \dots, n_k\} - k$ are not rank-perfect.

That means in particular: if we are able to provide such a set of $k+1$ webs for a certain value of k, then there exist only finitely many rank-perfect webs W_n^k. For $k = 3$, this follows from [18] where an infinite sequence of not rank-perfect webs with clique number 4 is presented, namely W_{33}^3, W_{42}^3, W_{51}^3, W_{60}^3, ... Hence, by Corollary 2, all webs W_n^3 with $n > 56$ are not rank-perfect and there exist only finitely many rank-perfect webs W_n^3. Similar results for all remaining values $k \geq 4$ are given in the companion paper [17]. Applying Corollary B.2.2 implies:

[2] By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

Theorem B.2.3. *[18, 17] For each $\omega \geq 4$, there are only finitely many rank-perfect webs with clique number ω, hence, almost all of them are not rank-perfect.*

The paper is organized as follows: subsection 2 is devoted to definitions and some general results which are frequently used in the sequel. The proof of Theorem B.2.1 is given in subsection 3 and we briefly discuss open problems in subsection 4.

Definitions and general results

Let $G = (V, E)$ be a graph. If V' is any subset of the vertex set V, we denote by $G[V']$ the subgraph of G induced by V'.

Recall that a web W_n^k is a graph with vertices $1, \ldots, n$ where ij is an edge if i and j differ by at most k (mod n) and $i \neq j$. Webs are natural generalizations of *odd holes* and *odd antiholes*, that are chordless odd cycles of length ≥ 5 and their complements. Perfect graphs are precisely the graphs without odd holes and odd antiholes as induced subgraphs [2].

The clique number of a web W_n^k is $k+1$ and the stability number is $\lfloor \frac{n}{k+1} \rfloor$. Unless stated otherwise, arithmetics are always performed modulo the number of vertices of the web involved in the computation. Let $1 \leq a, b \leq n$ be two vertices of a web W_n^k. We denote by $[a, b]$ the set of vertices $\{a, a+1, a+2, \ldots, b\}$, and by Q_a the maximum clique $[a, a+k]$.

Trotter characterized in [11] (Corollary 3.2) when a vertex subset V' of a web W_n^k induces a subweb $W_{|V'|}^{k'}$, namely, if and only if $|Q_i \cap V'| = k' + 1$ for any vertex $i \in V'$. We specify, for our purposes, the necessary condition as follows. For any finite set X, we denote its cardinality by $|X|$.

Lemma B.2.4. *Consider a web W with clique number ω and a set V' of vertices of W. Then V' induces a subweb W' of W with clique number ω'*

1. *if $|Q_i \cap V'| = \omega'$ for all $i \in V'$;*
2. *only if $|Q_i \cap V'| = \omega'$ for all $i \in V'$ and $|Q_i \cap V'| \geq \omega' - 1$ for all $i \notin V'$.*

Proof. The If-part is as Trotter's result in [11]. For the Only if-part, consider first $i \in V'$. Obviously $|Q_i \cap V'| \leq \omega'$ as $Q_i \cap V'$ is a clique of W'. If $j \notin Q_{i-\omega-1} \cup Q_i$ then j is not a neighbor of i in W'. Therefore the $2\omega' - 2$ neighbors of i in W' are exactly the set $((Q_{i-\omega-1} \cup Q_i) \cap V') - \{i\}$. Thus, $|Q_i \cap V'| = \omega'$. Now, consider $i \notin V'$. Let i' be the element of V' such that $[i'+1, i] \cap V' = \emptyset$. Since $Q_{i'} \cap V' \subseteq \{i'\} \cup Q_i$, we have $|Q_i \cap V'| \geq \omega' - 1$. □

Webs and line graphs belong to the classes of *quasi-line graphs* (the neighborhood of any vertex can be partitioned into two cliques) and *claw-free graphs* (the neighborhood of any vertex does not contain a stable set of size 3). The *line graph* $L(H)$ of a graph H is obtained by taking the edges of H as vertices of $L(H)$ and connecting two vertices in $L(H)$ iff the corresponding edges of H are incident. Webs and line graphs are relevant w.r.t. describing the stable set polytopes of those larger graph classes [8, 9, 15], see the next subsection.

Recall that the stable set polytope STAB(G) is the convex hull of the incidence vectors $\chi^{\mathbf{S}}$ of all stable sets S of G. We denote by $\mathbf{a}^T$ the transposed row vector of any column vector $\mathbf{a}$. An inequality $\mathbf{a}^T\mathbf{x} \leq b$ is said to be *valid* for STAB(G), if $\mathbf{a}^T\chi^{\mathbf{S}} \leq b$ holds for all stable sets S of G. A *root* of a valid inequality $\mathbf{a}^T\mathbf{x} \leq b$ is a stable set S such that $\mathbf{a}^T\chi^{\mathbf{S}} = b$. A valid inequality $\mathbf{a}^T\mathbf{x} \leq b$ for STAB(G) is a *facet* if and only if it has $|V(G)|$ roots with affinely independent incidence vectors (note that they have to be *linearly* independent if $b > 0$).

Let $G = (V, E)$ be a graph, F be a family of (at least three inclusion-wise) maximal cliques of G, $p \leq |F|$ be an integer, and define two sets as follows:

$$I(F,p) = \{i \in V : |\{Q \in F : i \in Q\}| \geq p\}$$
$$O(F,p) = \{i \in V : |\{Q \in F : i \in Q\}| = p - 1\}$$

Oriolo [15] showed that the *clique family inequality*

$$(p-r) \sum_{i \in I(F,p)} x_i + (p-r-1) \sum_{i \in O(F,p)} x_i \leq (p-r) \lfloor \frac{|F|}{p} \rfloor \tag{B.5}$$

is valid for the stable set polytope of *every* graph G where $r = |F| \bmod p$. A conjecture due to Ben Rebea says that the stable set polytopes of quasi-line graphs have clique family inequalities as only non-trivial facets, see [15].

All matrices in this paper have rational coefficients (in fact integer coefficients). If M is any square matrix, then $|M|$ stands for the determinant of M.

Rank-minimal facets of webs

Following Galluccio & Sassano [8], an inequality $\sum_{i \in V} x_i \leq \alpha(G)$ associated with a graph G with vertex set V and the graph G itself are called *rank-minimal* if and only if G is a clique or satisfies

1. $\sum_{i \in V} x_i \leq \alpha(G)$ defines a facet of STAB(G), i.e., G is rank facet-producing;
2. for each $V' \subset V$, the inequality $\sum_{i \in V'} x_i \leq \alpha(G)$ does not define a facet of STAB$(G[V'])$.

All rank-minimal claw-free graphs were described in [8]. In order to state the theorem, we need the following notations.

A graph G is said to be *partitionable* if there exist two integers p and q such that G has $pq + 1$ vertices and for every vertex v of G, the induced subgraph $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p. The webs $W^{\omega-1}_{\alpha\omega+1}$ with $\alpha, \omega > 1$ are examples of partitionable graphs, including all odd holes $W^1_{2\alpha+1}$ and all odd antiholes $W^{\omega-1}_{2\omega+1}$.

A graph H is called *hypomatchable* if it does not admit a perfect matching but $H - v$ does for all vertices $v \in V(H)$ (a matching is perfect if it meets all vertices of the graph). A hypomatchable graph H is called *critical* if $H - e$ is not hypomatchable anymore for all edges $e \in E(H)$.

Theorem B.2.5. *[8] Every rank-minimal claw-free graph is*

- *a clique,*
- *a partitionable web, or*
- *the line graph of a 2-connected, critical hypomatchable graph.*

We are interested in the question which rank-minimal graphs may occur as induced subgraphs of webs (recall: every web is in particular claw-free). It turns out that we essentially can exclude the third alternative of Theorem B.2.5 due to the next lemma:

Lemma B.2.6. *Let H be a 2-connected, critical hypomatchable graph. If its line graph $L(H)$ is an induced subgraph of a web, then $L(H)$ is a triangle or an odd hole.*

Proof. Consider a 2-connected, critical hypomatchable graph H. Since H is 2-connected, H has at least 3 vertices. Since H is critical hypomatchable, H must not admit parallel edges, i.e., H is simple. If $|H| = 3$, then H as well as $L(H)$ is a triangle. Hence assume $|H| \geq 5$ in the sequel (note: every hypomatchable graph has an odd number of vertices). We show that H as well as $L(H)$ is an odd hole if $L(H)$ is an induced subgraph of a web.

Due to Lovász [14], a graph H is hypomatchable if and only if there is a sequence $H_0, H_1, \ldots, H_k = H$ of graphs such that H_0 is a chordless odd cycle and for $1 \leq i \leq k$, H_i is obtained from H_{i-1} by adding a chordless odd path E_i that joins two (not necessarily distinct) vertices of H_{i-1} and has all internal vertices outside H_{i-1}. The odd paths $E_i = H_i - H_{i-1}$ are called *ears* for $1 \leq i \leq k$ and the sequence $H_0, H_1, \ldots, H_k = H$ an *ear decomposition* of H.

If a hypomatchable graph H is 2-connected and has at least 5 vertices, then H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$ s.t. every H_i is 2-connected for $0 \leq i \leq k$ by Cornuéjols & Pulleyblank [5] and H_0 is an odd hole (i.e. $|H_0| \geq 5$) by [20]. Moreover, in [20] is shown that we can always reorder the ears $E_1, \ldots E_k$ of a given decomposition s.t. the decomposition starts with all ears of length ≥ 3 and ends up with all ears of length one. Thus, every 2-connected hypomatchable graph H with $|V(H)| \geq 5$ has a *proper ear decomposition* $H_0, H_1, \ldots, H_k = H$ where H_0 has length ≥ 5, each H_i is 2-connected, and, if $k > 0$, there is an index j s.t. $E_1, \ldots, E_j$ have length ≥ 3 and $E_{j+1}, \ldots, E_k$ have length one.

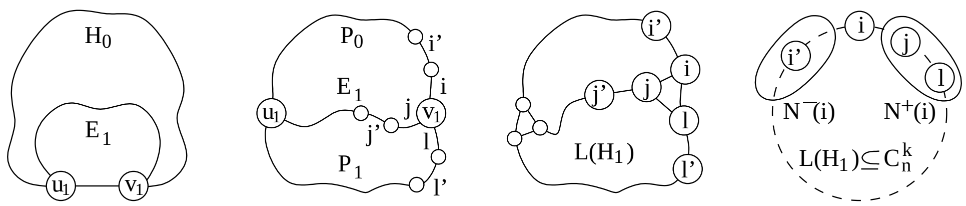
Consider a 2-connected hypomatchable graph H with $|V(H)| \geq 5$ and a proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H. We show in the next two claims: the decomposition of H has neither ears of length 1 nor of length ≥ 3 if H is critical and $L(H)$ is an induced subgraph of a web.

Claim 1. *If $H_0, H_1, \ldots, H_k = H$ contains an ear of length 1, then H is not critical hypomatchable.*

In that case, the last ear E_k of the proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H is a single edge. Removing the edge E_k from $H_k = H$ yields the hypomatchable graph H_{k-1} with the same vertex set. Thus, H is not critical hypomatchable. $\diamond$

Claim 2. *If $H_0, H_1, \ldots, H_k = H$ contains an ear of length ≥ 3, then H is not critical hypomatchable or $L(H)$ is not an induced subgraph of a web.*

In that case, the first ear E_1 of the proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H is a path of length ≥ 3. If the endvertices u_1 and v_1 of E_1 are adjacent in H_0 (see Fig. B.2(a)), then H admits a proper ear decomposition $H'_0, H'_1, \ldots, H'_k = H$ with $H'_0 = H_0 \cup E_1 - \{u_1v_1\}$ and $E_2, \ldots, E_k, \{u_1v_1\}$ as ear sequence (i.e. $H'_i = H'_{i-1} \cup E_{i+1}$ for $1 \leq i \leq k-1$ and $H'_k = H'_{k-1} \cup \{u_1v_1\}$). Thus, H admits an ear of length 1 and is not critical hypomatchable by Claim 1.



(a) (b) (c) (d)

Figure B.5: Some webs

If the endvertices u_1 and v_1 of E_1 are non-adjacent in H_0 (see Fig. B.2(b)), then there are 3 internally disjoint paths P_0, P_1, E_1 between u_1 and v_1 in H_1: P_0 with even length ≥ 2 and P_1, E_1 with odd length ≥ 3. Consider in H_1 the edges i, i', j, j', l, l' as shown in Fig. B.2(b). Then the edges i', j', l' are pairwise disjoint (note: u_1 may be an endvertex of i' but neither of j' nor of l' because of the parity of the paths).

Assume $L(H_1)$ is an induced subgraph of a web W_n^k. We have to find a respective order of the vertices i, i', j, j', l, l' in W_n^k (recall that the line operator transforms edges of H into vertices of $L(H)$, see Fig. B.2(c)). Moreover, recall that the neighborhood of every vertex x, denoted by $N(x)$, of a web W_n^k splits into two cliques $N^-(x) = \{x-k, \ldots, x-1\}$ and $N^+(x) = \{x+1, \ldots, x+k\}$ (where all indices are taken modulo n).

Consider $N(i)$ in W_n^k: we have $i', j, l \in N(i)$ where jl is an edge but neither $i'j$ nor $i'l$ (see Fig. B.2(c)). W.l.o.g. let $i' \in N^-(i)$. Then $j, l \in N^+(i)$ follows since both $N^-(i)$ and $N^+(i)$ are cliques. Furthermore, let $j < l$ (the case $l < j$ goes analogously due to $ij, il \in E$ but $ij', il' \notin E$), i.e., assume $i+1 \leq j < l \leq i+k$ (see Fig. B.2(d)).

Now, consider the vertex j'. We have $j' \in N(j)$ but $j' \notin N(i)$ (see Fig. B.2(c)). This implies $j' \in N^+(j)$ (since $N^-(j) \subseteq N(i)$ by $j \in N^+(i)$), i.e., we obtain $j' \in \{j+1, \ldots, j+k\}$. But $i+1 \leq j < l \leq i+k$ implies $N^+(j) \subseteq N(l)$, hence $j' \in N(l)$ in contradiction to j' and l non-adjacent (see Fig. B.2(c)). Thus, $L(H_1)$ cannot be an induced subgraph of a web W_n^k.

We conclude: if E_1 connects two adjacent vertices of H_0, then H is not critical, if E_1 connects two non-adjacent vertices of H_0, then $L(H)$ is not an induced subgraph of a web. $\diamond$

Hence, we have obtained that for every 2-connected, critical hypomatchable graph H holds the following. If H has 3 vertices, then H and its line graph $L(H)$ are triangles. Otherwise, H admits a proper ear decomposition $H_0, H_1, \ldots, H_k = H$ with and index j s.t. $E_1, \ldots, E_j$ have length ≥ 3 and $E_{j+1}, \ldots, E_k$ have length one. By Claim 1, there is no ear of length 1 (i.e. $j = k$). If the line graph of H is an induced subgraph of a web, then there is no ear of length ≥ 3 by Claim 1 and Claim 2 (i.e. $j = 0$). In conclusion, we obtain $k = 0$, thus H consists in the odd hole H_0 of length ≥ 5 only and $L(H)$ is an odd hole, too. $\square$

Remark. Claim 1 of Lemma B.2.6 shows: if the last ear E_k of a proper ear decomposition $H_0, H_1, \ldots, H_k = H$ of H has length one, then H is not critical hypomatchable. $L(H)$ is not rank-minimal by Theorem B.2.5 in particular. The reason is the following: the graph H_{k-1} obtained by removing the edge E_k from H is 2-connected and hypomatchable, hence $L(H_{k-1})$ is rank facet-producing by Edmonds & Pulleyblank [7]. Furthermore, $\frac{|H|-1}{2} = \alpha(L(H)) = \alpha(L(H_{k-1}))$ holds by $V(H) = V(H_{k-1})$, hence $L(H)$ cannot be rank-minimal.

Since odd holes are partitionable webs, Theorem B.2.5 and Lemma B.2.6 imply the following corollary:

Corollary B.2.7. *Every rank-minimal induced subgraph of a web is a clique or a partitionable web.*

Weak rank facets of webs

Recall that a facet $\mathbf{a}^T\mathbf{x} \leq c\alpha(G')$ of STAB(G) is a *weak rank facet* w.r.t. $G' \subseteq G$, if $a_i = c$ for every vertex i of G' and if G' is rank facet-producing.

Lemma B.2.8. *Let $\mathbf{a}^T\mathbf{x} \leq c\alpha(G[V'])$ be a weak rank facet of the stable set polytope of a web G. Then $c = \max\{a_i |\ i \in V(G)\}$.*

Proof. Let $\alpha' = \alpha(G[V'])$. By Corollary B.2.7, $G[V']$ contains a rank-minimal subgraph W with $\alpha(W) = \alpha'$, which is a clique or a partitionable web. If W is a clique then $\alpha' = 1$ and it follows that $a_i \leq c$ for every vertex i, due to the stable set $\{i\}$. Hence $c = \max\{a_i |\ i \in V(G)\}$.

If W is a partitionable web then let ω' be the clique number of W. We say that two vertices a and b of W are consecutive if $[a, b] \cap W = \{a, b\}$. Obviously, there is a labeling $\{w_1, \ldots, w_{|W|}\}$ of the vertices of W such that for every $1 \leq i \leq |W|$, w_i and w_{i+1} are consecutive (with arithmetics performed modulo $|W|$).

For every $1 \leq i \leq |W|$, let $S_i = \{w_{i+\omega'+1}, w_{i+2\omega'+1}, \ldots, w_{i+(\alpha'-1)\omega'+1}\}$ (with indices taken modulo $|W|$). Notice that S_i is a stable set of G (due to the labeling, if w_a and w_b are adjacent and $a \leq b$ then $w_a, w_{a+1}, \ldots, w_b$ is a clique of W). Since $|W| = \alpha'\omega' + 1$, we have that $w_i \notin N_G(S_i)$ and $w_{i+1} \notin N_G(S_i)$. It follows that for every u in $[w_i, w_{i+1}]$, the set $S_i' := S_i \cup \{u\}$ is a stable set of G. Since $\mathbf{a}^T\chi^{\mathbf{S_i'}} \leq c\alpha'$, we get $c(\alpha' - 1) + a_u \leq c\alpha'$. Thus $a_u \leq c$. Therefore, spanning all consecutive pairs of W we obtain $c = \max\{a_i |\ i \in V(G)\}$, as required. □

A general characterization of facets

The next lemma provides a characterization when a valid inequality $\mathbf{a}^T\mathbf{x} \leq b$ is a facet of the stable set polytope of a general graph G. For that we need the following notions. A pair i, j of vertices is $\mathbf{a}$-*critical* in G if there are two roots S_1 and S_2 of $\mathbf{a}^T\mathbf{x} \leq b$ such that $\{i\} = S_1 \setminus S_2$ and $\{j\} = S_2 \setminus S_1$. A subset V' of $V(G)$ is $\mathbf{a}$-*connected* if the graph with vertex set V' and edge set $\{ij |\ i, j \in V',\ ij\ \mathbf{a}\text{-critical in } G\}$ is connected.

Lemma B.2.9. *Let $\mathbf{a}^T\mathbf{x} \leq b$ be a valid inequality for STAB(G) with $b \neq 0$. Consider a partition $V_1, \ldots, V_p$ of $V(G)$ such that V_i is $\mathbf{a}$-connected for every $1 \leq i \leq p$. The inequality $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining if and only if there are p roots $S_1, \ldots, S_p$ with*

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0.$$

Proof. In order to prove the If-part, let $\mathbf{a'}^T\mathbf{x} \leq b'$ be a facet containing the face induced by the inequality $\mathbf{a}^T\mathbf{x} \leq b$. For every $1 \leq i \leq p$, the set V_i is $\mathbf{a}$-connected and so there exist λ_i such that $a_j = \lambda_i$ for all $j \in V_i$. Since for every stable set S, $\mathbf{a}^T\chi^{\mathbf{S}} = b$ implies that $\mathbf{a'}^T\chi^{\mathbf{S}} = b'$, V_i is $\mathbf{a}'$-connected. Therefore there exist λ_i' such that $a_j' = \lambda_i'$ for all $j \in V_i$. Hence we have for every $1 \leq i \leq p$:

$$\begin{array}{rcl} \lambda_1|S_i \cap V_1| + \ldots + \lambda_p|S_i \cap V_p| & = & b \\ \lambda_1'|S_i \cap V_1| + \ldots + \lambda_p'|S_i \cap V_p| & = & b' \end{array}$$

Since

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0$$

holds we get $\lambda_i' = \frac{b'}{b}\lambda_i$ for every $1 \leq i \leq b$. Thus $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining.

Now let us turn to the Only if-part. Since $\emptyset$ is not a root of the facet $\mathbf{a}^T\mathbf{x} \leq b$, there exist n roots $S_1, \ldots S_n$ whose incidence vectors are linearly independent. Let M be the matrix with the incidence vectors of $S_1, \ldots, S_n$ as rows. Let v_i be an element of V_i for $1 \leq i \leq p$. We add to the v_1-th column of M the other columns related to the other elements of V_1; we add to the v_2-th column of M the other columns related to the other elements of V_2 etc. This yields

$$\begin{vmatrix} . & |S_1 \cap V_1| & . & |S_1 \cap V_p| \\ . & \vdots & . & \vdots \\ . & |S_n \cap V_1| & . & |S_n \cap V_p| \end{vmatrix} \neq 0$$

and, thus, the (n, p)-matrix

$$\begin{pmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_n \cap V_1| & \cdots & |S_n \cap V_p| \end{pmatrix}$$

has p linearly independent rows, as required. □

Notice that Lemma B.2.9 generalizes the following well-known result of Chvátal [3] on critical edges which, in fact, inspirated Lemma B.2.9. An edge of a graph is *critical* if its deletion increases the stability number.

Theorem B.2.10. *[3] Let $G = (V, E)$ be a graph and E^* be the set of its critical edges. If $G^* = (V, E^*)$ is connected then G is rank facet-producing.*

The main result

In this subsection, we prove a more precise formulation of Theorem B.2.1.

Theorem B.2.11. *Let $\mathbf{a}^T\mathbf{x} \leq c\alpha_1$ be a proper weak rank facet of STAB(W_n^k). Then STAB(W_{n+k+1}^k) has the proper weak rank facet*

$$\sum_{1 \leq i \leq n} a_i x_i + \sum_{n < i \leq n+k+1} c\, x_i \;\leq\; c\,(\alpha_1 + 1) \tag{B.6}$$

Example. Consider the non-rank-perfect web with the least number of vertices, namely W_{25}^5. Its stable set polytope admits the following non-rank facet:

$$(2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2)^T\mathbf{x} \leq 6$$

Let V_1 be the set of vertices corresponding to the coefficients with value 2 (i.e., to the black vertices in Fig. B.6(a)). Notice that $G[V_1]$ is isomorphic to the partitionable web W_{10}^2 which is in particular rank facet-producing. Hence the above facet is a proper weak rank facet with $c = 2$, $\alpha(G[V'']) = 3$ and Theorem B.2.11 implies that

$$(2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,1,1,1,2,2,2,2,2,2,2)^T\mathbf{x} \leq 8$$

is a proper weak rank facet of STAB(W_{31}^5) (the vertices with coefficient 2 correspond to the black vertices in Fig. B.6(b)). We can, therefore, iteratively apply Theorem B.2.11 and obtain a sequence of non-rank-perfect webs: W_{25}^5, W_{31}^5, W_{37}^5, ...

Proof of Theorem B.2.11. By definition, the vertex set of W_n^k is $\{1, \ldots, n\}$ and the vertex set of W_{n+k+1}^k is $\{1, \ldots, n+k+1\}$. Hence we may use this convention to identify a vertex of W_n^k with the corresponding one of

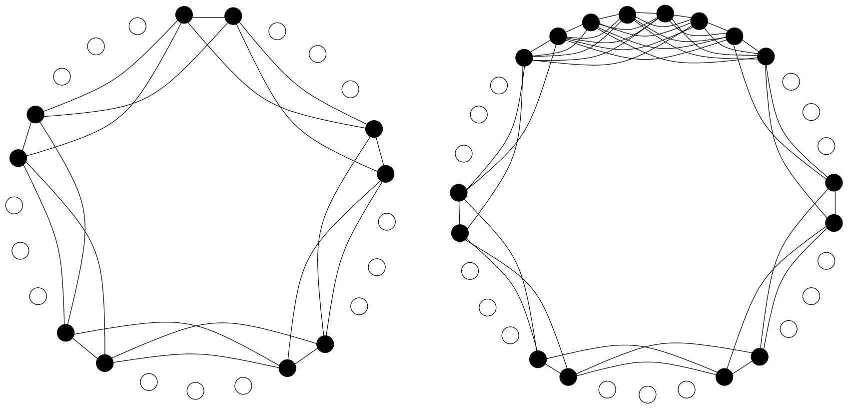


Figure B.6: Some webs

W^k_{n+k+1}. Denote by G^1 the web W^k_n and by G^2 the web W^k_{n+k+1}. Let $\omega = k+1$ be the clique number of both G^1 and G^2 and, for every $1 \le i \le n$ (resp. $1 \le i \le n+\omega$), let $Q^1_i = [i, i+k]$ (resp. $Q^2_i = [i, i+k]$) be the maximum clique of G^1 (resp. G^2) with 'first' element i.

Since $\mathbf{a}^T\mathbf{x} \le c\alpha_1$ is a proper weak rank facet of STAB(G^1), there exists a subset V_1 of vertices of G^1 such that $\alpha_1 = \alpha(G^1[V_1])$ and $G^1[V_1]$ is rank facet-producing. Moreover, $G^1[V_1]$ has a partitionable web with vertex set W_1, stability number α_1, and clique number $\omega_1 \ge 2$ as induced subgraph by Corollary B.2.7.

Notice that Q^1_{n-k} is the maximum clique $\{n-k, \dots, n\}$ of G^1. Let $w_1, \dots, w_h$ be the elements in increasing order of W_1 in Q^1_{n-k}. We have $h = \omega_1$ or $\omega_1 - 1$, by Lemma B.2.4. For every $1 \le i \le h$, let q_i be the element $w_i + \omega$ of Q^2_{n+1} and define:

$$W_2 = \begin{cases} W_1 \cup \{q_1, \dots, q_{\omega_1}\} & \text{if } h = \omega_1 \\ W_1 \cup \{n+1\} \cup \{q_1, \dots, q_{\omega_1 - 1}\} & \text{if } h = \omega_1 - 1 \end{cases}$$

Let $V_2 = V_1 \cup Q^2_{n+1} = V_1 \cup \{n+1, \dots, n+k+1\}$. Let $\mathbf{v}$ be the $(n+\omega)$-column vector $(a_1, \dots, a_n, c, \dots, c)$ and $\mathbf{y}$ be the $(n+\omega)$-column vector $(a_1, \dots, a_n, 0, \dots, 0)$.

Claim B.2.12. *Inequality (B.6) is valid for STAB(W^k_{n+k+1}).*

Let S be any stable set of G^2. Let l be the vertex of S such that $[l+1, n] \cap S = \emptyset$ and let t be the vertex of S such that $[n+1, t-1] \cap S = \emptyset$. Notice that $S \setminus \{t\}$ is a stable set of G^1. Hence we have $\mathbf{v}^T\chi^{\mathbf{S}} = (\mathbf{y} + c\chi^{\mathbf{Q^2_{n+1}}})^T\chi^{\mathbf{S}} \le c\alpha_1 + x_t \le c(\alpha_1 + 1)$ as $x_t \le c$ if $t \notin Q^2_{n+1}$ by Lemma B.2.8, and $x_t = c$ if $t \in Q^2_{n+1}$. $\diamond$

Claim B.2.13. *The set of vertices W_2 induces a partitionable web with stability number $\alpha_1 + 1$ and clique number ω_1.*

Let $1 \le v_1 \le v_2 \le \dots \le v_{n'} \le n$ be the vertices of W_1 in increasing order. We discuss the two cases $h = \omega_1$ and $h = \omega_1 - 1$.

If $h = \omega_1$ then let v be any vertex of W_2. If v is a vertex q_i of $\{q_1, \dots, q_{\omega_1}\}$ then the set of vertices Q^2_v meets W_2 exactly in the ω_1 vertices $\{q_i, \dots, q_{\omega_1}\} \cup \{v_1, \dots, v_{i-1}\}$, since W_1 induces a web of G^1 with clique number ω_1 by Lemma B.2.4 (see Fig. B.7). If v is a vertex w_i of $\{w_1, \dots, w_{\omega_1}\}$ then the set of vertices Q^2_v meets W_2 precisely in the ω_1 vertices $\{w_i, \dots, w_{\omega_1}\} \cup \{q_1, \dots, q_{i-1}\}$. If v is a vertex of $W_1 \setminus \{w_1, \dots, w_{\omega_1}\}$, we obviously have $|Q_v \cap W_2| = \omega_1$ since W_1 induces a web of G^1 with clique number ω_1 due to Lemma B.2.4.

If $h = \omega_1 - 1$ then notice that $w_1 \ne n-k$ (otherwise Lemma B.2.4 would imply $h = \omega_1$). Hence $n+1 \notin \{q_1, \dots, q_h\}$. Let v be any vertex of W_2. If v is a vertex q_i of $\{q_1, \dots, q_h\}$ then the set of vertices Q^2_v meets W_2 exactly in the ω_1 vertices $\{q_i, \dots, q_h\} \cup \{v_1, v_2, \dots, v_{i-1}, v_i\}$, since W_1 induces a web of G^1 with clique number

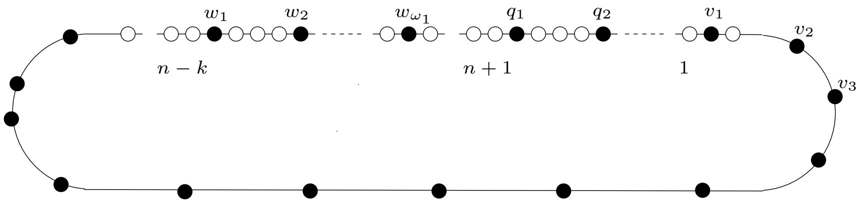


Figure B.7: Construction of W_2, case $h = \omega_1$ (vertices of W_2 are drawn in black)

ω_1 by Lemma B.2.4 (see Fig. B.8). If v is a vertex w_i of $\{w_1, \ldots, w_h\}$ then the set of vertices Q_v meets W_2 precisely in the ω_1 vertices $\{w_i, \ldots, w_h, n+1\} \cup \{q_1, \ldots, q_{i-1}\}$, as $w_h < n+1 < q_1$. If $v = n+1$ then the set of vertices Q_v meets W_2 exactly in the ω_1 vertices $\{n+1, q_1, \ldots, q_h\}$. If v is a vertex of $W_1 \setminus \{w_1, \ldots, w_h\}$, we obviously have $|Q_v \cap W_2| = \omega_1$ since W_1 induces a web of G^1 with clique number ω_1 due to Lemma B.2.4.

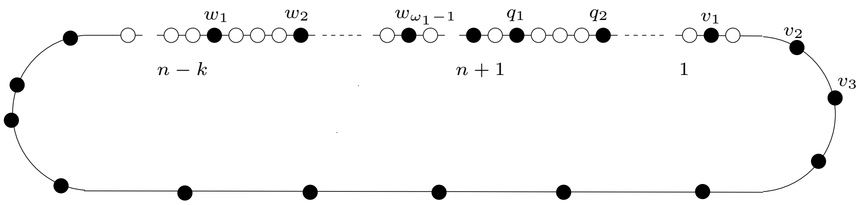


Figure B.8: Construction of W_2, case $h = \omega_1 - 1$ (vertices of W_2 are drawn in black)

Hence in both cases, W_2 induces a web with clique number ω_1 (Lemma B.2.4), with $|W|+\omega_1 = (\alpha_1+1)\omega_1+1$ vertices. Thus W_2 induces a partitionable web with stability number $\alpha_1 + 1$. $\diamond$

Claim B.2.14. *The vertex set $V_2 = W_2 \cup Q^2_{n+1}$ is $\mathbf{v}$-connected.*

We first show that W_2 is $\mathbf{v}$-connected. Since $\mathbf{a}^T\mathbf{x} \le c\alpha_1$ is a weak rank facet of STAB(G^1), we have by definition $a_i = c$ for every $i \in W_1$. Hence for every $i \in W_2$ follows $v_i = c$. Since W_2 is a partitionable web of stability number $\alpha_1 + 1$ by Claim B.2.13, this implies that W_2 is $\mathbf{v}$-connected.

Let $w_1 < w_2 < \ldots < w_{\omega_1}$ be the elements of W_2 in Q^2_{n+1} (by definition of W_2 there are exactly ω_1 of them). Let S be a maximum stable set of W_2 disjoint from Q^2_1 (S exists because $W_2 \cap Q^2_1$ is a subset of a maximum clique of W_2, and for every maximum clique Q of a partitionable graph, there exists a unique maximum stable set avoiding Q by [1]). Let s be the element of S with maximal index. Then for every $w_{\omega_1} \le q \le n+\omega$, the set $(S \setminus \{s\}) \cup \{q\})$ is obviously a root of inequality (B.6). Hence $W^2 \cup [w_{\omega_1}, n+\omega]$ is $\mathbf{v}$-connected. Likewise, the set $W^2 \cup [n+1, w_1]$ is $\mathbf{v}$-connected.

For every $1 \le i < \omega_1$, there exists a maximum stable set of W_2 disjoint from $Q^2_{w_{i+1}}$. Let s be the element of S with maximal index which is less than or equal to w_i. Then for every $w_i \le q \le w_{i+1}$, the set $(S \setminus \{s\}) \cup \{q\}$ is a root of inequality (B.6). Hence $W^2 \cup [w_i, w_{i+1}]$ is $\mathbf{v}$-connected and V_2 is $\mathbf{v}$-connected as well. $\diamond$

Let $p = n - |W_1|$ and $\{1, \ldots, n\} \setminus W_1 = \{y_1, \ldots, y_p\}$. Due to Lemma B.2.9, there are p roots $S_1, \ldots, S_p$ of $\mathbf{a}^T\mathbf{x} \le c\alpha_1$ such that the incidence vectors of their restriction to $\{1, \ldots, n\} \setminus W_1 = (\{1, \ldots, n\} \cup Q_n\}) \setminus V_2$ are linearly independent, that is

$$\begin{vmatrix} |S_1 \cap \{y_1\}| & \cdots & |S_1 \cap \{y_p\}| \\ \vdots & & \vdots \\ |S_{k'} \cap \{y_1\}| & \cdots & |S_{k'} \cap \{y_p\}|\} \end{vmatrix} \neq 0$$

Claim B.2.15. *For every $1 \le i \le p$, there exists a vertex q_i of G^2 such that $S'_i = S_i \cup \{q_i\}$ is a root of inequality (B.6).*

For every $1 \leq i \leq p$, let l_i (resp. t_i) be the element of S_i with minimal (resp. maximal) index. Let $q_i = t_i + \omega$. Obviously, q_i is not a neighbor of t_i in G^2. If q_i is a neighbor of l_i in G^2 then $q_i + \omega - 1 - (n + \omega) \geq l_i$. Thus $t_i + \omega - 1 - n \geq l_i$, which implies that t_i is a neighbor of l_i in G^1: a contradiction.

Hence $S'_i = S_i \cup \{q_i\}$ is a stable set of G^2. Since q_i is a vertex of the maximum clique Q_n, it follows that S'_i is a root of inequality (B.6), as required. $\diamond$

Since $G^2[W_2]$ has stability number $\alpha_1 + 1$ (Claim B.2.13), there is a stable set S'_0 of $G^2[V_2]$ which is a root of inequality (B.6).

For every $0 \leq i \leq p$ and $1 \leq j \leq p$, let $\delta_{i,j} = 1$ if $y_j \in S'_i$, 0 otherwise. By Claim B.2.12 and B.2.15, inequality (B.6) is a valid inequality with $p+1$ **v**-critical components $V_2, \{y_1\}, \ldots, \{y_p\}$, and $p+1$ roots $S'_0, S'_1, \ldots, S'_p$ such that

$$\begin{vmatrix} |S'_0 \cap V_2| & \delta_{0,1} & \cdots & \delta_{0,p} \\ |S'_1 \cap V_2| & \delta_{1,1} & \cdots & \delta_{1,p} \\ \vdots & \vdots & & \vdots \\ |S'_p \cap V_2| & \delta_{p,1} & \cdots & \delta_{p,p} \end{vmatrix} = \begin{vmatrix} \alpha_1 + 1 & 0 & \cdots & 0 \\ |S'_1 \cap V_2| & |S_1 \cap \{y_1\}| & \cdots & |S_1 \cap \{y_p\}| \\ \vdots & \vdots & & \vdots \\ |S'_p \cap V_2| & |S_p \cap \{y_1\}| & \cdots & |S_p \cap \{y_p\}| \end{vmatrix} \neq 0$$

Lemma B.2.9 implies that inequality (B.6) defines a facet of STAB(G^2). To finish the proof, it remains to show that it is a proper weak rank facet.

Claim B.2.16. *The set V_2 is rank facet-producing and $\alpha(G^2[V_2]) = \alpha_1 + 1$.*

We have $\alpha(G^2[V_2]) \leq \alpha(G^2[V_1]) + \alpha(Q_n) \leq \alpha_1 + 1$. Hence $\alpha(G^2[V_2]) = \alpha(G^2[W_2])$. Let v be any vertex of $V_2 \setminus W_2$. By the definition of V_2, v is an element of Q^2_{n+1}. Therefore $|N(v) \cap W_2| \geq \omega_1$ as $|W_2 \cap Q^2_{n+1}| = \omega_1$, by the definition of W_2. Let δ be the element of W_2 with maximal index.

If $v < \delta$ then $(\delta - \omega) \in N(v)$. As $\delta - \omega$ is an element of W_2 by the definition of W_2, we get $|N(v) \cap W_2| \geq \omega_1 + 1$. If $v \geq \delta$ then v has at least one neighbor in $Q^2_v \cap W^2$, as $|Q^2_v \cap W^2| \geq \omega_1 - 1 \geq 1$ (Lemma B.2.4). Hence $|N(v) \cap W_2| \geq \omega_1 + 1$.

Thus, in both cases, $|N(v) \cap W_2| \geq \omega_1 + 1$. Hence $\alpha(N(x) \cap W_2) = 2$ and therefore, $G^2[V_2]$ is rank facet-producing by Galluccio & Sassano [8] (recall that W^2 is a partitionable web by Claim B.2.13 and is, therefore, rank-minimal). $\diamond$

□

An immediate consequence of Theorem B.2.11 is the main result: if STAB(W^k_n) has a proper weak non-rank facet then STAB(W^k_{n+k+1}) has a proper weak non-rank facet (Theorem B.2.1).

Concluding remarks and open problems

The presented construction for non-rank facets of stable set polytopes of webs shows that we obtain, from every single proper weak non-rank facet in STAB(W^k_n), an infinite sequence W^k_n, $W^k_{n+(k+1)}$, $W^k_{n+2(k+1)}$, … of not rank-perfect webs (see Theorem B.2.1).

If there is a set of webs $W^k_{n_0}, \ldots, W^k_{n_k}$ such that STAB$(W^k_{n_i})$ has a proper weak non-rank facet and $n_i = i \pmod{k+1}$ then applying this construction implies that there exist only finitely many rank-perfect webs with clique number $k+1$ (Corollary B.2.2). Such sets of non-rank-perfect webs are presented for $k = 3$ in [18] and for all remaining values $k \geq 4$ in [17], implying that, for any $k \geq 3$, there exist only finitely many rank-perfect webs W^k_n.

According to Ben Rebea's Conjecture [15], the stable set polytopes of quasi-line graphs (and therefore of webs) have clique family inequalities as only non-trivial facets. This would particularly mean that all facets admit at most two non-zero coefficients. Notice that our construction of non-rank facets does not increase the number of non-zero coefficients. In particular, the non-rank facets presented in [18, 17] have coefficients equal to 2 and 1 only. On the other hand, Liebling et al. [13] found an infinite sequence of not rank-perfect webs where the non-rank facets admit coefficients a and $a+1$ for every $a \geq 1$. Hence we are still far from having a complete description of the stable set polytopes of webs.

References

[1] R.G. Bland, H.C. Huang, and L.E. Trotter. Graphical properties related to minimal imperfection. *Discrete Mathematics*, 27:11–22, 1979.

[2] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[3] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[4] V. Chvátal. On the strong perfect graph conjecture. *J. Comb. Theory Series B*, 20:139–141, 1976.

[5] G. Cornuéjols and W.R. Pulleyblank. Critical graphs, matchings, and tours of a hierarchy of relaxations for the traveling salesman problem. *Combinatorica*, 3:35–52, 1983.

[6] G. Dahl. Stable set polytopes for a class of circulant graphs. *SIAM Journal on Optimization*, 9(2):493–503, 1998.

[7] J.E. Edmonds and W.R. Pulleyblank. Facets of 1-matching polyhedra. In C. Berge and D.R. Chuadhuri, editors, *Hypergraph Seminar*, pages 214–242. Springer, 1974.

[8] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1):1–38, 1997.

[9] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[10] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[11] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12:373–388, 1975.

[12] J. Kind. *Mobilitätsmodelle für zellulare Mobilfunknetze: Produktformen und Blockierung*. PhD thesis, RWTH Aachen, 2000.

[13] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[14] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[15] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[16] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[17] A. Pêcher and A. Wagler. Almost all webs are not rank-perfect. *Mathematical Programming B*, 105(2-3):311–328, 2006.

[18] A. Pêcher and A. Wagler. On non-rank facets in stable set polytopes of webs with clique number four. *Discrete Applied Mathematics*, 154(9):1408–1415, jun 2006.

[19] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2:149–152, 2004.

[20] A.K. Wagler. *ritical Edges in Perfect Graphs*. PhD thesis, TU Berlin, 2000.

B.3 Almost all webs are not rank-perfect

par A. Pêcher et A. Wagler
Mathematical Programming Series B 105 (2006) [83]

Graphs with circular symmetry, called webs, are relevant w.r.t. describing the stable set polytopes of two larger graph classes, quasi-line graphs [7, 12] and claw-free graphs [6, 7]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [8]. Ben Rebea conjectured a description for quasi-line graphs, see [12]; Chudnovsky and Seymour [1] verified this conjecture recently for quasi-line graphs not belonging to the subclass of fuzzy circular interval graphs and showed that rank facets are required in this case only. Fuzzy circular interval graphs contain all webs and even the problem of finding all facets of their stable set polytopes is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [4, 9] while there are examples with clique number ≥ 4 having non-rank facets [10, 11, 12, 15].

In this paper we prove, building on a construction for non-rank facets from [14], that the stable set polytopes of almost all webs with clique number ≥ 5 admit non-rank facets. This adds support to the belief that these graphs are indeed the core of Ben Rebea's conjecture. Finally, we present a conjecture how to construct all facets of the stable set polytopes of webs.

Introduction

Graphs with circular symmetry of their maximum cliques and stable sets are called *webs*: a web W_n^k is a graph with vertices $1, \ldots, n$ where ij is an edge if i and j differ by at most k (mod n) and $i \neq j$. The webs W_9^k on nine vertices are depicted in Figure B.3. Notice that webs are also called circulant graphs C_n^k in [3] and that similar graphs $W(n,k)$ were introduced in [9].

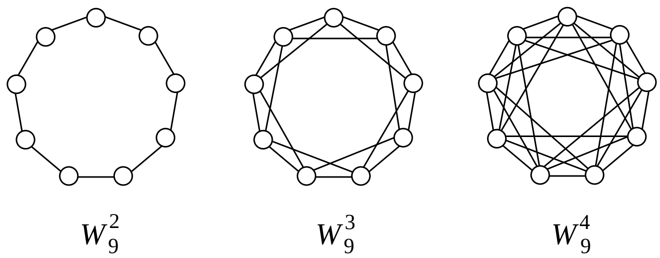


Figure B.9: Some webs

Webs and line graphs belong to the classes of quasi-line graphs and claw-free graphs and are relevant w.r.t. describing the stable set polytopes of those larger graph classes [6, 7, 12]. (The *line graph* of a graph H is obtained by taking the edges of H as nodes and connecting two nodes iff the corresponding edges of H are incident. A graph is *quasi-line* (resp. *claw-free*) if the neighborhood of any node can be partitioned into two cliques (resp. does not contain any stable set of size 3).)

The *stable set polytope* STAB(G) of G is defined as the convex hull of the incidence vectors of all stable sets of the graph G. In order to describe STAB(G) by means of facet-defining inequalities, the "trivial" facets $x_i \geq 0$ for all vertices i of G and the clique constraints

$$\sum_{i \in Q} x_i \leq 1$$

for all cliques $Q \subseteq G$ are necessary. These two types of facets are sufficient to describe STAB(G) for perfect graphs G only [2]. That are precisely the graphs without odd holes W_{2k+1}^1 and odd antiholes W_{2k+1}^{k-1} as induced subgraphs.

A natural way to generalize clique constraints is to investigate *rank constraints*, that are 0/1-constraints of the form

$$\sum_{i \in G'} x_i \;\leq\; \alpha(G')$$

associated with arbitrary induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G' (note $\alpha(G') = 1$ holds iff G' is a clique). A graph is *rank-perfect* if all non-trivial facets of its stable set polytope are rank constraints. The class of rank-perfect graphs contains all perfect graphs [2], odd holes and odd antiholes [13], line graphs [5], and the complements of webs [16].

A characterization of the *rank* facets in stable set polytopes of claw-free graphs was given by Galluccio and Sassano [6]. They showed that all rank facets can be constructed by means of standard operations from rank constraints associated with cliques, certain webs, and special line graphs. Finding *all* facets of their stable set polytopes is a long-standing problem (Grötschel, Lovász and Schrijver [8]), as claw-free graphs are not rank-perfect: Giles and Trotter [7], Oriolo [12], and Liebling et al. [11] found non-rank facets which occur even in the stable set polytopes of quasi-line graphs.

A famous conjecture due to Ben Rebea (see [12]) claims that the stable set polytopes of quasi-line graphs admit only one type of non-trivial facets, so-called clique family inequalities. Let $G = (V, E)$ be a graph, $\mathcal{F}$ be a family of (at least three inclusion-wise) maximal cliques of G, $p \leq |\mathcal{F}|$ be an integer, and define two sets as follows:

$$\begin{aligned} I(\mathcal{F}, p) = &\quad \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\} \\ O(\mathcal{F}, p) = &\quad \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p - 1\} \end{aligned}$$

The *clique family inequality* $(\mathcal{F}, p)$ is

$$(p - r) \sum_{i \in I(\mathcal{F},p)} x_i + (p - r - 1) \sum_{i \in O(\mathcal{F},p)} x_i \leq (p - r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \tag{B.7}$$

with $r = |\mathcal{F}| \bmod p$ and $r > 0$.

Oriolo [12] verified Ben Rebea's conjecture for line graphs and webs W_n^2 (note: the latter graphs are rank-perfect due to [4]). Chudnovsky and Seymour introduced recently the class of so-called fuzzy circular interval graphs and verified the conjecture for all quasi-line graphs which are *not* fuzzy circular interval graphs.

Let $\mathcal{C}$ be a circle and $\mathcal{I} = \{I_1, \ldots, I_m\}$ be a collection of intervals $I_k = [l_k, r_k]$ in $\mathcal{C}$ s.t. no interval in $\mathcal{I}$ is properly contained in another one and no two intervals in $\mathcal{I}$ share an endpoint. Moreover, let $V = \{v_1, \ldots, v_n\}$ be a finite multiset of points in $\mathcal{C}$ (i.e. $v_i \in \mathcal{C}$ may occur in V with a multiplicity > 1). The *fuzzy circular interval graph* $G(V, \mathcal{I}) = (V, E_1 \cup E_2)$ has node set V and edge set $E_1 \cup E_2$ where

$$\begin{aligned} E_1 &= \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i, v_j \in I_k \text{ and } \{v_i, v_j\} \neq \{l_k, r_k\}\} \\ E_2 &\subseteq \{v_i v_j : \exists I_k \in \mathcal{I} \text{ with } v_i = l_k, v_j = r_k\} \end{aligned}$$

(i.e., different endpoints of one interval are not necessarily joined by an edge).

Chudnovsky and Seymour recently proved that nonnegativity constraints, clique constraints, and rank constraints coming from clique family inequalities $(\mathcal{F}, 2)$ with $|\mathcal{F}|$ odd are the only necessary inequalities to describe stable set polytopes of quasi-line graphs which are not fuzzy circular interval graphs.

Webs are obviously quasi-line graphs as well as fuzzy circular interval graphs; the problem of describing their stable set polytopes is still open. So far, it is only known that webs W_n^1 are as holes perfect or rank-perfect [2, 13]; the webs W_n^2 are rank-perfect by Dahl [4]. On the other hand, Kind [10] found (by means of the PORTA software[3]) examples of webs with clique number > 4 which are *not* rank-perfect, e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9. Oriolo [12], Liebling et al. [11], and Pêcher and Wagler [15] presented further examples of such webs.

In this paper we prove, with the help of a construction for non-rank facets from [14], that there are only finitely many rank-perfect webs W_n^k for all $k \geq 4$. Together with a result from [15] showing the same for the case $k = 3$ we obtain that, for any $k \geq 3$, almost all webs W_n^k are not rank-perfect. This adds support to the belief that webs as subclass of fuzzy circular interval graphs are the core of Ben Rebea's conjecture.

[3]By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

The paper is organized as follows: the next subsection describes as our main results the construction of infinite sequences of not rank-perfect webs and discusses consequences. The three following subsections are devoted to the proofs of the three main theorems. We close with a conjecture which clique family inequalities give rise to facets in the stable set polytopes of webs.

Main results

For proving that almost all webs are not rank-perfect, we make use of a construction for non-rank facets from [14], introduced in the sequel. For that, we need the notion of *proper weak non-rank facets*. A facet $\mathbf{a}^T\mathbf{x} \leq c\alpha(G')$ of STAB(G) is a *weak rank facet* w.r.t. $G' \subseteq G$, if $a_i = c$ for every vertex i of G' and if G' is rank facet-producing (i.e., $\sum_{i \in V(G')} x_i \leq \alpha(G')$ defines a facet of STAB(G')); any rank facet is a particular weak rank facet (with $a_i = c = 1$ for every $i \in V(G')$ and $a_i = 0$ otherwise). A weak rank facet is *proper* if G' is not a clique and *non-rank* if it cannot be scaled to have 0/1-coefficients only (i.e., it is not a rank-constraint).

Theorem B.3.1. [14] *If STAB(W_n^k) possesses a proper weak non-rank facet then also STAB(W_{n+k+1}^k) has a proper weak non-rank facet.*

Therefore, if STAB(W_n^k) has a proper weak non-rank facet then all webs $W_{n+(k+1)}^k$ (≥ 0) are not rank-perfect, too. Hence Theorem B.3.1 implies the following corollary:

Corollary B.3.2. [14] *If there are $k+1$ webs $W_{n_0}^k, \ldots, W_{n_k}^k$ such that*

- *STAB$(W_{n_i}^k)$ has a proper weak non-rank facet*
- *$n_i = i \pmod{k+1}$ for $0 \leq i \leq k$*

then all webs W_n^k with $n \geq \max\{n_0, \ldots, n_k\} - k$ are not rank-perfect.

For $k = 3$, such a set W_{33}^3, W_{42}^3, W_{51}^3, W_{60}^3 is presented in [15], as consequence of the following theorem.

Theorem B.3.3. [15] *If $l = 2 \pmod 3$ and $l \geq 11$ then STAB(W_{3l}^3) has a proper weak non-rank facet.*

Thus, Theorem B.3.3 and Corollary B.3.2 imply that all webs W_n^3 with $n > 56$ are not rank-perfect. The aim of this paper is to provide such a set $W_{n_0}^k, \ldots, W_{n_k}^k$ of not rank-perfect webs for each value $k \geq 4$. For that, we consider special clique family inequalities giving rise to proper weak non-rank facets. A clique family inequality $(\mathcal{Q}, p)$ is *associated* with a proper subweb $W_{n'}^{k'}$ of a web W_n^k if $\mathcal{Q} = \{Q_i : i \in W_{n'}^{k'}\}$ is chosen as clique family and $p = k' + 1$, where $Q_i = \{i, \ldots, i+k\}$ denotes the maximum clique of W_n^k starting in vertex i. Note that the clique number of a web W_n^k is $k+1$ and the stability number is $\lfloor \frac{n}{k+1} \rfloor$.

Lemma B.3.4. [15] *Let $W_{n'}^{k'} \subset W_n^k$ be a proper induced subweb. The clique family inequality $(\mathcal{Q}, k'+1)$ associated with $W_{n'}^{k'}$*

$$(k'+1-r) \sum_{i \in I(\mathcal{Q},k'+1)} x_i + (k'-r) \sum_{i \in O(\mathcal{Q},k'+1)} x_i \leq (k'+1-r)\,\alpha(W_{n'}^{k'}) \tag{B.8}$$

where $r = n' \bmod (k'+1)$, $0 < r < k'+1$, is a valid inequality for STAB(W_n^k), such that $W_{n'}^{k'} \subseteq I(\mathcal{F}, p)$ holds.

For illustration, look at the smallest not rank-perfect web W_{25}^5. Its non-rank facets are clique family inequalities associated with induced subwebs $W_{10}^2 \subseteq W_{25}^5$ (note that the vertex sets 1, 2, 6, 7, 11, 12, 16, 17, 21, 22 and 1, 3, 6, 8, 11, 13, 16, 18, 21, 23 both induce a $W_{10}^2 \subseteq W_{25}^5$, see the black vertices in Figure B.3).

Choosing $\mathcal{Q} = \{Q_i : i \in W_{10}^2\}$ yields $p = \omega(W_{10}^2) = 3$ in both cases. All remaining vertices are covered 2 times, hence $O(\mathcal{Q}, p) = W_{25}^5 - W_{10}^2$ follows. The corresponding clique family inequality $(\mathcal{Q}, 3)$ is

$$2 \sum_{i \in W_{10}^2} x_i + 1 \sum_{i \notin W_{10}^2} x_i \leq 2\alpha(W_{10}^2)$$

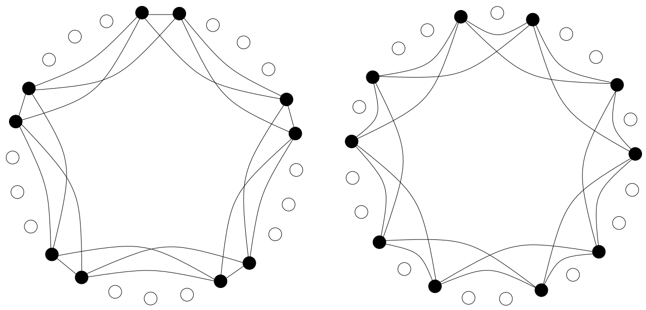


Figure B.10: The induced subwebs $W^2_{10} \subseteq W^5_{25}$

due to $r = |\mathcal{Q}| \bmod p = 1$ and yields a non-rank facet of STAB(W^5_{25}).

The main results of this paper prove that several clique family inequalities $(\mathcal{Q}, k'+1)$ associated with different regular subwebs $W^{k'}_{n'}$ induce proper weak non-rank facets (note that $(\mathcal{Q}, k'+1)$ is a proper weak non-rank constraint if $r < k'$). A subweb $W^{k'}_{n'} \subset W^k_n$ is called $(b_1, w_1, \dots, b_t, w_t)$*-regular*, if the vertices of $W^{k'}_{n'}$ occur in W^k_n in equal blocks where b_i consecutive vertices from $W^{k'}_{n'}$ alternate with w_i consecutive vertices outside $W^{k'}_{n'}$, for $1 \le i \le t$. The two subwebs $W^2_{10} \subseteq W^5_{25}$ presented in Figure B.3 show a (2,3)-regular and a (1,1,1,2)-regular subweb, resp. In subsection 3, we show the following:

Theorem B.3.5. *For any $k \ge 5$, consider a $(k', k-k')$-regular subweb $W^{k'}_{lk'} \subset W^k_{lk}$ with $2 \le k' \le k-3$ and odd $l \ge 3$. The clique family inequality*

$$2 \sum_{i \in W^{k'}_{lk'}} x_i + 1 \sum_{i \notin W^{k'}_{lk'}} x_i \le 2\alpha(W^{k'}_{lk'}) \tag{B.9}$$

associated with $W^{k'}_{lk'}$ is a proper weak non-rank facet of STAB(W^k_{lk}) if $l = 2 (\bmod\ k'+1)$ and $\alpha(W^{k'}_{lk'}) < \alpha(W^k_{lk})$.

As a consequence, we obtain many different infinite sequences of not rank-perfect webs, among them the required base sets for all even values of $k \ge 6$ (but not for the odd values $k \ge 5$ since all webs in the latter sequences have an odd number of vertices). For any even $k \ge 6$, choosing $k' = \frac{k}{2}$ if $k = 0 \pmod 4$ and $k' = \frac{k}{2} - 1$ if $k = 2 \pmod 4$ and $l = (k'+3) + (k'+1)2j$ for $j \ge 1$ in both cases as odd values of l with $l = 2 \pmod{k'+1}$ satisfies the precondition of Theorem B.3.5. Thus, we obtain the following infinite sequences of not rank-perfect webs:

Theorem B.3.6. *Let $k \ge 6$ be even. Then for every integer $j \ge 1$ holds*

- *STAB$\left(W^k_{\left(\frac{k+6}{2}+(k+2)j\right)k}\right)$ has a proper weak non-rank facet if $k = 0 (\bmod\ 4)$;*
- *STAB$\left(W^k_{\left(\frac{k+4}{2}+kj\right)k}\right)$ has a proper weak non-rank facet if $k = 2 (\bmod\ 4)$.*

That means for, e.g., $k = 6$ that there is an infinite sequence $W^6_{66}, W^6_{102}, W^6_{138}, W^6_{174}, W^6_{210}, W^6_{246}, W^6_{282}, \dots$ of not rank-perfect webs. Corollary B.3.2 implies, therefore, that all webs W^6_n with $n \ge 276$ are not rank-perfect. More generally, for every $1 \le j \le k+1$, we have

$$\left(\frac{k+6}{2} + (k+2)j\right) k = -\frac{k+6}{2} - j \pmod{k+1}$$

and

$$\left(\frac{k+4}{2} + kj\right) k = -\frac{k+4}{2} + j \pmod{k+1}$$

thus, the sequences contain the required base sets. Furthermore, if $k \geq 6$ then

$$\left(\frac{k+6}{2} + (k+2)(k+1)\right) k < 2(k+1)^3$$

follows and Theorem B.3.6 and Corollary B.3.2 imply together:

Corollary B.3.7. *For any even $k \geq 6$, all webs W_n^k with $n \geq 2(k+1)^3$ are not rank-perfect.*

It remains to construct the required base sets for $k = 4$ and all odd values of $k \geq 5$. The case $k = 4$ is treated in subsection B.3 by constructing sequences of clique family inequalities associated with regular subwebs $W_l^2 \subset W_{2l}^4$:

Theorem B.3.8. *The clique family inequality$(\mathcal{Q}, 3)$*

$$2 \sum_{i \in W_l^2} x_i + 1 \sum_{i \notin W_l^2} x_i \leq 2\,\alpha(W_l^2) \tag{B.10}$$

associated with a (1,1)-regular subweb $W_l^2 \subset W_{2l}^4$ is a proper weak non-rank facet of STAB(W_{2l}^4) if $l = 1 \pmod 3$ and $l \geq 13$.

Due to Theorem B.3.8, the stable set polytopes of the webs W_{26}^4, W_{32}^4, W_{38}^4, W_{44}^4, and W_{50}^4 have a proper weak non-rank facet. Hence Corollary B.3.2 implies that all webs W_n^4 with $n > 45$ are not rank-perfect.

For each odd $k \geq 5$, we extend the result for $k = 3$ from [15] by considering the clique family inequality associated with the $(k-1,1)$-regular subweb $W_{l(k-1)}^{k-1} \subset W_{lk}^k$:

Theorem B.3.9. *The clique family inequality $(\mathcal{Q}, k)$*

$$2 \sum_{i \in W_{l(k-1)}^{k-1}} x_i + 1 \sum_{i \notin W_{l(k-1)}^{k-1}} x_i \;\leq\; 2\,\alpha(W_{l(k-1)}^{k-1}) \tag{B.11}$$

associated with a $(k-1,1)$-regular subweb $W_{l(k-1)}^{k-1}$ is a proper weak non-rank facet of STAB(W_{lk}^k) for any odd $k \geq 5$ if $l = 3k+2$.

The sequence of the $k+1$ webs $W_{l'k+2}^k$ with $3 \leq k' \leq 3+k$ webs is the required base set for any odd $k \geq 5$, as

$$k(l'k+2) \bmod (k+1) \quad = \quad (l'-2) \bmod (k+1)$$

Thus, Theorem B.3.9 and Corollary B.3.2 imply together:

Corollary B.3.10. *W_n^k with $n \geq ((k+3)k+1)k$ is not rank-perfect for any odd $k \geq 5$.*

In summary, all the above results show:

Corollary B.3.11. *A web W_n^k is not rank-perfect if*

- *$k = 3$ and $n \geq 57$,*
- *$k = 4$ and $n \geq 46$,*
- *$k \geq 5$ is odd and $n \geq ((3+k)k+1)k$,*
- *$k \geq 6$ is even and $n \geq 2(k+1)^3$.*

Thus, for any $k \geq 3$ there are only finitely many rank-perfect webs W_n^k implying:

Corollary B.3.12. *Almost all webs with given clique size at least 4 are not rank-perfect.*

The following three subsections contain the proofs of the main results Theorem B.3.5, Theorem B.3.8, and Theorem B.3.9.

Proof of Theorem B.3.5

For any $k \geq 5$, let $W_{lk'}^{k'}$ be a $(k', k-k')$-regular subweb of W_{lk}^{k} with $2 \leq k' \leq k-3$ and odd $l \geq 3$. By assumption, we have $l = 2(\bmod\ k'+1)$ and $\alpha(W_{lk'}^{k'}) < \alpha(W_{lk}^{k})$.

In order to prove Theorem B.3.5, we have to establish that the inequality (B.9)

$$2 \sum_{i \in W_{lk'}^{k'}} x_i + 1 \sum_{i \notin W_{lk'}^{k'}} x_i \leq 2\alpha(W_{lk'}^{k'})$$

is valid and facet-inducing for STAB(W_{lk}^{k}).

Validity follows from Lemma B.3.4: since $l = 2 \pmod{k'+1}$, we have $lk' = -2 \pmod{k'+1}$ and therefore the remainder r of the division of lk' by $k'+1$ is equal to $k'-1$. Therefore the valid inequality (B.8) associated with the subweb $W_{lk'}^{k'}$ is

$$2 \sum_{i \in I(\mathcal{Q}, k'+1)} x_i + \sum_{i \in O(\mathcal{Q}, k'+1)} x_i \leq 2\,\alpha(W_{lk'}^{k'}) \tag{B.12}$$

where $W_{lk'}^{k'} \subseteq I(\mathcal{F}, p)$ holds. Therefore, inequality (B.9) is a valid inequality.

To prove that inequality (B.9) is facet-inducing, we may define the set of vertices V' of the $(k', k-k')$-regular subweb $W_{lk'}^{k'}$ w.l.o.g. as

$$V' = \bigcup_{0 \leq j < l} \{k \cdot j + 1, k \cdot j + 2, \ldots, k \cdot j + k'\}$$

(where $l \geq 5$ and $l = 2 \pmod{k'+1}$)

For convenience, we call the vertices in V' *black vertices* and all remaining vertices *white vertices*. A *black set* is a set of black vertices and likewise a *white set* is a set of white vertices.

The following lemma from [14] is essential for the proof. It provides a characterization when a valid inequality $\mathbf{a}^T\mathbf{x} \leq b$ is a facet of the stable set polytope of a general graph G. For that we need the following notions. A *root* of $\mathbf{a}^T\mathbf{x} \leq b$ is any stable set of G satisfying the inequality at equality. A pair i, j of vertices is $\mathbf{a}$-*critical* in G if there are two roots S_1 and S_2 of $\mathbf{a}^T\mathbf{x} \leq b$ such that $\{i\} = S_1 \setminus S_2$ and $\{j\} = S_2 \setminus S_1$. A subset V' of $V(G)$ is $\mathbf{a}$-*connected* if the graph with vertex set V' and edge set $\{ij|\ i, j \in V',\ ij\ \mathbf{a}\text{-critical in } G\}$ is connected.

All matrices in this paper have rational entries (in fact integer entries). If M is any square matrix, then $|M|$ stands for the determinant of M.

Lemma B.3.13. [14] *Let $\mathbf{a}^T\mathbf{x} \leq b$ be a valid inequality for STAB(G) with $b \neq 0$. Consider a partition $V_1, \ldots, V_p$ of $V(G)$ such that V_i is $\mathbf{a}$-connected for every $1 \leq i \leq p$. The inequality $\mathbf{a}^T\mathbf{x} \leq b$ is facet-defining if and only if there are p roots $S_1, \ldots, S_p$ with*

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0$$

If the involved inequality $\mathbf{a}^T\mathbf{x} \leq b$ is the full rank-constraint $\mathbf{1}^T\mathbf{x} \leq \alpha$, we use the terms α-critical and α-connected instead of $\mathbf{a}$-critical and $\mathbf{a}$-connected respectively.

Notice that Chvátal [2] called α-critical edges simply "critical" and that Lemma B.3.13 generalizes the well-known result of Chvátal [2] that a graph G is rank facet-producing if the set of its critical edges induces a conncted subgraph of G.

We now proceed to the proof that inequality (B.9) is facet-inducing,

Claim B.3.14. *The black set V' is $\mathbf{a}$-connected w.r.t. the valid inequality (B.9).*

Proof.

If $lk' = 0 \pmod{k'+1}$ then $-l = 0 \pmod{k'+1}$ and therefore $= 0 \pmod{k'+1}$, in contradiction with $= 2 \pmod{k'+1}$, as $k' \geq 2$. Hence $k'+1$ is not a divisor of lk'. Hence we have $lk' = \alpha(G[V'])(k'+1) + r$ with

$1 \leq r \leq k'$. Let $S_1 = \{1, 2+(k'+1), 2+2(k'+1), \ldots, 2+(\alpha(G[V'])-1)(k'+1)\}$ and $S_2 = \{2, 2+(k'+1), 2+2(k'+1), \ldots, 2+(\alpha(G[V'])-1)(k'+1)\}$. Since $2+(\alpha(G[V'])-1)(k'+1) = 2+(k'-r)-(k'+1) \leq k'-k'$, S_1 and S_2 are both maximum stable sets of $G[V']$. Hence, the edge $\{1,2\}$ of $G[V']$ is α-critical. By circular symmetry of $G[V']$, this implies that $G[V']$ is α-connected. Since $\alpha(G[V']) = \alpha'$, this implies that V' is **a**-connected. □

Claim B.3.15. *We have $k' > (\alpha' - 2)(k'+1) + 3k'$.*

Proof. Since $= 2 \pmod{k'+1}$, we have $k' = k' - 1 \pmod{k'+1}$. Hence $k' - \alpha'(k'+1) = k' - 1$. It follows that $k' - \alpha'(k'+1) > 3k' - 2(k'+1)$. Thus $k' > (\alpha'-2)(k'+1) + 3k'$. □

Claim B.3.16. *We obtain $\alpha(W_{k'}^{k'} \setminus [1, 3k']) \geq \alpha(W_{k'}^{k'}) - 1$.*

Proof. By the previous Claim, the set $S' := \{3k'+1, 3k'+(k'+1)+1, \ldots, 3k'+(\alpha'-2)(k'+1)+1\}$ is a stable set of size $\alpha' - 1$ of $W_{k'}^{k'} \setminus [1, 3k']$ and the result follows. □

Claim B.3.17. *For every $0 \leq i <$, the white set $V_i := ik + \{k'+1, \ldots, k\}$ is* **a***-connected.*

Proof. We are going to prove that V_1 is **a**-connected. By the previous Claim, there is a black stable set S' of size $\alpha' - 1$ in $G \setminus [k+1, 4k]$. For every $k + k' + 1 \leq j \leq 2k - 1$, the set $S_j := S' \cup \{3k, j\}$ is obviously a root of (B.9), hence the edges $\{k'+k+1, k'+k+2\}, \ldots, \{2k-2, 2k-1\}$, are **a**-critical (see Fig. B.11(a)).

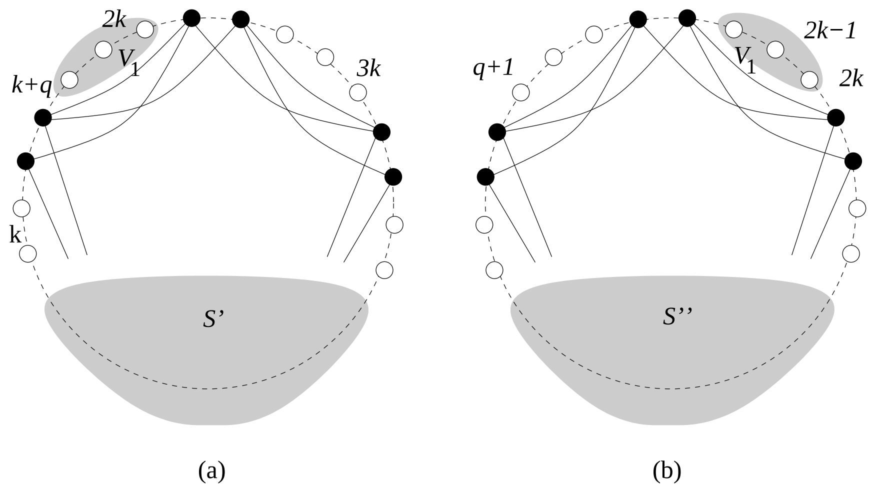


Figure B.11: : The roots for the proof of the fourth Claim with $k = 5$ and $k' = 2$: (a) the roots $S_j = S' \cup \{3k, j\}$ (b) the stable sets $S_1 = S'' \cup \{k'+1, 2k-1\}$ and $S_2 = S'' \cup \{k'+1, 2k\}$

It remains to show that the edge $\{2k-1, 2k\}$ is **a**-critical. By the previous Claim again, there exists a black stable set S'' of size $\alpha' - 1$ in $G \setminus [k' - k + 1, 3k]$. The set $S_1 := S'' \cup \{k'+1\} \cup \{2k-1\}$ is a root as $k' + 1 + k < 2k - 1$ (since $k' \leq k - 3$). The set $S_2 := S'' \cup \{k'+1\} \cup \{2k\}$ is also a root (see Fig. B.11(b)). Hence $\{2k-1, 2k\}$ is **a**-critical and, therefore, V_1 is **a**-connected.

Likewise, the sets $V_0, V_2 \ldots, V_{-1}$ are **a**-connected. □

Claim B.3.18. *For every $0 \leq i <$ there exists a stable set S_i such that S_i meets V' in exactly $\alpha' - 1$ vertices, V_i in exactly one vertex, and V_{i+1} in also exactly one vertex.*

Proof. For every $0 \leq i <$, there exists a black stable set S_i' of size $\alpha' - 1$ in $G \setminus [ik+1, (i+3)k]$. Let S_i be the stable set $S_i' \cup \{ik + k' + 1\} \cup \{(i+1)k + k' + 2\}$. Then $S_0, \ldots, S_{-1}$ give the result. □

Let S' be a maximum stable set of $G[V']$.

Hence we have

$$\begin{vmatrix} |S' \cap V'| & |S' \cap V_0| & \cdots & |S' \cap V_{-1}| \\ |S_0 \cap V'| & |S_0 \cap V_0| & \cdots & |S_0 \cap V_{-1}| \\ \vdots & \vdots & & \vdots \\ |S_{-1} \cap V'| & |S_{-1} \cap V_0| & \cdots & |S_{-1} \cap V_{-1}| \end{vmatrix} = \begin{vmatrix} \alpha' & 0 & \cdots & 0 \\ \alpha' - 1 & & & \\ \vdots & & C & \\ \alpha' - 1 & & & \end{vmatrix}$$

where C is the $(2, l)$-circulant matrix with top row $(1, 1, 0, \ldots, 0)$ of size . The matrix C is invertible as is odd. Hence the above determinant is non-zero.

