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par **Simone SCOTTI**

Applications of the Error Theory using Dirichlet Forms

Thèse soutenue le 16 octobre 2008 devant le jury composé de:

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Alla mia famiglia

Abstract

This thesis is devoted to the study of the applications of the error theory using Dirichlet forms. Our work is split into three parts. The first one deals with the models described by stochastic differential equations. After a short technical chapter, an innovative model for order books is proposed. We assume that the bid-ask spread is not an imperfection, but an intrinsic property of exchange markets instead. The uncertainty is carried by the Brownian motion guiding the asset. We find that spread evolutions can be evaluated using closed formulae and we estimate the impact of the underlying uncertainty on the related contingent claims. Afterwards, we deal with the PBS model, a new model to price European options. The seminal idea is to distinguish the market volatility with respect to the parameter used by traders for hedging. We assume the former constant, while the latter volatility being an erroneous subjective estimation of the former. We prove that this model anticipates a bid-ask spread and a smiled implied volatility curve. Major properties of this model are the existence of closed formulae for prices, the impact of the underlying drift and an efficient calibration strategy.

The second part deals with the models described by partial differential equations. Linear and non-linear PDEs are examined separately. In the first case, we show some interesting relations between the error and wavelets theories. When non-linear PDEs are concerned, we study the sensitivity of the solution using error theory. Except when exact solution exists, two possible approaches are detailed: first, we analyze the sensitivity obtained by taking “derivatives” of the discrete governing equations. Then, we study the PDEs solved by the sensitivity of the theoretical solutions. In both cases, we show that sharp and bias solve linear PDE depending on the solution of the former PDE itself and we suggest algorithms to evaluate numerically the sensitivities.

Finally, the third part is devoted to stochastic partial differential equations. Our analysis is split into two chapters. First, we study the transmission of an uncertainty, present on starting conditions, on the solution of SPDE. Then, we analyze the impact of a perturbation of the functional terms of SPDE and the coefficient of the related Green function. In both cases, we show that the sharp and bias verify linear SPDE depending on the solution of the former SPDE itself.

Key words: error calculus, Dirichlet form, carré du champ operator, bias, sensitivity, stochastic differential equation, financial model, liquidity model, bid-ask spread, partial differential equation, non-linear PDE, stochastic partial differential equation.

Resumé

Cette thèse est consacrée à l'étude des applications de la théorie des erreurs par formes de Dirichlet. Notre travail se divise en trois parties. La première analyse les modèles gouvernés par une équation différentielle stochastique. Après un court chapitre technique, un modèle innovant pour les carnets d'ordres est proposé. Nous considérons que le spread bid-ask n'est pas un défaut, mais plutôt une propriété intrinsèque du marché. L'incertitude est portée par le mouvement Brownien qui conduit l'actif. Nous montrons que l'évolution des spread peut être évalué grâce à des formules fermés et nous étudions l'impact de l'incertitude du sous-jacent sur les produits dérivés. En suite, nous introduisons le modèle PBS pour le pricing des options européennes. L'idée novatrice est de distinguer la volatilité du marché par rapport au paramètre utilisé par les traders pour se couvrir. Nous assumons la première constante, alors que le deuxième devient une estimation subjective et erronée de la première. Nous prouvons que ce modèle prévoit un spread bid-ask et un smile de volatilité. Les propriétés plus intéressantes de ce modèle sont l'existence de formules fermés pour le pricing, l'impact de la dérive du sous-jacent et une efficace stratégie de calibration.

La seconde partie s'intéresse aux modèles décrit par une équation aux dérivées partielles. Les cas linéaire et non-linéaire sont analysés séparément. Dans le premier nous montrons des relations intéressantes entre la théorie des erreurs et celui des ondelettes. Dans le cas non-linéaire nous étudions la sensibilité des solutions à l'aide de la théorie des erreurs. Sauf dans le cas d'une solution exacte, il y a deux approches possibles: On peut d'abord discrétiser l'EDP et étudier la sensibilité du problème discrétisé, soit démontrer que les sensibilités théoriques vérifient des EDP. Les deux cas sont étudiés, et nous prouvons que les sharp et le biais sont solutions d'EDP linéaires dépendantes de la solution de l'EDP originale et nous proposons des algorithmes pour évaluer numériquement les sensibilités.

Enfin, la troisième partie est dédiée aux équations stochastiques aux dérivées partielles. Notre analyse se divise en deux chapitres. D'abord nous étudions la transmission de l'incertitude, présente dans la condition initiale, à la solution de l'EDPS. En suite, nous analysons l'impact d'une perturbation dans les termes fonctionnelles de l'EDPS et dans le coefficient de la fonction de Green associée. Dans le deux cas, nous prouvons que le sharp et le biais sont solutions de deux EDPS linéaires dépendantes de la solution de l'EDPS originale.

Key words: calcul d'erreur, formes de Dirichlet, opérateur carré du champ, biais, sensibilité, équations différentielles stochastiques, modèles financières, modèles de liquidité, spread bid-ask équations aux dérivées partielles, EDP non-linéaires, équations stochastiques aux dérivées partielles

Riassunto

Questa tesi é dedicata allo studio delle applicazioni della teoria degli errori tramite forme di Dirichlet, il nostro lavoro si divide in tre parti. Nella prima vengono studiati i modelli descritti da un'equazione differenziale stocastica: dopo un breve capitolo con risultati tecnici viene descritto un modello innovativo per i libri d'ordini. La presenza dei differenziali denaro-lettera viene considerata non come un'imperfezione, bensì una proprietà intrinseca dei mercati. L'incertezza viene descritta come un rumore sul moto Browniano sottostante all'azione; dimostriamo che l'evoluzione di questi differenziali può essere valutata attraverso formule chiuse e stimiamo l'impatto dell'incertezza del sottostante sui prodotti derivati. In seguito proponiamo un nuovo modello, chiamato PBS, per il prezzaggio delle opzioni di tipo europeo: l'idea innovativa consiste nel distinguere la volatilità di mercato dal parametro usato dai trader per la copertura. Noi supponiamo la prima costante, mentre il secondo diventa una stima soggettiva ed erronea della prima. Dimostriamo che questo modello prevede dei differenziali lettera-denaro e uno smile di volatilità implicita. Le maggiori proprietà di questo modello sono l'esistenza di formule chiuse per il pricing, l'impatto del drift del sottostante e un'efficace strategia per la calibrazione.

La seconda parte è dedicata allo studio dei modelli descritti da delle equazioni alle derivate parziali. I casi lineare e non-lineare sono trattati separatamente. Nel primo caso mostriamo interessanti relazioni tra la teoria degli errori e quella delle wavelets. Nel caso delle EDP non-lineari studiamo la sensibilità della soluzione usando la teoria degli errori. Due possibili approcci esistono, salvo quando la soluzione è esplicita. Possiamo prima discretizzare il problema e studiare la sensibilità delle equazioni discretizzate, oppure possiamo dimostrare che le sensibilità teoriche verificano, a loro volta, delle EDP dipendenti dalla soluzione della EDP iniziale. Entrambi gli approcci sono descritti e vengono proposti degli algoritmi per valutare le sensibilità numericamente.

Infine, la terza parte è dedicata ai modelli descritti da un'equazione stocastica alle derivate parziali. La nostra analisi é divisa in due capitoli. Nel primo viene studiato l'impatto di un'incertezza, presente nella condizione iniziale, sulla soluzione dell'EDPS, nella seconda si analizzano gli impatti di una perturbazione dei termini funzionali dell'EDPS del coefficiente della funzione di Green associata. In entrambi i casi dimostriamo che lo sharp e la discrepanza sono soluzioni di due EDPS lineari dipendenti dalla soluzione dell'EDPS iniziale.

Key words: calcolo degli errori, forme di Dirichlet, operatore carré du champ, discrepanza, sensibilità, equazioni differenziali stocastiche, modelli finanziari, modelli di liquidità, differenziali denaro-lettera equazioni alle derivate parziali, EDP non-lineari, equazioni stocastiche alle derivate parziali.

SYNTHESE en FRANCAIS

La thèse que je présente ici est dédiée à l'étude et aux applications de la théorie des erreurs par formes de Dirichlet, cette technique a été introduite par Nicolas Bouleau dans l'article [7] en 2001, une analyse plus détaillée est décrite dans la monographie [8]. Quoique très récent, cette méthodologie présente déjà des intéressantes développements théoriques et appliqués dans plusieurs domaines des mathématiques appliquées, par exemple les travaux de Christophe Chorro dans sa thèse de doctorat [13] sur les liaisons entre les opérateurs en statistique et ceux de la théorie des erreurs, ou bien les vibrations non-linéaires dans les cristaux décrites dans le livre susmentionné.

Une présentation générale de la théorie des erreurs sera décrite dans l'introduction de cette thèse, même sans avoir un but exhaustif. J'aimerais quand même souligner quelques aspects: déjà le fait que l'innovation dans cette théorie soit inhérente à l'union de la théorie des erreurs de Gauss [23] et celle du potentiel qui est muni de deux opérateurs équipés des bonnes propriétés pour représenter la variance et le biais. A mon avis, ce dernier opérateur est le point innovant de la théorie; donc, entre les possibles axes de recherche, j'ai consacré le travail de ma thèse à ce sujet. J'ai décidé de répartir mes contributions dans trois parties, en les distinguant grâce à l'outil mathématique à la base du modèle analysé, c'est-à-dire les équations différentielles stochastiques, les équations aux dérivées partielles et les équations stochastiques aux dérivées partielles. Objectif

de cette section est celui de résumer une vue d'ensemble des résultats obtenues pendant cette thèse de doctorat.

Part II: Equations différentielles stochastiques et finance

Dans cette partie de la thèse nous appliquons la théorie des erreurs par formes de Dirichlet aux modèles dirigés par une équation différentielle stochastique, la modélisation en finance représente une des domaines les plus prolifiques. Cette partie se divise en 5 chapitres.

Le court chapitre 2 présente une analyse de l'opérateur de biais pour les structures d'erreurs de type Ornstein-Uhlenbeck pondéré dans l'espace de Wiener. Nous étendons quelques résultats sur le conditionnement, due à Nicolas Bouleau, à l'opérateur de biais. Les théorèmes démontrés ont des applications directes dans le chapitre suivant.

Le chapitre 3 résume le travail développé en collaboration avec Vathana Ly Vath, maître de conférence à l'université d'Evry. nous décrivons un modèle innovant pour la représentation des carnets d'ordres et des spread bid-ask toujours présents dans les prix des actifs financiers. En partant de l'hypothèse que ces spread ne sont pas un défaut, mais qui sont plutôt inhérents à la nature elle-même des échanges sur le marché, nous avons représenté ces incertitudes à l'aide d'un bruit sur le mouvement Brownien qui régit l'équation différentielle stochastique. L'aversion à l'incertitude et l'hypothèse de l'agent représentatif du marché nous permettent de justifier la présence des spread, alors que leur ampleur et leur évolution sont analysés à l'aide de la théorie des erreurs par formes de Dirichlet, qui nous permet d'évaluer les spread avec des formules semi-

fermés dans le cas d'un modèle à volatilité locale. Enfin, la robustesse de la théorie des erreurs nous autorise à évaluer l'impact des incertitudes du sous-jacent sur les prix des produits dérivés et de prévoir leur spread.

Les chapitres 4, 5 et 6 sont dédiés à l'étude d'un nouveau modèle pour l'évaluation des prix des options de type européenne. En particulier, le chapitre 4 présente le modèle dit Black Scholes perturbatif, l'idée est d'assumer que les prix des actions sous-jacentes suivent effectivement une diffusion log-normale (comme dans le cas du modèle Black Scholes), malheureusement les traders ne sont pas informés de la valeur des paramètres, en particulier de la volatilité, et ils ont obligés à faire des estimations. C'est bien évident que le résultat d'une telle statistique est une variable aléatoire, caractérisé pas une variance, généralement petite mais pas nulle. Nous étudions comment cette incertitude est transmise aux prix des options. L'argument clef réside dans l'évaluation du processus de profit et perte à la maturité qui, plutôt que être identiquement nulle, devient une variable aléatoire caractérisé par une variance et un petit biais, à cause de la convexité du payoff. La variance nous permet de justifier la présence d'une différence entre le prix d'achat et celui de vente, alors que le biais permet de reproduire le smile de volatilité. Une des plus intéressantes particularités du modèle Black Scholes perturbatif est le fait de préserver des formules fermés pour le pricing et les grecques de toutes les options qui ont la même caractéristique dans la diffusion de Black Scholes; calls, puts, options forward and une grand partie des options barrière rentrent dans cette catégorie. Ce chapitre fait l'objet de l'article [37].

Le chapitre 5 résume le travail développé en collaboration avec Luca Regis, PhD student à l'université de Turin. Nous montrons la présence d'un impact du paramètre de dérive du sous-jacent sur les prix des options, contrairement aux autres modèles actuellement utilisés en

finance. La différence entre les prix du modèle Black Scholes perturbative avec et sans dérive est évalué et on préserve le caractère fermé pour les formules du pricing. En particulier, on arrive à reproduire une courbe de volatilité implicite convexe (comme dans le cas du modèle sans dérive) mais aussi décroissante à la monnaie forward, en accord avec les statistiques du marché.

Enfin, le chapitre 6 décrit une stratégie efficace pour la calibration du modèle Black Scholes perturbative en utilisant les swap de variance. Nous étudions ces produits et le prix prévu par notre modèle; quelques caractéristiques de ces produits nous permettent de développer une méthodologie de calibration très robuste. Ce chapitre fait l'objet de l'article [39].

Part III: Equations aux dérivées partielles et physique

Dans cette partie nous appliquons la théorie des erreurs par formes de Dirichlet aux modèles décrites par une équation aux dérivées partielles; les applications sont nombreux, en particulier un grand nombre des modèles d'évolution en physique sont décrites par ce type d'équations, autres champs d'applications sont la finance et l'économie. Cette partie se divise en trois chapitres.

Le chapitre 7 est dédié à l'étude des équations aux dérivées partielles de type linéaire. La linéarité joue un rôle particulier dans plusieurs modèles physiques et elle nous permet de développer la solution en série. Dans notre analyse nous exploitons cette propriété en ayant recours à une base de représentation en ondelettes (wavelets), car ce type de base montre des intéressants phénomènes d'échelle. Nous étudions les interactions entre ondelettes, équations aux dérivées partielles et théorie des erreurs par formes de Dirichlet. Enfin nous analysons une application en

finance dans le cas des modèles dit de "volatilité à la Black". Ce chapitre fait l'objet de l'article [38].

Les chapitres 8 et 9 se concentrent sur l'étude des équations non-linéaires aux dérivées partielles. Le principal champ d'application est la mécanique des fluides, car l'équation originale dans ce domaine, c'est-à-dire les équations de Navier et Stokes, est de ce type. Pour décrire les applications possibles de la théorie des erreurs par formes de Dirichlet, nous avons considéré un cas moins complexe, c'est-à-dire les équations des eaux peu profondes, dites aussi équations de Saint-Venant. La non-linéarité de ces équations nous oblige à discrétiser les équations et trouver la solution sur un réseau. Deux stratégies sont envisageables: la première, analysé dans le chapitre 8, discrétise d'abord les équations aux dérivées partielles et, en suite, sur un problème réduit à une dimension finie, étudie la sensibilité de la solution à l'aide de la théorie des erreurs. Le deuxième approche, décrite dans le chapitre 9, démontre que le sharp et le biais de la solution théorique vérifient deux équations aux dérivées partielles de type linéaire et dépendantes de la solution du problème original lui-même. La discrétisation se passe seulement après ce stade pour pouvoir étudier, au même temps, la solution et sa sensibilité. Les résultats du chapitre 8 font l'objet de l'article [40].

Part IV: Equations stochastiques aux dérivées partielles et climatologie

Dans cette partie nous appliquons la théorie des erreurs par formes de Dirichlet aux modèles décrits par une équation stochastique aux dérivées partielles, l'étude de ce type d'équations est assai

récent, les applications sont nombreux et embrassent plusieurs domaines, comme la climatologie, avec les modèles stochastiques pour le climat (stochastic climate models), la physique appliqué ou la finance, pour la modélisation de la courbe des taux. Cette partie se divise en deux chapitres.

Le chapitre 10 commence avec une brève introduction à la théorie des équations stochastiques aux dérivées partielles. En suite, nous étudions l'impact d'une incertitude, présente dans la donnée initiale, sur la solution de l'équation stochastique aux dérivées partielles, en particulier nous soulignons que le sharp et le biais vérifient leur aussi deux équations stochastiques aux dérivées partielles, de type linéaire et dépendantes de la solution de l'EDPS originaire. Ce chapitre fait l'objet de l'article [41].

Le chapitre 11 analyse l'impact sur la solution d'une équation stochastique aux dérivées partielles, d'abord de la présence d'une incertitude dans les coefficients fonctionnelles, ensuite dans la constante de diffusion de la fonction de Green associé à l'EDPS. Dans le premier cas nous avons montré que le sharp et le biais sont solutions de deux équations stochastiques aux dérivées partielles, dépendantes de la solution même de l'EDPS initiale. Dans le deuxième cas nous avons étudié uniquement l'opérateur sharp; nous avons explicité la suite de Picard vérifié par cet opérateur et nous avons introduit une nouvelle type d'équation stochastique aux dérivées partielles vérifié par le sharp. Enfin, nous avons présenté quelques applications classiques des équations stochastiques aux dérivées partielles en climatologie, génétique, finance et théorie de l'assurance.

SINTESI in ITALIANO

La tesi che presento ha come oggetto lo studio e l'applicazione della teoria degli errori tramite l'uso delle forme di Dirichlet; questa tecnica viene descritta nell'originale articolo [7] di Nicolas Bouleau del 2001, che l'autore poi riprende con una più dettagliata analisi nella monografia [8] del 2003. Benché assai recente, tale metodologia mostra già interessanti sviluppi teorici ed applicazioni in vari ambiti della matematica applicata, quali ad esempio i lavori di Christophe Chorro nella sua tesi di dottorato [13] sulle relazioni tra alcuni operatori in statistica e quelli della teoria degli errori, oppure le vibrazioni non-lineari nei cristalli descritte nella sopracitata monografia.

Una descrizione generale, anche se non esaustiva, della teoria degli errori verrà presentata nell'introduzione. Vorrei però sottolinearne alcuni aspetti cruciali come il fatto che l'innovazione in questa teoria sia insita nell'unione tra la storica teoria degli errori di Gauss [23] e la moderna teoria del potenziale, che ha due operatori dotati delle buone proprietà per rappresentare la varianza e la discrepanza rispetto al valore atteso. Quest'ultimo operatore è, a mio parere, il più interessante risultato della metodologia; pertanto il mio lavoro di tesi, tra i vari possibili assi di ricerca, vi è dedicato. Ho scelto di dividere i contributi, svolti in questi anni, in tre parti, usando come elemento di distinzione lo strumento matematico alla base del modello analizzato, cioè le equazioni differenziali stocastiche, le equazioni alle derivate parziali e le equazioni stocastiche alle derivate parziali. Obiettivo di questa sezione è presentare, il più semplicemente possibile, una

veduta d'insieme dei risultati ottenuti durante il mio lavoro per la tesi di dottorato.

Part II: Equazioni differenziali stocastiche e finanza

In questa parte viene applicata la teoria degli errori tramite l'uso delle forme di Dirichlet ai modelli descritti da un'equazione differenziale stocastica; uno dei più prolifici campi di applicazioni sono i modelli finanziari per la descrizione dell'evoluzioni dei prezzi di attivi finanziari, come ad esempio azioni, obbligazioni, materie prime, etc..., valutazione e copertura dei relativi prodotti derivati. Questa sezione si divide in 5 capitoli.

Il breve capitolo 2 è dedicato all'analisi dell'operatore di discrepanza delle strutture d'errore di tipo Ornstein Uhlenbeck ponderato sullo spazio di Wiener: vengono estesi alcuni risultati di Nicolas Bouleau sul condizionamento all'operatore di discrepanza. I teoremi dimostrati hanno una applicazione diretta nel capitolo successivo.

Il capitolo 3 riassume il lavoro svolto in collaborazione con Vathana Ly Vath, professore all'Università di Evry Val d'Essonne. Qui viene descritto un innovativo modello per la descrizione dei differenziali lettera-denaro presenti nei prezzi degli attivi finanziari: Partendo dall'ipotesi che questi differenziali non rappresentino un difetto, ma che invece siano inerenti alla natura stessa del mercato, abbiamo descritto questa l'incertezza come un rumore presente sul moto Browniano. L'avversione all'incertezza e l'ipotesi dell'agente rappresentativo ci permettono di giustificare la presenza dei differenziali, il loro valore e la loro evoluzione viene invece studiata tramite la teoria degli errori che ci permette di valutare, con formule semichiusse, lo spread bid-ask nel caso di

un modello a volatilità locale. Infine la robustezza della teoria degli errori permette di valutare l'impatto sui prezzi e sugli spread dei prodotti derivati.

I capitoli 4, 5 e 6 descrivono uno nuovo modello per la valutazione dei prezzi delle opzioni di tipo europeo. Il capitolo 4, in particolare, presenta il modello detto “Black-Scholes perturbativo”, l'idea è di considerare che il prezzo delle azioni segua effettivamente una diffusione log-normale, come nel caso del modello di Black Scholes, ma che sfortunatamente però i trader non conoscano il valore dei parametri (in particolare la volatilità) e siano costretti a fare delle stime. Tali statistiche fan sì che la volatilità sia caratterizzata da una, seppur piccola, varianza; noi studiamo quindi come questa incertezza si trasferisca sulle opzioni.

L'argomento chiave utilizzato è il processo di profitto e perdita del trader valutato a maturità, che, invece di essere costantemente uguale a zero, è caratterizzato da una varianza e da una piccola discrepanza rispetto al valore teorico; la varianza ci permette di spiegare gli ampi differenziali denaro-lettera sulle opzioni, la discrepanza invece permette di riprodurre un effetto noto in finanza con il nome di smile di volatilità.

La più interessante peculiarità del modello Black Scholes perturbativo è l'avere formule chiuse per il pricing e per le greche di qualunque opzione che abbia la stessa caratteristica nella diffusione di Black Scholes: call, put, opzioni forward e alcune opzioni barriera rientrano in questa categoria. Questo capitolo è l'oggetto dell'articolo [\[37\]](#).

Il capitolo 5 riassume il lavoro svolto in collaborazione con Luca Regis, PhD student presso l'Università di Torino. Qui mostriamo la presenza, diversamente dal modello Black Scholes e dagli altri modelli finanziari, di un impatto legato al parametro di drift. La differenza nei prezzi è valutata e il pricing in questo modello, detto Black Scholes perturbativo con drift, ha ancora

formule chiuse e la volatilità implicita presenta una pendenza negativa alla moneta forward. Gran parte del capitolo è dedicata all'analisi numerica del modello.

Infine, il capitolo 6 descrive una strategia efficace per la calibrazione del modello Black Scholes perturbativo utilizzando gli Swap di varianza. Viene presentata l'analisi di tali prodotti e del loro prezzo previsto dal nostro modello e alcune caratteristiche precipue di questi prodotti ci permettono di sviluppare una metodologia di calibrazione molto robusta. Questo capitolo è l'oggetto dell'articolo [39].

Part III: Equazioni alle derivate parziali e fisica

In questa parte viene applicata la teoria degli errori tramite l'uso delle forme di Dirichlet ai modelli descritti da un'equazione alle derivate parziali. Le applicazioni sono molteplici, in particolare quasi ogni equazione d'evoluzione in fisica viene descritta da questo tipo di equazioni, che rappresentano un utile strumento anche in finanza ed economia. Questa sezione si divide in 3 capitoli.

Il capitolo 7 è dedicato allo studio delle equazioni lineari alle derivate parziali. La proprietà di linearità gioca un ruolo importante in molti modelli fisici ed economici e permette di riscrivere la soluzione sotto forma di uno sviluppo in serie, in cui ogni funzione verifica la stessa equazione di partenza. Nella nostra analisi sfruttiamo tale proprietà ricorrendo ad una base di rappresentazione in ondine (wavelets), poiché tale base presenta interessanti fenomeni di scala. Vengono studiate le interazioni tra wavelets, equazioni alle derivate parziali e teoria degli errori tramite forme di Dirichlet. Infine, viene presentata un'interessante applicazione in finanza nel caso dei modelli

detti di volatilità alla Black. Questo capitolo è l'oggetto dell'articolo [38].

I capitoli 8 e 9 sono dedicati allo studio delle equazioni non-lineari alle derivate parziali. Tali equazioni hanno numerose applicazioni in meccanica dei fluidi, basti ricordare a tal proposito le equazioni principe di Navier and Stokes. Per descrivere le possibili applicazioni della teoria degli errori tramite forme di Dirichlet abbiamo considerato un caso meno complesso, cioè le equazioni delle acque poco profonde, anche conosciute come equazioni di Saint Venant. La non linearità di queste equazioni ci obbliga a discretizzare la soluzione su un reticolo. Due strategie sono, a nostro avviso, possibili: la prima, analizzata nel capitolo 8, discretizza l'equazione alle derivate parziali e solo in seguito, su un problema ridotto a dimensione finita, studia la sensibilità della soluzione del problema discretizzato. Il secondo approccio invece, presentato nel capitolo 9, mostra che lo sharp e la discrepanza della soluzione teorica delle equazioni di Saint Venant verificano due equazioni alle derivate parziali di tipo lineare e dipendenti dalla soluzione del problema originario. La discretizzazione avviene solo a questo punto per poter risolvere il problema insieme alla sensibilità della soluzione. I risultati del capitolo 8 sono l'oggetto dell'articolo [40].

Part IV: Equazioni stocastiche alle derivate parziali e climatologia

In questa parte viene applicata la teoria degli errori tramite l'uso delle forme di Dirichlet ai modelli descritti da un'equazione stocastica alle derivate parziali. Lo studio di questo tipo di equazioni è abbastanza recente, le applicazioni sono molteplici e spaziano in vari campi, come ad esempio la climatologia, in cui esiste una classe di modelli di questo tipo detti modelli stocastici per il clima,

la fisica basso-energetica, per i problemi di interazione tra corpi, e la finanza, per la descrizione dell'evoluzione delle curve dei tassi. Questa sezione si divide in 2 capitoli.

Il capitolo 10 inizia con una breve presentazione della teoria delle equazioni stocastiche alle derivate parziali. Viene poi studiato l'impatto di una incertezza, presente nella condizione iniziale, sulla soluzione dell'equazione stocastica alle derivate parziali ed in particolare viene sottolineato che sia lo sharp, sia la discrepanza verificano a loro volta delle equazioni stocastiche alle derivate parziali di tipo lineare e dipendenti dalla soluzione. Questo capitolo è l'oggetto dell'articolo [41].

Il capitolo 11 analizza due tipi di perturbazione della soluzione di un'equazione stocastica alle derivate parziali, la prima dovuta a un'incertezza nei suoi coefficienti funzionali, la seconda nella costante di diffusione della funzione di Green associata all'EDPS.

Nel primo caso abbiamo dimostrato che sia lo sharp sia la discrepanza verificano due equazioni stocastiche lineari alle derivate parziali, dipendenti dalla soluzione dell'equazione originaria.

Nel secondo caso ci siamo interessati unicamente all'operatore di sharp, abbiamo trovato la serie di Picard verificata da quest'ultimo ed abbiamo formalmente introdotto un nuovo tipo di equazione stocastica alle derivate parziali verificata dallo sharp medesimo. Infine, abbiamo presentato qualche applicazione classica delle equazioni stocastiche alle derivate parziali ai modelli in climatologia, in genetica, in finanza ed in teoria assicurativa.

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Contents

I	Introduction	1
1.1	Gauss' error calculus	5
1.2	Error calculus with Dirichlet forms	11
1.3	Images and products of error structures	14
1.3.1	Images of error structures	14
1.3.2	Products of error structures	15
1.4	Error structures on Wiener space	17
1.4.1	Ornstein Uhlenbeck structure	18
1.4.2	Weighted Ornstein Uhlenbeck structure	21
1.5	Error structures on functional coefficients	23
1.6	Regularity of the bias operator	25
1.7	Thesis contributions	25
	Bibliography	29
II	Stochastic Differential Equations and Finance	33
2	Bias Operator and Conditional Expectation	35
2.1	Bias and filtration	35
2.2	Conditional expectation and semi-group	37
3	An Order Books Model	39
3.1	Introduction	39
3.2	The model	41
3.3	Uncertainty of an asset due to Brownian motion	42
3.3.1	Markov property	44
3.3.2	Closed form for $A[X_t]$	45
3.3.3	Covariance between the asset, its variance and its bias	45
3.4	Applications in finance	47
3.4.1	Black volatility models	48
3.5	Sensitivity of contingent claims	50
3.6	Conclusion	54
	Appendix 3.A Proof of theorems	55
	3.A.1 Theorem 3.2	55
	3.A.2 Theorem 3.3	59

3.A.3	Lemma 3.5	61
4	Perturbative Black Scholes Model	63
4.1	Introduction	63
4.2	Perturbative Black Scholes model	64
4.2.1	“Mismatch” on trading hedging	65
4.2.2	Role and choice of h functions and relative index	69
4.3	Call options case and volatility smile	70
4.3.1	Smile of volatility	73
4.3.2	Dupire’s formula and implicit local volatility model	74
4.4	Extension and relation with literature	75
4.5	Risk aversion	76
4.6	Conclusion and economics interpretation	78
5	Drift and Asymmetry Corrections	79
5.1	Introduction	79
5.2	Perturbative Black Scholes model	81
5.3	Impact of the drift term in security pricing	82
5.3.1	European Call options	84
5.4	Numerical Analysis	86
5.4.1	Parameter sensitivity	86
5.4.2	Spread Analysis	89
5.5	Conclusion	95
Appendix 5.A	Computation	95
6	Calibration	99
6.1	Notation and preliminaries	99
6.2	Perturbative Black Scholes model	100
6.3	Variance Swaps	103
6.4	Calibration	104
6.4.1	Variance Swap constraint	104
6.4.2	Bias estimation using ATM volatility	106
6.4.3	Calibration procedure	107
6.5	Conclusion	107
	Bibliography	109
III	Partial Differential Equations and Physics	113
7	Linear PDEs and Wavelets	115
7.1	Wavelets theory	116
7.1.1	Daubechies wavelets	117
7.2	Linear partial differential equation	118
7.2.1	Diffusionlets	119

7.3	Sensitivity of LPDE solution	122
7.3.1	Uncertainty on the solution	122
7.4	Applications to finance	124
7.5	Conclusion	124
8	Sensitivity of Non-Linear PDEs: Discrete Approach	127
8.1	Shallow water equations	128
8.1.1	Galerkin-Variational Approach	133
8.1.2	Solution of shallow water PDEs	135
8.2	Sensitivity	136
8.2.1	Deterministic approach	137
8.2.2	Probabilistic approach	137
8.2.3	Key theorem	138
8.3	Application to shallow water equations	138
8.3.1	Variance of shallow water PDEs solution	140
8.3.2	Bias of shallow water PDEs solution	144
8.3.3	Algorithm	144
8.4	Conclusion	145
	Appendix 8.A Generalization with implicit scheme	146
	8.A.1 Key theorem in general case	146
	8.A.2 Variance of shallow water PDEs solution with implicit scheme	148
	8.A.3 Bias of shallow water PDEs solution with implicit scheme	152
9	Sensitivity of Non-Linear PDEs: Continuum Approach	155
9.1	Introduction	155
9.2	Shallow water equations and uncertainties	157
9.3	Variance of the shallow water problem	158
9.3.1	PDE verified by the sharp operator	159
9.3.2	Characterization of the PDE	161
9.3.3	Formal solution via Laplace transform	161
9.3.4	PDE verified by the carré du champ operator	164
9.3.5	Initialisation using stationary processes	166
9.4	Analysis via characteristic form	167
9.5	Bias of the shallow water problem	170
9.5.1	PDE verified by the bias	170
9.5.2	Analysis of the PDE	171
9.6	Conclusion	173
	Bibliography	175
IV	Stochastic Partial Differential Equations and Climatology	177
10	Uncertainty on Starting Condition	179
10.1	Introduction	179

10.2	Stochastic partial differential equations	180
10.2.1	Gaussian noise	180
10.2.2	Particular cases	181
10.2.3	General case	182
10.3	Uncertainties in SPDEs due to the starting condition	184
10.3.1	Variance and covariance of solution	186
10.3.2	Bias of solution	190
10.4	Conclusion	194
11	Uncertainty on Diffusion Parameters	197
11.1	Parabolic stochastic partial differential equations	197
11.2	Uncertainties in SPDEs due to functional coefficients	199
11.2.1	Variance and covariance of solution	200
11.2.2	Bias of solution	206
11.3	Uncertainties in SPDEs Green function	210
11.3.1	Variance and covariance of solution	213
11.4	Applications of stochastic partial differential equations	218
11.4.1	Climatology: evolution models	218
11.4.2	Population genetics	219
11.4.3	Finance: interest rate theory	219
11.4.4	Insurance: mortality risk	220
11.5	Conclusion	221
	Bibliography	223

Part I
Introduction

In order to describe an event in physics, climatology, finance or economics, we use a model. A good model should be flexible enough to represent several states, therefore generally depending on few parameters, e.g. the viscosity of a fluid in physics, the density of the the volatility and the drift in the Black Scholes model in finance. All these parameters have to be estimated, calibrated in finance, thanks to some statistics. Unfortunately the precision of these estimations is compromised by errors due to the inaccuracies of the measurement device.

In physics, the errors are classified into two families, systematic and accidental errors. The first ones can be defined as the difference between a measured value and the true value that is caused by non-random fluctuations from an unknown source. The accidental errors are the difference between an estimated value and the theoretically correct value that is caused by random, and inherently unpredictable fluctuations in the measurement set. The first kind of errors are easy to handle: if the cause of the systematic error can be identified, then it can usually be eliminated *ex ante* or *ex post*. Instead, the accidental errors are inherent in each measure, they can be explained as a sum of large number of facts with small impact, e.g. atomic interactions in physics or individual transactions in finance, of which we have to consider the effects. However, we have to make some simplifying assumptions, since we cannot know the exact status.

These models are used to compute relevant quantities used to take some decisions, e.g. to define the hedging portfolio in finance or to define the thickness of a dam in engineering sciences. If the uncertainty on the parameters is propagated through the computation, all results are uncertain.

The choice of a mathematical framework to describe uncertainties and their propagation is an old subject, the pioneering work is due to Gauss [23], other works are due to Legendre and Laplace, see [24] and [21], a clear presentation of the Gauss' theory is given by Poincaré [35]. After these classical results, the stochastic nature of many objects are proved, e.g. in game theory. Wald shows the importance of optimization strategies in stochastic framework, see [44], and the 20th century has seen the rise of subjective probability and its application is economic theory, see De Finetti [17].

Nowadays, the models make use of more and more refined techniques to take into account more phenomena. As an example, the modern theory in finance, after the seminal paper of Black and Scholes [5], uses regularly stochastic differential equations. Partial differential equations play a crucial role in engineering science and physics. Following the work of Hasselmann [27] showing the fundamental importance of stochasticity in climatology, meteorologists have recourse to stochastic partial differential equations.

The complexity of approaches and methodologies requires a strict mathematical language for speaking about errors, whereas their variety needs suppleness. There are three approaches in literature:

- a) the probabilistic approach
- b) the infinitesimal approach
- c) the infinitesimal-probabilistic approach.

The first approach is to describe the errors as random variables, this is, theoretically, the most correct procedure, since the accidental errors are caused by random. However, this strategy has some shortfalls. First, in order to work with random variables, we need to know the probability

law and, in multi-dimensional case, the joint laws. Unfortunately, the errors are poorly known, statistics can only specify some properties of random variables, e.g. mean, variance or other moments, but the knowledge of the probability law is beyond us. Furthermore, models are generally non-linear with respect to parameters, so the study of transmission becomes rapidly too complex, since the problem to define the image of a probability law is theoretically solvable but numerically too expensive.

