A Gabriel Theorem for Coherent Twisted Sheaves and Picard Group and 2-factoriality of O'Grady's Examples of Irreducible Symplectic Varieties

Arvid Perego

UNIVERSITÉ DE NANTES
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UN THÉORÉME DE GABRIEL POUR LES FAISCEAUX COHÉRENTS TORDUS

et

GROUPE DE PICARD ET 2−FACTORIALITÉ DES EXEMPLES DE O'GRADY DE VARIÉTÉS IRRÉDUCTIBLES SYMPLECTIQUES

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Arvid PEREGO

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Président du jury	:	Manfred LEHN	Professeur (Universität Mainz)
Rapporteurs	:	Eyal MARKMAN	Professeur (University of Massachussets)
		Dimitri MARKUSHEVICH	Professeur (Université de Lille)
Examinateurs	:	Laurent EVAIN	Maître de Conférence (Université d'Angers)
		Vincent FRANJOU	Professeur (Université de Nantes)
		Manfred LEHN	Professeur (Universität Mainz)
		Dimitri MARKUSHEVICH	Professeur (Université de Lille)
		Christoph SORGER	Professeur (Université de Nantes)

Directeur de thèse : Christoph SORGER
Laboratoire : Laboratoire Jean Leray (UMR 6629 UN-CNRS-ECN)

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Introduction

Ce travail de thèse se compose de deux parties différentes, intitulées respectivement *Un théorème de Gabriel pour les faisceaux cohérents tordus* et *Groupe de Picard et factorialité locale des examples de O'Grady de variétés irréductibles symplectiques.*

Un théorème de Gabriel pour les faisceaux cohérents tordus

Le théorème de Gabriel est un des résultats les plus importants et, peut-être, éclatants de la géométrie algébrique des années '60. Il s'inscrit parfaitement dans l'esprit de la géométrie algébrique : à un objet géométrique X (un espace topologique, une variété, un schéma...) on associe un objet algébrique A (par example le groupe fondamental π_1, les groupes d'homologie H_i et de cohomologie H^i, une catégorie...), et on étudie les propriétés géométriques de X que l'objet A enregistre.

Le problème étudié dans le théorème de Gabriel est le suivant : à un schéma noethérien X on associe la catégorie abélienne $Coh(X)$ des faisceaux cohérents sur X. Donc, la question est de comprendre les propriétés de X qu'on peut recouvrir à partir de $Coh(X)$. Le résultat est le suivant :

Théorème 1. *(**Gabriel**, '62). Soit X un schéma noethérien. Alors :*

1. *le schéma X peut être reconstruit à partir de la catégorie $Coh(X)$;*
2. *si Y est un schéma noethérien et F est une équivalence entre les deux catégories $Coh(X)$ et $Coh(Y)$, alors F induit un isomorphisme de schémas entre X et Y.*

Le théorème de Gabriel a plusieurs conséquences. La première, et principale, est que connaître le schéma X est équivalent à connaître sa catégorie des faisceaux cohérents, donc $Coh(X)$ est un invariant géométrique très fort. Ceci a amené à l'étude de la catégorie dérivée (bornée) $D^b(X)$ de la catégorie $Coh(X)$, qui est un invariant géométrique moins fort que $Coh(X)$: il existe des variétés projective lisses X et X' qui ne sont pas isomorphes, mais dont les catégories dérivées $D^b(X)$ et $D^b(X')$ sont équivalentes.

Naturellement, il faut expliquer ce que signifie de reconstruire un schéma à partir de sa catégorie des faisceaux cohérents. L'idée de Gabriel est la suivante : d'abord, il définit une classe particulière de sous-catégories de $Coh(X)$, ce qu'on appelle les sous-catégories de Serre de type fini. Ce qu'il démontre est qu'il existe une correspondence bijective entre l'ensemble des sous-catégories de

Serre de type fini de $Coh(X)$ et celui des sous-ensembles fermés de X. En utilisant ce résultat, il définit une topologie sur l'ensemble E des sous-catégories de Serre irréductibles de type fini, et il démontre que cet espace topologique est homeomorphe à X. Le dernier passage est de définir sur E une structure d'espace annelé, et de démontrer que celui-ci est un schéma isomorphe à X : ceci est accompli en utilisant la notion de centre d'une catégorie.

Il est alors naturel de se demander si l'on peut démontrer un analogue du théorème de Gabriel pour des catégories abéliennes plus générales de la catégorie des faisceaux cohérents. La généralisation qu'on propose dans ce travail est celle des catégories des faisceaux cohérents tordus par un élément du groupe de Brauer cohomologique $Br'(X) = H^2_{ét}(X, \mathscr{O}_X^*)_{tors}$ d'un schéma X. Si on choisit $\alpha \in Br'(X)$, un faisceau $\mathscr{F}$ tordu par α est défini comme il suit : une fois choisi un recouvrement ouvert de X donné par des ouverts U_i, on fixe un faisceau $\mathscr{F}_i$ sur U_i, pour tout i, et on définit une donnée de recollement des $\mathscr{F}_i$ à l'aide de α. Ce qu'on obtient est donc localement (sur chaque U_i) un faisceau, mais en général $\mathscr{F}$ n'est pas globalement un faisceau.

Le résultat qu'on démontre est le suivant :

Théorème 2. *Soit X un schéma noethérien, et soit $\alpha \in Br'(X)$ un élément du groupe de Brauer cohomologique de X. Alors*

1. *le schéma X peut être reconstruit à partir de la catégorie $Coh(X, \alpha)$;*
2. *si Y est un schéma noethérien et $\beta \in Br'(Y)$, et si F est une équivalence de catégories entre $Coh(X, \alpha)$ et $Coh(Y, \beta)$, alors F induit un isomorphisme de schémas entre X et Y.*

La preuve adapte celle du théorème de Gabriel aux faisceaux tordus, et il faut donc résoudre plusieurs problèmes liés aux différences entre faisceaux et faisceaux tordus : d'abord, la correspondence bijective entre les sous-ensembles fermés de X et les sous-catégories de Serre de type fini de $Coh(X, \alpha)$. Dans le cas classique, cette correspondence est montrée grâce à deux propriétés des faisceaux cohérents qui ne sont pas si évidentes pour les faisceaux tordus : la première, est que tout faisceau cohérent défini sur un ouvert U du schéma X s'étend à un faisceau cohérent sur X ; la deuxième est qu'il existe un faisceau cohérent sur X dont le support est X.

Dans le cas des faisceaux tordus, la première propriété peut être facilement montrée à l'aide de la théorie des $\mathscr{O}_X^*-$gerbes. Ce résultat nous permet aussi de définir sans problèmes une structure d'espace annelé sur l'ensemble des sous-catégorie de Serre irréductibles de type fini de $Coh(X, \alpha)$, de la même façon que dans la preuve du théorème de Gabriel. La deuxième propriété est plus compliquée : premièrement, il est possible de montrer l'existence d'un faisceau coherent tordu dont le support est X dans le cas où X est un schéma noetherian réduit. Cette hypothèse ne serait pas nécessaire si l'on supposait $\alpha \in Br(X)$, car dans ce cas il existe un faisceau tordu localement libre. Après il faut montrer que pour avoir le Théorème 2 il suffit de se reconduire au cas où X est réduit.

Il y a plusieurs remarques qu'on peut ajouter au sujet du Théorème 2. La première est que si l'on choisit $\alpha \in Br(X)$, alors la catégorie $Coh(X, \alpha)$ est équivalente à la catégorie $Mod_{Coh}(\mathscr{A})$ dont les objets sont les faisceaux cohérents qui ont une structure de $\mathscr{A}-$module (et les morphismes sont morphismes de $\mathscr{A}-$modules), où $\mathscr{A}$ denote l'algèbre d'Azumaya dont la classe d'équivalence en $Br(X)$ est α. En particulier, $Mod_{Coh}(\mathscr{A})$ est une sous-catégorie de $Coh(X)$, donc on s'attend bien que le théorème soit vrai dans ce cas. En fait, on peut montrer que le théorème de Gabriel est vrai même pour la catégorie $Mod_{Coh}(\mathscr{A})$, pour n'importe quel faisceau cohérent $\mathscr{A}$ d'$\mathscr{O}_X-$algèbres dont le centre soit $\mathscr{O}_X$.

Ce qui est moins attendu est que dans l'énoncé du Théorème 2 on ne suppose que $\alpha \in Br'(X)$. En général, si le schéma X n'est pas quasi-projectif il peut y avoir des éléments $\alpha \in Br'(X)$ qui ne sont pas dans $Br(X)$: pour ces α, il n'y a pas de modèle cohérent pour la catégorie $Coh(X, \alpha)$. Ceci n'est vrai que si on considère n'importe quel sous-schéma affine de X : c'est bien cette propriété qui nous permet de montrer le Théorème 2.

Groupe de Picard et $2-$factorialité des examples de O'Grady de variétés irréductibles symplectiques

Un des objectifs de la géométrie algébrique complexe est la classification des variétés projectives lisses définies sur $\mathbb{C}$. Le premier invariant par rapport auquel on classifie ces variétés est la dimension. En dimension 1, les courbes algébriques ont été classifiées depuis longtemps à l'aide du diviseur canonique : si X est une courbe projective lisse définie sur $\mathbb{C}$, alors X est rationnelle si le diviseur canonique K_X a degré négatif (ce qui implique $-K_X$ ample) ; elle est elliptique si $deg(K_X) = 0$ et elle est de type général si $deg(K_X) > 0$, ce qui implique K_X ample.

Le diviseur canonique est un des moyens principaux pour classifier les variétés de dimension plus élevée. En particulier, l'invariant qu'on produit grâce au diviseur canonique est la dimension de Kodaira κ. Si X est une variété projective lisse définie sur $\mathbb{C}$, on a trois cas possibles. Le premier est $\kappa(X) = -\infty$, et dans ce cas la variété est dite de Fano ; le deuxième est $\kappa(X) = 0$, qui comprend le cas où $K_X = 0$; le dernier est $\kappa(X) > 0$, qui comprend le cas des variétés de type général. Dans ce travail, on ne considère que des variétés de dimension de Kodaira 0.

Dans la classe des variétés projective lisses de dimension de Kodaira 0, il y a une classe très importante, formée par les variétés dont la première classe de Chern est nulle. Elles ont été étudiées depuis longtemps, et classifiées à revêtement étale près à l'aide des groupes d'holonomie. En particulier, si X est une variété kählerienne dont la première classe de Chern est nulle, alors il existe un revêtement étale $X' \longrightarrow X$ tel que

$$X' = T \times \prod_{i=1}^{n} X_i \times \prod_{j=1}^{m} Y_j,$$

où

1. T est un tore complexe ;
2. X_i est une variété à holonomie spéciale ;
3. Y_j est une variété dont le groupe d'holonomie est $Sp(r)$.

Les tores complexes sont tous de la forme

$$T = \mathbb{C}^n/\Gamma,$$

pour un entier $n \in \mathbb{N}$ et un réseau Γ de rang maximal en $\mathbb{C}^n$. Les variétés à holonomie spéciale, appelées aussi variétés da Calabi-Yau, sont très nombreuses, est sont intéressantes en particulier en physique. Les variétés de la dernière classe sont appelées hyperkähleriennes irréductibles en géométrie différentielle, et irréductibles symplectiques en géométrie algébrique. En particulier, les variétés irréductibles symplectiques sont simplement connexes et admettent une unique (à multiplication par un élément de $\mathbb{C}^*$ près) forme symplectique.

Contrairement à ce qui arrive dans le cas des variétés à holonomie spéciale, il n'y a que cinq familles d'exemples connus, à deformation près, de variétés irréductibles symplectiques :

1. les surfaces K3 ;
2. les schémas de Hilbert $Hilb^n(X)$, où X est une surface K3 et $n \in \mathbb{N}$;
3. les variétés de Kummer généralisées $K^n(T)$, où T est un tore complexe et $n \in \mathbb{N}$;
4. l'exemple de O'Grady $\widetilde{M}_{10}$ en dimension 10 ;
5. l'exemple de O'Grady $\widetilde{M}_6$ en dimension 6.

Le problème de produire exemples de variétés irréductibles symplectiques est fortement lié au problème de construire variétés projectives de dimension élevée. Un des moyen les plus efficaces connus est celui des espaces des modules des faisceaux semistables sur une surface lisse S. Pour les définir, il faut fixer un diviseur ample H sur S et un vecteur de Mukai $v \in \widetilde{H}(S,\mathbb{Z}) := H^{2*}(S,\mathbb{Z})$. Ce dernier, appelé réseau de Mukai, est un $\mathbb{Z}-$module qui est réseau par rapport à une forme d'intersection $(.,.)$, appelée forme de Mukai. L'espace des modules des faisceaux $H-$semistables dont le vecteur de Mukai est v est noté M_v (on rappelle que le vecteur de Mukai d'un faisceau $\mathscr{F}$ sur S est défini comme $ch(\mathscr{F})\sqrt{td(S)}$). Si la surface S est projective, l'espace M_v est une variété projective qui peut être singulière. Un sous-ensemble ouvert très important dans M_v est M_v^s, qui paramètre les faisceaux $H-$stables. Dans le cas où S est une surface K3 projective ou une surface abélienne, M_v^s a les propriétés suivantes, due principalement à Mukai :

1. M_v^s est lisse ;
2. M_v^s admet une forme symplectique ;
3. la dimension de M_v^s est $2 + (v, v)$;
4. si le vecteur de Mukai v est primitif, i. e. v n'est pas divisible en $\widetilde{H}(S,\mathbb{Z})$, et le diviseur H est suffisamment générique, alors $M_v = M_v^s$.

Étant donnée une forme symplectique ω sur M_v^s, la question naturelle est donc s'il existe une résolution symplectique de M_v, c'est à dire une résolution des singularités

$$\pi_v : \widetilde{M}_v \longrightarrow M_v,$$

telle qu'il existe une forme symplectique $\widetilde{\omega}$ sur $\widetilde{M}_v$ dont la restriction à M_v^s est ω. De plus, s'il existe une telle résolution symplectique, il est naturel de se demander si $\widetilde{M}_v$ est une variété irréductible symplectique.

Si v est primitif et H est suffisamment générique, alors la question est triviale, car $M_v^s = M_v$. Le problème est donc si M_v est une variété irréductible symplectique. Un cas particulier est celui de $v = (1, 0, -n) \in \widetilde{H}(S,\mathbb{Z})$, si S est abélienne, où $v = (1, 0, 1-n)$ si S est K3 : dans ce cas, on a $M_v \simeq Hilb^n(S)$, le schéma de Hilbert qui paramètre les sous-schémas de S de dimension 0 et longueur n. Le résultat principal, due à Fujiki pour $n = 2$ et à Beauville en général, est le suivant :

Théorème 3. *Soit n un entier positif.*

1. *Si $S = X$ est une surface K3 projective, alors $Hilb^n(X)$ est une variété irréductible symplectique de dimension $2n$ dont le deuxième nombre de Betti est 23.*
2. *Si $S = J$ est une surface abélienne, alors il existe un morphisme*

$$Hilb^{n+1}(J) \longrightarrow J,$$

dont la fibre sur $0 \in J$ est notée $K^n(J)$ et appelée variété de Kummer généralisée. Alors $K^n(J)$ est une variété irréductible symplectique de dimension $2n$ dont le deuxième nombre de Betti est 8.

Ce résultat a été généralisé par plusieurs personnes (entre autres, Huybrechts-Göttsche, Mukai, Yoshioka, O'Grady), qui ont montré le résultat suivant :

Théorème 4. *Soit v un vecteur de Mukai primitif, et soit H une polarisation générique.*

1. *Si $S = X$ est une surface K3 projective et $(v, v) \geq 0$, alors M_v est une variété irréductible symplectique, qui est déformation d'un schéma de Hilbert $Hilb^n(X')$, pour une surface K3 X' et $n = 1 + \frac{(v,v)}{2}$.*
2. *Si $S = J$ est une surface abélienne et $(v, v) > 4$, alors il existe un morphisme $M_v \longrightarrow J \times \widehat{J}$ dont la fibre sur $(0, \mathscr{O}_J)$ est notée K_v. Alors K_v est une variété irréductible symplectique qui est déformation de $K^n(J')$, pour une surface abélienne J' et $n = \frac{(v,v)}{2} - 1$.*

Il ne reste donc qu'à étudier le cas où v n'est pas primitif. Dans ce cas, il existe un entier $m \in \mathbb{Z}$ et un vecteur de Mukai primitif $w \in H^2(S, \mathbb{Z})$ tels que $v = mw$. Le résultat principal sur l'existence de résolutions sympléctiques pour M_v a été montré par Kaledin, Lehn et Sorger, et est le suivant :

Théorème 5. *Soit $v = mw$ un vecteur de Mukai non-primitif tel que $(v, v) > 0$, et soit H une polarisation générique.*

1. *Si $m = 2$ et $(w, w) = 2$, alors il existe une résolution symplectique*

$$\pi_v : \widetilde{M}_v \longrightarrow M_v,$$

obtenue comme éclatement le long de la partie réduite du lieu singulier de M_v.

2. *Si $m > 2$ ou $(w, w) > 2$, alors il n'existe pas de résolution symplectique de M_v. En plus, les espaces de modules M_v sont localement factoriels.*

Grâce à ce théorème on peut construire les deux nouveaux exemples de O'Grady $\widetilde{M}_{10}$ et $\widetilde{M}_6$. En fait, O'Grady a démontré l'existence des deux nouveaux exemples avant que le Théorème 5 n'ait été montré.

Théorème 6. *(**O'Grady**). Soit $v = (2, 0, -2) \in \widetilde{H}(S, \mathbb{Z})$.*

1. *Si $S = X$ est une surface K3 projective telle que $Pic(X) = \mathbb{Z} \cdot H$ pour un diviseur ample H tel que $H^2 = 2$, alors la résolution symplectique $\widetilde{M}_{10}$ de l'espace de module $M_{10} := M_v$ du Théorème 5 est une variété irréductible symplectique de dimension 10 dont le deuxième nombre de Betti est 24.*
2. *Si $S = J$ est une surface abélienne telle que $NS(J) = \mathbb{Z} \cdot c_1(H)$ pour un diviseur ample H tel que $c_1^2(H) = 2$, alors l'espace de module M_v admet un morphisme $M_v \longrightarrow J \times \widehat{J}$ dont la fibre sur $(0, \mathscr{O}_J)$ est notée M_6. Alors la variété $\widetilde{M}_6 := \pi_v^{-1}(M_6)$ est une variété irréductible symplectique de dimension 6 dont le deuxième nombre de Betti est 8.*

La question qu'on se pose dans ce travail est la suivante : si v est primitif ou comme dans le point 2 du Théorème 5, l'espace des modules M_v est localement factoriel. Le seul cas restant est donc celui de v comme dans le point 1 du Théorème 5 : est-ce que M_v est localement factoriel ? Le résultat qu'on démontre est le suivant :

Théorème 7. *Les deux espaces de modules M_{10} et M_6 décrits dans l'énoncé du Théorème 6 sont $2-$factoriels.*

La preuve du Théorème 7 est basée sur l'analyse du groupe de Picard de $\widetilde{M}_{10}$ et $\widetilde{M}_6$, qui nous permet de calculer le groupe de Picard et le groupe des diviseurs de Weil (modulo équivalence linéaire) de M_{10} et de M_6. Dans le cas de M_{10}, ce dernier est isomorphe à $Pic(X) \oplus \mathbb{Z}[B]$, où B est le diviseur de Weil qui paramètre les faisceaux semistables qui ne sont pas localement libres. Grâce à un théorème du a Rapagnetta, on démontre que B ne peut pas être de Cartier, et grâce à une construction due à Le Potier, on démontre d'abord que M_{10} est $\mathbb{Q}-$factoriel, et après que $2B$ est un diviseur de Cartier, ce qui implique la $2-$factorialité de M_{10}.

Le cas de M_6 est plus compliqué. Le groupe des diviseurs de Weil (modulo équivalence linéaire) de M_6 est isomorphe à $NS(J) \oplus \mathbb{Z}[B] \oplus \mathbb{Z}/2\mathbb{Z}[D]$, où B est le diviseur de Weil qui paramètre les faisceaux semistables qui ne sont pas localement libres, et D est un diviseur de Weil dont le carré est nul. Comme dans le cas de M_{10}, un théorème de Rapagnetta nous permet de montrer que D n'est pas un diviseur de Cartier, mais on a toujours la $2-$factorialité de M_6 grâce à la construction de Le Potier.

Un autre point important dans la théorie des variétés irréductibles symplectiques est l'étude de la forme de Beauville-Bogomolov : si X est une variété irréductible symplectique, alors $H^2(X, \mathbb{Z})$ est un réseau par rapport à une forme d'intersection q, appelée forme de Beauville-Bogomolov. Cette forme est l'analogue pour les variétés irréductibles symplectiques de la forme d'intersection des surfaces K3. La forme de Beauville-Bogomolov a été déterminée pour les examples connus par Beauville et Rapagnetta. En particulier, si v est un vecteur de Mukai primitif et H est une polarisation générique, alors la forme de

Beauville-Bogomolov est déterminée par la forme de Mukai sur $\widetilde{H}(S,\mathbb{Z})$. Si $v^\perp \subseteq \widetilde{H}(S,\mathbb{Z})$ est l'orthogonal à v par rapport à la forme de Mukai, on a les résultats suivants :

1. Si X est une surface K3 projective et $(v,v)=0$, alors il existe une isométrie de structures de Hodge $v^\perp/v \longrightarrow H^2(M_v,\mathbb{Z})$.
2. Si X est une surface K3 projective et $(v,v)>0$, alors il existe une isométrie de structures de Hodge $v^\perp \longrightarrow H^2(M_v,\mathbb{Z})$.
3. Si J est une surface abélienne et $(v,v)>4$, alors il existe une isométrie de structures de Hodge $v^\perp \longrightarrow H^2(K_v,\mathbb{Z})$.

La question naturelle est si le même résultat est vrai pour $\widetilde{M}_{10}$ et $\widetilde{M}_6$. Le résultat qu'on démontre, en collaboration avec A. Rapagnetta, est le suivant :

Théorème 8. *Soit $v=(2,0,-2)$.*

1. *Soit X une surface K3 projective comme dans le point 1 du Théorème 6. Alors il existe un morphisme injectif de structures de Hodge*

$$v^\perp \longrightarrow H^2(\widetilde{M}_{10},\mathbb{Z}),$$

qui est une isométrie sur son image.

2. *Soit J une surface abélienne comme dans le point 2 du Théorème 6. Alors il existe un morphisme injectif de structures de Hodge*

$$v^\perp \longrightarrow H^2(\widetilde{M}_6,\mathbb{Z}),$$

qui est une isométrie sur son image.

A Gabriel's Theorem for Coherent Twisted Sheaves

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Contents

Introduction

Gabriel's Theorem is one of the main and, maybe, surprising results of the algebraic geometry of the '60s. It perfectly adheres to the spirit of algebraic geometry: to any geometrical object X (as a topological space, a manifold, a scheme...) we associate an algebraic object A (as the fundamental group π_1, the homology groups H_i, the cohomology groups H^i, a category...), and we study which geometrical properties of X are encoded by A.

Gabriel's Theorem deals with the following situation: to any Notherian scheme X we associate the abelian category $Coh(X)$ of coherent sheaves on X. The problem is to understand which properties of X can be recovered from $Coh(X)$. The result is the following:

Theorem 0.1. *(**Gabriel**, '62). Let X be any Noetherian scheme. Then:*

1. *the scheme structure of X can be recovered from $Coh(X)$;*
2. *if Y is a Noetherian scheme and F is an equivalence between $Coh(X)$ and $Coh(Y)$, then F induces an isomorphism of schemes between X and Y.*

Gabriel's Theorem has several consequences. The main one is that to know the scheme X is equivalent to know its category of coherent sheaves, meaning that $Coh(X)$ is a strong geometric invariant. This leaded to the study of the (bounded) derived category $D^b(X)$ of $Coh(X)$, a geometric invariant that is weaker than $Coh(X)$: indeed, there are smooth projective varieties X and X' which are not isomorphic, but which have equivalent derived categories $D^b(X)$ and $D^b(X')$.

Naturally, we need to explain what it means to recover the scheme structure of X from $Coh(X)$. Gabriel's idea is the following: first, he defines a particular class of subcategories of $Coh(X)$, called Serre's subcategories of finite type. He shows that there is a bijective correspondence between the set of Serre's subcategories of finite type of $Coh(X)$ and the set of closed subsets of X. Using this, he defines a topology on the set E of irreducible Serre's subcategories of finite type, and he shows that this topological space is homeomorphic to X. The last step is to define a ringed space structure on E, and to show that we obtain a scheme isomorphic to X: this is done using the notion of center of a category.

It is then natural to ask if one can show a similar theorem even for other abelian categories we can associate to a Noetherian scheme, but more general than the category of coherent sheaves. The generalization we present in this work is obtained for the abelian category of coherent sheaves twisted by an element of the cohomological Brauer group $Br'(X) = H^2_{ét}(X, \mathscr{O}_X^*)_{tors}$ of a Noetherian scheme X. Our result is the following:

Theorem 0.2. *Let X be a Noetherian scheme, and let $\alpha \in Br'(X)$ be an element of the cohomological Brauer group of X. Then:*

1. *the schematic structure of X can be recovered from $Coh(X, \alpha)$;*
2. *if Y is a Noetherian scheme and $\beta \in Br'(Y)$, and if F is an equivalence between $Coh(X, \alpha)$ and $Coh(Y, \beta)$, then F induces an isomorphism of schemes between X and Y.*

The proof is modelled on the one of Gabriel's Theorem, but clearly adapted to twisted sheaves, so that there are some problems one needs to fix, which are based on the differences between twisted and classical sheaves: first of all, one needs to show the existence of a bijective correspondence between the set of closed subsets of X and the set of Serre's subcategories of finite type of $Coh(X, \alpha)$. In the classical case, this was done using properties of coherent sheaves which are not so evident in the twisted case: the first one is the fact that any coherent sheaf on an open subset U of X extends to a coherent sheaf on X; the second one is the existence of coherent sheaf on X whose support if the whole X.

For twisted sheaves, the first property is shown using the theory of $\mathscr{O}_X^*-$gerbes. This result is even the main point to define a ringed space structure on the set of irreducible Serre's subcategories of finite type of $Coh(X, \alpha)$, exactly as in the proof of Gabriel's Theorem. The second property is more delicate: first of all, it is possible to show the existence of a twisted coherent sheaf whose support is the whole X if we suppose X to be reduced. This hypothesis is not necessary if one chooses $\alpha \in Br(X)$, since in this case there is a locally free twisted sheaf. Then, one needs to show that in order to get Theorem 2.1 it is sufficient to consider the reduced case.

There are more remarks one can add to Theorem 2.1. The first one is that if we choose $\alpha \in Br(X)$, then $Coh(X, \alpha)$ is equivalent to the category $Mod_{Coh}(\mathscr{A})$ of coherent sheaves admitting a structure of $\mathscr{A}-$module, where $\mathscr{A}$ is the Azumaya algebra whose equivalence class in $Br(X)$ is α. In particular, $Mod_{Coh}(\mathscr{A})$ is a full subcategory of $Coh(X)$, and one can expect the Theorem to be true. What is less expected is that in the statement of Theorem 2.1 we suppose $\alpha \in Br'(X)$. In general, if X is not quasi-projective, there is $\alpha \in Br'(X)$ which is not in $Br(X)$: for such an α, there is no coherent model for $Coh(X, \alpha)$. Anyway, this is true for any affine subscheme of X: this is the main argument to show Theorem 2.1.

Chapter 1

Generalities on Brauer groups and twisted sheaves

This first chapter is devoted to the introduction of basic facts on Brauer groups and twisted sheaves, in order to define the main tools of this work. The Brauer group of a scheme X can be defined in several ways. A first approach is to consider the group $H^2_{\acute{e}t}(X, \mathscr{O}_X^*)$, which is sometimes called the Brauer-Grothendieck group of X, and which can be interpreted as a higher analogue of the Picard group of a scheme. In general, the Brauer-Grothendieck group is rather complicated, and we consider only its torsion part, denoted $Br'(X)$ and called the cohomological Brauer group of X. If X is a regular scheme, then the cohomological Brauer group of X equals the Brauer-Grothendieck group, but in general this is not true.

The classical definition of the Brauer group is not cohomological: the Brauer group $Br(X)$ of X is the group of equivalence classes of Azumaya algebras on X, where the multiplication is given by the tensor product. However, it is a classical result that there is an injection of $Br(X)$ into $Br'(X)$. It is a rather complicated problem to understand when these two groups are equal, and the best results in this domain are due to de Jong and Gabber, who show that this is indeed the case for any quasi-projective scheme. Finally, there is another interpretation of the elements of the Brauer group of X, namely by means of $\mathscr{O}_X^*-$gerbes. This approach will be briefly resumed at the end of the first section.

The second topic is the introduction of the notion of sheaf twisted by an element of the Brauer-Grothendieck group, and of the definition of the abelian categories these objects form. In particular, for our purposes the most important among these is $Coh(X, \alpha)$, the category of coherent $\alpha-$twisted sheaves, for which we resume the most important and basic properties. In particular, we will present two definitions of twisted sheaf, the first one in stack-theoretic terms (which can be found, as instance, in Lieblich [Lie1], [Lie2], [Lie3], and in Donagi-Pantev [DP]), the second one in cohomological terms, (which can be found as instance in Căldăraru [Cal]).

1.1 Definitions of the Brauer group

In this first section, we recall the different notions of Brauer groups of a scheme. In the first part, we present three classical notions: the Brauer group, the cohomological Brauer group and the Brauer-Grothendieck group. In the second part, we briefly present the definition of gerbe and the relation between $\mathscr{O}_X^*$−gerbes and the elements in the Brauer group.

1.1.1 Brauer groups

In this section, let X be a scheme.

Definition 1.1. The *cohomological Brauer group* $Br'(X)$ of X is

$$Br'(X) := H^2_{\acute{e}t}(X, \mathscr{O}_X^*)_{tors} = H^2(X, \mathbb{G}_m)_{tors}.$$

Definition 1.2. The *Brauer group* $Br(X)$ of X is the group of equivalence classes of Azumaya algebras over X (see Section 5 of [Gro] or Chapter IV of [Mil] for the definition of Azumaya algebra).

Definition 1.3. The *Brauer-Grothendieck group of* X is

$$H^2_{\acute{e}t}(X, \mathscr{O}_X^*) = H^2(X, \mathbb{G}_m).$$

The main relation between these three groups is the following:

Proposition 1.1. *Let X be a scheme. We have the following chain of inclusions*

$$Br(X) \subseteq Br'(X) \subseteq H^2(X, \mathbb{G}_m).$$

Proof. The second inclusion is clear by definition. The first inclusion is Proposition 1.4 in [Gro]. □

The question of when these two groups are isomorphic is rather complicated, and we have the following results:

Proposition 1.2. *Let X be a scheme. The two groups $Br(X)$ and $Br'(X)$ are equal in one of the following cases:*

1. *the scheme X is noetherian of dimension 1;*
2. *the scheme X is regular of dimension 2;*
3. *the scheme X is quasi-compact, quasi-separated ad admits an ample line bundle.*

Proof. The first two points were shown by Grothendieck in [Gro2], Corollaire 2.2. The third one was first shown for the separated union of two affine schemes by Gabber in [Gabb]. Recently de Jong vastly generalized this result, obtaining point 3, in [deJ]. □

There are more results in this direction (see, for example, [HS]), but we won't need them. Another important result is the following:

Proposition 1.3. *Let X be a regular scheme. Then*

$$Br'(X) = H^2(X, \mathbb{G}_m).$$

Proof. See Example III.2.22 in [Mil], using the fact that the Brauer group of a field is torsion, as shown in [Se], Chapter 4, §4-§5. □

1.1.2 Gerbes

By definition, we have a sheaf-theoretic interpretation of the elements in the Brauer group of X: they are equivalence classes of Azumaya algebras over X. It is possible to give a similar interpretation for the elements in the Brauer-Grothendieck group: they are isomorphism classes of $\mathscr{O}_X^*-$gerbes, whose definition we are now going to recall briefly. Since this section is not supposed to be an introduction on this vast subject, we suggest to the reader a more detailed exposition on gerbes and twisted sheaves, as in [Lie1] or [Lie2]. In the following, we consider stacks for the étale topology.

Definition 1.4. A *gerbe* over X is a stack $\mathscr{X}$ over X such that the following conditions hold:

1. for any open U of X there is a covering V of U such that the fiber category $\mathscr{X}_V \neq \emptyset$;

2. for any open U of X and any $s, t \in obj(\mathscr{X}(U))$, there is a covering V of U and an isomorphism between s_V and t_V in the fiber category $\mathscr{X}_V$.

An $\mathscr{O}_X^*-$gerbe is a gerbe $\mathscr{X}$ over X along with an isomorphism between $\mathscr{O}_{\mathscr{X}}^*$ and the inertia stack $\mathscr{I}(\mathscr{X}) := \mathscr{X} \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X}$ (i. e. for any open $U \subseteq X$ and for any $s \in \mathscr{X}(U)$ we have fixed an isomorphism between $\mathscr{O}_X^*(U)$ and $Aut_{\mathscr{X}(U)}(s)$ compatible with pull-backs).

We have a more explicit description of $\mathscr{O}_X^*-$gerbes. Let $\mathscr{P}ic(X)$ denote the sheaf of Picard categories, i. e. the sheaf which assigns to any open subset U of X the category $\mathscr{P}ic(U)$, whose objects are line bundles on U, and $Hom_{\mathscr{P}ic(U)}(L, M) = Isom_U(L, M)$ for any $L, M \in Pic(U)$. Then, an $\mathscr{O}_X^*-$gerbe $\mathscr{X}$ assigns to any open subset U of X a $\mathscr{P}ic(U)$-torsor with compatibility of the assignments to different open subsets (see [DP], section 2.1.1). The relation between $\mathscr{O}_X^*-$gerbes and the Brauer-Grothendieck group of X is the following:

Proposition 1.4. *There is a natural bijection between the set of isomorphism classes of $\mathscr{O}_X^*-$gerbes and $H^2_{ét}(X, \mathscr{O}_X^*)$. In particular, the $\mathscr{O}_X^*-$gerbe associated to $\alpha \in H^2_{ét}(X, \mathscr{O}_X^*)$ will be denoted X_α.*

Proof. See [Mil], Proposition IV.2. □

Proposition 1.5. *Let $\alpha \in H^2_{ét}(X, \mathscr{O}_X^*)$ and let X_α be an $\mathscr{O}_X^*$−gerbe representing α. Then X_α is an algebraic stack locally of finite presentation. If X is quasi-separated, X_α is finitely presented. Moreover, X is (locally) noetherian if and only if X_α is.*

Proof. See Lemma 2.2.1.1 in [Lie1]. □

1.2 Twisted sheaves

In this section we introduce the notion of sheaf twisted by an element of the Brauer-Grothendieck group. We can define it in two equivalent ways. The first one is stack-theoretic, and uses the interpretation of the Brauer-Grothendieck group as the classification group of $\mathscr{O}_X^*$−gerbes. The second one is sheaf-theoretic, and is based on the notion of hypercoverings, which gives a way to describe the elements of the Brauer-Grothendieck group in cocycles terms. These two ways of understanding twisted sheaves present advantages and problems. For example, the stack-theoretic point of view is useful to study moduli problems, while the sheaf-theoretic is more friendly, and useful in particular in the quasi-projective case. Again, for the definitions in the stack-theoretic point of view, the main reference are [Lie1] and [Lie2].

Let $\alpha \in H^2_{ét}(X, \mathscr{O}_X^*)$, corresponding to an $\mathscr{O}_X^*$−gerbe X_α. Let

$$\mathscr{I}(X_\alpha) := X_\alpha \times_{X_\alpha \times X_\alpha} X_\alpha$$

be the *inertia stack of* X_α representing the functors of isomorphisms of objects. Given a sheaf $\mathscr{F}$ on X_α, there is a natural action $\mathscr{F} \times \mathscr{I}(X_\alpha) \longrightarrow \mathscr{F}$: for any U open subset of X, and for any $(f, s) \in \mathscr{F} \times \mathscr{I}(X_\alpha)(U)$, we get $s^* : \mathscr{F}(U) \longrightarrow \mathscr{F}(U)$ which is an isomorphism. We define the action as the one sending (f, s) to $s^*(f)$, and we call it *inertia action*.

Definition 1.5. An α−*twisted sheaf* (or *sheaf twisted by* α) is a sheaf of (left) $\mathscr{O}_{X_\alpha}$−modules such that the inertia action equals the natural action associated with the left $\mathscr{O}_{X_\alpha}$−module.

We denote $Sh(X, \alpha)$ the category of α−twisted sheaves, $QCoh(X, \alpha)$ and $Coh(X, \alpha)$ the subcategories of (quasi-)coherent α−twisted sheaves.

Since α is an element of $H^2_{ét}(X, \mathscr{O}_X^*)$, there is an hypercovering $U_\bullet$ of X and a cocycle $\underline{\alpha} \in \Gamma(U_2, \mathscr{O}_X^*)$ representing α in cohomology (see [Ver], Exposé V.7).

Definition 1.6. A *Căldăraru* − $\underline{\alpha}$−*twisted sheaf* (or *sheaf twisted by* $\underline{\alpha}$ *following Căldăraru*) is a couple $(\mathscr{F}, g)$ where $\mathscr{F}$ is a sheaf of $\mathscr{O}_{U_0}$−modules and $g : (pr_1^{U_1})^*\mathscr{F} \longrightarrow (pr_0^{U_1})^*\mathscr{F}$ is a gluing datum on U_1 such that $\delta(g) \in$

$Aut((pr_0^{U_2})^*\mathscr{F})$ equals the cocycle $\underline{\alpha}$. The couple $(\mathscr{F}, g)$ will be called *(quasi-) coherent* if $\mathscr{F}$ is. If $(\mathscr{F}, g), (\mathscr{F}', g')$ are two $C\breve{a}ld\breve{a}raru - \underline{\alpha}-$twisted sheaves, a morphism $f : (\mathscr{F}, g) \longrightarrow (\mathscr{F}', g')$ is a morphism of $\mathscr{O}_{U_0}-$modules between $\mathscr{F}$ and $\mathscr{F}'$ such that $(pr_0^{U_1})^* f \circ g = g' \circ (pr_1^{U_1})^* f$.

Remark 1.1. If X is a complex analytic space or a quasi-projective complex scheme, then we can take the hypercovering to be an open covering $\{U_i\}$ of X. Denoting $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ for any i, j, k, then a Căldăraru $\underline{\alpha}-$twisted sheaf is given by a family of $\mathscr{O}_{U_i}-$modules $\{\mathscr{F}_i\}$ and by a family of isomorphisms $g_{ij} : \mathscr{F}_{j|U_{ij}} \longrightarrow \mathscr{F}_{i|U_{ij}}$ such that $g_{ii} = id_{\mathscr{F}_i}$, $g_{ij} = g_{ji}^{-1}$ and $g_{ij} \circ g_{jk} \circ g_{ki} = \alpha_{ijk}$ for any i, j, k, where $\underline{\alpha} = \{\alpha_{ijk}\}$. This is the definition of twisted sheaf we can find in [Cal], where the two others are used in [Lie1], [Lie2], [Lie3], [deJ], and [DP].

We denote $Sh^C(X, U_\bullet, \underline{\alpha})$ the category of Căldăraru $\underline{\alpha}-$twisted sheaves. Note that while $Sh(X, \alpha)$ depends only on X and on α, the cohomology class of $\underline{\alpha}$, $Sh^C(X, U_\bullet, \underline{\alpha})$ depends on X, on the chosen hypercovering $U_\bullet$ and on the cocycle $\underline{\alpha}$ representing α. It can be checked (see [Cal], Lemma 1.2.3, Corollary 1.2.6 and Lemma 1.2.8) that if one changes hypercovering and cocycle, the category changes only by equivalence (which, in general, is non-canonical). This allows us to write it as $Sh^C(X, \alpha)$. In a similar way we define its full subcategories $QCoh^C(X, \alpha)$ and $Coh^C(X, \alpha)$.

Proposition 1.6. *There is a natural equivalence between the two categories* $Sh(X, \alpha)$ *and* $Sh^C(X, \alpha)$. *Moreover, this equivalence sends (quasi-)coherent* $\alpha-$*twisted sheaves to (quasi-)coherent Căldăraru twisted sheaves.*

Proof. See section 2.1.3 in [Lie3]. □

In the following, we will use notation $Sh(X, \alpha)$, $QCoh(X, \alpha)$, $Coh(X, \alpha)$ even to denote the categories of objects in Căldăraru's definitions. We will refer to the objects of these categories as (quasi-coherent, coherent) $\alpha-$twisted sheaves.

We are now interested in some useful properties of twisted sheaves. The first one corresponds to a similar one on classical sheaves: if X is a noetherian scheme, any quasi-coherent sheaf on X is the colimit of its coherent subsheaves. This implies that if U is an open subset of X, then any coherent sheaf on U extends to a coherent sheaf on X. The same property is true even for twisted sheaves:

Proposition 1.7. *Let* X *be a noetherian scheme, and let* $\alpha \in H^2_{ét}(X, \mathscr{O}_X^*)$. *Any quasi-coherent* $\alpha-$*twisted sheaf is colimit of its coherent* $\alpha-$*twisted subsheaves. In particular, if* U *is any open subset of* X, *any coherent* $\alpha-$*twisted sheaf on* U *extends to a coherent* $\alpha-$*twisted sheaf on* X.

Proof. See Proposition 2.2.1.5 in [Lie1], for the first part. The second works as in the classical case: if $\mathscr{F}$ is a coherent $\alpha_{|U}-$twisted sheaf on U, and if j_U is the inclusion of U in X, then $j_{U*}\mathscr{F}$ is a quasi-coherent $\alpha-$twisted sheaf on X. By the first part of the proposition, it is colimit of its coherent $\alpha-$twisted subsheaves. Since the restriction to U is coherent, this implies that there must be a coherent $\alpha-$twisted sheaf on X whose restriction to U is $\mathscr{F}$. □

Another important result is about the existence of a locally free $\alpha-$twisted sheaf of positive rank. The main point here is that in order to guarantee the existence of such a sheaf we must consider α to be an element of $Br(X)$.

Proposition 1.8. *Let X be a noetherian scheme, $\alpha \in H^2_{ét}(X, \mathscr{O}^*_X)$. There is a locally free $\alpha-$twisted sheaf $\mathscr{E}$ whose rank is constant and non-zero if and only if $\alpha \in Br(X)$.*

Proof. See Theorem 1.3.5 in [Cal]. □

This result can be used as a different definition of the Brauer group. Here is a result that links the two descriptions:

Proposition 1.9. *Let X be a noetherian scheme, $\alpha \in Br(X)$, $\mathscr{E}$ be a locally free $\alpha-$twisted sheaf. Then $\mathscr{A} := \mathscr{E}nd_{Sh(X,\alpha)}(\mathscr{E})$ is an Azumaya algebra representing α. Moreover, if $Mod(\mathscr{A})$ is the category of $\mathscr{A}-$modules on X, then the functor sending any $\alpha-$twisted sheaf $\mathscr{F}$ to $\mathscr{F} \otimes \mathscr{E}^\vee$ is an equivalence between $Sh(X, \alpha)$ and $Mod(\mathscr{A})$, which sends (quasi-)coherent objects to (quasi-)coherent objects.*

Proof. See Proposition 1.3.6 and Theorem 1.3.7 in [Cal]. □

Finally, we present an important notion similar to the one we have on classical sheaves.

