



Interpolation dans les algèbres de Hörmander

Myriam Ounaïes

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Interpolation in Hörmander Algebras

by

Myriam Ounaïes

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Introduction

We are dealing with interpolation problems with growth estimates on entire functions and some of their applications to harmonic analysis.

Let $\mathcal{A}(\mathbb{C})$ be the space of all entire functions, p a subharmonic positive function and $\mathcal{A}_p(\mathbb{C})$ the space of functions $f \in \mathcal{A}(\mathbb{C})$ for which there exist constants $A \geq 0$, $B \geq 0$ such that one has the estimate

$$|f(z)| \leq Ae^{Bp(z)} \quad \text{for all } z \in \mathbb{C}.$$

These spaces are clearly algebras under the ordinary product of functions, they are known as Hörmander algebras (see [2]).

Let $\{\alpha_j\}$ be a discrete sequence of complex numbers, $\{m_j\}$ a sequence of positive integers and $\{w_{j,l}\}_{j,0 \leq l < m_j}$ a doubly indexed sequence of complex numbers for which there exist constants $A' \geq 0$, $B' \geq 0$ such that

$$\sum_{l=0}^{m_j-1} |w_{j,l}| \leq A'e^{B'p(\alpha_j)}, \quad \text{for all } j.$$

The interpolation problems we are dealing with may be summarized as follows. Under what conditions does there exist a function $f \in \mathcal{A}_p(\mathbb{C})$ such that

$$\frac{f^{(l)}(\alpha_j)}{l!} = w_{j,l}, \quad \text{for all } j \text{ and for all } 0 \leq l < m_j ?$$

This problem is closely related to the fundamental principle in the solution space of homogeneous convolution equations, that is, with the expression of solutions of such an equation (called mean-periodic functions) through its elementary solutions, exponentials monomials.

The question of interpolation with growth estimate was studied by A.F. Leont'ev ([29, 30]) in the space $\text{Exp}(\mathbb{C})$ of all entire functions of exponential type, which corresponds to the case $p(z) = |z|$.

Another important example is $p(z) = |\text{Im } z| + \ln(1 + |z|^2)$. The resulting space $\mathcal{A}_p(\mathbb{C})$ is then the space $\hat{\mathcal{E}}'(\mathbb{R})$ of Fourier transforms of distributions with compact support on the real line. Interpolation problems in these spaces were studied by W.A. Squires in [44, 45].

We impose on p two conditions :

- (a) $\ln(1 + |z|^2) = O(p(z))$.
- (b) there exist constants $C_1, C_2 > 0$ such that $|z - w| \leq 1$ implies $p(w) \leq C_1 p(z) + C_2$.

The interest of these two conditions lies in their consequences. Condition (a) implies that $\mathcal{A}_p(\mathbb{C})$ contains all polynomials while condition (b) implies that $\mathcal{A}_p(\mathbb{C})$ is stable under differentiation.

Actually, thanks to condition (b), if $f \in \mathcal{A}_p(\mathbb{C})$ then the derivatives satisfy the stronger property

$$\sum_{l=0}^{\infty} \left| \frac{f^{(l)}(z)}{l!} \right| \leq A e^{Bp(z)}$$

for all $z \in \mathbb{C}$, where A and B are some positive constants.

Let $\mathcal{W}_p(\mathbb{C})$ be the space of measurable functions g satisfying, for some constant $C > 0$,

$$\int_{\mathbb{C}} |g(z)|^2 e^{-Cp(z)} d\lambda(z) < \infty,$$

where $d\lambda$ denotes the Lebesgue measure. As a consequence of condition (b), we have the equality

$$\mathcal{W}_p(\mathbb{C}) \cap \mathcal{A}(\mathbb{C}) = \mathcal{A}_p(\mathbb{C}).$$

This equality shows how the L^2 -estimates are related to the growth condition we are dealing with. We point out the paper [7] where C.A. Berenstein and B.A. Taylor showed that Hörmander's theorem about the existence of solutions of the inhomogeneous Cauchy-Riemann equation with L^2 -estimates was a powerful tool that could be used to simplify the study of interpolation problems in Hörmander's algebras. We will often use these techniques in our work.

The weights $p(z) = |z|$ and $p(z) = |\operatorname{Im} z| + \ln(1 + |z|^2)$ obviously satisfy conditions (a) and (b). Another important example of weight is $p(z) = |z|^\rho$, $\rho > 0$. In this case $\mathcal{A}_p(\mathbb{C})$ is the space of functions of order $\leq \rho$ and of finite type.

We may consider more rapidly growing weights, such as $p(z) = e^{|z|^\rho}$ with $0 < \rho \leq 1$. Note that because of condition (b), we must have $\ln p(z) = O(|z|)$.

Now consider a sequence $\{\alpha_j\}_{j \in \mathbb{N}}$ of distinct complex numbers with $|\alpha_j| \rightarrow \infty$ and attribute to each point α_j a multiplicity $m_j \in \mathbb{N}^*$. The collection of pairs $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ is called a multiplicity variety.

Let $\mathcal{A}(V)$ be the space of doubly indexed sequences $W = \{w_{j,l}\}_{j \in \mathbb{N}, 0 \leq l < m_j}$ of complex numbers. It may be seen as the space of holomorphic functions on V . There is a natural restriction map

$$\mathcal{R}_V : \mathcal{A}(\mathbb{C}) \rightarrow \mathcal{A}(V)$$

defined by

$$\mathcal{R}_V(f) = \left\{ \frac{f^{(l)}(\alpha_j)}{l!} \right\}_{j \in \mathbb{N}, 0 \leq l < m_j}.$$

As we mentioned before, for all $f \in \mathcal{A}_p(\mathbb{C})$, we have

$$\sum_{l=0}^{\infty} \left| \frac{f^{(l)}(\alpha_j)}{l!} \right| \leq A e^{Bp(\alpha_j)}.$$

In particular, if we denote by $\mathcal{A}_p(V)$ the space of all $W = \{w_{j,l}\}_{j, 0 \leq l < m_j} \in \mathcal{A}(V)$ for which there exist constants $A > 0$ and $B > 0$ such that one has the estimate

$$\sum_{l=0}^{m_j-1} |w_{j,l}| \leq A e^{Bp(\alpha_j)}, \quad \text{for all } j,$$

then

$$\mathcal{R}_V(\mathcal{A}_p(\mathbb{C})) \subset \mathcal{A}_p(V).$$

We will say that $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$ (or to simplify, that V is \mathcal{A}_p -interpolating) if for all $W = \{w_{j,l}\}_{j, 0 \leq l < m_j} \in \mathcal{A}_p(V)$, there exist $f \in \mathcal{A}_p(\mathbb{C})$ verifying $\frac{f^{(l)}(\alpha_j)}{l!} = w_{j,l}$, for all j and for all $0 \leq l < m_j$. In other words, V is \mathcal{A}_p -interpolating if and only if

$$\mathcal{R}_V(\mathcal{A}_p(\mathbb{C})) = \mathcal{A}_p(V).$$

We are mainly concerned with the two following questions :

Question 1 : Under what conditions on V is it true that it is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$?

Question 2 : What is the image of $\mathcal{A}_p(\mathbb{C})$ under the restriction map \mathcal{R}_V ?

We begin by giving answers to both questions in Chapter 2 in the particular case where V is a finite union of interpolating varieties.

Let us look more closely at Question 1. We have an analytic characterization of interpolating varieties given by C.A. Berenstein and B.Q. Li. They showed that a multiplicity variety $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ is \mathcal{A}_p -interpolating if and only if there exists $f \in \mathcal{A}_p(\mathbb{C})$ having every α_j as a zero of order m_j and verifying

$$\frac{|f^{(m_j)}(\alpha_j)|}{m_j!} \geq \varepsilon e^{-Cp(\alpha_j)}, \quad \text{for all } j,$$

with constants $\varepsilon, C > 0$ independent of j (see [3]).

We are specially interested in the problem of finding purely geometric conditions which depend only on the distribution of the points and that would enable us to decide whether a multiplicity variety is interpolating or not by direct computation.

In this direction, Berenstein and Li, in the same paper, showed that whenever p is radial (i.e. $p(z) = p(|z|)$) and doubling (i.e. $p(2z) \leq 2p(z)$) then $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ is \mathcal{A}_p -interpolating if and only if

$$\sum_{0 < |\alpha_k - \alpha_j| \leq |\alpha_j|} m_k \ln \frac{|\alpha_j|}{|\alpha_k - \alpha_j|} + m_j \ln |\alpha_j| = O(p(\alpha_j))$$

and

$$\sum_{0 < |\alpha_k| \leq r} m_k \ln \frac{r}{|\alpha_k|} = O(p(r)).$$

In the present work, we use Hörmander's L^2 -estimates to the solution of the $\bar{\partial}$ equation, as in the works by Berndtsson-Ortega-Cerdà [9] and Hartmann-Massaneda [21] to find a geometric answer to Question 1.

In Chapter 3, we apply this method to give geometric characterizations of \mathcal{A}_p -interpolating varieties for different classes of radial weights. In particular, we give a new proof in the case where the weight is radial and doubling and we solve the problem for $p(z) = e^{|z|}$ and more generally when $\ln p(e^r)$ is convex and $\ln p(r)$ is concave.

In Chapter 4, we obtain a geometric description of interpolating varieties for the weight $p(z) = |\operatorname{Im} z| + \ln(1 + |z|^2)$ and for more general Beurling weights.

Chapter 5 is dedicated to Question 2 in the case where p is radial and doubling. We use Hörmander's theorem with L^2 -estimates to give an explicit description of $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$. Given a multiplicity variety V (which is not a uniqueness set for $\mathcal{A}_p(\mathbb{C})$) and a doubly indexed sequence $W \in \mathcal{A}_p(V)$, the necessary and sufficient condition that $W \in \mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$ is obtained in terms of the growth of the divided differences with respect to V and to W .

As we said before, these interpolation problems are generally studied because of their application to harmonic analysis. We explore this aspect in Chapter 6 : Let θ be a Young function (for example $\theta(x) = x^\mu$, $\mu > 1$) and consider the space $\mathcal{F}_\theta(\mathbb{C})$ of all entire functions on \mathbb{C} with infra- θ -exponential growth. We are interested in the solutions $f \in \mathcal{F}_\theta(\mathbb{C})$ of the convolution equation $T \star f = 0$, called T -mean-periodic functions, where T is in the topological dual of $\mathcal{F}_\theta(\mathbb{C})$. We use the explicit description of the restriction map found in Chapter 5 to get an explicit representation formula for T -mean-periodic functions as a convergent series of exponential-polynomials solutions of the form $z^l e^{\alpha_j z}$, $0 \leq l < m_j$ where the α_j are the zeros of the Fourier-Borel transform of T and m_j are their order of multiplicity. These series converge after an Abel-summation process. When $V = \{(\alpha_j, m_j)\}_j$ is an interpolating variety, the convergence no longer requires an Abel-summation.

Finally, in Chapter 7, we give some results about Question 1 in the multivariate case, when V is a discrete sequence of \mathbb{C}^n . Berenstein and Li described the interpolating discrete sequences for $\mathcal{A}_p(\mathbb{C}^n)$ as zeros of an entire map $F = (f_1, \dots, f_n)$, $f_j \in \mathcal{A}_p(\mathbb{C}^n)$, where the jacobian determinant of F verifies a certain lower bound. Therefore, when looking for a control on the density of a discrete interpolating variety, we are led to the transcendental Bézout problem, that is, the problem of the existence of an upper bound on the zero set of an entire map from \mathbb{C}^n to \mathbb{C}^n . This problem was studied by B.Q. Li and B.A.Taylor (see [31]). We give a new look on the proof of the existence of such an upper bound when we count the zeros where the jacobian determinant of F is bounded below and of its corollary : a necessary condition on interpolating discrete varieties. On the other hand, we give a sufficient geometric condition when p is a radial and doubling weight growing more rapidly than $|z|^2$. In the case where $p(z) = |z|^2$, this condition is that the sequence is uniformly separated.

We conclude this introduction by pointing out that Chapters from 2 to 7 may be read independently while all preliminary definitions and useful known results about interpolation problems in $\mathcal{A}_p(\mathbb{C})$ are collected in Chapter 1.

CHAPTER 1

Preliminaries

1.1. Definitions and notations

DEFINITION 1.1.1. A subharmonic function $p : \mathbb{C} \longrightarrow \mathbb{R}_+$, is called a weight if it satisfies the two following conditions.

(w1) $\ln(1 + |z|^2) = O(p(z))$.

(w2) There exist constants $C_1, C_2 > 0$ such that $|z - w| \leq 1$ implies $p(w) \leq C_1 p(z) + C_2$.

Note that condition (w2) implies that $p(z) = O(e^{A|z|})$ for some $A > 0$.

DEFINITION 1.1.2. We say that the weight p is radial if $p(z) = p(|z|)$ and that is doubling if there exists $C > 0$ such that $p(2z) \leq C p(z)$ for all $z \in \mathbb{C}$.

REMARK 1.1.3. It is easy to see that property (w2) is satisfied whenever p is radial and doubling.

Let $\mathcal{A}(\mathbb{C})$ be the space of all entire functions, we consider the space

$$\mathcal{A}_p(\mathbb{C}) = \left\{ f \in \mathcal{A}(\mathbb{C}) : \forall z \in \mathbb{C}, |f(z)| \leq A e^{Bp(z)} \text{ for some } A > 0, B > 0 \right\}.$$

Note that $\mathcal{A}_p(\mathbb{C})$ can be seen as the union of the Banach spaces

$$\mathcal{A}_{p,B}(\mathbb{C}) = \{ f \in \mathcal{A}(\mathbb{C}), \|f\|_B := \sup_{z \in \mathbb{C}} |f(z)| e^{-Bp(z)} < \infty \}$$

and has a structure of an (LF)-space when endowed with the inductive limit topology. Let $\mathcal{W}_p(\mathbb{C})$ be the space of measurable functions g such that for some $C > 0$,

$$\int_{\mathbb{C}} |g(z)|^2 e^{-Cp(z)} d\lambda(z) < \infty,$$

where $d\lambda$ denotes the Lebesgue measure. The usefulness of conditions (w1) and (w2) lie in the following properties.

LEMMA 1.1.4.

- (i) $\mathcal{A}_p(\mathbb{C})$ contains all polynomials.
- (ii) For any $f \in \mathcal{A}_p(\mathbb{C})$, there are constants $A > 0$ and $B > 0$, such that

$$\sum_{l=0}^{\infty} \left| \frac{f^{(l)}(z)}{l!} \right| \leq A e^{Bp(z)} \text{ for all } z \in \mathbb{C}..$$

In particular, $\mathcal{A}_p(\mathbb{C})$ is stable under differentiation.

- (iii) $\mathcal{W}_p(\mathbb{C}) \cap \mathcal{A}(\mathbb{C}) = \mathcal{A}_p(\mathbb{C})$.

PROOF.

(i) This is an obvious consequence of condition (w1).

(ii) By the Cauchy inequality we have, for all $z \in \mathbb{C}$ and all $l \in \mathbb{N}$,

$$\left| \frac{f^{(l)}(z)}{l!} \right| \leq \frac{1}{2^l} \max_{|\zeta-z| \leq 2} |f(\zeta)|.$$

Condition (w2) implies the existence of constants $A > 0$ and $B > 0$ such that

$$\max_{|\zeta-z| \leq 2} |f(\zeta)| \leq A e^{Bp(z)}.$$

Property (ii) follows then immediately.

(iii) Let $f \in \mathcal{A}_p(\mathbb{C})$. Thanks to condition (w1) there exist constants $A > 0$ and $C > 0$ such that for all $z \in \mathbb{C}$,

$$|f(z)|^2 e^{-Cp(z)} \leq \frac{A}{(1+|z|^2)^2}.$$

We readily deduce that $f \in \mathcal{W}_p(\mathbb{C})$. Conversely, let $f \in \mathcal{W}_p(\mathbb{C}) \cap \mathcal{A}(\mathbb{C})$ and let $C > 0$ be such that

$$\int_{D(z,1)} |f(\zeta)|^2 e^{-Cp(\zeta)} d\lambda(\zeta) < \infty.$$

Using the mean-value inequality, then Cauchy-Schwarz inequality and finally condition (w2), we have

$$\begin{aligned} |f(z)| &\leq \frac{1}{\pi} \int_{D(z,1)} |f(\zeta)| d\lambda(\zeta) \\ &\leq \frac{1}{\pi} \left(\int_{D(z,1)} |f(\zeta)|^2 e^{-Cp(\zeta)} d\lambda(\zeta) \right)^{1/2} \left(\int_{D(z,1)} e^{Cp(\zeta)} d\lambda(\zeta) \right)^{1/2} \\ &\leq A e^{Bp(z)}, \end{aligned}$$

where A and B are positive constants. ■

Here are some examples of weights.

EXAMPLES 1.1.5.

- $p(z) = \ln(1 + |z|^2) + |\operatorname{Im} z|$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of Fourier transforms of distributions with compact support on the real line.
- $p(z) = \ln(1 + |z|^2)$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of all the polynomials.
- $p(z) = |z|$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of entire functions of exponential type.
- $p(z) = |z|^\alpha$, $\alpha > 0$. Then $\mathcal{A}_p(\mathbb{C})$ is the space of all entire functions of order $\leq \alpha$ and finite type.
- $p(z) = e^{|z|^\alpha}$, $0 < \alpha \leq 1$.

DEFINITION 1.1.6. We call a multiplicity variety V a collection of pairs (α_j, m_j) , where α_j are distinct points of \mathbb{C} and $m_j \in \mathbb{N}^*$ are the multiplicities at the points α_j and either the collection is finite or $|\alpha_j| \rightarrow \infty$. We say that $\zeta \in V$ to indicate that ζ is one of the points α_j .

For a non-zero function f , we denote by $Z(f)$ the collection of zeros of f with their respective multiplicities.

Let $V_1 = \{(\alpha_j, m_j)\}_{j \geq 0}$ and $V_2 = \{(\beta_k, n_k)\}_{k \geq 0}$ be two multiplicity varieties. We say that $V_1 \subset V_2$ if $\{\alpha_j\}_{j \geq 0}$ is a subsequence of $\{\beta_k\}_{k \geq 0}$ and, for the corresponding indices, we have $m_j \leq n_k$.

From now on, $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ will denote a multiplicity variety such that $|\alpha_j| \rightarrow \infty$. We consider the spaces

$$\mathcal{A}(V) = \{W = \{w_{j,l}\}_{j \in \mathbb{N}, 0 \leq l < m_j} \subset \mathbb{C}\},$$

$$\mathcal{A}_p(V) = \left\{ W \in \mathcal{A}(V), \quad \forall j \in \mathbb{N}, \quad \sum_{l=0}^{m_j-1} |w_{j,l}| \leq A e^{Bp(\alpha_j)} \text{ for some } A > 0, B > 0 \right\}.$$

Note that $\mathcal{A}_p(V)$ can also be seen as the union of the Banach spaces

$$\mathcal{A}_{p,B}(V) = \{W \in \mathcal{A}(V), \quad \|W\|_B := \sup_{j \geq 0} \sum_{l=0}^{m_j-1} |w_{j,l}| e^{-Bp(\alpha_j)} < \infty\}$$

and has a structure of an (LF)-space with the inductive limit topology. Define the restriction map \mathcal{R}_V by

$$\mathcal{R}_V : \mathcal{A}(\mathbb{C}) \longrightarrow \mathcal{A}(V)$$

$$f \longmapsto \left\{ \frac{f^{(l)}(\alpha_j)}{l!} \right\}_{j \in \mathbb{N}, 0 \leq l \leq m_j-1}.$$

In view of Lemma 1.1.4 (ii), it is clear that $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C})) \subset \mathcal{A}_p(V)$.

We now define the counting function and the integrated counting function that will be used to describe the geometric conditions.

DEFINITION 1.1.7. Let $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ be a multiplicity variety. For $z \in \mathbb{C}$ and $r > 0$ we set

$$n_V(z, r) = \sum_{|z - \alpha_j| \leq r} m_j$$

and

$$N_V(z, r) = \int_0^r \frac{n(z, t) - n(z, 0)}{t} dt + n(z, 0) \ln r$$

$$= \sum_{0 < |z - \alpha_j| \leq r} m_j \ln \frac{r}{|z - \alpha_j|} + n(z, 0) \ln r.$$

When there is no ambiguity, we will simply write $n(z, r) = n_V(z, r)$ and $N(z, r) = N_V(z, r)$. Throughout the manuscript, A, B and C will denote positive constants and their actual value may change from one occurrence to the next. $F(t) \lesssim G(t)$ means that there exist constants $A, B > 0$, not depending on t such that $F(t) \leq AG(t) + B$ while $F \simeq G$ means that $F \lesssim G \lesssim F$. The notation $D(z, r)$ will be used for the Euclidean disk of center z and radius r . We will write $\partial f = \frac{\partial f}{\partial z}, \bar{\partial} f = \frac{\partial f}{\partial \bar{z}}$. Then $\Delta f = 4\partial\bar{\partial}f$ denotes the laplacian of f .

1.2. Interpolating varieties

DEFINITION 1.2.1. We say that V is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$ (or that it is \mathcal{A}_p -interpolating) if for every doubly indexed sequence $\{w_{j,l}\}_{j \in \mathbb{N}, 0 \leq l < m_j}$ of complex numbers such that, for some positive constants A and B and for all $j \in \mathbb{N}$,

$$\sum_{l=0}^{m_j-1} |w_{j,l}| \leq A e^{Bp(\alpha_j)},$$

we can find an entire function $f \in \mathcal{A}_p(\mathbb{C})$, with

$$\frac{f^{(l)}(\alpha_j)}{l!} = w_{j,l}$$

for all $j \in \mathbb{N}$ and $0 \leq l < m_j$.

We may equivalently define the \mathcal{A}_p -interpolating varieties by the property that \mathcal{R}_V maps $\mathcal{A}_p(\mathbb{C})$ onto $\mathcal{A}_p(V)$.

We are collecting next several known properties about interpolating varieties, most of them are necessary conditions. We refer the reader to [2, 3, 7, 44, 45] for further details.

LEMMA 1.2.2. *Let f be a non zero function in $\mathcal{A}_p(\mathbb{C})$ and set $V = Z(f)$. Then for certain constants $A, B > 0$ we have*

$$N_V(0, R) \leq A + B \max_{|z|=R} p(z) \quad \text{for all } R > 0.$$

PROOF. We set $m = n_V(0, 0)$ and we apply Jensen's formula to the function $\frac{f(z)}{z^m}$ in the disk $D(0, R)$.

$$N_V(0, R) = \int_0^{2\pi} \ln |f(Re^{i\theta})| d\theta - \ln \left| \frac{f^{(m)}(0)}{m!} \right|.$$

As for all $\theta \in [0, 2\pi]$, we have $|f(Re^{i\theta})| \leq A e^{Bp(Re^{i\theta})}$, we readily deduce the desired estimate,

$$N_V(0, R) \leq A + B \max_{|z|=R} p(z).$$

■

LEMMA 1.2.3. *Let $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}}$ be an interpolating variety. Then there exists a non zero function $f \in \mathcal{A}_p(\mathbb{C})$ such that $V \subset Z(f)$.*

PROOF. As V is an interpolating variety, there exists $g \in \mathcal{A}_p(\mathbb{C})$ such that

- $g^{(l)}(\alpha_0) = 0$ for $0 \leq l < m_0$, except $g^{(m_0-1)}(\alpha_0) = 1$,
- $g^{(l)}(\alpha_j) = 0$ for all $j \neq 0$ and all $0 \leq l < m_j$.

Set $f(z) = (z - \alpha_0)g(z)$. Then $V \subset Z(f)$ and by property (w1) of the weight, it is clear that $f \in \mathcal{A}_p(\mathbb{C})$. ■

From Lemmas 1.2.2 and 1.2.3, we immediately deduce the following corollary.

COROLLARY 1.2.4. *Assume that V is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$. Then there exist constants $A, B > 0$ such that*

$$N_V(0, R) \leq A + B \max_{|z|=R} p(z), \quad \text{for all } R > 0.$$

REMARK 1.2.5. Whenever the weight p is radial, the latter necessary condition may be rewritten as follows. For some constants $A, B > 0$,

$$(1) \quad N_V(0, R) \leq A + B p(R), \quad \text{for all } R > 0.$$

A standard feature of the spaces $\mathcal{A}_p(\mathbb{C})$ is that the interpolation can be performed in a stable way. It is stated precisely in the following lemma.

LEMMA 1.2.6. *V is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$ if and only if, for all $B > 0$, there exist $A > 0$ and $B' > 0$ such that for all $W \in \mathcal{A}_{p,B}(V)$, there exist $f \in \mathcal{A}_{p,B'}(\mathbb{C})$ with $\mathcal{R}_V(f) = W$ and $\|f\|_{B'} \leq A\|W\|_B$.*

See for example [2, Lemma 2.2.6.] for a proof based on the Baire Category Theorem.

THEOREM 1.2.7. *Assume that $V = \{(\alpha_j, m_j)\}_j$ is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$. Let R_j be positive numbers satisfying*

$$(2) \quad |z - \alpha_j| \leq R_j \implies p(z) \leq C_1 + C_2 p(\alpha_j),$$

where C_1 and C_2 are positive constants not depending on j . Then the following condition holds. For some constants $A, B > 0$,

$$(3) \quad N_V(\alpha_j, R_j) \leq A + B p(\alpha_j), \quad \text{for all } j.$$

REMARK 1.2.8. In view of property (w2) of the weight p , we can always assume $R_j \geq e$ for all j . Thus, whenever V is an interpolating variety for $\mathcal{A}_p(\mathbb{C})$, the following condition is necessary. For some constants $A, B > 0$,

$$(4) \quad N_V(\alpha_j, e) \leq A + B p(\alpha_j), \quad \text{for all } j.$$

In particular, the multiplicities necessarily verify

$$(5) \quad m_j \leq A + B p(\alpha_j), \quad \text{for all } j.$$

In the case where p is doubling and radial, we are allowed the larger numbers $R_j = |\alpha_j|$. In that case, the following condition is necessary : for some constants $A, B > 0$,

$$(6) \quad N(\alpha_j, |\alpha_j|) \leq A + B p(\alpha_j), \quad \text{for all } j \in \mathbb{N}.$$

PROOF. Proof of Theorem 1.2.7

Lemma 1.2.6 gives a sequence $\{f_j\}_j$ such that

- (i) $f_j^{(l)}(\alpha_k) = 0$, for all k and for all $0 \leq l < m_k$, except $\frac{f_j^{(m_j-1)}(\alpha_j)}{(m_j-1)!} = 1$.
- (ii) for certain constants $A, B > 0$ not depending on j , $|f_j(z)| \leq A e^{Bp(z)}$.

Set $g_j(z) = (z - \alpha_j)f_j(z)$. In view of property (w2) there exist constants $A, B > 0$ (not depending on j) such that for all $z \in \mathbb{C}$, we have

$$|g_j(z)| \leq Ae^{B[p(z)+p(\alpha_j)]}.$$

Besides, $g_j^{(l)}(\alpha_k) = 0$ for all k and for all $0 \leq l < m_k$ and $\frac{g_j^{(m_j)}(\alpha_j)}{m_j!} = 1$. Applying Jensen's formula to the function $\frac{g_j(z)}{(z - \alpha_j)^{m_j}}$ in the disk $D(\alpha_j, R_j)$ we obtain

$$N_V(\alpha_j, R_j) = \int_0^{2\pi} \ln |g_j(\alpha_j + R_j e^{i\theta})| d\theta.$$

By definition of R_j , for all $\theta \in [0, 2\pi]$, $|g_j(\alpha_j + R_j e^{i\theta})| \leq Ae^{Bp(\alpha_j)}$ for certain constants $A, B > 0$. We readily deduce the estimate (3). \blacksquare

DEFINITION 1.2.9. We say that V is weakly separated if there exist constants $A > 0$ and $B > 0$ such that

$$(7) \quad \delta_j^{-m_j} \leq Ae^{Bp(\alpha_j)}$$

for all j , where

$$\delta_j := \inf \left\{ 1, \inf_{k \neq j} |\alpha_j - \alpha_k| \right\}$$

is called the separation radius.

REMARK 1.2.10. The disks $D(\alpha_j, \delta_j/2)$ are pairwise disjoint and because $\delta_j \leq 1$, there exist constants $A, B > 0$ such that for all $j \geq 0$ and for all z, ζ in $D(\alpha_j, \delta_j)$, we have $e^{p(z)} \leq Ae^{Bp(\zeta)}$.

LEMMA 1.2.11. *If V is an interpolating variety then it is weakly separated.*

PROOF. Lemma 1.2.6 gives a sequence of functions $\{f_j\}_j$ such that

- $f_j(\alpha_k) = 1$ for all $k \neq j$ and $f_j^{(l)}(\alpha_j) = 0$ for all $0 \leq l < m_j$.
- For all $z \in \mathbb{C}$, $|f_j(z)| \leq Ae^{Bp(z)}$, where $A, B > 0$ don't depend on j .

Set $g_j(z) = \frac{f_j(z)}{(z - \alpha_j)^{m_j}}$. If $|z - \alpha_j| \leq 1$, then by the maximum principle

$$|g_j(z)| \leq \sup_{|\zeta - \alpha_j|=1} |f_j(\zeta)| \leq Ae^{Bp(\alpha_j)},$$

using property (w2) of the weight. For all $k \neq j$ such that $|\alpha_k - \alpha_j| \leq 1$, we have

$$|\alpha_k - \alpha_j|^{-m_j} = |g_j(\alpha_k)| \leq Ae^{Bp(\alpha_j)}.$$

This concludes the proof of the lemma. \blacksquare

LEMMA 1.2.12. *If, for certain constants $A, B > 0$ we have $\delta_j^{-1} \leq Ae^{Bp(\alpha_j)}$ for all j , then there exists a constant $C > 0$ such that*

$$\sum_j e^{-Cp(\alpha_j)} < \infty.$$

In particular, this holds when V is weakly separated.

PROOF. Thanks to property (w1) of the weight there exists a constant $D > 0$ such that $\int_{\mathbb{C}} e^{-Dp(z)} d\lambda(z) < \infty$. In view of Remark 1.2.10, we have,

$$\begin{aligned} \int_{\mathbb{C}} e^{-Dp(z)} d\lambda(z) &\geq \sum_j \int_{D(\alpha_j, \delta_j/2)} e^{-Dp(z)} d\lambda(z) \\ &\geq \sum_j A e^{-DBp(\alpha_j)} \delta_j^2 \geq \sum_{j \geq 0} A e^{-Cp(\alpha_j)} \end{aligned}$$

where $C > 0$ is a certain constant. ■

LEMMA 1.2.13. *If (3) holds with some $R_j \geq 1$, then V is weakly separated.*

PROOF. Fix j and let $\alpha_l \neq \alpha_j$ be such that $|\alpha_j - \alpha_l| = \inf_{k \neq j} |\alpha_j - \alpha_k|$. If $|\alpha_j - \alpha_l| \geq 1$, then $\delta_j = 1$. Otherwise, $\delta_j = |\alpha_j - \alpha_l|$ and the following inequalities hold

$$N(\alpha_l, R_l) \geq \sum_{0 < |\alpha_k - \alpha_l| \leq R_l} m_k \ln \frac{R_l}{|\alpha_k - \alpha_l|} \geq m_j \ln \frac{R_l}{|\alpha_j - \alpha_l|} = m_j \ln \frac{R_l}{\delta_j} \geq \ln \frac{1}{\delta_j^{m_j}}.$$

Since by condition (3) and property (w2) of the weight,

$$N(\alpha_l, R_l) \leq A + Bp(\alpha_l) \leq A + Bp(\alpha_j),$$

we readily deduce the desired estimate. ■

Note that Lemma 1.2.11 may be obtained as a corollary of Theorem 1.2.7 and Lemma 1.2.13.

Let us now state an important theorem given by Berenstein and Li, which will be very useful throughout our work. It provides an analytic characterization of \mathcal{A}_p -interpolating varieties for general weights p .

THEOREM 1.2.14. (see [3, Corollary 3.5]) *A multiplicity variety $V = \{(\alpha_j, m_j)\}_j$ is \mathcal{A}_p -interpolating if and only if there exists $f \in \mathcal{A}_p(C)$ such that $V \subset \mathcal{Z}(f)$ and for some constants $\varepsilon, C > 0$*

$$\left| \frac{f^{(m_j)}(\alpha_j)}{m_j!} \right| \geq \varepsilon e^{-Cp(\alpha_j)} \quad \text{for all } j.$$

This leads us to the following observations.

- REMARK 1.2.15. (i) Let p and q be two weights such that $p \lesssim q$. If V is \mathcal{A}_p -interpolating then it is \mathcal{A}_q -interpolating (replace the defining function f in Theorem 1.2.14 by f^N).
- (ii) If $\{(\alpha_j, m_j)\}_{j \geq 0}$ is \mathcal{A}_p -interpolating then for any $N \in \mathbb{N}^*$, $\{(\alpha_j, Nm_j)\}_{j \geq 0}$ is also \mathcal{A}_p -interpolating.

When constructing interpolating entire functions in $\mathcal{A}_p(\mathbb{C})$ in Chapters 3, 4, 5 and 7, Hörmander's theorem giving the existence of a solution to the Cauchy-Riemann equation with L^2 -estimates will be a very strong tool. Let us conclude this chapter by stating this result.

THEOREM 1.2.16. (see [22]) *Let Ω be an open subset of \mathbb{C} , ϕ be a subharmonic function in Ω and v a measurable function in Ω such that*

$$\int_{\Omega} |v(z)|^2 e^{-\phi(z)} d\lambda(z) < \infty.$$

Then, there is function $u \in L^1_{loc}(\Omega)$ such that $\bar{\partial}u = v$, in the sense of distributions, and

$$\int_{\Omega} \frac{|u(z)|^2}{(1 + |z|^2)^2} e^{-\phi(z)} d\lambda(z) \leq \int_{\Omega} |v(z)|^2 e^{-\phi(z)} d\lambda(z).$$

CHAPTER 2

Finite union of interpolating varieties

Introduction

Recall that the weak separation is a necessary condition for a multiplicity variety to be interpolating (see Lemma 1.2.11). The union of two interpolating varieties is clearly not weakly separated in general thus it is not necessarily an interpolating variety. Nevertheless, if we assume that the union is weakly separated then it is interpolating. That is what we are going to prove in Section 2.1.

If V is an interpolating variety, then by definition, the trace of $\mathcal{A}_p(\mathbb{C})$ on V is the space $\mathcal{A}_p(V)$. How can we describe the trace of $\mathcal{A}_p(\mathbb{C})$ on a finite union of interpolating varieties ? That is the problem we are concerned with in Section 2.2. We will show that a discrete sequence V of the complex plane is the union of n interpolating sequences for the Hörmander algebras $\mathcal{A}_p(\mathbb{C})$ if and only if the trace of $\mathcal{A}_p(\mathbb{C})$ on V coincides with the space of functions on V for which the divided differences of order $n - 1$ are uniformly bounded. The analogous result holds in the unit disk for Korenblum-type algebras.

2.1. When a finite union of interpolating varieties is an interpolating variety

This result appear in [39] in the multivariate case with simple multiplicities.

THEOREM 2.1.1. *Assume that V_1, \dots, V_n are interpolating varieties for $\mathcal{A}_p(\mathbb{C})$. Then $V = V_1 \cup \dots \cup V_n$ is an interpolating variety if and only if it is weakly separated.*

PROOF. We may assume without loss of generality that $n = 2$. We will denote by Λ and Γ two discrete sequences of \mathbb{C} . We will denote by m_λ and m_γ the respective multiplicities affected to each $\lambda \in \Lambda$ and $\gamma \in \Gamma$. We will denote by $V_1 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ and $V_2 = \{(\gamma, m_\gamma)\}_{\gamma \in \Gamma}$ the corresponding multiplicity varieties. We assume that $V_1 \cup V_2$ is weakly separated. We may assume without loss of generality that the intersection of Λ and Γ is empty.

Let $W = \{w_{\lambda,l}\}_{\lambda \in \Lambda, 0 \leq l < m_\lambda} \in \mathcal{A}_p(V_1)$ and $Z = \{z_{\gamma,l}\}_{\gamma \in \Gamma, 0 \leq l < m_\gamma} \in \mathcal{A}_p(V_2)$ be the values to interpolate. We want to construct $f_1 \in \mathcal{A}_p(\mathbb{C})$ and $f_2 \in \mathcal{A}_p(\mathbb{C})$ such that

$$\begin{aligned} \frac{f_1^{(l)}}{l!}(\lambda) &= w_{\lambda,l}, \quad \lambda \in \Lambda, \quad 0 \leq l < m_\lambda, & \frac{f_1^{(l)}}{l!}(\gamma) &= 0, \quad \gamma \in \Gamma, \quad 0 \leq l < m_\gamma, \\ \frac{f_2^{(l)}}{l!}(\gamma) &= z_{\gamma,l}, \quad \gamma \in \Gamma, \quad 0 \leq l < m_\gamma, & \frac{f_2^{(l)}}{l!}(\lambda) &= 0, \quad \lambda \in \Lambda, \quad 0 \leq l < m_\lambda. \end{aligned}$$

Then setting $f = f_1 + f_2$, it is clear that

$$\frac{f^{(l)}}{l!}(\lambda) = w_{\lambda,l}, \quad \lambda \in \Lambda, \quad 0 \leq l < m_\lambda \quad \text{and} \quad \frac{f^{(l)}}{l!}(\gamma) = z_{\gamma,l}, \quad \gamma \in \Gamma, \quad 0 \leq l < m_\gamma,$$

in other words, $\mathcal{R}_{V_1 \cup V_2}(f) = W \cup Z$.

Let us show how to construct f_1 (the construction of f_2 will be of course similar reversing the roles of V_1 and V_2).

By Lemma 1.2.6, we can find a sequence $\{G_\lambda\}_{\lambda \in \Lambda}$ such that

- (i) $\frac{G_\lambda^{(l)}}{l!}(\gamma) = (-1)^l(\gamma - \lambda)^{-m_\lambda - l} \frac{(m_\lambda + l - 1)!}{(m_\lambda - 1)!l!}$, for all $\gamma \in \Gamma$ and $0 \leq l < m_\gamma$.
- (ii) $|G_\lambda(z)| \leq Ae^{B[p(\lambda) + p(z)]}$.

where $A, B > 0$ don't depend on λ .