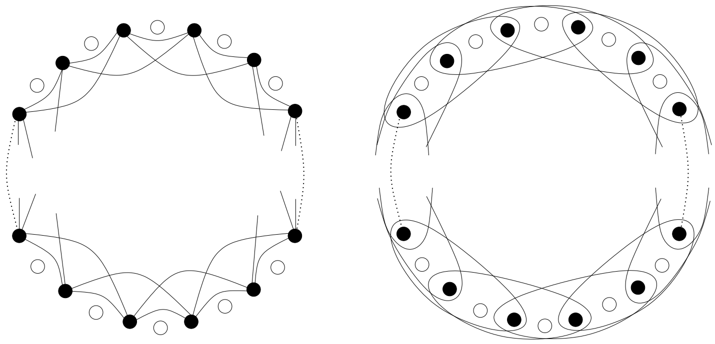
Therefore, the proof of Theorem B.3.5 is done due to Lemma B.3.13.

Proof of Theorem B.3.8

In order to obtain an infinite sequence of not rank-perfect webs W_n^4 we consider, for any even $n = 2l$, the (1,1)-regular subweb $W_l^2 \subset W_{2l}^4$. In the remaining part of this subsection, let $V = \{1, \ldots, 2l\}$ denote the vertex set of W_{2l}^4 and

$$\begin{aligned} V_o &= \{1, 3, \ldots, 2l - 1\} \\ V_e &= \{2, 4, \ldots, 2l\} \end{aligned}$$

denote the subsets of vertices with odd resp. even index in V. Then both sets V_o and V_e induce a subweb W_l^2 of W_{2l}^4, see Figure B.3(a).



(a) (b)

Figure B.12: The partition of W_{2l}^4

In the sequel, consider the clique family inequality $(\mathcal{Q}, 3)$ of STAB(W_{2l}^4) associated with the subweb W_l^2 induced by V_o. This means, we choose $\mathcal{Q} = \{Q_i : i \in V_o\}$ (the cliques of $\mathcal{Q}$ are represented in Figure B.3(b) where the vertices from V_o are drawn in black) and obtain $I(\mathcal{Q}, 3) = V_o$ and $O(\mathcal{Q}, 3) = V_e$. The clique family inequality $(\mathcal{Q}, 3)$ associated with $W_l^2 = W_{2l}^4[V_o]$ is, therefore,

$$(3-r) \sum_{i \in V_o} x_i + (2-r) \sum_{i \in V_e} x_i \leq (3-r)\, \alpha(W_l^2)$$

and it is a *non*-rank constraint if $r = 1$. Hence, $(\mathcal{Q}, 3)$ corresponds to the studied inequality (B.10) if $l = 1 \pmod 3$. We prove that it is a facet for all $l \in \{13, 16, 19, \ldots\}$.

For that, we have to present $2l$ roots of (B.10) whose incidence vectors are linearly independent. (Recall that a root of (B.10) is a stable set of W_{2l}^4 satisfying (B.10) at equality.) It follows from [9] that a web W_n^k produces the full rank facet $\sum_{i \in W_n^k} x_i \leq \alpha(W_n^k)$ if and only if $(k+1) \not| \, n$. Thus W_l^2 is facet-producing as $l = 1 \pmod 3$ and the maximum stable sets of W_l^2 yield, therefore, already l independent roots of (B.10). We need a set $\mathcal{S}$ of further l roots of (B.10) admitting vertices from V_o as well as from V_e, called *mixed roots*, and are independent, too.

We construct, for all $l \geq 13$ with $l = 1 \pmod 3$, a set $\mathcal{S}$ of l mixed roots S of size α_o+1 with $|S \cap V_o| = \alpha_o - 1$ and $|S \cap V_e| = 2$ where $\alpha_o = \alpha(W_l^2) = \lfloor \frac{l}{3} \rfloor$ (notice that $2(\alpha_o - 1) + 2 = 2\alpha_o$ according to the coefficients of (B.10)).

For that, we use the following representation of stable sets $S \subseteq W_{2l}^4$ of size $\alpha_o + 1$: choose a start vertex $i \in S$ and the distance vector $D = (d_1, \ldots, d_{\alpha_o+1})$ containing the distances between two consecutive vertices of S, i.e.,

$$S = S(i, D) = \{i, i + d_1, (i + d_1) + d_2, \ldots, (i + \sum_{j<\alpha_o} d_j) + d_{\alpha_o}\}$$

where

$$\sum_{j \leq \alpha_o+1} d_j = 2l \text{ (i.e. } i = i + \sum_{j \leq \alpha_o+1} d_j \bmod 2l)$$

and $d_j > k = 4$ for $1 \leq j \leq \alpha_o + 1$ (ensuring that S is a stable set in W_{2l}^4).

Claim B.3.19. *Let $D = (d_1, \ldots, d_{\alpha_o+1})$ be a distance vector such that D has 4 entries d_j equal to 5 and $\alpha_o - 3$ entries d_j equal to 6. Then $S(i, D)$ is a stable set of W_{2l}^4 for every vertex i for all $l = 1 \bmod 3$.*

Proof. We have to show that $\sum_{d_j \in D} d_j = 2l$ holds. Recalling $\alpha_o = \lfloor \frac{l}{3} \rfloor$ and $l = 1 \pmod 3$, we obtain

$$5 \cdot 4 + 6(\alpha_o - 3) = 2 + 6\alpha_o = 2 + 6\left\lfloor \frac{l}{3} \right\rfloor = 2 + 6\frac{l-1}{3} = 2 + 2(l-1) = 2l$$

as required. □

We define two different distance vectors with $\alpha_o + 1$ entries each by

$$D_1 = (5, 5, 5, 6, \ldots, 6, 5)$$
$$D_2 = (5, 6, 5, 5, 6, \ldots, 6, 5)$$

and show that they produce the studied mixed roots of (B.10).

Claim B.3.20. *For every $i \in V_e$, $S(i, D_1)$ (resp. $S(i, D_2)$) contains precisely the 2 vertices $i, i + 10$ (resp. $i, i + 16$) from V_e and $\alpha_o - 1$ vertices from V_o.*

Proof. By the choice of D_1 and D_2 and the previous claim, both stable sets have size $\alpha_o + 1$ and start with a vertex in V_e (due to $i \in V_e$). The parity of the distances d_j in D_1 and D_2 implies that the third vertex $i + 10$ of $S(i, D_1)$ and the fourth vertex $i + 16$ of $S(i, D_2)$ is in V_e again, whereas all remaining $\alpha_o - 1$ vertices belong to V_o (see Figure B.3, vertices in V_e (resp. V_o) are drawn in white (resp. black)). □

Hence, each set $S(i, D_1)$ and $S(i, D_2)$ with $i \in V_e$ is a mixed root of (B.10). We now have to choose a set $\mathcal{S}$ of l distinct mixed roots of (B.10) with linearly independent incidence vectors.

Assume that $\mathcal{S}$ is such a set and denote by $A_{\mathcal{S}}$ the square matrix containing the incidence vectors of l linearly independent maximum stable sets of $W_l^2 = W_{2l}^4[V_o]$ as first l rows and the incidence vectors of the l mixed roots in $\mathcal{S}$ as last l rows. Order the columns of $A_{\mathcal{S}}$ s.t. the first (resp. last) l columns correspond to the vertices in V_o (resp. V_e), both in increasing order. Then $A_{\mathcal{S}}$ has the block structure

$$A_{\mathcal{S}} = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right)$$

where the $l \times l$-matrix A_{11} is invertible since W_l^2 is facet-producing by [9] (in the considered case with $l = 1 \pmod 3$).

In the sequel, we provide a set $\mathcal{S}$ of l distinct mixed roots s.t. A_{22} (i.e. the intersection of the mixed roots with V_e) is an invertible $l \times l$-matrix (then $A_{\mathcal{S}}$ is invertible due to its block structure).

Claim B.3.21. *For every $l \geq 13$, there is a set $\mathcal{S}$ of l mixed roots of (B.10) containing precisely 2 vertices of V_e s.t. the $l \times l$-submatrix A_{22} of $A_{\mathcal{S}}$ is invertible.*

Proof. For any $i \in V_e$, both $S(i, D_1)$ and $S(i, D_2)$ are roots of (B.10) by the previous claim. Chose $S(i, D_1)$ with $i \in \{2, 4, \ldots, 2l-10\}$ as the first $l-5$ roots in $\mathcal{S}$ and $S(i, D_2)$ with $i \in \{2l-24, 2l-22, \ldots, 2l-16\}$ as the last 5 roots in $\mathcal{S}$. We have $S(i, D_1) \cap V_e = \{i, i+10\}$ and $S(i, D_2) \cap V_e = \{i, i+16\}$ by the previous claim.

Take the incidence vectors $\chi^{S(i,D_1)}$ for $i \in \{2, 4, \ldots, 2l-10\}$ as the first $l-5$ rows and $\chi^{S(i,D_2)}$ for $i \in \{2l-24, \ldots, 2l-16\}$ as the last 5 rows of $(A_{21}|A_{22})$. By construction, A_{22} is the $l \times l$-matrix shown in Figure B.14 (0-entries are dropped and the columns represent the vertices in V_e).

A_{22} has only 1-entries on the main diagonal (coming from the first vertices in V_e of $S(i, D_1)$ for $i \in \{2, 4, \ldots, 2l-10\}$ and from the second vertices in V_e of $S(i, D_2)$ for $i \in \{2l-24, \ldots, 2l-16\}$). The only non-zero entries of A_{22} below the main diagonal come from the first vertices in V_e of $S(i, D_2)$ for $i \in \{2l-24, \ldots, 2l-16\}$. Hence, A_{22} has the form

$$A_{22} = \left(\begin{array}{c|c} A'_{22} & \\ \hline 0 & A''_{22} \end{array} \right)$$

where both matrices A'_{22} and A''_{22} are invertible due to the following reasons. A'_{22} is an $(l-13) \times (l-13)$-matrix having 1-entries on the main diagonal and 0-entries below the main diagonal by construction; hence A'_{22} is clearly invertible. A''_{22} is the (2,13)-circulant matrix and, therefore, clearly invertible as well. (Note that $l = 13$ implies $A_{22} = A''_{22}$.)

This completes the proof that A_{22} is invertible for every $l \geq 13$ with $l = 1 \pmod 3$ if we choose the set $\mathcal{S}$ of l roots of (B.10) as constructed above. □

Remark. Note that there are no mixed roots of (B.10) in the case $l = 7$ (since $\alpha(W_7^2) = 2 = \alpha(W_{14}^4)$ implies that we cannot built stable sets of size $> \alpha(W_7^2)$ in W_{14}^4). In the case $l = 10$, there are only 5 mixed roots of size $> \alpha(W_{10}^2)$, namely $S(i, D_1)$ with $i \in \{2, 4, 6, 8, 10\}$ because of $S(i, D_1) = S(i+10, D_1)$; the sets $S(i, D_2)$ can be constructed only if $l \geq 13$. Hence, (B.10) is neither a facet of STAB(W_{14}^4) nor of STAB(W_{20}^2). Moreover, in the case $l = 13$, we would obtain the same set $\mathcal{S}$ by choosing the roots $S(i, D_1)$ with $i \in \{2l-8, \ldots, 2l\}$ instead of $S(i, D_2)$ with $i \in \{2l-24, \ldots, 2l-16\}$.

Hence, we have shown that, for every $l \geq 13$ with $l = 1 \pmod 3$, there are $2l$ roots of (B.10) whose incidence vectors are linearly independent, completeing the proof of Theorem B.3.8.

Proof of Theorem B.3.9

The aim of this subsection is to prove that the clique family inequality$(\mathcal{Q}, k)$ associated with any $(k-1, 1)$-regular subweb $W_{l(k-1)}^{k-1}$ induces the facet (B.11) of STAB(W_{lk}^k) for every odd $k \geq 5$ whenever $l \geq 3k+2$.

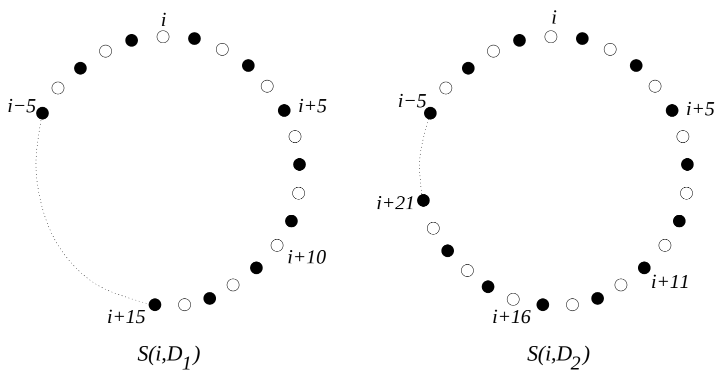


Figure B.13: The mixed roots $S(i, D_1)$ and $S(i, D_2)$

$$
\begin{array}{c|cccccc|ccccccc|}
 & 2 & \dots & 12 & \dots & 2l{-}26 & 2l{-}24 & \dots & 2l{-}16 & \dots & 2l{-}10 & 2l{-}8 & \dots & 2l \\
\hline
1 & 1 & & 1 & & & & & & & & & & \\
\vdots & & \ddots & & & \ddots & & & & & & & & \\
\vdots & & & & \ddots & & & \ddots & & & & & & \\
l{-}13 & & & & & 1 & & & 1 & & & & & \\
\hline
l{-}12 & & & & & & 1 & & & \ddots & & & & \\
\vdots & & & & & & & \ddots & & & & & & \\
\vdots & & & & & & & & & \ddots & & & \ddots & \\
l{-}5 & & & & & & & & & & 1 & & & 1 \\
l{-}4 & & & & & & 1 & & & & & 1 & & \\
\vdots & & & & & & & \ddots & & & & & \ddots & \\
l & & & & & & & & 1 & & & & & 1 \\
\hline
\end{array}
$$

Figure B.14: The $l \times l$-matrix A_{22}

For that, let $l = l'k + 2$ with $l' \geq 3$ and $n = lk = l'k^2 + 2k$. Denote the set of vertices of W_n^k by $V = \{1, \dots, n\}$ and the subset of all vertices $i \in V$ with $i \not| \, k$ by V', i.e., let $V' = V \setminus \{k, 2k, \dots, lk\}$ be the vertex set of $W_{l(k-1)}^{k-1}$. Then $\mathcal{Q} = \{Q_i : i \in V'\}$ obviously implies $I(\mathcal{Q}, k) = V'$. As

$$\alpha(W_{l(k-1)}^{k-1}) = \left\lfloor \frac{(k-1)l}{k} \right\rfloor = \left\lfloor l - l' - \frac{2}{k} \right\rfloor = l - l' - 1 = l'(k-1) + 1$$

holds, the clique family inequality $(\mathcal{Q}, k)$ reads

$$2 \sum_{i \in V'} x_i + \sum_{i \in V \setminus V'} x_i \leq 2l'(k-1) + 2$$

and is supposed to define a facet of STAB(W_{lk}^k) for any odd $k \geq 5$ if $l = l'k + 2$ and $l' \geq 3$.

In order to verify that we have to present lk roots, i.e., stable sets satisfying $(\mathcal{Q}, k)$ at equality, whose incidence vectors are linearly independent.

The maximum stable sets of $W_{l(k-1)}^{k-1}$ are independent by Trotter [9] as $k \not| \, l(k-1)$ by $l = 2 \pmod k$ and k odd.

This provides us already $l(k-1)$ independent roots of $(\mathcal{Q}, k)$ containing vertices from V' only. We are going to build l further mixed roots containing vertices from V' as well as from $V \setminus V'$ such that their incidence vectors are linearly independent, too.

For any vertex $i \in V \setminus V'$, denote by $D_i = \{i, i+1, \dots, i+k-1\}$ resp. $B_i = \{i, i+1, \dots, i+k^2-1\}$ the subset of V consisting of k resp. k^2 consecutive vertices starting in vertex i (with arithmetics performed modulo n). Furthermore, define $S(B_i) \subset B_i$ by

$$S(B_i) = \{i + 1 + j(k+1) : 0 \leq j \leq k-2\}$$

as the black vertices. By construction, $S(B_i)$ is obviously a stable set of W_{lk}^k consisting of nodes from V' only.

To build the mixed roots, we are going to use two types of partitions of $V = \{1, \dots, n\}$ into 2 subsets D_j of size k and l' subsets B_i of size k^2 (recall that $n = 2k + l'k^2$ holds). We pick the first vertex from each subset D_j and $S(B_i)$ from the involved subsets B_i and show that the so constructed vertex sets form roots of $(\mathcal{Q}, k)$.

For every $1 \leq i \leq l$, the vertex ki belongs to $V \setminus V'$ and

$$D_{ki} \cup B_{k(i+1)} \cup D_{k(i+1+k)} \cup \bigcup_{j=1,\dots,l'-1} B_{k(i+2+jk)}$$

forms a partition of $V = \{1, \dots, n\}$, as $k(i + 2 + (l' - 1)k) + k^2 - 1 = ki - 1 \pmod n$ holds. Let

$$S_{ki} = \{ki\} \cup S(B_{k(i+1)}) \cup \{k(i + 1 + k)\} \cup \bigcup_{j=1,\dots,l'-1} S(B_{k(i+2+jk)})$$

be the set consisting of the first vertices from D_{ki} and $D_{k(i+1+k)}$ and the stable sets $S(B_j)$ for the involved subsets B_j.

Furthermore, for every $1 \leq i \leq l$,

$$D_{ki} \cup \bigcup_{j=0,\dots,l'-3} B_{k(i+1+jk)} \cup D_{k(i+1+(l'-2)k)} \cup B_{k(i+2+(l'-2)k)} \cup B_{k(i+2+(l'-1)k)}$$

forms a second type of partition of $V = \{1, \dots, n\}$, as $k(i + 2 + (l' - 2)k) + k^2 - 1 = ki - 1 \pmod n$. Let

$$S'_{ki} = \{ki\} \cup \bigcup_{j=0,\dots,l'-3} S(B_{k(i+1+jk)}) \cup \{k(i + 1 + (l' - 2)k)\} \cup \bigcup_{j=l'-2,l'-1} S(B_{k(i+2+jk)})$$

be the corresponding set consisting of the first vertices from D_{ki} and $D_{k(i+1+(l'-2)k)}$ and the stable sets $S(B_j)$ for the involved subsets B_j. (Note that $S_{ki} = S'_{ki}$ iff $l' = 3$). We call ki the start vertex of S_{ki} resp. of S'_{ki}.

Claim B.3.22. *The sets S_{ki} and S'_{ki} are mixed roots of $(\mathcal{Q}, k)$ for $1 \leq i \leq l$.*

Proof. By construction, we have $ki, k(i + 1 + k) \in S_{ki}$ and $ki, k(i + 1 + (l' - 2)k) \in S'_{ki}$ (these vertices belong obviously to $V \setminus V'$). The remaining vertices of S_{ki} and S'_{ki} come from the sets $S(B_j)$ for each of the l' subsets B_j. Since $S(B_j)$ contains only $k - 1$ vertices from V' by construction, we obtain $|S_{ki} \cap V'| = |S'_{ki} \cap V'| = l'(k - 1)$ and $|S_{ki} \setminus V'| = |S'_{ki} \setminus V'| = 2$. Thus S_{ki} as well as S'_{ki} satisfy $(\mathcal{Q}, k)$ at equality for every $1 \leq i \leq l$.

It is left to show that S_{ki} and S'_{ki} are stable. For that, recall first that $S(B_j)$ is a stable set for any j. Second, the last k vertices of B_j do not belong to $S(B_j)$ by construction, thus B_j can be followed by any subset without introducing adjacencies in S_{ki} or S'_{ki}. Finally, consider a subset D_{kj} followed by $B_{k(j+1)}$. By construction, the last $k - 1$ vertices of D_{kj} as well as the first vertex of $B_{k(j+1)}$ do not belong to S_{ki} or S'_{ki}, thus no adjacencies are introduced again.

This implies that S_{ki} and S'_{ki} are stable sets satisfying $(\mathcal{Q}, k)$ at equality. □

We are now prepared to select a set of lk independent roots of $(\mathcal{Q}, k)$:

Claim B.3.23. *There are lk roots of $(\mathcal{Q}, k)$ whose incidence vectors are linearly independent: the $l(k - 1)$ maximum stable sets of $W^{k-1}_{l(k-1)}$ and the l stable sets S_{ki} for $1 \leq i \leq l - (k + 1)$ resp. S'_{ki} for $l - k \leq i \leq l$.*

Proof. The maximum stable sets of $W^{k-1}_{l(k-1)}$ are linearly independent as mentioned above. Moreover, they contain only vertices from V' whereas the stable sets S_{ki} and S'_{ki} contain vertices from V' as well as vertices from $V \setminus V'$. Thus, we are done if we can show that the incidence vectors of S_{ki} for $1 \leq i \leq l - (k + 1)$ and S'_{ki} for $l - k \leq i \leq l$ are linearly independent.

We construct an $(l \times lk)$-matrix M having the incidence vectors of $S_k, \dots, S_{k(l-(k+1))}$ as first $l - (k + 1)$ rows and the incidence vectors of $S'_{k(l-k)}, \dots, S'_{kl}$ as last $k + 1$ rows. We show that the $(l \times l)$-submatrix M' of M containing all columns corresponding to the vertices in $V \setminus V'$ is invertible. For that, choose an ordering of the columns of M s.t. the first l columns correspond to the vertices $k, \dots, lk$ in $V \setminus V'$ and the remaining $l(k - 1)$ columns correspond to the vertices in V' (see Figure B.3).

Each row of M' has a 1-entry on the main diagonal (since ki is the start vertex of S_{ki} as well as of S'_{ki} by construction), thus we have to discuss the second 1-entries of the rows coming from the vertices $k(i+1+k) \in S_{ki}$ resp. $k(i + 1 + (l' - 2)k) \in S'_{ki}$ (see Figure B.3).

Let $= l - (3k + 1)$ (we have ≥ 1 since $l = l'k + 2$ and $l' \geq 3$). We show that k is the first column with a 1-entry below the main diagonal, namely in the row corresponding to $S'_{k(l-k)}$.

$$
\begin{array}{l|ccc|cccccc|c}
 & k & \dots & k(-1) & k & \dots & k(+k) & k(+kl+1) & \dots & kl & V' \\
\hline
S_k & 1 & & & & & & & & & \\
\vdots & & \ddots & & & \ddots & & & & & \vdots \\
S_{k(-1)} & & & 1 & & & 1 & & & & \\
\hline
S_k & & & & 1 & & & 1 & & & \\
\vdots & & & & & \ddots & & & \ddots & & \vdots \\
S_{k(l-(k+1))} & & & & & & \ddots & & & 1 & \\
\cline{1-1}
S'_{k(l-k)} & & & & 1 & & & \ddots & & & \\
\vdots & & & & & \ddots & & & \ddots & & \vdots \\
S'_{kl} & & & & & & 1 & & & 1 & \\
\hline
\end{array}
$$

Figure B.15: The $(l \times lk)$-matrix M

The first $l-(k+1)$ rows of M' do not have any 1-entry below the main diagonal, since the second vertex of S_{kj} in $V \setminus V'$ is $k(j+1+k)$ and $k(j+1+k) \leq kl$ holds due to $j \leq l-(k+1)$. (In fact, the row corresponding to $S_{k(l-(k+1))}$ has its 1-entries in the columns $k(l-(k+1))$ and kl.)

Consider now the row $l-k$ corresponding to $S'_{k(l-k)}$. We have $S'_{k(l-k)} \setminus V' = \{k(l-k), k(l-k+1+(l'-2)k)\}$ where

$$k(l-k+1+(l'-2)k) = k(-3k+1+l'k) \bmod n = k$$

as $l = l'k+2$. Hence the row given by $S'_{k(l-k)}$ has indeed a 1-entry at column k.

The matrix M' is invertible, if its $(3k+2) \times (3k+2)$-submatrix M'' consisting of the colums $k, \dots, kl$ of the rows corresponding to $S_k, \dots, S_{k(l-(k+1))}, S'_{k(l-k)}, \dots, S'_{kl}$ is invertible (since the previous part of M' has 0-entries below the main diagonal only).

We complete the proof of this claim by showing that M'' is a $(3k+2, 2)$-circulant matrix: The 1-entries below the main diagonal start in column k, as seen above, and end in the last row in column $k(l-(2k+1))$ since

$$k(l+1+(l'-2)k) = kl + k(l'k+2-(2k+1)) \bmod n = k(l-(2k+1))$$

holds as $l = l'k+2$, whereas the 1-entries above the main diagonal start in column $k(l-2k)$ due to

$$k(+1+k) = k(l-(3k+1)+1+k) = k(l-2k)$$

and end with kl in the row corresponding to $S_{k(l-(k+1))}$ as shown above. This implies that M'' is a $(3k+2, 2)$-circulant matrix and, therefore, invertible as k is odd by our hypothesis. This complets the proof that the choosen stable sets are lk independent roots of $(\mathcal{Q}, k)$. □

Hence $(\mathcal{Q}, k)$ is, for any odd $k \geq 5$, a proper weak non-rank facet (B.11) of STAB(W^k_{kl}) if $l \geq 3k+2$ completing the proof of Theorem B.3.9.

Concluding remarks and open problems

In this paper, we presented infinite sequences of not rank-perfect webs W^k_n for $k = 4$ (Theorem B.3.8), all even $k \geq 6$ (Theorem B.3.6), and all odd $k \geq 5$ (Theorem B.3.9). Before, the case $k = 3$ was settled in [15]. Applying the construction from [14] yields that there are only finitely many rank-perfect webs W^k_n for all values of $k \geq 3$ (Corollary B.3.11), implying that almost all webs with fixed clique number at least 4 are not rank-perfect (Corollary B.3.12).

For our construction, we used clique family inequalities associated with certain subwebs yielding 1/2-valued facets; the construction from [14] does not change the involved coefficients and, therefore, the stable set polytopes of almost all webs admit 1/2-valued facets.

According to Ben Rebea's Conjecture [12], the stable set polytopes of quasi-line graphs (and therefore of webs) have clique family inequalities as only non-trivial facets. However, clique family inequalities constitute a large class of valid inequalities; even verifying Ben Rebea's Conjecture would provide no information about which inequalities are *essential* among them. The following conjecture addresses this problem for the subclass of webs.

Conjecture B.3.24. *Every facet of STAB(W_n^k) belongs to one of the following classes:*

(0) *nonnegativity constraints,*

(i) *clique constraints,*

(ii) *full rank constraint,*

(iii) *clique family inequalities* $(\mathcal{Q}, k'+1)$ *associated with proper subwebs* $W_{n'}^{k'}$ *where* $(k'+1)\not| n'$ *and* $\alpha(W_{n'}^{k'}) < \alpha(W_n^k)$.

All non-rank facets known so far for webs are of type (iii). Note that a web W_n^k usually has subwebs $W_{n'}^{k'}$ for all values $1 \leq k' < k$. Hence, we expect that the stable set polytope of W_n^k admits $(k-2)/(k-1)$-valued facets. The conjecture implies in particular, that the stable set polytopes of all webs W_n^3 have 1/2-valued facets only, where for all webs W_n^k with $k > 3$ larger coefficients are required. In fact, Liebling et al. [11] proved recently that, for any odd $k \geq 5$, the stable set polytope of $W_{k^2}^k$ has an $(k-2)/(k-1)$-valued facet. Hence, further effort is needed for having a complete description of stable set polytope of webs (and for the larger class of fuzzy circular interval graphs).

References

[1] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[2] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[3] V. Chvátal. On the strong perfect graph conjecture. *J. Comb. Theory Series B*, 20:139–141, 1976.

[4] G. Dahl. Stable set polytopes for a class of circulant graphs. *SIAM Journal on Optimization*, 9(2):493–503, 1998.

[5] J.E. Edmonds and W.R. Pulleyblank. Facets of 1-matching polyhedra. In C. Berge and D.R. Chuadhuri, editors, *Hypergraph Seminar*, pages 214–242. Springer, 1974.

[6] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1):1–38, 1997.

[7] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[8] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[9] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12:373–388, 1975.

[10] J. Kind. *Mobilitätsmodelle für zellulare Mobilfunknetze: Produktformen und Blockierung*. PhD thesis, RWTH Aachen, 2000.

[11] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[12] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[13] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[14] A. Pêcher and A. Wagler. A construction for non-rank facets of stable set polytopes of webs. *European Journal of Combinatorics*, 27(7):1172–1185, 2006.

[15] A. Pêcher and A. Wagler. On non-rank facets in stable set polytopes of webs with clique number four. *Discrete Applied Mathematics*, 154(9):1408–1415, jun 2006.

[16] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2:149–152, 2004.

B.4 On the circular chromatic number of circular partitionable graphs

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This paper studies the circular chromatic number of a class of circular partitionable graphs. We prove that an infinite family of circular partitionable graphs G has $\chi_c(G) = \chi(G)$. A consequence of this result is that we obtain an infinite family of graphs G with the rare property that the deletion of each vertex decreases its circular chromatic number by exactly 1.

Introduction

Suppose G is a graph and k and d, $k \geq 2d$, are positive integers. A (k,d)-coloring of G is a mapping $f : V(G) \to \mathbb{Z}_k$ such that for each edge xy of G, $d \leq |f(x) - f(y)| \leq k - d$. The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ has a } (k,d)\text{-coloring}\}.$$

As a $(k,1)$-coloring of G is equivalent to a k-coloring of G, we have $\chi_c(G) \leq \chi(G)$. On the other hand, it is known [11, 12] that $\chi_c(G) > \chi(G) - 1$. Therefore $\chi(G) = \lceil \chi_c(G) \rceil$. So the parameter $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

If G is vertex $\chi-$critical (i.e., the deletion of any vertex decreases its chromatic number), then deleting any vertex from G decreases its chromatic number by 1. However, for a vertex χ_c-critical graph G (i.e., the deletion of any vertex decreases its circular chromatic number), the decrease of the circular chromatic number by the deletion of one vertex could be any rational number $0 < r < 2$. Indeed, it seems very rare that a graph G has the property that the deletion of each vertex decrease its circular chromatic number by exactly 1. The question of characterizing such graphs was raised in [12]. Currently, complete graphs, the direct sum of vertex χ-critical graphs and a few isolated example graphs are known to have this property. In this paper, we study the circular chromatic number of partitionable graphs. We prove that for an infinite family of partitionable graphs G we have $\chi_c(G) = \chi(G)$. Since partitionable graphs G have the property for any vertex v, $\chi(G - v) = \omega(G - v) = \chi(G) - 1$, which implies that $\chi_c(G - v) = \chi(G) - 1$, it follows that for these partitionable graphs G, the deletion of any vertex decreases its circular chromatic number by exactly 1.

Another motivation for the study of the circular chromatic number of partitionable graphs concerns circular perfect graphs. Given positive integers k and d, $k \geq 2d$, the *circular complete graph* $K_{k/d}$ has vertex set $\mathbb{Z}_k$ in which ij is an edge if $d \leq |i - j| \leq k - d$. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y)$ is an edge of H whenever xy is an edge of G. Then a k-coloring of a graph G is equivalent to a homomorphism from G to K_k and a (k,d)-coloring of G is equivalent to a homomorphism from G to $K_{k/d}$. So in the study of circular chromatic number of graphs, the circular complete graphs $K_{k/d}$ play the role of complete graphs as in the study of chromatic number of graphs. The *circular clique number* of a graph G is defined as

$$\omega_c(G) = \sup\{k/d : K_{k/d} \text{ admits a homomorphism to } G\}.$$

It was shown in [13] that $\omega_c(G)$ is equal to the maximum of those k/d for which $K_{k/d}$ is an induced subgraph of G.

A graph G is called *circular perfect* [13] if for every induced subgraph H of G, $\chi_c(H) = \omega_c(H)$. Since $\omega(H) \leq \omega_c(H) \leq \chi_c(H) \leq \chi(H)$, every perfect graph is circular perfect. On the other hand, there are circular perfect graphs which are not perfect. In particular, odd cycles and the complement of odd cycles are circular perfect graphs. So the class of circular perfect graphs is strictly larger than the class of perfect graphs, but it is still a very restrictive class. A sufficient condition for a graph to be circular perfect is given in [13], and this sufficient condition is used to prove an analogue of Hajós Theorem for circular chromatic number. We call a graph G *minimal circular imperfect*, if G is not circular perfect but every proper induced subgraph of G is circular perfect. As an analogue to the study of perfect graphs, it would be nice to have an appealing conjecture on the structure of minimal circular imperfect graphs. Since all the minimal imperfect graphs are circular perfect, we need to search a larger area for minimal circular imperfect graphs. The class of partitionable graphs is a natural candidate. By studying the circular chromatic number of a subclass of circular partitionable graphs, we prove that all the partitionable graphs in this subclass are circular imperfect. The question whether they are minimal circular imperfect remains an open question.

Circular partitionable graphs and the main result

Suppose $p, q \geq 2$ are integers. A graph G is a (p, q)-*partitionable graph* if $|V(G)| = pq + 1$, and for each vertex v of G, $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p. A graph is *partitionable* if it is a (p, q)-partitionable graph for some $p, q \geq 2$. Partitionable graphs were introduced by Lovász [9] and Padberg [10] as a tool in the study of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, we have $\chi(H) = \omega(H)$. Here $\omega(H)$ is the *clique number* of G, which is the cardinality of a maximum clique of G. A graph G is *minimal imperfect* if G is not perfect, but every proper induced subgraph of G is perfect. The Strong Perfect Graph Theorem, which was conjectured by Berge [2] in 1961, proved by Chudnovsky, Robertson, Seymour and Thomas [5] in 2002, says that odd cycles of length at least 5 and their complements are the only minimal imperfect graphs. Before the proof of Berge's conjecture, it was shown by Lovász [9] and Padberg [10] that every minimal imperfect graph is a partitionable graph. Thus to prove Berge's conjecture, it suffices to show that none of the partitionable graph is a counterexample. Although the final proof of Berge's conjecture given by Chudnovsky, Robertson, Seymour and Thomas [5] takes a different route, the class of partitionable graphs has been studied thoroughly in the literature, and this turns out to be an interesting class of graphs. The understanding of the structure of this class of graphs may be helpful in the study of other graph theory problems. It is known (cf. [3]) that every (p, q)-partitionable graph has the following properties:

1. p is the maximum cardinality of a stable set of G, and q is the maximum cardinality of a clique of G;

2. G has exactly n stable sets of cardinality p and exactly n cliques of cardinality q, where n is the number of vertices of G;

3. For each maximum clique C of G, there is a unique maximum stable set S such that $C \cap S = \emptyset$; and similarly, for each maximum stable set S there is a unique maximum clique C such that $C \cap S = \emptyset$;

4. Each vertex belongs to exactly q maximum cliques, and belongs to p maximum stable sets.

In the study of partitionable graphs, some recursive constructions of sub-families of partitionable graphs are discussed in the literature [6, 4]. The class of circular partitionable graphs was introduced by Chvátal, Graham, Perold and Whitesides [6].

For two sets of integers X, Y, let $X+Y$ denote the set $\{x+y : x \in X,\ y \in Y\}$. If $X = \{x\}$ is a singleton, we write $x+Y$ instead of $\{x\}+Y$.

Suppose $m_i \geq 2$ ($i = 1, 2, \cdots, 2r$) are integers. Define integers μ_i (for $i = 0, 1, \cdots, 2r$), sets M_i (for $i = 1, 2, \cdots, 2r$), and sets C, S as follows:

$$\begin{aligned} \mu_i &= m_1 m_2 \cdots m_i \ (\mu_0 = 1), \\ M_i &= \{0, \mu_{i-1}, 2\mu_{i-1}, \cdots, (m_i - 1)\mu_{i-1}\}, \\ C &= M_1 + M_3 + \cdots + M_{2r-1}, \\ S &= M_2 + M_4 + \cdots + M_{2r}. \end{aligned}$$

Let $n = m_1 m_2 \cdots m_{2r} + 1$. We denote by $C[m_1, m_2, \cdots, m_{2r}]$ the circulant graph with vertex set $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$, where xy is an edge if and only if $x \neq y$ and $(x-y)$ modulo n is equal to the difference of two elements of C.

Note that $\mu_i > \sum_{j=1}^{i-1} \max M_j$. This implies that $|C| = m_1 m_3 \cdots m_{2r-1}$ and $|S| = m_2 m_4 \cdots m_{2r}$. Let $\omega = |C|$ and $\alpha = |S|$. Then $n = \omega\alpha + 1$. Suppose X is a subset of $\mathbb{Z}_n$. A *circular shift* of X is a set of the form $i + X = \{i + x \pmod{n} : x \in X\}$.

Theorem B.4.1. *[6] Suppose $m_i \geq 2$ are integers for $i = 1, 2, \cdots, 2r$. Then $G = C[m_1, m_2, \cdots, m_{2r}]$ is an (α, ω)-partitionable graph. Moreover, the n maximum cliques of G are the n circular shifts of C, and the n maximum stable sets of G are the n circular shifts of S.*

As an example, we consider the graph $C[2, 2, 2, 2]$. Then

$$\begin{aligned} \mu_i &= 2^i,\ \ i = 0, 1, 2, 3, 4; \\ M_1 &= \{0, 1\},\ M_2 = \{0, 2\},\ M_3 = \{0, 4\},\ M_4 = \{0, 8\}; \\ C &= \{0, 1, 4, 5\}; \\ S &= \{0, 2, 8, 10\}. \end{aligned}$$

The vertex set of $C[2, 2, 2, 2]$ is $\mathbb{Z}_{17}$, and ij is an edge if $|i-j| \in \{1, 3, 4, 5\}$. The graph is depicted in Figure B.4.

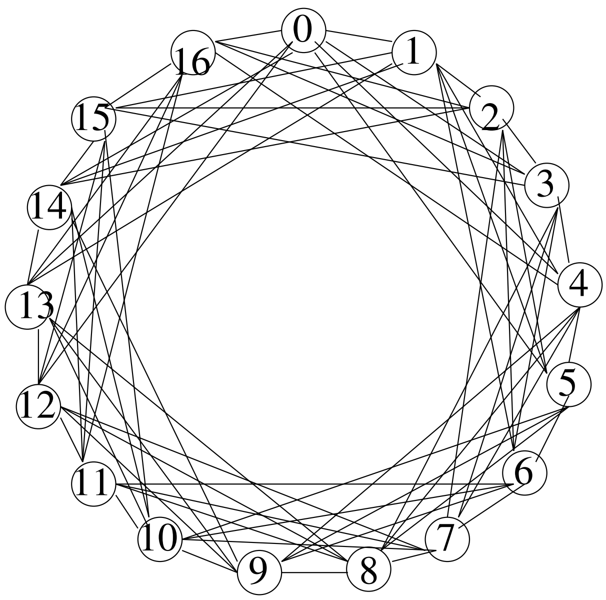

Figure B.16: The circular partitionable graph $C[2, 2, 2, 2]$

The following is the main result of this paper.

Theorem B.4.2. *Let $x_1, \dots, x_p$ $(p \geq 2)$ be integers such that $x_i \geq 2$ for every $1 \leq i \leq p$. Let $\delta = \max x_i$ and let $G = C[x_1, 2, \dots, x_p, 2]$. If $p = 2$ or $x_1 x_2 \dots x_p \geq 2^{p+1}\delta$ then $\chi_c(G) = \chi(G)$.*

The proof of Theorem B.4.2 is left to Section B.4.

Observe that for $G = C[x_1, 2, \dots, x_p, 2]$, $\alpha = 2^p$, $\omega = x_1 x_2 \cdots x_p$ and $|V(G)| = n = 2^p x_1 x_2 \dots x_p + 1$. Therefore

$$\chi_f(G) = \frac{n}{\alpha} = \omega + \frac{1}{2^p} < \chi_c(G) = \chi(G) = \omega + 1.$$

There are a few papers devoted to the study of the circular chromatic number of circulant graphs [7, 8]. It is known [12] that for any graph G, $\chi_f(G) \leq \chi_c(G) \leq \chi(G)$. A graph G is called *star extremal* if $\chi_f(G) = \chi_c(G)$. In [7, 8], classes of star extremal circulant graphs are investigated. Theorem B.4.2 provides a class of circulant graphs of another kind of extremality, i.e., circulant graphs with $\chi_c(G) = \chi(G)$.

Corollary B.4.3. *Suppose $G = C[x_1, 2, \dots, x_p, 2]$ satisfies the condition of Theorem B.4.2. Then G is circular imperfect.*

Proof. It is known [13] that $\omega(G) \leq \omega_c(G) < \omega(G) + 1$. As $\chi_c(G) = \chi(G) = \omega(G) + 1$, it follows that $\chi_c(G) > \omega_c(G)$.

■

It is unknown if any of the graphs $G = C[x_1, 2, \dots, x_p, 2]$ are minimal circular imperfect. A computer search shows that $C[2, 2, 2, 2]$ is not minimal circular imperfect. The subgraph of $C[2, 2, 2, 2]$ induced by the set $X = \{2, 3, 4, 5, 6, 9, 11, 12, 13, 14, 16\}$ has circular chromatic number 4 and circular clique number 3, and hence is circular imperfect.

Since the graphs $G = C[x_1, 2, x_2, 2, \cdots, x_p, 2]$ are partitionable, $\chi(G - x) = \omega(G - x) = \chi_c(G - x) = \chi(G) - 1$ for each vertex x. Therefore we have the following corollary.

Corollary B.4.4. *Suppose $G = C[x_1, 2, \dots, x_p, 2]$ satisfies the condition of Theorem B.4.2. Then $\chi_c(G - x) = \chi_c(G) - 1$ for each vertex x of G.*

In other words, the circulant graphs satisfying the condition of Theorem B.4.2 have the property that the deletion of each vertex decreases its circular chromatic number by exactly 1. Not many such graphs were known before, and the problem of characterizing and constructing such graphs was raised in [12].

Structural properties of G

In the remainder of this paper, $G = C[x_1, 2, x_2, \dots, x_p, 2]$. Let $\omega = x_1 \dots x_p$, $\alpha = 2^p$, $n = \alpha\omega + 1$.

In this section, we shall be interested in elements of $\mathbb{Z}_n$ only. If $a \equiv b \pmod{n}$, then a, b are treated as the same. However, by an abuse of notation, we also use the natural order of integers in the following sense: If $a < b$, we denote $[a, b]$ the set of integers $a \leq x \leq b$. Note that it is possible that $a, b \notin \{0, 1, \cdots, n-1\}$. However, $[a, b]$ always denote a subset of $\{0, 1, \cdots, n-1\}$, by means of taking modulo n. For example, $[-1, 1] = \{-1, 0, 1\} = \{n-1, 0, 1\}$. For a set A of integers, let $aA = \{ax : x \in A\}$, $a + A = \{a + x : x \in A\}$. Again the multiplications and additions are modulo n.

Let $c_0 = 1$, and for $i = 1, \dots, p$, let

$$c_i = 2^{i-1} x_1 x_2 \dots x_i, \text{ and } d_i = \sum_{j=1}^{i} c_j.$$

First we derive an explicit expression for the maximum cliques and stable sets of G. It follows from the definition that $\mu_0 = 1$ and for $1 \leq i \leq p$,

$$\mu_{2i-1} = 2^{i-1} x_1 x_2 \dots x_i = c_i, \text{ and } \mu_{2i} = 2^i x_1 x_2 \dots x_i = 2c_i.$$

Therefore for $1 \leq i \leq p$,

$$M_{2i-1} = 2c_{i-1}[0, x_i - 1], \text{ and } M_{2i} = \{0, c_i\}.$$

So

$$\begin{aligned} C &= \sum_{j=1}^{p} M_{2j-1} = \sum_{j=1}^{p} 2c_{j-1}[0, x_j - 1], \\ S &= \sum_{j=1}^{p} M_{2j} = \sum_{j=1}^{p} \{0, c_j\}. \end{aligned}$$

By Theorem B.4.1, the maximum cliques of G are the n circular shifts $C + i$ of C ($i \in \mathbb{Z}_n$), and the maximum stable sets of G are the n circular shifts $S + i$ of S ($i \in \mathbb{Z}_n$).

In the following, we consider the intersection $S \cap (S + i)$ of two maximum stable sets of G.

Lemma B.4.5. *For $2 \leq i \leq p$, $c_i \notin [-2d_{i-1}, 2d_{i-1}]$.*

Proof. Since $c_2 = 2x_1x_2 > 2x_1 = 2d_1$, and $n - c_2 \geq 4x_1x_2 + 1 - 2x_1x_2 > 2x_1x_2 > 2d_1$, we conclude that $c_2 \notin [-2d_1, 2d_1]$. Assume $i \geq 3$ and $c_{i-1} \notin [-2d_{i-2}, 2d_{i-2}]$. Since $c_i = 2x_ic_{i-1}$ and $x_i \geq 2$, we have

$$c_i \geq 2c_{i-1} + 2c_{i-1} > 2(2d_{i-2}) + 2c_{i-1} > 2d_{i-1}.$$

Furthermore,

$$n - c_i = 2^p x_1 x_2 \cdots x_p + 1 - c_i > 2c_i - c_i = c_i > 2d_{i-1}.$$

Therefore $c_i \notin [-2d_{i-1}, 2d_{i-1}]$. ∎

For $i = 1, 2, \cdots, p$, let

$$S_i = \sum_{j=1}^{i} M_{2j} = \sum_{j=1}^{i} \{0, c_j\}.$$

Lemma B.4.6. *For every x in $\mathbb{Z}_n$ and for every $2 \leq i \leq p$,*

$$\begin{aligned} S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)) \\ &\textit{or} \\ S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)). \end{aligned}$$

Proof. By definition, $S_i = S_{i-1} \cup (S_{i-1} + c_i)$. Hence $S_i + x = (S_{i-1} + x) \cup (S_{i-1} + c_i + x)$. Therefore

$$\begin{aligned} S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)) \\ &\quad \cup (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)). \end{aligned}$$

If $S_{i-1} \cap (S_{i-1} + x) \neq \emptyset$, then $x \in [-d_{i-1}, d_{i-1}]$. By Lemma B.4.5, $c_i \notin [-2d_{i-1}, 2d_{i-1}]$. Therefore $c_i + x, c_i - x \notin [-d_{i-1}, d_{i-1}]$. This implies that $S_{i-1} \cap (S_{i-1} + c_i + x) = \emptyset$ and $(S_{i-1} + c_i) \cap (S_{i-1} + x) = \emptyset$. Therefore

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)).$$

If $S_{i-1} \cap (S_{i-1} + x) = \emptyset$, then $(S_{i-1} + c_i) \cap (S_{i-1} + c_i + x) = \emptyset$, and hence

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)).$$

∎

For $1 \leq j \leq i \leq p$, let

$$B_{i,j} = \sum_{t \in \{1,2,\cdots,i\} \setminus \{j\}} M_{2t} = \sum_{t \in \{1,2,\cdots,i\} \setminus \{j\}} \{0, c_t\}.$$

Then $S_i = B_{i,j} + \{0, c_j\} = B_{i,j} \cup (B_{i,j} + c_j)$. For convenience, let $B_j = B_{p,j}$.

Lemma B.4.7. *For every $1 \leq i \leq p$ and for every $x \in \mathbb{Z}_n,\ x \neq 0$, we have $|S_i \cap (S_i + x)| \leq 2^{i-1}$. Moreover, if $|S_i \cap (S_i + x)| = 2^{i-1}$, then there is a unique index $j \leq i$ such that $x = \pm c_j$ and $S_i \cap (S_i + x) = B_{i,j}$ or $B_{i,j} + c_j$, depending on $x = -c_j$ or $x = c_j$. In particular, for any $x \in \mathbb{Z}_n$, $|S \cap (S + x)| \leq 2^{p-1}$, and if equality holds then there is a unique index j such that $x = \pm c_j$ and $S \cap (S + x) = B_j$ or $B_j + c_j$, depending on whether $x = -c_j$ or $x = c_j$.*

Proof. We prove this lemma by induction on i. It is obvious that for every $x \in \mathbb{Z}_n,\ x \neq 0$, we have $|S_1 \cap (S_1 + x)| \leq 1$, and equality holds only if $x = \pm c_1$. Let $i \geq 2$ and suppose the lemma is true for $i' \leq i - 1$.

By Lemma B.4.6,

$$\begin{aligned} S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)) \\ &\text{or} \\ S_i \cap (S_i + x) &= (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)) \end{aligned}$$

First we consider the case that

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x)).$$

By induction hypothesis, $|S_{i-1} \cap (S_{i-1} + x)| \leq 2^{i-1}$ and $|((S_{i-1} + c_i) \cap (S_{i-1} + c_i + x))| \leq 2^{i-1}$. Therefore

$$|S_i \cap (S_i + x)| \leq 2^{i-1} + 2^{i-1} = 2^i.$$

Moreover, if $|S_i \cap (S_i + x)| = 2^i$, then $|S_{i-1} \cap (S_{i-1} + x)| = 2^{i-1}$ and hence there is a unique index $j \leq i - 1$ such that $x = \pm c_j$ and $S_{i-1} \cap (S_{i-1} + x) = B_{i-1,j}$ or $B_{i-1,j} + c_j$. Then $S_i \cap (S_i + x) = B_{i-1,j} + \{0, c_i\} = B_{i,j}$ or $S_i \cap (S_i + x) = B_{i-1,j} + c_j + \{0, c_i\} = B_{i,j} + c_j$

Next we assume that

$$S_i \cap (S_i + x) = (S_{i-1} \cap (S_{i-1} + c_i + x)) \cup ((S_{i-1} + c_i) \cap (S_{i-1} + x)).$$

If $x \neq \pm c_i$ then the same argument as in the previous case works. Assume $x = c_i$. Then

$$|S_i \cap (S_i + x)| = |S_{i-1} + c_i| + |S_{i-1} \cap (S_{i-1} + 2c_i)|.$$

Note that if $i \leq p - 1$, then $2c_i \in C$. If $i = p$, then $2c_p = -1$. In any case, $2c_i$ is equal to the difference of two integers of C. Therefore $\{0, 2c_i\}$ is an edge of G. Hence $2c_i \notin S_{i-1} - S_{i-1}$ (as S_{i-1} is a stable set of G). This implies that

$$S_{i-1} \cap (S_{i-1} + 2c_i) = \emptyset.$$

Hence

$$|S_i \cap (S_i + x)| = |S_{i-1}| = 2^{i-1},$$

and

$$S_i \cap (S_i + x) = B_{i,i} + c_i$$

as $B_{i,i} \cap (B_{i,i} + 2c_i) = \emptyset$ ($B_{i,i}$ is a stable set and $\{0, 2c_i\}$ is an edge).

If $x = -c_i$, then the same argument shows that

$$S_i \cap (S_i + x) = B_{i,i}.$$

■

Proof of Theorem B.4.2

Assume that $\chi_c(G) = k/d$, where $\gcd(k,d) = 1$. Let f be a (k,d)-coloring of G, which is viewed as a homomorphism from G to $K_{k/d}$. For $i \in \mathbb{Z}_k$, let $X_i = f^{-1}(i)$ be the set of vertices of G of color i. Let $Y_i = X_i \cup X_{i+1} \cup \cdots \cup X_{i+d-1}$. Then Y_i is a stable set. It is known (see Lemma 1.3 of [12]) that for each i, $X_i \neq \emptyset$ and moreover, for each i, there is a vertex $x \in X_i$ and a vertex $y \in X_{i+d}$ such that xy is an edge of G. We need to prove that $d = 1$.

Lemma B.4.8. *If $\chi_c(G) = k/d$ and $\gcd(k,d) = 1$, then $d \leq 2$.*

Proof. Assume to the contrary that $d \geq 3$. We consider two cases.

- **Case 1:** $p = 2$

  Note that in this case $\alpha = 4$.

  Assume that there is an index i for which $|X_i| \geq 2$. Since $d \geq 3$, $X_i \cup X_{i+1} \cup X_{i+2} \subseteq Y_i$ and $X_{i-1} \cup X_i \cup X_{i+1} \subseteq Y_{i-1}$. However, $|Y_{i-1}|, |Y_i| \leq \alpha = 4$, and $|X_{i-1} \cup X_i \cup X_{i+1}|, |X_i \cup X_{i+1} \cup X_{i+2}| \geq 4$ (as $|X_i| \geq 2$, and each $|X_j| \geq 1$). Therefore $|Y_{i-1}| = |Y_i| = 4$, and hence Y_{i-1}, Y_i are maximum stable sets of G. However, $|Y_{i-1} \cap Y_i| = |X_i| + |X_{i+1}| \geq 3 > \alpha/2$, contrary to Lemma B.4.7.

  Hence, for every $i \in \mathbb{Z}_k$, we have $|X_i| = 1$. In particular, $n = k = 4\omega + 1$. From $\frac{k}{d} \leq \chi = \omega + 1$ and $\omega > 2$, we get $d \geq 4$. Since Y_0 is a stable set of size d, we have $d = 4$. Thus Y_0 and Y_1 are two maximum stable sets of G sharing $d - 1$ vertices, contrary again to Lemma B.4.7.

  This completes the proof of Case 1.

- **Case 2:** $\omega \geq 2^{p+1}\delta$

  If there exists $i \in \mathbb{Z}_k$ such that $|Y_i| + |Y_{i+1}| + |Y_{i+2}| = 3\alpha$ then $|Y_i| = |Y_{i+1}| = |Y_{i+2}| = \alpha$. By Claim B.4.7, we have $|Y_i \cap Y_{i+1}| \leq \frac{\alpha}{2}$. Hence $|Y_{i+1} \setminus Y_i| \geq \frac{\alpha}{2}$. As $Y_{i+1} \setminus Y_i = X_{i+d}$, we get $|X_{i+d}| \geq \frac{\alpha}{2}$. As $d \geq 3$ and $X_{i+2} \neq \emptyset$, we conclude that $|X_{i+2} \cup X_{i+d}| > \frac{\alpha}{2}$. However $X_{i+2} \cup X_{i+d} \subseteq Y_{i+1} \cap Y_{i+2}$. So Y_{i+1} and Y_{i+2} are two distinct maximum stable sets with $|Y_{i+1} \cap Y_{i+2}| > \frac{\alpha}{2}$, contrary to Lemma B.4.7.

  Thus for every $i \in \mathbb{Z}_k$, we have $|Y_i| + |Y_{i+1}| + |Y_{i+2}| < 3\alpha$. Since $\sum_{i \in \mathbb{Z}_k} |Y_i| = dn$, we have $\sum_{i \in \mathbb{Z}_k} (|Y_i| + |Y_{i+1}| + |Y_{i+2}|) = 3dn \leq (3\alpha - 1)k$.

  As $n = \alpha\omega + 1$ and $\frac{k}{d} < \omega + 1$, we have

  $$3n = 3(\alpha\omega + 1) < (3\alpha - 1)(\omega + 1).$$

  It follows that $\omega < 3\alpha - 4$. Hence

  $$2^{p+1}\delta \leq \omega < 3\alpha - 4,$$

  which is a contradiction, as $\alpha = 2^p$ and $\delta \geq 2$.

■

By Lemma B.4.8, we have $d = 1$ or 2. If $d = 1$, we are done. Thus we assume $d = 2$, and we shall derive a contradiction. As $\omega < \chi_c(G) \leq \chi(G) = \omega + 1$ and $d = 2$, we have $k = 2\omega + 1$.

Lemma B.4.9. *There exists an $i \in \mathbb{Z}_k$ such that $Y_i, Y_{i+1}, \ldots, Y_{i+2\delta}$ are all of size α.*

Proof.

- **Case 1:** $p = 2$

  If there exists $i \in \mathbb{Z}_k$ such that $|X_i| \geq 3$ then Y_i and Y_{i-1} are two maximum stable sets sharing $|X_i| > \alpha/2$ vertices, contrary to Lemma B.4.7. Thus, for every $i \in \mathbb{Z}_k$, we have $|X_i| \leq 2$. Since $\sum_{i \in \mathbb{Z}_k} |X_i| = n = 4x_1x_2 + 1 = 2k - 1$, it follows that there exists a unique $j \in \mathbb{Z}_k$, such that $|X_j| = 1$. Thus $Y_{j+1}, Y_{j+2}, \ldots, Y_{j+n-2}$ are $n - 2$ stable sets of size $\alpha = 4$. As $n - 2 = 4x_1x_2 - 1 > 2\delta + 1$, we are done.

- **Case 2:** $\omega \geq 2^{p+1}\delta$

  Assume to the contrary that for every $i \in \mathbb{Z}_k$, there exists one stable set in $Y_i, Y_{i+1}, \ldots, Y_{i+2\delta}$ of size strictly less than α, then
  $$\sum_{i=0}^{k-1}(|Y_i| + |Y_{i+1}| + \cdots + |Y_{i+2\delta}|) \leq ((2\delta+1)\alpha - 1)k.$$
  On the other hand, as $\sum_{i=0}^{k-1}|Y_i| = 2n$, we have
  $$\sum_{i=0}^{k}(|Y_i| + |Y_{i+1}| + \cdots + |Y_{i+2\delta}|) = (2\delta+1)2n.$$
  Therefore
  $$(2\delta+1)2n \leq ((2\delta+1)\alpha - 1)k.$$
  Since $k = 2\omega+1$ and $n = \omega\alpha+1$, straightforward calculation shows that
  $$4\delta + 2 \leq (2\delta+1)\alpha - 2\omega - 1.$$
  As $\omega \geq 2^{p+1}\delta$ and $\alpha = 2^p$, easy calculation derives a contradiction.

■

In the remainder of this section, let i be an index such that $Y_i, Y_{i+1}, \ldots, Y_{i+2\delta}$ are all of size α.

Lemma B.4.10. *For every $j = i+1, i+2, \ldots, i+2\delta$, we have $|X_j| = \frac{\alpha}{2}$.*

Proof. By Lemma B.4.7, $|X_j| = |Y_{j-1} \cap Y_j| \leq \frac{\alpha}{2}$. As $\alpha = |Y_j| = |X_j| + |X_{j+1}|$, it follows that $|X_j| = \alpha - |X_{j+1}| \geq \frac{\alpha}{2}$ if $i+1 \leq j \leq i+2\delta-1$. Hence $|X_j| = \frac{\alpha}{2}$. If $j = i+2\delta$, then since $\alpha = |Y_{j-1}| = |X_{j-1}| + |X_j|$, we also have $|X_j| = \frac{\alpha}{2}$.

■

Lemma B.4.11. *There exists an index $t \in \{1, 2, \cdots, p\}$ and $a \in \mathbb{Z}_n$, such that either for all $j = 1, 2, \ldots, 2\delta$,*
$$X_{i+j} = X_i + jc_t = B_t + a + jc_t, \text{ and } Y_{i+j-1} = S + a + (j-1)c_t$$
or for all $j = 1, 2, \ldots, 2\delta$,
$$X_{i+j} = X_i - jc_t = B_t + a - jc_t, \text{ and } Y_{i+j-1} = S + a - (j-1)c_t.$$

Proof. Let a be the element in $\mathbb{Z}_n$ such that $Y_i = S + a$.

By definition, for any $1 \leq t \leq p$, $S = B_t + \{0, c_t\} = B_t \cup (B_t + c_t)$. By Lemma B.4.10, $|Y_i \cap Y_{i+1}| = \frac{\alpha}{2}$. By Lemma B.4.7, $Y_{i+1} = Y_i \pm c_t$ for some $1 \leq t \leq p$. First we consider the case that $Y_{i+1} = Y_i + c_t$. Since $Y_i = S + a = (B_t + a) \cup (B_t + a + c_t)$, we have
$$X_{i+1} = (Y_i \cap Y_{i+1}) = (S \cap (S + c_t)) + a = B_t + c_t + a.$$
Moreover,
$$X_{i+2} = (Y_{i+1} \setminus Y_i) = ((S + c_t) \setminus S) + a = B_t + 2c_t + a.$$
Assume $3 \leq j \leq 2\delta$, $X_{i+j-1} = B_t + a + (j-1)c_t$ and $Y_{i+j-2} = S + a + (j-2)c_t$.

Assume $Y_{i+j-1} = S + j'$ for some $j' \in \mathbb{Z}_n$. Since
$$Y_{i+j-2} = S + a + (j-2)c_t$$
and
$$Y_{i+j-1} \cap Y_{i+j-2} = X_{i+j-1} = B_t + a + (j-1)c_t,$$

it follows that

$$S \cap (S + (j' - a - (j-2)c_t)) = B_t + c_t.$$

By Lemma B.4.7, $j' - a - (j-2)c_t = c_t$. Thus we conclude that $Y_{i+j-1} = S + a + (j-1)c_t$ and $X_{i+j} = B_t + a + jc_t$, as $X_{i+j} = Y_{i+j-1} \setminus X_{i+j-1}$.

If $Y_{i+1} = Y_i - c_t$, then the same argument shows that for $j = 1, 2, \cdots, 2\delta$, $X_{i+j} = B_t + a - jc_t$ and $Y_{i+j-1} = S + a - (j-1)c_t$.

∎

Now we derive the final contradiction. Let t be the index given in Lemma B.4.11. Without loss of generality, we assume that for $j = 1, 2, \cdots, 2\delta$, $X_{i+j} = B_t + a + jc_t$. If $t < p$, then $2x_{t+1}c_t = c_{t+1}$. Then $X_{i+2x_{t+1}} = X_i + 2x_{t+1}c_t = B_t + a + c_{t+1}$. By definition, $c_{t+1} \in B_t$. Thus $a + c_{t+1} \in B_t + a = X_i$. On the other hand, $0 \in B_t$, and hence $a + c_{t+1} \in X_{i+2x_{t+1}}$. This is a contradiction, as $2x_{t+1} \leq 2\delta < \omega < k$, which implies that $X_i \cap X_{i+2x_{t+1}} = \emptyset$. If $t = p$, then

$$2x_1c_t = (2^p x_1 x_2 \cdots x_p)x_1 = (n-1)x_1 = -x_1.$$

Then $X_{i+2x_1} = X_i + 2x_1c_t = B_t + a - x_1$. By definition, $x_1 \in B_t$. Thus $a \in B_t + a - x_1 = X_{i+2x_1}$. On the other hand, $0 \in B_t$, and hence $a \in B_t + a = X_i$. This is a contradiction, as $2x_1 < k$ and hence $X_i \cap X_{i+2x_1} = \emptyset$. This completes the proof of Theorem B.4.2.

Open question

Theorem 2 gives the circular chromatic number of some circular partitionable graphs such that their stability number is a power of two (these graphs are said to be of type 1 or 2 in [1]).

However, we believe that our result is likely to hold for most of the circular partitionable graphs: e.g., is it true that every graph $C[m_1, m_2, \ldots, m_{2r}]$ with $r \geq 2$ has its circular chromatic number equal to its chromatic number?

References

[1] G. Bacsó, E. Boros, V. Gurvich, F. Maffray, and M. Preissmann. On minimally imperfect graphs with circular symmetry. *J. of Graph Theory*, 29:209–225, 1998.

[2] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[3] R.G. Bland, H.C. Huang, and L.E. Trotter. Graphical properties related to minimal imperfection. *Discrete Mathematics*, 27:11–22, 1979.

[4] E. Boros, V. Gurvich, and S. Hougardy. Recursive generation of partitionable graphs. *J. of Graph Theory*, 41(4):259–285, 2002.

[5] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[6] V. Chvátal, R.L. Graham, A.F. Perold, and S.H. Whitesides. Combinatorial designs related to the perfect graph conjecture. *Applied Discrete Mathematics*, 21:197–206, 1984.

[7] Guogang Gao and Xuding Zhu. Star-extremal graphs and the lexicographic product. *Discrete Math.*, 152(1-3):147–156, 1996.

[8] K.-W. Lih, D. Liu, and X. Zhu. Star extremal circulant graphs. *SIAM J. Discrete Math.*, 12(4):491–499 (electronic), 1999.

[9] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[10] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[11] A. Vince. Star chromatic number. *J. of Graph Theory*, 12:551–559, 1988.

[12] X. Zhu. Circular chromatic number: a survey. *Discrete Mathematics*, 229:371–410, 2001.

[13] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3):186–209, 2005.

B.5 A note on the Chvátal rank of clique family inequalities

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RAIRO - Operations Research 41 (2007) [88]

Clique family inequalities $a\sum_{v\in W} x_v + (a-1)\sum_{v\in W'} x_v \leq a\delta$ form an intriguing class of valid inequalities for the stable set polytopes of all graphs. We prove firstly that their Chvátal-rank is at most a, which provides an alternative proof for the validity of clique family inequalities, involving only standard rounding arguments. Secondly, we strengthen the upper bound further and discuss consequences regarding the Chvátal-rank of subclasses of claw-free graphs.

For any polyhedron P, let P^I denote the convex hull of all integer points in P. Chvátal [2] (and implicitly Gomory [7]) introduced a method to obtain approximations of P^I outgoing from P as follows. If $\sum a_i x_i \leq b$ is valid for P and has integer coefficients only, then $\sum a_i x_i \leq \lfloor b \rfloor$ is a Chvátal-Gomory cut for P. Define P' to be the set of points satisfying all Chvátal-Gomory cuts for P, and let $P^0 = P$ and $P^{t+1} = (P^t)'$ for non-negative integers t. Obviously $P^I \subseteq P^t \subseteq P$ for every t. An inequality $\sum a_i x_i \leq b$ is said to have Chvátal-rank at most t if it is a valid inequality for the polytope P^t. Chvátal showed in [2] that for each polyhedron P there exists a finite $t \geq 0$ with $P^t = P^I$; the smallest such t is the *Chvátal-rank* of P.

The fractional matching polytope is a famous example of a polytope with Chvátal-rank one [2]. In this note, we consider the Chvátal-rank of the fractional stable set polytope $P = \text{QSTAB}(G)$. In particular, P^I is the stable set polytope STAB(G).

The *stable set polytope* STAB(G) of a graph G is defined as the convex hull of the incidence vectors of all its stable sets (in a stable set all nodes are mutually nonadjacent). A canonical relaxation of STAB(G) is the *fractional stable set polytope* QSTAB(G) given by all "trivial" facets, the nonnegativity constraints $x_v \geq 0$ for all nodes v of G, and by the *clique constraints* $\sum_{v\in Q} x_v \leq 1$ for all cliques $Q \subseteq G$ (in a clique all nodes are mutually adjacent). Clearly, STAB$(G) \subseteq$ QSTAB(G) and STAB(G)=QSTAB$(G)^I$ holds for all graphs G. We say that a graph class $\mathcal{G}$ has Chvátal-rank t if t is the minimum value such that QSTAB$(G)^t =$ STAB(G) for all $G \in \mathcal{G}$. We have STAB$(G) =$ QSTAB(G) if and only if G is perfect [3], that is perfect graphs form exactly the class of graphs with Chvátal-rank zero.

To describe the stable set polytopes of imperfect graphs, we consider two natural generalizations of clique constraints: 0/1-constraints associated with arbitrary induced subgraphs, and a/(a-1)-valued constraints associated with families of cliques. *Rank constraints* are 0/1-inequalities

$$\sum_{v\in G'} x_v \leq \alpha(G')$$

associated with induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G'. *Clique family inequalities* $(\mathcal{Q}, p)$

$$a\sum_{v\in V_p} x_v + (a-1)\sum_{v\in V_{p-1}} x_v \leq a\delta \tag{B.13}$$

rely on the intersection of cliques within a family $\mathcal{Q}$, where V_p (resp. V_{p-1}) contains all nodes belonging to at least p (resp. exactly $p-1$) cliques in $\mathcal{Q}$, and $a = p - r$ with $r = |\mathcal{Q}| \bmod p$ and $\delta = \lfloor \frac{|\mathcal{Q}|}{p} \rfloor$ holds.

Both types of inequalities are valid for the stable set polytopes of all graphs: rank constraints by the choice of the right hand side, and clique family inequalities by [11, 9].

It is known from [4] that the Chvátal-rank of rank constraints of a graph with n nodes is $\Omega((n/\log n)^{\frac{1}{2}})$ and from [5] that the split rank of clique family inequalities is one, that is, clique family inequalities are simple split cuts (split cuts were studied in [13]).

The aim of this note is to establish $\min\{r, p-r\}$ as upper bound of the Chvátal-rank for general clique family inequalities. We close with remarks regarding Chvátal-ranks of *quasi-line graphs* (where the neighbors of any node split into two cliques), as their stable set polytopes are completely described by nonnegativity, clique, and clique family inequalities [5].

The Chvátal-rank of clique family inequalities.

The following observation will be crucial for the proofs: summing up the clique inequalities corresponding to the cliques in $\mathcal{Q}$ and possibly adding nonnegativity constraints $-x_v \leq 0$ for those nodes $v \in V_p$ which are contained in more than p cliques, we obtain that

$$p \sum_{v \in V_p} x_v + (p-1) \sum_{v \in V_{p-1}} x_v \quad \leq \quad p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \tag{B.14}$$

is valid for $\mathrm{QSTAB}(G)$.

Theorem B.5.1. *Let $(\mathcal{Q}, p)$ be a clique family inequality and let $r = |\mathcal{Q}| \pmod{p}$. For every $1 \leq i \leq p-r$, the inequality $h(i)$*

$$i \sum_{v \in V_p} x_v + (i-1) \sum_{v \in V_{p-1}} x_v \leq i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

has Chvátal-rank at most i and, thus, $(\mathcal{Q}, p)$ has Chvátal-rank at most $p-r$.

Proof. For every $1 \leq i \leq p-r$, let $H(i)$ be the assertion : "The inequality $h(i)$ has Chvátal-rank at most i." The proof is performed by induction on i:

$H(1)$ is true: Inequality (B.14) implies that $\sum_{v \in V_p} x_v \leq \frac{|\mathcal{Q}|}{p}$ is valid for $\mathrm{QSTAB}(G)$, hence $\sum_{v \in V_p} x_v \leq \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$ has Chvátal-rank 1, as required.

Induction step: assume that $H(i)$ is true and $i < p-r$. To prove that $H(i+1)$ holds, we show that $h(i+1)$ is a Chvátal-Gomory cut from $h(i)$ and Inequality (B.14). Therefore, we have to find a pair of solutions (λ, μ) to the following system of equations:

$$\begin{aligned} \lambda i + \mu p &= i+1 \\ \lambda(i-1) + \mu(p-1) &= i \\ \left\lfloor \lambda i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \right) \right\rfloor &= (i+1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \end{aligned}$$

Indeed, $\lambda = \frac{p-i-1}{p-i}$, $\mu = \frac{1}{p-i}$ are solutions, as $\left\lfloor \lambda i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \mu \left(p \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r \right) \right\rfloor = \left\lfloor (\lambda i + \mu p) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \frac{r}{p-i} \right\rfloor = (i+1) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$, since $0 \leq r/(p-i) < 1$. ■

Note that the proof of Theorem B.5.1 yields an alternative proof for the validity of clique family inequalities for the stable set polytope of any graph, involving only standard rounding arguments.

Furthermore, we obtain that every *rank* clique family inequality has Chvátal-rank one. This is particularly nice, as neither general rank constraints nor general clique family inequalities have this property [4, 9], but the combination of both.

However, the upper bound established in Theorem B.5.1 gets weaker if r gets smaller; we therefore improve the upper bound for $r < p/2$.

Theorem B.5.2. *Every clique family inequality $(\mathcal{Q}, p)$ with $r = |\mathcal{Q}| \pmod p$ has Chvátal-rank at most r if $0 \le r < p - r$.*

Proof. For every $0 \le i \le r$, let $G(i)$ be the assertion : "The inequality $g(i)$: $(p-i)\sum_{v\in V_p} x_v + (p-i-1)\sum_{v\in V_{p-1}} x_v \le (p-i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i$ has Chvátal-rank at most i." The proof is performed by induction on i:

$G(0)$ is true due to Inequality (B.14).