Definition 1.7. Let $\alpha \in H^2_{ét}(X, \mathscr{O}^*_X)$ and let $\mathscr{F}$ be an $\alpha-$twisted sheaf. We call *support of* $\mathscr{F}$ the closed substack of X_α defined by the kernel of the map $\mathscr{O}_{X_\alpha} \longrightarrow \mathscr{E}nd_{X_\alpha}(\mathscr{F})$. The *schematic support of* $\mathscr{F}$ is the scheme-theoretic image in X of the support of $\mathscr{F}$, and will be denote $Supp(\mathscr{F})$.

Remark 1.2. The schematic support of any coherent $\alpha-$twisted sheaf on a noetherian scheme is closed.

We conclude this section with a brief remark about functors and properties of the categories of twisted sheaves. It is easy to show that $Sh(X, \alpha)$, $QCoh(X, \alpha)$ and $Coh(X, \alpha)$ are abelian categories. Moreover, $Sh(X, \alpha)$ and $QCoh(X, \alpha)$ have enough injective objects (see [Cal], Lemma 2.1.1).

Proposition 1.10. *Let X, Y be two noetherian schemes, $\alpha, \alpha' \in H^2_{ét}(X, \mathscr{O}_X^*)$, $\beta \in H^2_{ét}(Y, \mathscr{O}_Y^*)$ and let $f : X \longrightarrow Y$ be a morphism. Then we have the following functors:*

$$\mathscr{H}om(.,.) : Sh(X, \alpha) \times Sh(X, \alpha') \longrightarrow Sh(X, \alpha'\alpha^{-1})$$

$$\otimes : Sh(X, \alpha) \times Sh(X, \alpha') \longrightarrow Sh(X, \alpha\alpha')$$

$$f^* : Sh(Y, \beta) \longrightarrow Sh(X, f^*\beta)$$

$$f_* : Sh(X, f^*\beta) \longrightarrow Sh(Y, \beta).$$

Moreover f^* is left adjoint to f_*. All the statement remain true even if we pass to the categories of quasi-coherent twisted sheaves.

Proof. See [Cal], Propositions 1.2.10 and 1.2.13. □

Chapter 2

Reconstruction of a scheme

In this chapter we show that any noetherian scheme X can be recovered from the category $Coh(X,\alpha)$ of $\alpha-$twisted sheaves, for any $\alpha \in Br'(X)$. The idea is to give a ringed space structure to the set $E_{X,\alpha}$ of irreducible Serre subcategories of finite type of $Coh(X,\alpha)$, and then show that this is a scheme isomorphic to X.

In order to do so, we introduce the notion of Serre subcategory of an abelian category, and show that Serre subcategories of finite type of $Coh(X,\alpha)$ are in bijective correspondence with closed subsets of X. This allows us to put a topology on $E_{X,\alpha}$, which recovers the topology of X. The main result we show is the following

Theorem 2.1. *Let X be a noetherian scheme, and let $\alpha \in Br'(X)$. Then*

1. *the abelian category $Coh(X,\alpha)$ determines X;*

2. *if Y is a noetherian scheme and $\beta \in Br'(Y)$, then any equivalence between $Coh(X,\alpha)$ and $Coh(Y,\beta)$ induces an isomorphism between X and Y.*

The problem will be to give a good definition of a structure sheaf on $E_{X,\alpha}$, such that we can get an isomorphism between $E_{X,\alpha}$ and X. If in the statement of Theorem 2.1 one considers $\alpha = \beta = 1$, then $Coh(X,\alpha)$ is equivalent to $Coh(X)$ and $Coh(Y,\beta)$ is equivalent to $Coh(Y)$. Then Theorem 2.1 is a generalization of Gabriel's Theorem. Anyway, we remark that in the statement of Gabriel's Theorem one can drop the hypothesis of X being reduced.

Another important remark is the following: Theorem 2.1 states then any equivalence F between $Coh(X,\alpha)$ and $Coh(Y,\beta)$ induces an isomorphism f between X and Y. It seems natural to conjecture that $f^*(\beta) = \alpha$, but we are not able to show it. Anyway, this is true if we suppose X and Y to be smooth and projective, as shown in [CS].

2.1 Serre subcategories of an abelian category

For a reference on Serre subcategories and quotient categories see [Ga]. Let $\mathcal{A}$ be an abelian category.

Definition 2.1. A full subcategory $\mathcal{I}$ of $\mathcal{A}$ is called *Serre subcategory* if for every short exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have $B \in \mathcal{I}$ if and only if $A, C \in \mathcal{I}$.
We say that $\mathcal{I}$ is a *Serre subcategory of finite type* if it is a Serre subcategory of $\mathcal{A}$ generated by an element $A \in \mathcal{I}$, i. e. $\mathcal{I}$ is the smallest Serre subcategory of $\mathcal{A}$ containing A. Such an A will be called a *generator* for $\mathcal{I}$.
We say that $\mathcal{I}$ is an *irreducible Serre subcategory* if it is not generated (as Serre subcategory) by two proper Serre subcategories.

Example 2.1. Let $Coh_Z(X, \alpha)$ be the full subcategory of $Coh(X, \alpha)$ whose objects have support contained in the closed subset Z of X. It is easy to verify that this is a Serre subcategory of $Coh(X, \alpha)$.

Definition 2.2. If $\mathcal{I}$ is a subcategory of $\mathcal{A}$, the *quotient category* $\mathcal{A}/\mathcal{I}$ is the category having the same objects as $\mathcal{A}$ and whose morphisms are defined in the following way: if $A, B \in \mathcal{A}$, let

$$Hom_{\mathcal{A}/\mathcal{I}}(A, B) := \varinjlim Hom_{\mathcal{A}}(A', B')$$

where $i : A' \hookrightarrow A$ is a sub-object of A such that $coker(i) \in \mathcal{I}$ and $p : B \twoheadrightarrow B'$ is a quotient of B such that $ker(p) \in \mathcal{I}$.

If $\mathcal{I}$ is a Serre subcategory of $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is an abelian category. We have the two following lemmas:

Lemma 2.2. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor that admits a fully faithful right adjoint. Then $\ker(F)$ is a Serre subcategory of $\mathcal{A}$ and the induced functor $F : \mathcal{A}/\ker(F) \longrightarrow \mathcal{B}$ is an equivalence.*

Proof. It is easy to see that $\ker(F)$ is a Serre subcategory of $\mathcal{A}$: since F is exact, it takes a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

to a short exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0.$$

Then $F(A) = F(C) = 0$ if and only if $F(B) = 0$.

The induced functor is then an equivalence: let G be the fully faithful right adjoint of F. Let $A \in \mathcal{A}/\ker(F)$. From general properties of fully-faithful adjoint functors (see, for example, Proposition 1.5.6 in [KS]) there is a canonical isomorphism $F(G(A)) \simeq A$, so that F is essentially surjective. Let A, B be two objects in $\mathcal{A}/\ker(F)$. Since G is fully faithful, we have

$$Hom_{\mathcal{B}}(F(A), F(B)) = Hom_{\mathcal{A}}(G(F(A)), G(F(B))).$$

The canonical morphism $f : G(F(A)) \longrightarrow A$ has kernel and cokernel lying in $\ker(F)$, since F is exact and $F(f)$ is an isomorphism between $F(G(F(A)))$ and $F(A)$. This means that $G(F(A))$ is isomorphic to A in $\mathcal{A}/\ker(F)$. Doing the same for B we get

$$Hom_{\mathcal{A}}(G(F(A)), G(F(B))) = Hom_{\mathcal{A}/\ker F}(A, B),$$

showing that F is fully faithful, and we are done. □

Lemma 2.3. *Let $\mathcal{A}$ be an abelian category, $\mathcal{A}'$ a full abelian subcategory of $\mathcal{A}$ and $\mathcal{I}$ a Serre subcategory of $\mathcal{A}$. Suppose that for every $M \in \mathcal{A}'$, $N \in \mathcal{I}$, if N is a sub-object or a quotient of M, then $N \in \mathcal{I} \cap \mathcal{A}'$. Then the induced functor*

$$i : \mathcal{A}'/\mathcal{I} \cap \mathcal{A}' \longrightarrow \mathcal{A}/\mathcal{I}$$

is fully faithful.

Proof. Let $A, B \in \mathcal{A}'/\mathcal{I} \cap \mathcal{A}'$. By definition, a morphism f between A and B in $\mathcal{A}/\mathcal{I}$ is given by a limit of morphisms between A' and B', where $j : A' \hookrightarrow A$ is such that $coker(j) \in \mathcal{I}$ and $p : B \twoheadrightarrow B'$ is such that $ker(p) \in \mathcal{I}$. Since $A, B \in \mathcal{A}'$, by hypothesis $ker(p)$ and $coker(j)$ are in $\mathcal{A}' \cap \mathcal{I}$, so that any element of the limit to define f appears in the definition of a morphism f' between A and B in $\mathcal{A}'/\mathcal{I} \cap \mathcal{A}'$. The converse is clear, so that to define a morphism between A and B in $\mathcal{A}'/\mathcal{A}' \cap \mathcal{I}$ is the same as to define it in $\mathcal{A}/\mathcal{I}$. □

The following is the main result we will use about Serre subcategories and quotient categories:

Proposition 2.4. *Let X be a noetherian scheme, and let $\alpha \in H^2_{\acute{e}t}(X, \mathscr{O}_X^*)$. Moreover, let Z be a closed subset of X, whose complementary open subset is $U := X \setminus Z$, and let j_U be the inclusion of U in X. Then the functor*

$$j_U^* : Coh(X, \alpha)/Coh_Z(X, \alpha) \longrightarrow Coh(U, \alpha_{|U})$$

is an equivalence of abelian categories.

Proof. The pull-back functor

$$j_U^* : QCoh(X, \alpha) \longrightarrow QCoh(U, \alpha_{|U})$$

is an exact functor with a fully faithful right adjoint j_{U*}. Since $\ker j_U^* = QCoh_Z(X, \alpha)$, by Lemma 2.2 the functor

$$j_U^* : QCoh(X, \alpha)/QCoh_Z(X, \alpha) \longrightarrow QCoh(U, \alpha_{|U})$$

is an equivalence, and by Lemma 2.3 the induced functor

$$j_U^* : Coh(X, \alpha)/Coh_Z(X, \alpha) \longrightarrow Coh(U, \alpha_{|U})$$

is fully faithful. Since X is noetherian, by Proposition 1.7 we know that any coherent $\alpha_{|U}-$twisted sheaf on U extends to a coherent $\alpha-$twisted sheaf on X, so that j_U^* is essentially surjective, giving an equivalence between the two abelian categories $Coh(X, \alpha)/Coh_Z(X, \alpha)$ and $Coh(U, \alpha_{|U})$. □

2.2 Closed subsets and Serre subcategories

In this section we show that Serre's subcategories of finite type of $Coh(X, \alpha)$ (for any noetherian scheme X and any $\alpha \in H^2(X, \mathbb{G}_m)$) are in bijective correspondence with closed subsets of X. The first result we need is the following:

Lemma 2.5. *Let X be a noetherian reduced scheme, and let $\alpha \in H^2_{\acute{e}t}(X, \mathscr{O}_X^*)$. Then there is a coherent $\alpha-$twisted sheaf whose support is X.*

Proof. Since X is noetherian, by Proposition 1.7 there is a coherent $\alpha-$twisted sheaf $\mathscr{F}$ whose support is the whole X if and only if there is a coherent $\alpha_{|X_\eta}-$twisted sheaf on X_η, where X_η is the generic scheme of X. This means we can suppose X to be affine, so that by Proposition 1.2 we have $Br(X) = Br'(X)$. Now, since X is reduced the generic scheme of X is the spectrum of a finite product of fields. If we are able to produce such a sheaf on the spectrum of a field, we are done. Now, if K is a field, $H^2_{\acute{e}t}(Spec(K), \mathscr{O}^*_{Spec(K)}) = Br(K)$ is torsion, so that the cohomological Brauer group and the Brauer-Grothendieck group are the same. Since $Br(X) = Br'(X)$, by Proposition 1.8 there is a locally free $\alpha-$twisted sheaf on X, and we are done. See Lemma 3.1.3.2 in [Lie2] for more details. □

Lemma 2.6. *Let X be a noetherian scheme, Z a closed subscheme and Z_{red} the reduced scheme associated to Z. Moreover, let ι be the closed immersion of Z_{red} into X. If there is $\mathscr{F} \in Coh_Z(X, \alpha)$ such that $\langle \iota^*\mathscr{F} \rangle = Coh(Z_{red}, \iota^*\alpha)$, then $\langle \mathscr{F} \rangle = Coh_Z(X, \alpha)$.*

Proof. First, let us prove the Lemma for $Z = X$. Let $\mathscr{G} \in Coh(X, \alpha)$, and let $\mathscr{F} \in Coh(X, \alpha)$ be a generator for $Coh(X_{red}, \alpha)$. We need to show that $\mathscr{G}$ is in $\langle \mathscr{F} \rangle$. Now, the $\alpha-$twisted sheaf $\mathscr{G}$ admits a finite filtration

$$0 = \mathscr{G}_0 \subseteq \mathscr{G}_1 \subseteq ... \subseteq \mathscr{G}_n = \mathscr{G}$$

such that for any $i = 1, ..., n$ the subquotient $\mathscr{G}_i/\mathscr{G}_{i-1}$ is reduced (see Chapitre 15 in [LMB]), so that it is an element of $\langle\mathscr{F}\rangle$. Using the filtration and the fact that $\langle\mathscr{F}\rangle$ is a Serre subcategory of $Coh(X,\alpha)$, this implies $\mathscr{G} \in \langle\mathscr{F}\rangle$, and we are done.

Now, let Z be a closed subscheme of X, and let ι be the closed immersion of Z in X. Let $\mathscr{F} \in Coh_Z(X,\alpha)$ be such that $\langle\iota^*\mathscr{F}\rangle = Coh(Z_{red}, \iota^*\alpha)$. Then, by the first part of the lemma we have $\langle\iota^*\mathscr{F}\rangle = Coh(Z, \iota^*\alpha)$. Now, notice that $\iota_*(Coh(Z,\iota^*\alpha)) = Coh_Z(X,\alpha)$, and that $\iota_*\iota^*\mathscr{F} \in \langle\mathscr{F}\rangle$. This implies $Coh_Z(X,\alpha) \subseteq \langle\mathscr{F}\rangle$, and we are done since $\mathscr{F} \in Coh_Z(X,\alpha)$. $\square$

Using these two basic Lemmas, we are able to show the main point of the whole picture, which is the following proposition:

Proposition 2.7. *Let X be a noetherian scheme, $\alpha \in H^2_{\acute{e}t}(X, \mathscr{O}^*_X)$, and let Z be any closed subset of X. Then $Coh_Z(X,\alpha)$ is a Serre subcategory of finite type of $Coh(X,\alpha)$, generated by any $\mathscr{F} \in Coh(X,\alpha)$ such that $Supp(\mathscr{F}) = Z$.*

Proof. Notice that by Lemma 2.6 we just need to show that there is $\mathscr{F} \in Coh_Z(X,\alpha)$ such that $Supp(\mathscr{F}) = Z$, and any $\mathscr{F}$ of this sort is such that $\langle\iota^*\mathscr{F}\rangle = Coh(Z_{red}, \iota^*\alpha)$, where ι is the closed immersion of Z in X.

Moreover, notice that we are considering Z as a closed subset (or as a subscheme with the natural reduced structure) of X, so that the equality $Supp(\mathscr{F}) = Z$ is at the level of subschemes with the reduced structure. By Lemma 2.5, the existence of such a coherent twisted sheaf is granted.

In conclusion, we just need to show that for any noetherian reduced scheme X and any $\mathscr{F} \in Coh(X,\alpha)$ such that $Supp(\mathscr{F}) = X$ we have $Coh(X,\alpha) = \langle\mathscr{F}\rangle$. By Lemma 2.6, we can even suppose X to be integral. We proceed by induction on the dimension of X.

If $dim(X) = 0$, then $X = Spec(K)$ for some field K, and the Brauer-Grothendieck group of X is $H^2_{\acute{e}t}(Spec(K), \mathscr{O}^*_{Spec(K)}) = Br(K)$ is torsion. By Propositions 1.2 and 1.8 there is a locally free $\alpha-$twisted sheaf $\mathscr{E}$, and the abelian category $Coh(Spec(K),\alpha)$ is equivalent to the category of coherent sheaves which have the structure of $\mathscr{E}nd(\mathscr{E})-$modules (see Proposition 1.9). Since this last category is generated by $\mathscr{E}nd(\mathscr{E})$, $Coh(Spec(K),\alpha)$ is generated by $\mathscr{E}$, and we are done.

Now suppose that $dimX = n$ and that the proposition is true for all schemes of dimension less then or equal to $n-1$. Let Y be a proper closed subscheme of X. By induction $Coh_Y(X,\alpha) \subseteq \langle\mathscr{F}\rangle$.

Let $\mathscr{G} \in Coh(X,\alpha)$. We need to show that $\mathscr{G}$ is in $\langle\mathscr{F}\rangle$. First of all, it is sufficient to find an open subset U of X on which some direct sums $\mathscr{G}^s$ and $\mathscr{F}^r$ are isomorphic: indeed, if $j_U : U \longrightarrow X$ is the inclusion, $Y = X \setminus U$ and $j^*_U\mathscr{G}^s$ is isomorphic to $j^*_U\mathscr{F}^r$, then $\mathscr{G}^s$ and $\mathscr{F}^r$ are isomorphic in $Coh(X,\alpha)/Coh_Y(X,\alpha)$ by Proposition 2.4. Let f be an isomorphism in this category: by definition, this means that $\ker(f)$ and $\operatorname{coker}(f)$ are in $Coh_Y(X,\alpha)$ which is, by induction, contained in $\langle\mathscr{F}\rangle$. Since this is a Serre subcategory of $Coh(X,\alpha)$, we get that $\mathscr{G}$ is in $\langle\mathscr{F}\rangle$.

In conclusion, we just need to find an open subset U of X over which some direct sums $\mathscr{G}^s$ and $\mathscr{F}^r$ are isomorphic. Using the same argument of Lemma 2.5, we can consider U to be the generic scheme of X, so that there is a locally free $\alpha-$twisted sheaf on U. Shrinking U, we can suppose $j_U^*\mathscr{G}$ and $j_U^*\mathscr{F}$ to be locally free $\alpha-$twisted sheaves of rank r and s respectively, so that $j_U^*\mathscr{G}^s$ and $j_U^*\mathscr{F}^r$ are locally free $\alpha-$twisted sheaves of the same rank. Now, if two locally free $\alpha-$twisted sheaves $\mathscr{H}$, $\mathscr{H}'$ are of the same rank, there is an open covering given by open subsets V_i such that $\mathscr{H}_{|V_i} \simeq \mathscr{H}'_{|V_i}$ for every i. In conclusion, we have found an open subset U of X over which $j_U^*\mathscr{F}^r$ and $j_U^*\mathscr{G}^s$ are isomorphic. As we saw above, this implies $\mathscr{G}^s$ to be in $\langle\mathscr{F}\rangle$, getting $\mathscr{G} \in \langle\mathscr{F}\rangle$, and we are done. □

As a corollary we get the following:

Corollary 2.8. *Let X be a noetherian scheme, and let $\alpha \in H^2_{\acute{e}t}(X,\mathscr{O}_X^*)$. There is a bijective correspondence between the set $\mathcal{C}_X$ of closed subsets of X and the set $\mathcal{S}_{X,\alpha}$ of Serre subcategories of finite type of $Coh(X,\alpha)$. Moreover, this gives a bijective correspondence between the points of X and the set $E_{X,\alpha} \subseteq \mathcal{S}_{X,\alpha}$ of irreducible Serre subcategories of finite type of $Coh(X,\alpha)$.*

Proof. Define

$$p : \mathcal{C}_X \longrightarrow \mathcal{S}_{X,\alpha}, \qquad p(Z) := Coh_Z(X,\alpha),$$

and

$$q : \mathcal{S}_{X,\alpha} \longrightarrow \mathcal{C}_X, \qquad q(\langle\mathscr{F}\rangle) := Supp(\mathscr{F}).$$

In view of Proposition 2.7, it is straightforward to show p and q are well defined (i. e. that $Coh_Z(X,\alpha)$ is a Serre subcategory of finite type and that two different generators of the same Serre subcategory have the same support) and that $p = q^{-1}$.

Moreover, Z is irreducible if and only if $Coh_Z(X,\alpha)$ is irreducible as Serre subcategory of $Coh(X,\alpha)$: indeed if Z is reducible, then $Z = Z_1 \cup Z_2$, and it is clear that $Coh_Z(X,\alpha)$ is generated as Serre subcategory by $Coh_{Z_1}(X,\alpha)$ and $Coh_{Z_2}(X,\alpha)$. If $Coh_Z(X,\alpha)$ is reducible, then there are $\mathscr{F}_1,\mathscr{F}_2 \in Coh_Z(X,\alpha)$ such that $Coh_Z(X,\alpha)$ is generated as Serre subcategory by $\langle\mathscr{F}_1\rangle$ and $\langle\mathscr{F}_2\rangle$. Then it is clear that $Z = Supp\,\mathscr{F}_1 \cup Supp\,\mathscr{F}_2$.

Since the points of X are the generic points of irreducible subsets of X, p gives a bijective correspondence between X and $E_{X,\alpha}$. □

2.3 The reconstruction of X from $Coh(X, \alpha)$

In the previous section we have seen that we can recover the set of points of X from $Coh(X, \alpha)$. We can do the same for the topology of X. Indeed, on $E_{X,\alpha}$ we can define the following topology: let $\mathfrak{I}$ be a Serre subcategory of finite type of $Coh(X, \alpha)$, and write

$$D(\mathfrak{I}) := \{\mathfrak{J} \in E_{X,\alpha} \,|\, \mathfrak{J} \nsubseteq \mathfrak{I}\}.$$

This family of subsets forms a topology over $E_{X,\alpha}$ and the morphism

$$f := f_{X,\alpha} : E_{X,\alpha} \longrightarrow X, \qquad f(Coh_{\overline{\{x\}}}(X, \alpha)) := x$$

is a homeomorphism by Proposition 2.7 and Corollary 2.8. More precisely, if Z is a closed subset of X, $U = X \setminus Z$ and $\mathfrak{I} = Coh_Z(X, \alpha)$, then f gives a bijective correspondence between $D(\mathfrak{I})$ and U.

It remains to define a structure sheaf on $E_{X,\alpha}$, in order to make f an isomorphism of schemes. Let us recall the notion of center of a category.

Definition 2.3. Let $\mathcal{A}$ be a category. We call *center of* $\mathcal{A}$ the ring $Z(\mathcal{A})$ of endomorphisms of the identity functor of $\mathcal{A}$

$$Z(\mathcal{A}) := End_{\mathcal{A}}(id_{\mathcal{A}}).$$

Using this notion and those used above, we define the following presheaf on $E_{X,\alpha}$: for any Serre subcategory $\mathfrak{I}$ of finite type of $Coh(X, \alpha)$, let

$$\mathscr{O}_{E_{X,\alpha}}(D(\mathfrak{I})) := Z(Coh(U, \alpha_{|U})).$$

Lemma 2.9. *The presheaf $\mathscr{O}_{E_{X,\alpha}}$ is a sheaf on $E_{X,\alpha}$.*

Proof. Let U be an open subset of X, and let $\{U_i\}_{i\in I}$ be an open covering of U. Let $f \in Z(Coh(U, \alpha_{|U}))$, i. e. $f = \{f_{\mathscr{F}}\}_{\mathscr{F}\in Coh(U,\alpha_{|U})}$, where $f_{\mathscr{F}}$ is an endomorphism of $\mathscr{F}$ such that for any $\mathscr{G} \in Coh(U, \alpha_{|U})$ and any morphism $g : \mathscr{F} \longrightarrow \mathscr{G}$ we have $f_{\mathscr{G}} \circ g = g \circ f_{\mathscr{F}}$.

Let us suppose that for any $i \in I$ we have $f_{|U_i} = 0$. This means that for any $\mathscr{F} \in Coh(U, \alpha_{|U})$ we have $f_{\mathscr{F}|U_i} = 0$ in the abelian category $Coh(U_i, \alpha_{|U_i})$ for any $i \in I$. By Proposition 2.4 the category $Coh(U_i, \alpha_{|U_i})$ is equivalent to $Coh(U, \alpha_{|U})/Coh_{Z_i}(U, \alpha_{|U})$, where $Z_i := U \setminus U_i$.

Then, $f_{\mathscr{F}|U_i} = 0$ in the quotient category, so the image of $f_{\mathscr{F}|U_i}$ is contained in $Coh_{Z_i}(U, \alpha_{|U})$ for any $i \in I$, i. e. $Supp(im(f_{\mathscr{F}|U_i})) \subseteq Z_i$ for any $i \in I$. But this clearly implies $f_{\mathscr{F}} = 0$ for any $\mathscr{F} \in Coh(U, \alpha_{|U})$, getting $f = 0$.

Now, for any $i \in I$ consider an element $f_i \in Z(Coh(U_i, \alpha_{|U_i}))$, such that for any $i, j \in I$ we have $f_{i|U_{ij}} = f_{j|U_{ij}}$, where $U_{ij} := U_i \cap U_j$. In particular, for any $\mathscr{F} \in Coh(U, \alpha_{|U})$ we have a gluing datum $\{f_{i,\mathscr{F}}\}_{i\in I}$, so that there is a morphism $f_{\mathscr{F}} \in End(\mathscr{F})$ such that $f_{\mathscr{F}|U_i} = f_{i,\mathscr{F}}$ and such that for any

$\mathscr{G} \in Coh(U, \alpha_{|U})$ and any $g : \mathscr{F} \longrightarrow \mathscr{G}$ we have $f_{\mathscr{G}} \circ g = g \circ f_{\mathscr{F}}$. In conclusion, there is $f \in Z(Coh(U, \alpha_{|U}))$ such that $f_{|U_i} = f_i$ for any $i \in I$, and we are done. ☐

Finally, we define a morphism of sheaves $f^\natural : \mathscr{O}_X \longrightarrow f_* \mathscr{O}_{E_{X,\alpha}}$, given over every open subset U of X by

$$f^\natural(U) : \mathscr{O}_X(U) \longrightarrow Z(Coh(U, \alpha_{|U})), \qquad f^\natural(U)(s) := \cdot s.$$

In this way we have given to $E_{X,\alpha}$ the structure of ringed space, and we have defined a morphism of ringed spaces $(f, f^\natural) : E_{X,\alpha} \longrightarrow X$. We have now the following theorem, which shows the first part of Theorem 2.1:

Theorem 2.10. *Let X be a noetherian scheme over a field k, and let $\alpha \in Br'(X)$. The morphism of ringed spaces $(f, f^\natural) : E_{X,\alpha} \longrightarrow X$ defined above is an isomorphism. In particular, $E_{X,\alpha}$ is a noetherian $k-$scheme which depends only on $Coh(X, \alpha)$.*

Proof. We only need to show that $f^\natural$ is an isomorphism of rings. It suffices to show that it is an isomorphism on open affine subschemes of X. So, let us take $U = Spec\, A$ an open affine subscheme of X.

We know that $\mathscr{O}_X(U) \simeq A$, and we study $Z(Coh(U, \alpha_{|U}))$. Since $\alpha_{|U} \in Br'(U)$, from Proposition 1.2 this is an element of $Br(U)$, and by Proposition 1.8 there is a locally free $\alpha_{|U}-$twisted sheaf $\mathscr{E}$ of rank r. Moreover, by Proposition 1.9 there is an equivalence of categories

$$Coh(U, \alpha_{|U}) \stackrel{\sim}{\longrightarrow} Mod_{coh}(\mathscr{A}), \quad \mathscr{F} \mapsto \mathscr{F} \otimes \mathscr{E}^\vee,$$

where $\mathscr{A} := \mathscr{E}nd_{Coh(U,\, \alpha_{|U})}(\mathscr{E})$ is an Azumaya algebra on X, and $Mod_{coh}(\mathscr{A})$ is the category of coherent sheaves on X which have the structure of module over $\mathscr{A}$. Since we are on an affine scheme, taking global sections we get an equivalence between $Mod_{coh}(\mathscr{A})$ and $Mod_{ft}(End_{Coh(U,\, \alpha_{|U})}(\mathscr{E}))$. Finally we get an isomorphism

$$Z(Coh(U, \alpha_{|U})) \simeq Z(Mod_{ft}(End_{Coh(U, \alpha_{|U})}(\mathscr{E}))).$$

Now, we have the following lemma, which is classical.

Lemma 2.11. *Let A be a ring with unity, $Z(A)$ his center, and $Mod_{ft}(A)$ the category of modules of finite type over A. The canonical morphism*

$$Z(A) \longrightarrow Z(Mod_{ft}(A)), \quad a \mapsto \cdot a$$

is an isomorphism of commutative rings.

Proof. Injectivity is clear. If $f := \{f_M\} \in Z(Mod_{ft}(A))$, with $f_M \in End_A(M)$, then there is $a \in A$ such that $f_A = \cdot a$ or $f_A = a\cdot$. By definition of center of a category, for any $g \in End_A(A)$ we have $g \circ f_A = f_A \circ g$, so that $a \in Z(A)$. Now, since $f - (\cdot a) \in Z(Mod_{ft}(A))$ is trivial on A, and since, by noetherianity of A, any $A-$module of finite type is quotient of some A^r, we get $f - (\cdot a) = 0$, and we are done. □

By this Lemma, we finally get

$$Z(Coh(U, \alpha_{|U})) \simeq Z(End_{Coh(U,\alpha_{|U})}(\mathscr{E})).$$

Now, just use the following well-known result

Lemma 2.12. *Let X be a noetherian scheme, $\mathscr{A}$ an Azumaya algebra on X. The center of $\mathscr{A}$ is $\mathscr{O}_X$.*

Proof. This follows from Theorem IV.1.1 in [Mil]. □

This implies that

$$Z(End_{Coh(U,\alpha_{|U})}(\mathscr{E})) \simeq A,$$

and we are done. □

Remark 2.1. In the proof of various statements we make use of the existence of a locally free $\alpha-$twisted sheaf. First of all, looking at the proof of this fact in [Cal], Theorem 1.3.5, we see that to prove the existence of a locally free $\alpha-$twisted sheaf on X we use the cocycle $\underline{\alpha}$. Moreover, the choice of such a sheaf determines an Azumaya algebra, so that we use explicitly $\underline{\alpha}$. In conclusion, the isomorphism $f_{X,\alpha}$ depends on the choice of a representative cocycle $\underline{\alpha}$ for the class α.

We are now ready to conclude the proof of Theorem 2.1:

Proof. of Theorem 2.1. The first part is the content of Theorem 2.10. For the second part, let X, Y be two noetherian schemes, $\alpha \in Br'(X)$, $\beta \in Br'(Y)$. Let

$$F : Coh(X, \alpha) \longrightarrow Coh(Y, \beta)$$

be an equivalence. If $\mathfrak{I}$ is a (irreducible) Serre subcategory of finite type of $Coh(X, \alpha)$, then $F(\mathfrak{I})$ is a (irreducible) Serre subcategory of finite type of $Coh(Y, \beta)$. This gives a bijective correspondence

$$\tilde{f}_F : E_{X,\alpha} \longrightarrow E_{Y,\beta}, \qquad \tilde{f}_F(\mathfrak{J}) := F(\mathfrak{J}).$$

Moreover, $\tilde{f}_F$ is an homeomorphism and even an isomorphism of schemes: let $U := D(\mathfrak{I})$ be an open subset of $E_{X,\alpha}$. Then $\tilde{f}_F$ induces a bijective correspondence between $D(\mathfrak{I})$ and $W := D(F(\mathfrak{I}))$. Since F is an equivalence, the induced functor

$$F : Coh(X, \alpha)/Coh_{X\setminus U}(X, \alpha) \longrightarrow Coh(Y, \beta)/Coh_{Y\setminus W}(Y, \beta)$$

is an equivalence (use Lemma 2.2). By Proposition 2.4, this gives an equivalence

$$j_W^* \circ F \circ (j_U^*)^{-1} : Coh(U, \alpha_{|U}) \longrightarrow Coh(W, \beta_{|W}),$$

where j_U (resp. j_W) is the inclusion of U in X (resp. of W in Y). Thus we get an isomorphism

$$j_W^* \circ F \circ (j_U^*)^{-1} : Z(Coh(U, \alpha_{|U})) \longrightarrow Z(Coh(W, \beta_{|W})),$$

which defines an isomorphism of sheaves $\tilde{f}_F^\natural : \mathscr{O}_{E_{Y,\beta}} \longrightarrow \tilde{f}_{F*}\mathscr{O}_{E_{X,\alpha}}$. From Theorem 2.10, it follows that

$$f_F := f_{Y,\beta} \circ \tilde{f}_F \circ f_{X,\alpha}^{-1} : X \longrightarrow Y$$

is an isomorphism, and we are done. □

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Picard group and 2−factoriality of O'Grady's examples of irreducible symplectic varieties

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Contents

Introduction

The Kodaira dimension κ is one of the main geometric invariants one can associate to any smooth projective complex variety X. Roughly speaking, one can distinguish three possible cases. The first one is $\kappa(X) = -\infty$, incuding the case where the inverse of the canonical divisor K_X is ample, and the manifold is called Fano. The second case is $\kappa(X) = 0$, which includes those manifolds whose canonical divisor K_X is trivial. The last case is $\kappa(X) > 0$, including general type manifold.

The class of smooth projective varieties of Kodaira dimension 0 contains a very important class of manifolds, namely those whose first Chern class is trivial. These have been studied for a long time, and have been classified by means of the holonomy groups. In particular, if X is a Kähler manifold with trivial first Chern class, then there is an étale covering $X' \longrightarrow X$ such that

$$X' = T \times \prod_{i=1}^{n} X_i \times \prod_{j=1}^{m} Y_j,$$

where T is a complex torus, X_i and Y_j are simply-connected, X_i is a special holonomy manifold for any i, and Y_j is a manifold whose holonomy group is $Sp(r_j)$, where $4r_j$ is the real dimension of Y_j.

Complex tori are all of the form $T = \mathbb{C}^n/\Gamma$ for some $n \in \mathbb{N}$ and for some lattice Γ in $\mathbb{C}^n$ of maximal rank. Special holonomy manifolds, called Calabi-Yau manifolds, form a rather wide (and wild) family, and play an important role in physics. The manifolds in the last class are called irreducible hyperkähler in differential geometry, and irreducible symplectic in algebraic geometry. In particular, they admit a unique (up to multiplication by an element of $\mathbb{C}^*$) symplectic form.

Up to now, there are only five known deformation classes of irreducible symplectic manifolds:

1. K3 surfaces;
2. Hilbert schemes of points $Hilb^n(X)$, where X is a K3 surface and $n \in \mathbb{N}$;
3. generalized Kummer varieties $K^n(T)$, where T is a complex torus and $n \in \mathbb{N}$;
4. the $10-$dimensional O'Grady's example $\widetilde{M}_{10}$;
5. the $6-$dimensional O'Grady's example $\widetilde{M}_6$.

The problem of producing irreducible symplectic manifolds is strictly related to the one of producing higher dimensional varieties. One of the most important constructions for this purpose is that of moduli spaces of semistable sheaves on a smooth projective surface S. In order to define them, we need to fix an ample divisor H on S and a Mukai vector $v \in \widetilde{H}(S,\mathbb{Z}) := H^{2*}(S,\mathbb{Z})$. This last lattice, called *Mukai lattice*, is a $\mathbb{Z}-$module whose lattice structure is given by an intersection form $(.,.)$, called *Mukai form*. The moduli space of $H-$semistable sheaves whose Mukai vector is v is denoted M_v. If S is projective, the moduli space M_v is a projective variety, and it is a rather complicated geometrical object. In general, it is not reduced nor irreducible and can have rather wild singularities. Anyway, the main point in the study of moduli spaces of semistable sheaves is that their geometry has, somehow, to reflect geometrical features of the base surface.

A fundamental open subset of M_v is M_v^s, parameterizing $H-$stable sheaves. If S is a projective K3 or an abelian surface, then M_v^s, if non-empty, is smooth, carries a symplectic form and has dimension $2+(v,v)$. Moreover, if the Mukai vector v is primitive, i. e. it is not divisible in $\widetilde{H}(S,\mathbb{Z})$, and H is a sufficiently generic polarisation, then $M_v = M_v^s$.

If one fixes a symplectic form ω on M_v^s, a natural question is if there is a symplectic resolution of M_v, i. e. a resolution of the singularities

$$\pi_v : \widetilde{M}_v \longrightarrow M_v,$$

such that there is a symplectic form $\widetilde{\omega}$ on $\widetilde{M}_v$ whose restriction to M_v^s is ω. Moreover, if such a resolution exists, it seems natural to ask if $\widetilde{M}_v$ is an irreducible symplectic variety.

If v is primitive and H is sufficiently generic, then $M_v^s = M_v$. We then need to know if M_v is irreducible symplectic. A particular case is when $v = (1,0,-n) \in \widetilde{H}(S,\mathbb{Z})$ if S is an abelian surface, and $v = (1,0,1-n)$ if S is a K3: then $M_v \simeq Hilb^n(S)$, the Hilbert scheme parameterizing $0-$dimensional subschemes of S of length n. The main result, due to Fujiki for $n=2$ and to Beauville in general, is:

Theorem 0.0.1. *Let n be a positive integer.*

1. *If $S = X$ is a projective K3 surface, then $Hilb^n(X)$ is an irreducible symplectic variety of dimension $2n$ whose second Betti number is 23.*
2. *If $S = J$ is an abelian surface, then there is a morphism*

$$Hilb^{n+1}(J) \longrightarrow J,$$

whose fiber over $0 \in J$ is denoted $K^n(J)$ and called generalized Kummer variety. Then $K^n(J)$ as an irreducible symplectic variety of dimension $2n$ whose second Betti number is 8.

This result has been generalized in several steps by Huybrechts-Göttsche, Mukai, Yoshioka, O'Grady and others, obtaining the following:

Theorem 0.0.2. *Let v be a primitive Mukai vector, and let H be a $v-$generic polarization.*

1. *If $S = X$ is a projective K3 surface and $(v, v) \geq 0$, then M_v is an irreducible symplectic variety, which is deformation of $Hilb^n(X')$ for some K3 surface X' and $n = 1 + \frac{(v,v)}{2}$.*

2. *If $S = J$ is an abelian surface and $(v, v) > 4$, then there is a morphism*

$$M_v \longrightarrow J \times \widehat{J},$$

whose fiber over $(0, \mathscr{O}_J)$ is denoted K_v. Then K_v is an irreducible symplectic variety which is deformation of $K^n(J')$, for some abelian surface J' and for $n = \frac{(v,v)}{2} - 1$.

It then remains only to study the case of non-primitive v, so that there are $m \in \mathbb{Z}$ and a primitive Mukai vector $w \in H^2(S, \mathbb{Z})$ such that $v = mw$. O'Grady was the first one to exhibit two concrete examples of resolution of singularities of moduli spaces of semistable sheaves with non-primitive Mukai vector. In his works [OG2] and [OG3], he showed the following:

Theorem 0.0.3. ***(O'Grady)**. Let $v = (2, 0, -2) \in \widetilde{H}(S, \mathbb{Z})$.*

1. *Let $S = X$ be a projective K3 surface such that $Pic(X) = \mathbb{Z} \cdot H$ for an ample divisor H such that $H^2 = 2$, and let $M_{10} := M_v$. Then M_{10} admits a symplectic resolution $\widetilde{M}_{10}$, which is an irreducible symplectic variety of dimension 10 and whose second Betti number is 24.*

2. *If $S = J$ is an abelian surface such that $NS(J) = \mathbb{Z} \cdot c_1(H)$ for an ample divisor H such that $c_1^2(H) = 2$, then there is a morphism*

$$M_v \longrightarrow J \times \widehat{J}$$

whose fiber over $(0, \mathscr{O}_J)$ is denoted M_6. Then M_6 admits a symplectic resolution $\widetilde{M}_6$, which is irreducible symplectic variety of dimension 6 and second Betti number 8.

The main result on the existence of symplectic resolutions for M_v was shown by Kaledin, Lehn and Sorger, generalizing the previous result by O'Grady:

Theorem 0.0.4. *Let $v = mw$ be a non-primitive Mukai vector such that $(v, v) > 0$, and let H be a $v-$generic polarisation.*

1. *If $m = 2$ et $(w, w) = 2$, then there exist a symplectic resolution*

$$\pi_v : \widetilde{M}_v \longrightarrow M_v,$$

obtained as the blow-up of the reduced singular locus of M_v.

2. *If $m > 2$ or $(w, w) > 2$, then there is no symplectic resolution of M_v. Moreover, the moduli space M_v is locally factorial.*

From this Theorem, one can conclude the following: if v is primitive or as in point 2 of Theorem 0.0.4, then M_v is locally factorial. It seems natural to ask if the same property remains true even when the Mukai vector is chosen as in point 1 of Theorem 0.0.4. This is the problem we study in this work, and our main result is the following:

Theorem 0.0.5. *The moduli spaces M_{10} and M_6 described in Theorem 0.0.3 are $2-$factorial.*

The proof of Theorem 0.0.5 is based on the analysis of the Picard groups of $\widetilde{M}_{10}$ and $\widetilde{M}_6$, that allows us to calculate the Picard groups and the groups of Weil divisor (up to linear equivalence) of M_{10} and M_6. For M_{10}, this last is isomorphic to $Pic(X) \oplus \mathbb{Z}[B]$, where B is the Weil divisor parameterizing semistable non-locally free sheaves. Using a theorem due to Rapagnetta, we show that B is not a Cartier divisor. Then, using a construction due to Le Potier, we show first that $2B$ is a Cartier divisor, and we deduce from this the $2-$factoriality of M_{10}.

For M_6 the problem is more delicate. The group of Weil divisors (up to linear equivalence) of M_6 is isomorphic to $NS(J) \oplus \mathbb{Z}[B] \oplus \mathbb{Z}/2\mathbb{Z}[D]$, where B is the Weil divisor parameterizing semistable non-locally free sheaves, and D is a Weil divisor whose square is trivial. As for M_{10}, a result of Rapagnetta allows us to show that D is not a Cartier divisor. By the construction of Le Potier already used for the $10-$dimensional case, we finally show the $2-$factoriality of M_6.

Another important point in the theory of irreducible symplectic manifolds is the study of the Beauville-Bogomolov form: if X is irreducible symplectic, then $H^2(X, \mathbb{Z})$ is a lattice with respect to an intersection form q, called Beauville-Bogomolov form, which is the analogue of the intersection form on K3 surfaces. For the known examples, the Beauville-Bogomolov form has been calculated by Beauville and by Rapagnetta. In particular, if v is a primitive Mukai vector and H is a $v-$generic polarisation, let $v^\perp \subseteq \widetilde{H}(S, \mathbb{Z})$ be the orthogonal to v with respect to the Mukai form. Then

1. If X is a projective K3 surface and $(v, v) = 0$, then there is a Hodge isometry $v^\perp/v \longrightarrow H^2(M_v, \mathbb{Z})$.
2. If X is a projective K3 surface and $(v, v) > 0$, then there is a Hodge isometry $v^\perp \longrightarrow H^2(M_v, \mathbb{Z})$.
3. If J is an abelian surface and $(v, v) > 4$, then there is a Hodge isometry $v^\perp \longrightarrow H^2(K_v, \mathbb{Z})$.