The second approach is to take advantage of a peculiarity of errors, as their magnitude is generally small with respect to the estimated value of the respective parameters. Therefore, we can represent the errors as infinitely small quantities and use the classical differential calculus. This approach is very powerful, thanks to the tool in finite dimension, and is commonly used in engineering science. The generalization in infinite dimension is known as Gateaux derivatives. However, this strategy presents some drawbacks too, as a matter of fact, the errors are small but not infinitely small, so the result is an approximation. Another problem is the nature of objects. In this framework, the errors are not random variable. Therefore, it is hard to take into account the correlation between the quantities. But the crucial drawback is the loss of the biases. As a matter of fact, the result from computing the expectation of a non-linear function of a random variable is different from that of the computation of the function in the mean value of the random variable. This fact is known as Jensen's inequality, when the function is convex. This classical result hides a dramatic consequence in every single model, we can assume that our estimated parameters are unbiased, many statics provide them. However, if the model is not linear with respect to the parameters, the result can be biased, because of the previous result. This characteristic is intrinsic for a random error and the classical differential calculus cannot manage this fact.

The third approach combines the advantages of the two previous methodologies. The problem is to define a strict differential calculus when the objects are random variables, indeed, many types of convergence exists in probability, e.g. convergence in law, almost surely or in L^p -sense. A first study of this problem was done by Azencott [3] who has analyzed the Taylor expansion for a random variable. A new idea to study the uncertainties and their propagation is developed by Nicolas Bouleau in recent papers, see [8], [9] and [12], thanks to the powerful language of Dirichlet forms, see Albeverio [1], Bouleau and Hirsch [6] or Fukushima et al. [22]. Bouleau's idea is based on the following remark: two operators used in potential theory, i.e. the generator of semi-group and the carré du champ operator associated with the Dirichlet form, verify a chain rule that can be interpreted as the propagation of uncertainties.

This thesis produces many applications of the Bouleau's theory. These results can be divided into three macro classes depending on the mathematical tool used in the model:

- a) stochastic differential equations,
- b) partial differential equations,
- c) stochastic partial differential equations.

The applications of these tools range over the fields of economy, physics and atmospheric science. In particular stochastic differential equations (SDE) play an essential role in mathematical finance, partial differential equations (PDE) in engineering sciences, while stochastic partial differential equations (SPDE) are the basic tool for stochastic climate models. Thanks to this

classification, we divide the contributions of this thesis into three parts, with the exception of the survey of the subject in this introduction. In each part we present some original applications of Bouleau's theory.

Introduction scheme: The first section presents a survey of the seminal ideas behind errors theory: we assume that uncertainties are random and very small. Therefore we combine the powerful probability language with the infinitesimal calculus, rich in tools. Sections 2 and 3 are devoted to the introduction of the Error Theory using Dirichlet Forms and its tools. The goal is to give the reader an overall view of this approach as well as its flexibility, strength and describe a large class of means to compute the propagation of errors. We abstain to show the theory of Dirichlet forms, since a rich literature exists in this domain but we decide to concentrate our attention, in this survey, on the main problem. In sections 1.4 and 1.5, we present two applications of error theory in Wiener and functional spaces respectively, these two domains play a crucial role in the original works showed in this thesis. Section 1.6 focus on a mathematical difficulty on the bias operator. Section 1.7 resumes the original contributions of the thesis, whereas the two final sections summarize the thesis in French and Italian languages.

1.1 Gauss' error calculus

The first study about the propagation of errors is due to Gauss [23]. Following this error theory, when we consider a function $F(x_1, x_2, \dots)$, depending on several erroneous parameters x_1, x_2, \dots , we can estimate the quadratic error on the value of F knowing the quadratic errors of parameters x_1, x_2, \dots and, under the hypothesis of independence, the Gauss answer is that the quadratic error is given by the formula

$$\text{Var}[F(x_1, x_2, \dots)] = \sum_i \left(\frac{\partial F}{\partial x_i}(x_1, x_2, \dots) \right)^2 \text{Var}[x_i].$$

This formula is usually used in experimental physics to compute the uncertainties. The crucial advantage of this formula with respect to others is the coherence, indeed, if it exists a function G and a series of functions H_i such that

$$F(x_1, x_2, \dots) = G(H_1(x_1), H_2(x_2), \dots),$$

then the quadratic error of F does not depend on the way to compute it.

We think back over the proof of the Gauss result to understand the key idea that will be used in the sequel to extend this theory thanks to Dirichlet forms.

We consider a quantity c , that we suppose estimated thanks to a measure or a statistics. This quantity is characterized by an uncertainty, denoted with δc . Let F be a smooth function from \mathbb{R} to \mathbb{R} , we search to evaluate the uncertainty of $F(c)$, denoted $\Delta F(c)$, due to the error on c . The probabilistic approach, basic in Gauss theory, assume that the couple $(c, \Delta c)$ is a realization of a random variable $(C, \Delta C)$, defined in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and we suppose known the conditional mean $\mathbb{E}[\Delta C | C]$, generally called bias, and the conditional variance $\text{Var}[\Delta C | C]$ of ΔC given the value of C . This information is smaller than the conditional law of ΔC given C , since this joint law is practically unattainable. We assume that errors are small and the conditional bias and variance of errors are of the same order of magnitude, therefore we write

$\Delta C = \sqrt{\epsilon} X$ where X is supposed to be a random variable, bounded to sake of simplicity, and ϵ a calibration parameter of the errors. With these assumptions, we compute the conditional variance and bias of $F(C)$ given C thanks to a Taylor expansion.

$$\begin{aligned}
 F(C + \Delta C) &= F(C) + F'(C) \Delta C + \frac{1}{2} F''(C) (\Delta C)^2 + o(\epsilon) \\
 (1.1) \quad \mathbb{E}[F(C + \Delta C) - F(C) | C] &= F'(C) \mathbb{E}[\Delta C | C] + \frac{1}{2} F''(C) \mathbb{E}[(\Delta C)^2 | C] + o(\epsilon) \\
 \mathbb{E}[\{F(C + \Delta C) - F(C)\}^2 | C] &= [F'(C)]^2 \mathbb{E}[(\Delta C)^2 | C] + o(\epsilon)
 \end{aligned}$$

We can interpret the term $\mathbb{E}[F(C + \Delta C) - F(C) | C]$ as the bias of the estimation of F , while $\mathbb{E}[\{F(C + \Delta C) - F(C)\}^2 | C]$ is the variance. It is clear that some ambiguity exists in this definition of error variance, since the general definition assumes that the variance to be the expectation of the squared deviation from the mean value, i.e.

$$\mathbb{E}[\{F(C + \Delta C) - \mathbb{E}[F(C + \Delta C) | C]\}^2 | C]$$

however the difference

$$\mathbb{E}[\{F(C + \Delta C) - \mathbb{E}[F(C + \Delta C) | C]\}^2 | C] - \mathbb{E}[\{F(C + \Delta C) - F(C)\}^2 | C] = o(\epsilon^2)$$

is negligible. The previous result (1.1) hides an interesting effect.

Remark 1.1 *The uncertainty on the value F is no longer centered even if the estimation of the parameter C is unbiased, except for a linear application, the figure 1.1 show this effect.*

If we combine a series of smooth functions $(F_i)_{i \in \mathbb{N}^*}$, we can find a general chain rule for small errors:

$$(1.2) \quad \left\{ \begin{aligned}
 Var [F_n \circ F_{n-1} \circ \dots \circ F_1(C) | C] &= [F'_n(C)]^2 Var [F_{n-1} \circ \dots \circ F_1(C) | C] + o(\epsilon) \\
 Bias [F_n \circ F_{n-1} \circ \dots \circ F_1(C) | C] &= F'_n(C) Bias [F_{n-1} \circ \dots \circ F_1(C) | C] \\
 &\quad + \frac{1}{2} F''_n(C) Var [F_{n-1} \circ \dots \circ F_1(C) | C] + o(\epsilon).
 \end{aligned} \right.$$

This transport formula is the seminal result of the error theory using Dirichlet forms. When we search to compute the main term of two first central moments of a random variable, we have to start with the variances, i.e. the second moment, and after we can compute the biases. As a matter of fact, the chain rule for variance is a first order differential calculus that does not involve biases. Instead, the transport formula for biases is a second order differential calculus and involves both biases and variances.

Due to this fundamental peculiarity, Bouleau has studied the error variances in his book, see [8]. This thesis, on the contrary, is focused on the bias operator and its applications in physics, economics and life's sciences.

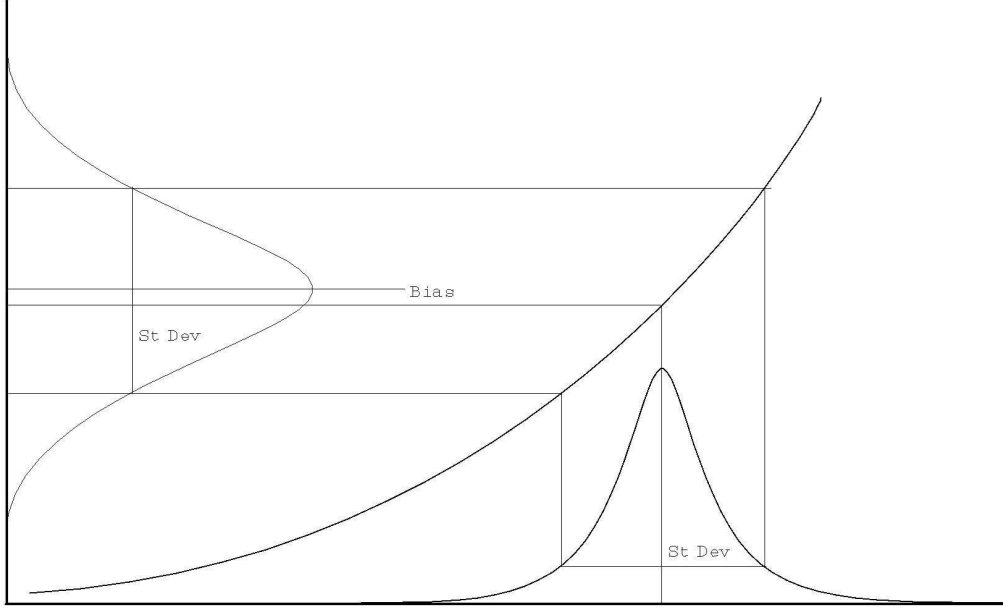


Figure 1.1: Impact of uncertainty on a parameter through a non-linear function.

When we consider the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_C)$ generated by the image of $(\Omega, \mathbb{F}, \mathbb{P})$ through the random variable C , we derive an operator Γ_C , called quadratic error operator, which, for any smooth function F , provides the conditional variance of the uncertainty on $F(C)$:

$$\Gamma_C[F] \circ C = \frac{\text{var} [\Delta(F \circ C) | C]}{\epsilon} \quad \mathbb{P}\text{-a.s.}$$

$$\Gamma_C[F](x) = \frac{\text{var} [\Delta(F \circ C) | C = x]}{\epsilon} \quad \mathbb{P}_C\text{-a.e..}$$

In a similar way, the operator Γ_C provides also the conditional covariance of the error on two functions, F and G :

$$\Gamma_C[F, G] \circ C = \frac{\text{covar} [\Delta(F \circ C) \Delta(G \circ C) | C]}{\epsilon} \quad \mathbb{P}\text{-a.s..}$$

This operator Γ_C acts on random variables, in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and satisfies the four following properties:

1. non-negativity, i.e. $\Gamma_C[F, F] \equiv \Gamma_C[F] \geq 0$,
2. symmetry, i.e $\Gamma_C[F, G] = \Gamma_C[G, F]$,
3. bi-linearity, i.e. $\Gamma_C \left[\sum_i \alpha_i F_i \sum_j \beta_j G_j \right] = \sum_{i,j} \alpha_i \beta_j \Gamma_C[F_i, G_j]$,

4. first order functional calculus on smooth functions

$$(1.3) \quad \Gamma_C [\varphi(F_1, \dots, F_n)] = \sum_{i,j} \frac{\partial \varphi}{\partial F_i}(F_1, \dots, F_n) \frac{\partial \varphi}{\partial F_j}(F_1, \dots, F_n) \Gamma[F_i, F_j].$$

In a similar way, we can define a second operator A_C which, for any smooth function F , provides the conditional bias of the uncertainty on $F(C)$:

$$\begin{aligned} A_C[F] \circ C &= \frac{\text{bias} [\Delta(F \circ C) | C]}{\epsilon} \quad \mathbb{P}\text{-a.s.} \\ A_C[F](x) &= \frac{\text{bias} [\Delta(F \circ C) | C = x]}{\epsilon} \quad \mathbb{P}_C\text{-a.e.} \end{aligned}$$

This operator A_C acts on random variables, in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and satisfies the following functional calculus on smooth functions.

$$(1.4) \quad \begin{aligned} A_C [\varphi(F_1, \dots, F_n)] &= \sum_i \frac{\partial \varphi}{\partial F_i}(F_1, \dots, F_n) A_C[F_i] \\ &+ \frac{1}{2} \sum_{i,j} \frac{\partial^2 \varphi}{\partial F_i \partial F_j}(F_1, \dots, F_n) \Gamma[F_i, F_j] \end{aligned}$$

In order to explain the meaning of quadratic error and bias operators, we consider a σ -algebra \mathcal{B} on Ω larger than \mathcal{F} such that the random variables and their errors are \mathcal{B} -measurable, whereas only the random variables are \mathcal{F} -measurable. In this sense, the previous conditional expectations are conditioned with respect to the σ -algebra \mathcal{F} .

$$\begin{aligned} \Gamma_C[F] \circ C &= \frac{\text{var} [\Delta(F \circ C) | \mathcal{F}]}{\epsilon} \quad \mathbb{P}\text{-a.s.} \\ A_C[F](x) &= \frac{\mathbb{E} [\Delta(F \circ C) | \mathcal{F}]}{\epsilon} \quad \mathbb{P}\text{-a.e.} \end{aligned}$$

Example 1.1 (GPS)

The Global Positioning System is a global navigation satellite system. Utilizing a constellation of satellites that transmit precise electromagnetic-wave signals, the system enables a GPS receiver to determine its location, speed, direction, and time. A typical GPS receiver calculates its position using the signals from four or more satellites, the receiver uses these measurements to solve an equation depending on the three spatial variables and the time. These values are then turned into more user-friendly forms, such as location on a map, then displayed to the user.

How works the GPS? Each satellite continuously broadcasts a message giving the time-of-day and its orbital position data. Knowing the position and the distance, computed using the time, of a satellite indicates that the receiver is located somewhere on the surface of an imaginary sphere centered on that satellite and whose radius is the distance to it.

Unfortunately, the measure of the distance between the satellite and the receiver is afflicted with an uncertainty. Many sources of noise exist: the speed of wave depends on the refractive index of the medium, different atmospheric conditions generate different refractive index; the precision of atomic clock inside the satellites; their synchronization; etc. These effects reduce the accuracy of the estimated position, resultant horizontal positional accuracies typically range between 10 and 30 meter and much larger for elevation measurements, for a technical analysis see [46].

We search to estimate the errors in this framework. We have three steps: first of all the choice of hypotheses about errors, the computation of the variances-covariances and, finally, the estimation of biases.

Hypotheses: For the sake of simplify, we consider a two dimensional space and two satellites (the local time is assumed known), we fix a Cartesian coordinate system, such that the two satellites have the coordinates $\left(\frac{d}{2}, 0\right)$ and $\left(-\frac{d}{2}, 0\right)$. Let L_1 and L_2 be the distance between the receiver and, respectively, the first and the second satellite, see figure 1.2.

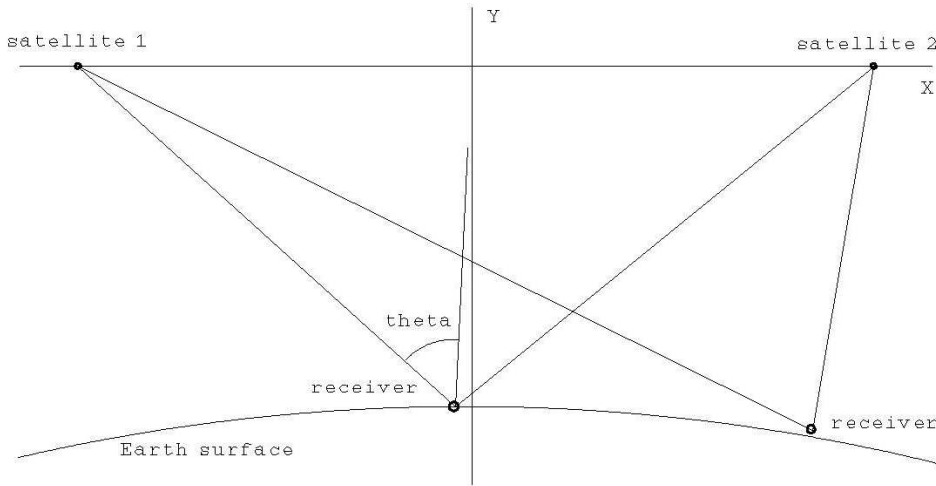


Figure 1.2: GPS working.

The estimated parameters L_1 and L_2 belong to a probability space that can be modeled as follows:

$$\left((0, \mathcal{L})^2, \mathcal{B}((0, \mathbb{L})^2), \frac{\lambda}{\mathcal{L}} \right)$$

where \mathcal{L} is a typical length, a.s. the maximal possible distance between a satellite and a receiver, i.e. 30000 km, and λ is the Lebesgue measure. Finally we assume that the quadratic error operator is modeled as follows:

$$(1.5) \quad \Gamma[f] = L_1^2 \left(\frac{\partial f}{\partial L_1} \right)^2 + L_2^2 \left(\frac{\partial f}{\partial L_2} \right)^2 + L_1 L_2 \frac{\partial f}{\partial L_1} \frac{\partial f}{\partial L_2}$$

This Gamma operator indicates that the errors on lengths are constant proportional and correlated. Finally we assume that length estimations are unbiased.

Variations: We compute the variance of uncertainties on the position of the receiver. the latitude/longitude (abscissa) and altitude (ordinate) of the receiver are given by

$$(1.6) \quad \begin{cases} X = \frac{L_2^2 - L_1^2}{2d} \\ Y = \frac{1}{2} \sqrt{2L_2^2 + 2L_1^2 - \frac{(L_2^2 - L_1^2)^2}{d^2} - d^2} \end{cases}$$

we use the functional calculus for Γ .

$$\begin{cases} \Gamma[X] = \frac{L_2^4 - L_1^2 L_2^2 + L_1^4}{d^2} \\ \Gamma[Y] = \frac{L_2^4 + L_1^2 L_2^2 + L_1^4 - 8X^2(L_2^2 + L_1^2)}{4Y^2} + \frac{X^2}{Y^2} \frac{L_2^4 - L_1^2 L_2^2 + L_1^4}{d^2} \\ \Gamma[X, Y] = \frac{X}{Y} \frac{d^2(L_2^2 + L_1^2) - L_1^4 + L_1^2 L_2^2 - L_2^4}{d^2} \end{cases}$$

An easy analysis proves that $X = 0$ is a minimum for the variance of X , in this case $L_1 = L_2 = \sqrt{Y^2 + d^2}$, so we find

$$\begin{cases} \Gamma[X]|_{X=0} = \left(\frac{d}{\sin \theta} \right)^2 \\ \Gamma[Y]|_{X=0} = \frac{3}{4} \left(\frac{d}{\sqrt{\sin \theta \cos \theta}} \right)^2 \\ \Gamma[X, Y]|_{X=0} = 0 \end{cases}$$

where θ is the semi-aperture of the cone generated by the two satellites with the receiver, see figure 1.2. However the covariance between X and Y is generally different to zero. Finally we remark that the latitude/longitude position is better estimated than the altitude when the angle θ is smaller than approximately 50 degrees.

Bias: We compute the bias of uncertainties on the position of the receiver.

$$\left\{ \begin{array}{l} \text{Bias}[X] = X \\ \text{Bias}[Y] = \left[\frac{d^2 + 2 d X - 2 L_1^2}{4 Y d^2} - \frac{L_1^2 (d + 2 X)^2}{8 Y^3 d^2} \right] L_1^2 \\ \quad + \left[\frac{d^2 - 2 d X - 2 L_2^2}{4 Y d^2} - \frac{L_2^2 (d - 2 X)^2}{8 Y^3 d^2} \right] L_2^2 \\ \quad + \frac{4 Y^2 - d^2 + 4 X^2}{8 Y^3 d^2} L_2^2 L_1^2 \end{array} \right.$$

We compute the value of the bias at the point $X = 0$ and we find

$$\left\{ \begin{array}{l} \text{Bias}[X]|_{X=0} = 0 \\ \text{Bias}[Y]|_{X=0} = -\frac{4 \cos^3 \theta + 3 \sin^2 \theta}{8 \cos^2 \theta \sin^3 \theta} d \end{array} \right.$$

therefore, we find that the estimation of latitude/longitude is unbiased, thanks to the symmetry of the problem, however the estimation of Y is negative biased.

This example shows the power of Gauss' formalism when we consider an explicit smooth function of parameters afflicted by uncertainties. Unfortunately, the great part of model models is defined via an implicit form, e.g. integrals, solutions of differential equations, PDEs, SDEs, SPDEs or fixed point problems. In these cases the classical Gauss' language cannot be applied and a refined theory is needed. A possible extension is proposed by Bouleau, in the next section we present a survey of this theory.

1.2 Error calculus with Dirichlet forms

The Bouleau's intuition is based on the remark that the two operators, Γ and A , i.e. the quadratic error and the bias, have the same chain rules of two operators existing in Dirichlet forms theory. Bouleau, in the paper [7], rewrite the Gauss' intuition in the rigorous frame of the Dirichlet form language. The basic tool is the error structure.

Definition 1.1 (Error structure)

An error structure is a term

$$\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma \right)$$

where

1. $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}} \right)$ is a probability space;

2. \mathbb{D} is a dense sub-vector space of $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$;
3. Γ is a positive symmetric bilinear application from $\mathbb{D} \times \mathbb{D}$ into $L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ satisfying the functional calculus of class $\mathcal{C}^1 \cap Lip$, i.e. if F and G are of class \mathcal{C}^1 and Lipschitzian, $\forall u$ and $v \in \mathbb{D}$, we have $F(u)$ and $G(v) \in \mathbb{D}$ and

$$(1.7) \quad \Gamma[F(u), G(v)] = F'(u)G'(v)\Gamma[u, v] \quad \tilde{\mathbb{P}} \text{ a.s.};$$

4. the bilinear form $\mathcal{E}[u, v] = \frac{1}{2}\tilde{\mathbb{E}}[\Gamma[u, v]]$ is closed, i.e. the space $(\mathbb{D}, \|u\|_{\mathbb{D}})$ is a Banach space, where the norm $\|u\|_{\mathbb{D}} = \sqrt{\|u\|_{L^2}^2 + \mathcal{E}[u, u]}$.
5. The constant function 1 belongs to \mathbb{D} , in this case the error structure is said Markovian.

For sake of simplicity, we will always write $\Gamma[u]$ for $\Gamma[u, u]$ and $\mathcal{E}[u]$ for $\mathcal{E}[u, u]$. First of all we have to emphasize the following crucial remark.

Remark 1.2 (Chain rule)

The function calculus (1.7) of the operator Γ is the same of the quadratic error operator, see equation (1.2). Therefore we can use the operator Γ to extend the Gauss' theory. Then, if $U = (U_1, \dots, U_n)$ belongs to \mathbb{D}^n , the matrix $\bar{\Gamma}[U] = [\Gamma[U_i, U_j]]_{0 \leq i, j \leq n}$ plays the role of the variance-covariance matrix of uncertainties on U given the value of U .

The bilinear form \mathcal{E} , introduced in the last definition, is known in the literature as a local Dirichlet form on $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, local means that for all $U \in \mathbb{D}$, let F and G be two smooth functions with non overlapped supports then $\mathcal{E}[F(U), G(U)] = 0$. The form \mathcal{E} possesses a carré du champ operator Γ , see Bouleau and Hirsch [6] page 16. This theory rise in the 20th century after the seminal work of Beurling and Deny [4] as a tool of potential theory, but it has a probabilistic interpretation in Markov process theory, see for example Fukushima et al. [22] or Silverstein [42].

Under some additional assumptions, see Bouleau and Hirsch [6] chapter 1 or Ma and Rockner [31], we can also associate a unique strongly-continuous contraction semi-group $(P_t)_{t \geq 0}$ on $L^2(\tilde{\mathbb{P}})$ with the error structure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma)$ via Hille-Yosida theorem, see Albeverio [1]. This semi-group is Markovian if and only if the property 5 in the definition 1.1 is verified, in this sense the error structure is said Markovian.

This semigroup has a generator $(\mathcal{A}, \mathcal{DA})$, with $\mathcal{DA} \subset \mathbb{D}$, i.e. a self-adjoint operator that satisfies, for all $F \in C^2$ with bounded derivatives, $U \in \mathcal{DA}$ and $\Gamma[U] \in L^2(\tilde{\mathbb{P}})$, $F(U)$ belongs to \mathcal{DA} and

$$(1.8) \quad \mathcal{A}[F(U)] = F'(U) \mathcal{A} + \frac{1}{2}F''(U) \Gamma[U] \quad \tilde{\mathbb{P}} - \text{a.s.},$$

a similar result exists when the function $F : \mathbb{R}^d \rightarrow \mathbb{R}$, see Bouleau [8] page 33.

We have defined the carré du champ Γ as a bilinear operator like variance-covariance. This characteristic is crucial in the error theory but, frequently, makes the tool very awkward to

perform computations. In the applications of classical Gauss' theory in physics, a squared root of the variance is often used, as known as the standard deviation. Is it possible to define a linear standard deviation operator in error theory using Dirichlet forms? The answer to this problem is positive under some constraints, when we consider random variables with values in an Hilbert space and if the domain \mathbb{D} of the carré du champ operator verifies the Mokobodzki hypothesis, see below. In this case, there are many linear versions of standard deviation of the error, called gradients, for more details see Bouleau [8] page 78. The existence of more than one gradient can be explain since the gradient is not intrinsic but is rather a derived concept depending on an exogenous space, as only Γ has an interpretation. However in the large class of gradients there is a preferable candidate called sharp.

Definition 1.2 (Sharp operator)

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma)$ be an error structure and $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ a copy of the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Under the Mokobodzki hypothesis, i.e. the space \mathbb{D} is separable, there exists an operator sharp $(\cdot)^\#$ with these three properties:

- $\forall u \in \mathbb{D}, u^\# \in L^2(\tilde{\mathbb{P}} \times \hat{\mathbb{P}})$;
- $\forall u \in \mathbb{D}, \Gamma[u] = \hat{\mathbb{E}} \left[(u^\#)^2 \right]$, where $\hat{\mathbb{E}}$ denotes the expectation under the probability $\hat{\mathbb{P}}$;
- $\forall u \in \mathbb{D}^n$ and $F \in \mathcal{C}^1 \cap Lip, (F(u_1, \dots, u_n))^\# = \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i} \circ u \right) u_i^\#$.

This operator is especially useful in Wiener space applications, see section 1.4. We conclude this section with some examples.

Example 1.2 (first error structure on \mathbb{R})

We consider the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R} and $\mathcal{N}(0, 1)$ denotes a reduced normal law.

We consider $\Gamma[u] \rightarrow (u')^2$ as the carré du champ operator, its domain will be $\mathbb{D} = H^1(\mathcal{N}(0, 1))$, i.e. the first Sobolev space with respect to the measure $\mathcal{N}(0, 1)$.

The term $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1), H^1(\mathcal{N}(0, 1)), (u')^2)$ is an error structure, because the operator $\Gamma[u] = (u')^2$ is the carré du champ operator of the Ornstein-Uhlenbeck Dirichlet form on \mathbb{R} . This identification supplies us the generator too, we have

$$\begin{aligned} \mathcal{A}[u] &= \frac{1}{2}u'' - \frac{1}{2}I \cdot u' \\ \mathcal{DA} &= \{u \in L^2(\mathcal{N}(0, 1)) \text{ with } u'' - x u' \text{ belongs to } L^2(\mathcal{N}(0, 1)) \text{ in the distribution sense}\} \end{aligned}$$

where I is the identity map on \mathbb{R} .

This example shows a powerful procedure in error theory using the language of Dirichlet forms. The Dirichlet forms theory is fifty-years-old, therefore, many Dirichlet forms have been identified. A classical approach in error theory is to recognize a known carré du champ to prove the closability of the related bilinear form, see hypothesis 4 in definition 1.1.

Example 1.3 (error structure on an interval)

Let $([0, 1], \mathcal{B}([0, 1]), \lambda)$ be our probability space where λ is the Lebesgue measure. Let

$$\begin{aligned}\Gamma[u] &= (u')^2 \\ \mathbb{D} &= H^1([0, 1]) = \{u \text{ and } u' \in L^2([0, 1], \lambda) \text{ in the distribution sense}\}\end{aligned}$$

be the carré du champ and its domain.

This term is an error structure, for the proof see Bouleau [8] pages 34-36, and the related generator is

$$\begin{aligned}\mathcal{A}[u] &= \frac{1}{2}u'' \\ \mathcal{DA} &= \{u \in C^2([0, 1]) \text{ with } u'(0) = u'(1) = 0\}\end{aligned}$$

More generally Hamza, see [26], has defined a necessary and sufficient condition, when the space is \mathbb{R} , for a couple, probability law and bilinear operator, to generate a Dirichlet form. However, this result cannot be generalized when the space has dimension bigger than one, see Fukushima et al [22] page 105.

Proposition 1.1 (Hamza 1975) *Let $(\mathbb{R}, \mathcal{F}, \mathbb{P})$ be a probability space on \mathbb{R} . We define $\Gamma[u](x) = (u'(x))^2 g(x)$, where $g(x)$ is a positive integrable function. The term $(\mathbb{R}, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ is an error structure, with \mathbb{D} suitable domain, if and only if*

1. *the measure $g \cdot \mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure*
2. *and its density, denoted $\eta(x)$ is worth zero a.e. on the set $\mathbb{R} \setminus \mathcal{R}(\eta)$ with*

$$\mathcal{R}(\eta) = \left\{ x \in \mathbb{R} \text{ such that } \exists \epsilon \int_{[x-\epsilon, x+\epsilon]} \frac{dy}{\eta(y)} < \infty \right\}.$$

1.3 Images and products of error structures

Errors structures are important objects but, when we use a model, we clash against two problems: first of all the greater part of models are non-linear with respect to parameters as well, generally, models depend on several parameters. This requires two operations that will be described in this section: the image of an error structure through a map and the product of error structures.

1.3.1 Images of error structures

Consider an error structure $S_1 = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ and let X be a random variable with codomain $\tilde{\Omega}$. Thanks to the classical probability theory, we can define a probability space

$$\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}} \right) = \left(\tilde{\Omega}, \tilde{\mathcal{F}}, (X_*\mathbb{P}) \right)$$

where $\check{\mathcal{F}}$ is a σ -algebra over the space $\check{\Omega}$ and $(X_*\mathbb{P})$ is the law of X , i.e. a non-negative measure on $\check{\Omega}$ such that $(X_*\mathbb{P})(E) = \mathbb{P}(X^{-1}(E))$ for all E belongs to $\check{\mathcal{F}}$. In order to define the error structure image of S_1 by X , we have only to set a coherent square of field operator and the related domain. We set

$$(1.9) \quad \begin{aligned} \Gamma_X[u](x) &= \mathbb{E}^{\mathbb{P}}[\Gamma[u \circ X] \mid X = x] \\ \mathbb{D}_X &= \{u \in L^2(X_*\mathbb{P}) \mid u \circ X \in \mathbb{D}\} \end{aligned}$$

where we use the classical notation for conditional expectation $\mathbb{E}^{\mathbb{P}}[Y \mid X] \equiv \mathbb{E}^{\mathbb{P}}[Y \mid \sigma(X)] = \phi(X)$ and $\mathbb{E}^{\mathbb{P}}[Y \mid X = x] = \phi(x)$.

Bouleau has proved the following proposition.

Proposition 1.2 (Image of Error Structure)

Let $S_2 = X_*S_1$ be the following term

$$S_2 = \left(\check{\Omega}, \check{\mathcal{F}}, X_*\mathbb{P}, \mathbb{D}_X, \Gamma_X[u](x) \right).$$

Then S_2 is an error structure. Moreover, if S_1 is Markovian then S_2 is Markovian too.

This proposition is proved in [8], pages 52-53. We show some examples:

Example 1.4 (constant proportional error)

We consider the classical Ornstein-Uhlenbeck structure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1), H^1(\mathcal{N}(0, 1)), u \rightarrow (u')^2)$ and the measurable function $x \rightarrow e^x$. The new probability space is $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), \nu)$, where ν is a log-normal measure, image of the law $\mathcal{N}(0, 1)$.

The new carré du champ operator is $\Gamma_{exp}[u](y) = y^2 (u'(y))^2$ and its domain is $\mathbb{D}_{exp} = \{u \in L^2(\nu) \mid y \rightarrow y^2 (u'(y))^2 \in L^1(\nu)\}$ where the belonging is in distribution sense.

This error structure, called homogeneous log-Ornstein-Uhlenbeck structure, is very useful in finance, when we consider a Black Scholes model for assets, see chapters 3 and 4.

1.3.2 Products of error structures

Another important tool in Bouleau's theory is the possibility to define an error structure thanks to a product of a finite or a countably infinite number of error structure. The infinite product is not allowed in Gauss theory, so the language of Dirichlet forms is useful to define errors for complex objects as integrals and functional objects.

We start with the analysis of the product between two error structures, the generalization when the number of error structures is bigger but finite is evident.

Proposition 1.3 (Finite Product of Error Structures)

Let $S_1 = (\Omega_1, \mathcal{F}_1, \mathbb{P}_1, \mathbb{D}_1, \Gamma_1)$ and $S_2 = (\Omega_2, \mathcal{F}_2, \mathbb{P}_2, \mathbb{D}_2, \Gamma_2)$ be two error structures, we define

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$$

$$\mathbb{D} = \left\{ u(x, y) \in L^2(\mathbb{P}_1 \times \mathbb{P}_2) \mid \forall x \mathbb{P}_1 - a.e. u(x, \cdot) \in \mathbb{D}_2; \forall y \mathbb{P}_2 - a.e. u(\cdot, y) \in \mathbb{D}_1 \right. \\ \left. \text{and } \int \left(\Gamma_1[u(\cdot, y)](x) + \Gamma_2[u(x, \cdot)](y) \right) d\mathbb{P}_1(x) d\mathbb{P}_2(y) < \infty \right\}$$

$$\Gamma[u](x, y) = \Gamma_1[u(\cdot, y)](x) + \Gamma_2[u(x, \cdot)](y)$$

Then the term $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ is an error structure, denoted $S = S_1 \times S_2$. Moreover, if S_1 and S_2 are both Markovian then S is Markovian too.

The product of two error structures describes the functions depending on two erroneous random variables but, implicitly, the two coordinate mappings and their uncertainties are independent. In order to describe correlated random variables with/or correlated uncertainties, we must consider the image structure through a function, this approach is usually applied in probability, in particular in stochastic processes theory.

The previous result holds the seminal argument to define an infinite product of error structures. As a matter of fact, an infinite product can be described as the limit of a finite product when the number of elements goes to infinity. If the factors have good properties the limit is well-defined, in the case of error structures we have the following proposition.

Proposition 1.4 (Infinite Product of Error Structures)

Let $S_n = (\Omega_n, \mathcal{F}_n, \mathbb{P}_n, \mathbb{D}_n, \Gamma_n)$ be a series of error structures, we define

$$(\Omega, \mathcal{F}, \mathbb{P}) = \left(\prod_{n=1}^{\infty} \Omega_n, \bigotimes_{n=1}^{\infty} \mathcal{F}_n, \prod_{n=1}^{\infty} \mathbb{P}_n \right)$$

$$\mathbb{D} = \left\{ u \in L^2(\mathbb{P}) \mid \forall k \text{ for almost every } x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots \text{ w.r.t. } \mathbb{P} \right. \\ \left. y \rightarrow u(x_1, x_2, \dots, x_{k-1}, y, x_{k+1}, \dots) \in \mathbb{D}_k \right. \\ \left. \text{and } \int \sum_{n=1}^{\infty} \Gamma_n[u] d\mathbb{P} < \infty \right\}$$

$$\Gamma[u] = \sum_{n=1}^{\infty} \Gamma_n[u]$$

Then the term $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ is an error structure, denoted $S = \prod_n S_n$. Moreover, if each S_n are Markovian then S is Markovian too.

This proposition is particularly useful when we search to define an error structure in an infinite dimensional space, e.g. the Wiener space.

1.4 Error structures on Wiener space

In order to define an error structure on Wiener space, we have to recall the definitions of Brownian motion and stochastic integral. Many approaches are practicable, however, to understand how to define an error structure on Wiener space, the best framework is the functional analysis in infinite dimension point of view, see Da Prato and Zabczyk [14].