Induction step: assume that $G(i)$ is true and $i < r$. To prove that $G(i+1)$ holds, we show that $g(i+1)$ is a Chvátal-Gomory cut from $g(i)$ and $h(i)$. Therefore, we have to find a pair of solutions (λ, μ) to the following system of equations:

$$\begin{aligned}
\lambda(p-i) + \mu i &= p - i - 1\\
\lambda(p-i-1) + \mu(i-1) &= p - i - 2\\
\left\lfloor \lambda\left[(p-i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i\right] + \mu i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \right\rfloor &= (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1
\end{aligned}$$

Indeed, $\lambda = \frac{p-2i-1}{p-2i}$, $\mu = \frac{1}{p-2i}$ are solutions as $\left\lfloor \lambda\left[(p-i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i\right] + \mu i \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \right\rfloor = \left\lfloor (\lambda(p-i) + \mu i)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + \lambda(r-i) \right\rfloor = \left\lfloor (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1 + \frac{p-i-r}{p-2i} \right\rfloor = (p-i-1)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r - i - 1$ since $0 \le \frac{p-i-r}{p-2i} < 1$. ■

Thus, Theorem B.5.1 and Theorem B.5.2 together imply:

Corollary B.5.3. *Every clique family inequality $(\mathcal{Q}, p)$ has Chvátal-rank at most $\min\{r, p-r\}$ where $r = |\mathcal{Q}| \pmod p$. In particular, a clique family inequality $(\mathcal{Q}, p)$ has Chvátal-rank at most $\frac{p}{2}$.*

Consequences for quasi-line graphs.

We now discuss consequences of the above results for quasi-line graphs, as all non-trivial, non-clique facets of their stable set polytopes are clique family inequalities according to [5].

Calling a graph G *rank-perfect* if STAB(G) has rank constraints as only non-trivial facets, Theorem B.5.1 implies that rank-perfect subclasses of quasi-line graphs have Chvátal-rank 1. This verifies Edmond's conjecture that the Chvátal-rank of claw-free graphs is one for the class of semi-line graphs, as they are rank-perfect [1].

A *semi-line graph* is a line graph or a quasi-line graph without a representation as fuzzy circular interval graph. A *line graph* $L(G)$ is obtained by turning adjacent edges of a root graph G into adjacent nodes of $L(G)$. Fuzzy circular interval graphs are defined as follows. Let $\mathcal{C}$ be a circle, $\mathcal{I}$ a collection of intervals in $\mathcal{C}$ without proper containments and common endpoints, and V a finite multiset of points in $\mathcal{C}$. The *fuzzy circular interval graph* $G(V, \mathcal{I})$ has node set V and two nodes are adjacent if both belong to one interval $I \in \mathcal{I}$, where edges between different endpoints of the same interval may be omitted.

As the only not rank-perfect quasi-line graphs are fuzzy circular interval, it suffices to restrict to this class in order to discuss the Chvátal-rank for quasi-line graphs. Giles and Trotter [6] exhibited a fuzzy circular interval graph with a clique family $\mathcal{Q}$ of size 37 such that $(\mathcal{Q}, 8)$ induces a facet. Oriolo noticed in [9] that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at least* 2. This example disproves Edmonds' conjecture for fuzzy circular interval graphs. On the other hand, Theorem B.5.1 shows that this clique family inequality $(\mathcal{Q}, 8)$ has Chvátal-rank *at most 3*, since $r = 5$ and so $p - r = 3$.

Furthermore, Giles and Trotter [6] introduced a sequence of fuzzy circular interval graphs G^k for $k \ge 1$ and showed that each of them admits a clique family facet $(\mathcal{Q}, k+2)$ with $|\mathcal{Q}| = 2k(k+2)+1$ and coefficients k and $k+1$; Theorem B.5.2 ensures that these facets have Chvátal-rank 1 since $r = 1$ holds in all cases.

Webs W_n^k are special fuzzy circular interval graphs with nodes $0, \dots, n-1$ and edges ij iff $\min\{|i-j|, n-|i-j|\} < k$. Liebling et al. [8] exhibited a sequence of webs $W_{(2a+3)^2}^{2(a+2)}$ for $a \ge 1$, each with a $(a+1)/a$-valued clique family facet $(\mathcal{Q}, a+2)$ with $|\mathcal{Q}| = (a+2)(2a+3)$. Since $(a+2)(2a+3) = 1 \pmod{a+2}$, Theorem B.5.2 shows that also these facets have Chvátal-rank 1.

The authors conjectured in [10] and Stauffer proved in [12] that all non-rank facets of webs W_n^k are clique family inequalities $(\mathcal{Q}, k'+1)$ associated with subwebs $W_{n'}^{k'} \subset W_n^k$ where the maximum cliques $\{i, \ldots, i+k\}$ of W_n^k starting in nodes i of the subweb $W_{n'}^{k'}$ yield the clique family $\mathcal{Q}$ of size n' where $(k'+1) \not| \, n'$ and $k' < k$. Thus, for any *fixed* k, the Chvátal-rank of all webs W_n^k is at most $\frac{k-1}{2}$. However, it is very likely that there exist sequences of webs inducing clique family facets $(\mathcal{Q}, p)$ with arbitrarily high p and $2p = |\mathcal{Q}|$ having Chvátal-rank $\frac{p}{2}$. Thus, also the Chvátal-rank of webs and, therefore, of quasi-line graphs could be arbitrarily large, as for general claw-free graphs [4].

References

[1] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[2] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.

[3] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[4] V. Chvátal, W. Cook, and M. Hartmann. On cutting-plane proofs in combinatorial optimization. *Linear Algebra and its Applications*, 114/115:455–499, 1989.

[5] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. Circular one matrices and the stable set polytope of quasi-line graph. *Lecture Notes in Computer Science*, 3509:291–305, 2005.

[6] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[7] R.E. Gomory. Outline of an algorithm for integer solutions to linear programs. *Bul. of the American Math. Soc.*, 64:275–278, 1958.

[8] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[9] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[10] A. Pêcher and A. Wagler. A conjecture on the stable set polytope for claw-free graphs. Preprint 11-2006, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik.

[11] A. Ben Rebea. *Etude des stables dans les graphes quasi-adjoints*. PhD thesis, Grenoble, 1981.

[12] G. Stauffer. *On the Stable Set Polytope of Claw-free Graphs*. PhD thesis, EPF Lausanne, 2005.

[13] R. Kannan W. Cook and A. Schrijver. Chvátal closures for mixed integer programming problems. *Math. Programming*, 47:155–174, 1990.

B.6 On classes of minimal circular-imperfect graphs

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Circular-perfect graphs form a natural superclass of perfect graphs: on the one hand due to their definition by means of a more general coloring concept, on the other hand as an important class of χ-bound graphs with the smallest non-trivial χ-binding function $\chi(G) \leq \omega(G) + 1$.

The Strong Perfect Graph Conjecture, recently settled by Chudnovsky et al. [4], provides a characterization of perfect graphs by means of forbidden subgraphs. It is, therefore, natural to ask for an analogous conjecture for circular-perfect graphs, that is for a characterization of all minimal circular-imperfect graphs.

At present, not many minimal circular-imperfect graphs are known. This paper studies the circular-(im)perfection of some families of graphs: normalized circular cliques, partitionable graphs, planar graphs, and complete joins. We thereby exhibit classes of minimal circular-imperfect graphs, namely, certain partitionable webs, a subclass of planar graphs, and odd wheels and odd antiwheels. As those classes appear to be very different from a structural point of view, we infer that formulating an appropriate conjecture for circular-perfect graphs, as analogue to the Strong Perfect Graph Theorem, seems to be difficult.

Introduction

Coloring the vertices of a graph is an important concept with a large variety of applications. Let $G = (V, E)$ be a graph with finite vertex set V and simple edge set E. A *k-coloring* of G is a mapping $f : V \to \{1, \dots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices of G receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* of G and denoted by $\chi(G)$. Calculating $\chi(G)$ is an NP-hard problem in general. In a set of k pairwise adjacent vertices, called *clique K_k*, all k vertices have to be colored differently. Thus the size of a largest clique in G, the *clique number $\omega(G)$*, is a trivial lower bound on $\chi(G)$. This bound can be arbitrarily bad [11] and is hard to evaluate as well.

Berge [1] proposed to call a graph G *perfect* if each induced subgraph $G' \subseteq G$ admits an $\omega(G')$-coloring. Perfect graphs turned out to be an interesting and important class of graphs with a rich structure, see [14] for a recent survey. In particular, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [6].

Recently, the famous Strong Perfect Graph Conjecture of Berge [1] on characterizing perfect graphs by means of forbidden subgraphs has been settled by Chudnovsky, Robertson, Seymour, and Thomas [4]: Berge [1] observed that chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, and their complements $\overline{C_{2k+1}}$, the *odd antiholes*, are imperfect as clique and chromatic number differ. (The *complement* $\overline{G}$ of a graph G has the same vertex set as G and two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.) Berge's famous conjecture was that odd holes and odd antiholes are the only minimal forbidden subgraphs in perfect graphs, i.e., the only *minimally imperfect graphs*. Considerable effort has been spent over the years to verify or falsify this conjecture revealing deep structural properties of minimally imperfect graphs [14]. Finally, Chudnovsky, Robertson, Seymour, and Thomas [4] succeeded in turning the conjecture into the Strong Perfect Graph Theorem and exhibited many structural properties of perfect graphs, that were not known before.

As generalization of perfect graphs, X. Zhu [21] introduced recently the class of circular-perfect graphs based on the following more general coloring concept.

Define a *(k, d)-coloring* of a graph G, as a mapping $f : V \to \{0, \dots, k-1\}$ such that for each edge xy of G, $d \leq |f(x) - f(y)| \leq k - d$. The *circular chromatic number* is:

$$\chi_c(G) = \inf \left\{ \frac{k}{d} : G \text{ has a } (k, d) - \text{coloring} \right\}$$

From the definition, we immediately obtain $\chi_c(G) \leq \chi(G)$ because a usual k-coloring of G is a $(k, 1)$-coloring. (Note that $\chi_c(G)$ is sometimes called the star chromatic number in the literature, see [3, 16, 20].)

In order to obtain a lower bound on $\chi_c(G)$, we generalize cliques as follows: Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k vertices $0, \dots, k-1$ and edges ij if and only if $d \leq |i - j| \leq k - d$. Such graphs $K_{k/d}$ are called *circular cliques* (note that they are also known as antiwebs in the literature, see [15, 17]). A circular clique $K_{k/d}$ with $\gcd(k, d) = 1$ is said to be *prime*. Prime circular cliques include all cliques $K_k = K_{k/1}$ as well as all odd antiholes $\overline{C_{2k+1}} = K_{2k+1/2}$ and all odd holes $C_{2k+1} = K_{2k+1/k}$, see Figure B.6. The *circular clique number* is

$$\omega_c(G) = \max\left\{\frac{k}{d} : K_{k/d} \subseteq G,\ \gcd(k,d) = 1\right\}$$

and we immediately obtain $\omega(G) \leq \omega_c(G)$.

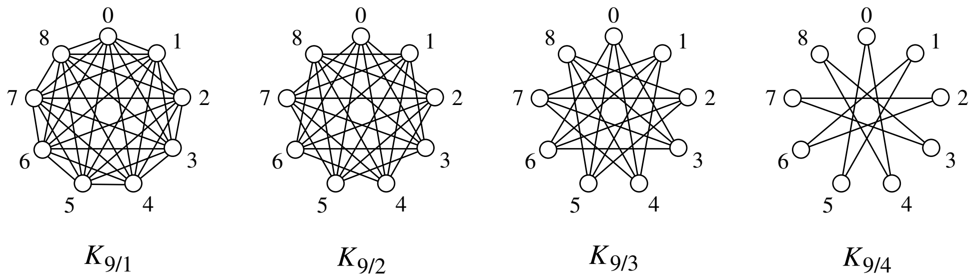

Figure B.17: The circular cliques on nine vertices.

Remark B.6.1. Colorings can also be interpreted as homomorphisms from a graph to a clique.

Let h be a homomorphism from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ where $h : V_1 \to V_2$ such that $h(u)h(v) \in E_2$ if $uv \in E_1$; we write $G_1 \preceq G_2$. Any k-coloring of a graph G is equivalent to a homomorphism from G to K_k. Then the circular chromatic number can be written as $\chi_c(G) = \inf\left\{\frac{k}{d} : G \preceq K_{k/d}\right\}$ and the circular clique number as $\omega_c(G) = \sup\left\{\frac{k}{d} : K_{k/d} \preceq G,\ \gcd(k,d) = 1\right\}$ [21].

Every circular clique $K_{k/d}$ clearly admits a (k,d)-coloring (simply take the vertex numbers as colors, as in Figure B.6) but no (k',d')-coloring with $\frac{k'}{d'} < \frac{k}{d}$ by [3]. Thus we obtain, for any graph G, the following chain of inequalities:

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{B.15}$$

A graph G is called *circular-perfect* if, for each induced subgraph $G' \subseteq G$, the circular clique number $\omega_c(G')$ and the circular chromatic number $\chi_c(G')$ coincide. Obviously, every perfect graph has this property by (1) as $\omega(G')$ equals $\chi(G')$. Moreover, it was proved in [21] that any circular clique is circular-perfect as well. Thus circular-perfect graphs constitute a proper superclass of perfect graphs. In contrary to perfect graphs, the class of circular-perfect graphs is not stable under complementation.

Another natural extension of perfect graphs was introduced by Gyárfás [7] as χ-bound graphs: A family $\mathcal{G}$ of graphs is called χ-bound with χ-binding function b if $\chi(G') \leq b(\omega(G'))$ holds for all induced subgraphs G' of $G \in \mathcal{G}$. Thus this concept uses functions in $\omega(G)$ as *upper* bound on $\chi(G)$. Since it is known for any graph G that $\omega(G) = \lfloor \omega_c(G) \rfloor$ by [21] and $\chi(G) = \lceil \chi_c(G) \rceil$ by [16], we obtain that circular-perfect graphs G satisfy the following Vizing-like property

$$\omega(G) \leq \chi(G) \leq \omega(G) + 1. \tag{B.16}$$

Thus, circular-perfect graphs are a class of χ-bound graphs with the smallest non-trivial χ-binding function. In particular, this χ-binding function is best possible for a *proper* superclass of perfect graphs implying that circular-perfect graphs admit coloring properties almost as nice as perfect graphs.

The aim of this paper is to look for other parallels between the classes of perfect and circular-perfect graphs. As analogue to the Strong Perfect Graph Theorem, one might be tempted to ask for an appealing conjecture on minimal forbidden subgraphs in circular-perfect graphs. We call a graph G *minimal circular-imperfect* if G is not circular-perfect but every proper induced subgraph is. The hope is to identify all classes of minimal circular-imperfect graphs in order to characterize circular-perfect graphs by means of forbidden subgraphs.

The main contribution of this paper is to characterize all minimal circular-imperfect graphs in the classes of normalized circular cliques, partitionable graphs, and complete joins, as well as to exhibit a class of minimal circular-imperfect planar graphs. However, at first sight there is no straightforward common structure in these graphs, hence formulating an analogue to the Strong Perfect Graph Theorem for circular-perfect graphs seems to be difficult.

Results

Normalized circular cliques and partitionable graphs

Given a graph G, an edge e of G is called *indifferent* if e is not contained in any maximum clique of G. The *normalized subgraph* $\mathrm{norm}(G)$ of G is obtained from G by deleting all indifferent edges.

A graph G is called (p, q)*-partitionable* if $|V(G)| = pq + 1$ and, for each vertex v of G, the subgraph $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p. A graph is *partitionable* if it is (p, q)-partitionable for some $p, q \geq 2$.

The complement of a circular clique (or antiweb) $K_{n/q}$ is a *web* C^q_n, and any circular clique $K_{n/q}$ (and its complement) with $n = \omega q + 1$ is a partitionable graph.

We characterize all circular cliques whose normalized subgraph is circular-imperfect, and show which of them are minimal with respect to this property.

Theorem B.6.2. *Let $K_{p/q}$ be any prime circular clique. Then $\mathrm{norm}(K_{p/q})$ is*

1. *circular-imperfect if and only if $p \not\equiv -1 \pmod{q}$ and $\lfloor p/q \rfloor \geq 3$;*
2. *minimal circular-imperfect if and only if $p = 3q + 1$ and $q \geq 3$;*
3. *isomorphic to $\overline{K}_{p/3}$ if $p = 3q + 1$ and $q \geq 3$.*

The above results imply:

Corollary B.6.3. *The partitionable webs C^3_{3q+1} are minimal circular-imperfect for all $q \geq 3$.*

Originally, Lovász [10] and Padberg [12] introduced partitionable graphs as a tool to study properties of minimal imperfect graphs, as every minimal imperfect graph is in particular partitionable. Since circular-perfect graphs include all perfect graphs and all minimal imperfect graphs, one might expect that some subclasses of partitionable graphs are circular-perfect. To support this feeling further, every partitionable graph G satisfies the Vizing-like proprty $\chi(G) \leq \omega(G) + 1$, as every circular-perfect graph. This motivates to study circular-(im)perfection of partitionable graphs.

The above corollary shows, however, that the circular cliques whose normalized subgraphs are minimal circular-imperfect, are partitionable graphs with clique number 3. Therefore, we cannot expect anymore the circular-perfection of all partitionable graphs. Even worse, Theorem B.6.4 below states that most partitionable graphs are in fact circular-imperfect:

Theorem B.6.4. *All partitionable graphs apart from circular cliques are circular-imperfect.*

This implies further:

Corollary B.6.5. *All normalized partitionable graphs apart from odd holes and odd antiholes are circular-imperfect.*

Planarity and Circular-perfection

Computer checks for small minimal circular-imperfect graphs showed that there exist planar ones (e.g. the 5-wheel); this suggests to check circular-(im)perfection of planar graphs.

Our first result introduces an interesting class of circular-perfect graphs: planar graphs where all vertices lie on the outer face, i.e., *outerplanar graphs*.

Theorem B.6.6. *Outerplanar graphs are circular-perfect.*

As a by-product of Theorem B.6.6, the circular chromatic number of an outerplanar graph is equal to 2 if all cycles have even size, or $2 + \frac{1}{d}$ where $2d + 1$ is the size of the smallest odd cycle. This gives a different proof of a recent result by Kemnitz and Wellmann [9].

Outgoing from the circular-perfection of outerplanar graphs, it is easy to introduce a simple class of minimal circular-imperfect planar graphs: for every positive integers k and l such that $(k, l) \neq (1, 1)$, let $T_{k,l}$ denote the planar graph with $2l + 1$ inner faces $F_1, F_2, \ldots F_{2l+1}$ of size $2k + 1$ arranged in a circular fashion around a central vertex, where all other vertices lie on the outer face, as depicted in Figure B.6. We show circular-imperfection, minimality follows from Theorem B.6.6 as the removal of any vertex yields an outerplanar graph.

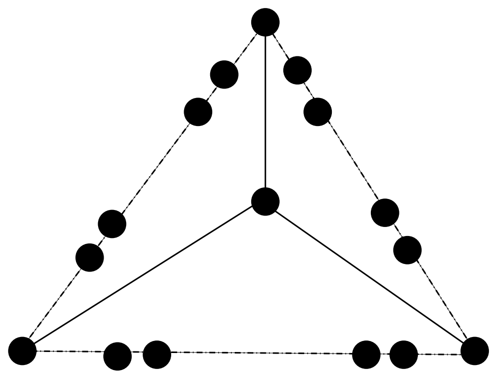

Figure B.18: Example of a graph $T_{k,1}$

Complete joins and circular-imperfection

At last, our third class of minimal circular-imperfect graphs involves *odd wheels* (complete joins of odd holes and *one* vertex) and *odd antiwheels* (complete joins of odd antiholes and *one* vertex); a complete join of two graphs G_1 and G_2 is the union of G_1 and G_2, and all edges between G_1 and G_2. We completely characterized complete joins w.r.t. circular-(im)perfection as follows:

Theorem B.6.7. *The complete join $G * G'$ of two graphs G and G' is*

(i) *circular-perfect if and only if both G and G' are perfect;*

(ii) *minimal circular-imperfect if and only if G is an odd hole or odd antihole and G' is a single vertex (or vice versa), that is if and only if $G * G'$ is an* odd wheel *or an* odd antiwheel.

Notice that odd wheels are the same as graphs $T_{1,l}$, that is a class of planar minimal circular-imperfect graphs. Odd antiwheels are examples of minimal circular-imperfect graphs with arbitrarily large clique and chromatic number.

Corollary B.6.8. *The complete join of more than two graphs is never minimal circular-imperfect.*

Normalized circular cliques and partitionable graphs

Proof of Theorem B.6.2

We shall prove that the normalized subgraph $\mathrm{norm}\left(K_{p/q}\right)$ of a prime circular clique $K_{p/q}$ is

- circular-imperfect iff $p \neq -1 \pmod{q}$ and $\lfloor p/q \rfloor \geq 3$ (assertion (i));

- minimal w.r.t. this property iff $p = 3q + 1$ for all $q \geq 3$ (assertion (ii));
- equal to $\overline{K}_{p/3}$ if $p = 3q + 1$ and $q \geq 3$ (assertion (iii)).

Given an integer p and a subset of integers S of $[0, p-1]$, the *circulant graph* $C(p, S)$ is the graph with vertex set $\{0, \ldots, p-1\}$ and edge set $\{ij|\ i - j \in S\}$ with arithmetics performed modulo p.

We first state the following observation which relates the normalized subgraph of a partitionable circular-clique to its complement.

Lemma B.6.9. *If* $p = \omega q + 1$*, then* $\mathrm{norm}(K_{p/q})$ *is isomorphic to the complement* $\overline{K}_{p/\omega} = C_p^{\omega}$ *of* $K_{p/\omega}$.

Proof. Both $\mathrm{norm}(K_{p/q})$ and $\overline{K}_{p/\omega}$ are circulant graphs on the vertex set $V = \{0, 1, \ldots, p-1\}$. The former has generating set

$$S = \{q, q+1, 2q, 2q+1, \ldots, (\omega-1)q, (\omega-1)q+1\}$$

and the latter has generating set

$$S' = \{1, 2, \ldots, \omega-1, p-1, p-2, \ldots, p-\omega+1\}.$$

It is easy to verify that $f : V \to V$ defined as $f(i) = iq \pmod{p}$ has the property $f(S') = S$. Hence f is an isomorphism from $\overline{K}_{p/\omega}$ to $\mathrm{norm}(K_{p/q})$.

■

We shall now proceed to the proof of Theorem B.6.2.

Proof. In the following, we denote by G the circular clique $K_{p/q}$ and let H denote the normalized subgraph $\mathrm{norm}\left(K_{p/q}\right)$ of G.

A *proper variant* of G is a subgraph H' of G obtained by removing a non-empty set of indifferent edges (i.e., any graph H' with $H \subseteq H' \subsetneq G$).

Let $p = \omega q + r$, where $0 \leq r \leq q-1$.

Claim B.6.10. *The normalized subgraph* H *of* G *is the circulant graph* $C[p, S]$*, where* $S = \{q, q+1, \cdots, q+r, 2q, 2q+1, \cdots, 2q+r, \cdots, (\omega-1)q, (\omega-1)q+1, \cdots, (\omega-1)q+r\}$.

Consider an edge $0t$. We have $t = kq + r'$, with $1 \leq k \leq \omega - 1$ and $0 \leq r' \leq q-1$.

If $0 \leq r' \leq r$, then the set $\{0, q+r', 2q+r', \cdots, (\omega-1)q+r'\}$ induces a maximum clique containing the edge $0t$, and so the edge $0t$ is not indifferent.

Conversely, if $r+1 \leq r' \leq q-1$, then let K be a clique containing $0, t$. The other vertices of K belong to the intervals $[q, (k-1)q+r']$ and $[(k+1)q+r', (\omega-1)q+r]$. Therefore, K has at most $\omega-1$ vertices, namely, at most $k-1$ vertices in the interval $[q, (k-1)q+r']$ and at most $\omega-k-2$ vertices from the interval $[(k+1)q+r', (\omega-1)q+r]$. Thus K is not a maximum clique and so $0t$ is an indifferent edge. $\diamond$

In particular, due to Lemma B.6.9 if $p = 3q+1$ then $\mathrm{norm}(K_{p/q})$ is isomorphic to $\overline{K}_{p/3}$, which proves assertion (iii).

Claim B.6.11. *Suppose* I *is a maximal stable set of* H *and* $i, i+t \in I$ *for some* $t \leq r+1$*. Then* $i+j \in I$ *for all* $0 \leq j \leq t$.

If x is adjacent to $i+j$ in H for some $0 \leq j \leq t$, then x is adjacent to either i or $i+t$ in H. $\diamond$

Claim B.6.12. *Suppose* I *is a stable set of* H*. There is a vertex* i *of* H *such that* $i+j \notin I$ *for any* $1 \leq j \leq r$.

Otherwise, Claim B.6.11 would imply that all vertices of H belong to a maximal stable set I' containing I, an obvious contradiction. $\diamond$

Claim B.6.13. *If* I *is a stable set of* H*, then* $|I| \leq q$.

As H is a circulant graph, by Claim B.6.12, we may assume without loss of generality that $S \cap I = \emptyset$, where $S = \{\omega q, \omega q+1, \cdots, \omega q+r-1\}$.

But $V(H) - S$ can be decomposed into the disjoint union of q cliques of H, namely, $K_i = \{i, i+q, i+2q, \cdots, i+(\omega-1)q\}$, for $i = 0, 1, \cdots, q-1$. As $|I \cap K_i| \leq 1$ for each $i \in \{0, 1, \cdots, q-1\}$, so $|I| \leq q$. $\diamond$

Claim B.6.14. *We have $\chi_c(H) = \chi_c\left(K_{p/q}\right) = p/q$.*

Since $\chi_c(K_{p/q}) = p/q$, we have $\chi_c(H) \leq p/q$. On the other hand, $\chi_c(H) \geq \chi_f(H) = |V(H)|/\alpha(H) \geq p/q$ due to Claim B.6.13 (where χ_f denotes the fractional chromatic number, a lower bound of the circular chromatic number [20]). So equality holds everywhere. $\diamond$

Therefore the removal of indifferent edges of a circular clique does not alter its circular chromatic number, but clearly its circular clique number. This implies that normalization destroys circular-perfection:

Claim B.6.15. *If $p \neq -1 \pmod q$ and $\lfloor p/q \rfloor \geq 3$ then $K_{p/q}$ is not normalized and every of its proper variants is circular-imperfect.*

We denote by $\Delta(G)$ the maximum degree of a graph G. We have $\Delta(K_{p/q}) = p-(2q-1)$ and $\Delta(H) = (r+1)(\omega-1)$, where $p = \omega q + r$ and r is the remainder modulo q, by Claim B.6.10. Therefore, if $K_{p/q}$ is normalized (i.e., if $K_{p/q} = \mathrm{norm}\left(\mathrm{K_{p/q}}\right)$) then $p-(2q-1) = (r+1)(\omega-1)$, that is $(\omega-2)q = (r+1)(\omega-2)$. Since $\omega = \lfloor p/q \rfloor \geq 3$, this implies that $r = q-1$, and so $p = -1 \pmod q$, a contradiction.

Hence $K_{p/q}$ is not normalized and the result follows from Claim B.6.14: if H' is any proper variant of $K_{p/q}$ then

$$\omega_c\left(H'\right) < p/q = \chi_c\left(H\right) = \chi_c(H').$$

$\diamond$

This completes the proof of the "if part" of Theorem B.6.2 (i). We now treat the "only if part" of assertion (i).

Claim B.6.16. *If $\lfloor p/q \rfloor < 3$ or $p = -1 \pmod q$ then $\mathrm{norm}\left(K_{p/q}\right)$ is circular-perfect.*

Notice that $\omega = \lfloor p/q \rfloor$ is the clique number of $K_{p/q}$. Therefore, if $\omega < 3$ then $\mathrm{norm}\left(K_{p/q}\right) = K_{p/q}$. Thus $\mathrm{norm}\left(K_{p/q}\right)$ is circular-perfect.

If $p = -1 \pmod q$ then $\mathrm{norm}\left(K_{p/q}\right) = K_{p/q}$ follows due to the description of $\mathrm{norm}\left(K_{p/q}\right)$ for general p and q in Claim B.6.10. Thus $\mathrm{norm}\left(K_{p/q}\right)$ is circular-perfect. $\diamond$

This completes the proof of Theorem B.6.2 (i). We now treat the "only if part" of assertion (ii).

Claim B.6.17. *If $p \neq 1, -1 \pmod q$ and $\omega = \lfloor p/q \rfloor \geq 3$ then $K_{p/q}$ has a circular clique $K_{(\omega q'+1)/q'}$ as an induced subgraph with at least one indifferent edge of $K_{p/q}$, and $q' \geq 3$.*

Let G denote the circular clique $K_{p/q}$ and let $2 \leq r \leq q-2$ such that $p = q\omega + r$. Notice that $q \neq 2r$ as p and q are relatively prime.

Case 1. If $r < \frac{q}{2}$ then let $q' = \lceil \frac{q}{r} \rceil$. We have $q' \geq 3$. For every $0 \leq i < \omega$, let $X_i = \{iq, iq+r, \ldots, iq+(q'-1)r\}$ and define $X = \left(\bigcup_{0 \leq i < \omega} X_i\right) \cup \{\omega q\}$. We first show that X induces a circular clique $K_{(\omega q'+1)/q'} \subseteq G$.

For every $0 \leq x < p$, we denote by S_x the maximum stable set $\{x, x+1, \ldots, x+q-1\}$ of G (arithmetics performed modulo p). Due to Trotter [8], it is enough to check that for every $x \in X$, S_x meets X in exactly q' vertices.

Let $x \in X$: by the definition of X, there exist $0 \leq i \leq \omega$ and $0 \leq \delta < q'$ such that $x = iq + \delta r$.

- If $i < \omega - 1$ then notice that $S_x \subseteq S_{iq} \cup S_{(i+1)q}$. Hence

$$\begin{aligned} S_x \cap X &= (S_{iq} \cap S_x \cap X) \cup (S_{(i+1)q} \cap S_x \cap X) \\ &= \{iq + \lambda r | \delta \leq \lambda < q'\} \cup \{(i+1)q + \lambda r | 0 \leq \lambda < \delta\} \end{aligned}$$

as for every $0 \leq \lambda < q'$, we have $(i+1)q + \lambda r \in S_x$ if and only if

$$0 \leq (i+1)q + \lambda r - x = q + (\lambda - \delta)r < q \text{ holds.}$$

Therefore S_x meets X in exactly q' vertices.

- If $i = \omega - 1$ and $\delta = 0$ then

$$S_x \cap X = S_{iq} \cap X = \{iq + \lambda r | 0 \leq \lambda < q'\}$$

holds and, again S_x meets X in exactly q' vertices.

- If $i = \omega - 1$ and $\delta > 0$ then $x = (\omega-1)q + \delta r$. We have $S_x = \{(\omega-1)q+\delta r, (\omega-1)q+\delta r+1, \ldots, (\omega-1)q + \delta r + q - 1\}$ (with arithmetics performed modulo p). Hence S_x is the disjoint union $S'_x \cup S"_x$ where $S'_x = \{(\omega-1)q+\delta r, (\omega-1)q+\delta r+1, \ldots, \omega q + r - 1\}$ and $S"_x = \{0, 1, \ldots, (\delta-1)r - 1\}$ ($S"_x = \emptyset$ if $\delta = 1$).

  We have

$$\begin{aligned} X \cap S_x &= (X_{\omega-1} \cup X_0 \cup \{\omega q\}) \cap S_x \\ &= (X_{\omega-1} \cap S'_x) \cup (X_0 \cap S"_x) \cup \{\omega q\} \end{aligned}$$

  and thus, $X \cap S$ is of size q' as

  – $X_{\omega-1} \cap S'_x = \{(\omega-1)q + \lambda r | \delta \leq \lambda < q'\}$ is of size $q' - \delta$;
  – $X_0 \cap S"_x = \{\lambda r | 0 \leq \lambda < \delta - 1\}$ is of size $\delta - 1$.

  Therefore S_x meets X in exactly q' vertices.

- If $i = \omega$ and $\delta = 0$ then $x = \omega q$. We have

$$\begin{aligned} S_x \cap X &= (\{\omega q, \omega q + 1, \ldots, \omega q + r - 1\} \cap X) \\ &\quad \cup (\{0, 1, \ldots, q - r - 1\} \cap X) \\ &= \{\omega q\} \cup \{\lambda r | 0 \leq \lambda r \leq q - r - 1 \text{ and } 0 \leq \lambda < q'\} \\ &= \{\omega q\} \cup \{\lambda r | 0 \leq \lambda \leq \lfloor q/r \rfloor - 1 = q' - 2 \text{ and } 0 \leq \lambda < q'\} \\ &= \{\omega q\} \cup \{\lambda r | 0 \leq \lambda \leq q' - 2\} \end{aligned}$$

  which also implies that S_x meets X in exactly q' vertices.

Hence S_x always meets X in exactly q' vertices and so X induces a circular clique $G' = K_{(\omega q'+1)/q'}$ of G according to [8]. As $\omega \geq 3$ and $0 < r < q/2$, we have $q + r < q + 2r < 2q$. Since $q' \geq 3$, the vertex $q + 2r$ belongs to G'. Hence the edge $\{0, q + 2r\}$ of G' is an indifferent edge of G by Claim B.6.10.

Case 2. If $r > \frac{q}{2}$ then we show that $K_{(3\omega+1)/3}$ is an induced subgraph of G.

For $j = 0, 1, \ldots, 3\omega$, let $x_j = \lfloor pj/(3\omega+1) \rfloor$. Let $X = \{x_0, x_1, \ldots, x_{3\omega}\}$.

We show that X induces a circular clique $K_{(3\omega+1)/3}$ of G: this is equivalent to show that for every $0 \leq i, j \leq 3\omega$, $\{x_i, x_j\}$ is an edge of G if and only if $3 \leq |i - j| \leq 3\omega - 2$.

To prove this, we shall use the following simple observation several times: if a and b are reals and δ is an integer such that $a - b \geq \delta$ then $\lfloor a \rfloor - \lfloor b \rfloor \geq \delta$.

- Let $0 \leq i, j \leq 3\omega$ such that $\{x_i, x_j\}$ is an edge of G and assume w.l.o.g. that $i < j$. We have $x_i < x_j$ and $q \leq x_j - x_i \leq p - q$.

  If $j - i \leq 2$, then $pj/(3\omega+1) - pi/(3\omega+1) \leq 2(q\omega+r)/(3\omega+1)$ follows. If $2(q\omega+r)/(3\omega+1) > q-1$ then as $\omega \geq 3$ and $q \geq r+2$, a short computation gives $r < 1$ a contradiction. Thus $2(q\omega+r)/(3\omega+1) \leq q-1$ and so $x_j - x_i \leq q - 1$, a contradiction. Hence $j - i \geq 3$.

  If $j - i \geq 3\omega - 1$, then $pj/(3\omega+1) - pi/(3\omega+1) \geq (3\omega-1)(q\omega+r)/(3\omega+1) \geq p - q + 1$ follows. Thus $x_j - x_i \geq p - q + 1$, a contradiction.

  Therefore, we infer $3 \leq j - i \leq 3\omega - 2$.

- Conversely, let $0 \le i, j \le 3\omega$ such that $3 \le j-i \le 3\omega-2$ and assume w.l.o.g. that $i < j$. We have $x_i < x_j$ and we need to check that $\{x_i, x_j\}$ is an edge of G.

  On the one hand, $j-i \ge 3$ and $3r \ge q$ imply

  $$pj/(3\omega+1) - pi/(3\omega+1) \ge 3(q\omega+r)/(3\omega+1) \ge q$$

  and, hence, $x_j - x_i \ge q$ follows.

  On the other hand, $j-i \le 3\omega-2$ yields

  $$pj/(3\omega+1) - pi/(3\omega+1) \le (3\omega-2)(q\omega+r)/(3\omega+1) \le p-q$$

  and shows $x_j - x_i \le p-q$.

  Therefore $\{x_i, x_j\}$ is an edge of G, as required; and X induces a circular clique $G' = K_{(3\omega+1)/3}$ of G.

At last, we need to exhibit an indifferent edge of G in G'.

By Claim B.6.10, the neighbours of 0 in norm(G) are the vertices in $S = \{q, q+1, \cdots, q+r, 2q, 2q+1, \cdots, 2q+r, \cdots, (\omega-1)q, (\omega-1)q+1, \cdots, (\omega-1)q+r\}$.

We have $2q - 5p/(3\omega+1) = (\omega q + 2q - 5r)/(3\omega+1) > 0$ as $\omega \ge 3$ and $r \le q-2$. Hence $x_5 < 2q$.

If $x_5 \ge q+r+1$ then $x_5 \notin S$ and $\{x_0, x_5\}$ is an edge of G' which is also an indifferent edge of G.

It remains to check the case $x_5 \le q+r$: identifying an edge of G' which is also an indifferent edge of G is more difficult to handle. We are going to exhibit one in an induced circular clique G'' sharing all vertices but one with G'.

For $t = 1, 2, \ldots, \omega-2$, let $\delta_t = x_{3t+2} - (tq+r+1)$. As $x_5 \le q+r$, we have $\delta_1 < 0$.

We first check that $\delta_{\omega-2} \ge 0$: we have $\frac{p(3\omega-4)}{3\omega+1} - (\omega-2)q - r - 1 = 2q - 1 - \frac{5p}{3\omega+1}$. If $5p/(3\omega+1) > 2q-1$ then $5q - 10 > \omega q - 3\omega + 2q - 1$ (as $r \le q-2$) which is equivalent to $0 > (q-3)(\omega-3)$. This is a contradiction as both q and ω are at least 3. Hence $\frac{p(3\omega-4)}{3\omega+1} - (\omega-2)q - r - 1 \ge 0$ and therefore $\delta_{\omega-2} \ge 0$.

Let t^* be the largest index such that $\delta_{t^*} < 0$: we have $1 \le t^* < \omega - 2$. Let $x'_{3t^*+2} = t^*q + r + 1$ and let $X' = (X - \{x_{3t^*+2}\}) \cup \{x'_{3t^*+2}\}$. Let G'' be the induced subgraph of G by X'. To prove that G'' is an induced circular clique $K_{(3\omega+1)/3}$ of G, we have to check that the neighborhood of x'_{3t^*+2} in G'' is the same than the one of x_{3t^*+2} in G', namely $\{x_0, x_1, \ldots, x_{3t^*-1}\} \cup \{x_{3t^*+5}, x_{3t^*+6}, \ldots, x_{3\omega}\}$.

If $\frac{(3t^*+5)p}{3\omega+1} - (t^*q+r+1) < q$ then we have $\frac{(3(t^*+1)+2)p}{3\omega+1} - ((t^*+1)q+r+1) < 0$. Thus we infer $\delta_{t^*+1} < 0$, in contradiction with the maximality of t^*. Hence $x_{3t^*+2} \le x'_{3t^*+2} \le x_{3t^*+5} - q$, and so x'_{3t^*+2} is adjacent to $\{x_0, x_1, \ldots, x_{3t^*-1}\} \cup \{x_{3t^*+5}, x_{3t^*+6}, \ldots, x_{3\omega}\}$ and x'_{3t^*+2} is not adjacent to x_{3t^*+3} and x_{3t^*+4}.

We have $t^*q + r + 1 - \frac{p3t^*}{3\omega+1} = r + 1 + \frac{t^*(q-3r)}{3\omega+1} < q$ as $r \le q-2$ and $r > q/3$. Hence x'_{3t^*+2} is not adjacent to x_{3t^*} and x_{3t^*+1}.

Therefore G'' induces a circular clique $K_{(3\omega+1)/3}$ of G. As $t^*q + r < x'_{3t^*+2} = t^*q + r + 1 < (t^*+1)q$ the edge $\{x_0, x'_{3t^*+2}\}$ of G'' is an indifferent edge of $K_{p/q}$. This finishes the second case.

Thus in both cases $K_{p/q}$ contains an induced circular clique $K_{(\omega q'+1)/q'}$ with $q' \ge 3$ and an indifferent edge of $K_{p/q}$. $\diamond$

Claim B.6.18. *If $H = \mathrm{norm}\left(\mathrm{K_{p/q}}\right)$ is minimal circular-imperfect then H is a partitionable web $C^{\omega}_{\omega q+1}$, and $q \ge 3$.*

Since H is circular-imperfect we have $p \neq -1 \pmod q$ and $\omega \ge 3$ due to Claim B.6.16.

If H is not partitionable then $p \neq 1 \pmod q$. By the previous claim, $K_{p/q}$ has an induced subgraph $K_{(\omega q'+1)/q'}$ with $q' \ge 3$ and vertex set W, containing an indifferent edge. As all non-indifferent edges of $K_{(\omega q'+1)/q'}$ are non-indifferent edges of $K_{p/q}$ (since these two graphs have same maximum clique size), the subgraph $H[W]$ of G, which is induced by W, is a proper variant of $K_{(\omega q'+1)/q'}$, and is, therefore, circular-imperfect by Claim B.6.15. Hence $K_{p/q} = K_{(\omega q'+1)/q'}$, and $q = q' \ge 3$.

This implies that H is partitionable.

It follows that $q \geq 3$ (as $q = 2$ implies that H is an odd antihole and, therefore, circular-perfect, a contradiction). Due to Claim B.6.9, this shows that H is a partitionable web $C^{\omega}_{\omega q+1}$ with $q \geq 3$. $\diamond$

Claim B.6.19. *A claw-free graph does not contain any circular cliques different from cliques, odd holes, and odd antiholes.*

Assume $K_{p/q}$ is a circular clique different from a clique, an odd hole, and an odd antihole. Then $q \geq 3$ and $p \geq 2q+2$. Thus $\{1, q+1, q+2, q+3\}$ induces a claw. $\diamond$

Claim B.6.20. *If H = norm$\big(K_{p/q}\big)$ is a minimal circular-imperfect graph, then H has clique number 3.*

We first recall the following result of Trotter [8]: let $C^{\omega'}_{n'}$ ($2k' \leq n'$) and C^{ω}_{n} ($2k \leq n$) be two webs, then $C^{\omega'}_{n'}$ is an induced subgraph of C^{ω}_{n} if and only if holds

$$\frac{\omega'-1}{\omega-1}n \quad \leq n' \leq \quad \frac{\omega'}{\omega}n \tag{B.17}$$

By Claim B.6.18, $H = \text{norm}\left(K_{p/q}\right)$ is a partitionable web $C^{\omega}_{\omega q+1}$, with $q \geq 3$. If $\omega \leq 2$ then H is a stable set or an odd hole and is therefore circular-perfect, a contradiction. Hence $\omega \geq 3$.

Assume that $\omega \geq 4$.

Due to Trotter's inequality (B.17), the web C^{3}_{3q-1} is an induced subweb of H if and only if holds

$$\frac{2}{\omega-1}(q\omega+1) \leq 3q-1 \leq \frac{3}{\omega}(q\omega+1)$$

Since the right inequality is always satisfied, this may be restated as $\frac{2}{\omega-1}(q\omega+1) \leq 3q-1$ which is equivalent to $1 + 4/(\omega-3) \leq q$.

If $q \geq 5$ (resp. $\omega \geq 5$) then $q \geq 1 + 4/(\omega-3)$ as $4/(\omega-3) \leq 4$ (resp. $q \geq 3$ and $4/(\omega-3) \leq 2$). Hence C^{3}_{3q-1} is a proper induced subweb of H. If C^{k}_{2k+1} is any induced odd antihole of C^{3}_{3q-1} then $k < 3$ due to Trotter's inequality (B.17). Hence the previous claim implies that $\omega_c(C^{3}_{3q-1}) = 3$. If C^{3}_{3q-1} is 3-colorable, then it admits a partition in 3 stables sets of size at most $q-1 = \lfloor (3q-1)/3 \rfloor$, a contradiction. Hence $\chi(C^{3}_{3q-1}) \geq 4$ and so $\chi_c(C^{3}_{3q-1}) > 3 = \omega_c(C^{3}_{3q-1})$. Thus C^{3}_{3q-1} is a proper induced circular-imperfect graph of H, a contradiction.

Therefore, $\omega = 4$ and ($q = 3$ or $q = 4$), that is $H = C^{4}_{13}$ or $H = C^{4}_{17}$:

- C^{4}_{13} is not minimal circular-imperfect as the subgraph induced by vertices $\{1, 2, 4, 5, 7, 9, 10, 12\}$ is circular-imperfect, since it has circular-clique number 3 and is not 3-colorable;
- C^{4}_{17} is not minimal circular-imperfect as the subgraph induced by vertices $\{1, 2, 3, 5, 6, 8, 9, 11, 13, 14, 16\}$ is circular-imperfect, since it has circular-clique number 3 and is not 3-colorable.

In both cases, we get a contradiction and infer, therefore, $\omega = 3$. $\diamond$

This completes the proof of the "only if part" of assertion (ii). We now proceed to the proof of the "if part".

Claim B.6.21. *Webs C^{3}_{3q+1} with $q \geq 3$ are minimal circular-imperfect.*

Let $q \geq 3$. The web C^{3}_{3q+1} is circular-imperfect by Claim B.6.15.

If C^{3}_{3q+1} is not minimal circular-imperfect, then there exists a proper induced subgraph W, which is minimal circular-imperfect. Let v be a vertex of C^{3}_{3q+1} not in W.

If $\omega(W) = 3$ then $\omega(W) = 3 \leq \omega_c(W) \leq \chi_c(W) \leq \chi(C^{3}_{3q+1} \setminus \{v\}) = 3$, a contradiction with the fact that W is minimal circular-imperfect.

If $\omega(W) = 2$ then let w be any vertex of W. If w is of degree at least 3 then w belongs to a triangle of W, as the neighborhood of any vertex of C^{3}_{3q+1} can be covered with only 2 cliques (i.e. C^{3}_{3q+1} is a quasi-line graph), a contradiction. Therefore, the degree of W is at most 2 and so W is a disjoint union of cycles and paths, and thus is circular-perfect, a contradiction.

Hence C^3_{3q+1} is minimal circular-imperfect. $\diamond$

This finally proves Theorem B.6.2. $\blacksquare$

Proof of Theorem B.6.4

Proof. Let G be a partitionable graph. We shall prove that G is circular-imperfect unless G is a circular clique. If $\omega_c(G) = \omega(G)$, then we have $\chi_c(G) > \omega(G) = \omega_c(G)$ by $\chi(G) = \omega(G) + 1$, therefore G is circular-imperfect.

Assume that $\omega_c(G) = p/q > \omega$ and let $\{0, \dots, p-1\}$ be the vertices of an induced circular clique $K_{p/q}$ (where the vertices are labeled the usual way). For every $0 \le i < \omega$, let Q_i be the maximum clique $\{jq | 0 \le j \le i\} \cup \{jq+1 | i < j < \omega\}$. Obviously $Q_0, \dots, Q_{\omega-1}$ are ω distinct maximum cliques of G containing the vertex 0.

If $p > \omega q + 1$ then the set $(Q_0 \setminus \{(\omega-1)q+1\}) \cup \{(\omega-1)q+2\}$ is another maximum clique containing 0, a contradiction as 0 belongs to exactly ω maximum cliques of G [2]. Hence $p = \omega q + 1$. This means that G contains the partitionable circular clique $K_{(\omega q+1)/q}$ as an induced subgraph. Hence G is the circular clique $K_{(\omega q+1)/q}$. $\blacksquare$

Proof of Corollary B.6.5

Proof. Let G be a circular-perfect normalized partitionable graph. We conclude that G is an odd hole or odd antihole. By Theorem B.6.4, G is a circular clique $K_{p/q}$. If $\omega(G) \ge 3$, since $p = 1 \pmod q$ (as G is partitionable) and G is circular-perfect, it follows from Theorem B.6.2 (i) that $p = -1 \pmod q$, and so $q = 2$. Hence G is an odd antihole. If $\omega(G) = 2$ then G is an odd hole.

$\blacksquare$

Some minimal circular-imperfect planar graphs

Proof of Theorem B.6.6

Proof. In order to show the circular-perfection of outerplanar graphs, we first discuss the circular clique number of planar graphs.

Claim B.6.22. *The circular clique number of a planar graph G is equal to*

- *1, if G is a stable set,*
- *2, if G is bipartite,*
- *4, if G has an induced K_4,*
- *else $2 + \frac{1}{d}$ where $2d+1$ is the odd girth of G, i.e. $2d+1$ is the size of a shortest chordless odd cycle in G.*

This claim follows from the easy to prove fact that the only planar circular cliques are odd holes and cliques of size at most 4. $\diamond$

It is well known that the identification of two disjoint perfect graphs G_1 and G_2 in a clique yields a perfect graph G again [5] (if $Q_1 \subseteq G_1 = (V_1, E_1)$ and $Q_2 \subseteq G_2 = (V_2, E_2)$ are two cliques of same size and ϕ is any bijection from Q_2 onto Q_1, the identification of G_1 and G_2 in Q_1 w.r.t. ϕ is the graph $G = (V, E)$ where $V = (V_1 \cup V_2) \setminus Q_2$ and $E = E_1 \cup (E_2 \setminus \{ij | \{i,j\} \cap Q_2 \neq \emptyset\}) \cup \{\phi(i)j | ij \in E_2, |i \in Q_2, j \notin Q_2\}$.

We prove that the same holds for circular-perfect planar graphs.

Claim B.6.23. *If G_1 and G_2 are two planar circular-perfect graphs, then identifying G_1 and G_2 in a clique K yields a circular-perfect graph G.*

If G_1 and G_2 are both bipartite then G is perfect and therefore circular-perfect. Hence we may assume that G_1 is not bipartite. In particular, $\omega(G) > 1$.

All we have to prove is that $\omega_c(G) = \chi_c(G)$.

If $\omega(G) = 4$ then $\omega_c(G) = \chi_c(G) = 4$ as $\omega(G) = 4 \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) = 4$. Hence we may assume that $\omega(G) \leq 3$.

If $\omega(G) = 3$ then G is 3-colorable as both G_1 and G_2 are 3-colorable. Hence $\omega_c(G) = \chi_c(G) = 3$ as $3 = \omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) = 3$.

It remains to handle the case $\omega(G) = 2$. Then the clique K is of size at most 2.

If G_2 is bipartite then it is homomorphic to an edge, and so G is homomorphic to G_1. Hence $\chi_c(G) \leq \chi_c(G_1)$ and so $\omega_c(G_1) \leq \omega_c(G) \leq \chi_c(G) \leq \chi_c(G_1) = \omega_c(G_1)$.

If G_2 is not bipartite then let $2d_1 + 1$ be the odd girth of G_1 and let $2d_2 + 1$ be the odd girth G_2. W.l.o.g. assume that $2d_1 + 1 \leq 2d_2 + 1$. There exists an homomorphism f_1 (resp. f_2) from G_1 (resp. G_2) into C_{2d_1+1}.

If K is of size 2 (resp. of size 1) then let q_1 and q_2 be the vertices of K (resp. let q be the vertex of K). Let σ be an automorphism of C_{2d_1+1} such that $f_1(q_1) = \sigma(f_2(q_1))$ and $f_1(q_2) = \sigma(f_2(q_2))$ (there is one as $\{f_1(q_1), f_1(q_2)\}$ and $\{f_2(q_1), f_2(q_2)\}$ are two edges of C_{2d_1+1}) (resp. such that $f_1(q) = \sigma(f_2(q))$. Then the application f which maps a vertex x of G onto $f_1(x)$ if $x \in G_1$, $\sigma(f_2(x))$ if $x \in G_2$ is a homomorphism from G into C_{2d_1+1}. Therefore, we have $\omega_c(G) = 2 + \frac{1}{d_1} \leq \chi_c(G) \leq 2 + \frac{1}{d_1}$. ◇

A connected outerplanar graph different from a cycle is always obtained by identifying two strictly smaller outerplanar graphs in one vertex or one edge. Therefore, the previous claim and the fact that cycles are circular-perfect imply circular-perfection of outerplanar graphs. ■

It remains to show that the graphs $T_{k,l}$ are minimal circular-imperfect.

Lemma B.6.24. *For every positive integers k and l such that $(k, l) \neq (1, 1)$, the graph $T_{k,l}$ is minimal circular-imperfect.*

Proof. If the graph $T_{k,l}$ has a $(2k+1, k)$-coloring then assume without loss of generality that the central vertex gets the color 0. Every neighbour of the central vertex is colored with k or $k + 1$, and two such neighbours belonging to a common inner face must have distinct colors (a $(2k + 1, k)$-coloring can be seen as a homomorphism h to the odd hole C_{2k+1}, see Remark B.6.1; the restriction of h to an odd hole of size $2k + 1$, e.g. any inner face of $T_{k,l}$, is bijective). Since the central vertex has an odd number of neighbours on the outer face, we get a contradiction. Hence graphs $T_{k,l}$ have $\omega_c(T_{k,l}) = 2 + 1/k$ (as $(k, l) \neq (1, 1)$) which is strictly less than $\chi_c(T_{k,l})$ and so are circular-imperfect.

Minimal circular-imperfection follows then from Theorem B.6.6 as the removal of any vertex yields an outerplanar graph. ■

Complete joins and minimal circular-imperfection

Proof of Theorem B.6.7

Proof. Our goal is to show that a complete join $G * G'$ is circular-perfect iff both G and G' are perfect and minimal circular-imperfect iff $G * G'$ is an odd wheel or odd antiwheel.

Claim B.6.25. *An odd wheel $C_{2k+1} * v$ is minimal circular-imperfect if $k \geq 2$.*

This follows from the fact that the odd wheels $C_{2k+1} * v$ are precisely the graphs $T_{1,k}$. ◇

Claim B.6.26. *An odd antiwheel $\overline{C}_{2k+1} * v$ is minimal circular-imperfect if $k \geq 2$.*

Since $\overline{C}_{2k+1}$ is an odd antihole for $k \geq 2$, we have $\omega(\overline{C}_{2k+1} * v) = k+1$ and $\chi(\overline{C}_{2k+1} * v) = k+2$. Moreover, $\omega_c(\overline{C}_{2k+1} * v) = \max\{k+1, k+\frac{1}{2}\} = k+1$ and $\chi_c(\overline{C}_{2k+1} * v) > \chi(\overline{C}_{2k+1} * v) - 1 = k+1$. Thus $\omega_c(\overline{C}_{2k+1} * v) < \chi_c(\overline{C}_{2k+1} * v)$ implies that $\overline{C}_{2k+1} * v$ is circular-imperfect. Minimality follows since removing any vertex yields a perfect graph or $\overline{C}_{2k+1}$, hence all proper induced subgraphs of $\overline{C}_{2k+1} * v$ are circular-perfect. $\diamond$

This implies the following for the complete joins of an imperfect graph with a single vertex:

Claim B.6.27. *If G is an imperfect graph, then $G * v$ is circular-imperfect and minimal if and only if G is an odd hole or odd antihole.*

Due to the Strong Perfect Graph Theorem, G contains an odd hole or odd antihole C as induced subgraph. Thus $G * v$ has $C * v$ as induced subgraph which is circular-imperfect by Claim B.6.25 or Claim B.6.26. $G * v$ is, therefore, circular-imperfect as well and minimal if and only if $C * v = G * v$ (i.e. $C = G$). $\diamond$

This proves assertion (ii), provided assertion (i) holds true.

Claim B.6.28. *If both graphs G and G' are imperfect, then $G * G'$ is circular-imperfect but never minimal.*

Let v' be a vertex of G'. Then $G * v'$ is a proper induced subgraph of $G * G'$ and circular-imperfect by Claim B.6.27. Thus $G * G'$ is circular-imperfect but never minimal. $\diamond$

Consider the complete join $G * G'$ of two graphs G and G'. If both graphs G and G' are perfect, then $G * G'$ is perfect as well. If one of G and G' is imperfect, then $G * G'$ is circular-imperfect by Claim B.6.27. This proves assertion (i). ■

Concluding remarks and further work

We shortly summarize the results obtained in this paper:

- Theorem B.6.2 studies the circular-imperfection of normalized circular cliques; we conclude that the webs C^3_{3q+1} with $q \geq 3$ are the only minimal circular-imperfect graphs in this class (Theorem B.6.2 and Corollary B.6.3).
- Theorem B.6.4 shows that no partitionable graphs different from circular cliques are circular-perfect.
- In Theorem B.6.6, we prove that outerplanar graphs are circular-perfect and use them to build our second class of minimal circular-imperfect graphs, the planar graphs $T_{k,l}$ with $(k,l) \neq (1,1)$.
- At last, in Theorem B.6.7, we study circular-imperfection of complete joins and prove that the minimal circular-imperfect complete joins are precisely odd wheels and odd antiwheels.

The last two families were independently found by B. Xu [18, 19]; since these results are easy consequences of our considerations on planar graphs and complete joins, we have included our (short) proofs in this paper.

At first sight there is no straightforward common structure in the presented families of minimal circular-imperfect graphs, hence formulating an analogue to the Strong Perfect Graph Theorem for circular-perfect graphs seems to be difficult.

The Strong Perfect Graph Conjecture is equivalent to "every minimal imperfect graph or its complement has clique number 2". As every known minimal circular-imperfect graph or its complement has clique number 2 or 3, one might be tempted to ask whether it holds for every minimal circular-imperfect graph. However, Pan and Zhu [13] found recently a way to construct minimal circular-imperfect graphs with arbitrarily large clique and stability number.

This adds further support to the believe that characterizing circular-perfect graphs by means of forbidden subgraphs is, indeed, a difficult task.

References

[1] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[2] R.G. Bland, H.C. Huang, and L.E. Trotter. Graphical properties related to minimal imperfection. *Discrete Mathematics*, 27:11–22, 1979.

[3] J.A. Bondy and P. Hell. A note on the star chromatic number. *J. of Graph Theory*, 14:479–482, 1990.

[4] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[5] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[6] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[7] A. Gyárfás. Problems for the world surrounding perfect graphs. *Zastos. Mat.*, XIX:413–441, 1987.

[8] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12:373–388, 1975.

[9] A. Kemnitz and P. Wellmann. Circular chromatic number of certain planar graphs and small graphs. *Congressus Numerantium*, 169:199–209, 2004.

[10] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[11] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[12] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[13] Z. Pan and X. Zhu. Minimal circular-imperfect graphs of large clique number and large independence number. manuscript, 2006.

[14] J. Ramirez-Alfonsin and B. Reed, editors. *Perfect Graphs*. Wiley, 2001.

[15] F.B. Shepherd. Applying Lehman's theorem to packing problems. *Mathematical Programming*, 71:353–367, 1995.

[16] A. Vince. Star chromatic number. *J. of Graph Theory*, 12:551–559, 1988.

[17] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2:149–152, 2004.

[18] B. Xu. On minimally circular-imperfect graphs. manuscript.

[19] B. Xu. Some results on circular perfect graphs and perfect graphs. *J. Syst. Sci. Complex.*, 18:167–173, 2005.

[20] X. Zhu. Circular chromatic number: a survey. *Discrete Mathematics*, 229:371–410, 2001.

[21] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3):186–209, 2005.

B.7 Partitionable graphs arising from near-factorizations of finite groups

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Discrete Mathematics 269 (2003) [79]

In 1979, two constructions for making partitionable graphs were introduced in [7].

The graphs produced by the second construction are called CGPWgraphs. A near-factorization (A, B) of a finite group is roughly speaking a non-trivial factorization of G minus one element into two subsets A and B. Every CGPWgraph with n vertices turns out to be a Cayley graph of the cyclic group $\mathbb{Z}_n$, with connection set $(A - A) \setminus \{0\}$, for a near-factorization (A, B) of $\mathbb{Z}_n$. Since a counter-example to the Strong Perfect Graph Conjecture would be a partitionable graph [13], any 'new' construction for making partitionable graphs is of interest.

In this paper, we investigate the near-factorizations of finite groups in general, and their associated Cayley graphs which are all partitionable. In particular we show that near-factorizations of the dihedral groups produce every CGPWgraph of even order. We present some results about near-factorizations of finite groups which imply that a finite abelian group with a near-factorization (A, B) such that $|A| \leq 4$ must be cyclic (already proved in [5]). One of these results may be used to speed up exhaustive calculations. At last, we prove that there is no counter-example to the Strong Perfect Graph Conjecture arising from near-factorizations of a finite abelian group of even order.

Introduction

In 1960, Claude Berge introduced the notion of perfect graphs : a graph is *perfect* if for every induced subgraph *H* of it, the chromatic number of *H* does not exceed the maximum number of pairwise adjacent vertices in *H*. A *hole* is a chordless cycle with at least four vertices. Berge conjectured that perfect graphs are exactly the graphs with no induced odd holes and no induced complement of an odd hole, or equivalently that *minimal imperfect graphs* are odd holes and their complements. This conjecture is often called the Strong Perfect Graph Conjecture and has motivated many works.

Lovász [11] and Padberg [13] gave some properties of minimal imperfect graphs. Following the paper of Bland, Huang and Trotter [3], a graph G is said to be *partitionable* if there exist two integers p and q such that G has $pq + 1$ vertices and for every vertex *v* of *G*, the induced subgraph $G \setminus \{v\}$ admits a partition in p cliques of cardinality q and also admits a partition in q stable sets of cardinality p. Let ω denote the maximum cardinality of a clique of G and α denote the maximum cardinality of a stable set of G. Then it is clear that $p = \alpha$ and $q = \omega$.

With this definition, Lovász [11] and Padberg [13] proved that every minimal imperfect graph is partitionable. Thus a counter-example to the Strong Perfect Graph Conjecture would lie in the class of partitionable graphs. Hence an approach to Berge's conjecture is to prove that a given class of partitionable graphs does not contain any minimal imperfect graph which is not an odd odd hole or anti-hole.

In 1979, Chvátal, Graham, Perold and Whitesides introduced two constructions for making partitionable graphs [7]. In 1996, Sebö proved that there is no counter-example to the Strong Perfect Graph Conjecture in the first one [14]. In 1984, Grinstead proved that there is no counter-example to the Strong Perfect Graph Conjecture in the second one [10]. A *variant* of a partitionable graph is a partitionable graph with the same vertices, the same maximum cliques and the same maximum stable sets. In 1998, Bacsó, Boros, Gurvich, Maffray and Preissmann [1] extended Grinstead's result to the wider class of the variants of the second construction.

A graph with n vertices is *circular* if there exists a cyclic numbering of its vertices (modulo n) such that, for every vertex x, for every maximum clique C and for every maximum stable set S, the set $\{(c + x) \pmod n \mid c \in C\}$ is a maximum clique and the set $\{(s + x) \pmod n \mid s \in S\}$ is a maximum stable set.

A *normalized* graph is a graph such that for every edge $\{i, j\}$, there exists a maximum clique containing both i and j.

A partitionable graph produced by the second construction due to Chvátal, Graham, Perold and Whitesides is called a *CGPW* graph, where CGPWgraph is the abbreviation of Chvátal-Graham-Perold-Whitesides graph. Any CGPWgraph appears to be a circular normalized partitionable graph. The converse is not established but Bacsó, Boros, Gurvich, Maffray and Preissmann conjectured that it holds:

Conjecture B.7.1. *[1] Every circular normalized partitionable graph is a* CGPW*graph.*

We call it *the circular partitionable graph conjecture*.

In 1984, Grinstead claimed, through a computer check, that this conjecture is true for graphs with a number of vertices at most fifty, or sixty-one [10]. In 1998, Bacsó, Boros, Gurvich, Maffray and Preissmann proved it for graphs with size of maximum cliques et most 5 [1].

Let G be a finite group of order n with operation $*$. Two subsets A and B of G of cardinality at least 2 are said to form a *near-factorization* of G if and only if $n = |A| \times |B| + 1$ and there is an element $u(A, B)$ of G such that $A * B = G \setminus \{u(A, B)\}$. Let S be a symmetric subset of G which does not contain the identity element e. The *Cayley graph* with *connection set* S is the graph with vertex set G and edge set $\{\{i, j\},\ i^{-1} * j \in S\}$. We denote by $\mathrm{Cay}(G, S)$ this graph. Notice that the definitions of a Cayley graph given in the literature may differ. The one we use in this paper is very close from the definition given in the book 'Algebraic Graph Theory' of Norman Biggs [2]. Since S is a symmetric set such that $e \notin S$, the graph $\mathrm{Cay}(G, S)$ is a simple graph without loops, as are all graphs in this paper.

Let Γ be any circular normalized partitionable graph with n vertices. Let C be a maximum clique of Γ and let S be a maximum stable set of Γ. Then it is easy to see that (C, S) is a near-factorization of the group $\mathbb{Z}_n$ and that Γ is the Cayley graph of the finite group $\mathbb{Z}_n$ with connection set $(C - C) \setminus \{0\}$. The converse is true: if (A, B) is a near-factorization of $\mathbb{Z}_n$ then the Cayley graph with connection set $(A - A) \setminus \{0\}$ is a circular normalized partitionable graph [1].

Due to this equivalence, the second construction of Chvátal, Graham, Perold and Whitesides had been first described by N.G. De Bruijn in 1956 [8], though in a different context.

If (A, B) is a near-factorization of a finite group then the Cayley graph with connection set $(A^{-1} * A) \setminus \{e\}$ is a normalized partitionable graph (Section 2). This observation has motivated this paper: the main aim is to produce near-factorizations of some finite groups, so as giving rise to 'new' partitionable graphs. We give 'new' near-factorizations for the dihedral groups but the associated Cayley graphs turn out all to be CGPWgraphs (Section 3). These near-factorizations produce all CGPWgraphs of even order. In Section 2, we give several results about near-factorizations for finite groups in general, which may be used to speed up exhaustive searches by computer. We give tools to explain why many groups do not have any near-factorization at all. We also prove that no Cayley graph associated to a near-factorization of an abelian group of even order is a counter-example to the Strong Perfect Graph Conjecture.

Near-factorizations of finite groups and partitionable graphs

A *group* is a non-empty set G with a closed associative binary operation $*$, an identity element e, and an inverse a^{-1} for every element $a \in G$. If G has a finite number of elements, then the cardinality of G is denoted by $|G|$ and is called the *order* of G. To avoid a conflict of notation, we use the symbol $\times$ to denote the standard multiplication between two integers. An *abelian* group is a group G such that $*$ is commutative, that is $g * g' = g' * g$ for all elements g and g' of G.

If X and Y are two subsets of G, we denote by $X * Y$ the set $\{x * y,\ x \in X,\ y \in Y\}$. With a slight abuse of notation, if g is an element of G and X is subset of G, we denote by gX the set $\{g\} * X$ and Xg the set $X * \{g\}$. Furthermore $|X|$ is the cardinality of X, that is the number of elements of X. The subset X is said to be *symmetric* if $X = X^{-1}$, where X^{-1} is the set $\{x^{-1},\ x \in X\}$.

Recall that two subsets A and B of cardinality at least 2 of a finite group G of order n form a *near-factorization* of G if and only if $n = |A| \times |B| + 1$ and there is an element $u(A, B)$ of G such that $A * B = G \setminus \{u(A, B)\}$: $u(A, B)$ is called the *uncovered element* of the near-factorization. Sometimes, we shall write simply u instead of $u(A, B)$. The condition about the cardinality of A and B is required to avoid the trivial case $A = G \setminus \{u\}$ and

$B = \{e\}$. Notice that every element x of G distinct from u may be written in a unique way as $x = a * b$ with $a \in A$ and $b \in B$. Hence a near-factorization (A, B) may be seen as a tiling of $G \setminus \{u(A, B)\}$ with proto tile A.

The *cyclic group* of order n is the group which is generated by an element x of order n. This group is denoted by $\mathbb{Z}_n$. For convenience, we use the following representation of $\mathbb{Z}_n$: the elements of $\mathbb{Z}_n$ are the integers between 0 and $n-1$ and the operation $*$ is defined by $x * y = (x + y) \pmod{n}$. Due to this definition of the operation of $\mathbb{Z}_n$, we denote this operation by $+$ rather than $*$.

Example B.7.2. *Let $\mathbb{Z}_{13}$ be the cyclic group of order 13,*

Let $A = \{0, 1, 2\}$ and $B = \{0, 3, 6, 9\}$.

Then $A + 0 = \{0, 1, 2\}$, $A + 3 = \{3, 4, 5\}$, $A + 6 = \{6, 7, 8\}$ and $A + 9 = \{9, 10, 11\}$. Thus $A + B = (Z_{13} \setminus \{12\})$, that is (A, B) is a near-factorization of $\mathbb{Z}_{13}$.

The following figure shows the tiling of $\mathbb{Z}_{13} \setminus \{12\}$ given by (A, B).

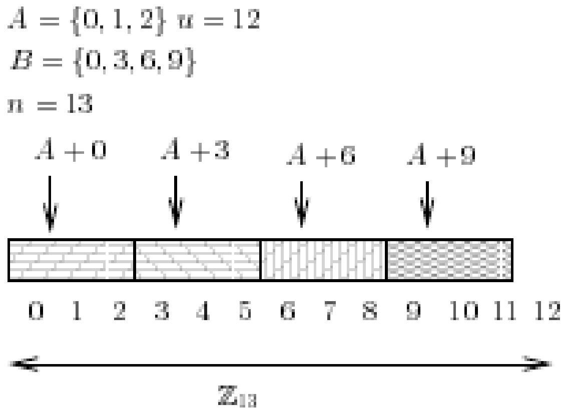


Figure B.19: Example of a near-factorization of $\mathbb{Z}_{13}$

Note that if A and B are seen as sets of integers and $+$ denotes the usual addition between integers, then $A + B$ is a tiling of the segment $[0, 11]$. This connection is somewhat detailed in page 145.

The *dihedral group* $\mathbb{D}_{2n}$ of even order $2 * n$ (with $n \geq 3$) is the non-abelian group generated by two elements r and s such that:

- r is of order n.
- s is of order 2.
- $s * r = r^{-1} * s$

The problem of characterizing the near-factorizations of the dihedral groups is addressed in Section 3.

Let $g_1, \ldots, g_n$ be the elements of the group G with $g_1 = e$. If R is any subset of G, we denote by $M(R)$ the square $n \times n$ $(0, 1)$-matrix defined by $M(R)_{i,j} = 1$ if and only if $g_j \in g_i R$.

Let I be the $n \times n$ identity matrix and J be the $n \times n$ matrix with all entries equal to 1. Then De Caen, Gregory, Hughes and Kreher [5] observed that (A, B) is a near-factorization of G with uncovered element e if and only if $M(A)M(B) = J - I$.

Since $M(A)M(B) = J - I$ implies that $M(B)M(A) = J - I$, we have the following property:

Lemma B.7.3. *[5] Let G be a finite group and A, B be two subsets of G. Then (A, B) is a near-factorization of G with $u(A, B) = e$ if and only if (B, A) is a near-factorization of G with $u(B, A) = e$.*

The hypothesis $u(A,B) = e$ is actually necessary: consider the dihedral group $\mathbb{D}_{16}$ of order 16. Let $A = \{e, r^5, sr^5\}$ and $B = \{e, s, r, sr, sr^7\}$. A small calculation shows that $A * B = D_{16} \setminus \{r^7\}$. Thus (A,B) is a near-factorization of $\mathbb{D}_{16}$, though (B,A) is not one as $sr^5 = e * sr^5 = s * r^5$.

The graph $G(A,B)$ associated with a near-factorization (A,B) is the Cayley graph with connection set $(A^{-1} * A) \setminus \{e\}$.

If Γ is a graph, we denote by $\omega(\Gamma)$ the maximum cardinality of a clique of Γ and $\alpha(\Gamma)$ the maximum cardinality of a stable set of Γ. We denote by $V(\Gamma)$ the vertex set of Γ and $E(\Gamma)$ the edge set of Γ.

The graph Γ with vertex set V is *isomorphic* to the graph Γ' with vertex set V' if there exists a bijective map f from V onto V' such that $\{i, j\}$ is an edge of Γ if and only if $\{f(i), f(j)\}$ is an edge of Γ'.

If e' is an edge of Γ we denote by $\Gamma - e'$ the subgraph of Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \setminus \{e'\}$. Likewise, if e' is a non-edge of Γ, we denote by $\Gamma + e'$ the graph with vertex set $V(\Gamma)$ and with edge set $E(\Gamma) \cup \{e'\}$. If v is any vertex of Γ, we denote by $\Gamma \setminus \{v\}$ the induced subgraph of Γ with vertex set $V(\Gamma) \setminus \{v\}$ and edge set $\{\{x,y\} \mid \{x,y\} \in V(\Gamma),\ x \neq v,\ y \neq v\}$.