In this work we show that a similar result holds even for $\widetilde{M}_{10}$ et $\widetilde{M}_6$:

Theorem 0.0.6. *Let $v = (2, 0, -2)$.*

1. *If X is a projective K3 surface as in point 1 of Theorem 0.0.3, then there is an injective Hodge morphism $v^\perp \longrightarrow H^2(\widetilde{M}_{10}, \mathbb{Z})$, which is an isometry on its image.*
2. *If J is an abelian surface as in point 2 of Theorem 0.0.3, then there is an injective Hodge morphism $v^\perp \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z})$, which is an isometry on its image.*

Chapter 1

Moduli spaces of sheaves

The theory of moduli spaces of semistable sheaves is one of the most powerful methods to produce higher dimensional varieties. Probably, this theory plays the most relevant role in the study of irreducible symplectic manifolds, since examples of this kind of manifolds can be obtained using moduli spaces of sheaves on surfaces. In [OG2] and [OG3], O'Grady introduced two moduli spaces of semistable sheaves with non-primitive Mukai vector, denoted M_{10} and M_6, which are the starting point for the construction of two new examples of irreducible symplectic manifolds. These two moduli spaces are the main object of this work, and will be the most important ingredient in chapters 2 and 3.

In this chapter we recall the basic steps in the construction of moduli spaces of semistable sheaves on projective varieties and the basic properties we will need. In the first section we recall the definitions and the fundamental properties we need for the theory of semistable sheaves, as torsion freeness, dual sheaves, Hilbert polynomials and slopes. In the third section we recall the basic properties of ($\mu-$)semistable sheaves, in particular the Harder-Narasimhan and the Jordan-Hölder filtrations, and the notions of polystability and of S-equivalence.

We then define the two basic notions for the construction of moduli spaces: the boundedness for families of sheaves, and the construction of the Grothendieck Quot-scheme. The construction of the moduli space works as follows: by the boundedness of the family of semistable sheaves with fixed Hilbert polynomial, we can define an open subscheme R in an appropriate Quot-scheme which parameterizes only semistable quotients. On R there is an action of the reductive group $GL(N)$ for some integer N. The main point is to show that there is a good quotient for this action: this will be the desired moduli space, which parameterizes S-equivalence classes of semistable sheaves.

Finally, and most important for our purposes, we introduce the notion of the Le Potier's determinant: using the very construction of the moduli space as a quotient of a Quot-scheme, Le Potier gives a method to associate a line bundle on a moduli space to a class in the topological Grothendieck group of the base surface.

There are many important topics in the theory of moduli spaces of semistable sheaves on surfaces which are not treated in this chapter, but that are fundamental for a good comprehension of the next two chapters. The best reference we suggest to the reader is the beautiful and complete introduction to moduli spaces of sheaves given in [H-L]. For the convenience of the reader, we decided to resume some of the basic topics in Appendix C.

1.1 Semistability for coherent sheaves

In this section we introduce the definition and the basic properties of semistable sheaves on Noetherian schemes. All the considered schemes will be of finite type over a field k.

1.1.1 Torsion-free sheaves

In the following, let X be a Noetherian scheme and let $Coh(X)$ denote the abelian category of coherent sheaves on X.

Definition 1.1.1. The *support* of $\mathscr{E} \in Coh(X)$ is defined as

$$Supp(\mathscr{E}) := \{x \in X \mid \mathscr{E}_x \neq 0\},$$

which is a closed subset in X. The *dimension* of $\mathscr{E}$ is the dimension of $Supp(\mathscr{E})$.

Definition 1.1.2. Let d be an integer. A coherent sheaf $\mathscr{E}$ is called *pure of dimension* d if $dim(\mathscr{F}) = d$ for any non-trivial coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$.

Definition 1.1.3. A *torsion filtration* of a coherent sheaf $\mathscr{E}$ is a filtration

$$0 \subseteq T_0(\mathscr{E}) \subseteq T_1(\mathscr{E}) \subseteq ... \subseteq T_d(\mathscr{E}) = \mathscr{E},$$

where $d = dim(\mathscr{E})$ and $T_i(\mathscr{E})$ is the maximal subsheaf of $\mathscr{E}$ of dimension $dim(T_i(\mathscr{E})) \leq i$. The sheaf $T_{d-1}(\mathscr{E})$ is called *torsion* of $\mathscr{E}$. A coherent sheaf $\mathscr{E}$ of dimension d is said to be *torsion-free* if $T_{d-1}(\mathscr{E}) = 0$.

Proposition 1.1.1. *Let $\mathscr{E}$ be a coherent sheaf. Then a torsion filtration exists and is unique. Moreover, $\mathscr{E}$ is pure if and only if it is torsion-free.*

Proof. See [H-L]. □

Let X be a Noetherian scheme of dimension n, and let $\mathscr{E}$ be a coherent sheaf of dimension $d \leq n$ on X.

Definition 1.1.4. The *dual sheaf* of $\mathscr{E}$ is the coherent sheaf

$$\mathscr{E}^\vee := \mathscr{E}xt^{n-d}(\mathscr{E}, \omega_X),$$

where ω_X is the dualizing sheaf of X.

Remark 1.1.1. If $d = n$, then $\mathscr{E}^\vee = \mathscr{E}^* \otimes \omega_X$, where $\mathscr{E}^* := \mathscr{H}om(\mathscr{E}, \mathscr{O}_X)$, and $\mathscr{E}^{\vee\vee} = \mathscr{E}^{**}$. If X is a K3 or an abelian surface, then $\mathscr{E}^\vee = \mathscr{E}^*$.

Lemma 1.1.2. *For any $\mathscr{E} \in Coh(X)$ there is a spectral sequence*

$$E_2^{p,q} := \mathscr{E}xt^p(\mathscr{E}xt^{-q}(\mathscr{E}, \omega_X), \omega_X) \Rightarrow \mathscr{E}.$$

In particular, there is a canonical morphism $\theta_{\mathscr{E}} : \mathscr{E} \longrightarrow E_2^{n-d,d-n} = \mathscr{E}^{\vee\vee}$.

Proof. See Lemma 1.1.8 in [H-L]. □

Definition 1.1.5. A coherent sheaf $\mathscr{E}$ is called *reflexive* if $\theta_{\mathscr{E}}$ is an isomorphism.

Proposition 1.1.3. *A coherent sheaf $\mathscr{E}$ is pure if and only if $\theta_{\mathscr{E}}$ is injective.*

Proof. This is the first part of Proposition 1.1.10 in [H-L]. □

Finally, we just want to recall the following proposition, whose proof relies on the study of Serre's conditions $S_{k,c}$ on smooth projective varieties (for the definition of these conditions, see [H-L]).

Proposition 1.1.4. *If X is a smooth projective surface, any 2-dimensional sheaf $\mathscr{E}$ is reflexive if and only if it is locally free. If it is pure, then the support of $\mathscr{E}^{**}/\mathscr{E}$ has codimension at least 2.*

Proof. See Proposition 1.1.6 and Example 1.1.16 in [H-L]. □

1.1.2 Reduced Hilbert polynomial

Let X be a projective scheme of dimension n over a field k, and let $\mathscr{O}_X(1)$ be a chosen ample line bundle on X. For any integer $m \in \mathbb{Z}$ let $\mathscr{O}_X(m) := \mathscr{O}_X(1)^{\otimes m}$.

Definition 1.1.6. The *Hilbert polynomial* of $\mathscr{E} \in Coh(X)$ is the function

$$P(\mathscr{E}) : \mathbb{Z} \longrightarrow \mathbb{Z}, \qquad P(\mathscr{E}, m) := \chi(\mathscr{E} \otimes \mathscr{O}_X(m)),$$

where $\chi(\mathscr{G}) := \sum_{i=0}^{n} (-1)^i h^i(X, \mathscr{G})$ for any $\mathscr{G} \in Coh(X)$.

Theorem 1.1.5. ***(Hirzebruch-Riemann-Roch)****. Let $\mathscr{E} \in Coh(X)$. Then*

$$\chi(\mathscr{E}) = \int_X ch(\mathscr{E}) td(X).$$

Proof. For the proof when $k = \mathbb{C}$, see [Hir]. □

The Hilbert polynomial of a coherent sheaf $\mathscr{E}$ depends, then, only on the Chern character (and hence, on the Chern classes) of $\mathscr{E}$, once a polarization has been chosen. It can be uniquely written as

$$P(\mathscr{E}, m) = \frac{\alpha_d(\mathscr{E})}{d!} m^d + \frac{\alpha_{d-1}(\mathscr{E})}{(d-1)!} m^{d-1} + ... + \alpha_1(\mathscr{E}) m + \alpha_0(\mathscr{E}),$$

where d is the dimension of $\mathscr{E}$.

Definition 1.1.7. The integer $\alpha_d(\mathscr{E})$ is called the *multiplicity* of $\mathscr{E}$. If $d = n$, then the *rank* of $\mathscr{E}$ is defined as

$$rk(\mathscr{E}) := \frac{\alpha_d(\mathscr{E})}{\alpha_d(\mathscr{O}_X)}.$$

Since we are going to work only on projective surfaces, we write down explicitly the form of the Hilbert polynomial in this case. If X is a projective surface, it can be easily shown, using the Hirzebruch-Riemann-Roch Theorem, that for any coherent sheaf $\mathscr{E}$ the Hilbert polynomial is

$$P(\mathscr{E}, m) = \frac{r}{2}h^2m^2 + \left(h\cdot c_1 - \frac{r}{2}c_1(K_X)\cdot h\right)m + r\chi(\mathscr{O}_X) - \frac{1}{2}c_1\cdot c_1(K_X) + \frac{1}{2}c_1^2 - c_2,$$

where $h = c_1(\mathscr{O}_X(1))$, K_X is the canonical line bundle, $r = rk(\mathscr{E})$, $c_1 = c_1(\mathscr{E})$, $c_2 = c_2(\mathscr{E})$. If X is a K3 surface, then $c_1(K_X) = 0$ and $\chi(\mathscr{O}_X) = 2$, so that the formula becomes

$$P(\mathscr{E}, m) = \frac{r}{2}h^2m^2 + (h\cdot c_1)m + 2r + \frac{c_1^2}{2} - c_2. \tag{1.1}$$

Definition 1.1.8. The *reduced Hilbert polynomial* of a coherent sheaf $\mathscr{E}$ is

$$p(\mathscr{E}, m) := \frac{P(\mathscr{E}, m)}{\alpha_d(\mathscr{E})}.$$

Definition 1.1.9. The *degree* of a coherent sheaf $\mathscr{E}$ of dimension $d = n$ is the number $deg(\mathscr{E}) := \alpha_{d-1}(\mathscr{E}) - rk(\mathscr{E})\alpha_{d-1}(\mathscr{O}_X)$. The *slope* of $\mathscr{E}$ is

$$\mu(\mathscr{E}) := \frac{deg(\mathscr{E})}{rk(\mathscr{E})}.$$

1.1.3 Semistability and $\mu-$semistability

Let X be a projective scheme of dimension n, and let $\mathscr{O}_X(1)$ be a chosen ample line bundle on X. For simplicity's sake, let $H := \mathscr{O}_X(1)$.

Definition 1.1.10. A coherent sheaf $\mathscr{E}$ is called $H-$*semistable* (or *semistable* if the polarization is clear) if it is pure and $p(\mathscr{F}) \leq p(\mathscr{E})$ for any $\mathscr{F} \subseteq \mathscr{E}$ (where $p(\mathscr{F}) \leq p(\mathscr{E})$ means that $p(\mathscr{F}, n) \leq p(\mathscr{E}, n)$ for $n \gg 0$). It is called $H-$*stable* if it is pure and $p(\mathscr{F}) < p(\mathscr{E})$ for any $\mathscr{F} \subsetneq \mathscr{E}$.

Definition 1.1.11. A coherent sheaf $\mathscr{E}$ is called $\mu-$*semistable* (with respect to H) if $T_{d-1}(\mathscr{E}) = T_{d-2}(\mathscr{E})$ and if $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$ for any $\mathscr{F} \subseteq \mathscr{E}$ such that $0 < rk(\mathscr{F}) < rk(\mathscr{E})$. It is called $\mu-$*stable* (with respect to H) if the same property holds with $\mu(\mathscr{F}) < \mu(\mathscr{E})$.

Remark 1.1.2. The reason why stability and $\mu-$stability are defined with respect to the line bundle H, is that the two notions are given by means of Hilbert polynomials, which depend on the ample line bundle fixed as polarization.

Proposition 1.1.6. *We have the following chain of implications for any pure sheaf:*

$$\mu-\textit{stability} \Rightarrow \textit{stability} \Rightarrow \textit{semistability} \Rightarrow \mu-\textit{semistability}.$$

Proof. By definition,

$$\mu(\mathscr{E}) = \frac{\alpha_{d-1}(\mathscr{E})}{\alpha_d(\mathscr{E})}\alpha_d(\mathscr{O}_X) - \alpha_{d-1}(\mathscr{O}_X).$$

If $\mu(\mathscr{F}) < \mu(\mathscr{E})$, then $\frac{\alpha_{d-1}(\mathscr{F})}{\alpha_d(\mathscr{F})} < \frac{\alpha_{d-1}(\mathscr{E})}{\alpha_d(\mathscr{E})}$. But these are the coefficients of the degree $d-1$ term in $p(\mathscr{F})$ and $p(\mathscr{E})$, so that $p(\mathscr{F}) < p(\mathscr{E})$. The remaining part of the proposition is similar. □

Remark 1.1.3. Any rank 1 torsion-free sheaf is $\mu-$stable with respect to any polarization.

In general, it is not trivial to show that a sheaf is semistable. Anyway, there is an important criterion on smooth projective surfaces that shows that in order to be semistable, a coherent sheaf must satisfy some numerical conditions.

Definition 1.1.12. Let $\mathscr{E}$ be a coherent sheaf of rank r, first Chern class c_1 and second Chern class c_2. The *discriminant* of $\mathscr{E}$ is $\Delta(\mathscr{E}) = 2rc_2 - (r-1)c_1^2$.

Theorem 1.1.7. ***(Bogomolov****, '78). Let X be a smooth projective surface, and let $\mathscr{E}$ be a torsion-free sheaf. If $\mathscr{E}$ is $\mu-$semistable with respect to some polarization, then $\Delta(\mathscr{E}) \geq 0$.*

Proof. For the original proof see [Bog2]. For different proofs see the one given by Gieseker in [Gie], or see Theorem 3.4.1 in [H-L]. □

Remark 1.1.4. In general, there is no hope to have an equivalence between semistability and stability, or between (semi)stability and $\mu-$(semi)stability. Anyway, this is possible in some cases: if $\mathscr{E}$ is $\mu-$semistable and $rk(\mathscr{E})$ and $deg(\mathscr{E})$ are coprime, then $\mathscr{E}$ is $\mu-$stable. Indeed, if $\mathscr{E}$ was not $\mu-$stable, then there should be a subsheaf $\mathscr{F}$ with $0 < rk(\mathscr{F}) < rk(\mathscr{E})$ such that the equality $deg(\mathscr{E}) \cdot rk(\mathscr{F}) = deg(\mathscr{F}) \cdot rk(\mathscr{E})$ holds. But this clearly contradicts the hypothesis on $rk(\mathscr{E})$ and $deg(\mathscr{E})$ of being coprime.

We have this important result, that we will use later.

Proposition 1.1.8. *Let $\mathscr{E}$ and $\mathscr{F}$ be two semistable sheaves on a projective scheme. If $p(\mathscr{E}) > p(\mathscr{F})$, then $Hom(\mathscr{E}, \mathscr{F}) = 0$.*

Proof. For an easy proof, see Proposition 1.2.7 in [H-L]. □

Definition 1.1.13. A coherent sheaf $\mathscr{E}$ is said to be *simple* if $End(\mathscr{E})$ is a $k-$vector space of dimension 1.

Corollary 1.1.9. *If the scheme X is defined over an algebraically closed field k, then any stable sheaf is simple.*

Proof. See [H-L], Corollary 1.2.8. □

An important tool in the study of semistable sheaves are the two filtrations of Harder-Narasimhan and Jordan-Hölder. The aim is to define an equivalence relation on semistable sheaves.

Definition 1.1.14. A *Harder-Narasimhan filtration* for a non-trivial sheaf $\mathscr{E}$ of dimension d is a filtration

$$HN: \qquad 0 = HN^0(\mathscr{E}) \subseteq HN^1(\mathscr{E}) \subseteq ... \subseteq HN^m(\mathscr{E}) = \mathscr{E},$$

where $gr_i^{HN}(\mathscr{E}) := HN^i(\mathscr{E})/HN^{i-1}(\mathscr{E})$ are semistable sheaves of dimension d for every i, and $p_1 > ... > p_m$ where $p_i = p(gr_i^{HN}(\mathscr{E}))$.

Proposition 1.1.10. *Any non-trivial pure sheaf of dimension d admits a unique Harder-Narasimhan filtration.*

Proof. See Theorem 1.3.4 in [H-L]. □

The Jordan-Hölder filtration is similar to that of Harder-Narasimhan, but is more strictly related to semistable sheaves.

Definition 1.1.15. A *Jordan-Hölder filtration* for a non-trivial sheaf $\mathscr{E}$ with reduced Hilbert polynomial p is a filtration

$$JH: \qquad 0 = JH^0(\mathscr{E}) \subseteq JH^1(\mathscr{E}) \subseteq ... \subseteq JH^l(\mathscr{E}) = \mathscr{E},$$

where $gr_i^{JH}(\mathscr{E}) = JH^i(\mathscr{E})/JH^{i-1}(\mathscr{E})$ is stable with reduced Hilbert polynomial p for every i.

In general, a Jordan-Hölder filtration is not unique. For example, consider the sheaf $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$, with $\mathscr{E}_1 \neq \mathscr{E}_2$: in this case, $\mathscr{E}$ admits at least two different Jordan-Hölder filtrations, one with $JH^1(\mathscr{E}) = \mathscr{E}_1$, and the other with $JH^1(\mathscr{E}) = \mathscr{E}_2$. Anyway, let us define

$$gr^{JH}(\mathscr{E}) := \bigoplus_{i=1}^{l} gr_i^{JH}(\mathscr{E}),$$

in order to state the following proposition:

Proposition 1.1.11. *Any semistable sheaf $\mathscr{E}$ admits a Jordan-Hölder filtration. If JH and JH' are two Jordan-Hölder filtrations for $\mathscr{E}$, then $gr^{JH}(\mathscr{E}) \simeq gr^{JH'}(\mathscr{E})$.*

Proof. See Proposition 1.5.2 in [H-L]. □

We have then the following definition:

Definition 1.1.16. Two semistable sheaves $\mathscr{F}$ and $\mathscr{G}$ are said to be *S-equivalent* if $gr^{JH}(\mathscr{F}) = gr^{JH}(\mathscr{G})$ for a Jordan-Hölder filtration JH.

Definition 1.1.17. A semistable sheaf is called *polystable* if it is direct sum of stable sheaves.

By Proposition 1.1.11, any semistable sheaf $\mathscr{E}$ with reduced Hilbert polynomial p is S-equivalent to a polystable sheaf whose direct summands have reduced Hilbert polynomial p.

1.1.4 Boundedness for the family of semistable sheaves with fixed Hilbert polynomial

In order to define the moduli space of semistable sheaves, we need to introduce an important property of families, namely the boundedness. Let X be a projective scheme, and let $\mathscr{O}_X(1)$ be a chosen ample line bundle on X.

Definition 1.1.18. Let m be an integer. Then $\mathscr{E} \in Coh(X)$ is said to be *$m-$regular* if $H^i(X, \mathscr{E} \otimes \mathscr{O}_X(m-i)) = 0$ for any $i > 0$.

The main result on $m-$regularity is the following:

Proposition 1.1.12. *Let $\mathscr{E} \in Coh(X)$ be an $m-$regular sheaf. The following are equivalent:*

1. *$\mathscr{E}$ is $m'-$regular for any $m' \geq m$;*
2. *$\mathscr{E} \otimes \mathscr{O}_X(m)$ is globally generated;*
3. *for any integer $n \geq 0$ the canonical map*

$$H^0(X, \mathscr{E} \otimes \mathscr{O}_X(m)) \otimes H^0(X, \mathscr{O}_X(n)) \longrightarrow H^0(X, \mathscr{E} \otimes \mathscr{O}_X(m+n))$$

   is surjective.

In particular, for any $\mathscr{E} \in Coh(X)$ of positive dimension there is $m \in \mathbb{Z}$ such that $\mathscr{E}$ is $m-$regular.

Proof. See [Mum] or [Kle]. □

This Proposition allows us to give the following important definition.

Definition 1.1.19. The *Mumford-Castelnuovo regularity* of a coherent sheaf $\mathscr{E}$ is $reg(\mathscr{E}) := \inf\{m \in \mathbb{Z} \,|\, \mathscr{E} \text{ is } m-\text{regular}\}$, with the convention $reg(0) = -\infty$.

The Mumford-Castelnuovo regularity is one of the basic tools in the study the boundedness of a family of sheaves, which is a natural property one needs to define the moduli space of a family of sheaves.

Definition 1.1.20. A family of isomorphism classes of coherent sheaves on a projective scheme X is *bounded* if there is scheme S of finite type and a coherent sheaf $\mathscr{F}$ on $S \times X$ such that the given family is contained in the set

$$\{\mathscr{F}_{|Spec(k(s))\times X} \mid s \in S \text{ is closed}\}.$$

Proposition 1.1.13. *Let $\{\mathscr{E}_i\}_{i\in I}$ be a family of coherent sheaves on X. The following are equivalent:*

1. *the family is bounded;*
2. *the set of Hilbert polynomials $\{P(\mathscr{E}_i)\}_{i\in I}$ is finite and there is $\rho \in \mathbb{N}$ such that $reg(\mathscr{E}_i) \leq \rho$ for all $i \in I$;*
3. *the set of Hilbert polynomials $\{P(\mathscr{E}_i)\}_{i\in I}$ is finite and there is a coherent sheaf $\mathscr{E}$ on X such that for any $i \in I$ there is a surjective morphism $\mathscr{E} \longrightarrow \mathscr{E}_i$.*

Proof. See [Kle]. □

The following result is one of the main properties we need for the construction of the moduli space of semistable sheaves.

Theorem 1.1.14. *The family of semistable sheaves with fixed Hilbert polynomial P on a projective scheme X is bounded.*

Proof. See Theorem 3.3.7 in [H-L]. □

Finally, we just want to recall some important properties of flat families of sheaves. Let X and S be two Noetherian $k-$schemes, and let $f : X \longrightarrow S$ be a morphism of finite type. For any $s \in S$, write $X_s := f^{-1}(s) = Spec(k(s)) \times_S X$ for the fiber of f over s. Any $\mathscr{F} \in Coh(X)$ is considered as a family of coherent sheaves parameterized by S. We write $\mathscr{F}_s := \mathscr{F}_{|X_s}$, the restriction of $\mathscr{F}$ to the fiber X_s.

Definition 1.1.21. An $S-$*flat family of sheaves* on X parameterized by S is an $S-$flat coherent sheaf on X.

Assume, from now on, that f is a projective morphism and that $\mathscr{O}(1)$ is a chosen $f-$ample line bundle on X, i.e. $\mathscr{O}(1)_s$ is ample on X_s for every $s \in S$.

Proposition 1.1.15. *Let $\mathscr{F} \in Coh(X)$. The following are equivalent:*

1. *$\mathscr{F}$ is an $S-$flat family;*
2. *$f_*(\mathscr{F} \otimes \mathscr{O}(m))$ is locally free for $m \gg 0$.*

If one of these is verified, then the function associating to any $s \in S$ the Hilbert polynomial $P(\mathscr{F}_s)$ is locally constant on S. If S is reduced, the converse is also true.

Proof. See Theorem III 9.9 in [Har]. □

We conclude this section with some properties of flat families.

Lemma 1.1.16. *Let*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of coherent sheaves on X. If $\mathscr{F}$ is $S-$flat, then $\mathscr{F}''$ is $S-$flat if and only if the canonical morphism $\mathscr{F}'_s \longrightarrow \mathscr{F}_s$ is injective for any $s \in S$. If $\mathscr{F}$ and $\mathscr{F}''$ are $S-$flat, then $\mathscr{F}'$ is $S-$flat.

Proof. See for example Theorem 49 in [Mat]. □

Theorem 1.1.17. *Let $f : X \longrightarrow S$ be a proper morphism, between two Noetherian schemes, and suppose S to be reduced. Let $i \in \mathbb{N}_0$, and let $\mathscr{F}$ be an $S-$flat sheaf. The following are equivalent:*

1. *The map sending any $s \in S$ to $h^i(X_s, \mathscr{F}_s)$ is constant;*
2. *the sheaf $\mathbb{R}^i f_* \mathscr{F}$ is locally free and for any $s \in S$ the canonical morphism*

$$(\mathbb{R}^i f_* \mathscr{F})_s \longrightarrow H^i(X_s, \mathscr{F}_s)$$

is an isomorphism.

If one of the two previous conditions is satisfied, then for any $s \in S$ the canonical morphism

$$(\mathbb{R}^{i-1} f_* \mathscr{F})_s \longrightarrow H^{i-1}(X_s, \mathscr{F}_s)$$

is an isomorphism.

Proof. See [Mum], Corollary 2 in Chapter II.5. □

For a family of sheaves we have even the following definition:

Definition 1.1.22. Let $f : X \longrightarrow S$ be a projective morphism between two Noetherian schemes of finite type over k, and let $\mathscr{O}(1)$ be a chosen $f-$ample line bundle. Let $P \in \mathbb{Q}[t]$. We say that the *Hilbert polynomial* of an $S-$flat family $\mathscr{F}$ is P if $P(\mathscr{F}_s) = P$ for any $s \in S$.

1.2 The construction of the moduli spaces

In this section we recall the construction of the moduli space of semistable sheaves. In order to do that, we need to introduce the notion of Quot-scheme, which is the starting point for various constructions on moduli spaces of sheaves: first of all, the moduli space of semistable sheaves on a projective variety is the quotient of an open subscheme of some Quot-scheme. This construction is the base for that of Le Potier's morphism we will present at the end of this chapter.

1.2.1 Construction of the Quot-scheme

Here we present the construction and the basic properties of the Grothendieck Quot-scheme. For the notion of universal family and representability of functors, see Appendix C.

Definition 1.2.1. A *quotient module* of a coherent sheaf $\mathscr{V}$ is an equivalence class $[q : \mathscr{V} \longrightarrow \mathscr{F}]$ of quotients, where two quotients $q_1 : \mathscr{V} \longrightarrow \mathscr{F}_1$ and $q_2 : \mathscr{V} \longrightarrow \mathscr{F}_2$ are said to be equivalent if $\ker(q_1) = \ker(q_2)$ (or, equivalently, if there is an isomorphism $\phi : \mathscr{F}_1 \longrightarrow \mathscr{F}_2$ such that $q_2 = \phi \circ q_1$).

Consider a projective morphism $f : X \longrightarrow S$, and let $\mathscr{O}(1)$ be an $f-$ample line bundle on X. Let $\mathscr{H} \in Coh(X)$, and let $P \in \mathbb{Q}[t]$ be a polynomial. Define the functor

$$\underline{Quot}_{X/S}(\mathscr{H}, P) : Sch(S)^{opp} \longrightarrow \mathcal{S}et$$

sending any $S-$scheme of finite type $f : T \longrightarrow S$ to the set of quotient moduli $[f^*\mathscr{H} \longrightarrow \mathscr{F}]$ such that $\mathscr{F}$ is $T-$flat and $P(\mathscr{F}) = P$. If $g : T' \longrightarrow T$ is a morphism of $S-$schemes of finite type, then $\underline{Quot}_{X/S}(\mathscr{H}, P)(g)$ sends a quotient module to its pull-back by g.

Theorem 1.2.1. *The functor $\underline{Quot}_{X/S}(\mathscr{H}, P)$ is represented by a projective $S-$scheme $Quot_{X/S}(\mathscr{H}, P)$, called Grothendieck Quot-scheme.*

Proof. This is Theorem 2.2.4 in [H-L]. □

As a corollary, by Proposition C.1.1 there is a universal family

$$[\widetilde{\rho} : p_X^*\mathscr{H} \longrightarrow \widetilde{\mathscr{F}}],$$

which is a quotient module on $Quot_{X/S}(\mathscr{H}, P) \times X$, where

$$p_X : Quot_{X/S}(\mathscr{H}, P) \times X \longrightarrow X$$

is the projection on X. In particular, this means that for any quotient module $[\rho : \mathscr{H} \longrightarrow \mathscr{F}]$ on X there is an isomorphism $\widetilde{\mathscr{F}}_{[\rho]} \simeq \mathscr{F}$. One of the most common ways to use the universal family $\widetilde{\mathscr{F}}$ is in the construction of line bundles on $Quot_{X/S}(\mathscr{H}, P)$. In section 2.5 we will present a general construction, due to Le Potier, but we introduce here a special case, which is important for the construction of moduli spaces. Let $r \in \mathbb{Z}$, and let p_Q, p_X be the projections from $Quot_{X/S}(\mathscr{H}, P) \times X$ to the two factors. Let

$$L_r := det(p_{Q*}(p_X^*\mathscr{O}_X(r) \otimes \widetilde{\mathscr{F}}) \in Pic(Quot_{X/S}(\mathscr{H}, P)).$$

Proposition 1.2.2. ***(Le Potier)**. For $r \gg 0$, the line bundle L_r is $S-$very ample.*

Proof. See Proposition 2.2.5 in [H-L]. □

Particular cases of Quot-schemes are obtained for $S = Spec(k)$ and $\mathscr{H} = \mathscr{O}_X$, where quotient modules correspond naturally to closed subschemes of X.

Definition 1.2.2. The *Hilbert scheme of closed subschemes of X with fixed Hilbert polynomial P* is defined as $Hilb^P(X) := Quot_{X/k}(\mathscr{O}_X, P)$.

In particular, the dimension of the closed subschemes parameterized by the Hilbert scheme is fixed, and is equal to $deg(P)$. If $deg(P) = 0$, then $P = n \in \mathbb{Z}$: $Hilb^n(X)$ parameterizes $0-$dimensional closed subschemes of length n.

Proposition 1.2.3. *Let X be a projective $k-$scheme, $\mathscr{H} \in Coh(X)$, $P \in \mathbb{Q}[t]$. Let $[q : \mathscr{H} \longrightarrow \mathscr{F}] \in Quot_{X/k}(\mathscr{H}, P)$, and let $\mathscr{K} = \ker(q)$. Then*

$$hom(\mathscr{K}, \mathscr{F}) - ext^1(\mathscr{K}, \mathscr{F}) \leq dim_{[q]}Quot_{X/k}(\mathscr{H}, P) \leq hom(\mathscr{K}, \mathscr{F}).$$

If equality holds in the first place, then $Quot_{X/k}(\mathscr{H}, P)$ is locally complete intersection near $[q]$. Moreover, if $ext^1(\mathscr{K}, \mathscr{F}) = 0$, then $[q]$ is a smooth point of $Quot_{X/k}(\mathscr{H}, P)$.

Proof. See Appendix 2.A in [H-L], in particular Proposition 2.A.11. □

Finally, we just want to mention an important property, which is a consequence of the existence of relative Quot-schemes.

Definition 1.2.3. Let $\mathscr{P}$ be a property of coherent sheaves on Noetherian schemes. We say that $\mathscr{P}$ is an *open property* if for any two Noetherian schemes X and S, for any projective morphism $f : X \longrightarrow S$, and for any $S-$flat family $\mathscr{F}$ on X parameterized by S, the set of $s \in S$ such that $\mathscr{F}_s$ verifies $\mathscr{P}$ is open in S.

Proposition 1.2.4. *The following properties of coherent sheaves are open: being locally free; ($\mu-$)semistability; ($\mu-$)stability.*

Proof. See the proof of Proposition 2.3.1 in [H-L]. □

1.2.2 Moduli spaces of semistable sheaves

In this section we recall the construction of the moduli space of semistable sheaves. For the notion of action of a group and of quotients, see Appendix C.

Let X be a projective $k-$scheme of finite type, and let $\mathscr{O}_X(1) = H$ be an ample line bundle on X. Moreover, let $P \in \mathbb{Q}[t]$. Consider the functor

$$\mathscr{M}'_H(P) : Sch(k)^{opp} \longrightarrow \mathcal{S}et$$

sending any $k-$scheme of finite type S to the set of isomorphism classes of $S-$flat families $\mathscr{F}$ of coherent sheaves on $S \times X$ such that $\mathscr{F}_s$ is $H-$semistable with Hilbert polynomial P for any $s \in S$. If $f : T \longrightarrow S$ is a morphism of $k-$schemes of finite type, $\mathscr{M}'_H(P)(f)(\mathscr{F}) = (f \times id_X)^*\mathscr{F}$ for any $\mathscr{F} \in \mathscr{M}'_H(S)$.

Definition 1.2.4. Let $\mathscr{F},\mathscr{G} \in \mathscr{M}'_H(P)(S)$. We say that $\mathscr{F}$ is equivalent to $\mathscr{G}$ (and we write $\mathscr{F} \sim \mathscr{G}$) if there is $L \in Pic(S)$ such that $\mathscr{F} \simeq \mathscr{G} \otimes p_S^* L$, where $p_S : S \times X \longrightarrow S$ is the projection.

We can now define the *moduli functor*:

$$\mathscr{M}_H(P) : Sch(k)^{opp} \longrightarrow \mathcal{S}et, \quad \mathscr{M}_H(P)(S) := \mathscr{M}'_H(P)(S)/\sim .$$

Theorem 1.2.5. *The moduli functor is universally corepresented by a projective $k-$scheme $M_H(P)$, called moduli space of $H-$semistable sheaves with Hilbert polynomial P.*

Proof. The proof is contained, for instance, in Chapter 4 in [H-L]. □

We recall here the construction of $M_H(P)$, since in the following we will need some elements of it. Consider the family of semistable sheaves on X (with respect to H) with Hilbert polynomial equal to P. By Theorem 1.1.14, this family is bounded, so that by Propositions 1.1.12 and 1.1.13 there is an integer $m \in \mathbb{Z}$ such that for any $\mathscr{E}$ in this family the sheaf $\mathscr{E}$ is $m-$regular, so that $\mathscr{E}(m) := \mathscr{E} \otimes \mathscr{O}_X(m)$ is generated by its global sections, i. e. there is a surjective morphism

$$H^0(X, \mathscr{E}(m)) \otimes \mathscr{O}_X(-m) \longrightarrow \mathscr{E}.$$

Notice that since $\mathscr{E}$ is $m-$regular, then $H^i(X,\mathscr{E}(m)) = 0$ for any $i > 0$. In particular, this implies $P(m) = P(\mathscr{E}, m) = h^0(X, \mathscr{E}(m))$, so that the dimension of $H^0(X, \mathscr{E}(m))$ does not depend on $\mathscr{E}$. Choosing a basis, we fix an isomorphism between $H^0(X, \mathscr{E}(m))$ and $V := \mathbb{C}^{P(m)}$. Let $\mathscr{H} := V \otimes \mathscr{O}_X(-m)$, which is a locally free sheaf on X of rank $P(m)$. We have then produced a surjective morphism

$$\rho : \mathscr{H} \longrightarrow \mathscr{E}$$

for any semistable sheaf $\mathscr{E}$ with Hilbert polynomial equal to P. Considering only the isomorphism classes of semistable sheaves, we have then a quotient module $[\rho : \mathscr{H} \longrightarrow \mathscr{E}]$, corresponding to a point of $Quot_{X/k}(\mathscr{H}, P)$.

Now, let $N := P(m)$. The reductive group $GL(N)$ acts on $Quot_{X/k}(\mathscr{H}, P)$. Indeed, $GL(N) \simeq Aut(\mathscr{H})$ and the action is (on $k-$rational points)

$$\sigma : GL(N) \times Quot_{X/k}(\mathscr{H}, P) \longrightarrow Quot_{X/k}(\mathscr{H}, P), \qquad \sigma(g, [\rho]) := [\rho \circ g].$$

Now, let $R \subseteq Quot_{X/k}(\mathscr{H}, P)$ be the subset parameterizing semistable quotients. By Proposition 1.2.4, R is an open subscheme of $Quot_{X/k}(\mathscr{H}, P)$, which is clearly $GL(N)-$invariant. In a similar way, let $R^s \subseteq R$ be the open subscheme parameterizing stable sheaves.

Let $[\widetilde{\rho} : p_X^* \mathscr{H} \longrightarrow \widetilde{\mathscr{F}}]$ be the universal quotient on $Quot_{X/k}(\mathscr{H}, P) \times X$. By Proposition 1.2.2 the line bundle L_r on $Quot_{X/k}(\mathscr{H}, P)$ is very ample for

$r \gg 0$. Moreover, it carries a natural $GL(N)-$linearization from the canonical one on $\widetilde{\mathscr{F}}$.

Theorem 1.2.6. *Let $m, r \in \mathbb{Z}$, and suppose them to be sufficiently big. Then $R = \overline{R}^{ss}(L_r)$ and $R^s = \overline{R}^{s}(L_r)$. Moreover, the closures of the orbits of two different points $[\rho_i : \mathscr{H} \longrightarrow \mathscr{F}_i]$ in R intersect each other if and only if $gr^{JH}(\mathscr{F}_1) \simeq gr^{JH}(\mathscr{F}_2)$. The orbit of a point is closed in R if and only if it represents a polystable sheaf.*

Proof. See Theorem 4.3.3 in [H-L]. □

By Theorem C.2.1, there is a projective $k-$scheme $M_H(P)$ which is a universal good quotient of R by the action of $GL(N)$. Let $p : R \longrightarrow M_H(P)$ be the quotient morphism. The open subset $M^s_H(P) := p(R^s)$ is a universal geometric quotient of R^s called *moduli space of $H-$stable sheaves with Hilbert polynomial P*. In view of Theorem 1.2.6, the moduli space $M_H(P)$ parameterizes polystable sheaves.

To conclude this section, we just want to resume the notations we will use in the following for moduli spaces of sheaves. By the Hirzebruch-Riemann-Roch Theorem, the Hilbert polynomial of a sheaf is determined by its Chern character, hence by its rank and its Chern classes. In the following we will be concerned only with moduli spaces of sheaves on a surface, and in order to fix the Hilbert polynomial P we will fix the rank r and the two first Chern classes $c_1 \in H^2(S, \mathbb{Z})$ and $c_2 \in H^4(S, \mathbb{Z})$. We will then denote the moduli space $M(P)$ (resp. $M^s(P)$) as $M(r, c_1, c_2)$ (resp. $M^s(r, c_1, c_2)$). Notice that instead of fixing c_1 we can restrict ourselves to the moduli space of semistable sheaves whose rank is r, whose second Chern class is c_2 and whose determinant is $\mathscr{L} \in Pic(S)$, where $c_1(\mathscr{L}) = c_1$, i. e. to the fiber over $\mathscr{L}$ of the determinant morphism

$$det : M(r, c_1, c_2) \longrightarrow Pic(S)$$

(see Appendix C). Such a fiber will be denoted $M(r, \mathscr{L}, c_2)$ (resp. $M^s(r, \mathscr{L}, c_2)$).

1.3 Line bundles on moduli spaces

In this section we recall a construction due to Joseph Le Potier, and we resume some important properties that will be useful in the following, concerning in particular the moduli space of $\mu-$semistable sheaves.

Le Potier's construction provides a way to produce line bundles on the moduli space of (semi)stable sheaves on any smooth projective surface X. The construction goes as follows: to any class in the topological Grothendieck group of X, we associate a line bundle on the open subscheme of the Quot-scheme whose quotient is the moduli space. Then we need to study conditions on the starting class in order to guarantee the descent of the obtained line bundle.

1.3.1 Le Potier's determinant

Before defining the Le Potier's morphism, it seems useful to recall construction and properties of the determinant. Details can be found in [K-M].

Let Y be any smooth projective variety, and let E be a vector bundle on Y of rank r. The determinant of E is defined as $det(E) := \Lambda^r E$, which is a line bundle on Y. In a more general way, if $E^\bullet$ is a bounded complex of vector bundles, then we can define

$$det(E^\bullet) := \bigotimes_i det(E^i)^{(-1)^i}.$$

If two bounded complexes $E^\bullet$ and $F^\bullet$ are quasi-isomorphic, then $det(E^\bullet) \simeq det(F^\bullet)$. Since any $\mathscr{E} \in Coh(Y)$ admits a finite resolution $E^\bullet$ of locally free sheaves, then one defines $det(\mathscr{F}) := det(E^\bullet)$.

Lemma 1.3.1. *Let*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be an exact sequence. Then $det(\mathscr{F}) \simeq det(\mathscr{F}') \otimes det(\mathscr{F}'')$. *If* $\mathscr{F}$ *and* $\mathscr{G}$ *are two coherent sheaves on* X, *then* $det(\mathscr{F} \otimes \mathscr{G}) \simeq det(\mathscr{F})^{rk(\mathscr{G})} \otimes det(\mathscr{G})^{rk(\mathscr{F})}$.

Proof. See [K-M]. □

In particular, if $K_{top}(Y)$ is the Grothendieck group of Y, we have the following morphism

$$det : K_{top}(Y) \longrightarrow Pic(Y)$$

sending any class $\alpha \in K_{top}(Y)$ to its determinant (the class α is represented by a sheaf, and $det(\alpha)$ is simply the determinant of this sheaf, which is well defined by Lemma 1.3.1). If Y is not smooth projective, but only a Noetherian $k-$scheme of finite type, we can define

$$det : K^0_{top}(Y) \longrightarrow Pic(Y)$$

in the same manner, where $K^0_{top}(Y)$ is the abelian group generated by locally free sheaves with relations $[E'] - [E] + [E'']$ for any short exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$.

Now, let X be any smooth projective surface, and let S be any $k-$scheme of finite type. Moreover, let $\mathscr{F}$ be an $S-$flat family of sheaves on $S \times X$. We can define the group morphism

$$\widetilde{\lambda} : K_{top}(X) \longrightarrow Pic(S), \quad \widetilde{\lambda}(\alpha) := det(p_{S!}(p_X^*\alpha \cdot [\mathscr{F}])),$$

where p_X and p_S are the two projections from $S \times X$. Indeed, $p_X^*\alpha$ and $[\mathscr{F}]$ are in $K^0_{top}(S \times X)$, so that $p_X^*\alpha \cdot [\mathscr{F}] \in K^0_{top}(S \times X)$. As the morphism p_S is smooth and projective, we have the morphism

$$p_{S!} : K^0_{top}(S \times X) \longrightarrow K^0_{top}(S),$$

(see Proposition 2.1.10 in [H-L]), so that $p_{S!}(p_X^*\alpha\cdot[\mathscr{F}])\in K^0_{top}(S)$. We can then use the definition of the determinant to get the morphism $\widetilde{\lambda}$.

Definition 1.3.1. The morphism $\widetilde{\lambda}$ is called *Le Potier's determinant* or *Le Potier's morphism.*

Let now H be an ample line bundle on X, and fix $r\in\mathbb{Z}$, $\mathscr{L}\in Pic(X)$ and $c_2\in H^4(X,\mathbb{Z})$. As seen in section 1.2.2, the moduli space $M(r,\mathscr{L},c_2)$ is obtained as a quotient of a scheme R by the action of $GL(N)$, for some $N\in\mathbb{N}$, where R is the open subscheme of $Quot(\mathscr{H},P)$ parameterizing semistable quotients. Let $q:p_X^*\mathscr{H}\longrightarrow\mathscr{F}$ be the universal family on $Quot(\mathscr{H},P)\times X$. Using the Le Potier's determinant, one associates to any class $\alpha\in K_{top}(X)$ a line bundle $\widetilde{\lambda}(\alpha)\in Pic(R)$. The natural question is if it descends to a line bundle on $M(r,\mathscr{L},c_2)$. Let us fix some notations: write $e:=[\mathscr{E}]\in K_{top}(X)$ for the class of a sheaf $\mathscr{E}$ parameterized by $M(r,\mathscr{L},c_2)$, and let $h:=[H]$. Consider

$$\xi:K_{top}(X)\times K_{top}(X)\longrightarrow\mathbb{Z},\quad \xi(\alpha,\beta):=\chi(\alpha\cdot\beta),$$

and for any $\beta\in K_{top}(X)$ let $\beta^\perp:=\ker(\xi(.,\beta))$.