The existence of the sequence of interpolating functions $\{G_\lambda\}_{\lambda \in \Lambda}$ needs a justification : according to condition (5), we have

$$\frac{(m_\lambda + l - 1)!}{(m_\lambda - 1)!l!} \leq 2^{m_\lambda + l} \leq 2^{m_\lambda + m_\gamma} \leq Ae^{B[p(\lambda) + p(\gamma)]}.$$

Besides, the weak separation of $V_1 \cup V_2$ means that for all $\lambda \in \Lambda$ and all $\gamma \in \Gamma$,

$$|\gamma - \lambda|^{-m_\lambda - l} \leq Ae^{B[p(\lambda) + p(\gamma)]}.$$

We have then verified that the coefficients interpolated by the functions G_λ has the correct growth and this completes the proof of the existence of the sequence $\{G_\lambda\}_{\lambda \in \Lambda}$.

Now, set $F_\lambda(z) = 1 - (z - \lambda)^{m_\lambda} G_\lambda$ for all $\lambda \in \Lambda$. We clearly have

$$F_\lambda(\lambda) = 1, \quad (F_\lambda)^{(l)}(\lambda) = 0, \quad 1 \leq l < m_\lambda.$$

Besides, notice that G_λ has the same derivatives as the function $(z - \lambda)^{-m_\lambda}$ in the points γ up to order $m_\gamma - 1$. Thus, F_λ verifies

$$F_\lambda^{(l)}(\gamma) = 0, \quad \forall \gamma \in \Gamma, \quad \forall 0 \leq l < m_\gamma.$$

According to Remark 1.2.15(ii), we know that $\{(\lambda, 2m_\lambda)\}$ is an interpolating variety. Thus there exists a function $H \in \mathcal{A}_p(\mathbb{C})$ such that

$$H^{(l)}(\lambda) = 0, \quad 0 \leq l < 2m_\lambda, \quad \text{except } H^{(m_\lambda)}(\lambda) = 1.$$

Then the function $H_\lambda(z) = \frac{H(z)}{(z - \lambda)^{m_\lambda}}$ satisfies the following conditions :

- (i) $H_\lambda(\lambda) = 1, \quad H_\lambda^{(l)}(\lambda) = 0, \quad 1 \leq l < m_\lambda,$
- (ii) $|H_\lambda(z)| \leq Ae^{Bp(z)}$ for some constants $A, B > 0$ not depending on λ .

Property (ii) comes from the fact that for $|z - \lambda| \geq 1$, it is clear that $|H_\lambda(z)| \leq |H(z)|$ and for $|z - \lambda| \leq 1$, by the maximum principle, we have $|H_\lambda(z)| \leq \sup_{|\zeta - \lambda|=1} Ae^{Bp(\zeta)} \leq Ae^{Bp(z)}$ by property (w2) of the weight.

We deduce that

- (i) $|(F_\lambda H_\lambda)(z)| \leq Ae^{B[p(\lambda) + p(z)]}$, for all $z \in \mathbb{C}$ where $A, B > 0$ don't depend on λ ,
- (ii) $(F_\lambda H_\lambda)(\lambda) = 1$ and $(F_\lambda H_\lambda)^{(l)}(\lambda) = 0$, for all $1 \leq l < m_\lambda$.

Applying once again Lemma 1.2.6, for all $M > 0$, there exists a sequence of functions $\{h_\lambda\}_{\lambda \in \Lambda}$ such that

- (i) $\frac{h_\lambda^{(l)}(\lambda)}{l!} = w_{\lambda,l} e^{Mp(\lambda)}, \quad 0 \leq l < m_\lambda$
- (ii) $h_\lambda^{(l)}(\lambda') = 0, \quad \lambda' \in \Lambda \setminus \{\lambda\}, \quad 0 \leq l < m_{\lambda'},$
- (iii) $|h_\lambda(z)| \leq A e^{Bp(z)}$. where A and B don't depend on λ .

Finally, we set

$$f_1 = \sum_{j \geq 0} F_\lambda H_\lambda h_\lambda e^{-Mp(\lambda)}$$

where M is chosen large enough so that the sum converges, according to Lemma 1.2.12. ■

2.2. The trace of $\mathcal{A}_p(\mathbb{C})$ on a finite union of interpolating varieties

This is a joint work with X. Massaneda and J. Ortega-Cerdà. It will appear in the proceedings of the international conference 'New Trends in Harmonic and Complex Analysis' held May 7-12, 2007 in Voss, Norway (see [35]).

We will restrict ourselves to the case of simple multiplicities though with similar techniques it should be possible to extend this result to an Hermite-type interpolation problem with multiplicities, along the lines of [41].

2.2.1. Divided differences.

DEFINITION 2.2.1. Let Λ be a discrete sequence in \mathbb{C} and ω a function given on Λ . The *divided differences* of ω are defined by induction as follows

$$\begin{aligned} \Delta^0 \omega(\lambda_1) &= \omega(\lambda_1), \\ \Delta^j \omega(\lambda_1, \dots, \lambda_{j+1}) &= \frac{\Delta^{j-1} \omega(\lambda_2, \dots, \lambda_{j+1}) - \Delta^{j-1} \omega(\lambda_1, \dots, \lambda_j)}{\lambda_{j+1} - \lambda_1} \quad j \geq 1. \end{aligned}$$

For any $n \in \mathbb{N}$, denote

$$\Lambda^n = \{(\lambda_1, \dots, \lambda_n) \in \Lambda \times \cdots \times \Lambda : \lambda_j \neq \lambda_k \text{ if } j \neq k\},$$

and consider the set $X_p^{n-1}(\Lambda)$ consisting of the functions in $\omega(\Lambda)$ with divided differences of order n uniformly bounded with respect to the weight p , i.e., such that for some $B > 0$

$$\sup_{(\lambda_1, \dots, \lambda_n) \in \Lambda^n} |\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)| e^{-B[p(\lambda_1) + \dots + p(\lambda_n)]} < +\infty.$$

REMARK 2.2.2. We have the inclusions $X_p^n(\Lambda) \subset X_p^{n-1}(\Lambda) \subset \dots \subset X_p^0(\Lambda) = \mathcal{A}_p(\Lambda)$.

To see this assume that $\omega(\Lambda) \in X_p^n(\Lambda)$, i.e., there exists $B > 0$ such that

$$\begin{aligned} C := \sup_{(\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}} & \left| \frac{\Delta^{n-1} \omega(\lambda_2, \dots, \lambda_{n+1}) - \Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)}{\lambda_{n+1} - \lambda_1} \right| \times \\ & \times e^{-B[p(\lambda_1) + \dots + p(\lambda_{n+1})]} < \infty. \end{aligned}$$

Then, given $(\lambda_1, \dots, \lambda_n) \in \Lambda^n$ and taking $\lambda_1^0, \dots, \lambda_n^0$ from a finite set (for instance the n first $\lambda_j^0 \in \Lambda$ different of all λ_j) we have

$$\begin{aligned} \Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n) &= \frac{\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n) - \Delta^{n-1}\omega(\lambda_1^0, \lambda_1, \dots, \lambda_{n-1})}{\lambda_n - \lambda_1^0}(\lambda_n - \lambda_1^0) + \\ &+ \frac{\Delta^{n-1}\omega(\lambda_1^0, \lambda_1, \dots, \lambda_{n-1}) - \Delta^{n-1}\omega(\lambda_2^0, \lambda_1^0, \dots, \lambda_{n-2})}{\lambda_{n-1} - \lambda_2^0}(\lambda_{n-1} - \lambda_2^0) + \dots + \\ &+ \frac{\Delta^{n-1}\omega(\lambda_{n-1}^0, \dots, \lambda_1^0, \lambda_1) - \Delta^{n-1}\omega(\lambda_n^0, \dots, \lambda_1^0)}{\lambda_1 - \lambda_n^0}(\lambda_1 - \lambda_n^0) + \Delta^{n-1}\omega(\lambda_n^0, \dots, \lambda_1^0) \end{aligned}$$

Then a direct estimate and (w1) show that for some $B > 0$ there is a constant $K(\lambda_1^0, \dots, \lambda_n^0)$ such that

$$\begin{aligned} |\Delta^{n-1}\omega(\lambda_1, \dots, \lambda_n)| &\leq C \left(e^{B[p(\lambda_1^0) + \dots + p(\lambda_n)]} + \dots + e^{B[p(\lambda_{n-1}^0) + \dots + p(\lambda_1)]} \right) \\ &\leq K(\lambda_1^0, \dots, \lambda_n^0) e^{B[p(\lambda_1) + \dots + p(\lambda_n)]}, \end{aligned}$$

and the statement follows.

The main result of this note is modelled after Vasyunin's description of the sequences Λ in the unit disk such that the trace of the algebra of bounded holomorphic functions H^∞ on Λ equals the space of (hyperbolic) divided differences of order n (see [46], [47]). The analogue in our context is the following.

THEOREM 2.2.3 (Main Theorem). *The identity $\mathcal{R}_\Lambda(\mathcal{A}_p(\mathbb{C})) = X_p^{n-1}(\Lambda)$ holds if and only if Λ is the union of n \mathcal{A}_p interpolating sequences.*

Note that the statement of the theorem when $n = 1$ is just the definition of an interpolating sequence.

2.2.2. General properties. We begin by showing that one of the inclusions of Theorem 2.2.3 is immediate.

PROPOSITION 2.2.4. *For all $n \in \mathbb{N}$, the inclusion $\mathcal{R}_\Lambda(\mathcal{A}_p(\mathbb{C})) \subset X_p^{n-1}(\Lambda)$ holds.*

PROOF. Let $f \in \mathcal{A}_p$. Let us show by induction on $j \geq 1$ that, for certain constants $A, B > 0$

$$|\Delta^{j-1}f(z_1, \dots, z_j)| \leq A e^{B[p(z_1) + \dots + p(z_j)]} \quad \text{for all } (z_1, \dots, z_j) \in \mathbb{C}^j.$$

As $f \in \mathcal{A}_p$, we have $|\Delta^0 f(z_1)| = |f(z_1)| \leq A e^{Bp(z_1)}$.

Assume that the property is true for j and let $(z_1, \dots, z_{j+1}) \in \mathbb{C}^{j+1}$. Fix z_1, \dots, z_j and consider z_{j+1} as the variable in the function

$$\Delta^j f(z_1, \dots, z_{j+1}) = \frac{\Delta^{j-1}f(z_2, \dots, z_{j+1}) - \Delta^{j-1}f(z_1, \dots, z_j)}{z_{j+1} - z_1}.$$

By the induction hypothesis,

$$\begin{aligned} |\Delta^{j-1}f(z_2, \dots, z_{j+1}) - \Delta^{j-1}f(z_1, \dots, z_j)| &\leq \\ &\leq A(e^{B[p(z_2) + \dots + p(z_{j+1})]} + e^{B[p(z_1) + \dots + p(z_j)]}) \leq 2A e^{B[p(z_1) + \dots + p(z_{j+1})]}. \end{aligned}$$

Thus, if $|z_{j+1} - z_1| \geq 1$, we easily deduce the desired estimate. For $|z_{j+1} - z_1| \leq 1$, by the maximum principle and (w2):

$$\begin{aligned} |\Delta^j f(z_1, \dots, z_{j+1})| &\leq 2A \sup_{|\xi - z_1|=1} e^{B[p(z_1) + \dots + p(z_j) + p(\xi)]} \\ &\leq Ae^{(B+D_0)[p(z_1) + \dots + p(z_j) + p(z_{j+1})]}. \end{aligned}$$

■

DEFINITION 2.2.5. A sequence Λ is *weakly separated* if there exist constants $\varepsilon > 0$ and $C > 0$ such that the disks $D(\lambda, \varepsilon e^{-Cp(\lambda)})$, $\lambda \in \Lambda$, are pairwise disjoint.

REMARK 2.2.6. If Λ is weakly separated then $X_p^0(V) = X_p^n(V)$, for all $n \in \mathbb{N}$.

To see this it is enough to prove (by induction) that $X_p^0(\Lambda) \subset X_p^n(\Lambda)$ for all $n \in \mathbb{N}$. For $n = 0$ this is trivial. Assume now that $X_p^0(\Lambda) \subset X_p^{n-1}(\Lambda)$. Given $\omega(\Lambda) \in X_p^0(\Lambda)$ we have

$$\begin{aligned} |\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})| &= \left| \frac{\Delta^{n-1}(\lambda_2, \dots, \lambda_{n+1}) - \Delta^{n-1}(\lambda_1, \dots, \lambda_n)}{\lambda_{n+1} - \lambda_1} \right| \\ &\leq \frac{2A}{\varepsilon} e^{(B+C)[p(\lambda_1) + \dots + p(\lambda_{n+1})]}. \end{aligned}$$

LEMMA 2.2.7. Let $n \geq 1$. The following assertions are equivalent:

- (a) Λ is the union of n weakly separated sequences,
- (b) There exist constants $\varepsilon > 0$ and $C > 0$ such that

$$\sup_{\lambda \in \Lambda} \#[\Lambda \cap D(\lambda, \varepsilon e^{-Cp(\lambda)})] \leq n.$$

- (c) $X_p^{n-1}(\Lambda) = X_p^n(\Lambda)$.

PROOF. (a) \Rightarrow (b). This is clear, by the weak separation.

(b) \Rightarrow (a). We proceed by induction on $j = 1, \dots, n$. For $j = 1$, it is again clear by the definition of weak separation. Assume the property true for $j - 1$. Let $1 \geq \varepsilon > 0$ and $C > 0$ be such that $\sup_{\lambda \in \Lambda} \#[\Lambda \cap D(\lambda, \varepsilon e^{-Cp(\lambda)})] \leq j$. Put $\varepsilon' = e^{-E_0 C} \varepsilon / 2$ and $C' = D_0 C$. By Zorn's Lemma, there is a maximal subsequence $\Lambda_1 \subset \Lambda$ such that the disks $D(\lambda, \varepsilon' e^{-C'p(\lambda)})$, $\lambda \in \Lambda_1$, are pairwise disjoint. In particular Λ_1 is weakly separated. For any $\alpha \in \Lambda \setminus \Lambda_1$, there exists $\lambda \in \Lambda_1$ such that

$$D(\lambda, \varepsilon' e^{-C'p(\lambda)}) \cap D(\alpha, \varepsilon' e^{-C'p(\alpha)}) \neq \emptyset,$$

otherwise Λ_1 would not be maximal. Then $\lambda \in D(\alpha, \varepsilon e^{-Cp(\alpha)})$, since

$$|\lambda - \alpha| < \varepsilon' e^{-C'p(\lambda)} + \varepsilon' e^{-C'p(\alpha)} < \varepsilon e^{-Cp(\alpha)},$$

by (w2). Thus $D(\alpha, \varepsilon e^{-Cp(\alpha)})$ contains at most $j - 1$ points of $\Lambda \setminus \Lambda_1$. We use the induction hypothesis to conclude that $\Lambda \setminus \Lambda_1$ is the union of $j - 1$ weakly separated sequences and, by consequence, Λ is the union of j weakly separated sequences.

(b) \Rightarrow (c). It remains to see that $X_p^{n-1}(\Lambda) \subset X_p^n(\Lambda)$. Given $\omega(\Lambda) \in X_p^{n-1}(\Lambda)$ and points $(\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}$, we have to estimate $\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})$. Under the assumption (b), at least one of these $n + 1$ points is not in the disk $D(\lambda_1, \varepsilon e^{-Cp(\lambda_1)})$. Note that Λ^n is invariant by

permutation of the $n + 1$ points, thus we may assume that $|\lambda_1 - \lambda_{n+1}| \geq \varepsilon e^{-Cp(\lambda_1)}$. Using the fact that $\omega(\Lambda) \in X_p^{n-1}(\Lambda)$, there are constants $A, B > 0$ such that

$$\begin{aligned} |\Delta^n \omega(\lambda_1, \dots, \lambda_{n+1})| &\leq \frac{|\Delta^{n-1} \omega(\lambda_2, \dots, \lambda_{n+1})| + |\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)|}{|\lambda_1 - \lambda_{n+1}|} \\ &\leq A e^{B[p(\lambda_1) + \dots + p(\lambda_{n+1})]}. \end{aligned}$$

(c) \Rightarrow (b). We prove this by contraposition. Assume that for all $C, \varepsilon > 0$, there exists $\lambda \in \Lambda$ such that $\#[\Lambda \cap D(\lambda, \varepsilon e^{-Cp(\lambda)})] > n$. Since Λ has no accumulation points, for any fixed $C > 0$, we can extract from Λ a weakly separated subsequence $\mathcal{L} = \{\alpha^l\}_{l \in \mathbb{N}}$ such that $\#[(\Lambda \setminus \mathcal{L}) \cap D(\alpha^l, 1/l e^{-Cp(\alpha^l)})] \geq n$ for all l . Let us call $\lambda_1^l, \dots, \lambda_n^l$ the points of $\Lambda \setminus \mathcal{L}$ closest to α^l , arranged by increasing distance. In order to construct a sequence $\omega(\Lambda) \in X_p^{n-1}(\Lambda) \setminus X_p^n(\Lambda)$, put

$$\begin{aligned} \omega(\alpha^l) &= \prod_{j=1}^{n-1} (\alpha^l - \lambda_j^l), \text{ for all } \alpha^l \in \mathcal{L} \\ \omega(\lambda) &= 0 \text{ if } \lambda \in \Lambda \setminus \mathcal{L}. \end{aligned}$$

To see that $\omega(\Lambda) \in X_p^{n-1}(\Lambda)$ let us estimate $\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n)$ for any given vector $(\lambda_1, \dots, \lambda_n) \in \Lambda^n$. We don't need to consider the case where the points are distant, thus, as \mathcal{L} is weakly separated, we may assume that at most one of the points is in \mathcal{L} . On the other hand, it is clear that $\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_n) = 0$ if all the points are in $\Lambda \setminus \mathcal{L}$. Then, taking into account that Δ^{n-1} is invariant by permutation, we will only consider the case where λ_n is some $\alpha^l \in \mathcal{L}$ and $\lambda_1, \dots, \lambda_{n-1}$ are in $\Lambda \setminus \mathcal{L}$. In that case,

$$\begin{aligned} |\Delta^{n-1} \omega(\lambda_1, \dots, \lambda_{n-1}, \alpha^l)| &= |\omega(\alpha^l)| \prod_{k=1}^{n-1} |\alpha^l - \lambda_k^l|^{-1} \\ &\leq 1, \end{aligned}$$

as desired.

On the other hand, a similar reasoning yields

$$|\Delta^n \omega(\lambda_1^l, \dots, \lambda_n^l, \alpha^l)| = |\omega(\alpha^l)| \prod_{k=1}^n |\alpha^l - \lambda_k^l|^{-1} = |\alpha^l - \lambda_n^l|^{-1} \geq l e^{Cp(\alpha^l)}.$$

Using (w2), for any constant $B > 0$, and choosing $C = B(nD_0 + 1)$, we have

$$|\Delta^n \omega(\lambda_1^l, \dots, \lambda_n^l, \alpha^l)| e^{-B(p(\lambda_1^l) + \dots + p(\lambda_n^l) + p(\alpha^l))} \geq l e^{-BnE_0} \rightarrow +\infty.$$

We finally conclude that $\omega(\Lambda) \notin X_p^n(\Lambda)$. ■

As a corollary we obtain again another proof of Lemma 1.2.11 stated in Chapter 1 :

COROLLARY 2.2.8. *If Λ is an interpolating sequence, then it is weakly separated.*

PROOF. If Λ is an interpolating sequence, then $\mathcal{R}_\Lambda(\mathcal{A}_p(\mathbb{C})) = X_p^0(\Lambda)$. On the other hand, by Proposition 2.2.4, $\mathcal{R}_\Lambda(\mathcal{A}_p(\mathbb{C})) \subset X_p^1(\Lambda)$. Thus $X_p^0(\Lambda) = X_p^1(\Lambda)$. We conclude by the preceding lemma applied to the particular case $n = 1$. ■

LEMMA 2.2.9. *Let $\Lambda_1, \dots, \Lambda_n$ be weakly separated sequences. There exist positive constants a, b, B_1, B_2 and $\varepsilon > 0$, a subsequence $\mathcal{L} \subset \Lambda_1 \cup \dots \cup \Lambda_n$ and disks $D_\lambda = D(\lambda, r_\lambda)$, $\lambda \in \mathcal{L}$, such that*

- (i) $\Lambda_1 \cup \dots \cup \Lambda_n \subset \bigcup_{\lambda \in \mathcal{L}} D_\lambda$
- (ii) $a\varepsilon e^{-B_1 p(\lambda)} \leq r_\lambda \leq b\varepsilon e^{-B_2 p(\lambda)}$ for all $\lambda \in \mathcal{L}$
- (iii) $\text{dist}(D_\lambda, D_{\lambda'}) \geq a\varepsilon e^{-B_1 p(\lambda)}$ for all $\lambda, \lambda' \in \mathcal{L}$, $\lambda \neq \lambda'$.
- (iv) $\#(\Lambda_j \cap D_\lambda) \leq 1$ for all $j = 1, \dots, n$ and $\lambda \in \mathcal{L}$.

PROOF. Let $0 < \varepsilon < 1$ and $C > 0$ be constants such that

$$(8) \quad |\lambda - \lambda'| \geq \varepsilon e^{-C/D_0(p(\lambda) - E_0)}, \quad \forall \lambda, \lambda' \in \Lambda_j, \lambda \neq \lambda', \quad \forall j = 1, \dots, n,$$

where $D_0 \geq 1$ and $E_0 \geq 0$ are given by (w2).

We will proceed by induction on $k = 1, \dots, n$ to show the existence of a subsequence $\mathcal{L}_k \subset \Lambda_1 \cup \dots \cup \Lambda_k$ and constants $C_k \geq C$, $B_k \geq 0$ such that:

- (i)_k $\Lambda_1 \cup \dots \cup \Lambda_k \subset \bigcup_{\lambda \in \mathcal{L}_k} D(\lambda, R_\lambda^k)$,
- (ii)_k $2^{-3k} e^{-C_k p(\lambda) - B_k} \varepsilon \leq R_\lambda^k \leq \varepsilon e^{-C p(\lambda)} \sum_{j=0}^{k-1} 2^{-(3j+2)} \leq 2/7 e^{-C p(\lambda)} \varepsilon$,
- (iii)_k $\text{dist}(D(\lambda, R_\lambda^k), D(\lambda', R_{\lambda'}^k)) \geq 2^{-3k} \varepsilon e^{-C_k p(\lambda) - B_k}$ for any $\lambda, \lambda' \in \mathcal{L}_k$, $\lambda \neq \lambda'$.

The constants C_k and B_k are chosen, in view of (w2), so that $C_k p(\lambda) + B_k \leq C_{k+1} p(\lambda') + B_{k+1}$ whenever $|\lambda - \lambda'| \leq 1$.

Then it suffices to choose $\mathcal{L} = \mathcal{L}_n$, $r_\lambda = R_\lambda^n$, $a = e^{-B_n} 2^{-3n}$, $b = 2/7$, $B_1 = C_n$ and $B_2 = C$. As $r_\lambda < e^{-C p(\lambda)} \varepsilon$, it is clear that $D(\lambda, r_\lambda)$ contains at most one point of each Λ_j , hence the lemma follows.

For $k = 1$, the property is clearly verified with $\mathcal{L}_1 = \Lambda_1$ and $R_\lambda^1 = e^{-C p(\lambda)} \varepsilon/4$.

Assume the property true for k and split $\mathcal{L}_k = \mathcal{M}_1 \cup \mathcal{M}_2$ and $\Lambda_{k+1} = \mathcal{N}_1 \cup \mathcal{N}_2$, where

$$\begin{aligned} \mathcal{M}_1 &= \{\lambda \in \mathcal{L}_k : D(\lambda, R_\lambda^k + 2^{-3k-2} \varepsilon e^{-C_k p(\lambda) - B_k}) \cap \Lambda_{k+1} \neq \emptyset\}, \\ \mathcal{N}_1 &= \Lambda_{k+1} \cap \bigcup_{\lambda \in \mathcal{L}_k} D(\lambda, R_\lambda^k + 2^{-3k-2} \varepsilon e^{-C_k p(\lambda) - B_k}), \\ \mathcal{M}_2 &= \mathcal{L}_k \setminus \mathcal{M}_1, \\ \mathcal{N}_2 &= \Lambda_{k+1} \setminus \mathcal{N}_1. \end{aligned}$$

Now, we put $\mathcal{L}_{k+1} = \mathcal{L}_k \cup \mathcal{N}_2$ and define the radii R_λ^{k+1} as follows:

$$R_\lambda^{k+1} = \begin{cases} R_\lambda^k + 2^{-3k-2} \varepsilon e^{-C_k p(\lambda) - B_k} & \text{if } \lambda \in \mathcal{M}_1, \\ R_\lambda^k & \text{if } \lambda \in \mathcal{M}_2, \\ 2^{-3k-3} \varepsilon e^{-C_{k+1} p(\lambda) - B_{k+1}} & \text{if } \lambda \in \mathcal{N}_2. \end{cases}$$

It is clear that

$$\Lambda_1 \cup \dots \cup \Lambda_{k+1} \subset \bigcup_{\lambda \in \mathcal{L}_{k+1}} D(\lambda, R_\lambda^{k+1})$$

and, by the induction hypothesis,

$$2^{-3k-3}\varepsilon e^{-C_{k+1}p(\lambda)+B_{k+1}} \leq R_{\lambda}^{k+1} \leq \varepsilon e^{-Cp(\lambda)} \sum_{j=0}^k 2^{-3j-2} \leq 2/7\varepsilon e^{-Cp(\lambda)}.$$

In order to prove $(iii)_k$ take now $\lambda, \lambda' \in \mathcal{L}_{k+1}$, $\lambda \neq \lambda'$. We will verify that

$$\text{dist}(D(\lambda, R_{\lambda}^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) = |\lambda - \lambda'| - R_{\lambda}^{k+1} - R_{\lambda'}^{k+1} \geq 2^{-3k-3}\varepsilon e^{-C_{k+1}p(\lambda)-B_{k+1}}$$

by considering different cases.

If $\lambda, \lambda' \in \mathcal{L}_k$ and $p(\lambda) \leq p(\lambda')$, then

$$\begin{aligned} \text{dist}(D(\lambda, R_{\lambda}^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) &\geq |\lambda - \lambda'| - R_{\lambda}^k - R_{\lambda'}^k - 2^{-3k-1}\varepsilon e^{-C_k p(\lambda)-B_k} \\ &\geq 2^{-3k-1}\varepsilon e^{-C_k p(\lambda)-B_k}. \end{aligned}$$

Assume now $\lambda, \lambda' \in \mathcal{N}_2$ and $p(\lambda) \leq p(\lambda')$. Condition (1.2.10) implies $|\lambda - \lambda'| \geq \varepsilon e^{-Cp(\lambda)}$, hence

$$\text{dist}(D(\lambda, R_{\lambda}^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) \geq (1 - 2^{-3k-2})\varepsilon e^{-Cp(\lambda)}.$$

If $\lambda \in \mathcal{M}_1$ and $\lambda' \in \mathcal{N}_2$ there exists $\beta \in \mathcal{N}_1$ such that $|\lambda - \beta| \leq R_{\lambda}^{k+1}$. There is no restriction in assuming that $|\lambda - \lambda'| \leq 1$. Then, using (1.2.10) on $\beta, \lambda' \in \Lambda_{k+1}$, we have

$$|\lambda - \lambda'| \geq |\beta - \lambda'| - |\lambda - \beta| \geq \varepsilon e^{-C/D_0(p(\beta)-E_0)} - R_{\lambda}^{k+1} \geq \varepsilon e^{-Cp(\lambda)} - R_{\lambda}^{k+1}.$$

The definition of $R_{\lambda'}^{k+1}$ together with the estimate $R_{\lambda}^{k+1} \leq 2/7\varepsilon e^{-Cp(\lambda)}$ yield

$$\begin{aligned} \text{dist}(D(\lambda, R_{\lambda}^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) &\geq \varepsilon e^{-Cp(\lambda)} - 2R_{\lambda}^{k+1} - R_{\lambda'}^{k+1} \\ &\geq \varepsilon e^{-Cp(\lambda)} - 2R_{\lambda}^k - 2^{-3k-1}\varepsilon e^{-C_k p(\lambda)-B_k} - 2^{-3k-3}\varepsilon e^{-C_{k+1}p(\lambda')-B_{k+1}} \\ &\geq \varepsilon e^{-Cp(\lambda)} - \frac{4}{7}\varepsilon e^{-Cp(\lambda)} - 2^{-3k}\varepsilon e^{-C_k p(\lambda)-B_k} \geq \varepsilon e^{-Cp(\lambda)}(3/4 - 2^{-3k}), \end{aligned}$$

as required.

Finally, if $\lambda \in \mathcal{M}_2$ and $\lambda' \in \mathcal{N}_2$, again, assuming that $|\lambda - \lambda'| \leq 1$, we have

$$\begin{aligned} \text{dist}(D(\lambda, R_{\lambda}^{k+1}), D(\lambda', R_{\lambda'}^{k+1})) &= |\lambda - \lambda'| - R_{\lambda}^k - 2^{-3k-3}\varepsilon e^{-C_{k+1}p(\lambda')-B_{k+1}} \\ &\geq 2^{-3k-2}\varepsilon e^{-C_k p(\lambda)-B_k} - 2^{-3k-3}\varepsilon e^{-C_k p(\lambda)-B_k} \\ &\geq 2^{-3k-3}\varepsilon e^{-Cp(\lambda)}. \end{aligned}$$

■

2.2.3. Proof of Theorem 2.2.3. Necessity. Assume $\mathcal{R}_{\Lambda}(\mathcal{A}_p(\mathbb{C})) = X_p^{n-1}(\Lambda)$, $n \geq 2$. Using Proposition 2.2.4, we have $X_p^{n-1}(V) = X_p^n(V)$, and by Lemma 2.2.7 we deduce that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_n$, where $\Lambda_1, \dots, \Lambda_n$ are weakly separated sequences. We want to show that each Λ_j is an interpolating sequence.

Let $\omega(\Lambda_j) \in \mathcal{A}_p(\Lambda_j) = X_p^0(\Lambda_j)$. Let $\cup_{\lambda \in \mathcal{L}} D_\lambda$ be the covering of Λ given by Lemma 2.2.9. We extend $\omega(\Lambda_j)$ to a sequence $\omega(\Lambda)$ which is constant on each $D_\lambda \cap \Lambda_j$ in the following way:

$$\omega|_{D_\lambda \cap \Lambda} = \begin{cases} 0 & \text{if } D_\lambda \cap \Lambda_j = \emptyset \\ \omega(\alpha) & \text{if } D_\lambda \cap \Lambda_j = \{\alpha\}. \end{cases}$$

We verify by induction that the extended sequence is in $X_p^{k-1}(\Lambda)$ for all k . It is clear that it belongs to $X_p^0(\Lambda)$. Assume that $\omega \in X_p^{k-2}(\Lambda)$ and consider $(\alpha_1, \dots, \alpha_k) \in \Lambda^k$. If all the points are in the same D_λ then $\Delta^{k-1}\omega(\alpha_1, \dots, \alpha_k) = 0$, so we may assume that $\alpha_1 \in D_\lambda$ and $\alpha_k \in D_{\lambda'}$ with $\lambda \neq \lambda'$. Then we have

$$|\alpha_1 - \alpha_k| \geq a\varepsilon e^{-B_1 p(\lambda)},$$

by Lemma 2.2.9 (iii). With this and the induction hypothesis it is clear that for certain constants $A, B > 0$

$$\begin{aligned} |\Delta^{k-1}\omega(\alpha_1, \dots, \alpha_k)| &= \left| \frac{\Delta^{k-2}\omega(\alpha_2, \dots, \alpha_k) - \Delta^{k-2}\omega(\alpha_1, \dots, \alpha_k)}{\alpha_1 - \alpha_k} \right| \\ &\leq A e^{B[p(\alpha_1) + \dots + p(\alpha_k)]}. \end{aligned}$$

In particular $\omega(\Lambda) \in X_p^{n-1}(\Lambda)$, and by assumption, there exist $f \in \mathcal{A}_p(\mathbb{C})$ interpolating the values $\omega(\Lambda)$. In particular f interpolates $\omega(\Lambda_j)$.

2.2.4. Proof of Theorem 2.2.3. Sufficiency. According to Proposition 2.2.4 we only need to see that $X_p^{n-1}(\Lambda) \subset \mathcal{R}_\Lambda(\mathcal{A}_p(\mathbb{C}))$.

Assume that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_n$ where $\Lambda_1, \dots, \Lambda_n$ are interpolating sequences. Recall that each Λ_j is weakly separated (see Corollary 2.2.8 or Lemma 1.2.11). Consider also the covering of Λ given by Lemma 2.2.9.

LEMMA 2.2.10. *There exist constants $A, B > 0$ and a sequence $\{F_\lambda\}_{\lambda \in \mathcal{L}} \subset \mathcal{A}_p(\mathbb{C})$ such that*

$$\begin{aligned} F_\lambda(\alpha) &= \begin{cases} 1 & \text{if } \alpha \in \Lambda \cap D_\lambda \\ 0 & \text{if } \alpha \in \Lambda \cap D_{\lambda'}, \lambda' \neq \lambda \end{cases} \\ |F_\lambda(z)| &\leq A e^{B(p(\lambda) + p(z))} \quad \text{for all } z \in \mathbb{C}. \end{aligned}$$

PROOF. Fix $\lambda \in \mathcal{L}$ and define $\omega(\Lambda)$ by

$$\omega(\alpha) = \begin{cases} \prod_{\beta \in \Lambda \cap D_\lambda} (\alpha - \beta)^{-1} & \text{if } \alpha \notin \Lambda \cap D_\lambda \\ 0 & \text{if } \alpha \in \Lambda \cap D_\lambda. \end{cases}$$

By Lemma 2.2.9 (iii), we have $|\alpha - \beta| \geq c\varepsilon e^{-Cp(\alpha)}$ whenever $\alpha \notin \Lambda \cap D_\lambda, \beta \in \Lambda \cap D_\lambda$. Since $\#(\Lambda \cap D_\lambda) \leq n$ we deduce that

$$|\omega(\alpha)| \leq (c\varepsilon)^{-n} e^{nCp(\alpha)}$$

Recall that Λ_j is an interpolating sequence for all $j = 1, \dots, n$, thus, using Lemma 1.2.6, there exist a n -indexed sequence $\{f_{\lambda,j}\}_{\lambda \in \mathcal{L}, j \in [[1,n]]} \subset \mathcal{A}_p(\mathbb{C})$ such that for all $z \in \mathbb{C}$,

$$|f_{\lambda,j}(z)| \leq Ae^{Bp(z)}$$

$$f_{\lambda,j}(\alpha) = \prod_{\beta \in \Lambda \cap D_\lambda} (\alpha - \beta)^{-1} \text{ if } \alpha \notin \Lambda_j \cap D_\lambda,$$

with the constants A and B independent of λ .

The sequence of functions $\{F_\lambda\}_{\lambda \in \mathcal{L}}$ defined by

$$F_\lambda(z) = \prod_{j=1}^n \left[1 - \prod_{\beta \in \Lambda \cap D_\lambda} (z - \beta) f_{\lambda,j}(z) \right]$$

has the desired properties. ■

LEMMA 2.2.11. *For all $D > 0$, there exist $D' > 0$ and a sequence $\{G_\lambda\}_{\lambda \in \mathcal{L}} \subset \mathcal{A}_p(\mathbb{C})$ such that*

$$G_\lambda(\alpha) = e^{Dp(\lambda)} \text{ if } \alpha \in \Lambda \cap D_\lambda.$$

$$|G_\lambda(z)| \leq Ae^{Bp(\lambda)} e^{D'p(z)} \text{ for all } z \in \mathbb{C},$$

where $A, B > 0$ do not depend on D .

PROOF. In this proof D' denotes a constant depending on D but not on λ , and its actual value may change from one occurrence to the other.

Let $\lambda \in \mathcal{L}$. Assume, without loss of generality, that $D_\lambda \cap \Lambda_j = \{\alpha_{\lambda,j}\}$ for all j . As Λ_1 is an interpolating sequence and $e^{Dp(\lambda)} \leq Ae^{D'p(\alpha_{\lambda,1})}$, by Lemma 1.2.6 there exists a sequence $\{h_{\lambda,1}\}_\lambda \subset \mathcal{A}_p(\mathbb{C})$ such that

$$h_{\lambda,1}(\alpha_{\lambda,1}) = e^{Dp(\lambda)}$$

$$|h_{\lambda,1}(z)| \leq Ae^{D'p(z)} \text{ for all } z \in \mathbb{C}.$$

Setting $H_{\lambda,1}(z) = h_{\lambda,1}(z)$, we have $H_{\lambda,1}(\alpha_{\lambda,1}) = e^{Dp(\lambda)}$. Now, as Λ_2 is \mathcal{A}_p -interpolating and

$$\frac{|e^{Dp(\lambda)} - H_{\lambda,1}(\alpha_{\lambda,2})|}{|\alpha_{\lambda,2} - \alpha_{\lambda,1}|} = \frac{|H_{\lambda,1}(\alpha_{\lambda,1}) - H_{\lambda,1}(\alpha_{\lambda,2})|}{|\alpha_{\lambda,2} - \alpha_{\lambda,1}|} \leq Ae^{D'p(\alpha_{\lambda,2})},$$

there exists a sequence $\{h_{\lambda,2}\}_\lambda \subset \mathcal{A}_p(\mathbb{C})$ such that

$$h_{\lambda,2}(\alpha_{\lambda,2}) = \frac{e^{Dp(\lambda)} - H_{\lambda,1}(\alpha_{\lambda,2})}{\alpha_{\lambda,2} - \alpha_{\lambda,1}}$$

$$|h_{\lambda,2}(z)| \leq Ae^{D'p(z)} \text{ for all } z \in \mathbb{C}.$$

Setting $H_{\lambda,2}(z) = h_{\lambda,1}(z) + h_{\lambda,2}(z)(z - \alpha_{\lambda,1})$. We have

$$H_{\lambda,2}(\alpha_{\lambda,1}) = H_{\lambda,2}(\alpha_{\lambda,2}) = e^{Dp(\lambda)}.$$

We proceed by induction to construct a sequence of functions $\{h_{\lambda,k}\}_\lambda \subset \mathcal{A}_p(\mathbb{C})$ such that

$$\begin{aligned} h_{\lambda,k}(\alpha_{\lambda,k}) &= \frac{e^{Dp(\lambda)} - H_{\lambda,k-1}(\alpha_{\lambda,k})}{(\alpha_{\lambda,k} - \alpha_{\lambda,1}) \cdots (\alpha_{\lambda,k} - \alpha_{\lambda,k-1})} \\ |h_{\lambda,k}(z)| &\leq Ae^{D'p(z)} \quad \text{for all } z \in \mathbb{C}. \end{aligned}$$

Then the function defined by $H_{\lambda,k}(z) = H_{\lambda,k-1}(z) + h_{\lambda,k}(z)(z - \alpha_{\lambda,1}) \cdots (z - \alpha_{\lambda,k-1})$ verifies

$$H_{\lambda,k}(\alpha_{\lambda,1}) = \cdots = H_{\lambda,k}(\alpha_{\lambda,k}) = e^{Dp(\lambda)}.$$

Finally, we set $G_\lambda = H_{\lambda,n}$. ■

To proceed with the proof of the inclusion $X_p^{n-1}(\Lambda) \subset \mathcal{R}_\Lambda(\mathcal{A}_p(\mathbb{C}))$, let $\omega(\Lambda) \in X_p^{n-1}(\Lambda)$.