A *perfect matching* in a graph with $2n$ vertices is a set of n node-disjoint edges.

Obviously, distinct near-factorizations of a given group may give rise to the same graph. In particular, we may left-shift A and right-shift B without altering the associated graph:

Lemma B.7.4. *Let x and y be two elements of G. Then (xA, By) is a near-factorization of G such that $u(xA, By) = x * u(A,B) * y$ and $G(xA, By)$ is isomorphic to $G(A,B)$.*

Proof. The proof is straightforward. ■

We say that (xA, By) is *shift-isomorphic* to (A,B).

Thus due to Lemma B.7.4, we may always assume that the uncovered element is e, without altering the associated graph.

In the case of abelian groups, De Caen, Gregory, Hughes and Kreher gave a useful property of near-factorizations:

Lemma B.7.5. *[5] Let G be an abelian group and (A,B) be a near-factorization of G. Then there exist two elements x and y of G such that xA is symmetric and that By is symmetric.*

An *automorphism* of G is a bijective map h of G onto itself such that $h(x * y) = h(x) * h(y)$ for all x and y of G. An *inner-automorphism* h of G is an automorphism of G such that there exists an element g of G which satisfies $h(x) = g * x * g^{-1}$ for all x of G.

Then we have this obvious Lemma:

Lemma B.7.6. *Let Cay(G,S) be a Cayley graph with connection set S of a group G. Let h be any automorphism of G. Then the Cayley graph Cay(G,h(S)) is isomorphic to Cay(G,S).*

If y is any element of G, we denote by $\langle y \rangle$ the cyclic subgroup of G generated by y. The *order* of y is the smallest integer k such that $y^k = e$ and is denoted by $o(y)$. An *involution* of G is an element of G of order 2. The *center* of G is the set of all elements in G which commute with every element of G.

Let H be any subgroup of G and (A,B) be a near-factorization of G with uncovered element u.

A *right coset* of H is any subset Hx with $x \in G$. A *left coset* of H is any subset xH with $x \in G$. The proof of Lagrange's Theorem asserts that for any subgroup H of G, there exists a unique partition of G in right cosets of H. Likewise there exists a unique partition in left cosets of H. A subgroup H of G is *normal* if for every g of G, we have $gH = Hg$.

A *right-tile* of A is the trace of A onto a right-coset of H, that is the subset T is a right-tile of A if and only if there exists g in G such that $T = A \cap Hg$. A *left-tile* of A is the trace of A onto a left-coset of H.

The unique partition of G in right cosets of H induces a unique partition of A in right-tiles: let $\{Hg_1, \ldots, Hg_d\}$ be the partition of G in right-cosets, then the set of right-tiles of A is $\{A \cap Hg_1, \ldots, A \cap Hg_d\}$. If T is a right-tile of A which is equal to a whole right-coset, then T is called a *H-right-coset*.

Let τ be the partition of A in right-tiles induced by a given subgroup H. Clearly $\{Tb,\ T \in \tau,\ b \in B\}$ is a partition of $G \setminus \{u\}$. Hence, given the subgroup H, a near-factorization (A, B) may be seen as a tiling of $G \setminus \{u\}$ with the right-tiles of A as tiles. Let K be any such tile and b be any element of B. Notice that Kb lies entirely in a right-coset of H. Thus this tiling of $G \setminus \{u\}$ induces a tiling for every right-coset of H distinct from Hu and induces a tiling of $(Hu) \setminus \{u\}$. Let Hg be any right coset of H: we shall say that the right-tile K is used to cover Hg if there exists an element b of B such that $Kb \subseteq Hg$. The trick of many proofs in this paper is to collect enough informations about the tiling of every right-coset of H so as being able to get informations about the near-factorization (A, B).

Example B.7.7. *Let (A, B) be the near-factorization of the dihedral group $\mathbb{D}_{16}$ given by $A = \{e, r^5, sr^5\}$ and $B = \{s, r, sr, r^2, sr^2\}$.*

Let $H := \{e, s\}$ be the cyclic subgroup of $\mathbb{D}_{16}$ generated by s. Then $\{H, Hr, Hr^2, \ldots, Hr^7\}$ is the partition of $\mathbb{D}_{16}$ in right cosets of H. Hence A splits in exactly two right-tiles T_1 and T_2 with

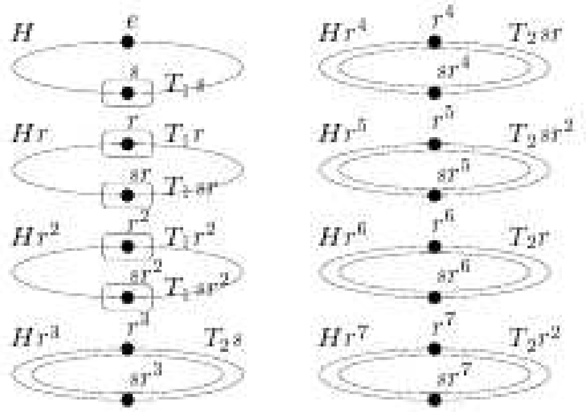
$$T_1 = \{e\} = A \cap H$$

$$T_2 = \{r^5, sr^5\} = A \cap Hr^5$$

The tile T_2 is a H-right-coset. The set B has 5 elements, this implies that T_2 is used to cover 5 of the 8 right-cosets of H, namely the right-cosets Hr^3, Hr^6, Hr^4, Hr^7 and Hr^5 because $Hr^3 = T_2 s$, $Hr^6 = T_2 r$, $Hr^4 = T_2 sr$, $Hr^7 = T_2 r^2$ and $Hr^5 = T_2 sr^2$.

The tile T_1 is used exactly twice to cover the right-coset Hr as $Hr = \{r, sr\} = T_1 r \cup T_1 sr$. The tile T_1 is used exactly twice to cover the right-coset Hr^2 as $Hr^2 = \{r^2, sr^2\} = T_1 r^2 \cup T_1 sr^2$. The last time T_1 is used, it is to cover $H \setminus \{e\}$ as $H \setminus \{e\} = \{s\} = T_1 s$.

The following figure represents this tiling of the right-cosets of H.



The unique partition of G in left cosets of H also induces a unique partition of A in left-tiles. If T is a left-tile of A which is equal to a whole left-coset, then T is called a *H-left-coset*.

When the uncovered element is e, we know that (B, A) is a near-factorization of G too. Thus we get a tiling of $G \setminus \{e\}$ with the left-tiles of A as tiles. Let K be any such tile and b be any element of B. Notice that bK lies entirely in a left-coset of H. Hence we have a tiling for every left-coset of H distinct from He and a tiling of $(He) \setminus \{e\}$. Let gH be any left coset of H: we shall say that the left-tile K is used to cover gH if there exists an element b of B such that $bK \subseteq gH$.

Example B.7.8. *We consider again the near-factorization (A, B) of the dihedral group $\mathbb{D}_{16}$ given by $A = \{e, r^5, sr^5\}$ and $B = \{s, r, sr, r^2, sr^2\}$ and the cyclic subgroup H of $\mathbb{D}_{16}$ generated by s.*

As $u(A, B) = e$, we know that (B, A) is a near-factorization of $\mathbb{D}_{16}$ too.

Notice that $\{H, rH, r^2H, \ldots, r^7H\}$ *is the partition of* $\mathbb{D}_{16}$ *in left cosets of* H*. Hence* A *splits in exactly three left-tiles* T_1*,* T_2 *and* T_3 *with*

$$T_1 = \{e\} = H \cap A$$

$$T_2 = \{r^5\} = r^5H \cap A$$

$$T_3 = \{sr^5\} = r^3H \cap A$$

Thus no left-tile of A *is a left-coset. This means that the tiling induced by* (B, A) *is actually different of the one induced by* (A, B)*.*

Let Hg_1, Hg_2, …, Hg_d be a partition of G in right-cosets of H. Let X be any subset of G. We define the integer $\mathrm{disp}^r_H(X)$ as

$$\mathrm{disp}^r_H(X) := |\{i,\ 1 \leq i \leq d,\ \emptyset \subsetneq Hg_i \cap X \subsetneq Hg_i\}|$$

The counter $\mathrm{disp}^r_H(X)$ is the number of right-cosets of H which meet X and are not a subset of X.

Let $\mathrm{disp}^l_H(X)$ be the number of left-cosets of H which meet X and are not a subset of X. When H is a normal subgroup then we use rather the notation $\mathrm{disp}_H(X)$ instead of $\mathrm{disp}^r_H(X)$ or $\mathrm{disp}^l_H(X)$. The notation disp_H is related to the word 'dispersion'.

Let y be any element of G. A subset W of G is a *left-y-chain* (respectively *right-y-chain*) if $|W| \neq |\langle y \rangle|$ and W can be written $w * \{e,\ y,\ \ldots,\ y^{|W|-1}\}$ (respectively $\{e,\ y,\ \ldots,\ y^{|W|-1}\} * w$).

If H is a cyclic subgroup $\langle y \rangle$, then it is useful to subdivide any tile of A in right-y-chains. For conveniency, these right-y-chains will be considered again as tiles. Let $T := \{e,\ y,\ \ldots,\ y^{|T|-1}\} * t$ and $T' := \{e,\ y,\ \ldots,\ y^{|T'|-1}\} * t'$ be two maximal right-y-chains of A not necessarily distinct. Let b and b' be two elements of B. The tile $T'b'$ is said to be used after the tile Tb if and only if $t' * b' = y^{|T|} * t * b$. This implies that $t'^{-1} * y^{|T|} * t = b' * b^{-1}$ is an element of $B * B^{-1}$. When this relation is all we need, we say simply that the tile T' is used after the tile T (see figure B.20).

The fact that $G(A, B)$ is a normalized partitionable graph may be deduced from [7] and [5]. We give here a direct proof which shows how the near-factorization (A, B) and the partitionable graph are closely related, by exhibiting the partition in maximum cliques and the partition in maximum stable sets of $G(A, B) \setminus \{x\}$ for every x:

Lemma B.7.9. *If* (A, B) *is a near-factorization of a finite group* G *such that* $A * B = G \setminus \{e\}$*, then the graph* $G(A, B)$ *is a normalized partitionable graph with maximum cliques* $\{xA,\ x \in G\}$ *and maximum stable sets* $\{xB^{-1},\ x \in G\}$*.*

Proof.

Claim B.7.10. *For every* x *of* G*,* xA *is a clique of* $G(A, B)$

Let x_1 and x_2 be two distinct elements of xA: there exist a_1 and a_2 of A such that $x_1 = x * a_1$ and $x_2 = x * a_2$. Then ${x_1}^{-1} * x_2 = {a_1}^{-1} * a_2$ is an element of $(A^{-1} * A) \setminus \{e\}$. Thus $\{x_1, x_2\}$ is an edge of $G(A, B)$, and so xA is a clique of $G(A, B)$

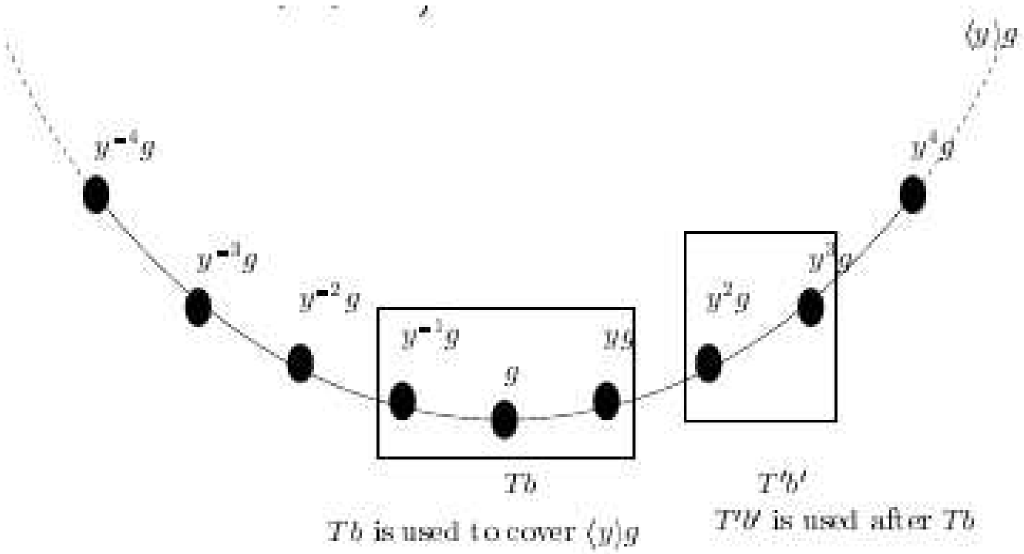
Claim B.7.11. *For every* x *of* G*,* xB^{-1} *is a stable set of* $G(A, B)$*.*

Let x_1 and x_2 be two distinct elements of xB^{-1}: there exist b_1 and b_2 of B such that $x_1 = x * {b_1}^{-1}$ and $x_2 = x * {b_2}^{-1}$.

If $\{x_1, x_2\}$ is an edge of $G(A, B)$, then ${x_1}^{-1} * x_2 = b_1 * b_2^{-1}$ is an element of $A^{-1} * A$. Thus there exist a_1 and a_2 in A such that $b_1 * b_2^{-1} = {a_1}^{-1} * a_2$. Hence $a_1 * b_1 = a_2 * b_2$. Since (A, B) is a near-factorization, this implies that $a_1 = a_2$ and $b_1 = b_2$. Thus $x_1 = x_2$, a contradiction.

Hence $\{x_1, x_2\}$ is not an edge of $G(A, B)$. This implies that xB^{-1} is a stable set of $G(A, B)$.

$$\left.\begin{array}{rcl} T & = & \{e,y,y^2\}*t \\ T' & = & \{e,y\}*t' \end{array}\right\} \text{are two right}-y-\text{chains of A}$$



$$\left.\begin{array}{rcl} tb & = & y^{-1}g \\ t'b' & = & y^2 g \end{array}\right\} \Rightarrow b' = t'^{-1}y^3tb$$

Figure B.20: Fragment of the tiling of the coset $\langle y\rangle g$

Claim B.7.12. *For every x of G, $G(A,B)\setminus\{x\}$ is partitioned by the $|B|$ cliques $\{xbA,\ b\in B\}$ and is also partitioned by the $|A|$ stable sets $\{xa^{-1}B^{-1},\ a\in A\}$. Hence $G(A,B)$ is a partitionable graph with $\omega=|A|$ and $\alpha=|B|$.*

If there exists b in B such that $x\in xbA$ then there is an element a in A such that $x=x*b*a$ thus $e=b*a$, hence $b=a^{-1}$ and so $a*b=e$ in contradiction with the hypothesis $A*B=G\setminus\{e\}$. Hence $\bigcup_{b\in B} xbA\subset G\setminus\{x\}$. If $xbA\cap xb'A\neq\emptyset$ with b and b' in B, then there are a and a' in A such that $x*b*a=x*b'*a'$ thus $b*a=b'*a'$. This implies with Lemma B.7.3 again that $a=a'$ and $b=b'$. Hence $|\bigcup_{b\in B} xbA|=\sum_{b\in B}|xbA|=|B|*|A|=|G\setminus\{x\}|$. Thus $\bigcup_{b\in B} xbA=G\setminus\{x\}$ and $\{xbA,\ b\in B\}$ is a partition of $G\setminus\{x\}$.

If there exists a in A such that $x\in xa^{-1}B^{-1}$ then there is an element b in B such that $x=x*a^{-1}*b^{-1}$ thus $e=a^{-1}*b^{-1}$ and so $e=b*a$: contradiction. Hence $\bigcup_{a\in A} xa^{-1}B^{-1}\subset G\setminus\{x\}$. If $xa^{-1}B^{-1}\cap xa'^{-1}B^{-1}\neq\emptyset$ with a and a' in A, then there are b and b' in B such that $x*a^{-1}*b^{-1}=x*a'^{-1}*b'^{-1}$ thus $a^{-1}*b^{-1}=a'^{-1}*b'^{-1}$ and so $b*a=b'*a'$. This implies that $a=a'$ and $b=b'$. Hence $|\bigcup_{a\in A} xa^{-1}B^{-1}|=\sum_{a\in A}|xa^{-1}B^{-1}|=|B|*|A|=|G\setminus\{x\}|$. Thus $\bigcup_{a\in A} xa^{-1}B^{-1}=G\setminus\{x\}$ and $\{xa^{-1}B^{-1},\ a\in A\}$ is a partition of $G\setminus\{x\}$.

Claim B.7.13. *For every maximum clique Q of $G(A,B)$, there is an element x of G such that $Q=xA$, hence the set of the n maximum cliques is $\{xA,\ x\in G\}$. Likewise the set of the n maximum stable sets of $G(A,B)$ is $\{xB^{-1},\ x\in G\}$.*

Since $G(A,B)$ is a partitionable graph, we know that $G(A,B)$ has exactly n maximum cliques. Thus we are done if we show that for every pair of elements x and y of G such that $x\neq y$, we have $xA\neq yA$. This is equivalent to show that if $A=zA$ then $z=e$. Suppose $A=zA$. Then for every element a of A, we have that $z*a$ is an element of A. Thus A admits a partition in $\langle z\rangle$-right-cosets. Hence $\omega=0\pmod{o(z)}$ where $o(z)$ is the order of z. Thus $n=1\pmod{o(z)}$. As $o(z)$ divides the number of elements of G, we also have $n=0\pmod{o(z)}$. Therefore $o(z)=1$ and so $z=e$. This proof also works for the maximum stable sets.

Claim B.7.14. *$G(A, B)$ is a normalized graph.*

Let $\{x, y\}$ be any edge of $G(A, B)$. Then $x^{-1} * y \in A^{-1} * A$, thus there exists $a \in A$ such that $y \in xa^{-1}A$. Obviously $x \in xa^{-1}A$. Hence $G(A, B)$ is a normalized graph.

■

Since the cardinality of a maximum clique of $G(A, B)$ is equal to $|A|$, we denote by ω the value of $|A|$. Likewise, we denote by α the value of $|B|$.

A graph $\Gamma = (V, E)$ on $\alpha\omega + 1$ vertices is called a *web*, if the maximum cardinality of a clique of Γ is ω, the maximum cardinality of a stable set of Γ is α, and there is a cyclical order of V so that every set of ω consecutive vertices in this cyclical order is an ω-clique. Equivalently, normalized webs with n vertices are graphs induced by any near-factorization (A, B) of $\mathbb{Z}_n$ such that A is an interval.

In 1979, V. Chvátal, R.L. Graham, A.F. Perold and S.H. Whitesides [7] introduced a method to produce a large class of near-factorizations of the cyclic groups $\mathbb{Z}_n$.

Two subsets A_1 and B_1 of $\mathbf{N}$ are said to form a *near-factorization in integers* if and only if $A_1 + B_1 = [0..(|A_1| \times |B_1| - 1)]$. Obviously, a near-factorization in integers induces a near-factorization of $\mathbb{Z}_{|A_1|\times|B_1|+1}$.

Let (A_1, B_1) be a near-factorization in integers such that $A_1 + B_1 = [0..n_1 - 2]$. Let k, k' be any positive integers.

One may obtain a near-factorization in integers (A_2, B_2) such that $A_2 + B_2 = [0..n_2 - 2]$ with

$$n_2 := (|A_1| \times k) \times (|B_1| \times k') + 1$$

by defining:

$$A_2 := A_1 + (n_1 - 1) \times [0..k - 1] \text{ and } B_2 := B_1 + (n_1 - 1) \times k \times [0..k' - 1]$$

A *CGPW graph* is a graph $G(A, B)$ where (A, B) is obtained with a finite number of applications of this method starting from a *basic factorization*, that is a near-factorization (A_1, B_1) such that $A_1 = [0..|A_1| - 1]$ and $B_1 = |A_1| \times [0..|B_1| - 1]$.

Explicitly, the CGPWgraph G given by $2p$ positive integers $k_1, \ldots, k_{2p}$ is constructed in this way :

- take $A_1 = [0..k_1 - 1]$ and $B_1 = k_1 \times [0..k_2 - 1]$. Set $n_1 = k_1 \times k_2 + 1$.
- take $k = k_3$ and $k' = k_4$ then calculate A_2 and B_2. Set $n_2 = k_1 \times k_2 \times k_3 \times k_4 + 1$.
- take $k = k_5$ and $k' = k_6$ then calculate A_3 and B_3 starting from A_2 and B_2. Set $n_3 = k_1 \times k_2 \times k_3 \times k_4 \times k_5 \times k_6 + 1$.
- ...
- until $k = k_{2p-1}$ and $k' = k_{2p}$.

G is $G(A_p, B_p)$ and is denoted by $C[k_1, \ldots, k_{2p}]$. By construction, $|A_p| = k_1 \times k_3 \times \ldots \times k_{2p-1} = \omega$, $|B_p| = k_2 \times k_4 \times \ldots \times k_{2p} = \alpha$ and $n_p = k_1 \times k_2 \times \ldots \times k_{2p} + 1 = \alpha \times \omega + 1$.

Notice that normalized webs are CGPWgraphs such that $p = 1$.

Following [1], a near-factorization produced by this method is called a *De Bruijn near-factorization*.

Let X be any subset of the group G. We set

$$\mathrm{INT}(X) = \max_{x\in G,\ y\in G,\ x\neq y} \{|xX \cap yX|\}$$

Notice that $\mathrm{INT}(A)$ denotes the maximum cardinality of the intersection between two distinct ω-cliques of $G(A, B)$ and that $\mathrm{INT}(B^{-1})$ denotes the maximum cardinality of the intersection between two distinct α-stable sets.

An edge e of a graph Γ is said to be an *α-critical edge* if and only if $\alpha(\Gamma - e) > \alpha(\Gamma)$. Similarly, a non-edge e' is said to be *co-critical* if and only if $\omega(\Gamma + e') > \omega(\Gamma)$. It is easy to check that a graph $G(A, B)$ has a co-critical non-edge (respectively α-critical edge) if and only if $\mathrm{INT}(A) = \omega - 1$ (respectively $\mathrm{INT}(B^{-1}) = \alpha - 1$).

Lemma B.7.15.
$$\mathrm{INT}(X) = \max_{g \in G \setminus \{e\}} \{|X \cap gX|\}$$

Proof. The proof is straightforward. ■

Next lemma will be used in the proofs of this article:

Lemma B.7.16. *Let G be a finite group having a near-factorization (A, B). Let H be any normal subgroup of G. If there is a H-coset (Ha) in A, then in every coset of H, a tile T of A may be used at most once.*

Proof. Let T be any tile of A: there exists y of G such that $T = A \cap Hy$. Let g be any element of G and let B_g be the set $\{b \in B,\ Tb \subseteq Hg\}$. We want to show that $|B_g| \leq 1$.

If $|B_g| \geq 2$ then there exist two distinct elements b and b' of B such that $Tb \subseteq Hg$ and $Tb' \subseteq Hg$. From $T \subseteq Hy$, we get $Hg = Hyb$ and $Hg = Hyb'$. Then $Hab = ay^{-1}Hyb$ because H is a normal subgroup. Thus $Hab = ay^{-1}Hg = ay^{-1}Hyb' = Hab'$. Since (A, B) is a near factorization and $Ha \subseteq A$, $\{b, b'\} \subseteq B$, this implies that $b = b'$: a contradiction. Hence $|B_g| \leq 1$. ■

Notice that Example B.7.7 shows that the hypothesis that H must be normal is actually needed.

We are now ready to state the main result of this paper.

Theorem B.7.17. *Let G be a finite group admitting a near-factorization (A, B). Let H be a non-trivial proper subgroup of G. Then*

1. $\mathrm{disp}^r_H(A) > 0$ *and* $\mathrm{disp}^l_H(A) > 0$.
2. *if* $\mathrm{disp}^r_H(A) = 1$ *or* $\mathrm{disp}^l_H(A) = 1$ *then* $|H| = 2$.
3. *if H is a normal subgroup,* $\mathrm{disp}_H(A) = 2$ *and* $|A| \neq 2$, *then* $|H| = \frac{n}{2}$.

Proof. Since no special property is required for B, we may assume that $u(A, B) = e$ since otherwise all we have to do is to right-shift B by $u(A, B)^{-1}$. Hence we have $A * B = G \setminus \{e\} = B * A$ (Lemma B.7.3).

1. If $\mathrm{disp}^r_H(A) = 0$, then every right-tile of A is a H-right-coset. Let T be a right-tile of A which is used to cover the right-coset He. There exists b of B such that $Tb \subseteq He$. Since T is a H-right-coset, we have $Tb = He$. Hence $e \in A * B$, a contradiction. Thus $\mathrm{disp}^r_H(A) > 0$.

   Likewise, we have $\mathrm{disp}^l_H(A) > 0$.

2. Suppose that $\mathrm{disp}^r_H(A) = 1$. Let $Hg_1, Hg_2, \ldots, Hg_d$ be a partition of G in right-cosets of H. Since $\mathrm{disp}^r_H(A) = 1$ there exists a unique integer p between 1 and d such that $\emptyset \subsetneq A \cap Hg_p \subsetneq Hg_p$. Let $A' := A \cap Hg_p$. Thus the set of right-tiles of A is A' and some H-right-cosets.

   Let b be an element of B such that $A'b \subseteq He$. Then we have $Hg_p b = He$, which implies that $(g_p * b) \in He$. Thus, if for every b in B, we have $A'b \subseteq He$, then $g_p B \subseteq He$. We know that (B, A) is a near-factorization with $u(B, A) = e$. Hence $(g_p B, A)$ is a near-factorization with uncovered element g_p. As $g_p B \subseteq He$, $g_p B$ has only one right-tile. Since H is a proper subgroup of G, there exists a right coset Hx distinct from He. Thus $|Hx| = 0 \pmod{|g_p B|} = 0 \pmod{\alpha}$, which implies $n = 0 \pmod{\alpha}$, contradicting the relation $n = \alpha \times \omega + 1$.

   Hence there exists b in B such that $A'b$ lies in a coset Hx distinct from He. Obviously A' is the only tile of A which can be used to cover Hx because the other tiles are H-right-cosets thus $|Hx| = 0 \pmod{|A'|}$. The tile A' is again the only tile which can be used to cover He, thus $|He| = 1 \pmod{|A'|}$. Hence $|A'| = 1$.

Let H' be the conjugate subgroup $g_p^{-1}Hg_p$ of H. Let $H'g'_1, H'g'_2, \ldots, H'g'_d$ be a partition of G in right-cosets of H'. For every i between 1 and d, let $B_i := B \cap H'g'_i$. Then for every i between 1 and d, we have $(A' * B_i) \subseteq (Hg_p * g_p^{-1}Hg_pg'_i) = Hg_pg'_i$.

Let i be any integer between 1 and d. If $B_i \neq \emptyset$ then A' is used at least once to cover $Hg_pg'_i$. Thus $Hg_pg'_i$ is covered with the right-tile A' only. Hence we have $(Hg_pg'_i) \setminus \{e\} = \cup_{b \in B,\ A'b \subseteq Hg_pg'_i} A'b$. Let b be any element of B and let j be the integer such that $b \in B_j$. Thus $A'b \subseteq Hg_pg'_j = g_pH'g'_j$. Hence, if b is not in B_i then $A'b$ is not a subset of $Hg_pg'_i$. Thus we have $A' * B_i = (Hg_pg'_i) \setminus \{e\}$. Since $|A'| = 1$, we must have $|B_i| = |(Hg_pg'_i) \setminus \{e\}|$.

Hence we have for all i between 1 and d, $|B_i| = 0$ or $|B_i| = |Hg_pg'_i \setminus \{e\}|$. Thus $\mathrm{disp^r}_{H'}(B) \leq 1$. We know that $\mathrm{disp^r}_{H'}(B) = 0$ is impossible according to the first section of the proof of this Theorem. Therefore we have $\mathrm{disp^r}_{H'}(B) = 1$. There exists a unique integer p' between 1 and d such that $B_{p'} \neq \emptyset$ and $B_{p'} \neq H'g'_{p'}$. We set $B' := B_{p'}$. Then we get $|B'| = 1$ as we have seen for A'.

We have $A' * B' = (Hg_pg'_{p'}) \setminus \{e\}$. If $Hg_pg'_{p'} \neq He$, then we have $|H| = |A' * B'| = 1$, hence H is the trivial subgroup: a contradiction. Thus $Hg_pg'_{p'} = He$, which implies $|H| = 2$ as required.

If $\mathrm{disp}^l_H(A) = 1$ then the same proof may be applied to the quasi-factorization (B, A) by working with the left-cosets of H.

3. Notice that H is assumed to be normal.

   Since $\mathrm{disp}_H(A) = 2$, there exist two distinct cosets Hg_1 and Hg_2 of G such that $\emptyset \subsetneq A \cap Hg_1 \subsetneq Hg_1$ and $\emptyset \subsetneq A \cap Hg_2 \subsetneq Hg_2$. Let $A_1 := A \cap Hg_1$ and $A_2 := A \cap Hg_2$.

   If there is a H-coset in A then by Lemma B.7.16, A_1 (and A_2) cannot be used twice on the same coset. Thus A_1 is used at least once on a coset distinct from He otherwise we would have $\alpha \leq 1$. Let Hv be such a coset. Obviously Hv is not covered with only A_1 because A_1 is not a H-coset. Hence A_1 and A_2 are used exactly once to cover Hv. Thus $|Hv| = |A_1| + |A_2|$. Hence $n = 0 \pmod{|A_1| + |A_2|}$. If C is any H-coset of A, we have $|C| = |H| = |A_1| + |A_2|$. Thus $\omega = 0 \pmod{|A_1| + |A_2|}$. From $n = \alpha \times \omega + 1$, we get $n = 1 \pmod{|A_1| + |A_2|}$ contradicting $n = 0 \pmod{|A_1| + |A_2|}$. Therefore there is no H-coset in A.

   Thus $A = A_1 \cup A_2$. As H is a proper subgroup of G, there exists x such that $He \cap Hx = \emptyset$.

   If $|A_1| = |A_2|$, then due to the cover of Hx, we get $n = 0 \pmod{|A_1|}$. From $n = \alpha \times \omega + 1$, we have $n = 1 \pmod{|A_1|}$. Thus $|A_1| = 1$. This means that $|A| = 2$, which is contradictory to the hypothesis of the Theorem. Hence $|A_1| \neq |A_2|$ and we may assume that $|A_1| > |A_2|$.

   If z is any element of G, let $n_z(A_1)$ (respectively $n_z(A_2)$) be the number of times the tile A_1 (respectively A_2) is used to cover the coset Hz, that is $n_z(A_1) = |\{b \in B |\ A_1b \subseteq Hz\}|$ (respectively $n_z(A_2) = |\{b \in B |\ A_2b \subseteq Hz\}|$). Let $n_{\max}(A_1) := \max_{z \in G}\{n_z(A_1)\}$, $n_{\min}(A_1) := \min_{z \in G}\{n_z(A_1)\}$, $n_{\max}(A_2) := \max_{z \in G}\{n_z(A_2)\}$ and $n_{\min}(A_2) := \min_{z \in G}\{n_z(A_2)\}$.

   Claim B.7.18.

   $$n_{\max}(A_1) = n_{\max}(A_2)$$

   $$n_{\min}(A_1) = n_{\min}(A_2)$$

   Proof. Let b be any element of B and z be any element of G.

   If $A_1b \subseteq Hz$ then $b \in Hg_1{}^{-1}z$ as $A_1 \subseteq Hg_1$ and H is a normal subgroup of G. From $A_2 \subseteq Hg_2$, we get $A_2b \subseteq Hg_2Hg_1{}^{-1}z = Hg_2g_1{}^{-1}z$.

   Likewise, if $A_2b \subseteq Hg_2g_1{}^{-1}z$ then $A_1b \subseteq Hz$. Hence $A_1 \subseteq Hz$ if and only if $A_2b \subseteq Hg_2g_1{}^{-1}z$. And so for any z in G, there exists z' and z'' such that $n_z(A_1) = n_{z'}(A_2)$ and $n_z(A_2) = n_{z''}(A_1)$.

   Thus $n_{\min}(A_1) = n_{\min}(A_2)$ and $n_{\max}(A_1) = n_{\max}(A_2)$. Let $n_{\max} := n_{\max}(A_1)$ and $n_{\min} := n_{\min}(A_1)$.

   ■

Claim B.7.19.
$$n_{\max} > n_{\min}$$

Proof. If $n_{\max} = n_{\min}$ then $|Hx| = n_{\min} \times (|A_1| + |A_2|)$ and so $n = 0 \pmod{\omega}$, contradicting $n = \alpha \times \omega + 1$.

■

To simplify the notation, let $a_1 = |A_1|$ and let $a_2 = |A_2|$.

Claim B.7.20. $n_{\max} = n_{\min} + 1$, $a_1 = a_2 + 1$ *and* $|H| = n_{\max} a_1 + n_{\min} a_2$.

Proof. If g is any element of G, we set $\epsilon(g) = 1$ if $Hg = H$ and we set $\epsilon(g) = 0$ otherwise.

Let z be an element of G such that $n_z(A_2) = n_{\max}$ (by definition such an element exists), we first show that $n_z(A_1) = n_{\min}$.

By definition there exists g in G such that $n_g(A_1) = n_{\min}$. Let $k \geq n_{\min}$ and $l \leq n_{\max}$ be integers such that $|Hz| = ka_1 + n_{\max} a_2 + \epsilon(z) = |Hg| = n_{\min} a_1 + la_2 + \epsilon(g)$. We get that $(k - n_{\min})a_1 = (l - n_{\max})a_2 + \epsilon(g) - \epsilon(z)$. Since $k - n_{\min} \geq 0$, $a_1 > a_2 \geq 1$, $1 - n_{\max} \leq 0$, $\epsilon(g) - \epsilon(z) \leq 1$, we get that $k = n_z(A_1) = n_{\min}$.

Now let h be an element of G such that $n_h(A_1) = n_{\max}$.

We have $|Hz| = n_{\min} a_1 + n_{\max} a_2 + \epsilon(z) = |Hh| \geq n_{\max} a_1 + n_{\min} a_2 + \epsilon(h)$ and so $\epsilon(z) - \epsilon(h) \geq (n_{\max} - n_{\min})(a_1 - a_2)$. Since $n_{\max} > n_{\min} \geq 0$, $a_1 > a_2 \geq 0$ and $\epsilon(z) - \epsilon(h) \leq 1$, we get $n_{\max} = n_{\min} + 1$, $a_1 = a_2 + 1$, $\epsilon(z) = 1$, $\epsilon(h) = 0$ and $n_h(A_2) = n_{\min}$. Notice that from these equalities $|H| = n_{\max} a_1 + n_{\min} a_2 = n_{\min} a_1 + n_{\max} a_2 + 1$.

■

Claim B.7.21. *H is of cardinality $\frac{n}{2}$*

Proof. Let z be any element of G. From what precedes it is not possible that $n_z(A_1) = n_z(A_2) = n_{\max}$ or $n_z(A_1) = n_z(A_2) = n_{\min}$, so either $n_z(A_1) = n_{\max}$, $n_z(A_2) = n_{\min}$ and $Hz \neq He$, or $n_z(A_1) = n_{\min}$, $n_z(A_2) = n_{\max}$ and $Hz = He$. Let d be the number of cosets of H, then $|B| = \sum_{i=1,\ldots,d} n_{g_i}(A_1) = \sum_{i=1,\ldots,d} n_{g_i}(A_2) = (d-1)n_{\max} + n_{\min} = (d-1)n_{\min} + n_{\max}$. Since $n_{\max} \neq n_{\min}$, this implies that $d = 2$.

■

■

Example B.7.22. *Let (A, B) be the near-factorization of $\mathbb{D}_{16}$ introduced in Example B.7.7: $A = \{e, r^5, sr^5\}$ and $B = \{r, r^2, s, sr, sr^2\}$.*

Let $H_1 := \{e, sr^5\}$. Since $\mathrm{disp}_{H_1}(A) = 1$, H_1 must be of cardinality 2.

Let $H_2 := \{e, r, r^2, r^3, r^4, r^5, r^6, r^7\}$. Since $\mathrm{disp}_{H_2}(A) = 2$, $|A| \neq 2$ and H_2 is normal, H_2 must be of cardinality $\frac{16}{2} = 8$.

Theorem B.7.17 may be used to decrease the number of cases to be investigated when looking for a near-factorization for a given group with the help of a computer. From the list of all subsets A of G of cardinality ω, we may keep only those satisfying Theorem B.7.17 and then for every of these A check if there exists a subset B of cardinality α such that (A, B) is a near-factorization. For every group of small order (that is less than 1000), it is quite easy to get the list of all subgroups of G and the list of all normal subgroups of G using *GAP* [9] for instance. Theorem B.7.17 is an interesting filter because it may be applied to any group. Our implementation revealed that it performs quite well when ω or α is small as one might expect. In some groups, there are no subsets at all satisfying Theorem B.7.17 with the required cardinality. For instance, the only groups of order 16 with a subset A of cardinality 3 satisfying Theorem B.7.17 are the dihedral groups and cyclic groups.

We will use Theorem B.7.17 to derive Lemma B.7.24 and Lemma B.7.28.

Lemma B.7.23. *l If $\omega = 3$, A is symmetric and n is odd then $G(A, B)$ is a web.*

Proof. Since n is odd, there is no involution in G. This implies with $A = A^{-1}$ that there is a in G such that $A = \{a^{-1}, e, a\}$. Let H be the cyclic subgroup generated by a. Notice that $A \subseteq H$, thus $\mathrm{disp}^r_H(A) = \mathrm{disp}^l_H(A) = 1$. If H is distinct from G then by Theorem B.7.17, we must have $|H| = 2$, which is impossible as n is odd. Thus G is a cyclic group. Since $\omega = 3$, $G(A, B)$ is a web [1].

■

András Sebö proved in [14] that the minimal imperfect graphs containing certain configurations of two α-critical edges and one co-critical non-edge are exactly the odd holes or anti-holes.

S. Markossian, G. Gasparian, I. Karapetian and A. Markosian also studied in [12] such edges and non-edges in conjunction with the Strong Perfect Graph Conjecture.

Recall that a graph $G(A, B)$ has a co-critical non-edge if and only if $\mathrm{INT}(A) = \omega - 1$. Next Lemma partially characterizes graphs $G(A, B)$ with a co-critical non-edge.

Lemma B.7.24. *Let G be a finite group such that every involution z commutes with every element of G. If (A, B) is a near-factorization of G such that $\mathrm{INT}(A) = \omega - 1$ then G is a cyclic group and $G(A, B)$ is a web.*

Proof. Since $\mathrm{INT}(A) = \omega - 1$, by Lemma B.7.15 there exists an element y of G such that $|A \cap yA| = \omega - 1$. Let H be the cyclic subgroup of G generated by y. Notice that A admits a unique partition in maximal right-y-chains and H-right-cosets. Let k be the number of maximal right-y-chains in this partition. Then we have $|A \cap yA| = \omega - k$. Thus there is exactly one maximal right-y-chain in A. Let $T := \{e,\ y,\ y^2,\ \ldots,\ y^{|T|-1}\} * t$ be this maximal right-y-chain. Notice that T is a subset of a H-right coset. Therefore we have $\mathrm{disp}^r_H(A) = 1$, as the right-tiles of A are T and H-right-cosets,

Obviously $y \neq e$, hence H is not the trivial subgroup of G. Thus by Theorem B.7.17, we have $H = G$ or $|H| = 2$.

If $|H| = 2$ then y is an involution of G distinct from e, and we must have $|T| = 1$. Hence there must be some H-right-cosets in A. The element y commutes with every element of G, hence H is a normal subgroup of G. If T is used only on the coset $Hu(A, B)$, then $\alpha \leq 1$, which is impossible. Therefore T is used in the cover of another coset Hx. As only T is used on Hx, it is used at least twice, which is in contradiction with Lemma B.7.16 because H is a normal subgroup of G.

Therefore $H = G$, that is G is a cyclic group.

Hence $A = T$ and $G(t^{-1}A, B)$ is a web. Thus $G(A, B)$ which is isomorphic to $G(t^{-1}A, B)$ is a web.

■

Lemma B.7.24 is not true if the hypothesis that every involution is in the center of G is not assumed. Indeed the dihedral groups are examples of non-cyclic groups having near-factorizations (A, B) and $\mathrm{INT}(A) = \omega - 1$ (see Section 3). Besides we give in Section 4, a graph $G(A, B)$ with 50 vertices such that $\mathrm{INT}(A) = \omega - 1$, which is not a web.

Corollary B.7.25. *If G is a non-cyclic finite abelian group then it admits no near-factorization (A, B) such that* $\mathrm{INT}(A) = \omega - 1$.

Corollary B.7.26. *If G is a non-cyclic finite group of odd order then it admits no near-factorization (A, B) such that* $\mathrm{INT}(A) = \omega - 1$.

Proof. Indeed there is no involution in a group of odd order. ■

Example B.7.27. *Let G be any group of order $3 \times p + 1$ (p a prime) such that its center contains all its involutions, with a symmetric near-factorization (A, B). We may assume that $|A| = 3$. Since $|A|$ is odd and A is symmetric, there must be an element w in A such that $w^2 = e$. Let a be another element in A. Thus $\{a,\ w\} \subseteq A \cap awA$ and so $\mathrm{INT}(A) \geq 2$. Then by Lemma B.7.24, G must be cyclic. This implies for instance that 7 groups, out of the 14 groups of order 16, have no symmetric near-factorizations.*

There are many non-abelian groups containing in their center all their involutions: according to GAP [9] there are 58 such groups out of the 267 groups of order 64, and 52 such groups out of the 231 groups of order 96. Notice that for $n = 64$ or 96, ω or α must be prime, hence any CGPW graph of these orders is a web. Thus if any of these groups has a near-factorization (A, B) then the graph $G(A, B)$ is not a CGPW graph. Notice that for $n = 64$, these groups do not have any symmetric near-factorization (A, B) such that $|A| = 3$.

Lemma B.7.28. *Let G be a finite group such that all its cyclic subgroups are normal and admitting a near-factorization (A, B) such that $\mathrm{INT}(A) = \omega - 2$. Then*

- *If G is abelian then G is cyclic.*
- *If G is not abelian then the order of G is a multiple of 4, G has an element y of order $\frac{n}{2}$ and $y^{\frac{n}{4}}$ is the only involution of G .*

Proof. Since $\mathrm{INT}(A) = \omega - 2$, we have $\omega \geq 3$ and there exists an element y of G such that $|A \cap yA| = \omega - 2$. Let $T_1 := \{e,\ y,\ y^2,\ \ldots,\ y^{|T_1|-1}\} * t_1$ and $T_2 := \{e,\ y,\ y^2,\ \ldots,\ y^{|T_2|-1}\} * t_2$ be the two maximal right-y-chains of A. Let u be the uncovered element. Let H be the cyclic subgroup generated by the element y. Hence by assumption on G, H is a non-trivial normal subgroup of G:

If $G = H$ then G is abelian and cyclic, thus we are done. Hence we may assume that $H \subsetneq G$.

Since A is made of T_1, T_2 and some H-cosets, we have $\mathrm{disp}^r_H(A) \leq 2$. By Theorem B.7.17, we have $\mathrm{disp}^r_H(A) > 0$. If $\mathrm{disp}^r_H(A) = 1$ then by Theorem B.7.17 again, we get $|H| = 2$. Since $\mathrm{disp}^r_H(A) = 1$, T_1 and T_2 must lie in the same right-coset of H. Thus $T_1 \cup T_2$ is a H-coset, and this implies that $\mathrm{disp}^r_H(A) = 0$, a contradiction.

Hence $\mathrm{disp}^r_H(A) = 2$ and by Theorem B.7.17 again, H has cardinality $\frac{n}{2}$. Therefore y is an element of order $\frac{n}{2}$ and there is no H-coset in A.

Claim B.7.29. *We have $|T_1| \neq |T_2|$.*

Proof. Suppose that $|T_1| = |T_2|$. As there is no H-coset in A, we have $|H| = 1 \pmod{|T_1|}$ due to the cover of the coset $Hu(A, B)$. Then we also have $|H| = 0 \pmod{|T_1|}$ due to the cover of the other coset. Hence $|T_1| = 1$. This implies that $|A| = 2$. This is impossible as $\omega \geq 3$.

■

Thus $|T_1| \neq |T_2|$ and we may assume that $|T_2| < |T_1|$.

Claim B.7.30. *The pair $\{Ht_1,\ Ht_2\}$ is a partition of G in right cosets.*

Proof. If t_1 and t_2 lie in the same right coset then $\mathrm{disp}^r_H(A) \leq 1$, contradicting $\mathrm{disp}^r_H(A) = 2$. Thus $Ht_1 \cap Ht_2 = \emptyset$. As $|H| = \frac{n}{2}$, we are done. ■

Claim B.7.31. *We have $(Ht_1)^{-1} = Ht_1$ and $(Ht_2)^{-1} = Ht_2$.*

Proof. Suppose that $H = Ht_1$ then we obviously have $(Ht_1)^{-1} = Ht_1$. Since the inversion map is a bijective map, this implies that $(Ht_2)^{-1} = Ht_2$. The proof for the case $H = Ht_2$ is similar. ■

Claim B.7.32. *If G is abelian then G is a cyclic group.*

Proof. If G is abelian then let b be any element of B distinct from $t_2^{-1} * y^{-|T_2|} * u$, that is, $T_2 b$ is not followed by the uncovered element u. Hence $T_2 b$ is followed by a tile $T_2 b'$ or by a tile $T_1 b'$, that is $t_2 * b' = y^{|T_2|} * t_2 * b$ or $t_1 * b' = y^{|T_2|} * t_2 * b$. Thus $b' = y^{|T_2|} * b$ or $b' = y^{|T_2|} * t_1^{-1} * t_2 * b$. If $b' = y^{|T_2|} * b$ then $t_1 * b' = t_1 * y^{|T_2|} * b$. Since $|T_2| < |T_1|$, $y^{|T_2|} * t_1$ is an element of T_1. Thus $y^{|T_2|} * t_1$ is an element of A and we have a contradiction. Therefore $b' = y^{|T_2|} * t_1^{-1} * b * t_2$. Let $y' := y^{|T_2|} * t_1^{-1} * t_2$. We have seen that for every element b of B except maybe one, $y'b$ is an element of B. Thus $\mathrm{INT}(B) = \alpha - 1$. Since G is abelian, (B, A) is obviously a near-factorization of G. Hence by Lemma B.7.24, G must be cyclic. ■

Claim B.7.33. *If G is not abelian then n is a multiple of 4 and $y^{\frac{n}{4}}$ is the only involution of G.*

Proof. By assumption, G is not abelian.

Let q be an element of G such that $Hq \neq H$.

If n is not a multiple of 4 then $|H|$ is odd. Hence due to Fact B.7.31 there exists at least one element z in Hq such that $z^2 = e$. Since $\langle z \rangle$ is a normal subgroup of G, z must commute with every element of G and in particular with y. Since z is an element of Hq, there exists an integer i such that $z = y^i * q$. From $z * y = y * z$, we get $y^i * q * y = y^{i+1} * q$. Thus $q * y = y * q$. Due to Fact B.7.30, G must be abelian, which is impossible. Thus n is a multiple of 4 and so $y^{\frac{n}{4}}$ is an involution of G.

Obviously in the coset H there are exactly two involutions: the elements e and $y^{\frac{n}{4}}$. Thus if there is another involution in G then there must be an involution z in Hq, and we have seen that in this case G must be abelian, which is impossible. Hence we are done.

■

■

Corollary B.7.34. *If (A, B) is a near-factorization of a finite abelian group G such that $|A| \leq 4$ then G is cyclic [5] and $G(A, B)$ is a CGPW graph.*

Proof. Let (A, B) be a near-factorization of G such that $|A| \leq 4$. Since G is abelian, we use the additive notation $+$ to denote the operation of G.

If $|A| \leq 3$ then obviously $\mathrm{INT}(A) \geq \omega - 2$. Thus G is cyclic by Lemma B.7.28 and Corollary B.7.25. Then it is proved in [1] that $G(A, B)$ must be a CGPWgraph.

If $|A| = 4$ then n is odd and there is no involution in G. By Lemma B.7.5, there exist x and y in G such that $(x+A, B+y)$ is a symmetric near-factorization. Let $A' := x+A$. Since $A' = -A'$ and there is no involution, there are a and a' in G such that $A' = \{a, a', -a, -a'\}$. Then $\{a, a'\} \subseteq A' \cap A' + (a+a')$. Hence $\mathrm{INT}(A') \geq \omega - 2$. By Lemma B.7.28 and Corollary B.7.25, G must be the cyclic group. Thus $G(A, B) \sim G(A', B')$ is a CGPWgraph [1].

■

Example B.7.35. *The Quaternion group Q_8 of order 8 is an example of a non-abelian finite group such that all its cyclic subgroups are normal.*

There does not seem to be many non-abelian groups such that all their cyclic subgroups are normal. According to GAP, there is only one (out of 267) such group of order 64: the 262^{th} group. As it has no element of order 32, we know that is has no near-factorization (A, B) such that $|A| = 7$ and $\mathrm{INT}(A) \geq 5$. There is also only one (out of 231) such group of order 96: the 222^{th} group. This group does not have any element of order 48.

In the remaining of this section, we study the problem of characterizing the minimal imperfect graphs in the class of the graphs produced by near-factorizations of finite groups. We first need to recall some results about minimal imperfect graphs.

A *small transversal* is a subset of vertices T such that T is of cardinality at most $\omega + \alpha - 1$ and T meets every maximum clique and every maximum stable set.

In 1976, V. Chvátal found a very useful property of minimal imperfect graphs which states that a minimal imperfect graph contains no small transversal [6].

In 1998, G. Bacsó, E. Boros, V. Gurvich, F. Maffray and M. Preissmann [1] introduced a sufficient condition for partitionable graphs to have a small transversal called the 'Parents Lemma'. A maximum clique K of G is a *mother* of a vertex $x \in K$ if every maximum clique K' containing x satisfies $|K \cap K'| \geq 2$. Similarly, a maximum stable set S of G is a *father* of a vertex $x \in S$ if every maximum stable set S' containing x satisfies $|S \cap S'| \geq 2$.

Lemma B.7.36. 'The Parents Lemma' *[1] If a vertex of a partitionable graph has a father and a mother then the graph has a small transversal.*

Then we have the following result:

Lemma B.7.37. *Let G be a finite group of even order such that every involution y commutes with every element of G. If (A, B) is any symmetric near-factorization of G then $G(A, B)$ has a small transversal, hence is not minimal imperfect.*

Proof. Since n is even, ω and α are necessarily odd.

As ω is odd, there is an element y of A such that $y^2 = e$. We are going to show that A is a mother of y. Let pA be any ω-clique containing y distinct from A. Hence there is a in A such that $y = p * a$. If $a^{-1} = y$ then $p = y * a^{-1} = y^2 = e$ and so $pA = A$, a contradiction. Thus a^{-1} is not equal to y. We have $a^{-1} = y * p = p * y$ because y commutes with p. Thus a^{-1} is an element of $p * A$. Hence $\{a^{-1},\ y\} \subset A \cap pA$. This means that A is a mother of y.

Likewise there exists an element x of B such that $x^2 = e$ and $B = B^{-1}$ is a father of x. Hence $yx^{-1}B = yx^{-1}B^{-1}$ is a father of y. By applying the Parents Lemma, we see that the graph $G(A, B)$ has a small transversal.

■

Corollary B.7.38. *Let G be a finite abelian group of even order. If (A, B) is any near-factorization of G then $G(A, B)$ is not minimal imperfect.*

Near-factorizations of the dihedral groups

In this section, we show how to carry any near-factorization of a cyclic group of even order to the dihedral group of the same order.

We begin by introducing a map ϕ from $\mathbb{Z}_{2n}$ into $\mathbb{D}_{2n}$.

An *even* element of $\mathbb{Z}_{2n}$ is an element of $2\mathbb{Z}_{2n}$. The *odd* elements are the other elements of $\mathbb{Z}_{2n}$. Notice that if x is an even element of $\mathbb{Z}_{2n}$, then there exists a unique integer y between 0 and $(n-1)$ such that $x = 2 \times y$. We denote by $\frac{x}{2}$ this integer.

If x and y are two even elements of $\mathbb{Z}_{2n}$ then we have $\frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} \pmod{n}$ and if x is any element of $\mathbb{Z}_{2n}$ then we have $\frac{2x}{2} = x \pmod{n}$.

Let ϕ be the bijective map of $\mathbb{Z}_{2n}$ onto $\mathbb{D}_{2n}$ defined by:

$$\begin{array}{cccc} \phi: & \mathbb{Z}_{2n} & \to & D_{2n} \\ & x \text{ is even} & \mapsto & r^{\frac{x}{2}} \\ & x \text{ is odd} & \mapsto & sr^{\frac{x-1}{2}} \end{array}$$

We now state some properties of ϕ which are useful for the proofs:

Lemma B.7.39. *For every x and y of $\mathbb{Z}_{2n}$, we have*

- *if y is even, $\phi(x) * \phi(y)^{-1} = \phi(x-y)$ and $\phi(x+y) = \phi(x) * \phi(y)$.*
- *if y is odd, $\phi(x) * \phi(y)^{-1} = \phi(y-x)$.*

Proof. If x and y are even then we have $\phi(x+y) = r^{\frac{x+y}{2}} = r^{\frac{x}{2}+\frac{y}{2}} = r^{\frac{x}{2}} * r^{\frac{y}{2}} = \phi(x) * \phi(y)$ and $\phi(x-y) = r^{\frac{x-y}{2}} = r^{\frac{x}{2}+\frac{-y}{2}} = r^{\frac{x}{2}} * r^{\frac{-y}{2}} = \phi(x) * \phi(y)^{-1}$.

If x is odd and y is even then we have $\phi(x+y) = sr^{\frac{x+y-1}{2}} = sr^{\frac{x-1}{2}+\frac{y}{2}} = sr^{\frac{x-1}{2}} * r^{\frac{y}{2}} = \phi(x) * \phi(y)$ and $\phi(x-y) = sr^{\frac{x-y-1}{2}} = sr^{\frac{x-1}{2}-\frac{y}{2}} = sr^{\frac{x-1}{2}} * r^{\frac{-y}{2}} = \phi(x) * \phi(y)^{-1}$.

Hence, if y is even then we have $\phi(x+y) = \phi(x) * \phi(y)$ and $\phi(x) * \phi(y)^{-1} = \phi(x-y)$.

If x is even and y is odd then we have $\phi(x) * \phi(y)^{-1} = r^{\frac{x}{2}} * (sr^{\frac{y-1}{2}})^{-1} = sr^{\frac{y-x-1}{2}} = \phi(y-x)$.

If x is odd and y is odd then we have $\phi(x) * \phi(y)^{-1} = sr^{\frac{x-1}{2}} * (sr^{\frac{y-1}{2}})^{-1} = r^{\frac{y-x}{2}} = \phi(y-x)$.

Hence, if y is odd then we have $\phi(x) * \phi(y)^{-1} = \phi(y-x)$. ∎

From a near-factorization (A, B) of $\mathbb{Z}_{2n}$, we get a near-factorization of $\mathbb{D}_{2n}$ this way:

Input: a near-factorization (A, B) of $\mathbb{Z}_{2n}$
Output: a near-factorization (A', B') of $\mathbb{D}_{2n}$
 Step 1: find an element x of $\mathbb{Z}_{2n}$ such that $A + x$ is symmetric and let $A_1 := A + x$ (exists by Lemma B.7.5).
 Step 2: take an element a_1 of A_1 and let $A_2 := A_1 + a_1$.
 Step 3: let B_0 be the set of the even elements of B and B_1 be the set of the odd elements of B. Then take $A' := \phi(A_2)$ and $B' := \phi(B_0) \cup \phi(B_1)r^{a_1}$.

Algorithme B.1: Carrying a near-factorization of $\mathbb{Z}_{2n}$ into $\mathbb{D}_{2n}$

We say that (A', B') is a *dihedral near-factorization associated to* (A, B). We call *De Bruijn dihedral near-factorization* any dihedral near-factorizations associated to a De Bruijn near-factorization.

Obviously one may get several distinct near-factorizations of $\mathbb{D}_{2n}$ through this algorithm from one near-factorization of $\mathbb{Z}_{2n}$ as x is not uniquely defined in **Step** 1 and neither is a_1 in **Step** 2.

We first prove that any couple (A', B') produced by this algorithm is indeed a near-factorization of $\mathbb{D}_{2n}$.

Theorem B.7.40. *Let (A, B) be a near-factorization of $\mathbb{Z}_{2n}$. Let (A', B') be an output of algorithm B.1 with input (A, B). Then (A', B') is a near-factorization of $\mathbb{D}_{2n}$.*

Proof. Recall that due to the algorithm, we have $A' = \phi(A_2)$ and $A_2 = A_1 + a_1$ where A_1 is symmetric and a_1 is an element of A_1.

Claim B.7.41. *For every b of B, there exists b' in B' such that $\phi(A_2 + b) = A'b'$.*

Proof. If b is even then let a be any element of A_2. By Lemma B.7.39, we have $\phi(a + b) = \phi(a) * \phi(b)$. Hence $\phi(A_2 + b) \subseteq \phi(A_2) * \phi(b)$. Since ϕ is a bijective map, we get $\phi(A_2 + b) = \phi(A_2) * \phi(b)$ with $\phi(b) \in B'$. Thus we are done.

If b is odd then let a be any element of A_2. By definition of A_2, $a - a_1$ is an element of A_1, which is a symmetric set. Hence $a_1 - a$ is an element of A_1. Thus $2a_1 - a$ is an element of A_2. Notice that $2a_1 + b$ is odd. Let $b' := \phi(2a_1 + b)$. As $\phi(2a_1 + b) = sr^{a_1 + \frac{b-1}{2}} = sr^{\frac{b-1}{2}} * r^{a_1}$, b' is an element of B'. If a is even then $\phi(2a_1 - a) * b' = r^{a_1 - \frac{a}{2}} * sr^{a_1 + \frac{b-1}{2}} = sr^{\frac{a+b-1}{2}} = \phi(a + b)$. Hence $\phi(a + b) \in A'b'$. If a is odd then $\phi(2a_1 - a) * b' = sr^{\frac{2a_1 - a - 1}{2}} * sr^{\frac{2a_1 + b - 1}{2}} = r^{\frac{a+b}{2}} = \phi(a + b)$. Thus $\phi(a + b) \in A'b'$. Therefore we have $\phi(A_2 + b) \subseteq A'b'$. This implies that $\phi(A_2 + b) = A'b'$ because ϕ is a bijective map.

∎

Claim B.7.42. *The couple (A', B') is a near-factorization of $\mathbb{D}_{2n}$.*

Proof. We have seen that $\{\phi(A_2 + b),\ b \in B\} \subseteq \{A'b',\ b' \in B'\}$. Since ϕ is a bijective map, there exists u in $\mathbb{D}_{2n}$ such that $\{\phi(A_2 + b),\ b \in B\}$ is a partition of $\mathbb{D}_{2n} \setminus \{u\}$. As B and B' are of equal cardinality, we get that $\{A'b',\ b' \in B'\}$ is a partition of $\mathbb{D}_{2n} \setminus \{u\}$. Therefore (A', B') is a near-factorization of $\mathbb{D}_{2n}$.

∎

∎

Example B.7.43.

$$\begin{aligned} A_2 &= \{0,1,2,9,10,11,18,19,20\} \\ B &= \{0,3,6,27,30,33,54,57,60\} \\ A' &= \{e,s,r,sr^4,r^5,sr^5,r^9,sr^9,r^{10}\} \\ B' &= \{e,r^3,sr^{11},r^{15},sr^{23},sr^{26},r^{27},r^{30},sr^{38}\} \end{aligned}$$

The couple (A', B') is a near-factorization of $\mathbb{D}_{82}$ induced by the near-factorization (A_2, B) of $\mathbb{Z}_{82}$

We now prove that the graph $G(A', B')$ is not altered by the choice of x in **S**tep 2 or by the choice of a_1 in **S**tep 3.

Lemma B.7.44. *Let (A, B) be a near-factorization of $\mathbb{Z}_{2n}$. Let $(A'\ ,B')$ and $(A'',\ B'')$ be two dihedral near-factorizations associated to $(A,\ B)$. Then the graph $G(A',\ B')$ is isomorphic to the graph $G(A'',\ B'')$.*

Proof. By construction, there exist two elements x and y of $\mathbb{Z}_{2n}$ such that $A' = \phi(A+x)$ and $A'' = \phi(A+y)$.

We have

$$\begin{aligned} A' &= \phi(A+x) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A+x\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A+x\} \end{aligned}$$

and

$$\begin{aligned} A'' &= \phi(A+y) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A+y\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A+y\} \end{aligned}$$

If $y-x$ is even then by taking the unique integer j between 0 and $n-1$ such that $2j = 2i+x-y \pmod{2n}$, we get

$$\begin{aligned} A'' &= \left\{r^{j+\frac{y-x}{2}} \mid 0 \le j \le n-1,\ 2j \pmod{2n} \in A+x\right\} \\ &\quad \bigcup \left\{sr^{j+\frac{y-x}{2}} \mid 0 \le j \le n-1,\ 2j+1 \pmod{2n} \in A+x\right\} \end{aligned}$$

Hence, $A'' = A'r^{\frac{y-x}{2}}$. Thus we have $A''^{-1}A'' = r^{-\frac{y-x}{2}}A'^{-1}A'r^{\frac{y-x}{2}}$. This means that the connecting set $(A''^{-1}A'') \setminus \{e\}$ is the image of $(A'^{-1}A') \setminus \{e\}$ under the inner automorphism $g \mapsto r^{-\frac{y-x}{2}}gr^{\frac{y-x}{2}}$. Then Lemma B.7.6 implies that the Cayley graph $G(A'', B'')$ is isomorphic to the Cayley graph $G(A', B')$.

The case $y-x$ is odd is slightly trickier.

Let k be an element of $\mathbb{Z}_{2n}$ such that $A+k$ is symmetric. Let $A_{\text{sym}} := A+k$. We have $A' = \phi(A_{\text{sym}}+(x-k))$ and $A'' = \phi(A_{\text{sym}}+(y-k))$. Thus

$$\begin{aligned} A' &= \phi(A_{\text{sym}}+(x-k)) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}}+(x-k)\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A_{\text{sym}}+(x-k)\} \end{aligned}$$

and

$$\begin{aligned} A'' &= \phi(A_{\text{sym}}+(y-k)) \\ &= \{r^i \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}}+(y-k)\} \\ &\quad \cup \{sr^i \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A_{\text{sym}}+(y-k)\} \end{aligned}$$

For every integer p between 0 and $n-1$, we have:

$$\begin{aligned}
A'sr^p &= \{sr^{p-i} \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}} + (x-k)\} \\
&\quad \bigcup \{r^{p-i} \mid 0 \le i \le n-1,\ 2i+1 \pmod{2n} \in A_{\text{sym}} + (x-k)\} \\
&= \{sr^{p+i} \mid 0 \le i \le n-1,\ 2i \pmod{2n} \in A_{\text{sym}} + (k-x)\} \\
&\quad \bigcup \{r^{p+i} \mid 0 \le i \le n-1,\ 2i-1 \pmod{2n} \in A_{\text{sym}} + (k-x)\} \\
&= \{sr^{p+i} \mid 0 \le i \le n-1, \\
&\quad 2i + x - 2k + y \pmod{2n} \in A_{\text{sym}} + (y-k)\} \\
&\quad \bigcup \{r^{p+i} \mid 0 \le i \le n-1, \\
&\quad 2i - 1 + x - 2k + y \pmod{2n} \in A_{\text{sym}} + (y-k)\}
\end{aligned}$$

Thus by taking $p = -k + \frac{(y+x)-1}{2} \pmod{n}$, we have $A'sr^p = A''$. Hence $A''^{-1}A'' = sr^pA'^{-1}A'sr^p$. Therefore the connecting set $(A''^{-1}A'') \setminus \{e\}$ is the image of $(A'^{-1}A') \setminus \{e\}$ under the inner automorphism $g \mapsto sr^pgsr^p$. This implies that the Cayley graph $G(A'', B'')$ is isomorphic to the Cayley graph $G(A', B')$.

■

Thus from a near-factorization (A, B) of $\mathbb{Z}_{2n}$, we get a unique partitionable graph $G(A', B')$ where (A', B') is any dihedral near-factorization associated to (A, B). It remains to know if we may get some 'new' partitionable graphs this way. We have not succeeded in proving that in general the graph $G(A', B')$ is isomorphic to $G(A, B)$ when (A, B) is any near-factorization of the cyclic group.

Nevertheless, in Theorem B.7.45 we prove that this is true for all the graphs $G(A, B)$ on cyclic groups known so far.

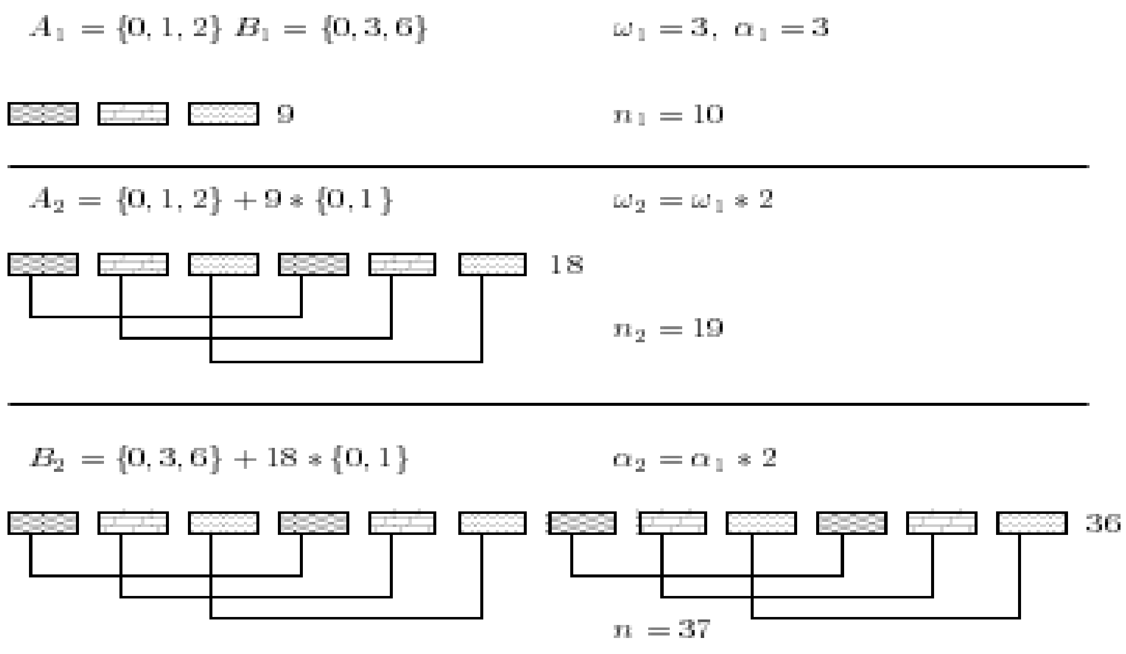


Figure B.21: The De Bruijn near-factorization given by $a_1 = 3$, $a_2 = 3$, $a_3 = 2$, $a_4 = 1$, $a_5 = 1$ and $a_6 = 2$

Theorem B.7.45. *If (A, B) is a De Bruijn near-factorization of $\mathbb{Z}_{2n}$ then the graph $G(A, B)$ is isomorphic to the graph $G(A', B')$ where (A', B') is a dihedral near-factorization associated to (A, B).*

Proof. We first calculate a dihedral near-factorization (A', B') associated to (A, B). Notice that due to Lemma B.7.44, we may proceed without having to fear any loss of generality.

Let $k_1, \ldots, k_{2p}$ be the parameters of the graph $G(A, B)$, that is $G(A, B) = C[k_1, \ldots, k_{2p}]$. As $2n$ is even, $|A|$ and $|B|$ must be odd. This implies that the $2p$ parameters k_i are all odd. Thus for every j between 1 and p, $n_j = k_1 * k_2 * k_3 * \cdots * k_{2j} + 1$ is even. We set $n_0 := 2$ in order to avoid a special case in the proof.

Let $a^+ := (k_1 - 1) + \sum_{j=1}^{p-1} \left(\Pi_{i=1}^{2j} k_i\right)(k_{2j+1} - 1)$. Notice that a^+ is the greatest element of A seen as a set of integers and that it is an even element of A such that $A - \frac{a^+}{2}$ is symmetric. Thus in **Step** 1, we may take $x = -\frac{a^+}{2}$.

Since $-x$ is an element of $A - \frac{a^+}{2}$, we may take $A_2 := A$ in **Step** 2. Hence by taking $A' := \phi(A)$ and B' as defined in **Step** 3, we get a dihedral near-factorization associated to (A, B).

Claim B.7.46. *We have $A' * {A'}^{-1} = \phi(A - A)$.*

Proof. We have to prove that $\phi(A) * \phi(A)^{-1} = \phi(A - A)$.

We first prove the inclusion $\phi(A) * \phi(A)^{-1} \subseteq \phi(A - A)$. Let w be any element of $\phi(A) * \phi(A)^{-1}$: there exist a and a' in A such that $w = \phi(a) * \phi(a')^{-1}$. Hence by Lemma B.7.39, we have $w = \phi(a - a')$ or $\phi(a' - a)$. In both cases, w is an element of $\phi(A - A)$. Thus $\phi(A) * \phi(A)^{-1} \subseteq \phi(A - A)$.

We now prove the converse inclusion. Let w be any element of $\phi(A - A)$; there exist a and a' in A such that $w = \phi(a' - a)$.

If a' is even then $w = \phi(a) * \phi(a')^{-1}$ hence it is an element of $\phi(A) * \phi(A)^{-1}$.

If a' is odd, then due to the definition of A, there exist integers $\delta_0,\ \delta_1,\ \ldots, \delta_{p-1}$ and $\delta'_0,\ \delta'_1,\ \ldots, \delta'_{p-1}$ such that $a = \delta_0 + (n_1 - 1)\delta_1 + \ldots + (n_p - 1)\delta_{p-1}$ and $a' = \delta'_0 + (n_1 - 1)\delta'_1 + \ \ldots + (n_p - 1)\delta'_{p-1}$ with $0 \leq \delta_i,\ \delta'_i \leq (k_{2i+1} - 1)$ for every i between 0 and $p - 1$. Since a' is odd, there must be an integer j between 0 and $p - 1$ such that $0 < \delta'_j < (k_{2j+1} - 1)$ because all the $k_{2i+1} - 1$ are even. Thus $k_{2j+1} > 1$.

If $\delta_j = 0$ then $a + (n_j - 1)$ is an element of A and $a' + (n_j - 1)$ is an element of A. Then $w = \phi(a - a') = \phi((a + n_j - 1) - (a' + n_j - 1)) = \phi(a + n_j - 1) * \phi(a' + n_j - 1)^{-1}$ because $a' + n_j - 1$ is even as $n_j = a_1 * a_2 * a_3 * \ldots * a_{2j} + 1$ is even. Therefore w is an element of $\phi(A) * \phi(A)^{-1}$.

If $\delta_j > 0$ then $a - (n_j - 1)$ is an element of A and $a' - (n_j - 1)$ is an element of A. Then $w = \phi(a - a') = \phi((a - n_j + 1) - (a' - n_j + 1)) = \phi(a - n_j + 1) * \phi(a' - n_j + 1)^{-1}$ because $a' - n_j + 1$ is even. Hence w is an element of $\phi(A) * \phi(A)^{-1}$.

Thus $\phi(A - A) \subseteq \phi(A) * \phi(A)^{-1}$.

Therefore $\phi(A - A) = \phi(A) * \phi(A)^{-1}$.