Theorem 1.3.2. ***(Le Potier)****. Let $\alpha\in K_{top}(X)$.*

1. *The line bundle $\widetilde{\lambda}(\alpha)_{|R^s}$ descends to $\lambda^s(\alpha)\in Pic(M^s(r,\mathscr{L},c_2))$ if $\alpha\in e^\perp$, and we get a morphism $\lambda^s:e^\perp\longrightarrow Pic(M^s(r,\mathscr{L},c_2))$.*
2. *The line bundle $\widetilde{\lambda}(\alpha)$ descends to a line bundle $\lambda(\alpha)\in Pic(M(r,\mathscr{L},c_2))$ if $\alpha\in e^\perp\cap\{1,h,h^2\}^{\perp\perp}$, and we get a morphism*
$$\lambda:e^\perp\cap\{1,h,h^2\}^{\perp\perp}\longrightarrow Pic(M(r,\mathscr{L},c_2)).$$
3. *For any $\alpha\in e^\perp\cap\{1,h,h^2\}^{\perp\perp}$, we have $\lambda(\alpha)_{|M^s(r,\mathscr{L},c_2)}\simeq\lambda^s(\alpha)$.*
4. *Let $\mathscr{S}$ be an $S-$flat family of semistable sheaves of rank r, determinant $\mathscr{L}$ and second Chern class c_2 on $S\times X$, and let $f_{\mathscr{S}}:S\longrightarrow M(r,\mathscr{L},c_2)$ be the induced morphism. Let $\lambda_{\mathscr{S}}:K_{top}(X)\longrightarrow Pic(S)$ be the Le Potier's determinant associated to $\mathscr{S}$. Then $f^*_{\mathscr{S}}\circ\lambda=\lambda_{\mathscr{S}}$. The same is true for an $S-$flat family $\mathscr{S}$ of stable sheaves, and for the corresponding morphism $f_{\mathscr{S}}:S\longrightarrow M^s(r,\mathscr{L},c_2)$.*

Proof. See [LP] or Theorem 8.1.5 in [H-L]. □

Remark 1.3.1. If on $M(r,\mathscr{L},c_2)\times X$ there is a universal family $\mathscr{F}$, then for any $\alpha\in e^\perp$ we have $\lambda(\alpha)=det(p_{M!}(p_X^*\alpha\cdot[\mathscr{F}]))$. The choice of α in $e^\perp$ is not necessary to get an element of $Pic(M(r,\mathscr{L},c_2))$, but implies that the definition of $\lambda(\alpha)$ does not depend on the choice of the universal family $\mathscr{F}$. In general, the two morphisms λ^s and λ do not depend on the universal family chosen to define $\widetilde{\lambda}$.

1.3.2 Moduli space of $\mu-$semistable sheaves

In this section we apply the construction of the Le Potier's determinant to describe the moduli space of $\mu-$semistable sheaves. First, we briefly want to sketch how to construct such a moduli space. Let X be a smooth projective surface, H a fixed ample line bundle on X and let $r \in \mathbb{Z}$, $\mathscr{L} \in Pic(X)$ and $c_2 \in H^4(X, \mathbb{Z})$. Let $R^{\mu-ss} \subseteq Quot(\mathscr{H}, P)$ be the open subscheme parameterizing quotients $[q : \mathscr{H} \longrightarrow \mathscr{E}]$, where $\mathscr{E}$ is $\mu-$semistable. Using the notations of the previous section, let $u := -rh + \chi(e \cdot h)[\mathbb{C}_p] \in K_{top}(X)$ for some $p \in X$. By Theorem 1.3.2, $\widetilde{\lambda}(u)$ descends to a line bundle $\lambda(u) \in Pic(M(r, \mathscr{L}, c_2))$.

Proposition 1.3.3. *There is $m \in \mathbb{Z}$ such that $\widetilde{\lambda}(u)^m$ is generated by global sections invariant with respect to the action of $SL(N)$.*

Proof. See Proposition 8.2.3 in [H-L]. □

Remark 1.3.2. Proposition 1.3.3 implies that the line bundle $\lambda(u)^m$ is generated by its global sections. The morphism

$$\phi : M(r, \mathscr{L}, c_2) \longrightarrow \mathbb{P}(H^0(M(r, \mathscr{L}, c_2), \lambda(u)^m)^*) \tag{1.2}$$

associated to the complete linear system $|\lambda(u)^m|$ has no base points.

Proposition 1.3.4. *There is an integer $k \in \mathbb{Z}$ such that the graded ring $\bigoplus_{l \geq 0} H^0(R^{\mu-ss}, \widetilde{\lambda}(u)^{lkm})^{SL(N)}$ is finitely generated.*

Proof. See the proof of Proposition 8.2.6 in [H-L]. □

Definition 1.3.2. Let $k \in \mathbb{Z}$ be as in Proposition 1.3.4. The *moduli space of $\mu-$semistable sheaves* with rank r, determinant $\mathscr{L}$ and second Chern class c_2 is defined as

$$M^{\mu-ss}(r, \mathscr{L}, c_2) := Proj\left(\bigoplus_{l \geq 0} H^0(R^{\mu-ss}, \widetilde{\lambda}(u)^{lkm})^{SL(N)} \right).$$

By definition, then, the image of ϕ in (1.2) is exactly $M^{\mu-ss}(r, \mathscr{L}, c_2)$.

Proposition 1.3.5. *The restriction of ϕ to the open subscheme $M^{\mu,lf}(r, \mathscr{L}, c_2)$ of $M(r, \mathscr{L}, c_2)$ parameterizing $\mu-$stable locally free sheaves is an embedding.*

Proof. See Corollary 8.2.16 in [H-L]. □

By this Proposition, the moduli space $M^{\mu-ss}$ naturally contains an open subscheme parameterizing $\mu-$stable locally free sheaves, so that it can be considered as a compactification of $M^{\mu,lf}$, different from the one given by M.

Let $M^{\mu-poly}(r, \mathscr{L}, c)$ be the moduli space of $\mu-$polystable sheaves on X of rank r, determinant $\mathscr{L}$ and second Chern class c.

Proposition 1.3.6. *The moduli space $M^{\mu-ss}(r,\mathscr{L},c_2)$ admits (set-theoretically) a stratification*

$$M^{\mu-ss}(r,\mathscr{L},c_2)=\coprod_{l\geq 0} M^{\mu-poly}(r,\mathscr{L},c_2-l)\times S^l(X).$$

Proof. See [F-M] or [Li]. □

Example 1.3.1. Let $n\in\mathbb{Z}$. By Example C.3.1, the moduli space $M(1,0,n)$ is isomorphic to $Hilb^n(X)$. Moreover, $M^{\mu-ss}(1,0,n)\simeq S^n(X)$ and the morphism ϕ is simply the Hilbert-Chow morphism ρ_n (see Example 8.2.9 in [H-L]).

An important element we will use in the following is the Donaldson's morphism. Consider a smooth projective surface X, a scheme S and an $S-$flat family $\mathscr{F}$ on $S\times X$.

Definition 1.3.3. The *Donaldson's morphism* is defined as

$$\mu_D: H^2(X,\mathbb{Z})\longrightarrow H^2(S,\mathbb{Z}),\quad \mu_D(\alpha):=c_2(\mathscr{F})/\alpha,$$

where $c_2(\mathscr{F})/\alpha$ is the slant product.

Proposition 1.3.7. *Let $\Sigma=M_\Sigma\times S^l(X)$ be a stratum of $M^{\mu-ss}(r,L,c_2)$. Let $\alpha\in H^2(X,\mathbb{Z})$. Then*

$$\mu_D(\alpha)_{|\Sigma}=\mu_{M_\Sigma}(\alpha)\otimes 1+1\otimes\alpha\in H^2(\Sigma,\mathbb{Z}),$$

for some $\mu_{M_\Sigma}(\alpha)\in H^2(M_\Sigma,\mathbb{Z})$.

Proof. See Proposition 6.5 in [F-M]. □

1.4 Moduli spaces on K3 or abelian surfaces

In this section we introduce the main notations and properties of moduli spaces of semistable sheaves on K3 or abelian surfaces. In Appendix D we briefly present the main results in this subject, but here we resume some important properties and notations, in order to introduce the problems we will study in the two next chapters. Let S be an abelian or a projective K3 surface, and let $\mathscr{E}\in Coh(S)$. Let

$$\widetilde{H}(S,\mathbb{Z}):=H^{2*}(S,\mathbb{Z}),$$

and define

$$(.,.):\widetilde{H}(S,\mathbb{Z})\times\widetilde{H}(S,\mathbb{Z})\longrightarrow\mathbb{Z},\qquad (\alpha,\beta):=-\int_X\alpha^\vee\cdot\beta,$$

where if $\alpha=(\alpha_0,\alpha_2,\alpha_4)$, then $\alpha^\vee:=(\alpha_0,-\alpha_2,\alpha_4)$. The integral form $(.,.)$ is non-degenerate, and $\widetilde{H}(S,\mathbb{Z})$ with this form is called *Mukai lattice*. On the

Mukai lattice of any smooth projective surface one can define a weight 2 Hodge structure in the following way:

$$\widetilde{H}^{2,0}(S) := H^{2,0}(S), \quad \widetilde{H}^{1,1}(S) := H^0(S,\mathbb{C}) \oplus H^{1,1}(S) \oplus H^4(S,\mathbb{C}).$$

Definition 1.4.1. The *Mukai vector* of $\mathscr{E}$ is $v(\mathscr{E}) := ch(\mathscr{E}) \cdot \sqrt{td(S)}$. An element $v \in \widetilde{H}(S,\mathbb{Z})$ will be called *Mukai vector.* If $v = (v_0, v_2, v_4)$, for $v_i \in H^i(S,\mathbb{Z})$, then v_0 will be called the *rank* of v and v_2 will be called the *first Chern class* of v.

If S is a K3 surface, then $td(S) = (1,0,2)$ and $\sqrt{td(S)} = (1,0,1)$, so that for any $\mathscr{E} \in Coh(S)$ we have

$$v(\mathscr{E}) = \left(rk(\mathscr{E}), c_1(\mathscr{E}), \frac{c_1^2(\mathscr{E})}{2} - c_2(\mathscr{E}) + rk(\mathscr{E}) \right). \tag{1.3}$$

If S is abelian, then $td(S) = (1,0,0)$, and $v(\mathscr{E}) = ch(\mathscr{E})$. In conclusion, the Mukai vector of $\mathscr{E}$ is determined by its rank and its Chern classes (and viceversa). If v is the Mukai vector associated to r, c_1 and c_2, then the moduli space $M(r, c_1, c_2)$ will be denoted $M(v)$ (the same for the moduli space of stable sheaves $M^s(v)$). By Theorem C.3.1, $M^s(v)$ is smooth and has dimension $2 + (v,v)$. Moreover, by Theorem C.5.1 it carries a symplectic form coming from the one on S.

In general, the moduli space $M^s(v)$ can be empty. Indeed, as $dim(M^s(v)) = 2 + (v,v)$, in order to have $M^s(v) \neq \emptyset$ we must have $(v,v) \geq -2$. From now on we suppose $rk(v) \geq 0$, $c_1(v) \in NS(S)$ and $(v,v) \geq -2$, or $rk(v) = 0$, $c_1(v) \in NS(S)$ is the class of an ample line bundle and $v_4 \neq 0$. In general we have $M^s(v) \neq M(v)$, but we have some special cases where the equality is satisfied, as shown in the following:

Proposition 1.4.1. *Let X be an abelian or a projective K3 surface, H an ample divisor and $v \in \widetilde{H}(X,\mathbb{Z})$. If there is a connected component Y of $M(v)$ such that $Y \subseteq M^s(v)$, then $Y = M(v)$. In particular $M^s(v) = M(v)$.*

Proof. This proposition was first shown by Mukai for isotropic v, later generalized in [K-L-S], Theorem 4.1. □

The two main definitions we need are the following:

Definition 1.4.2. Let S be a connected algebraic surface, and let v be a Mukai vector of rank r. A numerical class $\xi \in NS(S)$ is called *of type v* if

$$-\frac{r}{4}\Delta(v) \leq \xi^2 \leq 0$$

(see Definition 1.1.12 for the notion of $\Delta(v)$). An ample line bundle $H \in Pic(S)$ is called $v-$*generic* if it satisfies the following condition: for any $\xi \in NS(S)$ of type v, if $\xi \cdot c_1(H) = 0$ then $\xi = 0$.

Definition 1.4.3. A Mukai vector $v \in \widetilde{H}(X,\mathbb{Z})$ is called *primitive* if it is not divisible in $\widetilde{H}(X,\mathbb{Z})$.

In particular, if $v \in \widetilde{H}(X,\mathbb{Z})$ is a Mukai vector, then there are a unique $m \in \mathbb{Z}$ and a unique primitive Mukai vector $w \in \widetilde{H}(X,\mathbb{Z})$ such that $v = mw$. Primitive Mukai vectors and $v-$generic polarizations play an important role in the theory of moduli spaces. One of the main results is the following, due to Mukai.

Proposition 1.4.2. *If v is a primitive Mukai vector and H is a $v-$generic polarization, then $M(v) = M^s(v)$.*

Proof. See for example Theorem 4.C.3 in [H-L] or [Muk2]. □

The existence of strictly semistable sheaves introduces a non-empty closed locus $M(v) \setminus M^s(v)$ which may give rise to singularities. This implies that $M(v)$ is not a good candidate to be an irreducible symplectic manifold. However, by Theorem C.5.1 on the open subscheme $M^s(v)$ there is a symplectic structure. It is then natural to ask if there is a resolution of the singularities of $M(v)$ on which one can find a symplectic structure extending the one on $M^s(v)$. This is the sense of the following definition:

Definition 1.4.4. Let Y be a normal scheme, let Y^s be its regular part and let ω be a symplectic structure on Y^s. A *symplectic resolution* of the couple (Y,ω) is a triple $(\widetilde{Y},\pi,\widetilde{\omega})$ where $\pi : \widetilde{Y} \longrightarrow Y$ is a resolution of singularities, and $\widetilde{\omega}$ is a symplectic structure on $\widetilde{Y}$ such that $i^*\widetilde{\omega} = \pi^*\omega$, where $i : \pi^{-1}(Y^s) \longrightarrow \widetilde{Y}$ is the natural inclusion.

The first example of moduli space admitting a symplectic resolution was described by O'Grady in [OG2], where he considered the moduli space M_{10} of semistable sheaves with Mukai vector $(2,0,-2)$ on a generic projective K3 surface. He showed that M_{10} admits a symplectic resolution $\widetilde{M}_{10}$ which is a $10-$dimensional irreducible symplectic manifold with $b_2(\widetilde{M}_{10}) \geq 24$: it was a new example of irreducible symplectic manifold. This construction will be the main object of the next chapter, where it will be more precisely described. O'Grady even studied moduli spaces of semistable sheaves with Mukai vector $(2,0,-2-2c)$ with $c \in \mathbb{N}$, but he was not able to conclude as in the former case.

O'Grady's examples motivated an extensive investigation on the subject, which leaded to the following results.

Theorem 1.4.3. *(**Kaledin-Lehn-Sorger**, '06). Let v be a Mukai vector of the form $v = mw$, and let H be a $v-$generic ample line bundle. If $m > 2$ or $(w,w) > 2$, then the moduli space $M(v)$ is locally factorial.*

Proof. See Theorem 5.3 in [K-L-S]. □

Remark 1.4.1. The same conclusion holds if $m = 1$, $rk(v) > 0$, $c_1(v) \in NS(X)$ and $(v,v) \geq -2$, since in this case $M(v)$ is either a reduced point or a smooth projective variety. Moreover, if $m > 1$ we have two possible cases where the same property holds: if $(w,w) = -2$, then $M(v)$ is reduced to a point; if $(w,w) = 0$ then $M(v) \simeq S^m M(w)$, where $M(w)$ is a K3 surface. For the details, see Appendix D.

Proposition 1.4.4. ***(Kaledin-Lehn-Sorger**, '06). Let v be a Mukai vector of the form $v = mw$, and let H be a $v-$generic polarization. If $m > 2$ or $(w,w) > 2$, then $M(v)$ does not admit a symplectic resolution of singularities.*

Proof. The proof of this theorem is the content of [K-L-S]. □

Remark 1.4.2. In the case of rank 2 sheaves, Y. Kiem and J. Choy give another proof of Proposition 1.4.4 in [C-K], using a completely different approach. Anyway, we have chosen to follow Kaledin-Lehn-Sorger approach, since they show the local factoriality of $M(v)$ for $v = mw$, with $m > 2$ or $(w,w) > 2$.

The only case it remains to study is the one of Mukai vectors $v = 2w$, for $(w,w) = 2$. Here is the general result:

Theorem 1.4.5. ***(Lehn-Sorger**). Let $v = 2w$, for $(w,w) = 2$. Then the moduli space $M(v)$ admits a symplectic resolution $\widetilde{M}(v)$, which is obtained by blowing-up $M(v)$ along the reduced part of its singular locus Σ_v. Moreover, the codimension of Σ_v in $M(v)$ is 2.*

Proof. See Théorème 1.1 in [L-S]. □

Chapter 2

The $10-$dimensional O'Grady's example $\widetilde{M}_{10}$

As seen in the previous chapter, if the Mukai vector v is not of the form $2w$ with w primitive and $(w, w) = 2$, then every moduli space of semistable sheaves $M(v)$ on a K3 surface is not necessarily smooth, but is always locally factorial. One might expect that the same holds even for the moduli spaces $M(v)$ admitting a symplectic resolution $\widetilde{M}(v)$. The first example we analyze is the moduli space M_{10} of semistable sheaves with Mukai vector $v := (2, 0, -2)$. It was introduced by O'Grady in [OG2], where he was able to show that M_{10} admits a symplectic resolution $\widetilde{M}_{10}$ (before [K-L-S] and [L-S]), and that $\widetilde{M}_{10}$ is an irreducible symplectic manifold with $b_2 \geq 24$. Our main result is the following:

Theorem 2.0.6. *The moduli space M_{10} is $2-$factorial.*

This chapter provides the proof of this theorem. First we recall the basic results on M_{10} and $\widetilde{M}_{10}$ we need, in particular those obtained by Rapagnetta, and we show that M_{10} cannot be locally factorial. Once this done, we study further characteristics of the sheaves parameterized by M_{10}: in particular, we calculate their cohomology and show that any sheaf parameterized by M_{10} is $\mu-$stable if and only if it is locally free. We then calculate the Picard group of $\widetilde{M}_{10}$ analyzing the relation between the Mukai's and the Donaldson's morphisms.

Finally, we conclude by showing that M_{10} is $2-$factorial. This is done by describing the Picard group of M_{10}, which is shown to be isomorphic to $Pic(X) \oplus \mathbb{Z} \cdot \beta$ for some line bundle β. Then, we show that β has to be $2B$, where B is the Weil divisor of M_{10} parameterizing non-locally free sheaves. As a corollary to our construction, we show that there is an isometry between $v^{\perp} \subseteq \widetilde{H}(X, \mathbb{Z})$ and its image in $H^2(\widetilde{M}_{10}, \mathbb{Z})$. This is the generalization of Theorem D.3.9 for moduli spaces $M(v)$, with v primitive.

2.1 Generalities on M_{10}

In this section we recall the basic facts about the construction of the moduli space M_{10} described in [OG2]. Let X be a projective K3 surface such that $Pic(X) = \mathbb{Z} \cdot H$, where H is an ample line bundle such that $H^2 = 2$. Consider the Mukai vector $v = (2, 0, -2)$, and let $M_{10} := M(v)$ be the moduli space of $H-$semistable sheaves of rank 2, trivial determinant and second Chern class equal to 4. In particular, notice that $v = 2w$, with $(w, w) = 2$. The scheme M_{10} is a $10-$dimensional projective variety containing the moduli space M^s_{10}, the subset parameterizing only stable sheaves with Mukai vector v. In particular, M^s_{10} is open in M_{10} by Proposition 1.2.4.

Proposition 2.1.1. *Let Σ be the singular locus of M_{10}. Then Σ parameterizes sheaves of the form $\mathscr{I}_Z \oplus \mathscr{I}_W$, for $Z, W \in Hilb^2(X)$. In particular, Σ is isomorphic to $S^2(Hilb^2(X))$ and $codim_{M_{10}}(\Sigma) = 2$.*

Proof. See Lemma 1.1.5. in [OG2]. □

Corollary 2.1.2. *Let M^{lf}_{10} be the moduli space of locally free semistable sheaves with Mukai vector v. Then $M^{lf}_{10} \subseteq M^s_{10}$.*

Another property, which will be important in the following, is:

Proposition 2.1.3. *Let $\mathscr{E}$ be a stable sheaf defining a point in M^s_{10}. If it is not locally free, then $\mathscr{E}^{**} = \mathscr{O}_X \oplus \mathscr{O}_X$.*

Proof. See Claim 4.2 in [Lehn]. □

This proposition allows us to study in detail the moduli space $M^{\mu-ss}_{10}$ of $\mu-$semistable sheaves with Mukai vector v. Following Section 1.3.2, there is a surjective morphism

$$\phi : M_{10} \longrightarrow M^{\mu-ss}_{10}.$$

Moreover, as shown in [OG2] we have $M^{\mu-ss}_{10} = M^{lf}_{10} \coprod S^4(X)$. By Proposition 2.1.3, we can describe the morphism ϕ: first of all, it is an isomorphism on M^{lf}_{10}; if $\mathscr{E}$ defines a point in $M^s_{10} \setminus M^{lf}_{10}$, then $\mathscr{E}^{**} = \mathscr{O}_X \oplus \mathscr{O}_X$ and the singular locus of $\mathscr{E}$ has length 4. Then let $\phi(\mathscr{E}) := Supp(Sing(\mathscr{E}))$. Finally, if $\mathscr{E}$ defines a point in Σ, then $\mathscr{E}$ is S-equivalent to $\mathscr{I}_Z \oplus \mathscr{I}_W$, and $\phi(\mathscr{E}) := Supp(Z) + Supp(W)$. The most important property of ϕ is the following:

Proposition 2.1.4. *Let B be the closed subset of M_{10} parameterizing non-locally free sheaves.*

1. *Let $S^4_s(X)$ be the smooth part of $S^4(X)$. Then the restriction of ϕ to $\phi^{-1}(S^4_s(X))$ is a $\mathbb{P}^1-$bundle whose generic fiber is denoted γ'.*

2. *B is an irreducible Weil divisor of M_{10}.*

Proof. See Theorem 4.1 and Claim 4.3 in [Lehn]. □

In his paper, O'Grady shows the following:

Theorem 2.1.5. *The moduli space M_{10} admits a symplectic resolution*

$$\pi : \widetilde{M}_{10} \longrightarrow M_{10}.$$

The smooth symplectic variety $\widetilde{M}_{10}$ is an irreducible symplectic manifold of dimension 10 and such that $b_2(\widetilde{M}_{10}) \geq 24$.

Proof. This is the content of [OG2]. □

Remark 2.1.1. The previous theorem shows that $\widetilde{M}_{10}$ is a new example of irreducible symplectic manifold, since in dimension 10 there are only two other known deformation classes of irreducible symplectic varieties: $Hilb^5(X)$ (where X is a K3 surface) with $b_2 = 23$, and $K^5(A)$ (where A is an abelian surface) with $b_2 = 7$. Since $b_2(\widetilde{M}) \geq 24$, the projective variety $\widetilde{M}_{10}$ cannot be deformed to any of the other examples.

Remark 2.1.2. The symplectic resolution $\widetilde{M}_{10}$ of M_{10} was also described by Lehn and Sorger in [L-S]: it is the blow-up of Σ with reduced scheme structure. This is an important point, since it simplifies the construction originally given by O'Grady.

In the following, we will write $\widetilde{\Sigma}$ for the exceptional divisor of π, and $\widetilde{B}$ for the proper transform of B under π.

Proposition 2.1.6. *Let Σ^0 be the smooth locus of Σ.*

1. *The restriction of π to $\pi^{-1}(\Sigma^0)$ is a $\mathbb{P}^1-$bundle whose generic fiber is denoted δ.*

2. *The restriction of π to $\pi^{-1}(\phi^{-1}(S^4_s(X)))$ is a $\mathbb{P}^1-$bundle whose generic fiber is denoted γ.*

Proof. The first part is Proposition 2.3.1 in [OG2], and the second is just Proposition 2.1.4 together with Corollary 2.1.2. □

The basic result we recall in this section is the following:

Theorem 2.1.7. *(**Rapagnetta**, '07). The second Betti number of $\widetilde{M}_{10}$ is 24. Let $\mu_D : H^2(X, \mathbb{Z}) \longrightarrow H^2(M^{\mu ss}_{10}, \mathbb{Z})$ be the Donaldson's morphism.*

1. *The morphism $\widetilde{\mu} := \pi^* \circ \phi^* \circ \mu_D : H^2(X, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_{10}, \mathbb{Z})$ is injective.*

2. *We have the following equalities:*

$$c_1(\widetilde{\Sigma}) \cdot \delta = -2, \qquad c_1(\widetilde{B}) \cdot \delta = 1$$

$$c_1(\widetilde{\Sigma}) \cdot \gamma = 3, \qquad c_1(\widetilde{B}) \cdot \gamma = -2.$$

3. *The second integral cohomology of $\widetilde{M}_{10}$ is*

$$H^2(\widetilde{M}_{10}, \mathbb{Z}) = \widetilde{\mu}(H^2(X, \mathbb{Z})) \oplus \mathbb{Z} \cdot c_1(\widetilde{\Sigma}) \oplus \mathbb{Z} \cdot c_1(\widetilde{B}).$$

4. *Let q be the Beauville-Bogomolov form of $\widetilde{M}_{10}$. The lattice $(H^2(\widetilde{M}_{10}, \mathbb{Z}), q)$ is isomorphic to $\Lambda_{K3} \oplus T$, where Λ_{K3} is the lattice of the K3 surface X and T is the lattice $\mathbb{Z} \cdot c_1(\widetilde{\Sigma}) \oplus \mathbb{Z} \cdot c_1(\widetilde{B})$, where*

$$q(c_1(\widetilde{\Sigma}), c_1(\widetilde{\Sigma})) = -6, \quad q(c_1(\widetilde{\Sigma}), c_1(\widetilde{B})) = 3,$$
$$q(c_1(\widetilde{B}), c_1(\widetilde{\Sigma})) = 3, \quad q(c_1(\widetilde{B}), c_1(\widetilde{B})) = -2.$$

Proof. The proof of this theorem is contained in [Rap]. The calculation of the second Betti number of $\widetilde{M}_{10}$ is Theorem 1.1. Items 1, 2 and 3 are shown in Theorem 3.1. The rest is contained in Theorem 4.3. □

2.2 The local factoriality of M_{10}

The aim of this section is to show that the moduli space M_{10} considered by O'Grady cannot be locally factorial. In particular, using Theorem 2.1.7 one shows that the Weil divisor B defined in Proposition 2.1.4 is not Cartier. At the end of the section, we show that the divisor B can be interpreted as the locus parameterizing sheaves with non-trivial cohomology.

2.2.1 The moduli space M_{10} is not locally factorial

A first application of Theorem 2.1.7 is the following (for a recall on local factoriality, see Appendix A):

Proposition 2.2.1. *If there is $n \in \mathbb{N}$ such that nB is a Cartier divisor, then n must be even. In particular, M_{10} is not locally factorial.*

Proof. Let $n \in \mathbb{N}$ be such that nB is Cartier. Then, we can consider its pull-back to $\widetilde{M}_{10}$, which will be

$$\pi^*(nB) = n\widetilde{B} + m\widetilde{\Sigma}$$

for some $m \in \mathbb{Z}$, since $\widetilde{B}$ is the proper transform of B. Notice that by the projection formula we have $c_1(\pi^*(nB)) \cdot \delta = 0$, as δ is contracted by π. In particular, by point 2 of Theorem 2.1.7 we get

$$0 = c_1(\pi^*(nB)) \cdot \delta = nc_1(\widetilde{B}) \cdot \delta + mc_1(\widetilde{\Sigma}) \cdot \delta = n - 2m.$$

As $m \in \mathbb{Z}$, this equation forces n to be even.

Finally, this implies that M_{10} cannot be locally factorial. We can suppose M_{10} to be $\mathbb{Q}-$factorial (otherwise there is nothing to prove), so that there must be $n \in \mathbb{Z}$ such that nB is Cartier. But then n must be even by the first part of the Proposition, so that M_{10} cannot be locally factorial. □

Remark 2.2.1. Notice that Theorem 2.1.7 implies even that $Pic(M_{10})$ has no torsion. Indeed, suppose there is $L \in Pic(M_{10})$ which is torsion of period $t \in \mathbb{N}$, and let $\widetilde{L}$ be its proper transform under π. Then $\pi^*(L) = \widetilde{L} + m\widetilde{\Sigma}$ for some $m \in \mathbb{Z}$, so that

$$0 = \pi^*(qL) = q(\widetilde{L} + m\widetilde{\Sigma}). \tag{2.1}$$

Now, $\widetilde{M}_{10}$ is an irreducible symplectic manifold, so that the first Chern class morphism $c_1 : Pic(\widetilde{M}_{10}) \longrightarrow H^2(\widetilde{M}_{10}, \mathbb{Z})$ is injective. By point 3 of Theorem 2.1.7, this implies that $Pic(\widetilde{M}_{10})$ has no torsion. Then, equation (2.1) implies $\widetilde{L} = -m\widetilde{\Sigma}$, so that $L = 0$.

The same proof shows that $\pi^* : Pic(M_{10}) \longrightarrow Pic(\widetilde{M}_{10})$ is injective. Moreover, this implies that $c_1 : Pic(M_{10}) \longrightarrow H^2(M_{10}, \mathbb{Z})$ is injective. Indeed, let $L, L' \in Pic(M_{10})$ be such that $c_1(L) = c_1(L')$, then

$$c_1(\pi^*(L)) = \pi^*(c_1(L)) = \pi^*(c_1(L')) = c_1(\pi^*(L')),$$

getting $\pi^*(L) = \pi^*(L')$ as c_1 is injective at the level of $\widetilde{M}_{10}$. As π^* is injective, this implies $L = L'$, and we are done.

2.2.2 Properties of the Weil divisor B

In this section we show that the Weil divisor B is the locus parameterizing sheaves with non-trivial cohomology.

Lemma 2.2.2. *Let E be a locally free sheaf of rank 2 and trivial determinant. Then $E \simeq E^*$.*

Proof. Consider the canonical morphism $E \otimes E \longrightarrow E \wedge E$ sending a local section $\alpha \otimes \beta$ to $\alpha \wedge \beta$. But $E \wedge E = det(E) \simeq \mathscr{O}_X$ by hypothesis, so that the morphism above is

$$E \otimes E \longrightarrow \mathscr{O}_X.$$

It is an easy calculation to show that this is a perfect pairing, so that E is isomorphic to E^*. □

Another important property is the following:

Lemma 2.2.3. *Let $\mathscr{E}$ be a sheaf defining a point in M_{10}. Then $H^0(X, \mathscr{E}) = 0$ and $h^1(X, \mathscr{E}) = h^2(X, \mathscr{E})$.*

Proof. The Chern character of $\mathscr{E}$ is $ch(\mathscr{E}) = (2, 0, -4)$ since $\mathscr{E} \in M_{10}$. By the Hirzebruch-Riemann-Roch Theorem, for any $n \in \mathbb{Z}$ we have

$$P(\mathscr{E}, n) = \chi(\mathscr{E} \otimes \mathscr{O}_X(n)) = \int_X ch(\mathscr{E}) ch(\mathscr{O}_X(n)) td(X) = 2n^2,$$

since $\mathscr{O}_X(1) = H$ and $H^2 = 2$. In particular, $\chi(\mathscr{E}) = 0$, so that

$$h^1(X, \mathscr{E}) = h^0(X, \mathscr{E}) + h^2(X, \mathscr{E}),$$

and we just need to show that $h^0(X, \mathscr{E}) = 0$. By definition, the reduced Hilbert polynomial of $\mathscr{E}$ is $p(\mathscr{E}, n) = n^2$, and it is easy to see that $p(\mathscr{O}_X, n) = n^2 + 2$. As $\mathscr{O}_X$ and $\mathscr{E}$ are semistable and $p(\mathscr{O}_X) > p(\mathscr{E})$, by Proposition 1.1.8 we get $Hom(\mathscr{O}_X, \mathscr{E}) = 0$. As $Hom(\mathscr{O}_X, \mathscr{E}) = H^0(X, \mathscr{E})$, we are done. □

Before going on with the cohomology of sheaves parameterized by B, we show this proposition, that can be seen as a corollary of the two lemmas above. This will not be used in the following, but we insert it here as an observation.

Proposition 2.2.4. *Let $M_{10}^{\mu} \subseteq M_{10}$ be the open subset parameterizing $\mu-$stable sheaves. Then $M_{10}^{\mu} = M_{10}^{lf}$.*

Proof. We begin with the inclusion $M_{10}^{\mu} \subseteq M_{10}^{lf}$: let $\mathscr{E}$ be a $\mu-$stable sheaf which is not locally free. Then $[\mathscr{E}] \in B \cap M_{10}^{s}$, so that $\mathscr{E}^{**} \simeq \mathscr{O}_X \oplus \mathscr{O}_X$ by Proposition 2.1.3. Since $\mu-$stability is preserved by duality, $\mathscr{E}^{**}$ has to be $\mu-$stable, then simple by Corollary 1.1.9, and we get a contradiction.

Now we show the opposite inclusion, i. e. $M_{10}^{lf} \subseteq M_{10}^{\mu}$. Let E be a vector bundle defining a point in M_{10}^{lf}. We need to show that it is $\mu-$stable, i. e. that for any $\mathscr{F} \subseteq E$ of rank 1, we have $\mu(\mathscr{F}) < \mu(E)$. Notice that $\mu(E) = 0$ since $c_1(E) = 0$, and $c_1(\mathscr{F}) = fc_1(H)$ for some $f \in \mathbb{Z}$, since $Pic(X) = \mathbb{Z} \cdot H$. In conclusion, $\mu(\mathscr{F}) = 2f$, and we need to show that $f < 0$.

Let $c_2(\mathscr{F}) = d \in \mathbb{Z}$. As we have seen in the proof of Lemma 2.2.3, the reduced Hilbert polynomial of E is $p(E, n) = n^2$. It is an easy calculation to show that

$$p(\mathscr{F}, n) = n^2 + 2fn + f^2 + 2 - d.$$

As E locally free, then E is stable (see Corollary 2.1.2), so that $p(\mathscr{F}) < p(E)$. We have then two possible cases: the first one is $f < 0$, so that $\mu(\mathscr{F}) < \mu(E)$ and we are done. The second case is $f = 0$ and $d > 2$, getting $\mu(\mathscr{F}) = \mu(E)$. We need to show that this second possibility cannot be verified.

Now, suppose there is a subsheaf $\mathscr{F}$ of E of rank 1, trivial determinant and $c_2 > 2$. As E is locally free and X is a surface, $\mathscr{F}$ has to be torsion-free. It is then of the form $\mathscr{I}_Z$ for a 0−dimensional subscheme Z of length $d > 2$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow E \longrightarrow \mathscr{G} \longrightarrow 0$$

and apply the functor $\mathscr{H}om(., \mathscr{O}_X)$, getting the exact sequence

$$0 \longrightarrow \mathscr{G}^* \longrightarrow E^* \xrightarrow{f} \mathscr{O}_X$$

since $\mathscr{I}_Z^*$ is a line bundle with trivial first Chern class. As $rk(\mathscr{F}) > rk(\mathscr{G})$, the morphism f cannot be trivial, so that $Hom(E^*, \mathscr{O}_X) \neq 0$. By Serre's duality and Lemma 2.2.2, this implies $H^0(X, E) \neq 0$. But this is not possible, since by Lemma 2.2.3 we have $H^0(X, E) = 0$. In conclusion, for any rank 1 subsheaf $\mathscr{F}$ of E, we have $\mu(\mathscr{F}) < \mu(E)$, and we are done. □

We can finally present the results on the cohomology of the sheaves parameterized by M_{10}.

Proposition 2.2.5. *Let E be a sheaf defining a point in M_{10}^{lf}. Then for any $i = 0, 1, 2$ we have $H^i(X, E) = 0$.*

Proof. By Lemma 2.2.3 we have $H^0(X, E) = 0$ and $h^1(X, E) = h^2(X, E)$. To conclude, apply Serre's duality to E, getting $h^2(X, E^*) = 0$. By Lemma 2.2.2, this implies $h^2(X, E) = 0$, and we are done. □

As a final calculation, we show the following:

Proposition 2.2.6. *Let $\mathscr{E}$ be a non-locally free semistable sheaf defining a point in M_{10}. Then $H^0(X, \mathscr{E}) = 0$ and $h^1(X, \mathscr{E}) = h^2(X, \mathscr{E}) \neq 0$.*

Proof. By Lemma 2.2.3, any sheaf $\mathscr{E}$ has no global sections and $h^1(X, \mathscr{E}) = h^2(X, \mathscr{E})$. We just then need to show that $h^2(X, \mathscr{E}) \neq 0$. The proof is divided in three cases.

Case 1: $\mathscr{E}$ is stable, so that it defines a point in $B \cap M_{10}^s$. By Proposition 2.1.3, the bidual of $\mathscr{E}$ is $\mathscr{O}_X \oplus \mathscr{O}_X$, and we have a short exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X \oplus \mathscr{O}_X \longrightarrow \mathscr{G} \longrightarrow 0, \tag{2.2}$$

since $\mathscr{E}$ is torsion free, where $\mathscr{G}$ is a rank 0 sheaf supported on a finite number of points. This and the exact sequence (2.2) give $h^2(X, \mathscr{E}) = h^2(X, \mathscr{O}_X \oplus \mathscr{O}_X)$. As X is a K3 surface, by Serre's duality we have $h^2(X, \mathscr{O}_X) = 1$, so that $h^2(X, \mathscr{E}) = 2$.

Case 2: $\mathscr{E}$ is strictly polystable. Then it is of the form $\mathscr{I}_Z \oplus \mathscr{I}_W$, for some $Z, W \in Hilb^2(X)$ by Proposition 2.1.1. The short exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Z \longrightarrow 0$$

gives $h^2(X, \mathscr{I}_Z) = 1$, so that $h^2(X, \mathscr{E}) = 2$.

Case 3: $\mathscr{E}$ is strictly semistable but not polystable. By Proposition 2.1.1, $\mathscr{E}$ fits into an exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_W \longrightarrow 0$$

for some $Z, W \in Hilb^2(X)$, so that there is a surjective morphism

$$H^2(X, \mathscr{E}) \longrightarrow H^2(X, \mathscr{I}_W).$$

As seen in the previous case, $h^2(X, \mathscr{I}_W) = 1$, so that $h^2(X, \mathscr{E}) \neq 0$, and we are done. □

2.3 The Picard group of M_{10}

In this section we show that the Picard group of the irreducible symplectic variety $\widetilde{M}_{10}$ is isomorphic to $Pic(X) \oplus \mathbb{Z}[\widetilde{\Sigma}] \oplus \mathbb{Z}[\widetilde{B}]$. This could be concluded by point 2 of Theorem 2.1.7 and the fact that the Donaldson's morphism preserves the Hodge decomposition, but we present a different proof, based on the analysis of the relation between Donaldson's morphism and Le Potier's morphism. In particular, using Le Potier's determinant we define a morphism $\overline{\lambda}$ from $Pic(X)$ to $Pic(\widetilde{M}_{10})$, and we show that for any line bundle $L \in Pic(X)$ we have $c_1(\overline{\lambda}(L)) = \widetilde{\mu}(c_1(L))$. The main reason to use this approach is that it leads to a natural and more explicit way of producing line bundles on M_{10}, which is our goal.

Finally, we show that the formula for the Picard group of $\widetilde{M}_{10}$ implies that the moduli space M_{10} is either $2k-$factorial for some integer k or it is not even $\mathbb{Q}-$factorial. This will be done by comparing the Picard group of M_{10} and the group of Weil divisors (up to linear equivalence) on M_{10} (see Appendix A).

2.3.1 Construction of flat families

Before giving the proof of the formula for the Picard group of $\widetilde{M}_{10}$, we recall some basic facts about flat families that we will apply in the next section and later.

In the following, let S be an algebraic surface, and let T be a proper scheme. Moreover, let $p_S : T \times S \longrightarrow S$ be the projection on S and $p_T : T \times S \longrightarrow T$ be the projection on T. Let $\mathscr{V}$ and $\mathscr{W}$ be two $T-$flat coherent sheaves on $T \times S$, and suppose that the sheaf $p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W})$ is a vector bundle on T. Let

$$p : \mathbb{P}(p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W})) := Y \longrightarrow T$$

be the associated projective bundle, and let $\mathscr{T}$ be the tautological line bundle on Y. By [Har], Chapter II, Proposition 7.11, there is a canonical injective morphism

$$f : \mathscr{T} \longrightarrow p^* p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W}),$$

corresponding to a global section

$$\sigma \in H^0(Y, p^* p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W}) \otimes \mathscr{T}^{-1}).$$

Now, let $q_Y : Y \times S \longrightarrow Y$ be the projection on Y and $q_S : Y \times S \longrightarrow S$ be the projection on S, so that we have the following commutative diagram:

$$\begin{array}{ccccc} Y & \xleftarrow{q_Y} & Y \times S & \xrightarrow{q_S} & S \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p \times id_S} & & \| \\ T & \xleftarrow[p_T]{} & T \times S & \xrightarrow[p_S]{} & S \end{array}$$

and the following relations hold:

$$p^* p_{T*} \mathcal{H}om(\mathcal{V}, \mathcal{W}) = q_{Y*}(p \times id_S)^* \mathcal{H}om(\mathcal{V}, \mathcal{W}) =$$

$$= q_{Y*} \mathcal{H}om((p \times id_S)^* \mathcal{V}, (p \times id_S)^* \mathcal{W}).$$

Then, σ defines a global section

$$\sigma' \in H^0(Y, q_{Y*} \mathcal{H}om((p \times id_S)^* \mathcal{V}, (p \times id_S)^* \mathcal{W}) \otimes \mathcal{T}^{-1}).$$

Using the projection formula, we get a global section

$$\sigma'' \in H^0(Y \times S, \mathcal{H}om((p \times id_S)^* \mathcal{V} \otimes q_Y^* \mathcal{T}, (p \times id_S)^* \mathcal{W})),$$

corresponding to a morphism

$$\widetilde{f} : (p \times id_S)^* \mathcal{V} \otimes q_Y^* \mathcal{T} \longrightarrow (p \times id_S)^* \mathcal{W}.$$

We can apply this construction to the two following examples. The first one will be used in the next section, the second will be one of the main technical tools in the proof of the $2-$factoriality of M_{10}.