Fix $\lambda \in \mathcal{L}$ and let $\Lambda \cap D_\lambda = \{\alpha_1, \dots, \alpha_k\}$, $k \leq n$. We first consider a polynomial interpolating the values $\omega(\alpha_1), \dots, \omega(\alpha_k)$:

$$P_\lambda(z) = \Delta^0 \omega(\alpha_1) + \Delta^1 \omega(\alpha_1, \alpha_2)(z - \alpha_1) + \cdots + \Delta^{k-1} \omega(\alpha_1, \dots, \alpha_k) \prod_{j=1}^{k-1} (z - \alpha_j).$$

Notice that $P_\lambda \in \mathcal{A}_p$, since $\omega(\Lambda) \in X_p^{n-1}(\Lambda)$ and by properties (w1) and (w2) we have

$$|P_\lambda(z)| \leq A|z|^k e^{B[p(\alpha_1) + \cdots + p(\alpha_k)]} \leq Ae^{B'[p(z) + p(\lambda)]}.$$

Now, define

$$f = \sum_{\lambda \in \mathcal{L}} F_\lambda G_\lambda P_\lambda e^{-Dp(\lambda)},$$

where D is a large constant to be chosen later on.

By the preceding estimates on G_λ and P_λ , there exist constants $A, B > 0$ not depending on D and a constant $D'' > 0$ such that, for all $z \in \mathbb{C}$, we have

$$|f(z)| \leq Ae^{D''p(z)} \sum_{\lambda \in \mathcal{L}} e^{(B-D)p(\lambda)}.$$

In view of Lemma 1.2.12, choosing D large enough, the latter sum converges and $f \in \mathcal{A}_p(\mathbb{C})$.

To verify that f interpolates $\omega(\Lambda)$, let $\alpha \in \Lambda$ and let λ be the (unique) point of \mathcal{L} such that $\alpha \in D_\lambda$. Then, $f(\alpha) = G_\lambda(\alpha)P_\lambda(\alpha)e^{-Dp(\alpha)} = P_\lambda(\alpha) = \omega(\alpha)$, as desired.

2.2.5. Similar results in the disk. The previous definitions and proofs can be adapted to produce analogous results in the disk. To do so one just needs to replace the Euclidean distance used in \mathbb{C} by the pseudo-hyperbolic distance

$$\rho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right| \quad z, \zeta \in \mathbb{D},$$

and the Euclidean divided differences by their hyperbolic version

$$\begin{aligned} \delta^0 \omega(\lambda_1) &= \omega(\lambda_1), \\ \delta^j \omega(\lambda_1, \dots, \lambda_{j+1}) &= \frac{\Delta^{j-1} \omega(\lambda_2, \dots, \lambda_{j+1}) - \Delta^{j-1} \omega(\lambda_1, \dots, \lambda_j)}{\frac{\lambda_{j+1} - \lambda_1}{1 - \lambda_1 \lambda_{j+1}}} \quad j \geq 1. \end{aligned}$$

In this context a function $\phi : \mathbb{D} \longrightarrow \mathbb{R}_+$ is a *weight* if

(wd1) There is a constant $K > 0$ such that $\phi(z) \geq K \ln\left(\frac{1}{1-|z|}\right)$.

(wd2) There are constants $D_0 > 0$ and $E_0 > 0$ such that whenever $\rho(z, \zeta) \leq 1$ then

$$\phi(z) \leq D_0 \phi(\zeta) + E_0.$$

The model for the associated spaces

$$A_\phi = \{f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f(z)| e^{-B\phi(z)} < \infty \text{ for some } B > 0\},$$

is the Korenblum algebra $A^{-\infty}$, which corresponds to the choice $e^{-\phi(z)} = 1 - |z|$. The interpolating sequences for this and similar algebras have been characterised in [13] and [32].

With these elements, and replacing the factors $z - \alpha$ by $\frac{z - \alpha}{1 - \bar{\alpha}z}$ when necessary, we can follow the proofs above and, mutatis mutandis, show that Theorem 2.2.3 also holds in this situation.

The only point that requires further justification is the validity of Lemmas 1.2.6 and 1.2.12 for the weights ϕ . Lemma 1.2.6 is a standard consequence of the open mapping theorem for (LF)-spaces applied to the restriction map \mathcal{R}_Λ , and the same proof as in [2, Lemma 2.2.6] holds. Applying this Lemma to the sequences $\omega_\lambda(\Lambda)$ defined by

$$\omega_\lambda(\lambda') = \begin{cases} 1 & \text{if } \lambda' = \lambda \\ 0 & \text{if } \lambda' \neq \lambda \end{cases}$$

we have functions $f_\lambda \in A_\phi$ interpolating these values and with growth control independent of λ . Since $1 = |f_\lambda(\lambda) - f_\lambda(\lambda')|$, an estimate on the derivative of f_λ shows that for some $C > 0$ and $\varepsilon > 0$ the pseudohyperbolic disks $D_H(\lambda, \varepsilon e^{-C\phi(\lambda)}) = \{z \in \mathbb{D} : \rho(z, \lambda) < e^{-C\phi(\lambda)}\}$ are pairwise disjoint. In particular the sum of their areas is finite, hence

$$\sum_{\lambda \in \Lambda} (1 - |\lambda|)^2 e^{-2C\phi(\lambda)} < +\infty.$$

From this and condition (wd1) we finally obtain Lemma 1.2.6.

CHAPTER 3

Geometric conditions on interpolating varieties for radial weights

The results of this chapter are published in Journal of Geometric Analysis (See [40]).

Introduction

We are interested in finding a geometric description on an interpolating variety, depending only on the distribution of its points, which would enable us to decide whether it is \mathcal{A}_p -interpolating by a direct calculation.

The geometric conditions will be given in terms of $N_V(z, r)$, the integrated counting function of the points of V (see Definition 1.1.7).

When p is radial and doubling (see Definition 1.1.2), C.A. Berenstein and B.Q. Li [3] gave a geometric characterization of \mathcal{A}_p -interpolating varieties, namely,

- (i) $N(\alpha_j, |\alpha_j|) = O(p(\alpha_j))$ when $j \rightarrow \infty$;
- (ii) $N(0, r) = O(p(r))$ when $r \rightarrow \infty$.

A. Hartmann and X. Massaneda (see [21]) gave a proof of this theorem based on the L^2 -estimates for the solution to the $\bar{\partial}$ -equation, provided that

$$(9) \quad p(z) = O(|z|^2 \Delta p(z)).$$

Note that we can always regularize p into a smooth function (see Remark 3.0.12 below).

We will use the L^2 -techniques to give a proof of the same result without the assumption (9) (see Theorem 3.1.3). When the condition (9) is satisfied, we will prove that (ii) is no longer needed and that (i) is necessary and sufficient (see Theorem 3.1.4).

In [3], Berenstein and Li also studied rapidly growing radial weights, allowing infinite order functions in $\mathcal{A}_p(\mathbb{C})$, as $p(z) = e^{|z|}$, and more generally weights such that $\ln p(e^r)$ is convex. They gave sufficient conditions as well as necessary ones.

We will give a characterization of interpolating varieties for the weight $p(z) = e^{|z|}$ and more generally for weights p such that $p(z) = O(\Delta p(z))$ (see Theorem 3.1.6) and also for radial p when $\ln p(e^r)$ is convex and $\ln p(r)$ is concave for large r (see Theorem 3.1.7).

In particular, we will show that V is $\mathcal{A}_{e^{|z|}}$ -interpolating if and only if

$$N(\alpha_j, e) = O(e^{|\alpha_j|}), \text{ when } j \rightarrow \infty.$$

The difficult part in each case is the sufficiency. As in [9, 21], we will follow a Bombieri-Hörmander approach based on L^2 -estimates on the solution to the $\bar{\partial}$ -equation. The scheme will be the following: the condition on the density gives a smooth interpolating function F with a good growth such that the support of $\bar{\partial}F$ is far from the points $\{\alpha_j\}$ (see Lemma 3.2.1). Then we are led to solve the $\bar{\partial}$ -equation: $\bar{\partial}u = -\bar{\partial}F$ with L^2 -estimates. To do so, we need to construct

a subharmonic function U with a convenient growth and with prescribed singularities on the points α_j . Following Bombieri [11], the fact that e^{-U} is not summable near the points $\{\alpha_j\}$ forces u to vanish on the points α_j and we are done by defining the interpolating entire function by $u + F$.

The delicate point of the proof is the construction of the function U . It is done in two steps: first we construct a function U_0 behaving like $\ln |z - \alpha_j|^2$ near α_j with a good growth and with a control on ΔU_0 (the laplacian of U_0), thanks to the conditions on the density and the hypothesis on the weight itself. Then we add a function W such that ΔW is large enough so that $U = U_0 + W$ is subharmonic.

Throughout this chapter, we will assume that the weight p is radial.

REMARK 3.0.12. The weight p may be regularized as in [21, Remark 2.3] by replacing p by its average over the disc $D(z, 1)$. Thus we may suppose p to be of class \mathcal{C}^2 when needed.

We have collected the statements of the main results in Section 3.1. In Section 3.2, we show how to construct the smooth interpolating function F and how the problem reduces in constructing the subharmonic function U . Finally, in Section 3.3, we give the proofs of the main results and we show how to actually construct the function U .

3.1. Main results

We begin by giving a sufficient condition for a multiplicity variety V to be interpolating.

THEOREM 3.1.1. *If condition (4) holds and for some constants $A, B > 0$,*

$$(10) \quad \int_0^R n(0, t) dt \leq A p(R) + B, \quad \text{for all } R > 0,$$

then V is interpolating for $\mathcal{A}_p(\mathbb{C})$.

REMARK 3.1.2. In fact, we have

$$\int_0^R n(0, t) dt = \sum_{|\alpha_j| \leq R} m_j(R - |\alpha_j|) \leq RN(0, R).$$

Consequently, the necessary condition (1) implies that for some constants $A, B > 0$,

$$(11) \quad \int_0^R n(0, t) dt \leq ARp(R) + B, \quad \text{for all } R > 0.$$

Note that the necessary condition ((11) and (4)) differ from the sufficient condition ((10) and (4)) by a factor R in (11).

Adapting our method to the doubling case we find the characterization given by Berenstein and Li [3, Corollary 4.8]:

THEOREM 3.1.3. *Assume p to be radial and doubling.*

V is interpolating for $\mathcal{A}_p(\mathbb{C})$ if and only if conditions (1) and (6) hold.

The theorem holds if we replace $N(\alpha_j, |\alpha_j|)$ by $N(\alpha_j, C|\alpha_j|)$ for any constant $C > 0$. Note that radial and doubling weights satisfy $p(r) = O(r^\alpha)$ for some $\alpha > 0$. In other words, they have at most a polynomial growth. Examples of radial and doubling weights are $p(z) = |z|^\alpha (\ln(1 + |z|^2))^\beta$, $\alpha > 0, \beta \geq 0$, but for $p(z) = |z|^\alpha$, we may give a better result:

THEOREM 3.1.4. *Assume that $p(z) = O(|z|^2 \Delta p(z))$ and (w3) there exists constants $C_1, C_2 > 0$, such that $|z - w| \leq |z|$ implies $p(w) \leq C_1 p(z) + C_2$. Then V is interpolating for $\mathcal{A}_p(\mathbb{C})$ if and only if condition (6) holds.*

REMARK 3.1.5. (i) We don't need to assume that p is radial in this theorem.
(ii) It is easy to see that radial and doubling weights satisfy condition (w3).
(iii) Theorem 3.1.4 applies to $p(z) = |z|^\alpha$, $\alpha > 0$. For this weight and with the extra assumption that there is a function $f \in \mathcal{A}_p(\mathbb{C})$ such that $V \subset Z(f)$, it was shown in ([44, Theorem 3]) that condition (6) is sufficient and necessary.

Next we are interested in the case where p grows rapidly, allowing infinite order functions in $\mathcal{A}_p(\mathbb{C})$. A fundamental example is $p(z) = e^{|z|}$.

In [3], Berenstein and Li studied this weight and more generally those for which $\ln p(e^r)$ is convex. They gave sufficient conditions (Corollaries 5.6 and Corollary 5.7) as well as necessary ones (Theorem 5.14, Corollary 5.15).

The following result gives a characterization in particular for the weight $p(z) = e^{|z|}$.

THEOREM 3.1.6. *Assume that $p(z) = O(\Delta p(z))$. Then V is interpolating for $\mathcal{A}_p(\mathbb{C})$ if and only if condition (4) holds.*

The next theorem will give a characterization when p is radial, $q(r) = \ln p(e^r)$ is convex and $\frac{r}{q'(\ln r)} = \frac{p(r)}{p'(r)}$ is increasing (for large r). If we set $u(r) = \ln p(r)$, we have $\frac{p(r)}{p'(r)} = \frac{1}{u'(r)}$. Thus, the last condition means that $u(r)$ is concave for large r . We recall that the convexity of q implies that $p(r) \geq Ar + B$, for some $A, B > 0$ (see [3, Lemma 5.8]).

The weights $p(z) = |z|^\alpha$, $\alpha > 0$ and $p(z) = e^{|z|}$ satisfy these conditions. Examples of weights for which Theorem 3.1.7 applies but not any of Theorems 3.1.3, 3.1.4 or 3.1.6 are $p(z) = e^{|z|^\alpha}$, $0 < \alpha < 1$ and $p(z) = e^{[\log(1+|z|^2)]^\beta}$, $\beta > 1$.

THEOREM 3.1.7. *Assume that p is a radial weight and that for a certain $r_0 > 0$ it satisfies the following properties*

- $q(r) := \ln p(e^r)$ is convex on $[\ln r_0, \infty[$;
- $q'(\ln r_0) > 0$ and $\frac{r}{q'(\ln r)}$ is increasing on $[r_0, \infty[$.

Then V is interpolating for $\mathcal{A}_p(\mathbb{C})$ if and only if the following condition holds:

$$(12) \quad \exists A > 0, \exists B > 0, \quad \forall |\alpha_j| \geq r_0, N(\alpha_j, \max(\frac{|\alpha_j|}{q'(\ln |\alpha_j|)}, e)) \leq Ap(\alpha_j) + B.$$

The theorem holds if we replace $\frac{|\alpha_j|}{q'(\ln |\alpha_j|)}$ by $\frac{C|\alpha_j|}{q'(\ln |\alpha_j|)}$ for any constant $C > 0$.

When $p(z) = |z|^\alpha$, conditions (12) and (6) are the same and when $p(z) = e^{|z|}$, conditions (12) and (4) are the same.

Specializing Theorem 3.1.7, we get the following corollaries.

COROLLARY 3.1.8. *Let $p(z) = e^{|z|^\alpha}$, $0 < \alpha < 1$. V is interpolating for $\mathcal{A}_p(\mathbb{C})$ if and only if the following condition holds:*

$$(13) \quad \exists A > 0, \exists B > 0, \quad \forall j, \quad N(\alpha_j, |\alpha_j|^{1-\alpha}) \leq Ap(\alpha_j) + B.$$

COROLLARY 3.1.9. *Let $p(z) = e^{[\ln(1+|z|^2)]^\beta}$, $\beta \geq 1$. V is interpolating for $\mathcal{A}_p(\mathbb{C})$ if and only if the following condition holds:*

$$(14) \quad \exists A > 0, \exists B > 0, \quad \forall j, \quad N(\alpha_j, |\alpha_j|[\ln(1+|\alpha_j|^2)]^{1-\beta}) \leq Ap(\alpha_j) + B.$$

3.2. Preliminary results

We first construct a smooth interpolating function with the right growth.

LEMMA 3.2.1. *Suppose $V = \{(\alpha_j, m_j)\}_j$ is weakly separated. Given $W = \{w_{j,l}\}_{j \in \mathbb{N}, 0 \leq l \leq m_j-1} \in \mathcal{A}_p(V)$, there exists a smooth function F such that*

- (i) *for some constants $A > 0$ and $B > 0$, $|F(z)| \leq Ae^{Bp(z)}$, $|\bar{\partial}F(z)| \leq Ae^{Bp(z)}$ for all $z \in \mathbb{C}$;*
- (ii) *The support of $\bar{\partial}F$ is contained in the union of the annuli*

$$A_j = \{z \in \mathbb{C} : \frac{\delta_j}{2} \leq |z - \alpha_j| \leq \delta_j\};$$

- (iii) $\frac{F^{(l)}(\alpha_j)}{l!} = w_{j,l}$ for all $j \in \mathbb{N}$, $0 \leq l \leq m_j - 1$.

A suitable function F is of the form

$$F(z) = \sum_j \mathcal{X}\left(\frac{|z - \alpha_j|^2}{\delta_j^2}\right) \sum_{l=0}^{m_j-1} w_{j,l} (z - \alpha_j)^l,$$

where \mathcal{X} is a smooth cut-off function with $\mathcal{X}(x) = 1$ if $|x| \leq 1/4$ and $\mathcal{X}(x) = 0$ if $|x| \geq 1$. See [21] for details of the proof.

Now, when looking for a holomorphic interpolating function of the form $f = F + u$, we are led to the $\bar{\partial}$ -problem

$$\bar{\partial}u = -\bar{\partial}F.$$

The interpolation problem is then reduced to the following:

LEMMA 3.2.2. *If $V = \{(\alpha_j, m_j)\}_j$ is weakly separated and if there exists a subharmonic function U satisfying for certain constants $A, B > 0$,*

- (i) $U(z) \leq Ap(z) + B$ for all $z \in \mathbb{C}$;
- (ii) $-U(z) \leq Ap(z) + B$ for z in the support of $\bar{\partial}F$;
- (iii) $U(z) \simeq m_j \ln |z - \alpha_j|^2$ near α_j ,

then V is \mathcal{A}_p -interpolating.

PROOF. Let $W = \{w_{j,l}\}_{j \in \mathbb{N}, 0 \leq l \leq m_j-1}$ be in $\mathcal{A}_p(V)$. As V is weakly separated, there exists a smooth interpolating function F satisfying the properties in Lemma 3.2.1. Using properties (ii) in Lemma 3.2.2, (ii) in Lemma 3.2.1 and (w1), we see that there exists a constant $E > 0$ such that

$$\int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-U(z)-Ep(z)} d\lambda(z) < \infty.$$

Hörmander's theorem 1.2.16 gives a \mathcal{C}^∞ function u such that $\bar{\partial} u = -\bar{\partial} F$ and

$$(15) \quad \int_{\mathbb{C}} \frac{|u(z)|^2 e^{-U(z)-Ep(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \frac{1}{2} \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-U(z)-Ep(z)} d\lambda(z).$$

Setting $f = u + F$, it is clear that $f \in \mathcal{A}(\mathbb{C})$. Moreover, by condition (iii) in Lemma 3.2.2, near α_j , $e^{-U(z)}(z - \alpha_j)^l$ is not summable for $0 \leq l \leq m_j - 1$, so we have necessarily $u^{(l)}(\alpha_j) = 0$ for all j and $0 \leq l \leq m_j - 1$ and consequently $f := u + F$ is $\frac{f^{(l)}(\alpha_j)}{l!} = w_{j,l}$.

In view of property (iii) in Lemma 1.1.4, it only remains to show that $f \in \mathcal{W}_p(\mathbb{C})$. It suffices to show that $F \in \mathcal{W}_p(\mathbb{C})$ and $u \in \mathcal{W}_p(\mathbb{C})$.

Condition (ii) in Lemma 3.2.1 gives constants $A, B > 0$ such that, for all $z \in \mathbb{C}$, $|F(z)| \leq Ae^{Bp(z)}$. By condition (w1), choosing a constant $C > 0$ large enough, we obtain

$$\int_{\mathbb{C}} |F(z)|^2 e^{-Cp(z)} d\lambda(z) < \infty,$$

in other words, $F \in \mathcal{W}_p(\mathbb{C})$.

By properties (i) in Lemma 3.2.2 and (w1), there exist a constant $D > 0$ such that, for all $z \in \mathbb{C}$,

$$e^{-Dp(z)} \lesssim \frac{e^{-U(z)}}{(1+|z|^2)^2}.$$

Therefore,

$$\int_{\mathbb{C}} |u(z)|^2 e^{-(D+E)p(z)} d\lambda(z) \lesssim \int_{\mathbb{C}} \frac{|u(z)|^2 e^{-U(z)-Ep(z)}}{(1+|z|^2)^2} d\lambda(z) < \infty.$$

This shows that $u \in \mathcal{W}_p(\mathbb{C})$ and concludes the proof. ■

3.3. Proofs of the main theorems

We will use a smooth cut-off function \mathcal{X} with $\mathcal{X}(x) = 1$ if $|x| \leq 1/4$ and $\mathcal{X}(x) = 0$ if $|x| \geq 1$.

REMARK 3.3.1. In the proofs of the sufficiency part, we may need to assume that for all j , we have $|\alpha_j| \geq a$ for a suitable $a > 0$. This will be done without loss of generality up to a linear transform and in view of property (b) of the weight.

PROOF OF THEOREM 3.1.1.

By Lemma 1.2.13, condition (4) implies the weak separation. So we are done if we construct a function U satisfying the conditions of Lemma 3.2.2.

Set $\mathcal{X}_j(z) = \mathcal{X}(|z - \alpha_j|^2)$.

In order to construct the desired function we begin by defining

$$U_0(z) = \sum_j m_j \mathcal{X}_j(z) \ln |z - \alpha_j|^2.$$

Note that there is locally a finite number of non vanishing terms in the sum and that each term (and consequently U_0) is nonpositive. It is also clear that $U_0(z) - m_j \ln |z - \alpha_j|^2$ is continuous near α_j .

We want to estimate $-U_0$ on the support of $\bar{\partial}F$, and the “lack of subharmonicity” of U_0 , then we will add a correcting term to obtain the function U of the lemma.

Suppose z is in the support of $\bar{\partial}F$. We want to show that $-U_0(z) \lesssim p(z)$. Let k be the unique integer such that $\frac{\delta_k}{2} \leq |z - \alpha_k| \leq \delta_k$. Then

$$-U_0(z) \leq 2 \sum_{|z - \alpha_j| \leq 1} m_j \ln \frac{1}{|z - \alpha_j|} = 2m_k \ln \frac{1}{|z - \alpha_k|} + 2 \sum_{j \neq k, |z - \alpha_j| \leq 1} m_j \ln \frac{1}{|z - \alpha_j|}.$$

Using that $|z - \alpha_k| \geq \frac{\delta_k}{2}$ and that for $j \neq k$ we have

$$|\alpha_k - \alpha_j| \leq |z - \alpha_j| + |z - \alpha_k| \leq 2|z - \alpha_j|,$$

we obtain that

$$(16) \quad -U_0(z) \leq 2 \ln \frac{1}{\delta_k^{m_k}} + 2N(\alpha_k, 2) \lesssim p(\alpha_k) \lesssim p(z).$$

The last inequalities follows from condition (4), the weak separation (7) and property (b) of the weight p .

Now we want to get a lower bound on $\Delta U_0(z)$. We have

$$\begin{aligned} \Delta U_0(z) &= \sum_j m_j \mathcal{X}_j(z) \Delta \ln |z - \alpha_j|^2 \\ &\quad + 8 \operatorname{Re} \left(\sum_j m_j \bar{\partial} \mathcal{X}_j(z) \partial \ln |z - \alpha_j|^2 \right) + 4 \sum_j m_j \partial \bar{\partial} \mathcal{X}_j(z) \ln |z - \alpha_j|^2. \end{aligned}$$

The first sum is a positive measure and on the supports of $\bar{\partial} \mathcal{X}_j$ and $\partial \bar{\partial} \mathcal{X}_j$, we see that $1/2 \leq |z - \alpha_j| \leq 1$. Consequently, for a certain constant $\gamma > 0$ we have

$$\Delta U_0(z) \geq -\gamma(n(z, 1) - n(z, 1/2)) \geq -\gamma n(z, 1) \geq -\gamma(n(0, |z| + 1) - n(0, |z| - 1)).$$

We set $n(0, t) = 0$ if $t < 0$,

$$f(t) = \int_{t-1}^{t+1} n(0, s) ds, \quad g(t) = \int_0^t f(s) ds \text{ and } W(z) = g(|z|).$$

Let us compute the Laplacian of W , taking the derivatives in the sense of distributions

$$\Delta W(z) = \frac{1}{|z|} g'(|z|) + g''(|z|) \geq g''(|z|) = f'(|z|) = n(0, |z| + 1) - n(0, |z| - 1).$$

The function U defined by

$$U(z) = U_0(z) + \gamma W(z)$$

is then clearly subharmonic. On the other hand, using condition (10) and (b) we have the following inequalities:

$$f(s) \leq 2n(0, s+1), \quad W(z) = g(|z|) \leq 2 \int_1^{|z|+1} n(0, s) ds \lesssim p(z).$$

Since $U_0 \leq 0$ it is clear that U satisfies condition (i) of Lemma 3.2.2. Using the estimate (16) and the fact that W is nonnegative we see that U satisfies condition (ii). Finally as condition (iii) is also already fulfilled by U_0 and W is continuous, it is also fulfilled by U . \blacksquare

PROOF OF THEOREM 3.1.3.

Necessity. In view of Remark 3.1.5 and Remark 1.2.8, we apply Theorem 1.2.7 with $R_j = |\alpha_j|$ and we readily obtain the necessity of (6). Condition (1) is necessary by Corollary 1.2.4.

Sufficiency. By Lemma 1.2.13, condition (6) implies the weak separation. We will proceed as in Theorem 3.1.1, constructing a function U satisfying (i), (ii) and (iii) from Lemma 3.2.2. Thanks to the doubling condition, we can control the weight p in discs $D(\alpha_j, |\alpha_j|)$ instead of just $D(\alpha_j, e)$ in the general case. We will construct U_0 as in the previous theorem, except that we now take \mathcal{X}_j 's with supports of radius $\simeq |\alpha_j|$:

Set

$$\mathcal{X}_j(z) = \mathcal{X}\left(\frac{16|z - \alpha_j|^2}{|\alpha_j|^2}\right),$$

and introduce the negative function

$$U_0(z) = \sum_j m_j \mathcal{X}_j(z) \ln \frac{16|z - \alpha_j|^2}{|\alpha_j|^2}.$$

When z is in the support of $\bar{\partial}F$, let k be the unique integer such that $\frac{\delta_k}{2} \leq |z - \alpha_k| \leq \delta_k$. Repeating the estimate on $-U_0(z)$, we have

$$-U_0(z) \leq 2 \sum_{|z - \alpha_j| \leq \frac{|\alpha_j|}{4}} m_j \ln \frac{|\alpha_j|}{4|z - \alpha_j|} \leq 2m_k \ln \frac{|\alpha_k|}{\delta_k} + 2 \sum_{0 < |\alpha_k - \alpha_j| \leq \frac{|\alpha_j|}{2}} m_j \ln \frac{|\alpha_j|}{2|\alpha_k - \alpha_j|}.$$

We have $\frac{|\alpha_j|}{2} \leq |\alpha_k|$ whenever $|\alpha_k - \alpha_j| \leq \frac{|\alpha_j|}{2}$. We deduce the inequalities

$$(17) \quad -U_0(z) \leq 2 \ln \frac{1}{\delta_k^{m_k}} + 2N(\alpha_k, |\alpha_k|) \lesssim p(\alpha_k) \lesssim p(z).$$

Again, the last inequalities follow from condition (4), the weak separation (7) and property (b) of the weight p .

We estimate $\Delta U_0(z)$ as before except that now $|\bar{\partial}\mathcal{X}_j(z)| \lesssim \frac{1}{|\alpha_j|}$ and $|\partial\bar{\partial}\mathcal{X}_j(z)| \lesssim \frac{1}{|\alpha_j|^2}$. On the support of these derivatives, $\frac{|\alpha_j|}{8} \leq |z - \alpha_j| \leq \frac{|\alpha_j|}{4}$ and $\frac{|z|}{2} \leq |\alpha_j| \leq 2|z|$. We deduce that

$$\Delta U_0(z) \gtrsim -\frac{n(0, 2|z|) - n(0, \frac{|z|}{2})}{|z|^2}.$$

To construct the correcting term W set

$$f(t) = \int_0^t n(0, s)ds, \quad g(t) = \int_0^t \frac{f(s)}{s^2}ds \text{ and } W(z) = g(2|z|).$$

Finally, to estimate the Laplacian of W , we write $t = 2|z|$. We have

$$\Delta W(z) = 4 \left(\frac{1}{t} g'(t) + g''(t) \right) = \frac{4}{t^2} \left(f'(t) - \frac{f(t)}{t} \right)$$

and

$$f(t) = \int_0^t n(0, s)ds = \int_0^{\frac{t}{4}} n(0, s)ds + \int_{\frac{t}{4}}^t n(0, s)ds \leq \frac{t}{4} n\left(0, \frac{t}{4}\right) + t \left(1 - \frac{1}{4}\right) n(0, t).$$

Thus,

$$f'(t) - \frac{f(t)}{t} = n(0, t) - \frac{f(t)}{t} \geq \frac{1}{4} \left(n(0, t) - n\left(0, \frac{t}{4}\right) \right)$$

and

$$\Delta W(z) \gtrsim \frac{n(0, 2|z|) - n(0, \frac{|z|}{2})}{|z|^2}.$$

Now, the desired function will be of the form

$$U(z) = U_0(z) + \gamma W(z),$$

where γ is a positive constant sufficiently large. The following inequalities are easy to see:

$$f(t) \leq tn(0, t), \quad g(t) \leq \int_0^t \frac{n(0, s)}{s}ds = N(0, s).$$

Thus, by condition (1) and the doubling condition,

$$0 \leq W(z) \leq N(0, 2|z|) \lesssim p(2z) \lesssim p(z).$$

We conclude that U satisfies all the desired conditions. ■

PROOF OF THEOREM 3.1.4.

Necessity. Recalling Remark 1.2.8, we apply once again Theorem 1.2.7 to deduce the necessity of condition (6).

Sufficiency. The proof is the same as for Theorem 3.1.3, we only change the estimate on ΔU_0 and the correcting term W . Let us have a new look at $\Delta U_0(z)$.

$$(18) \quad \Delta U_0(z) \gtrsim - \sum_{|z - \alpha_j| \leq \frac{|\alpha_j|}{4}} \frac{m_j}{|z|^2}.$$

If the sum is not empty, let α_k be the point appearing in the sum with the largest modulus. For all α_j such that $|z - \alpha_j| \leq \frac{|\alpha_k|}{4}$, we have

$$|\alpha_j - \alpha_k| \leq |z - \alpha_k| + |z - \alpha_j| \leq \frac{|\alpha_k|}{4} + \frac{|\alpha_j|}{4} \leq \frac{|\alpha_k|}{2}.$$

We deduce that

$$\Delta U_0(z) \gtrsim -\frac{n(\alpha_k, \frac{|\alpha_k|}{2})}{|z|^2}.$$

Besides,

$$n(\alpha_k, \frac{|\alpha_k|}{2}) \leq m_k + \frac{1}{\ln 2} \sum_{0 < |\alpha_j - \alpha_k| \leq \frac{|\alpha_k|}{2}} m_j \ln \frac{|\alpha_k|}{|\alpha_j - \alpha_k|} \lesssim N(\alpha_k, |\alpha_k|) \lesssim p(\alpha_k).$$

Note that $|z - \alpha_k| \leq \frac{|\alpha_k|}{4}$ implies that $|z - \alpha_k| \leq |z|$. Thus by condition (w3) we have $p(\alpha_k) \lesssim p(z)$. Finally we get

$$\Delta U_0(z) \gtrsim -\frac{p(z)}{|z|^2} \gtrsim -\Delta p(z).$$

Then we take

$$U(z) = U_0(z) + \gamma p(z)$$

where γ is a positive constant chosen large enough. ■

PROOF OF THEOREM 3.1.6. We already know by Theorem 1.2.7 that condition (4) is necessary.

Let us consider the function U_0 that we constructed in the proof of Theorem 3.1.1. Again, we only change the estimate on ΔU_0 and the correcting term W . We find

$$(19) \quad \Delta U_0(z) \gtrsim -n(z, 1).$$

If $n(z, 1) \neq 0$, let α_k be in $D(\alpha_k, 1)$. Then

$$n(z, 1) \leq n(\alpha_k, 2) \leq m_k + \frac{1}{1 - \ln 2} \sum_{0 < |\alpha_k - \alpha_j| < 2} m_j \ln \frac{e}{|\alpha_k - \alpha_j|} \lesssim N(\alpha_k, e) \lesssim p(\alpha_k) \lesssim p(z).$$

The function

$$U(z) = U_0(z) + \gamma p(z),$$

with $\gamma > 0$ large enough has the desired properties. ■

PROOF OF THEOREM 3.1.7.

We set $c := \inf(q'(\ln r_0), 1)$ and $\psi(r) = \frac{r}{q'(\ln r)}$.

CLAIM 3.3.2.

(i) *Let $r \geq 2r_0$. Then $c\psi(r) \leq r$ and*

$$|x| \leq c \frac{\psi(r)}{2} \text{ implies that } \frac{\psi(r)}{2} \leq \psi(r+x) \leq 2\psi(r);$$

(ii) For all $r \geq r_0$,

$$p(r + \psi(r)) \leq ep(r).$$

Assuming this claim true for the moment, let us proceed with the proof of Theorem 3.1.7. *Necessity.* In view of Theorem 1.2.7, it suffices to show that $R_j = \psi(|\alpha_j|)$ satisfy condition (2). Let $|\alpha_j| \geq r_0$ and w be such that $|w - \alpha_j| \leq R_j$. Thus $|w| \leq |\alpha_j| + \psi(|\alpha_j|)$ and as a consequence of (ii) of the claim, we obtain

$$p(w) \leq p(|\alpha_j| + \psi(|\alpha_j|)) \leq ep(|\alpha_j|).$$

Sufficiency. We may assume that $|\alpha_j| \geq r_0$ for all j (see Remark 3.3.1). We apply Lemma 1.2.13 to deduce that V is weakly separated. We repeat the proof of Theorems 3.1.4 and 3.1.3, replacing $|\alpha_j|$ by $c\psi(|\alpha_j|)$. More precisely, we set

$$\mathcal{X}_j(z) = \mathcal{X}\left(\frac{16|z - \alpha_j|^2}{c^2\psi(|\alpha_j|)^2}\right)$$

and we define the negative function

$$U_0(z) = \sum_j m_j \mathcal{X}_j(z) \ln \frac{16|z - \alpha_j|^2}{c^2\psi(|\alpha_j|)^2}.$$

We use (i) of Claim 3.3.2 to obtain that whenever $|z - \alpha_j| \leq \frac{c\psi(|\alpha_j|)}{2}$, we have

$$\begin{aligned} \psi(|z|) &\leq \psi(|\alpha_j| + |z - \alpha_j|) \leq 2\psi(|\alpha_j|); \\ \frac{\psi(|\alpha_j|)}{2} &\leq \psi(|\alpha_j| - |z - \alpha_j|) \leq \psi(|z|). \end{aligned}$$

Adapting the inequality (17) when $\frac{\delta_k}{2} \leq |z - \alpha_k| \leq \delta_k$ and applying condition (12) we find

$$-U_0(z) \leq 2 \ln \frac{1}{\delta_k^{m_k}} + 2N(\alpha_k, c\psi(|\alpha_k|)) \lesssim p(\alpha_k) \lesssim p(z).$$

Let us now find a lower bound for ΔU_0 on \mathbb{C} . By analogy to (18) and the inequalities following we obtain

$$\Delta U_0(z) \gtrsim - \sum_{|z - \alpha_j| \leq \psi(|\alpha_j|)c/4} \frac{1}{\psi(|\alpha_j|)^2} \gtrsim - \frac{N(\alpha_l, c\psi(|\alpha_l|))}{(\psi(|z|))^2}$$

where α_l is one of the points appearing in the sum with the largest modulus. Recall that

$$|z - \alpha_l| \leq \psi(|\alpha_l|)c/4 \leq \psi(|z|)c/2 \leq \psi(|z|)$$

and consequently from (ii) of Claim 3.3.2 we deduce that

$$p(\alpha_l) \leq p(|z| + |z - \alpha_l|) \leq p(|z| + \psi(|z|)) \leq ep(z).$$

Finally, we apply condition (12)

$$\Delta U_0(z) \gtrsim - \frac{p(z)}{\psi(|z|)^2}.$$

Let us compute the Laplacian of $p(z) = e^{q(\ln |z|)}$ in terms of the convex function q . Setting $r = |z|$,

$$\Delta p(z) = \frac{p'(r)}{r} + p''(r) = \frac{[q'(\ln r)]^2}{r^2} p(r) + \frac{q''(\ln r)}{r^2} p(r) \geq \frac{[q'(\ln r)]^2}{r^2} p(r) = \frac{p(r)}{[\psi(r)]^2}.$$

We readily deduce that

$$\Delta U_0(z) \geq -\gamma \Delta p(z)$$

for some $\gamma > 0$. As in the preceding proofs, the function $U(z) = U_0(z) + \gamma p(z)$ satisfies the properties stated in Lemma 3.2.2. ■

PROOF OF CLAIM 3.3.2. Let $r > r_0$. A computation gives

$$\psi'(r) = \frac{q'(\ln(r)) - q''(\ln r)}{[q'(\ln r)]^2}.$$

Recall that q'' is nonnegative. Besides as q' is increasing we have $q'(\ln r) \geq q'(\ln r_0) \geq c$. We deduce that $0 \leq \psi'(r) \leq \frac{1}{c}$. Note that we also have the inequality $c\psi(r) \leq r$.