We are given the Hilbert space $H = (L^2(\mathbb{R}^+), \mathcal{B}(L^2(\mathbb{R}^+)), \lambda)$, equipped with the inner product $\langle \cdot, \cdot \rangle$, a complete orthonormal basis ξ_n in H and a sequence of independent identically distributed reduced normal random variables β_n in a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We can define easily an homomorphism W from $L^2(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), \lambda)$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$, thanks to

$$W(f) = \sum_{n=1}^{\infty} \langle f, \xi_n \rangle \beta_n.$$

In this framework the image of the indicator function over the interval $[0, t]$ is a standard Brownian motion stopped at time t , denoted by B_t . For a complete proof that B_t verifies the properties of a Brownian motion see Da Prato [15] pages 35-39.

More generally $W(\cdot)$ is a linear operator called the stochastic (Wiener) integral and denoted

$$W(f) = \int_0^{\infty} f(s) dB_s,$$

for a complete analysis see Da Prato [15] pages 42-47.

Now we fix the space $(\Omega, \mathcal{F}, \mathbb{P})$ equal to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, where μ is a reduced Gaussian measure. A particular basis in the space $L^2(\mathbb{R}, \mu)$ can be defined in terms of the Hermite polynomials

$$(1.10) \quad H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{\frac{1}{2}x^2} \frac{\partial^n}{\partial x^n} e^{-\frac{1}{2}x^2} \quad \forall n \in \mathbb{N}.$$

We denote the closed subspaces $L_n^2(\mathbb{R}, \mu)$ of $L^2(\mathbb{R}, \mu)$ spanned by

$$\{H_n(W(f)) \text{ such that } f \in L^2(\mathbb{R}^+, \lambda) \text{ and } \langle f, f \rangle = 1\}.$$

In particular the space $L_0^2(\mathbb{R}, \mu)$ contains all non-stochastic functions, whereas $L_1^2(\mathbb{R}, \mu)$ is the space of gaussian random variables.

Proposition 1.5 (Wiener-Ito decomposition)

The system of the subspaces $L_n^2(\mathbb{R}, \mu)$ is orthonormal and we have

$$(1.11) \quad L^2(\mathbb{R}, \mu) = \bigoplus_{n=0}^{\infty} L_n^2(\mathbb{R}, \mu)$$

Formula (1.11) is called the chaos decomposition of $L^2(\mathbb{R}, \mu)$ and $L_n^2(\mathbb{R}, \mu)$ is the n-th component of the decomposition, for proofs and properties see Da Prato [15] chapter 9. This decomposition is very useful, when we work with a particular class of error structures in Wiener space,

called structures of the generalized Mehler type, see Bouleau [8] page 113. The n -th chaos can be described as the subspace spanned by

$$\int_{0 < t_1 < t_2 < \dots < t_n} h(t_1, t_2, \dots, t_n) dB_{t_1} dB_{t_2} \dots dB_{t_n}.$$

The previous description of chaos decomposition can be generalized when the space Ω is \mathbb{R}^d , see Da Prato and Zabczyk [14], or in more abstract space, see Gross [25].

1.4.1 Ornstein Uhlenbeck structure

In order to define the error structure, we can use the series β_n as a coordinate mappings of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, therefore we have

$$(\Omega, \mathcal{F}, \mathbb{P}) = \prod_{n=1}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1)).$$

If we take into account an uncertainty in each factor of the previous product, we can define an error structure via an infinite product of error structures, see section 1.3.2.

$$(1.12) \quad (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n=1}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1), \mathbf{d}_n, \gamma_n)$$

Where the domain \mathbb{D} contains a function $f(x_1, x_2, \dots, x_n, \dots)$ if and only if for any n the function

$$y \rightarrow f(x_1, x_2, \dots, x_{n-1}, y, x_{n+1}, \dots)$$

belongs to \mathbf{d}_n for all x_1, x_2, \dots \mathbb{P} -a.e. and

$$\Gamma[f] = \sum_n \gamma_n[f] \text{ belongs to } L^1(\mathbb{P})$$

where the operator γ_n acts only on the n -th variable of f .

The previous approach gives birth to a large class of error structures. As a matter of fact, we can choose the error structure in each factor, i.e. the bilinear form γ_n and its domain, as well as the correlation between two terms. Thanks to the basis β_n , our choice is equivalent to the definition of $\Gamma[\beta_n]$ and $\Gamma[\beta_n, \beta_m]$ for any n and $m \in \mathbb{N}$. In order to specify a particular error structure, we introduce two hypotheses easily to justify.

Hypothesis 1.1 (Independence)

Let u_n and u_m be two random variables belonging, respectively to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1), \mathbf{d}_n, \gamma_n)$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0, 1), \mathbf{d}_m, \gamma_m)$, two distinguished error structures in the product (1.12), then $\Gamma[u_n, u_m] = 0$

Hypothesis 1.2 (Isotropy)

Let β_n be the n -th element of the orthonormal basis of the probability space in the error structure (1.12), then $\Gamma[\beta_n]$ cannot depend on n .

The hypothesis of independence says that each original error structure is not correlated with another one, as the isotropy guarantees that all factors are indiscernible, since their work with the same weight together in the final error structure.

Moreover, if we suppose that the error structure on each factor is of type Ornstein-Uhlenbeck, see example 1.2, we find an error structure called Ornstein-Uhlenbeck error structure on Wiener space.

Proposition 1.6 (Ornstein Uhlebeck structure)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}_{OU}, \Gamma_{OU})$ be the error structure defined in equation (1.12), under hypotheses 1.1, 1.2 and that the carré du champ operator is of type Ornstein-Uhlenbeck in each factor, we obtain

$$\begin{aligned}
 \Gamma_{OU}[\beta_n, \beta_m] &= \delta_{n,m} \\
 \Gamma_{OU} \left[\int_0^\infty f(s) dB_s \right] &= \sum_{n=1}^\infty \langle f, \xi_n \rangle^2 = \|f\|_2^2 \\
 \Gamma_{OU} \left[F \left(\int_0^\infty f_1(s) dB_s \dots \int_0^\infty f_n(s) dB_s \right) \right] &= \sum_{i,j=1}^n \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \langle f_i, f_j \rangle_2
 \end{aligned}
 \tag{1.13}$$

where $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle_2$ denote, respectively, the norm and the inner product in $L^2(\mathbb{R}^+)$ space, we have also assumed that f_i belongs to $L^2(\mathbb{R}^+)$ and $F \in C^1$ and Lipschitz.

This result is an easy consequence of the properties of the Ornstein-Uhlenbeck error structure and the definition of stochastic integral, for more details see Bouleau [8] pages 101-102.

In order to define the sharp operator, we consider $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ a copy of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a series of independent identically distributed reduced normal random variables $\widehat{\beta}_n$, we denote \widehat{B}_t the related Brownian motion. Thanks to the definition 1.2 of sharp operator, we have

$$\begin{aligned}
\beta_n^\# &= \widehat{\beta}_n \\
B_t^\# &= \widehat{B}_t \\
[F(\beta_1, \dots, \beta_n, \dots)]^\# &= \sum_{i=1}^{\infty} \frac{\partial F}{\partial \beta_i} \widehat{\beta}_i \\
(1.14) \quad \left[\int_0^\infty f(s) dB_s \right]^\# &= \int_0^\infty f(s) d\widehat{B}_s \\
\forall X \in \mathbb{D}_{OU} \quad X^\#(\omega, \widehat{\omega}) &= \int_0^\infty \sum_n \frac{\partial X}{\partial \beta_n} \xi_n(s) d\widehat{B}_s \\
\Gamma_{OU}[X] &= \mathbb{E}^{\widehat{\mathbb{P}}} \left[(X^\#)^2 \right]
\end{aligned}$$

The Ornstein Uhlenbeck error structure on the Wiener space has been extensively studied, e.g. see Ikeda and Watanabe [29], Bouleau and Hirsch [6], Nualart [34], Malliavin [33], Ustunel and Zakai [43], etc. In particular, this structure preserves the Wiener chaos decomposition, see proposition 1.5, in the sense that if $X \in L^2(\mathbb{P})$ and

$$X = \sum_{n=0}^{\infty} X_n$$

where X_n are elements of $L_n^2(\mathbb{P})$ then

$$\begin{aligned}
\mathbb{D}_{OU} &= \left\{ X \in L^2(\mathbb{P}) \text{ such that } \sum_{n=0}^{\infty} n \mathbb{E}[X_n^2] < \infty \right\} \\
\Gamma_{OU}[X] &= \sum_{n=0}^{\infty} \sqrt{n} X_n \\
(1.15) \quad \mathcal{DA}_{OU} &= \left\{ X \in L^2(\mathbb{P}) \text{ such that } \sum_{n=0}^{\infty} n^2 \mathbb{E}[X_n^2] < \infty \right\} \\
\mathcal{A}_{OU}[X] &= - \sum_{n=0}^{\infty} \frac{n}{2} X_n
\end{aligned}$$

We remark that $\mathcal{DA} \subset \mathbb{D}$ and each subspace $L^2(\mathbb{P})$ is an eigenspace for Γ_{OU} and \mathcal{DA}_{OU} , the eigenvalues are, respectively n and $-\frac{n}{2}$.

1.4.2 Weighted Ornstein Uhlenbeck structure

The Ornstein-Uhlenbeck structure is very useful but it is not the unique error structure in Wiener space that exploits the construction (1.12). Let $X = F(\beta_1, \dots, \beta_n, \dots)$ be a bounded random variable where β_i represent the coordinate map into the previous decomposition. We define the operator P_t as

$$P_t(X) = \widehat{\mathbb{E}} \left[F \left(\beta_1 e^{-\frac{1}{2}\alpha_1 t} + \widehat{\beta}_1 \sqrt{1 - e^{-\alpha_1 t}}, \dots, \beta_n e^{-\frac{1}{2}\alpha_n t} + \widehat{\beta}_n \sqrt{1 - e^{-\alpha_n t}}, \dots \right) \right],$$

where $\widehat{\beta}_i$ are the coordinate mappings of a probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \widehat{\mathbb{P}})$, copy of the original space, $\widehat{\mathbb{E}}$ denotes the expectation under probability $\widehat{\mathbb{P}}$ and α_n are positive numbers, it is easy to prove that P_t is a strongly continuous Markov semi-group, see Bouleau [8], pages 113-114.

Let us define the following Dirichlet form and its domain.

$$\left\{ \begin{array}{l} \mathbb{D}_W = \left\{ X \in L^2(\mathbb{P}) \text{ such that } \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[(X - P_t(X)) X]}{t} < \infty \right\} \\ \mathcal{E}_W[X] = \lim_{t \rightarrow 0^+} \frac{\mathbb{E}[(X - P_t(X)) X]}{t} \end{array} \right.$$

This Dirichlet form generates the error structure

$$(1.16) \quad (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}_W, \Gamma_W) = \prod_{n=1}^{\infty} \left(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), u \rightarrow a_n (u')^2 \right),$$

where we can rewrite the domain and the carré du champ operator as

$$\left\{ \begin{array}{l} \mathbb{D}_W = \left\{ X = F(\beta_1, \beta_2, \dots) \in L^2(\mathbb{P}) \text{ such that } \forall n \frac{\partial F}{\partial \beta_n} \in H^1(\mu) \right. \\ \quad \left. \text{and } \sum_{n=1}^{\infty} a_n \left(\frac{\partial F}{\partial \beta_n} \right)^2 \in L^1(\mathbb{P}) \right\} \\ \Gamma_W[X] = \sum_{n=1}^{\infty} a_n \left(\frac{\partial F}{\partial \beta_n} \right)^2. \end{array} \right.$$

To work with this type of error structure, the following expression, called generalized Mehler formula, is useful.

$$(1.17) \quad P_t(F) = \widehat{\mathbb{E}} \left[F \left(\sum_n \langle \mathbb{I}_{[0, u]}, \xi_n \rangle e^{-\frac{1}{2}\alpha_n t} \beta_n + \sum_n \langle \mathbb{I}_{[0, u]}, \xi_n \rangle \sqrt{1 - e^{-\alpha_n t}} \widehat{\beta}_n \right) \right] \quad \forall F \in L^2(\Omega)$$

In the Ornstein-Uhlenbeck case, we have $a_n = 1$ for all n , so we find

$$P_t(B_s) = e^{-\frac{1}{2}t} B_s + \sqrt{1 - e^{-t}} \widehat{B}_s.$$

The class of error structures (1.16) is very large, we can distinguish a subclass, called the Weighted Ornstein-Uhlenbeck case.

Proposition 1.7 (Weighted Ornstein Uhlenbeck structure)

Let $\alpha(u)$ be a positive function belonging to $L^1(\mathbb{R}^+, \lambda)$ and let

$$(1.18) \quad P_t(F) = \widehat{\mathbb{E}} \left[F \left(\int_0^\infty e^{-\frac{1}{2}\alpha(u)t} dB_u + \int_0^\infty \sqrt{1 - e^{-\alpha(u)t}} d\widehat{B}_u \right) \right]$$

be the action of the semi-group, thanks to the Mehler formula. Then, the related error structure $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}_{WOU}, \Gamma_{WOU})$, called *Weighted Ornstein Uhlenbeck error structure with weight α* , acts on a stochastic integral in the following way.

$$(1.19) \quad \Gamma_{WOU} \left[\int_0^\infty f(t) dB_t \right] = \int_0^\infty \alpha(t) f^2(t) dt \quad \forall F \in L^2(\mathbb{R}^+)$$

This proposition is a direct consequence of the Mehler formula (1.18). The following proposition gives the sharp of the weighted Ornstein Uhlenbeck structure and the Ornstein Uhlenbeck one as a particular case.

Proposition 1.8 (Sharp on Weighted Ornstein Uhlebeck case)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}_{WOU}, \Gamma_{WOU})$ be a weighted Ornstein Uhlenbeck structure with weight $\alpha(t)$, then for all functions u belong to $L^2(\mathbb{R}^+, (1 + \alpha) dt)$ and for all adapted square integrable functions H_t we have

$$(1.20) \quad \begin{aligned} \left(\int_0^\infty u(s) dB_s \right)^\# &= \int_0^\infty \sqrt{\alpha(s)} u(s) d\widehat{B}_s \\ \left(\int_0^\infty H_s dB_s \right)^\# &= \int_0^\infty H_s^\# dB_s + \int_0^\infty \sqrt{\alpha(s)} H_s d\widehat{B}_s. \end{aligned}$$

The proof of this proposition comes from the definition of the sharp operator, in particular its linearity, and the representation of Γ_{WOU} , see (1.19), for more details see Bouleau [8] pages 165-167.

The two following propositions are helpful when we work with conditional expectations, e.g. in finance.

Proposition 1.9 (conditional expectation and domain \mathbb{D})

The conditional expectation operator $\mathbb{E}[\cdot | \mathcal{F}_t]$ maps \mathbb{D} into \mathbb{D} and is an orthogonal projector in \mathbb{D} .

Proposition 1.10 (conditional expectation and Γ)

Let us define Γ_t by

$$\Gamma_t \left[\int_0^\infty u(s) dB_s \right] = \Gamma_{WOU} \left[\int_0^\infty \mathbb{I}_{[0,t]}(s) u(s) dB_s \right]$$

and let $U \rightarrow U^{\#t}$ the sharp operator associated with Γ_t , then $\forall U \in \mathbb{D}$

$$(\mathbb{E}[U | \mathcal{F}_t]) = \mathbb{E}[U^{\#t} | \mathcal{F}_t]$$

The two previous properties of the structures $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}_{WOU}, \Gamma_{WOU})$ hide an important characterization, the error is *orthogonal* with respect to the time, i.e. the perturbation of the Brownian path does not involve the future trajectory, the proof of this fact is straight-forward using the Mehler formula (1.18). This advantage of Weighted Ornstein-Uhlenbeck is very useful in many applications, e.g. when markets are efficient in finance. However, a more complicate class of error structures is needed when we search to perturb the time, e.g. anisotropy in information diffusion in finance or colored noise in physics. The complexity of this problem induces us to restrict our work in this thesis to Weighted Ornstein-Uhlenbeck structure.

The study of Bouleau, about the carré du champ and the sharp of the Weighted Ornstein-Uhlenbeck structure, can be extended to bias operator, the chapter 2 studies this problem.

1.5 Error structures on functional coefficients

After the Wiener space, a second interesting space has a relevant importance in this work, the functional spaces. As a matter of fact, a large part of model depends on functions, e.g. in finance, the local volatility models depend on a functional coefficient, the volatility; in physics and climatology all boundary and starting conditions are functions, frequently measured and, therefore, afflicted by uncertainties.

We consider a function $f(\cdot)$ and we search to define an error structure to represent its uncertainty. The first plain idea is to introduce an external parameter α , supposed equal to one, and a function F such that $F(\cdot, 1) = f(\cdot)$, afterwards we define an error structure for α and we consider the function $F(\cdot, \alpha)$. Thanks to the image property, see section 1.3.1, we can define an error structure for the function $F(\cdot, \alpha)$, that is the function $f(\cdot)$. This approach is, probably, the easiest strategy to define an error structure for a function but it presents several drawbacks, the two principal ones are: First of all, all computed uncertainties depend on a single error structure, the one of the parameter α , therefore all uncertainties are perfectly correlated. Secondly, the result depends crucially on the form of the function $F(\cdot, \alpha)$ with respect to α , when the function is too plain, e.g. a re-scaling factor, there are no interesting effects, if the function is too complicate, it is impossible to compute the image error structure.

A second practicable strategy is to remark that the function f belongs to a functional space, e.g. the space $L^1(\mu)$ of integrable functions with respect to the measure μ , the space C^1 of differentiable functions, etc. Generally, the considered space is a vector space equipped with a non-unique frame ξ_n , i.e. an ordered basis; then we can expand our function in series with respect to this basis,

$$(1.21) \quad f(\cdot) = \sum_n a_n \xi_n$$

where a_n are the coordinates. As a example, we consider the space $C^\infty(\mathbb{R})$ of smooth functions in \mathbb{R} and let ξ_n be the Hermite polynomials, see equation (1.10). The coordinates are defined as a linear isomorphism ϕ from the space $\mathbb{R}^{\mathbb{N}}$ into $C^\infty(\mathbb{R})$.

In order to define an error structure on the considered space, a good idea is to use the isomorphism ϕ . We can define an error structure in each space \mathbb{R} , then we take the infinite product of these error structures to have an error structure on $\mathbb{R}^{\mathbb{N}}$. Finally, the isomorphism ϕ gives the error structure on the considered space, i.e. the closure of the space $C^\infty(\mathbb{R})$.

In practice, we randomize the coefficients of the decomposition of the function f into the basis ξ_n rather than the function itself. This approach allows an high number of degrees of freedom (theoretically infinite) but this strategy asks many questions. The crucial problem is the choice of the error structures in each subspace \mathbb{R} , we have not only to choose the carré du champ operator, but also the law of probability. This choice is not irrelevant, since the law of probability contributes to define the probability space into the error structure and cooperates with the Dirichlet form to the form of the bias operator and the domain \mathbb{D} of the carré du champ, in particular we need that all interesting (for our problem) functions belong to \mathbb{D} .

In order to precise an error structure, it is important to analyze the particular problem we have to study, since the error structure is a part of the model that must be fitted on the particular situation. In order to understand this point, we show some examples.

Example 1.5 (Engineering science applications)

A large class of engineering problems is described as the solution of a non-linear differential equation. Frequently the solution of the equation cannot be found explicitly. A classical strategy is to force the solution to have a particular shape, e.g. a spline function see Schoenberg [36], and to search which function verifies a weak condition, obtained from the original problem.

In this case, it is evident that the choosing of an error structure generated by an infinite product is self-defeating, a good choice is to use the particular basis of spline. In chapter 8 we study a problem of this kind.

Example 1.6 (Wavelets bases)

Many recent papers in physics take advantage of a recent class of bases, the wavelets, see Daubechies [16] or Mallat [32]. One of interesting properties is the capacity to catch the large part of any function with a very low number of terms. Thanks to this property, wavelets are used in MPEG4 format to store movies.

We can use this property when we define an error structure in a functional space using wavelets as the basis. We can decide either to randomize a finite number of coefficients a_n or to assume that the high orders coefficients are "less erroneous" in order to facilitate the convergence of the sum of the carré du champ operator in proposition 1.4. The result is a larger domain \mathbb{D} .

The two previous examples show that, as usual, the problem to define a model is an equilibrium question between many requirements, a model must have a sufficient high number of degrees of freedom, to allow a certain elasticity, but not too many with respect to data. The error structure is a part of the model, therefore an arrangement is needed. However, is it hard sometimes to choose ex ante, i.e. from the very beginning of a study, how many and what coefficients have to be randomized. An adaptive strategy may be interesting, Bouleau in [8] pages 84 and 172 has proposed a possible solution:

Result 1.11 (Structure with finite (but random) number of coefficients)

We consider that the error structure is generate by the infinite product

$$(1.22) \quad (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty} (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_n, \mathbf{d}_n, \gamma_n)$$

where μ_n, γ_n are defined later. We choose the a priori probability measure

$$(1.23) \quad \mu_n = \vartheta_n \mu + (1 - \vartheta_n) \delta_0$$

where μ is a probability measure on \mathbb{R} absolutely continuous with respect to the Lebesgue measure, δ_0 denotes the Dirac mass at zero and we assume $\vartheta_n \in (0, 1)$ and $\sum_n \vartheta_n < \infty$. Under these hypotheses only a finite number of coefficients a_n are different to zero and the rescaling $(a_0, \dots, a_n, \dots) \rightarrow (\lambda a_0, \dots, \lambda a_n, \dots)$ gives $\mathbb{P} = \otimes_n \mu_n$ a measure absolutely continuous with respect to Lebesgue one.

The second choice concerns the carré du champ operators on each subspace \mathbb{R} generated by the coordinate maps a_n . The argument of Hamza, see proposition 1.1 or [26], imposes that $\Gamma[a_n](x) = 0$ when $x = 0$.

An acceptable choosing for Γ can be to verify the following proportionality hypothesis:

Hypothesis 1.3

$$\Gamma[a_n, a_m] = \kappa(n, m) a_n a_m$$

1.6 Regularity of the bias operator

When using the functional calculus for the bias operator \mathcal{A} under the form

$$\mathcal{A}[F(h_1, \dots, h_n)] = \sum_i F'_i(h_1, \dots, h_n) \mathcal{A}[h_i] + \frac{1}{2} \sum_{i,j} F''_{ij}(h_1, \dots, h_n) \Gamma[h_i, h_j]$$

for $F \in C^2(\mathbb{R}^n)$ with bounded derivatives F''_{ij} and $H_i \in \mathcal{DA}$, a specific verification has to be made to ensure $F(h_1, \dots, h_n)$ belongs to the domain in the L^2 -sense (and not only in the L^1 -sense) and that $\Gamma[h_i, h_j] \in L^2$ (and not only L^1), see Bouleau and Hirsh [6] corollary 6.14 chapter I and exercise 6.2.

This verification involves the specific definition of the Dirichlet structure and depends on its reference measure and its carré du champs operator. We call this verification **assumption of regularity on the bias** (ARB). It will be supposed to be fulfilled in the whole thesis. That means that, when we study the bias operator and its propagation through models or equations, we suppose that the errors on the data are not the most general ones but satisfy this regularity condition.

1.7 Thesis contributions

In this section, we summarize our contributions delineated in this thesis. The analysis is divided into three parts, in accordance with the mathematical framework used, i.e. stochastic differential equations, partial differential equations and stochastic partial differential equations.

Part II: Stochastic differential equations and finance

In this part, we will apply the error theory using Dirichlet forms to models described by a stochastic differential equation; finance is one of the most prolific domains of application. This part consists of five chapters.

Chapter 2 is devoted to the analysis of bias operator when a weighted Ornstein Uhlenbeck error structure is mapped on the Wiener space. We will extend to the bias theorem some results about conditioning, due to Nicolas Bouleau. The theorems have an immediate application in the next chapter.

Chapter 3 summarizes the work developed with Vathana Ly Vath, an assistant professor at Evry university. In this chapter, an innovative model for order books is delineated. We assume that the bid-ask spread is not an imperfection of the market, but an intrinsic property of exchange markets instead. We describe this uncertainty adding a noise to the Brownian motion guiding the asset. The aversion with respect to uncertainty and representative agent's hypothesis allow us to justify the presence of a bid-ask spread, while its wideness and evolution are analyzed by means of error theory using Dirichlet forms. We find that spread evolutions can be evaluated using closed formulae, when a local volatility model is used. Finally, the robustness of the error theory enable us to evaluate the impact of the underlying uncertainty on the related contingent claims prices and to foresee their spreads.

Chapters 4, 5 and 6 are devoted to the study of a new model to evaluate European contingent claims. Chapter 4 deals with the so called perturbative Black Scholes model. The idea is to assume that the underlying prices really follow a log-normal diffusion (as in the Black Scholes model), but, that unfortunately, market traders do not know the true value for the parameters, such as the volatility, and they are forced to estimate it. It is evident that the result of this statistics is a random variable, characterized by a non-vanished variance. We study how this uncertainty is transferred to option prices. The key argument is the evaluation of the profit and loss process at maturity time, which is identically zero in the Black Scholes framework, while it becomes a random variable in our analysis, characterized by a variance and a small bias, caused by the non-linearity of the payoff. We use the variance in order to justify the presence of a bid-ask spread, while bias modifies the implied volatility curve turning it in a convex function with respect to the strike. One of the most interesting properties of the perturbative Black Scholes model is the existence of closed formulae for the prices and greeks when they exist in Black Scholes one; calls, puts, forward options and a large bunch of barrier derivatives belong to this class. Results of this chapter are summarized in submitted article [37].

Chapter 5 summarizes the work developed with Luca Regis, a PhD student at Torino University. We show the impact of the underlying drift parameter on the prices of contingent claims, contrary to all other models used in finance at the present time. The difference on prices between the perturbative Black Scholes models with or without drift is computed and we show that pricing formulae remain closed. In particular, we can reproduce a downsloping implied volatility curve with respect to the strike.

Finally, chapter 6 describes an efficient strategy to calibrate the perturbative Black Scholes model using Variance Swaps. We study these contingent claims and their prices forecast by our model. Some properties of these options enable us to develop a robust calibration methodology. Results of this chapter are summarized in [39].

Part III: Partial differential equations and physics

In this part, we apply the error theory using Dirichlet forms to models described by a partial differential equation; these have a large number of applications, e.g. in physics, economics and finance. This part is split into three chapters.

Chapter 7 is dedicated to the study of linear partial differential equations. Linearity plays a leading role in many physical models and it enable us to expand the solution into a series. In our analysis, we exploit this property using a wavelet basis, since this type of basis has interesting scaling effects. We study interactions among wavelets, partial differential equations and the error theory using Dirichlet forms. Finally, we introduce an application to finance in the framework known as “Black volatility models”. The work of this chapter is presented in details in the article [38].

Chapters 8 and 9 focus on the analysis of non-linear partial differential equations. The main field of applications is fluid mechanics, since the primary equation in this domain, the Navier-Stokes equations, are non-linear. In order to describe possible applications of the error theory using Dirichlet forms, we have considered the elementary problem of the shallow water equations, also known as the Saint Venant equations. The non-linearity of these equations forces us to discretize to solve them on a lattice. Two strategies are applicable: the first one, detailed in chapter 8, is to first discretize the partial differential equations, then, on the equivalent discrete problem, study the sensibility of the solution using error theory. The second approach, detailed in chapter 9, shows that the sharp and bias operators of the theoretical solution solve two linear partial differential equations depending on the solution itself. The discretisation is used only after this stage, in order to evaluate numerically the solution with its sensitivity. Results of chapter 8 are presented in article [40].

Part IV: Stochastic partial differential equations and climatology

In this part, we apply the error theory using Dirichlet forms to models driven by a stochastic partial differential equation. The study of this type of equations is rather recent, applications are numerous and span many fields, e.g. climatology with stochastic climate models, applied physics or finance for interest rate curve models. This part is split into two chapter.

Chapter 10 draws, shortly, the theory of stochastic partial differential equation, analytic approach, see Da Prato [14], is preferred to the martingale one, see Walsh [45], for sake of simplicity. We study the impact of uncertainty, present on starting conditions, on the solution of stochastic partial differential equations. In particular, we underline that the sharp and bias operators solve two stochastic partial differential equations too. These SPDEs are linear and depending on the solution of the former SPDE itself. Results of this chapter are been presented in article [41].

Chapter 11 analyze the impact of two types of uncertainty, i.e. an error on the functional coefficients of stochastic partial differential equation, and a perturbation of the diffusion coefficient of the related Green function. In the first case, we have showed that the sharp and bias operators solve a linear stochastic partial differential equation depending on the solution of the former SPDE itself. In the second case, we have restricted our analysis to the sharp operator; we have proved that it verifies a Picard series that solve a new type of stochastic partial differential equation. This chapter ends with some applications of stochastic partial differential equations in climatology, genetics, finance and insurance.

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Part II

Stochastic Differential Equations and Finance

Chapter 2

Bias Operator and Conditional Expectation

In this small chapter we have separated a theoretical study of the bias operator, in particular we investigate how we can compute the bias of an erroneous stochastic process, defined by means of a conditional expectation. We start from the work of Bouleau, that has studied the relation between the conditional expectation operator and the carré du champ one, see Bouleau [12] pages 166-171 or section 1.4.

The case of the bias operator is more easy, thanks to the linearity, but there are some precautions to take. For sake of simplicity, we consider only the case of a Weighted Ornstein-Uhlenbeck structure, see section 1.4. The chapter is divided into two sections: In the first one, we analyze how to compute the bias of a stochastic process defined via a conditional expectation, While in section 3, we show that the semi-group and the conditional expectation operator commute.

2.1 Bias and filtration

We study the bias of a random process defined by the following relation:

$$Y_t = \mathbb{E}[X | \mathcal{F}_t]$$

where X is a random variable belonging to $L^2(\mathbb{P})$ and we search to generalize this definition when X is erroneous, i.e. it exists an error structure for X , see Bouleau [12] pages 166-167. We distinguish two problems:

1. to show, under some hypotheses, that if $X \in \mathcal{DA}$ then $Y_t \in \mathcal{DA}$, where \mathcal{DA} is the domain of the bias operator related with the error structure, see section 1.2;
2. to find the operator A_t such that $A[\mathbb{E}[X | \mathcal{F}_t]] = \mathbb{E}[A_t[X] | \mathcal{F}_t]$ and the relation with the operator Γ_t , see proposition 1.10.

We analyze the two previous questions separately into two propositions.

Proposition 2.1 (Orthogonal projection) *Let the Ornstein-Uhlenbeck structure, then the operators of conditional expectation with respect to the Brownian filtration $\{\mathcal{F}_t\}_{t \leq 0}$ are orthogonal projections in \mathbb{D} and preserve \mathcal{DA} .*

Proof: Since X is a random variable in L^2 ; we can expand X thanks to the Wiener chaos decomposition, see Bouleau and Hirsch [11] or section 1.4.

$$(2.1) \quad X = \sum_n X_n \text{ where } X_n \in L_n^2(\mathbb{P}) \text{ and } X_n = \int_{0 < t_1 < \dots < t_n} f_n(t_1, t_2, \dots, t_n) dB_{t_1} \dots dB_{t_n}$$

We know that $L_n^2(\mathbb{P})$ are eigenspaces for the generator and the Dirichlet form of Ornstein-Uhlenbeck, see Bouleau and Hirsch [11] pages 109-110, in particular:

$$(2.2) \quad \begin{aligned} X \in \mathcal{DA}_{OU} &\iff \sum_n n^2 \mathbb{E}[X_n^2] < \infty \\ X \in \mathbb{D}_{OU} &\iff \sum_n n \mathbb{E}[X_n^2] < \infty \end{aligned}$$

and we have

$$(2.3) \quad \begin{aligned} A_{OU}[X] &= -\sum_n \frac{n}{2} X_n \\ \Gamma_{OU}[X] &= \sum_n n X_n^2. \end{aligned}$$

If we take the conditional expectation, we find

$$(2.4) \quad \begin{aligned} Y_t = \mathbb{E}[X|\mathcal{F}_t] &= \sum_n \int_{0 < t_1 < \dots < t_n < t} f_n(t_1, t_2, \dots, t_n) dB_{t_1} \dots dB_{t_n} \\ &= \sum_n \int_{0 < t_1 < \dots < t_n} \mathbb{I}_{t_n < t} f_n(t_1, t_2, \dots, t_n) dB_{t_1} \dots dB_{t_n} = \sum_n (Y_t)_n \end{aligned}$$

where $(Y_t)_n$ is the projection of Y_t on the n -chaos of Wiener. We know that

$$\mathbb{E}[(Y_t)_n^2] \leq \mathbb{E}[X_n^2],$$

therefore we have a control on each term of the chaos expansion, then $Y_t \in \mathcal{DA}_{OU}$ since the series (2.2) are convergent by hypothesis. □

Now we want define an operator A_t such that

$$(2.5) \quad A[\mathbb{E}[X|\mathcal{F}_t]] = \mathbb{E}[A_t[X]|\mathcal{F}_t]$$

Thanks to equations (2.3) and (2.4) we have the following definition for A_t .

$$(2.6) \quad A_t[X] = -\sum_n \frac{n}{2} \int_{0 < t_1 < \dots < t_n} \mathbb{I}_{t_n < t} f_n(t_1, t_2, \dots, t_n) dB_{t_1} \dots dB_{t_n}$$

This is similar to the definition of operator Γ_t , we can make an interesting remark.

Remark 2.1 We remark that the operator A_t coincide with the operator A , in the sense that $\mathbb{E}[A_t[X]|\mathcal{F}_t] = \mathbb{E}[A[X]|\mathcal{F}_t]$, we maintain the notation A_t to mean that A_t is the operator associated at Γ_t , that is different to the classical operator Γ .

Now, we want study the relation between the conditional variance operator Γ_t , see [12] page 167, and A_t defined by the equation (2.5). We assume that the conditional expectation operators are orthogonal projections into \mathbb{D} . We recall that

$$\Gamma_t \left[\int_0^\infty f(s) dB_s \right] = \Gamma \left[\int_0^\infty \mathbb{I}_{s < t} f(s) dB_s \right]$$

and we remark that we can define a sharp operator coherent with Γ_t

$$(\mathbb{E}[U|\mathcal{F}_t])^\# = \mathbb{E}[U^{\#t}|\mathcal{F}_t].$$

Finally, we define the operator δ^t adjoint of the sharp $(\cdot)^{\#t}$. We know that if $X \in \mathcal{D}\widehat{A}_t$, where \widehat{A}_t denotes the generator associated with the carré du champ Γ_t , thanks to Hille-Yosida theorem, see Albeverio [1]. Then we have

$$\widehat{A}_t[X] = -\frac{1}{2}\delta^t(X^{\#t}).$$

We remark that $\widehat{A}_t = A_t$, if the error structure is of type Weighted Ornstein-Uhlenbeck. Therefore, A_t is well the generator associated with the carré du champ Γ_t .

2.2 Conditional expectation and semi-group

In this section we prove the following theorem:

Theorem 2.2 (Commutation between conditional expectation and semigroup)

Let $(P_s)_{s \geq 0}$ be the semi-group of Weighted Ornstein-Uhlenbeck, and let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by the Brownian motion.