■

Claim B.7.47. *Let Γ be the graph with vertex set $\mathbb{D}_{2n}$ and with edge set $\{\{x, y\},\ x * y^{-1} \in (A' * {A'}^{-1}) \setminus \{e\}\}$. Then $G(A, B)$ is isomorphic to Γ.*

Proof. Let $\{i, j\}$ be any edge of $G(A, B)$. Then $i - j \in (A - A) \setminus \{0\}$. Thus $j - i \in (A - A) \setminus \{0\}$. Hence $\phi(i - j) \in \phi((A - A) \setminus \{0\})$ and $\phi(j - i) \in \phi((A - A) \setminus \{0\})$. Thus $\phi(i)\phi(j)^{-1} \in \phi((A - A) \setminus \{0\})$. So $\phi(i)\phi(j)^{-1} \in (\phi(A)\phi(A)^{-1}) \setminus \{e\}$. Therefore $\{\phi(i), \phi(j)\}$ is an edge of Γ.

Let $\{\phi(i), \phi(j)\}$ be any edge of Γ. Then $\phi(i)\phi(j)^{-1} \in (\phi(A)\phi(A)^{-1}) \setminus \{e\}$. Since $\phi(i)\phi(j)^{-1}$ is equal to $\phi(i - j)$ or $\phi(j - i)$, we get $\phi(i - j) \in \phi((A - A) \setminus \{0\})$ or $\phi(j - i) \in \phi((A - A) \setminus \{0\})$, by Fact B.7.46. Hence $i - j \in (A - A) \setminus \{0\}$, that is $\{i, j\}$ is an edge of $G(A, B)$.

■

Claim B.7.48. *There exists an element g such that gA' is a symmetric subset of $\mathbb{D}_{2n}$*

Proof. Let k be an element of $\mathbb{Z}_{2n}$ such that $A + k$ is a symmetric subset of $\mathbb{Z}_{2n}$.

Let A_0 be the set of the even elements of A and let A_1 be the set of the odd elements of A. Let H be the subgroup of $\mathbb{D}_{2n}$ generated by r.

If k is even then $r^{\frac{k}{2}}A' = r^{\frac{k}{2}}\phi(A) = r^{\frac{k}{2}}\phi(A_0) \cup r^{\frac{k}{2}}\phi(A_1) = \phi(A_0 + k) \cup r^{\frac{k}{2}}\phi(A_1)$. Then $r^{\frac{k}{2}}\phi(A_1)$ is a subset of sH, thus it is a symmetric subset of $\mathbb{D}_{2n}$ as every of its elements is an involution. The set $\phi(A_0 + k)$ is a symmetric subset of $\mathbb{D}_{2n}$ because $A_0 + k$ is a symmetric subset of $\mathbb{Z}_{2n}$. Hence $r^{\frac{k}{2}}A'$ is symmetric.

If k is odd then $sr^{-\frac{k+1}{2}}A' = sr^{-\frac{k+1}{2}}\phi(A_0) \cup sr^{-\frac{k+1}{2}}\phi(A_1)$. The set $sr^{-\frac{k+1}{2}}\phi(A_0)$ is a symmetric subset of $\mathbb{D}_{2n}$ as it is a subset of sH. We have $\phi(A + k) = sr^{\frac{k-1}{2}}\phi(A_0) \cup sr^{-\frac{k+1}{2}}\phi(A_1)$, hence $sr^{-\frac{k+1}{2}}\phi(A_1) = H \cap \phi(A+k) = \phi(A_1 + k)$. Since $A_1 + k$ is a symmetric subset of $2\mathbb{Z}_n$, this implies that $\phi(A_1 + k)$ is symmetric, thus $sr^{-\frac{k+1}{2}}\phi(A_1)$ is symmetric. Therefore $sr^{-\frac{k+1}{2}}A'$ is symmetric.

■

Claim B.7.49. *The graph $G(A', B')$ is isomorphic to the graph $G(A, B)$.*

Proof. All we have to show is that $G(A', B')$ is isomorphic to Γ.

Let g be an element of $\mathbb{D}_{2n}$ such that gA' is symmetric and let $A'' := gA'$.

Obviously, $G(A', B')$ is isomorphic to $G(A'', B')$. Let Γ' be the graph with vertex set $\mathbb{D}_{2n}$ and with edge set $\{\{x, y\},\ x * y^{-1} \in (A'' * A''^{-1}) \setminus \{e\}\}$.

Let `inv` be the bijective map of $\mathbb{D}_{2n}$ onto itself which maps an element onto its inverse. $\{x, y\}$ is an edge of $G(A'', B')$ if and only if $x^{-1} * y \in (A''^{-1} * A'') \setminus \{e\}$, that is if and only if $\mathrm{inv}(x)\mathrm{inv}(y)^{-1} \in (A'' * A''^{-1}) \setminus \{e\}$ as $A'' = A''^{-1}$, hence if and only if $\{\mathrm{inv}(x),\ \mathrm{inv}(y)\}$ is an edge of Γ'. Hence $G(A'', B')$ is isomorphic to Γ'.

Let h denote the inner automorphism of $\mathbb{D}_{2n}$ which maps an element x onto $g^{-1}xg$. Then $\{x, y\}$ is an edge of Γ' if and only if $\{h(x), h(y)\}$ is an edge of Γ. Thus Γ' is isomorphic to Γ.

Therefore $G(A', B')$ is isomorphic to Γ.

■

■

In 1990, D. De Caen, D.A. Gregory, I.G. Hughes and D.L. Kreher [5] described a class of near-factorizations of the dihedral groups: if ω is any divisor of $2n - 1$, then let $\alpha := \frac{2n-1}{\omega}$ and define

$$A := \left\{r^i,\ 1 \le i \le \frac{\omega - 1}{2}\right\} \cup \left\{sr^i,\ 0 \le i \le \frac{\omega - 1}{2}\right\}$$

$$B := \left\{r^{i\omega},\ 0 \le i \le \frac{\alpha - 1}{2}\right\} \cup \left\{sr^{i\omega},\ 1 \le i \le \frac{\alpha - 1}{2}\right\}$$

The graphs associated to these near-factorizations are a strict subset of the CGPWgraphs of even order:

Lemma B.7.50. *The graphs $G(A, B)$ produced by this method are webs.*

Proof. We have $A = \left\{s, r, sr, r^2, sr^2, \ldots, r^{\frac{\omega-1}{2}}, sr^{\frac{\omega-1}{2}}\right\}$.

Consider the De Bruijn near-factorization of $\mathbb{Z}_{2n}$ given by $A_0 := \{0, 1, \ldots, \omega - 1\}$ and by $B_0 := \omega * \{0, \ldots, \alpha - 1\}$. Let $A' := \phi(A_0)$. We know that there exists B' such that (A', B') is a near-factorization of $\mathbb{D}_{2n}$ with $G(A', B')$ isomorphic to $G(A_0, B_0)$. We have $A' = \left\{e, s, r, \ldots, r^{\frac{\omega-1}{2}}\right\}$. Thus $A' = Asr^{\frac{\omega-1}{2}}$. Hence $A'^{-1}A' = sr^{\frac{\omega-1}{2}}A^{-1}Asr^{\frac{\omega-1}{2}}$. This means that the connection set of $G(A, B)$ is the image under an inner automorphism of $\mathbb{D}_{2n}$ of the connection set of $G(A', B')$. Thus $G(A, B)$ is isomorphic to $G(A', B')$. As $G(A', B')$ is isomorphic to $G(A_0, B_0)$ which is a web, we are done.

■

Some open questions

This paper gives rise to several questions. We first recall the circular partitionable graph conjecture:

Conjecture B.7.51. *If (A, B) is a near-factorization of the cyclic group $\mathbb{Z}_n$ then there exists a De Bruijn near-factorization (A', B') such that $G(A, B)$ is isomorphic to $G(A', B')$.*

Grinstead has verified by computer this conjecture for groups of order less than 50, and Bacsó, Boros, Gurvich, Maffray and Preissmann have proved it when A is of cardinality at most 5.

We do not know any near-factorization (A, B) of the dihedral groups whose associated graph $G(A, B)$ is not a CGPWgraph. Thus we ask this question, which may be seen as the circular partitionable graph conjecture in dihedral groups:

Problem B.7.52. *If (A, B) is a near-factorization of the dihedral group $\mathbf{D}_{2n}$, is $G(A, B)$ always isomorphic to a graph $G(A', B')$ with (A', B') a De Bruijn dihedral near-factorization ?*

We believe that this is not true because in a dihedral group, a tile may be used 'backwards', which is not possible in the cyclic group. Hence a tiling of $\mathbb{D}_{2n} \setminus \{u\}$ does not behave in the same way than a tiling of $\mathbb{Z}_{2n} \setminus \{u\}$, whereas a positive answer to Problem B.7.52 would suggest the opposite.

With the help of Theorem B.7.17, an exhaustive search by computer revealed that the only groups of order strictly less than 64 having a symmetric near-factorization are the cyclic groups and the dihedral groups. Hence this leads to this natural question:

Problem B.7.53. *Are the cyclic groups and the dihedral groups the only groups having symmetric near-factorizations ?*

Recently, Boros, Gurvich and Hougardy [4] introduced a construction of partitionable graphs generalizing the first construction of Chvátal, Graham, Perold and Whitesides. Let us call BGH-graphs the partitionable graphs produced by this new method. All the BGH-graphs contain a critical ω-clique, that is an ω-clique Q such that the critical edges of Q induce a tree covering all vertices of Q.

Our computer experiments revealed that the group $\mathbb{D}_{10} \times \mathbb{Z}_5$ has a near-factorization (A, B) below, such that the graph $G(A, B)$ does not have any critical ω-clique. We denote this graph by Γ_{50}.

$$\begin{aligned} A &= \{(e,0),(s,0),(e,3),(s,3),(r,4),(sr,4),(r^2,4)\} \\ B &= \{(s,1),(r,1),(sr^2,1),(sr^3,3),(r^4,3),(sr^3,4),(r^4,4)\} \end{aligned}$$

Lemma B.7.54. *The graph Γ_{50} does not have any critical edge, whereas the critical edges of $\overline{\Gamma_{50}}$ form a perfect matching of $\overline{\Gamma_{50}}$.*

Proof. If Γ_{50} has a critical edge then there exists an element y such that $|B^{-1} \cap yB^{-1}| = 6$. Let H be the cyclic subgroup generated by y. By Theorem B.7.17 applied to the near-factorization (B^{-1}, A^{-1}), we have $|H| = 2$, thus y must be an involution.

The set of involutions is $\{(s,0),(sr,0),(sr^2,0),(sr^3,0),(sr^4,0)\}$. A quick computation shows that y can not be any of these 5 values, thus we have a contradiction: Γ_{50} does not have any critical edge.

$\{i, j\}$ is a critical edge of $\overline{\Gamma_{50}}$ if and only if there exist k and k' such that $\{i\} = kA \setminus k'A$ and $\{j\} = k'A \setminus kA$. Thus $|A \cap k^{-1}k'A| = 6$ and by Theorem B.7.17 we get that $k^{-1}k'$ must be an involution. Then it is clear that $k^{-1}k'$ must be equal to $(s, 0)$. Thus if $\{i, j\}$ is a critical edge then there exists k such that $\{i\} = kA \setminus k(s,0)A$ and $\{j\} = k(s,0)A \setminus kA$, that is $i = k(r^2, 4)$ and $j = k(sr^2, 4)$. This implies that $j = i(sr^4, 0)$.

Hence any critical edge of $\overline{\Gamma_{50}}$ is a left coset of the subgroup H' generated by the involution $(sr^4, 0)$. As any left coset of H' form a critical edge of $\overline{\Gamma_{50}}$, we have that the critical edges of $\overline{\Gamma_{50}}$ form the perfect matching of $\overline{\Gamma_{50}}$ given by the left cosets of H'.

■

Thus this graph, as well as its complement, does not have any critical ω-clique. Therefore it is not a BGH-graph, and neither is it a CGPW-graph. Hence near-factorizations of finite groups do produce 'new' partitionable graphs.

Problem B.7.55. *Is it possible to describe a class of near-factorizations of a sequence of finite groups, whose associated graphs are 'new' partitionable graphs?*

References

[1] G. Bacsó, E. Boros, V. Gurvich, F. Maffray, and M. Preissmann. On minimally imperfect graphs with circular symmetry. *J. of Graph Theory*, 29:209–225, 1998.

[2] N. Biggs. *Algebraic Graph Theory*. Syndics of the Cambridge University Press, 1974.

[3] R.G. Bland, H.C. Huang, and L.E. Trotter. Graphical properties related to minimal imperfection. *Discrete Mathematics*, 27:11–22, 1979.

[4] E. Boros, V. Gurvich, and S. Hougardy. Recursive generation of partitionable graphs. *J. of Graph Theory*, 41(4):259–285, 2002.

[5] D. De Caen, D.A. Gregory, I.G. Hughes, and D.L. Kreher. Near-factors of finite groups. *Ars Combinatoria*, 29:53–63, 1990.

[6] V. Chvátal. On the strong perfect graph conjecture. *J. Comb. Theory Series B*, 20:139–141, 1976.

[7] V. Chvátal, R.L. Graham, A.F. Perold, and S.H. Whitesides. Combinatorial designs related to the perfect graph conjecture. *Applied Discrete Mathematics*, 21:197–206, 1984.

[8] N.G. DeBruijn. On number systems. *Nieuw Arch. Wisk.*, 3(4):15–17, 1956.

[9] The GAP Group, Aachen, St Andrews. *GAP – Groups, Algorithms, and Programming, Version 4.1*, 1999. (`http://www-gap.dcs.st-and.ac.uk/~gap`).

[10] C.M. Grinstead. On circular critical graphs. *Discrete Mathematics*, 51:11–24, 1984.

[11] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[12] S. Markossian, G. Gasparian, I. Karapetian, and A. Markosian. On essential components and critical sets of a graph. *Disc. Math.*, 178:137–153, 1998.

[13] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[14] A. Sebő. On critical edges in minimal imperfect graphs. *J. of Comb. Theory*, 67:62–85, 1996.

B.8 Cayley partitionable graphs

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In this paper we investigate the class of Cayley partitionable graphs. This investigation is motivated by the Strong Perfect Graph Conjecture. Cayley partitionable graphs are Cayley Graphs which are closely related to near-factorizations of finite groups.

We prove that near-factorizations satisfy a strong structural property. We used it to speed up exhaustive computations, which revealed a Cayley partitionable graph with fifty vertices, which is not generated by all constructions of partitionable graphs known so far.

Introduction

In 1960, Claude Berge introduced the notion of perfect graphs and conjectured that perfect graphs are exactly the graphs with no induced odd hole and no induced complement of an odd hole, or equivalently that *minimal imperfect graphs* are odd holes and their complements. This well-known open conjecture is called the Strong Perfect Graph Conjecture.

Padberg [10] proved that every minimal imperfect graph is partitionable. Thus a minimal imperfect graph contradicting the Strong Perfect Graph Conjecture would lie in the class of partitionable graphs.

In 1979, Chvátal, Graham, Perold and Whitesides introduced two constructions for making partitionable graphs [7]. Due to the initials of the names of these four authors, we call CGPW_1-graphs the graphs produced by the first method and CGPW_2-graphs the graphs produced by the second one.

In 1984, Grinstead proved that there is no counter-example to the Strong Perfect Graph Conjecture in the class of normalized CGPW_2-graphs [9]. In 1996, Sebö proved that no CGPW_1-graph is a counter-example to the Strong Perfect Graph Conjecture [12].

Boros, Gurvich and Hougardy described last year a recursive generation of partitionable graphs extending the first method of Chvátal, Graham, Perold and Whitesides. We call BGH-graphs the graphs generated by this new construction. It is unknown whether there is a counter-example to the Strong Perfect Graph Conjecture in this wider class.

The primary motivation of this paper was to find out a generalization of the second construction of Chvátal, Graham, Perold and Whitesides. It was noticed that every CGPW_2-graph is a normalized Cayley partitionable graph of a cyclic group [1]. Grinstead conjectured in 1984 that the converse is true, that is, every normalized Cayley partitionable graph is a CGPW_2-graph. This question is still open, though Bacsó, Boros, Gurvich, Maffray and Preissman gave a positive answer for graphs with size of maximum cliques less than 5 in 1998 [1].

One way to extend the class of CGPW_2-graphs is to study Grinstead's conjecture with the hope that there are counter-examples. This appears to be quite difficult...

We took another approach, based on the relaxation of the class of groups considered. The class of Cayley partitionable graphs of finite groups is indeed a natural extension of the class of Cayley partitionable graphs of cyclic groups.

If (A, B) is a near-factorization of a finite group then the Cayley graph $G(A, B)$ with connection set $(A^{-1} * A) \setminus \{e\}$ is a normalized partitionable graph [11]. Conversely, if Γ is any Cayley partitionable graph on a group G, then there exists a near-factorization (A, B) of G such that $G(A, B)$ is the normalized graph of Γ (Lemma B.8.5).

This equivalence motivated this paper: we wanted to produce near-factorizations of some finite groups, so as giving rise to 'new' partitionable graphs. Beyond the class of the cyclic groups, dihedral groups have near-factorizations. In this paper, we give a basic result explaining the relationship between near-factorizations of the cyclic groups and near-factorizations of the dihedral groups, for half of the dihedral groups.

There does not seem to be very much other groups having near-factorizations. In fact in the abelian case, it is known that such a group would be of order at least 92 [5]! Exhaustive computations to find out the near-factorizations of a given group are very expensive. We present a result (Theorem B.8.9) stating a structural property of all near-factorizations. It may be used to speed up computations. In the last section, we present the smallest groups which are not cyclic and not dihedral, admitting near-factorizations. The partitionable graphs associated to the near-factorizations of these groups do not belong to the three constructions abovementioned.

Definitions

Following the paper of Bland, Huang and Trotter [2], a graph Γ is said to be *partitionable* if there exist two integers p and q (with $p \geq 2$ and $q \geq 2$) such that Γ has $pq + 1$ vertices and for every vertex v of G, the induced subgraph $G \setminus \{v\}$ admits a partition into p cliques of cardinality q and also admits a partition into q stable sets of cardinality p.

If Γ is a graph, we denote by $\omega(\Gamma)$ the maximum cardinality of a clique of Γ and $\alpha(\Gamma)$ the maximum cardinality of a stable set of Γ. We denote by $V(\Gamma)$ the vertex set of Γ and $E(\Gamma)$ the edge set of Γ. When any confusion is unlikely on the graph considered, we write simply ω instead of $\omega(\Gamma)$ and α instead of $\alpha(\Gamma)$.

A *determined* edge is an edge $\{i, j\}$ such that there exists a maximum clique containing both i and j. A *normalized* graph is a graph such that every of its edges is a determined edge. The *normalized graph* of a graph Γ is the graph with vertex set $V(\Gamma)$ and edge set the set of determined edges of Γ.

The graph Γ with vertex set V is *isomorphic* to the graph Γ' with vertex set V' if there exists a bijective map f from V onto V' such that $\{i, j\}$ is an edge of Γ if and only if $\{f(i), f(j)\}$ is an edge of Γ'.

If e' is an edge of Γ we denote by $\Gamma - e'$ the subgraph of Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \setminus \{e'\}$. If v is any vertex of Γ, we denote by $\Gamma \setminus \{v\}$ the induced subgraph of Γ with vertex set $V(\Gamma) \setminus \{v\}$ and edge set $\{\{x, y\} \mid \{x, y\} \in V(\Gamma),\ x \neq v,\ y \neq v\}$. An edge e of a graph Γ is said to be an *α-critical edge* if and only if $\alpha(\Gamma - e) > \alpha(\Gamma)$.

It is known that in a partitionable graph Γ, a maximum clique intersects at least $2\omega - 2$ other maximum cliques. Following [3], we call *critical clique* any maximum clique which intersects exactly $2\omega - 2$ other maximum cliques. Sebö [12] proved that a maximum clique Q of a partitionable graph is a critical clique if and only if the critical edges in Q form a spanning tree of Q.

Let Γ be any partitionable graph with a critical clique Q and let T be the tree made of the critical edges of Q. Boros, Gurvich and Hougardy [3] noticed that for every edge e of T, there exist two maximum cliques Q' and Q'' such that $Q \cap Q'$ is equal to one of the two connected components of $T - e$ and that $Q \cap Q''$ is equal to the other connected component of $T - e$. Thus if e is any edge containing a leaf of T, one of the maximum clique Q' and Q'' shares $\omega(\Gamma) - 1$ vertices with Q. Furthermore, there exist obviously two maximum stable sets of Γ sharing $\alpha(\Gamma) - 1$ vertices, as there is at least one critical edge in Γ.

Every BGH-graph has a critical clique [3].

A *small transversal* of a graph Γ is a set of vertices T such that $|T| \leq \alpha(T) + \omega(T) - 1$ and that T meets every maximum clique and every maximum stable set of Γ. Chvátal proved in [6] that no minimal imperfect graph contains a small transversal.

If z is a complex number, we denote its modulus by $|z|$ and by $\overline{z}$ its conjugate. Every matrix in this paper is a matrix with complex entries. If M is any such matrix, we denote by ${}^tM$ the transposed matrix of M, and by $\overline{M}$ the conjugate matrix of M.

A *group* is a non-empty set G with a closed associative binary operation $*$, an identity element e, and an inverse a^{-1} for every element $a \in G$. The number of elements of a finite group G is called the *order* of G. To avoid a conflict of notations, we use the symbol $\times$ to denote the standard multiplication between two integers and also to denote the direct product of two groups.

If X and Y are two subsets of G, we denote by $X * Y$ the set $\{x * y,\ x \in X,\ y \in Y\}$. With a slight abuse of notation, if g is an element of G and X is a subset of G, we denote by gX the set $\{g\} * X$ and Xg the set $X * \{g\}$. Furthermore $|X|$ is the cardinality of X, i.e. the number of elements of X. The subset X is said to be *symmetric* if $X = X^{-1}$, where X^{-1} is the set $\{x^{-1},\ x \in X\}$.

If y is any element of G, we denote by $\langle y \rangle$ the subgroup of G generated by y. The *order* of y is the smallest integer k such that $y^k = e$ and is denoted by $o(y)$. An *involution* of G is an element of G of order 2. The *exponent* of the group G is the smallest positive integer k such that $g^k = e$ for every element g of G.

The *cyclic group* of order n is the group which is generated by an element x of order n. This group is denoted by $\mathbb{Z}_n$.

The *dihedral group* D_{2n} of even order $2 * n$ (with $n \geq 3$) is the non-abelian group generated by two elements r and s such that:

- r is of order n.
- s is of order 2.
- $s * r = r^{-1} * s$

Let H be any subgroup of G. A *right coset* of H is any subset Hx with $x \in G$. The proof of Lagrange's Theorem asserts that for any subgroup H of G, there exists a unique partition of G in right cosets of H. A subgroup H of G is *normal* if for every g of G, we have $gH = Hg$. When H is a normal subgroup, right cosets of H are simply called *cosets* of H.

Let H be a normal subgroup of G. Let Ψ be the set of the cosets of H. If H_1 and H_2 are two elements of Ψ then $H_1 * H_2$ is an element of Ψ. With this operation, Ψ is a group called the *quotient group* of G by H, and is denoted by G/H.

An *homomorphism* from a group G with operation $*_G$ into a group G' with operation $*_{G'}$ is a map ϕ from G into G' such that for all elements x and y of G, we have $\phi(x *_G y) = \phi(x) *_{G'} \phi(y)$. Two groups G and G' are said to be *isomorphic* if there exists a bijective homomorphism from G into G'. An *automorphism* of a group G is a bijective homomorphism from G into itself. With the composition law as operation, the set of automorphisms of a group G is a group denoted by $\mathrm{AUT}(G)$.

Let $(G,*_G)$ and $(G',*_{G'})$ be two groups and let $\Theta :\ h \mapsto \Theta_h$ be an homomorphism from G' into $\mathrm{AUT}(G)$. The *semidirect product* of G by G' determined by Θ is the group, denoted by $(G \times_\Theta G', *_{G\times_\Theta G'})$, whose elements are the elements of the cartesian product $G \times G'$, with operation $*_{G\times_\Theta G'}$ given by:

$$\forall (g,h) \in G \times G',\ \forall (g',h') \in G \times G',\ (g,h) *_{G\times_\Theta G'} (g',h') := (g *_G \Theta_h(g'), h *_{G'} h')$$

Two subsets A and B of cardinality at least 2 of a finite group G of order n form a *near-factorization* of G if and only if $|G| = |A| \times |B| + 1$ and there is an element $u(A,B)$ of G such that $A * B = G \setminus \{u(A,B)\}$. The element $u(A,B)$ is called the *uncovered element* of the near-factorization. The condition about the cardinality of A and B is required to avoid the trivial case $A = G \setminus \{u(A,B)\}$ and $B = \{e\}$. Notice that every element x of G distinct from $u(A,B)$ may be written in a unique way as $x = a * b$ with $a \in A$ and $b \in B$. Hence a near-factorization (A,B) may be seen as a tiling of $G \setminus \{u(A,B)\}$ with proto tile A.

A near-factorization (A,B) is said to be a *symmetric near-factorization* if both A and B are symmetric.

We have the following property:

Lemma B.8.1. *[5] Let G be a finite group and A, B be two subsets of G. Then (A,B) is a near-factorization of G with $u(A,B) = e$ if and only if (B,A) is a near-factorization of G with $u(B,A) = e$.*

Let G be a finite group with operation $*$. Let S be a symmetric subset of G which does not contain the identity element e. The *Cayley graph* with *connection set* S is the simple graph with vertex set G and edge set $\{\{i,j\},\ i^{-1} * j \in S\}$. The graph $G(A,B)$ associated with a near-factorization (A,B) is the Cayley graph with connection set $(A^{-1} * A) \setminus \{e\}$.

Obviously, distinct near-factorizations of a given group may give rise to the same graph. In particular, we may left-shift A and right-shift B without altering the associated graph:

Lemma B.8.2. *Let x and y be two elements of G. Then (xA, By) is a near-factorization of G such that $u(xA,By) = x * u(A,B) * y$ and $G(xA,By)$ is isomorphic to $G(A,B)$.*

Proof. The proof is straightforward.

Thus due to Lemma B.8.2, we may always assume that the uncovered element is e, without altering the associated graph.

In the case of abelian groups, De Caen, Gregory, Hughes and Kreher gave a useful property of near-factorizations:

Lemma B.8.3. *[5] Let G be an abelian group and (A,B) be a near-factorization of G. Then there exist two elements x and y of G such that xA is symmetric and that By is symmetric.*

Next Lemma exhibits the relationship between Cayley partitionable graphs and near-factorizations of groups.

Lemma B.8.4. *If (A,B) is a near-factorization of a finite group G such that $A * B = G \setminus \{e\}$, then the graph $G(A,B)$ is a normalized partitionable graph ([7]+[5], or [11] for a direct proof) with maximum cliques $\{xA,\ x \in G\}$ and maximum stable sets $\{xB^{-1},\ x \in G\}$ ([11]).*

The converse statement is true: we have

Lemma B.8.5. *Let Γ be a Cayley partitionable graph for a group G. Then G has a near-factorization (A, B) such that $G(A, B)$ is the normalized graph of Γ.*

Proof. Let A be any maximum clique of Γ. Since Γ is a Cayley graph, xA is a maximum clique of Γ for every element x of G.

Let x and y be two elements of G such that $xA = yA$. Thus $A = x^{-1}yA$. Let H be the cyclic subgroup of G generated by $x^{-1}y$. Then A is an union of right-cosets of H. Therefore $|G| = |A|\alpha(\Gamma) + 1 = 1 \pmod{|H|}$. Since $|H|$ divides $|G|$, we must have $|H| = 1$. Hence $x = y$. The graph Γ is a partitionable graph, therefore Γ has exactly $|G|$ maximum cliques. Thus for every maximum clique Q of Γ, there exists an element x of G such that $Q = xA$.

Let $A_1, \dots, A_\alpha$ be a partition in maximum cliques of $\Gamma \setminus \{e\}$. Let $b_1, \dots, b_\alpha$ be the elements of G such that $A_1 = b_1A, \dots, A_\alpha = b_\alpha A$. Let $B = \{b_1, \dots, b_\alpha\}$. Then (B, A) is obviously a near-factorization of G, such that $u(B, A) = e$. Hence (A, B) is a near-factorization of G such that $u(A, B) = e$ (Lemma B.8.1). As the maximum cliques of $G(A, B)$ are the maximum cliques of Γ, $G(A, B)$ is the normalized graph of Γ.

Since the cardinality of a maximum clique of $G(A, B)$ is equal to $|A|$, we denote by ω the value of $|A|$. Likewise, we denote by α the value of $|B|$.

A graph $\Gamma = (V, E)$ on $\alpha\omega + 1$ vertices is called a *web*, if the maximum cardinality of a clique of Γ is ω, the maximum cardinality of a stable set of Γ is α, and there is a cyclical order of V so that every set of ω consecutive vertices in this cyclical order is an ω-clique. Equivalently, normalized webs with n vertices are graphs induced by any near-factorization (A, B) of Z_n such that A is an interval.

Let X be any subset of the group G. We set

$$\mathrm{INT}(X) = \max_{x \in G,\ y \in G,\ x \neq y} \{|xX \cap yX|\}$$

Notice that $\mathrm{INT}(A)$ denotes the maximum cardinality of the intersection between two distinct ω-cliques of $G(A, B)$ and that $\mathrm{INT}(B^{-1})$ denotes the maximum cardinality of the intersection between two distinct α-stable sets.

It is easy to check that a graph $G(A, B)$ has a α-critical edge if and only if $\mathrm{INT}(B^{-1}) = \alpha - 1$.

Carrying a symmetric near-factorization of a cyclic group $\mathbb{Z}_{2n}$ to the dihedral group $\mathbb{D}_{2n}$

Let G be any abelian group. Let Θ be the homomorphism from $\mathbb{Z}_2$ into the automorphism group $\mathrm{AUT}(G)$ of G given by

$$\begin{array}{cccc} \Theta: & \mathbb{Z}_2 & \to & \mathrm{AUT}(G) \\ & 0 & \mapsto & (\mathrm{id} : x \mapsto x) \\ & 1 & \mapsto & (\mathrm{inv} : x \mapsto x^{-1}) \end{array}$$

Then we have the following Lemma:

Lemma B.8.6. *Let G be an abelian group. Let X and Y be two symmetric subsets of the direct product of G by $\mathbb{Z}_2$, whose operation is denoted by $*$. Then X and Y are two symmetric subsets of the semidirect product $G \times_\Theta \mathbb{Z}_2$, of G by $\mathbb{Z}_2$ determined by Θ, as defined above. Furthermore we have $X * Y = X *_\Theta Y$.*

Proof. We denote by $*_G$ the operation of G, by $*_{\mathbb{Z}_2}$ the operation of $\mathbb{Z}_2$, by $*$ the operation of $G \times \mathbb{Z}_2$ and by $*_\Theta$ the operation of $G \times_\Theta \mathbb{Z}_2$. If x is an element of G or $\mathbb{Z}_2$, we denote by x^{-1} its inverse with respect to $*_G$ or $*_{\mathbb{Z}_2}$.

We first prove that $X * Y \subseteq X *_\Theta Y$: let (x_1, x_2) be any element of X and let (y_1, y_2) any element of Y. We have to prove that $(x_1, x_2) * (y_1, y_2)$ is an element of $(X *_\Theta Y)$, i.e. $(x_1 *_G y_1, x_2 *_{\mathbb{Z}_2} y_2) \in (X *_\Theta Y)$.

If $x_2 = 0$ then $(x_1 *_G y_1, y_2 *_{\mathbb{Z}_2} y_2) = (x_1 *_G \mathrm{id}(y_1), x_2 *_{\mathbb{Z}_2} y_2) = (x_1, x_2) *_\Theta (y_1, y_2)$, thus $(x_1 *_G y_1, x_2 *_{\mathbb{Z}_2} y_2) \in (X *_\Theta Y)$.

If $x_2 = 1$ then $(x_1 *_G y_1, x_2 *_{\mathbb{Z}_2} y_2) = (x_1 *_G \text{inv}({y_1}^{-1}), x_2 *_{\mathbb{Z}_2} {y_2}^{-1})$ (because ${y_2}^{-1} = y_2$ in the group $\mathbb{Z}_2$), thus $(x_1 *_G y_1, x_2 *_{\mathbb{Z}_2} y_2) = (x_1, x_2) *_\Theta ({y_1}^{-1}, {y_2}^{-1})$. Since Y is symmetric with respect to $*$, $({y_1}^{-1}, {y_2}^{-1})$ is an element of Y. Thus $(x_1 *_G y_1, x_2 *_{\mathbb{Z}_2} y_2) \in (X *_\Theta Y)$.

Therefore we have $X * Y \subseteq X *_\Theta Y$.

We now prove the converse inclusion: let (x_1, x_2) be any element of X and let (y_1, y_2) be any element of Y:

If $x_2 = 0$ then $(x_1, x_2) *_\Theta (y_1, y_2) = (x_1 *_G y_1, x_2 *_{\mathbb{Z}_2} y_2) = (x_1, x_2) * (y_1, y_2) \in X * Y$.

If $x_2 = 1$ then $(x_1, x_2) *_\Theta (y_1, y_2) = (x_1 *_G {y_1}^{-1}, x_2 *_{\mathbb{Z}_2} y_2) = (x_1, x_2) * ({y_1}^{-1}, {y_2}^{-1}) \in X * Y$ because $y_2 = {y_2}^{-1}$ and Y is symmetric with respect to $*$.

It remains to prove that X and Y are both symmetric with respect to $*_\Theta$. Let (x, y) be any element of $G \times_\Theta \mathbb{Z}_2$: its symmetric with respect to $*_\Theta$ is (x^{-1}, y) if $y = 0$, and is (x, y) if $y = 1$. Let (x_1, x_2) be any element of X: if $x_2 = 0$ then $({x_1}^{-1}, 0)$ is an element of X as X is symmetric with respect to $*$; if $x_2 = 1$ then $(x_1, 1)$ is obviously an element of X, thus X is symmetric with respect to $*_\Theta$. The set Y is likewise symmetric with respect to $*_\Theta$.

Corollary B.8.7. *Let (A, B) be any symmetric near-factorization of the direct product $G \times Z_2$ of the group G by the group $\mathbb{Z}_2$. Then (A, B) is a symmetric near-factorization of the semidirect product $G \times_\Theta Z_2$, of the group G by the group $\mathbb{Z}_2$ determined by Θ.*

Proof. This results from Lemma B.8.6.

Thus we have a process to carry a symmetric near-factorization of $G \times \mathbb{Z}_2$ into $G \times_\Theta Z_2$. It remains to know if this alters the associated partitionable graph $G(A, B)$. This is not the case, as stated below:

Corollary B.8.8. *The graph Γ associated to the near-factorization (A, B) as being a symmetric near-factorization of $G \times \mathbb{Z}_2$ is isomorphic to the graph Γ' associated to (A, B) as being a symmetric near-factorization of $G \times_\Theta \mathbb{Z}_2$.*

Proof. If z is any element of $G \times \mathbb{Z}_2$ then we denote by z^{-1} the inverse of z with respect to $*$, and we denote by z^{-1_Θ} its inverse with respect to $*_\Theta$.

Due to Lemma B.8.6, we have $(A * A) \setminus \{e\} = (A *_\Theta A) \setminus \{e\}$.

If x and y are two adjacent vertices of Γ then we have $x^{-1} * y \in (A * A) \setminus \{e\}$, since A is symmetric with respect to $*$. If $x_2 = 0$ then $x^{-1} * y = x^{-1_\theta} *_\Theta y$ and therefore the vertices x and y are two adjacent vertices of Γ'. If $x_2 = 1$ then $x^{-1_\Theta} *_\Theta y = y^{-1} * x$ because G is abelian. But $y^{-1} * x = (x^{-1} * y)^{-1}$, thus $x^{-1_\Theta} *_\Theta y = y^{-1} * x \in ((A * A) \setminus \{e\})^{-1} = (A * A) \setminus \{e\} = (A *_\Theta A) \setminus \{e\}$. Hence the vertices x and y are two adjacent vertices of Γ'.

If x and y are two adjacent vertices of Γ' then $x^{-1_\Theta} *_\Theta y \in (A *_\Theta A) \setminus \{e\}$, since A is symmetric with respect to $*_\Theta$. As $x^{-1_\Theta} *_\Theta y = x^{-1} * y$ or $x^{-1_\Theta} *_\Theta y = y^{-1} * x$, we have $x^{-1} * y \in (A * A) \setminus \{e\}$ or $y^{-1} * x \in (A * A) \setminus \{e\}$. Therefore $x^{-1} * y \in (A * A) \setminus \{e\}$, which means that x and y are two adjacent vertices of Γ.

We may now give a relationship between near-factorizations of cyclic groups and near-factorizations of dihedral groups. Before proceeding, we need to recall the definition of $\mathbb{D}_{2n}$ as a semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_2$.

Let Θ be the homomorphism from $\mathbb{Z}_2$ into $\text{AUT}(\mathbb{Z}_n)$ given by

$$\begin{array}{cccc} \Theta: & \mathbb{Z}_2 & \to & \text{AUT}(\mathbb{Z}_n) \\ & 0 & \mapsto & (\text{id} : x \mapsto x) \\ & 1 & \mapsto & (\text{inv} : x \mapsto -x) \end{array}$$

Then the group $\mathbb{Z}_n \times_\Theta \mathbb{Z}_2$ is isomorphic to the dihedral group of order $2n$.

If n is odd then $\mathbb{Z}_{2n}$ is isomorphic to the group $\mathbb{Z}_n \times \mathbb{Z}_2$. Let (A, B) be any near-factorization of $\mathbb{Z}_{2n}$. We may assume that this near-factorization is symmetric without altering the associated partitionable graph by Lemma B.8.3. Thus by Corollary B.8.7, (A, B) induces a symmetric near-factorization of D_{2n}. In particular by Corollary B.8.8, all CGPW$_2$-graphs of order $2n$ with n odd may be seen as graphs associated to some near-factorizations of dihedral groups. If n is even, no near-factorization of $\mathbb{Z}_n \times \mathbb{Z}_2$ is known so far.

In fact, near-factorizations of the dihedral groups give rise to every CGPW$_2$-graphs of even order [11]. We do not know any near-factorization (A, B) of a dihedral group, whose associated graph is not a CGPW$_2$-graph.

A near-factorization splits 'equally' in some cosets.

Let H be any subgroup of G and X be any subset of G. If Z is any right coset of H, we denote by $\mathrm{n}_Z(X)$ the number of elements of X which lie in the right coset Z, i.e. $\mathrm{n}_Z(X) := |X \cap Z|$. We denote by X_Z the set $X \cap Z$.

Theorem B.8.9. *Let H be any normal subgroup of G such that the quotient group G/H is abelian and of exponent at most 4. Let d denote the index of H. If (A, B) is a near-factorization of G then for every coset Z of H, we have*

$$\left\lfloor \frac{|A|}{d} \right\rfloor \leq \mathrm{n}_Z(A) \leq \left\lfloor \frac{|A|}{d} \right\rfloor + 1$$

Proof. Notice that if G/H is of exponent 1 then $d = 1$ and there is nothing to prove. Thus we shall assume that G/H is of exponent 2, 3 or 4.

As there is no hypothesis on B, we may assume without loss of generality that the uncovered element is e, due to Lemma B.8.2.

Let $g_1, \ldots, g_d$ be an enumeration of the elements of the quotient group G/H, with the convention that $g_1 = H$, that is, g_1 is the identity element of G/H.

Let $\overrightarrow{s}$ be the d-dimensional column vector with $s_1 := \frac{n}{d} - 1$ and $s_i := \frac{n}{d}$ for every i between 2 and d.

Let M be the $d \times d$-matrix defined by

$$M \quad := \quad \begin{pmatrix} \mathrm{n}_{g_1 g_1^{-1}}(B) & \mathrm{n}_{g_1 g_2^{-1}}(B) & \cdots & \mathrm{n}_{g_1 g_d^{-1}}(B) \\ \mathrm{n}_{g_2 g_1^{-1}}(B) & \mathrm{n}_{g_2 g_2^{-1}}(B) & \cdots & \mathrm{n}_{g_2 g_d^{-1}}(B) \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{n}_{g_d g_1^{-1}}(B) & \mathrm{n}_{g_d g_2^{-1}}(B) & \cdots & \mathrm{n}_{g_d g_d^{-1}}(B) \end{pmatrix}$$

Let $\overrightarrow{x}$ be the d-dimensional column vector with $x_i := \mathrm{n}_{g_i}(A)$ for every i between 1 and d.

Claim B.8.10. *We have*

$$M\overrightarrow{x} = \overrightarrow{s}$$

Let i be an integer between 1 and d, we are going to show that

$$g_i \setminus \{e\} \quad = \bigcup_{1 \leq j,k \leq d,\ g_j g_k = g_i} \mathrm{A}_{g_j} \mathrm{B}_{g_k} \tag{B.18}$$

Let z be any element of $g_i \setminus \{e\}$. There exist a in A and b in B such that $z = ab$. Let g_j be the coset containing a and let g_k be the coset containing b. Then $z \in g_j g_k$. Thus $g_j g_k = g_i$. Hence z is an element of the right part of (B.18). Thus

$$g_i \setminus \{e\} \quad \subseteq \bigcup_{1 \leq j,k \leq d,\ g_j g_k = g_i} \mathrm{A}_{g_j} \mathrm{B}_{g_k}$$

Let z be any element of the right part of (B.18): there exist j and k, a in A_{g_j}, b in B_{g_k} such that $z = ab$ and $g_j g_k = g_i$. Since $a \in g_j$ and $b \in g_k$, we have $z = ab \in g_j g_k = g_i$. Obviously, z is distinct of the uncovered element e. Hence z is an element of the left part of (B.18).

Therefore we have:

$$g_i \setminus \{e\} \quad = \bigcup_{1 \leq j,k \leq d,\ g_j g_k = g_i} \mathrm{A}_{g_j} \mathrm{B}_{g_k}$$

We now prove that the right part of (B.18) is a disjoint union: let i,j,k,l and m be indices such that $g_j g_k = g_l g_m = g_i$. If $\mathrm{A}_{g_j} \mathrm{B}_{g_k} \cap \mathrm{A}_{g_l} \mathrm{B}_{g_m} \neq \emptyset$, then there exist $a_j \in \mathrm{A}_{g_j}$, $b_k \in \mathrm{B}_{g_k}$, $a_l \in \mathrm{A}_{g_l}$ and $b_m \in \mathrm{B}_{g_m}$ such that $a_j * b_k = a_l * b_m$. Thus $a_j = a_l$ and so $j = l$. Similarly $k = m$.

Hence we have for every i between 1 and d,

$$\begin{aligned}
|g_i \setminus \{e\}| &= \sum_{1\leq j,k\leq d,\ g_jg_k=g_i} |\mathrm{A}_{g_j}\mathrm{B}_{g_k}| \\
&= \sum_{1\leq j,k\leq d,\ g_jg_k=g_i} |\mathrm{A}_{g_j}| \times |\mathrm{B}_{g_k}| \\
&= \sum_{1\leq j\leq d} \mathrm{n}_{g_j}(A) \times \mathrm{n}_{g_j{}^{-1}g_i}(B) \\
&= \sum_{1\leq j\leq d} \mathrm{n}_{g_j}(A) \times \mathrm{n}_{g_ig_j{}^{-1}}(B) \quad (G/H \text{ is abelian})
\end{aligned}$$

We need now some results belonging to the theory of linear representations of finite groups. The scope of this theory is far beyond what we need, thus we simply recall some basic facts. A detailed exposition is given in the book 'Représentations linéaires des groupes finis' by J-P Serre [13] or in the book 'Groupes' by A. Bouvier and D. Richard [4], for instance.

If W is any finite abelian group, a character τ of W is an homomorphism from W into C^*, the multiplicative group of the non-zero complex numbers. Notice that for every w of W, we have $|\tau(w)| = 1$ as $\tau(w)^{o(w)} = \tau(w^{o(w)}) = \tau(e_W) = 1$. We denote by $\ker(\tau)$ the kernel of τ, i.e. the set of elements of W that τ maps onto the complex number 1.

Since G/H is abelian, there exist d distinct characters $\tau_1, \ldots, \tau_d$ of G/H (Theorem 7 [13]) such that for every i and j between 1 and d,

$$\frac{1}{d}\sum_{1\leq k\leq d} \tau_i(g_k)\overline{\tau_j(g_k)} = 1 \text{if} i = j \tag{B.19}$$

$$0 \text{if} i \neq j \qquad \text{(Theorem 3 [13])} \tag{B.20}$$

For every k between 1 and d, let

$$\lambda_k = \sum_{g\in G/H} \tau_k(g^{-1})\mathrm{n}_g(B)$$

and let $\overrightarrow{V_k}$ be the d-dimensional column vector $\overrightarrow{(\tau_k(g_j))}_{1\leq j\leq d}$.

Let P be the $d\times d$-matrix whose column vectors are given by the $\overrightarrow{V_i}$, i.e.

$$P = \begin{pmatrix} \tau_1(g_1) & \cdots & \tau_d(g_1) \\ \cdots & \cdots & \cdots \\ \tau_1(g_d) & \cdots & \tau_d(g_d) \end{pmatrix}$$

Claim B.8.11. *The matrix P is invertible and the elements of P^{-1} are of modulus $\frac{1}{d}$.*

Due to the equations (B.19), we have $P^{-1} = \frac{1}{d}{}^t\overline{P}$ (where $\overline{P}$ is the conjugate matrix of P). Since the elements of P are of modulus 1, the proof is over.

Claim B.8.12. *For every i between 1 and d, we have $M\overrightarrow{V_i} = \lambda_i\overrightarrow{V_i}$. In particular the complex numbers $\lambda_1, \cdots, \lambda_d$ are the eigenvalues of M.*

For every k between 1 and d, we have

$$\begin{aligned}
(\lambda_i \overrightarrow{V_i})_k &= \left(\sum_{g \in G/H} \tau_i(g^{-1}) \mathrm{n}_g(B) \right) \tau_i(g_k) \\
&= \sum_{g \in G/H} \tau_i(g^{-1} g_k) \mathrm{n}_g(B) \\
&= \sum_{g \in G/H} \tau_i(g) \mathrm{n}_{g_k g^{-1}}(B)
\end{aligned}$$

This means that $(M\overrightarrow{V_i})_k = (\lambda_i \overrightarrow{V_i})_k$.

Thus $M\overrightarrow{V_i} = \lambda_i \overrightarrow{V_i}$.

Claim B.8.13. *The non-zero eigenvalues of M are of modulus at least 1.*

Let λ_k be a non-zero eigenvalue of M.

If G/H is of exponent 2 or 4 then there exist two integers a and b such that $\lambda_k = a + bi$. Thus $|\lambda_k| = \sqrt{a^2 + b^2} \geq \max(|a|, |b|) \geq 1$.

If G/H is exponent 3 then there exist three integers a,b and c such that $\lambda_k = a + bj + cj^2$, where j is the complex number $e^{\frac{2i\pi}{3}}$. Thus $|\lambda_k|^2 = (a+bj+cj^2)\overline{(a + bj + cj^2)} = a^2+b^2+c^2-ab-ac-bc$. Hence $|\lambda_k| = 0$ or $|\lambda_k| \geq 1$. Since $\lambda_k \neq 0$, this implies that $|\lambda_k| \geq 1$.

Claim B.8.14. *The complex eigenvalues of M are of modulus at least 1.*

Due to Claim B.8.13, we only have to prove that every eigenvalue of M is distinct of 0.

Let λ_k be some eigenvalue such that $\lambda_k = 0$. Thus we have

$$\sum_{g \in G/H} \tau_k(g^{-1}) \mathrm{n}_g(B) = 0 \tag{B.21}$$

Notice that we have

$$\sum_{g \in G/H} \mathbf{1}(g^{-1}) \mathrm{n}_g(B) = \alpha \tag{B.22}$$

where $\mathbf{1}$ is the trivial character which maps every element of G/H onto the complex number 1.

If τ is any character of G/H, we call the order of τ the smallest strictly positive integer r such that $\tau^r = \mathbf{1}$, where τ^r denotes the character of G/H, which maps every element g onto $\tau(g^r)$. Obviously the order of a character τ divides the exponent of G/H.

As G/H is of exponent at most four, every character of G/H is of order at most 4.

If τ_k is of order 2 then G/H must be of exponent 2 or 4, and therefore n is even. For every element g of G/H, we have $\tau_k(g) = 1$ or -1. Hence we have

$$\sum_{g \in G/H} \mathbf{1}(g^{-1}) \mathrm{n}_g(B) + \sum_{g \in G/H} \tau_k(g^{-1}) \mathrm{n}_g(B) = 2 \sum_{g \in \ker(\tau_k)} \mathrm{n}_g(B)$$

Thus $\alpha + 0 = 2 \sum_{g \in \ker(\tau_k)} \mathrm{n}_g(B)$. Hence $n = \alpha \times \omega + 1$ must be odd: contradiction.

If τ_k is of order 3 then G/H must be of exponent 3, and therefore n is a multiple of 3. For every element g of G/H, we have $\tau_k(g) = 1$, j or j^2. Hence we have

$$\begin{aligned}
\sum_{g\in G/H} \mathbf{1}(g^{-1})\mathrm{n}_g(B) &= \alpha \\
\sum_{g\in G/H} \tau_k(g^{-1})\mathrm{n}_g(B) &= 0 \\
\sum_{g\in G/H} \overline{\tau_k(g^{-1})}\mathrm{n}_g(B) &= 0 \quad \text{(by conjugacy)}
\end{aligned}$$

Thus $\alpha + 0 + 0 = 3\sum_{g\in\ker(\tau_k)} \mathrm{n}_g(B)$. Hence $n = \alpha \times \omega + 1 = 1 \pmod 3$: contradiction.

If τ_k is of order 4 then G/H must be of exponent 4, and therefore n is a multiple of 4. For every element g of G/H, we have $\tau_k(g) = 1$, i, -1 or $-i$. Hence we have

$$\begin{aligned}
\sum_{g\in G/H} \mathbf{1}(g^{-1})\mathrm{n}_g(B) &= \alpha \\
\sum_{g\in G/H} \tau_k(g^{-1})\mathrm{n}_g(B) &= 0
\end{aligned}$$

Hence we get

$$\alpha + 0 = 2\sum_{g\in\ker(\tau_k)} \mathrm{n}_g(B) + (1+i)\times \sum_{g,\ \tau_k(g^{-1})=i} \mathrm{n}_g(B) + (1-i)\times \sum_{g,\ \tau_k(g^{-1})=-i} \mathrm{n}_g(B)$$

Since α is real, we have $\sum_{g,\ \tau_k(g^{-1})=i} \mathrm{n}_g(B) = \sum_{g,\ \tau_k(g^{-1})=-i} \mathrm{n}_g(B)$. Therefore we get

$$\alpha = 2\sum_{g\in\ker(\tau_k)} \mathrm{n}_g(B) + 2\sum_{g,\ \tau_k(g^{-1})=i} \mathrm{n}_g(B)$$

Thus α is even, which implies that n is odd, in contradiction with the fact that 4 divides n.

Hence the proof of Claim B.8.14 is over.

Due to Claim B.8.14, the equation $M\overrightarrow{X} = \overrightarrow{s}$ has $\overrightarrow{x}$ for unique solution.

If $\overrightarrow{y}$ is any vector, we denote by $\|\overrightarrow{y}\|_\infty$ the maximum modulus of its elements.

Claim B.8.15. *Let $\overrightarrow{1}$ be the $d-$dimensional vector with all entries equal to 1. Then we have*

$$\overrightarrow{x} = \frac{n}{\alpha * d}\overrightarrow{1} + \overrightarrow{c}$$

where $\overrightarrow{c}$ is a $d-$dimensional vector such that $\|\overrightarrow{c}\|_\infty \leq 1$.

We have $M\overrightarrow{x} = \overrightarrow{s}$. Let $\overrightarrow{z}$ be the unique solution of $M\overrightarrow{Y} = \frac{n}{d}\overrightarrow{1}$ and let $\overrightarrow{c}$ be the unique solution of $M\overrightarrow{Y} = \overrightarrow{s'}$ with $\overrightarrow{s'} := \overrightarrow{s} - \frac{n}{d}\overrightarrow{1}$. Obviously we have $\overrightarrow{x} = \overrightarrow{z} + \overrightarrow{c}$ and $\overrightarrow{z} = \frac{n}{\alpha * d}\overrightarrow{1}$.

We have the equation $P^{-1}MP = M'$, where M' is the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_d \end{pmatrix}$$

with the eigenvalues of M for diagonal entries.

We have $\overrightarrow{c} = PM'^{-1}P^{-1}\overrightarrow{s'}$. Notice that $P^{-1}\overrightarrow{s'}$ is a column of P^{-1}. Let $\overrightarrow{c'}$ be that column vector. We denote by c'_1 the element of the first row, ..., c'_d the element of the last row. Since M'^{-1} is the diagonal matrix of diagonal $(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_d})$, we have the equation

$$M'^{-1}c' = \begin{pmatrix} \frac{c'_1}{\lambda_1} \\ \cdots \\ \frac{c'_d}{\lambda_d} \end{pmatrix}$$

Since the elements of P are of modulus 1, we get

$$\| \overrightarrow{c} \|_\infty \leq \left(\frac{|c'_1|}{|\lambda_1|} + \cdots + \frac{|c'_d|}{|\lambda_d|} \right)$$

By Claim B.8.11, we have $\| \overrightarrow{c'} \|_\infty \leq \frac{1}{d}$, hence

$$\| \overrightarrow{c} \|_\infty \leq \frac{1}{d} \left(\frac{1}{|\lambda_1|} + \cdots + \frac{1}{|\lambda_d|} \right)$$

By Claim B.8.14, we get $\| \overrightarrow{c} \|_\infty \leq 1$ as required.

Claim B.8.16. *Let g be any element of G/H. Then*

$$\left\lfloor \frac{\omega}{d} \right\rfloor \leq \mathrm{n}_g(A) \leq \left\lfloor \frac{\omega}{d} \right\rfloor + 1$$

We know that $\mathrm{n}_g(A)$ must be one of the elements of $\overrightarrow{x}$. Thus $\mathrm{n}_g(A) \geq \frac{\omega}{d} + \frac{1}{\alpha * d} - 1 \geq \left\lfloor \frac{\omega}{d} \right\rfloor + \frac{1}{\alpha * d} - 1 > \left\lfloor \frac{\omega}{d} \right\rfloor - 1$. Hence $\mathrm{n}_g(A) \geq \left\lfloor \frac{\omega}{d} \right\rfloor$.

Let k be the remainder of ω modulo d: we have $\frac{\omega}{d} = \left\lfloor \frac{\omega}{d} \right\rfloor + \frac{k}{d}$ with $0 \leq k \leq d-1$. Then $\mathrm{n}_g(A) \leq \frac{\omega}{d} + \frac{1}{\alpha * d} + 1 \leq \left\lfloor \frac{\omega}{d} \right\rfloor + \frac{k}{d} + \frac{1}{\alpha * d} + 1 \leq \left\lfloor \frac{\omega}{d} \right\rfloor + \frac{d-1}{d} + \frac{1}{\alpha * d} + 1 \leq \left\lfloor \frac{\omega}{d} \right\rfloor + 1 + \frac{1-\alpha}{\alpha * d} + 1 < \left\lfloor \frac{\omega}{d} \right\rfloor + 2$ as $\alpha \geq 2$. Hence $\mathrm{n}_g(A) \leq \left\lfloor \frac{\omega}{d} \right\rfloor + 1$.

Corollary B.8.17. *Let H be any normal subgroup of G such that the quotient group G/H is abelian and of exponent at most 4. Let d denote the index of H. If (A, B) is a near-factorization of G then for every coset Z of H, we have*

$$\left\lfloor \frac{|B|}{d} \right\rfloor \leq \mathrm{n}_Z(B) \leq \left\lfloor \frac{|B|}{d} \right\rfloor + 1$$

Proof. As there is no hypothesis on A, we may assume without loss of generality that the uncovered element is e, due to Lemma B.8.2. Hence we have $A * B = G \setminus \{e\} = B * A$ (Lemma B.8.1). Then Theorem B.8.9 may be applied to the near-factorization (B, A).

Some applications

If k is any integer, we say that $\mathrm{part}(k)$ is odd if $k = \pm 3 \pmod 8$. Let G be a group with a near-factorization (A, B). If G has a normal subgroup of index 4, then we know by Theorem B.8.9 that there exists an integer q such that $|A| = 4q \pm 1$. Then we have that q is odd if and only if $\mathrm{part}(|A|)$ is odd.

Lemma B.8.18. *Let G' be any finite group. Let p and q be any strictly positive integers. The group $G := \mathbb{Z}_{2p} \times \mathbb{Z}_{4q} \times G'$ does not have any symmetric near-factorization (A, B) such that* $\mathrm{part}(|A|)$ *is odd or* $\mathrm{part}(|B|)$ *is odd.*

Proof. We have $\mathrm{involutions}(G) = \{0, p\} \times \{0, 2q\} \times \mathrm{involutions}(G')$.

Let H be the subgroup $(2\mathbb{Z}_{2p}) \times (2\mathbb{Z}_{4q}) \times G'$ of G. H is a subgroup of G of index 4. Let H_0, H_1, H_2, H_3 be the four right-cosets with respect to H defined as: $H_0 := H$, $H_1 := H(1, 0, e_{G'})$, $H_2 := H(0, 1, e_{G'})$, $H_3 := H(1, 1, e_{G'})$.

Notice that each of these right-cosets is symmetric and that $\mathrm{involutions}(G) \subseteq (H_0 \cup H_1)$.

Suppose G has a symmetric near-factorization (A, B) such that $\mathrm{part}(|A|)$ is odd.

Let $A_0 := H_0 \cap A$, $A_1 := H_1 \cap A$, $A_2 := H_2 \cap A$ and $A_3 := H_3 \cap A$.

Let $n_0 := |A_0|$, $n_1 := |A_1|$, $n_2 := |A_2|$ and $n_3 := |A_3|$. Since $|A|$ is odd, three of these four values are equal by Theorem B.8.9.

Since $\mathrm{part}(|A|)$ is odd, they are equal to an odd number. Thus at least one of n_2 and n_3 is odd. If n_2 is odd then, since A_2 is symmetric as A and H_2 are both symmetric, there is at least one element a in A_2 such that $a = a^{-1}$. This is in contradiction with $\mathrm{involutions}(G) \subseteq H_0 \cup H_1$. If n_3 is odd, we get the same contradiction. Thus there is no symmetric near-factorization (A, B) such that $\mathrm{part}(|A|)$ is odd.

Due to Lemma B.8.18 there is only one non-cyclic abelian group of order 16 which is likely to have a near-factorization because $\mathrm{part}(3)$ is odd: this group is $\mathbb{Z}_2^4$. Likewise there is only one non-cyclic abelian group of order 64 which is likely to have a near-factorization (A, B) such that $|A| = 3$ and $|B| = 21$: this group is $\mathbb{Z}_2^6$. Lemma B.8.18 does not give informations about the case $|A| = 7$ and $|B| = 9$ as $\mathrm{part}(7)$ and $\mathrm{part}(9)$ are both even. Since $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^6$ are groups of exponent 2, they do not have any near-factorizations [5].

The group $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{273}$ does not have any near-factorization as $2 * 4 * 273 - 1 = 37 * 59$, 37 and 59 are prime, and $\mathrm{part}(37)$ is odd.

Notice that Lemma B.8.18 does not make full use of Theorem B.8.9 as the proof involves only *one* subgroup of G.

Using Theorem B.8.9 as a filter

Theorem B.8.9 may be used to decrease the number of cases to be investigated when looking for a near-factorization for a given group with the help of a computer. From the list of all subsets A of G of cardinality ω, we may keep only those satisfying the inequalities in Theorem B.8.9 and then for every of these A, check if there exists a subset B of cardinality α such that (A, B) is a near-factorization. We used GAP [8] to get the list of all normal subgroups inducing a quotient group of exponent at most 4.

Our computations revealed that, besides the two groups of order 50 mentioned in the next section of this paper, the only groups of order at most 64 admitting near-factorizations are the cyclic groups and the dihedral groups. Theorem B.8.9 turned out to be a total filter for more than one half of the 267 groups of order 64.

In [5], D. de Caen, D.A. Gregory, I.G. Hughes and D.L. Kreher proved that all non-cyclic abelian groups of order at most 100 failed to have any near-factorization, except may be for the three following groups: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{19}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{23}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$. With the help of Theorem B.8.9, an exhaustive check by computer established that these groups have no symmetric near-factorization, hence the only abelian groups of order at most 100 having near-factorizations are the cyclic groups.

A 'new' partitionable graph with fifty vertices

Exhaustive computations showed that the smallest group admitting a near-factorization which is neither a cyclic group, nor a dihedral group, has order fifty. In fact there are two such groups of order fifty. The near-factorizations of these two groups give rise to a unique partitionable graph (and its complement). This partitionable graph is not a CGPW_2-graph and is not a BGH-graph.

The group $\mathbb{D}_{10} \times \mathbb{Z}_5$.

Since the associated graphs to the near-factorizations of this group turn out to be isomorphic to a particular partitionable graph or to its complement, there is no need to give several near-factorizations of this group: one is enough.

Let

$$\begin{aligned} A &= \{(e,0),(s,0),(e,3),(s,3),(r,4),(sr,4),(r^2,4)\} \\ B &= \{(s,1),(r,1),(sr^2,1),(sr^3,3),(r^4,3),(sr^3,4),(r^4,4)\} \end{aligned}$$

Then it is straightforward to check that (A,B) is indeed a near-factorization of $\mathbb{D}_{10} \times \mathbb{Z}_5$. We denote by Γ_{50} this graph $G(A,B)$.

We have $\mathrm{INT}(A) = 6$ and $\mathrm{INT}(B^{-1}) = 4$.

Since $\omega = 7$ is prime and $\omega = \alpha$, we know that there is a unique CGPW_2-graph of order fifty: the web with fifty vertices. Since $\mathrm{INT}(B^{-1}) = 4 \neq 6$, the graph Γ_{50} is obviously not a CGPW_2-graph and is not a BGH-graph.

A computer assisted calculation revealed that Γ_{50} has a quite nice property: the critical edges of the complement of Γ_{50} form a perfect matching, i.e. they form a set of disjoint vertices such that every vertex of the graph belongs to exactly one of these edges. We believe this is the first known partitionable graph with this property.

At last, the graph Γ_{50} is not a counter-example to the Strong Perfect Graph Conjecture as it has a small transversal.

The group $(\mathbb{Z}_5 \times \mathbb{Z}_5) \times_\Theta \mathbb{Z}_2$

The group $(\mathbb{Z}_5 \times \mathbb{Z}_5) \times_\Theta \mathbb{Z}_2$ is the semidirect product of $(\mathbb{Z}_5 \times \mathbb{Z}_5)$ determined by Θ as defined in the second section.

The graphs associated to the near-factorizations of this group turn out to be isomorphic to the graph Γ_{50} of the preceding section, or to its complement. Thus we shall only give one of these near-factorizations.

Let

$$\begin{aligned} A &= \{((0,0),0),((0,0),1),((1,0),0),((0,1),0),((4,0),1),((0,4),1),((2,2),0)\} \\ B &= \{((2,0),1),((0,2),1),((3,3),1),((4,1),0),((1,4),0),((2,2),1),((4,4),0)\} \end{aligned}$$

Then it is straightforward to check that (A,B) is indeed a near-factorization of $(\mathbb{Z}_5 \times \mathbb{Z}_5) \times_\Theta \mathbb{Z}_2$.

A property shared by all partitionable graphs?

Every web and every BGH-graph has a critical clique. Thus in any of these graphs, there exist two distinct cliques sharing $\omega - 1$ vertices.

Every CGPW_2-graph which is not a web, has no critical clique, though every CGPW_2-graphs has two distinct maximum cliques sharing at least one half of their vertices [11].

As both the graph Γ_{50} and its complement have two distinct maximum cliques sharing at least one half of their vertices, this leads to the following natural problem:

Problem

Is it true that every partitionable graph has two distinct maximum cliques sharing at least one half of their vertices?

If so, then to prove the Strong Perfect Graph Conjecture, it would be enough to prove the following one:

Conjecture B.8.19. *Every partitionable graph, with two distinct maximum cliques sharing at least one half of their vertices, fails to invalidate the Strong Perfect Graph Conjecture.*

Conjecture B.8.19 is proved in [11] for the class of circular partitionable graphs, including all CGPW_2-graphs.

I am deeply indebted to C. Delorme who pointed out to me that using linear representations of groups would improve a former version of Theorem B.8.9, which involved only normal subgroups H of index at most 4.

References

[1] G. Bacsó, E. Boros, V. Gurvich, F. Maffray, and M. Preissmann. On minimally imperfect graphs with circular symmetry. *J. of Graph Theory*, 29:209–225, 1998.

[2] R.G. Bland, H.C. Huang, and L.E. Trotter. Graphical properties related to minimal imperfection. *Discrete Mathematics*, 27:11–22, 1979.

[3] E. Boros, V. Gurvich, and S. Hougardy. Recursive generation of partitionable graphs. *J. of Graph Theory*, 41(4):259–285, 2002.

[4] A. Bouvier and D. Richard. *Groupes*. Hermann, 1974.

[5] D. De Caen, D.A. Gregory, I.G. Hughes, and D.L. Kreher. Near-factors of finite groups. *Ars Combinatoria*, 29:53–63, 1990.

[6] V. Chvátal. A semi-strong perfect graph conjecture. *Ann.Discrete Math.*, 21:279–280, 1984.

[7] V. Chvátal, R.L. Graham, A.F. Perold, and S.H. Whitesides. Combinatorial designs related to the perfect graph conjecture. *Applied Discrete Mathematics*, 21:197–206, 1984.

[8] The GAP Group, Aachen, St Andrews. *GAP – Groups, Algorithms, and Programming, Version 4.1*, 1999. (`http://www-gap.dcs.st-and.ac.uk/~gap`).

[9] C.M. Grinstead. On circular critical graphs. *Discrete Mathematics*, 51:11–24, 1984.

[10] M.W. Padberg. Perfect zero-one matrices. *Mathematical Programming*, 6:180–196, 1974.

[11] A. Pêcher. Partitionable graphs arising from near-factorizations of finite groups. *Discrete Mathematics*, 269(1-3):191–218, 2003.

[12] A. Sebő. On critical edges in minimal imperfect graphs. *J. of Comb. Theory*, 67:62–85, 1996.

[13] J.-P. Serre. *Représentations linéaires des groupes finis*. Hermann, 2 edition, 1971.

Annexe C

Articles à paraître

Sommaire

C.1 Triangle-Free Strongly Circular-Perfect Graphs

par S. Coulonges, A. Pêcher et A. Wagler

A paraître dans Discrete Mathematics [28]

Zhu [15] introduced circular-perfect graphs as a superclass of the well-known perfect graphs and as an important χ-bound class of graphs with the smallest non-trivial χ-binding function $\chi(G) \leq \omega(G) + 1$. Perfect graphs have been recently characterized as those graphs without odd holes and odd antiholes as induced subgraphs [4]; in particular, perfect graphs are closed under complementation [8]. In contrary, circular-perfect graphs are not closed under complementation and the list of forbidden subgraphs is unknown.

We study strongly circular-perfect graphs: a circular-perfect graph is strongly circular-perfect if its complement is circular-perfect as well. This subclass entails perfect graphs, odd holes, and odd antiholes. As main result, we fully characterize the triangle-free strongly circular-perfect graphs, and prove that, for this graph class, both the stable set problem and the recognition problem can be solved in polynomial time.

Moreover, we address the characterization of strongly circular-perfect graphs by means of forbidden subgraphs. Results from [10] suggest that formulating a corresponding conjecture for circular-perfect graphs is difficult; it is even unknown which triangle-free graphs are minimal circular-imperfect. We present the complete list of all triangle-free minimal not strongly circular-perfect graphs.

Introduction

Coloring the vertices of a graph is an important concept with a large variety of applications. Let $G = (V, E)$ be a graph with vertex set V and edge set E, then a *k-coloring* of G is a mapping $f : V \to \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* $\chi(G)$; calculating $\chi(G)$ is NP-hard in general. In a set of k pairwise adjacent vertices, called *clique* K_k, all k vertices have to be colored differently. Thus the size of a largest clique in G, the *clique number* $\omega(G)$, is a trivial lower bound on $\chi(G)$; this bound is hard to evaluate as well.

Berge [2] proposed to call a graph G *perfect* if each induced subgraph $G' \subseteq G$ admits an $\omega(G')$-coloring. Perfect graphs have been recently characterized as those graphs without chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, and their complements $\overline{C}_{2k+1}$, the *odd antiholes*, as induced subgraphs (Strong Perfect Graph Theorem [4]). (The *complement* $\overline{G}$ of a graph G has the same vertex set as G and two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.) In particular, the class of perfect graphs is closed under complementation [8]. Perfect graphs turned out to be an interesting and important class with a rich structure, see [11] for a recent survey. For instance, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [5].

Strongly circular-perfect graphs

As a generalization of perfect graphs, Zhu [15] introduced recently the class of circular-perfect graphs based on the following more general coloring concept. For integers $k \geq 2d$, a *(k, d)-circular coloring* of a graph $G = (V, E)$ with at least one edge is a mapping $f : V \to \{0, \ldots, k-1\}$ with $|f(u)-f(v)| \geq d \bmod k$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k, d)-circular colorings of G; we have $\chi_c(G) \leq \chi(G)$ since every $(k, 1)$-circular coloring is a usual k-coloring of G. (Note that $\chi_c(G)$ is sometimes called the star chromatic number [3, 13].) The circular chromatic number of a stable set is set to be 1.

In order to obtain a lower bound on $\chi_c(G)$, we generalize cliques as follows: Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k vertices $0, \ldots, k-1$ and edges ij iff $d \leq |i-j| \leq k-d$. Such graphs $K_{k/d}$ are called *circular cliques* (or sometimes antiwebs [12, 14]) and are said to be *prime* if $\gcd(k, d) = 1$. Circular cliques include all cliques $K_t = K_{t/1}$, all odd antiholes $\overline{C}_{2t+1} = K_{(2t+1)/2}$, and all odd holes $C_{2t+1} = K_{(2t+1)/t}$, see Figure 1. The *circular clique number* is defined as $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G, \gcd(k, d) = 1\}$, and we immediately obtain that $\omega(G) \leq \omega_c(G)$. (Note: in this paper, we always denote an induced subgraph G' of G by $G' \subseteq G$.)

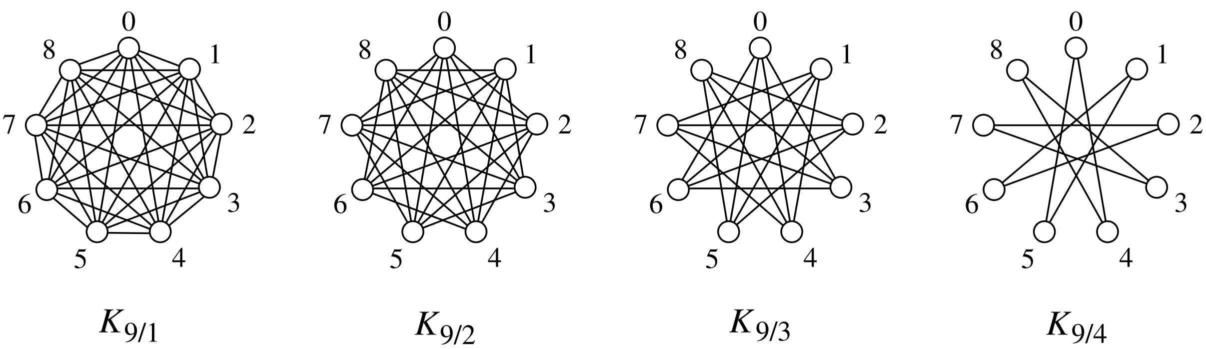


Figure C.1: The circular cliques on nine vertices

Every circular clique $K_{k/d}$ clearly admits a (k, d)-circular coloring (simply take the vertex numbers as colors, as in Figure 1), but no (k', d')-circular coloring with $\frac{k'}{d'} < \frac{k}{d}$ by [3]. Thus we obtain, for any graph G, the following chain of inequalities:

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{C.1}$$

A graph G is called *circular-perfect* if, for each induced subgraph $G' \subseteq G$, circular clique number $\omega_c(G')$ and circular chromatic number $\chi_c(G')$ coincide. Obviously, every perfect graph has this property by (C.1) as $\omega(G')$ equals $\chi(G')$. Moreover, any circular clique is circular-perfect as well [15, 1]. Thus circular-perfect graphs constitute a proper superclass of perfect graphs.