Assume now $r \geq 2r_0$ and let $|x| \leq \frac{c\psi(r)}{2}$. Then

$$|r + x| \geq r - \frac{c\psi(r)}{2} \geq \frac{r}{2} \geq r_0.$$

We can now use the mean value theorem and the preceding estimates to write

$$|\psi(r + x) - \psi(r)| \leq \frac{|x|}{c} \leq \frac{\psi(r)}{2}.$$

We easily deduce (i).

To prove (ii), put $u(t) = q(\ln t) = \ln p(t)$. We have $u'(t) = \frac{1}{\psi(t)} \leq \frac{1}{\psi(r)}$ for all $t \geq r \geq r_0$ because ψ is increasing on $[r_0, \infty[$. Applying the mean value theorem again, we obtain $u(r + \psi(r)) - u(r) \leq 1$. We deduce that

$$p(r + \psi(r)) = e^{u(r + \psi(r)) - u(r)} p(r) \leq e p(r). \quad \blacksquare$$

CHAPTER 4

A geometric characterisation of interpolating varieties for Beurling weights

The results of this chapter are joint work with X. Massaneda and J. Ortega-Cerdà. They were published in Transactions of the American Mathematical Society ([34]).

Introduction

Let $\mathcal{E}(\mathbb{R})$ be the space of smooth functions in \mathbb{R} and let $\mathcal{E}'(\mathbb{R})$ be its dual, the space of distributions with compact support on \mathbb{R} . It is well known that the space $\hat{\mathcal{E}}'(\mathbb{R})$ of Fourier transforms of distributions in $\mathcal{E}'(\mathbb{R})$ coincides with the algebra of entire functions f such that

$$|f(z)| \leq C(1 + |z|)^A e^{B|\operatorname{Im} z|},$$

where $A, B, C > 0$ may depend on f (see [2, Theorem 1.4.15]).

The origin of the interest in $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolation lies in its relationship with convolution equations and, in particular, with the density of exponential families $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ in the space of solutions $g \in \mathcal{E}(\mathbb{R})$ of equations of type $\mu \star g = 0$, $\mu \in \mathcal{E}'(\mathbb{R})$. Any solution g to the convolution equation is the limit of linear combinations of $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ where Λ is the zero set of $\hat{\mu}$. If moreover the sequence Λ is $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating then the series that represents g enjoys better convergence properties. For more on this relationship see [18] or [2, Chapter 6] (in particular Theorem 6.1.11).

For the weight $p(z) = |\operatorname{Im} z| + \ln(1 + |z|^2)$, Ehrenpreis and Malliavin gave a necessary geometric condition which turns out to be sufficient provided that Λ is a zero sequence of a slowly decreasing function (see [18, Theorem 4]). Later Squires, probably unaware of Ehrenpreis and Malliavin's result (which was stated in terms of solutions to convolution equations), proved the same result [45, Theorem 2].

In this chapter, we are giving a geometric characterization for $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating sequences (Theorem 4.1.2). The characterization shows in particular that the geometric condition given by Ehrenpreis & Malliavin and Squires is also sufficient whenever the sequence is contained in the region

$$|\operatorname{Im} z| \leq C \log(1 + |z|^2).$$

In general, however, their condition alone is not sufficient.

A similar characterization is obtained for the more general Beurling weights. These weights appear naturally in the context of convolution equations when one replaces distributions with compact support with Beurling-Björck ultradistributions of compact support (see [10]). They are not necessarily subharmonic, but we will prove that they are equivalent to a subharmonic weight (see Lemma 4.3.2).

This chapter is organised as follows. In Section 4.1 we give the definition and some properties of Beurling weights then we state our main result. In Section 4.2 we prove that the geometric

conditions of Theorem 4.1.2 are necessary, while in Section 4.3 we show that they are also sufficient.

4.1. Main result

DEFINITION 4.1.1. A Beurling weight is a function

$$p(z) = |\operatorname{Im} z| + \omega(|z|),$$

where $\omega(t)$ is a subadditive increasing continuous function, normalized with $\omega(0) = 0$ and such that:

$$(c) \log(1+t) \lesssim \omega(t) \text{ for } t > 1.$$

$$(d) \int_0^\infty \frac{\omega(t)}{1+t^2} dt < \infty.$$

Canonical examples of such weights are given by $\omega(t) = \log(1+t^2)$ and $\omega(t) = t^\gamma$, $\gamma \in (0, 1)$.

Beurling weights satisfy the following additional properties:

- (e) For every $c > 0$ there exists $C > 0$ such that $p(\zeta) \leq Cp(z)$ if $\zeta \in D(z, cp(z))$.
- (f) For $\varepsilon > 0$ small enough, there exists $C(\varepsilon) > 0$ such that if $z \in D(\zeta, \varepsilon p(\zeta))$ then $p(\zeta) \leq C(\varepsilon)p(z)$. Also, $C(\varepsilon)$ tends to 1 as ε goes to 0.
- (g) For $x \in \mathbb{R}^+$ big enough, the function $\omega(x)$ does not oscillate too much. More precisely, for fixed $C > 0$, if $y \in (x - C\omega(x), x + C\omega(x))$ then $1/2 \leq \omega(y)/\omega(x) \leq 2$ for x big enough.

Properties (e) and (f) follow easily from the subadditivity of ω . Property (g) follows from the subadditivity and the fact that $\omega(x) = o(|x|/\log|x|)$ (see [10, Lemma 1.2.8]): for any $y \in (x - C\omega(x), x + C\omega(x))$

$$\begin{aligned} \omega(x - C\omega(x)) &\leq \omega(y) \leq \omega(x + C\omega(x)) \leq \omega(x - C\omega(x)) + \omega(2C\omega(x)) \leq \\ &\leq \omega(x - C\omega(x)) + \omega(2Cx/\log x) \leq 2\omega(x - C\omega(x)). \end{aligned}$$

We are ready to state our main result. For convenience, a multiplicity variety will be denoted by $V = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ where Λ is a discrete sequence of complex numbers.

THEOREM 4.1.2. A multiplicity variety $V = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is \mathcal{A}_p -interpolating if and only if:

(i) There is $C > 0$ such that

$$(20) \quad N_V(\lambda, p(\lambda)) \leq Cp(\lambda) \quad \forall \lambda \in \Lambda.$$

(ii) The following Carleson-type condition holds

$$(21) \quad \sup_{x \in \mathbb{R}} \sum_{\lambda \in \Lambda, |\operatorname{Im} \lambda| > \omega(|\lambda|)} m_\lambda \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2} < \infty.$$

Since the Poisson kernel at λ in the corresponding half-plane (upper half-plane if $\operatorname{Im} \lambda > 0$ and lower half-plane when $\operatorname{Im} \lambda < 0$) is $P(\lambda, x) = \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2}$, a restatement of condition (ii) is that the measure $\sum_{\lambda: |\operatorname{Im} \lambda| > \omega(|\lambda|)} m_\lambda \delta_\lambda$ has bounded Poisson balayage.

REMARK 4.1.3. Notice that for sequences Λ within the region $|\operatorname{Im} z| \leq \omega(|z|)$, condition (20) (shown to be necessary by Ehrenpreis & Malliavin and Squires) provides a complete characterization. However, this is not the case in general, i.e. condition (21) does not follow from (20), as it is shown in the following example. Take the sequence Λ contained in the sector $\mathcal{A} = \{z \in \mathbb{C}; |\operatorname{Re} z| < \operatorname{Im} z\}$ and having in each segment $\{\operatorname{Im} z = 2^n\} \cap \mathcal{A}$ exactly 2^n equispaced points. Then Λ satisfies condition (20) (basically $n(\lambda, t) \leq t$ for $t \leq p(\lambda)$) for all $\lambda \in \Lambda$, but it does not satisfy (21) (it is not even a Blaschke sequence).

4.2. Necessary conditions

Let $V = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ be a multiplicity variety.

PROPOSITION 4.2.1. *If V is \mathcal{A}_p -interpolating, then condition (20) holds.*

PROOF. Property (e) of the weight shows that we may consider $R_j = p(\alpha_j)$ in Theorem 1.2.7. Thus this is a rephrasing of condition (3) in Theorem 1.2.7. \blacksquare

The necessity of condition (21) is an immediate consequence of the following proposition. Assume that $\Lambda \cap \mathbb{R} = \emptyset$; otherwise move the horizontal line so that it does not touch any of the points in Λ . Let \mathbb{H} denote the upper half-plane.

PROPOSITION 4.2.2. *If V is \mathcal{A}_p -interpolating, then there exists $C > 0$ such that*

$$\sum_{\substack{\lambda' \in \Lambda \cap \mathbb{H} \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1} \leq Cp(\lambda) \quad \text{for all } \lambda \in \Lambda \cap \mathbb{H}.$$

Of course an analogous result could be given for any upper ($\{z : \operatorname{Im} z > a\}$) or lower ($\{z : \operatorname{Im} z < a\}$) half plane.

PROOF. Let $z = x + iy$ and consider the Poisson transform of $\omega(|t|)$:

$$u(z) := P[\omega](z) = \int_{\mathbb{R}} \frac{y \omega(|t|)}{(x-t)^2 + y^2} dt,$$

which converges by (d). Define $H = \exp(u + i\tilde{u})$, where \tilde{u} is a harmonic conjugate of u .

By Lemma 1.2.6, there exist a sequence of functions $\{f_\lambda\}_{\lambda \in \Lambda}$ such that

- (i) $f_\lambda^{(l)}(\lambda') = 0$ for all $\lambda' \in \Lambda \setminus \{\lambda\}$ and all $0 \leq l < m'_\lambda$, except $f_\lambda(\lambda) = 1$.
- (ii) For all $z \in \mathbb{C}$, $|f_\lambda(z)| \leq Ae^{Bp(z)}$ where $A, B > 0$ don't depend on λ .

Given $\lambda \in \Lambda \cap \mathbb{H}$, define

$$h_\lambda(z) = \frac{f_\lambda(z)e^{iM_1z}}{(H(z))^{M_2}},$$

with M_1, M_2 to be chosen. It is clear that h_λ is holomorphic in \mathbb{H} . On the other hand, for all z in the upper half plane $|\log |H(z)| - \omega(|\operatorname{Re} z|)| \leq A + B|\operatorname{Im} z|$, see [10, Lemma 1.3.11]. Moreover $|\omega(|\operatorname{Re} z|) - \omega(|z|)| \leq \omega(|\operatorname{Im} z|) \leq A + B|\operatorname{Im} z|$, thus $|\log |H(z)| - \omega(|z|)| \leq A + B|\operatorname{Im} z|$. Therefore, if M_1 and M_2 are big enough, h_λ is bounded in \mathbb{H} by a constant which does not depend on λ :

$$|h_\lambda(z)| \leq Ce^{Mp(z) - M_1 \operatorname{Im} z - M_2 \log |H(z)|} \lesssim 1.$$

Also,

$$|h_\lambda(\lambda)| = e^{-M_1 \operatorname{Im} \lambda - M_2 \log |H(\lambda)|} \geq e^{-Cp(\lambda)}.$$

Now apply Jensen's Formula in the half-plane to the function h_λ :

$$\log |h_\lambda(\lambda)| = \int_{\mathbb{R}} P(\lambda, x) \log |h_\lambda(x)| dx - \int_{\mathbb{H}} G(\lambda, \zeta) \Delta \log |h_\lambda(\zeta)|,$$

where $P(\lambda, x)$ denotes the Poisson kernel and $G(\lambda, \zeta) = \log \left| \frac{\lambda - \zeta}{\lambda - \bar{\zeta}} \right|^{-1}$ is the Green function in \mathbb{H} with pole in λ .

Since h_λ vanishes on $\Lambda \setminus \{\lambda\}$, Jensen's Formula and the estimates above yield

$$\sum_{\substack{\lambda' \in \Lambda \cap \mathbb{H} \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1} \leq \sup_{\mathbb{R}} \log |h_\lambda| - \log |h_\lambda(\lambda)| \lesssim p(\lambda).$$

■

■

REMARK 4.2.3. The necessary condition of Proposition 4.2.2 can be seen as a Carleson type condition; it can be rewritten as

$$|B_\lambda(\lambda)| \geq \delta e^{-Cp(\lambda)} \quad \lambda \in \Lambda \cap \mathbb{H},$$

where B denotes the Blaschke product in \mathbb{H} of $\{(\lambda, m_\lambda)\}_{\lambda \in \Lambda \cap \mathbb{H}}$, and

$$B_\lambda(z) = B(z) \left(\frac{z - \bar{\lambda}}{z - \lambda} \right)^{m_\lambda}.$$

It can also be seen as density conditions for the counting function associated to the hyperbolic metric in the half-plane. Letting $\nu = \sum_{\lambda \in \Lambda \cap \mathbb{H}} m_\lambda \delta_\lambda$ and using the distribution function we have

$$\sum_{\lambda \in \Lambda \cap \mathbb{H}} m_\lambda \log \left| \frac{z - \lambda}{z - \bar{\lambda}} \right|^{-1} = \int_{\mathbb{H}} \log \left| \frac{z - \zeta}{z - \bar{\zeta}} \right|^{-1} d\nu(\zeta) = \int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt,$$

where

$$D_{\mathbb{H}}(z, t) = \left\{ \zeta : \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| < t \right\}, \quad \text{and} \quad n_{\mathbb{H}}(z, t) := \nu(D_{\mathbb{H}}(z, t))$$

is the number of points of Λ in the pseudohyperbolic disk of “center” z and “radius” t (actually the true disk of center $\operatorname{Re} z + i \frac{1+t^2}{1-t^2} \operatorname{Im} z$ and radius $\frac{2t}{1-t^2} \operatorname{Im} z$).

PROPOSITION 4.2.4. *If V is \mathcal{A}_p -interpolating, then condition (21) holds.*

PROOF. Define $\Lambda_+ = \Lambda \cap \{\operatorname{Im} z > \omega(|z|)\}$. Given $x \in \mathbb{R}$ consider $\lambda \in \Lambda_+$ such that $|x - \lambda| = \inf_{\Lambda_+} |x - \lambda|$. Then

$$|\lambda - \bar{\lambda}'| \leq |\lambda - x| + |x - \bar{\lambda}'| = |\lambda - x| + |x - \lambda'| \leq 2|x - \lambda'|,$$

and therefore

$$\sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\operatorname{Im} \lambda'|}{|x - \bar{\lambda}'|^2} \leq 2 \sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\operatorname{Im} \lambda'|}{|\lambda - \bar{\lambda}'|^2}.$$

The estimate $\log t^{-1} \geq 1 - t$ for $t \in (0, 1)$ shows that

$$\sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \neq \lambda}} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|\lambda - \bar{\lambda}'|^2} \leq \sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1}.$$

Since $p(\lambda) \simeq |\operatorname{Im} \lambda|$ for $\lambda \in \Lambda_+$, it is clear, using Proposition 4.2.2, that this implies condition (20). ■

4.3. Sufficient conditions

We split the sequence into three pieces, according to the non-isotropy of the weight p . Consider the regions

$$\begin{aligned} \Omega_0 &= \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \omega(|z|)\} \\ \Omega_+ &= \{z \in \mathbb{C} : \operatorname{Im} z > \omega(|z|)\} \\ \Omega_- &= \{z \in \mathbb{C} : \operatorname{Im} z < -\omega(|z|)\}, \end{aligned}$$

and define $\Lambda_0 = \Lambda \cap \Omega_0$, $\Lambda_+ = \Lambda \cap \Omega_+$ and $\Lambda_- = \Lambda \cap \Omega_-$. Let also $V_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0}$, $V_+ = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_+}$ and $V_- = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_-}$.

Recall that condition (20) implies that V is weakly separated (see Lemma 1.2.13). Thus, it is enough to prove that each piece V_+ , V_- , V_0 of the variety V is \mathcal{A}_p -interpolating. This is because a weakly separated union of a finite number of \mathcal{A}_p -interpolating varieties is also \mathcal{A}_p -interpolating (see Theorem 2.1.1 in Chapter 2). It is also clear that the varieties V^+ and V^- can be dealt with similarly.

We start with another easy consequence of condition (20).

LEMMA 4.3.1. *If condition (20) holds, then there exists $\varepsilon, C > 0$ such that*

$$n_V(z, \varepsilon p(z)) \leq Cp(z), \forall z \in \mathbb{C}.$$

PROOF. When $z = \lambda \in \Lambda$, this is immediate from the estimate

$$\int_{1/2p(\lambda)}^{p(\lambda)} \frac{n_V(\lambda, 1/2p(\lambda)) - 1}{t} dt \leq N_V(\lambda, p(\lambda)).$$

When $z \notin \Lambda$, then let $\varepsilon > 0$ be such that $\zeta \in D(z, \varepsilon p(z))$ implies

$$D(z, \varepsilon p(z)) \subset D(\zeta, 1/2p(\zeta)),$$

which exists by property (f) of the weight. Take $\lambda \in D(z, \varepsilon p(z))$ (if there is no such λ the estimate is obviously true). Then, by the previous case and property (e) of the weight

$$n_V(z, \varepsilon p(z)) \leq n_V(\lambda, 1/2p(\lambda)) \lesssim p(\lambda) \lesssim p(z).$$

■

4.3.1. Case of Λ_0 . We would like to prove that $V_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0}$ is \mathcal{A}_p -interpolating using a $\tilde{\delta}$ -scheme as in Chapter 3 and 4. This is easier if we can regularize the weight in the following way.

LEMMA 4.3.2. *There exists \tilde{p} subharmonic in \mathbb{C} such that $p(z) \simeq \tilde{p}(z)$ and*

$$(22) \quad 1/\tilde{p}(z) \lesssim \Delta \tilde{p}(z) \quad \text{if} \quad |\operatorname{Im} z| \leq 2\omega(|z|).$$

The fact that $p \simeq \tilde{p}$ clearly implies that $\mathcal{A}_p(\mathbb{C}) = \mathcal{A}_{\tilde{p}}(\mathbb{C})$ and the interpolating varieties for $\mathcal{A}_p(\mathbb{C})$ and $\mathcal{A}_{\tilde{p}}(\mathbb{C})$ are the same.

PROOF. We will construct $\tilde{p}(z) = |\operatorname{Im} z| + r(z)$, where r satisfies the following properties:

- (i) $r \geq 0$ and \tilde{p} is subharmonic in \mathbb{C} ,
- (ii) $r(z) = 0$ if $|\operatorname{Im} z| \geq 10\omega(|z|)$.
- (iii) $1/p(z) \lesssim \Delta \tilde{p}(z)$ and $r(z) \simeq \omega(|z|)$ if $|\operatorname{Im} z| \leq 2\omega(|z|)$.

In order to construct r , we partition the real line into intervals I_n defined in the following way.

Let $x_1 > 1$, $x_{n+1} = x_n + \omega(x_n)$ for $n \geq 1$ and $x_n = -x_{-n}$ for $n \leq -1$. Set $I_0 = [x_{-1}, x_1]$, $I_n = [x_n, x_{n+1}]$ for $n \geq 1$ and $I_n = [x_{n-1}, x_n]$ for $n \leq -1$. Denote by ω_n the length of I_n .

We consider two measures in \mathbb{C} . The first one is the usual length measure $d\nu$ in \mathbb{R} , which we split $d\nu = \sum_n d\nu_n$, with $d\nu_n = dx|_{I_n}$. The second one is defined as a sum of convolutions of the $d\nu_n$'s: let

$$d\mu_n(z) = \left(\frac{1}{100\pi\omega_n^2} \int_{I_n} \chi_{D_n}(z-x) dx \right) dm(z),$$

where $D_n = D(0, 10\omega_n)$, and define $d\mu = \sum_n d\mu_n$.

Notice that when z is at a distance of I_n smaller than $2\omega_n$, we can use property (g) of the Beurling weights to deduce that $d\mu(z) \simeq 1/\omega(|z|) \simeq 1/p(z)$. Hence $d\mu(z) \simeq dm(z)/p(z)$.

Define

$$r(z) = \int_{\mathbb{C}} \log |z-w| (d\mu(w) - d\nu(w)).$$

Since $\Delta |\operatorname{Im} z| = d\nu$ we have $\Delta \tilde{p} = d\mu \geq 0$.

Let S_n denote the support of μ_n . Let

$$r_n(z) := \int_{\mathbb{C}} \log |z-w| (d\mu_n(w) - d\nu_n(w)) = \int_{S_n} \log |z-w| d\mu_n(w) - \int_{I_n} \log |z-x| dx$$

Using the definition of μ_n and reversing the order of integration we get

$$r_n(z) = \int_{I_n} M(x) dx,$$

where

$$M(x) = \frac{1}{100\pi\omega_n^2} \int_{D(x, 10\omega_n)} \log |z-w| dm(w) - \log |z-x| \geq 0.$$

In particular, r is non-negative in \mathbb{C} .

If $z \notin S_n$ and $x \in I_n$, $\log |z-w|$ is harmonic in $D(x, 10\omega_n)$, hence $r_n(z) = 0$

Suppose now $z \in D(x_n, 3\omega_n)$. Then, for each $x \in I_n$, $|z - x| \leq 4\omega_n$ and

$$M(x) \geq \frac{1}{100\pi\omega_n^2} \int_{9\omega_n \leq |w-x| \leq 10\omega_n} \log \frac{|z-w|}{|z-x|} dm(w) \gtrsim 1.$$

Thus, $r_n(z) \gtrsim \omega_n \gtrsim \omega(|z|)$.

If $z \in S_n$, using that μ_n and ν_n have the same mass $\omega(x_n)$, we obtain

$$\begin{aligned} \int_{\mathbb{C}} \log |z-w| (d\mu_n(w) - d\nu_n(w)) &\leq \int_{\mathbb{C}} \left| \log \frac{|z-w|}{\omega(x_n)} \right| (d\mu_n(w) + d\nu_n(w)) \lesssim \\ \int_{\mathbb{C}} \left| \log \frac{|x_n-w|}{\omega(x_n)} \right| (d\mu_n(w) + d\nu_n(w)) &\lesssim \omega(|z|). \end{aligned}$$

Since $|\operatorname{Im} z| \leq 2\omega(|z|)$, z belongs at most to a finite number of S_n 's and at least to one $D(x_n, 10\omega_n)$, by property (g) of the Beurling weights, we are done. \blacksquare

Let us prove now that V_0 is A_p -interpolating. In view of Lemma 4.3.2, we assume that there exist $C > 0$ such that, for all z with $|\operatorname{Im} z| \leq 2\omega(|z|)$,

$$\frac{1}{p(z)} \leq C\Delta p(z).$$

Consider the separation radii δ_λ given by Remark 1.2.10.

Given a sequence of values $W = \{w_{\lambda,l}\}_{\lambda \in \Lambda, 0 \leq l < m_\lambda} \in \mathcal{A}_p(V)$, define the smooth interpolating function

$$F(z) = \sum_{\lambda \in \Lambda_0} p_\lambda(z) \mathcal{X}\left(\frac{|z-\lambda|^2}{\delta_\lambda^2}\right),$$

where $p_\lambda(z) = \sum_{l=0}^{m_\lambda-1} w_{\lambda,l} (z-\lambda)^l$ and \mathcal{X} is a smooth cut-off function with $|\mathcal{X}'| \lesssim 1$, $\mathcal{X}(x) = 1$ if $|x| \leq 1$ and $\mathcal{X}(x) = 0$ if $|x| \geq 2$.

It is clear that $F^{(l)}(\lambda)/l! = w_{\lambda,l}$, and that F has the characteristic growth of $\mathcal{A}_p(\mathbb{C})$ functions: the support of F is contained in $\cup_\lambda D_\lambda$ and for $z \in D_\lambda$

$$|F(z)| \leq \sum_{l=0}^{m_\lambda-1} |w_{\lambda,l}| \leq C e^{\alpha p(\lambda)} \lesssim e^{Kp(z)}.$$

There is also a good estimate on $\bar{\partial}F$. Its support is the union of the annuli

$$C_\lambda = \{z \in \mathbb{C} : \delta_\lambda \leq |z-\lambda| \leq 2\delta_\lambda\},$$

and for $z \in C_\lambda$,

$$\left| \frac{\partial F}{\partial \bar{z}}(z) \right| \lesssim \sum_{l=0}^{m_\lambda-1} |w_{\lambda,l}| |\mathcal{X}'| \frac{1}{\delta_\lambda} \lesssim e^{Cp(\lambda)} \lesssim e^{Kp(z)},$$

for K big enough.

Altogether, there exists $\gamma > 0$ such that

$$(23) \quad \int_{\mathbb{C}} |F(z)|^2 e^{-\gamma p(z)} < \infty \quad , \quad \int_{\mathbb{C}} |\bar{\partial}F(z)|^2 e^{-\gamma p(z)} < \infty .$$

Now, when looking for a holomorphic interpolating function of the form $f = F - u$, we are led to the $\bar{\partial}$ -problem

$$\bar{\partial}u = \bar{\partial}F ,$$

which we solve using Hörmander's theorem 1.2.16. We apply Hörmander's theorem with

$$\psi_\beta(z) = \beta p(z) + v(z) ,$$

where $\beta > 0$ will be chosen later on and

$$v(z) = \sum_{\lambda \in \Lambda_0} m_\lambda \left[\log |z - \lambda|^2 - \frac{1}{\pi \varepsilon^2 p^2(\lambda)} \int_{D(\lambda, \varepsilon p(\lambda))} \log |z - \zeta|^2 dm(\zeta) \right] .$$

Here ε is a fixed small constant to be determined later on.

Integrating by parts the equality

$$\int_0^{2\pi} \log |a - re^{i\theta}|^2 \frac{d\theta}{2\pi} = \begin{cases} \log |a|^2 & \text{if } |a| > r \\ \log r^2 & \text{if } |a| \leq r \end{cases}$$

one sees that for $a \in \mathbb{C}$ and $r > 0$:

$$\log |a|^2 - \frac{1}{\pi r^2} \int_{D(a, r)} \log |\zeta|^2 dm(\zeta) = \begin{cases} \log \left| \frac{a}{r} \right|^2 + 1 - \left| \frac{a}{r} \right|^2 & \text{if } |a| \leq r \\ 0 & \text{if } |a| > r \end{cases} .$$

Thus

$$v(z) = \sum_{\lambda: |\lambda - z| \leq \varepsilon p(\lambda)} m_\lambda \left[\log \frac{|z - \lambda|^2}{\varepsilon^2 p^2(\lambda)} + 1 - \frac{|z - \lambda|^2}{\varepsilon^2 p^2(\lambda)} \right] .$$

In particular $v \leq 0$ and $\Delta v(z) = 0$ if $z \notin \cup_\lambda D(\lambda, \varepsilon p(\lambda))$. For $z \in \cup_\lambda D(\lambda, \varepsilon p(\lambda))$ we have $|\operatorname{Im} z| \leq 2\omega(|z|)$ and

$$\Delta v(z) \geq \sum_{\lambda: |\lambda - z| \leq \varepsilon p(\lambda)} \frac{-m_\lambda}{\varepsilon^2 p^2(\lambda)} \gtrsim \sum_{\lambda: |\lambda - z| \leq C(\varepsilon)p(z)} \frac{-m_\lambda}{p^2(z)} = - \frac{n(z, C(\varepsilon)p(z))}{p^2(z)} .$$

As observed in Lemma 4.3.1, with ε small enough $n(z, C(\varepsilon)p(z)) \lesssim p(z)$, thus $\Delta v(z) \gtrsim -1/p(z)$. This and (22) show that ψ_β is subharmonic if β is chosen big enough.

Also, we deduce from (c) that for any $\beta' > \beta$:

$$\int_{\mathbb{C}} |u|^2 e^{-\beta' p} dm \lesssim \int_{\mathbb{C}} |u|^2 \frac{e^{-\psi_\beta}}{(1 + |z|^2)^2} dm \lesssim \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-\psi_\beta} dm .$$

We need to control ψ_β on the support of $\bar{\partial}F$. For $z \in C_\lambda$,

$$\begin{aligned} |\psi_\beta(z) - \beta p(z)| &\leq \sum_{\lambda: |\lambda-z| \leq \varepsilon p(\lambda)} m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z - \lambda|^2} \\ &\simeq m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z - \lambda|^2} + \sum_{\substack{\lambda': |z-\lambda'| \leq \varepsilon p(\lambda') \\ \lambda' \neq \lambda}} m_{\lambda'} \log \frac{\varepsilon^2 p^2(\lambda')}{|z - \lambda'|^2} \\ &\lesssim p(\lambda) + \sum_{\substack{\lambda': |\lambda'-z| \leq C(\varepsilon)p(z) \\ \lambda' \neq \lambda}} m_{\lambda'} \log \frac{C(\varepsilon)^2 p^2(z)}{|z - \lambda'|^2} \lesssim p(z) + N(z, C(\varepsilon)p(z)) \end{aligned}$$

CLAIM 4.3.3. *For ε small enough $N(z, C(\varepsilon)p(z)) \lesssim p(z)$ for all $z \in \text{supp}(\bar{\partial}F)$.*

Assuming the claim we have $|\psi_\beta(z) - \beta p(z)| \leq Kp(z)$ on $\text{supp}(\bar{\partial}F)$. Therefore, for β large enough

$$\int_{\mathbb{C}} |u|^2 e^{-\beta' p} dm \lesssim \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-\psi_\beta} dm \leq \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-\gamma p} dm < \infty.$$

This shows that $u \in \mathcal{W}_p(\mathbb{C})$ and we already know that $F \in \mathcal{W}_p(\mathbb{C})$. Therefore, $f := F - u \in \mathcal{W}_p(\mathbb{C}) \cap \mathcal{A}_p(\mathbb{C}) = \mathcal{A}_p(\mathbb{C})$ (see Lemma 1.1.4).

Since $e^{-\psi_\beta} \simeq |z - \lambda|^{-2m_\lambda}$ around each λ , also $u^{(l)}(\lambda) = 0$ for all $\lambda \in \Lambda$, $l = 0, \dots, m_\lambda - 1$, and therefore $f^{(l)}(\lambda)/l! = F^{(l)}(\lambda)/l! = w_{\lambda,l}$, as required.

Proof of the claim: Assume $z \in C_\lambda$ and observe that $n(z, t) = 0$ for $t < \delta_\lambda$ and that $n(z, t) \leq m_\lambda$ for $\delta_\lambda \leq t < 2\delta_\lambda$. Since $D(z, t) \subset D(\lambda, t + 2\delta_\lambda)$ and $|z| < |\lambda| + 2\delta_\lambda$, we have (changing into $s = t + 2\delta_\lambda$)

$$\begin{aligned} N(z, C(\varepsilon)p(z)) &\leq \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^{C(\varepsilon)p(z)} \frac{n(z, t) - m_\lambda}{t} dt \leq \\ &\leq p(\lambda) + \int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s - 2\delta_\lambda} ds \lesssim \\ &\lesssim p(\lambda) + \int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s/2} ds \lesssim p(\lambda) + N(\lambda, C'(\varepsilon)p(\lambda)). \end{aligned}$$

From the properties of the weight and the hypothesis we have finally that for ε small $N(z, C(\varepsilon)p(z)) \lesssim p(\lambda) \lesssim p(z)$.

4.3.2. Case of Λ^+ . According to Theorem 1.2.14, it is enough to construct a function $G \in \mathcal{A}_p(\mathbb{C})$ such that $V_+ \subset Z(G)$ and

$$\frac{|G^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq \varepsilon e^{-Kp(\lambda)} \quad \lambda \in \Lambda_+$$

for some constants $\varepsilon, k > 0$. In fact, the hypotheses of Theorem 1.2.14 require the weight p to be subharmonic, and our weights are not necessarily so. Nevertheless, by Lemma 4.3.2, there exists a subharmonic weight \tilde{p} equivalent to p , and we may apply Theorem A to \tilde{p} .

Take any entire function F such that $Z(F) = V_+$. Since the necessary conditions imply that V_+ satisfies the Blaschke condition in \mathbb{H} , we can consider also the Blaschke product

$$B(z) = \prod_{\lambda \in \Lambda_+} \left(\frac{z - \lambda}{z - \bar{\lambda}} \right)^{m_\lambda}, \quad z \in \mathbb{H}.$$

Define

$$\phi(z) = \begin{cases} \log \left| \frac{F(z)}{B(z)} \right| & \text{Im } z > 0 \\ \log |F(z)| & \text{Im } z \leq 0. \end{cases}$$

LEMMA 4.3.4. *ϕ is harmonic outside the real axis, subharmonic on \mathbb{C} and its Laplacian is uniformly bounded.*

PROOF. It is clear, by definition, that ϕ is harmonic on $\mathbb{C} \setminus \mathbb{R}$. In order to prove that ϕ is subharmonic on \mathbb{C} , it is enough to check the mean inequality for $x \in \mathbb{R}$. We have

$$\phi(x) = \log |F(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(x + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(x + re^{i\theta}) d\theta.$$

Since $\Delta \log |F| \equiv 0$ around \mathbb{R} , it is enough to compute the Laplacian of

$$\psi(z) = \begin{cases} \log \frac{1}{|B(z)|} & \text{Im } z > 0 \\ 0 & \text{Im } z \leq 0. \end{cases}$$

Being

$$\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{\lambda \in \Lambda^+} m_\lambda \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2,$$

it will be enough to compute the Laplacian of each term

$$\psi_\lambda(z) = \begin{cases} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2 & \text{Im } z > 0 \\ 0 & \text{Im } z \leq 0. \end{cases}$$

It is clear that $\partial \psi_\lambda / \partial x = 0$ on \mathbb{R} , hence $\Delta \psi_\lambda = \partial^2 \psi_\lambda / \partial y^2$. Since ψ_λ is continuous around \mathbb{R} , this Laplacian has a magnitude equivalent to the jump of the first derivative of ψ_λ . The derivative of the Green function on the half-plane with respect to the normal direction y is the Poisson kernel:

$$\frac{\partial}{\partial y} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2 \Big|_{y=0} = \frac{4 \operatorname{Im} \lambda}{|x - \lambda|^2}.$$

Therefore

$$\Delta \phi(x) = 4 \sum_{\lambda \in \Lambda^+} m_\lambda \frac{\operatorname{Im} \lambda}{|x - \lambda|^2} dx,$$

which is bounded by hypothesis. ■

Define

$$\Psi(z) = N|\operatorname{Im} z| - \phi(z).$$

Observe that $\Delta\Psi(z) = N dx - \Delta\phi(x) dx$, thus according to the previous Lemma $\Delta\Psi \simeq dx$ when $N \in \mathbb{N}$ is big enough. In this situation, according to [42, Lemma 3], there exists a multiplier associated to Ψ , i.e., an entire function h such that:

- (a) $Z(h)$ is a separated sequence contained in \mathbb{R}
- (b) Given any $\varepsilon > 0$, $|h(z)| \simeq \exp(\Psi(z))$ for all points z such that $d(z, Z(h)) > \varepsilon$.

Define now $G = hF$. It is clear that $G \in \mathcal{A}_p(\mathbb{C})$:

$$|G(z)| \lesssim e^{\Psi(z) + \log|F(z)|} \leq e^{\Psi(z) + \phi(z)} \leq e^{Np(z)} \quad z \in \mathbb{C}.$$

It is also clear that $V_+ \subset Z(G)$, since $V_+ \subset Z(F)$.

In order to prove that there exist $\varepsilon, C > 0$ such that

$$(24) \quad \left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \varepsilon e^{-Cp(\lambda)}$$

consider then the disjoint disks $D_\lambda = D(\lambda, \delta_\lambda)$, $\delta_\lambda = \delta e^{-C\frac{p(\lambda)}{m_\lambda}}$ given by Lemma 4.3.1(i). Since Λ_+ is far from $Z(h)$, the estimate

$$|G(z)| = |h(z)|e^{\phi(z)}|B(z)| \simeq e^{N|\operatorname{Im} z|}|B(z)| \quad z \in \partial D_\lambda$$

holds.

CLAIM 4.3.5. *There exists $C > 0$ such that $|B(z)| \geq \varepsilon e^{-Cp(z)}$, $z \in \partial D_\lambda$.*

Assuming this we have $|G(z)| \gtrsim e^{-Cp(z)}$ for all $z \in \partial D_\lambda$. Define then $g(z) = G(z)/(z - \lambda)^{m_\lambda}$. It is clear that g is holomorphic, non-vanishing in D_λ , and $|g(z)| \gtrsim e^{-cp(\lambda)}$ for $z \in \partial D_\lambda$. By the minimum principle

$$\left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| = |g(0)| \gtrsim e^{-cp(\lambda)},$$

as desired.