Then

$$(2.7) \quad \mathbb{E}[P_s[F(\omega)]|\mathcal{F}_t] = P_s[\mathbb{E}[F(\omega)|\mathcal{F}_t]] \quad \forall \omega, F \in \mathcal{F}_\infty.$$

Proof:

Thanks to the monotone class theorem we can assume that F is of the type

$$F = G \left(\int_0^\infty f_1(u)dB_u, \dots, \int_0^\infty f_k(u)dB_u \right),$$

where $f_i \in L^2([0, \infty])$ and $G \in C^\infty$ with compact support. The Mehler formula, see equation (1.18) gives how the semi-group P_t acts on F .

$$P_s \left[\int_0^\infty g(u) dB_u \right] = \widehat{\mathbb{E}} \left[\int_0^\infty g(u) \sqrt{e^{-\alpha(u)s}} dB_u + \int_0^\infty g(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u \right]$$

where \widehat{B}_t is an independent Brownian motion and $\widehat{\mathbb{E}}$ is the related expectation. Therefore we have

$$P_s[F] = \widehat{\mathbb{E}} \left[G \left(\int_0^\infty f_1(u) \sqrt{e^{-\alpha(u)s}} dB_u + \int_0^\infty f_1(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u, \dots \right. \right. \\ \left. \left. \dots, \int_0^\infty f_k(u) \sqrt{e^{-\alpha(u)s}} dB_u + \int_0^\infty f_k(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u \right) \right].$$

Now we take the conditional expectation and we find

$$(2.8) \quad \mathbb{E} [P_s[F] | \mathcal{F}_t] = \widetilde{\mathbb{E}} \left[\widehat{\mathbb{E}} \left[G \left(\int_0^t f_1(u) \sqrt{e^{-\alpha(u)s}} dB_u + \int_t^\infty f_1(u) \sqrt{e^{-\alpha(u)s}} d\widetilde{B}_u \right. \right. \right. \\ \left. \left. + \int_0^\infty f_1(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u, \dots \dots, \int_0^t f_k(u) \sqrt{e^{-\alpha(u)s}} dB_u \right. \right. \\ \left. \left. + \int_t^\infty f_k(u) \sqrt{e^{-\alpha(u)s}} d\widetilde{B}_u + \int_0^\infty f_k(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u \right) \right] \right]$$

where \widetilde{B}_t is another independent Brownian motion and $\widetilde{\mathbb{E}}$ is the related expectation. On other hand, if we take the conditional expectation of F we have

$$\mathbb{E} [F | \mathcal{F}_t] = \widetilde{\mathbb{E}} \left[G \left(\int_0^t f_1(u) dB_u, \dots, \int_t^\infty f_1(u) d\widetilde{B}_u \right) \right]$$

and if we apply the semi-group we find

$$(2.9) \quad \mathbb{E} [P_s[F] | \mathcal{F}_t] = \widehat{\mathbb{E}} \left[\widetilde{\mathbb{E}} \left[G \left(\int_0^t f_1(u) \sqrt{e^{-\alpha(u)s}} dB_u + \int_0^t f_1(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u \right. \right. \right. \\ \left. \left. + \int_t^\infty f_1(u) d\widetilde{B}_u, \dots, \int_0^t f_k(u) \sqrt{e^{-\alpha(u)s}} dB_u \right. \right. \\ \left. \left. + \int_0^t f_k(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u + \int_t^\infty f_k(u) d\widetilde{B}_u \right) \right] \right]$$

If we compare equations (2.8) and (2.9), we know that the two Brownian motions \widehat{B} and \widetilde{B} are independents, we can exchange the two expectation, and the problem to prove the relation (2.7) became to prove the following evident identity in law for all $f \in L^2[0, \infty]$.

$$\int_0^t f(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u + \int_t^\infty f(u) d\widetilde{B}_u \stackrel{\mathcal{L}}{=} \int_t^\infty f(u) \sqrt{e^{-\alpha(u)s}} d\widetilde{B}_u \\ + \int_0^\infty f(u) \sqrt{1 - e^{-\alpha(u)s}} d\widehat{B}_u$$

□

Chapter 3

An Order Books Model

Joint work with Vathana Ly Vath.

In this chapter, we propose a methodology to model order books (OB) when the underlying follows a local volatility model. We start with a local volatility diffusion but we assume that the Brownian motion is uncertain, in a sense that we explain. The uncertainty on the Brownian motion generates a noise on the trajectories of the underlying and we use this noise to expound the presence of a bid-ask spread, besides we prove that this noise has an impact also on mid-price. We enrich our analysis with a numerical simulation when the volatility is a power function of the asset price. Finally, we investigate the impact of this uncertainty on option prices.

3.1 Introduction

A common problem for stock traders consists in the presence of a difference between the prices asked to sell and to buy an asset. The difference between the lowest buy price and the highest sell one is called bid-ask spread, the presence of this spread is intrinsic on markets, since when a new buy (sell) order appears, it either adds to the bid (ask) book if it is below (above) the ask (bid) price, or generates a trade if it is above (below) or equal. However, classical models in finance, e.g. Black Scholes model see [10], describe the price of an asset at a fixed time as a single value.

Bid-ask spread and order books play a crucial role in three financial problems, i.e. the unwinding large block order of shares for large investors, the portfolio management for small investors and the hedging strategy of options for traders. Problems of this type were investigated by Bertsimas and Loo [8], Almgren et al. [3] [4] [5], Obizhaeva and Wang [46], Alfonsi et al. [2], to mention only a few. However, all previous models assume really simplified evolutions, e.g. Obizhaeva and Wang suppose that the underlying price has a Bachelier's dynamics, while, in the paper of Alfonsi et al., the underlying is described by a martingale process; in addition, the time evolution is often discretized.

In our analysis, we concentrate us on a possible description of the evolution of asset prices and order books, and on the study of the impact on contingent claims prices, sooner than the study of optimal execution strategies for block market orders. Accordingly, our model assumes a large class of processes as the fundamental diffusion, i.e. the class of local volatility models currently

used by merchant banks in order to evaluate their positions.

The seminal idea is to explain the presence of many sell and buy prices and the difference between them as the presence of an uncertainty on markets. It is well-known that the price of an asset can be interpreted as the sum of forecast coming returns, that are random. Thus, the value of an asset is suffering from an uncertainty. We can represent this uncertainty via a random variable, so the price of an asset at a fixed time are characterized by a mean value, but also by a variance (with the addition of the high orders). In conformity with this point of view, the presence of many sell and buy prices can be explained thanks to different agents with different risk aversions; in order to clarify this idea, we consider an agent that can settle to buy or sell an asset, he knows the distribution of possible asset values, given by market information, a coherent decision is to send a buy (sell) offer with a price lower (higher) with respect to the asset mean value such that this difference justify the risk. Of course, many agents with different risk aversions generate many prices; therefore, order books draw, implicitly, the market evaluation about uncertainties, i.e. great uncertainty implies large spreads and vice versa.

The mathematical formulation of such problems relies on the specifying of a coherent framework to describe the remaining randomness on prices. As a matter of fact, in our problem the asset value must depend on two random sources: the first one describes the evolution of the asset mean value, the second one delineates the shape of asset (sell-buy) prices when the time is fixed. The coupling of the two probability spaces, with the relative filtration, requires complex tools and represents the principal drawback of this kind of approach. Therefore, we choose a different strategy based on error theory using Dirichlet forms formalism. The advantages of this approach are inherent on its elasticity and its powerful tools, see the introduction of this thesis for a survey. Order books framework justifies automatically many assumptions of error theory, e.g. bid-ask spreads are almost always very lower with respect to the mid price, that admits the limited expansion approach.

An important peculiarity of order books, and a drawback too, is the lack of information. As a matter of fact, order books are not completely public, e.g, in France, the market regulator restricts the known part of a book to the five best prices; this kind of decision is taken to avoid market abuses. However, it is impossible to define the shape of prices randomness with only ten data¹, this fact justifies the restriction of our analysis to a gaussian behavior, another hypothesis of error theory. All the same, this assumption conforms with empirical observations made by Biais et al. [9] and Potters et al. [49], that show a decreasing shape of order books as price goes away from the mid-price, yet the maximum is reached near but not always precisely on the first best ask (bid) price. This shifting on the maximum is hard to justify in a gaussian framework, since the maximum for gaussian density is reached at the mean. Error theory relieves us as it foresees a bias with respect to the theoretical mean. This discrepancy can explain the shifting on the maximum.

A further advantage of error theory approach is the possibility to transfer the uncertainty through assets till the related derivatives, i.e. we can evaluate the bid-ask spread on contingent claims prices induced by a spread on underlying asset.

The chapter is organized as follow. In section 2, we introduce the economic model for order books. In section 3, we present the analysis of prices variance and bias. In section 4, we interpret the result in financial framework. Section 5 is devoted to the evaluation of the impact of underlying

¹The five best sell prices and the five buy ones.

uncertainty on contingent claims prices. Finally, section 6 resumes and concludes. All technical proofs are placed in appendix A.

3.2 The model

In this section, we aim at modeling the dynamics of the mid-price and its two order books. In order to show the powerful of our approach, we consider a very general model of diffusion, i.e. a local volatility model:

$$(3.1) \quad dX_t = X_t r_t dt + \sigma(t, X_t) dW_t$$

where X_t represents the asset price, r is the interest rate, $\sigma(t, X_t)$ is the volatility, function of time and asset price. Thus, the volatility is a random process, but it is known under the knowledge of asset price X_t ; in this sense the model is called a local volatility model. Further in this chapter, we state a precise class of local volatility models introduced by Hagan and Woodward, see [36], models characterized by the shape of the volatility function that becomes a power of the asset price X_t . In this particular case, we show some numerical simulation in section 3.4.1.

The goal, in financial applications, is to take into account the presence of a difference between the ask and the bid price of an asset. We assume that X_t represents asset mid-price, while the ask and the bid prices are a consequence of the presence of an uncertainty on Brownian motion W_t , we model this uncertainty thanks to error theory using Dirichlet forms. Thus, we fix an error structure

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is the Wiener space where the Brownian motion W_t lives, while Γ is a weighted Ornstein-Uhlenbeck carré du champ operator with weight $\alpha(t)$ (see section 1.4). Mehler formula, see Bouleau [12] page 116, provides an intuitive interpretation of this uncertainty, there is a second independent Brownian motion with a very small wideness that perturb the first Brownian motion W_t ; the result is a noise around the mid price X_t .

Now we consider the presence of many agents on the market, all informed about the economic evolution of the mid-price X_t but without money-market intelligence about the residual information drawn by the perturbation. All agents are risk adverse and they can estimate the distribution of the uncertainty of asset price at each fixed time t . It stands to reason that, at each time t , it exists an agent with minimal risk aversion with respect to his colleagues. This agent accepts to buy the asset at a price B_t smaller than X_t , owing to risk aversion, but bigger with respect to each proposal of his colleagues. Thus, B_t is the bid price. A symmetric analysis generate the ask price A_t .

Let us assume, for sake of simplicity, that there exists a representative agent that proposes always the best buy and sell prices and we assume that this agent accepts to buy the asset at a price B_t such that the risk of overvaluing of the asset is equal to a supportable risk probability $\chi < 0.5$, clearly the "risk" of undervaluing is $1 - \chi > 0.5$, therefore the agent take the risk against the expected earnings, see figure 3.1.

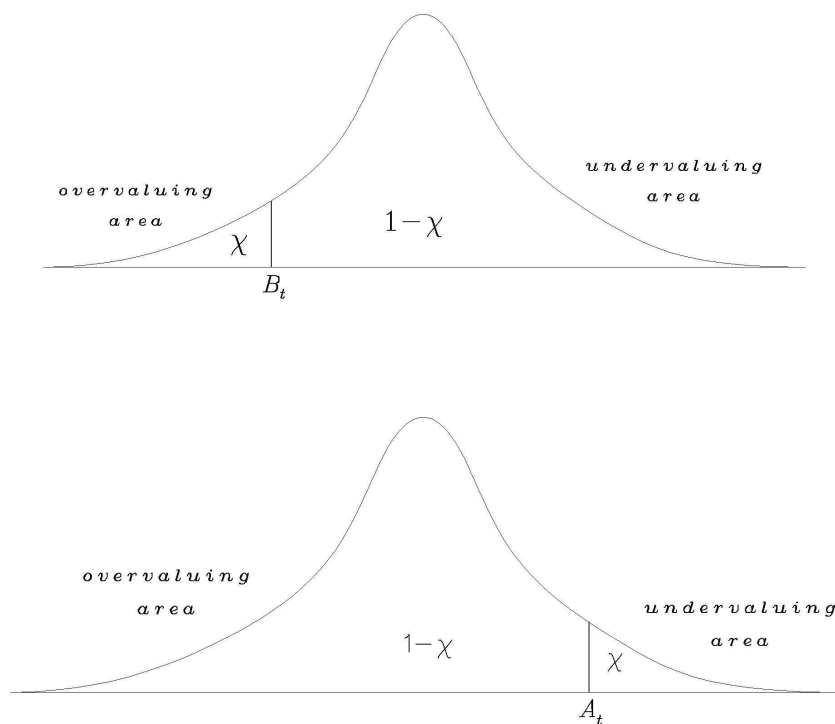


Figure 3.1: Bid and ask prices definition, defined by a risk probability χ .

The definition of ask price A_t is symmetric. Since the law of residual uncertainty is always gaussian, the definition of the supportable risk is equivalent to the definition of the trader utility function, see chapter 4 for an analysis of relations between error theory and utility functions. For sake of simplicity, we fix the same supportable risk for sell and buy proposals.

Finally, error theory foresees a bias that we have to evaluate, this bias shift the mid-price with respect to the theoretical price X_t . In the following section, we show all mathematical results of our analysis.

3.3 Uncertainty of an asset due to Brownian motion

In this section, we study the sensitivity of an asset due to a perturbation on Brownian motion. We start recalling a result of Bouleau, see [12] page 167.

Theorem 3.1 (Bouleau)

Let X_t be the solution of SDE (3.1), we assume that this solution exists and it is unique.

We consider that the Brownian motion W_t in equation (3.1) is erroneous with an error structure $(\Omega, \mathbb{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ of type Weighed Ornstein Uhlenbeck with weigh $\alpha(s) \in L^1$ and non-negative, and, finally, we assume $x \sigma(t, x)$ belongs to $C^1(\Omega) \cap L^2(\Omega)$, bounded and Lipschitz with its first derivative.

Then, the uncertainty effect on process X_t is characterized by the following variance-covariance:

$$\begin{aligned}
(3.2) \quad \Gamma[X_t] &= M_t^2 \int_0^t \frac{X_s^2 \sigma^2(s, X_s)}{M_s^2} \alpha(s) ds \\
\Gamma[X_t, X_s] &= M_t M_s \int_0^{t \wedge s} \frac{X_u^2 \sigma^2(u, X_u)}{M_u^2} \alpha(u) du
\end{aligned}$$

where

$$\begin{aligned}
M_t &= \exp \left\{ \int_0^t K_s dW_s - \frac{1}{2} \int_0^t K_s^2 ds + \int_0^t r_s ds \right\} \\
K_t &= \sigma(t, X_t) + X_t \frac{\partial \sigma}{\partial x}(t, X_t)
\end{aligned}$$

Bouleau has studied the transmission of variance. In this paper instead, we focus on the bias of process X_t , since the variance can explain the bid-ask spread phenomenon whereas the bias permits to perturb the predictions of a model; in particular, we search to compute the bias in order to explain many phenomenons in financial data using classical simple models where we consider the uncertainty. An example of this approach is the Perturbative Black Scholes model, that uses the bias due to an uncertainty on volatility, estimated by traders, to explain the smile of implied volatility, see Scotti [53] or chapter 4.

Theorem 3.2 (SDE for the bias due to Brownian motion)

Let X_t be the solution of SDE (3.1), we assume that this solution exists and it is unique. We consider that Brownian motion W_t in equation (3.1) is erroneous with an error structure $(\Omega, \mathbb{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ of type Weighted Ornstein Uhlenbeck with weigh $\alpha(s) \in L^2$ and non-negative.

We assume that the error structure $(\Omega, \mathbb{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ is so that the process $\Gamma[X_t]$, defined in Bouleau [12], belongs to $L^2(\Omega)$; we assume that, for all t , X_t belongs to \mathcal{DA} , i.e. the domain of the operator A given by Hille Yosida theorem², we suppose the ARB condition hold, see section 1.6 and, finally, we assume $x \sigma(t, x)$ belongs to $C^2(\Omega) \cap L^2(\Omega)$, bounded and Lipschitz with its first and second derivatives.

Then we have the following SDE for the bias.

$$\begin{aligned}
(3.3) \quad A[X_t] &= -\frac{1}{2} \int_0^t X_s \sigma(s, X_s) \alpha(s) dB_s + \int_0^t \left\{ \sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right\} A[X_s] dW_s \\
&+ \int_0^t \left\{ \frac{\partial \sigma}{\partial x}(s, X_s) + \frac{1}{2} X_s \frac{\partial^2 \sigma}{\partial x^2}(s, X_s) \right\} \Gamma[X_s] dW_s + \int_0^t A[X_s] r_s ds
\end{aligned}$$

We prove this theorem, in a more general framework, in appendix 3.A. Now, we make a remark.

Remark 3.1 (Linearity) SDE (3.3) is linear. Therefore, under the hypothesis of the explicit knowledge of processes X_t and $\Gamma[X_t]$ we know, explicitly, process $A[X_t]$. In section 3.3.2, we present this computation.

²see Albeverio [1].

If we analyze SDE (3.3) an obvious question is when this SDE has a unique solution, the following theorem gives us an answer.

Theorem 3.3 (Existence and uniqueness of bias process)

Under the same hypotheses of theorem 3.2 and if there exist two constants C and D such that

$$|x \sigma(t, x)| \leq C (D + |x|)$$

Then the stochastic differential equation (3.3) has an almost surely unique continuous solution.

The proof of this fact is quite classical, we give a proof in appendix 3.A. In following subsections, we analyze some properties of processes $\Gamma[X_t]$ and $A[X_t]$

3.3.1 Markov property

It is clear that the process $A[X_t]$ is not Markovian, but we can prove the following proposition

Proposition 3.4 (Markov property)

The three-dimensional process $(X_t, \Gamma[X_t], A[X_t])$ is Markovian

In order to prove this proposition we need the following lemma.

Lemma 3.5 (SDE for the couple variance-bias)

Under the same hypotheses assumed in theorem 3.2, the couple $(\Gamma[X_t], A[X_t])$ verifies the following stochastic differential equation.

$$\begin{aligned}
 (3.4) \quad \begin{pmatrix} \Gamma[X_t] \\ A[X_t] \end{pmatrix} &= \int_0^t \begin{bmatrix} 2 \frac{\partial(x \sigma)}{\partial x}(s, X_s) & 0 \\ \frac{1}{2} \frac{\partial^2(x \sigma)}{\partial x^2}(s, X_s) & \frac{\partial(x \sigma)}{\partial x}(s, X_s) \end{bmatrix} \begin{pmatrix} \Gamma[X_s] \\ A[X_s] \end{pmatrix} dW_s \\
 &+ \int_0^t \begin{bmatrix} 2r(s) + \left[\frac{\partial(x \sigma)}{\partial x}(s, X_s) \right]^2 & 0 \\ 0 & r(s) \end{bmatrix} \begin{pmatrix} \Gamma[X_s] \\ A[X_s] \end{pmatrix} ds \\
 &+ \int_0^t \begin{pmatrix} 0 \\ -\frac{1}{2} X_s \sigma(s, X_s) \alpha(s) \end{pmatrix} dW_s + \int_0^t \begin{pmatrix} \sigma^2(s, X_s) X_s^2 \alpha(s) \\ 0 \end{pmatrix} ds
 \end{aligned}$$

The proof of this lemma is in appendix 3.A.

Now we can prove easily proposition 3.4.

Proof: Since the process X_t is a semi-martingale, it is a Markov process. Using SDE (3.4) for the couple we have that the process $(\Gamma[X_t], A[X_t])$ is a diffusion depending only on process X_t and on Brownian motion W_t . Therefore, the triplet $(X_t, \Gamma[X_t], A[X_t])$ is a Markov process thanks to the properties of Ito processes.

□

3.3.2 Closed form for $A[X_t]$

In this subsection, we show an explicit closed form for the process $A[X_t]$, we have the following theorem.

Theorem 3.6 (Closed form for bias)

Under the assumptions of theorems 3.2 and 3.3, we have the following relation for the bias of a local volatility diffusion, when we assume³ $A[X_0] = 0$.

$$(3.5) \quad A[X_t] = M_t \int_0^t \frac{P_s}{M_s} (dW_s - K_s ds)$$

where

$$(3.6) \quad \begin{aligned} K_s &= \sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \\ dM_s &= M_s (K_s dW_s + r_s ds) \\ P_s &= -\frac{1}{2} X_s \sigma(s, X_s) \alpha(s) + \frac{1}{2} \left(2 \frac{\partial \sigma}{\partial x}(s, X_s) + X_s \frac{\partial^2 \sigma}{\partial x^2}(s, X_s) \right) \Gamma[X_s] \end{aligned}$$

Proof: The proof is based on the method of variation of parameters; in order to solve a more general class of SDE, see Flandoli and Russo [30] or Protter [50] chapter V section 9. We start with a remark: we can rewrite SDE (3.3) according to notations (3.6).

$$A[X_t] = \int_0^t \frac{A[X_s]}{M_s} dM_s + \int_0^t P_s dW_s$$

Now, we study the object of the first integral; we find, thanks to Ito formula,

$$dQ_t = d \left(\frac{A[X_t]}{M_t} \right) = \frac{P_t}{M_t} dW_t - \frac{P_t}{M_t} K_t dt$$

and we conclude with an integration and using the identity $A[X_t] = M_t Q_t$.

□

3.3.3 Covariance between the asset, its variance and its bias

In this subsection, we compute the covariance matrix of vector $(X_t, \Gamma[X_t], A[X_t])$.

Proposition 3.7 (Covariance matrix)

Under the same assumptions of theorems 3.2 and 3.3, we have the following variance-covariance matrix for the process $(X_t, \Gamma[X_t], A[X_t])$.

³The general case has a similar, but more complicate, closed form.

$$\begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} & \Sigma_{1,3} \\ \Sigma_{2,1} & \Sigma_{2,2} & \Sigma_{2,3} \\ \Sigma_{3,1} & \Sigma_{3,2} & \Sigma_{3,3} \end{bmatrix}$$

where

$$\begin{aligned} \Sigma_{1,1} &= \int_0^t X_s^2 \sigma^2(s, X_s) ds \\ \Sigma_{1,2} &= \Sigma_{2,1} = 2 \int_0^t X_s \Gamma[X_s] \sigma(s, X_s) K_s ds \\ \Sigma_{1,3} &= \Sigma_{3,1} = \int_0^t X_s \sigma(s, X_s) \left\{ K_s A[X_s] + P_s \right\} ds \\ \Sigma_{2,2} &= \int_0^t \left\{ \Gamma[X_s] \right\}^2 K_s^2 ds \\ \Sigma_{2,3} &= \Sigma_{3,2} = 2 \int_0^t \Gamma[X_s] K_s \left\{ K_s A[X_s] + P_s \right\} ds \\ \Sigma_{3,3} &= \int_0^t \left\{ K_s A[X_s] + P_s \right\}^2 ds \end{aligned}$$

Proof: The variance part is easy to compute. We study the covariation between the spot and the bias, we find, easily, using Ito lemma and equations (3.1) and (3.3), the bracket

$$d[X_t, A[X_t]] = X_t \sigma(t, X_t) \left\{ K_t A[X_t] + P_t \right\} dt$$

gives the covariance between X_t and $A[X_t]$. Similarly, we can find the covariation with the variance $\Gamma[X_t]$, using the SDE (3.4).

$$\begin{aligned} d[X_t, \Gamma[X_t]] &= 2 X_t \Gamma[X_t] \sigma(t, X_t) K_t dt \\ d[\Gamma[X_t], A[X_t]] &= 2 \Gamma[X_t] K_t \left\{ K_t A[X_t] + P_t \right\} dt \end{aligned}$$

□

3.4 Applications in finance

In this section, we present an application in finance of the results obtained in the previous section.

We start with a remark: there is no asset model, in our knowledge, that take into account the existence of bid-ask spreads endogenously; all asset models work with the mid-price and add a (symmetric) bid-ask spread at the end of computations, since bid-ask spreads are considered like a market defect. Then, we search to define an asset model that consider bid-ask spreads like an inherent part of asset price evolution and we search to evaluate the consequence of this effect on contingent claims prices. Following ideas stated in section 3.2. We enunciate the following results.

Result 3.8 (Bid-Ask Spread) *The uncertainty on Brownian motion is transmitted to the stochastic process X_t , that represents the asset price. Therefore, each realization $\bar{\omega}$ of process X_t at time t is not a fixed value $X_t(\bar{\omega})$ but it is a random variable described by*

$$(3.7) \quad X_t(\bar{\omega}) + \epsilon A[X_t(\bar{\omega})] + \sqrt{\epsilon \Gamma[X_t(\bar{\omega})]} \tilde{\mathcal{N}}(0, 1)$$

where $\Gamma[X_t]$ is given by equation (3.2), $A[X_t]$ is given by (3.5) and $\tilde{\mathcal{N}}(0, 1)$ is an independent reduced gaussian random variable.

Risk aversion theory permits to define a supportable risk probability $\chi < 0.5$ such that an agent accepts to buy the stock at price

$$(\text{Bid price})(t, \bar{\omega}) = X_t(\bar{\omega}) + \epsilon A[X_t(\bar{\omega})] + \sqrt{\epsilon \Gamma[X_t(\bar{\omega})]} \tilde{\mathcal{N}}(\chi)$$

and, by symmetry, a supportable risk probability $\bar{\chi} = 1 - \chi$ such that an agent accepts to sell the stock at price

$$(\text{Ask price})(t, \bar{\omega}) = X_t(\bar{\omega}) + \epsilon A[X_t(\bar{\omega})] + \sqrt{\epsilon \Gamma[X_t(\bar{\omega})]} \tilde{\mathcal{N}}(1 - \chi)$$

Moreover, we have the following corollary.

Result 3.9 (Mid price) *The Mid price is given by*

$$(3.8) \quad (\text{Mid price})(t, \bar{\omega}) = X_t(\bar{\omega}) + \epsilon A[X_t(\bar{\omega})].$$

Therefore, the Mid-price is different from X_t .

We conclude with a remark:

Remark 3.2 (Non-linear effect) *If we take into account an uncertainty on Brownian motion in order to explain, endogenously, bid-ask spreads of asset prices in a local volatility model, the mid-price presents a bias with respect to the process X_t . Therefore, the presence of a bid-ask spread induces a bias.*

3.4.1 Black volatility models

We consider a particular case of local volatility models, the Black volatility models, introduced by Hagan et al. [36], i.e. we suppose that the volatility in diffusion (3.1) is of the form

$$\sigma(t, x) = \vartheta(t) x^\nu$$

where ν is a constant and $\vartheta(t)$ a deterministic positive function, diffusion (3.1) becomes

$$(3.9) \quad dX_t = X_t r_t dt + \vartheta(t) X_t^{\nu+1} dW_t$$

and we can write the value of variance $\Gamma[X_t]$ and bias $A[X_t]$ thanks to closed forms (3.2) and (3.5).

$$(3.10) \quad \begin{aligned} \Gamma[X_t] &= M_t^2 \int_0^t \frac{X_s^2 \sigma^2(s, X_s)}{M_s^2} \alpha(s) ds \\ A[X_t] &= M_t \int_0^t \frac{P_s}{M_s} (dW_s - K_s ds) \\ K_s &= (\nu + 1) \vartheta(s) X_s^\nu \\ dM_s &= M_s (K_s dW_s + r_s ds) \\ P_s &= -\frac{1}{2} \vartheta(s) X_s^{(\nu+1)} \alpha(s) + \frac{\nu(\nu+1)}{2} \vartheta(s) X_s^{(\nu-1)} \Gamma[X_s] \end{aligned}$$

In the following graphs, we represent a trajectory of X_t , the related variance and bias and the evolution of the bid, mid and ask prices. We have chosen the following values for the parameters: $X_0 = 100$, $\vartheta(t) = 2$, $r = 0.02$, $\nu = -0.3$, $T = 27$ days, discretization 30 steps per day, $\epsilon = 0.1$ and one standard deviation to define the semi-spread bid-ask. It is clear that the value of ϵ is very big, but to draw the evolution of prices, we have preferred a large variance in order to distinguish the four trajectories.

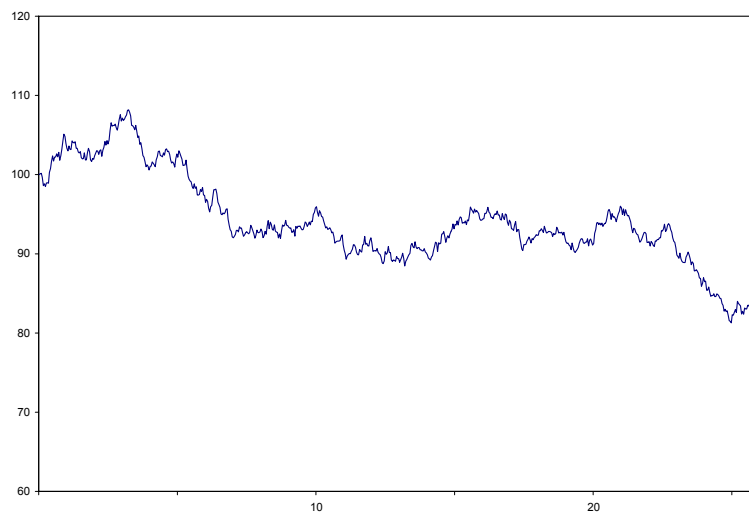


Figure 3.2: A trajectory of the diffusion (3.9) over 27 days for $ATM = 100$, $\vartheta = 2$, $r = 0.02$ and $\nu = -0.3$.

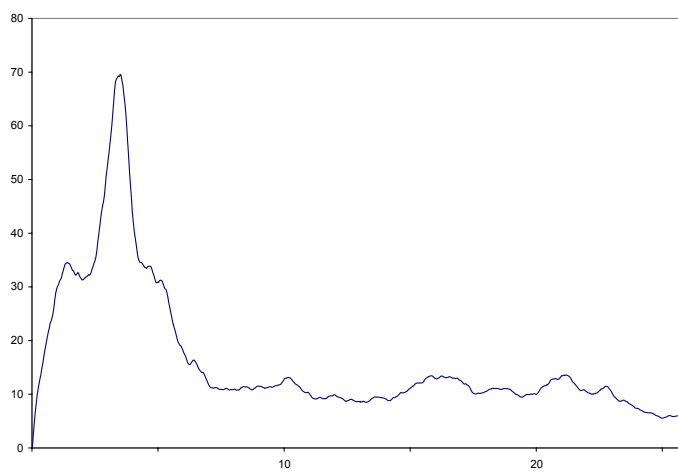


Figure 3.3: The related Gamma.

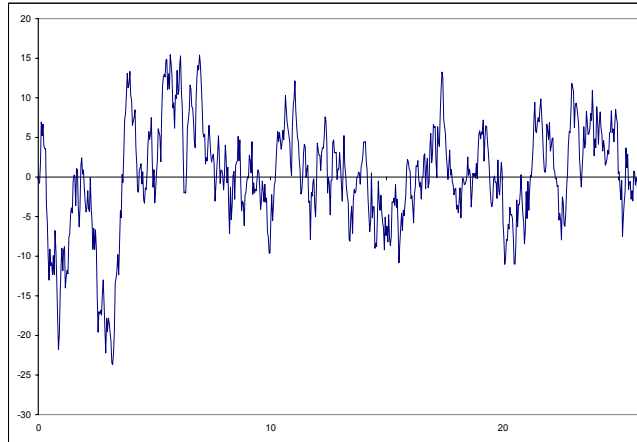


Figure 3.4: The related Bias.

3.5 Sensitivity of contingent claims

One of the applications of stochastic models in finance is the pricing of contingent claims. It is clear that if the underlying presents an uncertainty, prices of related options are erroneous. In particular, the presence of a bid-ask spread in underlying prices gives birth to a bid-ask spread on securities prices. In this section, we evaluate the transmission of this uncertainty from the underlying to the contingent claims. We consider an underlying, of which the price is described by the stochastic process X_t , we assume that X_t follows the SDE (3.1) with an uncertainty on Brownian motion, see the three previous sections. We analyze an European contingent claim with payoff function $f(X_T)$, at the maturity T . The price at time t of this security, if the underlying is non-erroneous, is given by

$$(3.11) \quad V_t = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} f(X_T) \mid \mathcal{F}_t \right]$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation under risk neutral probability \mathbb{Q} , r_s is the short interest rate at time s and \mathcal{F}_t is the filtration at time t , see Lamberton and Lapeyre [42].

Bouleau, see [12] pages 165-171, gives the following theorem for the gamma operator applied on prices (3.11).

Theorem 3.10 (Bouleau)

Let us suppose that the function f belongs to $C^1 \cap Lip$. Then V_t belongs to \mathbb{D} and we have

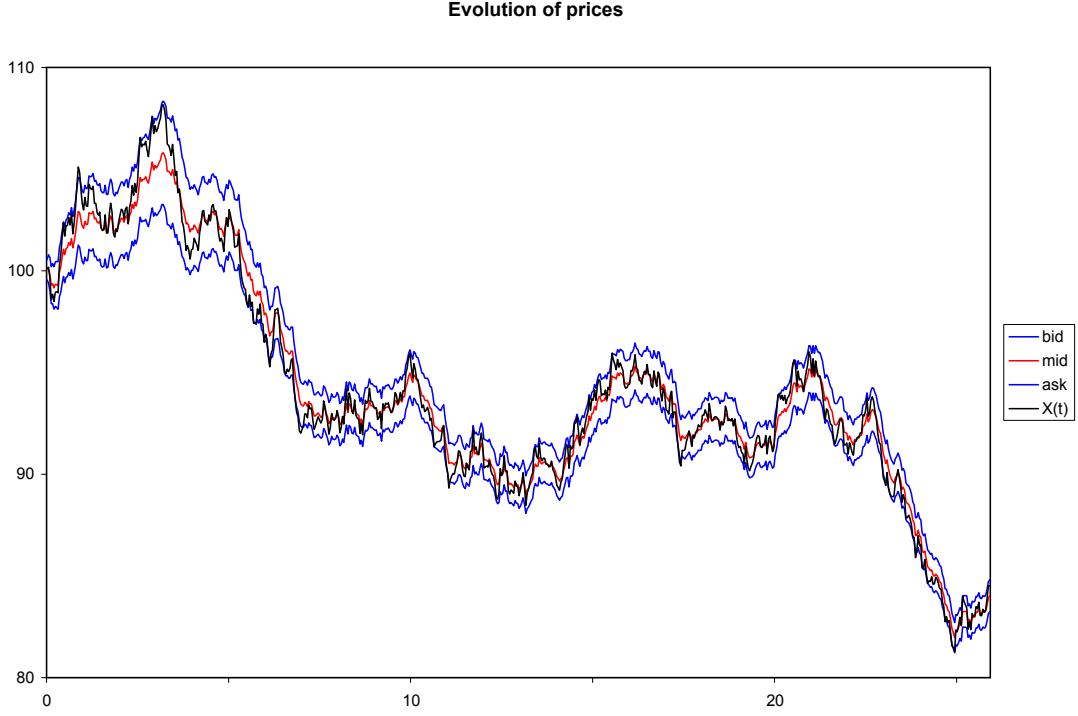


Figure 3.5: The evolution of the process X_t in black, the mid price in red and the bid and the ask prices in blue, when $\epsilon = 0, 1$.

(3.12)

$$\begin{aligned} \Gamma[V_t] &= \exp\left(-2 \int_t^T r_s ds\right) \left\{ \mathbb{E}^{\mathbb{Q}}[f'(X_T) M_T | \mathcal{F}_t] \right\}^2 \int_0^t \frac{\alpha(s) X_s^2 \sigma(s, X_s)}{M_s^2} ds \\ \Gamma[V_t, V_s] &= \exp\left(-\int_t^T r_u du - \int_s^T r_v dv\right) \mathbb{E}^{\mathbb{Q}}[f'(X_T) M_T | \mathcal{F}_t] \mathbb{E}^{\mathbb{Q}}[f'(X_T) M_T | \mathcal{F}_s] \\ &\quad * \int_0^{t \wedge s} \frac{\alpha(u) X_u^2 \sigma(u, X_u)}{M_u^2} du \end{aligned}$$

where M_t is given by (3.6).

We study the bias of security prices, we have the following theorem.

Theorem 3.11 (Bias of option price)

Let us suppose that f belongs to C^2 with bounded derivatives and let X_t be a stochastic process that verifies hypotheses of theorem 3.2, then the bias of option price (3.11) is given by

$$(3.13) \quad A[V_t] = e^{-\int_t^T r_s ds} \mathbb{E}^{\mathbb{Q}} \left[f'(X_T) A_t[X_T] + \frac{1}{2} f''(X_T) \Gamma_t[X_T] \middle| \mathcal{F}_t \right],$$

where

$$(3.14) \quad \begin{aligned} A_t[X_T] &= M_T \int_0^T \frac{\mathcal{P}_s}{M_s} (dW_s - K_s ds) \\ \Gamma_t[X_T] &= M_T^2 \int_0^t \frac{X_s^2 \alpha(s) \sigma^2(s, X_s)}{M_s^2} ds \end{aligned}$$

where the process \mathcal{P}_s is given by

$$(3.15) \quad \begin{aligned} \mathcal{P}_s &= -\frac{1}{2} X_s \sigma(s, X_s) \alpha(s) + \frac{1}{2} \left(2 \frac{\partial \sigma}{\partial x}(s, X_s) + X_s \frac{\partial^2 \sigma}{\partial x^2}(s, X_s) \right) \Gamma_t[X_s] \\ \Gamma_t[X_s] &= M_s^2 \int_0^{t \wedge s} \frac{X_u^2 \alpha(u) \sigma^2(u, X_u)}{M_u^2} du \end{aligned}$$

while M_t and K_t are given by relations (3.6).