Another natural extension of perfect graphs was introduced by Gyárfás [6] as follows: A family $\mathcal{G}$ of graphs is called χ-bound with χ-binding function b if $\chi(G') \leq b(\omega(G'))$ holds for all induced subgraphs G' of $G \in \mathcal{G}$. Thus, this concept uses functions in $\omega(G)$ as *upper* bound on $\chi(G)$. Since it is known for any graph G that $\omega(G) = \lfloor \omega_c(G) \rfloor$ by [15] and $\chi(G) = \lceil \chi_c(G) \rceil$ by [13], we obtain that circular perfect graphs G satisfy the following Vizing-like property

$$\omega(G) \leq \chi(G) \leq \omega(G) + 1. \tag{C.2}$$

Thus, the class of circular-perfect graphs is χ-bound with the smallest non-trivial χ-binding function. In particular, this χ-binding function is best possible for a *proper* superclass of perfect graphs implying that circular-perfect graphs admit coloring properties almost as nice as perfect graphs. In contrary to perfect graphs, circular-perfect graphs are not closed under complementation and the list of forbidden subgraphs is unknown.

In this paper, we study *strongly circular-perfect graphs*: a circular-perfect graph is strongly circular-perfect if its complement is circular-perfect as well. We address the problem of finding the minimal not strongly circular-perfect graphs and provide complete answers in the triangle-free case.

Summary of results

We first address the problem which circular cliques occur in strongly circular-perfect graphs, see Section C.1. For that we fully characterize which circular cliques have a circular-perfect complement (Theorem C.1.3).

Section C.1 deals with triangle-free strongly circular-perfect graphs. A graph G is said to be an *interlaced odd hole* if and only if the vertex set of G admits a suitable partition $((A_i)_{1 \leq i \leq 2p+1}, (B_i)_{1 \leq i \leq 2p+1})$ into $2p+1$ (with $p \geq 2$) non-empty sets $A_1, \ldots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \ldots, B_{2p+1}$ such that

1. $\forall 1 \leq i \leq 2p+1$, $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (indices modulo $2p+1$),
2. $\forall 1 \leq i \leq 2p+1$, $B_i \neq \emptyset$ implies $|A_i| = 1$,

and the edge set of G is equal to $\cup_{i=1,\ldots,2p+1}(E_i \cup E'_i)$, where E_i (resp. E'_i) denotes the set of all edges between A_i and A_{i+1} (resp. between A_i and B_i); see Figure C.1 for an example (the sets of vertices in B_i are grey).

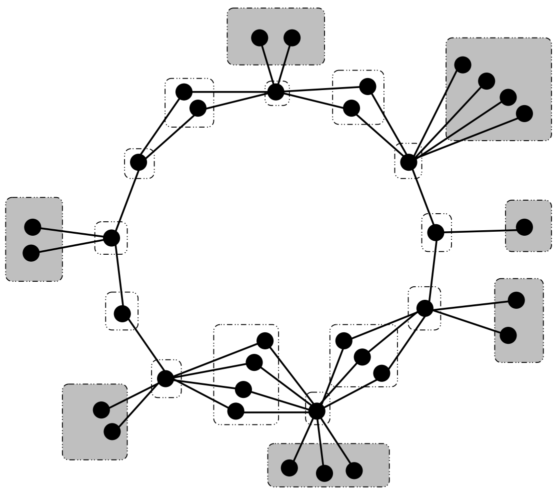


Figure C.2: An interlaced odd hole

We prove that a graph G is triangle-free strongly circular-perfect if and only if G is bipartite or an interlaced odd hole (Theorem C.1.15). We use this characterization of triangle-free strongly circular-perfect graphs to exhibit that both the stable set problem and the recognition problem can be solved in polynomial time for such graphs (see Theorem C.1.15 and Algorithm C.1).

In Section C.1, we finally address, motivated by the Strong Perfect Graph Theorem, the problem of finding all forbidden subgraphs for the class of strongly circular-perfect graphs. Results in [10] indicate that even formulating an appropriate conjecture for circular-perfect graphs is difficult, e.g., it is unknown which triangle-free graphs are not circular-perfect. We present the complete list of all triangle-free graphs which are minimal not strongly circular-perfect (Theorem C.1.22).

Circular cliques in strongly circular-perfect graphs

In this section, we solve the problem which prime circular cliques occur as induced subgraphs of a strongly circular-perfect graph. As the class of strongly circular-perfect graphs is closed under complementation, this is equivalent to ask which circular cliques have a circular-perfect complement.

The complement of a circular clique is called a *web* and we denote by $C_n^{\omega-1}$ the web $\overline{K_{n/\omega}}$, that is the graph with vertices $0, \ldots, n-1$ and edges ij such that i and j differ by at most $\omega - 1 \pmod{n}$, and $i \neq j$. In particular, the maximum clique size of $C_n^{\omega-1}$ is ω.

For that, we use the following result on claw-free graphs (note that webs are claw-free as the neighborhood of any node splits into two cliques).

Lemma C.1.1. *[10] A claw-free graph does not contain any prime antiwebs different from cliques, odd antiholes, and odd holes.*

This immediately implies for circular clique numbers of claw-free graphs:

Corollary C.1.2. *Let G be a claw-free graph.*

1. *If $\omega(G) = 2$, then $\omega_c(G) = 2$ follows iff G is perfect and $\omega_c(G) = 2 + \frac{1}{k}$ iff G is imperfect and C_{2k+1} is the shortest odd hole in G.*
2. *If $\omega(G) \geq 3$, then $\omega_c(G) = \max\{\omega(G), k' + \frac{1}{2}\}$ where $\overline{C}_{2k'+1}$ is the longest odd antihole in G.*

This enables us to completely characterize the circular-(im)perfection of webs as follows (note that the proof of assertion (3) is given in [10]).

Theorem C.1.3. *The web C_n^k is*

1. *circular-perfect if $k = 1$ or $n \leq 2(k+1) + 1$,*
2. *circular-perfect if $k = 2$ and $n \equiv 0 \pmod 3$,*
3. *minimal circular-imperfect if $k = 2$ and $n \equiv 1 \pmod 3$,*
4. *circular-imperfect if $k = 2$ and $n \equiv 2 \pmod 3$,*
5. *circular-imperfect if $k \geq 3$ and $n \geq 2(k+2)$.*

Proof. For that, we prove the following sequence of claims.

Claim C.1.4. *Any web C_n^k with $k = 1$ or $n \leq 2(k+1) + 1$ is circular-perfect.*

The webs C_n^1 are obviously all circular-perfect. Moreover, C_n^k is perfect if $n \leq 2(k+1)$ and an odd antihole if $n = 2(k+1) + 1$, thus C_n^k is circular-perfect if $n \leq 2(k+1) + 1$. $\diamond$

Thus Claim C.1.4 verifies already assertion (1). In the sequel, we have to consider webs C_n^k with $k \geq 2$ and $n \geq 2(k+2)$ only. In [10] it is shown that the webs $C_{3\alpha+1}^2$ are minimal circular-imperfect for $\alpha \geq 3$; this already ensures assertion (3). In order to show circular-perfection for the webs $C_{3\alpha}^2$ with $\alpha \geq 3$ and circular-imperfection for all remaining webs, we need the following.

Claim C.1.5. *C_n^k with $k \geq 2$, $n \geq 2(k+2)$ is circular-perfect only if $\omega(C_n^k) = \chi(C_n^k)$.*

We have $\omega(C_n^k) \geq 3$ and Corollary C.1.2(2) implies $\omega_c(C_n^k) = \max\{k+1, k'+\frac{1}{2}\}$ taken over all odd antiholes $C_{2k'+1}^{k'-1}$ in C_n^k. As $C_{n'}^l \subset C_n^k$ holds only if $l < k$ due to Trotter [7], we obtain that $k+1 > k'+\frac{1}{2}$ for any odd antihole $C_{2k'+1}^{k'-1}$ in C_n^k. Thus, $\omega(C_n^k) = k+1 = \omega_c(C_n^k)$ holds, implying the assertion by $\lceil \chi_c(C_n^k) \rceil = \chi(C_n^k)$. $\diamond$

Claim C.1.6. *For a web C_n^k with $n \geq 2(k+2)$, we have $\omega(C_n^k) < \chi(C_n^k)$ if and only if $(k+1) \not| \, n$.*

For any non-complete web C_n^k, it is well-known that $\chi(C_n^k) = \lceil \frac{n}{\alpha} \rceil$ holds where $\alpha = \alpha(C_n^k) = \lfloor \frac{n}{k+1} \rfloor$. Assuming $n = \alpha(k+1) + r$ with $r < k+1$ we obtain

$$\chi(C_n^k) = \left\lceil \frac{n}{\alpha} \right\rceil = \left\lceil \frac{\alpha(k+1)+r}{\alpha} \right\rceil = k+1+\left\lceil \frac{r}{\alpha} \right\rceil$$

implying $k+1 = \omega(C_n^k) < \chi(C_n^k)$ whenever $r > 0$, i.e., whenever $(k+1) \not| \, n$. $\diamond$

Combining Claim C.1.5 and Claim C.1.6 proves assertion (4); the only possible circular-perfect webs C_n^k satisfy $(k+1)|n$. This is obviously true for the webs $C_{3\alpha}^2$. In order to show their circular-perfection, we have to ensure that none of them contains a minimal circular-imperfect induced subgraph. By $\omega(C_{3\alpha}^2) = 3 = \chi(C_{3\alpha}^2)$, every induced subgraph G' of $C_{3\alpha}^2$ is clearly 3-colorable. Thus, $\omega(G') = 3$ implies $\omega_c(G') = \chi_c(G')$. The next claim also excludes the occurrence of minimal circular-imperfect induced subgraphs with less clique number:

Claim C.1.7. *No web C_n^2 contains a (minimal) circular-imperfect graph with clique number 2 as induced subgraph.*

Suppose $G' \subset C_n^2$ is triangle-free. Then G' does not admit any vertex of degree 3 (since every vertex of C_n^2 together with three of its neighbors contains a triangle). The assertion follows since all graphs with maximal degree 2 are collections of paths and cycles, and are thus circular-perfect. $\diamond$

Hence, assertion (2) is true.

For the last assertion (5), it is left to show that every web C_n^k with $k \geq 3$ and $(k+1)|n$ contains a circular-imperfect induced subgraph.

Claim C.1.8. *Any web $C_{\alpha(k+1)}^k$ with $k, \alpha \geq 3$ is circular-imperfect.*

We show that all those webs $C_{\alpha(k+1)}^k$ contain a circular-imperfect web as induced subgraph. Claim C.1.6 implies that $C_{\alpha k-1}^{k-1}$ is circular-imperfect as $k \not| \, (\alpha k - 1)$. We show $C_{3\alpha-1}^2 \subseteq C_{\alpha(k+1)}^k$ if $\alpha < k$ and $C_{\alpha k-1}^{k-1} \subseteq C_{\alpha(k+1)}^k$ if $\alpha \geq k$ with the help of the following result of Trotter [7]:

$$C_{n'}^{k'} \subseteq C_n^k \text{ if and only if } \frac{k'}{k}n \leq n' \leq \frac{k'+1}{k+1}n$$

Hence, we have $C_{3\alpha-1}^2 \subseteq C_{\alpha(k+1)}^k$ for $\alpha < k$ since

$$\frac{2}{k}\alpha(k+1) = 2\alpha + \frac{2\alpha}{k} \leq 3\alpha - 1 \leq \frac{3}{k+1}\alpha(k+1) = 3\alpha$$

holds: the first inequality is satisfied by $2\frac{\alpha}{k} < 2 \leq \alpha - 1$ if $\alpha < k$ and $\alpha \geq 3$; the second one is trivial. Moreover, $C_{\alpha k-1}^{k-1} \subseteq C_{\alpha(k+1)}^k$ follows for $\alpha \geq k$ since

$$\frac{k-1}{k}\alpha(k+1) = \alpha(k-1) + \frac{\alpha(k-1)}{k} \leq \alpha k - 1 \leq \frac{k}{k+1}\alpha(k+1) = \alpha k$$

holds: the first inequality is satisfied since $\frac{\alpha(k-1)}{k} \leq \alpha - 1$ is true due to $\alpha \geq k$; the second inequality obviously holds again. $\diamond$

Thus, a web C_n^k with $k \geq 3$ and $n > 2(k+1)+1$ is circular-imperfect: if $(k+1) \not| \, n$ by Claim C.1.6 and if $(k+1)|n$ by Claim C.1.8, finally verifying assertion (5). $\square$

Corollary C.1.9. *The induced prime circular cliques of a strongly circular-perfect graph are cliques, odd antiholes and odd holes.*

Corollary C.1.10. *A circular clique is strongly circular-perfect if and only if it is a clique, an odd antihole, an odd hole, a stable set, or of the form $K_{3k/3}$ with $k \geq 3$.*

We end this section with two lemmas discussing the adjacency of odd (anti)holes in strongly circular-perfect graphs and the behaviour under multiplying vertices. We call an induced subgraph $G' \subseteq G$ *dominating* (resp. *antidominating*) if every vertex in $G - G'$ has at least one neighbor (resp. non-neighbor) in G'.

Lemma C.1.11. *Every odd hole or odd antihole in a strongly circular-perfect graph is dominating as well as antidominating.*

Proof. We know from [10] that no vertex of a circular-perfect graph G is totally joined to any odd hole or odd antihole C in G, thus C is antidominating. If G is strongly circular-perfect, then the same applies to $\overline{G}$ and C is also dominating. □

Let $G_{v,S}$ be the graph obtained by multiplication of a vertex v in G by a stable set S (i.e., v is replaced by $|S|$ vertices having exactly the same neighbors as v in G) and let G_{v+w} be the graph obtained by adding a node w to G, whose only neighbour is v.

Lemma C.1.12.

(i) $G_{v,S}$ is circular-perfect if and only if G is circular-perfect;

(ii) G_{v+w} is circular-perfect if and only if G is circular-perfect.

Proof. Notice that both graphs $G_{v,S}$ and G_{v+w} contain G as an induced subgraph, so we only have to prove the *if* part of both assertions. Hence assume that G is circular-perfect.

The $|S|$ copies of the vertex v in $G_{v,S}$ are pairwise non-adjacent and have the same neighbors. Thus, $G_{v,S}$ cannot contain any new circular cliques and $\omega_c(G_{v,S}) = \omega_c(G)$ follows. Furthermore, all copies of v can receive the same color, namely the previous color of v, implying $\chi_c(G_{v,S}) = \chi_c(G)$. The same is obviously true for all induced subgraphs. Hence, as multiplication of vertices does neither change the circular clique nor the circular chromatic number, the graph $G_{v,S}$ is circular-perfect.

If G is a stable set then G_{v+w} is perfect and therefore circular-perfect. If G is not a stable set then adding the leaf w does neither change the circular clique number nor the circular chromatic number. Therefore G_{v+w} is circular-perfect. □

Triangle-free strongly circular-perfect graphs

The aim of this section is to fully characterize the triangle-free strongly circular-perfect graphs and to address stable set and recognition problem for these graphs.

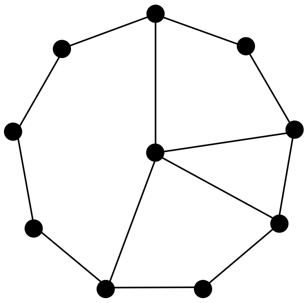
Corollary C.1.9 implies that the only prime circular cliques in a triangle-free strongly circular-perfect graph are cliques and odd holes; we first consider shortest odd holes in triangle-free strongly circular-perfect graphs.

Lemma C.1.13. *Every vertex outside a shortest odd hole $\mathcal{O}$ of a triangle-free graph has at most two neighbours in $\mathcal{O}$. Furthermore, if x has two such neighbours y_1 and y_2 then y_2 has a common neighbour with y_1 in $\mathcal{O}$.*

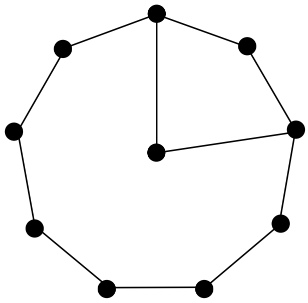
Proof. Let x be a vertex outside a shortest odd hole $\mathcal{O}$. W.l.o.g. assume that the vertices of $\mathcal{O}$ are labelled in the canonical cyclic order as $\{1, \ldots, 2p+1\}$ and let $x_1 < \ldots < x_k$ be the neighbours of x in $\mathcal{O}$. For every $2 \leq i \leq k$, let $r_i = x_i - x_{i-1} - 1$ and let $r_1 = x_1 + 2p + 1 - x_k - 1$ (see Fig. 2a). We have

$$2p + 1 = |\mathcal{O}| \quad = \quad k + \sum_{i=1,\ldots,k} r_i \tag{C.3}$$

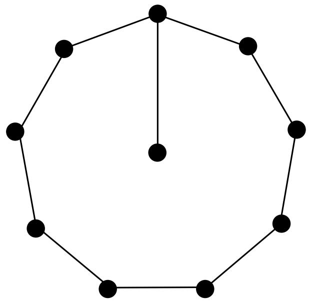
Since G is triangle-free, we have $r_i > 0, \forall 1 \leq i \leq k$. As $|\mathcal{O}| = 2p + 1$ is odd, Eq. (C.3) implies that there exists j such that r_j is even. As $\mathcal{O}$ is a shortest odd hole, this implies that $r_j = |\mathcal{O}| - 1$ or $r_j = |\mathcal{O}| - 3$. As all r_i are positive, Eq. (C.3) implies $k = 1$ (resp. $k = 2$) if $r_j = |\mathcal{O}| - 1$ (see Fig. 2c) (resp. $r_j = |\mathcal{O}| - 3$ (see Fig. 2b)). □



$r_1 = 1$, $r_2 = 0$, $r_3 = 1$, $r_4 = 3$
Fig. 2a



$r_1 = 1$, $r_2 = 6$
Fig. 2b



$r_1 = 8$
Fig. 2c

Lemma C.1.14. *Let G be a strongly circular-perfect graph with a shortest odd hole $\mathcal{O}$. Then every edge is incident to the odd hole $\mathcal{O}$.*

Proof. Suppose that there is an edge xy which is not incident to $\mathcal{O}$. Let $2p+1$ be the size of $\mathcal{O}$. Then the subgraph H induced by $\mathcal{O}$ and the vertices x and y is a strongly circular-perfect graph, with stability number at most $p+1$. Since x has at most 2 neighbours in $\mathcal{O}$, the vertex x does not see at least one maximum stable set of $\mathcal{O}$. Thus H has stability number $p+1$. Due to Theorem C.1.3, this implies that the circular clique number of $\overline{H}$ is $p+1$. As $\overline{H}$ is circular-perfect, we have $\chi_c(\overline{H}) = p+1$. Since $\chi(\overline{H})$ is the upper integer part of $\chi_c(\overline{H})$, the graph $\overline{H}$ is $(p+1)$-colorable. Hence H admits a covering with at most $p+1$ cliques $Q_1, \dots Q_{p+1}$. Let Q_x (resp. Q_y) be the clique containing x (resp. y). Then at least one of Q_x and Q_y meets $\mathcal{O}$ in two consecutive vertices, and has therefore at least 3 vertices. This implies that one of x and y belongs to a triangle: a contradiction. □

We are now prepared to prove the following characterization:

Theorem C.1.15. *A triangle-free graph G is strongly circular-perfect if and only if G is bipartite or an interlaced odd hole.*

Proof. ***Only if.*** Let G be a triangle-free strongly circular-perfect graph. If G is perfect then G is bipartite and we have nothing to prove. If G is not perfect, then G contains an induced odd hole or antihole by the Strong Perfect Graph Theorem. Since G is triangle-free, this means that G contains at least one induced odd hole $\mathcal{O}$. Let $2k+1$ be the size of this shortest odd hole.

The proof is by induction on the number of vertices: let $H(p,n)$ be the hypothesis "Every triangle-free strongly circular-perfect graph with a shortest odd hole of size $2p+1$ and at most n vertices is an interlaced odd hole".

Let n be the number of vertices of G: we have $n \geq 2p+1$. $H(p, 2p+1)$ is obviously true, hence assume that $n > 2p+1$ and that $H(p, n-1)$ is true.

There exists a vertex x outside the shortest odd hole $\mathcal{O}$. By induction hypothesis, $G - x$ is an interlaced odd hole and there exists a suitable partition of $G - x$ into $2p+1$ non-empty sets $A_1, \dots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \dots, B_{2p+1}$, i.e.,

1. $\forall 1 \leq i \leq 2p+1$, $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (with indices modulo $2p+1$),
2. $\forall 1 \leq i \leq 2p+1$, $B_i \neq \emptyset$ implies $|A_i| = 1$,

and the edge set of $G - x$ is equal to $\cup_{i=1,\dots,2p+1}(E_i \cup E'_i)$, where E_i (resp. E'_i) denotes the set of all edges between A_i and A_{i+1} (resp. between A_i and B_i).

By Lemma C.1.14 and Lemma C.1.13, x is of degree 1 or 2.

If x is of degree 1 then the neighbour y of x belongs to $\mathcal{O}$ due to Lemma C.1.14 again. Since y belongs to an odd hole of $G - x$, there exits a set A_j such that $y \in A_j$. For every $1 \leq i \leq 2p+1$ with $i \neq j$, let $B'_i = B_i$ and let $B'_j = B_j \cup \{x\}$. Then obviously $A_1, \dots, A_{2p+1}$ and $B'_1, \dots, B'_{2p+1}$ is a suitable partition of G. Thus G is an interlaced odd hole.

If x is of degree 2 then the neighbours y_1 and y_2 of x belong to $\mathcal{O}$ due to Lemma C.1.14. By Lemma C.1.13, there exists an index j such that y_1 belongs to A_{j-1} and y_2 belongs to A_{j+1} (or vice-versa). If A_{j-1} has at least two vertices, then there exists a shortest odd hole such that xy_1 is not incident to it, a contradiction to Lemma C.1.14. Hence $|A_{j-1}| = |A_{j+1}| = 1$. Let $\mathcal{O}'$ be the shortest odd hole $(\mathcal{O} \cup x) \setminus A_j$. If $B_j \neq \emptyset$ then there are no edges between B_j and $\mathcal{O}'$: a contradiction to Lemma C.1.11 . Thus $B_j = \emptyset$.

For every $1 \leq i \leq 2p+1$ with $i \neq j$, let $A'_i = A_i$ and let $A'_j = A_j \cup \{x\}$. Then obviously $A'_1, \dots, A'_{2p+1}$ and $B_1, \dots, B_{2p+1}$ is a suitable partition of G. Thus G is an interlaced odd hole. $\diamond$

If.

Let G be a bipartite graph or an interlaced odd hole. If G is bipartite then G is perfect and therefore strongly circular-perfect. If G is an interlaced odd hole, then G is circular-perfect due to Lemma C.1.12.

It remains to show that $\overline{G}$ is circular-perfect as well. The proof is by contradiction: assume that $\overline{G}$ is not circular-perfect and take an induced subgraph H of G such that $\overline{H}$ is a minimal circular-imperfect graph. We have $\omega_c(\overline{H}) < \chi_c(\overline{H})$.

Notice that H is not perfect as $\overline{H}$ is circular-imperfect. Since H is an induced subgraph of G this implies that H is an interlaced odd hole and is not an odd hole. H admits a suitable partition into $2p+1$ non-empty sets $A_1, \dots, A_{2p+1}$ and $2p+1$ possibly empty sets $B_1, \dots, B_{2p+1}$.

Claim C.1.16. *We have $\omega_c(\overline{H}) = \alpha(H)$.*

By construction, $2p+1$ is the size of every odd hole of H. As H is triangle-free, $\overline{H}$ is claw-free and the prime induced circular-cliques of $\overline{H}$ are stable sets, cliques, odd holes and odd antiholes due to Lemma C.1.1. Thus $\omega_c(\overline{H}) = \max\{p+1/2, \omega(\overline{H}) = \alpha(H)\}$. As H is not an odd hole, there exists a set A_i with at least 2 vertices or a non-empty set B_i, and in both cases, $\alpha(H) \geq p+1$. Therefore $\omega_c(\overline{H}) = \alpha(H)$ as required. $\diamond$

Claim C.1.17. *H does not have any vertex of degree 1.*

Assume that H has a vertex x of degree 1 and let y be the neighbour of x: the removal of y yields a bipartite graph. Hence $H \setminus \{x, y\}$ has a covering $\mathcal{Q}$ with $\alpha(H \setminus \{x, y\})$ cliques. Notice that if S is any maximum stable set of $H \setminus \{x, y\}$ then $S \cup \{x\}$ is a stable set of H. Hence $\alpha(H) > \alpha(H \setminus \{x, y\})$. Therefore $\mathcal{Q} \cup \{\{x, y\}\}$ is a covering with at most $\alpha(H)$ cliques of H. Thus

$$\alpha(H) = \omega_c(\overline{H}) \leq \chi_c(\overline{H}) \leq \chi(\overline{H}) \leq \alpha(H)$$

yields $\omega_c(\overline{H}) = \chi_c(\overline{H})$, a contradiction. $\diamond$

Claim C.1.18. *For every $1 \leq i \leq 2p+1$, the set A_i is a singleton.*

The proof is similar to the proof of Claim C.1.17. Assume that there is a set Q_i with at least two vertices $\{x, x'\}$. The vertex x has two neighbours y and z. The removal of the set of vertices $\{x, y, z\}$ yields a bipartite graph. Hence $H \setminus \{x, y, z\}$ has a covering $\mathcal{Q}$ with $\alpha(H \setminus \{x, y, z\})$ cliques. We have $\alpha(H) > \alpha(H \setminus \{x, y, z\})$.

Notice that x' is isolated in $H \setminus \{x, y, z\}$. Hence $\{x'\} \in \mathcal{Q}$. Thus $(\mathcal{Q} \setminus \{\{x'\}\}) \cup \{\{x', y\}, \{x, z\}\}$ is a covering with at most $\alpha(H)$ cliques of H. Since $H - x$ is strongly circular-perfect, this implies that H is strongly circular-perfect, a contradiction. $\diamond$

Therefore, every set B_i is empty due to Claim C.1.17 and every set A_i is a singleton due to Claim C.1.18. Thus H is an odd hole, a final contradiction. $\square$

In order to treat the stable set problem for triangle-free strongly circular-perfect graphs, we show that they belong to a subclass of the well-known t-perfect graphs for which a maximum weight stable set can be found in polynomial time [5]. A graph is *almost-bipartite* if it has a vertex v such that $G - v$ is bipartite; such graphs are t-perfect (see [5]).

Lemma C.1.19. *Interlaced odd holes are almost-bipartite.*

Proof. Let G be an interlaced odd hole and $((A_i)_{1\leq i\leq 2p+1}, (B_i)_{1\leq i\leq 2p+1})$ be a suitable partition of G. Obviously at least one of the sets A_i is a singleton $\{v\}$ and $G-v$ is bipartite, as v belongs to all odd holes of G. $\square$

As bipartite graphs are almost-bipartite, Lemma C.1.19 and Theorem C.1.15 imply:

Corollary C.1.20. *In a triangle-free strongly circular-perfect graph, a maximum weight stable set can be found in polynomial time.*

Remark. Interlaced odd holes are also near-bipartite (for every vertex v, $G-N(v)$ is bipartite), nearly-bipartite planar (a planar graph such that at most two faces are bounded by an odd number of edges), series-parallel (it does not contain a subdivision of K_4), strongly t-perfect (it does not contain a subdivision of K_4 such that all four circuits corresponding to triangles in K_4 are odd).

It is an open question whether there exists a polynomial time algorithm to recognize strongly circular-perfect graphs (resp. circular-perfect graphs). However, it is easy to derive such an algorithm for *triangle-free* strongly circular-perfect graphs from our characterization (see Algorithm C.1).

Input: a graph G
Output: boolean true if and only if G is triangle-free circular-perfect.
1: **Si** G is bipartite **Alors**
2: return TRUE
3: compute a shortest odd cycle $\mathcal{O} = (o_1, \dots, o_{2p+1})$. (Note that a triangle is an odd cycle and that computing a shortest odd cycle is much easier than finding out a shortest odd hole).
From now on, indices are modulo $2p+1$.
4: **Si** p=1 **Alors**
5: return FALSE
6: **Pour** $i \in 1 \dots 2p+1$ **:**
7: $B_i := \{v | \deg(v) = 1, | vo_i \in E(G)\}$
8: $A_i := \{v | vo_{i-1} \in E(G),\ vo_{i+1} \in E(G)\} \cup \{o_i\}$
9: **Pour** $i \in 1 \dots 2p+1$ **:**
10: **Si** $(|A_i| > 1$ and $(|A_{i+1}| > 1$ or $|A_{i-1}| > 1))$ or $(B_i \neq \emptyset$ and $|A_i| > 1)$ **Alors**
11: return FALSE
12: $V := \emptyset$; $E := \emptyset$
13: **Pour** $i \in 1 \dots 2p+1$ **:**
14: $V := V \cup A_i \cup B_i$
15: $E_i := A_i \times A_{i+1}$; $E'_i := A_i \times B_i$; $E := E \cup E_i \cup E'_i$
16: **Si** $V \neq V(G)$ or $E \neq E(G)$ **Alors**
17: return FALSE
18: return TRUE

Algorithme C.1: A polynomial time recognition algorithm for triangle-free strongly circular-perfect graphs

Theorem C.1.21. *Algorithm C.1 works correct in polynomial time.*

Sketch of the proof.

1-3 Recognizing a bipartite graph in polynomial time is a standard exercise.

4 The graph is not bipartite. If it is triangle-free without an odd hole then it is perfect, and therefore bipartite, a contradiction. Hence the graph has a triangle or a shortest odd hole. In both cases, there exists a shortest odd cycle $\mathcal{O}$ which can be exhibited in polynomial time [9].

5-7 If a shortest odd cycle has size 3 then the graph is not triangle-free.

8-11 The graph is triangle-free. With every vertex o_i of the shortest odd hole $\mathcal{O}$, we define the set B_i as the set of neighbours of o_i of degree 1, and A_i as the union of o_i and vertices of degree two with neighbours o_{i-1} and o_{i+1}.

12- Notice that if the graph is an interlaced odd hole, then the sets A_i and B_i should be a suitable partition of the vertex set of the graph. This is tested in the remaining part of the algorithm. □

Triangle-free minimal strongly circular-imperfect graphs

By the Strong Perfect Graph Theorem, triangle-free minimal imperfect graphs are odd holes. We prove a similar result for strongly circular-perfectness: triangle-free strongly circular-imperfect graphs are some odd holes with at most 2 extra-vertices. To be more precise, let us say that a graph G is an *extended odd hole* if it admits a proper partition into an induced odd hole $\mathcal{O} = \{o_1, \ldots, o_{2p+1}\}$ and a pair of vertices $\{x, y\}$ which is connected to $\mathcal{O}$ in one of the following ways:

(a) $\{o_1x, xy, o_4y\}$

(b) $\{o_1x, xy, o_2y\}$

(c) $\{o_1x, o_3x, xy, o_4y\}$

(d) $\{o_1x, o_3x, xy, o_2y\}$

(e) $\{o_1x, o_3x, xy, o_2y, o_4y\}$

(f) $\{o_1x, o_3x, o_2y, o_4y\}$

Theorem C.1.22. *A triangle-free graph G is minimal strongly circular-imperfect if and only if G is either the disjoint union of an odd hole and a singleton or an extended odd hole.*

Proof. ***Only if.*** Let G be a triangle-free minimal strongly circular-imperfect graph. If G does not have any induced odd hole then G is perfect, a contradiction. Let $\mathcal{O}$ be a shortest induced odd hole of G. Notice that $\mathcal{O} \subsetneq G$. Let $\{o_1, \ldots, o_{2p+1}\}$ be a labeling of the vertices of $\mathcal{O}$ the usual way (o_io_{i+1} is an edge of $\mathcal{O}$ for every $1 \le i \le 2p+1$, and indices modulo $2p+1$).

Claim C.1.23. *If there is a vertex x of degree 0 then G is the disjoint union of $\mathcal{O}$ and the singleton x.*

If x is of degree 0 then $x \notin \mathcal{O}$. By Lemma C.1.11, the induced subgraph $\mathcal{O} \cup \{x\}$ is strongly circular-imperfect, hence $G = \mathcal{O} \cup \{x\}$. ◇

Claim C.1.24. *If there is a unique vertex x of G outside $\mathcal{O}$ then G is the disjoint union of $\mathcal{O}$ and the singleton x.*

If x is not isolated, G is an interlaced odd hole by Lemma C.1.13, a contradiction. ◇

Thus, we may assume from now on, that G has at least two vertices outside $\mathcal{O}$. We have to prove that G is an extended odd hole.

Claim C.1.25. *Every vertex of G is of degree at least 2.*

By Claim C.1.23, every vertex is of degree at least 1. If there exists a vertex v in G of degree 1, then obviously $v \notin \mathcal{O}$. Notice that $G' = G - v$ is triangle-free strongly circular-perfect. Hence by Theorem C.1.15, G' is bipartite or an interlaced odd hole. The case G' bipartite is excluded, otherwise G would be also bipartite. Let $((A_i)_{i=1..2p+1}, (B_i)_{i=1..2p+1})$ be a suitable partition of G'. The neighbour w of v belongs obviously to $\mathcal{O}$ (if not, $\mathcal{O} \cup \{v\}$ would be a proper induced strongly circular-imperfect graph). Thus there exists an index i such that $w \in A_i$. If A_i is of size 1 then G is an interlaced odd hole, a contradiction. Hence there exists $t \in A_i \setminus \{w\}$. Thus $((\mathcal{O} \setminus \{w\}) \cup \{t\}) \cup \{v\}$ is a proper induced subgraph of G which is the disjoint union of an odd hole and a singleton, and is therefore strongly circular-imperfect, a final contradiction. ◇

Claim C.1.26. *If G has at least 3 vertices outside $\mathcal{O}$ then $G \setminus \mathcal{O}$ is a stable set and for every vertex v of G outside $\mathcal{O}$, there exists an index $f(v)$ such that $N_G(v) \cap \mathcal{O} = \{o_{f(v)}, o_{f(v)+2}\}$ (with indices modulo $2p+1$).*

Assume that there is an edge ab which is not incident to $\mathcal{O}$ and let c be a third vertex outside $\mathcal{O}$. Then $G - c$ is an interlaced odd hole with the edge ab which is not incident to the odd hole $\mathcal{O}$, a contradiction to Lemma C.1.14. Hence $G \setminus \mathcal{O}$ is a stable set. Let v be a vertex of G outside $\mathcal{O}$. Let w be another vertex of G outside $\mathcal{O}$. Since $G - w$ is an interlaced odd hole and $v \notin \mathcal{O}$, this implies with Claim C.1.25 that v has exactly two neighbours on $\mathcal{O}$, and that there exists an index $f(v)$ such that $N_G(v) \cap \mathcal{O} = \{o_{f(v)}, o_{f(v)+2}\}$ (with indices modulo $2p + 1$). $\diamond$

Claim C.1.27. *There are exactly two vertices of G outside $\mathcal{O}$.*

Assume that there are at least 3 vertices outside $\mathcal{O}$. Hence Claim C.1.26 applies: for every $v \notin \mathcal{O}$, let $f(v)$ be the index such that $N_G(v) \cap \mathcal{O} = \{o_{f(v)}, o_{f(v)+2}\}$ (with indices modulo $2p + 1$).

For every $1 \leq i \leq 2p + 1$, let A_i be the set of vertices $\{o_i\} \cup \{v | \, v \notin \mathcal{O},\ f(v) = i - 1\}$. Notice that the edge set of G is precisely $\cup_{i=1,\dots,2p+1} E_i$, where E_i denotes the set of all edges between A_i and A_{i+1}. Every set A_i is obviously non-empty. If there exists i such that A_i and A_{i+1} are both of size at least 2, then let $a_i \in A_i \setminus \{o_i\}$ and let $a_{i+1} \in A_{i+1} \setminus \{o_{i+1}\}$. Since there are at least 3 vertices outside $\mathcal{O}$, there is also a vertex z outside $\mathcal{O}$, distinct of a_i and a_{i+1}. Then $G - z$ is an interlaced odd hole, with a shortest odd hole $\mathcal{O}' = (\mathcal{O} \setminus \{o_i\}) \cup \{a_i\}$ and an edge $o_i a_{i+1}$ which is not incident to $\mathcal{O}'$: a contradiction with Lemma C.1.14. Hence $\forall 1 \leq i \leq 2p + 1$, $|A_i| > 1$ implies $|A_{i-1}| = |A_{i+1}| = 1$, (with indices modulo $2p + 1$).

Therefore G is an interlaced odd hole and is circular-perfect: a contradiction. Hence there are exactly two 2 vertices outside $\mathcal{O}$. $\diamond$

From now on, assume that x and y are the two distinct vertices of G outside $\mathcal{O}$. Since $G - y$ (resp. $G - x$) is an interlaced odd hole and $x \notin \mathcal{O}$ (resp. $y \notin \mathcal{O}$), this implies that there exists an index $f(x)$ (resp. $f(y)$) such that $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ or $N_G(x) \cap \mathcal{O} = \{o_{f(x)}\}$ (resp. $N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$ or $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$) (with indices modulo $2p + 1$).

Claim C.1.28. *If x is not adjacent to y then G is an extended odd hole of type f.*

Due to Claim C.1.25, we have $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$. Notice that if $f(x) \neq f(y) \pm 1 \pmod{2p + 1}$ then G is an interlaced odd hole, a contradiction. Hence $f(x) = f(y) \pm 1$ $(\mathrm{mod}\ 2p + 1)$ and G is an extended odd hole of type f. $\diamond$

In the following, we assume that x is adjacent to y. We have to prove that G is an extended odd hole of type a, b, c, d or e.

Claim C.1.29. *If $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$ then G is an extended odd hole of type e.*

Let $z = o_{f(y)+1}$. Notice that $\mathcal{O}' = (\mathcal{O} \setminus \{z\}) \cup \{y\}$ is an induced odd hole of $G - z$. If $z \neq o_{f(x)}$ or $o_{f(x)+2}$ then x is a vertex of $G - z$ outside $\mathcal{O}'$ with 3 neighbours in $\mathcal{O}'$. Hence $G - z$ is not an interlaced odd hole, a contradiction as it is not bipartite. Thus $f(x) = f(y) \pm 1$ and G is an extended odd hole of type e. $\diamond$

Claim C.1.30. *If ($N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$) or ($N_G(y) \cap \mathcal{O} = \{o_{f(y)}, o_{f(y)+2}\}$ and $N_G(x) \cap \mathcal{O} = \{o_{f(x)}\}$) then G is an extended odd hole of type c or d.*

Assume w.l.o.g. that $N_G(x) \cap \mathcal{O} = \{o_{f(x)}, o_{f(x)+2}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$. Let $z = o_{f(x)+1}$. Notice that $\mathcal{O}' = (\mathcal{O} \setminus \{z\}) \cup \{x\}$ is an induced odd hole of $G - z$. If $z = o_{f(y)}$ then G is an extended odd hole of type d. If $z \neq f(y)$ then y has two neighbours in $\mathcal{O}'$, and one of them is x. Since $G - z$ is an interlaced odd hole, this implies that $o_{f(y)}$ is at distance 2 in $\mathcal{O}'$ from x. Hence $f(y) = f(x) + 3$ of $f(y) = f(x) - 1$. In both cases, G is an extended odd hole of type c. $\diamond$

Claim C.1.31. *If $N_G(x) \cap \mathcal{O} = \{o_{f(x)}\}$ and $N_G(y) \cap \mathcal{O} = \{o_{f(y)}\}$ then G is an extended odd hole of type a or b.*

Assume w.l.o.g. that $f(x) \geq f(y)$. The case $f(x) = f(y)$ is excluded as G is triangle-free. If $f(x) - f(y)$ is even, notice that
$\{x, y, o_{f(y)}, o_{f(y)+1}, \dots, o_{f(x)}\}$ induces an odd hole. If $f(x) = f(y) + 2p$ then $f(x) = 1$, $f(y) = 2p + 1$ and G is an extended odd hole of type b. If $f(x) < f(y) + 2p$ then the subgraph $G \setminus \{o_{f(x)+1}\}$ is an interlaced odd hole. Hence $f(x) + 2$ is adjacent to the odd hole $\{x, y, f(y), f(y) + 1, \dots, f(x)\}$. Thus $(f(x) + 2) + 1 = f(y)$

$\pmod{2p+1}$. This implies that G is an extended odd hole of type a as $f(y) = f(x)+3 \pmod{2p+1}$. If $f(x)-f(y)$ is odd, notice that $\{x,y\} \cup \{1,2\ldots,o_{f(y)}\} \cup \{o_{f(x)}, o_{f(x)+1}, \ldots, 2p+1\}$ induces an odd hole. If $f(x) = f(y)+1$ then G is an extended odd hole of type b. If $f(x) > f(y)+1$ then the subgraph $G \backslash \{o_{f(y)+1}\}$ is an interlaced odd hole. Hence $o_{f(y)+2}$ is adjacent to the odd hole $\{x,y\} \cup \{1,2\ldots,o_{f(y)}\} \cup \{o_{f(x)}, o_{f(x)+1}, \ldots, 2p+1\}$. Thus $(f(y)+2)+1 = f(x) \pmod{2p+1}$. This implies that G is an extended odd hole of type a as $f(x) = f(y)+3 \pmod{2p+1}$. $\diamond$

If. The disjoint union of an odd hole and a singleton is strongly circular-imperfect due to Lemma C.1.11. If G is an extended odd hole, then G is strongly circular-imperfect as no extended odd hole is an interlaced odd hole. Let v be a vertex of G. It is straightforward to check that $G-v$ is bipartite or an interlaced odd hole, and therefore strongly circular-perfect, whatever the type (a, b, c, d, e or f) of G is. $\square$

References

[1] J. Bang-Jensen and J. Huang. Convex-round graphs are circular-perfect. *J. of Graph Theory*, 40:182–184, 2002.

[2] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[3] J.A. Bondy and P. Hell. A note on the star chromatic number. *J. of Graph Theory*, 14:479–482, 1990.

[4] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[5] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[6] A. Gyárfás. Problems for the world surrounding perfect graphs. *Zastos. Mat.*, XIX:413–441, 1987.

[7] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12:373–388, 1975.

[8] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[9] B. Monien. The complexity of determining a shortest cycle of even length. *Comput.*, 31:355–369, 1983.

[10] A. Pêcher, A. Wagler, and X. Zhu. Three classes of minimal circular imperfect graphs. In *Proceedings of GRACO05*, volume 19 of *Electonic Notes in Discrete Mathematics*, pages 9–15. Elsevier, avril 2005.

[11] J. Ramirez-Alfonsin and B. Reed, editors. *Perfect Graphs*. Wiley, 2001.

[12] F.B. Shepherd. Applying Lehman's theorem to packing problems. *Mathematical Programming*, 71:353–367, 1995.

[13] A. Vince. Star chromatic number. *J. of Graph Theory*, 12:551–559, 1988.

[14] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2:149–152, 2004.

[15] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3):186–209, 2005.

C.2 Characterizing and bounding the imperfection ratio for some classes of graphs

par S. Coulonges, A. Pêcher et A. Wagler

A paraître dans Mathematical Programming Series A [27]

Perfect graphs constitute a well-studied graph class with a rich structure, reflected by many characterizations with respect to different concepts. Perfect graphs are, for instance, precisely those graphs G where the stable set polytope STAB(G) equals the fractional stable set polytope QSTAB(G). The dilation ratio $\min\{t : \text{QSTAB}(G) \subseteq t\ \text{STAB}(G)\}$ of the two polytopes yields the imperfection ratio of G. It is NP-hard to compute and, for most graph classes, it is even unknown whether it is bounded. For graphs G such that all facets of STAB(G) are rank constraints associated with antiwebs, we characterize the imperfection ratio and bound it by $3/2$. Outgoing from this result, we characterize and bound the imperfection ratio for several graph classes, including near-bipartite graphs and their complements, namely quasi-line graphs, by means of induced antiwebs and webs, respectively.

Introduction

Graph coloring is an important concept with a large variety of applications; calculating the minimum number $\chi(G)$ of required colors is NP-hard in general. The size $\omega(G)$ of a largest clique in G is a trivial lower bound on $\chi(G)$, but is also hard to evaluate and can be arbitrarily bad [10].

Berge [1] proposed to call a graph G *perfect* if the two parameters coincide for each induced subgraph $G' \subseteq G$. He conjectured that they are precisely the graphs without induced chordless cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$ (the complement $\overline{G}$ has the same node set as G but two nodes are adjacent in $\overline{G}$ iff they are non-adjacent in G). This famous characterization was recently achieved by Chudnovsky et al. [3],

Perfect graphs turned out to be an important class with a rich structure (see [11] for surveys on many aspects of perfect graphs). In particular, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [9]. The latter result relies on the characterization of the *stable set polytope* STAB(G) of perfect graphs by means of facet-inducing inequalities. STAB(G) is defined as the convex hull of the incidence vectors of all stable sets of G. A canonical relaxation of STAB(G) is the *fractional stable set polytope* QSTAB(G) given by all "trivial" facets $x_i \geq 0$ for all nodes i of G and by the *clique constraints* $\sum_{i \in Q} x_i \leq 1$ for all cliques $Q \subseteq G$. We have STAB$(G) \subseteq$ QSTAB(G) for any graph but equality for perfect graphs only [5]. In principle, computing the stability number $\alpha(G)$ for a perfect graph G could, therefore, be done by solving the linear program $\max \mathbb{1}^T x, x \in$ QSTAB(G). However, this does not work directly since the clique constraints cannot be separated in polynomial time [9], but only via a detour, the famous theta-body TH(G) sandwiched between STAB(G) and QSTAB(G): for perfect graphs G, the stability number equals $\vartheta(G) = \max\{\mathbb{1}^T x : x \in \text{TH}(G)\}$ which can be evaluated in polynomial time for any graph [9].

For all imperfect graphs G we have that STAB$(G) \subset$ QSTAB(G). It is natural to use the difference between the two polytopes to determine how far a certain imperfect graph is away from being perfect. Gerke and McDiarmid introduced in [7] the *imperfection ratio* $\text{imp}(G)$ of a graph G as the maximum ratio of the fractional chromatic number $\chi_f(G)$ and the clique number $\omega(G)$ in their weighted versions, taken over all positive weight vectors. They showed that $\text{imp}(G)$ is invariant under taking complements and can be expressed as the dilation ratio

$$\text{imp}(G) = \min\{t : \text{QSTAB}(G) \subseteq t\,\text{STAB}(G)\}$$

of STAB(G) and QSTAB(G). This immediately implies $\text{imp}(G) = 1$ for any perfect graph G, and graphs with small imperfection ratios can be seen as close to perfection: Such graphs admit nice coloring properties according to the original definition of $\text{imp}(G)$, and the value $\vartheta(G)$ yields a good approximation for the stability number by $\vartheta(G) \leq \text{imp}(G)\ \alpha(G)$. In addition, such graphs often admit well-described stable set polytopes (see below) and behave nicely in the context of graph entropy (see Chapter 13 in [11]).

However, evaluating $\text{imp}(G)$ is NP-hard and there does not exist a general upper bound on the imperfection ratio [7, 8]. So far, upper bounds are only known for some classes, among them minimally imperfect, h-perfect, and line graphs whose stable set polytopes have certain *rank constraints* as only non-trivial facets. Rank constraints $x(G') = \sum_{i \in G'} x_i \leq \alpha(G')$ associated with arbitrary induced subgraphs $G' \subseteq G$ are natural generalizations of clique constraints as $\alpha(G') = 1$ holds iff G' is a clique. We call a graph G *rank-perfect* if STAB(G) is given by trivial and rank constraints only. By construction, perfect and h-perfect graphs are rank-perfect (as clique resp. clique and rank constraints associated with odd holes suffice to describe the stable set polytopes).

A *line graph* $L(G)$ is obtained by turning adjacent edges of a root graph G into adjacent nodes of $L(G)$, thus Edmonds' result on the matching polytope [6] yields a description of STAB$(L(G))$, implying that line graphs are rank-perfect.

An *antiweb* $K_{n/k}$ is a graph with n nodes $0, \ldots, n-1$ and edges ij iff $k \leq |i-j| \leq n-k$, where $n \geq 2k$. Antiwebs are also known as circular cliques, rational complete graphs or circular-cliques in the litterature (see [15] for instance). Antiwebs include all cliques $K_k = K_{k/1}$, odd antiholes $\overline{C}_{2k+1} = K_{2k+1/2}$, and odd holes $C_{2k+1} = K_{2k+1/k}$, see Figure C.3.

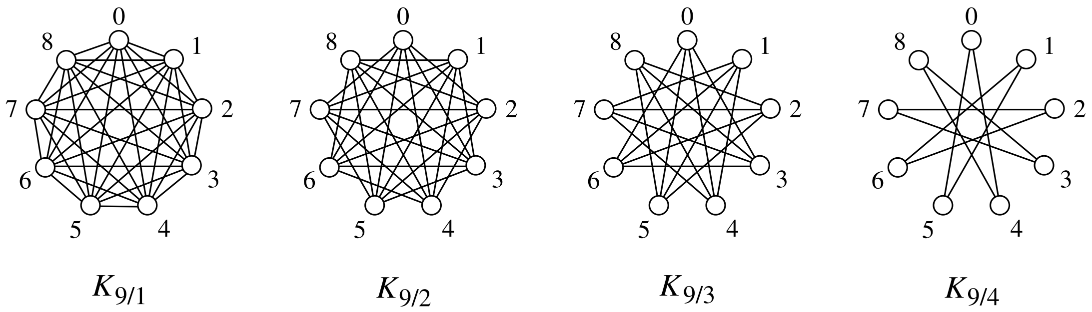


Figure C.3: The antiwebs on nine nodes.

Line graphs and antiwebs are further examples of rank-perfect graphs.

In [13], it is shown that the only non-trivial facets of stable set polytopes of antiwebs are rank constraints associated with cliques and *prime* antiwebs, i.e., with antiwebs $K_{n/\alpha}$ where $\gcd(n, \alpha) = 1$. We call a graph with such a description of its stable set polytope *a-perfect*. By [14], the complements of fuzzy circular interval graphs are a-perfect as well. The latter are defined as follows. Let $\mathcal{C}$ be a circle, $\mathcal{I}$ a collection of intervals in $\mathcal{C}$ without proper containments and common endpoints, and V a multiset of points in $\mathcal{C}$. The *fuzzy circular interval graph* $G(V, \mathcal{I})$ has node set V and two nodes are adjacent if both belong to one interval $I \in \mathcal{I}$, where edges between different endpoints of the same interval may be omitted.

Gerke and McDiarmid showed in [7] that the imperfection ratio of any h-perfect or line graph G relies on its shortest odd hole which implies $\operatorname{imp}(G) \leq \frac{5}{4}$ (see next section for details); the aim of this paper is to extend this result further.

On the one hand, we establish the same upper bound for the imperfection ratio of all *semi-line graphs*, that are either line graphs or quasi-line graphs without representation as a fuzzy circular interval graph. A *quasi-line graph* has the property that the neighborhood of any node splits into two cliques. Semi-line graphs form a superclass of line graphs as there exist quasi-line graphs neither being fuzzy circular interval nor line, see for instance the graphs depicted in Figure C.2 (the gray-filled nodes induce an obstruction for line graphs, the squared nodes an obstruction for fuzzy circular interval graphs).

On the other hand, we characterize the imperfection ratio of a-perfect graphs G by means of induced prime antiwebs which implies $\operatorname{imp}(G) < \frac{3}{2}$ (Section C.2). We discuss in Section C.2 how this result can be extended to other graph classes, namely, to quasi-line graphs and their complements, the *near-bipartite graphs* (where the non-neighbors of every node split into two stable sets).

Finally, we close with some concluding remarks.

Generalizing known results on the imperfection ratio

From $\operatorname{imp}(G) = \min\{t : \text{QSTAB}(G) \subseteq t\,\text{STAB}(G)\}$ due to [7], we obtain an immediate consequence:

Lemma C.2.1. *For any graph G,*

$$\operatorname{imp}(G) \quad = \quad \max\{a^T y : a \in \mathcal{F}(G), y \in \text{QSTAB}(G)\} \tag{C.4}$$

where $\mathcal{F}(G) = \{a \in \mathbb{R}^{|G|} : a^T x \leq 1$ *is a non-trivial facet of* $\text{STAB}(G)\}$.

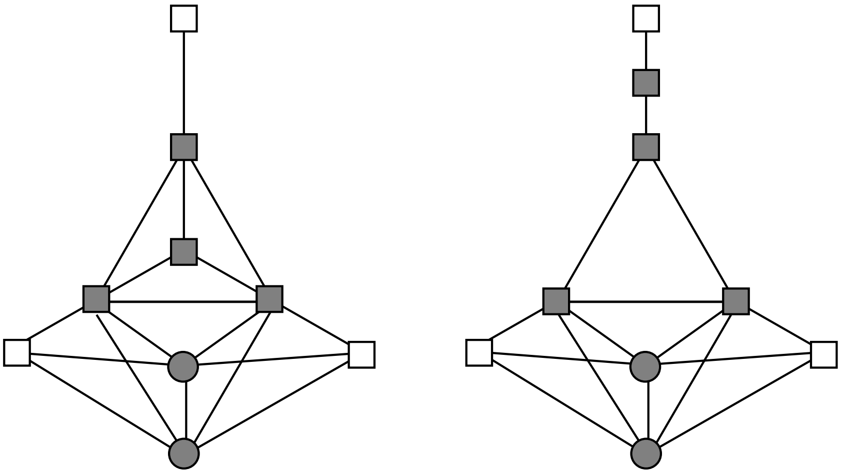


Figure C.4: Quasi-line graphs not being fuzzy circular interval or line graphs

Proof. We have $\operatorname{imp}(G) = \min\{t : (\frac{1}{t}y) \in \mathrm{STAB}(G) \forall\, y \in \mathrm{QSTAB}(G)\}$, i.e., $(\frac{1}{t}y)$ has to satisfy all non-trivial facet defining inequalities $a^T x \leq 1$ of $\mathrm{STAB}(G)$ (scaled to have right hand side 1) or, equivalently,

$$\operatorname{imp}(G) \quad = \quad \min\{t : a^T(\tfrac{1}{t}y) \leq 1 \;\forall\, a \in \mathcal{F}(G), \forall\, y \in \mathrm{QSTAB}(G)\}$$

which finally implies $\operatorname{imp}(G) = \min\{t : a^T y \leq t\} = \max\{a^T y\}$ taken over all $a \in \mathcal{F}(G)$ and all vectors y in $\mathrm{QSTAB}(G)$. $\square$

This suggests that knowledge on the facet-defining system of $\mathrm{STAB}(G)$ should help to determine $\operatorname{imp}(G)$. Moreover, the following two lower bounds for the imperfection ratio of any graph are known from [7]

$$\begin{aligned} \operatorname{imp}(G) &\geq \tfrac{|G|}{\alpha(G)\omega(G)} \\ \operatorname{imp}(G) &\geq \operatorname{imp}(G') \;\forall G' \subseteq G \end{aligned}$$

and we obtain

$$\operatorname{imp}(G) \quad \geq \quad \max\left\{\tfrac{|G'|}{\alpha(G')\omega(G')} : G' \subseteq G\right\} \tag{C.5}$$

by combining the two bounds. It is, therefore, natural to ask for which graphs we can identify the crucial subgraphs to obtain equality in (C.5). In [7] it is shown that

$$\operatorname{imp}(G) \quad = \quad \max\{\tfrac{2k+1}{2k} : C_{2k+1} \subseteq G\}$$

whenever G is a line graph or h-perfect and

$$\operatorname{imp}(G) \quad = \quad \max\{\tfrac{2k+1}{2k} : C_{2k+1}, \overline{C}_{2k+1} \subseteq G\}$$

for all graphs G where STAB(G) is given by rank constraints associated with cliques, odd holes, and odd antiholes only. Since C_5 is the shortest odd (anti)hole, the imperfection ratio of all such graphs is at most $\frac{5}{4}$.

We are going to exhibit that the crucial subgraphs for a-perfect graphs are prime antiwebs, thereby generalizing the above results from [7] as cliques, odd holes, and odd antiholes are special antiwebs.

Theorem C.2.2. *For any a-perfect graph G, the imperfection ratio is given by*

$$\operatorname{imp}(G) \quad = \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n'/\alpha'} \subseteq G \text{ is prime}\}$$

where $\omega' = \lfloor n'/\alpha' \rfloor$ holds.

Proof. The stable set polytope of any a-perfect graph has as non-trivial facets only rank constraints associated with cliques and prime antiwebs, i.e.,

$$\text{STAB}(G) = \text{QSTAB}(G) \cap \{x \in \mathbb{R}^n : x(K_{n'/\alpha'}) \leq \alpha' \; \forall K_{n'/\alpha'} \subseteq G \text{ prime}\}$$

where n stands for the number of nodes in G. In particular,

$$\mathcal{F}(G) \quad = \quad \{\tfrac{1}{\alpha'}\chi^{K_{n'/\alpha'}} : K_{n'/\alpha'} \subseteq G,\ \gcd(n', \alpha') = 1\}$$

follows, where $\chi^{K_{n'/\alpha'}}$ stands for the incidence vector of $K_{n'/\alpha'}$. Thus, Lemma C.2.1 yields that

$$\text{imp}(G) \quad = \quad \max\{\tfrac{1}{\alpha'}y(K_{n'/\alpha'}) : K_{n'/\alpha'} \subseteq G \text{ is prime}, y \in \text{QSTAB}(G)\}$$

as we clearly have

$$(\chi^{K_{n'/\alpha'}})^T y = \sum_{i \in K_{n'/\alpha'}} y_i = y(K_{n'/\alpha'}).$$

Furthermore, $y(K_{n'/\alpha'}) \leq \frac{n'}{\omega'}$ follows as each node of $K_{n'/\alpha'}$ can be covered ω' times by the n' maximum cliques $Q_i = \{i, i+\alpha', \ldots, i+(\omega'-1)\alpha'\}$ for $1 \leq i \leq n'$ of $K_{n'/\alpha'}$ (all these cliques are distinct as $\gcd(\alpha', n') = 1$). Thus,

$$\text{imp}(G) \quad \leq \quad \max\{\tfrac{1}{\alpha'}\tfrac{n'}{\omega'} : K_{n'/\alpha'} \subseteq G \text{ is prime}\}$$

and combining this with inequality (C.5)

$$\text{imp}(G) \quad \geq \quad \max\{\tfrac{n'}{\alpha'\omega'} : K_{n'/\alpha'} \subseteq G\}$$

from above finally yields equality, as required. □

As a consequence, we know the imperfection ratios of all subclasses of a-perfect graphs, including antiwebs [13], co-fuzzy circular interval graphs [14], and their complementary classes. In addition, we obtain the following upper bound:

Corollary C.2.3. *For any a-perfect G, we have* $\text{imp}(G) < \frac{3}{2}$.

Proof. Let $K_{n'/\alpha'}$ be a prime antiweb and $n' = \alpha'\omega' + r'$ with $0 \leq r' < \alpha'$ and $\alpha', \omega' \geq 2$. We have $\frac{n'}{\alpha'\omega'} = 1 + \frac{r'}{\alpha'\omega'} < 1 + \frac{1}{\omega'}$. □

Remark C.2.4. This bound is best possible, as for instance

$$\text{imp}(K_{3\alpha-1/\alpha}) = \frac{3\alpha - 1}{2\alpha} \to \frac{3}{2} \text{ if } \alpha \to \infty$$

holds, see also [7].

We shall extend this result further by proving a general result for complete joins of facet-producing subgraphs. The *complete join* $G_1 * G_2$ of two disjoint graphs G_1 and G_2 is obtained by joining every node of G_1 and every node of G_2 by an edge. For a collection $\mathcal{C}$ of graphs, we call a graph G *joined $\mathcal{C}$-perfect* if every facet-inducing subgraph of G is the complete join of graphs in $\mathcal{C}$.

Theorem C.2.5. *For every joined $\mathcal{C}$-perfect graph G, we have that*

$$\text{imp}(G) = \max\{\text{imp}(G_i) : G_i \subseteq G, G_i \in \mathcal{C}\}.$$

Proof. Since $\text{imp}(G) = \max\{a^T y : a \in \mathcal{F}(G), y \in \text{QSTAB}(G)\}$ holds by Lemma C.2.1, we conclude

$$\text{imp}(G) = \max\{\text{imp}(G_a) : a \in \mathcal{F}(G)\}$$

where G_a denotes the subgraph of G induced by all nodes i with $a_i \neq 0$. Since G is joined $\mathcal{C}$-perfect, every such graph G_a is the complete join of graphs $G_1, \ldots, G_k$ in $\mathcal{C}$. We have

$$\mathrm{imp}(G_1 * \ldots * G_k) = \max\{\mathrm{imp}(G_1), \ldots, \mathrm{imp}(G_k)\}$$

by [7] (who proved this relation for the disjoint union of graphs and the invariance of the imperfection ratio under taking complements, thus the same applies to complete joins). Combining both facts yields

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(G_i) : G_i \subseteq G, G_i \in \mathcal{C}\}$$

which proves the assertion. □

We call a graph *joined a-perfect* if all facet-inducing subgraphs are complete joins of antiwebs.

Corollary C.2.6. *For any joined a-perfect G, the imperfection ratio is*

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(K_{n'/\alpha'}) : K_{n'/\alpha'} \subseteq G \text{ prime}\}$$

and it is less than $\frac{3}{2}$.

Consequences for quasi-line and near-bipartite graphs

The aim of this section is to discuss consequences of the previous results for quasi-line graphs and near-bipartite graphs by characterizing and bounding their imperfection ratio. The following structural result on quasi-line graphs was recently obtained by Chudnovsky and Seymour [4].

Theorem C.2.7. [4] *A connected quasi-line graph G is either a fuzzy circular interval graph or STAB(G) is given by trivial, clique, and rank constraints $(\mathcal{Q}, 2)$*

$$\sum_{i \in I(\mathcal{Q},2)} x_i \leq \frac{|\mathcal{Q}|-1}{2}$$

associated with clique families $\mathcal{Q}$ where $|\mathcal{Q}|$ is odd and $I(\mathcal{Q}, 2)$ contains all nodes belonging to at least two cliques in $\mathcal{Q}$.

Lemma C.2.8. *For every semi-line graph G, we have* $\mathrm{imp}(G) \leq \frac{5}{4}$.

Proof. The stable set polytope of such a graph G has as non-trivial facets only clique constraints and rank constraints $(\mathcal{Q}, 2)$ with $|\mathcal{Q}|$ odd by Theorem C.2.7.

If $|\mathcal{Q}| = 3$ then $(\mathcal{Q}, 2)$ is a clique constraint as the intersection of some cliques is either empty or a clique; thus $I(\mathcal{Q}, 2)$ is a clique and $\frac{|\mathcal{Q}|-1}{2} = 1$. I.e.,

$$\mathrm{STAB}(G) = \mathrm{QSTAB}(G) \cap \{x \in \mathbb{R}^n : x(I(\mathcal{Q}, 2)) \leq \tfrac{|\mathcal{Q}|-1}{2},\ |\mathcal{Q}| > 3 \text{ is odd}\}$$

where n stands for the number of nodes in G. Consider a vector $x \in \mathrm{QSTAB}(G)$. For any clique family $\mathcal{Q}$ of G, we have $x(I(\mathcal{Q}, 2)) \leq \frac{|\mathcal{Q}|}{2}$ as each node of $I(\mathcal{Q}, 2)$ is covered at least twice by the cliques in $\mathcal{Q}$. Let $y = \frac{4}{5}\,x$. Note that y belongs to QSTAB(G) as $\frac{4}{5} < 1$ holds. We are going to show $y \in \mathrm{STAB}(G)$ by verifying $y(I(\mathcal{Q}, 2)) \leq \frac{|\mathcal{Q}|-1}{2}$ for all clique families $\mathcal{Q}$ with $|\mathcal{Q}| \geq 5$. Indeed, we obtain

$$y(I(\mathcal{Q}, 2)) = \tfrac{4}{5}\, x(I(\mathcal{Q}, 2)) \leq \tfrac{4}{5} \tfrac{|\mathcal{Q}|}{2} \leq \tfrac{|\mathcal{Q}|-1}{|\mathcal{Q}|} \tfrac{|\mathcal{Q}|}{2} = \tfrac{|\mathcal{Q}|-1}{2}$$

by $|\mathcal{Q}| \geq 5$, as required. From $y = \frac{4}{5}\, x \in \mathrm{STAB}(G)$ follows $x \in \frac{5}{4}\,\mathrm{STAB}(G)$, implying $\mathrm{QSTAB}(G) \subseteq \frac{5}{4}\,\mathrm{STAB}(G)$ as $x \in \mathrm{QSTAB}(G)$ was chosen arbitrarily. □

Corollary C.2.9. *For every quasi-line graph G, we have* $\mathrm{imp}(G) < \frac{3}{2}$.

Proof. Consider a quasi-line graph G and any of its connected components G'. If G' is a fuzzy circular interval graph, then $\overline{G}'$ is in particular a-perfect due to [14], hence $\mathrm{imp}(G') < \frac{3}{2}$ follows from Theorem C.2.2 and the invariance of the imperfection ratio under complementation. Otherwise, Lemma C.2.8 shows $\mathrm{imp}(G') \leq \frac{5}{4}$. As $\mathrm{imp}(G)$ clearly equals the maximum taken over the imperfection ratios of all its connected components G', we finally obtain $\mathrm{imp}(G) < \frac{3}{2}$. □

The invariance of the imperfection ratio under complementation implies the same bound for near-bipartite graphs. In addition, the description of the stable set polytope for such graphs from Shepherd [12] allows us to characterize their imperfection ratio by means of its induced antiwebs.

Theorem C.2.10. [12] For any near-bipartite graph G, the only non-trivial facets of their stable set polytope are constraints

$$\sum_{i \leq k} \frac{1}{\alpha_i} x(K_{n_i/\alpha_i}) \leq 1$$

associated with the complete join of prime antiwebs $K_{n_1/\alpha_1}, \ldots, K_{n_k/\alpha_k}$.

Thus, near-bipartite graphs are particularly joined a-perfect graphs and we immediatly obtain from Corollary C.2.6:

Corollary C.2.11. *For any near-bipartite graph G, the imperfection ratio is given by*

$$\mathrm{imp}(G) = \max\{\mathrm{imp}(K_{n'/\alpha'}) : K_{n'/\alpha'} \subseteq G \text{ prime}\}$$

and it is less than $\frac{3}{2}$.

Shepherd [12] showed for the subclass of co-line graphs further that the only prime antiwebs are odd antiholes. Combining this result and Corollary C.2.6 reproves Gerke and McDiarmid's result for (co-)line graphs [7].

Concluding remarks and open problems

In this paper, we extended the results of Gerke and McDiarmid [7] that the imperfection ratio of any line or h-perfect graph depends on its shortest odd hole and is at most $\frac{5}{4}$ in two ways: On the one hand, we established the same bound for a superclass of line graphs, the semi-line graphs. On the other hand, we extended the result on h-perfect graphs to a-perfect and joined a-perfect graphs by proving that their imperfection ratios rely on prime antiwebs only and are less than $\frac{3}{2}$. Finally, we discussed consequences for the imperfection ratios of near-bipartite and quasi-line graphs, see the chart in Figure C.2 for a summary of the obtained bounds. The bound $\mathrm{imp}(G) < \frac{3}{2}$ is tight for the graphs in those classes, as there exist antiwebs with an imperfection ratio arbitrarily close to $\frac{3}{2}$, see Remark C.2.4 (complements of antiwebs are quasi-line).

Our results imply that all such graphs can be considered as close to perfection. In particular, we obtain an approximation ratio of $\vartheta(G) < \frac{3}{2}\,\alpha(G)$ for any such graph. This is interesting for a-perfect and joined a-perfect graphs: as antiweb constraints cannot be separated in polynomial time [2], we cannot simply compute their stability number $\alpha(G)$ as a linear program, although the facet-description of their stable set polytope is known. In addition, it is open to find the crucial prime antiwebs in those graphs, in order to compute the imperfection ratio and to know the facets explicitly.

Finally, it is open for which other graph classes prime antiwebs are the (only) crucial subgraphs. It is very likely for circular-arc graphs, unit disc graphs might be other candidates, whereas it is not true for claw-free graphs (they certainly constitute a superclass of quasi-line graphs, but there is no upper bound for the imperfection ratio of claw-free graphs–as they contain all graphs with stability number two and, thus, graphs with arbitrarily high imperfection ratio [7]).

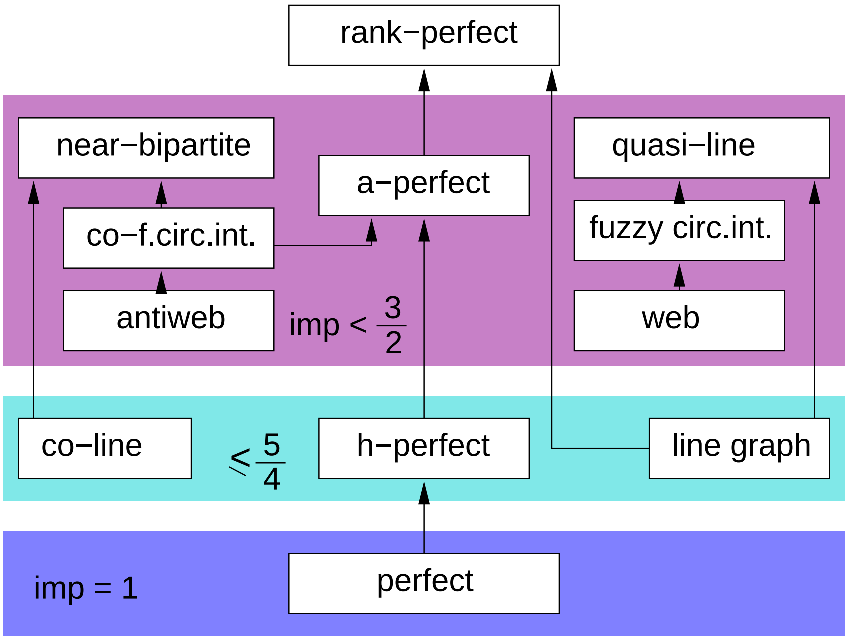


Figure C.5: Inclusion relations and bounds for the imperfection ratio

References

[1] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[2] E. Cheng and S. de Vries. Antiweb inequalities: Strength and intractability. *Congr. Numeration*, 152:5–19, 2001.

[3] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[4] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[5] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[6] J.E. Edmonds. Maximum matching and a polyhedron with (0,1) vertices. *J. Res. Nat. Bur. Standards*, 69(B):125–130, 1965.

[7] S. Gerke and C. McDiarmid. Graph imperfection. *J. of Combinatorial Theory B*, 83:58–78, 2001.

[8] S. Gerke and C. McDiarmid. Graph imperfection ii. *J. of Combinatorial Theory B*, 83:59–101, 2001.

[9] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[10] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[11] J. Ramirez-Alfonsin and B. Reed, editors. *Perfect Graphs*. Wiley, 2001.

[12] F.B. Shepherd. Applying Lehman's theorem to packing problems. *Mathematical Programming*, 71:353–367, 1995.

[13] A. Wagler. Antiwebs are rank-perfect. *4OR*, 2:149–152, 2004.

[14] A. Wagler. On rank-perfect subclasses of near-bipartite graphs. *4OR*, pages 329–336, 2005.

[15] X. Zhu. Circular chromatic number: a survey. *Discrete Mathematics*, 229:371–410, 2001.