Proof of the claim: As observed in Remark 4.2.3(b), the estimate we want to prove is equivalent to

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim p(z) \quad z \in \partial D_\lambda.$$

This is proved like Claim 4.3.3, except replacing the Euclidean disks by the hyperbolic ones. We have

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z, t) - m_\lambda}{t} dt.$$

The first term is controlled by $p(\lambda)$. In order to control the second term observe that $D_{\mathbb{H}}(z, t) \subset D_{\mathbb{H}}(\lambda, \frac{t+\delta_\lambda}{1+t\delta_\lambda})$; hence changing the variable into $s = \frac{t+\delta_\lambda}{1+t\delta_\lambda}$ we get

$$\int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z, t) - m_\lambda}{t} dt \leq \int_{\frac{3\delta_\lambda}{1+2\delta_\lambda^2}}^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s - \delta_\lambda} \frac{1 - \delta_\lambda^2}{(1 - \delta_\lambda)^2} ds.$$

There is no restriction in assuming that $\delta_\lambda < 1/2$. Then $\frac{3\delta_\lambda}{1+2\delta_\lambda^2} > 2\delta_\lambda$ and therefore $s - \delta_\lambda > s/2$. With this and condition (ii) in Theorem 4.1.2 we obtain

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim p(\lambda) + \int_0^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds.$$

Since $p(\lambda) \lesssim p(z)$, we will be done as soon as we prove that

$$\int_0^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds \lesssim p(\lambda).$$

There exists $\delta > 0$ (independent of λ) such that $D_{\mathbb{H}}(\lambda, \delta) \subset D(\lambda, p(\lambda))$. Then

$$\begin{aligned} \int_0^\delta \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds &= \sum_{0 < |\frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'}| < \delta} m_{\lambda'} \log \frac{\delta}{|\frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'}|} \leq \sum_{0 < |\frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'}| < \delta} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \\ &\lesssim \sum_{0 < |\lambda - \lambda'| < p(\lambda)} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \leq N(\lambda, p(\lambda)) \lesssim p(\lambda). \end{aligned}$$

For the remaining part we use condition (ii) in Theorem 4.1.2 and the estimate $\log t^{-1} \simeq 1 - t$ for $\delta < t < 1$. Taking $x = \operatorname{Re} \lambda$ we have

$$\begin{aligned} \int_\delta^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds &\lesssim \sum_{\lambda \neq \lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|\lambda - \bar{\lambda}'|^2} \lesssim \\ &\lesssim \sum_{\lambda \neq \lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|x - \lambda'|^2} \lesssim |\operatorname{Im} \lambda| \simeq p(\lambda). \end{aligned}$$

CHAPTER 5

The trace of $\mathcal{A}_p(\mathbb{C})$ on a multiplicity variety when p is radial and doubling

The results of this section are to appear in Michigan Mathematical Journal.

Introduction

Let $V = \{(\lambda_j, m_j)\}_j$ be a multiplicity variety and let $W = \{w_{j,l}\}_{j, 0 \leq l < m_j}$ be a doubly indexed sequence of complex values. We assume that the weight p is radial and doubling. In this chapter, we are concerned with the following question : Under what conditions on W does there exist an entire function $f \in \mathcal{A}_p(\mathbb{C})$ such that

$$\frac{f^{(l)}(\alpha_j)}{l!} = w_{j,l}, \quad \forall j, \quad \forall 0 \leq l < m_j?$$

In other words, what is the image of $\mathcal{A}_p(\mathbb{C})$ by the restriction map \mathcal{R}_V ?

In [7], Berenstein and Taylor described the space $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$ in the case where there exists a function $g \in \mathcal{A}_p(\mathbb{C})$ such that $V = Z(g)$. They used groupings of the points of V with respect to the connected components of the set $\{|g(z)| \leq \varepsilon \exp(-Bp(z))\}$, for some $\varepsilon, B > 0$ and the divided differences with respect to this grouping.

The main aim of this chapter is to determine more explicitly the space $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$ in the more general case where condition (1) is satisfied. We know that (1) is verified whenever V is not a uniqueness set for $\mathcal{A}_p(\mathbb{C})$, that is, when $V \subset Z(f)$ for a non-zero function f in $\mathcal{A}_p(\mathbb{C})$ (see Lemma 1.2.2). We refer to [12] and [28] for similar results in the case where $p(z) = |z|^\alpha$.

As in [7] and [12], the divided differences will be important tools. Our condition will involve the divided differences with respect to the intersections of V with discs centered at the origin. To be more precise, the main theorem, stated in the case of simple multiplicities, for the sake of simplicity, is the following :

THEOREM 5.0.6. *Assume that V verifies condition (1). Then $W = \{w_j\}_j \in \mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$ if and only if for all $R > 0$,*

$$\left| \sum_{|\alpha_k| < R} w_k \prod_{|\alpha_n| < R, n \neq k} R/(\alpha_k - \alpha_n) \right| \leq A e^{Bp(R)},$$

where $A, B > 0$ are positive constants only depending on V and W .

We will denote by $\tilde{\mathcal{A}}_p(V)$ the space of sequences $W = \{w_j\}_j$ satisfying the above condition. We will show that in general $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C})) \subset \tilde{\mathcal{A}}_p(V)$, thus, we may consider the restriction map $\mathcal{R}_V : \mathcal{A}_p(\mathbb{C}) \rightarrow \tilde{\mathcal{A}}_p(V)$. In this context, the theorem states that condition (1) implies the surjectivity of \mathcal{R}_V .

On the other hand, we will prove that condition (1) is actually equivalent to saying that V is not a uniqueness set or, in other words, it is equivalent to the non-injectivity of \mathcal{R}_V .

As a corollary of the main theorem, we will find the sufficiency in the geometric characterization of interpolating varieties given in Theorem 3.1.3.

As in Chapter 3, the difficult part of the proof of the main theorem is the sufficiency and we will again follow a Bombieri-Hörmander approach based on L^2 -estimates on the solution to the $\bar{\partial}$ -equation. The condition on W gives a smooth interpolating function F with a good growth, using a partition unity and Newton polynomials (see Lemma 5.3.5). Then we are led to solve the $\bar{\partial}$ equation : $\bar{\partial}u = -\bar{\partial}F$ with L^2 -estimates, using Hörmander theorem 1.2.16. To do so, we need to construct a subharmonic function U with a convenient growth and with prescribed singularities on the points α_j (see Lemma 5.3.6). Following Bombieri [11], the fact that e^{-U} is not summable near the points $\{\alpha_j\}$ forces u to vanish on the points α_j and we are done by defining the interpolating entire function by $u + F$.

This chapter is organised as follows ; In Section 5.1, we give some preliminary definitions about divided differences and Newton polynomials, then we define the various spaces that we will need to describe the trace of $\mathcal{A}_p(\mathbb{C})$ on V . We state our main results in Section 5.2 and we prove them in Section 5.3.

5.1. Preliminaries and definitions

We will throughout this chapter assume that p is radial and doubling (see Definition 1.1.2).

Let $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}^*}$ be a multiplicity variety. We will say that V is a uniqueness set for $\mathcal{A}_p(\mathbb{C})$ if there is no function $f \in \mathcal{A}_p(\mathbb{C})$, except the zero function, such that $V \subset Z(f)$.

It is clear by Lemma 1.2.2 that whenever V is not a uniqueness set for $\mathcal{A}_p(\mathbb{C})$, condition (1) holds. We will lately show that the converse property holds.

It is also clear that the restriction map

$$\begin{aligned} \mathcal{R}_V : \mathcal{A}_p(\mathbb{C}) &\longrightarrow \mathcal{A}_p(V) \\ f &\longmapsto \left\{ \frac{f^l(\alpha_j)}{l!} \right\}_{j, 0 \leq l < m_j} \end{aligned}$$

is injective if and only if V is a uniqueness set for $\mathcal{A}_p(\mathbb{C})$.

Our aim is to describe the trace of $\mathcal{A}_p(\mathbb{C})$ on V or, in other words, the space $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$, provided that condition (1) is verified.

5.1.1. Divided differences and Newton polynomials. To any $W = \{w_{j,l}\}_{j \in \mathbb{N}^*, 0 \leq l < m_j} \in \mathcal{A}(V)$, we associate the sequence of divided differences $\Phi(W) = \{\phi_{j,l}\}_{j, 0 \leq l < m_j}$ defined by induction as follows :

We will denote by

$$\begin{aligned} \Pi_q(z) &= \prod_{k=1}^q (z - \alpha_k)^{m_k}, \text{ for all } q \geq 1. \\ \phi_{1,l} &= w_{1,l}, \text{ for all } 0 \leq l \leq m_1 - 1, \\ \phi_{q,0} &= \frac{w_{q,0} - P_{q-1}(z_q)}{\Pi_{q-1}(z_q)}, \end{aligned}$$

$$\phi_{q,l} = \frac{w_{q,l} - \frac{P_{q-1}^{(l)}(z_q)}{l!} - \sum_{j=0}^{l-1} \frac{1}{(l-j)!} \Pi_{q-1}^{(l-j)}(z_q) \phi_{q,j}}{\Pi_{q-1}(z_q)} \quad \text{for } 1 \leq l \leq m_q - 1$$

where

$$P_{q-1}(z) = \sum_{j=1}^{q-1} \left(\sum_{l=0}^{m_j-1} \phi_{j,l}(z - \alpha_j)^l \prod_{t=1}^{j-1} (z - z_t)^{m_t} \right).$$

REMARK 5.1.1. Actually, P_q is the polynomial interpolating the values $w_{j,l}$ at the points α_j with multiplicity m_j , for $1 \leq j \leq q$. It is the unique polynomial of degree $m_1 + \dots + m_q - 1$ such that

$$\frac{P_q^{(l)}(\alpha_j)}{l!} = w_{j,l}$$

for all $1 \leq j \leq q$ and $0 \leq l \leq m_j - 1$.

EXAMPLES 5.1.2. • Let $W_0 = \{\delta_{1,j} \delta_{l,m_1-1}\}_{j,0 \leq l < m_j}$.

Using the fact that $P_j(z)$ must coincide with $(z - \alpha_1)^{m_1-1} \prod_{k=2}^{j-1} (z - \alpha_k)^{m_k}$ and identifying the coefficient in front of $z^{m_1+\dots+m_{j-1}+l-1}$, we find :

$$\phi_{1,1} = \phi_{1,2} = \dots = \phi_{1,m_1-2} = 0, \quad \phi_{1,m_1-1} = 1,$$

and, for $j \geq 2$, $0 \leq l \leq m_j - 1$,

$$\phi_{l,j} = (\alpha_1 - \alpha_j)^{-(l+1)} \prod_{k=2}^{j-1} (\alpha_1 - \alpha_k)^{-m_k}.$$

• In the special case where $m_j = 1$ for all j and $W = \{w_j\}_j$, we have for all $j \geq 1$,

$$\phi_j = \sum_{k=1}^j w_k \prod_{1 \leq l \leq j, l \neq k} (\alpha_k - \alpha_l)^{-1}.$$

To compute the coefficients, we may use the fact that $P_j(z)$ must coincide with the Lagrange polynomial $\sum_{n=1}^j w_n \prod_{1 \leq k \leq j, k \neq n} \frac{(z - \alpha_k)}{(\alpha_n - \alpha_k)}$ and identify the coefficient in front of z^{j-1} .

5.1.2. The spaces $\tilde{\mathcal{A}}_p(V)$ and $\tilde{\tilde{\mathcal{A}}}_p(V)$. Let us denote by $\tilde{\mathcal{A}}_p(V)$ the subspace of $\mathcal{A}(V)$ consisting of the elements $W \in \mathcal{A}(V)$ verifying : for all $n \in \mathbb{N}$, for all α_j such that $|\alpha_j| \leq 2^n$,

$$(25) \quad \forall 0 \leq l \leq m_j - 1, \quad |\phi_{j,l}| 2^{n(l+m_1+\dots+m_{j-1})} \leq A \exp(Bp(2^n)),$$

where A and B are positive constants only depending on V and W .

LEMMA 5.1.3. Assume $\alpha_1 = 0$. Then, condition (I) holds if and only if

$$W_0 = \{\delta_{1,j} \delta_{l,m_1-1}\}_{j,0 \leq l < m_j} \in \tilde{\mathcal{A}}_p(V).$$

PROOF. Suppose that (1) is verified. Let $n \in \mathbb{N}$, $0 < |\alpha_j| \leq 2^n$ and $0 \leq l \leq m_j - 1$. We have by definition,

$$N(0, 2^n) = \sum_{0 < |\alpha_k| \leq 2^n} m_k \ln \frac{2^n}{|\alpha_k|} + m_1 \ln(2^n) \geq \ln \left(2^{n(m_1 + \dots + m_j)} \prod_{k=2}^j |\alpha_k|^{-m_k} \right),$$

$$|\phi_{j,l}| = |\alpha_j|^{m_j-l-1} \prod_{k=2}^j |\alpha_k|^{-m_k} \leq 2^{n(m_j-l-1)} \prod_{k=2}^j |\alpha_k|^{-m_k} \leq \exp(N(0, 2^n)) 2^{-n(m_1 + \dots + m_{j-1} + l + 1)}.$$

We readily obtain the estimate (25), using that $N(0, 2^n) \leq A + B p(2^n)$.

Conversely, let n be an integer. Using the estimate (25) when $j \geq 2$ is the number of distinct points $\{\alpha_k\}$ in $D(0, 2^n)$ and $l = m_j - 1$, we have

$$N(0, 2^n) = \ln \left(2^{n(m_1 + \dots + m_j)} \prod_{k=2}^j |\alpha_k|^{-m_k} \right) = \ln(2^{n(m_1 + \dots + m_j)} |\phi_{j,m_j-1}|) \leq A + B p(2^n).$$

Then, we deduce the estimate for $N(0, R)$ using the above one with $2^{n-1} \leq R < 2^n$ and the doubling property of p . \blacksquare

Now, let us denote by $\tilde{\mathcal{A}}_p(V)$ the subspace of $\mathcal{A}(V)$ consisting of the elements $W \in \mathcal{A}(V)$ verifying

$$(26) \quad \forall j, \quad \forall 0 \leq l \leq m_j - 1, \quad |\phi_{j,l}| |\alpha_j|^{(l+m_1+\dots+m_{j-1})} \leq A \exp(Bp(\alpha_j)),$$

where A and B are positive constants only depending on V and W .

If we define the norm

$$\|W\|_B = \sup_j \sup_{0 \leq l \leq m_j-1} |\phi_{j,l}| |\alpha_j|^{(l+m_1+\dots+m_{j-1})} \exp(-Bp(\alpha_j)).$$

Then the space $\tilde{\mathcal{A}}_p(V)$ can also be seen as an (LF)-space that is an inductive limit of the Banach spaces

$$\tilde{A}_{p,B}(V) = \{W \in \mathcal{A}(V), \quad \|W\|_B < \infty\}.$$

LEMMA 5.1.4. *We have the following inclusion*

$$\tilde{\mathcal{A}}_p(V) \subset \tilde{\tilde{\mathcal{A}}}_p(V).$$

If condition (1) is satisfied then

$$\tilde{\mathcal{A}}_p(V) = \tilde{\tilde{\mathcal{A}}}_p(V).$$

PROOF. Let $W \in \tilde{\mathcal{A}}_p(V)$ and let $n \in \mathbb{N}$ be such that $2^{n-1} \leq |\alpha_j| \leq 2^n$. Then for all $0 \leq l < m_j$, we have

$$\begin{aligned} |\phi_{j,l}| |\alpha_j|^{(l+m_1+\dots+m_{j-1})} &\leq |\phi_{j,l}| 2^{n(l+m_1+\dots+m_{j-1})} \\ &\leq A \exp(Bp(2^n)) \leq A \exp(Bp(\alpha_j)). \end{aligned}$$

This proves the first inclusion. To prove the second part of the lemma, observe that for all $n \in \mathbb{N}$, for all $|\alpha_j| \leq 2^n$ and for all $0 \leq l < m_j$, we have

$$(27) \quad N(0, 2^n) = \sum_{|\alpha_k| \leq 2^n} m_k \log \frac{2^n}{|\alpha_k|} \geq \sum_{|\alpha_k| \leq |\alpha_j|} m_k \log \frac{2^n}{|\alpha_k|} \geq \log \left(\frac{2^n}{|\alpha_j|} \right)^{l+m_1+\dots+m_{j-1}}.$$

Assume now that condition (1) is fulfilled and that $W \in \tilde{\mathcal{A}}_p(V)$, then

$$\begin{aligned} |\phi_{j,l}| 2^{n(l+m_1+\dots+m_{j-1})} &\leq |\phi_{j,l}| |\alpha_j|^{l+m_1+\dots+m_{j-1}} \exp(N(0, 2^n)) \\ &\leq A \exp(Bp(\alpha_j)) \exp(Bp(2^n)) \leq A \exp(Bp(2^n)). \end{aligned}$$

We have proved that $W \in \tilde{\mathcal{A}}_p(V)$ ■

We are now ready to state the main results.

5.2. Main results

Let $V = \{(\alpha_j, m_j)\}_{j \in \mathbb{N}^*}$ be a multiplicity variety.

PROPOSITION 5.2.1. *The restriction operator \mathcal{R}_V maps $\mathcal{A}_p(\mathbb{C})$ continuously into $\tilde{\mathcal{A}}_p(V)$.*

PROPOSITION 5.2.2. *Under the assumption of condition (1), $\tilde{\mathcal{A}}_p(V)$ is a subspace of $\mathcal{A}_p(V)$.*

PROPOSITION 5.2.3. *If conditions (1) and (6) are verified, then $\tilde{\tilde{\mathcal{A}}}_p(V) = \tilde{\mathcal{A}}_p(V) = \mathcal{A}_p(V)$.*

THEOREM 5.2.4. *If condition (1) holds, then*

$$\tilde{\tilde{\mathcal{A}}}_p(V) = \tilde{\mathcal{A}}_p(V) = \mathcal{R}_V(\mathcal{A}_p(\mathbb{C})).$$

The combination of Proposition 5.2.3 and Theorem 5.2.4 shows easily the sufficiency in Theorem 3.1.3.

Using the results given so far, we may already deduce next theorem :

THEOREM 5.2.5. *The following assertions are equivalent :*

- (i) V is not a uniqueness set for $\mathcal{A}_p(\mathbb{C})$.
- (ii) The map \mathcal{R}_V is not injective.
- (iii) V verifies condition (1).
- (iv) The sequence $W_0 = \{\delta_{1,j} \delta_{l, m_1-1}\}_{j, 0 \leq l < m_j}$ belongs to $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$.

In particular, it shows that condition (1) is equivalent to the existence of a function $f \in \mathcal{A}_p(\mathbb{C})$ such that $V \subset Z(f)$. Combined with Theorem 5.2.4, it shows that, if \mathcal{R}_V is not injective, then it is surjective and that, if the image contains W_0 , then it contains the whole $\tilde{\mathcal{A}}_p(V)$.

PROOF OF THEOREM 5.2.5. As we mentioned before, it is clear that (i) is equivalent to (ii) and that (i) implies (iii).

(iv) implies (i) : We have a function $f \in \mathcal{A}_p(\mathbb{C})$ not identically equal to 0 such that $f^{(l)}(\alpha_j) = 0$ for all $j \neq 1$ and for all $0 \leq l < m_j$. The function g defined by $g(z) = (z - \alpha_1)f(z)$ belongs

to $\mathcal{A}_p(\mathbb{C})$, thanks to property (w1) of the weight p , and vanishes on every α_j with multiplicity at least m_j .

(iii) implies (iv) :

Up to a translation, we may suppose that $\alpha_1 = 0$. By Lemma 5.1.3, we know that $W_0 \in \tilde{\mathcal{A}}_p(\mathbb{C})$. By Theorem 5.2.4, $W_0 \in \mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$. ■

5.3. Proof of the main results

PROOF OF PROPOSITION 5.2.1. We will first recall some definitions about the divided differences and about Newton polynomials. We refer the reader to [2, Chapter 6.2] or [23, Chapter 6] for more details.

Let $f \in \mathcal{A}(\mathbb{C})$ and x_1, \dots, x_q be distinct points of \mathbb{C} . The q th divided difference of the function f with respect to the points x_1, \dots, x_q is defined by

$$\Delta^{q-1}f(x_1, \dots, x_q) = \sum_{j=1}^q f(\alpha_j) \prod_{1 \leq k \leq q, k \neq j} (x_j - x_k)^{-1}$$

and the Newton polynomial of f of degree $q - 1$ is

$$P(z) = \sum_{j=1}^q \Delta^{j-1}f(x_1, \dots, x_j) \prod_{k=0}^{j-1} (z - x_k).$$

It is the unique polynomial of degree $q - 1$ such that $P_q(z) = f(x_j)$ for all $1 \leq j \leq q$.

When x_j , $1 \leq j \leq q$ are each one repeated l_j times, the divided differences are defined by

$$\begin{aligned} & \Delta^{l_1 + \dots + l_q - 1} f(\underbrace{x_1, \dots, x_1}_{l_1}, \dots, \underbrace{x_{q-1}, \dots, x_{q-1}}_{l_{q-1}}, \underbrace{x_q, \dots, x_q}_{l_q}) \\ &= \frac{1}{l_1! \dots l_q!} \frac{\partial^{l_1 + \dots + l_q}}{\partial x_1^{l_1} \dots \partial x_q^{l_q}} \Delta^{q-1} f(x_1, \dots, x_q). \end{aligned}$$

The corresponding Newton polynomial is the unique polynomial of degree $l_1 + \dots + l_q - 1$ such that, for all $0 \leq j \leq q$ and $0 \leq l \leq l_j - 1$,

$$P^{(l)}(x_j) = f^{(l)}(x_j).$$

We have the following estimate

LEMMA 5.3.1. [2, Lemma 6.2.9.]

Let $f \in \mathcal{A}(\mathbb{C})$, Ω an open set of \mathbb{C} , $\delta > 0$ and x_1, \dots, x_k in $\Omega_0 = \{z \in \Omega : d(z, \Omega^c) > \delta\}$. Then

$$|\Delta^{k-1}f(x_1, \dots, x_k)| \leq \frac{2^{k-1}}{\delta^{k-1}} \sup_{z \in \Omega} |f(z)|.$$

Let $B > 0$ be fixed and $f \in \mathcal{A}_{p,B}(\mathbb{C})$.

Let n be a fixed integer. Let $|\alpha_j| \leq 2^n$ and $0 \leq l \leq m_j - 1$. We consider the divided differences of f with respect to the points $\alpha_1, \dots, \alpha_j$, each α_k , $1 \leq k \leq j-1$ repeated m_k times and α_j repeated l times.. Denote by $M_{j,l} = m_1 + \dots + m_{j-1} + l$, the divided differences are

$$\phi_{j,l} = \Delta^{M_{j,l}} f(\underbrace{\alpha_1, \dots, \alpha_1}_{m_1 \text{ times}}, \dots, \underbrace{z_{j-1}, \dots, z_{j-1}}_{m_{j-1} \text{ times}}, \underbrace{\alpha_j, \dots, \alpha_j}_{l+1 \text{ times}}).$$

Using Lemma 5.3.1 with $\Omega = D(0, 2^{n+2})$, $\delta = 2^{n+1}$, $k = M_{j,l} + 1$, we have

$$|\phi_{j,l}| \leq 2^{-nM_{j,l}} \|f\|_B \exp(Bp(2^{n+2})) \leq 2^{-nM_{j,l}} \|f\|_B \exp(B'p(2^n)).$$

Thus,

$$\|\mathcal{R}_V(f)\|_{B'} \leq \|f\|_B$$

and this concludes the proof of Proposition 5.2.1. ■

Before proceeding with the proofs of the main results, we need the following lemmas :

LEMMA 5.3.2. *Condition (1) implies that there exist constants $A, B > 0$ such that, for all $R > 0$,*

$$n(0, R) \leq A + B p(R).$$

PROOF. Using the fact that the weight p is doubling, we have

$$n(0, R) \leq 2 \int_R^{2R} \frac{n(0, t)}{t} dt \leq 2N(0, 2R) \leq A + B p(2R) \leq A + B p(R). ■$$

LEMMA 5.3.3. *Let W be an element of $\mathcal{A}(V)$ and q be in \mathbb{N}^* . We suppose that for all $1 \leq j \leq q$, for all $n \in \mathbb{N}$ such that $|z_q| \leq 2^n$ and for all $0 \leq l \leq m_j - 1$, we have*

$$|\phi_{j,l}| 2^{n(l+m_1+\dots+m_{j-1})} \leq A \exp(Bp(2^n)),$$

where A and B are positive constants only depending on V and W .

Then, there exist constants $A, B > 0$ only depending on V and W , such that, for all $n \in \mathbb{N}$ and $|z| \leq 2^n$,

$$\begin{aligned} \sum_{l=0}^{+\infty} \frac{|P_q^{(l)}(z)|}{l!} &\leq A \exp(Bp(2^n)) \sum_{j=1}^q 2^{2(m_1+\dots+m_j)}, \\ \sum_{l=0}^{+\infty} \frac{|\Pi_q^{(l)}(z)|}{l!} &\leq 2^{(n+2)(m_1+\dots+m_q)}. \end{aligned}$$

PROOF. If $|z| \leq 2^{n+1}$, then for $j = 1, \dots, q$, $|z - \alpha_j| \leq 2^{n+2}$,

$$\begin{aligned} |P_q(z)| &\leq \sum_{j=1}^q 2^{(n+2)(m_1+\dots+m_{j-1})} \sum_{l=0}^{m_j-1} |\phi_{j,l}| 2^{(n+2)l} \\ &\leq A \exp(Bp(2^n)) \sum_{j=1}^q 2^{2(m_1+\dots+m_j)} \end{aligned}$$

and

$$|\Pi_q(z)| = \prod_{j=1}^q |z - \alpha_j|^{m_j} \leq 2^{(n+2)(m_1+\dots+m_q)}.$$

Now for $|z| \leq 2^n$, if $|z - w| \leq 2$, then $|w| \leq 2^{n+1}$. By the preceding inequalities and Cauchy inequalities, for all $l \geq 0$,

$$\frac{|P_q^{(l)}(z)|}{l!} \leq \frac{1}{2^l} \max_{|z-w| \leq 2} |P_q(w)| \leq \frac{1}{2^l} A \exp(Bp(2^n)) \sum_{j=1}^q 2^{2(m_1+\dots+m_j)}.$$

We readily obtain the desired estimate for P_q . Using Cauchy estimates once again for the function Π_q we obtain the second inequality. \blacksquare

PROOF OF PROPOSITION 5.2.2. We assume that condition (1) holds. Let $W = \{w_{j,l}\}_{j,0 \leq l \leq m_j-1} \in \tilde{\mathcal{A}}_p(V)$. Let $q \geq 1$ and n be the integer such that $2^{n-1} \leq |z_q| < 2^n$. We know that $\frac{P_q^{(l)}(z_q)}{l!} = w_{q,l}$ for every $0 \leq l \leq m_{q-1}$. By the preceding lemma,

$$\sum_{l=0}^{m_q-1} |w_{q,l}| \leq \sum_{l=0}^{+\infty} \frac{|P_q^{(l)}(z_q)|}{l!} \leq A \exp(Bp(2^n)) \sum_{j=1}^q 2^{2(m_1+\dots+m_j)}.$$

By Lemma 5.3.2, $m_1 + \dots + m_j \leq n(0, |\alpha_j|) \leq Ap(\alpha_j) + B$. Using that $q \leq n(0, |z_q|) \leq Ap(z_q) + B$, we obtain

$$\sum_{l=0}^{m_q-1} |w_{q,l}| \leq A \exp(Bp(2^n)) \leq A \exp(Bp(z_q)),$$

that is $W \in \mathcal{A}_p(V)$. \blacksquare

PROOF OF THEOREM 5.2.3. We assume that conditions (1) and (6) are fulfilled. We already have $\tilde{\mathcal{A}}_p(V) \subset \mathcal{A}_p(V)$ by Proposition 5.2.2.

Before proving the reverse inclusion, we need some useful consequences of (1) and (6) :

LEMMA 5.3.4. *There exist constants $A, B > 0$ such that, for all $j \in \mathbb{N}^*$ and for all $n \in \mathbb{N}$ such that $|\alpha_j| \leq 2^n$, we have*

- (i) $2^{nm_j} \leq A |\alpha_j|^{m_j} \exp(Bp(2^n))$, $2^{n(m_1+\dots+m_j)} \leq A |\alpha_j|^{m_1+\dots+m_j} \exp(Bp(2^n))$.
- (ii) $|\alpha_j|^{m_j} \leq A \exp(Bp(\alpha_j))$,
- (iii) $\prod_{k=1}^{j-1} |\alpha_j - \alpha_k|^{-m_k} \leq A \exp(Bp(2^n)) 2^{-n(m_1+\dots+m_{j-1})}$.

PROOF. (i) It's a direct consequence of condition (1) and inequality (27).

(ii) It is a simple consequence of condition (6) :

$$m_j \ln |\alpha_j| \leq N(\alpha_j, |\alpha_j|) \leq Ap(\alpha_j) + B.$$

(iii) Using condition (6), we have

$$\sum_{k=1}^{j-1} m_k \ln \frac{|\alpha_j|}{|\alpha_j - \alpha_k|} \leq \sum_{0 < |\alpha_k - \alpha_j| \leq |\alpha_j|} m_k \ln \frac{|\alpha_j|}{|\alpha_j - \alpha_k|} = N(\alpha_j, |\alpha_j|) \leq Ap(\alpha_j) + B.$$

We deduce that

$$\begin{aligned} \prod_{k=1}^{j-1} |\alpha_j - \alpha_k|^{-m_k} &\leq A \exp(Bp(\alpha_j)) |\alpha_j|^{-(m_1 + \dots + m_{j-1})} \\ &\leq A 2^{-n(m_1 + \dots + m_{j-1})} \exp(Bp(2^n)) \end{aligned}$$

using (i). ■

Let $W = \{w_{j,l}\}_{j,0 \leq l \leq m_j-1}$ be in $\mathcal{A}_p(V)$. In order to show that W verifies (25), we are going to use Lemma 5.3.3 and show by induction on $q \geq 1$ the following property :

For all $n \in \mathbb{N}$ such that $|z_q| \leq 2^n$ and for all $0 \leq l \leq m_q - 1$,

$$|\phi_{q,l}| 2^{n(l+m_1+\dots+m_{q-1})} \leq A \exp(Bp(2^n)),$$

where A and B are positive constants only depending on V and W .

$q = 1$: for $|\alpha_1| \leq 2^n$ and $0 \leq l \leq m_1 - 1$, we have

$$|\phi_{1,l}| = |w_{1,l}| \leq A \exp(Bp(\alpha_1)) \leq A \exp(Bp(\alpha_1)) 2^{-nl} 2^{nm_1} \leq A \exp(Bp(2^n)) 2^{-nl}$$

using Lemma 5.3.4, (i) and (ii).

Suppose the property true for $1 \leq j \leq q-1$. Let $n \in \mathbb{N}$ be such that $|z_q| \leq 2^n$.

Again, we proceed by induction on l , $0 \leq l \leq m_q - 1$.

$l = 0$: by Lemmas 5.3.3 and 5.3.2, we have

$$|P_{q-1}(z_q)| \leq A \exp(Bp(2^n)) \sum_{j=1}^{q-1} 2^{2(m_1+\dots+m_j)} \leq (q-1) 2^{2(m_1+\dots+m_{q-1})} \leq A \exp(Bp(2^n)).$$

By Lemma 5.3.4 (iii),

$$|\Pi_{q-1}(z_q)|^{-1} = \prod_{k=1}^{q-1} |z_q - \alpha_k|^{-m_k} \leq A \exp(Bp(2^n)) 2^{-n(m_1+\dots+m_{q-1})}$$

We deduce that

$$|\phi_{q,0}| \leq A \exp(Bp(2^n)) 2^{-n(m_1+\dots+m_{q-1})}.$$

Suppose the estimate true for $0 \leq j \leq l-1$, using both inequalities of Lemma 5.3.3 and Lemma 5.3.2, we have

$$\sum_{j=0}^{l-1} \left| \frac{\Pi_{q-1}^{(l-j)}(z_q)}{(l-j)!} \phi_{q,j} \right| \leq A \exp(Bp(2^n))$$

and

$$\left| \frac{P_{q-1}^{(l)}(z_q)}{l!} \right| \leq A \exp(Bp(2^n)).$$

As for $l = 0$, we use Lemma 5.3.4 (iii) to complete the proof. ■

PROOF OF THEOREM 5.2.4. We already showed the necessity in Proposition 5.2.1. Let us prove the sufficiency.

We assume condition (1). Let $W = \{w_{j,l}\}_{j,0 \leq l \leq m_j-1}$ be an element of $\tilde{\mathcal{A}}_p(V)$.

Let \mathcal{X} be a smooth cut-off function such that $\mathcal{X}(x) = 1$ if $|x| \leq 1$ and $\mathcal{X}(x) = 0$ if $|x| \geq 4$.

Set $\mathcal{X}_n(z) = \mathcal{X}(\frac{|z|^2}{2^{2n}})$, for $n \in \mathbb{N}$, $\rho_0 = \mathcal{X}_0$ and $\rho_{n+1} = \mathcal{X}_{n+1} - \mathcal{X}_n$. It is clear that the family $\{\rho_n\}_n$ form a partition of unity, that the support of \mathcal{X}_n is contained in the disk $|z| \leq 2^{n+1}$ and that the support of ρ_n is contained in the annulus $\{2^{n-1} \leq |z| \leq 2^{n+1}\}$ for $n \geq 1$.

We will denote by q_n the number of distinct points α_j in $D(0, 2^n)$, that is : $q_n = \sum_{|\alpha_j| \leq 2^n} 1$.

LEMMA 5.3.5. *There exists a \mathcal{C}^∞ function F on \mathbb{C} such that, for certain constants $A, B > 0$,*

- (i) $\frac{F^{(l)}(\alpha_j)}{l!} = w_{j,l}$ for all $j \in \mathbb{N}$, $0 \leq l \leq m_j - 1$.
- (ii) for all $z \in \mathbb{C}$, $|F(z)| \leq A e^{Bp(z)}$,
- (iii) $\bar{\partial}F = 0$ on $D(0, 1)$ and for any $n \geq 2$ and $2^{n-2} \leq |z| < 2^{n-1}$,

$$|\bar{\partial}F(z)| \leq A 2^{-n(m_1 + \dots + m_{q_n})} \prod_{k=1}^{q_n} |z - \alpha_k|^{m_k} e^{Bp(2^n)}.$$

PROOF. We set

$$F(z) = \sum_{n \geq 2} \rho_{n-2}(z) P_{q_n}(z).$$

where

$$P_q(z) = \sum_{j=1}^q \left(\sum_{l=0}^{m_j-1} \phi_{j,l}(z - \alpha_j)^l \right) \prod_{k=1}^{j-1} (z - \alpha_k)^{m_k}.$$

It is the Newton polynomial we mentioned in Remark 5.1.1.

(i) : For all $j \geq 1$ and $0 \leq l \leq m_j - 1$, if α_j is in the support of ρ_{n-2} , then $P_{q_n}^{(l)}(\alpha_j) = l! w_{j,l}$.
Thus

$$\begin{aligned} F^{(l)}(\alpha_j) &= \sum_{n \geq 2} \left(\sum_{k=0}^l C_l^k \rho_{n-2}^{(l-k)}(\alpha_j) k! w_{j,k} \right) \\ &= \sum_{k=0}^l C_l^k k! w_{j,k} \left(\sum_n \rho_n \right)^{(l-k)}(\alpha_k) = l! w_{j,l}. \end{aligned}$$

(ii) : For $z \geq 1$, let $n \geq 2$ be the integer such that $2^{n-2} \leq |z| < 2^{n-1}$. Then, we have :

$$F(z) = \rho_{n-2}(z)P_{q_n}(z) + \rho_{n-1}(z)P_{q_{n+1}}(z).$$

For all $0 \leq j \leq q_n$, we have $|\alpha_j| \leq 2^n$ and $|z - \alpha_j| \leq 2^{n+1}$. Using Lemmas 5.3.3 and condition (1), we have

$$|P_{q_n}(z)| \lesssim \exp(Bp(2^n)) \leq A \exp(Bp(2^n)) \leq A \exp(Bp(z)).$$

The same estimation holds for $P_{q_{n+1}}$ thus,

$$|F(z)| \lesssim \exp(Bp(z)).$$

(iii) Now, we want to estimate $\bar{\partial}F$.

It is clear that $F(z) = P_{q_2}(z)$ on $D(0, 1)$.

Let $|z| \geq 1$ and n the integer such that $2^{n-2} \leq |z| < 2^{n-1}$. We have

$$\bar{\partial}F(z) = \bar{\partial}\rho_{n-2}(z)P_{q_n}(z) + \bar{\partial}\rho_{n-1}(z)P_{q_{n+1}}(z).$$

Since z is outside the supports of $\bar{\partial}\mathcal{X}_{n-3}$ and of $\bar{\partial}\mathcal{X}_{n-1}$, we have

$$\bar{\partial}F(z) = -\bar{\partial}\mathcal{X}_{n-2}(z)(P_{q_{n+1}}(z) - P_{q_n}(z)) = \prod_{k=1}^{q_n} (z - \alpha_k)^{m_k} G_n(z)$$

where

$$G_n(z) = -\bar{\partial}\mathcal{X}_{n-2}(z) \sum_{j=q_n+1}^{q_{n+1}} \prod_{k=q_n+1}^{j-1} (z - \alpha_k)^{m_k} \left(\sum_{l=0}^{m_j-1} \phi_{j,l}(z - \alpha_j)^l \right).$$

For $k \leq q_{n+1}$, $|z - \alpha_k| \leq 2^{n+2}$, thus, using the estimate given by (25) then Lemma 5.3.2, we show that

$$\begin{aligned} |G_n(z)| &\leq A \exp(Bp(2^n)) 2^{-n(m_1+\dots+q_n)} \sum_{j=q_n+1}^{q_{n+1}} 2^{m_{q_n+1}+\dots+m_j} \\ &\lesssim \exp(Bp(2^n)) 2^{-n(m_1+\dots+m_{q_n})}. \end{aligned}$$

We readily obtain the desired estimate. ■

Now, when looking for a holomorphic interpolating function of the form $f = F + u$, we are led to the $\bar{\partial}$ -problem

$$\bar{\partial}u = -\bar{\partial}F.$$

The interpolation problem is then reduced to the following lemma.

LEMMA 5.3.6. *Let F be the function given by Lemma 5.3.5. There exists a subharmonic function U such that, for certain constants $A, B > 0$,*

- (i) $U(z) \simeq m_j \log |z - \alpha_j|^2$ near α_j ,
- (ii) $U(z) \leq Ap(z) + B$ for all $z \in \mathbb{C}$.
- (iii) $|\bar{\partial}F(z)|^2 e^{-U(z)} \leq Ae^{B(p(z))}$ for all $z \in \mathbb{C}$.

Admitting this lemma for a moment, we proceed with the proof of Theorem 5.2.4.