Proof: The bias operator is linear and characterized by the chain rule (1.8); therefore, the form of equation (3.13) is correct. However, the gamma operator is bilinear, so we cannot exchange directly the bias with the expectation, but we have to define two operators A_t and Γ_t that are the equivalent operators of bias and variance under conditional expectation. Definitions of Γ_t and A_t require the two following lemmas. □

Lemma 3.12 (Gamma operator (Bouleau))

The conditional expectation maps the domain \mathbb{D} of the form into itself, this map is an orthogonal projection in \mathbb{D} and its range is an error sub-structure. Besides, let Γ_t be defined by

$$\Gamma_t \left[\int g(s) dW_s \right] = \Gamma [\mathbb{I}_{s \leq t} g(s) dW_s]$$

and let $(\cdot)^{\#t}$ be the associated sharp operator, then

$$(\mathbb{E}[U | \mathbb{F}_t])^{\#} = \mathbb{E}[U^{\#t} | \mathcal{F}_t]$$

for all $U \in \mathbb{D}$. Therefore, in our particular case

$$\Gamma_t[X_s] = M_s^2 \int_0^{t \wedge s} \frac{X_u^2 \alpha(u) \sigma^2(u, X_u)}{M_u^2} du$$

This result is due to Bouleau, see [12] pages 165-171 for more details.

Lemma 3.13 (Bias operator)

Given the operator Γ_t , it exists a unique generator A_t associated with Γ_t and this operator acts on the process X_s , in accord with the following stochastic differential equation.

$$(3.16) \quad \begin{aligned} A_t[X_s] = & -\frac{1}{2} \int_0^s X_u \sigma(u, X_u) \alpha(u) dB_u \\ & + \int_0^s \left\{ \sigma(u, X_u) + X_u \frac{\partial \sigma}{\partial x}(u, X_u) \right\} A_t[X_u] dW_u \\ & + \int_0^s \left\{ \frac{\partial \sigma}{\partial x}(u, X_u) + \frac{1}{2} X_u \frac{\partial^2 \sigma}{\partial x^2}(u, X_u) \right\} \Gamma_t[X_u] dW_u + \int_0^s A[X_u] r_u du \end{aligned}$$

and the solution of this SDE is given by formula (3.14).

Proof: The existence and the uniqueness of A_t is a consequence of Hille-Yosida theorem, see Albeverio [1]. SDE (3.14) is a consequence of the definition of Γ_t , the linearity of bias operator A_t and the theorem 3.16 in appendix 3.A. The proof of SDE solution (3.14) follows the proposition 3.6, for more details about the relation between bias and conditional expectation see chapter 2.

□

This lemma concludes the proof of theorem 3.11.

We remark that the bias of a contingent claim can be separated into two terms, the first of witch depends on the bias of underlying at maturity.

$$(3.17) \quad A[V_t] |_{\text{Bias}} = e^{-\int_t^T r_s ds} \mathbb{E}^{\mathbb{Q}} \left[f'(X_T) M_T \int_0^T \frac{\mathcal{P}_s}{M_s} (dW_s - K_s ds) \middle| \mathcal{F}_t \right]$$

Proposition 3.14 (Bias at maturity)

The bias $A[V_t] |_{\text{Bias}}$ given by formula (3.17) converges to $f'(X_T) A[X_T]$ when $t \rightarrow T$ in L^2 -norm.

Proof: This fact is an easy consequence of the properties of conditional expectation, we have only to prove that $A_t[X_T]$ converges to $A[X_T]$ when $t \rightarrow T$ in L^2 -norm, whose demonstrations are relatively straight-forward using the definition of Γ_t and equation (3.16).

□

We make two remarks on option pricing.

Remark 3.3 (Vanilla options) Although vanilla options do not verify the hypotheses assumed in this chapter, proposition 3.14 has an easy interpretation. The term $f'(X_T)$ is equal to zero or (minus) one in call (put) case, this represents the fact that the option is exerted. When a call is given up, the bias is worth zero; on the contrary, when the call is exerted, the bias is equal to the bias of underlying. Therefore, the buyer of the security receives X_T plus the bias of X_T , this sum is, in fact, the mid price of the underlying at time T , see section 3.4. The put case is symmetric.

Remark 3.4 (exotic options) *The interpretation of proposition 3.14 in vanilla case can be generalized for all exotic options. The term $f'(X_T)$ represent the delta position of hedging portfolio at maturity. Therefore, the value of an exotic option is corrected by the bias of X_T multiplied by the sensitivity of the payoff with respect to the underlying.*

At the end, the correction given by the proposition 3.14 has an easy financial interpretation. However there is a second term in relation (3.13) that depends on the variance.

$$(3.18) \quad A[V_t] |_{\text{Variance}} = \frac{1}{2} e^{-\int_t^T r_s ds} \mathbb{E}^{\mathbb{Q}} [f''(X_T) \Gamma_t[X_T] | \mathcal{F}_t]$$

We have an other proposition about the limit of this term

Proposition 3.15 (Bias at maturity)

bias $A[V_t] |_{\text{Variance}}$ given by the formula converges to $\frac{1}{2} f''(X_T) \Gamma[X_T]$ when $t \rightarrow T$ in L^2 -norm.

Proof: Proof proceeds along the same idea of proposition 3.14.

□

We have two remarks

Remark 3.5 (Vanilla options) *In the case of Call and Put options, the bias $A[V_t] |_{\text{Variance}}$ vanishes for all options excepting when $K = X_T$; in this case, the bias diverges to plus (minus) infinity when the option is a call (put). The financial interpretation is that, near to this point, the hedging portfolio is put through an high variation of delta position, the presence of a bid-ask spread becomes relevant and the risk related with an up or downgrade cannot be hedged.*

Remark 3.6 (Exotic options) *When we consider an exotic option the correction gives by proposition 3.15 is similar to the correction presents in Ito's formula. Therefore, we can explain this effect thanks to fluctuations of underlying X_t . As an example, we consider the logarithm contract, see Neuberger [44], the logarithm has a positive first derivative and a negative second one. The correction for a log-contract is given by*

$$(3.19) \quad A[V_T^{\log}] = \frac{A[X_T]}{X_T} - \frac{1}{2} \frac{\Gamma[X_T]}{X_T^2}$$

3.6 Conclusion

In this chapter, we have applied error theory using Dirichlet forms to model order books in order to describe evolution of bid-ask spreads.

We have supposed that the basic model is a local volatility model but we have assumed that the Brownian motion, that guides the stochastic differential equation, has an uncertainty, in the sense that, even if we know the trajectory of Brownian motion, it exists a noise on values taken by this process. We have model this uncertainty thanks to an error structure on Wiener space.

We have proposed to use this noise to model the bid-ask spread of asset prices thanks to a simple risk aversion argument; besides the uncertainty on Brownian motion generates, at the same time, a bias on the mid-price with respect to the basic diffusion, this bias has an easy interpretation, since the stochastic differential equation is not linear; Jensen's inequality says that the mean of a non-linear function is different to the function evaluated at the mean. Clearly, this bias has an impact on prices of contingent claims.

This chapter is just a starting work; as a matter of fact, the evaluation of derivatives is not the aim of our analysis but this model is the first attempt, in our knowledge, that consider bid-ask spreads as an endogenous characteristic of asset prices evolution. Up to now, bid-ask spread was explained with a liquidity problem and all model added this effect at the end of computations.

Our model shows that it exists a bias between the mid-price and the price given by the basic diffusion; therefore, in this framework, it is not correct to work with the basic diffusion, since this do not represent the evolution of mid-price.

We have added an application in a particular case when the volatility is proportional to a power of asset price, this model, studied by Hagan et. al [36] exhibits interesting properties.

Appendix 3.A Proof of theorems

In this appendix, we give the proofs of the complex theorems.

3.A.1 Theorem 3.2

We start with theorem 3.2, but we consider the following more general case:

Theorem 3.16 (Bias due to Brownian motion)

Let X_t be the solution of SDE

$$(3.20) \quad dX_t = H(t, X_t) dW_t + J(t, X_t) dt,$$

we assume that this SDE has a unique solution. We consider that Brownian motion W_t in equation (3.20) is erroneous with an error structure $(\Omega, \mathbb{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ of type Weighted Ornstein Uhlenbeck with weight $\alpha(s) \in L^2$ and non-negative. We assume that error structure $(\Omega, \mathbb{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ is so that the process $\Gamma[X_t]$, defined in Bouleau [12] chapter VII, belongs to $L^2(\Omega)$, we assume that, for all t , X_t belongs to \mathcal{DA} , i.e. the domain of semi-group A given by the Hille Yosida theorem⁴. Finally, we assume $H(t, x)$ and $J(t, x)$ belong to $C^2(\Omega) \cap L^2(\Omega)$, bounded and Lipschitz with its first and second derivatives.

Then we have the following SDE for the bias.

⁴see Albeverio [1].

$$\begin{aligned}
(3.21) \quad A[X_t] &= -\frac{1}{2} \int_0^t H(s, X_s) \alpha(s) dB_s \\
&+ \int_0^t \frac{\partial H}{\partial x}(s, X_s) A[X_s] dW_s + \int_0^t \frac{\partial J}{\partial x}(s, X_s) A[X_s] ds \\
&+ \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial x^2}(s, X_s) \Gamma[X_s] dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 J}{\partial x^2}(s, X_s) \Gamma[X_s] ds
\end{aligned}$$

Proof:

First of all, we remark that $H(t, x)$ and $J(t, x)$ belongs to L^2 ; therefore, we can consider two series of simple functions, that converge to $H(t, x)$ and $J(t, x)$ in L^2 -norm.

$$\begin{aligned}
h(t, x) &= \sum_{i=0}^n H(t_i, x) 1_{t_i \leq t < t_{i+1}} \\
j(t, x) &= \sum_{i=0}^n J(t_i, x) 1_{t_i \leq t < t_{i+1}}
\end{aligned}$$

where $t_0 = 0 < t_1 < \dots < t_n = t$ is a partition of the interval $[0, t]$, and we take the limit when $n \rightarrow \infty$ and the step of partition goes to zero. We study the following SDE

$$Y_t = \int_0^t h(s, Y_s) dB_s + \int_0^t j(s, Y_s) ds = \sum_{i=0}^n H(t_i, Y_{t_i}) (W_{t_{i+1}} - W_{t_i}) + \sum_{i=0}^n J(t_i, Y_{t_i}) (t_{i+1} - t_i)$$

that converges, see Da Prato [25] chapter 4, to X_t in L^2 -norm. Now we compute the bias of the previous relation.

$$\begin{aligned}
(3.22) \quad A[Y_t] &= A \left[\sum_{i=0}^n H(t_i, Y_{t_i}) (W_{t_{i+1}} - W_{t_i}) + \sum_{i=0}^n J(t_i, Y_{t_i}) (t_{i+1} - t_i) \right] \\
&= \sum_{i=0}^n \left\{ A[H(t_i, X_{t_i})] (W_{t_{i+1}} - W_{t_i}) + H(t_i, X_{t_i}) A[W_{t_{i+1}} - W_{t_i}] \right. \\
&\quad \left. + \Gamma[H(t_i, X_{t_i}), (W_{t_{i+1}} - W_{t_i})] + A[J(t_i, X_{t_i})] (t_{i+1} - t_i) \right\}
\end{aligned}$$

We study each term separately, in order to prove the convergence to SDE (3.21). We start with the term generated by the bias of $H(t, x)$, we analyze

$$(3.23) \quad A[H(t_i, Y_{t_i})] (W_{t_{i+1}} - W_{t_i}) = \left\{ \frac{\partial H}{\partial x}(t_i, Y_{t_i}) A[Y_{t_i}] + \frac{1}{2} \frac{\partial^2 H}{\partial x^2}(t_i, Y_{t_i}) \Gamma[Y_{t_i}] \right\} (W_{t_{i+1}} - W_{t_i})$$

since $H \in C^2$ with bounded first and second derivative w.r.t. the variable x . $A[Y_{t_i}]$ and $\Gamma[Y_{t_i}]$ are \mathcal{F}_{t_i} -measurable, since the first is given by the previous relation, while $\Gamma[Y_{t_i}]$ comes from the sharp $Y_{t_i}^\#$ that is adapted. Now, we sum and we have to prove that

$$(3.24) \quad \sum_{i=0}^n A[H(t_i, Y_{t_i})] (W_{t_{i+1}} - W_{t_i}) \xrightarrow{L^2} \int_0^t \left\{ \frac{\partial H}{\partial x}(s, X_s) A[X_s] + \frac{1}{2} \frac{\partial^2 H}{\partial x^2}(s, X_s) \Gamma[X_s] \right\} dW_s.$$

We consider the first term in right-hand side of previous equation and the same one in equation (3.23), and we study the L^2 -norm of the difference.

$$\begin{aligned} & \mathbb{E} \left[\left\{ \sum_{i=0}^n \frac{\partial H}{\partial x}(t_i, Y_{t_i}) A[Y_{t_i}] (W_{t_{i+1}} - W_{t_i}) - \int_{t_i}^{t_{i+1}} \frac{\partial H}{\partial x}(s, X_s) A[X_s] dW_s \right\}^2 \right] \leq \\ & \leq \sum_{i=0}^n \int_{t_i}^{t_{i+1}} \mathbb{E} \left[\left\{ \frac{\partial H}{\partial x}(t_i, Y_{t_i}) (A[Y_{t_i}] - A[X_s]) + A[Y_{t_i}] \left[\frac{\partial H}{\partial x}(t_i, Y_{t_i}) - \frac{\partial H}{\partial x}(s, X_s) \right] \right\}^2 \right] ds \leq \\ & \leq 2 \sum_{i=0}^n \int_{t_i}^{t_{i+1}} \left\{ \mathbb{E} \left[\left(\frac{\partial H}{\partial x}(t_i, Y_{t_i}) \right)^2 \right] \mathbb{E} [\{A[Y_{t_i}] - A[X_s]\}^2] \right. \\ & \quad \left. + \mathbb{E} [\{A[Y_{t_i}]\}^2] \mathbb{E} \left[\left\{ \frac{\partial H}{\partial x}(t_i, Y_{t_i}) - \frac{\partial H}{\partial x}(s, X_s) \right\}^2 \right] \right\} ds \end{aligned}$$

but the process Y_t converges to X_t ; thus, the second term in the previous relation goes to zero, thanks to Lipschitz hypothesis on the first derivative of $H(t, x)$ and thanks to Gronwall lemma. First term is controlled by the coefficient that bounds the first derivative of $H(t, x)$ multiplied by $\mathbb{E}[(A[Y_{t_i}] - A[X_s])^2]$, that converges to zero by continuity.

For the same reasons, the second term of relation (3.23) converges to the second term of relation (3.24), we exploit the fact that $\Gamma[X_t] \in L^2$, the boundary and Lipschitz control on the second derivative of $H(t, x)$ and, finally, the following control.

$$\begin{aligned} & \sum_{i=0}^n \int_{t_i}^{t_{i+1}} \mathbb{E}[(\Gamma[Y_{t_i}] - \Gamma[X_s])^2] ds \\ & \leq 2 \sum_{i=0}^n \int_{t_i}^{t_{i+1}} \mathbb{E} [(\Gamma[Y_{t_i}] - \Gamma[Y_{t_i}, X_s])^2 + (\Gamma[Y_{t_i}, X_s] - \Gamma[X_s])^2] ds \\ & \leq 2 \sum_{i=0}^n \int_{t_i}^{t_{i+1}} \mathbb{E} [3(\Gamma[Y_{t_i} - X_s, Y_{t_i}])^2 + 2(\Gamma[Y_{t_i} - X_s])^2] ds \end{aligned}$$

where the first term is controlled by $\Gamma[Y_{t_i} - X_s] \Gamma[Y_{t_i}]$, $\Gamma[Y_{t_i}]$ is constant on the interval $[t_i, t_{i+1}[$ and $\Gamma[(Y_{t_i} - X_s)]$ is controlled by the discretization step, see Bouleau [12]. The second term is controlled by the fact that $\Gamma[X_t] \in L^2$ and $Y_{t_i} - X_s$ converges to zero with the discretization step. Therefore, the previous relation goes to zero with the discretization step.

The term generated by the bias of $J(t, x)$ has a similar analysis and we can, easily, prove that

$$(3.25) \quad \sum_{i=0}^n A[J(t_i, Y_{t_i})] (t_{i+1} - t_i) \xrightarrow{L^2} \int_0^t \left\{ \frac{\partial J}{\partial x}(s, X_s) A[X_s] + \frac{1}{2} \frac{\partial^2 J}{\partial x^2}(s, X_s) \Gamma[X_s] \right\} ds.$$

Now, we have to study the term given by the bias of Brownian motion:

$$(3.26) \quad \sum_{i=0}^n H(t_i, Y_{t_i}) A[W_{t_{i+1}} - W_{t_i}] = \sum_{i=0}^n H(t_i, Y_{t_i}) (A[W_{t_{i+1}}] - A[W_{t_i}]).$$

We recall the hypothesis that our error structure is of type Weighted Ornstein-Uhlenbeck. So, we have the following property, see Bouleau [12] pages 116 and 165-167.

$$(3.27) \quad A[W_t] = -\frac{1}{2} \int_0^t \alpha(s) dW_s$$

Therefore, relation (3.26) converges to the equivalent of the first term of equation (3.21), since we can remark that

$$\begin{aligned} & \mathbb{E} \left[\left\{ \sum_{i=0}^n \left(H(t_i, Y_{t_i}) A[W_{t_{i+1}} - W_{t_i}] + \frac{1}{2} \int_{t_i}^{t_{i+1}} H(s, X_s) \alpha(s) dW_s \right) \right\}^2 \right] \\ &= \mathbb{E} \left[\frac{1}{4} \sum_{i=0}^n \left\{ -H(t_i, Y_{t_i}) \int_{t_i}^{t_{i+1}} \alpha(s) dW_s + \int_{t_i}^{t_{i+1}} H(s, X_s) \alpha(s) dW_s \right\}^2 \right] \\ &\leq \frac{1}{4} \sum_{i=0}^n \int_{t_i}^{t_{i+1}} \mathbb{E} [\{H(t_i, Y_{t_i}) - H(s, X_s)\}^2] \alpha^2(s) ds \xrightarrow{L^2} 0. \end{aligned}$$

We have used the fact that the function $H(t, x)$ belongs to L^2 .

The last term is given by

$$(3.28) \quad \begin{aligned} \sum_{i=0}^n \Gamma [H(t_i, Y_{t_i}), (W_{t_{i+1}} - W_{t_i})] &= \frac{\partial H}{\partial x}(t_i, Y_{t_i}) \Gamma [Y_{t_i}, (W_{t_{i+1}} - W_{t_i})] \\ &= \frac{\partial H}{\partial x}(t_i, Y_{t_i}) \left\langle Y_{t_i}^\#, (W_{t_{i+1}} - W_{t_i})^\# \right\rangle, \end{aligned}$$

but we know that

$$\left(\int_0^t f(s) dW_s \right)^\# = \int_0^t f(s) \sqrt{\alpha(s)} d\widehat{W}_s.$$

So, we can prove that the equation (3.28) is worth zero because

$$\begin{aligned} \Gamma [H(t_i, Y_{t_i}), (W_{t_{i+1}} - W_{t_i})] &= \frac{\partial H}{\partial x}(t_i, Y_{t_i}) \widehat{\mathbb{E}} \left[Y_{t_i}^\# (W_{t_{i+1}} - W_{t_i})^\# \right] \\ &= \frac{\partial H}{\partial x}(t_i, Y_{t_i}) \widehat{\mathbb{E}} \left[\widehat{\mathbb{E}} \left[Y_{t_i}^\# (W_{t_{i+1}} - W_{t_i})^\# \mid \widehat{\mathcal{F}}_{t_i} \right] \right] \\ &= \frac{\partial H}{\partial x}(t_i, Y_{t_i}) \widehat{\mathbb{E}} \left[Y_{t_i}^\# \widehat{\mathbb{E}} \left[(W_{t_{i+1}} - W_{t_i})^\# \mid \widehat{\mathcal{F}}_{t_i} \right] \right] = 0, \end{aligned}$$

where we have applied the fact that $Y_{t_i}^\#$ is $\widehat{\mathcal{F}}_{t_i}$ -measurable. □

3.A.2 Theorem 3.3

Now we prove the existence and the uniqueness of the solution of SDE (3.21).

Theorem 3.17 (Existence and uniqueness of bias process)

Under the same hypotheses of previous theorem and if there exist two constant C and D such that

$$\begin{aligned} |H(t, x)| &\leq C(D + |x|) \\ |J(t, x)| &\leq C(D + |x|) \end{aligned}$$

Then the stochastic differential equation (3.21) has an almost surely unique continuous solution.

Proof: Uniqueness

We consider Y_t and Z_t two solutions of SDE (3.20) with the same starting condition $Y_0 = Z_0$ and with the same bias and variance at the time $t = 0$, i.e. $\Gamma[Y_0] = \Gamma[Z_0]$ and $A[Y_0] = A[Z_0]$. A classical theorem, see Karatzas and Shreve [41] page 290, assures the existence and the uniqueness of the solution, i.e. $Y_t = Z_t$. Bouleau has proved, see [12] page 167, the existence and the uniqueness for the variance operator; therefore, we have $\Gamma[Y_t] = \Gamma[Z_t]$ almost surely for all time t . If we analyze the difference $A[Y_t] - A[Z_t]$, SDE (3.21) gives us

$$\begin{aligned} A[Y_t] - A[Z_t] &= -\frac{1}{2} \int_0^t H(s, Y_s) \alpha(s) dW_s + \frac{1}{2} \int_0^t H(s, Z_s) \alpha(s) dW_s \\ &\quad + \int_0^t \frac{\partial H}{\partial x}(s, Y_s) A[Y_s] dW_s - \int_0^t \frac{\partial H}{\partial x}(s, Z_s) A[Z_s] dW_s \\ &\quad + \int_0^t \frac{\partial J}{\partial x}(s, Y_s) A[Y_s] ds - \int_0^t \frac{\partial J}{\partial x}(s, Z_s) A[Z_s] ds \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial x^2}(s, Y_s) \Gamma[Y_s] dW_s - \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial x^2}(s, Z_s) \Gamma[Z_s] dW_s \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 J}{\partial x^2}(s, Y_s) \Gamma[Y_s] ds - \frac{1}{2} \int_0^t \frac{\partial^2 J}{\partial x^2}(s, Z_s) \Gamma[Z_s] ds. \end{aligned}$$

We can simplify, using the facts that $Y_t = Z_t$ and $\Gamma[Y_t] = \Gamma[Z_t]$, and we find

$$A[Y_t] - A[Z_t] = \int_0^t \frac{\partial H}{\partial x}(s, Y_s) \{A[Y_s] - A[Z_s]\} dW_s + \int_0^t \frac{\partial J}{\partial x}(s, Y_s) \{A[Y_s] - A[Z_s]\} ds,$$

we control the difference in L^2 -norm:

$$\begin{aligned} \mathbb{E} \left[(A[Y_t] - A[Z_t])^2 \right] &\leq 2 \max_{0 \leq s \leq t} \mathbb{E} \left[\left(\frac{\partial H}{\partial x}(s, Y_s) \right)^2 \right] \int_0^t \mathbb{E} \left[(A[Y_u] - A[Z_u])^2 \right] du \\ &\quad + 2T \max_{0 \leq s \leq t} \left\{ \mathbb{E} \left[\frac{\partial J}{\partial x}(s, Y_s) \right]^2 \right\} \int_0^t \mathbb{E} \left[(A[Y_u] - A[Z_u])^2 \right] du \\ &\leq 2K^2(T+1) \int_0^t \mathbb{E} \left[(A[Y_u] - A[Z_u])^2 \right] du \end{aligned}$$

where $t \leq T$ and K is the bigger Lipschitz constant between the functions $H_x(t, x)$ and $J_x(t, x)$, we can conclude $\mathbb{E} \left[(A[Y_t] - A[Z_t])^2 \right]$, thanks to Gronwall lemma. □

Proof: Existence

The proof is similar to the classical case, i.e. the convergence of Picard iteration. We build a sequence of stochastic processes $\{A[X_n(t)]\}_{n \in \mathbb{N}}$.

$$\begin{aligned} A[X_0(t)] &= A[X_0] \\ A[X_{n+1}(t)] &= -\frac{1}{2} \int_0^t H(s, X_s) \alpha(s) dW_s \\ &\quad + \int_0^t \frac{\partial H}{\partial x}(s, X_s) A[X_n(s)] dW_s + \int_0^t \frac{\partial J}{\partial x}(s, X_s) A[X_n(s)] ds \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial x^2}(s, X_s) \Gamma[X_s] dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 J}{\partial x^2}(s, X_s) \Gamma[X_s] ds \end{aligned} \tag{3.29}$$

we can compute the difference $A[X_{n+1}(t)] - A[X_n(t)]$ and, thanks to the same argument used for the uniqueness, we have the following control.

$$\begin{aligned} \mathbb{E} \left[\left(A[X_{n+1}(t)] - A[X_n(t)] \right)^2 \right] &\leq 2(T+1)K^2 \int_0^t \mathbb{E} \left[\left(A[X_{n+1}(s)] - A[X_n(s)] \right)^2 \right] ds \\ &\leq 2^n (T+1)^n K^{2n} \frac{t^{n-1}}{(n-1)!} \max_{0 \leq s \leq T} \mathbb{E} \left[\left(A[X_1(t)] - A[X_0(t)] \right)^2 \right] \end{aligned}$$

Thanks to the hypothesis about the linear growth and the fact that the term

$$-\frac{1}{2} \int_0^t H(s, X_s) \alpha(s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 H}{\partial x^2}(s, X_s) \Gamma[X_s] dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 J}{\partial x^2}(s, X_s) \Gamma[X_s] ds$$

belongs to L^2 , we have that sequence (3.29) is a Cauchy series.

□

Remark 3.7 *It is clear that the SDE verified by the bias is linear. Therefore, the hypothesis of linear growth of $H(t, x)$ and $J(t, x)$ is used to force the existence and the uniqueness of X_t , if we prove that the SDE verified by X_t has a unique solution we can relax the above-mentioned hypothesis.*

3.A.3 Lemma 3.5

We prove the lemma 3.5.

Proof: It is clear that the second SDE in equation (3.4) is the SDE (3.3). Therefore, we have only to prove that the first SDE of (3.4) is the SDE verified by the carré du champ operator.

We consider SDE (3.1) for the process X_t and we can write the SDE for the sharp process, see Bouleau [12] page 167.

$$X_t^\# = \int_0^t \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right] X_s^\# dW_s + \int_0^t \sigma(s, X_s) X_s \sqrt{\alpha(s)} d\widehat{W}_s + \int_0^t r(s) X_s^\# ds$$

We study $(X_t^\#)^2$, thanks to Ito formula:

$$\begin{aligned} (X_t^\#)^2 &= 2 \int_0^t X_s^\# dX_s^\# + \int_0^t d[X_s^\#] \\ &= 2 \int_0^t (X_s^\#)^2 \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right] dW_s \\ &\quad + 2 \int_0^t X_s^\# \sigma(s, X_s) X_s \sqrt{\alpha(s)} d\widehat{W}_s + 2 \int_0^t (X_s^\#)^2 r(s) ds \\ &\quad + \int_0^t \left\{ \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right]^2 (X_s^\#)^2 + \sigma^2(s, X_s) X_s^2 \alpha(s) \right\} ds \end{aligned}$$

and, thanks to properties of sharp operator, $\Gamma[X_t] = \widehat{\mathbb{E}}[(X_t^\#)^2]$, so we find

$$\begin{aligned} \widehat{\mathbb{E}} \left[(X_t^\#)^2 \right] &= 2\widehat{\mathbb{E}} \left[\int_0^t (X_s^\#)^2 \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right] dW_s \right] \\ &\quad + 2\widehat{\mathbb{E}} \left[\int_0^t X_s^\# \sigma(s, X_s) X_s \sqrt{\alpha(s)} d\widehat{W}_s \right] + 2\widehat{\mathbb{E}} \left[\int_0^t (X_s^\#)^2 r(s) ds \right] \\ &\quad + \widehat{\mathbb{E}} \left[\int_0^t \left\{ \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right]^2 (X_s^\#)^2 + \sigma^2(s, X_s) X_s^2 \alpha(s) \right\} ds \right] \end{aligned}$$

where we can exchange the expectation with the integral, and, using the martingale property, in second term. Therefore, we find the SDE verified by the carré du champ operator.

$$\begin{aligned} \Gamma[X_t] &= 2 \int_0^t \Gamma[X_s] \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right] dW_s + \int_0^t \sigma^2(s, X_s) X_s^2 \alpha(s) ds \\ (3.30) \quad &+ \int_0^t \Gamma[X_s] \left\{ 2r(s) + \left[\sigma(s, X_s) + X_s \frac{\partial \sigma}{\partial x}(s, X_s) \right]^2 \right\} ds \end{aligned}$$

□

Chapter 4

Perturbative Black Scholes Model

In this chapter, we study the point of transition between complete and incomplete financial models thanks to Dirichlet Forms methods. We apply the error theory using Dirichlet forms to hedging procedures in order to perturb parameters and stochastic processes, in the case of a volatility parameter fixed but uncertain for traders. We call this model Perturbed Black Scholes (PBS) Model. We show that this model can reproduce at the same time a smile effect and a bid-ask spread; we exhibit the volatility function associated to the local-volatility model equivalent to PBS model when vanilla options are concerned. Lastly, we present a connection between Error Theory using Dirichlet Forms and Utility Function Theory.

4.1 Introduction

In classical theory of financial mathematics, we assume that all market securities have a definite price. Indeed, the hypothesis of completeness of the market (see Lamberton et al. [42]) forces a single price for a contingent claim. If we take into account an uncertainty on a parameter, we find that the price of the contingent claim is not unique but we have many possible prices, therefore we can reproduce the bid-ask spread by means of a utility function related to the uncertainty on prices. If the uncertainty on parameter is small, we may neglect orders higher than the second, so we choose to work with Gaussian distributions.

Historical Black Scholes model for asset pricing assumes that the diffusion process for asset price is log-normal with a constant volatility; however, many works on empirical market data present a skewed structure of market implied volatilities with respect to the strike; this effect is called smile of volatility or volatility skew (see Renault et al. [51] and Rubinstein [52]).

Implied volatility is convex as a function of the strike and generally exhibits a slope with respect to the strike at forward money (see Perignon et al. [48]); to take this into account, we propose a new model based on BS model, characterized by an uncertain volatility parameter, called Perturbed Black-Scholes model. This is a *subjective* volatility model with closed forms for option pricing, the attribute subjective means that the volatility is split in the sense that two volatilities exist, the market volatility and the estimation of the trader, where the later is subjective. We study some constraints to force a smile on implied volatility and define the local volatility model, by means of its volatility function, equivalent to the PBS model for vanilla options.

Summarizing, we propose a new financial model for securities pricing based on the Black Scholes model with a random variable as volatility, we use a perturbative approach to preserve closed forms for options prices and greeks; this model permits to reproduce a smile on implied volatility and generate automatically a bid-ask spread.

The chapter is organized as follows: In section 2, we present the PBS model and study the effect of uncertainty on volatility for an underlying following Black Scholes model without drift. In section 3, we investigate the relations with the literature, while section 4 presents an interpretation of the relative index defined in section 2 by means of utility functions theory. Finally section 5 resumes and concludes.

4.2 Perturbative Black Scholes model

We start with the classical Black Scholes model (see Black et al. [10] and Lamberton et al. [42]); let $(\Omega, \mathcal{F}, \mathbb{P})$ be the historical probability space and B_t the associated Brownian motion, we suppose that the dynamic of the risky asset under historical probability \mathbb{P} is given by the following BS diffusion without drift¹:

$$\begin{aligned} dS_t &= S_t \sigma_0 dB_t \\ S_t &= S_0 e^{\sigma_0 B_t - \frac{1}{2} \sigma_0^2 t} \end{aligned}$$

In this framework, the price of a European vanilla option is well known (see Lamberton et al. [42]).

The BS model presents many advantages; in particular, the pricing depends only on volatility and we find closed forms for premium and greeks of vanilla options; unluckily, the BS model cannot reproduce the market price of call options for all strikes at the same volatility, this effect is called smile. We propose to consider a perturbation of this model by means of an error structure on volatility.

We make three hypotheses:

1. the real market follows a BS model with fixed and non perturbed volatility σ_0 ;
2. the trader has to estimate the volatility, so its volatility contains intrinsic inaccuracies, we model this ambiguity by means of an error structure; nonetheless we assume that the stock price S_t is not erroneous. We evaluate the impact of the perturbation, generated by the trader mishandling, on the profit and loss process used by trader to hedge the vanilla option;
3. the trader knows this perturbation and he wants to modify the option prices to take into account the bias induced by the perturbation on volatility.

¹We can remark further in this analysis that the presence of a drift term has an impact otherwise from the classical BS model.

4.2.1 “Mismatch” on trading hedging

We consider a trader that uses an “official” BS asset model in order to hedge vanilla options; he uses the market price to determine the “fair” values of parameters in his model (in this case, only the level of flat volatility σ_0) by inversion of pricing formula. The trader finds an observed volatility process ς_t , usually known as implied volatility. He hedges his portfolio according to his volatility, so the price of an option, that pays a payoff Φ , is $F(\varsigma_0, x, 0)$.

We study the profit and loss process associated to the hedging position. The profit and loss process at the maturity of a trader that follows the strategy associated with his volatility ς_t is given by:

$$(4.1) \quad P\&L = F(\varsigma_0, x, 0) + \int_0^T \frac{\partial F}{\partial x}(\varsigma_t, S_t, t) dS_t - \Phi(S_T)$$

We make two remarks:

Remark 4.1 *The profit and loss process is stochastic, due to two random sources:*

- *First of all, the stochastic “real” model, since the trader cannot use the correct hedging portfolio.*
- *Second, the stochastic process ς_t , that can depend on a random component independent to the Brownian motion B_t .*

Remark 4.2 *The profit and loss process must be studied on historical probability \mathbb{P} , therefore the presence of a drift on the BS diffusion modifies the second term of equation (4.1). Without drift this term is a martingale and this fact simplifies the computation. The case of BS model with drift will be dealt in chapter 5.*

In order to analyze the law of P&L process, it is sufficient to study the expectation on a class of regular test functions $h(P\&L)$ and the error on them. We will come back on the role and choice of a particular function h in the next subsection. We suppose, for simplicity, that the trader volatility ς_t is a time independent random variable:

$$(4.2) \quad \varsigma_t = \sigma.$$

We define an error structure for the volatility σ , therefore this volatility admits the following expansion:

$$(4.3) \quad \sigma_0 \rightarrow \sigma_0 + \epsilon A[\sigma](\sigma_0) + \sqrt{\epsilon \Gamma[\sigma](\sigma_0)} \tilde{\mathcal{N}}$$

where $\tilde{\mathcal{N}}$ is a standard gaussian variable defined in a space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ independent to Ω . Furthermore, if the volatility has been estimated by means of a statistic on market data, we can specify the functional Γ . In fact, thanks to a result of Bouleau and Chorro [14], Γ is related with the inverse of the Fisher information matrix.