C.3 On facets of stable set polytope of claw-free graphs with stability number three

par A. Pêcher et A. Wagler
A paraître dans Discrete Applied Mathematics [90]

Providing a complete description of the stable set polytopes of claw-free graphs is a long-standing open problem. Eisenbrand et al. recently achieved a breakthrough for the subclass of quasi-line graphs. As a consequence, every non-trivial facet of their stable set polytope has at most two different, but arbitrarily high left hand side coefficients. For the graphs with stability number two, Cook showed that all their non-trivial facets are 1/2-valued. For claw-free but not quasi-line graphs with stability number at least four, Stauffer conjectured that the same holds true. In contrary, there are known claw-free graphs with stability number three which induce facets with up to 8 different left hand side coefficients. We prove that the situation is even worse: for every positive integer b, we exhibit a claw-free graph with stability number three inducing a facet with b different left hand side coefficients.

Introduction

The *stable set polytope* $\text{STAB}(G)$ of a graph G is defined as the convex hull of the incidence vectors of all its stable sets. For standard definitions and general bacground on linear programming, we refer to Grötschel, Lovász and Schrijver's textbook [6]. The description of $\text{STAB}(G)$ by means of facet-defining inequalities is unknown for most graphs. A graph is *claw-free* if the neighbourhood of any vertex does not contain any stable set of size 3. Providing a complete description of the stable set polytopes of claw-free graphs is a long-standing open problem [6]. A characterization of the 0/1-valued facets, called *rank* facets, in stable set polytopes of claw-free graphs was given by Galluccio and Sassano [4]. However, claw-free graphs have non-rank facets in general and even conjectures regarding their non-rank facets were formulated only recently [8, 11].

There are subclasses of claw-free graphs where rank facets suffice, e.g., the *line graphs*, obtained by taking the edges of a root graph H as vertices and connecting two vertices of the line graph if and only if the corresponding edges of H are incident. All facets of their stable set polytopes are known from matching theory [2], namely, clique constraints and certain rank constraints coming from odd set inequalities.

Ben Rebea [9] generalized the odd set inequalities to so-called clique family inequalities. He claimed and Eisenbrand et al. [3] recently proved that clique family inequalities suffice for *quasi-line graphs*, that are graphs where the neighbourhood of any vertex can be partitioned into two cliques. As a consequence, every non-trivial facet of a quasi-line graph is of the form $k\sum_{v\in V_1} x_v + (k+1)\sum_{v\in V_2} x_v \leq b$ for some positive integers k and b, and non-empty sets of vertices V_1 and V_2. This implies that their facets have *at most two different* left hand side coefficients.

Cook showed (see [10]) that for a graph G with stability number $\alpha(G) = 2$, all non-trivial facets of $\text{STAB}(G)$ are of the form $1\sum_{v\in V_1} x_v + 2\sum_{v\in V_2} x_v \leq 2$. Stauffer conjectured [11] that all non-rank facets of claw-free but not quasi-line graphs G with $\alpha(G) \geq 4$ are also of this type. As a consequence, every non-trivial facet of a claw-free but not quasi-line graph G with $\alpha(G) \neq 3$ would be 1/2-valued.

We have a different situation in the case of graphs G with stability number $\alpha(G) = 3$. It is already known that such graphs have facets with up to 8 different left hand side coefficients [5, 7, 8], and all the difficult facets of claw-free graphs occur in this case (provided Stauffer's conjecture is true).

In this paper, we support the feeling further that the case with $\alpha(G) = 3$ is indeed the difficult one. For that we exhibit, for every integer $b \geq 4$, a claw-free graph with stability number three whose stable set polytope has a facet with b different left hand side coefficients.

Theorem C.3.1. *For every positive integer $b \geq 5$, there exists a claw free graph G_b with maximum stable set size 3 and a partition of its vertex set into $b-1$ non-empty subsets $V_1, V_2, \ldots, V_{b-1}$ such that*

$$\sum_{v \in V_1} x_v + 2 \sum_{v \in V_2} x_v + \ldots + (b-1) \sum_{v \in V_{b-1}} x_v \quad \leq \quad b$$

is a facet of its stable set polytope.

Proof of Theorem 1

We now proceed to give a description of the graph G_b. The result is based on [8] where we proved that all non-rank facets associated with claw-free graphs with stability number 3 belong to only one class of inequalities: so-called co-spanning 1-forest constraints. A *co-spanning 1-forest constraint* of a graph G with n vertices is a facet of $\mathrm{STAB}(G)$ whose n tight independent stable sets, called *roots*, correspond in the complementary graph to the following cliques: the edges of a 1-forest H (consisting of tree and 1-tree components, where a 1-tree is a tree with an extra edge) and as many triangles as H has tree-components. (Note that all edges of a tree are linearly independent; as we have in a tree one edge less than vertices, we need to compensate this missing root for each tree by one triangle. A 1-tree has as many edges as vertices, and all its edges are independent provided its unique cycle is an odd hole, i.e., a chordless cycle of odd length ≥ 5.)

Our construction of G_b uses in fact a 1-forest consisting of tree components only. For that, we proceed in 2 steps:

- firstly, we define a graph H_b with clique number 3 and n maximal cliques where n is the number of vertices of H_b. We have to assign appropriate weights and to ensure that all those cliques are independent, in order to obtain the studied roots of the facet;

- secondly, we define a graph H'_b by adding an appropriate set of edges in order to make the complementary graph G_b of H'_b claw-free, with the co-spanning 1-forest constraint associated to H_b.

Let $k = b - 2$ and $n = 5k - 3$. We define the graph H_b as follows (see Fig. 1): the vertex set V of H_b consists of the n vertices $\{x_0, x_1, \ldots, x_{4k-6}\} \cup \{w_0, w_1, \ldots, w_{k-1}\} \cup \{y_0, y_1\}$, and the graph H_b has exactly the following n maximal cliques:

- $k-1$ triangles $U_1, U_2, \ldots, U_{k-1}$ where $U_j = \{w_j, x_{4j-1}, x_{4j}\}$ for every $1 \leq j \leq k-2$ and $U_{k-1} = \{w_0, w_{k-1}, x_0\}$;

- the edges of a spanning forest made of $k-1$ trees $T_1, T_2, \ldots, T_{k-1}$ where for every $1 \leq i \leq k-3$, the tree T_i is the chain of size 4 $\{x_{4i}, x_{4i+1}, x_{4i+2}, x_{4i+3}\}$, the tree T_{k-2} is the chain of size 7 $\{x_{4(k-2)}, x_{4(k-2)+1}, x_{4k-6}, x_0, x_1, x_2, x_3\}$, and the remaining tree T_{k-1} has vertices $\{w_0, w_1, \ldots, w_{k-1}\} \cup \{y_0, y_1\}$ with edges $\{y_0w_0, y_0w_1, \ldots, y_0w_{\lfloor k/2 \rfloor}\} \cup \{y_1w_{\lfloor k/2 \rfloor}, y_1w_{\lfloor k/2 \rfloor+1}, \ldots, y_1w_{k-1}\}$.

Let c be the weight function from the vertices of H_b into $\mathbb{N}$ defined as follows (see Fig. 1 again):

$$\begin{array}{rcl} c & \rightarrow & \mathbb{N} \\ i = 0, 2, \ldots, 4k-6 : \; x_i & \mapsto & k - \lfloor i/4 \rfloor \\ i = 1, 3, \ldots 4k-7 : \; x_i & \mapsto & \lfloor i/4 \rfloor + 2 \\ w_0, w_1, \ldots, w_{k-1} & \mapsto & 1 \\ y_0, y_1 & \mapsto & k+1 \end{array}$$

We partition the vertex set of H_b as follows:

$$\begin{array}{rcl} V_1 & = & \{w_0, w_1, \dots, w_{k-1}\} \\ V_2 & = & \{x_1, x_3, x_{4k-8}, x_{4k-6}\} \\ V_3 & = & \{x_5, x_7, x_{4k-12}, x_{4k-10}\} \\ \vdots & \vdots & \vdots \\ V_i & = & \{x_{4i-7}, x_{4i-5}, x_{4k-4i}, x_{4k-4i+2}\} \\ \vdots & \vdots & \vdots \\ V_{k-1} & = & \{x_{4k-11}, x_{4k-9}, x_4, x_6\} \\ V_k & = & \{x_0, x_2, x_{4k-7}\} \\ V_{k+1} & = & \{y_0, y_1\} \end{array}$$

Each of the sets $V_1, \dots, V_{k+1}$ is non-empty and we have the following propery, whose straightforward proof is omitted:

Claim C.3.2. *For every $j \in \{1 \dots k+1\}$ and every $v \in V_j$, we have $c(v) = j$.*

Claim C.3.3. *The inequality*

$$\sum_{v \in V_1} x_v + 2 \sum_{v \in V_2} x_v + \dots + (b-1) \sum_{v \in V_{b-1}} x_v \quad \leq \quad b \tag{C.6}$$

is a facet of STAB($\overline{H_b}$).

Proof. Notice that for every edge xy of a tree T_i ($i \in \{1 \dots k-1\}$), we have $c(x) + c(y) = k + 2 = b$ and for every triangle $U_j = \{x, y, z\}$ ($j \in \{1 \dots k-1\}$), we have $c(x) + c(y) + c(z) = k + 2 = b$. Hence every maximal stable set of $\overline{H_b}$ is tight, and the inequality (C.6) is valid.

It remains to check that the incidence vectors of the n maximal stable sets of $\overline{H_b}$ are linearly independent. Let M be the square matrix of the incidence vectors of the n maximal cliques of H_b.

We have to show that M is invertible. Let $\mathbf{c}$ be the column vector with entries $c(v)$ ($v \in V$). Notice that $\mathbf{c}$ is a solution of the equation

$$M\mathbf{x} \quad = \quad b\mathbf{1} \tag{C.7}$$

where $\mathbf{1}$ is the column vector with all entries equal to 1.

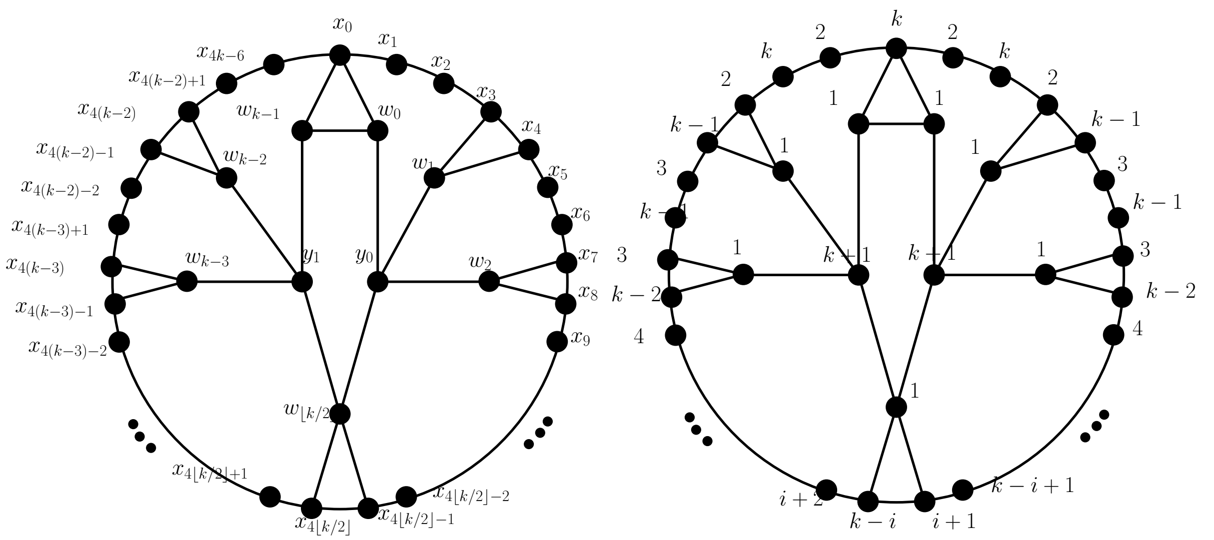
We are going to show that $\mathbf{c}$ is the unique solution of (C.7). Let $\mathbf{c}'$ be a solution of (C.7). Assume that one vertex v of T_{k-1} is such that $c(v) \neq c'(v)$. Let $\delta = c(v) - c'(v)$. Notice that

- either for every leaf u of T_{k-1}, $c'(u) = c(u) + \delta$;
- either for every leaf u of T_{k-1}, $c'(u) = c(u) - \delta$.

Assume w.l.o.g. that for every leaf u of T_{k-1}, we have $c'(u) = c(u) + \delta$. On the one hand, $c'(x_0) = c(x_0) - 2\delta$, $c'(x_4) = c(x_4) - 3\delta$, ..., $c'(x_{4(k-2)}) = c(x_{4(k-2)}) - k\delta$. On the other hand, $c'(x_{4(k-2)}) = c(x_{4(k-2)}) + 2\delta$. Hence $\delta = 0$, a contradiction.

Therefore, for every vertex v of T_{k-1}, we have $c(v) = c'(v)$. Notice that $H_b \setminus T_{k-1}$ is an odd hole such that for every edge xy, we have $c'(y) - c(y) = c(x) - c'(x)$. Therefore if $c'(x) \neq c(x)$ for some vertex x not in T_{k-1}, we get a contradiction due to the odd number of vertices of $H_b \setminus T_{k-1}$.

Thus $\mathbf{c}' = \mathbf{c}$ and so M is invertible. ∎



graph H_b coefficients $c(v)$

Figure C.6: The spanning 1-forest H_b and the coefficients of the associated facet

We now proceed to the second step: we need to add edges to H_b in order to make the complementary graph claw-free without violating the inequality (C.6) or increasing the clique number.

Let H_b' be the graph with vertex set V and edge set $E' \supset E$ defined as follows (Fig. 2 depicts the additional edges for $i = 2$ below):

$$
\begin{aligned}
E' = E \quad & \textstyle\bigcup_{i=1,\dots,k-2} && \{x_{4i}x_1, x_{4i}x_3, \dots, x_{4i}x_{4i-3}\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{x_{4i}w_{i+1}, x_{4i}w_{i+2}, \dots, x_{4i}w_{k-1}\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{x_{4i-1}x_{4i+2}, x_{4i-1}x_{4i+4}, \dots, x_{4i-1}x_{4k-6}\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{x_{4i-1}w_0, x_{4i-1}w_1, \dots, x_{4i-1}w_{i-1}\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{w_ix_0, w_ix_2, \dots, w_ix_{4i-4}\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{w_ix_{4i+3}, w_ix_{4i+5}, \dots, w_ix_{4k-7}\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{w_iy_0, w_iy_1\} \\
& \textstyle\bigcup_{i=1,\dots,k-2} && \{x_{4i-2}w_{k-1}, x_{4i+1}w_0\}
\end{aligned}
$$

We say that an edge e is a new edge if $e \in E' \setminus E$.

Claim C.3.4. *If T is a triangle of H_b' then T is a triangle of H_b.*

Proof. Let $T = \{x, y, z\}$ be a triangle of H_b'. Assume that T is not a triangle of H_b: we are going to show that this leads to a contradiction.

We denote by X the set of vertices $\{x_0, x_1, \dots, x_{4k-6}\}$, by W the set of vertices $\{w_0, \dots, w_{k-1}\}$ and by Y the set of the two remaining vertices $\{y_0, y_1\}$.

The neighbours of w_0 in H_b' are $\{x_0, w_{k-1}\} \cup \{x_3, x_7, \dots, x_{4k-9}\} \cup \{x_5, x_9, \dots, x_{4k-7}\} \cup \{y_0\}$ and the neighbours of w_{k-1} in H_b' are $\{x_0, w_0\} \cup \{x_4, x_8, \dots, x_{4k-8}\} \cup \{x_2, x_6, \dots, x_{4k-10}\} \cup \{y_1\}$. Thus if w_0w_{k-1} is an edge of T then $x_0 \in T$, a contradiction.

Therefore w_0w_{k-1} is not an edge of T.

By the definition, no edge of E' has

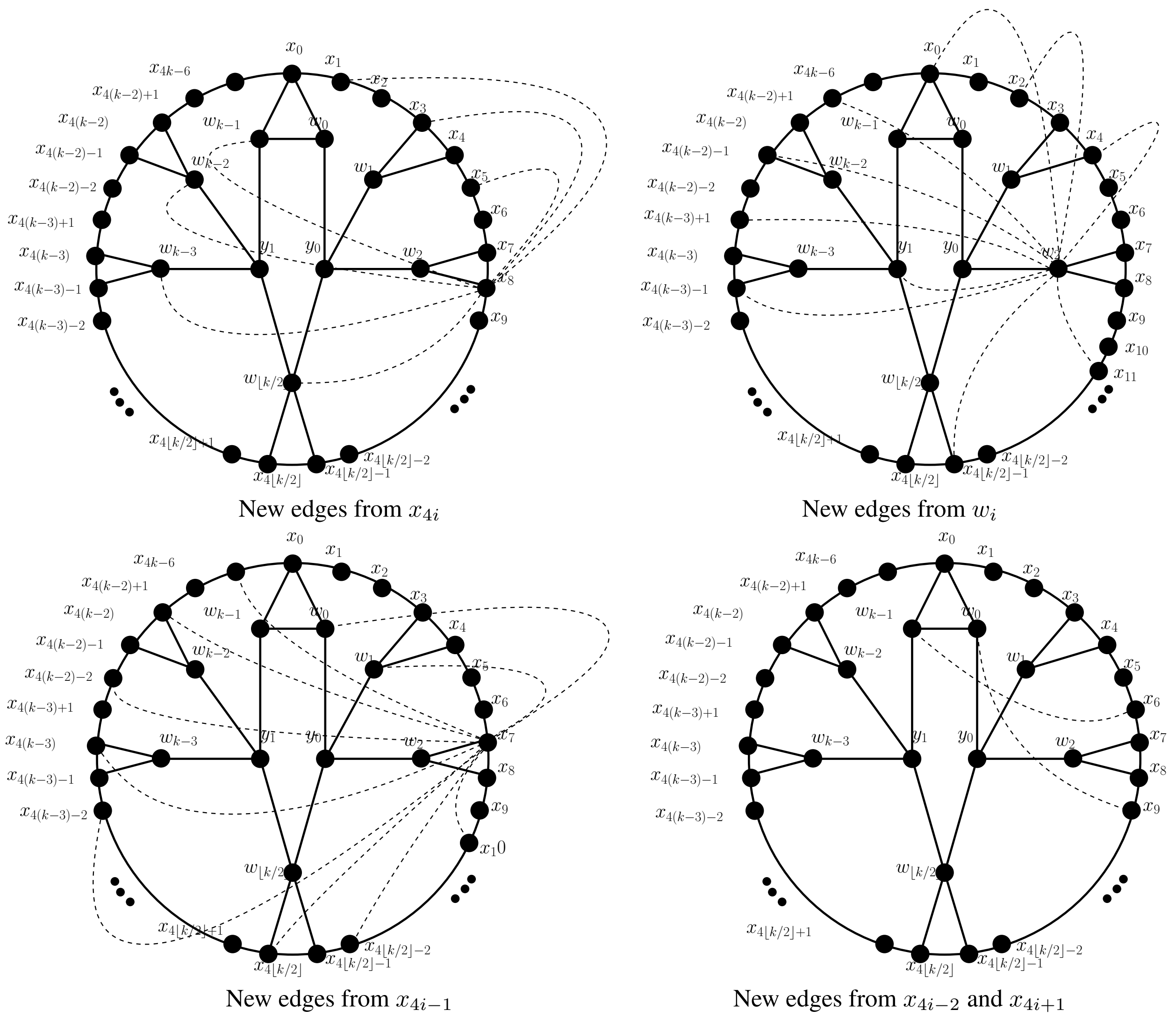


Figure C.7: New edges for $i = 2$ (new edges are dashed edges)

(i) one vertex in X and the other in Y, or

(ii) both vertices in W, or

(iii) both vertices in Y.

If T has a vertex in Y then T has exactly one vertex in Y by (iii) and no vertex in X by (i). Thus T has two vertices in W, contradicting (ii).

Therefore T does not have any vertex in Y. It follows from (ii) that T has at least two vertices, say x and y, in X. Notice that the indices of the vertices of edges of E' with both vertices in X do not have same parity. Therefore $z \notin X$, thus $z \in W$. Let $x = x_i$ and $y = x_j$, and assume w.l.o.g. that i is odd and j is even. Let r be the index of $z = w_r$ ($0 \leq r \leq k-1$).

If $r = 0$ then $j = 0$ and $3 \leq i \leq 4k-7$ as $\{x_0\} \cup \{x_3, x_7, \ldots, x_{4k-9}\} \cup \{x_5, x_9, \ldots, x_{4k-7}\}$ are the neighbours of w_0 in X. This is a contradiction as x_0 has only two neighbours in X, namely x_1 and x_{4k-6}.

If $r = k-1$ then we have a contradiction with i being odd as $\{x_0\} \cup \{x_4, x_8, \ldots, x_{4k-8}\}$ $\cup \{x_2, x_6, \ldots, x_{4k-10}\}$ are the neighbours of w_{k-1} in X.

Therefore $1 \leq r \leq k-2$.

If xz (resp. yz) is not a new edge then $x = x_{4r-1}$ (resp. $y = x_{4r}$). The neighbours of x (resp. y) in X are $\{x_{4r+2}, x_{4r+4}, \dots, x_{4k-6}\} \cup \{x_{4r}\}$ (resp. $\{x_1, x_3, \dots, x_{4r-3}\} \cup \{x_{4r-1}\}$). Notice that the neighbourhood of w_r is $\{x_0, x_2, \dots, x_{4r-4}\} \cup \{x_{4r+3}, x_{4r+5}, \dots, x_{4k-7}\} \cup \{x_{4r}, x_{4r-1}\}$. Hence $y \in (\{x_0, x_2, \dots, x_{4r-4}\} \cup \{x_{4r}\}) \cap (\{x_{4r+2}, x_{4r+4}, \dots, x_{4k-6}\} \cup \{x_{4r}\}) = \{x_{4r}\}$ (resp. $x \in (\{x_{4r+3}, x_{4r+5}, \dots, x_{4k-7}\} \cup \{x_{4r-1}\}) \cap (\{x_1, x_3, \dots, x_{4r-3}\} \cup \{x_{4r-1}\}) = \{x_{4r-1}\}$). Thus T is a triangle of H_b, a contradiction.

Therefore xz and yz are new edges. We have

$$x \in \{x_{4r+3}, x_{4r+5}, \dots, x_{4k-7}\} \quad \text{(C.8)}$$
$$y \in \{x_0, x_2, \dots, x_{4r-4}\} \quad \text{(C.9)}$$

It follows that xy is also a new edge (otherwise, x and y would have consecutive coefficients modulo $4k-5$). Hence, there is an index $1 \leq t \leq k-2$ such that ($x = x_{4t-1}$ and $y \in \{x_{4t+2}, \dots, x_{4k-6}\}$) or ($y = x_{4t}$ and $x \in \{x_1, x_3, \dots, x_{4t-3}\}$).

If $x = x_{4t-1}$ and $y \in \{x_{4t+2}, \dots, x_{4k-6}\}$ then from (C.8), we get $t \geq r+1$ and from the equality (C.9), we get $t \leq r-2$, a contradiction.

If $y = x_{4t}$ and $x \in \{x_1, x_3, \dots, x_{4t-3}\}$ then from (C.9), we get $t \leq r-1$ and from the equality (C.8), we get $t \geq r+2$, a final contradiction. ∎

Claim C.3.5. *The graph $\overline{H'_b}$ is a claw-free graph with stability number 3.*

Proof. Due to Claim C.3.4, the graph $\overline{H'_b}$ has stability number 3 as $\overline{H_b}$ does. It remains to check that $\overline{H'_b}$ is claw-free, that is to check that every vertex of H'_b is adjacent to every triangle of H'_b.

Let T be a triangle of H'_b. Due to Claim C.3.4, T is one of $U_1, U_2, \dots, U_{k-1}$ where $U_j = \{w_j, x_{4j-1}, x_{4j}\}$ for every $j \in \{1 \dots k-2\}$ and $U_{k-1} = \{w_0, w_{k-1}, x_0\}$. Let v be any vertex of H'_b.

If $v \in Y$ then v is adjacent to T by the definition of H_b and due to the new edges $y_0 w_i$ and $y_1 w_i$ for $i \in \{1 \dots k-2\}$.

If $v \in W$ then $v = w_r$ for some $r \in \{0 \dots k-1\}$. If $T = U_{k-1}$ then v is adjacent to the vertex x_0 of T. If $T = U_j$ for some $j \in \{1 \dots k-2\}$ then if $r \leq j$ (resp. $r > j$), the vertex v is adjacent to the vertex x_{4j-1} (resp. x_{4j}) of T.

If $v \in X$ then $v = x_r$ for some $r \in \{0 \dots 4k-6\}$. If $T = U_{k-1}$ then if r is odd (resp. even), v is adjacent to the vertex w_0 (resp. w_{k-1}) of T. If $T = U_j$ for some $j \in \{1 \dots k-2\}$ then if r is odd (resp. even), v is adjacent to the vertex x_{4j} (resp. x_{4j-1}) of T, if $r \leq 4j+1$ (resp. $4j-2 \leq r$) or to the vertex w_j of T if $4j+3 \leq r$ (resp. $r \leq 4j-4$).

∎

Claim C.3.6. *The inequality (C.6)*

$$\sum_{v \in V_1} x_v + 2 \sum_{v \in V_2} x_v + \dots + (b-1) \sum_{v \in V_{b-1}} x_v \leq b$$

is a valid inequality for the stable set polytope of $\overline{H'_b}$

Proof. Due to Claim C.3.4, we only have to check that for every edge $e = xy$ of $E' \setminus E$, we have $c(e) = c(x) + c(y) \leq b = k+2$.

Let e be any edge of $E' \setminus E$. By the definition of E', one of the following 8 cases occur:

- if $e \in \{x_{4i}x_1, x_{4i}x_3, \dots, x_{4i}x_{4i-3}\}$ for some $i \in \{1 \dots k-2\}$ then $c(e) \leq k-i+(i-1)+2 = k+1 < b$;
- if $e \in \{x_{4i}w_{i+1}, x_{4i}w_{i+2}, \dots, x_{4i}w_{k-1}\}$ for some $i \in \{1 \dots k-2\}$ then $c(e) \leq k-i+1 < b$;

- if $e \in \{x_{4i-1}x_{4i+2}, x_{4i-1}x_{4i+4}, \ldots, x_{4i-1}x_{4k-6}\}$ for some $i \in \{1 \ldots k-2\}$ then $c(e) \leq i+1+k-i = k+1 < b$;
- if $e \in \{x_{4i-1}w_0, x_{4i-1}w_1, \ldots, x_{4i-1}w_{i-1}\}$ for some $i \in \{1 \ldots k-2\}$ then $c(e) \leq i+1+1 < b$;
- if $e \in \{w_i x_0, w_i x_2, \ldots, w_i x_{4i-4}\}$ for some $i \in \{1 \ldots k-2\}$ then $c(e) \leq 1+k < b$;
- if $e \in \{w_i x_{4i+3}, w_i x_{4i+5}, \ldots, w_i x_{4k-7}\}$ for some $i \in \{1 \ldots k-2\}$ then $c(e) \leq 1+k-2+2 < b$;
- if $e \in \{w_i y_0, w_i y_1\}$ for some$i \in \{1 \ldots k-2\}$ then $c(e) = 1+k+1 = b$;
- if $e \in \{x_{4i-2}w_{k-1}, x_{4i+1}w_0\}$ for some $i \in \{1 \ldots k-2\}$ then $c(e) \leq 1+k < b$.

Hence $c(e) \leq b$. ■

Therefore $G_b = \overline{H'_b}$ is a claw-free graph with stability number 3 by Claim C.3.5, such that the inequality (C.6) is a facet of its stable set polytope by Claims C.3.6 and C.3.3, where for every $j \in \{1 \ldots k+1\}$, the set V_j is not empty. This finally proves Theorem C.3.1.

Concluding remarks

In this paper, we exhibit that, for every positive integer b, there is a claw-free graph with stability number three whose stable set polytope has a facet with b different left hand side coefficients. This supports the feeling further that the case with $\alpha(G) = 3$ is indeed the difficult one. In fact, if Stauffer's conjecture for the case with $\alpha(G) \geq 4$ is true, then all facets of claw-free graphs with stability number $\alpha(G) \neq 3$ have at most two different left hand side coefficients, whereas we showed that claw-free graphs with stability number $\alpha(G) = 3$ have arbitrarily high and arbitrarily many different left hand side coefficients.

Chudnovsky and Seymour recently achieved a decomposition of claw-free graphs, whose lengthy proof is split in several papers. In the process, they identified several subfamilies of claw-free graphs and it is worth to mention that graphs G_b exhibited in this paper belong to the so-called *antiprismatic* graphs (a graph G is said to be antiprismatic if for every stable set S of size 3 and every vertex v not in S, v has a unique non-neighbour in S), which "seem to form a subclass of claw-free graphs that is very different from the others" and difficult to cope with: "understanding antiprismatic graphs was probably the most difficult part of the project" [1]. It does not mean however that graphs G_b presented in our construction have to be antiprismatic: indeed, when defining E', we introduced a minimal subset of edges to be added in order to ensure that the complementary graph is claw-free. Proceeding this way simplified the proof, since we could prove that these extra edges do not create new triangles.

References

[1] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In B.S. Webb, editor, *Surveys in combinatorics*, volume 327 of *London. Math. Soc. Lecture Notes*, pages 153–171, 2005.

[2] J.E. Edmonds. Maximum matching and a polyhedron with (0,1) vertices. *J. Res. Nat. Bur. Standards*, 69(B):125–130, 1965.

[3] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. Circular one matrices and the stable set polytope of quasi-line graph. *Lecture Notes in Computer Science*, 3509:291–305, 2005.

[4] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1):1–38, 1997.

[5] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[6] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[7] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[8] A. Pêcher and A. Wagler. A conjecture on the stable set polytope for claw-free graphs. Preprint 11-2006, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik.

[9] A. Ben Rebea. *Etude des stables dans les graphes quasi-adjoints*. PhD thesis, Grenoble, 1981.

[10] F.B. Shepherd. Near-perfect matrices. *Mathematical Programming*, 64:295–323, 1994.

[11] G. Stauffer. *On the Stable Set Polytope of Claw-free Graphs*. PhD thesis, EPF Lausanne, 2005.

Annexe D

Manuscripts

Sommaire

D.1 On stable set polytopes of circular-perfect and strongly circular-perfect graphs

par S. Coulonges, A. Pêcher et A. Wagler

Présenté à la 5ème édition d'ALIO/EURO conference on combinatorial optimization (2005) [24]

Perfect graphs constitute a well-studied graph class with a rich structure. Circular-perfect graphs as introduced by Zhu are a natural superclass of perfect graphs defined by means of a more general coloring concept and form an important class of χ-bound graphs with the smallest non-trivial χ-binding function $\chi(G) \leq \omega(G) + 1$. Apart from perfect graphs, circular-perfect graphs include all convex-round graphs and outerplanar graphs. A linear description of facets of stable set polytopes of circular-perfect graphs is still unknown, though convex-round graphs and outerplanar-graphs are rank-perfect.

In this paper, we exhibit an infinite class of *non* rank-perfect circular-perfect graphs. We introduce *strongly circular-perfectness*: a circular-perfect graph is said to be strongly circular-perfect if its complement is also circular-perfect. This subclass of circular-perfect graphs includes perfect graphs, odd-holes and anti-holes. We show that there are infinitely many *non* rank-perfect strongly circular-perfect graphs and fully characterize triangle-free minimal strongly circular-imperfect graphs: it turns out that these graphs are very close to odd-holes.

Introduction

Coloring the nodes of a graph is an important concept with a large variety of applications. Let $G = (V, E)$ be a graph with finite node set V and simple edge set E, then a *k-coloring* of G is a mapping $f : V \to \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent nodes of G receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* $\chi(G)$; calculating $\chi(G)$ is a NP-hard problem in general. In a set of k pairwise adjacent nodes, called *clique* K_k, all k nodes have to be colored differently. Thus the size of a largest clique in G, the *clique number* $\omega(G)$, is a trivial lower bound on $\chi(G)$; this bound can be arbitrarily bad [5] and is hard to evaluate as well.

Perfect graphs turned out to be an interesting and important class with a rich structure, see [7] for a recent survey. In particular, both parameters $\omega(G)$ and $\chi(G)$ can be determined in polynomial time if G is perfect [4].

As generalization of perfect graphs, Zhu [10] introduced recently the class of circular-perfect graphs based on the following more general coloring concept. Define a (k, d)-*circular coloring* of a graph $G = (V, E)$ as a mapping $f : V \to \{0, \ldots, k-1\}$ with $d \leq |f(u) - f(v)| \geq k - d$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k, d)-circular colorings of G; we immediate obtain $\chi_c(G) \leq \chi(G)$ since every $(k, 1)$-circular coloring is a usual k-coloring of G. Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k nodes $0, \ldots, k-1$ and edges ij iff $d \leq |i - j| \leq k - d$. Such graphs $K_{k/d}$ are called *circular-cliques*. A circular clique $K_{k/D}$ is said to be *be prime* if $(k, d) = 1$, that is k and d are relatively primes. The *circular-clique number* is $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G,\ (k, d) = 1\}$ and we have $\omega(G) \leq \omega_c(G)$.

Every circular-clique $K_{k/d}$ clearly admits a (k, d)-circular coloring but no (k', d')-circular coloring with $\frac{k'}{d'} < \frac{k}{d}$ by [1]. Thus we have, for any graph G, the following chain of inequalities:

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{D.1}$$

A graph G is called *circular-perfect* if, for each induced subgraph $G' \subseteq G$, circular-clique number $\omega_c(G')$ and circular chromatic number $\chi_c(G')$ coincide. Obviously, every perfect graph has this property by (1) as $\omega(G')$ equals $\chi(G')$. Moreover, any circular-clique is circular-perfect as well [10]. Thus circular-perfect graphs constitute a proper superclass of perfect graphs. A graph G is said to be *strongly circular-perfect* if both G and its complement are circular-perfect. A graph is said to be *minimal strongly circular-imperfect* if it is not circular-perfect but every of its proper induced subgraph is.

Since it is known for any graph G that $\omega(G) = \lfloor \omega_c(G) \rfloor$ by [10] and $\chi(G) = \lceil \chi_c(G) \rceil$ by [8], we obtain that circular-perfect graphs G satisfy the following Vizing-like property

$$\omega(G) \leq \chi(G) \leq \omega(G) + 1. \tag{D.2}$$

Thus, circular-perfect graphs are a class of χ-bound graphs with the smallest non-trivial χ-binding function.

Results

Formulating an analogue to the Strong Perfect Graph Theorem for circular-perfect graphs seems to be difficult, as [6] provides three, rather simple, classes of minimal circular-imperfect graphs with no straightforward common structure in these graphs.

The *stable set polytope* $\text{STAB}(G)$ of a graph $G = (V, E)$ is defined as the convex hull of the incidence vectors of all stable sets of G. A canonical relaxation of $\text{STAB}(G)$ is the *clique constraint polytope* $\text{QSTAB}(G)$ given by all "trivial" facets, the *nonnegativity constraints*

$$x_i \geq 0 \tag{D.3}$$

for all nodes i of G and by the *clique constraints*

$$\sum_{i \in Q} x_i \leq 1 \tag{D.4}$$

for all cliques $Q \subseteq G$. We have $\text{STAB}(G) \subseteq \text{QSTAB}(G)$ for any graph but equality for *perfect* graphs only [2].

A natural way to generalize clique constraints is to investigate *rank constraints*

$$\sum_{i \in G'} x_i \leq \alpha(G') \tag{D.5}$$

associated with *arbitrary* induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G'.

A canonical relaxation of $\text{QSTAB}(G)$ is, therefore, the *rank constraint polytope* $\text{RSTAB}(G)$ given by the nonnegativity constraints (D.3) and all rank constraints (D.5); a graph G is *rank-perfect* by [9] if and only if $\text{STAB}(G)$ and $\text{RSTAB}(G)$ coincide.

We introduce the class of *a-perfect graphs* as those graphs whose stable set polytopes are given by nonnegativity constraints and rank constraints associated with cliques and prime circular-cliques only. So far, little is known about the stable set polytope of circular-perfect graphs, though outerplanar graphs and convex-round graphs are a-perfect [3]. We show that circular-cliques are not enough in general to get a description of all facets of stable set polytopes of circular-perfect graphs. Hence circular-perfect graphs do not have as nice polyhedral properties as perfect graphs:

Theorem D.1.1. *Let G be a graph, $k \geq 0$ an integer and $H(G,k)$ be the graph obtained by adding a universal node v to G and replacing every edge adjacent to v by a chain of k vertices is a circular-perfect graph. If $k = 1$ then for every clique Q of G,*

$$(|Q| - 1)x_v + \sum_{u \in Q} x_u + \sum_{u \in N(v) \cap N(Q)} x_u \leq |Q| \tag{D.6}$$

is a facet of $STAB(H(G,1))$. If G is a clique then $H(G,k)$ is circular-perfect.

In particular, for every clique Q of size at least 3, $H(Q,1)$ is a non rank-perfect circular-perfect graph.

Proof.

Claim D.1.2. *If $k = 1$ then for every clique $Q \in G$, In. (D.6) is a facet of $STAB(H(G,1))$*

Let S be any stable set of $H(G,1)$. If $v \in S$ then $S \cap (N(v) \cap N(Q)) = \emptyset$ and S meets Q in at most one vertex, thus S satisfy In. (D.6). If $v \notin S$ then S satifies satisfies In. (D.6), as $N(v) \cap N(Q)$ is of size Q and contains Q in its neighborhood. Thus In. (D.6) is a valid inequality.

Let $S = N(v) \cap N(Q)$ and let $\{1, \ldots, |Q|\}$ be the nodes of Q. For every $1 \leq i \leq |Q|$ denote by i' the neighbour of i in S.

Define $S_0 = S$ and for every $1 \leq i \leq |Q|$, $S_i = \{i, v\}$ and $S_{i+|Q|} = \{1', \ldots, n', i\} \setminus \{i'\}$. Notice that these sets of vertices are $2|Q| + 1$ roots of In. (D.6). We are going to prove that they are linearly independent, i.e. $\sum_{i=0}^{2n} \lambda_i \chi_{S_i} = 0$ implies $\lambda_i = 0$, for all i.

From $\sum_{i=0}^{2n} \lambda_i \chi_{S_i} = 0$, we get

- the equation

  $$\sum_{j=1}^{|Q|} \lambda_j = 0$$

  since v belongs to the stable sets $S_1, S_2, S_|Q|$ only;

- the $|Q|$ equations

  $$\lambda_i + \lambda_{i+|Q|} = 0$$

  for every $1 \leq i \leq |Q|$, since the vertex i belongs to the stable sets S_i and $S_{i+|Q|}$ only;

- the $|Q|$ equations

$$\lambda_0 + \sum_{j=|Q|+1}^{2|Q|} \lambda_j - \lambda_{i+|Q|} \quad = \quad 0$$

for every $1 \le i \le |Q|$, since the vertex i' belongs to the stable sets S_0, $\{S_j||Q|+1 \le j \le 2|Q|,\ j \ne i+|Q|$.

Hence we get for every $1 \le i \le |Q|$,

$$\lambda_0 - \sum_{j=1}^{|Q|} \lambda_j + \lambda_i = 0$$

Thus for every $1 \le i \le |Q|$

$$\lambda_0 + \lambda_i \quad = \quad 0 \tag{D.7}$$

Summing up this $|Q|$ last equations, we get $\lambda_0 = 0$, and so $\lambda_i = 0$ for $1 \le i \le |Q|$.

To complete our set of roots, define for every vertex $x \notin \{v\} \cup S \cup Q$, the set of vertices S_x as $S \cup \{x\}$: the set S_x is obviously a root of In. (D.6), and it is straightforward to check that $\{S_0, S_1, \ldots, S_{2|Q|}\} \cup \{S_x | x \notin \{v\} \cup S \cup Q\}$ is a set of n roots linearly independant of In. (D.6) where n denotes the number of vertices of $H(G, 1)$.

$\diamond$

Let $k \ge 0$ and Q be a clique. Let G' denote the graph $H(Q, k)$. Let $S_0 = \{v\}$ and for every $1 \le i \le k$, denote by S_i the set of vertices from G' adjacent to S_{i-1}. We have to prove that G' is circular-perfect.

Notice that if $k = 0$ then G' is a clique, thus G' is perfect. Assume that $k \ge 1$. If Q is of size 1 then G is a pair of adjacent vertices and is perfect. Assume that Q is of size at least 2.

Claim D.1.3. *If $|Q| \ge 3$ then $\chi(G') = |Q|$.*

Let $j = |Q|$ and assume that Q is colored with colors from $\{c_1, \ldots, c_j\}$. We have to extend the coloring of G to the set of vertices $S_0 \cup S_1 \cup \ldots \cup S_k$. Let T_k be the subset of S_k adjacent to vertices of G colored with color c_1. Then we extend the coloring as follows:

- vertices of T_k get the same color c_2;
- vertices of $S_k \setminus T_k$ get the same color c_1;
- for every $1 \le i \le k-1$, for every vertex v in S_i, if the neighbour of v in S_{i+1} is colored with c_1 (resp. c_2), v is colored with c_2 (resp. c_1);
- v gets the color c_3.

$\diamond$

Let H be any subgraph of G'. We have to prove that $\omega_c(H) = \chi_c(H)$. Let Z be the induced subgraph $H \cap Q$ of G. If Z is of size at least 3, then H is $|Z|$-colorable by Claim D.1.3, hence $|Z| = \omega_c(H) = \chi_c(H)$. If Z is a singleton then H is perfect an we have nothing to prove. If Z is a pair of adjacent vertices then H is outerplanar and is therefore circular-perfect.

$\square$

At last, we show that there are infinitely many *non* rank-perfect strongly circular-perfect graphs: let $G_{n,\delta}$, $1 \le \delta \le n$, be the graph with $V(G_n) = \{0, 1, \ldots, n\}$ s.t. the subgraph induced by $\{1, \ldots, n\}$ is the hole of n vertices, and the vertex 0 is adjacent to the nodes $1, \ldots, \delta$.

Theorem D.1.4. *Let $p \ge 2$. The graph $G_{2p+1,\delta}$ is strongly circular-perfect if and only if $\delta < 2p+1$. Furthermore, if $\delta \ge 5$, then $G_{2p+1,\delta}$ is not rank-perfect as $\left\lfloor \frac{\delta-1}{2} \right\rfloor x_0 + \sum_{i=1}^{2p+1} x_i \le p$ is a facet of $STAB\,(G_{2p+1,\delta})$.*

Proof.

Only if

Let $0 \leq \delta \leq 2p+1$ and assume that $G_{2p+1,\delta}$ is strongly circular-perfect. If $\delta = 0$ (resp. $2p+1$), then $\overline{G_{2p+1,\delta}}$ (resp. $G_{2p+1,\delta}$) is the complete join of an odd antiwheel and a singleton (resp. odd wheel and a singleton), which is circular imperfect by [6], a contradiction. Hence $0 < \delta < 2p+1$.

If

Assume that $0 < \delta < 2p+1$ and let $G = G_{2p+1,\delta}$.

Claim D.1.5. *If $\delta = 1$ then G is strongly circular-perfect.*

Notice that G is outerplanar and therefore circular-perfect [6]. To prove that $\overline{G}$ is circular-perfect, it is enough to show that $\omega_c(\overline{G}) = \chi_c(\overline{G})$ as any proper induced subgraph of $\overline{G}$ is perfect or an odd anti-hole. It is straightforward to check that $\omega_c(\overline{G}) = \omega(\overline{G}) = p+1 = \chi(\overline{G})$.

$\diamond$

Claim D.1.6. *If $2 \leq \delta < 2p+1$ then G is circular-perfect.*

Notice that $3 \leq \omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) = 3$ as G as a triangle (since $2 \leq \delta$ and admits a 3-coloration with colors a, b, c as follows:

- give color a to 0 and $2p+1$;
- give color b to vertices i such that $0 < i < 2p+1$ and i odd;
- give color b to vertices i such that $0 < i < 2p+1$ and i even.

Hence $\omega_c(G) = \chi_c(G) = 3$. Let H be a proper induced subgraph of G. If H does not contain the vertex 0 then H is circular perfect as an induced subgraph of an odd hole. If H contains the vertex 0 and a triangle then $\omega_c(H) = \chi_c(H) = 3$ as G is 3-colorable. If H does not contain a triangle then H is a hole. Thus $\omega_c(H) = \chi_c(H)$. Therefore G is circular-perfect.

$\diamond$

Claim D.1.7. *If $\delta = 2$ then $\overline{G}$ is circular-perfect.*

Notice that $\alpha(G) = p+1$ and that G has a cover with $p+1$ cliques. Therefore $p+1 = \alpha(G) = \omega(\overline{G}) \leq \omega_c(\overline{G}) \leq \chi_c(\overline{G}) \leq \chi(\overline{G}) = p+1$. Thus $\omega_c(G) = \chi_c(G)$. Let H be a proper induced subgraph of G. If H does not contain the vertex 0 then $\overline{H}$ is circular-perfect as an induced subgraph of an odd anti hole. If H contains the vertex 0 then H is perfect and so $\overline{H}$ is circular-perfect.

Therefore $\overline{G}$ is circular-perfect.

Claim D.1.8. *If $3 \leq \delta < 2p+1$ then $\overline{G}$ is circular-perfect.*

We have $p+\frac{1}{2} \leq \omega_c(\overline{G}) \leq \chi_c(\bar{(G)})$. Notice that the neighbours of the vertex 0 in $\overline{G}$ are $\delta+1, \delta+2, \ldots, 2p+1$. Denote by A the odd anti hole of $\overline{G}$ induced by the set of vertices $\{1, \ldots, 2p+1\}$. Notice that sending the vertex 0 onto 2 yields an homomorphism from $\overline{G}$ into A (as $\delta \geq 3$). Hence $\chi_c(\overline{G}) \leq p+\frac{1}{2}$. Thus $\omega_c(\overline{G}) = \chi_c(\overline{G}$.

Let H be a proper induced subgraph of G. If H does not contain the vertex 0 then $\overline{H}$ is circular-perfect as an induced subgraph of an odd anti hole.

If H contains the vertex 0 and δ is even then notice that H is perfect, and so $\overline{H}$ is circular-perfect.

Assume that δ is odd. If H does not contain at least one of the vertices $\{\delta, \delta+1, \ldots, 2p+1\} \cup \{1\}$ then H is perfect and so $\overline{H}$ is circular-perfect. Otherwise $\{\delta, \delta+1, \ldots, 2p+1\} \cup \{1\} \cup \{0\}$ induces an odd hole $\mathcal{O}$ in G of size $(2p+1) - \delta + 3$. If $H = \mathcal{O}$ then $\overline{H}$ is circular-perfect. If not, let v be a vertex of H outside $\mathcal{O}$. Let $H' = H \setminus \{0\}$. Notice that H' is perfect and thus admits a cover with $\alpha(H')$ cliques. Let Q_v be a clique of this cover containing v. Notice that $Q_v \cup 0$ is a clique of H. Hence H admits a cover with $\alpha(H')$ cliques. Therefore we have

$$\alpha(H') \leq \omega(\overline{H'}) \leq \omega(\overline{H}) \leq \omega_c(\overline{H}) \leq \chi_c(\overline{H}) \leq \chi(\overline{H}) \leq \alpha(\overline{H'})$$

Thus $\omega_c(\overline{H}) = \chi_c(\overline{H})$ Hence $\overline{G}$ is circular-perfect.

◇

Claim D.1.9. *The inequality* $\lfloor \frac{\delta-1}{2} \rfloor x_0 + \sum_{i=1}^{2p+1} x_i \leq p$ *is a facet of* $STAB(G)$.

Since every stable set containing the vertex 0 is of cardinality at most $\lfloor (2p+3-\delta)/2 \rfloor$, the inequality $\lfloor \frac{\delta-1}{2} \rfloor x_0 + \sum_{i=1}^{2p+1} x_i \leq p$ is valid for the stable set polytope of G.

Let $\mathcal{O}$ be the odd hole of size $2p+1$ induced by the set of vertices $\{1, 2, \ldots 2p+1\}$ The $2p+1$ maximum stable sets $S_1, \ldots, S_{2p+1}$ of $\mathcal{O}$ are $2p+1$ roots linearly independent .

Let $S = \{0\} \cup \{\delta+1, \delta+3, \ldots, \delta+2k\}$ where $k = \lfloor \frac{2p+1-(\delta+1)}{2} \rfloor$. Then it is straightforward to check that S is a root containing 0. Thus $S, S_1, \ldots, S_{2p+1}$ are $2p+2$ roots linearly independent, and so $\lfloor \frac{\delta-1}{2} \rfloor x_0 + \sum_{i=1}^{2p+1} x_i \leq p$ is a facet of $STAB(G)$.

◇

□

References

[1] J.A. Bondy and P. Hell. A note on the star chromatic number. *J. of Graph Theory*, 14:479–482, 1990.

[2] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[3] S. Coulonges, A. Pêcher, and A. Wagler. On the imperfection ratio of a-perfect graphs and some consequences. In *CTW05 - Cologne-Twente Workshop on Graphs and Combinatorial Optimization*, mai 2005.

[4] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[5] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[6] A. Pêcher, A. Wagler, and X. Zhu. Three classes of minimal circular imperfect graphs. In *Proceedings of GRACO05*, volume 19 of *Electonic Notes in Discrete Mathematics*, pages 9–15. Elsevier, avril 2005.

[7] J. Ramirez-Alfonsin and B. Reed, editors. *Perfect Graphs*. Wiley, 2001.

[8] A. Vince. Star chromatic number. *J. of Graph Theory*, 12:551–559, 1988.

[9] A. Wagler. Rank-perfect and weakly rank-perfect graphs. *Mathematical Methods of Operations Research*, 95:127–149, 2002.

[10] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3):186–209, 2005.

D.2 Results and Conjectures on the stable set polytope of claw-free graphs

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Soumis dans Discrete Mathematics [87]

The question of a polyhedral description for claw-free graphs remains one of the interesting open problems in polyhedral combinatorics. Galluccio and Sassano [8] characterized the rank facets of claw-free graphs, but regarding their non-rank facets there was even no conjecture at hand so far. For the subclass of quasi-line graphs, Ben Rebea claimed and Chudnovsky and Seymour [1] and Eisenbrand et al. [6] proved recently that all non-trivial facets of their stable set polytopes belong to only one class, the clique family inequalities. However, for claw-free but not quasi-line graphs clique family inequalities do not suffice: We show that the clique neighborhood constraints describing the stable set polytopes of graphs G with $\alpha(G) = 2$ (see [18]) are no clique family inequalities. In [19] it is conjectured that certain clique neighborhood constraints are the only non-rank facets for claw-free but not quasi-line graphs G with $\alpha(G) \geq 4$. Further non-rank facets for the case $\alpha(G) = 3$ are presented in [9, 11] and, in fact, all the known difficult facets of claw-free graphs occur if $\alpha(G) = 3$. We exhibit that these facets are no clique family inequalities and show, as our main result, that all of them belong to only one class, the co-spanning 1-forest constraints. It turns out that clique neighborhood constraints are special co-spanning 1-forest constraints. Combining all those results enables us to formulate a conjecture on the non-rank facets for general claw-free graphs, stating that all of them are inequalities of two types only, namely either clique family inequalities (for quasi-line graphs) or co-spanning 1-forest constraints (for all other claw-free graphs).

Introduction

In this paper, we consider the stable set polytopes of *claw-free graphs*, that are graphs having no node with a stable set of size three in its neighborhood (in a stable set all nodes are mutually nonadjacent).

For members of this class, the stable set problem can be solved in polynomial time [12, 13, 17]; the existing algorithms are extensions of Edmonds' matching algorithm [4]. This implies that also the optimization problem over the *stable set polytope* STAB(G) of a claw-free graph G is solvable in polynomial time [10], where STAB(G) is the convex hull of the incidence vectors of all stable sets. As it is possible to optimize over the stable set polytope of a claw-free graph in polynomial time, the stable set polytope of claw-free graphs is, in this respect, under control. However, no explicit description by means of a facet-defining system is known for the stable set polytope of claw-free graphs yet. This apparent asymmetry between the algorithmic and the polyhedral status of the stable set problem in claw-free graphs gives rise to the challenging problem of providing a complete description of the facets of claw-free graphs, a long-standing open problem originally posed in [10]; our goal is to formulate an appealing conjecture.

For all graphs G, the *nonnegativity constraints* $x_i \geq 0$ for all nodes i of G and the *clique constraints*

$$x(Q) := \sum_{i \in Q} x_i \leq 1 \tag{D.8}$$

for all maximal cliques $Q \subseteq G$ are facets of STAB(G), but suffice for perfect graphs only [2] (in a clique, all nodes are mutually adjacent). For claw-free graphs G, only a characterization of the *rank facets* was given by Galluccio and Sassano [8] who showed that they are the 0/1-valued constraints

$$x(G') := \sum_{i \in G'} x_i \leq \alpha(G') \tag{D.9}$$

which are basically associated with certain line graphs and webs $G' \subseteq G$ ($\alpha(G')$ stands for the size of a maximum stable set in G'). More precisely, all rank facets of claw-free graphs can be obtained by means of standard techniques from rank constraints associated with cliques, line graphs of hypomatchable graphs, and partitionable webs.

However, claw-free graphs have *non-rank* facets in general and even a conjecture regarding their non-rank facets was not at hand yet.

Claw-free graphs contain all *line graphs*, obtained by taking the edges of a root graph H as nodes and connecting two nodes iff the corresponding edges of H are adjacent. All facets of the stable set polytope of line graphs are known from matching theory [3], namely, rank constraints associated with cliques and line graphs of hypomatchable graphs [5] (a graph H is hypomatchable if H does not admit a perfect matching, but $H - v$ does for every node v). In contrary to the algorithmic aspect, this description for the matching polytope could not be extended to the stable set polytopes of claw-free graphs, but only for the intermediate class of *quasi-line graphs*, where the neighborhood of any node can be partitioned into two cliques.

Let $G = (V, E)$ be a graph, $\mathcal{Q}$ be a family of at least three inclusion-wise maximal cliques of G, $p \leq |\mathcal{Q}|$ be an integer, and $r = |\mathcal{Q}| (\bmod p) > 0$. The *clique family inequality* $(\mathcal{Q}, p)$ is defined as

$$(p-r) \sum_{i \in V_p} x_i + (p-r-1) \sum_{i \in V_{p-1}} x_i \leq (p-r) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor \tag{D.10}$$

where V_p (resp. V_{p-1}) contains all nodes that belong to at least p (resp. exactly $p-1$) cliques in $\mathcal{Q}$. Clique family inequalities are valid for the stable set polytope of every graph [14]; Ben Rebea (see [14]) claimed and Chudnovsky and Seymour [1] and Eisenbrand et al. [6] proved that they suffice for quasi-line graphs.

However, already the smallest not quasi-line graph, the 5-wheel in Figure D.1(a), has a facet not associated with a clique family. Due to Cook, all facets for graphs G with $\alpha(G) = 2$ are *clique-neighborhood constraints* $F(Q)$

$$2x(Q) + 1x(N'(Q)) \leq 2 \tag{D.11}$$

where $Q \subseteq G$ is a clique and $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$ (see [18]) and are no clique family inequalities if they are non-rank (see subsection D.2). Stauffer [19] conjectures that the only non-rank facets of claw-free graphs G with $\alpha(G) \geq 4$ are clique neighborhood constraints as well.

For claw-free but not quasi-line graphs G with $\alpha(G) = 3$, non-rank facets with up to five different non-zero coefficients are presented in [9, 11] (see Figure D.1(b),(c) for two examples). So far, the structure of the latter facets was not well-understood; it has been only observed that all the known difficult facets of claw-free graphs occur if $\alpha(G) = 3$. We analyze these facets in subsection D.2 and show, as our main result, that all of them belong to only one class of constraints, the co-spanning 1-forest constraints. We prove, in particular, that clique neighborhood constraints are special co-spanning 1-forest constraints.

Combining all those results enables us to formulate a conjecture on the non-rank facets for claw-free graphs, stating that all of them are inequalities of *two types* only, namely, either clique family inequalities (for quasi-line graphs) or co-spanning 1-forest constraints (for all other claw-free graphs).

Known facets for general claw-free graphs

In the following subsubsections, we survey results on non-rank facets of claw-free graphs from [9, 11] and discuss why the clique family inequalities do not cover such kinds of facets.

Graphs G with $\alpha(G) = 2$

Cook showed (see [18]) that for graphs G with $\alpha(G) = 2$, all non-trivial facets of STAB(G) are clique-neighborhood constraints $F(Q)$ of the form $2x(Q) + 1x(N'(Q)) \leq 2$ s.t. the subgraph $\overline{G}[N'(Q)]$ of the complement $\overline{G}$ has no bipartite component (recall that $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$). Clearly, a clique-neighborhood constraint $F(Q)$ is

- the clique constraint associated with Q if and only if Q is a maximal clique of G (and, therefore, $N'(Q) = \emptyset$),
- the full rank constraint associated with G if and only if $Q = \emptyset$ (and, therefore, $N'(Q) = G$), or
- a non-rank constraint if both parts Q and $N'(Q)$ are non-empty.

By definition of $N'(Q)$, the subgraph $G[Q \cup N'(Q)]$ is the complete join of Q and $N'(Q)$. If $F(Q)$ is a facet, then each component of $\overline{G}[N'(Q)]$ contains an *odd hole* C_{2k+1} with $k \geq 2$ (since odd cycles of length three are excluded by $\omega(\overline{G}) = 2$). If $F(Q)$ is non-rank, then $G[Q \cup N'(Q)]$ contains an *odd antiwheel* $\overline{C}_{2k+1} * v$, that is the complete join of a single node v and an odd antihole $\overline{C}_{2k+1}$ (note that the 5-wheel is also the 5-antiwheel by $C_5 = \overline{C}_5$).

Thus, any non-rank clique-neighborhood facet is a lifting of an odd antiwheel constraint $x(\overline{C}_{2k+1}) + 2x_v \leq 2$. We have:

Lemma D.2.1. *Odd antiwheel constraints $x(\overline{C}_{2k+1}) + 2x_v \le 2$ are no clique family inequalities.*

Proof. Assume to the contrary that the constraint is a clique family inequality $(\mathcal{Q}, p)$, then we have $V_p = \{v\}$, $V_{p-1} = \overline{C}_{2k+1}$, $(p-r) = 2$, and $(p-r)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 2$, and hence $\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 1$. We count how often the nodes of $\overline{C}_{2k+1}$ are covered by the cliques in $\mathcal{Q}$, and we obtain

$$(2k+1)(p-1) \;=\; \sum_{Q \in \mathcal{Q}} |Q \cap \overline{C}_{2k+1}| \;\le\; |\mathcal{Q}| \cdot k$$

by $V_{p-1} = \overline{C}_{2k+1}$ and $k = \omega(\overline{C}_{2k+1})$. This implies $\frac{2k+1}{k} \le \frac{|\mathcal{Q}|}{p-1}$ and we have to bound the latter further. By definition, we know $|\mathcal{Q}| = p\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor + r$, thus $|\mathcal{Q}| = p + r$ follows by $\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 1$ and further $|\mathcal{Q}| = 2p - 2$ by $p - r = 2$. This implies

$$2 + \frac{1}{k} \;=\; \frac{2k+1}{k} \;\le\; \frac{|\mathcal{Q}|}{p-1} \;=\; \frac{2p-2}{p-1} \;=\; 2$$

which yields the final contradiction. □

As a clique neighborhood constraint $F(Q)$ defines a non-rank facet only if it is a lifted odd antiwheel constraint, we obtain that no non-rank clique neighborhood constraint is a clique family inequality.

Moreover, if a claw-free graph contains a complete join $G_1 * G_2$, then both parts have at most stability number two (as already the complete join of a single node and a graph with a stable set of size 3 would contain a claw) and, thus, $G_1 * G_2$ has stability number two as well. This implies further:

Lemma D.2.2. *All complete join facets of claw-free graphs are clique neighborhood constraints.*

Claw-free graphs G with $\alpha(G) = 3$

Here we collect non-rank facets for claw-free graphs G with $\alpha(G) = 3$ from the literature and exhibit that all of them are no clique family inequalities.

Giles and Trotter introduced a class of claw-free graphs G with $\alpha(G) = 3$, called *wedges*, and showed that they produce non-rank facets of the form

$$1x(A) + 2x(B) \le 3$$

for node sets A and B (see subsection D.2 for more details and Figure D.1(b) for one example). It is easy to see that no such facet is a clique family inequality:

Lemma D.2.3. *No facet $1x(A) + 2x(B) \le 3$ is a clique family inequality.*

Proof. In a clique family inequality $(\mathcal{Q}, p)$, we would have $3 = (p-r)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$ as right hand side. As 3 is a prime number, this is possible only if either $p - r = 1$ and $\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 3$ or $\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor = 1$ and $p - r = 3$ holds, which is both not possible as the highest coefficient $p - r$ equals 2. □

The claw-free graph G with $\alpha(G) = 3$ in Figure D.1(c) induces by [9] the following facet $3(x_1 + x_2) + 1(x_3 + x_4 + x_5) + 2(x_6 + \ldots + x_{10}) \le 4$ and the two different non-zero coefficients of a clique family inequality are clearly not enough. Liebling et al. [11] found two further such graphs producing facets with up to five consecutive non-zero coefficients which cannot be clique family inequalities due to the same reason.

Claw-free graphs G with $\alpha(G) \ge 4$

Fouquet [7] showed that a connected claw-free graph with $\alpha(G) \ge 4$ is quasi-line if and only if G does not contain a 5-wheel (in particular, in such graphs can neither occur longer odd antiwheels by [7] nor longer odd wheels by claw-freeness). Stauffer exhibited in [19] that the 5-wheels play indeed a central role to describe the facets of such graphs, as 5-wheel constraints can be sequentially lifted to more general inequalities of the form $1x(A) + 2x(B) \leq 2$ with suitable node subsets A and B. This led Stauffer conjecture:

Conjecture D.2.4. [19] *The stable set polytope of a claw-free but not quasi-line graph G with $\alpha(G) \geq 4$ is given by non-negativity constraints, rank constraints, and lifted 5-wheel constraints.*

Note that this conjecture claims in particular that a claw-free but not quasi-line graph G with $\alpha(G) \geq 4$ does even not contain a non-rank facet-producing *sub*graph G' with $\alpha(G') = 3$ and that the only non-rank facet-producing subgraphs G' with $\alpha(G') = 2$ rely on 5-wheels. Lifted 5-wheel constraints $1x(A) + 2x(B) \leq 2$ are obviously special clique neighborhood constraints, thus combining this conjecture with Cook's result yields:

Conjecture D.2.5. *Any non-trivial, non-clique facet of the stable set polytope of a claw-free graph G with $\alpha(G) \neq 3$ is either a clique neighborhood constraint or a clique family inequality.*

The graphs with stability number three

The results from the previous subsection imply that all difficult facets of claw-free graphs occur in the case $\alpha(G) = 3$. Our goal is to describe the stable set polytope for general graphs G with $\alpha(G) = 3$ completely, and to discuss consequences for the non-rank facets of the claw-free case.

Let $G = (V, E)$ be a graph with $\alpha(G) = 3$ and $a^T x \leq b$ a facet of STAB(G). Then there is a collection $\mathcal{S}$ of $n = |V|$ independent tight stable sets $S_1, \ldots, S_n$, called *roots*, with $a^T \chi^{S_i} = b$ for $1 \leq i \leq n$. By $\alpha(G) = 3$, every root contains at most 3 nodes. If a root is one single node $S_i = \{s\}$, then $a_s = b$ would follow and s must be completely joined to any other node v of G with $a_v > 0$. According to Chvátal [2], it is known how to built complete join facets, so we restrict ourself to the case where all roots of $a^T x \leq b$ are of size 2 or 3, i.e., when $a^T x \leq b$ is not a complete join facet.

In the complement $\overline{G}$, the roots of $a^T x \leq b$ in $\mathcal{S}$ correspond to cliques of size 2 or 3; denote by $R_{e,\mathcal{S}}(a, b)$ (resp. $R_{\triangle,\mathcal{S}}(a, b)$) the set of those edges (resp. triangles) in $\overline{G}$. Let further $F_{\mathcal{S}}(a, b) = (V, R_{e,\mathcal{S}}(a, b))$ be the subgraph of $\overline{G}$ containing the edge-roots from $\mathcal{S}$. We show that $F_{\mathcal{S}}(a, b)$ is an *odd 1-forest*, having trees and odd 1-trees as only components (where a *1-tree* is a tree with one additional edge, and is *odd* if its only cycle is an odd hole).

Lemma D.2.6. *Let G be a graph with $\alpha(G) = 3$ and $a^T x \leq b$ a non-complete join facet of STAB(G) with a collection $\mathcal{S}$ of roots. Then $F_{\mathcal{S}}(a, b)$ is an odd 1-forest.*

Proof. Any component T of $F_{\mathcal{S}}(a, b)$ must have at least $|T| - 1$ edges (since it is connected), but at most $|T|$ edges (since all its edges correspond to roots in $\mathcal{S}$ and are linearly independent by construction). Thus, any component T of $F_{\mathcal{S}}(a, b)$ is either a tree (if it has $|T| - 1$ edges) or a 1-tree (if it has $|T|$ edges). Note that every 1-tree T has as many edges as nodes, but all edges are independent only if the only cycle of T has odd length; this odd cycle must have length > 3, otherwise its edges would form a triangle and could not be roots in $\mathcal{S}$. □

A facet $a^T x \leq b$ of STAB(G) is a *co-spanning 1-forest constraint* if in all collections of roots $\mathcal{S}$, the roots correspond to edges of a spanning 1-forest $F_{\mathcal{S}}(a, b)$ consisting of trees and odd 1-trees and as many triangles of $\overline{G}$ as $F_{\mathcal{S}}(a, b)$ has tree-components. We further distinguish two special cases: a co-spanning 1-forest constraint $a^T x \leq b$ is called *co-spanning forest constraint* if there is a collection $\mathcal{S}$ s.t. $F_{\mathcal{S}}(a, b)$ is a forest (i.e., has tree-components only) and *co-spanning tree constraint* if $F_{\mathcal{S}}(a, b)$ is a tree.

Example D.2.7.

(a) Giles and Trotter [9] call a claw-free graph $G = (A \cup B, E)$ *wedge* if $\overline{G}$ has a unique triangle $\Delta \subseteq A$, a spanning tree T with two or three spokes, and additional edges linking inner nodes of T to exactly one node of Δ (to avoid claws in G), see Figure D.1(b) for an example. They showed that each wedge G has a non-rank facet

$$1x(A) + 2x(B) \leq 3$$

whose roots correspond in $\overline{G}$ to the edges of T and the triangle Δ. Thus, the facets induced by wedges are co-spanning tree constraints.

(b) Liebling et al. [11] showed that the claw-free graph "fish in a net" induces the non-rank facet

$$1x(\circ) + 2x(\bullet) + 3x(\square) + 4x(\otimes) \leq 5$$

whose roots correspond in $\overline{G}$ to the edges of two trees and two triangles (see Figure D.2(a)) and, thus, is a co-spanning forest facet. With the help of linear programming techniques, it is possible to extend the structure of "fish in a net" in order to obtain an infinite sequence of claw-free graphs inducing co-spanning forest facets with arbitrary many different left hand side coefficients and arbitrary high right hand sides (see [15] for the construction). Two graphs from this series are shown in Figure D.2(b),(c) and the corresponding facets are

$$1x(\circ) + 2x(\bullet) + 3x(\square) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) \leq 7$$

and

$$1x(\circ) + 2x(\bullet) + 3x(\square) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) + 7x(\diamond) + 8x(\oplus) \leq 9.$$

(c) Giles and Trotter [9] showed that the claw-free graph in Figure D.1(c) induces the non-rank facet

$$1x(\circ) + 2x(\bullet) + 3x(\square) \leq 4$$

which is a co-spanning 1-forest facet, as its roots correspond in $\overline{G}$ to the edges of a path and an odd hole and one triangle (see Figure D.3(a)). In fact, it is easy to construct similar examples by, e.g., using longer odd holes, extending the odd hole to a more general 1-tree, or attaching further branches to the tree.

As a new example, the complement of the graph in Figure D.3(b) has

$$2x(\bullet) + 3x(\square) + 4x(\otimes) \leq 6$$

as facet. It is obtained by joining the facets associated with the 5-hole and the smallest wedge (in fact, joining appropriate scalings of two facets, e.g., of any odd co-1-tree and any wegde, is a general method to obtain new facets).

Liebling et al. [11] showed that the claw-free graph "fish in a net with bubble" induces the non-rank facet

$$2x(\bullet) + 3x(\square) + 4x(\otimes) + 5x(\triangle) + 6x(\odot) \leq 8$$

which is a co-spanning 1-forest facet as well (see Figure D.3(c)).

Thus, all the previously known non-rank facets of claw-free graphs G with $\alpha(G) = 3$ are co-spanning 1-forest constraints. The main result of this paper is that *all* non-clique, non-complete join facets of (general) graphs G with $\alpha(G) = 3$ are of this type:

Theorem D.2.8. *Let G be a graph with $\alpha(G) = 3$ and $a^T x \leq b$ a non-clique, non-complete join facet of STAB(G). Then $a^T x \leq b$ is a co-spanning forest constraint if b is odd and a co-spanning 1-forest constraint if b is even.*

Proof. Consider a collection $\mathcal{S}$ of $n = |G|$ linearly independent roots of $a^T x \leq b$. By the assumption $\alpha(G) = 3$, all roots in $\mathcal{S}$ have size at most 3; as $a^T x \leq b$ is a non-clique, non-complete join facet, roots of size 1 are excluded. Thus, all roots in $\mathcal{S}$ have size 2 or 3; denote by $R_{e,\mathcal{S}}(a,b)$ (resp. $R_{\triangle,\mathcal{S}}(a,b)$) the set of edges (resp. triangles) in $\overline{G}$ which correspond to the roots in $\mathcal{S}$.

By Lemma D.2.6, the subgraph $F_{\mathcal{S}}(a,b) = (V, R_{e,\mathcal{S}}(a,b)) \subseteq \overline{G}$ is an odd 1-tree and contains $|G| - k$ roots from $\mathcal{S}$, where k is the number of tree-components of $F_{\mathcal{S}}(a,b)$, and $R_{\triangle,\mathcal{S}}(a,b)$ contains the remaining k roots from $\mathcal{S}$. Thus, $a^T x \leq b$ is a co-spanning 1-forest constraint.

Now suppose that b is odd, but that $F_{\mathcal{S}}(a,b)$ has an odd 1-tree component T and consider the odd hole $H \subseteq T$ of length $2l + 1$.

For every edge $v_i v_{i+1}$ of H, we have $a_i + a_{i+1} = b$. In particular, $a_i + a_{i+1} = a_{i+1} + a_{i+2} = b$ implies that all nodes in H with odd (resp. even) index should have the same weight. As also $a_1 + a_{2l+1} = b$ holds, we infer that all nodes v_i of H must have the same weight $a_i = \frac{b}{2}$. This is possible only if $2|b$.

Hence, whenever $a^T x \leq b$ with $b \geq 3$ odd is a non-rank, non-complete join facet of a graph G with $\alpha(G) \geq 3$, then $a^T x \leq b$ must be a co-spanning forest constraint. ■

In order to obtain the complete description for stable set polytopes of graphs G with $\alpha(G) = 3$, we combine Theorem D.2.8 with Chvátal's construction of complete join facets [2]. Chvátal proved that all nontrivial facets of $\text{STAB}(G_1 * G_2)$ are of the form

$$b_2 \cdot a_1^T x + b_1 \cdot a_2^T x \leq b_1 \cdot b_2$$

where $a_1^T x \leq b_1$ and $a_2^T x \leq b_2$ are facets of $\text{STAB}(G_1)$ and $\text{STAB}(G_2)$, resp. In particular, the set of roots for the complete join facet is the union of sets of roots from the original facets. For the case of graphs G with $\alpha(G) = 3$, this implies that a nontrivial facet has a single node as root only if it is a clique constraint or the complete join of a clique constraint and a co-spanning 1-forest constraint (the complete join of two co-spanning 1-forest constraints yields again a co-spanning 1-forest constraint, see Figure D.3(b) for an example). This implies:

Corollary D.2.9. *All nontrivial facets of graphs G with $\alpha(G) = 3$ are complete joins of clique constraints and co-spanning 1-forest constraints.*

We can further describe the rank facets of graphs G with $\alpha(G) = 3$.