Example 2.3.1. Let X be a projective K3 surface with $Pic(X) = \mathbb{Z} \cdot H$, where H is an ample line bundle such that $H^2 = 2$. Fix three different points $x_1, x_2, x_3 \in X$. We can define the following morphism

$$i : X \longrightarrow S^4(X), \quad i(x) := x + x_1 + x_2 + x_3,$$

which is a closed immersion. Let $T := i(X)$, which is a surface isomorphic to X. Notice that $T \subseteq M_{10}^{\mu-ss}$. Consider a surjective morphism

$$\varphi : \mathcal{O}_X^2 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{C}_{x_i}$$

as in Proposition 3.0.5 in [OG2], and let $\mathcal{K} := \ker(\varphi)$. This is a rank 2 sheaf with trivial determinant and second Chern class equal to 3. Notice that any sheaf defining a point in $\phi^{-1}(T)$ is the kernel of a surjective morphism from $\mathcal{K}$ to $\mathbb{C}_x$ for a point $x \in X$ (see Proposition 3.0.5 in [OG2]).

Let $\Delta \subseteq T \times X$ be the diagonal (up to the natural isomorphism between T and X). By Theorem 1.1.17 the sheaf $p_{T*} \mathcal{H}om(p_X^* \mathcal{K}, \mathcal{O}_\Delta)$ is a rank 2 vector bundle on T and for any $x \in T$ the canonical morphism

$$p_{T*} \mathcal{H}om(p_X^* \mathcal{K}, \mathcal{O}_\Delta)_x \longrightarrow Hom(\mathcal{K}, \mathbb{C}_x)$$

is an isomorphism. Let $Y := \mathbb{P}(p_{T*} \mathcal{H}om(p_X^* \mathcal{K}, \mathcal{O}_\Delta))$, and let

$$p : Y \longrightarrow T.$$

Clearly, p is a $\mathbb{P}^1-$bundle. Using the construction described in the previous section, we have a canonical morphism

$$\widetilde{f}: q_X^*\mathscr{K} \otimes q_Y^*\mathscr{T} \longrightarrow (p \times id_X)^*\mathscr{O}_\Delta.$$

Let $\mathscr{H} := \ker(\widetilde{f})$.

Lemma 2.3.1. *Let $\mathscr{E}$ be a sheaf defining a point in B and whose singular locus is given by x, x_1, x_2, x_3. Then $\mathscr{E}$ defines a point $[f_\mathscr{E}] \in Y$, and*

$$\mathscr{H}_{[f_\mathscr{E}]} \simeq \mathscr{E}.$$

Moreover, the morphism $\widetilde{f}$ is surjective and $\mathscr{H}$ is a $Y-$flat family.

Proof. The sheaf $\mathscr{E}$ is the kernel of a surjective (hence non-zero) morphism

$$f_\mathscr{E}: \mathscr{K} \longrightarrow \mathbb{C}_x,$$

defining a point $[f_\mathscr{E}] \in p^{-1}(x)$ since $p^{-1}(x) \simeq \mathbb{P}(Hom(\mathscr{K}, \mathbb{C}_x))$. By definition of $\widetilde{f}$, we have $\widetilde{f}_{|q_Y^{-1}([f_\mathscr{E}])} = f_\mathscr{E}$.

Now, $\widetilde{f}$ is surjective: indeed, coker$(\widetilde{f})$ is trivial if and only if it is trivial on the fibers of q_Y. Now, let $t \in Y$, which corresponds to a surjective morphism $f_\mathscr{E}$. Then coker$(\widetilde{f})_{|q_Y^{-1}(t)} =$ coker$(f_\mathscr{E}) = 0$, and we are done.

Since $\widetilde{f}$ is surjective, the family $\mathscr{H}$ is $Y-$flat. Now, since $q_X^*\mathscr{K} \otimes q_Y^*\mathscr{T}$ and $(p \times id_X)^*\mathscr{O}_\Delta$ are $Y-$flat, by Lemma 1.1.16 for any $t \in Y$ the canonical morphism

$$\mathscr{H}_t \longrightarrow (q_X^*\mathscr{K} \otimes q_Y^*\mathscr{T})_t \simeq \mathscr{K}$$

is injective. This implies that $\mathscr{H}_t$ is the kernel of the morphism $\widetilde{f}_{|q_Y^{-1}(t)}$. As seen above, any $t \in Y$ corresponds to a unique surjective morphism $f_\mathscr{E}: \mathscr{K} \longrightarrow \mathbb{C}_x$, where $x = p(t)$, whose kernel is $\mathscr{E}$, and $\widetilde{f}_{|q_Y^{-1}(t)} = f_\mathscr{E}$. Then $\mathscr{H}_{[f_\mathscr{E}]} \simeq \mathscr{E}$, and we are done. □

Example 2.3.2. We describe another example that will be used later. Let $x, x_1, x_2, x_3 \in X$ be four different points in X. Consider the morphism

$$\varphi: \mathscr{O}_X^2 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{C}_{x_i}$$

as in the previous example, and let again $\mathscr{K} := \ker(\varphi)$.

In this example, let $T = \{x\}$ and let the two sheaves be $p_X^*\mathscr{K}$ and $i_*\mathbb{C}_x$, where i is the inclusion of x in X. In this situation we have $Y \simeq \mathbb{P}^1$, and the tautological line bundle is $\mathscr{T} = \mathscr{O}(-1)$. By the general construction, we find a morphism

$$\widetilde{f}: (p \times id_X)^* p_X^*\mathscr{K} \longrightarrow (p \times id_X)^* i_*\mathbb{C}_x \otimes q_Y^*\mathscr{O}(1),$$

where $p : Y \longrightarrow \{x\}$ is simply the canonical morphism. In particular, notice that $(p \times id_X)^* p_X^* \mathscr{K} = q_X^* \mathscr{K}$ and

$$(p \times id_X)^* i_* \mathbb{C}_x \otimes q_Y^* \mathscr{O}(1) = j_* \mathscr{O}(1),$$

where $j : \mathbb{P}^1 \times \{x\} \longrightarrow \mathbb{P}^1 \times X$ is the inclusion. In conclusion, we have

$$\widetilde{f} : q_X^* \mathscr{K} \longrightarrow j_* \mathscr{O}(1).$$

Finally, let $\mathscr{H} := \ker(\widetilde{f})$.

Lemma 2.3.2. *Let $\mathscr{E}$ be a sheaf defining a point in B whose singular locus is given by x, x_1, x_2, x_3, and let $[f_{\mathscr{E}}]$ be the point of Y defined by $\mathscr{E}$. Then*

$$\mathscr{H}_{[f_{\mathscr{E}}]} \simeq \mathscr{E},$$

and $\widetilde{f}$ is a surjective morphism. Moreover, the family $\mathscr{H}$ is $Y-$flat.

Proof. The proof works as the one of Lemma 2.3.1. □

2.3.2 The Picard group of $\widetilde{M}_{10}$

Let R be the open subscheme of a Quot-scheme whose quotient is M_{10}. In section 1.3.1 we have defined the Le Potier's determinant

$$\widetilde{\lambda} : K_{top}(X) \longrightarrow Pic(R).$$

By Theorem 1.3.2, the line bundle $\widetilde{\lambda}(\alpha)$ descends to a line bundle $\lambda(\alpha) \in Pic(M_{10})$ if $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$. Here, $e = [\mathscr{E}]$ is the class of a sheaf parameterized by M_{10}, and $h = [H]$. First of all, we describe the elements in $e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$.

Lemma 2.3.3. *Let $\alpha \in K_{top}(X)$. Then $\alpha \in \{1, h, h^2\}^{\perp\perp}$ if and only if $c_1(\alpha)$ is the first Chern class of a line bundle on X.*

Proof. First of all, we describe the classes $\beta \in \{1, h, h^2\}^{\perp}$. By definition, β is in $\{1, h, h^2\}^{\perp}$ if and only if $\chi(\beta) = \chi(\beta \cdot h) = \chi(\beta \cdot h^2) = 0$. By the Hirzebruch-Riemann-Roch Theorem it is easy to see that this happens if and only if $v(\beta) = (0, b, 0)$, where $b \in H^2(X, \mathbb{Z})$ is such that $b \cdot c_1(H) = 0$. By Proposition B.2.5 this implies that $\beta \in H^2(X, \mathbb{Z}) \cap (H^{2,0}(X) \oplus H^{0,2}(X))$, since $Pic(X) = \mathbb{Z} \cdot H$.

Now, let $\alpha \in K_{top}(X)$. Then $\alpha \in \{1, h, h^2\}^{\perp\perp}$ if and only if $\chi(\alpha \cdot \beta) = 0$ for any $\beta \in \{1, h, h^2\}^{\perp}$. By the first part of the lemma, the Mukai vector of these β is of the form $v(\beta) = (0, b, 0)$ for $b \in H^2(X, \mathbb{Z}) \cap (H^{2,0}(X) \oplus H^{0,2}(X))$, so that $\chi(\alpha \cdot \beta) = c_1(\alpha) \cdot b$. As this must be trivial, then $c_1(\alpha)$ has to be the first Chern class of a line bundle on X by Proposition B.2.5, and we are done. □

Lemma 2.3.4. *Let $\alpha \in K_{top}(X)$. Then $\alpha \in e^{\perp}$ if and only if $ch_2(\alpha) = 0$.*

Proof. Recall that the Mukai vector of e is $(2,0,-2)$. By the Hirzebruch-Riemann-Roch Theorem, the condition $\chi(\alpha\cdot v)=0$ is

$$0=-\int_X v(\alpha)^\vee v=-2ch_2(\alpha),$$

and we are done. □

Now, let $p\in X$ and define

$$u:Pic(X)\longrightarrow K_{top}(X),\qquad u(L):=[\mathscr{O}_X-L]+\frac{c_1^2(L)}{2}[\mathbb{C}_p].$$

Proposition 2.3.5. *There is a group morphism*

$$\lambda\circ u:Pic(X)\longrightarrow Pic(M_{10}).$$

Proof. The only point we need to verify is that $u(L)\in e^\perp\cap\{1,h,h^2\}^{\perp\perp}$. The Mukai vector of $u(L)$ is

$$v(u(L))=(0,-c_1(L),0)$$

so that by Lemma 2.3.4 we have $u(L)\in e^\perp\cap\{1,h,h^2\}^{\perp\perp}$. By Theorem 1.3.2, the Le Potier's determinant defines a line bundle $\lambda(u(L))\in Pic(M_{10})$.

It remains to show that $\lambda\circ u$ is a group morphism. Consider $L_1,L_2\in Pic(X)$. Then we have

$$u(L_1\otimes L_2)=[\mathscr{O}_X-L_1\otimes L_2]+\frac{c_1^2(L_1\otimes L_2)}{2}[\mathbb{C}_p].$$

Notice that $v(u(L_1\otimes L_2))=v(u(L_1)+u(L_2))$, so that

$$u(L_1\otimes L_2)=u(L_1)+u(L_2)$$

by Proposition C.4.3. In conclusion u, hence $\lambda\circ u$, is a group morphism. □

We start now to study the morphism $\lambda\circ u$. First of all, we have this:

Proposition 2.3.6. *Let $L\in Pic(X)$. Then $c_1(\lambda(u(L)))\cdot\gamma'=0$.*

Proof. As $Pic(X)=\mathbb{Z}\cdot H$, we need to verify the statement only for H. By Proposition 8.2.3 in [H-L] there is a positive integer m such that $\lambda(u(H))^{\otimes m}$ is generated by its global sections, and by definition the canonical map

$$\phi:M_{10}\longrightarrow\mathbb{P}(H^0(M_{10},\lambda(u(H))^{\otimes m})^*)$$

has $M_{10}^{\mu-ss}$ as image. In particular, $\phi^*\mathscr{O}(1)=\lambda(u(H))^{\otimes m}$, so that

$$mc_1(\lambda(u(H)))\cdot\gamma'=c_1(\lambda(u(H))^{\otimes m})\cdot\gamma'=c_1(\phi^*\mathscr{O}(1))\cdot\gamma'=0,$$

as γ' is contracted by ϕ. Finally, this implies $c_1(\lambda(u(H)))\cdot\gamma'=0$, and we are done. □

The main result we need to show is the following:

Proposition 2.3.7. *The following diagram*

$$\begin{array}{ccc} Pic(X) & \xrightarrow{\lambda\circ u} & Pic(M_{10}) \\ {\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle c_1} \\ H^2(X,\mathbb{Z}) & \xrightarrow[\phi^*\circ\mu_D]{} & H^2(M_{10},\mathbb{Z}) \end{array}$$

is commutative, i. e. $c_1(\lambda(u(L))) = \phi^*(\mu_D(c_1(L)))$ *for any* $L \in Pic(X)$*.*

Proof. The proof of this proposition is done in two steps: first we show that these two classes are equal when restricted to a well-chosen subvariety; then we show that the equality on this restriction implies the equality everywhere.

Step 1. In this step we use the same notations we introduced in Example 2.3.1. Let $i : T \longrightarrow M_{10}^{\mu-ss}$ be the natural inclusion, and let $j : Y \longrightarrow M_{10}$ be the morphism induced by the family $\mathscr{H}$. By Lemma 2.3.1, the morphism j is injective and its image is $\phi^{-1}(T)$, so that the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & M_{10} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \phi} \\ T & \xrightarrow[i]{} & M_{10}^{\mu-ss} \end{array}$$

is commutative. In particular, for any line bundle $L \in Pic(X)$ we have

$$j^*\phi^*(\mu_D(c_1(L))) = p^*i^*(\mu_D(c_1(L))).$$

By Proposition 1.3.7 we have, up to the natural isomorphism between X and T, that

$$i^*(\mu_D(c_1(L))) = c_1(L) \in Pic(T).$$

We need to show that $j^*c_1(\lambda(u(L))) = p^*(c_1(L))$. Clearly

$$j^*c_1(\lambda(u(L))) = c_1(j^*\lambda(u(L))),$$

and by Theorem 1.3.2 and Lemma 2.3.1 we have

$$j^*\lambda(u(L)) = det(q_{Y!}(q_X^*u(L) \cdot [\mathscr{H}])),$$

since j is the morphism induced by $\mathscr{H}$. We have then to calculate the first Chern class of $q_{Y!}(q_X^*u(L) \cdot [\mathscr{H}])$. By the Grothendieck-Riemann-Roch Theorem, this is

$$[ch(q_{Y!}(q_X^*u(L) \cdot [\mathscr{H}]))]_1 = [q_{Y!}(ch(q_X^*u(L) \cdot [\mathscr{H}])q_X^*td(X)^{-1})]_1.$$

Since the fibers of q_Y are of dimension 2, the last term is

$$q_{Y*}[q_X^*(ch(u(L))td(X)^{-1}) \cdot ch(\mathscr{H})]_3.$$

Notice that $ch(u(L))td(X)^{-1} = (0, -c_1(L), 0)$, so that

$$[q_X^*(ch(u(L))td(X)^{-1}) \cdot ch(\mathscr{H})]_3 = -q_X^*(c_1(L)) \cdot ch_2(\mathscr{H}).$$

We have then to calculate $ch_2(\mathscr{H})$. First of all, by Lemma 2.3.1 we have

$$ch(\mathscr{H}) = q_X^* ch(\mathscr{K}) \cdot q_Y^* ch(\mathscr{T}) - (p \times id_X)^* ch(\mathscr{O}_\Delta),$$

where

$$ch(\mathscr{K}) = ch(\mathscr{O}_X^2) - ch\left(\bigoplus_{i=1}^{3} \mathbb{C}_{x_i}\right) = (2, 0, -3).$$

In conclusion, we get

$$-q_X^*(c_1(L)) \cdot ch_2(\mathscr{H}) =$$

$$= 3q_X^*(c_1(L)) \cdot q_X^*[p] - q_X^*(c_1(L)) \cdot q_Y^*(c_1^2(\mathscr{T})) + q_X^*(c_1(L)) \cdot (p \times id_X)^*[\Delta].$$

Now, notice that

$$q_{Y*}(3q_X^*(c_1(L) \cdot [p])) = q_{Y*}(q_X^*(c_1(L)) \cdot q_Y^*(c_1^2(\mathscr{T}))) = 0,$$

so that

$$c_1(j^*(\lambda(u(L)))) = q_{Y*}(q_X^* c_1(L) \cdot (p \times id_X)^*[\Delta]) =$$

$$= p^*(p_{T*}(p_X^*(c_1(L)) \cdot [\Delta])) = p^*(c_1(L)),$$

and we are done.

Step 2. Let $L \in Pic(X)$, and consider

$$\beta := \phi^* \mu_D(c_1(L)) - c_1(\lambda(u(L))) \in H^2(M_{10}, \mathbb{Z}).$$

We need to show that $\beta = 0$. By Step 1, $j^*\beta = 0$. Moreover, $\beta \cdot \gamma' = 0$: indeed $\phi^*\mu_D(c_1(L)) \cdot \gamma' = 0$ since γ' is contracted by ϕ, and $c_1(\lambda(u(L))) \cdot \gamma' = 0$ by Proposition 2.3.6.

Consider $\pi^*\beta \in H^2(\widetilde{M}_{10}, \mathbb{Z})$. By point 3 of Theorem 2.1.7, there are $\alpha \in H^2(X, \mathbb{Z})$ and $n, m \in \mathbb{Z}$ such that

$$\pi^*\beta = \pi^*\phi^*\mu_D(\alpha) + nc_1(\widetilde{\Sigma}) + mc_1(\widetilde{B}).$$

By point 2 of Theorem 2.1.7, we get

$$0 = \pi^*\beta \cdot \delta = \widetilde{\mu}(\alpha) \cdot \delta + nc_1(\widetilde{\Sigma}) \cdot \delta + mc_1(\widetilde{B}) \cdot \delta = m - 2n$$

as δ is contracted by π, and

$$0 = \pi^*\beta \cdot \gamma = \widetilde{\mu}(\alpha) \cdot \delta + nc_1(\widetilde{\Sigma}) \cdot \delta + mc_1(\widetilde{B}) \cdot \delta = 3n - 2m$$

since $\pi^*\beta \cdot \gamma = \beta \cdot \gamma' = 0$ by the projection formula. In conclusion, since $m = 2n$, the integer n has to satisfy the equality $3n = 4n$, so that $n = 0$, implying even $m = 0$.

Finally, we get $\beta = \phi^*\mu_D(\alpha)$: indeed, $\beta \in c_1(Pic(M_{10}))$, so that $\widetilde{\mu}(\alpha) = \pi^*(\beta) \in c_1(Pic(\widetilde{M}_{10}))$. This implies $\phi^*\mu_D(\alpha) \in c_1(Pic(M_{10}))$. By Remark 2.2.1, the equality $\pi^*(\beta) = \pi^*(\phi^*\mu_D(\alpha))$ implies $\beta = \phi^*\mu_D(\alpha)$. If we restrict β to Y we get

$$0 = j^*\beta = j^*\phi^*\mu_D(\alpha) = p^*(\alpha),$$

the last equality coming from Proposition 1.3.7. To conclude, simply notice that $p^* : Pic(T) \longrightarrow Pic(Y)$ is injective, as Y is a $\mathbb{P}^1-$bundle on T, so that we finally get $\alpha = 0$. But this implies $\beta = 0$, and we are done. $\square$

We conclude this section with the following corollary of Proposition 2.3.7.

Corollary 2.3.8. *The morphism $\lambda \circ u$ is injective. Moreover, we have*

$$Pic(\widetilde{M}_{10}) \simeq \pi^* \circ \lambda \circ u(Pic(X)) \oplus \mathbb{Z}[\widetilde{\Sigma}] \oplus \mathbb{Z}[\widetilde{B}].$$

Proof. The injectivity of $\pi^* \circ \lambda \circ u$ follows from point 1 of Theorem 2.1.7 and Proposition 2.3.7. Indeed, suppose $\pi^*(\lambda(u(L))) = \mathscr{O}_{\widetilde{M}_{10}}$ for some $L \in Pic(X)$. This implies

$$0 = c_1(\pi^*(\lambda(u(L)))) = \widetilde{\mu}(c_1(L)))$$

by Proposition 2.3.7. By point 1 of Theorem 2.1.7, the morphism $\widetilde{\mu}$ is injective, so that $\pi^*(\phi^*(\mu_D(c_1(L)))) = 0$ implies $c_1(L) = 0$. But this implies $L = \mathscr{O}_X$ as X is a K3 surface. The injectivity of $\lambda \circ u$ follows from that of $\pi^* \circ \lambda \circ u$.

Now, let $L \in Pic(\widetilde{M}_{10})$, so that $c_1(L) \in H^2(\widetilde{M}_{10}, \mathbb{Z})$. By point 3 of Theorem 2.1.7, there are $\alpha \in H^2(X, \mathbb{Z})$, $n, m \in \mathbb{Z}$ such that

$$c_1(L) = \widetilde{\mu}(\alpha) + nc_1(\widetilde{\Sigma}) + mc_1(\widetilde{B}).$$

We just need to show that $\alpha \in Pic(X)$. Now, $\widetilde{\mu}(\alpha)$ is a line bundle, so that $\phi^*\mu_D(\alpha)$ has to be a line bundle, and $j^*\phi^*\mu_D(\alpha) \in Pic(Y)$. By Proposition 1.3.7 we have $j^*\phi^*\mu_D(\alpha) = p^*(\alpha)$, so that $\alpha \in Pic(X)$, and we are done. $\square$

2.3.3 The Picard group of M_{10}

As an application of the description of the Picard group of $\widetilde{M}_{10}$, we show that the moduli space M_{10} is either $2k-$factorial, for some integer k, or it is not $\mathbb{Q}-$factorial. First of all we show the following:

Lemma 2.3.9. *If there is $n \in \mathbb{Z}$ such that nB is a Cartier divisor, then*

$$c_1(nB) \cdot \gamma' = -\frac{n}{2}.$$

Proof. The rational curve γ' intersects Σ in three different points: indeed, let $x := (x_1, x_2, x_3, x_4) \in S^4_s(X)$ be given by four distinct points. Then $\phi^{-1}(x) \cap \Sigma$ is given by three points, which are $[\mathscr{I}_{x_1,x_2} \oplus \mathscr{I}_{x_3,x_4}]$, $[\mathscr{I}_{x_1,x_3} \oplus \mathscr{I}_{x_2,x_4}]$ and $[\mathscr{I}_{x_1,x_4} \oplus \mathscr{I}_{x_2,x_3}]$.

This implies that $\pi^*(\gamma') = \gamma + l\delta$ for some rational number l, coming from the three points of the intersection of γ' with Σ. Since these points are singular for M_{10}, l need not to be an integer. Indeed, by point 2 of Theorem 2.1.7 we have

$$3 = c_1(\widetilde{\Sigma}) \cdot \gamma = c_1(\widetilde{\Sigma}) \cdot \pi^*(\gamma') - l(c_1(\widetilde{\Sigma}) \cdot \delta) = 2l,$$

since by the projection formula we have $c_1(\widetilde{\Sigma}) \cdot \pi^*(\gamma') = 0$. In conclusion

$$\pi^*(\gamma') = \gamma + \frac{3}{2}\delta.$$

Now, suppose there is $n \in \mathbb{Z}$ such that nB is a Cartier divisor. By Proposition 2.2.1 such a n has to be even. As nB is a Cartier divisor, we have

$$c_1(nB) \cdot \gamma' = nc_1(\widetilde{B}) \cdot \pi^*(\gamma')$$

by the projection formula. In conclusion we get

$$c_1(nB) \cdot \gamma' = nc_1(\widetilde{B}) \cdot \gamma + \frac{3n}{2} c_1(\widetilde{B}) \cdot \delta = -2n + \frac{3n}{2} = -\frac{n}{2},$$

by Theorem 2.1.7, and we are done. □

Remark 2.3.1. We could conclude that B cannot be a Cartier divisor even from Lemma 2.3.9: if B was Cartier, then $B \cdot \gamma' \in \mathbb{Z}$, which is clearly not the case. Moreover, by Lemma 2.3.9 it follows even that M is not locally factorial. Indeed, suppose $n \in \mathbb{Z}$ be such that nB is Cartier. As in the proof of Proposition 2.2.1, there is $m \in \mathbb{Z}$ such that

$$\pi^*(nB) = n\widetilde{B} + m\widetilde{\Sigma}.$$

We can now intersect with γ, getting

$$c_1(\pi^*(nB)) \cdot \gamma = -2n + 3m$$

by point 2 of Theorem 2.1.7, and we need to calculate $c_1(\pi^*(nB)) \cdot \gamma$. Following the proof of Lemma 2.3.9, we have $\gamma = \pi^*(\gamma') - \frac{3}{2}\delta$, so that

$$c_1(\pi^*(nB)) \cdot \gamma = c_1(\pi^*(nB)) \cdot \pi^*(\gamma') - \frac{3}{2} c_1(\pi^*(nB)) \cdot \delta = -\frac{n}{2}$$

by projection formula and Lemma 2.3.9, since δ is contracted by π. We finally get

$$-\frac{n}{2} = -2n + 3m,$$

implying $n = 2m$, so that n must be even. Finally, for the local factoriality we can conclude just as in the proof of Proposition 2.2.1.

Proposition 2.3.10. *Let $A^1(M_{10})$ be the group of Weil divisors of M_{10} modulo linear equivalence. Then*

$$A^1(M_{10}) = \lambda(u(Pic(X))) \oplus \mathbb{Z}[B],$$

and $Pic(M_{10}) \subsetneq A^1(M_{10})$.

Proof. We have the following equalities:

$$A^1(M_{10}) = A^1(M_{10}^s) = Pic(M_{10}^s) = Pic(\pi^{-1}(M_{10}^s)).$$

Indeed, the first equality follows from Lemma A.0.6, as Σ has codimension 2 in M_{10}. The second equality follows from Proposition A.0.4 as M_{10}^s is smooth by Remark C.3.1. The last equality follows from the definition of π.

Now, let us consider the sequence

$$0 \longrightarrow \mathbb{Z} \cdot [\widetilde{\Sigma}] \longrightarrow Pic(\widetilde{M}_{10}) \longrightarrow Pic(\pi^{-1}(M_{10}^s)) \longrightarrow 0.$$

We claim that this sequence is exact: $\mathbb{Z} \cdot [\widetilde{\Sigma}]$ injects into $Pic(\widetilde{M}_{10})$ in the obvious way, the restriction morphism is surjective (see Lemma A.0.6), and clearly the restriction of $\widetilde{\Sigma}$ to $\pi^{-1}(M_{10}^s)$ is trivial.

It remains to show that if a line bundle $L \in Pic(\widetilde{M}_{10})$ has trivial restriction to $\pi^{-1}(M_{10}^s)$, then it is a multiple of $\widetilde{\Sigma}$. As $L \in Pic(\widetilde{M}_{10})$, by Corollary 2.3.8 there are $M \in Pic(X)$ and $n, m \in \mathbb{Z}$ such that

$$L = \pi^*(\lambda(u(L))) + n\widetilde{B} + m\widetilde{\Sigma}.$$

As $L_{|\pi^{-1}(M_{10}^s)} = 0$, we get

$$0 = L_{|\pi^{-1}(M_{10}^s)} = \lambda(u(M)) + nB.$$

In conclusion, we have $nB = -\lambda(u(M))$, so that nB is a line bundle on M_{10}. By Lemma 2.3.9 and Proposition 2.3.6, we have

$$-\frac{n}{2} = nB \cdot \gamma' = -\lambda(u(M)) \cdot \gamma' = 0,$$

so that $n = 0$ and $M = \mathscr{O}_X$ (by Proposition 2.3.7).

In conclusion, we get

$$A^1(M_{10}) = Pic(\pi^{-1}(M_{10}^s)) \simeq Pic(\widetilde{M}_{10})/\mathbb{Z} \cdot [\widetilde{\Sigma}],$$

so that

$$A^1(M_{10}) = \lambda \circ u(Pic(X)) \oplus \mathbb{Z} \cdot [B].$$

Finally, the Picard group of M_{10} is contained in $A^1(M_{10})$ since M_{10} is irreducible (see Proposition A.0.7), and we are done. □

Since, by Corollary 2.3.8, $\lambda \circ u(Pic(X))$ is contained in $Pic(M_{10})$, the only Weil divisors that might not be Cartier are the multiples of B. We have indeed the following:

Proposition 2.3.11. *The moduli space M_{10} is either $2k-$factorial for some positive integer k, or it is not $\mathbb{Q}-$factorial. In particular, the moduli space M_{10} is $2k-$factorial if and only if there is an injection*

$$Pic(X) \oplus \mathbb{Z} \longrightarrow Pic(M_{10}).$$

Proof. By Corollary 2.3.8 and Proposition 2.3.10, there is the following chain of inclusions:

$$\lambda \circ u(Pic(X)) \subseteq Pic(M_{10}) \subseteq \lambda \circ u(Pic(X)) \oplus \mathbb{Z} \cdot [B].$$

We have then only two possibilities: the first one is that $Pic(M_{10}) \simeq Pic(X)$, which implies that neither B nor any of its multiples is in $Pic(M_{10})$. In this case, M_{10} is not $\mathbb{Q}-$factorial.

The second possibility is that $Pic(M_{10}) = \lambda \circ u(Pic(X)) \oplus \mathbb{Z} \cdot \beta$ for some line bundle β. In particular, such a β must be of the form $\mathscr{O}_{M_{10}}(nB)$ for some $n \in \mathbb{Z}$ in view of Proposition 2.3.10. This implies that nB is Cartier, and by Proposition 2.2.1 we have $n = 2k$ for some integer k. In particular, this implies that M_{10} is $2k-$factorial, and we are done. □

2.3.4 $\mathbb{Q}-$factoriality of M_{10}

In this section we prove that M_{10} is $2k-$factorial for some positive integer k. In section 2.3.2 we presented the construction of the morphism

$$\lambda \circ u : Pic(X) \longrightarrow Pic(M_{10}),$$

in order to produce line bundles on M_{10} starting from those on X. Anyway, to produce line bundles on $Pic(M_{10})$ we just need to start from a class $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$. Here is the main choice for such a class.

Lemma 2.3.12. *Let $n \in \mathbb{Z}$. Then $n[\mathscr{O}_X] \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$, so that $\lambda(n[\mathscr{O}_X])$ is a line bundle on M_{10}. Moreover, for any $n \in \mathbb{Z} \setminus \{0\}$, we have $v(n[\mathscr{O}_X]) \neq v(u(L))$ for any $L \in Pic(X)$.*

Proof. It is a trivial fact that $v(n[\mathscr{O}_X]) = (n, 0, n)$. Then, it is sufficient to apply Lemmas 2.3.3 and 2.3.4. □

The natural question is on what kind of line bundle $\lambda([\mathscr{O}_X])$ is. The main ingredient is the following:

Theorem 2.3.13. *Let γ' be as in Proposition 2.1.4. Then*

$$c_1(\lambda([\mathscr{O}_X])) \cdot \gamma' = -1.$$

Proof. Notice that $c_1(\lambda([\mathscr{O}_X])) \cdot \gamma' = c_1(\lambda(\mathscr{O}_X)_{|\gamma'})$. Using the family $\mathscr{H}$ defined in Example 2.3.2, we have

$$deg(\lambda([\mathscr{O}_X])_{|\gamma'}) = deg(q_{Y!}(q_X^*[\mathscr{O}_X] \cdot [\mathscr{H}]))$$

(here we use the same notations as in Example 2.3.2). Using the Grothendieck-Riemann-Roch Theorem, we finally get

$$c_1(\lambda([\mathscr{O}_X])) \cdot \gamma' = q_{Y*}[q_X^*(ch(\mathscr{O}_X)td(X)^{-1}) \cdot ch(\mathscr{H})]_3$$

since the fibers of the morphism q_Y are of dimension 2. Moreover $ch(\mathscr{O}_X)td(X)^{-1}$ is simply $td(X)^{-1}$, so that

$$[q_X^*(ch(\mathscr{O}_X)td(X)^{-1})\cdot ch(\mathscr{H})]_3 = -2q_X^*[y]\cdot ch_1(\mathscr{H}) + ch_3(\mathscr{H}),$$

where $[y]$ is the class of a point in X. In conclusion we need to calculate $ch_i(\mathscr{H})$ for $i = 1, 3$. By Lemma 2.3.2 we have

$$ch_i(\mathscr{H}) = q_X^* ch_i(\mathscr{K}) - ch_i(j_*\mathscr{O}(1)),$$

where

$$ch_i(\mathscr{K}) = ch_i(\mathscr{O}_X^2) - ch_i\left(\bigoplus_{i=1}^{3}\mathbb{C}_{x_i}\right) = (2, 0, -3).$$

These two equalities imply

$$ch_1(\mathscr{H}) = -ch_1(j_*\mathscr{O}(1)), \quad ch_3(\mathscr{H}) = -ch_3(j_*\mathscr{O}(1)).$$

Now, by Grothendieck-Riemann-Roch Theorem, we have

$$ch(j_*\mathscr{O}(1)) = j_*(ch(\mathscr{O}(1)))\cdot q_X^* td(X)^{-1}$$

so that $ch_1(j_*\mathscr{O}(1)) = [j_* ch(\mathscr{O}(1))]_1$ and

$$ch_3(j_*\mathscr{O}(1)) = [j_* ch(\mathscr{O}(1))]_3 - 2q_X^*[y]\cdot[j_* ch(\mathscr{O}(1))]_1.$$

Now, simply remark that $[j_* ch(\mathscr{O}(1))]_i = q_Y^*(ch_{i-2}(\mathscr{O}(1)))$ since the codimension of $\mathbb{P}^1$ in $\mathbb{P}^1\times X$ is 2. In conclusion, we get

$$ch_1(\mathscr{H}) = 0, \quad ch_3(\mathscr{H}) = -q_Y^*[p],$$

where $[p]$ is the class of a point in $\mathbb{P}^1$, since $ch(\mathscr{O}(1)) = (1, 1)$. Finally, this implies that

$$c_1(\lambda([\mathscr{O}_X]))\cdot\gamma' = q_{Y*}(-q_Y^*[p]) = -1,$$

and we are done. □

Here is the main result of this section:

Proposition 2.3.14. *The moduli space M_{10} is $2k-$factorial for some $k\in\mathbb{N}$, and*

$$Pic(M_{10}) = \lambda(u(Pic(X)))\oplus\mathbb{Z}\cdot[2kB].$$

Proof. By Proposition 2.3.11, the moduli space is $2k-$factorial for some $k\in\mathbb{N}$ if there is an injection of $Pic(X)\oplus\mathbb{Z}$ in $Pic(M_{10})$. We claim that the morphism

$$Pic(X)\oplus\mathbb{Z}\longrightarrow e^\perp\cap\{1, h, h^2\}^{\perp\perp}, \qquad (L, n)\mapsto u(L) + n[\mathscr{O}_X]$$

is an isomorphism.

For the injectivity, let $(L,n),(M,m) \in Pic(X) \oplus \mathbb{Z}$ be such that $u(L) + n[\mathscr{O}_X] = u(M) + m[\mathscr{O}_X]$. Then

$$(n, -c_1(L), n) = v(u(L) + n[\mathscr{O}_X]) = v(u(M) + m[\mathscr{O}_X]) = (m, -c_1(M), m).$$

This implies $n = m$ and $c_1(L) = c_1(M)$, so that $L = M$ in $Pic(X)$, and injectivity is shown.

For the surjectivity, consider $\alpha \in e^\perp \cap \{1, h, h^2\}^{\perp\perp}$: by Lemmas 2.3.3 and 2.3.4, we have

$$v(\alpha) = (r, c_1(L), r)$$

for some $r \in \mathbb{Z}$ and $L \in Pic(X)$. We have then $v(\alpha) = v(u(L^{-1}) + r[\mathscr{O}_X])$, so that by Proposition C.4.3 we have $\alpha = u(L^{-1}) + r[\mathscr{O}_X]$, and surjectivity is done.

In conclusion, we get a morphism

$$Pic(X) \oplus \mathbb{Z} \simeq e^\perp \cap \{1, h, h^2\}^{\perp\perp} \xrightarrow{\lambda} Pic(M_{10}).$$

This morphism is injective. Indeed, let $L, M \in Pic(X)$ and $n, m \in \mathbb{Z}$ be such that $\lambda(u(L) + n[\mathscr{O}_X]) = \lambda(u(M) + m[\mathscr{O}_X])$. By Theorem 2.3.13 and Proposition 2.3.6 we have

$$-n = c_1(\lambda(u(L) + n[\mathscr{O}_X])) \cdot \gamma' = c_1(\lambda(u(M) + m[\mathscr{O}_X])) \cdot \gamma' = -m,$$

so that $m = n$ and $\lambda(u(L)) = \lambda(u(M))$. This last equality implies $L = M$ in $Pic(X)$ by Proposition 2.3.7, and we are done. □

2.4 The 2−factoriality of M_{10}

This section is devoted to the proof of the 2−factoriality of M_{10}, that will be proven showing that the Weil divisor $2B$ is Cartier. As a final result, we show that the Donaldson's morphism defines an Hodge isometry between $v^\perp \subseteq \widetilde{H}(X, \mathbb{Z})$ and a sublattice of rank 23 in $H^2(\widetilde{M}_{10}, \mathbb{Z})$.

2.4.1 Line bundles on M_{10}

The Weil divisor B cannot be obtained as $\lambda(u(L))$ for any $L \in Pic(X)$, since it is not a Cartier divisor by Proposition 2.2.1. But we have more: for any integer $n \in \mathbb{Z} \setminus \{0\}$, we have $nB \notin \lambda(u(Pic(X)))$. Indeed, if nB was in the image of $\lambda \circ u$, then there will be an $L \in Pic(X)$ such that $nB = \lambda(u(L))$. By Lemma 2.3.9 and Proposition 2.3.6, intersecting with γ' we get

$$-\frac{n}{2} = c_1(nB) \cdot \gamma' = c_1(\lambda(u(L))) \cdot \gamma' = 0,$$

which is not possible since $n \neq 0$.

Theorem 2.4.1. *There exists a line bundle $L \in Pic(X)$ such that*

$$2B = \lambda([\mathscr{O}_X] + u(L)).$$

In particular, the moduli space M_{10} is 2−factorial.

Proof. By Proposition 2.3.14, there are $L \in Pic(X)$ and $n \in \mathbb{Z}$ such that

$$\lambda([\mathscr{O}_X]) = \lambda(u(L)) + 2nkB.$$

Now, intersecting with γ' we get

$$-1 = c_1(\lambda([\mathscr{O}_X])) \cdot \gamma' = c_1(\lambda(u(L))) \cdot \gamma' + c_1(2nkB) \cdot \gamma' = -nk$$

by Theorem 2.3.13, Proposition 2.3.6 and Lemma 2.3.9. But $n, k \in \mathbb{Z}$ and $k > 0$, so that this implies $n = k = 1$, and we are done. □

Up to now we don't have any information on the line bundle L in the statement of Theorem 2.4.1. We show that it is trivial, and in order to do this, we study in deeper details the line bundle $\lambda([\mathscr{O}_X])$. By definition of $\widetilde{\lambda}$ we have $\widetilde{\lambda}([\mathscr{O}_X]) = det(\mathbb{R}p_{R*}(\mathscr{F}))$, where $\mathscr{F}$ is a universal family on $R \times X$. By Theorem 2.4.1 we have

$$det(\mathbb{R}p_{R*}(\mathscr{F})) = p^*(2B) \otimes \widetilde{\lambda}(u(L)),$$

where $p : R \longrightarrow M_{10}$ is the quotient morphism. Consider the universal quotient module

$$0 \longrightarrow \mathscr{G} \longrightarrow p_X^*\mathscr{H} \xrightarrow{\rho} \mathscr{F} \longrightarrow 0, \tag{2.3}$$

where $\mathscr{F}$ is a universal family on $Quot_X(\mathscr{H}, P) \times X$ and p_X is the projection on X. Moreover, $\mathscr{H} := H^0(X, \mathscr{E}(NH)) \otimes \mathscr{O}_X(-NH)$ for $N \in \mathbb{Z}$ sufficiently big, where $\mathscr{E}$ is any sheaf parameterized by M_{10}. In particular, $\mathscr{H}$ is locally free and such that $H^0(X, \mathscr{H}) = H^1(X, \mathscr{H}) = 0$. In the following, we will write $\mathscr{G}$, $p_X^*\mathscr{H}$ and $\mathscr{F}$ even for their restrictions to $R \times X$. Notice that any $s \in R$ corresponds to an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{H} \xrightarrow{f_{\mathscr{E}}} \mathscr{E} \longrightarrow 0. \tag{2.4}$$

Since $\mathscr{F}$ and $p_X^*\mathscr{H}$ are R−flat, then $\mathscr{G}$ is R−flat and for any $s \in R$ we have

$$\mathscr{G}_s \simeq \ker((p_X^*\mathscr{H})_s \longrightarrow \mathscr{F}_s) = \ker(f_{\mathscr{E}}) = \mathscr{K},$$

where $\mathscr{G}_s$ (resp. $(p_X^*\mathscr{H})_s$, $\mathscr{F}_s$) denotes the restriction of $\mathscr{G}$ (resp. $p_X^*\mathscr{H}$, $\mathscr{F}$) to the fiber of the projection $p_R : R \times X \longrightarrow R$ over the point $s \in R$.

Proposition 2.4.2. *We have the following properties:*

1. *For any $i \in \mathbb{Z}$ the sheaves $\mathbb{R}^i p_{R*}\mathscr{G}$ and $\mathbb{R}^i p_{R*}(p_X^*\mathscr{H})$ are locally free of rank $h^i(X, \mathscr{H})$.*

2. For any $s \in R$ and for any $i \in \mathbb{Z}$, the canonical morphism

$$(\mathbb{R}^i p_{R*}\mathscr{F})_s \longrightarrow H^i(p_R^{-1}(s), \mathscr{F}_s) \simeq H^i(X, \mathscr{E})$$

is an isomorphism, where $\mathscr{E}$ is a sheaf corresponding to the point $s \in R$.

Proof. As the fibers of p_R are of dimension 2, we only need to show the proposition for $i = 0, 1, 2$, since for other values of i we have $\mathbb{R}^i p_{R*}(.) = 0$. We begin with $\mathbb{R}^i p_{R*}(p_X^*\mathscr{H})$. For any $s \in R$ we have $(p_X^*\mathscr{H})_s \simeq \mathscr{H}$, so that $H^i(p_R^{-1}(s), (p_X^*\mathscr{H})_s) \simeq H^i(X, \mathscr{H})$. This implies that the function sending $s \in R$ to $h^i(p_R^{-1}(s), (p_X^*\mathscr{H})_s)$ is constant. By Theorem 1.1.17, the sheaf $\mathbb{R}^i p_{R*}(p_X^*\mathscr{H})$ is then locally free of rank $h^i(X, \mathscr{H})$. In particular, as $h^0(X, \mathscr{H}) = h^1(X, \mathscr{H}) = 0$, the sheaves $\mathbb{R}^0 p_{R*}(p_X^*\mathscr{H})$ and $\mathbb{R}^1 p_{R*}(p_X^*\mathscr{H})$ are trivial.