We want to estimate the variance and bias error of $\mathbb{E}[h(P\&L)]$. To perform the calculus, we assume that $\sigma = \sigma_0$ is the right value of the random variable in the sense that $\varsigma_t = \sigma_0$ and $P\&L(\sigma_0) = 0$. We have the following relation for the sharp of volatility:

$$\varsigma_t^\# = \sigma^\#$$

Then we can state

Theorem 4.1

Under the condition (ARB), see section 1.6, we have the following bias and variance:

$$(4.4) \quad A[\mathbb{E}[h(P\&L)]] = h'(0)\Upsilon_1^{BS}(\sigma_0) + \frac{1}{2}h''(0)\Upsilon_2^{BS}(\sigma_0)$$

$$(4.5) \quad \Gamma[\mathbb{E}[h(P\&L)]] = [h'(0)]^2 \Lambda^{BS}(\sigma_0)$$

where

$$\begin{aligned} \Upsilon_1^{BS}(\sigma_0) &= \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)A[\sigma](\sigma_0) + \frac{1}{2} \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0)\Gamma[\sigma](\sigma_0) \right\} \\ \Upsilon_2^{BS}(\sigma_0) &= \left\{ \left[\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) \right]^2 + \sigma_0^2 \int_0^T \mathbb{E} \left[S_t^2 \left(\frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_t, t) \right)^2 \right] dt \right\} \Gamma[\sigma](\sigma_0) \\ \Lambda^{BS}(\sigma_0) &= \left\{ \mathbb{E} \left[\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) \right]^2 \right\} \Gamma[\sigma](\sigma_0) \end{aligned}$$

and we have the following truncated expansion:

$$(4.6) \quad \mathbb{E}[h(P\&L)] \approx \epsilon h'(0) \Upsilon_1^{BS}(\sigma_0) + \epsilon \frac{1}{2} h''(0) \Upsilon_2^{BS}(\sigma_0) + \sqrt{\epsilon [h'(0)]^2 \Lambda^{BS}} \tilde{\mathcal{N}}(0, 1)$$

Proof:

We start with the study of the variance. A computation yields

$$(\mathbb{E}[h(P\&L)])^\# = \mathbb{E} \left[h'(P\&L) \left(\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right) \sigma^\# \right]$$

thus the quadratic error is equal to:

$$\Gamma[\mathbb{E}[h(P\&L)]] = [h'(0)]^2 \left\{ \mathbb{E} \left[\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right]^2 \right\} \Gamma[\sigma](\sigma_0)$$

and the second term vanishes, since the stock price is a martingale, here the hypothesis on drift is crucial.

The study of the bias is more complicated, we start with the remark that the bias is a linear operator:

$$A[\mathbb{E}[h(P\&L)]] = \mathbb{E}[A[h(P\&L)]] = \mathbb{E}\left[h'(P\&L)A[P\&L] + \frac{1}{2}h''(P\&L)\Gamma[P\&L]\right]$$

We study the two terms separately; for the first, we find the expectation of quadratic error in the case $\varsigma = \sigma_0$:

$$\mathbb{E}[\Gamma[P\&L]] = \left\{ \mathbb{E}\left[\left(\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)\right)^2\right] + \int_0^T \mathbb{E}\left[\sigma_0^2 S_s^2 \left(\frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s)\right)^2 ds\right] \right\} \Gamma[\sigma](\sigma_0).$$

We study the bias operator; we must evaluate the expectation of bias of profit and loss process, and we find the following result always in the case $\varsigma = \sigma_0$:

$$\mathbb{E}[A[P\&L]] = \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)A[\sigma](\sigma_0) + \frac{1}{2}\frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0)\Gamma[\sigma](\sigma_0)$$

Finally the bias of expectation of a function of profit and loss process:

$$\begin{aligned} A[\mathbb{E}[h(P\&L)]] &= h'(0) \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)A[\sigma](\sigma_0) + \frac{1}{2}\frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0)\Gamma[\sigma](\sigma_0) \right\} \\ &\quad + \frac{1}{2}h''(0) \left\{ \left[\frac{\partial F}{\partial \sigma}(\sigma_0, x, 0)\right]^2 + \right. \\ &\quad \left. + \sigma_0^2 \int_0^T \mathbb{E}\left[S_t^2 \left(\frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_t, t)\right)^2\right] dt \right\} \Gamma[\sigma](\sigma_0) \end{aligned}$$

The proof ends with the truncated expansion that is a consequence of the error theory using Dirichlet Forms (see Bouleau [15] and [16])

□

In order to interpret this result in finance, we consider that the trader knows the presence of errors in his procedure and wants to neutralize this effect.

We associate:

- the variance of $h(P\&L)$ process to the bid-ask spread of options;
- the bias of $h(P\&L)$ process to a shift of prices of options asked by the trader to the buyer.

Indeed, in the classical theory of financial mathematics we assume that all market securities have a single price, with the probability theory language we can associate at any derivative securities a Dirac distribution for its price. If we take into account uncertainty on volatility, we have found that the price of the contingent claim is not unique but we have many possible prices; thus the Dirac distribution changes into a continuous distribution, characterized by a variance and a shift of the mean with respect to the previous Dirac distribution (see figure 4.1).

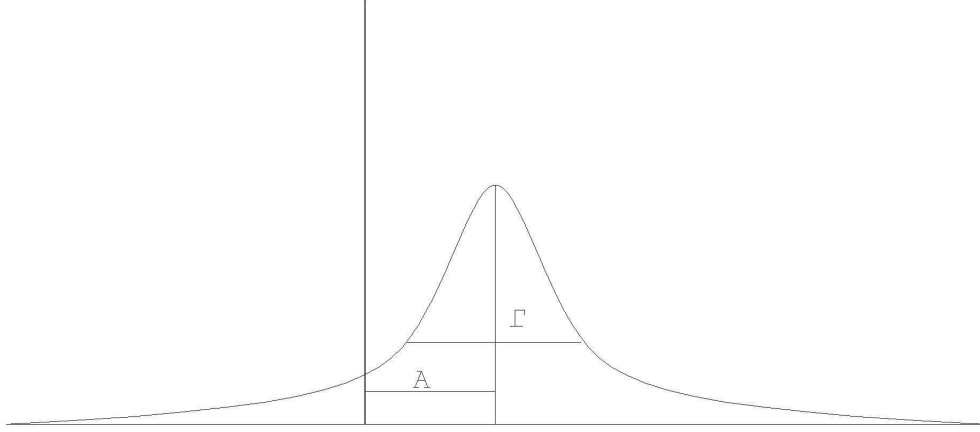


Figure 4.1: Impact of ambiguity: the Dirac distribution of price X becomes a continuous distribution; the mean shifts of $\epsilon A[X]$ and the variance is $\epsilon \Gamma[X]$.

Theorem 4.2 *If the perturbation in the volatility is small, we can neglect orders higher than the second, so we always work with Gaussian distributions; the trader must modify his prices in order to take into account the two previous effects, namely the variance and the bias, then he fixes a supportable risk probability $\alpha < 0.5$ and accepts to buy the option at the price*

$$(\text{Bid Premium}) = (\text{BS Premium}) + \epsilon A [\mathbb{E}[h(P\&L)]] + \sqrt{\epsilon \Gamma [\mathbb{E}[h(P\&L)]]} \mathcal{N}_\alpha$$

where \mathcal{N}_α is the α -quantile of the reduced normal law. Likewise, the trader accepts to sell the option at the price

$$(\text{Ask Premium}) = (\text{BS Premium}) + \epsilon A [\mathbb{E}[h(P\&L)]] + \sqrt{\epsilon \Gamma [\mathbb{E}[h(P\&L)]]} \mathcal{N}_{1-\alpha}$$

Remark 4.3 *We remark that the two previous prices are symmetric, since $\mathcal{N}_\alpha + \mathcal{N}_{1-\alpha} = 0$; therefore the mid-premium is*

$$(\text{Mid Premium}) = (\text{BS Premium}) + \epsilon A [\mathbb{E}[h(P\&L)]] .$$

We emphasize that with our model we can reproduce a bid-ask spread and we can associate its width to the trader's risk aversion (the probability α) and the volatility uncertainty (the term $\sqrt{\epsilon \Gamma [\mathbb{E}[h(P\&L)]]}$).

In the rest of this analysis we work directly with the mid premium, but all results represent the center of a normal distribution, in order to reproduce the bid and the ask premium we need to specify the probability α .

We conclude this subsection with a remark.

Remark 4.4 *The presence of the perturbation induces a problem on the completeness of the market; the market with perturbed volatility is not complete, since the volatility depends on a second random source orthogonal to Ω , and the presence of a bid-ask spread is a direct consequence of this fact. On the other hand the enforcement that σ to be equal σ_0 cancels the impact of the second random source. This apparent contradiction is due to the fact that the argument acts precisely at the boundary between complete and incomplete markets.*

4.2.2 Role and choice of h functions and relative index

In this subsection, we discuss on the choice of function h in equation (4.6), because function h defines the magnitude of correction on prices; this choice becomes simpler since we have to specify only the first and second derivative in zero; therefore we can consider that the function h is a parabola that passes through the origin. Owing to the two degrees of freedom associated with ϵ and α , we can take $h'(0) = 1$: this is easy to understand from the economical point of view because the trader wants to balance his portfolio, i.e. the $P\&L$ process. If we look at equation (4.6), we find that the choice of $h'(0) = 1$ defines completely the term of variance. Since the second derivative of h has an impact only on the bias and the coefficient $\Upsilon_2^{BS}(\sigma_0)$ is positive, this impact is a shift of the mean, as in the following figure.

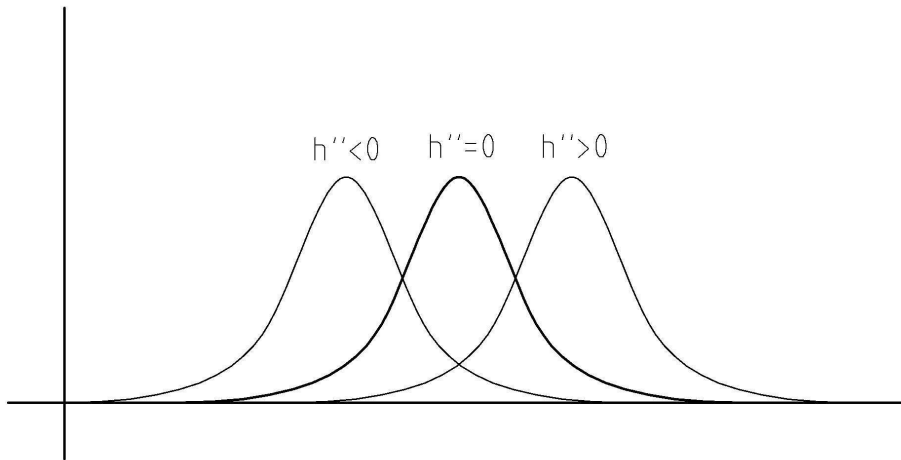


Figure 4.2: Impact of ambiguity: the convexity (resp. concavity) of function h raises (resp. reduces) the mean of prices but leaves the variance unchanged.

We suggest to interpret this impact as an asymmetry of the balance between supply and demand. Indeed, if $h''(0) = 0$ we find that the function h is the identity: this means that the trader uses directly the process of profit and loss, and the bias is “neutral”, i.e. that we find the same result if we consider the buyer’s point of view (it is enough to take minus identity function as h). A surplus of the demand of an option with respect to the supply induces a raising of the prices, this is the classical case of market where banks sell options and private investors buy. We model this perturbation with a positive second derivative for h and we consider that if $h''(0) > 0$

(resp. $h''(0) < 0$) the demand (resp. supply) exceeds the supply (resp. demand). We define the following index of asymmetry of balance between supply and demand:

$$r_{S/D} = \frac{h''(0)}{h'(0)}$$

We can identify this index by means of the classical utility theory: if we interpret h as a utility function, $r_{S/D}$ is known as the absolute index of h . In the next part, we study the bias of profit and loss process in the case of a call option; to simplify matters, we suppose that h is the identity function, i.e. $r_{S/D} = 0$, but before we must introduce an other index, very important in the continuation of this article.

We concentrate our attention on a problem; after the perturbation of a parameter σ_0 , we have:

$$\sigma_0 \rightarrow \sigma = \sigma_0 + \epsilon A[\sigma] + \sqrt{\epsilon \Gamma[\sigma]} \mathcal{N} \text{ with } \mathcal{N} \sim N(0, 1)$$

However, ϵ is generally unknown: the Error Theory via Dirichlet Forms cannot define this parameter. In order to deal with this question, we propose to renormalize this problem; we consider the ratio between the bias and the variance, since the variance is almost surely strictly positive, therefore that the dependence on ϵ is canceled:

$$\frac{\text{Bias } X}{\text{Variance } X} = \frac{\epsilon A[X]}{\epsilon \Gamma[X]} = \frac{A[X]}{\Gamma[X]}$$

This ratio is not homogeneous, because the generator is linear and the operator “carre du champ” is bilinear, so we define a relative index by:

$$(4.7) \quad r_r(X) = 2 \frac{XA[X]}{\Gamma[X]}$$

The factor 2 will be justified in section 4.5. where we show a relation between Dirichlet forms and utility theory, since we can interpret $r_r(X)$ as a relative index of an exogenous utility function.

4.3 Call options case and volatility smile

We concentrate on call option and we study the bias and its derivatives in order to determine some sufficient condition to force the presence of a smile on implied volatility. We know the premium of a call option (see Lamberton et al. [42]) with strike K and spot value x , and its hedging strategy:

$$\begin{aligned} C(\sigma_0, x, 0) &= F(\sigma_0, x, 0) = x \mathcal{N}(d_1) - K \mathcal{N}(d_2) \\ \text{Delta} &= \frac{\partial F}{\partial x}(\sigma_0, x, 0) = \mathcal{N}(d_1) \\ \text{where} \quad d_1 &= \frac{\ln x - \ln K + \frac{\sigma_0^2}{2} T}{\sigma_0 \sqrt{T}} \text{ and } d_2 = d_1 - \sigma_0 \sqrt{T}. \end{aligned}$$

The following results are classical (see [42]):

$$(4.8) \quad \begin{aligned} \frac{\partial F}{\partial \sigma_0}(\sigma_0, x, 0) &= x\sqrt{T} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \\ \frac{\partial^2 F}{\partial \sigma_0^2}(\sigma_0, x, 0) &= \frac{x\sqrt{T}}{\sigma_0} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} d_1 d_2 \\ \frac{\partial^2 F}{\partial K^2}(\sigma_0, x, 0) &= \frac{x}{K^2 \sigma_0 \sqrt{T}} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}. \end{aligned}$$

Then the bias of the call premium is given by

$$(4.9) \quad A[C]|_{\sigma=\sigma_0} = x \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ A[\sigma\sqrt{T}]|_{\sigma=\sigma_0} + \frac{d_1 d_2}{2\sigma_0 \sqrt{T}} \Gamma[\sigma\sqrt{T}]|_{\sigma=\sigma_0} \right\}.$$

We can compute the first derivative with respect to the strike:

$$\begin{aligned} \frac{\partial A[C]}{\partial K} \Big|_{\sigma=\sigma_0} &= \frac{x}{K \sigma_0 \sqrt{T}} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ d_1 A[\sigma\sqrt{T}]|_{\sigma=\sigma_0} - \frac{d_1 + d_2 - d_1^2 d_2}{2\sigma\sqrt{T}} \Gamma[\sigma\sqrt{T}]|_{\sigma=\sigma_0} \right\} \\ &= \frac{d_1 A[C]|_{\sigma=\sigma_0}}{K \sigma_0 \sqrt{T}} - \frac{x}{2K \sigma_0^2 T} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} (d_1 + d_2) \Gamma[\sigma\sqrt{T}]|_{\sigma=\sigma_0} \end{aligned}$$

We find that the first derivative vanishes at the forward money² if and only if the bias of call vanishes at the same strike.

$$\begin{aligned} A[C]|_{K=x, \sigma=\sigma_0} &= x \frac{e^{-\frac{\sigma_0^2 T}{8}}}{\sqrt{2\pi}} \left\{ A[\sigma\sqrt{T}]|_{\sigma=\sigma_0} - \frac{\sigma_0 \sqrt{T}}{8} \Gamma[\sigma\sqrt{T}]|_{\sigma=\sigma_0} \right\} \\ \frac{\partial A[C]}{\partial K} \Big|_{K=x, \sigma=\sigma_0} &= \frac{1}{2x} A[C]|_{K=x, \sigma=\sigma_0} \end{aligned}$$

We can remark that the bias and its first derivative are positive (resp negative) at the money if and only if

$$(4.10) \quad r_r^{BS}(\sigma\sqrt{T}) \Big|_{\sigma=\sigma_0} = 2\sigma_0 \sqrt{T} \frac{A[\sigma\sqrt{T}]}{\Gamma[\sigma\sqrt{T}]} \Big|_{\sigma=\sigma_0} > (\text{resp. } <) \frac{\sigma_0^2 T}{4}$$

We find three cases:

1. if $r_r(\sqrt{\sigma_0^2 T}) < \frac{1}{4} \sigma_0^2 T$, then the bias of call and his first derivative are negative at the money.

²In interest-free case the forward money is for $K = x$, in this case we have $d_1 = -d_2$.

2. if $r_r \left(\sqrt{\sigma_0^2 T} \right) = \frac{1}{4} \sigma_0^2 T$, then the bias of call and his first derivative vanish at the money.
3. if $r_r \left(\sqrt{\sigma_0^2 T} \right) > \frac{1}{4} \sigma_0^2 T$, then the bias of call price and his first derivative are positive at the money.

Remark 4.5 *This bound increases with maturity; if we suppose a constant relative index (CRI) we can define a bound on maturity \mathcal{T}_{bias} . Therefore, if we study an option with maturity smaller (resp. greater) than \mathcal{T}_{bias} we have that the bias associated to the hedging "profit and loss" process and his first derivative are positive (resp. negative).*

We calculate the second derivative:

$$\frac{\partial^2 A[C]}{\partial K^2} \Big|_{\sigma=\sigma_0} = \frac{d_2}{K \sigma_0 \sqrt{T}} \frac{\partial A[C]}{\partial K} \Big|_{\sigma=\sigma_0} - \frac{x}{K^2 \sigma_0^2 T} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ A \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} + \frac{d_1^2 + 2d_1 d_2 - 2}{2\sigma \sqrt{T}} \Gamma \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} \right\}$$

and we evaluate the second derivative at the forward money.

$$\frac{\partial^2 A[C]}{\partial K^2} \Big|_{K=x, \sigma=\sigma_0} = -\frac{1}{K} \frac{e^{-\frac{\sigma_0^2 T}{8}}}{\sqrt{2\pi}} \left\{ \left(\frac{1}{4} + \frac{1}{\sigma_0^2 T} \right) A \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} - \left[\frac{\sigma_0^2 T}{32} + \frac{1}{8} + \frac{1}{\sigma_0^2 T} \right] \frac{\Gamma \left[\sigma \sqrt{T} \right]}{\sigma \sqrt{T}} \Big|_{\sigma=\sigma_0} \right\}.$$

If we force the bias of the call to be convex, we find:

$$(4.11) \quad r_r^{BS} \left(\sigma \sqrt{T} \right) \Big|_{\sigma=\sigma_0} = 2\sigma \sqrt{T} \frac{A \left[\sigma \sqrt{T} \right]}{\Gamma \left[\sigma \sqrt{T} \right]} \Big|_{\sigma=\sigma_0} < \frac{\sigma_0^4 T^2 + 4\sigma_0^2 T + 32}{4\sigma_0^2 T + 16} = \Theta \left(\sigma_0 \sqrt{T} \right).$$

Remark 4.6 *Previous bound $\Theta(x)$ is strictly positive, decreasing in $[0, 4\sqrt{2}]$ and increasing if $x > 4\sqrt{2}$, and we find that $\Theta(4\sqrt{2}) = \frac{7}{2}\sqrt{2} - 3$.*

If the relative index is constant (CRI) we have an always convex bias if $r_r < \Theta(4\sqrt{2})$

In the previous relations we note that the constraint depends on the volatility by means of cumulated volatility $\sigma_0 \sqrt{T}$. For more generality in this study we can assume that the erroneous parameter is not the volatility but the cumulated variance $\int_0^T \sigma_0^2(s) ds$ that appears in general Black & Scholes model when the volatility is deterministic but depends on time. Now we study the evolution of slope as a function of maturity at the money.

$$\begin{aligned} \frac{\partial A}{\partial T} \Big|_{\sigma=\sigma_0} &= \frac{x}{2T} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \left\{ (1 + d_1 d_2) A \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} + \frac{4d_1^2 d_2^2 - 3\sigma_0^2 T - (d_1 + d_2)^2}{8\sigma_0 \sqrt{T}} \Gamma \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} \right\} \\ \frac{\partial A}{\partial T} \Big|_{K=x, \sigma=\sigma_0} &= \frac{x e^{-\frac{\sigma_0^2 T}{8}}}{8T \sqrt{2\pi}} \left\{ (4 - \sigma_0^2 T) A \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} + \frac{\sigma_0 \sqrt{T}}{8} (\sigma_0^2 T - 12) \Gamma \left[\sigma \sqrt{T} \right] \Big|_{\sigma=\sigma_0} \right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 A}{\partial K \partial T} \Big|_{\sigma=\sigma_0} &= -\frac{x}{2\sigma_0 T^{\frac{3}{2}} K \sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \left\{ d_2(1-d_1^2) A[\sigma\sqrt{T}] \Big|_{\sigma=\sigma_0} + \right. \\ &\quad \left. + d_1 \frac{4d_1^3 d_2^2 - 3d_1 \sigma_0^2 T + (d_1 + d_2)(4 - 9d_1 d_2 - d_1^2)}{8\sigma_0 \sqrt{T}} \Gamma[\sigma\sqrt{T}] \Big|_{\sigma=\sigma_0} \right\} \\ \frac{\partial^2 A}{\partial K \partial T} \Big|_{K=x, \sigma=\sigma_0} &= -\frac{e^{-\frac{\sigma_0^2 T}{8}}}{16T \sqrt{2\pi}} \left\{ (\sigma_0^2 T - 4) A[\sigma\sqrt{T}] \Big|_{\sigma=\sigma_0} - \sigma_0 \sqrt{T} \frac{\sigma_0^2 T - 12}{8} \Gamma[\sigma\sqrt{T}] \Big|_{\sigma=\sigma_0} \right\}. \end{aligned}$$

In order to find a slope that increases with increasing maturity, we have to impose:

$$(4.12) \quad r_r(\sigma\sqrt{T}) \Big|_{\sigma=\sigma_0} \geq \frac{\sigma_0^2 T}{4} \frac{\sigma_0^2 T - 12}{4 - \sigma_0^2 T} \quad \text{with } \sigma_0^2 T < 4.$$

We study the evolution of smile as a function of maturity.

$$\frac{\partial^3 A}{\partial K^2 \partial T} \Big|_{K=x, \sigma=\sigma_0} = \frac{e^{-\frac{\sigma_0^2 T}{8}}}{x \sigma_0^2 T^2 \sqrt{2\pi}} \left\{ \frac{16 + \sigma_0^4 T^2}{32} A[\sigma\sqrt{T}] \Big|_{\sigma=\sigma_0} - \frac{\sigma_0^2 T (\sigma_0^2 T - 4)^2 + 128}{256} \frac{\Gamma[\sigma\sqrt{T}]}{\sigma\sqrt{T}} \Big|_{\sigma=\sigma_0} \right\}$$

This term is positive if and only if

$$(4.13) \quad r_r(\sigma\sqrt{T}) \Big|_{\sigma=\sigma_0} > \frac{1}{4} \frac{\sigma_0^2 T (\sigma_0^2 T - 4)^2 + 128}{16 + \sigma_0^4 T^2}.$$

4.3.1 Smile of volatility

Now we study a particular case, in order to reproduce the smile of volatility showed in market data.

Theorem 4.3 *We fix the relative index at*

$$(4.14) \quad r_r^{BS}(\sigma\sqrt{T}) \Big|_{\sigma=\sigma_0} = \frac{\sigma_0^2 T}{4}$$

This choice fix the values of the bias and its derivatives at the money, in particular this choice force the bias and its first derivative to be zero at the money (see equation 4.10); the second derivative becomes positive at the money thanks to equation 4.11 for any maturity T . Therefore the bias is strictly convex around the money and vanishes at the money, therefore it is positive in a neighborhood of the money. Now if the bias vanishes at the money the implied ATM volatility is σ_0 , but, since the bias is positive around, the implied volatility becomes greater than σ_0 around the money³; Finally we have reproduced a smile effect around the money.

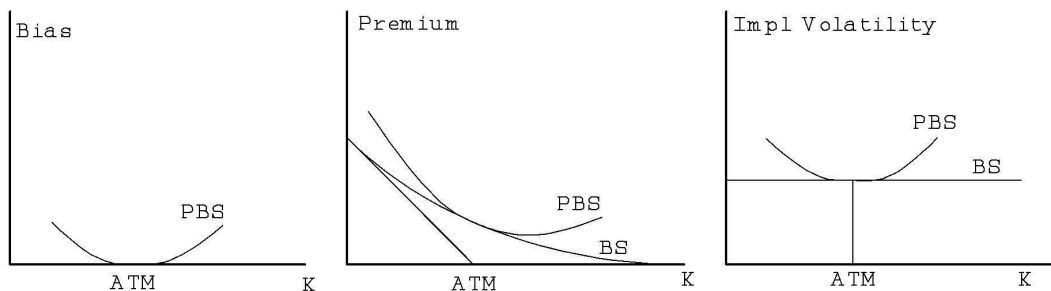


Figure 4.3: Representation of the smile in PBS model.

We conclude this example with a remark on the evolution of smile as function of maturity.

Remark 4.7 (evolution of smile) *If we assume that the relative index verifies the relation (4.14) and if the cumulated variance $\sigma_0^2 T$ is smaller⁴ than 16, then the second derivative of the implied volatility with respect of the strike decreases as the maturity increases, in accord with the market's data, see Hagan et al. [35] figure 2.1.*

It is sufficient to see the relation (4.13), the second derivative of bias is a decreasing function with respect to maturity T , and the argument of theorem 4.3 gives the result.

4.3.2 Dupire's formula and implicit local volatility model

In this section, we want to specify the Local Volatility Model equivalent to Perturbed Black Scholes Model. We know that the knowledge of prices of options for all strikes and maturities defines a single local volatility model that reproduces these prices; the Dupire formula (see Dupire [28]) defines the local volatility function:

$$(4.15) \quad \sigma_{imp}^2(T, K) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}$$

Our model is a perturbation of BS model, so we can consider the following expansion:

³We recall that the vega is positive for call options.

⁴This hypothesis is absolutely realistic since the annual cumulated volatility is smaller than 100% for all blue chips.

$$\begin{array}{rcl}
& \text{BS Term} & \text{Perturbation} \\
C(\sigma, x, K, t, T) & = & C(\sigma_0, x, K, t, T) + \epsilon A[C](\sigma, x, K, t, T)|_{\sigma=\sigma_0} \\
\frac{\partial C}{\partial T}(\sigma, x, K, t, T) & = & \frac{\partial C}{\partial T}(\sigma_0, x, K, t, T) + \epsilon \frac{\partial A[C]}{\partial T}(\sigma, x, K, t, T) \Big|_{\sigma=\sigma_0} \\
\frac{\partial^2 C}{\partial K^2}(\sigma, x, K, t, T) & = & \frac{\partial^2 C}{\partial K^2}(\sigma_0, x, K, t, T) + \epsilon \frac{\partial^2 A[C]}{\partial K^2}(\sigma, x, K, t, T) \Big|_{\sigma=\sigma_0}
\end{array}$$

But in fact, if ϵ vanishes, the model is a Black Scholes model with volatility σ_0 . Thanks to equation (4.8), we can rewrite:

$$\sigma_{imp}^2(T, K) \approx \sigma_0^2 + \frac{\epsilon}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}} \left[\frac{\partial A[C]}{\partial T} \Big|_{\sigma=\sigma_0} - \frac{1}{2}K^2 \sigma_0^2 \frac{\partial^2 A[C]}{\partial K^2} \Big|_{\sigma=\sigma_0} \right]$$

Theorem 4.4 *The local volatility model, equivalent to the PBS model for vanilla options, has the following local volatility function:*

$$(4.16) \quad \sigma_{imp}^2(T, K) \approx \sigma_0^2 \left\{ 1 + \frac{\epsilon}{2} \left[4 \frac{A[\sigma\sqrt{T}]|_{\sigma=\sigma_0}}{\sigma_0\sqrt{T}} - \left[\sigma_0^2 T + 2 - \frac{4 \ln\left(\frac{x}{K}\right)^2}{\sigma_0^2 T} \right] \frac{\Gamma[\sigma\sqrt{T}]|_{\sigma=\sigma_0}}{\sigma_0^2 T} \right] \right\}$$

Remark 4.8 *Local volatility $\sigma(T, K)$ has a minimum at forward money $K = x$, and presents a logarithmic behavior as K approaches zero and infinity. We must preserve the positivity of the square of volatility, so we fix the following constraint:*

$$\sigma_0^2 T + 2 - 2r_r \left(\sigma\sqrt{T} \right) \Big|_{\sigma=\sigma_0} < \left\{ \epsilon \frac{\Gamma[\sigma\sqrt{T}]|_{\sigma=\sigma_0}}{\sigma_0^2 T} \right\}^{-1}$$

4.4 Extension and relation with literature

We have proved that the PBS model can reproduce, at the same time, the bid-ask spread and the volatility smile; but in literature many authors (see Perignon et al. [48], Renault et al. [51] and Rubinstein [52]) have remarked that, generally, the volatility presents a skewed structure (the graph of implied volatility is downward sloping); besides in this chapter we have limited the study of the PBS model at the martingale case, when the extra returns of the stock are zero; however, a simple argument of risk aversion induce leads us to suppose that the parameter μ (the extra-returns term in the BS model) must be positive. In chapter 5 we will show that the extra returns term has an impact in PBS model, contrary to BS model, and we can use this effect to generate a slope in implied volatility. We want emphasize that the PBS model uses a perturbative approach, i.e. we start with a simple model (the Black Scholes model) and we adjust the model at the market data through a perturbation of the principal parameter, i.e. the volatility.

Clearly in literature, some authors introduce a perturbative approach in finance, we recall the papers of Hagan et al. [35] and the book of Fouque et al. [31]; our approach is however different, it can relate the bid-ask spread and the volatility smile through the simple economic argument of the existence of uncertainty in the market.

The principal difference between our approach with respect to SABR model or Fouque et al. approach consists in the nature of the volatility. In the papers of Hagan et al. and Fouque et al. the volatility is a stochastic process, in our model they are two volatility, one completely flat for the market and the second one is an estimation of the first used, by the trader, in order to hedging. This volatility is a random variable. Therefore this model is not a stochastic volatility model, but, more properly a "subjective" volatility model.

This subjectiveness explain the bid-ask spread and the smile of volatility thanks to a risk aversion argument, in the following section we show the relation between theory of utility functions and error theory. This chapter represents a starting point, many generalizations can be treated, we can divide them between the mathematical extensions and the linked financial consequences:

1. the impact of a risk premium, that transforms the profit and loss process in a sub-martingale;
2. the impact of an asymmetry between the supply and the demand, that can be handled with a non linear function h ;
3. we can release the hypothesis of a constant trader's volatility, therefore we find that this volatility becomes a stochastic processes;
4. finally, the problem of the calibration.

The first question is analyzed in chapter 5, whereas chapter 6 is focused on the last problem.

4.5 Risk aversion

In this section, we make some recalls on the theory of utility functions and we show a connection with error theory using Dirichlet forms. We consider a utility function $U(x) : \mathbb{R} \rightarrow \mathbb{R}$ and $U(x) \in C^2$: let ρ be defined by the relation

$$(4.17) \quad \mathbb{E}[U(X)] = U(\mathbb{E}[X] - \rho).$$

We find, in the case of a small variance, that:

$$(4.18) \quad \rho = -\frac{\sigma^2}{2} \frac{U''(\mathbb{E}[X])}{U'(\mathbb{E}[X])} = \frac{\sigma^2}{2} r_a(X).$$

In a similar way, we define $\hat{\rho}$ so that:

$$(4.19) \quad \begin{aligned} \mathbb{E}[U(X)] &= U\{\mathbb{E}[X](1 - \hat{\rho})\} : \text{ and we find} \\ \hat{\rho} &= -\frac{\sigma^2}{2} \mathbb{E}[X] \frac{U''(\mathbb{E}[X])}{U'(\mathbb{E}[X])} = \frac{\sigma^2}{2} r_r(X). \end{aligned}$$

We can name the three objects:

1. ρ is the risk price;
2. $r_a(X)$ is the absolute index of aversion at the wealth X ;
3. $r_r(X)$ is the relative index of aversion at the wealth X .

We write the relations of error and bias, given an error structure on X :

$$\begin{aligned} A[U(X)] &= U'(X)A[X] + \frac{1}{2}U''(X)\Gamma[X] \\ \Gamma[U(X)] &= (U'(X))^2\Gamma[X] \end{aligned}$$

We observe that the bias of $U(X)$, $A[U(X)]$ is zero if and only if $A[X] = \rho$; in this case we find

$$(4.20) \quad \begin{aligned} A[X] &= \frac{r_a}{2}\Gamma[X] \\ \frac{A[X]}{X} &= \frac{r_r}{2}\frac{\Gamma[X]}{X^2}. \end{aligned}$$

We have found a relation between the utility function theory and the error calculus using Dirichlet Forms. We suppose that all traders buy and sell according to their risk aversion, and this aversion is represented via a utility function $U(x)$; then the utility function is defined by its relative index of aversion. We make an hypothesis:

Hypothesis 4.1 *For a trader the bias of the utility of a traded wealth vanishes.*

We can interpret this hypothesis from an economics point of view in two ways:

1. the utility function of traders, supposed to be known, is like a lens traders look the market through. Traders don't add any effect to balance their aversion;
2. the vector (X, \mathcal{A}, Γ) is supposed to be known, we can define the utility function of a trader as function $U(x)$ that cancels the bias of $U(X)$ where X is the considered wealth.

Under hypothesis 4.1, we have two relations between the utility function and Dirichlet Forms:

1. $A[X] = \rho(X)$, where ρ is the risk price;
2. $r_r(X) = 2\frac{XA[X]}{\Gamma[X]}$

Remark 4.9 *Thanks to relation 4.14, we can define the class of utility function that preserves vanishing bias and slope at the money.*

$$\begin{aligned}
r_r(\sigma\sqrt{T}) &= \frac{\sigma^2 T}{4} \\
X \frac{U''(X)}{U'(X)} &= -\frac{X^2}{4} \\
U'(X) &= e^{-\frac{X^2}{8}} \\
U(X) &= \mathcal{N}\left(\frac{X}{2}\right)
\end{aligned}$$

where $\mathcal{N}(X)$ is the distribution function of the normal law. This utility function is concave if the wealth is positive and convex otherwise.

4.6 Conclusion and economics interpretation

In this chapter we have studied the impact of a perturbation on volatility in Black-Scholes model in absence of drift and term structure; in particular, we have dealt with the problem of call hedging.

We have proposed a new model for option pricing, called Perturbed Black-Scholes model; the basic idea is to take into account the effect of uncertainty of volatility value in order to reproduce, at the same time, the spread bid-ask and the smile on implied volatility, in this sense this is a "subjective" volatility model, since the market's volatility is flat, like in Black Scholes, but the volatility used by the traders is an statistical approximation of this volatility, i.e. the trader's volatility is a random variable.

The mainly advantage of this model is that it is based on the classical Black Scholes model and the price of an option in PBS model is the BS price plus a small perturbation that depends only on the greeks founded with BS formula, therefore the computation of PBS price is given by a closed form.

The PBS model depends on four parameters, naturally on the volatility of stock, but also on the variance of the estimated volatility $\epsilon \Gamma[\sigma](\sigma_0)$, on a relative index $r_r(\sigma)$, that represents the ratio between the bias and the variance of the estimated volatility, and, finally, on the drift μ , contrary to Black Scholes model, that represents the extra returns of stock. The impact of the drift rate μ will be studies in chapter 5. In particular, if we set the relative index to be equal to $\frac{1}{4}\sigma_0^2 T$, we have proved that the implied volatility shows a smile around the money.

Finally, we have defined a Local Volatility Model equivalent to Perturbed Black Scholes Model as far as vanilla options are concerned; the related local volatility function is defined by Dupire formula

$$\sigma^2(T, K) \approx \sigma_0^2 \left\{ 1 + \frac{\epsilon}{2} \left[4 \frac{A[\sigma\sqrt{T}]}{\sigma_0\sqrt{T}} - \left[\sigma^2 T + 2 - \frac{4 \ln\left(\frac{x}{K}\right)^2}{\sigma_0^2 T} \right] \frac{\Gamma[\sigma\sqrt{T}]}{\sigma_0^2 T} \right] \right\}.$$

Chapter 5

Drift and Asymmetry Corrections

Joint work with Luca Regis

We study the risk premium impact in the Perturbative Black Scholes model. The Perturbative Black Scholes model is a subjective volatility model based on the classical Black Scholes one, where the volatility used by the trader is an estimation of the market one and contains measurement errors. In this chapter, we analyze the correction to the pricing formulae due to the presence of an underlying drift different from the risk free return. We prove that, under some hypothesis on the parameters, if the asset price is a sub-martingale under historical probability, then the implied volatility presents a skewed structure, and the position of the minimum depends on the risk premium.