Lemma D.2.10. *Any rank facet of the stable set polytope of a graph G with $\alpha(G) = 3$ is either*

- *a clique constraint,*
- *the rank constraint $x(G') \leq 2$ associated with a proper subgraph $G' \subset G$ with $\alpha(G') = 2$, or*
- *the full rank constraint $x(G) \leq 3$ of G itself.*

Proof. For any rank facet $a^T x \leq b$ of $\text{STAB}(G)$, we have $b \in \{1, 2, 3\}$ by $\alpha(G) = 3$. Let G' be the subgraph of G induced by the node subset $V' = \{v : a_v = 1\}$.

If $b = 1$, then $\alpha(G') = 1$ implies that G' is a clique and, thus, $a^T x \leq b$ a clique constraint.

If $b = 2$, then $\alpha(G') = 2$ implies that $a^T x \leq b$ is a clique-neighborhood constraint $2x(Q) + 1x(N'(Q)) \leq 2$ with $Q = \emptyset$ and $N'(Q) = V'$. In this case, there are no roots of size 1. All roots of size two have both endnodes in G'; all roots of size 3 are (1,1,0)-valued, containing two nodes from G' and one node from $G - G'$. As G contains at least one stable set S of size 3 and $|S \cap G'| \leq 2$ holds, we have that $G' \subset G$ is a proper subgraph.

If $b = 3$, then $\alpha(G') = 3$ implies that all roots must be (1,1,1)-valued and of size 3. In particular, every node occurs in some root and, thus, no node v can have $a_v = 0$, which implies that $a^T x \leq b$ is the full rank constraint $x(G) \leq 3$. (Note that for any collection $\mathcal{S}$ of roots, $F_{\mathcal{S}}(a, b) = (V, \emptyset)$ is a forest with n single-node-trees and $R_{\triangle,\mathcal{S}}(a, b))$ contains $|G|$ independent triangles.) ∎

The previous lemma shows that also clique neighborhood facets occur in graphs G with $\alpha(G) = 3$. More precisely, we have:

Lemma D.2.11.

- *Every rank clique neighborhood facet is either a clique constraint or a co-spanning 1-forest constraint.*
- *Every non-rank clique neighborhood constraint is the complete join of a clique constraint and a co-spanning 1-forest constraint.*

Proof. Consider a clique neighborhood constraint $F(Q)$ of the form

$$2x(Q) + 1x(N'(Q)) \leq 2$$

of $\text{STAB}(G)$ and let $V' = \{v \in V(G) : a_v > 0\}$.

$F(Q)$ is a rank constraint only if either Q is maximal and $N'(Q) = \emptyset$ or Q is empty and $N'(Q) = V'$. In the latter case, each component of $\overline{G}[N'(Q)]$ is non-bipartite and, thus, contains an odd hole by $\omega(\overline{G}) = 2$, if the constraint defines a facet. The roots correspond, therefore, in $\overline{G}[V']$ to the edges of a spanning 1-forest consisting of odd 1-trees only. (Note that every tree-component would particularly require a triangle, which is not present in $\overline{G}[V']$ by assumption.)

$F(Q)$ is a non-rank constraint only if both Q and $N'(Q)$ are non-empty. By definition, Q and $N'(Q)$ are completely joined, thus $F(Q)$ is the complete join of the clique constraint associated with Q and the rank constraint associated with $N'(Q)$. ∎

Calling a facet *joined co-spanning 1-forest constraint* if it is a complete join of a clique constraint and a co-spanning 1-forest constraint (where one of the parts can be empty), we finally obtain:

Corollary D.2.12. *Every nontrivial facet of the stable set polytope of a graph G with $\alpha(G) \leq 3$ is a joined co-spanning 1-forest constraint.*

Consequences for claw-free graphs

The problem of characterizing the stable set polytope of claw-free graphs has been open for almost two decades and is still open. Only their *rank* facets are well-understood [8], but it was open so far which *non*-rank facets are required; even no conjecture was at hand.

In order to formulate a conjecture regarding the types of non-rank facets for claw-free graphs G, we combine Cook's result in the case $\alpha(G) = 2$, our results from the previous subsection for the case $\alpha(G) = 3$, and the conjecture of Stauffer [19] for the case $\alpha(G) \geq 4$.

Conjecture D.2.13. *Let G be claw-free, $a^T x \leq b$ a non-rank facet of STAB(G), and G' the subgraph induced by nodes v with $a_v > 0$. Then $a^T x \leq b$ is a*

- *clique neighborhood constraint if $\alpha(G') = 2$;*
- *co-spanning 1-forest constraint if $\alpha(G') = 3$;*
- *clique neighborhood or clique family inequality if $\alpha(G') \geq 4$.*

This conjecture is true if and only if Stauffer's conjecture from [19] is. An affirmative answer to this conjecture would verify that indeed all complicated facets associated with claw-free graphs occur in the case $\alpha(G') = 3$ only, as clique-neighborhood constraints are 0,1,2-valued and clique family inequalities have also only two different non-zero coefficients. But even if the conjecture turns out to be false, it is an interesting question whether for claw-free but not quasi-line graphs G with $\alpha(G) \geq 4$ some "easy" facets suffice to describe STAB(G) (e.g., facets with only two consecutive non-zero coefficients, possibly even 0,1,2-valued).

Moreover, we can specify the above conjecture further, taking into account that clique neighborhood constraints are joined co-spanning 1-forest constraints.

Conjecture D.2.14. *A non-rank facet of the stable set polytope of a claw-free graph G is either a clique family inequality (if G is quasi-line) or a joined co-spanning 1-forest constraint (otherwise).*

If true, the latter conjectures provide an answer to the long-time open problem which types of non-rank facets are required for the stable set polytopes of claw-free graphs.

In a companion paper [16], we address further the problem whether co-spanning 1-forest constraints can be interpreted as some more general clique family inequalities. We present first results in this direction and conjecture that this is, in fact, true for all co-spanning 1-forest facets of claw-free graphs. An affirmative answer to this conjecture would imply that all facets of claw-free graphs are certain types of general clique family inequalities. Hence, also the polyhedral aspect of the stable set problem for claw-free graphs would be an extension of Edmonds' description of the matching polytope, as every such clique family inequality generalizes Edmonds' odd set inequalities for the matching polytope.

List of figures

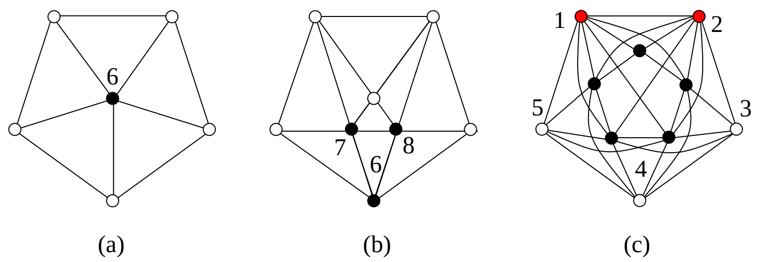

Figure D.1: Graphs with facets different from clique family inequalities

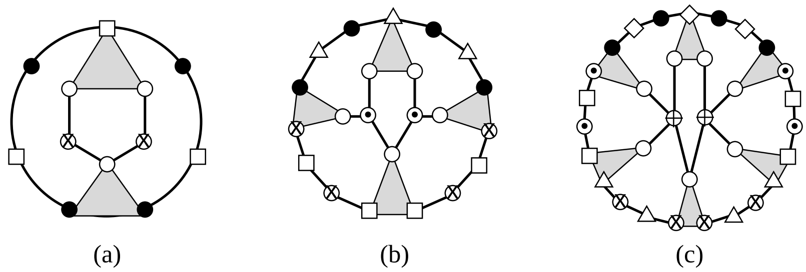

Figure D.2: Complements of graphs with co-spanning forest facets

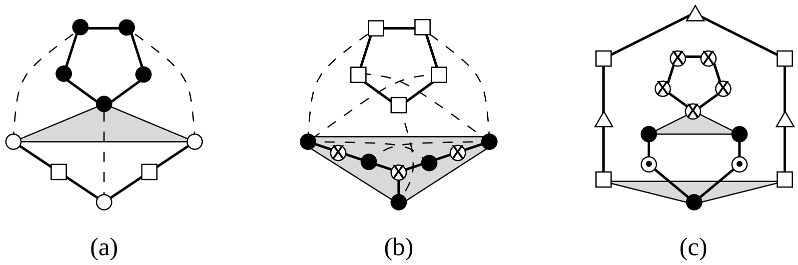

Figure D.3: Complements of graphs with co-spanning 1-forest facets

References

[1] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[2] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[3] J.E. Edmonds. Maximum matching and a polyhedron with (0,1) vertices. *J. Res. Nat. Bur. Standards*, 69(B):125–130, 1965.

[4] J.E. Edmonds. Paths, trees, and flowers. *Canad. J. Math.*, 17:449–467, 1965.

[5] J.E. Edmonds and W.R. Pulleyblank. Facets of 1-matching polyhedra. In C. Berge and D.R. Chuadhuri, editors, *Hypergraph Seminar*, pages 214–242. Springer, 1974.

[6] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. Circular one matrices and the stable set polytope of quasi-line graph. *Lecture Notes in Computer Science*, 3509:291–305, 2005.

[7] J. L. Fouquet. A strengthening of Ben Rebea's lemma. *J. Comb. Theory Ser. B*, 59(1):35–40, 1993.

[8] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1):1–38, 1997.

[9] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[10] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[11] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[12] G.J. Minty. On maximal independent sets of vertices in claw-free graphs. *J. Comb. Theory Series B*, 28:284–304, 1980.

[13] D. Nakamura and A. Tamura. A revision of Minty's algorithm for finding a maximum weighted stable set of a claw-free graph. *Journal of the OR Society of Japan*, 44:194–204, 2001.

[14] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[15] A. Pêcher, P. Pesneau, and A. Wagler. On facets of stable set polytope of claw-free graphs with maximum stable set size three. In *CS06 - Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications*, jul 2006.

[16] A. Pêcher and A. Wagler. Generalized clique family inequalities for claw-free graphs. In Elsevier, editor, *CTW06 - Cologne-Twente Workshop on Graphs and Combinatorial Optimization*, volume 25, pages 117–121. Electronic Notes in Discrete Mathematics, jun 2006.

[17] N. Sbihi. Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile. *Disc. Math.*, 29:53–76, 1980.

[18] F.B. Shepherd. Near-perfect matrices. *Mathematical Programming*, 64:295–323, 1994.

[19] G. Stauffer. *On the Stable Set Polytope of Claw-free Graphs*. PhD thesis, EPF Lausanne, 2005.

D.3 Generalized clique family inequalities for claw-free graphs

par A. Pêcher et A. Wagler
Soumis dans Discrete Applied Mathematics [85]

Providing a complete description of the stable set polytopes of claw-free graphs is a long-standing open problem [8]. For the subclass of quasi-line graphs, Ben Rebea claimed and Eisenbrand et al. [4] recently proved that all non-trivial facets belong to only one class, the so-called clique family inequalities. For general claw-free graphs, however, more complex facets are required to describe the stable set polytope [11]. We introduce a generalization of clique family inequalities, and prove that several facet-defining inequalities for general claw-free graphs are of this type. We conjecture that all non-rank factes of claw-free graphs can be interpreted that way.

Introduction

The *stable set polytope* STAB(G) of a graph G is defined as the convex hull of the incidence vectors of all its stable sets. The description of STAB(G) by means of facet-defining inequalities is unknown for the most graphs. The *nonnegativity constraints* $x_i \geq 0$ for all nodes i of G and the *clique constraints*

$$x(Q) := \sum_{i \in Q} x_i \leq 1$$

for all maximal cliques $Q \subseteq G$ are always facets of STAB(G), but suffice for *perfect graphs* only [2]. According to a recent characterization [1], that are precisely those graphs without any chordless cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$, as induced subgraph.

A graph is *claw-free* if the neighborhood of any node does not contain any stable set of size 3. Providing a complete description of the stable set polytopes of claw-free graphs is a long-standing open problem [8]. A characterization of the 0/1-valued facets, called *rank* facets, in stable set polytopes of claw-free graphs was given by Galluccio and Sassano [6], who proved that their rank facets rely on certain line graphs and webs. However, claw-free graphs have *non-rank* facets in general and even a conjecture regarding their non-rank facets was formulated only recently [11].

Claw-free graphs contain all *line graphs*, that are graphs obtained by taking the edges of a root graph H as nodes and connecting two nodes of the line graph iff the corresponding edges of H are adjacent. All facets of the stable set polytope of line graphs are known from matching theory [3], namely, clique constraints and certain rank constraints coming from odd set inequalities. Ben Rebea [12] generalized the odd set inequalities for the matching polytope to clique family inequalities for the stable set polytopes of all graphs as follows. Let $G = (V, E)$ be a graph, $\mathcal{Q}$ be a family of at least three inclusion-wise maximal cliques of G, $p \leq |\mathcal{Q}|$ be an integer, and consider the two sets

$$\begin{aligned} V_p = & \quad \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| \geq p\}, \\ V_{p-1} = & \quad \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| = p - 1\}. \end{aligned}$$

The clique family inequality $(\mathcal{Q}, p)$ is defined as

$$(p-r) \sum_{i \in V_p} x_i + (p-r-1) \sum_{i \in V_{p-1}} x_i \leq (p-r) \left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor$$

with $r = |\mathcal{Q}| (\bmod p)$ and $r > 0$. Clique family inequalities are valid for the stable set polytope of *every* graph [10]; the question is for which graphs do they suffice. Ben Rebea [12] claimed and Eisenbrand et al. [4] recently proved that this is the case for *quasi-line graphs*, that are graphs where the neighborhood of any node can be partitioned into two cliques. However, even for the small claw-free but not quasi-line graphs depicted in Figure D.4 clique family inequalities do not suffice to describe the stable set polytope:

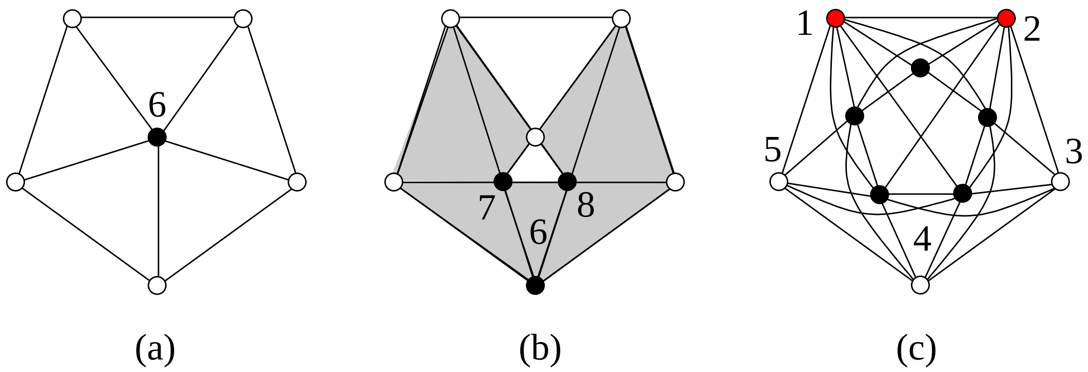

Figure D.4: Graphs with facets different from clique family inequalities.

The 5-wheel in Figure D.4(a) induces the facet $1x(\circ) + 2x(\bullet) \leq 2$. The only clique family inequality with $V_p = V_\bullet$ and $V_{p-1} = V_\circ$ is $(\mathcal{Q}, 3)$ with all five maximal cliques in $\mathcal{Q}$, but $r = 2$ yields only $1x(V_p) + 0x(V_{p-1}) \leq 1$.

The wedge in Figure D.4(b) has the facet $1x(\circ) + 2x(\bullet) \leq 3$ by [7]. A clique family inequality with $V_p = V_\bullet$, $V_{p-1} = V_\circ$ is $(\mathcal{Q}, 3)$ using the 7 grey triangles, which yields $r = 1$ but $1x(\circ) + 2x(\bullet) \leq 4$ with a too weak right hand side.

The graph in Figure D.4(c) induces $1x(\circ) + 2x(\bullet) + 3x(\Box) \leq 4$ as facet by [7] which is clearly no clique family inequality since it involves more than two different non-zero coefficients.

This motivated us to extend the concept of clique family inequalities by

- defining more than the two sets V_p and V_{p-1};
- allowing values $r < |\mathcal{Q}| \bmod p$;
- strengthening the right hand side appropriately.

Let $G = (V, E)$ be a graph and $\mathcal{Q}$ be a family of at least three cliques of G. Choose integers $p \leq |\mathcal{Q}|$, r with $0 \leq r \leq R = |\mathcal{Q}| \bmod p$, and J with $0 \leq J \leq p - r$. Define different types of sets as

$$\begin{aligned} V_p &= \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| \geq p\}, \\ V_{p-j} &= \{i \in V : |\{Q \in \mathcal{Q} : i \in Q\}| = p - j \end{aligned}$$

for $1 \leq j \leq J$ (some of the sets $V(\mathcal{Q}, p-j)$ might be empty). We define the *general clique family inequality* $(\mathcal{Q}, p, r, J, b)$ by

$$\sum_{0 \leq j \leq J} (p - r - j)\, x(V_{p-j}) \;\leq\; b$$

which is valid by an appropriate choice of the right hand side b. In this context, two questions arise, namely, which choices of the parameters p, r, J, b guarantee validity and for which graphs do such inequalities suffice. We exhibit several valid inequalities of this type in Section D.3.

Our further goal is to express the non-rank facets of claw-free but not quasi-line graphs as certain types of general clique family inequalities, see Section D.3. For that, we have to consider two types of inequalities, namely, clique neighborhood constraints and co-spanning 1-forest constraints.

For any clique Q of a graph G, the *clique neighborhood constraint* $F(Q)$ is

$$2x(Q) + 1x(N'(Q)) \leq 2$$

where $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$ is the clique neighborhood. Cook showed (see [13]) that for a graph G with $\alpha(G) = 2$, all non-trivial facets of STAB(G) are clique neighborhood constraints; Stauffer [14] conjectured that special clique neighborhood constraints are the only non-rank facets of claw-free but not quasi-line graphs G with $\alpha(G) \geq 4$. Clique neighborhood constraints are no clique family inequalities [11], but we show in Section D.3 that they are general clique family inequalities $(\mathcal{Q}, k+1, k-1, 1, 2)$.

For the graphs G with $\alpha(G) = 3$, some examples of facets where known from [7, 9]. They can admit more than two different non-zero coefficients, and all the difficult facets of claw-free graphs occur if $\alpha(G) = 3$ (provided Stauffer's conjecture for the case $\alpha(G) \geq 4$ is true). In [11] we showed that all these facets belong to only one class, the co-spanning 1-forest constraints. A 1-forest F consists of tree and 1-tree components, where a 1-tree is a tree with an extra edge. A facet of the stable set polytope of a graph G is called *co-spanning 1-forest constraint*, if all tight stable sets correspond in the complementary graph to the edges of a 1-forest F and as many triangles as F has tree components. Co-spanning 1-forest constraints are no clique family inequalities [11], but we show in Section D.3 that some special classes are general clique family inequalities of the form $(\mathcal{Q}, p, R, J, p)$.

We conjecture that all co-spanning 1-forest facets of claw-free graphs can be interpreted as general clique family inequalities. If this conjecture and Stauffer's conjecture for the case $\alpha(G) \geq 4$ are both true, then all facets of claw-free graphs would be certain types of general clique family inequalities. In particular, also the polyhedral aspect of the stable set problem for claw-free graphs would be an extension of Edmonds' description of the matching polytope, as every general clique family inequality extends Edmonds' odd set inequalities for the matching polytope, see Section D.3.

Some valid general clique family inequalities

We first note that the original proof of validity of usual clique family inequalities can be easily extended to handle more than 2 left coefficients.

Lemma D.3.1. *Let $(\mathcal{Q}, p)$ be a family of n cliques of a graph G, $R = n(\text{mod} p)$ and $0 \leq J \leq p - R$. Every $(\mathcal{Q}, p, R, J, b)$ general clique family inequality with $b \geq (p-R)\lfloor n/p \rfloor$ is valid.*

Proof. Let S be any stable set. For every $0 \leq j \leq J$, let s_j be the number of vertices of S in V_{p-j}. Let $s = \sum_{0 \leq j \leq J} s_j$.

If $s \leq \lfloor n/p \rfloor$ then $\sum_{0 \leq j \leq J}(p-R-j)s_j \leq (p-R)s \leq (p-R)\lfloor n/p \rfloor$.

If $s \geq \lfloor n/p \rfloor + 1$ then, since $\sum_{0 \leq j \leq J}(p-j)s_j \leq n = p\lfloor n/p \rfloor + R$, we have $\sum_{0 \leq j \leq J}(p-R-j)s_j \leq p\lfloor n/p \rfloor + R - Rs \leq (p-R)\lfloor n/p \rfloor$.

Hence in both cases, $\sum_{0 \leq j \leq J}(p-R-j)s_j \leq (p-R)\lfloor n/p \rfloor \leq b$. ■

Notice that Lemma D.3.1 with $J = 2$ proves again the validity of usual clique family inequalities. However, in contrary to clique family inequalities for quasi-line graphs, this right hand side is not strong enough to get all facets of claw-free graphs. For instance, the graph in Figure D.4 (c) induces a facet with right hand side 4 which is not a multiple of the biggest left coefficient. Hence to get this facet, we need to lower the right hand side further.

It is not in general possible to improve Lemma D.3.1 as the right hand side is tight for many graphs. Therefore, additional requirements are needed to lower the right hand side.

Lemma D.3.2. *Let $(\mathcal{Q}, p)$ be a family of n cliques of a graph G, $R = n(\text{mod} p)$ and $0 \leq J \leq p - R$. Let $0 \leq \delta \leq \min\{R, p-R\}$.*

If for every stable set with $\lfloor n/p \rfloor$ or $\lfloor n/p \rfloor + 1$ vertices in $V(\mathcal{Q}) = \cup_{Q \in \mathcal{Q}} Q$ the following inequality holds

$$(p-R)x(V_p) + \ldots + (p-R-J)x(V_{p-J}) \quad \leq \quad (p-R)\lfloor n/p \rfloor - \delta \tag{D.12}$$

then the general clique family inequality $(\mathcal{Q}, p, R, J, (p-R)\lfloor n/p \rfloor - \delta)$ is valid for STAB(G).

Proof. Let S be a stable set. For every $0 \leq j \leq J$, let s_j be the number of vertices of S in V_{p-j}. Let $s = \sum_{0 \leq j \leq J} s_j$. We are going to show that the incidence vector of S satisfies In. (D.12), by pushing the arguments of the proof of Lemma D.3.1 a little bit further.

If $\lfloor n/p \rfloor \leq s \leq \lfloor n/p \rfloor + 1$, we have nothing to prove.

If $s < \lfloor n/p \rfloor$ then $\sum_{0 \leq j \leq J}(p-R-j)s_j \leq (p-R)s \leq (p-R)(\lfloor n/p \rfloor - 1) \leq (p-R)\lfloor n/p \rfloor - \delta$.

If $s \geq \lfloor n/p \rfloor + 2$ then, since $\sum_{0 \leq j \leq J}(p-j)s_j \leq n = p\lfloor n/p \rfloor + R$, we have $\sum_{0 \leq j \leq J}(p-R-j)s_j \leq p\lfloor n/p \rfloor + R - Rs \leq (p-R)\lfloor n/p \rfloor - R \leq (p-R)\lfloor n/p \rfloor - \delta$.

Hence in both cases, $\sum_{0 \leq j \leq J}(p-R-j)s_j \leq (p-R)\lfloor n/p \rfloor - \delta$.

■

For instance, Lemma D.3.2 may be used to get the good value of the right hand side of the facet of the graph in Figure D.4 (c) : label clockwise the nodes of the outer C_5 from 1 to 5 (starting from the left top node) and label clockwise the nodes of the inner C_5 from 6 to 10 (starting from the left top node), then it is straightforward to check that the facet $1x(\circ) + 2x(\bullet) + 3x(\square) \leq 4$ is in fact the general clique family inequality $(\mathcal{Q}, p, R, J, b)$ where $\mathcal{Q}$ is the family of the 12 cliques $\{\{1,2,6\}, \{1,2,6,7\}, \{1,2,6,10\}, \{1,2,7,8\}, \{1,2,9,10\}, \{3,4,7,8\}, \{4,5,9,10\}, \{3,4,8,9\}, \{5,9,10\}, \{3,7,8\}, \{5\}, \{6\}\}$ and $p = 5$, $R = 3$, $J = 2$ and $b = (p-R)\lfloor n/p \rfloor - \delta$, with $\delta = 2 = \min\{p-R, R\}$.

In Theorem D.3.12, we give another formulation of this facet as a general clique family inequality using only 9 cliques.

General clique family facets for claw-free graphs

Our goal is to express the non-rank facets of claw-free but not quasi-line graphs as certain types of general clique family inequalities.

The clique neighborhood constraints

Recall that for the graphs G with $\alpha(G) = 2$ (including the 5-wheel), all facets are clique neighborhood constraints by Cook (see [13]):

Theorem D.3.3. *For a graph G with $\alpha(G) = 2$, all non-trivial facets of STAB(G) are clique-neighborhood constraints $F(Q)$*

$$2x(Q) + 1x(N'(Q)) \leq 2$$

where $Q \subseteq G$ is a clique and $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$ s.t. $\overline{G}[N'(Q)]$ has no bipartite component.

Clearly, a clique-neighborhood constraint $F(Q)$ is

- the clique constraint associated with Q if and only if Q is a maximal clique of G (and, therefore, $N'(Q) = \emptyset$),
- the full rank constraint associated with G if and only if $Q = \emptyset$ (and, therefore, $N'(Q) = G$), or
- a non-rank constraint if both parts Q and $N'(Q)$ are non-empty.

By definition of $N'(Q)$, the subgraph $G[Q \cup N'(Q)]$ is the complete join of Q and $N'(Q)$. If $F(Q)$ is a facet, then each component of $\overline{G}[N'(Q)]$ contains an odd hole (since odd cycles of length three are excluded by $\omega(\overline{G}) = 2$). If $F(Q)$ is non-rank, then $G[Q \cup N'(Q)]$ contains an *odd antiwheel* $\overline{C}_{2k+1} * v$, that is the complete join of a single node v and an odd antihole $\overline{C}_{2k+1}$ (note that the 5-wheel is also the 5-antiwheel by $C_5 = \overline{C}_5$).

A claw-free graph G is quasi-line if and only if it does not contain an odd antiwheel [5], and clique neighborhood constraints associated with odd antiwheels are no clique family inequalities by [11]. As a clique neighborhood constraint $F(Q)$ defines a non-rank facet only if it is a lifted odd antiwheel constraint, we obtain that no non-rank clique neighborhood constraint is a clique family inequality. In this section, we prove that every clique neighborhood constraint is a general clique family inequality.

Any non-rank facet $F(Q)$ of a graph G with $\alpha(G) = 2$ is the complete join of the clique constraint associated with Q and the full rank constraint associated with $N'(Q)$. As a first step towards our goal we prove the following more general assertion on full rank constraints of graphs with stability number two, which is interesting for its own.

Theorem D.3.4. *Let G be a graph with $\alpha(G) = 2$ and $\overline{C}_{2k+1}$ be a shortest odd antihole in G. Then the rank constraint $x(G) \leq 2$ is a general clique family inequality $(\mathcal{Q}, k, r, 1, 2)$ for some $0 \leq r \leq R$.*

Proof. As G has stability number 2, all of its minimal imperfect subgraphs are odd antiholes [1]. Every odd antihole $\overline{C}_{2l+1}$ has $2l+1$ cliques of maximum size l, namely $Q(i) = \{i, \ldots, i+l-1\}$, for $1 \leq i \leq 2l+1$ (indices are taken modulo $2l+1$); in particular, each node i of $\overline{C}_{2l+1}$ belongs to the l maximum cliques $Q(i-l+1), \ldots, Q(i)$. In order to present the studied clique family $\mathcal{Q}$, we show that the maximum cliques of a *shortest* odd antihole $\overline{C}_{2k+1} \subseteq G$ can be extended in such a way that every node $v \in G \setminus \overline{C}_{2k+1}$ is covered at least k times (possibly using more than $2k+1$ cliques).

Claim D.3.5. *For any $C_{2l+1} \subseteq G$ and $v \in G \setminus \overline{C}_{2l+1}$, the set of non-neighbors of v on $\overline{C}_{2l+1}$ induces a clique.*

Otherwise, $\alpha(G[\overline{C}_{2l+1} \cup \{v\}]) = 3$. $\diamond$

Claim D.3.6. *For any $\overline{C}_{2l+1} \subseteq G$, each node $v \in G \setminus \overline{C}_{2l+1}$ is adjacent to at least $l+t$ consecutive nodes of $\overline{C}_{2l+1}$ where $t \geq 1$; in particular, v is completely joined to $t+1$ maximum cliques of $\overline{C}_{2l+1}$.*

Denote the maximum interval of consecutive neighbors of v on $\overline{C}_{2l+1}$ by $1,\ldots,l+t$. Then v's non-neighbors are among the nodes $l+t+1,\ldots,2l+1$. As those non-neighbors induce a clique by Claim D.3.5, we have $l+t+1 \geq l+2$, i.e., $t \geq 1$ follows. Thus, v is completely joined to the $t+1$ consecutive maximum cliques $Q(1)=\{1,\ldots,l\},\ldots,Q(1+t)=\{1+t,\ldots,l+t\}$ of $\overline{C}_{2l+1}$. $\diamond$

Claim D.3.7. *Let $v \in G \setminus \overline{C}_{2l+1}$ and $1,\ldots,l+t$ be the maximum interval of consecutive neighbors of v on $\overline{C}_{2l+1}$. If $t+1 < l$ then $G[\overline{C}_{2l+1} \cup \{v\}]$ contains a shorter odd antihole $\overline{C}_{2(t+1)+1}$.*

In this case, v is certainly not adjacent to $l+t+1$ and $2l+1$ (but v might be adjacent to nodes in between). We show that the node subset $V' = \{2l+1, 1, \ldots, t, v, l+1, \ldots, l+1+t\}$ induces a $\overline{C}_{2(t+1)+1}$ in G. Note that we can rewrite V' as $2l+1,\ldots,(2l+1)+t, v, k+1,\ldots,(l+1)t$. By construction, a node $x \in V'$ has exactly the following non-neighbors in V':

$x \in V'$	$\overline{N}_{V'}(x)$		
v	$l+t+1$	and	$2l+1$
$l+t', 1 \leq t' \leq t$	$2l+t'$	and	$2l+t'+1$
$(l+1)+t$	$2l+t+1$	and	v
$2l+1$	v	and	$l+t$
$(2l+1)+t', 1 \leq t' \leq t$	$l+t'$	and	$l+t'+1$

as required. $\diamond$

Claim D.3.8. *If $\overline{C}_{2k+1} \subseteq G$ is a* shortest *odd antihole in G, then each code $v \in G \setminus \overline{C}_{2k+1}$ is completely joined to at least k maximum cliques of $\overline{C}_{2k+1}$.*

This follows directly from Claim D.3.6 and Claim D.3.7. $\diamond$

Thus, extending the maximum cliques of $\overline{C}_{2k+1}$ appropriatly, we can construct a clique family $\mathcal{Q}$ with $|\mathcal{Q}| \geq 2k+1$ s.t. each node in G is covered at least k times by $\mathcal{Q}$. Choosing $p=k$ yields $V(\mathcal{Q},k)=V(G)$ and, hence, the clique family inequality $(\mathcal{Q},k)$ reads as

$$(p-r)x(G) \leq (p-r)\alpha(G) \leq (p-r)\left\lfloor \frac{|\mathcal{Q}|}{k} \right\rfloor$$

wich finally yields $x(G) \leq 2$ by $\lfloor \frac{|\mathcal{Q}|}{k} \rfloor \geq 2$ due to $|\mathcal{Q}| \geq 2k+1$, for any choice of r with $0 \leq r \leq R$. ■

With the help of this result, we interprete the clique-neighborhood constraint $F(Q)$ as follows:

- In a first step, we express the rank constraint $x(N'(Q)) \leq 2$ as general clique family inequality $(\mathcal{Q}', k, r, 1, 2)$ associated with a shortest odd antihole $\overline{C}_{2k+1}$ in $N'(Q)$.
- In a second step, we consider the complete join of Q and $N'(Q)$. For that, we adjust the family $\mathcal{Q}'$ in such a way that each node of $N'(Q)$ is covered *exactly* k times (by reducing the cliques in $\mathcal{Q}'$ to appropriate non-maximal cliques); we add Q to each such clique and obtain a new clique family $\mathcal{Q}$. By construction, each node in $N'(Q)$ is covered exactly k times by the cliques in $\mathcal{Q}$, and each node in Q exactly $|\mathcal{Q}|$ times. Choosing $p=k+1$ and $J=1$ yields $V(\mathcal{Q},p)=Q$ and $V(\mathcal{Q},p-1)=N'(Q)$. By choosing $r=k-1$ we finally obtain the general clique family inequality $(\mathcal{Q},k+1,k-1,1,2)$

$$\begin{array}{lll} 2\,x(V(\mathcal{Q},k+1)) & +1\,x(V(\mathcal{Q},k)) & \leq 2\left\lfloor \frac{2k+1}{k+1} \right\rfloor \\ 2\,x(Q)) & +1\,x(N'(Q))) & \leq 2 \end{array}$$

  as required.

Thus, starting from the above theorem we have obtained:

Theorem D.3.9. *Let G be a graph with $\alpha(G) = 2$, Q be a non-maximal clique of G, and $\overline{C}_{2k+1}$ be a shortest odd antihole in $N'(Q)$. The clique-neighborhood constraint $F(Q)$ is a general clique family inequality $(\mathcal{Q}, k+1, k-1, 1, 2)$.*

Note that *all* complete join facets of claw-free graphs are of type $F(Q)$ (as already the complete join of a single node and a graph with a stable set of size 3 contains a claw). Thus, the previous theorem implies the more generally:

Corollary D.3.10. *All complete join facets of claw-free graphs and all facets associated with graphs of stability number two are general clique family inequalities.*

If Stauffer's conjecture is true, then all non-rank facets for claw-free but not quasi-line graphs G with $\alpha(G) \neq 3$ are general clique family inequalities.

Special co-spanning 1-forest constraints

It remains to treat the case of claw-free but not quasi-line graphs G with $\alpha(G) = 3$. Recall that all non-rank facets of such graphs are co-spanning 1-forest constraints by [11], including all the difficult facets with many different non-zero coefficients. Hence, it is rather involved to figure out whether facets of this type are general clique family inequalities. We discuss this issue for co-spanning 1-forest constraints where the 1-forest F has a particular structure.

First, we observed in [11] that *rank* clique neighborhood facets are special co-spanning 1-forest constraints, where the 1-forest F consists of 1-tree components only (the condition when a clique neighborhood constraint $F(Q)$ is facet-inducing is equivalent to require that $N'(Q)$ is a 1-forest in the complementary graph: $\overline{G}[N'(Q)]$ has no bipartite component). Hence, Theorem D.3.4 shows that such facets are general clique family inequalities $(\mathcal{Q}, k, r, 1, 2)$ associated with the shortest odd antihole $\overline{C}_{2k+1}$ in $N'(Q)$.

We shall discuss two further classes of co-spanning 1-forest constraints, namely those, where the 1-forest F has only one tree and the graph is *basic* (that is, its complement contains no unneccessary edges but only those which are either required for roots or for the claw-freeness). If the 1-forest F is exactly one tree, we call the associated facet a *co-spanning tree constraint*. As observed in [11], every such facet is (1,2)-valued with right hand side 3. We have:

Theorem D.3.11. *For a claw-free graph G with $\alpha(G) = 3$, every basic co-spanning tree facet*

$$1x(\circ) + 2x(\bullet) \leq 3$$

is a general clique family inequality $(\mathcal{Q}, p, R, p-2, p)$ with $|\mathcal{Q}| = 7$ and $p = 3$.

Proof. The roots of this facet correspond in the complementary graph $\overline{G}$ to one triangle $\Delta = \{1, 2, 3\}$ and the edges of a spanning tree T, where the triangle Δ consists of $\circ$-nodes only and $\circ$-nodes and $\bullet$-nodes alternate in the tree T. Note that we have the following in $\overline{G}$:

(i) all inner $\circ$-nodes of T form a stable set (since G is basic);

(ii) all $\bullet$-nodes form a stable set (as G contains no stable set of weight > 3);

(iii) each node outside Δ has exactly one neighbor in Δ (at least one neighbor in Δ since G is claw-free and at most one since G is basic).

Let $N_\circ(i)$ (resp. $N_\bullet(i)$) denote the set of all $\circ$-neighbors (resp. $\bullet$-neighbors) of node i in G. We construct a clique family $\mathcal{Q}$ in G as follows: we choose $Q_{i,\circ} = \{i\} \cup N_\circ(i)$ and $Q_{i,\bullet} = \{i\} \cup N_\bullet(i)$ for each $i \in \Delta$ and $Q_\bullet$ consisting of all $\bullet$-nodes. Each set $Q_{i,\circ}$ is a clique by (i) and the definition of $N_\circ(i)$; all three cliques $Q_{1,\circ}$, $Q_{2,\circ}$, and $Q_{3,\circ}$ cover the nodes in Δ once and all inner $\circ$-nodes of T twice by (iii). Similarly, each set $Q_{i,\bullet}$ is a clique by (ii) and the definition of $N_\bullet(i)$; all three cliques $Q_{1,\bullet}$, $Q_{2,\bullet}$, and $Q_{3,\bullet}$ cover the nodes in Δ once and all $\bullet$-nodes twice by (iii). $Q_\bullet$ is a clique by (ii) and covers all $\bullet$-nodes once.

In total, each ∘-node is covered twice and each •-node three times. We have $|\mathcal{Q}| = 7$ and choose $p = 3$, $R = 1$, $J = 1$, and $b = (p - R)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor - J = p$. Thus, we obtain the studied facet

$$\sum_{0 \le j \le 1} (2-j)x(V(\mathcal{Q}, 3-j)) = 1x(\circ) + 2x(\bullet) \le 2\left\lfloor \frac{7}{3} \right\rfloor - 1 \le 3$$

as a general clique family inequality $(\mathcal{Q}, p, R, p-2, p)$. ■

Note that all wegdes induce basic co-spanning tree constraints which are all general clique family inequalities by the above result. With the help of similar techniques we show further that (1,2,3)-valued basic co-spanning 1-forest constraints with right hand side 4 are general clique family inequalities:

Theorem D.3.12. *For a claw-free graph G with exactly one stable set of size 3, every basic co-spanning 1-forest facet*

$$1x(\circ) + 2x(\bullet) + 3x(\square) \le 4$$

is a general clique family inequality $(\mathcal{Q}, p, R, p-2, p)$ with $|\mathcal{Q}| = 9$ and $p = 4$.

Proof. The roots of this facet correspond in the complement $\overline{G}$ to the only triangle $\Delta = \{1, 2, 3\}$ and the edges of a spanning 1-forest having exactly one tree component (and arbitrarily many 1-tree components). The triangle Δ consists of two ∘-nodes 1,2 and one •-node 3, ∘-nodes and □-nodes alternate in the tree T, all 1-trees consists of •-nodes. We have the following in $\overline{G}$:

(i) all inner ∘-nodes of T form a stable set (since G is basic);

(ii) a □-node is neither adjacent to a •-node nor to another □-node (as G contains no stable set of weight > 4);

(iii) each node outside Δ has exactly one neighbor in Δ (at least one neighbor in Δ since G is claw-free and at most one since G is basic), where this neighbor of a □-node is not node 3.

Let $N_\circ(i)$ (resp. $N_\bullet(i)$, $N_\square(i)$) denote the set of all ∘-neighbors (resp. •-neighbors, □-neighbors) of node i in G. Further, let $\overline{N}_\bullet(i)$ be the set of all •-neighbors of node i in $\overline{G}$. Note that $N_\bullet(i)$ is the union of all •-neighbors of the other two nodes from Δ in $\overline{G}$ and the union of $\overline{N}_\bullet(1), \overline{N}_\bullet(2), \overline{N}_\bullet(3)$ covers all •-nodes but node 3. Finally, let $Q_\square$ denote the set consisting of all □-nodes. We construct a clique family $\mathcal{Q}$ in G as follows: we choose $Q_{i,a} = \{i\} \cup N_\circ(i) \cup \overline{N}_\bullet(i+1)$ and $Q_{i,b} = \{i\} \cup N_\square(i) \cup \overline{N}_\bullet(i+1)$ for $1 \le i \le 3$, as well as $Q_{1,c} = Q_\square \cup \{3\} \cup \overline{N}_\bullet(1)$, $Q_{2,c} = Q_\square \cup \overline{N}_\bullet(2)$, $Q_{3,c} = Q_\square \cup \overline{N}_\bullet(3)$.

Each set $Q_{i,a}$ is a clique by (i), (iii), and since G is basic; all three cliques $Q_{1,a}$, $Q_{2,a}$, and $Q_{3,a}$ cover the nodes in Δ and all •-nodes once, and all inner ∘-nodes of T twice by (iii). Each set $Q_{i,b}$ is a clique by (ii) and (iii); all three cliques $Q_{1,b}$, $Q_{2,b}$, and $Q_{3,b}$ cover the nodes in Δ and all •-nodes once, and all □-nodes twice by (iii). The sets $Q_{i,c}$ are cliques by (ii); all three cliques $Q_{1,c}$, $Q_{2,c}$, and $Q_{3,c}$ cover all □-nodes three times and all •-nodes (including node 3) once. In total, each ∘-node is covered twice, each •-node three times, and each □-node five times. We have $|\mathcal{Q}| = 9$ and choose $p = 4$, $J = 2$, $r = R = 1$, and $b = (p - R)\left\lfloor \frac{|\mathcal{Q}|}{p} \right\rfloor - J = p$. Thus, we obtain the studied facet

$$\sum_{0 \le j \le 2} (3-j)x(V(\mathcal{Q}, 4-j)) = 1x(\circ) + 2x(\bullet) + 3x(\square) \le 3\left\lfloor \frac{9}{4} \right\rfloor - 2 \le 4$$

as a general clique family inequality $(\mathcal{Q}, p, R, p-2, p)$. ■

This result includes the facet of the graph from Figure D.4(c) and all facets of similar type. In addition, analogue to the composition of a rank clique neighborhood constraint and a clique constraint in Section 3.1, we can compose rank clique neighborhood constraints and basic co-spanning tree constraints to general clique family inequalities $(\mathcal{Q}, p, R, J, p)$. In fact, we conjecture that all co-spanning 1-forest facets of claw-free graphs can be interpreted that way.

Concluding remarks

If the latter conjecture is true, then all non-rank facets of claw-free graphs G with $\alpha(G) = 3$ are general clique family inequalities. If Stauffer's conjecture is true, then all non-rank facets of claw-free but not quasi-line graphs G with $\alpha(G) \geq 4$ are certain clique neighborhood constraints and, therefore, all non-rank facets for claw-free but not quasi-line graphs G with $\alpha(G) \neq 3$ are general clique family inequalities. Thus, if both conjectures are true, then all facets of claw-free graphs would be certain types of general clique family inequalities. In particular, also the polyhedral aspect of the stable set problem for claw-free graphs would be an extension of Edmonds' description of the matching polytope, as every general clique family inequality extends Edmonds' odd set inequalities for the matching polytope.

References

[1] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[2] V. Chvátal. On certain polytopes associated with graphs. *J. of Combinatorial Theory B*, 18:138–154, 1975.

[3] J.E. Edmonds. Maximum matching and a polyhedron with (0,1) vertices. *J. Res. Nat. Bur. Standards*, 69(B):125–130, 1965.

[4] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. Circular one matrices and the stable set polytope of quasi-line graph. *Lecture Notes in Computer Science*, 3509:291–305, 2005.

[5] J. L. Fouquet. A strengthening of Ben Rebea's lemma. *J. Comb. Theory Ser. B*, 59(1):35–40, 1993.

[6] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Theory, Ser. B*, 69(1):1–38, 1997.

[7] R. Giles and L.E. Trotter jr. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Theory B*, 31:313–326, 1981.

[8] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, 1988.

[9] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods Oper. Res.*, 59(1):25–35, 2004.

[10] G. Oriolo. On the stable set polytope for quasi-line graphs. In *Special issue on stability problems*, volume 132, pages 185–201. Discrete Applied Mathematics, 2003.

[11] A. Pêcher and A. Wagler. A conjecture on the stable set polytope for claw-free graphs. Preprint 11-2006, Otto-von-Guericke-Universität Magdeburg, Fakultät für Mathematik.

[12] A. Ben Rebea. *Etude des stables dans les graphes quasi-adjoints*. PhD thesis, Grenoble, 1981.

[13] F.B. Shepherd. Near-perfect matrices. *Mathematical Programming*, 64:295–323, 1994.

[14] G. Stauffer. *On the Stable Set Polytope of Claw-free Graphs*. PhD thesis, EPF Lausanne, 2005.

D.4 Claw-free circular-perfect graphs

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The circular chromatic number of a graph is a well-studied refinement of the chromatic number. Circular-perfect graphs is a superclass of perfect graphs defined by means of this more general coloring concept. This paper studies claw-free circular-perfect graphs. A consequence of the strong perfect graph theorem is that minimal imperfect graphs G have $\min\{\alpha(G), \omega(G)\} = 2$. In contrast to this result, it is shown in [10] that minimal circular-imperfect graphs G can have arbitrarily large independence number and arbitrarily large clique number. In this paper, we prove that claw-free minimal circular-imperfect graphs G have $\min\{\alpha(G), \omega(G)\} \leq 3$.

Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Then a k-*coloring* of G is a mapping $f : V \to \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* of G and denoted by $\chi(G)$. The *clique number* $\omega(G)$ of G is the order of a largest clique of G, i.e., the maximum number of pairwise adjacent vertices of G.

The circular chromatic number and circular clique number of graphs are refinements of the chromatic number and the clique number. Suppose $G = (V, E)$ is a graph, and $k \geq d$ are positive integers. A (k, d)-*circular coloring* of G is a mapping $f : V \to \{0, \ldots, k-1\}$ with $d \leq |f(u) - f(v)| \leq k - d$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k, d)-circular colorings of G. Since every $(k, 1)$-circular coloring is a usual k-coloring of G, we have $\chi_c(G) \leq \chi(G)$. On the other hand, it is known [16] and easy to see that for any graph G, $\chi_c(G) > \chi(G) - 1$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$.

Let $K_{k/d}$ with $k \geq 2d$ or $k = d$ denote the graph with k vertices $0, \ldots, k-1$ and edges ij such that $d \leq |i - j| \leq k - d$. The graphs $K_{k/d}$ are called *circular cliques*. Circular cliques include all cliques $K_t = K_{t/1}$, all chordless odd cycles, termed odd holes $C_{2t+1} = K_{(2t+1)/t}$, and all their complements, termed odd anti-holes, $\overline{C}_{2t+1} = K_{(2t+1)/2}$. The *circular clique number* is defined as $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G, \gcd(k, d) = 1\}$. It follows from the definition that $\omega(G) \leq \omega_c(G)$. It is also known [20] that for any graph G, $\omega_c(G) < \omega(G) + 1$, and hence $\omega(G) = \lfloor \omega_c(G) \rfloor$. Therefore $\omega_c(G)$ is a refinement of $\omega(G)$.

Obviously $\omega(G)$ is a lower bound for $\chi(G)$. A graph G is *perfect* if each induced subgraph $G' \subseteq G$ has $\omega(G') = \chi(G')$. Similarly, $\omega_c(G)$ is a lower bound for $\chi_c(G)$. A graph G is called *circular-perfect* [20] if each induced subgraph $G' \subseteq G$ has $\chi_c(G') = \omega_c(G)$.

Perfect graphs have been studied extensively since the concept and two conjectures (the weak and the strong perfect graph conjectures) were proposed by Berge [1] in 1961. The weak perfect graph conjecture was settled by Lovász [9] in 1972. Recently, the strong perfect graph conjecture has been settled by Chudnovsky, Robertson, Seymour an Thomas in [3], which gives a characterization of perfect graphs by means of forbidden induced subgraphs: A graph G is perfect if and only if G contains neither odd holes, nor odd antiholes.

It follows from the definitions that for any graph G,

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G). \tag{D.13}$$

Therefore every perfect graph is circular-perfect. However, odd cycles and their complements are circular-perfect graphs but not perfect graphs. So the class of circular-perfect graphs is a proper superclass of the class of perfect graphs.

Is there a simple characterization of circular-perfect graphs by means of forbidden induced subgraphs? It is shown in [17] that the line graph $L(G)$ of a cubic graph G is circular-perfect if and only if G is 3-edge colourable. Thus a characterization of claw-free circular-perfect graphs by means of forbidden induced subgraphs implies a characterization of critically non-3-edge colourable cubic graphs, which is known to be a difficult problem. So it is unlikely that there is a simple forbidden induced subgraph characterization of circular-perfect graphs. Some sufficient conditions for a graph to be circular-perfect were obtained in [19, 20]. Classes of minimal circular-imperfect graphs were constructed in [10, 13, 14, 18]. Minimal circular imperfect line graphs were studied in [17]. In this paper, we study claw-free circular-perfect graphs.

A graph G is *claw-free* if $K_{1,3}$ is not an induced subgraph of G. Claw-free graphs is a superclass of line graphs and has been studied extensively in the literature. Recently, Chudnovsky and Seymour [5, 4] presented a structural characterization of claw-free graphs. In this paper, we use the characterization of a quasi-line graph to prove a structural property of minimal circular-imperfect graphs. One consequence of the strong perfect graph theorem is that minimal imperfect graphs G have $\min\{\omega(G), \alpha(G)\} = 2$. It was asked in [13] whether $\min\{\omega(G), \alpha(G)\}$ is bounded for minimal circular-imperfect graphs G. This question was answered in the negative in [10], where it is proved that for any positive integer k, there is a minimal circular imperfect graph G with $\min\{\omega(G), \alpha(G)\} \geq k$. This paper shows that if restricted to claw-free graphs, the question above has a positive answer. We prove that if G is a claw free minimal circular-imperfect graph, then $\min\{\omega(G), \alpha(G)\} \leq 3$.

Main result

This section proves the following theorem, which is the main result of the paper.

Theorem D.4.1. *If G is a minimal circular-imperfect graph, then* $\min\{\omega(G), \alpha(G)\} \leq 3$.

Suppose G is a claw-free graph with independence number at least 3. It was proved by Fouquet [7] that for any vertex x of G, the neighborhood $N_G(x)$ of x either contains an induced C_5, or can be covered with two cliques. If G is circular-perfect, then $N_G(x)$ does not contain an induced C_5, for otherwise G contains the odd wheel W_5, which is circular-imperfect. A graph G for which the neighbourhood of each vertex can be covered by two cliques is called a *quasi-line graph*. Thus we have the following observation.

If G is a claw-free circular-perfect graph with independence number at least 3*, then G is a quasi-line graph.*

We shall need the following two easy lemmas in our proofs.

Lemma D.4.2. *For every claw-free graph G with clique number at least 3,* $\omega_c(G) = \omega(G)$ *or* $\omega(G) + 1/2$.

Proof. A claw-free circular clique, is a clique, an odd hole, or an odd anti-hole. ∎

Lemma D.4.3. *If G is a connected claw-free circular-perfect graph which contains an induced odd antihole H of size at least* 7 *then for every vertex $x \in V(G-H)$ and for every maximum stable set S of H, $N_G(x) \cap S \neq \emptyset$.*

Proof. Let $h_0, h_1 \ldots, h_{2p}$ be the vertices of H (where $p \geq 3$), with the usual ordering ($\{h_i, h_{i+1}\}$ is a stable set for every index i modulo $2p+1$). Let $x \in V(G-H)$. We shall show that for index i, x has at least one neighbour in $\{h_i, h_{i+1}\}$ (summation in indices are modulo $2p+1$). First we assume that x is adjacent to at least one vertex of H. Assume to the contrary that x has no neighbour in $\{h_0, h_1\}$. Then for every index $3 \leq i \leq 2p-1$, xh_i is not an edge, for otherwise $\{x, h_0, h_1, h_i\}$ is a claw. Since x is in the neighborhood of H, this implies that x has at least one neighbour in $\{h_2, h_{2p}\}$. Assume without loss of generality that h_2 is a neighbour of x. Then $\{h_2, x, h_4, h_5\}$ is a claw, which is a contradiction.

To prove Lemma D.4.3, it remains to show that every vertex $x \in V(G-H)$ is adjacent to some vertices of H. Assume to the contrary that G has at least one vertex which is not adjacent to any vertex of H. Since G is connected, there is a edge xy such that y is not adjacent to any vertex of H but x is adjacent to some vertices of H. By the previous paragraph, for any i, x is adjacent to at least one of $\{h_i, h_{i+1}\}$. This implies that there is an index i such that x is adjacent to both h_i and h_{i+1}. Then $\{x, y, h_i, h_{i+1}\}$ induces a claw, in contrary to our assumption. This completes the proof of Lemma D.4.3. $\diamond$ ∎

Theorem D.4.4. *If G is a connected claw-free circular-perfect graph with an induced odd antihole H of size at least* 7 *then $G \setminus H$ is a clique. Furthermore $\alpha(G) = 2$.*

Proof. Let $h_0, \ldots, h_{2p}$ be the vertices of H (where $p \geq 3$), with the usual ordering ($\{h_i, h_{i+1}\}$ is a stable set for every index i modulo $2p+1$).

Assume to contrary that $x, x' \in V(G-H)$ and $x \not\sim x'$. Let $G' = G[H \cup \{x, x'\}]$. If $\omega_c(G') > p+1$ then by Lemma D.4.2, $\omega_c(G') = \frac{2p+3}{2}$ which implies that $G' = K_{(2p+3)/2}$. This is impossible as $K_{(2p+3)/2}$ does not contain $K_{(2p+1)/2}$ as an induced subgraph. Thus $\omega_c(G') \leq p+1$ and hence G' is $(p+1)$-colourable. Let ϕ be a $(p+1)$-colouring of G'. Then each colour class contains two vertices of H, except one color class which contains only one vertex of H. Without loss of generality, we may assume that $\phi(h_0) = 0$, and for $i = 1, 2, \ldots, p$, $\phi(h_{2i-1}) = \phi(h_{2i}) = i$. By Lemma D.4.3, each of x and x' has a neighbour in $\{h_{2i-1}, h_{2i}\}$ for $i = 1, 2, \ldots, p$.

Therefore $\phi(x) = \phi(x') = 0$. If there is an index $2 \leq i \leq 2p-1$ such that $h_i \sim x$ and $h_i \sim x'$, then $\{h_i, h_0, x, x'\}$ induces a claw, in contrary to the assumption that G is claw-free. On the other hand, by Lemma D.4.3, each of x, x' has a neighbour in $\{h_i, h_{i+1}\}$ for any i. Thus we may assume, without loss of generality, that x is adjacent to $h_1, h_2, h_4, h_6, \ldots, h_{2p}$ and x' is adjacent to $h_1, h_3, h_5, \ldots, h_{2p-1}, h_{2p}$. Then the subgraph G'' of G induced by $\{x, h_0, h_1, h_2, h_3, h_{2p-1}, h_{2p}\}$ (as depicted in Figure D.4) has $\omega_c(G'') = 3$ and $\chi(G'') = 4$, in contrary to the assumption that G is circular-perfect. Thus $G \setminus H$ is a clique. By Lemma D.4.3, we have $\alpha(G) = 2$.

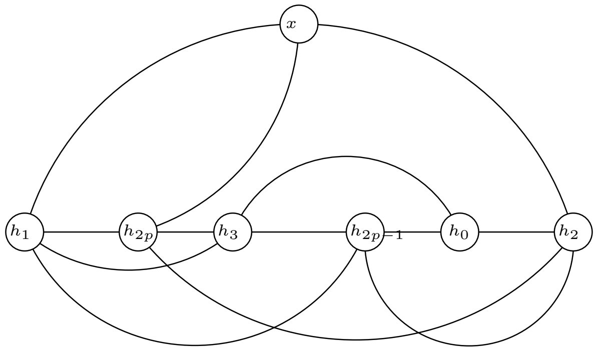

Figure D.5: The circular-imperfect graph G''

◇

■

Since every minimal circular-imperfect graph is 2-connected, we have the following corollary.

Corollary D.4.5. *If G is a claw-free minimal circular-imperfect graph and contains an induced odd antihole H of size at least 7, then $\alpha(G) \leq 3$.*

In the remainder of this section, we study claw-free circular perfect graphs G that do not contain an odd antihole of order at least 7. Due to Theorem D.4.4, G has independence number at least 3, and is therefore, as mentioned above, quasi-line. We shall establish that if G has clique number at least 4 then G has an independent set I that intersects each maximum clique.

We shall prove a stronger statement in Theorem D.4.8: a connected quasi-line graph G such that $\chi(G - v) = \omega(G - v) = \omega(G)$ for each vertex v and with clique number at least 4 is the complement of a circular-clique or has an independent set I that intersects each maximum clique.

As a consequence, a claw-free graph G with $\omega(G) = k \geq 4$ and $\alpha(G) \geq 4$ can not be minimal circular-imperfect. Because otherwise, G is not the complement of a circular-clique [6], and G is quasi-line since it does not contain the odd wheel W_5 (which is already minimal circular-imperfect). Hence there is an independent set I intersecting each maximum clique. Since $G - I$ is circular-perfect, we have $\omega_c(G - I) = \chi_c(G - I)$. Due to Corollary D.4.5, G, and thus $G - I$, does not contain $K_{(2p+1)/2}$ for $p \geq 3$. It follows from Lemma D.4.2 that $\omega_c(G - I) = \omega(G - I) = \omega(G) - 1$ and hence $\chi(G - I) = \omega(G) - 1$. But then $\chi(G) = \omega(G)$, and hence G is circular-perfect.

We first state a basic lemma, which is used in the proof of Theorem D.4.8.

Lemma D.4.6. *Suppose G is a graph with $\omega(G) = k$. If x is a vertex of G such that $G - x$ is k-colorable, and $y \neq x$ is a vertex of G such that every maximum clique of G containing x also contains y, then G has a stable set which intersects every maximum clique of G.*

Proof. Color $G - x$ with k colors, and let I be the color class containing y. If Q is a maximum clique of G not containing x, then Q is a k-clique of $G - x$ and hence intersects each of the k color classes. In particular, $Q \cap I \neq \emptyset$. If Q contains x, then by assumption $y \in Q$ and hence $Q \cap I \neq \emptyset$. ■

It is proved by Chudnovsky and Seymour [4] that every quasi-line graph belongs to one of two classes: the first is the class of the so-called “fuzzy circular interval graphs”, and the second is “compositions of fuzzy linear interval strips”. Below is the precise definition of the two classes of graphs.

Let Σ be a circle and let $F_1, F_2, \ldots, F_k$ be intervals of Σ. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which x, y are adjacent if and only if $x, y \in F_i$ for some i. Such a graph is called a *circular interval graph*. A *linear interval graph* is constructed in the same manner except that Σ is the real line instead of a circle.

Suppose G is a circular interval graph. An edge xy of G is a *maximal edge* of G if for every interval F_i containing both x, y, the two vertices x, y are the two ends of F_i. Let xy be a maximal edge of G. Replace x and y by cliques A and B, respectively, such that every member of A has the same neighbours as x in $V \setminus \{x, y\}$, and every member of B has the same neighbours as y in $V \setminus \{x, y\}$. The edges between A and B are arbitrary. The resulting graph is said to be obtained from G by *replacing the edge xy by a homogeneous pair*. Choose a matching M of maximal edges of G. Let H be obtained from G by replacing each edge of M by a homogeneous pair. Then H is called a *fuzzy circular interval graph*. A homogeneous pair (A, B) of H is called *non-trivial* if $G[A \cup B]$ contains an induced C_4.

A *linear interval strip* is a triple (S, a, b) such that S is an interval graph and a, b are two distinct *simplicial vertices*, i.e., each of $N_S(a)$ and $N_S(b)$ induces a clique. A *fuzzy linear interval strip* is defined in the same manner as above (because each linear interval graph is also a circular interval graph). We emphasize that in a fuzzy linear interval strip (S, a, b), the two vertices a, b must be simplicial vertices. Suppose (S, a, b) and(S', a', b') are fuzzy linear interval strips. Let A, B, A', B' be the sets of neighbours of a, b, a', b', respectively. Take the disjoint union of $S \setminus \{a, b\}$ and $S' \setminus \{a', b'\}$. Join every vertex of A to every vertex of A' by an edge, and join every vertex of B to every vertex of B' by an edge. The resulting graph is a *composition* of (S, a, b) and (S', a', b').

Let S_0 be a graph which is the disjoint union of complete graphs with $V(S_0) = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$. For $i = 1, 2, \ldots, n$, let (S'_i, a'_i, b'_i) be a fuzzy linear interval strip. Let S_i be the composition of (S_{i-1}, a_i, b_i) and (S', a'_i, b'_i). The resulting graph G is called a *composition of the fuzzy linear interval strips* (S'_i, a'_i, b'_i). If (A, B) is a non-trivial homogeneous pair of one of the fuzzy linear interval strips (S'_i, a'_i, b'_i), then we also call (A, B) a non-trivial homogenous pair of G.

Theorem D.4.7. *[5] [2] Let G be a connected, quasi-line graph. Then G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

Chudnovsky and Seymour observed that if G is a fuzzy circular interval graph with no non-trivial homogeneous pair then G is a circular interval graph.

Theorem D.4.8. *If G is a connected quasi-line graph, $\omega(G) = k \geq 4$ and for every vertex x, $G - x$ has a k-colouring, then either G is the complement of a circular clique or G has a stable set which intersects every maximum clique of G.*

Proof. Due to Theorem D.4.7, a connected quasi-line graph without a non-trivial homogeneous pair of cliques is a circular interval graph or a composition of linear interval strips.

First we consider the case that G has a non-trivial homogeneous pair (A, B) of cliques. Let C be the set of vertices of G that are A-complete and B-complete (i.e., $C = \{x : A \cup B \subseteq N_G(x)\}$). It follows from the definition of fuzzy circular interval graphs and fuzzy linear interval strips that C is a clique. Since (A, B) is non-trivial, both sets A and B are of size at least 2.

Let H be the subgraph of G induced by $A \cup B \cup C$. Then $\bar{H}$ is a bipartite graph, and hence H is perfect and has $\chi(H) = \omega(H)$. Assume $\omega(H) < k$. If Q is a maximum clique of G, then $Q \nsubseteq (A \cup B \cup C)$. Note that if $v \notin A \cup B \cup C$ and v is adjacent to some vertex of A, then v is not adjacent to any vertex of B. So for any $x, y \in A$, any maximum clique Q of G containing x does not intersect B, and hence also contains y. By Lemma D.4.6, G has a stable set which intersects every maximum clique of G.

Assume $\omega(H) = k$. If there is a vertex $y \in A$ such that for any $x \in A$, every maximum clique of G containing x contains y, then by Lemma D.4.6, we are done again. Thus we assume that for any $y \in A$, $\omega(H - y) = k$. In particular, $H - y$ is not $(k-1)$-colorable. By symmetry, we may assume $H - y$ is not $(k-1)$-colorable for all

$y \in B$. Let f be a k-coloring of H. Then each color class contains one vertex of $A \cup C$ and one vertex of $B \cup C$. Thus both $A \cup C$ and $B \cup C$ are maximum cliques and there is a one-to-one correspondence $\phi : A \to B$ such that for each $y \in A$, $f(y) = f(\phi(y))$.

Assume there exist $y_1, y_2 \in A$ such that y_1 is not adjacent to $\phi(y_2)$. We claim that any maximum clique of G containing y_1 also contains y_2. Indeed, if Q is a maximal clique of G that contains y_1 but does not contain y_2, then Q intersects B and hence is a clique of H. Since Q contains y_1 and y_1 is not adjacent to $\phi(y_2)$, we have $Q \cap \{y_2, \phi(y_2)\} = \emptyset$. But $H \setminus \{y_2, \phi(y_2)\}$ is $(k-1)$-colorable, and hence Q is not a maximum clique of G. This proves the claim and hence the conclusion of the theorem follows from Lemma D.4.6.

Assume for each $y \in A$, y is adjacent to every vertex of B except $\phi(y)$. Let $x \in A$, and let g be a k-coloring of $G - x$. Then for any $y \in A \setminus \{x\}$, $g(y) = g(\phi(y))$, because both $B \cup C$ and $y \cup (B \setminus \{\phi(y)\}) \cup C$ are k-cliques. Let I be a color class of g which contains y for some $y \in A \setminus \{x\}$. We shall show that I intersect every maximum clique of G. Assume to the contrary that Q is a k-clique of G for which $I \cap Q = \emptyset$. Then $x \in Q$, and $y \notin Q$. This implies that Q intersects B and hence Q is a clique of H. But $\phi(y) \notin Q$ (because $\phi(y) \in I$) and $H \setminus \{y, \phi(y)\}$ is $(k-1)$-colorable. This is a contradiction.

We now assume that G has no non-trivial homogeneous pair (A, B) of cliques. Thus G is either a circular interval graph or a composition of linear interval strips. First we consider the case that G is a circular interval graph. Assume G is not the complement of a circular clique. Then there exists a vertex x of degree less than $2\omega(G) - 2$. Let Q_1 and Q_2 be the two cliques covering $N_G(x)$, where Q_1 consists of neighbours of x on the "left" side", and Q_2 consists of the neighbours of x on the "right side" (recall that the vertices of G lie on a circle). Assume $Q_1 \cup \{x\}$ is not a maximum clique. Let y be the first neighbour of x on the right side of x. Then it is easy to see that every maximum clique containing x also contains y. The conclusion follows from Lemma D.4.6.

Next we consider the case that G is the composition of linear interval strips. Assume that G is the composition of linear interval strips (S'_i, a'_i, b'_i) $(1 \leq i \leq n)$.

The vertices of S'_i lie on an interval, and a'_i, b'_i are the first and last vertex on the interval, respectively. Assume firstly that a'_n has more than one neighbour in S'_n. Let x, y be the second and third vertices on the interval. Then $N(x) \subseteq N(y) \cup \{y\}$ due to the definition of composition of strips.

Let Q be a maximum clique containing x. If y does not belong to Q then we have $Q \subseteq N(x) \subseteq N(y)$, and so $\{y\} \cup Q$ is a clique, contradicting that Q is a maximum clique. Therefore, every maximum clique containing x also contains y, and we are done by Lemma D.4.6.

Assume a'_n has a unique neighbour x in S'_n. Let A_n, B_n be the sets of neighbours of a_n, b_n in S_{n-1}, and let B'_n be the set of neighbours of b'_n in S'_n. By the definition of composition of strips, x belongs to two cliques in G: if $x \notin B'_n$ then x is contained in cliques $Q_1 = A_n \cup \{x\}$ and $Q_2 = (N_{S'_n}(x) \cup \{x\}) \setminus \{a'_n\}$, if $x \in B'_n$, then x is contained in cliques $Q_1 = A_n \cup \{x\}$ and $Q_2 = B_n \cup B'_n$. In the later case, any clique of G containing x is contained in Q_1 or Q_2. Hence there are at most 2 maximum cliques of G which contains x. In the former case, any clique of G containing x is contained in Q_1 or Q_2 or $Q_3 = (N_{S'_n}(x) \cap B'_n) \cup (B_n \cap A_n)$. Thus x is contained in at most 3 maximum cliques.

Let ϕ be a k-colouring of $G - x$. If Q is a k-clique of G containing x, then exactly one color class of ϕ does not intersect Q. Since x belongs to at most three maximum cliques Q_1, Q_2, Q_3 of G and there are $k \geq 4$ color classes, there is a color class I of ϕ that intersects each of Q_1, Q_2, Q_3. Then I intersects every maximum clique of G. ■

From Theorem D.4.8, we derive, as mentioned above, our main result:

Corollary D.4.9. *If G is a claw-free minimal circular-imperfect graph then*

$$\min\{\alpha(G), \omega(G)\} \leq 3$$

Before the Strong Perfect Graph Conjecture becomes a theorem, the conjecture was confirmed for claw-free graphs in [8][11]. Our result above implies an alternative proof of the result that claw-free minimal imperfect graphs are odd hole or odd antihole, of course without making use of the Strong Perfect Graph Theorem [3].

Corollary D.4.10. *[8][11] If G is claw-free minimal imperfect then G is an odd hole or odd antihole.*

Proof. Let G be a claw-free minimal imperfect graph.

If G is circular-perfect then G is partitionable circular-perfect, and is therefore a circular-clique [12]. Hence G is a claw-free circular-clique. Thus G is an odd hole or odd antihole.

If G is circular-imperfect then G is minimal circular-imperfect. Due to Co. D.4.9, we have $\min\{\alpha(G), \omega(G)\} \leq 3$. Hence G is an odd hole or odd antihole [15]. ■

References

[1] C. Berge. Farbung von graphen deren sietliche bzw. deren ungerade kreise starr sind. *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg*, pages 114–115, 1961.

[2] M. Chudnovsky and A. Ovetsky. Coloring quasi-line graphs. *J. Graph Theory*, 54(1):41–50, 2007.

[3] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[4] M. Chudnovsky and P. Seymour. Claw-free graphs vi. the structure of quasi-line graphs. *manuscript*, 2004.

[5] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. In B.S. Webb, editor, *Surveys in combinatorics*, volume 327 of *London. Math. Soc. Lecture Notes*, pages 153–171, 2005.

[6] S. Coulonges, A. Pêcher, and A. Wagler. On strongly circular-perfectness. In *VIIth International Conference on Graph Theory*, volume 22 of *Electonic Notes in Discrete Mathematics*, pages 369–374. Elsevier, oct 2005.

[7] J. L. Fouquet. A strengthening of Ben Rebea's lemma. *J. Comb. Theory Ser. B*, 59(1):35–40, 1993.

[8] R. Giles, L.E. Trotter jr, and A. Tucker. The strong perfect graph theorem for a class of partitionable graphs. *Annals of Discrete Mathematics*, 21:161–167, 1984.

[9] L. Lovász. A characterization of perfect graphs. *J. of Combinatorial Theory B*, 13:95–98, 1972.

[10] Z. Pan and X. Zhu. Minimal circular-imperfect graphs of large clique number and large independence number. manuscript, 2006.

[11] K.R. Parthasarathy and G. Ravindra. The strong perfect graph conjecture is true for $k_{1,3}$-free graphs. *J. Combin. Theory Ser. B*, 21:212–223, 1976.

[12] A. Pêcher and A. Wagler. On classes of minimal circular-imperfect graphs. *Discrete Applied Mathematics*, 156:998–1010, 2008.

[13] A. Pêcher, A. Wagler, and X. Zhu. Three classes of minimal circular imperfect graphs. In *Proceedings of GRACO05*, volume 19 of *Electonic Notes in Discrete Mathematics*, pages 9–15. Elsevier, avril 2005.

[14] A. Pêcher and X. Zhu. On the circular chromatic number of circular partitionable graphs. *Journal of Graph Theory*, 52(4):294–306, aug 2006.

[15] Alan Tucker. Critical perfect graphs and perfect 3-chromatic graphs. *J. Combinatorial Theory Ser. B*, 23(1):143–149, 1977.

[16] A. Vince. Star chromatic number. *J. of Graph Theory*, 12:551–559, 1988.

[17] B. Xu. On minimally circular-imperfect graphs. manuscript.

[18] B. Xu. Some results on circular perfect graphs and perfect graphs. *J. Syst. Sci. Complex.*, 18:167–173, 2005.

[19] X. Zhu. Perfect graphs for generalized colouring—circular perfect graphs. In *Graphs, morphisms and statistical physics*, volume 63 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 177–193. Amer. Math. Soc., Providence, RI, 2004.

[20] X. Zhu. Circular perfect graphs. *J. of Graph Theory*, 48(1–3):186–209, 2005.