Using properties (iii) in Lemma 5.3.6 and (w1), we see that there exists a constant $E > 0$ such that

$$\int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-U(z)-Ep(z)} d\lambda(z) < \infty.$$

Hörmander's theorem 1.2.16 gives a \mathcal{C}^∞ function u such that $\bar{\partial}u = -\bar{\partial}F$ and

$$(28) \quad \int_{\mathbb{C}} \frac{|u(z)|^2 e^{-U(z)-Ep(z)}}{(1+|z|^2)^2} d\lambda(z) \leq \frac{1}{2} \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-U(z)-Ep(z)} d\lambda(z).$$

Setting $f = u + F$, it is clear that $f \in \mathcal{A}(\mathbb{C})$. Moreover, by condition (i) in Lemma 5.3.6, near α_j , $e^{-U(z)}(z - \alpha_j)^l$ is not summable for $0 \leq l \leq m_j - 1$, so we have necessarily $u^{(l)}(\alpha_j) = 0$ for all j and $0 \leq l \leq m_j - 1$ and consequently $f := u + F$ is $\frac{f^{(l)}(\alpha_j)}{l!} = w_{j,l}$.

In view of property (iii) in Lemma 1.1.4, it only remains to show that $f \in \mathcal{W}_p(\mathbb{C})$. It suffices to show that $F \in \mathcal{W}_p(\mathbb{C})$ and $u \in \mathcal{W}_p(\mathbb{C})$.

Condition (ii) in Lemma 5.3.5 gives constants $A, B > 0$ such that, for all $z \in \mathbb{C}$, $|F(z)| \leq Ae^{Bp(z)}$. By condition (w1), choosing a constant $C > 0$ large enough, we obtain

$$\int_{\mathbb{C}} |F(z)|^2 e^{-Cp(z)} d\lambda(z) < \infty,$$

in other words, $F \in \mathcal{W}_p(\mathbb{C})$.

By properties (ii) in Lemma 5.3.6 and (w1), there exist a constant $D > 0$ such that, for all $z \in \mathbb{C}$,

$$e^{-Dp(z)} \lesssim \frac{e^{-U(z)}}{(1+|z|^2)^2}.$$

Therefore,

$$\int_{\mathbb{C}} |u(z)|^2 e^{-(D+E)p(z)} d\lambda(z) \lesssim \int_{\mathbb{C}} \frac{|u(z)|^2 e^{-U(z)-Ep(z)}}{(1+|z|^2)^2} d\lambda(z) < \infty.$$

This shows that $u \in \mathcal{W}_p(\mathbb{C})$ and concludes the proof of Theorem 5.2.4. ■

PROOF OF LEMMA 5.3.6. For the sake of simplicity and up to a homotethy, we may assume that $|\alpha_k| > 2$ for all $\alpha_k \neq 0$. Besides, in the definition of the following functions V_n , we will assume $\alpha_1 \neq 0$, otherwise, we may add the term $m_1 \ln |z|$ to each V_n . We set

$$V_n(z) = \sum_{0 < |\alpha_j| \leq 2^n} m_j \log \frac{|z - \alpha_j|^2}{|\alpha_j|^2},$$

then

$$V(z) = \sum_{n \geq 2} \rho_{n-2}(z) V_n(z).$$

First, we will show that V verifies (i), (ii) and (iii). Then, we will estimate ΔV from below and add a correcting term W . The subharmonic function U will be of the form $V + W$.

(i) Let $|\alpha_k|$ be such that $2^{m-1} < |\alpha_k| < 2^{m+1}$. For $2^{m-1} < |z| < 2^{m+1}$,

$$V(z) = \rho_{m-1}(z)V_{m+1}(z) + \rho_m(z)V_{m+2}(z) + \rho_{m+1}(z)V_{m+3}(z).$$

As the ρ_n 's form a partition of unity, it is clear that $V(z) - m_k \ln |z - \alpha_k|^2$ is continuous in a neighborhood of α_k .

Note that V is smooth on $\{|z| \leq 2\}$ since we have assumed that all $|\alpha_j| > 2$.

(ii) Let $n \geq 2$ and $2^{n-2} \leq |z| < 2^{n-1}$. then

$$V(z) = \rho_{n-2}(z)V_n(z) + \rho_{n-1}(z)V_{n+1}(z).$$

For all $|\alpha_j| < 2^n$, we have $|z - \alpha_j| < 2^{n+1}$. Thus,

$$V_n(z) \leq \sum_{|\alpha_j| \leq 2^n} m_j \log \frac{2^{n+1}}{|\alpha_j|} \leq N(0, 2^{n+1}).$$

Finally, we obtain that

$$V(z) \leq N(0, 2^{n+1}) + N(0, 2^{n+2}) \lesssim p(2^n) \lesssim p(z)$$

by condition (1) and the doubling property of the weight p .

(iii) We have

$$-V(z)/2 = \sum_{|\alpha_j| \leq 2^n} m_j \ln \frac{|\alpha_j|}{|z - \alpha_j|} + \rho_{n+1}(z) \sum_{2^n < |\alpha_j| \leq 2^{n+1}} m_j \ln \frac{|\alpha_j|}{|z - \alpha_j|}.$$

Note that for all $2^n < |\alpha_j| \leq 2^{n+1}$, we have $|z - \alpha_j| > 2^n - 2^{n-1} = 2^{n-1}$. We obtain

$$\begin{aligned} -V(z)/2 &\leq \sum_{|\alpha_j| \leq 2^n} m_j \ln \frac{2^n}{|z - \alpha_j|} + \ln 4 \sum_{2^n < |\alpha_j| \leq 2^{n+1}} m_j \\ (29) \quad &\leq \ln \left(2^{n(m_1 + \dots + m_{q_n})} \prod_{j=1}^{q_n} |z - \alpha_j|^{-m_j} \right) + \ln(A \exp(Bp(2^n))) \end{aligned}$$

for certain constants $A, B > 0$ using Lemma 5.3.2. Finally, combining this inequality with (iii) of Lemma 5.3.5, we obtain

$$|\bar{\partial}F(z)| \exp(-V(z)/2) \lesssim \exp(Bp(2^n)) \lesssim \exp(Bp(z)).$$

Now, in order to get a lower bound of the laplacian, we compute $\Delta V(z)$:

$$\Delta V = \sum_{n \geq 2} \rho_{n-2} \Delta V_n + 2 \operatorname{Re} \left(\sum_n \bar{\partial} \rho_{n-2} \partial V_n \right) + \sum_{n \geq 2} \partial \bar{\partial} \rho_{n-2} V_n.$$

The first sum is positive since every V_k is subharmonic.

Let us estimate the second and the third sums, that we will denote respectively by $B(z)$ and $C(z)$. For $n \geq 2$ and $2^{n-2} \leq |z| < 2^{n-1}$, since z is outside the supports of $\bar{\partial} \mathcal{X}_{n-3}$ and of $\bar{\partial} \mathcal{X}_{n-1}$, we have

$$\begin{aligned} B(z) &= 2 \operatorname{Re} [\bar{\partial} \mathcal{X}_{n-2}(z) \partial (V_n(z) - V_{n+1}(z))] , \\ C(z) &= \partial \bar{\partial} \mathcal{X}_{n-2}(z) (V_n(z) - V_{n+1}(z)) . \end{aligned}$$

$$\begin{aligned} V_n(z) - V_{n+1}(z) &= \sum_{2^n < |\alpha_j| \leq 2^{n+1}} m_j \log \frac{|z - \alpha_j|^2}{|\alpha_j|^2} , \\ \partial (V_n(z) - V_{n+1}(z)) &= \sum_{2^n < |\alpha_j| \leq 2^{n+1}} m_j \frac{1}{z - \alpha_j} , \end{aligned}$$

and

$$|\bar{\partial} \mathcal{X}_{n-2}(z)| \lesssim \frac{1}{2^n}, \quad |\partial \bar{\partial} \mathcal{X}_{n-2}(z)| \lesssim \frac{1}{2^{2n}}.$$

For z in the support of $\bar{\partial} \mathcal{X}_{n-2}$, we have $|z| \leq 2^{n-1}$, and for $2^n \leq |\alpha_j| < 2^{n+1}$, $2^{n-1} \leq |z - \alpha_j| \leq 2^{n+2}$. Thus, we obtain that

$$|\partial \bar{\partial} \mathcal{X}_{n-2}(z) (V_{n+1}(z) - V_n(z))| \lesssim \frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}},$$

and

$$|\bar{\partial} \mathcal{X}_{n-2}(z) \partial (V_{n+1}(z) - V_n(z))| \lesssim \frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}}.$$

Finally,

$$\Delta V(z) \gtrsim -\frac{n(0, 2^{n+1}) - n(0, 2^n)}{2^{2n}} \gtrsim -\frac{n(0, 2^3|z|) - n(0, 2|z|)}{|z|^2}.$$

To construct the correcting term, W , we begin by putting

$$f(t) = \int_0^t n(0, s) ds, \quad g(t) = \int_0^t \frac{f(s)}{s^2} ds \quad \text{and} \quad W(z) = g(2^3|z|).$$

The following inequalities are easy to see :

$$f(t) \leq t n(0, t), \quad g(t) \leq \int_0^t \frac{n(0, s)}{s} ds = N(0, s).$$

Thus, by condition (1),

$$W(z) \leq N(0, 2^3|z|) \lesssim p(2^3z) \lesssim p(z)$$

Finally, to estimate the laplacian of W , we will denote $t = 2^3|z|$.

$$\Delta W(z) = \frac{1}{t} g'(t) + g''(t) = \frac{1}{t^2} (f'(t) - \frac{f(t)}{t}).$$

$$f(t) = \int_0^t n(0, s) ds = \int_0^{\frac{t}{4}} n(0, s) ds + \int_{\frac{t}{4}}^t n(0, s) ds \leq \frac{t}{4} n(0, \frac{t}{4}) + t(1 - \frac{1}{4}) n(0, t).$$

Thus,

$$f'(t) - \frac{f(t)}{t} = n(0, t) - \frac{f(t)}{t} \geq \frac{1}{4} (n(0, t) - n(0, \frac{t}{4}))$$

and

$$\Delta W(z) \gtrsim \frac{n(0, 2^3|z|) - n(0, 2|z|)}{|z|^2}.$$

Now, the desired function will be of the form

$$U(z) = V(z) + \alpha W(z),$$

where α is a positive constants chosen big enough. ■

CHAPTER 6

Expansion in series of exponential polynomials of mean-periodic functions

This chapter is a joint work with H. Ouerdiane. The main results are announced in C.R. Acad. Sci. Sér. I Math. (see [36]).

Introduction

A periodic function f with period t may be defined in terms of convolution equation as a function satisfying the convolution equation

$$(\delta_t - \delta_0) \star f = 0,$$

while a function with zero average over an interval of length $t > 0$ satisfies the convolution equation

$$\mu \star f = 0,$$

where μ is defined by $\langle \mu, f \rangle = \frac{1}{t} \int_{-t/2}^{t/2} f(x) dx$. From the observation that the second notion is more natural from the point of view of experimental physics, Delsartes generalized the concept of periodic functions by introducing in [15] the notion of "mean-periodic" functions as the solutions of homogeneous convolution equations.

In this paper, we are dealing with the fundamental principle, that is, with the expression of mean-periodic function through series of exponentials polynomials. We refer the interested reader to [5, 6, 8, 18, 43] for more about this subject.

Let θ be a Young function and θ^* its Legendre transform (see Definitions 6.1.1 and 6.1.2 below). A canonical example of a Young function is $\theta(x) = x^\mu$, $\mu > 1$, then $\theta^*(x) = x^\nu$ where $\frac{1}{\mu} + \frac{1}{\nu} = 1$.

The mean-periodic functions will lie in the space $\mathcal{F}_\theta(\mathbb{C})$ of all functions $f \in \mathcal{A}(\mathbb{C})$ such that

$$(30) \quad \sup_{z \in \mathbb{C}} |f(z)| e^{-\theta^*(m|z|)} < \infty,$$

for all constants $m > 0$.

We will also consider the limit case where $\theta(x) = x$. In this case, the associated conjugate function θ^* is formally infinite. Therefore, no growth condition of the type (30) is involved and we put $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{A}(\mathbb{C})$.

We will say that $f \in \mathcal{F}_\theta(\mathbb{C})$ is a mean-periodic function if, for a certain non zero analytic functional $T \in \mathcal{F}'_\theta(\mathbb{C})$, f verifies the convolution equation

$$(31) \quad T \star f = 0.$$

In this case, we say that f is T -mean-periodic.

For example, if we denote by $\{\alpha_k\}_k$ the zeros of the Fourier-Borel transform of T and m_k their order of multiplicity, then all exponential monomials $z^j e^{\alpha_k z}$, with $j < m_k$ are T -mean-periodic functions (see Lemma 6.2.4). Then, every convergent series whose general term is a linear combination of such exponential monomials is also a T -mean-periodic function.

Our main result (see Theorem 6.2.5) states roughly that the converse holds, provided that we apply an Abel-summation procedure in order to make the sum convergent. In fact, we prove that any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(32) \quad f(z) = \sum_k \sum_{l=0}^{m_k-1} c_{k,l} \left[\sum_{j=1}^k e^{z\alpha_j} P_{k,j,l}(z) \right],$$

where $P_{k,j,l}$ are polynomials of degree $< m_j$, explicitly given by (46) and (47) in terms of $V = \{(\alpha_k, m_k)\}_k$. Moreover, the coefficients $c_{k,l}$ verify the growth condition (37) and can be explicitly computed in terms of f and T . As we will see, the key result used here is the explicit description of the image of the restriction operator on V obtained in Chapter 5.

When $V = \{(\alpha_k, m_k)\}_k$ is an interpolating variety, the convergence of the series doesn't require any Abel-summation procedure, we simply obtain that any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(33) \quad f(z) = \sum_k e^{z\alpha_k} \sum_{l=0}^{m_k-1} d_{k,l} \frac{z^j}{j!},$$

where the coefficients $d_{k,l}$ verify the growth estimate (49) (see Theorem 6.4.2).

These results generalize those obtained in [7], where the authors considered the case $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{A}(\mathbb{C})$. In fact, they showed that, given $T \in \mathcal{H}'(\mathbb{C})$, there exists a sequence of indices $k_1 = 1 < k_2 < \dots$ such that any T -mean periodic function $f \in \mathcal{A}(\mathbb{C})$ admits a unique expansion, convergent in $\mathcal{A}(\mathbb{C})$, of the form

$$(34) \quad f(z) = \sum_{n \geq 1} \sum_{k_n \leq k < k_{n+1}} e^{\alpha_k z} \sum_{j=0}^{m_k-1} d_{k,j} \frac{z^j}{j!}.$$

In (34), the sum converges by grouping the terms rather than by an Abel-summation process. But in general, the sequence $\{k_n\}_n$ is not explicit, except in the case when V is an interpolating variety, where the sequence $k_n = n$ works, thus formula (34) leads to (33).

This chapter is organized as follows : Section 6.1 is devoted to preliminary definitions and useful results from functional analysis. The main results are stated in section 6.2 and the proof of the main theorem is given in section 6.3. Finally, in section 6.4, we study the particular case when V is an interpolating variety.

6.1. Preliminaries and definitions

DEFINITION 6.1.1. A function $\theta : [0, +\infty[\rightarrow [0, +\infty[$ is called a Young function if it is convex, continuous, increasing and verifies $\theta(0) = 0$ and $r = o(\theta(r))$ when $r \rightarrow +\infty$.

DEFINITION 6.1.2. Let θ be a Young function. The Legendre transform θ^* of θ is the function defined by

$$\theta^*(x) = \sup_{t \geq 0} (tx - \theta(t)).$$

Note that the Legendre transform of a Young function is a Young function and $\theta^{**} = \theta$. We refer the reader to [26] for further details.

Throughout the paper, θ will denote either the function $\theta(x) = x$ or a Young function.

For any $m > 0$, consider $E_{\theta,m}(\mathbb{C})$, the Banach space of all functions $f \in \mathcal{A}(\mathbb{C})$ such that

$$\|f\|_{\theta,m} := \sup_{z \in \mathbb{C}} |f(z)| e^{-\theta(m|z|)} < +\infty$$

and define

$$\mathcal{G}_\theta(\mathbb{C}) = \cup_{p \in \mathbb{N}^*} E_{\theta,p}(\mathbb{C})$$

endowed with the inductive limit topology. It is clear that $\mathcal{G}_\theta(\mathbb{C})$ is an algebra under the ordinary multiplication of functions.

REMARK 6.1.3. When $\theta(x) = x^\mu$, $\mu \geq 1$, the space $\mathcal{G}_\theta(\mathbb{C})$ coincides with $\mathcal{A}_p(\mathbb{C})$ where $p(z) = |z|^\mu$. It is the space of all entire functions of order $\leq \mu$ and of finite type. In particular, when $\mu = 1$, $\mathcal{G}_\theta(\mathbb{C})$ is the space of all entire functions of exponential type, usually denoted by $\text{Exp}(\mathbb{C})$. Note that when $\theta(x) = x^\mu$, $\mu > 1$, then $\theta^*(x) = x^\nu$ where $\frac{1}{\mu} + \frac{1}{\nu} = 1$.

We define the space $\mathcal{F}_\theta(\mathbb{C})$ as follows :

(i) In the case where $\theta(x) = x$, we put $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{A}(\mathbb{C})$, the space of all entire functions endowed with the topology of uniform convergence on every compact of \mathbb{C} . It is a Fréchet-Schwartz space (see [2]).

(ii) In the case where θ is a Young function, we denote

$$\mathcal{F}_\theta(\mathbb{C}) = \cap_{p \in \mathbb{N}^*} E_{\theta^*,1/p}(\mathbb{C})$$

endowed with the projective limit topology. The space $\mathcal{F}_\theta(\mathbb{C})$ is a nuclear Fréchet space (see [19]), hence it is a Fréchet-Schwartz space.

For any fixed $\xi \in \mathbb{C}$, and $l \in \mathbb{N}$, we will denote by $M_{l,\xi}$ the exponential monomial $z \rightarrow z^l e^{\xi z}$. It is easy to see that $M_{l,\xi} \in \mathcal{F}_\theta(\mathbb{C})$. In the next we denote by $\mathcal{F}'_\theta(\mathbb{C})$ the strong topological dual of $\mathcal{F}_\theta(\mathbb{C})$.

Let us recall some definitions and properties from functional analysis. We refer to [2] for further details in the case (i) and to [19] for the case (ii).

To any fixed $u \in \mathbb{C}$, define the translation operator τ_u on $\mathcal{F}_\theta(\mathbb{C})$ by

$$(\tau_u f)(z) = f(z + u), \text{ for all } f \in \mathcal{F}_\theta(\mathbb{C}) \text{ and } z \in \mathbb{C}.$$

It's easy to see that $\mathcal{F}_\theta(\mathbb{C})$ is invariant under these translation operators.

For all $S \in \mathcal{F}'_\theta(\mathbb{C})$ and $f \in \mathcal{F}_\theta(\mathbb{C})$, the function $z \rightarrow \langle S, \tau_z f \rangle$, where \langle, \rangle denotes the duality bracket, is an element of $\mathcal{F}_\theta(\mathbb{C})$. Therefore, for any $S \in \mathcal{F}'_\theta(\mathbb{C})$, the map $S \star : \mathcal{F}_\theta(\mathbb{C}) \rightarrow \mathcal{F}_\theta(\mathbb{C})$ defined by

$$S \star f(z) = \langle S, \tau_z f \rangle$$

is a convolution operator, i.e., it is linear, continuous and commute with any translation operator.

For any $S \in \mathcal{F}'_\theta(\mathbb{C})$, the Fourier-Borel transform of S , denoted by $\mathcal{L}(S)$ is defined by

$$\mathcal{L}(S)(\xi) = \langle S, e^{\xi \cdot} \rangle,$$

where $e^{\xi \cdot} = M_{0,\xi}$ is the function $z \in \mathbb{C} \rightarrow e^{\xi z}$.

For any two elements S and U of $\mathcal{F}'_\theta(\mathbb{C})$, the convolution product $S \star U \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\forall f \in \mathcal{F}_\theta(\mathbb{C}), \quad \langle S \star U, f \rangle = \langle S, U \star f \rangle.$$

Moreover for any $S, U \in \mathcal{F}'_\theta(\mathbb{C})$

$$\mathcal{L}(S \star U) = \mathcal{L}(S)\mathcal{L}(U)$$

Under this convolution, $\mathcal{F}'_\theta(\mathbb{C})$ is a commutative algebra admitting δ_0 , the Dirac measure at the origin, as unit.

PROPOSITION 6.1.4. *The Fourier-Borel transform \mathcal{L} is a topological isomorphism between the algebras $\mathcal{F}'_\theta(\mathbb{C})$ and $\mathcal{G}_\theta(\mathbb{C})$.*

6.2. Main results

Throughout the rest of the chapter, let T be a fixed non-zero element of $\mathcal{F}'_\theta(\mathbb{C})$. Our main goal in this section is to show that any function $f \in \mathcal{F}_\theta(\mathbb{C})$ satisfying the equation

$$(35) \quad T \star f = 0$$

can be represented as convergent series of exponential-polynomials which are them-selves solution of (35).

DEFINITION 6.2.1. We say that a function $f \in \mathcal{F}_\theta(\mathbb{C})$ is T -mean-periodic if it satisfies the equation (35).

Denote by Φ the entire function in $\mathcal{G}_\theta(\mathbb{C})$ defined by $\Phi = \mathcal{L}(T)$. Before going further, let us show the following division property :

LEMMA 6.2.2. *Let $h \in \mathcal{A}(\mathbb{C})$ and $g \in \mathcal{G}_\theta(\mathbb{C})$. If g is not identically zero and if $f = gh \in \mathcal{G}_\theta(\mathbb{C})$, then $h \in \mathcal{G}_\theta(\mathbb{C})$.*

PROOF. Up to a translation, we may assume that $g(0) \neq 0$. Let us apply the minimum modulus theorem and it's corollary given in [2, Lemma 2.2.11] to the function g in the disc of center 0 and radius $2^{n+1}e$, where n is any positive integer.

As $g \in \mathcal{G}_\theta(\mathbb{C})$, there exists $p \in \mathbb{N}^*$ and $C_p > 0$ (not depending on n) such that

$$\max_{|\xi| \leq 2^{n+3}e} |g(\xi)| \leq C_p e^{\theta(p2^n)}.$$

Thus, there exists $\varepsilon_p > 0$ (not depending on n) and $R_n, 2^n \leq R_n \leq 2^{n+1}$ such that

$$\min_{|\xi|=R_n} |g(\xi)| \geq \varepsilon_p e^{-\theta(p2^n)}.$$

Let $n \in \mathbb{N}$ and $|\xi| = R_n$. As $f \in \mathcal{G}_\theta(\mathbb{C})$, there exists $q > 0$ and $C_q > 0$ (not depending on n), such that

$$|f(\xi)| \leq C_q e^{\theta(q2^n)}.$$

Using the convexity of θ and the fact that $\theta(0) = 0$, we have

$$\theta(p2^n) = \theta\left(\frac{1}{2}p2^{n+1}\right) \leq \frac{1}{2}\theta(p2^{n+1}) + \frac{1}{2}\theta(0) = \frac{1}{2}\theta(p2^{n+1}).$$

If we assume, for example, that $p \geq q$, we deduce that

$$|h(\xi)| = |f(\xi)| \frac{1}{|g(\xi)|} \leq \frac{C_q}{\varepsilon_p} e^{\theta(q2^n) + \theta(p2^n)} \leq B_p e^{\theta(p2^{n+1})}.$$

Now let $z \in \mathbb{C}$, be such that $2^{n-1} \leq |z| \leq 2^n < R_n$. By the maximum modulus theorem,

$$|h(z)| \leq B_p e^{\theta(p2^{n+1})} \leq B_p e^{\theta(4p|z|)}.$$

This proves that $h \in \mathcal{G}_\theta(\mathbb{C})$. ■

COROLLARY 6.2.3. *In the case where Φ has no zeros, the only mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ is the zero function.*

PROOF. Assume that Φ has no zeros. Then $\frac{1}{\Phi} \in \mathcal{A}(\mathbb{C})$ and, by Lemma 6.2.2, $\frac{1}{\Phi} \in \mathcal{G}_\theta(\mathbb{C})$. By Proposition 6.1.4, $S = (\mathcal{L})^{-1}(\frac{1}{\Phi}) \in \mathcal{F}'_\theta(\mathbb{C})$. Then, we have $S \star T = T \star S = \delta_0$. If we assume $T \star f = 0$, then $\delta_0 \star f = f = 0$. ■

We will throughout the rest of the paper assume that Φ has zeros, and denote them by $|\alpha_0| \leq |\alpha_1| \leq \dots \leq |\alpha_k| \leq \dots$, $\alpha_k \neq \alpha_{k'}$ if $k \neq k'$.

We will denote by m_k be the order of multiplicity of Φ at α_k and we will consider the multiplicity variety $V = Z(\Psi) = \{(\alpha_k, m_k)\}_{k \in \mathbb{N}}$.

LEMMA 6.2.4. (i) *For all $\xi \in \mathbb{C}$ and $l \in \mathbb{N}$, we have $\langle T, M_{l,\xi} \rangle = \Phi^{(l)}(\xi)$.*
(ii) *Each exponential monomial M_{l,α_k} , for $0 \leq l < m_k$ is T -mean-periodic.*

PROOF. To prove (i), we proceed by induction on $l \geq 0$. The property is true for $l = 0$ by definition of the Fourier-Borel transform of T .

Suppose the property true for l . Let $\xi \in \mathbb{C}$ be fixed. Let us verify that the function $\frac{M_{l,\xi+u} - M_{l,\xi}}{u}$ converges in $\mathcal{F}_\theta(\mathbb{C})$ to $M_{l+1,\xi}$ when u tends to 0. For all $z \in \mathbb{C}$ and $u \leq 1$, we have

$$\left| \frac{e^{uz} - 1}{u} - z \right| = \left| uz^2 \sum_{n \geq 2} \frac{(uz)^{n-2}}{n!} \right| \leq |u||z|^2 e^{|z|}.$$

This implies that

$$\left| \frac{M_{l,\xi+u}(z) - M_{l,\xi}(z)}{u} - M_{l+1,\xi}(z) \right| = |z|^l e^{\xi z} \left| \frac{e^{uz} - 1}{u} - z \right| \leq |u||z|^{l+2} e^{(1+|\xi|)|z|}.$$

Therefore, $\frac{M_{l,\xi+u} - M_{l,\xi}}{u}$ converges to $M_{l+1,\xi}$ for the topology of $\mathcal{F}_\theta(\mathbb{C})$ when u tends to 0.

From this, we obtain

$$\begin{aligned} \Phi^{(l+1)}(\xi) &= \lim_{u \rightarrow 0} \frac{\Phi^{(l)}(\xi + u) - \Phi^{(l)}(\xi)}{u} = \lim_{u \rightarrow 0} \frac{\langle T, M_{l,\xi+u} \rangle - \langle T, M_{l,\xi} \rangle}{u} \\ &= \lim_{u \rightarrow 0} \langle T, \frac{M_{l,\xi+u} - M_{l,\xi}}{u} \rangle = \langle T, M_{l+1,\xi} \rangle, \end{aligned}$$

by continuity of T . This completes the proof of (i).

In order to prove (ii), it is sufficient to see that

$$T \star M_{l,\alpha_k}(z) = \langle T, \tau_z M_{l,\alpha_k} \rangle = e^{\alpha_k z} \sum_{n=0}^l C_l^n z^{l-n} \langle T, M_{n,\alpha_k} \rangle = e^{\alpha_k z} \sum_{n=0}^l C_l^n z^{l-n} \Phi^{(n)}(\alpha_k).$$

■

Our main theorem states roughly that T -mean-periodic functions are series of linear combinations of the exponential monomials M_{l,α_k} :

THEOREM 6.2.5. (i) Any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(36) \quad f(z) = \sum_{k \geq 0} \sum_{l=0}^{m_k-1} c_{k,l} \left[\sum_{j=1}^k e^{z\alpha_j} P_{k,j,l}(z) \right],$$

where $P_{k,j,l}$ are the polynomials of degree $< m_j$ given by (46) and (47). The coefficients $c_{k,l}$ verify the following estimate

$$(37) \quad \forall m > 0, \quad \sum_{k \geq 0} e^{\theta(m|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)} \right) < +\infty$$

and are given by

$$c_{k,l} = \langle S_{k,l}, f \rangle$$

where $S_{k,l} \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(S_{k,l})(\xi) = (\xi - \alpha_k)^l \prod_{n=1}^{k-1} (\xi - \alpha_n)^{m_n}.$$

(ii) Conversely, any such serie whose coefficients $c_{n,l}$ satisfy the estimate (37) converges in $\mathcal{F}_\theta(\mathbb{C})$ to a function f solving the equation (35).

COROLLARY 6.2.6. Assume that all the multiplicities m_k are equal to 1. Then

(i) any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(38) \quad f(z) = \sum_{k \geq 0} c_k \left[\sum_{j=0}^k e^{z\alpha_j} \prod_{0 \leq n \leq k, n \neq j} (\alpha_j - \alpha_n)^{-1} \right],$$

where the coefficients c_k satisfy the following estimate

$$(39) \quad \forall m > 0, \quad \sum_{k \geq 0} e^{\theta(m|\alpha_k|)} |c_k| (|\alpha_k| + 1)^{-(k-1)} < +\infty$$

and are given by

$$c_k = \langle S_k, f \rangle$$

where $S_k \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(S_k)(\xi) = \prod_{n=0}^{k-1} (\xi - \alpha_n)^{m_n}.$$

(ii) Conversely, any such series whose coefficients c_k satisfy the estimate (39) converges in $\mathcal{F}_\theta(\mathbb{C})$ to a function f solving the equation (35).

6.3. Proof of the main theorem

It will be a crucial point in the proof of the main theorem, to determine the image of the restriction operator ρ defined on $\mathcal{G}_\theta(\mathbb{C})$ by

$$\rho(g) = \left\{ \frac{g^l(\alpha_k)}{l!} \right\}_{k \geq 0, 0 \leq l < m_k}, \quad g \in \mathcal{G}_\theta(\mathbb{C}).$$

We choose to denote this operator in this chapter by ρ rather than \mathcal{R}_V to simplify the notations. As an immediate consequence of Lemma 6.2.2, we have the following lemma.

LEMMA 6.3.1. *The kernel of the restriction operator ρ is the ideal generated by Φ in $\mathcal{G}_\theta(\mathbb{C})$, i.e.,*

$$\text{Ker } \rho = \{\Phi g, g \in \mathcal{G}_\theta(\mathbb{C})\}.$$

We are going to use a characterization, obtained in [37], of the elements $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_{k-1}}$ belonging to $\rho(\mathcal{G}_\theta(\mathbb{C}))$. This characterization is given in terms of growth conditions involving the divided differences (see [23] for further details about divided differences).

To any discrete doubly indexed sequence $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ of complex numbers, we associate the sequence of divided differences $\Psi(a) = \{b_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$. We recall that they are the coefficients of the Newton polynomials,

$$(40) \quad Q_q(\xi) = \sum_{k=0}^q \prod_{n=0}^{k-1} (\xi - \alpha_n)^{m_n} \left(\sum_{l=0}^{m_k-1} b_{k,l} (\xi - \alpha_k)^l \right),$$

defined, for any $q \geq 1$, as the unique polynomial of degree $m_0 + \dots + m_q - 1$ such that

$$\frac{Q_q^{(l)}(\alpha_k)}{l!} = a_{k,l}, \quad \text{for } 0 \leq k \leq q \text{ and } 0 \leq l \leq m_k - 1.$$

When all the multiplicities $m_k = 1$, we may give a simple formula for the coefficients b_k :

$$b_k = \sum_{j=0}^k a_j \prod_{0 \leq n \leq k, n \neq j} (\alpha_j - \alpha_n)^{-1}.$$

In the general case, see [37] we define them by induction :

$$b_{1,l} = a_{1,l}, \text{ for all } 0 \leq l \leq m_1 - 1,$$

$$b_{k,0} = \frac{a_{k,0} - Q_{k-1}(\alpha_k)}{\prod_{k-1}(\alpha_k)},$$

$$b_{k,l} = \frac{a_{k,l} - \frac{Q_{k-1}^{(l)}(\alpha_k)}{l!} - \sum_{n=0}^{l-1} \frac{1}{(l-n)!} \Pi_{k-1}^{(l-k)}(\alpha_j) b_{k,n}}{\Pi_{k-1}(\alpha_k)} \quad \text{for } 1 \leq l \leq m_k - 1$$

where we have denoted by

$$\Pi_k(\xi) = \prod_{n=0}^k (\xi - \alpha_n)^{m_n}, \text{ for all } k \geq 0.$$

In [37], we showed that the elements $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ belonging to $\rho(\mathcal{G}_\theta(\mathbb{C}))$ are precisely those verifying the growth condition

$$\sup_{k \geq 0} \sup_{0 \leq l < m_k} |b_{k,l}| (|\alpha_k| + 1)^{m_1 + \dots + m_{k-1} + l} e^{-\theta(m|\alpha_k|)} < +\infty,$$

for a certain $m > 0$, where $b = \{b_{k,l}\}_{k, 0 \leq l < m_k} = \Psi^{-1}(a)$.

In order to give a topological structure to this space, let us denote by $\mathcal{B}_{\theta,m}(V)$ the Banach space of all doubly indexed sequences of complex numbers $b = \{b_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that

$$\|b\|_{\theta,m} = \sup_{k \in \mathbb{N}} \sup_{0 \leq l < m_k} |b_{k,l}| (|\alpha_k| + 1)^{m_1 + \dots + m_{k-1} + l} e^{-\theta(m|\alpha_k|)} < +\infty.$$

Let us consider the space $\mathcal{A}_{\theta,m}(V) = \Psi^{-1}(\mathcal{B}_{\theta,m}(V))$, that is, the space of all doubly indexed sequences of complex numbers $a = \{a_{k,l}\}_{k \in \mathbb{N}^*, 0 \leq l < m_k}$ such that

$$\|\Psi(a)\|_{\theta,m} < +\infty.$$

It is easy to see that $\mathcal{A}_{\theta,m}(V)$ endowed with the norm $\|a\|_{\theta,m} = \|\Psi(a)\|_{\theta,m}$ is a Banach space and that Ψ is an isometry from $\mathcal{A}_{\theta,m}(V)$ into $\mathcal{B}_{\theta,m}(V)$.

Now, we define the spaces

$$\mathcal{A}_\theta(V) = \cup_{p \in \mathbb{N}^*} \mathcal{A}_{\theta,p}(V) \text{ and } \mathcal{B}_\theta(V) = \cup_{p \in \mathbb{N}^*} \mathcal{B}_{\theta,p}(V)$$

endowed with the topology of inductive limit of Banach spaces.

We define the linear map

$$\alpha = \Psi \circ \rho \circ \mathcal{L} : \mathcal{F}'_\theta(\mathbb{C}) \rightarrow \mathcal{B}_\theta(V).$$

PROPOSITION 6.3.2. *The map α is continuous and surjective.*

PROOF. By Proposition 6.1.4, we know that $\mathcal{L} : \mathcal{F}'_\theta(\mathbb{C}) \rightarrow \mathcal{G}_\theta(\mathbb{C})$ is a topological isomorphism.

The fact that the operator

$$\rho : \mathcal{G}_\theta(\mathbb{C}) \rightarrow \mathcal{A}_\theta(V)$$

is continuous and surjective are easy consequences of Proposition 5.2.1 and Theorem 5.2.4.

Finally, by construction, it is clear that

$$\Psi : \mathcal{A}_\theta(V) \rightarrow \mathcal{B}_\theta(V)$$

is a topological isomorphism. ■

Recall that $\mathcal{F}_\theta(\mathbb{C})$ is a Fréchet-Schwartz space, therefore it is reflexive. Then, the transpose α^t of α is defined from the strong dual of $\mathcal{B}_\theta(V)$, denoted by $\mathcal{B}'_\theta(V)$, into $\mathcal{F}_\theta(\mathbb{C})$.

In the next lemma, we will characterize the dual space $\mathcal{B}'_\theta(V)$.

LEMMA 6.3.3. *The space $\mathcal{B}'_\theta(V)$ is topologically isomorphic through the canonical bilinear form*

$$\langle c, b \rangle = \sum_{k=0}^{+\infty} \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}$$

to the space $\mathcal{C}_\theta(V) = \bigcap_{p \in \mathbb{N}^*} \mathcal{C}_{\theta,p}(V)$ endowed with the projective limit topology, where, for all p , $\mathcal{C}_{\theta,p}(V)$ is the Banach space of the sequences $c = \{c_{k,l}\}_{k \geq 0, 0 \leq l < m_k}$ such that

$$(41) \quad \|c\|'_{\theta,p} := \sum_{k \geq 0} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)} \right) < +\infty.$$

Moreover, $\mathcal{C}_\theta(V)$ is a Fréchet-Schwartz space.

PROOF. Let us show that $\beta : \mathcal{C}_\theta(V) \rightarrow \mathcal{B}'_\theta(V)$ defined by

$$\beta(c)(b) = \langle c, b \rangle = \sum_{k=0}^{+\infty} \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}$$

is a topological isomorphism.

Let $c = \{c_{k,l}\}_{k, 0 \leq l < m_k}$ be an element of $\mathcal{C}_\theta(V)$ and $b = \{b_{k,l}\}_{k, 0 \leq l < m_k} \in \mathcal{B}_{\theta,p}(V)$, for a certain p . For any $k \geq 0$, we have, by definition of $\|b\|_{\theta,p}$,

$$\sum_{l=0}^{m_k-1} |b_{k,l} c_{k,l}| \leq e^{\theta(p|\alpha_k|)} \|b\|_{\theta,p} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)}.$$

Using the estimate (41), we see that the sum converges (absolutely) and that

$$|\langle c, b \rangle| \leq \|c\|'_{\theta,p} \|b\|_{\theta,p}.$$

This shows the continuity of β . Let $B^{k,l}$ be the doubly indexed sequence of \mathbb{C} defined by (using the Kronecker symbols) :

$$(42) \quad B^{k,l} = \{\delta_{kj} \delta_{ln}\}_{j, 0 \leq n < m_j}.$$

We easily see that $B^{k,l} \in \mathcal{B}_{\theta,p}(V)$. For all k and $0 \leq l < m_k$, we have $c_{k,l} = \langle \beta(c), B^{k,l} \rangle$. It is then clear that β is injective. Conversely, to an element $\nu \in \mathcal{B}'_\theta(V)$, consider the doubly indexed sequence $c = \{c_{k,l}\}_{k, 0 \leq l < m_k}$ defined by

$$c_{k,l} = \langle \nu, B^{k,l} \rangle$$

We verify that $c \in \mathcal{C}_\theta(V)$. let $p \in \mathbb{N}^*$ be fixed and define $\tilde{b} = \{\tilde{b}_{k,l}\}_{k, 0 \leq l < m_k}$ by

$$\tilde{b}_{k,l} = e^{\theta(p|\alpha_k|)} \frac{\bar{c}_{k,l}}{|c_{k,l}|} (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)} \text{ if } c_{k,l} \neq 0, \quad \tilde{b}_{k,l} = 0 \text{ otherwise.}$$

It is clear that $\tilde{b} \in \mathcal{B}_{\theta,p}(V)$ and that $\|\tilde{b}\|_{\theta,p} \leq 1$.