The next step is to study $\mathbb{R}^i p_{R*}\mathscr{G}$. As $h^0(X, \mathscr{H}) = h^1(X, \mathscr{H}) = 0$, we just need to show that $\mathbb{R}^0 p_{R*}\mathscr{G} = \mathbb{R}^1 p_{R*}\mathscr{G} = 0$ and that $\mathbb{R}^2 p_{R*}\mathscr{G}$ is a vector bundle of rank $h^2(X, \mathscr{H})$. Applying the functor $\mathbb{R}p_{R*}$ to the exact sequence (2.3), we get $\mathbb{R}^0 p_{R*}\mathscr{G} = 0$ and $\mathbb{R}^1 p_{R*}\mathscr{G} \simeq \mathbb{R}^0 p_{R*}\mathscr{F}$, as $\mathbb{R}^0 p_{R*}(p_X^*\mathscr{H}) = \mathbb{R}^1 p_{R*}(p_X^*\mathscr{H}) = 0$. We need to show that $\mathbb{R}^0 p_{R*}(\mathscr{F}) = 0$. Consider any $\mathscr{E}$ parameterized by M_{10}, and consider a corresponding point $s \in R$. Then $\mathscr{F}_s \simeq \mathscr{E}$, and the map sending s to $H^0(X, \mathscr{F}_s)$ is constant and trivial by Lemma 2.2.3. The canonical morphism

$$(\mathbb{R}^0 p_{R*}(\mathscr{F}))_s \longrightarrow H^0(X, \mathscr{F}_s) = 0$$

is then an isomorphism by Theorem 1.1.17, so that $\mathbb{R}^0 p_{R*}(\mathscr{F}) = 0$. It the remains to show that $\mathbb{R}^2 p_{R*}\mathscr{G}$ is a vector bundle of rank $h^2(X, \mathscr{H})$. To do so, consider a point $s \in R$ and the associated exact sequence (2.4). The long exact sequence induced by this and Lemma 2.2.3 imply that $h^2(X, \mathscr{G}_s) = h^2(X, \mathscr{H})$. Then the function sending $s \in R$ to $h^2(X, \mathscr{G}_s)$ is constant. By Theorem 1.1.17, this implies that $\mathbb{R}^2 p_{R*}\mathscr{G}$ is a vector bundle of rank $h^2(X, \mathscr{H})$, and for any $s \in R$ the canonical morphism $(\mathbb{R}^2 p_{R*}\mathscr{G})_s \longrightarrow H^2(X, \mathscr{G}_s) \simeq H^2(X, \mathscr{K})$ is an isomorphism.

Finally, we need to study $\mathbb{R}^i p_{R*}\mathscr{F}$. We have already shown that for any $s \in R$ the canonical morphism

$$(\mathbb{R}^0 p_{R*}\mathscr{F})_s \longrightarrow H^0(X, \mathscr{F}_s)$$

is an isomorphism, and we need to show the same property for $\mathbb{R}^1 p_{R*}\mathscr{F}$ and $\mathbb{R}^2 p_{R*}\mathscr{F}$. As $\mathbb{R}^3 p_{R*}\mathscr{F} = 0$, by Theorem 1.1.17 the canonical morphism

$$(\mathbb{R}^2 p_{R*}\mathscr{F})_s \longrightarrow H^2(X, \mathscr{E})$$

is an isomorphism. Moreover, let $\xi : \mathbb{R}^1 p_{R*}\mathscr{F} \longrightarrow \mathbb{R}^2 p_{R*}\mathscr{G}$ be the morphism induced by the exact sequence (2.3). Since $\mathbb{R}^1 p_{R*}(p_X^*\mathscr{H}) = 0$ by the first part of the proof, ξ is injective. In particular, for any $s \in R$ the morphism ξ_s is injective, so that

$$(\mathbb{R}^1 p_{R*}(\mathscr{F}))_s \simeq \ker((\mathbb{R}^2 p_{R*}(\mathscr{G}))_s \xrightarrow{\delta} (\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H}))_s).$$

But $(\mathbb{R}^2 p_{R*}\mathscr{G})_s \simeq H^2(X,\mathscr{K})$, $(\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H}))_s \simeq H^2(X,\mathscr{H})$ and the morphism δ is simply the morphism $H^2(X,\mathscr{K}) \longrightarrow H^2(X,\mathscr{H})$ induced by the exact sequence (2.4), by the previous part of the proof. Since $H^1(X,\mathscr{H}) = 0$, we have $\ker(\delta) \simeq H^1(X,\mathscr{E})$, and we are done. □

Corollary 2.4.3. *We have the following exact sequence*

$$0 \longrightarrow \mathbb{R}^1 p_{R*}(\mathscr{F}) \longrightarrow \mathbb{R}^2 p_{R*}(\mathscr{G}) \xrightarrow{\beta} \mathbb{R}^2 p_{R*}(p_X^*\mathscr{H}) \longrightarrow \mathbb{R}^2 p_{R*}(\mathscr{F}) \longrightarrow 0.$$

In particular, $det(\mathbb{R}p_{R*}\mathscr{F}) \simeq det(\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H})) \otimes det(\mathbb{R}^2 p_{R*}\mathscr{G})^{-1}$.

Proof. Applying the functor $\mathbb{R}p_{R*}$ to the exact sequence (2.3), by point 1 of Proposition 2.4.2 we get the exact sequence in the statement. By this exact sequence we get

$$det(\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H})) \otimes det(\mathbb{R}^2 p_{R*}\mathscr{G})^{-1} \simeq det(\mathbb{R}^2 p_{R*}\mathscr{F}) \otimes det(\mathbb{R}^1 p_{R*}\mathscr{F})^{-1}.$$

As $\mathbb{R}^0 p_{R*}\mathscr{F} = 0$ by point 2 of Proposition 2.4.2, this implies the statement. □

We are finally able to show the following

Proposition 2.4.4. *The line bundle $\lambda([\mathscr{O}_X])$ is isomorphic to $2B$.*

Proof. Consider $det(\beta) : det(\mathbb{R}^2 p_{R*}\mathscr{G}) \longrightarrow det(\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H}))$, that gives a section s of $det(\mathbb{R}p_{R*}(\mathscr{F}))$ by Corollary 2.4.3, whose zero locus is given by the set where $det(\beta)$ is not an isomorphism. By Lemma 2.2.3, Proposition 2.2.6 and point 2 of Proposition 2.4.2 this locus is exactly $p^{-1}(B)$, and we are done. □

2.4.2 Description of $H^2(\widetilde{M}_{10},\mathbb{Z})$

As a consequence of the description we have given for the construction of line bundles on M_{10}, we have the following theorem, which is a generalization of Theorem D.3.9, and that was pointed out to me by Rapagnetta.

Theorem 2.4.5. *Let $v = (2,0,-2) \in \widetilde{H}(X,\mathbb{Z})$. There is a morphism of Hodge structures*

$$f : v^{\perp} \longrightarrow H^2(\widetilde{M}_{10},\mathbb{Z}),$$

which is an isometry between $v^{\perp}$, viewed as a sublattice of the Mukai lattice $\widetilde{H}(X,\mathbb{Z})$, and its image in $H^2(\widetilde{M}_{10},\mathbb{Z})$, being a lattice with the Beauville-Bogomolov form q.

Proof. It is an easy calculation to show that a Mukai vector w is orthogonal to v if and only if it is of the form

$$w = (r,c,r) \in \widetilde{H}(X,\mathbb{Z}),$$

for $r \in \mathbb{Z}$ and $c \in H^2(X,\mathbb{Z})$, so that there is an isomorphism

$$v^\perp \longrightarrow H^2(X,\mathbb{Z}) \oplus \mathbb{Z}, \quad w \mapsto (c,r).$$

Let

$$f : v^\perp \longrightarrow H^2(\widetilde{M}_{10},\mathbb{Z}), \qquad f((r,c,r)) := \widetilde{\mu}(c) + 2rc_1(\widetilde{B}) + rc_1(\widetilde{\Sigma}).$$

The morphism f is clearly an isomorphism of Hodge structures on its image, since the Donaldson's morphism is an injective morphism of Hodge structures. Moreover, it is an isometry: indeed for any two classes $c,d \in H^2(X,\mathbb{Z})$ and for any two $r,s \in \mathbb{Z}$ we have:

$$q(f(r,c,r), f(s,d,s)) = cd - 2rs$$

by point 4 of Theorem 2.1.7, and

$$((r,c,r),(s,d,s)) = cd - 2rs$$

by definition of the Mukai pairing. □

Using Proposition 2.4.4 we can give a geometric interpretation of the definition of f. It is the Donaldson's morphism on $H^2(X,\mathbb{Z})$, so that by Proposition 2.3.7 it is the Le Potier's morphism on $Pic(X)$. It seems then natural to define f as the Le Potier's morphism on the remaining part of $v^\perp$. Indeed, by Lemma 2.3.12 we have $(r,0,r) = v(r[\mathscr{O}_X])$, and by Proposition 2.4.4 $\lambda(r[\mathscr{O}_X]) = 2rB$. As $\pi^*(2rB) = 2r\widetilde{B} + n\widetilde{\Sigma}$ for some integer n, by point 2 of Theorem 2.1.7 we get

$$0 = \pi^*(2rB) \cdot \delta = 2r - 2n,$$

since δ is contracted by π. In conclusion, we get $n = r$ and

$$\pi^*\lambda(r[\mathscr{O}_X]) = 2r\widetilde{B} + r\widetilde{\Sigma}.$$

In particular, we see that

$$f(r,0,r) = c_1(\pi^*\lambda(r[\mathscr{O}_X])).$$

Chapter 3

The $6-$dimensional O'Grady's example $\widetilde{M}_6$

This last chapter is devoted to the study of the $6-$dimensional O'Grady example, which provides the last known example of irreducible symplectic manifold which is not deformation of an Hilbert scheme of points (on a K3 surface) or of a generalized Kummer variety (over an abelian surface).

The idea behind the construction of this $6-$dimensional variety mixes those of the constructions of $\widetilde{M}_{10}$ and of the generalized Kummer varieties. Namely, as seen in Chapter 3, the moduli spaces of semistable sheaves over an abelian surface whose Mukai vector is primitive give rise to deformations of generalized Kummer varieties: it is sufficient to consider a fiber of the Albanese map. The natural idea, already behind the construction of $\widetilde{M}_{10}$, is to consider the moduli space of semistable sheaves with non-primitive Mukai vector, and to look at the fibers of the Albanese map. Since the variety obtained in this way is expected to be singular, we need to understand if it admits a symplectic resolution. As seen in section 3.3.2, a symplectic resolution exists if and only if the Mukai vector is of the form $2w$, for w a primitive Mukai vector such that $(w, w) = 2$.

In [OG3], O'Grady chooses the Mukai vector as $v = (2, 0, -2)$, as he does in the $10-$dimensional example, and considers the $6-$dimensional fiber M_6 of the Albanese map. Then, he shows the existence of a symplectic resolution $\widetilde{M}_6$ of M_6, which is shown to be an irreducible symplectic manifold whose second Betti number is 8: since the other known irreducible symplectic manifolds in dimension 6 have second Betti number 23 and 7, we get a new example.

The goal of this chapter is to show that M_6 is a $2-$factorial scheme: this will be done using the same ideas as in the previous chapter to prove the $2-$factoriality of M_{10}, with some important differences. Namely, one shows the existence of a Weil divisor D on M_6 which is not Cartier, but such that $2D = 0$. This does not exist on M_{10} as the exceptional divisor $\widetilde{\Sigma}$ on $\widetilde{M}_{10}$ is not divisible by 2, as it is in the case of $\widetilde{M}_6$. Once this shown, one proceeds using Le Potier's morphism, as in the 10−dimensional case, to show the 2−factoriality.

3.1 Generalities on M_6

In this section we recall the construction of M_6 and of $\widetilde{M}_6$, and we resume the basic properties we need for the proof of the 2−factoriality. In the following, let C be a smooth projective complex curve of degree 2, and let $J := Pic^0(C)$ be its jacobian variety. Since the degree of C is 2, the variety J is an abelian surface. Moreover, we suppose that there is an ample line bundle H on J such that $NS(J) = \mathbb{Z} \cdot c_1(H)$ and $c_1^2(H) = 2$. Finally, let $\widehat{J} := Pic^0(J)$ be the abelian surface dual to J. Let $p_0 \in C$, and let

$$i : C \longrightarrow J, \qquad i(p) := [p - p_0]$$

be the Abel-Jacobi map, which is injective. Moreover, let $\Theta := i(C)$, which is an irreducible divisor on J called the *theta divisor*. For any $\alpha \in J$ we can even define $\Theta_\alpha := \Theta + \alpha$, the divisor obtained as translation of Θ by the point α. The composition of i with the translation by α is denoted i_α.

Let $v \in \widetilde{H}(J, \mathbb{Z})$ be the Mukai vector $v := (2, 0, -2)$, and let M_v be the moduli space of H−semistable sheaves on J whose Mukai vector is v. The moduli space M_v parameterizes rank 2 sheaves with trivial first Chern class and second Chern class equal to 2 (since J is an abelian surface, so that $td(J) = 1$). With the usual notations, let M_v^s the the open subset of M_v parameterizing H−stable sheaves, and M_v^{lf} be the open subset of M_v parameterizing H−semistable locally free sheaves. By Theorem C.3.1 M_v^s is smooth, and by Theorem C.5.1 it carries a symplectic structure coming from the one we have on J. Moreover, M_v^s has dimension 10. The first result is the analogue of Proposition 2.1.1, on the singularities of M_v:

Proposition 3.1.1. *Let Σ_v be the singular locus of M_v. Then Σ_v parameterizes sheaves of the form*

$$(\mathscr{I}_{x_1} \otimes L_1) \oplus (\mathscr{I}_{x_2} \otimes L_2),$$

where $x_1, x_2 \in J$ and $L_1, L_2 \in \widehat{J}$. In particular, we have $\Sigma_v \simeq S^2(J \times \widehat{J})$, and $codim_{M_v}(\Sigma_v) = 2$.

Proof. See Lemma 2.1.2 in [OG3]. □

Now, let $\overline{B}_v$ be the closed subset of M_v parameterizing non-locally free sheaves. As a corollary of Proposition 3.1.1, we have $\Sigma_v \subseteq \overline{B}_v$. Moreover, let

$$\Omega_v := \{[(\mathscr{I}_{x_1} \otimes L_1) \oplus (\mathscr{I}_{x_2} \otimes L_2)] \in \Sigma_v \,|\, x_1 = x_2,\ L_1 \simeq L_2\} \subseteq \Sigma_v.$$

Consider the blow-up of M_v along Σ_v with reduced scheme structure, i. e.

$$\pi_v : \widetilde{M}_v \longrightarrow M_v,$$

which is a symplectic resolution of M_v. Let then $\widetilde{\Sigma}_v$ be the exceptional divisor, and let $\widetilde{B}_v$ be the proper transform of $\overline{B}_v$. Moreover, let $\widetilde{\Omega}_v := \pi_v^{-1}(\Omega_v)$. One of the main technical tools O'Grady uses in [OG3] is a moduli-theoretic description of $\widetilde{M}_v \setminus \widetilde{\Omega}_v$: this can be described as the set of equivalence classes of simple semistable sheaves on J whose Mukai vector is v, where the equivalence relation is described as follows. Let $\mathscr{E}$ be a simple semistable sheaf on J with Mukai vector v. If $\mathscr{E}$ is strictly semistable, then there is a short exact sequence

$$0 \longrightarrow \mathscr{I}_{x_1} \otimes L_1 \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_{x_2} \otimes L_2 \longrightarrow 0,$$

defining the extension class $e_{\mathscr{E}} \in Ext^1(\mathscr{I}_{x_2} \otimes L_2, \mathscr{I}_{x_1} \otimes L_1)$. Since $\mathscr{E}$ is simple, this extension class is not trivial, and is well-defined up to $\mathbb{C}^*$. Moreover, it is an easy calculation to show that since $\mathscr{E}$ is simple we must have $L_1 \not\simeq L_2$ or $x_1 \neq x_2$ and that $Ext^1(\mathscr{I}_1 \otimes L_1, \mathscr{I}_2 \otimes L_2)$ has dimension 2. By Serre's duality, the Yoneda coupling

$$Ext^1(\mathscr{I}_{x_2} \otimes L_2, \mathscr{I}_{x_1} \otimes L_1) \times Ext^1(\mathscr{I}_{x_1} \otimes L_1, \mathscr{I}_{x_2} \otimes L_2) \longrightarrow \mathbb{C}$$

is a perfect pairing, so that one can define $e_{\mathscr{E}}^{\perp}$, a generator for the one-dimensional subspace of $Ext^1(\mathscr{I}_{x_1} \otimes L_1, \mathscr{I}_{x_2} \otimes L_2)$ annihilating $e_{\mathscr{E}}$.

Definition 3.1.1. Let $\mathscr{E}$ and $\mathscr{F}$ be two simple semistable sheaves on J with Mukai vector v. Then $\mathscr{E}$ and $\mathscr{F}$ are called $\widetilde{S}-$equivalent if and only if one of the two following conditions is satisfied:

1. $\mathscr{E}$ and $\mathscr{F}$ are stable and S-equivalent;
2. $\mathscr{E}$ and $\mathscr{F}$ are strictly semistable and $\mathscr{E}$ is isomorphic either to $\mathscr{F}$ or to the extension given by $e_{\mathscr{F}}^{\perp}$.

The following result is Proposition 2.2.10 in [OG3]:

Proposition 3.1.2. *Let E_v be the set of $\widetilde{S}-$equivalence classes of torsion-free simple $H-$semistable sheaves on J with Mukai vector v. Then there is a natural bijection $\widetilde{M}_v \setminus \widetilde{\Omega}_v \longrightarrow E_v$.*

Another important tool is the following proposition:

Proposition 3.1.3. *Let $\mathscr{E}$ be any simple semistable torsion-free sheaf on J whose $\widetilde{S}-$equivalence class defines a point in $\widetilde{B}_v$, and let $E := \mathscr{E}^{**}$. Then there is a short exact sequence*

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0,$$

where $L_1, L_2 \in \widehat{J}$, and the sequence is split if and only if $L_1 \not\simeq L_2$. Moreover, the length of $Sing(\mathscr{E})$ is 2.

Proof. See Lemma 4.3.3 in [OG3]. □

We are finally able to define the main object of this chapter: let

$$a_v : M_v \longrightarrow J \times \widehat{J}, \qquad a_v(\mathscr{E}) := \left(\sum c_2(\mathscr{E}), det(\mathscr{E}) \right),$$

and let

$$M_6 := a_v^{-1}(0, \mathscr{O}_J), \qquad \widetilde{M}_6 := \pi_v^{-1}(M_6).$$

Let $\pi := \pi_{v|\widetilde{M}_6}$, and let $\Sigma := \Sigma_v \cap M_6$. In particular, π is the blow up of M_6 along Σ with reduced scheme structure. Finally, let $\overline{B} := \overline{B}_v \cap M_6$, $\widetilde{\Sigma} := \pi^{-1}(\Sigma)$ (the exceptional divisor of π) and $\widetilde{B} := \widetilde{B}_v \cap \widetilde{M}_6$ (the proper transform of $\overline{B}$ under π). The main result is the following

Theorem 3.1.4. *(**O'Grady**, '01). The variety $\widetilde{M}_6$ is an irreducible symplectic manifold of dimension 6 and second Betti number $b_2(\widetilde{M}_6) = 8$.*

Proof. The proof of this theorem is contained in [OG3]. □

In particular, $\widetilde{M}_6$ is a new example of irreducible symplectic manifold of dimension 6, the others being deformations of $Hilb^3(X)$ for a K3 surface X (whose second Betti number is 23) and $K^3(A)$ for an abelian surface A (whose second Betti number is 7). As seen in the previous chapter, it is important to determine the existence of $\mathbb{P}^1-$fibrations on $\widetilde{M}_6$. In order to state the main result, we need the following definitions: let $J^0 := J \setminus J[2]$, where $J[2]$ is the set of the $2-$torsion points on J. For any $\alpha \in J$ we can define the following

$$Z_\alpha := \{[\mathscr{E}] \in M_6 \,|\, \mathscr{E}_{|\Theta_\alpha} \text{ is not locally free semistable}\}.$$

Moreover, let $\widetilde{Z}_\alpha := \pi^{-1}(Z_\alpha)$, $\Sigma_\alpha := \Sigma \cap Z_\alpha$, $\widetilde{\Sigma}_\alpha := \pi^{-1}(\Sigma_\alpha)$ and $\widetilde{B}_\alpha := \widetilde{B} \cap \widetilde{Z}_\alpha$. For any $\alpha \in J^0$, we can consider the following map

$$C \longrightarrow J \times J, \qquad p \mapsto (i_\alpha(p), -i_\alpha(p)),$$

whose image is denoted C_α. Finally, let $M(2)$ be the moduli space of semistable locally free sheaves on J with rank 2, trivial first and second Chern classes and $hom(E, E) = 2$. It is an easy fact to see that there is a bijective correspondence between $M(2)$ and $Hilb^2(\widehat{J})$ (see section 5.1 in [OG3]). Under this correspondence, we can identify $K^2(\widehat{J})$, the Kummer surface of $\widehat{J}$, with the fiber over $\mathscr{O}_J$ of the morphism

$$det : M(2) \longrightarrow \widehat{J}$$

sending any $E \in M(2)$ to its determinant. Here is the main result:

Proposition 3.1.5. *Let $\alpha \in J^0$.*

1. *Let $f := \pi_{|\widetilde{\Sigma}_\alpha}$. Then*
$$f : \widetilde{\Sigma}_\alpha \longrightarrow \Sigma_\alpha$$
is a $\mathbb{P}^1-$fibration whose generic fiber is denoted δ. In particular, $\widetilde{\Sigma}$ is an irreducible divisor on $\widetilde{M}_6$.

2. *Let*
$$g : \widetilde{B}_\alpha \longrightarrow C_\alpha \times K^2(\widehat{J}), \qquad g([\mathscr{E}]) := ((x, -x), E),$$
where $[\mathscr{E}]$ is the $\widetilde{S}-$equivalence class of $\mathscr{E}$, whose bidual is E and whose singular locus is given by x and $-x$ (see Proposition 3.1.3). Then g is a $\mathbb{P}^1-$fibration whose generic fiber is denoted γ. In particular, $\widetilde{B}$ is an irreducible divisor on $\widetilde{M}_6$.

Proof. See section 4.1 in [OG3] for the first item. The second item is contained in section 5.1 of [OG3]. □

Using this proposition, we can finally state the following:

Theorem 3.1.6. ***(Rapagnetta**, '06). Let μ_D be the Donaldson's morphism, and let $\phi : M_6 \longrightarrow M_6^{\mu-ss}$.*

1. *The morphism $\widetilde{\mu} := \pi^* \circ \phi^* \circ \mu_D : H^2(J, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z})$ is injective.*

2. *There is a line bundle $A \in Pic(\widetilde{M}_6)$ such that $c_1(\widetilde{\Sigma}) = 2c_1(A)$.*

3. *We have the following equalities:*
$$c_1(A) \cdot \delta = -1, \qquad c_1(\widetilde{B}) \cdot \delta = 1,$$
$$c_1(A) \cdot \gamma = 1, \qquad c_1(\widetilde{B}) \cdot \gamma = -2.$$

4. *The second integral cohomology of $\widetilde{M}_6$ is*
$$H^2(\widetilde{M}_6, \mathbb{Z}) = \widetilde{\mu}(H^2(J, \mathbb{Z})) \oplus \mathbb{Z} \cdot c_1(A) \oplus \mathbb{Z} \cdot c_1(\widetilde{B}).$$

5. *Let q be the Beauville-Bogomolov form of $\widetilde{M}_6$. Then, q equals the intersection form of J on $H^2(J, \mathbb{Z})$, and we have*
$$q(c_1(A), c_1(A)) = -2, \qquad q(c_1(A), c_1(\widetilde{B})) = 2,$$
$$q(c_1(\widetilde{B}), c_1(A)) = 2, \qquad q(c_1(\widetilde{B}), c_1(\widetilde{B})) = -4.$$
Finally, we have $q(\widetilde{\mu}(\alpha), c_1(\widetilde{B})) = q(\widetilde{\mu}(\alpha), c_1(A)) = 0$.

Proof. Item 1 is proven by O'Grady in [OG3], Proposition 7.3.3. The proof of the other points is contained in [Rap2]: more precisely, Item 2 is Theorem 3.3.1, items 3 and 4 are contained in Theorem 3.4.1 and item 5 is Theorem 3.5.1. □

This theorem is the analogue of Theorem 2.1.7, and will be the starting point of our investigation. Notice that in this case, as it happens for Hilbert schemes and generalized Kummer varieties, the exceptional divisor is divisible by 2. This is not the case for $\widetilde{M}_{10}$, and this is one of the main differences between these two varieties.

3.2 The local factoriality of M_6

As a consequence of Rapagnetta's Theorem, we have the following property:

Lemma 3.2.1. *There is a non-trivial irreducible Weil divisor $D \in A^1(M_6)$ such that $2D = 0$. If $\widetilde{D}$ is the proper transform of D under π, there is $m \in \mathbb{Z}$ such that*

$$A = \widetilde{D} + m\widetilde{\Sigma}$$

in the group $Div(\widetilde{M}_6)$ of Weil divisors of $\widetilde{M}_6$.

Proof. Since π is the blow-up of M_6 along Σ, and since Σ has codimension 2 in M_6, we have $A^1(M_6) \simeq Pic(\pi^{-1}(M_6^s))$ (analogously to what has been proven in Proposition 2.3.10). The restriction of A to $\pi^{-1}(M_6^s)$ defines then an irreducible Weil divisor $D \in A^1(M_6)$. In particular, by point 2 in Theorem 3.1.6 we have

$$2D = 2A_{|\pi^{-1}(M_6^s)} = \widetilde{\Sigma}_{|\pi^{-1}(M_6^s)} = 0.$$

Now, the Weil divisor $\widetilde{\Sigma}$ is a prime divisor, so it is a generator for the group $Div(\widetilde{M}_6)$. Since A is a line bundle on $\widetilde{M}_6$, it defines an element of $Div(\widetilde{M}_6)$, so that there are $m, m_1, ..., m_n \in \mathbb{Z}$ and prime divisors $D_1, ..., D_n$ such that

$$A = m\widetilde{\Sigma} + \sum_{i=1}^{n} m_i D_i.$$

As $A_{|\pi^{-1}(M_6^s)} = \sum_{i=1}^n m_i D_{i|\pi^{-1}(M_6^s)}$, we have $\sum_{i=1}^n m_i D_i = \widetilde{D}$, and we are done. It remains to show that D is not trivial: if $D = 0$, then $\widetilde{D} = 0$, so that $A = m\widetilde{\Sigma} = 2mA$ (by point 2 of Theorem 3.1.6). This implies that A is torsion in $Pic(\widetilde{M}_6)$, so that $c_1(A)$ is torsion in $H^2(\widetilde{M}_6, \mathbb{Z})$. By point 4 of Theorem 3.1.6 this is not possible, and we are done. □

Moreover, we have the following proposition:

Proposition 3.2.2. *The Weil divisor D is not Cartier, and M_6 is not locally factorial.*

Proof. If D was a Cartier divisor, then $\pi^*(D) = \widetilde{D} + kA$, for some $k \in \mathbb{Z}$. By Lemma 3.2.1 we have $\widetilde{D} = A - m\widetilde{\Sigma} = (1 - 2m)A$, so that

$$\pi^*(D) = (1 - 2m + k)A.$$

The integer $1-2m+k$ is odd: indeed, if there was $n \in \mathbb{Z}$ such that $2n = 1-2m+k$, then $\pi^*(D) = n\widetilde{\Sigma}$ and we would have

$$D = \pi^*(D)_{|\pi^{-1}(M_6^s)} = n\widetilde{\Sigma}_{|\pi^{-1}(M_6^s)} = 0,$$

which is not possible since D is non-trivial.

In particular, $1-2m+k \neq 0$. By point 3 of Theorem 3.1.6 and the fact that δ is contracted by π, one gets

$$0 = c_1(\pi^*(D)) \cdot \delta = (1-2m+k)c_1(A) \cdot \delta = 2m-k-1.$$

As $2m-k-1 \neq 0$, we get a contradiction, and D cannot be a Cartier divisor. Finally, this clearly implies that M_6 cannot be locally factorial. □

Remark 3.2.1. As a consequence of the previous proposition, we can show that $Pic(M_6)$ has no torsion. Indeed, suppose there is $L \in Pic(M_6)$ which is torsion of period t, and let $\widetilde{L}$ be its proper transform under π. Then $\pi^*(L) = \widetilde{L} + kA$ for some $k \in \mathbb{Z}$, and $t(\widetilde{L} + kA) = 0$. As $Pic(\widetilde{M}_6)$ has no torsion by point 4 of Theorem 3.1.6, we get $\widetilde{L} = -kA$, and

$$L = \widetilde{L}_{|\pi^{-1}(M_6^s)} = -kA_{|\pi^{-1}(M_6^s)} = -kD.$$

As $L \in Pic(M_6)$, we get $kD \in Pic(M_6)$, so that k has to be even by Proposition 3.2.2 and Lemma 3.2.1. In conclusion $L = 0$, and we are done.

The same proof even shows that $\pi^* : Pic(M_6) \longrightarrow Pic(\widetilde{M}_6)$ is injective. As in Remark 2.2.1, from this one can deduce that $c_1 : Pic(M_6) \longrightarrow H^2(M_6, \mathbb{Z})$ is injective.

3.3 The Picard group of M_6

In this section we calculate the Picard groups of $\widetilde{M}_6$ and of M_6, using the same arguments as in the calculation of the Picard groups of $\widetilde{M}_{10}$ and of M_{10}. Notice that it follows easily by Theorem 3.1.6 that

$$Pic(\widetilde{M}_6) = \widetilde{\mu}(NS(J)) \oplus \mathbb{Z} \cdot [A] \oplus \mathbb{Z} \cdot [\widetilde{B}].$$

Anyway, as we want to produce line bundles on M_6 it seems useful to understand the line bundles produced by the Le Potier's morphism starting from the topological Grothendieck group $K_{top}(J)$ of J.

As in section 2.3, we need some flat families describing subvarieties of $\widetilde{M}_6$: constructions of such families are provided in the first subsection. Then we proceed with the calculations of the two Picard groups and with the study of the Le Potier's morphism.

3.3.1 Construction of flat families

In the following, we will use the next lemma:

Lemma 3.3.1. *Let $\alpha, x \in J^0$ and let $E \in M(2)$ define a point $K^2(\widehat{J})$. The fiber of g over $((x,-x),E) \in C_\alpha \times K^2(\widehat{J})$ is in bijective correspondence with the set of $\widetilde{S}-$equivalence classes of sheaves $\mathscr{E}$ fitting into an exact sequence of the form*

$$0 \longrightarrow \mathscr{E} \longrightarrow E \longrightarrow \mathbb{C}_x \oplus \mathbb{C}_{-x} \longrightarrow 0.$$

Proof. See the proof of the Claim in section 5.1 of [OG3]. □

Using this, we are abe to describe two examples of flat families we will need in what follows.

Example 3.3.1. Let $E \in M(2)$ be a rank 2 vector bundle on J with trivial first and second Chern classes and such that $hom(E,E) = 2$. This vector bundle defines a point in $Hilb^2(\widehat{J})$, so that we can choose it to be in the smooth part of $S^2(\widehat{J})$: this means that E fits into an exact sequence of the form

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

with $L_1, L_2 \in \widehat{J}$ such that $L_1 \not\simeq L_2$ (by Proposition 3.1.3). Moreover, let $y \in J^0$ and fix a surjective morphism

$$\varphi : E \longrightarrow \mathbb{C}_y.$$

Let $\mathscr{K} := \ker(\varphi)$: by Proposition 3.1.3, any sheaf defining a point in $\widetilde{B}_v$ is the kernel of a surjective morphism from $\mathscr{K}$ to $\mathbb{C}_x$ for some point $x \in J$. Let $p_1, p_2 : J \times J \longrightarrow J$ be the two projections. As in Example 2.3.1, by Theorem 1.1.17 the sheaf $p_{1*}\mathscr{H}om(p_2^*\mathscr{K}, \mathscr{O}_\Delta)$ is a vector bundle of rank 2, and for any $x \in J$ the canonical morphism

$$(p_{1*}\mathscr{H}om(p_2^*\mathscr{K}, \mathscr{O}_\Delta))_x \longrightarrow Hom(\mathscr{K}, \mathbb{C}_x)$$

is an isomorphism. Let $Y := \mathbb{P}(p_{1*}\mathscr{H}om(p_2^*\mathscr{K}, \mathscr{O}_\Delta)) \xrightarrow{p} J$. Using the same construction and notations as in section 3.1, there is a tautological morphism

$$\widetilde{f} : q_J^*\mathscr{K} \otimes q_Y^*\mathscr{T} \longrightarrow (p \times id_J)^*\mathscr{O}_\Delta,$$

whose kernel is denoted $\mathscr{H}$.

Lemma 3.3.2. *Let $\mathscr{E}$ be a sheaf defining a point in $\widetilde{B}_v$ whose bidual is E and whose singular locus is given by $x, y \in J$. Let $f_\mathscr{E} : \mathscr{K} \longrightarrow \mathbb{C}_x$ be the surjective morphism whose kernel is $\mathscr{E}$. Then $f_\mathscr{E}$ defines a point $[f_\mathscr{E}] \in Y$, and $\mathscr{H}_{[f_\mathscr{E}]} \simeq \mathscr{E}$. Moreover, $\mathscr{H}$ is a $Y-$flat family and the morphism $\widetilde{f}$ is surjective.*

Proof. The proof is the same as the one of Lemma 2.3.1. □

Example 3.3.2. Let E be as in the previous example, with the further property that $det(E) \simeq \mathscr{O}_J$: this means that E defines a point in $K^2(\widehat{J})$. Let

$$\varphi : E \longrightarrow \mathbb{C}_{-x}$$

be a surjective morphism, and let $\mathscr{K}$ be its kernel. In this example, let

$$Y := \mathbb{P}(p_{x*}\mathscr{H}om(p_J^*\mathscr{K}, \mathbb{C}_x)) \xrightarrow{p} \{x\},$$

where $p_J : \{x\} \times J \longrightarrow J$ and $p_x : \{x\} \times J \longrightarrow \{x\}$ are the two projections. Then, Y is a $\mathbb{P}^1$, and its points correspond to surjective morphisms from $\mathscr{K}$ to $\mathbb{C}_x$, i. e. points in $g^{-1}((x, -x), E)$ by Lemma 3.3.1. As in section 4.2.1, we get a tautological morphism

$$\widetilde{f} : q_J^*\mathscr{K} \longrightarrow j_*\mathscr{O}_{\mathbb{P}^1}(1),$$

where $j : \mathbb{P}^1 \times \{x\} \longrightarrow \mathbb{P}^1 \times J$ is the immersion. Let $\mathscr{H} := \ker(\widetilde{f})$.

Lemma 3.3.3. *Let $\mathscr{E}$ be a sheaf defining a point in $\widetilde{B}$ whose bidual is E and whose singular locus is given by $x, -x \in J$. Let $f_{\mathscr{E}} : \mathscr{K} \longrightarrow \mathbb{C}_x$ be the surjective morphism whose kernel is $\mathscr{E}$. Then $f_{\mathscr{E}}$ defines a point $[f_{\mathscr{E}}] \in Y$, and $\mathscr{H}_{[f_{\mathscr{E}}]} \simeq \mathscr{E}$. Moreover, $\mathscr{H}$ is a $Y-$flat family and the morphism $\widetilde{f}$ is surjective.*

Proof. Again, the proof is the same as that of Lemma 2.3.1. □

3.3.2 The Picard group of $\widetilde{M}_6$

Using the flat families constructed in the previous section, we are able to calculate the Picard group of $\widetilde{M}_6$. The key point is the study of the Le Potier's morphism, as we did in the 10−dimensional case. The first result we need is the following:

Lemma 3.3.4. *Let $e := [\mathscr{E}] \in K_{top}(J)$ be the class of a sheaf $\mathscr{E}$ parameterized by M_6, and let $h := [H] \in K_{top}(J)$. Let $\alpha \in K_{top}(J)$. Then $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$ if and only if $c_1(\alpha) \in c_1(H)^{\perp\perp}$ and $ch_2(\alpha) = rk(\alpha)\eta_J \in H^4(J, \mathbb{Z})$, where η_J is the fundamental class of J.*

Proof. The proof works exactly as those of Lemmas 2.3.3 and 2.3.4. □

This lemma allows us to show the following: let $p \in J$ and let

$$u : Pic(J) \longrightarrow K_{top}(J), \qquad u(L) := [\mathscr{O}_J - L] + \frac{c_1^2(L)}{2}[\mathbb{C}_p].$$

Proposition 3.3.5. *Let $i : M_6 \longrightarrow M_v$ be the immersion. There is a group morphism*

$$\lambda \circ u : Pic(J) \longrightarrow Pic(M_v).$$

In particular, the morphism $\widetilde{\lambda}_6 := i^* \circ \lambda \circ u : Pic(J) \longrightarrow Pic(M_6)$ is a group morphism. Moreover, if $L, L' \in Pic(J)$ are such that $c_1(L) = c_1(L')$, then $\widetilde{\lambda}_6(L) \simeq \widetilde{\lambda}_6(L')$.

Proof. The existence of the maps $\lambda \circ u$ and $\widetilde{\lambda}_6$ is implied by Lemma 3.3.4 and Theorem 1.3.2. The fact that $\lambda \circ u$, and hence $\widetilde{\lambda}_6$, is a group morphism is easily done: for any two line bundles $L_1, L_2 \in Pic(J)$ we have

$$v(u(L_1 \otimes L_2)) = v(u(L_1) + u(L_2)),$$

so that u is a group morphism by Proposition C.4.3 (this proposition is stated only for K3s, but the same statement is true even for abelian surfaces replacing $H^*(X, \mathbb{Z})$ with $H^{2*}(J, \mathbb{Z})$). The same proposition allows us to show that if $c_1(L_1) = c_1(L_2)$, then $u(L_1) = u(L_2)$, and we are done. □

Remark 3.3.1. As in the 10−dimensional example, one would like to argue that $\widetilde{\lambda}_6$ is injective: by Proposition 3.3.5, this is surely not the case, since taken two line bundles $L, L' \in Pic(J)$, the equality $c_1(L) = c_1(L')$ does not imply $L \simeq L'$, as J is an abelian surface. In order to get injectivity, one has then toconsider the Néron-Severi group $NS(J)$ of J. In the following, we will then consider the morphism

$$\lambda_6 : NS(J) \longrightarrow Pic(M_6)$$

induced by $\widetilde{\lambda}_6$.

First of all, we need to show the following:

Lemma 3.3.6. *Let $L \in Pic(J)$. Then*

$$c_1(\pi^*\widetilde{\lambda}_6(L)) \cdot \gamma = c_1(\pi^*\widetilde{\lambda}_6(L)) \cdot \delta = 0.$$

Proof. The equality $c_1(\pi^*\widetilde{\lambda}_6(L)) \cdot \delta = 0$ is trivial, since δ is a curve contracted by π. It then remains to show the other: since

$$c_1(\pi^*\widetilde{\lambda}_6(L)) \cdot \gamma = c_1(\pi^*\widetilde{\lambda}_6(L)_{|\gamma}),$$

by Theorem 1.3.2 and Lemma 3.3.3 we just need to show that

$$c_1(\lambda_{\mathscr{H}}(u(L))) = 0,$$

where $\lambda_{\mathscr{H}}$ is the Le Potier's morphism defined using the flat family $\mathscr{H}$ of Example 3.3.2. Using the Grothendieck-Riemann-Roch Theorem, we have

$$c_1(\lambda_{\mathscr{H}}(u(L))) = [q_{Y!}(q_J^*(ch(u(L))td(J)) \cdot ch(\mathscr{H}))]_1 \in H^2(\mathbb{P}^1, \mathbb{Z}).$$

Now, $ch(u(L)) = (0, -c_1(L), 0)$ and $td(J) = (1, 0, 0)$. Moreover, the fibers of $q_Y : \mathbb{P}^1 \times J \longrightarrow \mathbb{P}^1$ are of dimension 2, so that

$$c_1(\lambda_{\mathscr{H}}(u(L))) = -q_{Y*}(q_J^*(c_1(L)) \cdot ch_2(\mathscr{H})) = 0,$$

and we are done. □

This lemma is the analogue of Proposition 2.3.6. Using this, we are finally able to show the following result:

Proposition 3.3.7. *The following diagram*

$$\begin{array}{ccc} Pic(J) & \xrightarrow{\pi^*\circ\widetilde{\lambda}_6} & Pic(\widetilde{M}_6) \\ {\scriptstyle c_1}\big\downarrow & & \big\downarrow{\scriptstyle c_1} \\ H^2(J,\mathbb{Z}) & \xrightarrow[\widetilde{\mu}]{} & H^2(\widetilde{M}_6,\mathbb{Z}) \end{array}$$

is commutative, i. e. $c_1(\pi^*\widetilde{\lambda}_6(L)) = \widetilde{\mu}(c_1(L)))$ *for any* $L \in Pic(J)$. *In particular, the morphism* λ_6 *is injective.*

Proof. The proof of this proposition is almost the same as that of Proposition 2.3.7. Let $L \in Pic(J)$ and let Y and $\mathscr{H}$ be as in Example 3.3.1. Then, $\mathscr{H}$ is a $Y-$flat family of sheaves on $Y \times J$ inducing an injection of Y into $\widetilde{M}_v$. Using the same argument as in Step 1 of the proof of Proposition 2.3.7, we have

$$c_1(\pi^*(\lambda(u((L))))_{|Y} = \widetilde{\mu}(c_1(L))_{|Y}. \tag{3.1}$$

Now, let $Y_6 := Y \cap \widetilde{M}_6$, and let

$$\beta := c_1(\pi^*\widetilde{\lambda}_6(L)) - \widetilde{\mu}(c_1(L)) \in H^2(\widetilde{M}_6, \mathbb{Z}).$$

By equation (3.1), we have $\beta_{|Y_6} = 0$, and by Lemma 3.3.6 and definition of $\widetilde{\mu}$ we have $\beta \cdot \gamma = \beta \cdot \delta = 0$. Following Step 2 of the proof of Proposition 2.3.7, these two properties imply $\beta = 0$, and we are done.

It remains to prove that λ_6 is injective: by the first part of the proposition, the morphism $c_1 \circ \pi^* \circ \widetilde{\lambda_6}$ is injective. Now, $c_1 \circ \pi^* = \pi^* \circ c_1$, and by Remark 3.3.1 and Proposition 3.3.5 this implies that $\pi^* \circ \lambda_6$ is injective, so that λ_6 has to be injective. □

Corollary 3.3.8. *We have the following equality:*

$$Pic(\widetilde{M}_6) = \pi^*\lambda_6(NS(J)) \oplus \mathbb{Z} \cdot [A] \oplus \mathbb{Z} \cdot [\widetilde{B}].$$

Proof. Using Proposition 3.3.7, we can apply the same method used to prove Corollary 2.3.8. □

3.3.3 The Picard group of M_6

Using Corollary 3.3.8, we are able to calculate te Picard group of M_6 and to study its local factoriality. Before doing this, we need to add a remark. The proper transform of $\overline{B}$ is an irreducible Weil divisor in $\widetilde{M}_6$, and this implies that $\overline{B}$ has to be of the form

$$\overline{B} = \Sigma \cup B$$

for some irreducible Weil divisor B of M_6, and the proper transform of $\overline{B}$ equals the proper transform of B. In particular, $B \cap \Sigma \neq \emptyset$, since $\widetilde{B} \cdot \delta \neq 0$, but we don't know if $\Sigma \subseteq B$ or not. Anyway, there is an irreducible Weil divisor $B \in A^1(M_6)$ whose proper transform is $\widetilde{B}$.