5.1 Introduction

It is now common knowledge that the Black and Scholes model, which worked well before the 1987 crash, is nowadays unable to price options correctly. As can be deduced by comparing the two papers by Rubinstein [52] and Jackwerth and Rubinstein [40], something has evidently changed in the market for options after that event. The most shared explanation for the failure in which BS model incurs today is usually thought to reside in the fact that the constant volatility parameter it proposes is not a good representation of reality anymore. Empirical evidence shows that the underlying stock volatility, for example, is not time invariant during the life of an option.

Moreover, while before 1987 the lognormal distribution of stock prices implied by the Black and Scholes model seemed to be a good approximation of the real one and volatility observed across strike prices had a moderately pronounced smile, from that date onwards the implied volatility curve appears to be steeper and generally skewed to the left. Jackwerth and Rubinstein [40], recovering stock price distribution from observed prices, empirically find a “fatter” left tail phenomenon. Constantinides et al. [23], in the context of an equilibrium model, find stochastic dominance violations on both tails of the implied volatility curve. Christensens and Prabhala [20], for example, suggest that a regime switch has occurred after the crash.

The most natural solution to the problem of pricing options more correctly seems then to let volatility change with time. Many pricing models have then been proposed, with different formulations for the stochastic process driving volatility, e.g. local volatility models, see Dupire [28], or stochastic volatility models. Then, an option pricing model is characterized by a system of differential equations, since two different processes are specified, one for the stock price and one for its underlying volatility.

The first authors to solve the problem of pricing options with stochastic volatility were Hull and White [39]. However, they were able to obtain closed form solutions only for the case of uncorrelated volatilities and stock prices, while Heston [38], using a new technique, managed to find exact prices also for the correlated case. In this stream of literature, one of the most used model in practice is probably the SABR one, introduced in Hagan et al. [35], which provides excellent fitting for interest rates derivatives. More sophisticated models include for example the possibility of jumps in the stock price evolution, for instance see Brigo et al. [17], or directly in the process for volatility, for instance see Eraker et al. [29].

However, stochastic volatility models like the ones described above are usually complex and characterized by a large number of parameters and, unless in special cases (SABR model [35], Heston [38]), they do not provide closed form solutions for vanilla options prices.

The PBS model (see chapter 4) introduced a new category of stochastic volatility pricing models, being founded on the notion of subjective volatility. Using error theory through Dirichlet forms, the PBS model generalizes the standard Black and Scholes one, imposing an error structure on volatility. Thus, rather than specifying a possible pattern of evolution for volatility through time, the perturbative approach deals with the concept of measurement errors present in the estimation procedure performed by the trader.

One of the most important issues of this model lies in the possibility of obtaining closed forms solutions for European vanilla option prices and for each kind of derivative which has a closed form solution using the classical Black and Scholes model such as Asian and barrier options. This important framework, joint with the flexibility of the model, permits us to calibrate it to different markets and fit them, even if they imply opposite behaviors of the implied volatility curve. PBS can reproduce a right-tailed or a left-tailed skewness effect, as well as sharper or flatter slopes, obviously depending on the calibration of parameters. In section 4.3, sufficient conditions for the presence of a smile are derived in the case with no drift term in the stock price dynamics.

This chapter studies a natural implication of the PBS model: the dependence of option prices on risk premia. Dependence of option prices on the expected excess return on the stock is ruled out in the classical Black and Scholes model, as a result of the lognormality assumption. Lo and Wang [43], starting from the evidence that the predictability naturally captured by the expected return on stocks affects option prices, assume an Ornstein-Uhlenbeck process for stock prices and show that the impact of a drift term is not negligible anymore.

In the PBS model we maintain the assumption of lognormality of stock prices. However, we show that the presence of measurement errors in the estimation of parameters induces market incompleteness and lets the hedging position of a trader not invariant to the expected excess return on stock. We analyze the implications of this fact and we show that taking into account the impact of a risk premium on stocks it is possible to reproduce the most commonly observed behaviors of the market for options in terms of implied volatility. The PBS model is able to generate the usual smile and skew effects pointed out by the empirical literature we addressed

above.

This chapter is organized as follows: Section 2 recalls the most important features of the PBS model. Section 3 studies the impact of the drift term on the general profit and loss function and in the particular case of the price of a European call option. Section 4 analyzes the sensitivity of the volatility implied by the model to some parameters and above all to risk premium. Section 5 resumes and concludes. Finally, the appendix provides technical computations.

5.2 Perturbative Black Scholes model

In this section, we recall the key features of the Perturbative Black Scholes model introduced in chapter 4.

The PBS model is based on the classical Black Scholes model, (Black and Scholes [10]). If we assume that the interest rate is worth zero or, from an economic point of view, that all assets are priced in terms of the money market, then the underlying stock price follows the SDE

$$(5.1) \quad dS_t = \mu S_t dt + \sigma_0 S_t dW_t$$

where μ is the return on the stock, σ_0 is the volatility and W_t is a Brownian motion.

In the BS model pricing formulae depend on the diffusion term only and not on μ ; we find closed forms expressions for the premium and the greeks of vanilla options. In contrast with its simplicity, unluckily the BS model cannot reproduce the so called smile effect: the volatility implied by the BS model is constant across strike prices, while the observed one is usually u-shaped.

We introduce the notation of risk premium λ as the ratio between the expected return on the stock and market volatility.

$$\lambda = \frac{\mu}{\sigma_0}$$

The PBS model lies on three main hypotheses:

1. the stock price follows a geometrical Brownian motion with fixed and non perturbed volatility σ_0 ;
2. the trader has to estimate the volatility parameter, and the value of his own estimation contains intrinsic inaccuracies. The model reproduces this fact through an error structure; nonetheless we assume that the stock price S_t is not erroneous. We evaluate the impact of the perturbation generated by those measurement errors on the profit and loss process used by trader to hedge a position on a vanilla option;
3. the trader knows the existence of the perturbation described above and wants to modify his own offered prices in order to take into account the bias present on volatility and, as a consequence, on the hedged position.

Summarizing, all traders use a geometric Brownian motion to model the stock price process and they hold some positions involving vanilla options; they use observed market prices to determine

the values of parameters by inversion of pricing formulae. Thus, they find an observed volatility process ς_t , usually known as implied volatility, they take it as a forecast for future volatility and hedge their portfolio accordingly.

Since we made as an assumption that the trader knows the existence of errors in his estimation procedure, volatility is incorporated into the model in two different ways. It has a “market” value, the classical parameter used to set up standard pricing formulae, and a subjective one. The former is denoted with σ_0 and it is supposed to be a constant parameter as in the Black Scholes model. The “subjective” volatility notion comes from the intuition that in the real world, when an operator deals with the problem of option pricing, he does not know the precise value volatility will assume during its life. Hence, he has to estimate it from market observations. The value he gets from this procedure, as pointed out above, will obviously be subject to measurement errors¹, captured in the PBS model by the error structure form. We assume that the stock volatility “market” value is also the mean value of volatility in the erroneous estimation procedure performed by the trader. The volatility estimated by the trader is then a random variable, and is “subjective”, since it can assume a different value in the expectation of each operator.

The profit and loss process of a trader has a key role in the PBS model. The value of this process at maturity is given by:

$$(5.2) \quad P\&L = F(\varsigma_0, S_0, 0) + \int_0^T \frac{\partial F}{\partial x}(\varsigma_t, S_t, t) dS_t - \Phi(S_T)$$

where $F(\varsigma_0, S_0, 0)$ is the security premium, the integral term represents the hedging strategy, $\Phi(S_T)$ is the Payoff and S_t follows Black Scholes SDE (5.1).

5.3 Impact of the drift term in security pricing

In this section, we study the impact of a non zero drift term in the diffusive process assumed for stock prices on prices determined with the PBS model.

As we have shown previously, the expected profit and loss function from the hedging position is then in turn a random variable, characterized by a bias and a variance term, which make it different from the one implied by the BS model. We make an important remark:

Remark 5.1 (Drift impact) *In the PBS model, the profit and loss process defined in equation (5.2) depends crucially on the drift rate μ , which is the expected excess return on the stock. As a matter of fact, the integral term in (5.2) depends on the diffusive process described in (5.1) where μ plays a role.*

In the Black Scholes model, instead, the price of an option does not depend on the drift term. In that case, in fact, the P&L process is worth zero almost surely; as a consequence, we can change the probability measure without altering the result. If, as in the PBS model, we assume that the volatility ς_t used by the trader is σ and not σ_0 , the profit and loss process is not worth zero a.s.; on the contrary, it becomes a stochastic process characterized by two random sources:

- *the Brownian motion which describes the evolution of the stock price and*

¹Measurement errors arise from the uncertainty expected using the central limit theorem.

- the process ς_t , the trader's volatility, which depends on an independent probability space.

As a consequence, we cannot change the probability measure without changing the value of the profit and loss process at maturity.

We suppose that the trader's volatility ς_t is the time independent random variable σ we defined in the previous section. Using the language of Dirichlet forms, we derive the following expansion for the volatility estimation:

$$\sigma_0 \rightarrow \sigma_0 + \epsilon \mathcal{A}[\sigma](\sigma_0) + \sqrt{\epsilon \Gamma[\sigma](\sigma_0)} \tilde{\mathcal{N}}$$

where $\tilde{\mathcal{N}}$ is a standard Gaussian random variable. Moreover, we assume that this error structure admits a sharp operator.

We estimate the variance and bias of the error on $\mathbb{E}[P\&L]$. In the computation we assume that $\sigma = \sigma_0$ is the right value of the random variable, in the sense that if $\varsigma_t = \sigma_0$, then $P\&L(\sigma_0) = 0$ almost surely. Notice that, however, this does not mean that the trader believes the BS model to be correct.

Then we can prove, see chapter 4, that we have the following bias and variance terms:

$$\begin{aligned} \mathcal{A}[\mathbb{E}[P\&L]] &= \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \mathbb{E} \left[\int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right] \right\} \mathcal{A}[\sigma](\sigma_0) \\ (5.3) \quad &+ \frac{1}{2} \left\{ \frac{\partial^2 F}{\partial \sigma^2}(\sigma_0, x, 0) + \mathbb{E} \left[\int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) dS_s \right] \right\} \Gamma[\sigma](\sigma_0) \\ \Gamma[\mathbb{E}[P\&L]] &= \left\{ \frac{\partial F}{\partial \sigma}(\sigma_0, x, 0) + \mathbb{E} \left[\int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right] \right\}^2 \Gamma[\sigma](\sigma_0) \end{aligned}$$

These values represent the inaccuracies that the trader knows to be present in his estimates. We can give the following interpretation to this error structure:

- the bias in the $P\&L$ process represents a deviation in security prices asked by the trader to the buyer.
- the variance of the $P\&L$ process naturally generates a bid-ask spread on security prices. The width of the bid-ask spread depends both on the traders' risk aversion and on the perceived uncertainty on volatility.

As a consequence of the presence of the error structure, the price of a security is thus not unique², but it can be represented, at each instant in time, as a distribution, whose characteristics depend on the parameters which characterize the error structure³.

² When, as in the classical B-S formulation, prices are unique, risk-neutral arguments can be formulated in order to solve the partial differential equations which rule the pricing of assets. In this sense, at each instant in time, price can be represented through a Dirac distribution.

³As pointed out in chapter 4, PBS model induces market incompleteness.

Therefore, we have shown that the trader must modify his prices in order to take into account the two previous effects, namely the variance and the bias on his expected profit and loss process. Thus, he fixes a supportable risk probability $\alpha < 0.5$ and accepts to buy the option at a certain price

$$(\text{Bid Premium}) = (\text{BS Premium}) + \epsilon \mathcal{A} [\mathbb{E}[P\&L]] + \sqrt{\epsilon \Gamma [\mathbb{E}[P\&L]]} \mathcal{N}_\alpha$$

where \mathcal{N}_α is the α -quantile of the reduced normal law. Analogously, the trader accepts to sell the option at the price

$$(\text{Ask Premium}) = (\text{BS Premium}) + \epsilon \mathcal{A} [\mathbb{E}[P\&L]] + \sqrt{\epsilon \Gamma [\mathbb{E}[P\&L]]} \mathcal{N}_{1-\alpha}$$

Since $\mathcal{N}_\alpha + \mathcal{N}_{1-\alpha} = 0$; the mid-premium is

$$(5.4) \quad (\text{Mid Premium}) = (\text{BS Premium}) + \epsilon \mathcal{A} [\mathbb{E}[P\&L]]$$

and the bid-ask spread is

$$(5.5) \quad \text{Bid-Ask spread} = 2\sqrt{\epsilon \Gamma [\mathbb{E}[P\&L]]} \mathcal{N}_\alpha$$

5.3.1 European Call options

We now focus our attention on European call options and we study the bias and its derivatives in order to derive some sufficient conditions for the presence of a smiled behavior on implied volatility. We know the premium of a call option (see Lamberton et al. [42]) with strike K , spot price x , volatility σ_0 and maturity T , and we know its hedging strategy in the usual Black Scholes setting:

$$\begin{aligned} F(\sigma_0, x, 0) &= x\mathcal{N}(d_1) - K\mathcal{N}(d_2) \\ \text{Delta} &= \frac{\partial F}{\partial x}(\sigma_0, x, 0) = \mathcal{N}(d_1) \\ \text{where} \quad d_1 &= \frac{\ln x - \ln K + \frac{\sigma_0^2}{2}T}{\sigma_0\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_0\sqrt{T}. \end{aligned}$$

The following results are classical (see [42]):

$$(5.6) \quad \begin{aligned} \frac{\partial F}{\partial \sigma_0}(\sigma_0, x, 0) &= x\sqrt{T} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \\ \frac{\partial^2 F}{\partial \sigma_0^2}(\sigma_0, x, 0) &= \frac{x\sqrt{T}}{\sigma_0} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} d_1 d_2 \\ \frac{\partial^2 F}{\partial K^2}(\sigma_0, x, 0) &= \frac{x}{K^2\sigma_0\sqrt{T}} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}. \end{aligned}$$

and we can easily prove that:

$$(5.7) \quad \begin{aligned} \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) &= -\frac{1}{\sqrt{2\pi} \sigma_0} d_2(S_s, s) e^{-\frac{1}{2} d_1^2(S_s, s)} \\ \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) &= \frac{d_1(S_s, s) + d_2(S_s, s) - d_1(S_s, s) d_2^2(S_s, s)}{\sqrt{2\pi} \sigma_0^2} e^{-\frac{1}{2} d_1^2(S_s, s)} \end{aligned}$$

$$\text{where} \quad d_1(S_s, s) = \frac{\ln S_s - \ln K + \frac{\sigma_0^2}{2}(T-s)}{\sigma_0 \sqrt{T-s}}$$

$$\text{and} \quad d_2(S_s, s) = d_1(S_s, s) - \sigma_0 \sqrt{T-s}$$

We apply the Perturbative Black Scholes model to find the corrections it imposes on the expected profit and loss process for a trader who is hedging a short position on a plain vanilla European call option. Then the bias on the call premium is given by two terms. The first one is the bias when $\mu = 0$. This case is accurately studied in chapter 4.

$$(5.8) \quad A_{\mu=0}[C]|_{\sigma=\sigma_0} = x \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ A[\sigma \sqrt{T}]|_{\sigma=\sigma_0} + \frac{d_1 d_2}{2 \sigma_0 \sqrt{T}} \Gamma[\sigma \sqrt{T}]|_{\sigma=\sigma_0} \right\}$$

Now, if we assume that the drift term of the stock price process is non zero, this correction is not sufficient in order to hedge the position correctly. We have to study another term, which is the correction when $\mu \neq 0$. While when $\mu = 0$ the stochastic integrals in equation (5.3) are martingales, if $\mu > 0$ we have to evaluate their expectations:

$$\begin{aligned} A_{\text{cor}}[C]|_{\sigma=\sigma_0} &= \mathbb{E} \left[\int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right] \mathcal{A}[\sigma](\sigma_0) \\ &\quad + \frac{1}{2} \mathbb{E} \left[\int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) dS_s \right] \Gamma[\sigma](\sigma_0) \end{aligned}$$

In appendix 5.A, we compute the two integrals and we find

$$(5.9) \quad \begin{aligned} A_{\text{cor}}[C]|_{\sigma=\sigma_0} &= K \left\{ \frac{\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)}{\mathcal{L}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \right\} A[\sigma \sqrt{T}]|_{\sigma=\sigma_0} \\ &\quad - \frac{K}{\sigma_0 \sqrt{T}} \left\{ \left[\sigma_0 \sqrt{T} \left[\frac{3}{\mathcal{L}^2} + \left(1 + \frac{d_2}{\mathcal{L}}\right)^2 \right] - \frac{8}{\mathcal{L}} - 6 \frac{d_2}{\mathcal{L}^2} \right] [\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)] \right. \\ &\quad \left. + \frac{1}{\sqrt{2\pi}} \left[d_2^2 + 4 - 2 \frac{d_2}{\mathcal{L}} + \frac{8}{\mathcal{L}^2} + \sigma_0 \sqrt{T} \left(d_2 - \frac{4}{\mathcal{L}} - \frac{d_2}{\mathcal{L}^2} \right) \right] e^{-\frac{1}{2}d_2^2} \right. \\ &\quad \left. + \frac{1}{\sqrt{2\pi}} \left[\frac{\sigma_0 \sqrt{T}}{\mathcal{L}} \left(1 + \frac{d_2}{\mathcal{L}} \right) - \frac{8}{\mathcal{L}^2} \right] e^{-\frac{1}{2}(d_2 + \mathcal{L})^2} \right\} \Gamma[\sigma \sqrt{T}]|_{\sigma=\sigma_0} \end{aligned}$$

where \mathcal{L} is the cumulated risk premium:

$$(5.10) \quad \mathcal{L} = \lambda \sqrt{T} = \frac{\mu T}{\sigma_0 \sqrt{T}}$$

Remark 5.2 *We remark that the first correction (5.8) derives from the bias of the option price. It is then an uncertainty coming from the error on estimating the value of volatility.*

The second correction, (5.9) is a consequence of the presence of a bias on the strategy, which introduces uncertainty on the hedging procedure also.

It is easy to compute the value of the variance term of the error structure for a call option:

$$(5.11) \quad \Gamma[Call] = \left\{ \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} + K \left[\frac{\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)}{\mathcal{L}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \right] \right\}^2 \Gamma[\sigma\sqrt{T}]$$

We assume that the ask price is then simply the mid price increased by a standard deviation, symmetrically the bid price is the mid price decreased by a standard deviation. The spread is simply given by

$$(5.12) \quad \text{Bid-Ask spread}[Call] = 2 \sqrt{\epsilon \Gamma[Call]}$$

5.4 Numerical Analysis

In this section, we explore the sensitivity of the PBS model to some parameters and, through numerical analysis, we give evidence of the fact that the model is able to reproduce all the observed behaviors of the implied volatility curve. The existence of closed form solutions to the pricing formulae allows us to make some comparative static exercises in order to analyze the dependence of the implied volatility curve on time horizon and, above all, on the drift term.

5.4.1 Parameter sensitivity

First of all, we point out that the corrections obtained with respect to Black and Scholes prices depend on the choice of the parameters A and Γ and on the magnitude⁴ of the epsilon. A captures the bias introduced on the profit and loss function, while Γ is a variance term. ϵ is just a scale factor. It must be small enough to make the higher order expansion terms be negligible. Let us analyze the sensitivity of the volatility implied by the PBS model to the choice of these parameters. Thus, we fix a value for σ and the other real world parameters and we let the coefficients of the error structure vary.

⁴ Since we have three parameters, a possible way to calibrate the model to the market behavior is by using instruments which price variance (e.g. variances swaps). Such derivatives permit to find an implicit link between the bias and the variance term; by fixing an epsilon, it is possible then to calibrate the model on just one parameter, see chapter 6. Another possibility is instead to fix an arbitrary epsilon small and use the implied spread to calibrate the two coefficients A and Γ .

Figure 5.1 shows the effect of an increase in the absolute value of the coefficient of the bias term. As we will explain below, there are good reasons for considering a negative bias. Then, the lower the coefficient, the more the curve shifts downwards and the point of minimum variance to the right. Increasing the value of the coefficient of the variance term, instead, clearly “opens” up the smile, which becomes more pronounced. Moreover, as can be seen in figure 5.2, a higher coefficient is associated with a more pronounced skew effect which makes the implied volatility higher for out-of-the money options, compared to in-the-money ones.

A change in epsilon, instead, combines the two effects described above. Notice that this parameter must be small enough to justify expansion (1.2) in our case. A higher epsilon, thus, produces both a downward shift of the curve and more pronounced smile and skew effects, see figure 5.3.

From now on, we fix the values of these three parameters to make some comparative static analysis of the parameters that capture the real world features. We set epsilon to 0.02 for convenience.

The coefficient on the variance term is set, by a normalization argument⁵, to

$$(5.13) \quad \Gamma[\sigma]_{\sigma_0} = \sigma_0^2$$

implying that

$$(5.14) \quad \text{Variance} = \varepsilon \sigma_0^2$$

The bias coefficient is instead set to

$$(5.15) \quad A[\sigma]_{\sigma_0} = -5 \sigma_0$$

leading to

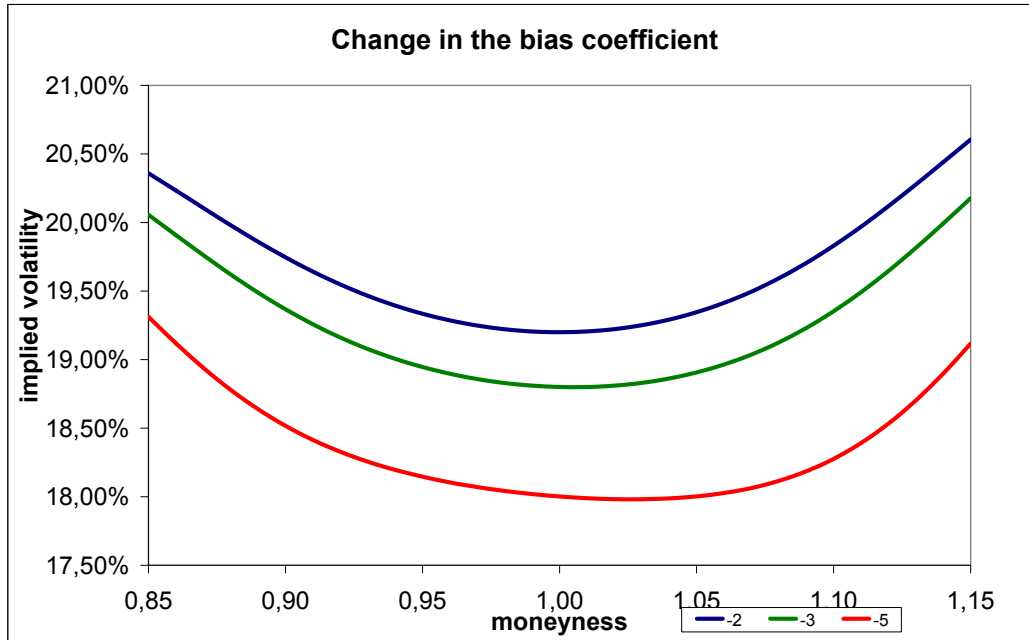
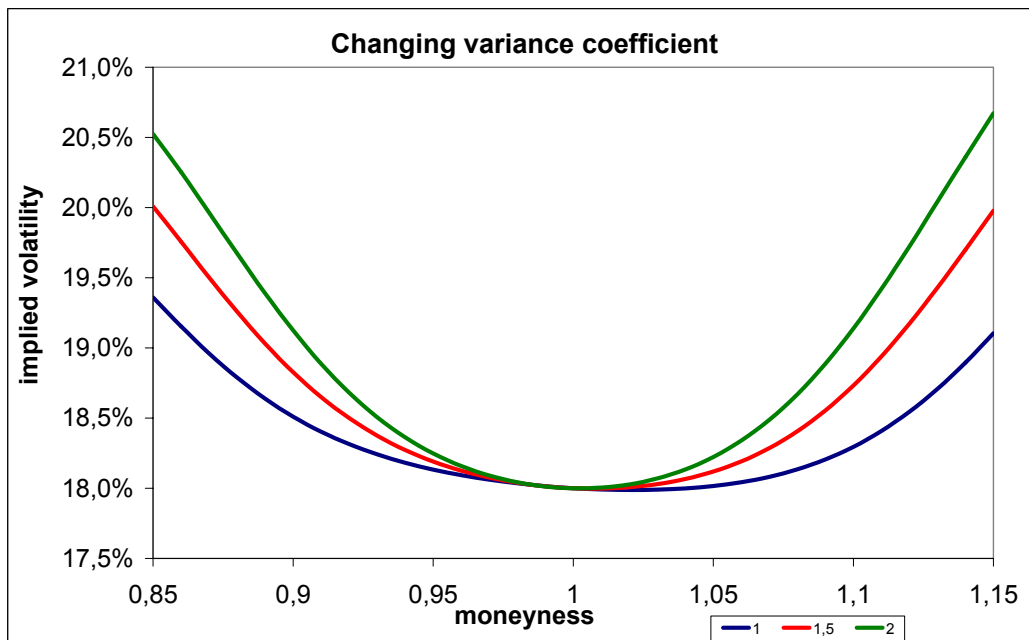
$$(5.16) \quad \text{Bias} = -5 \varepsilon \sigma_0$$

This last choice is made in order to reproduce a precautionary effect. The hypothesis we make is that for some reason, the trader believes he overestimated volatility in his procedure. This feeling can be justified by two reasons, one mathematical and one economic.

The mathematical explanation lies in the analysis of the usual formula used to estimate historical volatility under the hypothesis of lognormality of stock prices. Since it is a concave function, it is more likely that the approximated value found by the estimation procedure is an overestimation of the true one.

The economic explanation lies in the way volatility is usually described in models. Markets are opened for 8 hours a day only. However, the flow of information does not stop when markets are closed: variability accumulates even if securities are not traded. Then, in almost every pricing

⁵Obviously, this coefficient can not be negative. As shown in Figure 5.2, higher values of this coefficient lead to more pronounced smiles.

Figure 5.1: Implied volatilities curves depending on $A[\sigma]$.Figure 5.2: Implied volatilities curves depending on $\Gamma[\sigma]$.

model, the asset is described as a continuous process. This is of course a simplification, but seems nevertheless reasonable. However, it has been shown by some authors, for instance see Stoll and Whaley [55] that overnight volatility is consistently lower than intra day one. Hence, it is straightforward to believe that usual models overestimate volatility.

Let us consider the PBS model prediction on a one-month European call option with the parameters we set above.

First, we keep the risk premium measure fixed and we analyze how implied volatility changes with different maturities. Our finding is that we obtain curves which are flatter as long as the option time horizon becomes longer. This behavior is consistent with empirical evidence on almost every derivative market, see Hagan et al. [35]. Figure 5.4 shows that the implied volatility curve is skewed to the left for each maturity; the point of minimum variability shifts towards higher strikes for longer time horizons.

As we showed in the previous section, the use of our perturbative approach implies that option prices and, thus, implied volatilities are affected by changes in the risk premium. If we let λ , the risk premium we defined previously, change and we fix the parameters that characterize the error structure, we can observe and analyze this sensitivity.

Figures 5.5 and 5.6 show the behavior of the implied volatility curve on a 1 month European call option for an expected excess return on stock term that ranges from 0 to 0.2. The curve evidently shifts to the right side of the graph as the risk premium term increases. With almost every value up to 0.2, i.e $\lambda = 1$, there is a skew effect towards lower maturities. The lower the risk premium, the higher is the value of implied volatility for deep in the money options and the lower for options which are far out of the money. As the value of μ increases the curve appears to become steeper on the right side. In particular, for this parameter choice, for a very high risk premium, there is a slight tendency to change the skew direction⁶. Curves cross approximately at the money, around 102.

These findings are consistent with those obtained by other stochastic volatility models, such as the SABR one (Hagan et al. [35]). The authors of that model, fitting it on prices of Eurodollar options, obtained those behaviors of the implied volatility curve, under the hypothesis that asset prices and volatilities are correlated.

5.4.2 Spread Analysis

Up to now, we have considered the mid price only. As pointed out before, the perturbative approach used by the PBS model can naturally generate a spread on prices and volatilities.

The spread on implied volatility is then obtained by inversion of the pricing formula.

For the same set of parameters described above, we can thus analyze the effect of changing the drift term on a theoretical bid-ask spread. Figures 5.7 and 5.8 give an example of price and volatility spread behavior for a chosen value of μ . It is straightforward to notice that higher prices imply higher volatility. Figure 5.9 shows the magnitude of the spread on implied volatility⁷ for three different values of μ . For low strikes, the spread is higher when there is no risk premium; it

⁶ For $\mu = 0.2$ the implied volatility at moneyness 1.15 is slightly higher than at 0.85. Unreported simulations show that this behavior is common to every choice of parameter. This could suggest that in periods of high risk premia, volatility should tend to be higher for out of the money options.

⁷Notice that the analysis of the relative spread leads to the same conclusions.

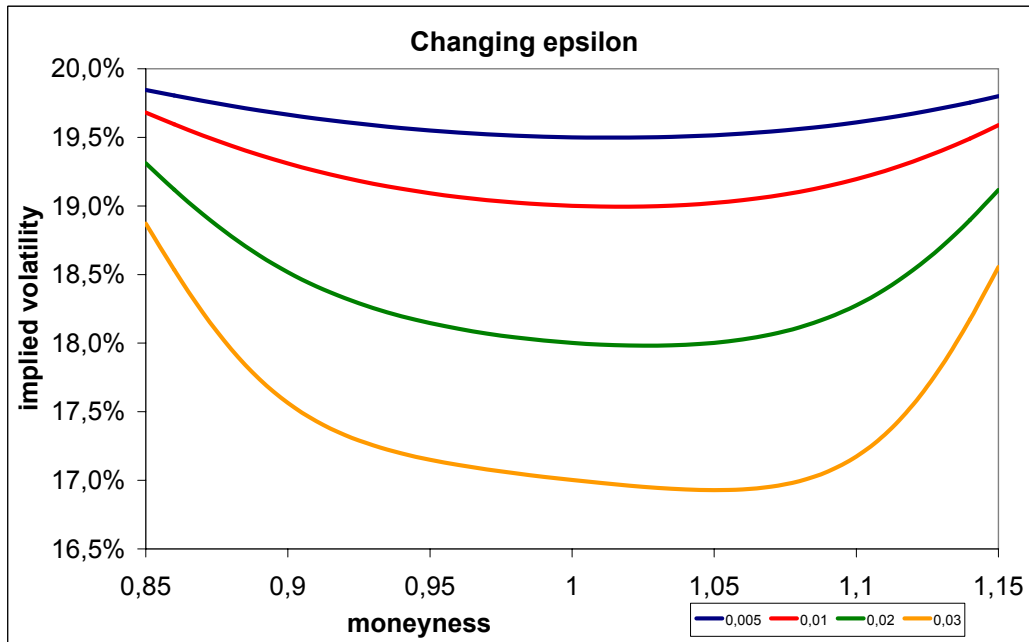


Figure 5.3: Implied volatilities curves depending on ϵ .

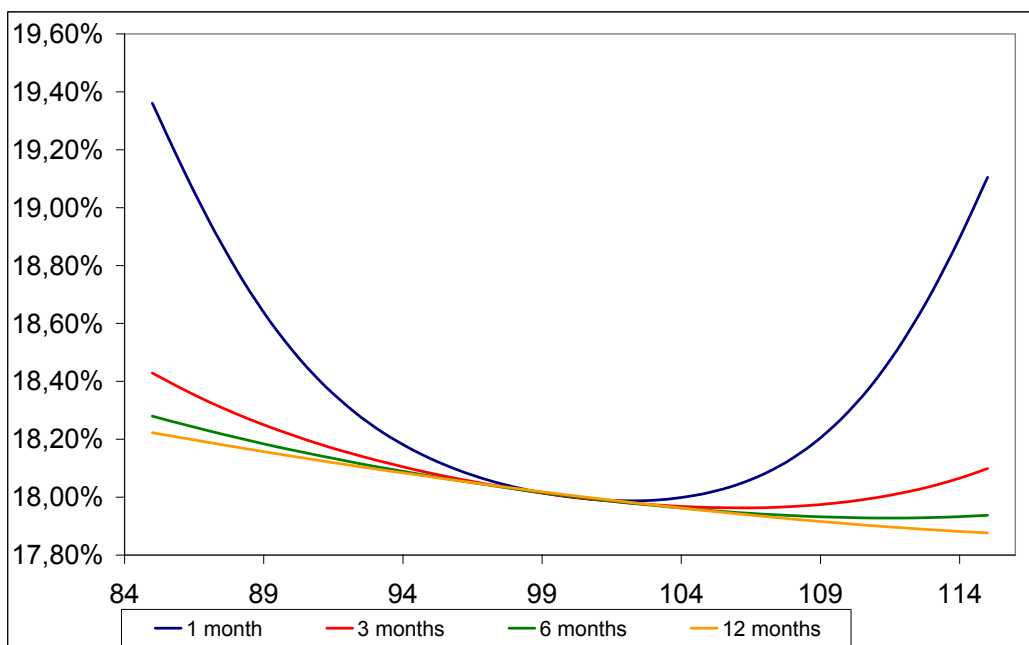


Figure 5.4: Skewed implied volatility depending on maturity.

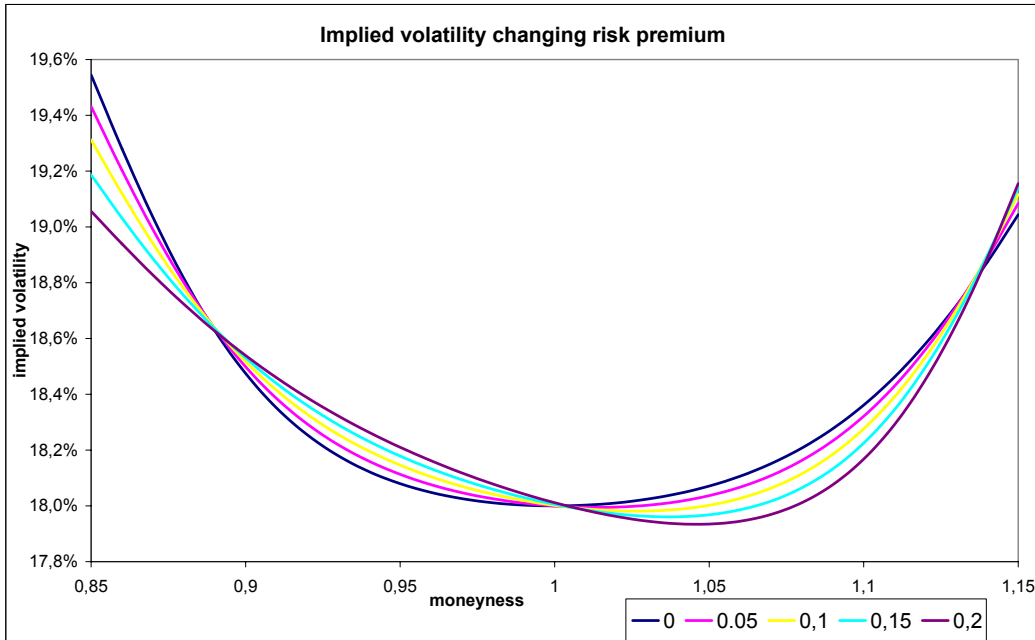


Figure 5.5: Implied volatility depending on λ .