Therefore, all the finite sequences $\tilde{b}^K = \sum_{k=0}^K \sum_{l=0}^{m_k-1} \tilde{b}_{k,l} B^{k,l}$ satisfy

$$\|\tilde{b}^K\|_{\theta,p} \leq 1.$$

Denoting by $\|\nu\|_p$ the operator norm, we have, for all K ,

$$|\langle \nu, \tilde{b}^K \rangle| \leq \|\nu\|_p \|\tilde{b}^K\|_{\theta,p} \leq \|\nu\|.$$

On the other hand,

$$\langle \nu, \tilde{b}^K \rangle = \sum_{k=0}^K \sum_{l=0}^{m_k-1} \tilde{b}_{k,l} \langle \nu, B^{k,l} \rangle = \sum_{k=0}^K e^{\theta(p|\alpha_k|)} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)},$$

by definition of $\tilde{b}_{k,l}$. Letting K tend to infinity, we obtain that $c \in \mathcal{C}_{\theta,p}(V)$ and that

$$(43) \quad \|c\|_{\theta,p} \leq \|\nu\|_p.$$

Consider now an element $b = \{b_{k,l}\}_{k,0 \leq l < m_k}$ of $\mathcal{B}_{\theta,p}(V)$ and put $b^K = \sum_{k=0}^K \sum_{l=0}^{m_k-1} b_{k,l} B^{k,l}$. Let q be an integer strictly larger than p . Note that by convexity of θ , for all k the following inequality holds

$$(44) \quad -\theta(q|\alpha_k|) + \theta(p|\alpha_k|) \leq -(1 - p/q)\theta(q|\alpha_k|).$$

Using this inequality, we find

$$\|b - b^K\|_{\theta,q} \leq \|b\|_{\theta,p} \sup_{k > K} e^{-\theta(q|\alpha_k|) + \theta(p|\alpha_k|)} \leq \|b\|_{\theta,p} e^{-(1-p/q)\theta(q|\alpha_K|)}.$$

We readily deduce that b^K converges to b when K tends towards infinity and that $\beta(c) = \nu$.

The continuity of β^{-1} is a direct consequence of the inequality (43). We have proved that β is a topological isomorphism.

In order to prove that $\mathcal{C}_\theta(V)$ is a Fréchet-Schwartz space, in view of [2, Proposition 1.4.8.], it is sufficient to see that, for any $p \in \mathbb{N}^*$, the canonical injection

$$i_p : \mathcal{C}_{\theta,p+1}(V) \rightarrow \mathcal{C}_{\theta,p}(V)$$

is compact. Let $\{c^n\}_n$ be a sequence of elements in $\mathcal{C}_{\theta,p+1}(V)$ such that, for all n , $\|c^n\|_{\theta,p+1} \leq 1$. It suffices to show that one can extract a subsequence of $\{c^n\}_n$ converging in $\mathcal{C}_{\theta,p}(V)$.

It is easy to see that, for all $k \in \mathbb{N}$ and $0 \leq l < m_k$ the sequence $\{c_{k,l}^n\}_n$ is bounded. Thus, up to taking a subsequence, we may assume without loss of generality that $c_{k,l}^n$ converges to a certain $c_{k,l} \in \mathbb{C}$. Putting $c = \{c_{k,l}\}_{k,0 \leq l < m_k}$, we readily see that $c \in \mathcal{C}_{\theta,p+1}(V)$ and $\|c\|_{\theta,p+1} \leq 1$.

Let us verify that $\|c^n - c\|_{\theta,p}$ tends to 0 when n tends to infinity. We assume that $|\alpha_k| \rightarrow \infty$, otherwise, the result is trivial. Then, again using inequality (44) we find that $e^{\theta(p|\alpha_k|) - \theta((p+1)|\alpha_k|)}$ tends to 0 when k tends towards infinity. Let $\varepsilon > 0$. For a certain $K \in \mathbb{N}$ and for all $k \geq K$,

$e^{\theta(p|\alpha_k|)-\theta((p+1)|\alpha_k|)} < \frac{\varepsilon}{4}$. Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned} & \sum_{k \geq K} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}^n - c_{k,l}| (|\alpha_k| + 1)^{-(m_1+\dots+m_{k-1}+l)} \right) \\ & \leq \frac{\varepsilon}{4} \sum_{k \geq K} e^{\theta((p+1)|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}^n - c_{k,l}| (|\alpha_k| + 1)^{-(m_1+\dots+m_{k-1}+l)} \right) \\ & \leq \frac{\varepsilon}{4} \|c^n - c\|_{\theta,p+1} \leq \frac{\varepsilon}{4} (\|c^n\|_{\theta,p+1} + \|c\|_{\theta,p+1}) \leq \frac{\varepsilon}{2}. \end{aligned}$$

Moreover, for a certain $N \in \mathbb{N}$ and for all $n \geq N$, we have

$$\sum_{k=0}^{K-1} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |c_{k,l}^n - c_{k,l}| (|\alpha_k| + 1)^{-(m_1+\dots+m_{k-1}+l)} \right) \leq \frac{\varepsilon}{2}.$$

Finally, for $n \geq N$, $\|c^n - c\|_{\theta,p} < \varepsilon$. ■

From now on, we will identify $\mathcal{B}'_{\theta}(V)$ with the space $\mathcal{C}_{\theta}(V)$. The duality is given by, for all $c = \{c_{k,l}\}_{k,0 \leq l < m_k} \in \mathcal{C}_{\theta}(V)$ and for all $b = \{b_{k,l}\}_{k,0 \leq l < m_k} \in \mathcal{B}_{\theta}(V)$,

$$(45) \quad \langle c, b \rangle = \sum_{k=0}^{+\infty} \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}.$$

The next step is to prove the following lemma :

LEMMA 6.3.4. (i) α^t is a topological isomorphism onto its image and $\text{Im } \alpha^t = (\text{Ker } \alpha)^{\circ}$, the orthogonal space of $\text{Ker } \alpha$.

(ii) $\text{Ker } \alpha = \{T \star U, U \in \mathcal{F}'_{\theta}(\mathbb{C})\}$.

(iii) $(\text{Ker } \alpha)^{\circ} = \text{Ker } T \star = \{f \in \mathcal{F}_{\theta}(\mathbb{C}) \mid T \star f = 0\}$.

PROOF. (i) From Proposition 6.3.2, α is a surjective continuous linear map. Therefore, α^t is a topological isomorphism onto its image and $\text{Im } \alpha^t = (\text{Ker } \alpha)^{\circ}$ (see [2, Proposition 1.4.12]).

(ii) Recalling Remark 6.3.1, we have

$$\text{Ker } \alpha = \text{Ker } (\rho \circ \mathcal{L}) = \mathcal{L}^{-1}(\text{Ker } \rho) = \{T \star \mathcal{L}^{-1}(g), g \in \mathcal{G}_{\theta}(\mathbb{C})\} = \{T \star U, U \in \mathcal{F}'_{\theta}(\mathbb{C})\}.$$

(iii) Let f be an element of $(\text{Ker } \alpha)^{\circ}$. For all $z \in \mathbb{C}$,

$$(T \star f)(z) = \langle T, \tau_z f \rangle = \langle T, \delta_z \star f \rangle = \langle T \star \delta_z, f \rangle = 0,$$

using the fact that $T \star \delta_z \in \text{Ker } \alpha$.

Conversely, let $f \in \mathcal{F}_{\theta}(\mathbb{C})$ be such that $T \star f = 0$ and let $U \in \mathcal{F}'_{\theta}(\mathbb{C})$. We have

$$\langle T \star U, f \rangle = \langle U, T \star f \rangle = 0.$$

This shows that $f \in (\text{Ker } \alpha)^{\circ}$ and concludes the proof of the lemma. ■

Let us proceed with the proof of Theorem 6.2.5.

(i) Let $f \in \mathcal{F}_\theta(\mathbb{C})$ be a T -mean-periodic function, that is, $f \in \text{Ker } T_\star$. From Lemmas 6.3.4 and 6.3.3, there is a unique sequence $c \in \mathcal{C}_\theta(V)$ such that $f = \alpha^t(c)$.

For $z \in \mathbb{C}$, denoting by δ_z the Dirac measure at z , we have

$$f(z) = \langle \delta_z, f \rangle = \langle \delta_z, \alpha^t(c) \rangle = \langle c, \alpha(\delta_z) \rangle = \langle c, \Psi(\rho(g_z)) \rangle$$

where we have denoted by $g_z = \mathcal{L}(\delta_z)$, that is, the function in $\mathcal{G}_\theta(\mathbb{C})$ defined by $g_z(\xi) = e^{z\xi}$.

Let us compute $\Psi(\rho(g_z)) = b(z) = \{b_{k,l}(z)\}_{k,0 \leq l < m_k}$, which is an element of $\mathcal{B}_\theta(V)$. By well know formulas about Newton polynomials (See, for example [2, Definition 6.2.8]), we have, for $k \in \mathbb{N}$, and denoting by

$$\partial_j^m = \frac{1}{m!} \frac{\partial^m}{\partial \alpha_j^m},$$

for $0 \leq l < m_k$,

$$b_{k,l}(z) = \partial_1^{m_1-1} \dots \partial_{k-1}^{m_{k-1}-1} \partial_k^l \left(\sum_{j=0}^k e^{z\alpha_j} \prod_{0 \leq n \leq k, n \neq j} (\alpha_j - \alpha_n)^{-1} \right) = \sum_{j=0}^k e^{z\alpha_j} P_{k,j,l}(z),$$

where we have denoted by

$$(46) \quad P_{k,j,l}(z) = \sum_{i=0}^{m_k-1} \frac{z^i}{i!} \partial_i^{m_j-1-i} \left(\prod_{0 \leq n \leq k-1, n \neq j} (\alpha_j - \alpha_n)^{-m_n} (\alpha_j - \alpha_k)^{-(l+1)} \right)$$

and, for $0 \leq l < m_k$,

$$(47) \quad P_{k,k,l}(z) = \sum_{i=0}^l \frac{z^i}{i!} \partial_k^{l-i} \left(\prod_{0 \leq n \leq k-1} (\alpha_k - \alpha_n)^{-m_n} \right).$$

Thus,

$$f(z) = \sum_{k \geq 0} \left(\sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right) = \sum_{k \geq 0} \left(\sum_{l=0}^{m_k-1} c_{k,l} \sum_{j=0}^k e^{z\alpha_j} P_{k,j,l}(z) \right).$$

So, the equality (38) is established.

Let us verify the convergence in $\mathcal{F}_\theta(\mathbb{C})$ of the series.

Case where $\theta(x) = x$. Here, $\mathcal{F}_\theta(\mathbb{C}) = \mathcal{A}(\mathbb{C})$. We have to verify that the serie converges uniformly on every compact of \mathbb{C} . Let $p \in \mathbb{N}^*$ and $z \in \mathbb{C}$, $|z| \leq p$.

We have, for all $\xi \in \mathbb{C}$, $|g_z(\xi)| = |e^{z\xi}| \leq e^{p|\xi|}$, that is,

$$\|g_z\|_{\theta,p} \leq 1.$$

Thus, by continuity of $\Psi \circ \rho$, there exists $p' \in \mathbb{N}^*$ and $C_p > 0$ such that

$$\|b(z)\|_{\theta,p'} \leq C_p \|g_z\|_{\theta,p} \leq C_p.$$

For all $k \geq 0$, we have

$$\left| \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right| \leq \|b(z)\|_{\theta,p'} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)} e^{\theta(p'|\alpha_n|)}$$

We obtain

$$\sup_{|z| \leq p} \left| \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right| \leq C_p \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)} e^{\theta(p'|\alpha_k|)}$$

Recalling that $c \in C'_\theta(V)$, the right term is the general term of a convergent serie, thus, the right-hand side of (38) is convergent in $\mathcal{F}_\theta(\mathbb{C})$. Moreover,

$$\sup_{|z| \leq p} \sum_{k \geq 0} \left| \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right| \leq C_p \|c\|'_{\theta, p'}.$$

Case where θ is a Young function.

For any $p \in \mathbb{N}^*$, observe that

$$\|g_z\|_{\theta, p} \leq e^{\theta^*(\frac{1}{p}|z|)}.$$

Thus, by continuity of $\Psi \circ \rho$, there exists $p' \in \mathbb{N}^*$ and $C_p > 0$ such that

$$\|b(z)\|_{\theta, p'} \leq C_p \|g_z\|_p \leq C_p e^{\theta^*(\frac{1}{p}|z|)}.$$

For all $k \geq 0$ and $z \in \mathbb{C}$, we have

$$\left| \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right| \leq \|b(z)\|_{\theta, p'} \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{m_1 + \dots + m_{k-1} + l} e^{-\theta(p'|\alpha_k|)}$$

We obtain

$$\sup_{z \in \mathbb{C}} \left| \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right| e^{-\theta^*(\frac{1}{p}|z|)} \leq C_p \sum_{l=0}^{m_k-1} |c_{k,l}| (|\alpha_k| + 1)^{m_1 + \dots + m_{k-1} + l} e^{-\theta(p'|\alpha_k|)}.$$

As in the previous case, we deduce that the right-hand side of (38) is absolutely convergent in $\mathcal{F}_\theta(\mathbb{C})$. Moreover,

$$\sup_{z \in \mathbb{C}} \sum_{k \geq 0} \left| \sum_{l=0}^{m_k-1} c_{k,l} b_{k,l}(z) \right| \leq C_p \|c\|'_{\theta, p'}.$$

In order to find an explicit formula for the coefficients $c_{n,l}$, consider the elements $B^{k,l}$ of $\mathcal{B}_\theta(\mathbb{C})$ defined by (42) and observe that, by the definition of the Newton polynomials (see (40)) with respect to the coefficients of $B^{k,l}$, for all $q \geq k$, we have

$$Q_q(\xi) = (\xi - \alpha_k)^l \prod_{l=0}^{k-1} (\xi - \alpha_l)^{m_l}$$

and for $q < k$, $Q_q = 0$. We readily deduce that $\alpha(S_{k,l}) = \Psi \circ \rho \circ \mathcal{L}(S_{k,l}) = B^{k,l}$.

Now, for all $k \in \mathbb{N}$ and $0 \leq l < m_k$,

$$\langle S_{k,l}, f \rangle = \langle S_{k,l}, \alpha^t(c) \rangle = \langle \alpha(S_{k,l}), c \rangle = \langle B^{k,l}, c \rangle = c_{k,l}.$$

(ii) The converse part is easily deduced from the proof of (i) and Lemma 6.2.4.

6.4. Case of an interpolating variety

DEFINITION 6.4.1. We say that V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$ if, for any doubly indexed sequence $a = \{a_{k,l}\}_{k \in \mathbb{N}^*, 0 \leq l < m_k}$ such that, for a certain $m > 0$,

$$\sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| e^{-\theta(m|\alpha_k|)} < +\infty,$$

there exists a function $g \in \mathcal{G}_\theta(\mathbb{C})$ such that, for all k and all $0 \leq l < m_k - 1$,

$$\frac{g^l(\alpha_k)}{l!} = a_{k,l}.$$

We assume from now on that $V = Z(\Phi)$ is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$. Then we have the following result :

THEOREM 6.4.2. (i) Any T -mean-periodic function $f \in \mathcal{F}_\theta(\mathbb{C})$ admits the following expansion as a convergent series in $\mathcal{F}_\theta(\mathbb{C})$

$$(48) \quad f(z) = \sum_{k \geq 0} e^{z\alpha_k} \sum_{l=0}^{m_k-1} d_{k,l} \frac{z^l}{l!},$$

where the coefficients $a_{k,l}$ verify the following estimate :

$$(49) \quad \sum_{k \geq 0} e^{\theta(m|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |d_{k,l}| \right) < +\infty$$

for every $m > 0$. Moreover, for all $k \in \mathbb{N}$ and $0 \leq l < m_k$, we have the equality

$$d_{k,l} = \langle T_{k,l}, f \rangle$$

where $T_{k,l} \in \mathcal{F}'_\theta(\mathbb{C})$ is defined by

$$\mathcal{L}(T_{k,l})(\xi) = \Phi(\xi)(\xi - \alpha_k)^{l-m_k} (m_k! / \Phi^{(m_k)}(\alpha_k)),$$

(ii) Conversely, any such series whose coefficients $d_{k,l}$ satisfy these estimate (49) converges in $\mathcal{F}_\theta(\mathbb{C})$ to a function f solving the equation (31).

Note that $\mathcal{L}(T_{k,l}) \in \mathcal{G}_\theta(\mathbb{C})$ by Proposition 6.2.2.

REMARK 6.4.3. In the case where $\theta(x) = x$, this is also a consequence of [2, Theorem 6.2.6.].

We will denote by $\mathcal{A}_{\theta,m}(V)$ the space of all doubly indexed sequences of complex numbers $a = \{a_{k,l}\}_{k \in \mathbb{N}, 0 \leq l < m_k}$ such that

$$\|a\|_{\theta,m} := \sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| e^{-\theta(m|\alpha_k|)} < +\infty$$

and

$$\mathcal{A}_\theta(V) = \cup_{p \in \mathbb{N}^*} \mathcal{A}_{\theta,p}(V)$$

endowed with the strict inductive limit of Banach spaces.

We define the linear map

$$\alpha = \rho \circ \mathcal{L} : \mathcal{F}'_\theta(\mathbb{C}) \rightarrow \mathcal{A}_\theta(V).$$

PROPOSITION 6.4.4. *The map α is continuous and surjective.*

PROOF. It is sufficient to show that the map $\rho : \mathcal{G}_\theta(\mathbb{C}) \rightarrow \mathcal{A}_\theta(V)$ is surjective and continuous. The surjectivity follows from the fact that V is an interpolating variety.

In order to show the continuity, let $g \in \mathcal{G}_{\theta,p}(\mathbb{C})$ and let $z \in \mathbb{C}$. By the Cauchy estimates applied to the disc of center z and radius 2, for all $l \in \mathbb{N}$,

$$\left| \frac{g^l(z)}{l!} \right| \leq \frac{1}{2^l} \sup_{|\xi-z| \leq 2} |g(\xi)|.$$

For $|\xi - z| \leq 2$, we have

$$|g(\xi)| \leq \|g\|_{\theta,p} e^{\theta(p|\xi|)} \leq \|g\|_{\theta,p} e^{\theta(2p+p|z|)} \leq \|g\|_{\theta,p} e^{1/2\theta(4p)} e^{1/2\theta(2p|z|)}$$

by convexity of θ . Thus,

$$\sum_{l=0}^{\infty} \left| \frac{g^l(z)}{l!} \right| \leq 2 \|g\|_{\theta,p} e^{1/2\theta(2p)} e^{\theta(2p|z|)}.$$

In particular, we deduce that $\rho(g) \in \mathcal{A}_{\theta,2p}(V)$ and that

$$\|\rho(g)\|_{\theta,2p} \leq 2 \|g\|_{\theta,p} e^{1/2\theta(2p)}.$$

The continuity of ρ follows from the last inequality and the surjectivity from the definition 6.4.1. ■

We need to characterize $\mathcal{A}'_\theta(V)$, the strong dual of $\mathcal{A}_\theta(V)$.

LEMMA 6.4.5. *The space $\mathcal{A}'_\theta(V)$ is topologically isomorphic to the space $\mathcal{D}_\theta(V) = \cap_{p \in \mathbb{N}^*} \mathcal{C}_{\theta,p}(V)$ endowed with the projective limit topology, where, for all p , $\mathcal{D}_{\theta,p}(V)$ is the Banach space of the sequences $d = \{d_{k,l}\}_{k,0 \leq l < m_k}$ such that*

$$(50) \quad \|d\|'_{\theta,p} := \sum_{k \geq 0} e^{\theta(p|\alpha_k|)} \left(\sum_{l=0}^{m_k-1} |d_{k,l}| (|\alpha_k| + 1)^{-(m_1 + \dots + m_{k-1} + l)} \right) < +\infty.$$

Moreover, $\mathcal{D}_\theta(V)$ is a Fréchet-Schwartz space.

The proof is similar to the one of Lemma 6.3.3. Lemma 6.3.1 still holds with the new definition of α .

We are now ready to prove Theorem 6.4.2.

By Lemma 6.3.1, any T -mean-periodic function f is the image by α^t of a unique $d \in \mathcal{A}'_\theta(V)$. We have, for all $z \in \mathbb{C}$,

$$f(z) = \langle \delta_z, f \rangle = \langle \delta_z, \alpha^t(d) \rangle = \langle d, \alpha(\delta_z) \rangle = \langle d, \rho(g_z) \rangle = \sum_{k \geq 0} e^{z\alpha_k} \sum_{l=0}^{m_k-1} \frac{z^l}{l!} d_{k,l}.$$

To compute the coefficients $d_{k,l}$:

$$\langle T_{k,l}, f \rangle = \langle T_{k,l}, \alpha^t(d) \rangle = \langle d, \alpha(T_{k,l}) \rangle = d_{k,l}.$$

The last equality follows from the observation that

$$\alpha(T_{k,l}) = B^{k,l}.$$

The rest of the proof is similar to the one of Theorem 6.2.5.

Let us recall some results about interpolating varieties that enables one to determine whether V is interpolating or not. The following is a rephrasing of the analytic characterization given in 1.2.14. The spaces of entire functions considered are slightly different, but it is clear how to adapt these results to our spaces.

THEOREM 6.4.6. *Let $V = Z(\Phi) = \{(\alpha_k, m_k)\}_k$. Then V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$ if and only if, there are constants $\varepsilon > 0$ and $m > 0$ such that, for all k ,*

$$\left| \frac{\Phi^{(m_k)}(\alpha_k)}{m_k!} \right| \geq \varepsilon e^{-\theta(m|\alpha_k|)}.$$

We also have a geometric characterization of interpolating varieties, in terms of the distribution of the points $\{(\alpha_k, m_k)\}_k$.

Adapting [3, Corollary 4.8] or Theorem 3.1.3 to our spaces :

THEOREM 6.4.7. *V is an interpolating variety for $\mathcal{G}_\theta(\mathbb{C})$ if and only if conditions*

$$(51) \quad \exists A > 0, \exists m > 0 \quad \forall R > 0, \quad N(0, R) \leq A + \theta(mR)$$

and

$$(52) \quad \exists A > 0, \exists m > 0 \quad \forall k \in \mathbb{N}, \quad N(\alpha_k, |\alpha_k|) \leq A + \theta(m|\alpha_k|)$$

hold.

Actually, since in our case $V = Z(\Phi)$ and $\Phi \in \mathcal{G}_\theta(\mathbb{C})$, condition (51) is necessarily verified (see Lemma 1.2.2). Thus, V is an interpolating variety if and only if condition (52) holds.

REMARK 6.4.8. We can obtain Theorem 6.4.2 as a corollary of Theorem 6.2.5, using the density condition (52). This second proof is rather technical, we will skip it here. Let us just give the correspondence between the coefficients $c_{k,l}$ and $d_{k,l}$:

$$(53) \quad \begin{aligned} d_{k,l} = & \sum_{i=l}^{m_k-1} c_{k,i} \partial_k^{i-l} \left(\prod_{0 \leq n \leq k-1} (\alpha_k - \alpha_n)^{-m_n} \right) \\ & + \sum_{j=k+1}^{\infty} \sum_{i=0}^{m_j-1} c_{j,i} \partial_k^{m_k-1-l} \left(\prod_{0 \leq n \leq j-1, n \neq k} (\alpha_k - \alpha_n)^{-m_n} (\alpha_k - \alpha_j)^{-(i+1)} \right), \end{aligned}$$

the convergence of the second sum being a consequence of conditions (52) and (51).

In the case where all $m_k = 1$, we have

$$d_k = \sum_{j=k}^{\infty} c_j \prod_{0 \leq n \leq j, n \neq k} (\alpha_k - \alpha_n)^{-1}.$$

CHAPTER 7

The multivariate case

Introduction

The results of this chapter are published in [38] and [39].

Here, we are concerned with the interpolation problem in several variables. C.A. Berenstein and B.Q. Li described the interpolating discrete sequences for $\mathcal{A}_p(\mathbb{C}^n)$ as zeros of an entire map $F = (f_1, \dots, f_n)$, $f_j \in \mathcal{A}_p(\mathbb{C}^n)$, where the jacobian determinant of F verifies a certain lower bound (see Theorem 7.2.1).

B.Q. Li and B.A. Taylor, in [31], proved that any \mathcal{A}_p -interpolating discrete sequence $V = \{\alpha_j\}_j$ verifies $n(r, V) = O((\sup_{|z|=r} p(z))^n)$ where $n(r, V)$ denotes the number of points of V within the ball of center 0 and radius r . This makes the link between the interpolation problem and the transcendental Bézout problem, that is, the problem of the existence of an upper bound for the zero set of an entire map.

The classical Bézout theorem states that if A_1, \dots, A_n are algebraic varieties then the degree of the intersection $A_1 \cap \dots \cap A_n$ is smaller than the product of the degrees.

When f is an entire function of one variable, an application of Jensen's formula shows that for any $\alpha > 1$, there exists a constant $C > 0$ such that

$$n(r, Z(f)) \leq C \log \sup_{|z|=\alpha r} |F(z)|.$$

By analogy with the number of zeros of a polynomial, this suggests that $\log \sup_{|z|=\alpha r} |F(z)|$ plays the role of the degree. The transcendental Bézout problem is the natural question, in the case where $F = (f_1, \dots, f_n)$ is an entire map from \mathbb{C}^n to \mathbb{C}^n , whether

$$n(r, F^{-1}(0)) \leq C \left(\log \sup_{|z|=\alpha r} |F(z)| \right)^n.$$

A counter-example by Cornalba and Shiffman (see [14]) shows that this is not the case in general. They proved that for any positive function $S(r)$ such that $S(r) \uparrow +\infty$, there exists a map $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of order zero, with

$$\frac{n(r, F^{-1}(0))}{S(r)} \rightarrow +\infty.$$

However, if we only count the non-degenerate zeros, that is, those where the jacobian determinant is not too small, then it is possible to find an upper bound in terms of the growth of the function. This was proved in [31]. We are going to give a new look on the proof using the techniques developed in [37]. Combining this result with the analytic characterisation of \mathcal{A}_p -interpolating discrete sequences given by Berenstein and Li, this will enable us to give a new look on the

proof of the necessary condition on an \mathcal{A}_p -interpolating sequence given by Li and Taylor (see Corollary 7.4.3). This is the object of Section 7.4.

But before getting to that, we give some definitions in Section 7.1 and we state some general properties about interpolating sequences in Section 7.2. They are often an extension of the one variable case. In Section 7.3, we give a sufficient geometric condition for a discrete sequence in \mathbb{C}^n to be interpolating for $\mathcal{A}_p(\mathbb{C}^n)$ when p is a radial and doubling weight growing more rapidly than $|z|^2$. In the case where $p(z) = |z|^2$, this condition is that the sequence is uniformly separated. To our knowledge, there is no geometric sufficient and necessary condition known yet in several variables, even in the case where the weight p is radial and doubling.

7.1. Definitions and notations

We will use the following standard notations. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, let $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$. We denote the standard Kähler form by $\beta = \frac{i}{2} \partial \bar{\partial} |z|^2$ and $\beta_k = \frac{1}{k!} \beta^k$. We denote by $B(a, r)$ the euclidian ball $\{z \in \mathbb{C}^n / \|z - a\| < r\}$.

A plurisubharmonic function $p : \mathbb{C}^n \rightarrow [0, \infty)$ is called a weight if it satisfies the following conditions :

(W1) $\log(1 + |z|^2) = O(p(z))$.

(W2) There exist constants C_1 and C_2 such that $p(z) \leq C_1 p(\omega) + C_2$ whenever $|z - \omega| \leq 1$.

Let $A(\mathbb{C}^n)$ be the space of all of entire functions on \mathbb{C}^n and

$$\mathcal{A}_p(\mathbb{C}^n) = \{f \in A(\mathbb{C}^n) : \exists A, B > 0 / |F(z)| \leq A \exp(Bp(z)), \forall z \in \mathbb{C}^n\}.$$

Let $V = \{\alpha_j\}_{j \in \mathbb{N}}$ be a discrete variety in \mathbb{C}^n i.e., a sequence of \mathbb{C}^n such that $|\alpha_j| \rightarrow \infty$ when $j \rightarrow \infty$. Let us define

$$\mathcal{A}_p(V) = \{W = \{w_j\}_{j \in \mathbb{N}} \subset \mathbb{C} : \exists A, B > 0 / |w_j| \leq A \exp(Bp(\alpha_j)), \forall j \in \mathbb{N}\}.$$

DEFINITION 7.1.1. We will say that V is \mathcal{A}_p -interpolating if for every $W \in \mathcal{A}_p(V)$, there is a function $f \in \mathcal{A}_p(\mathbb{C}^n)$ such that $f(\alpha_j) = w_j$ for all $j \in \mathbb{N}$.

We denote by \mathcal{R}_V the restriction map :

$$\begin{aligned} \mathcal{R}_V : \mathcal{A}_p(\mathbb{C}^n) &\rightarrow \mathcal{A}_p(V) \\ f &\rightarrow \{f(\alpha_j)\} \end{aligned}$$

In other terms, V is \mathcal{A}_p -interpolating if and only if \mathcal{R}_V maps $\mathcal{A}_p(\mathbb{C}^n)$ onto $\mathcal{A}_p(V)$.

DEFINITION 7.1.2. Let $V = \{\alpha_j\}_{j \in \mathbb{N}}$ be a discrete variety of \mathbb{C}^n .

We will say that $\{\alpha_j\}$ is weakly separated if there exist $\varepsilon, C > 0$ such that for all $k \neq j$,

$$|\alpha_j - \alpha_k| \geq \varepsilon \exp(-Bp(\alpha_j)).$$

and we will say that V is uniformly separated if there exists $\varepsilon > 0$ such that for all $k \neq j$,

$$|\alpha_j - \alpha_k| \geq \varepsilon.$$

For $B > 0$, we set

$$\mathcal{A}_{p,B}(\mathbb{C}^n) = \{f \in A(\mathbb{C}^n) / \|f\|_B < +\infty\}$$

where

$$\|f\|_B = \sup_{z \in \mathbb{C}^n} |f(z)| \exp(-B p(z)).$$

Then $\mathcal{A}_{p,B}(\mathbb{C}^n)$ with this norm is a Banach space and we have

$$\mathcal{A}_p(\mathbb{C}^n) = \bigcup_{j \in \mathbb{N}} \mathcal{A}_{p,B}(\mathbb{C}^n).$$

The space $\mathcal{A}_p(\mathbb{C}^n)$, endowed with the topology of inductive limit, has a structure of an (LF)-space (see [25] or [27]).

Now, let $V = \{\alpha_j\}$ be a discrete variety of \mathbb{C}^n and set

$$\mathcal{A}_{p,B}(V) = \{W = \{w_j\}_j \subset \mathbb{C} : \|W\|_B < +\infty\}$$

where

$$\|W\|_B = \sup_{j \in \mathbb{N}} |w_j| \exp(-B p(\alpha_j)).$$

In the same way, $\mathcal{A}_{p,B}(V)$ with this norm is a Banach space and

$$\mathcal{A}_p(V) = \bigcup_{j \in \mathbb{N}} \mathcal{A}_{p,B}(V).$$

$\mathcal{A}_p(V)$ endowed with the topology of inductive limit is another (LF)-space.

For a holomorphic map $F = (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, we denote by $J_F(z)$ the determinant of the jacobian matrix of F in the point z , $Z(F)$ the pre-image of 0 by F and

$$M_F(r) = \sup_{z \in B(0,r)} |F(z)|.$$

If V is a discrete subset \mathbb{C}^n , we denote by $n(r, V)$ the number of points of the set $V \cap B(0, r)$.

C_n will denote a constant only depending on n . Its actual value may vary from one occurrence to another.

7.2. General properties of interpolating discrete varieties

Berenstein and Li gave an analytic characterization of interpolating discrete varieties generalising Theorem 1.2.14 :

THEOREM 7.2.1. ([4, Corollary 2.7]) *Let $V = \{\alpha_j\}_{j \in \mathbb{N}}$ be a discrete variety of \mathbb{C}^n . Then V is an interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$ if and only if there exist n functions f_1, \dots, f_n in $\mathcal{A}_p(\mathbb{C}^n)$ such that, denoting by $F = (f_1, \dots, f_n)$,*

$$V \subset Z(F)$$

and for some $\varepsilon, B > 0$,

$$|J_F(\alpha_j)| \geq \varepsilon \exp(-B p(\alpha_j)), \text{ for all } j \in \mathbb{N}.$$

REMARK 7.2.2. Theorem 7.2.1 implies that whenever V is an interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$, it is also an interpolating variety for $\mathcal{A}_q(\mathbb{C}^n)$ if $p(z) = O(q(z))$.

LEMMA 7.2.3. *V is an interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$ if and only if, for all $B > 0$, there exist $A > 0$ and $B' > 0$ such that for all $W \in \mathcal{A}_{p,B}(V)$, there exist $f \in \mathcal{A}_{p,B'}(\mathbb{C}^n)$ with $\mathcal{R}_V(f) = W$ and $\|f\|_{B'} \leq A\|W\|_B$.*

See [13], page 456, for a proof based on the open mapping theorem for (LF)-spaces or [4], page 169, for a proof based on the Baire-category theorem.

Thanks to property (W2), $\mathcal{A}_p(\mathbb{C}^n)$ is closed under differentiation and more precisely :

LEMMA 7.2.4. *If $f \in \mathcal{A}_{p,B}(\mathbb{C}^n)$, then for $k = 1, \dots, n$, $\frac{\partial f}{\partial z_k} \in \mathcal{A}_{p,BC_1}$ and*

$$\left\| \frac{\partial f}{\partial z_k} \right\|_{BC_1} \leq \exp(B C_2) \|f\|_B.$$

PROOF. Let $z \in \mathbb{C}^n$. Using Cauchy formula,

$$\begin{aligned} \left| \frac{\partial f}{\partial z_k}(z) \right| &= \left| \frac{1}{2i\pi} \int_{|u|=1} \frac{f(z_1, \dots, z_{k-1}, z_k + u, z_{k+1}, \dots, z_n)}{u^2} du \right| \\ &\leq \max_{|z-\xi| \leq 1} |f(\xi)| \leq \|f\|_B \exp(B C_1 p(z) + B C_2). \end{aligned}$$

■

LEMMA 7.2.5. *If V is a discrete interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$, then V is weakly separated.*

PROOF. Using Lemma 7.2.3, we can find a sequence $\{f_j\}_{j \geq 0}$ such that $f_j(\alpha_j) = 1$, $f_j(\alpha_k) = 0$ for all $k \neq j$ and

$$|f_j(z)| \leq A e^{Bp(z)}, \quad z \in \mathbb{C}^n,$$

with $A, B > 0$ not depending on j .

For $j \neq k$, suppose $|a_j - \alpha_j| < 1$, then

$$1 = |f_j(\alpha_j) - f_j(\alpha_k)| \leq |\alpha_j - \alpha_k| \sup_{z \in B(\alpha_j, 1)} \max_{l=1, \dots, n} \left| \frac{\partial f}{\partial z_l}(z) \right|.$$

For each $l = 1, \dots, n$,

$$\left| \frac{\partial f}{\partial z_l}(z) \right| \leq A e^{Bp(z)} \leq A e^{Bp(\alpha_j)}$$

by property (W2).

■

LEMMA 7.2.6. *If $V = \{\alpha_j\}$ is weakly separated, then there exists $C > 0$ such that*

$$\sum_{j \geq 0} e^{-C p(\alpha_j)} < \infty.$$

The proof is the same as in one variable (See Lemma 1.2.12).

THEOREM 7.2.7. ([39])

Let V_1, \dots, V_n be discrete interpolating varieties for $\mathcal{A}_p(\mathbb{C}^n)$. Then $V = V_1 \cup \dots \cup V_n$ is an interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$ if and only if V is weakly separated.

PROOF. We may assume without loss of generality that $n = 2$. and that the intersection of V_1 and V_2 is empty. Let $W = \{w_\lambda\}_{\lambda \in V_1} \in \mathcal{A}_p(V_1)$ and $Z = \{z_\gamma\}_{\gamma \in V_2} \in \mathcal{A}_p(V_2)$ be the values to interpolate. We want to construct $f_1 \in \mathcal{A}_p(\mathbb{C}^n)$ and $f_2 \in \mathcal{A}_p(\mathbb{C}^n)$ such that

$$f_1(\lambda) = w_\lambda, \quad \lambda \in V_1, \quad \text{and} \quad f_1(\gamma) = 0, \quad \gamma \in V_2,$$

$$f_2(\gamma) = z_\gamma, \quad \gamma \in V_2, \quad \text{and} \quad f_2(\lambda) = 0, \quad \lambda \in V_1.$$

Then setting $f = f_1 + f_2$, it is clear that

$$f(\lambda) = w_\lambda, \quad \lambda \in V_1 \quad \text{and} \quad f(\gamma) = z_\gamma, \quad \gamma \in V_2,$$

in other words, $\rho_{V_1 \cup V_2}(f) = W \cup Z$.

Let us show how to construct f_1 (the construction of f_2 will be of course similar reversing the roles of V_1 and V_2).