Proposition 3.3.9. *There is an inclusion*

$$Pic(M_6) \subsetneq A^1(M_6) = \lambda_6(NS(J)) \oplus \mathbb{Z} \cdot [B] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [D].$$

Proof. We only need to show the formula for $A^1(M_6) \simeq Pic(\pi^{-1}(M_6^s))$. We have a short sequence

$$0 \longrightarrow \mathbb{Z} \cdot [\widetilde{\Sigma}] \longrightarrow Pic(\widetilde{M}_6) \longrightarrow Pic(\pi^{-1}(M_6^s)) \longrightarrow 0,$$

where the first map sends the spanning class to the line bundle $2A$. We claim that this sequence is exact: the only thing to prove is that if $L \in Pic(\widetilde{M}_6^s)$ has trivial restriction to $\pi^{-1}(M_6^s)$, then it is a multiple of $\widetilde{\Sigma}$. By Corollary 3.3.8, there are $M \in Pic(J)$ and $n, m \in \mathbb{Z}$ such that

$$L = \pi^*(\lambda_6(c_1(M))) + n\widetilde{B} + mA.$$

By Lemma 3.2.1, the restriction of L to $\pi^{-1}(M_6^s)$ is then of the form

$$L_{|\pi^{-1}(M_6^s)} = \lambda_6(c_1(M)) + nB + mD \in A^1(M_6).$$

As $L_{|\pi^{-1}(M_6^s)} = 0$, then $2L_{|\pi^{-1}(M_6^s)} = 0$, so that $2nB = \lambda_6(2c_1(M))$, since $2mD = 0$ by Lemma 3.2.1. In particular, their proper transforms are equal, getting $2n\widetilde{B} = \pi^*(\lambda_6(2c_1(M)))$, so that

$$-4n = 2nc_1(\widetilde{B}) \cdot \gamma = \widetilde{\mu}(2c_1(M)) \cdot \gamma = 0,$$

by point 3 of Theorem 3.1.6, so that $n = 0$ and $c_1(M) = 0$ (as $\pi^* \circ \lambda_6$ is injective and $NS(J)$ has no torsion). In conclusion $L = mA$ for some $m \in \mathbb{Z}$, so that

$$0 = L_{|\pi^{-1}(M_6^s)} = mA_{|\pi^{-1}(M_6^s)} = mD.$$

By Lemma 3.2.1, then, m is even and L is a multiple of $\widetilde{\Sigma}$. □

Corollary 3.3.10. *One of the two following possibilities is verified:*

1. $Pic(M_6) = \lambda_6(NS(J))$*, so that* M_6 *is not* $\mathbb{Q}-$*factorial.*
2. $Pic(M_6) = \lambda_6(NS(J)) \oplus \mathbb{Z}$*, so that there is an even integer* $n \in \mathbb{N}$ *such that* M_6 *is* $n-$*factorial.*

Proof. By Proposition 3.2.2, the Weil divisor D is not Cartier, so that by Proposition 3.3.9 we have

$$Pic(M_6) \subseteq \lambda_6(NS(J)) \oplus \mathbb{Z}.$$

This implies the two possible forms of $Pic(M_6)$ in the statement: by Proposition 3.3.7 we have $\lambda_6(NS(J)) \subseteq Pic(M_6)$, so that $Pic(M_6)$ can only be either $\lambda_6(NS(J))$, or $\lambda_6(NS(J)) \oplus \mathbb{Z}\beta$, for some Cartier divisor β.

Now, if $Pic(M_6) = \lambda_6(NS(J)) \oplus \mathbb{Z}\beta$, then there are $m \in \mathbb{Z}$, $m \neq 0$ and $t \in \mathbb{Z}/2\mathbb{Z}$ such that

$$\beta = mB + tD.$$

If $t = 0$, then mB is a Cartier divisor, so that M_6 is at least $2m-$factorial (if m is odd, $mD = D$, which is not Cartier, but $2mD = 0$). If $t = 1$, then $mB + D$ is Cartier. Since D is not Cartier, this implies that mB is not Cartier. Anyway $2(mB + D) = 2mB$ is Cartier, so that M_6 is $2m-$factorial. Now set $n = 2m$, and we are done. □

3.4 The $2-$factoriality of M_6

This section is devoted to the proof of the $2-$factoriality of M_6. This will be shown with the same techniques as those used in Chapter 2 to prove the $2-$factoriality of M_{10}, namely to show that $2B$ is a Cartier divisor.

3.4.1 Line bundles on M_6

As the Weil divisor D is not Cartier, then $D \notin \lambda_6(NS(J))$. Moreover, for any $n \in \mathbb{Z} \setminus \{0\}$ the Weil divisor nB is not in $\lambda_6(NS(J))$. Indeed, if it was the case, then there should be a line bundle $L \in NS(J)$ such that $nB = \widetilde{\lambda}_6(L)$. Their proper transforms are then equal, so that by point 3 of Theorem 3.1.6 we get

$$n = nc_1(\widetilde{B}) \cdot \delta = c_1(\pi^* \widetilde{\lambda}_6(L)) \cdot \delta = 0,$$

as δ is contracted by π. But this is clearly not possible, as $n \neq 0$, and we are done. The key point is the following.

Proposition 3.4.1. *Let $\alpha := [\mathscr{O}_J] + [\mathbb{C}_p] \in K_{top}(J)$ for some point $p \in J$. Then $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$, so that $\lambda(\alpha)_{|M_6} \in Pic(M_6)$ and*

$$c_1(\pi^*(\lambda(\alpha)_{|M_6})) \cdot \gamma = -1.$$

Proof. The proof is the same as that of Theorem 2.3.13: let us consider γ as the fiber of the morphism g defined in Proposition 3.1.5 over the point $((x, -x), E) \in C_a \times K^2(\widehat{J})$, for some $a \in J^0$. Moreover, let Y and $\mathscr{H}$ be as in Example 3.3.2: then, $\mathscr{H}$ is a $Y-$flat family of sheaves on $Y \times J$ giving an inclusion of $Y \simeq \mathbb{P}^1$ into $\widetilde{M}_6$, whose image is γ. By Theorem 1.3.2 we then have

$$c_1(\pi^*(\lambda(\alpha)_{|M_6})) \cdot \gamma = c_1(\pi^* \lambda(\alpha)_{|Y}) = c_1(\lambda_{\mathscr{H}}(\alpha)) \in H^2(\mathbb{P}^1, \mathbb{Z}),$$

where $\lambda_{\mathscr{H}}$ is the Le Potier's morphism defined using the family $\mathscr{H}$. We can then apply the same method used in the proof of Theorem 2.3.13: the calculations are the same, and we are done. □

This proposition allows us to conclude:

Theorem 3.4.2. *There is a line bundle $L \in Pic(J)$ and $t \in \mathbb{Z}/2\mathbb{Z}$ such that*

$$B + tD = \lambda(\alpha)_{|M_6} + \widetilde{\lambda}_6(L),$$

where $\alpha \in K_{top}(J)$ is as in Proposition 3.4.1. In particular, M_6 is 2−factorial.

Proof. By Lemma 3.3.4 and Proposition 3.3.9 there are $L \in Pic(J)$, $n \in \mathbb{Z}$ and $t \in \mathbb{Z}/2\mathbb{Z}$ such that

$$\lambda(\alpha)_{|M_6} = \widetilde{\lambda}_6(L^{-1}) + nB + tD \in A^1(M_6).$$

In particular, then, we have $nB + tD \in Pic(M_6)$, and to get the statement we just need to show that $n = 1$. Taking the pull-back of $nB + tD$ to $\widetilde{M}_6$ there is $m \in \mathbb{Z}$ such that

$$n\widetilde{B} + mA = \pi^*(nB + tD) = \pi^*(\lambda(\alpha)_{|M_6}) + \pi^*\widetilde{\lambda}_6(L).$$

By point 3 of Theorem 3.1.6 we get

$$0 = \pi^*(nB + tD) \cdot \delta = n\widetilde{B} \cdot \delta + mA \cdot \delta = n - m,$$

as δ is contracted by π, and

$$-2n + m = n\widetilde{B} \cdot \gamma + mA \cdot \gamma = \pi^*(\lambda(\alpha)_{|M_6}) \cdot \gamma + \pi^*\widetilde{\lambda}_6(L) \cdot \gamma = -1$$

by Proposition 3.4.1 and Lemma 3.3.6. In conclusion, $n = 1$ and we are done.

It remains to show that M_6 is 2−factorial: since $B + tD$ is a Cartier divisor, we have

$$\lambda_6(NS(J)) \oplus \mathbb{Z}[B + tD] \subseteq Pic(M_6).$$

By Corollary 3.3.10, this implies that M_6 is n−factorial for some even integer n. We have then two possibilities: the first one is if $t = 0$, so that B is Cartier. In this case, the only Weil divisor which is not Cartier is D, but since $2D = 0$, we have that M_6 is 2−factorial. The second case is if $t = 1$, so that $B + D$ is Cartier. Since D is not Cartier, this implies that neither B is Cartier. As before, we have $2D = 0$, so that

$$2B = 2B + 2D = 2(B + D) \in Pic(M_6),$$

and M_6 is 2−factorial. □

Remark 3.4.1. As seen in the proof, one has

$$\pi^*(\lambda(\alpha)_{|M_6}) = \widetilde{B} + A + \pi^*\lambda_6(L)$$

for some line bundle $L \in Pic(J)$. As it was pointed out to me by Rapagnetta, using our construction one can easily show that there is a line bundle $A \in Pic(\widetilde{M_6})$ such that $2A = \widetilde{\Sigma}$. Indeed, as shown in [OG2], we have

$$H^2(\widetilde{M}_6, \mathbb{Q}) = \widetilde{\mu}(H^2(J, \mathbb{Q})) \oplus \mathbb{Q} \cdot c_1(\widetilde{B}) \oplus \mathbb{Q} \cdot c_1(\widetilde{\Sigma}),$$

so that there are $\beta \in H^2(J, \mathbb{Q})$ and $n, m \in \mathbb{Q}$ such that

$$c_1(\pi^*(\lambda(\alpha)_{|M_6})) = \widetilde{\mu}(\beta) + n\widetilde{B} + m\widetilde{\Sigma}.$$

By equation 7.3.5 in [OG2] one gets

$$0 = c_1(\pi^*(\lambda(\alpha)_{|M_6})) \cdot \delta = n - 2m$$

and

$$-1 = c_1(\pi^*(\lambda(\alpha)_{|M_6})) \cdot \gamma = -2n + 2m.$$

In conclusion $m = 1/2$ and $n = 1$. Now, $c_1(\pi^*(\lambda(\alpha)_{|M_6})) \in H^2(\widetilde{M}_6, \mathbb{Z})$, so that if $\widetilde{\Sigma}$ was a generator for $H^2(\widetilde{M}_6, \mathbb{Z})$, we would have $m \in \mathbb{Z}$, which is clearly not the case. Then, there must be a line bundle $A \in Pic(\widetilde{M_6})$ such that $2c_1(A) = c_1(\widetilde{\Sigma})$, and we are done.

3.4.2 Description of $H^2(\widetilde{M_6}, \mathbb{Z})$

In this last section, we prove an analogue of Theorem 2.4.5, about the Beauville-Bogomolov form of $\widetilde{M}_6$. Here is the result:

Theorem 3.4.3. *Let $v = (2, 0, -2) \in \widetilde{H}(J, \mathbb{Z})$. There is a morphism of Hodge structures*

$$f : v^{\perp} \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z}),$$

which is an isometry between $v^{\perp}$, viewed as a sublattice of the Mukai lattice $\widetilde{H}(J, \mathbb{Z})$, and its image in $H^2(\widetilde{M}_6, \mathbb{Z})$, being a lattice with respect to the Beauville-Bogomolov form q.

Proof. As in Theorem 2.4.5, a Mukai vector w is orthogonal to v if and only if $w = (r, c, r)$ for $r \in \mathbb{Z}$ and $c \in H^2(J, \mathbb{Z})$, so that $v^{\perp} \simeq H^2(J, \mathbb{Z}) \oplus \mathbb{Z}$. Let

$$f : v^{\perp} \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z}), \qquad f((r, c, r)) := \widetilde{\mu}(c) + rc_1(\widetilde{B}) + rc_1(A).$$

The morphism f is an injective morphism of Hodge structures. Moreover, it is an isometry: indeed for any two classes $c, d \in H^2(J, \mathbb{Z})$ and for any two $r, s \in \mathbb{Z}$ we have:

$$q(f(r, c, r), f(s, d, s)) = cd - 2rs$$

by point 5 of Theorem 2.1.7, and

$$((r,c,r),(s,d,s)) = cd - 2rs$$

by definition of the Mukai pairing. □

As seen in Remark 3.4.1 we have

$$c_1(\pi^*(\lambda(r\alpha)_{|M_6})) = rc_1(\widetilde{B}) + rc_1(A) + \widetilde{\mu}(c_1(L))$$

for some $L \in Pic(J)$. Notice that $f((r,0,r)) = c_1(\pi^*(\lambda(r\alpha)_{|M_6}))$ if and only if $c_1(L) = 0$. This is the case for M_{10}, so that one would expect the same even here, but up to now we have not been able to show it.

Appendix A

Local factoriality

In this appendix, we recall the two basic notions of Weil and Cartier divisor and the relation between them and with line bundles. We will conclude with the definitions of $\mathbb{Q}-$factorial and $n-$factorial schemes, as generalizations of locally factorial schemes. In the following, X will be a scheme defined over a field k.

Definition A.0.1. X is called *locally factorial* if for any $x \in X$ the local ring $\mathscr{O}_x$ is a unique factorization domain.

Proposition A.0.4. *If the scheme X is regular, then it is locally factorial.*

The notion of local factoriality has many relations with the one of divisor. The first definition of divisor appearing in literature was given by Weil. Suppose X to be regular in codimension one, i. e. for any point $x \in X$, if $dim(\mathscr{O}_x) = 1$ then $\mathscr{O}_x$ is a regular ring.

Definition A.0.2. Any integral closed subscheme of codimension 1 in X is called *prime divisor*. A *Weil divisor* on X is an element of the free abelian group $Div(X)$ generated by all prime divisors.

If Y is a prime divisor, and η_Y is its generic point, then $\mathscr{O}_{\eta_Y}$ is a discrete valuation ring, whose valuation morphism is denoted v_Y.

Lemma A.0.5. *Let f be a non-zero function on X. Then $v_Y(f) = 0$ for all except finitely many prime divisors on X.*

Proof. See Chapter II, Lemma 6.1 in [Har]. □

This lemma allows us to define an equivalence relation on the set of all Weil divisors of X, in the following way.

Definition A.0.3. Let f be a function on X. The *principal Weil divisor associated to* f is the Weil divisor defined as

$$(f) = \sum_Y v_Y(f)Y,$$

where Y runs over the set of all prime divisors. Two Weil divisors D_1, D_2 are said to be *linearly equivalent* if there is a function f on X such that $D_1 = D_2 + (f)$.

We can finally define the group $A^1(X)$ of Weil divisors on X as the quotient of $Div(X)$ by linear equivalence.

Lemma A.0.6. *Let Z be a proper closed subset of X, and let U be its complementary open subset.*

1. *There is a surjective morphism $A^1(X) \longrightarrow A^1(U)$ sending any Weil divisor $D \in A^1(X)$ to its restriction to U.*
2. *If Z has codimension at least 2 in X, then $A^1(X) = A^1(U)$.*
3. *If Z is irreducible of codimension 1, we have an exact sequence*

$$\mathbb{Z} \longrightarrow A^1(X) \longrightarrow A^1(U) \longrightarrow 0$$

where the first map sends 1 to Z.

Proof. See Proposition 6.5 in [Har]. □

The second definition of divisor was given by Cartier.

Definition A.0.4. Let $\mathscr{K}_X$ be the sheaf of total quotient rings of $\mathscr{O}_X$ (that is, the sheaf associated to the presheaf that to each open subset U of X associates $S(U)^{-1}\mathscr{O}_X(U)$, where $S(U)$ is the set of all non zero divisors of $\mathscr{O}_X(U)$), and let $\mathscr{K}_X^*$ be its subsheaf of invertible elements. A *Cartier divisor* is a global section of $\mathscr{K}_X^*/\mathscr{O}_X^*$.

Having fixed an open cover $\{U_i\}_{i\in I}$ of X, a Cartier divisor is then defined by an element $f_i \in \mathscr{K}_X^*(U_i)$ for any $i \in I$, such that $f_i/f_j \in \mathscr{O}_X^*(U_i \cap U_j)$ for any $i, j \in I$. As for Weil divisors, on the set of all Cartier divisors we can define an equivalence relation as follows.

Definition A.0.5. A *principal Cartier divisor* is a Cartier divisor in the image of the natural morphism $H^0(X, \mathscr{K}_X^*) \longrightarrow H^0(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$. Two Cartier divisors are called *linearly equivalent* if their difference is a principal Cartier divisor.

The quotient group of classes of Cartier divisors by linear equivalence is denoted $Car(X)$. We can now begin to study the relations among Weil divisors, Cartier divisors and line bundles.

Proposition A.0.7. *Let X be a normal separated Noetherian scheme. There is an injective map $Car(X) \longrightarrow A^1(X)$ which is an isomorphism if X is locally factorial.*

Proof. See Proposition 6.11 in [Har]. □

Proposition A.0.8. *On any scheme X there is an injective morphism*

$$Car(X) \longrightarrow Pic(X).$$

If X is integral, then it is an isomorphism.

Proof. See Corollary 6.14 and Proposition 6.15 in [Har]. □

In conclusion, if X is a normal integral variety we have an injective morphism

$$Pic(X) \longrightarrow A^1(X)$$

which is an isomorphism if X is locally factorial. If X is singular, there may be Weil divisors which are not Cartier. This is the reason for the following definition:

Definition A.0.6. A normal separated Noetherian scheme is called *locally $\mathbb{Q}-$factorial* (or simply *$\mathbb{Q}-$factorial*) if the usual injective morphism

$$Pic(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow A^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism, or, equivalently, if for any Weil divisor D on X there is $n \in \mathbb{Z}$ (that might depend on D) such that $nD \in Pic(X)$.

Definition A.0.7. Let $n \in \mathbb{Z}$. A normal separated Noetherian scheme is called *locally $n-$factorial* (or, equivalently, *$n-$factorial*) if $nD \in Pic(X)$ for any $D \in A^1(X)$.

Appendix B

Irreducible symplectic manifolds

In this appendix we resume the basic facts in the theory of irreducible symplectic manifolds, which arise as one of the three ingredients in the description of compact Kähler manifolds with vanishing first Chern class (the others being complex tori and special unitary manifolds). The structure of compact Kähler manifolds with $c_1 = 0$ follows from the theory of holonomy groups, that will be the main object of the first section. The structure of the holonomy groups encodes important geometrical properties of Riemannian manifolds, such as the existence of a Kähler metric or of symplectic forms. Irreducible symplectic manifolds are compact Kähler manifolds whose holonomy group is the symplectic group.

In the second section we resume some important properties of K3 surfaces, which give all the examples of $2-$dimensional irreducible symplectic manifolds. Moreover, K3 surfaces are the basic tool for the construction of higher dimensional examples of irreducible symplectic manifolds, together with abelian surfaces. In the third section, we introduce one of the main invariants of irreducible symplectic manifolds: the Beauville-Bogomolov form. This is a non-degenerate bilinear form on the second integral cohomology of any irreducible symplectic manifold, that generalizes the intersection form given by the cup product on the second integral cohomology of any K3 surface.

B.1 Irreducible symplectic manifolds

Let (M, g) be a Riemannian manifold of dimension n. Let $p, q \in M$ be two points, and let γ be a path on M from p to q. By parallel transport, we associate to γ an isometry (with respect to the metric g)

$$\varphi_\gamma : T_p(M) \longrightarrow T_q(M)$$

between the two tangent spaces to M in p and in q. A particular case is when p equals q: in this situation, to any loop based at p we associate an isometry of $T_p(M)$, getting a morphism

$$\varphi : \Omega(p) \longrightarrow O(T_p(M)),$$

where $\Omega(p)$ is the set of loops based at p.

Definition B.1.1. The *holonomy group* of M at p is $Hol_p(M) := \varphi(\Omega(p))$. If $\Omega_0(p)$ is the subset of $\Omega(p)$ of those loops based at p which are homotopically equivalent to the trivial loop, then $Hol_p^0(M) := \varphi(\Omega_0(p))$ is called *restricted holonomy group* of M at p.

Remark B.1.1. If M is connected, then $Hol_p(M) \simeq Hol_q(M)$ for any $p, q \in M$. Then, it makes sense to consider $Hol(M)$, the *holonomy group* of M.

Remark B.1.2. The restricted holonomy group $Hol_p^0(M)$ is a compact Lie subgroup of $SO(T_p(M))$, and we have a natural representation

$$\rho : Hol_p^0(M) \longrightarrow SO(n).$$

Definition B.1.2. Any Riemannian manifold (M, g) is said to be *irreducible* if the representation of the holonomy group is irreducible.

For our purposes, the main result in the theory of holonomy groups is the following theorem.

Theorem B.1.1. ***(Berger**, '55). Let (M, g) be any Riemannian manifold of dimension n, and suppose it to be not locally symmetric. Then $Hol^0(M)$ is isomorphic to one of the following subgroups of $SO(n)$:*

$$SO(n); \quad U(m), \text{ for } 2m = n; \quad SU(m), \text{ for } 2m = n;$$

$$Sp(r), \text{ for } 4r = n; \quad Sp(1) \cdot Sp(r), \text{ for } 4r = n; \quad Spin(9), \ n = 16;$$

$$Spin(7), \ n = 8; \quad G_2, \ n = 7.$$

Proof. See [Ber]. □

The main cases we need to study are those whose holonomy group is isomorphic to $U(m)$, $SU(m)$ or $Sp(r)$, corresponding to those manifolds admitting a Kähler metric. We begin with the notion of symplectic manifold.

Definition B.1.3. Let X be a complex manifold. A *symplectic structure* on X is a non-degenerate holomorphic closed $2-$form on X. Any complex manifold admitting a symplectic structure is called *symplectic*.

Proposition B.1.2. *Let (M, g) be a connected Riemannian manifold.*

1. *If $Hol^0(M) \subseteq U(m)$, then M admits a Kähler metric.*

2. *If $Hol^0(M) \subseteq SU(m)$, then (M, g) is Kähler and Ricci-flat.*

3. *If $Hol^0(M) \subseteq Sp(r)$, then (M, g) is Kähler and symplectic.*

Proof. See Examples 1, 2 and 3 in [Beau]. □

As a consequence of this, any Riemannian manifold (M, g) with $Hol^0(M)$ isomorphic to a subgroup of $U(m)$ admits a complex structure for which g is Hermitian. From now on, we fix such a complex structure, and we consider (M, g) to be a complex manifold X.

Theorem B.1.3. *Let X be a compact Ricci-flat Kähler manifold. Then there is an étale covering X' of X isomorphic, as Kähler manifold, to a product $T \times \prod_i V_i \times \prod_j W_j$, where T is a complex torus with the standard Kähler metric, V_i is a compact simply connected Kähler manifold with $Hol(V_i) = SU(m_i)$, and W_j is a compact simply connected Kähler manifold with $Hol(W_j) = Sp(r_j)$.*

Proof. See the proof of Théorème 1 in [Beau]. □

Remark B.1.3. By Calabi-Yau's theorem (see [Yau]), a compact Kähler manifold is Ricci-flat if and only if its first Chern class is trivial.

Theorem B.1.3 implies that in order to study compact Kähler manifolds with trivial first Chern class, we just need to study two classes of complex manifolds, namely those whose holonomy group are $SU(m)$ or $Sp(r)$.

Definition B.1.4. Let X be a complex manifold. If $Hol(X) = SU(m)$, then X is called *special unitary*. If $Hol(X) \subseteq Sp(r)$, then X is called *hyperkähler*, and if $Hol(X) = Sp(r)$, then X is called *irreducible hyperkähler*.

Remark B.1.4. The name hyperkähler is referred to the fact that these manifolds admit three independent complex structures I, J, K for which g is Kähler (see [H-L]).

The class of special unitary manifolds (sometimes called *Calabi-Yau manifolds*) contains several examples. A fundamental one is the following: if V is any Fano variety, i.e. the line bundle $-K_V$ is ample, then any smooth hypersurface X in the complete linear system $|-K_V|$ is special unitary. Another class of examples is given by smooth complete intersection of r hypersurfaces of degree $d_1, ..., d_r$ in $\mathbb{P}^n$, where $d_1 + ... + d_r = n + 1$.

Definition B.1.5. An *irreducible symplectic manifold* is a compact connected complex manifold which is Kähler, simply connected and $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \omega$ for a symplectic form ω.

Proposition B.1.4. *Let X be a compact Kähler manifold. Then X is hyperkähler if and only if it admits a symplectic structure. If one of these conditions is verified, then the canonical line bundle K_X is trivial. Moreover, X is irreducible hyperkähler if and only if it is irreducible symplectic.*

Proof. See Proposition 3 and 4 in [Beau]. □

While special unitary manifolds form a rather big and wild class, irreducible symplectic manifolds are known in a very few number of cases. In particular, any irreducible symplectic manifold has even complex dimension and trivial canonical bundle.

B.2 K3 surfaces

In this section we recall some basic facts about K3 surfaces, which are the only possible irreducible symplectic surfaces, and are one of the basic tools to produce higher dimensional examples. For a more complete treatment of this interesting subject, see [Geo].

Definition B.2.1. A *K3 surface* is a compact complex surface with $b_1 = 0$ and whose canonical line bundle is trivial.

Proposition B.2.1. *Any K3 surface is simply connected.*

Proof. See Exposé VI in [Geo]. □

Theorem B.2.2. *(**Siu**, '83). Any K3 surface is Kähler.*

Proof. See [Siu] or Exposé XII in [Geo]. □

A consequence of this, we have the following:

Proposition B.2.3. *A compact complex surface is irreducible symplectic if and only if it is a K3 surface.*

Proof. An irreducible symplectic surface has trivial canonical bundle. By the birational classification of surfaces, then it can only an abelian or a K3 surface. Since abelian surfaces are not simply connected, an irreducible symplectic surface must be a K3.

For the converse, by Siu's Theorem any K3 surface X is Kähler. Moreover, X is simply connected by Proposition B.2.1 and $h^2(X, \mathscr{O}_X) = 1$ by Serre's duality and the fact that K_X is trivial. □

In the remaining part of this section, we recall some basic properties of K3 surfaces that we will use in this work.

Remark B.2.1. If X is a K3 surface, then $h^{1,0}(X) = h^{0,1}(X) = 0$ and $h^{2,0}(X) = h^{0,2}(X) = 1$. Moreover, by the exponential sequence the canonical morphism

$$Pic(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

is an injection (this is true for any variety whose first Betti number is trivial, since in this case $H^1(X, \mathscr{O}_X) = 0$).

Proposition B.2.4. *Let X be a K3 surface. Then $H_1(X, \mathbb{Z}) = 0$ and $H^2(X, \mathbb{Z})$ is a rank 22 free $\mathbb{Z}-$module.*

Proof. See Exposé IV, Proposition 1.1 and Corollaire 1.2 in [Geo]. □

Proposition B.2.5. *Let $\alpha \in H^2(X, \mathbb{Z})$. The following are equivalent:*

1. *there is $L \in Pic(X)$ such that $\alpha = c_1(L)$;*
2. *the class α is in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$;*
3. *if ω is a generator for $H^0(X, \Omega^2_X)$, then $\alpha \cdot \omega = 0$.*

Proof. See Exposé IV, Théorème 2.3 in [Geo]. □

The cup product on $H^2(X, \mathbb{Z})$ defines an even unimodular quadratic form (see for example [Sha]), and the lattice $H^2(X, \mathbb{Z})$ with this intersection product is isometric to the lattice $\Lambda_{K3} := 3H \oplus 2E_8(-1)$, where H is the rank 2 lattice with the hyperbolic intersection form and E_8 is the lattice associated to the corresponding Dynkin diagram. The signature of such a lattice is $(3, 19)$ (see Exposé IV, Corollaire 1.3.2 in [Geo]). Let $\Lambda_{\mathbb{C}} = \Lambda_{K3} \otimes \mathbb{C}$.

Definition B.2.2. A *marked K3 surface* is a couple (X, σ), where X is a K3 surface and $\sigma : H^2(X, \mathbb{Z}) \longrightarrow \Lambda_{K3}$ is an isometry. The *period* of a marked K3 surface (X, σ) is the line $\sigma_{\mathbb{C}}(H^{2,0}(X)) \subseteq \Lambda_{\mathbb{C}}$.

The period of a marked K3 surface (X, σ) defines a point in $\Omega \subseteq \mathbb{P}(\Lambda_{\mathbb{C}})$, where

$$\Omega := \{[\omega] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,|\, \omega^2 = 0,\ \omega \cdot \overline{\omega} > 0\}$$

is the *period domain*. The main result in the theory of K3 surfaces is the following, which is known as Torelli Theorem.

Theorem B.2.6. ***(Burns, Rapoport**, '75). Let X, X' be two K3 surfaces. If $u : X \longrightarrow X'$ is an isomorphism, then $u^* : H^2(X', \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z})$ is an effective Hodge isometry. Conversely, if $\varphi : H^2(X', \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z})$ is an effective Hodge isometry, then there is a unique isomorphism $u : X \longrightarrow X'$ such that $u^* = \varphi$.*

Proof. See [B-R] or Exposés VIII and IX in [Geo]. □

Corollary B.2.7. *Let (X,σ), (X',σ') be two marked K3 surfaces. Then X and X' are isomorphic if and only if (X,σ) and (X',σ') have the same period.*

Let $\mathscr{M}$ be the set of isomorphism classes of marked K3 surfaces, and let

$$\beta : \mathscr{M} \longrightarrow \Omega, \quad \beta(X,\sigma) := [\sigma_{\mathbb{C}}(H^{2,0}(X))]$$

be the period map. Then we have the following result:

Theorem B.2.8. *Any element of Ω is the period of a marked K3 surface.*

Proof. The surjectivity of the period map is shown in Exposé X in [Geo]. For the rest of the statement, see Exposé VI, Théorème 1 in [Geo]. □

Corollary B.2.9. *Any two K3 surfaces are diffeomorphic.*

An important example of K3 surface consists of any smooth quartic X in $\mathbb{P}^3$. These are all compact Kähler surfaces which have trivial canonical line bundle by adjunction formula. They are all simply connected by Lefschetz's Theorem in homotopy (see Exposé VI, Proposition 1 in [Geo]). Another important example is described in the following:

Theorem B.2.10. *If S is any smooth sextic in $\mathbb{P}^2$, then the double cover of $\mathbb{P}^2$ branched over S is a K3 surface.*

Conversely, if X is a K3 surface and there is an integral curve C on X such that $h^1(C,\mathscr{O}_C) = 2$, then the morphism

$$\varphi_C : X \longrightarrow \mathbb{P}(H^0(X,\mathscr{O}_X(C))^*)$$

associated to the line bundle $\mathscr{O}_X(C)$ is a double cover of $\mathbb{P}^2$ branched over a (not necessarily smooth) sextic.

Proof. See Exposé IV, Corollaire 3.6 in [Geo]. □

B.3 The Beauville-Bogomolov form

In this section we resume some basic facts concerning the Beauville-Bogomolov form, which is the analogue of the cup product on K3 surfaces. The first result we need is the following:

Proposition B.3.1. *Let $f : Y \longrightarrow B$ be a smooth proper morphism.*

1. *Let $b_0 \in B$ such that $Y_0 := f^{-1}(b_0)$ is a symplectic Kähler manifold. Then there exists $U \subseteq B$ an analytic neighborhood of b_0 such that $Y_b := f^{-1}(b)$ is symplectic Kähler for any $b \in U$.*

2. *If Y_0 is an irreducible symplectic manifold, then Y_b is symplectic for any $b \in B$.*

Proof. See §8, Proposition 1 and Remarque 1 in [Beau]. □

Consider an irreducible symplectic manifold X of complex dimension $2n$, and let $f : \mathscr{X} \longrightarrow \mathscr{M}$ be the Kuranishi family of X. In particular, $\mathscr{M}$ is a local universal deformation space for X: there is a point $0 \in \mathscr{M}$ such that $\mathscr{X}_0$ is isomorphic to X, and for any $m \in \mathscr{M}$, the fiber $\mathscr{X}_m$ is a deformation of X.

Theorem B.3.2. ***(Bogomolov**, '78). The point $0 \in \mathscr{M}$ is smooth in $\mathscr{M}$.*

Proof. See [Bog]. □

This theorem implies that there is no obstruction to the deformation of X. Shrinking $\mathscr{M}$, we can suppose it to be smooth and connected. Moreover, by Proposition B.3.1 we can suppose that for any $m \in \mathscr{M}$ the fiber $\mathscr{X}_m$ is a symplectic Kähler manifold with a unique (up to scalar) symplectic structure $\omega_m \in H^0(\mathscr{X}_m, \Omega^2_{\mathscr{X}_m})$, and that there is a diffeomorphism $d : X \times \mathscr{M} \longrightarrow \mathscr{X}$ such that $f \circ d = p_2$. In particular, for any $m \in \mathscr{M}$ we have a diffeomorphism $d_m : X \longrightarrow \mathscr{X}_m$.

Definition B.3.1. The *period map* for X is the morphism

$$p : \mathscr{M} \longrightarrow \mathbb{P}(H^2(X, \mathbb{C})), \qquad p(m) := [d_m^*(\omega_m)].$$

Let ω be the symplectic structure on X, so that $\omega = \omega_0$. For any $\alpha \in H^2(X, \mathbb{C})$ let

$$q'_X(\alpha) := \frac{n}{2}\int_X (\omega\overline{\omega})^{n-1}\alpha^2 + (1-n)\int_X \omega^{n-1}\overline{\omega}^n\alpha \cdot \int_X \omega^n\overline{\omega}^{n-1}\alpha.$$

Theorem B.3.3. ***(Beauville**, '83). The quadratic form q'_X is non-degenerate. Moreover:*

1. *There is integral quadratic form $q_X : H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$ whose signature is $(3, b_2(X) - 3)$, and there is $\lambda \in \mathbb{R}_{\geq 0}$ such that $q_X(\alpha) = \lambda q'_X(\alpha)$ for any $\alpha \in H^2(X, \mathbb{Z})$.*

2. *Let $\Omega \subseteq \mathbb{P}^{b_2(X)-1}_{\mathbb{C}}$ be the subset defined as*

$$\Omega := \{\alpha \in \mathbb{P}(H^2(X, \mathbb{C})) \,|\, q_X(\alpha) = 0,\ q_X(\alpha + \overline{\alpha}) > 0\}.$$

   Then $p(\mathscr{M}) \subseteq \Omega$ and $p : \mathscr{M} \longrightarrow \Omega$ is a local isomorphism.

Proof. See the proof of §8, Théorème 5 in [Beau]. □

Definition B.3.2. The integral quadratic form q_X is called *Beauville-Bogomolov form* of the irreducible symplectic manifold X.

Remark B.3.1. If X is an irreducible symplectic surface, i. e. a K3 surface, then for any $\alpha \in H^2(X, \mathbb{Z})$ we have $q_X(\alpha) = (\alpha, \alpha)$, since in this case $n = 1$. Theorem B.3.3 can then be seen as a generalization of the Torelli Theorem for K3 surfaces. However, the Torelli Theorem is stronger: the period map is an actual isomorphism, whereas in the more general case of irreducible symplectic manifolds of any dimension it is only a local isomorphism.

As in the case of K3 surfaces, the Beauville-Bogomolov form is related to the cup product on $H^2(X, \mathbb{Z})$:

Theorem B.3.4. ***(Fujiki**, '87). There is a unique rational number $c_X \in \mathbb{Q}$ such that*

$$\int_X \alpha^{2n} = c_X q_X(\alpha)^n$$

for any $\alpha \in H^2(X, \mathbb{Z})$.

Proof. See [Fuj]. □

Definition B.3.3. The rational number c_X is called the *Fujiki constant* of X.

Remark B.3.2. By Remark B.3.1, the Fujiki constant of any K3 surface X is $c_X = 1$. The Beauville-Bogomolov form and the Fujiki constant have been determined for every known example of irreducible symplectic manifold.

Appendix C

Complements on moduli spaces of semistable sheaves

In this appendix we resume some elements of the construction of moduli spaces of sheaves we did not insert in Chapter 2. The main subjects will be the notion of representability of functors and of group actions and quotients. Moreover, we resume important properties as smoothness, dimension, existence of universal families and of symplectic structures, following [H-L].

C.1 Representability of functors

In the following, let $\mathcal{C}$ be any category, and write $\mathcal{C}^o$ for its opposite category, i. e. $obj(\mathcal{C}^o) := obj(\mathcal{C})$ and $Hom_{\mathcal{C}^o}(A, B) := Hom_{\mathcal{C}}(B, A)$ for any $A, B \in \mathcal{C}$. Moreover, let $\mathcal{C}'$ be the category of functors from $\mathcal{C}^o$ to $\mathcal{S}et$.

Definition C.1.1. A functor $F : \mathcal{C}^o \longrightarrow \mathcal{S}et$ is said to be *represented* by an object $X \in \mathcal{C}$ if F is isomorphic to the functor $Hom_{\mathcal{C}}(., X)$. The functor F is said to be *representable* if there is an object $X \in \mathcal{C}$ representing F.

Definition C.1.2. A functor $F : \mathcal{C}^o \longrightarrow \mathcal{S}et$ is said to be *corepresented* by an object $X \in \mathcal{C}$ if $Hom_{\mathcal{C}}(X, Y) = Hom_{\mathcal{C}'}(F, Hom_{\mathcal{C}}(., Y))$ for any $Y \in \mathcal{C}$. The functor F is said to be *universally corepresented* by X if for any morphism $Hom_{\mathcal{C}}(., T) \longrightarrow Hom_{\mathcal{C}}(., X)$, the functor $Hom_{\mathcal{C}}(., T) \times_{Hom_{\mathcal{C}}(.,X)} F$ is corepresented by T.

It follows from the Yoneda's Lemma that any functor represented by X is universally corepresented by X.

Definition C.1.3. We say that there is a *universal object* for a functor $F \in \mathcal{C}'$ if there is an object $X \in \mathcal{C}$ and an element $f \in F(X)$ such that for any $Y \in \mathcal{C}$ and for any $g \in F(Y)$ there is a unique $h \in Hom_{\mathcal{C}}(Y, X)$ such that $h^*(f) = g$.

Proposition C.1.1. *A universal object for a functor $F \in \mathcal{C}'$ exists if and only if F is representable.*

Proof. Suppose F to represented by an object X. Then $F(X) \simeq Hom_{\mathcal{C}}(X, X)$, and define the universal object to be the element of $F(X)$ corresponding to id_X. Then for any $Y \in \mathcal{C}$, any element $g \in F(Y)$ corresponds to a morphism $g : Y \longrightarrow X$, and if $h \in Hom(Y, X)$, then $h^*(f) = f \circ h$ for any $f \in F(X)$. Then simply choose $h = g$, so that $g^*(id_X) = g$, and id_X is a universal object.

If there is a universal object $f \in F(X)$, then X represents F. Indeed, consider the morphism of functors sending any $g \in F(Y)$ to the unique $h \in Hom_{\mathcal{C}}(Y, X)$ such that $h^*(f) = g$. Such a morphism is readily seen to be an isomorphism of functors. □

C.2 Group actions

In the following, let k be a field, X be a Noetherian scheme on k and G be an algebraic group on k.

Definition C.2.1. A *(left) action* of G on X is a morphism $\sigma : G \times X \longrightarrow X$ such that for any $k-$scheme T, the induced morphism

$$\sigma(T) : Hom(T, G) \times Hom(T, X) \longrightarrow Hom(T, X)$$

is an action of the group $Hom(T, G)$ on the set $Hom(T, X)$. If σ is the projection, then we say that it *is the trivial action* of G on X.

Definition C.2.2. If X, Y are two $k-$schemes on which G acts with actions σ_X and σ_Y, then a morphism $f : X \longrightarrow Y$ is called *$G-$equivariant* if the following diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{f \times 1_G} & Y \times G \\ {\scriptstyle\sigma_X}\downarrow & & \downarrow{\scriptstyle\sigma_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative. If σ_Y is the trivial action, then f is said to be *$G-$invariant.*

Definition C.2.3. Let σ be an action of G on X, and let $x \in X$ be a closed point. We call *orbit of x* the image of $\sigma_x := \sigma_{|G \times \{x\}}$. The *stabilizer of x* is

$$G_x := \sigma_x^{-1}(x).$$

The orbit of a closed point is a locally closed subscheme, which in general is not closed. The idea of the quotient space is that of a space which parameterizes the orbits of the action. The closest notion to this one is the following:

Definition C.2.4. A morphism $f : X \longrightarrow Y$ is a *good quotient* for an action σ of an algebraic group G on X if the following are satisfied:

1. f is a $G-$invariant surjective affine morphism;
2. a subset $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open in X;
3. the natural morphism $\mathscr{O}_Y \longrightarrow (f_*\mathscr{O}_X)^G$ is an isomorphism, where $(f_*\mathscr{O}_X)^G$ is the $G-$invariant subsheaf of $f_*\mathscr{O}_X$;
4. if $W \subseteq X$ is a $G-$invariant closed subset of X, then $f(W)$ is closed in Y;
5. if $W_1, W_2 \subseteq X$ are two $G-$invariant closed subsets and $W_1 \cap W_2 = \emptyset$, then $f(W_1) \cap f(W_2) = \emptyset$.

The morphism f is a *universal good quotient* if for any morphism of $k-$schemes $Y' \longrightarrow Y$, the morphism $Y' \times_Y X \longrightarrow Y'$ is a good quotient.

Definition C.2.5. A morphism $f : X \longrightarrow Y$ is a *geometric quotient* for an action σ of an algebraic group G on X if it is a good quotient and its geometric fibers are the orbits of geometric points of X. The morphism f is a *universal geometric quotient* if for any morphism of $k-$schemes $Y' \longrightarrow Y$, the induced morphism $Y' \times_Y X \longrightarrow Y'$ is a geometric quotient.

Up to now, we have only considered actions of algebraic groups on schemes. For the construction of moduli spaces we need to define the action of an algebraic group on line bundles.

Definition C.2.6. Let X be a $k-$scheme of finite type, G an algebraic group over k and σ an action of G on X. Let $\mathscr{F}$ be a coherent sheaf on X. A *$G-$linearization* of $\mathscr{F}$ is an isomorphism $\varphi : \sigma^*\mathscr{F} \longrightarrow p_2^*\mathscr{F}$ of $\mathscr{O}_{G\times X}-$modules, where $p_2 : G \times X \longrightarrow X$ is the projection, such that

$$(\mu \times id_X)^*\varphi = p_{23}^*\varphi \circ (id_G \times \sigma),$$

where $p_{23} : G \times G \times X \longrightarrow G \times X$ is the projection and $\mu : G \times G \longrightarrow G$ is the multiplication morphism.

Definition C.2.7. Let X be a projective $k-$scheme of finite type, and let G be an algebraic group acting on X. Let $L \in Pic(X)$ be a $G-$linearized ample line bundle. We say that $x \in X$ is *semistable* with respect to L if there is $n \in \mathbb{Z}$ and a $G-$invariant global section $s \in H^0(X, L^n)^G$ such that $s(x) \neq 0$. The point x is *stable* if it is semistable, the stabilizer G_x of x is finite and the $G-$orbit of x is closed in the open set of all semistable points. The point x is called *strictly semistable* if it is semistable but not stable.