Recall that by the weak separation, there exist constants $\varepsilon, C > 0$ such that for all $\lambda \in V_1$ and $\gamma \in V_2$, we have

$$|\lambda - \gamma| \geq \varepsilon e^{-Cp(\gamma)}.$$

Using Lemma 1.2.6, we can find a sequence of functions $\{G_{\lambda,l}\}_{\lambda \in \Lambda, 0 \leq l \leq n}$ such that

- (i) $G_{\lambda,l}(\gamma) = \frac{1}{(\gamma - \lambda)_l}$ if $|(\gamma - \lambda)_l| \geq \varepsilon/\sqrt{n}e^{-Cp(\gamma)}$.
- (ii) $|G_{\lambda,l}(z)| \leq Ae^{Bp(z)}$ where $A, B > 0$ don't depend on λ (or l).

We have denoted by $(\gamma - \lambda)_l$ the l -th coordinate of $\gamma - \lambda$. Now Set

$$F_\lambda(z) = \prod_{l=1}^n (1 - (z - \lambda)_l G_{\lambda,l}).$$

For all $\gamma \in \Gamma$, there exists $l \in \{1, \dots, n\}$ such that $(\gamma - \lambda)_l G_{\lambda,l}(\gamma) = 1$. Thus, $F_\lambda(\gamma) = 0$. Besides, it is clear that $F_\lambda(\lambda) = 1$ and that, for certain constants $A, B > 0$, we have

$$|G_{\lambda,l}(z)| \leq Ae^{B[p(z)+p(\lambda)]}$$

using property (W1) of the weight.

Applying again Lemma 1.2.6, for all $M > 0$, there exists a sequence of functions $\{h_\lambda\}$ such that

- (i) $h_\lambda(\lambda) = w_\lambda e^{Mp(\lambda)}$,
- (ii) $h_\lambda(\lambda') = 0$, for all $\lambda' \in V_1, \lambda' \neq \lambda$,
- (iii) $|h_\lambda(z)| \leq Ae^{Bp(z)}$, for all $z \in \mathbb{C}^n$, where A and B don't depend on λ .

Finally, we set

$$f_1 = \sum_{j \geq 0} F_\lambda h_\lambda e^{-Mp(\lambda)}$$

where M is chosen large enough so that the sum converges according to Lemma 7.2.6. ■

7.3. A sufficient condition

Let $V = \{\alpha_j\}$ be a discrete variety on \mathbb{C}^n . We consider the separation radius

$$\delta_k = \inf_{j \neq k} (|\alpha_k - \alpha_j|, 1).$$

THEOREM 7.3.1. *Let $\rho \geq 2$, p a weight such that $|z|^\rho = O(p(z))$ and $V = \{\alpha_j\}$ a discrete variety on \mathbb{C}^n .*

If V satisfies the following condition : there exist two constants $D_0 > 0$ and $D_1 > 0$ such that

$$\frac{1}{\delta_k^2} \log \frac{4}{\delta_k} \leq D_0 + D_1 |\alpha_j|^{\rho-2} \quad \text{for all } k \in \mathbb{N},$$

then V is an interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$.

REMARK 7.3.2. It's obvious that this condition implies the weak separation. Actually, it means that the points α_j have to be "separated enough".

Of course, by the previous result, the theorem is still true if V is weakly separated and is a finite reunion of varieties satisfying this condition.

An immediate corollary of this theorem is

COROLLARY 7.3.3. *Let p be a weight such that $|z|^2 = O(p(z))$.*

Any uniformly separated discrete variety on \mathbb{C}^n is an interpolating variety for $\mathcal{A}_p(\mathbb{C}^n)$.

In particular, \mathbb{N}^n is interpolating for $A_{|z|^2}(\mathbb{C}^n)$.

REMARK 7.3.4. This corollary is also a consequence of a result due to P. Thomas and X. Massaneda : [33, Corollary 5.2].

They showed the following result which is more precise in the case where $p(z) = |z|^2$:

if $V = \{\alpha_j\}$ is uniformly separated, then there exists $B_0 > 0$ such that for $B \geq B_0$, the restriction map

$$\begin{aligned} \rho_B : A_{|z|^2, B}(\mathbb{C}^n) &\rightarrow A_{|z|^2, B}(V) \\ f &\rightarrow \{f(\alpha_j)\} \end{aligned}$$

is onto.

In particular, that implies that V is an interpolating variety for $A_{|z|^2}(\mathbb{C}^n)$ and therefore, for $\mathcal{A}_p(\mathbb{C}^n)$ if $|z|^2 = O(p(z))$.

On the other hand, as a consequence of [1, Theorem 3.1.1], we can see that \mathbb{N}^n is a uniqueness set for the space, that we will denote by $\mathcal{A}_{|z|^2}^\tau$, of holomorphic functions on \mathbb{C}^n such that for some $A > 0$,

$$|f(z)| \leq A \exp(\tau |z|)$$

where $\tau < \pi$ is a fixed real number. Whenever $p(z) = o(|z|)$, $\mathcal{A}_p(\mathbb{C}^n) \subset \mathcal{A}_{|z|^2}^\tau$. Then \mathbb{N}^n is a uniqueness set for $\mathcal{A}_p(\mathbb{C}^n)$ and therefore cannot be \mathcal{A}_p -interpolating.

Let us mention another sufficient condition given by A. Hartmann and X. Massaneda.

THEOREM 7.3.5. [21, Corollary 6.3.] *Let p be a doubling and radial weight such that $p(z)\beta \leq c|z|^2(i\partial\bar{\partial}p(z))$ for some $c > 0$. Let $V = \{\alpha_j\}_j$ a discrete variety in \mathbb{C} . If there exist a constant $C > 0$ such that*

- (i) $N(0, R) \leq Cp(R)$.
- (ii) $N(\alpha_j, |\alpha_j|) \leq Cp(\alpha_j)$ for all $j \in \mathbb{N}$. Then V is \mathcal{A}_p -interpolating.

This sufficient condition and ours are different. For example, for the weight $p(z) = |z|^2$, a uniformly separated sequence verifies the hypothesis of Theorem 7.3 but not those of Theorem 7.3.5. On the other hand, it is possible to find a non-uniformly separated sequence verifying the hypothesis of Theorem 7.3.5 but not those of Theorem 7.3 (see [21, Remarks 1 and 2]). In particular, neither of these two sufficient conditions are necessary.

PROOF OF THEOREM 7.3.1. Let η be a C^∞ function satisfying

- (i) $0 \leq \eta(z) \leq 1 \quad \forall z \in \mathbb{C}^n$,
- (ii) $\eta(z) = 0$ if $|z| \geq \frac{1}{2}$,
- (iii) $\eta(z) = 1$ if $|z| \leq \frac{1}{4}$.

For a (p, q) -form $\gamma(z) = \sum_{|I|=p, |J|=q} \gamma_{I,J}(z) dz_I \wedge d\bar{z}_J$, $|\gamma|$ will denote $\max_{I,J} |\gamma_{I,J}(z)|$.

If g is a C^∞ function on \mathbb{C}^n , we set

$$\begin{aligned} \partial g(z) &= \sum_{i=1, \dots, n} \frac{\partial g}{\partial z_i}(z) dz_i, \\ \bar{\partial} g(z) &= \sum_{j=1, \dots, n} \frac{\partial g}{\partial \bar{z}_j}(z) d\bar{z}_j, \\ \partial \bar{\partial} g(z) &= \sum_{i,j=1, \dots, n} \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j}(z) dz_i \wedge d\bar{z}_j. \end{aligned}$$

Let $M > 0$ such that

$$\sup_{z \in \mathbb{C}^n} |\partial \eta(z)| \leq M, \sup_{z \in \mathbb{C}^n} |\bar{\partial} \eta(z)| \leq M \text{ and } \sup_{z \in \mathbb{C}^n} |\partial \bar{\partial} \eta(z)| \leq M.$$

The function $\eta_j(z) = \eta\left(\frac{z - \alpha_j}{\delta_j}\right)$ for $j \in \mathbb{N}$ will satisfy :

- (i) $\eta_j(\alpha_j) = 1, \quad \eta_j(\alpha_k) = 0 \quad \forall k \neq j$,
- (ii) $\eta_j(z) = 1$ if $|z - \alpha_j| \leq \frac{\delta_j}{4}$,
- (iii) $\eta_j(z) = 0$ if $|z - \alpha_j| \geq \frac{\delta_j}{2}$.

LEMMA 7.3.6. Let $V = \{\alpha_j\}$ be a p -weakly separated discrete variety on \mathbb{C}^n and $\{w_k\} \in \mathcal{A}_p(V)$.

Then, we can find a positive C^∞ function F on \mathbb{C}^n satisfying

- (i) $F(\alpha_j) = w_j \quad \forall j \in \mathbb{N}$,
- (ii) $\exists A, B > 0$ / $|F(z)| \leq Ae^{Bp(z)}, \quad |\bar{\partial} F(z)| \leq Ae^{Bp(z)}$ for $z \in \mathbb{C}^n$,
 - for $z \in \mathbb{C}^n \setminus \bigcup_{j \in \mathbb{N}} B(\alpha_j, \frac{\delta_j}{2})$, $\bar{\partial} F(z) = 0$,
 - for $z \in \bigcup_{k \in \mathbb{N}} B(\alpha_k, \frac{\delta_k}{4})$, $\bar{\partial} F(z) = 0$.

PROOF. There exist constants $\epsilon_1 > 0$ and $B_1 > 0$ such that, for all $j \in \mathbb{N}$,

$$1 \geq \delta_j \geq \epsilon_1 e^{-B_1 p(\alpha_j)},$$

and constants $A_2 > 0$ and $B_2 > 0$ such that, for all $j \in \mathbb{N}$,

$$|w_j| \leq A_2 e^{B_2 p(\alpha_j)}.$$

Put $F(z) = \sum_{j \in \mathbb{N}} \alpha_j(z) w_j$. Then obviously, $F(\alpha_j) = w_j$ for all $j \in \mathbb{N}$.

Note that the balls $B(\alpha_j, \frac{\delta_j}{2})$ are disjoint.

If $z \in \mathbb{C}^n \setminus \bigcup_{j \in \mathbb{N}} B(\alpha_j, \frac{\delta_j}{2})$, then $\alpha(z) = 0$ and $\bar{\partial}\alpha(z) = 0$.

If $\exists j \in \mathbb{N}$ such that $z \in B(\alpha_j, \frac{\delta_j}{2})$ (j is then unique), we have

$$\begin{aligned} |\alpha(z)| &= \alpha_j(z) |w_j| \leq |w_j| \leq A_2 e^{B_2 p(\alpha_j)} \\ &\text{with property (W2), } \leq A_2 e^{B_2 C_2} e^{B_2 C_1 p(z)}. \end{aligned}$$

We also have

$$\begin{aligned} |\bar{\partial}\alpha(z)| &= |\bar{\partial}\alpha_j(z) w_j| \leq M \frac{|w_j|}{\delta_j} \leq \frac{M A_2}{\epsilon_1} \exp((B_1 + B_2) p(\alpha_j)) \\ &\leq \frac{M A_2}{\epsilon_1} \exp((B_1 + B_2) C_2) e^{(B_1 + B_2) C_1 p(z)}. \end{aligned}$$

Besides, if $|z - \alpha_j| \leq \frac{\delta_j}{4}$, $\bar{\partial}\alpha(z) = 0$. ■

Let us proceed with the proof of the theorem.

Let $W = \{w_j\} \in \mathcal{A}_p(V)$. The preceding lemma gives an interpolating \mathcal{C}^∞ function F satisfying the desired growth condition. The next step will be to solve the $\bar{\partial}$ equation $\bar{\partial}u = -\bar{\partial}F$ with β satisfying the same growth condition and vanishing on V . Then it suffices to take $f = F + u$ to have the desired interpolating function.

To solve the $\bar{\partial}$ equation, we use Hörmander's theorem :

THEOREM 7.3.7. (see [22]) *Let ψ be a plurisubharmonic function in \mathbb{C}^n . For every $(0, 1)$ -form γ with \mathcal{C}^∞ coefficients on \mathbb{C}^n and such that $\bar{\partial}\gamma = 0$, there is a \mathcal{C}^∞ function β such that $\bar{\partial}\beta = \gamma$ and*

$$\int_{\mathbb{C}^n} |\beta(z)|^2 e^{-\psi(z)} (1 + |z|^2)^{-2} d\lambda(z) \leq \int_{\mathbb{C}^n} |\gamma(z)|^2 e^{-\psi(z)} d\lambda(z).$$

We will apply this theorem with $\gamma = -\bar{\partial}\alpha$. The essential point is to find a plurisubharmonic function ψ not summable on the points α_j of V and such that the right integral is finite. This will force the solution β to vanish on V .

To begin, put

$$\tilde{\phi}(z) = \sum_j \alpha_j(z) \log |z - \alpha_j|^{2n}.$$

This function has almost the desired properties except that it is not plurisubharmonic. We will estimate its "lack of plurisubharmonicity" by estimating its Levi-form:

$$\langle \mathcal{L}(\tilde{\phi})\lambda, \lambda \rangle = \sum_{k,l} \frac{\partial^2 \tilde{\phi}(z)}{\partial z_k \partial \bar{z}_l} \lambda \alpha_j \bar{\lambda}_l, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n.$$

We want a lower bound of this term.

$$\begin{aligned} \langle \mathcal{L}(\tilde{\phi})\lambda, \lambda \rangle &= n \sum_j \alpha_j(z) \langle \mathcal{L}(\log |z - \alpha_j|^2)\lambda, \lambda \rangle \\ &\quad + 2n \operatorname{Re} \left[\sum_j \sum_{k,l} \frac{\partial \alpha_j(z)}{\partial z_k} \frac{\partial \log |z - \alpha_j|^2}{\partial \bar{z}_l} \lambda_j \bar{\lambda}_l \right] \\ &\quad + n \sum_j \log |z - \alpha_j|^2 \sum_{k,l} \frac{\partial^2 \alpha_j(z)}{\partial z_k \partial \bar{z}_l} \lambda_j \bar{\lambda}_l. \end{aligned}$$

For every j , $\log |z - \alpha_j|$ is plurisubharmonic so the first term is positive.

Let us denote by $A(z)$ the second term and by $B(z)$ the third term.

If $z \in \mathbb{C}^n \setminus \bigcup_j B(\alpha_j, \frac{\delta_j}{2})$ or if $z \in \bigcup_j B(\alpha_j, \frac{\delta_j}{4})$, then $A(z) = B(z) = 0$.

If $z \in B(\alpha_j, \frac{\delta_j}{2}) \setminus B(\alpha_j, \frac{\delta_j}{4})$, that is $\frac{\delta_j}{4} \leq |z - \alpha_j| \leq \frac{\delta_j}{2}$, then

$$\begin{aligned} \left| \frac{\partial \alpha_j(z)}{\partial z_k} \right| &\leq \frac{M}{\delta_j} \quad \text{for } k = 1, \dots, n, \\ \left| \frac{\partial \log |z - \alpha_j|^2}{\partial \bar{z}_l} \right| &= \frac{|(z - \alpha_j)_l|}{|z - \alpha_j|^2} \leq \frac{1}{|z - \alpha_j|} \leq \frac{4}{\delta_j} \quad \text{for } l = 1, \dots, n. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |A(z)| &\leq \frac{8nM}{\delta_j^2} |\lambda|^2 \leq 8nM(D_0 + D_1 |\alpha_j|^{\rho-2}) |\lambda|^2 \\ &\leq 8nM(D_0 + 2^{\rho-2} D_1) (1 + |z_k|^{\rho-2}) |\lambda|^2. \end{aligned}$$

Let us estimate $B(z)$.

$$\begin{aligned} \left| \frac{\partial^2 \alpha_j(z)}{\partial z_k \partial \bar{z}_l} \right| &\leq \frac{M}{\delta_j^2}, \quad \text{for } k, l = 1, \dots, n, \\ |\log |z - \alpha_j|^2| &= 2 \log \frac{1}{|z - \alpha_j|} \leq 2 \log \frac{4}{\delta_j}. \end{aligned}$$

We deduce that

$$\begin{aligned} |B(z)| &\leq 2n M \frac{1}{\delta_j^2} \log \frac{4}{\delta_j} |\lambda|^2 \leq 2n M (D_0 + D_1 |\alpha_j|^{\rho-2}) |\lambda|^2 \\ &\leq 2n M (D_0 + 2^{\rho-2} D_1) (1 + |\alpha_j|^{\rho-2}) |\lambda|^2. \end{aligned}$$

Finally, we obtain that

$$\langle \mathcal{L}(\tilde{\phi})\lambda, \lambda \rangle \geq -K (|z|^{\rho-2} + 1) |\lambda|^2 \text{ where } K = 8nM(D_0 + 2^{\rho-2}D_1).$$

Now, we add a function with a Levi-form "positive enough" to make the sum positive. Set

$$\phi = \tilde{\phi} + K (|z|^\rho + |z|^2).$$

A simple calculation shows that

$$\langle \mathcal{L}(|z|^\rho)\lambda, \lambda \rangle \geq |z|^{\rho-2} |\lambda|^2 \text{ and } \langle \mathcal{L}(|z|^2)\lambda, \lambda \rangle = |\lambda|^2 \text{ for } \lambda \in \mathbb{C}^n.$$

Therefore, from [24, Theorem 2.9.11], ϕ is almost everywhere equal to a plurisubharmonic function. We can assume that ϕ is plurisubharmonic.

ϕ satisfies the following conditions :

$$\begin{aligned} \phi(z) &\leq K (|z|^\rho + |z|^2) \leq \tilde{K} p(z), \\ \exp(-\phi(z)) &\sim \frac{1}{|z - \alpha_j|^{2n}} \text{ in a neighborhood of } \alpha_j. \end{aligned}$$

Let us take the function F given by Lemma 7.3.6 and consider $|\bar{\partial}\alpha(z)|^2 \exp(-\phi(z))$. If z belongs to the support of $\bar{\partial}F$, there is a (unique) j such that

$$\frac{\delta_j}{4} \leq |z - \alpha_j| \leq \frac{\delta_j}{2}$$

then,

$$-\tilde{\phi}(z) = \alpha_j(z) \log \frac{1}{|z - \alpha_j|^{2n}} \leq 2n \log \frac{4}{\delta_j} \leq K_1 p(\alpha_j) \leq K_1 C_1 p(z) + K_1 C_2$$

so $\exp(-\phi(z)) \leq \exp(-\tilde{\phi}(z)) \leq K_2 \exp(K_3 p(z))$ and finally,

$$|\bar{\partial}F(z)|^2 \exp(-\phi(z)) \leq A^2 K_2 \exp((2B + K_3)p(z)) \text{ for all } z \in \mathbb{C}^n.$$

By property (0.1), $\log(1 + |z|^2) = O(p(z))$ thus, if we choose B_1 large enough,

$$\int_{\mathbb{C}^n} |\bar{\partial}F(z)|^2 \exp(-\phi(z)) \exp(-B_1 p(z)) d\lambda(z) < +\infty.$$

From Hörmander theorem, we can find a \mathcal{C}^∞ function β such that $\bar{\partial}\beta = -\bar{\partial}F$ and

$$\int_{\mathbb{C}^n} \frac{|\beta|^2 e^{-\phi(z)} e^{-B_1 p(z)}}{(1 + |z|^2)^2} d\lambda(z) < +\infty,$$

i.e: taking B_2 large enough,

$$I = \int_{\mathbb{C}^n} |\beta|^2 e^{-\phi(z)} e^{-B_2 p(z)} d\lambda(z) < +\infty.$$

Recall that locally near every α_j , $e^{-\phi(z)} \sim \frac{1}{|z - \alpha_j|^{2n}}$. So we have necessarily $\beta(\alpha_j) = 0$ for all j .

Set $f = F + \beta$. f is then holomorphic on \mathbb{C}^n , and $f(\alpha_j) = F(\alpha_j) = C_j$ for all j .

The last step is to verify that f has the desired growth condition.

By the mean value inequality,

$$\begin{aligned} |F(z)| &\leq \frac{1}{v_n} \int_{B(z,1)} |f(w)| d\lambda(w) \leq \frac{1}{v_n} \int_{B(z,1)} |F(w)| d\lambda(w) \\ &\quad + \frac{1}{v_n} \int_{B(z,1)} |\beta(w)| d\lambda(w). \end{aligned}$$

Let us estimate the two integrals that we denote I_1 and I_2 . For $w \in B(z, 1)$,

$$|F(w)| \leq A e^{B p(w)} \leq A e^{B C_2} e^{B C_1 p(z)}.$$

Then,

$$I_1 \leq A e^{B C_2} e^{B C_1 p(z)}.$$

To estimate I_2 , we use Cauchy-Schwarz inequality,

$$I_2 \leq J_1 J_2$$

where

$$\begin{aligned} J_1 &= \left[\int_{B(z,1)} |\beta(w)|^2 e^{-\phi(w) - B_2 p(w)} d\lambda(w) \right]^{\frac{1}{2}} \\ J_2 &= \frac{1}{v_n} \left[\int_{B(z,1)} e^{\phi(w) + B_2 p(w)} d\lambda(w) \right]^{\frac{1}{2}}. \\ J_1 &\leq \left[\int_{\mathbb{C}^n} |\beta(w)|^2 e^{-\phi(w) - B_2 p(w)} d\lambda(w) \right]^{\frac{1}{2}} < +\infty. \end{aligned}$$

For $w \in B(z, 1)$, using property (W2),

$$e^{\phi(w) + B_2 p(w)} \leq e^{(\tilde{K} + B_2) p(w)} \leq e^{(\tilde{K} + B_2) C_2} e^{(\tilde{K} + B_2) C_1 p(z)}.$$

then there are two positive constants A_1 and B_3 such that $J_2 \leq A_1 e^{B_3 p(z)}$.

Finally, taking $A_2 = \max(A e^{B C_2}, I_1^{1/2} A_1)$ and $B_4 = \max(B C_1, B_3)$, we have

$$|F(z)| \leq A_2 e^{B_4 p(z)}.$$

Then f is in $\mathcal{A}_p(\mathbb{C}^n)$. ■

7.4. On the transcendental Bézout problem

THEOREM 7.4.1. *Let F be an entire map from \mathbb{C}^n to \mathbb{C}^n . Let ϕ be a positive function on \mathbb{C}^n . Setting $\tilde{\phi}(r) = \sup_{|z| \leq r} \phi(z)$, we assume that $\tilde{\phi}$ is increasing and*

$$M_F(r) \leq A \left(\tilde{\phi}(r) \right)^B, \quad \text{where } A \text{ and } B \text{ are positive constants.}$$

Let $Z_\phi(F) = \{a_j\}_{j \geq 0}$ be the zeros of F verifying

$$|J_F(a_j)| \geq \frac{1}{\phi(a_j)}.$$

Then, if r is large enough so that $\log \tilde{\phi}(r) \geq 1$ et $r' > r$, we have

$$n(r, Z_\phi(F)) \leq C_n \frac{r'^{2n}}{(r' - r)^{2n}} (r' - r)^{\frac{-n}{\log \tilde{\phi}(r')}} (\log \tilde{\phi}(r'))^n.$$

An immediate application of Theorem 7.4.1, we find the following results that were proved previously by Li and Taylor (see [31]).

COROLLARY 7.4.2. *Let $\tilde{Z}(F) = \{a_j\}_{j \geq 0}$ be the zeros of F verifying*

$$|J_F(a_j)| \geq \varepsilon M_F(|a_j|)^{-B}.$$

Then for all $\alpha > 1$, there exists a constant $C_\alpha > 0$ such that

$$n(r, V) \leq C_\alpha \log(M_F(\alpha r))^n.$$

We are also led to the following necessary condition for interpolating discrete varieties.

COROLLARY 7.4.3. *Let $V = \{a_j\}_{j \geq 0}$ be an interpolating discrete variety for $\mathcal{A}_p(\mathbb{C}^n)$. Then, for all $\alpha > 1$, there exists $C_\alpha > 0$ such that*

$$n(r, V) \leq C_\alpha (p(\alpha r))^n.$$

Proof of Theorem 7.4.1.

Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an entire map non identically equal to zero. Let $r' > r > 0$ (r will be chosen large enough when needed).

LEMMA 7.4.4. *There exists a constant $C_n > 0$ such that, for all $\gamma > 0$ (γ may depend on r),*

$$\int_{B(0,r)} i\partial\bar{\partial} \log |F|^2 \wedge (i\partial\bar{\partial} |F|^{2\gamma})^{n-1} \leq C_n \frac{r'^{2n}}{(r' - r)^{2n}} M_F(r')^{2\gamma(n-1)} \log M_F(r').$$

PROOF. We set $r_k = r' - k \frac{r'-r}{n}$ for $k \in 1, 2, \dots, n$ and

$$I_k = \int_{B(0,r_k)} i\partial\bar{\partial} \log |F|^2 \wedge (i\partial\bar{\partial} |F|^{2\gamma})^{k-1} \wedge \beta_{n-k}.$$

Using Lemma I.2.2,[37], we obtain the estimates

$$\begin{aligned}
I_k &\leq \frac{1}{(r_{k-1}^2 - r_k^2)^2} \int_{B(0, r_{k-1})} (r_{k-1}^2 - |z|^2)^2 i\partial\bar{\partial} \log |F|^2 \wedge (i\partial\bar{\partial} |F|^{2\gamma})^{k-1} \beta_{n-k} \\
&= \frac{1}{(r_{k-1}^2 - r_k^2)^2} \int_{B(0, r_{k-1})} |F|^{2\gamma} i\partial\bar{\partial} \log |F|^2 \wedge i\partial\bar{\partial} (r_{k-1}^2 - |z|^2)^2 \wedge (i\partial\bar{\partial} |F|^{2\gamma})^{k-2} \wedge \beta_{n-k} \\
&\leq \frac{8r_{k-1}^2(n-k+1)}{(r_{k-1}^2 - r_k^2)^2} \int_{B(0, r_{k-1})} |F|^{2\gamma} i\partial\bar{\partial} \log |F|^2 \wedge (i\partial\bar{\partial} |F|^{2\gamma})^{k-2} \wedge \beta_{n-k+1} \\
&\leq 8n^3 \frac{M_F(r')^{2\gamma}}{(r' - r)^2} I_{k-1}.
\end{aligned}$$

After iteration, we get

$$I_n \leq (8n^3)^{n-1} \frac{M_F(r')^{2\gamma(n-1)}}{(r' - r)^{2(n-1)}} I_1.$$

We still have to estimate I_1 . Repeating the above argument we have

$$I_1 \leq 8n^3 \frac{1}{(r'^2 - r^2)} \int_{B(0, r')} |\log |F|^2| \beta_n.$$

Thus

$$\begin{aligned}
|\log |F|^2| &= \log^+ |F|^2 + \log^- |F|^2 \\
\text{and } \log |F|^2 &= \log^+ |F|^2 - \log^- |F|^2 \\
\text{with } \log^+ |F|^2 &= \sup(\log |F|^2, 0) \\
\text{and } \log^- |F|^2 &= -\inf(\log |F|^2, 0).
\end{aligned}$$

We may assume up to a translation that $F(0) \neq 0$. Then

$$\log |F(0)|^2 \leq \frac{n!}{\pi^n r'^{2n}} \int_{B(0, r')} \log |F(z)|^2 d\lambda(z).$$

We deduce that

$$\int_{B(0, r')} \log^- |F(z)|^2 d\lambda(z) \leq \int_{B(0, r')} \log^+ |F(z)|^2 d\lambda(z) - \frac{r'^{2n} \pi^n}{n!} \log |F(0)|$$

and consequently,

$$\begin{aligned}
\int_{B(0, r')} |\log |F(z)|| d\lambda(z) &\leq 2 \int_{B(0, r')} \log^+ |F(z)|^2 d\lambda(z) - \frac{r'^{2n} \pi^n}{n!} \log |F(0)| \\
&\leq C_n \log M_F(r') r'^{2n}.
\end{aligned}$$

Finally,

$$I_n \leq C_n \frac{r'^{2n}}{(r' - r)^{2n}} M_F(r')^{2\gamma(n-1)} \log M_F(r').$$

■

REMARK 7.4.5. We may also find this kind of estimates in [16].

Let us recall the following theorem which gives a quantitative inverse function theorem.

THEOREM 7.4.6. [37] *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an entire map and let $r' > r > 0$. Let $z_0 \in B(0, r)$ with $J_F(z_0) \neq 0$. Then F is injective on $B(z_0, S)$ where*

$$S = C_n(r' - r)^{n+1} M_F(r')^{-n} |J_F(z_0)|$$

and $F(B(z_0, \frac{S}{2}))$ contains a ball $B(F(z_0), S')$ where

$$S' = C'_n(r' - r)^{2n} M_F(r')^{-2n+1} |J_F(z_0)|^2.$$

PROPOSITION 7.4.7. *For $\gamma > 0$, we have*

$$\sum_{a \in F^{-1}(0) \cap B(0, r)} |J_F(a)|^{4(n-1)\gamma} \leq C_n \gamma^{n-1} M_F(r')^{4n(n-1)\gamma} \log M_F(r') (r' - r)^{-4n(n-1)\gamma} \frac{r'^{2n}}{(r' - r)^{2n}}.$$

PROOF. Let $r_1 = \frac{r'-r}{2}$. A computation shows that the measures

$$i\partial\bar{\partial} \log |z|^2 \wedge (i\partial\bar{\partial} |z|^{2\gamma})^{n-1}$$

and

$$\gamma^n |z|^{-2n+2(n-1)\gamma} d\lambda(z)$$

are the same up to a constant. Using Lemma 7.4.4, we deduce that

$$(54) \quad \int_{B(0, r_1)} |F|^{-2n+2(n-1)\gamma} |J_F(z)|^2 d\lambda(z) \leq C_n \gamma^{-n} \frac{r'^{2n}}{(r' - r)^{2n}} M_F(r')^{2\gamma(n-1)} \log M_F(r').$$

For all $a \in F^{-1}(0) \cap B(0, r)$, by Theorem 7.4.6, F is injective on $B(a, S)$. In particular, the balls $B(a, \frac{S}{2})$ are pairwise disjoint. Moreover, $F(B(a, \frac{S}{2}))$ contains $B(0, S')$. Thus

$$\begin{aligned} \int_{B(0, r_1)} |F|^{-2n+2(n-1)\gamma} |J_F(z)|^2 d\lambda(z) &\geq \sum_{a \in F^{-1}(0) \cap B(0, r)} \int_{B(a, \frac{S}{2})} |F|^{-2n+2(n-1)\gamma} |J_F|^2 d\lambda(z) \\ &\geq \sum_{a \in F^{-1}(0) \cap B(0, r)} \int_{B(0, S')} |z|^{-2n+2(n-1)\gamma} d\lambda(z) \\ &\geq C_n \gamma^{-1} \sum_{a \in F^{-1}(0) \cap B(0, r)} S'^{2(n-1)\gamma}. \end{aligned}$$

The desired results follows by replacing S' by its value given in Theorem 7.4.6 and using the inequality (54). ■

To complete the proof of Theorem 7.4.1, we only need to apply Proposition 7.4.7 with

$$\gamma = \frac{1}{4(n-1) \log \tilde{\phi}(r')}.$$

We obtain

$$n(r, Z_\phi(f)) \leq e^{-1} \sum_{a \in V \cap B(0, r)} |J_F(a)|^{\frac{1}{\log \tilde{f}(r')}} \leq C_n (\log \tilde{f}(r'))^n (r' - r)^{-\frac{n}{\log \tilde{f}(r')}} \frac{r'^{2n}}{(r' - r)^{2n}}.$$

Further questions

On the space $\hat{\mathcal{E}}'(\mathbb{R})$

The trace of $\hat{\mathcal{E}}'(\mathbb{R})$ on a variety V

Let us put $p(z) = |\operatorname{Im} z| + \ln(1 + |z|^2)$. We know that $\mathcal{A}_p(\mathbb{C}) = \hat{\mathcal{E}}'(\mathbb{R})$, the space of Fourier transforms of distributions with compact support on the real line.

In Chapter 4, we gave a complete geometric description of \mathcal{A}_p -interpolating varieties. We now want to know more about Question 2, that is, to describe the image of $\mathcal{A}_p(\mathbb{C})$ by the restriction map \mathcal{R}_V .

In [7], Berenstein and Taylor considered the case where $V = Z(f)$ with $f \in \mathcal{A}_p(\mathbb{C})$ and f is slowly decreasing, i.e. there is a constant $A > 0$ such that for all $x \in \mathbb{R}$,

$$\max\{|f(x+t)| : t \in \mathbb{R}, |t| \leq A \ln(1 + |x|)\} \leq (A + |x|)^{-A}.$$

They described the trace $\mathcal{R}_V(\mathcal{A}_p(\mathbb{C}))$ in terms of divided differences after grouping the points of $V = Z(f)$ in connected components $\{V_\alpha\}_\alpha$ of the set $\{|g(z)| \leq \varepsilon \exp(-Bp(z))\}$, for some $\varepsilon, B > 0$.

We would like to find an explicit description, like we did for radial and doubling weights in Chapter 3. To apply a similar method, we need to impose some natural geometric conditions on V .

One of these conditions would be that V lies in the logarithmic strip about the real axis, i.e., there is a constant $A > 0$ such that

$$|\operatorname{Im} \alpha_j| \leq A \ln(1 + |\alpha_j|) \quad \forall j \in \mathbb{N}$$

(such a variety is called hyperbolic). This includes the case where all the points lie on the real axis. Note that when V is hyperbolic, $p(\alpha_j) \simeq \ln(1 + |\alpha_j|)$ and Theorem 4.1.2 may be re-written in simpler terms : V is \mathcal{A}_p -interpolating if and only if There is $C > 0$ such that

$$N(\alpha_j, \ln(1 + |\alpha_j|)) \leq C \ln(1 + |\alpha_j|) \quad \forall j \in \mathbb{N}.$$

The other condition would be a control on the number of points lying in $V \cap D(x, \ln(1 + |x|))$, $x \in \mathbb{R}$ by $\ln(1 + |x|)$. This property holds for zeros of slowly decreasing functions in $\mathcal{A}_p(\mathbb{C})$. In the case of radial and doubling weights (Chapter 5), we covered the complex plane by increasing disks centered at the origin. If we want to apply the same techniques in the present case, we shall need an adapted covering of the part of the complex plane lying near the real axis.

Application to mean-periodic functions

By Paley-Wiener' theorem, the Fourier transform, that we will denote by \mathcal{F} , is an isomorphism between $\mathcal{E}'(\mathbb{R})$ and $\mathcal{A}_p(\mathbb{C})$.

Let $\mu \in \mathcal{E}'(\mathbb{R})$. We say that $f \in \mathcal{E}(\mathbb{R})$ is μ -mean -periodic if it verifies the convolution equation

$$\mu \star f = 0.$$

Denote by $V = \{(\alpha_j, m_j)\} = Z(\mathcal{F}(\mu))$. As in [7], knowing the trace of $\hat{\mathcal{E}}'(\mathbb{R})$ give a representation of μ -mean-periodic functions in series of exponential monomial that converges after grouping with respect to the components $\{V_\alpha\}_\alpha$. A geometric description of $\mathcal{R}_V(\mathbb{C})$ would give a more explicit series, on the model of the ones given in Chapter 6. Let us mention that by the general theory of convolution operators, $\mathcal{F}(\mu)$ being slowly decreasing (or invertible) is equivalent to the surjectivity of $T\star$. Also, V being hyperbolic is equivalent to the existence of a continuous right inverse for $T\star$.

Non-homogeneous convolution equations

In Chapter 6, we studied homogeneous convolution equations

$$T \star f = 0, T \in \mathcal{F}'_\theta(\mathbb{C}), T \neq 0, f \in \mathcal{F}_\theta(\mathbb{C}).$$

In our situation, the Fourier-Borel transform Φ of T is invertible (see Lemma 6.2.2). Therefore, it is known from the general theory that the operator $T\star$ maps $\mathcal{F}'_\theta(\mathbb{C})$ onto itself. In other words, for any $g \in \mathcal{F}_\theta(\mathbb{C})$, the equation

$$T \star f = g$$

has at least one solution $f \in \mathcal{F}_\theta(\mathbb{C})$. As we described the kernel of $T\star$ in Theorem 6.2.5, we would completely solve the non-homogenous equation if we found a way to construct a particular solution. In a joint work project with H. Ouerdiane, we use a series representation of $T \star f(z)$ to try to construct an explicit solution. We already have some partial results in the particular cases where g is the monomial z^k , $k \in \mathbb{N}$. The idea is then to deduce a solution in the general case $g(z) = \sum_{k \in \mathbb{N}} g_k z^k$ by linearity and convergence.

The multivariate case

Let us consider the case of radial and doubling weights. In Chapter 7, we discussed discrete varieties in \mathbb{C}^n and we gave sufficient geometric conditions as well as necessary ones in terms of the counting functions.

These counting functions still have a sense in several variables. According to the work by A. Hartmann and X. Massaneda, the conditions $N(0, R) = 0(p(R))$ and $N(\alpha_j, |\alpha_j|) = 0(p(\alpha_j))$ that are necessary and sufficient in one variable turn out to be still sufficient but not necessary in several variables (see [21], Chapters 3 and 7). Can we find similar conditions in several variables that would be necessary and sufficient? What would play the role of the counting functions? The analytic characterization given by Berenstein and Li (see Theorem 7.2.1) shows that whenever $V = \{\alpha_j\}$ is \mathcal{A}_p -interpolating, then for all $u \in \mathbb{C}^n$, $|u| = 1$ and for all j , $N_u(\alpha_j, |\alpha_j|) \leq A + Bp(\alpha_j)$, where $A, B > 0$ are independent of j and of u and where we have denoted by

$$N_u(\alpha_j, |\alpha_j|) = \sum_{\alpha_k = \alpha_j + u\zeta_{j,k}} \ln \frac{|\alpha_j|}{|\zeta_{j,k}|}.$$

This condition is not sufficient since it clearly doesn't imply the necessary condition $n(0, R) = O(p(R)^n)$ but how far is it from being sufficient ?

A simpler problem is to consider the particular case where the points of V are regularly distributed (in a sense that remains to be explicated). The fact that a uniformly separated discrete variety is \mathcal{A}_p -interpolating when $p(z) = |z|^2$ suggests that a condition in the more general case $p(z) = |z|^\rho$, $\rho \geq 2$ would be a separation of order $|\alpha_j|^{1-\rho/2}$.

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