Let $X^{ss}(L)$ be te set of $L-$semistable points in X, and $X^s(L)$ be the set of $L-$stable points. The main theorem of this section is the following:

Theorem C.2.1. *If G is reductive, then there is a projective scheme Y and a morphism $\pi : X^{ss}(L) \longrightarrow Y$ which is a universal good quotient for the action of G, and such that there is an open subset $Y^s \subseteq Y$ such that $X^s(L) = \pi^{-1}(Y^s)$ and Y^s is a universal geometric quotient of $X^s(L)$.*

Proof. See Theorem 1.10 and Remark 1.11 in [M-F]. □

C.3 Smoothness and dimension of moduli spaces

In general, the moduli spaces of semistable sheaves are rather wild, and have complicated singularities. Let X be a smooth projective scheme, and let E be a locally free sheaf. Let $tr_E : \mathscr{E}nd(E) \longrightarrow \mathscr{O}_X$ be the trace map, inducing maps

$$tr_E^j : Ext^j(E,E) \longrightarrow H^j(X, \mathscr{O}_X)$$

for any j. We can generalize this construction to the case where $E^\bullet$ is a bounded complex of locally free sheaves, so that for every coherent sheaf $\mathscr{E}$ on X and for every j we have trace maps $tr_{\mathscr{E}}^j : Ext^j(\mathscr{E},\mathscr{E}) \longrightarrow H^j(X,\mathscr{O}_X)$ (since X is projective and smooth, so that any $\mathscr{E} \in Coh(X)$ is quasi-isomorphic to a bounded complex $E^\bullet$ of locally free sheaves on X). Let $Ext^j(\mathscr{E},\mathscr{E})_0 := \ker(tr_{\mathscr{E}}^j)$, and let $ext^j(\mathscr{E},\mathscr{E})_0$ be its dimension.

Now, if S be a $k-$scheme of finite type and $\mathscr{F}$ is an $S-$flat family of coherent sheaves on $S \times X$, then the family $det(\mathscr{F})$ is an $S-$flat family of line bundles on $S \times X$. By the universal property of $M(P)$, we get a morphism

$$det : M(P) \longrightarrow Pic(X)$$

sending any semistable sheaf to its determinant. If $\mathscr{L} \in Pic(X)$, we denote $M(P,\mathscr{L}) := det^{-1}(\mathscr{L})$, the moduli space of semistable sheaves on X with Hilbert polynomial P and determinant $\mathscr{L}$.

Theorem C.3.1. *Let X be a smooth projective variety, and let $\mathscr{E}$ be a stable sheaf defining a point $[\mathscr{E}] \in M^s(P,\mathscr{L})$. The Zariski tangent space $T_{[\mathscr{E}]}(M(P,\mathscr{L}))$ is canonically isomorphic to $Ext^1(\mathscr{E},\mathscr{E})_0$. If $Ext^2(\mathscr{E},\mathscr{E})_0 = 0$, then $[\mathscr{E}]$ is a smooth point in $M(P,\mathscr{L})$. Moreover, we have*

$$ext^1(\mathscr{E},\mathscr{E})_0 - ext^2(\mathscr{E},\mathscr{E})_0 \leq dim_{[\mathscr{E}]}M(P,\mathscr{L}) \leq ext^1(\mathscr{E},\mathscr{E})_0.$$

Proof. See Theorem 4.5.4 in [H-L]. □

If X is a smooth projective surface, we can be more explicit. In this case, the Hilbert polynomial of a sheaf $\mathscr{E}$ is determined by $r := rk(\mathscr{E})$, $c_1 := c_1(\mathscr{E})$ and $c_2 := c_2(\mathscr{E})$. The moduli space of semistable sheaves whose Hilbert polynomial is determined by r, c_1 and c_2 is denoted $M(r,c_1,c_2)$ (similarly, we use the notation $M^s(r, c_1, c_2)$ for the open subset of $M(r, c_1, c_2)$ parameterizing stable sheaves). Using this and Hirzebruch-Riemann-Roch Theorem, we get

$$ext^1(\mathscr{E}, \mathscr{E})_0 - ext^2(\mathscr{E}, \mathscr{E})_0 = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathscr{O}_X).$$

This number is the *expected dimension* of $M(r, c_1, c_2)$. Let $M(r, \mathscr{L}, c_2) := det^{-1}(\mathscr{L})$.

Remark C.3.1. If the canonical line bundle K_X is trivial, then $M^s(r, \mathscr{L}, c_2)$ is smooth: by Serre duality $Ext^2(\mathscr{E}, \mathscr{E}) \simeq Hom(\mathscr{E}, \mathscr{E})^*$ for any $\mathscr{E} \in Coh(X)$. If $\mathscr{E}$ is stable, by Corollary 1.1.9 it is simple, so that $hom(\mathscr{E}, \mathscr{E}) = 1$. As $h^2(X, \mathscr{O}_X) = 1$ by Serre duality, we get $Ext^2(\mathscr{E}, \mathscr{E})_0 = 0$. By Theorem C.3.1, $M^s(r, \mathscr{L}, c_2)$ is smooth.

Remark C.3.2. Let X be a projective K3 surface, and $\mathscr{L} \in Pic(X)$. Since the first Chern character is injective, for any $\mathscr{E} \in Coh(X)$ we have $det(\mathscr{E}) = \mathscr{L}$ if and only if $c_1(\mathscr{E}) = c_1(\mathscr{L})$. This implies $M(r, \mathscr{L}, c_2) = M(r, c_1, c_2)$.

Example C.3.1. Let X be a K3 surface, and let $n \in \mathbb{N}$. Then there is a natural isomorphism

$$f : Hilb^n(X) \longrightarrow M(1, 0, -n).$$

Indeed, $Hilb^n(X)$ parameterizes ideal sheaves $\mathscr{I}_Z$ of $0-$dimensional subschemes Z of length n in X. As $rk(\mathscr{I}_Z) = 1$, it is stable by Remark 1.1.3. Moreover, $c_1(\mathscr{I}_Z) = 0$ and $c_2(\mathscr{I}_Z) = n$. Then $\mathscr{I}_Z$ defines a point in $M(1, 0, -n)$. Moreover, if $\mathscr{E}$ is a sheaf parameterized by $M(1, 0, -n)$, then $\mathscr{E}$ is torsion free, so that $\mathscr{E} \subseteq \mathscr{E}^{**}$. Since $\mathscr{E}^{**}$ is a line bundle and $c_1(\mathscr{E}^{**}) = c_1(\mathscr{E}) = 0$, then $\mathscr{E}^{**} \simeq \mathscr{O}_X$. Then $\mathscr{E}$ is a torsion free rank 1 subsheaf of $\mathscr{O}_X$ with trivial first Chern class, so that there is a subscheme $Z \subseteq X$ of dimension 0 such that $\mathscr{E} \simeq \mathscr{I}_Z$. Moreover, since $c_2(\mathscr{E}) = n$, Z has length n.

C.4 Universal and quasi-universal families

By Theorem 1.2.5, the moduli space $M(P)$ is a projective scheme universally corepresenting the moduli functor. In general, $M(P)$ does not represent it: as it parameterizes S-equivalence classes of semistable sheaves, two different S-equivalent strictly semistable sheaves correspond to the same point in $M(P)$ (see Lemma 4.1.2 in [H-L]). Anyway, one might ask if there is a universal family on $M^s(P)$. The main result is the following, at least on surfaces:

Proposition C.4.1. *Let X be a smooth projective surface, and let $r \in \mathbb{Z}$, $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z})$. If*

$$g.c.d.(r, c_1 \cdot c_1(H), \frac{c_1}{2} \cdot (c_1 - c_1(K_X)) - c_2) = 1,$$

then $M(r, c_1, c_2) = M^s(r, c_1, c_2)$ and there is a universal family on the product $M(r, c_1, c_2) \times X$.

Proof. See Corollary 4.6.7 in [H-L]. □

In general there is no way to find a universal family even on $M^s(r, c_1, c_2)$, but we can always find a more general object, whose definition is the following.

Definition C.4.1. A *quasi-universal family* on $M^s(P) \times X$ is a $M^s(P)-$flat family $\mathscr{F}$ of coherent sheaves on X such that for any scheme S and any $S-$flat family $\mathscr{F}'$ of stable sheaves on X with Hilbert polynomial P, there is a vector bundle W such that $\mathscr{F}' \otimes p_S^* W \simeq f^* \mathscr{F}$, where $f : S \longrightarrow M^s(P)$ is the morphism induced by $\mathscr{F}'$.

In particular, for any stable sheaf $\mathscr{E}$ defining a point $[\mathscr{E}] \in M^s(P)$ there is an integer $r \in \mathbb{Z}$ such that $\mathscr{F}_{[\mathscr{E}]} \simeq \mathscr{E}^r$. This integer r does not depend on $\mathscr{E}$, and is called the *similitude* of the quasi-universal family $\mathscr{F}$. A universal family is then a quasi-universal family of similitude 1.

Proposition C.4.2. *Let X be a smooth projective surface. For any $r, c_2 \in \mathbb{Z}$ and $\mathscr{L} \in Pic(X)$, there is a quasi-universal family on the product $M^s(r, \mathscr{L}, c_2) \times X$.*

Proof. See Proposition 4.6.2 in [H-L]. □

As a final point in this section, we define the Mukai morphism, which is one of the main tools in the study of the integral cohomology of irreducible symplectic surfaces. First, we have the following:

Proposition C.4.3. *Let X be an abelian or projective K3 surface. The group morphism*

$$K_{top}(X) \longrightarrow H^{2*}(X, \mathbb{Z}), \qquad \alpha \mapsto v(\alpha)$$

is an isomorphism, where $v(\alpha) = ch(\alpha) \cdot \sqrt{td(X)}$.

Proof. See [Kar] and [Mar]. □

Definition C.4.2. The *Mukai morphism* associated to the quasi-universal family $\mathscr{F}$ is the morphism

$$\mu_M : \widetilde{H}(X, \mathbb{Z}) \longrightarrow H^2(M^s(P), \mathbb{Z}), \quad \mu_M(\alpha) := [p_{M!}(p_X^* \alpha^\vee \cdot [\mathscr{F}] \cdot p_X^* td(X)^{-1})]_1.$$

Remark C.4.1. If there is no universal family, we can always use a quasi-universal family $\mathscr{F}$ of similitude ρ, but the definition changes: for any $\alpha \in \widetilde{H}(X, \mathbb{Z})$ let

$$\mu_M(\alpha) := \frac{1}{\rho}[p_{M!}(p_X^* \alpha^\vee \cdot [\mathscr{F}] \cdot p_X^* td(X)^{-1})]_1.$$

See [Mar] for further details.

C.5 Symplectic structures

Let X be a smooth projective surface, and let H be a fixed ample line bundle on X. Fix $r \in H^0(X, \mathbb{Z})$, $\mathscr{L} \in Pic(X)$ and $c_2 \in H^4(X, \mathbb{Z})$. Let us denote $M^0(r, \mathscr{L}, c_2)$ the open subscheme of $M(r, \mathscr{L}, c_2)$ parameterizing sheaves $\mathscr{E}$ such that $Ext^2(\mathscr{E}, \mathscr{E})_0 = 0$. By Theorem C.3.1, $M^0(r, \mathscr{L}, c_2)$ is a smooth quasi-projective variety. In this section we recall a method, originally due to Mukai, used to produce $2-$forms on $M^0(r, \mathscr{L}, c_2)$. Before doing this, we need some definitions.

The first one is that of Yoneda coupling. Let $E^\bullet$, $F^\bullet$ and $G^\bullet$ be three bounded complexes of locally free sheaves on X, and let $i, j \in \mathbb{Z}$. It is a well known fact that

$$Ext^i(E^\bullet, F^\bullet) \simeq Hom_{D^b(X)}(E^\bullet, F^\bullet[i]), \tag{C.1}$$

where $F^\bullet[i]$ is the complex defined by $F^j[i] := F[i + j]$, and $D^b(X)$ is the bounded derived category of coherent sheaves on X. Notice that if we consider two morphisms $f : E^\bullet \longrightarrow F^\bullet[i]$ and $g : F^\bullet \longrightarrow G^\bullet[j]$, then $g[i] \circ f$ is in $Hom_{D^b(X)}(E^\bullet, G^\bullet[i + j])$. Using this and the identification (C.1), we get a morphism

$$Y_{i,j} : Ext^i(E^\bullet, F^\bullet) \times Ext^j(F^\bullet, G^\bullet) \longrightarrow Ext^{i+j}(E^\bullet, G^\bullet).$$

When $E^\bullet = F^\bullet = G^\bullet$, the morphism $Y_{i,j}$ is called *Yoneda coupling*. This construction allows us to define the Yoneda coupling for any $\mathscr{E} \in Coh(X)$. Indeed, since X is smooth and projective, there is a bounded complex $E^\bullet$ of locally free sheaves which is quasi-isomorphic to $\mathscr{E}$, so that $Ext^i(\mathscr{E}, \mathscr{E}) = Ext^i(E^\bullet, E^\bullet)$. The Yoneda coupling for $E^\bullet$ is then the Yoneda coupling $\mathscr{E}$.

The second definition is the Atiyah class. First of all, let $p_1, p_2 : X \times X \longrightarrow X$ be the projections on the two factors, and let $\Delta \subseteq X \times X$ be the diagonal. Moreover, let $\mathscr{I}$ be the ideal sheaf of Δ, and let $\mathscr{O}_{2\Delta} := \mathscr{O}_{X \times X}/\mathscr{I}^2$. Since $\mathscr{O}_\Delta$ is p_2-flat, the exact sequence

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{O}_{2\Delta} \longrightarrow \mathscr{O}_\Delta \longrightarrow 0$$

remains exact when tensoring by p_2^*F for any locally free sheaf F on X. Applying the functor p_{1*} we get the exact sequence

$$0 \longrightarrow F \otimes \Omega_X \longrightarrow p_{1*}(p_2^*F \otimes \mathscr{O}_{2\Delta}) \longrightarrow F \longrightarrow 0.$$

The corresponding extension class $A(F) \in Ext^1(F, F \otimes \Omega_X)$ is called the *Atiyah class of* F. In a similar way, we can define the Atiyah class of any bounded complex $F^\bullet$ of locally free sheaves, which is $A(F^\bullet) \in Ext^1(F^\bullet, F^\bullet \otimes \Omega_X)$. If $F^\bullet$ is quasi-isomorphic to $E^\bullet$, then the induced isomorphism between the two vector spaces $Ext^1(F^\bullet, F^\bullet \otimes \Omega_X)$ and $Ext^1(E^\bullet, E^\bullet \otimes \Omega_X)$ identifies $A(F^\bullet)$ and $A(E^\bullet)$. This allows us to define the Atiyah class $A(\mathscr{E})$ for any $\mathscr{E} \in Coh(X)$.

Let now $i \in \mathbb{N}$ be a positive integer, and let $\mathscr{E} \in Coh(X)$. Moreover, let $A(\mathscr{E})^{\otimes i} \in Ext^i(\mathscr{E}, \mathscr{E} \otimes \Omega_X^{\otimes i})$ be the composition of i copies of $A(\mathscr{E})$. The canonical morphism $\Omega_X^{\otimes i} \longrightarrow \Omega_X^i$ induces a morphism

$$Ext^i(\mathscr{E}, \mathscr{E} \otimes \Omega_X^{\otimes i}) \longrightarrow Ext^i(\mathscr{E}, \mathscr{E} \otimes \Omega_X^i).$$

The image of $A(\mathscr{E})^{\otimes i}$ under this morphism is denoted $A^i(\mathscr{E})$.

Definition C.5.1. The $i-$th *Newton class* of $\mathscr{E}$ is

$$\gamma_i(\mathscr{E}) := tr^i(A^i(\mathscr{E})) \in H^i(X, \Omega_X^i).$$

We are now able to define $2-$forms on $M^0(r, L, c_2)$. We do this in general, supposing S smooth and that there is a universal family $\mathscr{F}$ on $S \times X$. Let

$$\tau_{\mathscr{F}} : H^0(X, K_X) \simeq H^2(X, \mathscr{O}_X)^* \longrightarrow H^0(S, \Omega_S^2)$$

be the morphism corresponding to $\gamma'(\mathscr{F})$, the component of $\gamma_2(\mathscr{F})$ lying in $H^0(S, \Omega_S^2) \otimes H^2(X, \mathscr{O}_X)$. If there is no universal family on S, we can use quasi-universal families. In particular, on $M^s(r, \mathscr{L}, c_2) \times X$ there is always a quasi-universal family $\mathscr{F}$, and $M^0 := M^0(r, \mathscr{L}, c_2) \cap M^s(r, \mathscr{L}, c_2)$ is smooth.

Definition C.5.2. Let $\omega \in H^0(X, K_X)$ be a $2-$form on X, and let $\mathscr{F}$ be a quasi-universal family on $M^0 \times X$. The $2-$form

$$\tau(\omega) := \frac{1}{rk(\mathscr{F})} \tau_{\mathscr{F}}(\omega) \in H^0(M^0, \Omega_{M^0}^2)$$

is called the $2-$*form associated to* ω.

Theorem C.5.1. *(**Mukai**, '84). For any $\omega \in H^0(X, K_X)$, the $2-$form $\tau(\omega)$ on M^0 does not depend on the quasi-universal family $\mathscr{F}$ used in the definition. Moreover, $\tau(\omega)$ is non-degenerate on M^0 if and only if the morphism*

$$\omega_* : Ext^1(\mathscr{E}, \mathscr{E}) \longrightarrow Ext^1(\mathscr{E}, \mathscr{E} \otimes K_X)$$

is an isomorphism for any $\mathscr{E} \in M^0$.

Proof. The original proof is due to Mukai, and is contained in [Muk]. See also Theorem 10.4.3 in [H-L]. □

Remark C.5.1. By Remark C.3.1, if X is an abelian or a projective K3 surface, then $M^0 = M^s(r, \mathscr{L}, c_2)$. Moreover, $H^0(X, K_X) \simeq \mathbb{C} \cdot \omega$ for a symplectic $2-$form ω. The morphism ω_* of Theorem C.5.1 is clearly an isomorphism, so that $\tau(\omega)$ is a symplectic $2-$form on $M^s(r, \mathscr{L}, c_2)$.

Appendix D

Moduli spaces of sheaves on K3 and abelian surfaces

In this appendix we resume the basic results on the known examples of irreducible symplectic manifolds. These are all contructed using the theory of moduli spaces of sheaves on K3 or abelian surfaces. The first result in this subject is due to Fujiki, who showed that the Hilbert scheme $Hilb^2(X)$ on a projective K3 surface X is an irreducible symplectic variety of dimension 4. This result was vastly generalized by Beauville, who showed that for any integer n, the Hilbert scheme $Hilb^n(X)$ over a K3 surface X is an irreducible symplectic manifold of dimension $2n$. In the same work, Beauville introduced the generalized Kummer varieties, giving another family of examples of irreducible symplectic manifolds.

The next step was to to study moduli spaces of stable sheaves on K3 (and abelian) surfaces. In the middles of the nineties it was shown that these are deformation equivalent to Hilbert schemes or to generalized Kummer varieties. In 1999 and 2003 O'Grady presented two new deformation classes of irreducible symplectic manifolds, using singular moduli spaces of semistable sheaves: these are the main object of study of this work, and were described in Chapters 2 and 3. The study of irreducible symplectic varieties coming from moduli spaces of semistable sheaves was concluded by Kaledin, Lehn and Sorger, who showed that the two examples provided by O'Grady are the only new examples of irreducible symplectic variety we can produce using the theory of moduli spaces of semistable sheaves on surfaces.

D.1 Hilbert schemes of points

Let X be a smooth projective surface, and let $n \in \mathbb{N}$. In section 2.2 we defined $Hilb^n(X)$ as the projective scheme parameterizing $0-$dimensional subschemes of X of length n, and we have seen that there is a universal subscheme Ξ_n on $Hilb^n(X) \times X$, i.e. such that for any $[Z] \in Hilb^n(X)$, we have $\Xi_{n|[Z]\times X} \simeq Z$.

There is an alternative and more geometrical construction for $Hilb^n(X)$. Let X^n be the product of n copies of X, and let Σ_n act on X^n by permutation of factors. Let $X^n_d := \{(x_1, ..., x_n) \in X^n | x_i \neq x_j \text{ for } i \neq j\}$, and let $\Delta_n := X^n \setminus X^n_d$.

Definition D.1.1. The $n-th$ *symmetric product* of X is the projective scheme $S^n(X) := X^n/\Sigma_n$. The $n-th$ *isospectral Hilbert scheme* is the blow-up $B^n(X)$ of X^n along Δ_n.

We complete the definition with some notations: let $\pi_n : X^n \longrightarrow S^n(X)$ be the quotient morphism, and let $\phi_n : B^n(X) \longrightarrow X^n$ be the blow-up. If $(x_1, ..., x_n) \in X^n$, we let $\pi_n(x_1, ..., x_n) =: \sum_{i=1}^n x_i$. Notice that the action of Σ_n on X^n is free on X^n_d, and it extends to an action of Σ_n on $B^n(X)$. Let $D_n := \pi_n(\Delta_n)$.

Proposition D.1.1. *The projective scheme $S^n(X)$ is singular along D_n. Moreover, $codim_{S^n(X)}(D_n) = 2$.*

Proof. The codimension of D_n in $S^n(X)$ is easily seen to be 2. The rest is done by a local calculation, as shown in Proposition 2.2 in [Fog]. □

Let $b_n : Bl_{D_n}(S^n(X)) \longrightarrow S^n(X)$ be the blow-up of $S^n(X)$ along D_n. By definition, there is a morphism $p_n : B^n(X) \longrightarrow Bl_{D_n}(S^n(X))$, which is the quotient of $B^n(X)$ under the action of Σ_n.

Definition D.1.2. The *Hilbert-Chow morphism* is defined as

$$\rho_n : Hilb^n(X) \longrightarrow S^n(X), \qquad \rho_n([Z]) := \sum_{x\in X} l(\mathscr{O}_{Z,x})x,$$

where $l(\mathscr{O}_{Z,x})$ is the length of $\mathscr{O}_Z$ at x.

Theorem D.1.2. ***(Fogarty**, '73). The Hilbert scheme $Hilb^n(X)$ is a smooth irreducible projective variety of dimension $2n$ and the Hilbert-Chow morphism ρ_n is a resolution of singularities of $S^n(X)$. Moreover, $Hilb^n(X) \simeq Bl_{D_n}(S^n(X))$ and $b_n = \rho_n$.*

Proof. This is shown in Section 5 in [Fog]. □

The main result on the Hilbert schemes of points on a K3 surface is the following:

Theorem D.1.3. ***(Beauville**, '83). Let X be a projective K3 surface, and let $n \in \mathbb{N}$ be a natural number. Then $Hilb^n(X)$ is an irreducible symplectic variety of dimension $2n$ and second Betti number $b_2(Hilb^n(X)) = 23$.*

This result was shown in [Beau] (with more general hypothesis: one can drop the projectivity hypothesis, and just consider any K3 surface; in this case, one has to use Douady spaces instead of Hilbert schemes, but all the argument goes through).

D.2 Generalized Kummer varieties

Here we introduce the second example of irreducible symplectic manifold, constructed starting from the Hilbert scheme of points on an abelian surface.

Definition D.2.1. A $2-$*dimensional complex torus* T is the quotient of $\mathbb{C}^2$ by a maximal lattice $\Gamma \subseteq \mathbb{C}^2$. If T is projective, then it is called *abelian surface*.

Any $2-$dimensional complex torus is equipped with a unique (up to multiplication by a complex number) symplectic structure, namely the standard one on $\mathbb{C}^2$: if z_1, z_2 are linear coordinates on $\mathbb{C}^2$, then the symplectic structure $dz_1 \wedge dz_2$ is translation invariant, so that it descends to a symplectic structure on $\mathbb{C}^2/\Gamma = T$. Anyway, a $2-$dimensional complex torus cannot be an irreducible symplectic surface since $\pi_1(T) \simeq \Gamma \simeq \mathbb{Z}^4$, so that T is not simply connected.

Theorem D.2.1. *Let* $T = \mathbb{C}^2/\Gamma$ *be a* $2-$*dimensional complex torus. Then* $H^i(T,\mathbb{Q}) \simeq \Lambda^i Hom(\Gamma,\mathbb{Z})$ *and* $H^0(T,\Omega^i_T) = \Lambda^i\mathbb{C}\langle dz_1, dz_2\rangle$, *where* z_1, z_2 *are coordinates on* $\mathbb{C}^2$.

Proof. See for example Corollary 1.3.2 and Theorem 1.4.1 in [B-L]. □

Even if abelian surfaces are not simply connected, they are an important starting point for the construction of irreducible symplectic varieties. The basic construction is that of the Kummer surface associated to any abelian surface. From now on, we will write A for an abelian surface. Let

$$\iota : A \longrightarrow A, \qquad x \mapsto -x$$

be the natural involution on A, and consider the quotient A/ι. In particular, any point $x \in A$ is fixed by ι if and only if it is a $2-$torsion point for A, and any of these points gives a singular point for A/ι. It is well-known that on any abelian surface A there are 16 different $2-$torsion points (see [B-L]), so that A/ι has 16 different singular points, each one of type A_1.

Definition D.2.2. The *Kummer surface associated to* A is the blow-up

$$f : K(A) \longrightarrow A/\iota$$

of A/ι along its 16 singular points.

Theorem D.2.2. *The Kummer surface* $K(A)$ *associated to any abelian surface* A *is a K3 surface.*

There is an equivalent way to construct the Kummer surface associated to an abelian surface A. Consider $A \times A$, and let $\Delta \subseteq A \times A$ be the diagonal. On $A \times A$ acts the group Σ_2 simply interchanging the two factors. The quotient $S(A) := A \times A/\Sigma_2$ is the symmetric product of A. If we write $\pi : A \times A \longrightarrow S(A)$ for the quotient morphism, then $S(A)$ is singular along $D := \pi(\Delta)$. Notice that $codim_{S(A)}(D) = 2$. Consider

$$\rho : \widetilde{S(A)} \longrightarrow S(A),$$

the blow-up of $S(A)$ along its singular locus D. Moreover, consider

$$s : S(A) \longrightarrow A, \quad s([a_1, a_2]) = a_1 + a_2,$$

where $[a_1, a_2] := \pi(a_1, a_2) \in S(A)$.

Proposition D.2.3. *For any abelian surface A, we have $K(A) \simeq \rho^{-1}(s^{-1}(0))$.*

Proof. It is clear that $s^{-1}(0) = \{[a, -a] \in S(A) \,|\, a \in A\}$, so that $s^{-1}(0) \simeq A/\iota$. Moreover, $D \cap s^{-1}(0) = \{[a, -a] \in S(A) \,|\, 2a = 0\}$, and it corresponds to the singular points of A/ι. □

We generalize now this construction to the higher dimensional case. Let A be an abelian surface, and let $n \in \mathbb{N}$ be a positive integer. Now, consider the Hilbert-Chow morphism

$$\rho_{n+1} : Hilb^{n+1}(A) \longrightarrow S^{n+1}(A),$$

that we compose with the following sum morphism

$$s_{n+1} : S^{n+1}(A) \longrightarrow A, \quad s_{n+1}\left(\sum_{i=1}^{n+1} a_i\right) := a_1 + ... + a_{n+1}.$$

Definition D.2.3. The *generalized Kummer variety* $K^n(A)$ associated to an abelian surface A is

$$K^n(A) := \rho_{n+1}^{-1}(s_{n+1}^{-1}(0)).$$

Remark D.2.1. By Proposition D.2.3, we have $K^1(A) = K(A)$, the Kummer surface associated to A. This is why these varieties are called generalized Kummer.

Theorem D.2.4. *For any abelian surface A and for any $n \in \mathbb{N}$, the generalized Kummer variety $K^n(A)$ is an irreducible symplectic variety of dimension $2n$, whose second Betti number is $b_2(K^n(A)) = 7$.*

Proof. See §7, Théorème 4 in [Beau]. □

This theorem is the analogue of Theorem D.1.3. Since for any $n > 1$ we have $b_2(Hilb^n(X)) \neq b_2(K^n(A)$, we get two different examples of irreducible symplectic varieties on any possible dimension.

D.3 Moduli spaces of stable sheaves on K3

The next step in the study of irreducible symplectic varieties is to look at moduli spaces of sheaves on projective K3 surfaces. The main reason for this comes by analogy with Hilbert schemes: for any projective K3 surface X, the Hilbert scheme $Hilb^n(X)$ is isomorphic to the moduli space $M(1,0,-n)$ of rank 1 semistable sheaves on X with trivial determinant and second Chern class equal to n, and provides an example of irreducible symplectic variety of dimension $2n$.

D.3.1 $0-$dimensional moduli spaces

Definition D.3.1. Any Mukai vector v of rank $r > 0$ is called *exceptional* if $(v,v) = -2$. A vector bundle E on X is called *exceptional* if it is simple and $Ext^1(E,E) = 0$.

Remark D.3.1. Notice that $\chi(E,E) = 2$ for any exceptional vector bundle E. Moreover, $\chi(E,E) = -(v(E),v(E))$, so that the Mukai vector of an exceptional bundle is exceptional.

Theorem D.3.1. *(**Kuleshov**, '89). Let v be an exceptional Mukai vector, and H be any polarization.*

1. *There is a $\mu-$semistable exceptional bundle E such that $v(E) = v$.*

2. *If $H^2 \geq 4$ and X is generic (in the moduli space of polarized K3 surfaces (X,H') with $(H')^2 = H^2$), then there is a stable exceptional bundle E such that $v(E) = v$.*

Proof. See Theorem 2.1 in [Kul]. □

Remark D.3.2. Notice that if v is an exceptional vector bundle with rank 1, then there is only one exceptional bundle whose Mukai vector is v: this is the unique line bundle L such that $c_1(L) = c_1(v)$. In particular, such an exceptional bundle is stable. This is not true in general: it $r(v) > 1$ and $\rho(X) > 1$, then there can be non-isomorphic exceptional bundles with Mukai vector v (see the example after Corollary 1.11 in [Kul]). Theorem D.3.2 below will then imply the existence of unstable exceptional bundles.

Theorem D.3.2. *(**Mukai**). Let X be a projective K3 surface, and v be an exceptional Mukai vector on X. If H is a $v-$generic polarisation and $M^s(v) \neq \emptyset$, then $M(v) = M^s(v) = \{[E]\}$, where E is a stable exceptional bundle.*

Proof. See [Muk2] or Theorem 6.1.6 in [H-L]. □

D.3.2 2−dimensional moduli spaces

The next case to look at is that of moduli spaces of semistable sheaves $M(v)$ where v is an isotropic Mukai vector, i.e. $(v,v)=0$. In this case, if $M^s(v)\neq\emptyset$ then it is a surface. Notice that in this case for any $m\in\mathbb{Z}$ the Mukai vector mv is isotropic. Let then v be a primitive isotropic Mukai vector and choose a $v-$generic polarization on X, so that $M^s(v)=M(v)$ is a smooth surface. By Theorem C.5.1 the moduli space $M(v)$ is a symplectic surface and by Proposition C.4.2 there is a quasi-universal family $\mathscr{F}$ on $M(v)\times X$ of similitude ρ. Let

$$f_{\mathscr{F}}: H^*(X,\mathbb{Q})\longrightarrow H^*(M(v),\mathbb{Q}),\qquad g_{\mathscr{F}}: H^*(M(v),\mathbb{Q})\longrightarrow H^*(X,\mathbb{Q}),$$

be the Mukai morphisms defined using the quasi-universal family $\mathscr{F}$. In it is a universal family, we have an important result:

Theorem D.3.3. ***(Mukai)****. Let v be a primitive isotropic Mukai vector and let $\mathscr{F}$ be a universal family on $M(v)\times X$. Then $M(v)$ is a K3 surface and the morphism $f_{\mathscr{F}}$ defines an isometry of Hodge structures between $\widetilde{H}(X,\mathbb{Z})$ and $\widetilde{H}(M(v),\mathbb{Z})$.*

Proof. See [Muk2] or Theorem 6.1.13 in [H-L]. □

Now, let $v^{\perp}\subseteq\widetilde{H}(X,\mathbb{Z})$ be the orthogonal of the Mukai vector v under the Mukai pairing. Since v is isotropic, then $v\in v^{\perp}$. Moreover, if v is primitive, the $\mathbb{Z}-$module $v^{\perp}/\mathbb{Z}\cdot v$ is free of rank 22.

Theorem D.3.4. ***(Mukai)****. Let v a primitive isotropic Mukai vector, and let $\mathscr{F}$ be a quasi-universal family on $M(v)\times X$. The Mukai morphism $f_{\mathscr{F}}$ gives an isometry of Hodge structures*

$$f_{\mathscr{F}}: v^{\perp}/\mathbb{Z}\cdot v\longrightarrow H^2(M(v),\mathbb{Z})$$

which is independent on the chosen quasi-universal family.

Proof. See [Muk2] or Theorem 6.1.14 in [H-L]. □

In order to conclude this section, we study the case $v=mw$ where $m\in\mathbb{Z}$ and w is primitive isotropic. As a corollary of Proposition **??** we have:

Corollary D.3.5. *Let $v=mw$ be a Mukai vector such that $m\in\mathbb{N}$ and w is primitive, and let H be a $v-$generic polarization. Then $M(v)\simeq S^mM(w)$.*

By Theorem D.3.3, $M(w)$ is a K3 surface, so that $M(v)=S^mM(w)$ is the symmetric product of m copies of a K3 surface. In conclusion, by Theorem D.1.2 the moduli space $M(v)$ admits a symplectic resolution of singularities

$$Hilb^m(M(w))\longrightarrow M(v),$$

which is an irreducible symplectic manifold of dimension $2m$.

D.3.3 Higher dimensional moduli spaces

In this section, we study the case of primitive Mukai vector v such that $(v,v) \geq 2$. The first result we need is the following:

Theorem D.3.6. *(**Yoshioka**, '00). Let v be a primitive Mukai vector such that $rk(v) > 0$, $c_1(v) \in NS(X)$ or $rk(v) = 0$, $c_1(v) \in NS(X)$ is the first Chern class of an effective line bundle and $v_4 \neq 0$. Moreover, let H be a $v-$generic polarization. Then, the moduli space $M(v)$ is non-empty if and only if $(v,v) \geq -2$. Under this hypothesis, $M(v)$ is an irreducible normal projective variety.*

Proof. This is contained in Theorem 0.1 and Theorem 8.1 in [Yo1]. □

Corollary D.3.7. *Let v be an arbitrary Mukai vector such that $rk(v) > 0$ and $c_1(v) \in NS(X)$, and let H be a $v-$generic polarization. The moduli space $M(v)$ is non-empty if and only if $v = mw$ for $m \in \mathbb{Z}$ and for a primitive Mukai vector $w \in \widetilde{H}(X,\mathbb{Z})$ such that $(w,w) \geq -2$.*

If v is primitive and $(v,v) > 0$, by Remark C.3.1 the moduli space $M(v)$ is a smooth projective variety of even complex dimension $(v,v) + 2 \geq 4$, and by Theorem C.5.1 it admits a symplectic structure. It is then natural to ask if $M(v)$ is an irreducible symplectic variety.

Theorem D.3.8. *(**Yoshioka**, '99). Let X be a projective K3 surface, v be a primitive Mukai vector such that $rk(v) > 0$, $c_1(v) \in NS(X)$ and $(v,v) \geq 2$, and let H be a $v-$generic polarization. Then the moduli space $M(v)$ is an irreducible symplectic variety which is deformation equivalent to $Hilb^{\frac{(v,v)}{2}+1}(S)$ for some projective K3 surface S.*

Proof. This theorem has been shown in different steps: the first one is due to Mukai, dealing only with $rk(v) = 2$ (see [Muk]). In [G-H], Göttsche and Huybrechts show that the moduli spaces of stable sheaves on a K3 surface have the same Hodge numbers of the Hilbert scheme. Another step is due to O'Grady, in [OG1], dealing with primitive Mukai vectors v such that $c_1(v)$ is not divisible in $NS(X)$. Then Yoshioka proves it in [Yo3], when $(v,v) > 2l^2$, for $l := g.c.d.(r, c_1(v))$, meaning $r = lr'$ for some $r' \in \mathbb{Z}$ and $c_1 = lc$ for some $c \in NS(X)$. The final result is Theorem 0.1 in [Yo2]. □

This theorem concludes the investigation on moduli spaces of stable sheaves: if v is a primitive Mukai vector and the polarization is $v-$generic, the moduli space $M(v)$ is reduced to a point if $(v,v) = -2$, it is a K3 surface if $(v,v) = 0$ and it is (up to deformation) the Hilbert scheme of points on some K3 surface if $(v,v) \geq 2$. Another important result is on the second integral cohomology of

$M(v)$. By Proposition C.4.2, over $M(v) \times X$ there is a quasi-universal family $\mathscr{F}$ of similitude ρ. We can then define the Mukai's morphism

$$\mu_M : v^{\perp} \longrightarrow H^2(M(v), \mathbb{Z}),$$

which is independent on the choice of $\mathscr{F}$ since μ_M is defined over $v^{\perp}$.

Theorem D.3.9. *The Mukai's morphism defines an isometry of Hodge structures between $v^{\perp}$ and $H^2(M(v), \mathbb{Z})$, the first being the sublattice of the Mukai lattice of X, the second being a lattice with the Beauville-Bogomolov form.*

Proof. Again, this is Theorem 0.1 in [Yo2]. □

Remark D.3.3. In the statement of Theorem D.3.9, the $\mathbb{Z}-$module $H^2(M(v), \mathbb{Z})$ has a lattice structure given by the Beauville-Bogomolov form. In the case treated in Theorem D.3.4, the lattice structure on $H^2(M(v), \mathbb{Z})$ is given by the natural intersection product on $M(v)$ which is a projective K3 surface. By Remarks B.3.1 and B.3.2, this intersection form is the Beauville-Bogomolov form, so that Theorem D.3.9 is a generalization of Theorem D.3.4.

D.4 Moduli spaces of sheaves on abelian surfaces

As in the case of K3 surfaces, one can study moduli spaces of semistable sheaves on abelian surfaces in order to provide new examples of irreducible symplectic manifolds. So, let A be an abelian surface, and let H be an ample line bundle on A. In the following, we will write $\widehat{A}$ for the abelian surface dual to A, i. e. $\widehat{A} \simeq Pic(A)$.

Fix a Mukai vector $v \in H^{2*}(A, \mathbb{Z}) =: \widetilde{H}(A, \mathbb{Z})$, and let $v = (v_0, v_2, v_4)$, where $v_i \in H^i(A, \mathbb{Z})$. We will consider only Mukai vectors v such that $v_0 > 0$ and $v_2 \in NS(X)$ or $v_0 = 0$, $v_2 \in NS(X)$ is the numerical class of an effective divisor and $v_4 \neq 0$. Let $M(v)$ be the moduli space of $H-$semistable sheaves with Mukai vector v, and let $M^s(v)$ be the open subset of $M(v)$ parameterizing stable sheaves. As $td(A) = (1, 0, 0)$, the Mukai vector of any sheaf $\mathscr{E}$ on A equals its Chern character, so that if $\mathscr{E}$ has rank r, first Chern class c_1 and second Chern class c_2, then $v(\mathscr{E}) = (r, c_1, \frac{c_1^2}{2} - c_2)$ and

$$expdim(M(v)) = dim(M^s(v)) = 2 + (v, v).$$

Notice that since A is an abelian surface, then Theorem C.3.1 implies that $M^s(v)$ is smooth, and Theorem C.5.1 implies the existence of a symplectic structure on $M^s(v)$.

An important tool in the study of moduli spaces of sheaves on abelian surfaces is the map

$$a_v : M(v) \longrightarrow A \times \widehat{A}$$

defined in the following way: fix a sheaf $\mathscr{E}_0$ defining a point in $M(v)$, and let $\mathscr{P}$ be the Poincaré line bundle on $A \times \widehat{A}$. Then

$$a_v(\mathscr{E}) := (det(p_{\widehat{A}!}(p_X^*[\mathscr{E} - \mathscr{E}_0] \cdot [\mathscr{P} - \mathscr{O}_{A\times\widehat{A}}])), det(\mathscr{E}) \otimes det(\mathscr{E}_0)^{-1}).$$

Remark D.4.1. Using the Grothendieck-Riemann-Roch Theorem, it is easy to show that $det(p_{\widehat{A}!}(p_X^*[\mathscr{E} - \mathscr{E}_0] \cdot [\mathscr{P} - \mathscr{O}_{A\times\widehat{A}}])) \in Pic^0(A) \simeq A$.

The main result on the map a_v is the following:

Theorem D.4.1. ***(Mukai, Yoshioka)****. Let A be an abelian surface, v a primitive Mukai vector and let H be a $v-$generic ample line bundle.*

1. *If $(v, v) = 0$, then $M(v)$ is an abelian surface, and the morphism a_v is an immersion.*
2. *If $(v, v) = 2$, then a_v is an isomorphism.*
3. *If $(v, v) \geq 4$, then a_v is the Albanese map of $M(v)$, and the fiber*

$$K(v) := a_v^{-1}(0, 0)$$

is an irreducible symplectic manifold deformation equivalent to the generalized Kummer variety $K^{\frac{(v,v)}{2}-1}(A)$. Moreover, the Mukai morphism

$$\mu_M : v^{\perp} \longrightarrow H^2(K(v), \mathbb{Z})$$

is an isometry of Hodge structures.

Proof. The proof of the first item is contained in [Muk4]. The second item is shown by Mukai in [Muk3] and by Yoshioka in [Yo4], Proposition 4.2. Finally, the last part of the statement is shown by Yoshioka in two different papers: the final result is Theorem 0.2 in [Yo1]. □

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Résumé : Cette thèse se compose de deux parties : dans la première on démontre une généralisation du théorème de Gabriel sur les faisceaux cohérents au cas des faisceaux cohérents tordus. Plus précisément, on démontre que tout schéma noethérien X peut être reconstruit à partir de sa catégorie abélienne $Coh(X, \alpha)$ des faisceaux cohérents tordus par un élément α du groupe de Brauer cohomologique de X. Dans la deuxième partie on étudie les deux espaces des modules M_{10} et M_6 introduits par O'Grady, qu'il utilise pour obtenir ses deux nouveaux examples de variétés irréductibles symplectiques de dimension 10 et 6 respectivement. On calcule les groupes de Picard de M_{10} et M_6, et on démontre que ces deux variétés ne sont pas localement factorielles, mais $2-$factorielles. Ceci est accompli en utilisant les résultats de Rapagnetta sur la cohomologie et la forme de Beauville-Bogomolov de M_{10} et M_6, et en étudiant les propriétés du morphisme de Le Potier dans ces deux cas.

Mots clés : faisceaux cohérents tordus, catégories abéliennes, variétés irréductibles symplectiques, espaces des modules de faisceaux sur des surfaces algébriques.

Summary : This PhD thesis is divided in two parts : in the first one, we present a generalization of Gabriel's Theorem on coherent sheaves to twisted coherent sheaves. More precisely, we show that any Noetherian scheme X can be reconstructed from its abelian category $Coh(X, \alpha)$ of coherent sheaves twisted by an element α of the cohomological Brauer group of X. In the second part we study the two moduli spaces M_{10} and M_6 introduced by O'Grady, which he uses to obtain his two new examples of irreducible symplectic varieties in dimension 10 and 6. We calculate the Picard group of M_{10} and M_6, and we show that these two varieties are not locally factorial, but $2-$factorial. This is done using the results obtained by Rapagnetta on the cohomology and the Beauville-Bogomolov form of M_{10} and M_6, and studying the properties of the Le Potier's morphism in these two cases.

Key words : Twisted coherent sheaves, Abelian categories, Irreducible symplectic manifolds, Moduli spaces of sheaves on algebraic surfaces.