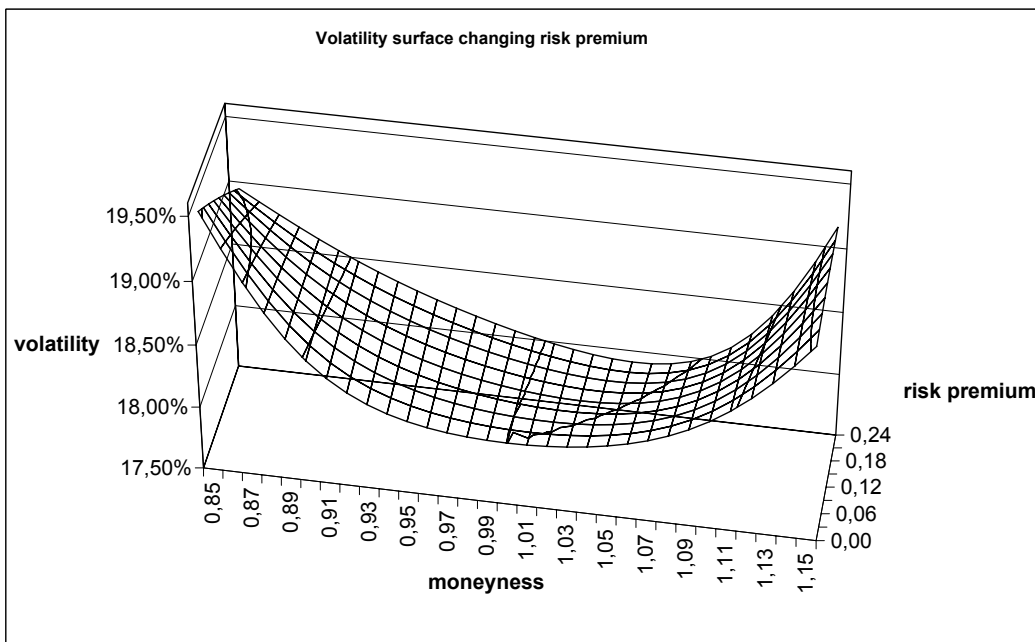


Figure 5.6: Implied volatility depending on λ .

reaches a minimum around the money, then it starts increasing. For out of the money options, the behavior is reversed: the spread is higher the higher the risk premium. The main difference we find with the standard $\mu = 0$ setting is that the spread has no longer its point of minimum variance around the money. The variance spread becomes indeed wider as the strike increases. As shown in figure 5.10, the relative spread on prices (spread-mid price) is almost zero for in deep in the money options, then increases sharply with both strike and risk premium for out of the money calls.

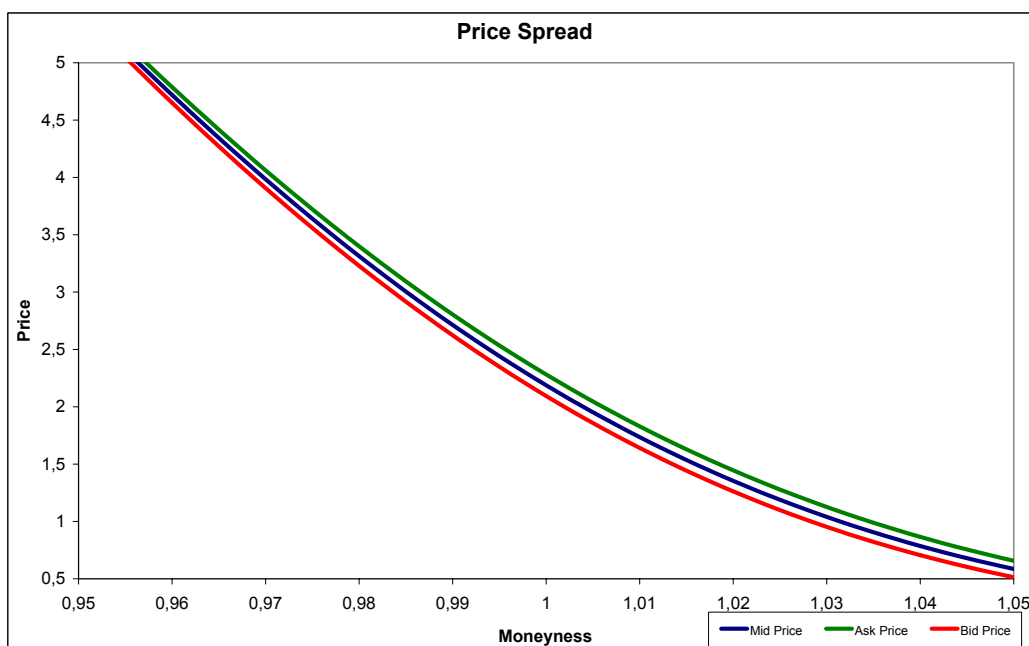


Figure 5.7: Price spread.

Let us finally consider directly the effect of the addition of the correction in equation (5.9) to the PBS model implied volatility curve. Figure 5.11 clearly shows that the presence of a risk premium skews the implied volatility curve toward higher strikes.

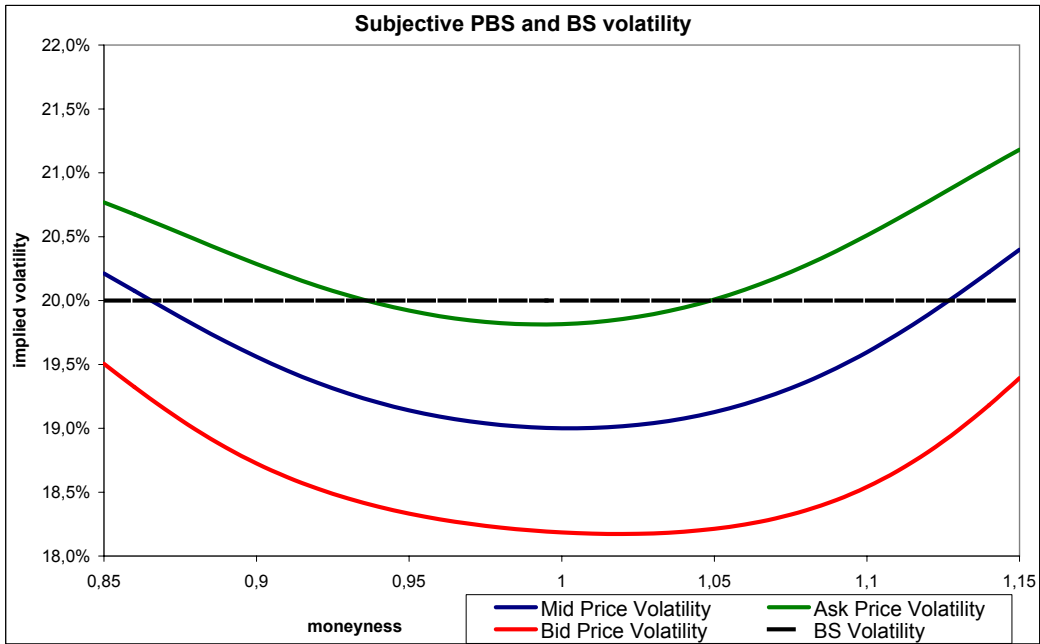


Figure 5.8: Percentual price spread.

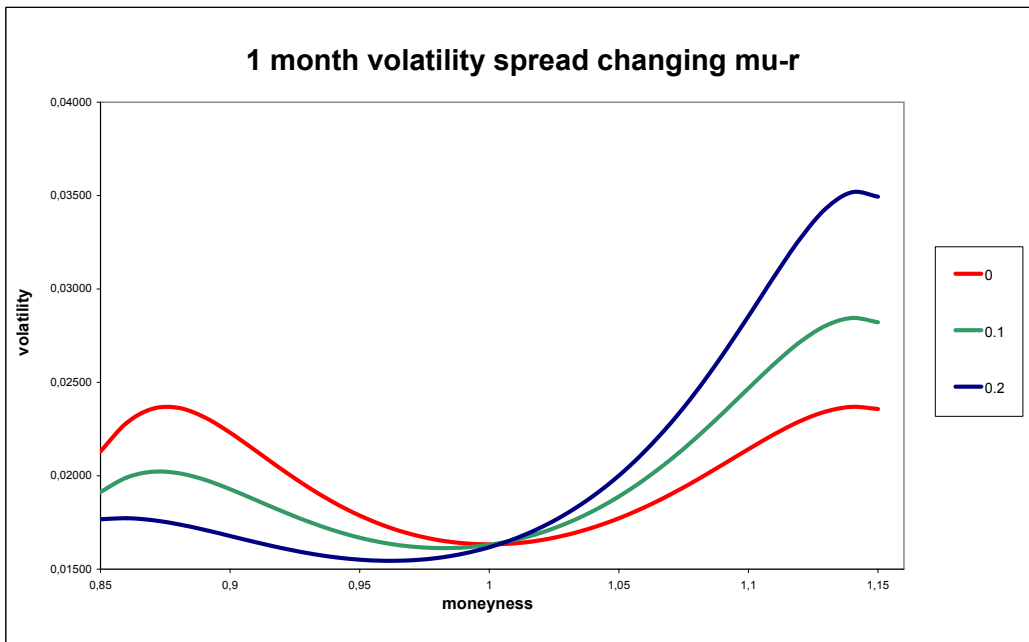


Figure 5.9: Volatility spread depending on λ .

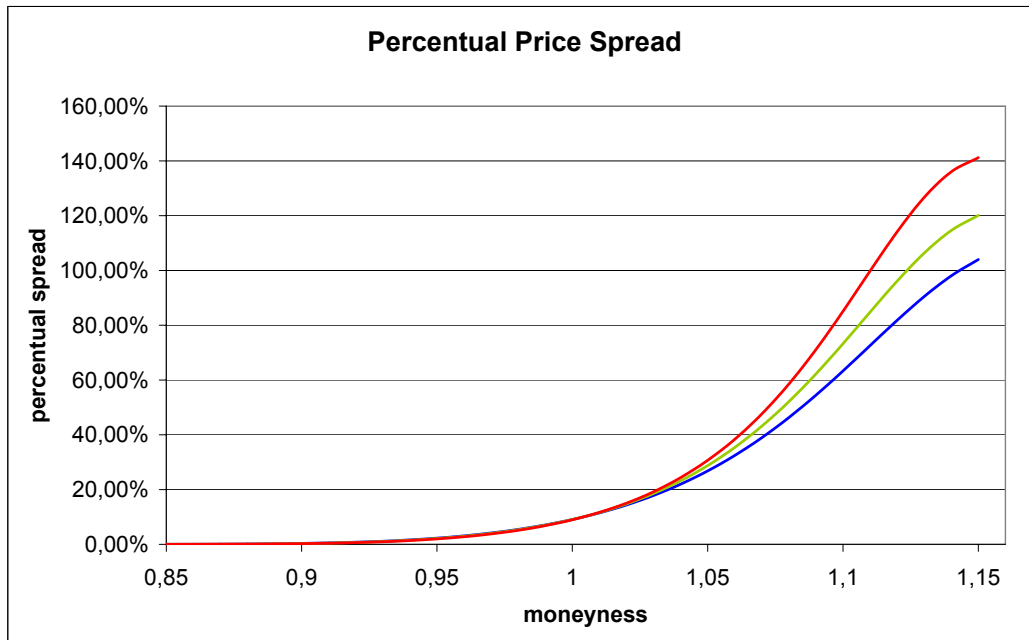


Figure 5.10: Implied volatility in PBS model with and without drift.

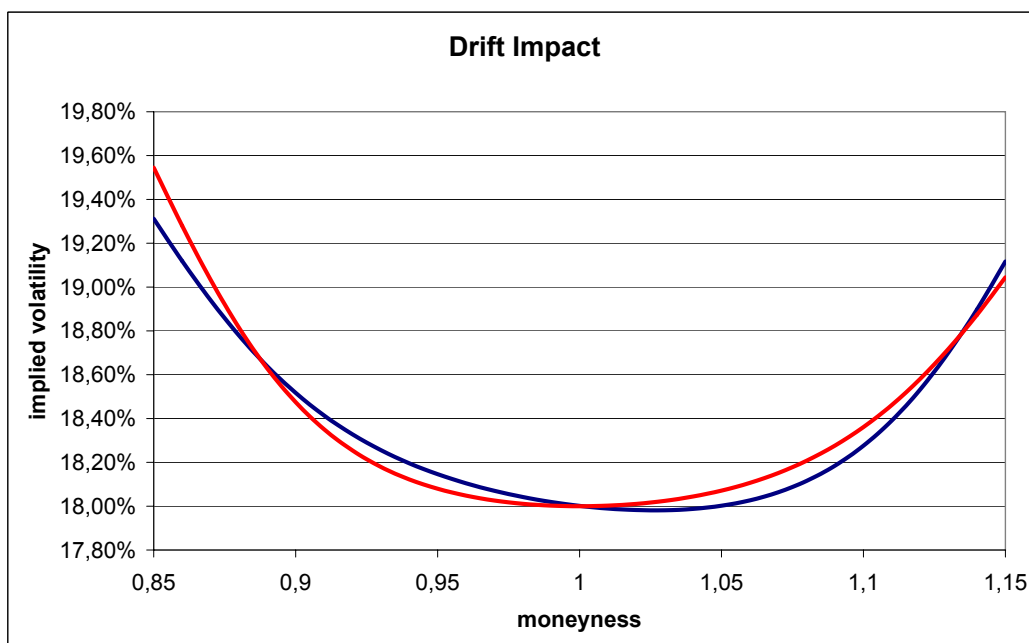


Figure 5.11: Bid, ask, mid implied volatilities in PBS model.

5.5 Conclusion

In this chapter, we have studied the Perturbative Black Scholes model, introduced in chapter 4, when we drop the hypothesis that the underlying is a martingale under the historical probability. Without imposing any behavior of volatility through time, we showed that the hedging procedure of a trader who estimates it depends on the expected excess return on stocks.

We then introduced a correction with respect to the model described in chapter 4, when the drift term of the diffusive process for the stock price is different from the risk free rate. We found a closed form solution for the pricing of a European vanilla call option. This formula depends on the same parameters of the classical Black Scholes model, i.e. the volatility σ_0 , on the two parameters of the PBS model, the variance $\Gamma[\sigma]$ and the bias $A[\sigma]$, which characterize the error structure of the volatility estimated by the trader. Since the PBS model induces market incompleteness, pricing formulas depend also on the cumulated risk premium \mathcal{L} , as shown by equation (5.9).

We analyzed how a simple risk aversion argument forces the underlying price to be a sub-martingale and we studied the dependence of implied volatility on the parameters of the model. We numerically studied the most case in which the volatility used by traders is an overestimation of the true value and we showed that higher risk premia tend to increase the skewness and the smile of the implied volatility curve, since the distribution of stock prices at maturity is shifted towards higher values.

We finally found out that the Perturbative Black Scholes model with drift can reproduce the behavior of the implied volatility curve after the 1987 crash.

Appendix 5.A Computation

We have to compute

$$(5.17) \quad \mathbb{E} \left[\int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right]$$

where S_t follows the Black Scholes diffusion (5.1) and $F(\sigma_0, S_s, s)$ is the price of a call option with strike K , starting at time s , when the spot value is S_s and the volatility is σ_0 .

$$\begin{aligned}
\mathbb{E} \left[\int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right] &= -\frac{\mu}{\sqrt{2\pi} \sigma_0} \int_0^T \mathbb{E} \left[d_2(S_s, s) e^{-\frac{1}{2} d_1^2(S_s, s)} S_s \right] ds \\
&= -\frac{\mu S_0}{\sqrt{2\pi} \sigma_0} \int_0^T \mathbb{E} \left[\frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 W_s - \frac{\sigma_0^2(T-s)}{2}}{\sigma_0 \sqrt{T-s}} \right. \\
&\quad \left. \star e^{-\frac{1}{2} \left\{ \frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 W_s + \frac{\sigma_0^2(T-s)}{2}}{\sigma_0 \sqrt{T-s}} \right\}^2} e^{\sigma_0 W_s} \right] e^{(\mu - \frac{1}{2} \sigma_0^2) s} ds \\
&= -\frac{\mu S_0}{2\pi \sigma_0} \int_0^T \int_{\mathbb{R}} \frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 \sqrt{s} y - \frac{\sigma_0^2(T-s)}{2}}{\sigma_0 \sqrt{T-s}} \\
&\quad \star e^{-\frac{1}{2} \left\{ \frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 \sqrt{s} y + \frac{\sigma_0^2(T-s)}{2}}{\sigma_0 \sqrt{T-s}} \right\}^2} e^{\sigma_0 \sqrt{s} y - \frac{1}{2} y^2 + (\mu - \frac{1}{2} \sigma_0^2) s} dy ds \\
&= -\frac{\mu K}{\sqrt{2\pi} \sigma_0} \int_0^T \frac{T-s}{T} \left[\frac{\ln \frac{S_0}{K} + \mu s}{\sigma_0 \sqrt{T}} - \frac{1}{2} \sigma_0 \sqrt{T} \right] \\
&\quad \star e^{-\frac{1}{2} \left\{ \frac{\ln \frac{S_0}{K} + \mu s}{\sigma_0 \sqrt{T}} - \frac{1}{2} \sigma_0 \sqrt{T} \right\}^2} ds \\
&= -K \frac{\mu T}{\sqrt{2\pi} \sigma_0} \int_0^1 (1-u) \left[\frac{\mu T}{\sigma_0 \sqrt{T}} u + d_2 \right] \\
&\quad \star \exp \left\{ -\frac{1}{2} \left[\frac{\mu T}{\sigma_0 \sqrt{T}} u + d_2 \right]^2 \right\} ds
\end{aligned}$$

We integrate by part and we find

$$(5.18) \quad \mathbb{E} \left[\int_0^T \frac{\partial^2 F}{\partial \sigma \partial x}(\sigma_0, S_s, s) dS_s \right] = -\frac{K \sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{2} d_2^2} + \frac{K \sqrt{T}}{\mathcal{L}} [\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2)]$$

where \mathcal{N} is the cumulated distribution function of a reduced gaussian random variable.

The second term that we have to compute in equation (5.9) is

$$(5.19) \quad \mathbb{E} \left[\int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) dS_s \right]$$

We can compute this term following the same steps we used for the first term (5.17).

$$\begin{aligned}
\mathbb{E} \left[\int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) dS_s \right] &= \frac{\mu}{\sqrt{2\pi} \sigma_0^2} \int_0^T \mathbb{E} \left[e^{-\frac{1}{2} d_1^2(S_s, s)} \left\{ d_1(S_s, s) + d_2(S_s, s) \right. \right. \\
&\quad \left. \left. - d_1(S_s, s) d_2^2(S_s, s) \right\} S_s \right] ds \\
&= \frac{\mu S_0}{\sqrt{2\pi} \sigma_0^2} \int_0^T \mathbb{E} \left[e^{-\frac{1}{2} \left[\frac{B_s}{\sqrt{T-s}} + \Theta + \Lambda \right]^2} e^{\mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 B_s} \right. \\
&\quad \star \left\{ 2 \left[\frac{B_s}{\sqrt{T_s}} + \Theta - \Lambda \right] + 2\Lambda - \left[\frac{B_s}{\sqrt{T_s}} + \Theta - \Lambda \right]^3 \right. \\
&\quad \left. \left. - 2\Lambda \left[\frac{B_s}{\sqrt{T_s}} + \Theta - \Lambda \right]^2 \right\} \right] ds \\
&= \frac{\mu S_0}{\sqrt{2\pi} \sigma_0^2} \int_0^T \int_{\mathbb{R}} e^{-\frac{1}{2} \left[\sqrt{\frac{s}{T-s}} y + \Theta + \Lambda \right]^2} e^{\mu s - \frac{1}{2} \sigma_0^2 s + \sigma_0 \sqrt{s} y} \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi}} \\
&\quad \star \left\{ 2 \left[\sqrt{\frac{s}{T-s}} y + \Theta - \Lambda \right] + 2\Lambda - \left[\sqrt{\frac{s}{T-s}} y + \Theta - \Lambda \right]^3 \right. \\
&\quad \left. \left. - 2\Lambda \left[\sqrt{\frac{s}{T-s}} y + \Theta - \Lambda \right]^2 \right\} dy ds \\
&= -\frac{\mu T K}{\sqrt{2\pi} \sigma_0^2} \int_0^1 e^{-\frac{1}{2} \left(d_2 + \frac{\mu \sqrt{T}}{\sigma_0} x \right)^2} \left\{ 2(1-x) \left(d_2 + \frac{\mu \sqrt{T}}{\sigma_0} x \right) \right. \\
&\quad \left. + \sigma_0 \sqrt{T} (1-x)^2 \left(d_2 + \frac{\mu \sqrt{T}}{\sigma_0} x \right)^2 \right. \\
&\quad \left. + (1-x)^2 \left(d_2 + \frac{\mu \sqrt{T}}{\sigma_0} x \right)^3 \right\} dx \\
&= -\frac{K \sqrt{T}}{\sqrt{2\pi} \sigma_0} \int_{d_2}^{d_2 + \frac{\mu \sqrt{T}}{\sigma_0}} e^{-\frac{1}{2} y^2} \left\{ 2 \left[1 - \frac{\sigma_0}{\mu \sqrt{T}} (y - d_2) \right] y \right. \\
&\quad \left. + \sigma_0 \sqrt{T} \left[1 - \frac{\sigma_0}{\mu \sqrt{T}} (y - d_2) \right]^2 y^2 \right. \\
&\quad \left. + \left[1 - \frac{\sigma_0}{\mu \sqrt{T}} (y - d_2) \right]^2 y^3 \right\} dy
\end{aligned}$$

where

$$\begin{aligned}\Theta &= \frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2}\sigma_0^2 s}{\sigma_0 \sqrt{T-s}} \\ \Lambda &= \frac{1}{2}\sigma_0 \sqrt{T-s} \\ \sqrt{\frac{T-s}{T}} (\Theta - \Lambda) &= \frac{\ln \frac{S_0}{K} + \mu s - \frac{1}{2}\sigma_0^2 T}{\sigma_0 \sqrt{T}} = d_2 + \frac{\mu}{\sigma_0 \sqrt{T}} s\end{aligned}$$

Finally, we integrate by parts three times and we find

$$\begin{aligned}(5.20) \quad \mathbb{E} \left[\int_0^T \frac{\partial^3 F}{\partial \sigma^2 \partial x}(\sigma_0, S_s, s) dS_s \right] &= -\frac{K \sqrt{T}}{\sigma_0 \sqrt{2\pi}} \left\{ \frac{\sigma_0 \sqrt{T}}{\mathcal{L}} \left(1 + \frac{d_2}{\mathcal{L}} \right) - \frac{8}{\mathcal{L}^2} \right\} e^{-\frac{1}{2}(d_2 + \mathcal{L})^2} \\ &\quad - \frac{K \sqrt{T}}{\sigma_0 \sqrt{2\pi}} \left\{ d_2^2 + 4 - 2 \frac{d_2}{\mathcal{L}} + \frac{8}{\mathcal{L}^2} \right. \\ &\quad \left. + \sigma_0 \sqrt{T} \left[d_2 - \frac{4}{\mathcal{L}} - \frac{d_2}{\mathcal{L}^2} \right] \right\} e^{-\frac{1}{2}d_2^2} \\ &\quad - \frac{K \sqrt{T}}{\sigma_0} \left\{ \sigma_0 \sqrt{T} \left[\frac{3}{\mathcal{L}^2} + \left(1 + \frac{d_2}{\mathcal{L}} \right)^2 \right] - \frac{8}{\mathcal{L}} - 6 \frac{d_2}{\mathcal{L}^2} \right\} \\ &\quad \star \left[\mathcal{N}(d_2 + \mathcal{L}) - \mathcal{N}(d_2) \right]\end{aligned}$$

Chapter 6

Calibration

This chapter is dedicated to a strategy for the calibration of perturbative Black Scholes model, using Variance Swaps securities. A Variance Swap is a financial security whose payoff is equal to the difference between the realized variance over a span of time and a fixed quantity, known as variance strike, chosen in order to cancel the derivative premium.

Perturbative Black Scholes model (PBS), see chapter 4, is a new model based on Black Scholes model (see Black and Scholes [10]), characterized by an uncertain volatility parameter; this uncertainty is treated thanks to an error structure. We suppose that the uncertainty on volatility parameter is small, hypothesis that justifies a perturbative approach.

The problem of calibration is a key question in finance. As a matter of fact, all financial models depend on a few parameters and these are to be chosen in order to reproduce prices of some specific contingent claims that are considered exchanged enough to assume their market prices fair, e.g. calls and puts are currently used as far as equities and currencies markets are concerned. This problem is often complicated because financial models have no closed forms for options prices and a calibration requires lots of prices evaluations, thus many numerical strategies are developed nowadays in order to define effective calibration procedures, e.g. see Cont and Tankov [24] for a general analysis. On the contrary, Perturbative Black Scholes model preserves the closed form for pricing peculiar to Black Scholes model. We overwork this advantage of PBS model together with a distinctive property of Variance Swap securities, that is the absence of model risk innate in these contingent claims.

This chapter is organized as follows:

In section 1, we resume notations used in this chapter. Section 2 is dedicated to a short summary of PBS model and its generalization when the volatility is time-dependent. In section 3, we analyze Variance Swap securities. In section 4, we explain how to calibrate PBS model in concordance with Variance Swap options, their bid-ask spreads and at-the-money implied volatility. Finally section 5 resumes and concludes.

6.1 Notation and preliminaries

To make this chapter self-contained, we resume classical notations on probability, mathematical finance, and error theory with Dirichlet forms used in this chapter. According to classical financial theory, we consider a market composed by two assets, one riskless and another risky. We use the

following notations:

- $(\Omega, \mathcal{F}, \mathbb{P})$ is the historical probability space, denoted with Ω for the sake of brevity;
- $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a filtration on the probability space Ω .
- $\{B_t\}_{0 \leq t \leq T}$ is a Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$;
- T is a fixed number that represents the maturity of derivatives, $\Phi(\cdot)$ denotes the generic Payoff of a security, while $F(\cdot)$ denotes the price of this security;
- S_t denotes the price of risky asset at time t , this asset follows a Black Scholes model with volatility $\sigma_0(t)$ depending only on time, $\bar{\sigma}_0(T)$ is the average, in L^2 -norm, of volatility over the interval $[0, T]$;
- $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is another probability space, used to represent the uncertainty on volatility parameter, denoted with $\tilde{\Omega}$ for the sake of brevity;
- $\mathbb{E}[\cdot]$ and $\mathbb{E}[\cdot | \mathcal{F}_t]$ denote, respectively, the expectation and the conditional expectation under probability \mathbb{P} , while $\tilde{\mathbb{E}}[\cdot]$ denotes the expectation under the probability $\tilde{\mathbb{P}}$;
- $(P_t)_{t \geq 0}$ denotes a strongly continuous contraction semi-group, \mathcal{A} its generator, with domain \mathcal{DA} , and Γ the "carré du champ" operator associated with the Dirichlet form of the semi-group, with domain \mathbb{D} ;
- thus, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma)$ is an error structure used to define the uncertainty on volatility parameter in error theory using Dirichlet form framework;
- ς_t is the volatility at time t estimated by the trader, $\bar{\varsigma}_T$ is its average, in L^2 -norm, over the interval $[0, T]$.

6.2 Perturbative Black Scholes model

To make this chapter self-contained, we give a quick survey on Perturbed Black Scholes model and its generalization when the volatility is time-dependent, for a complete analysis see chapter 4. PBS model is based on classical Black Scholes one without drift, see Black and Scholes [10], underlying price follows the stochastic differential equation (SDE)

$$(6.1) \quad dS_t = \sigma_0(t) S_t dW_t$$

where $\sigma_0(t)$ is called the volatility and W_t is a Brownian motion.

Black Scholes model has many advantages; in particular, pricing depends only on volatility and we find closed forms for premium and greeks of vanilla options. Besides, Black Scholes formula

can be easily generalized when the volatility is a deterministic function of time t . It is sufficient to replace the volatility with

$$(6.2) \quad \bar{\sigma}_0(T) = \sqrt{\frac{1}{T} \int_0^T \sigma_0^2(t) dt},$$

$\bar{\sigma}_0(T)$ is well an average volatility between 0 and T .

Unluckily, Black Scholes model cannot reproduce market prices of call and put options for all strikes with the same volatility, this effect is called volatility smile, e.g. see Renault and Touzi [51] or Rubinstein [52]. The basic idea of PBS model is to consider a perturbation of this model by means of an error structure on volatility in order to reproduce a volatility smile together with a bid-ask spread. We make three hypotheses:

1. asset price S_t follows a BS model with a non perturbed volatility $\sigma_0(t)$;
2. every trader has to estimate the volatility, but, as a consequence, its estimation contains intrinsic inaccuracies. We model this ambiguity by means of an error structure; nonetheless, we assume that the stock price S_t is not erroneous. We evaluate the impact of perturbation, generated by the trader mishandling, on the profit and loss process used by trader to hedge vanilla options;
3. trader knows this perturbation and he wants to modify its option prices in order to take into account the bias induced by this perturbation on his volatility.

All traders use an "official" BS asset model in order to hedge vanilla options; they use market prices to determine the "fair" values of parameters by inversion of pricing formula. Traders find an observed volatility process ς_t , usually known as implied volatility and hedge their portfolio according to this volatility.

Trader's profit and loss process has a key role in PBS model, its value at the maturity is given by:

$$(6.3) \quad P\&L = F(\varsigma_0, S_0, 0) + \int_0^T \frac{\partial F}{\partial x}(\varsigma_t, S_t, t) dS_t - \Phi(S_T)$$

where $F(\varsigma_0, S_0, 0)$ is the security premium, the integral term represents the hedging strategy, $\Phi(S_T)$ is the Payoff and S_t follows the Black Scholes SDE (6.1). For sake of simplicity, we suppose that the difference between the trader average volatility $\bar{\varsigma}_T$, defined via a tantamount relation to (6.2), and the true average volatility $\bar{\sigma}_0(T)$ is a time independent random variable and we define an error structure for this volatility; therefore, this average volatility admits the following expansion according to the language of Dirichlet forms:

$$\bar{\varsigma}_T \rightarrow \bar{\sigma}_0(T) + \epsilon \mathcal{A}[\bar{\varsigma}_T](\bar{\sigma}_0(T)) + \sqrt{\epsilon \Gamma[\bar{\varsigma}_T](\bar{\sigma}_0(T))} \tilde{\mathcal{N}}$$

where $\tilde{\mathcal{N}}$ is a standard Gaussian random variable and, finally, we assume that this error structure admits a sharp operator.

We estimate variance and bias of $\mathbb{E}[P\&L]$. To perform the calculus, we follow a perturbative approach in the sense that we assume $\bar{\sigma}_0(T)$ as the true value of $\bar{\zeta}_T$ and we find, according to this hypothesis, that $P\&L(\bar{\sigma}_0(T)) = 0$ almost surely.

Then we can prove, see chapter 4, that we have the following bias and variance:

$$(6.4) \quad \begin{aligned} \mathcal{A}[\mathbb{E}[P\&L]] &= \left\{ \frac{\partial F}{\partial \sigma}(\bar{\sigma}_0(T), x, 0) \mathcal{A}[\bar{\zeta}_T](\bar{\sigma}_0(T)) + \frac{1}{2} \frac{\partial^2 F}{\partial \sigma^2}(\bar{\sigma}_0(T), x, 0) \Gamma[\bar{\zeta}_T](\bar{\sigma}_0(T)) \right\} \\ \Gamma[\mathbb{E}[P\&L]] &= \left\{ \mathbb{E} \left[\frac{\partial F}{\partial \sigma}(\bar{\sigma}_0(T), x, 0) \right] \right\}^2 \Gamma[\bar{\zeta}_T](\bar{\sigma}_0(T)) \end{aligned}$$

The financial interpretation of this result is that the trader knows the presence of errors in his procedure and wants to neutralize this effect.

We associate:

- the variance of $P\&L$ process to the bid-ask spread of options;
- the bias of $P\&L$ process to a shift of options prices asked by the trader to the buyer.

In classical theory of financial mathematics, we assume that all market securities have a single price, if we take into account uncertainty on volatility, we have founded that the price of a contingent claim is not unique but we have many possible prices.

Thus, the trader must modify his prices in order to take into account the two previous effects, i.e. the variance and the bias, then he fixes a bearable risk probability $\alpha < 0.5$ and accepts to buy the option at the price

$$(\text{Bid Premium}) = (\text{BS Premium}) + \epsilon \mathcal{A}[\mathbb{E}[P\&L]] + \sqrt{\epsilon \Gamma[\mathbb{E}[P\&L]]} \mathcal{N}_\alpha,$$

where \mathcal{N}_α is the α -quantile of the reduced normal law. Likewise, the trader accepts to sell the option at the price

$$(\text{Ask Premium}) = (\text{BS Premium}) + \epsilon \mathcal{A}[\mathbb{E}[P\&L]] + \sqrt{\epsilon \Gamma[\mathbb{E}[P\&L]]} \mathcal{N}_{1-\alpha}.$$

Since $\mathcal{N}_\alpha + \mathcal{N}_{1-\alpha} = 0$, the mid-premium is given by

$$(6.5) \quad (\text{Mid Premium}) = (\text{BS Premium}) + \epsilon \mathcal{A}[\mathbb{E}[P\&L]]$$

and the bid-ask spread is

$$(6.6) \quad \text{Bid-Ask spread} = 2\sqrt{\epsilon \Gamma[\mathbb{E}[P\&L]]} \mathcal{N}_\alpha$$

Now, we concentrate on vanilla options and we study the relative bias. We consider a call option with strike K and maturity T ; thus, the payoff is $(S_T - K)^+$. We have proved, see chapter 4, that the bias of this option is given by:

$$(6.7) \quad \mathcal{A}[C]|_{\bar{s}_T=\bar{\sigma}_0(T)} = S_0 \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ \mathcal{A} \left[\bar{s}_T \sqrt{T} \right] |_{\bar{s}_T=\bar{\sigma}_0(T)} + \frac{d_1 d_2}{2\bar{\sigma}_0(T)\sqrt{T}} \Gamma \left[\bar{s}_T \sqrt{T} \right] |_{\bar{s}_T=\bar{\sigma}_0(T)} \right\}.$$

where $d_{1,2} = \frac{\ln S_0 - \ln K}{\bar{\sigma}_0(T)\sqrt{T}} \pm \bar{\sigma}_0(T)\sqrt{T}$.

Thanks to Call-Put parity, see Lamberton and Lapeyre [42], the bias of a put option is the same.

6.3 Variance Swaps

In this section, we give a survey on Variance Swap securities, for more details see Neuberger [44] and Demeterfi et al. [26]. A Variance Swap is a forward contract on variance of a stock, called the underlying. Its payoff is equal to

$$(6.8) \quad N \left\{ \int_u^T \sigma^2(\dots, s) ds - \mathbb{E} \left[\int_u^T \sigma^2(\dots, s) ds | \mathcal{F}_u \right] \right\}$$

where N is a nominal, u is the signature date of the swap (or a future date when we consider a forward starting Variance Swap), T is the expiration date, while $\sigma(\cdot)$ is the spot volatility of underlying estimated at the maturity thanks to the approximation

$$(6.9) \quad \int_t^T \sigma^2(\dots, s) ds \simeq \sum_{n=1}^M \left[\frac{S_n \frac{T-t}{M} - S_{(n-1)} \frac{T-t}{M}}{S_{(n-1)} \frac{T-t}{M}} \right]^2,$$

where M is, according to the financial agreement, the number of days between t and T . Now, we have to evaluate the term

$$\mathbb{E} \left[\int_t^T \sigma^2(\dots, s) ds \middle| \mathcal{F}_t \right].$$

We assume the following hypothesis.

Hypothesis 6.1 (continuous path) *The underlying evolution is a semi-martingale with continuous paths.*

Therefore, underlying price S_t follows a stochastic differential equation of type

$$(6.10) \quad dS_t = S_t \mu(\dots, t) dt + S_t \sigma(\dots, t) dW_t$$

where W_t is a Brownian motion, while $\mu(\dots, t)$ and $\sigma(\dots, t)$ are adapted functions, that can depend on underlying price S_t and realization $\omega \in \Omega$. A consequence of hypothesis 6.1 is that we assume the stock pays no dividends; as a matter of fact, many securities are priced with a

future contract as underlying, then hypothesis 6.1 is not so restrictive. By applying Ito's lemma at equation (6.10), it is easy to find the following relation, see Derman [27].

$$(6.11) \quad \frac{1}{2} \int_u^T \sigma^2(\omega, s) ds = \int_u^T \frac{dS_s}{S_s} - \ln S_T + \ln S_u$$

First integral can be hedge with a shares position continuously rebalanced to be worth one currency. Second term is a short position on a log-contract. For hedging reasons, we want to replicate the log-contract using vanilla options, because these are more liquid. The following identity suggests the decomposition into a combination of out-of-the-money puts and calls, and forwards.

$$(6.12) \quad \begin{aligned} \ln S_u - \ln S_T &= -\frac{S_T - S_u}{S_u} && \text{forward contract} \\ &+ \int_0^{S_u} \frac{1}{K^2} (K - S_T)^+ dK && \text{put options} \\ &+ \int_{S_u}^{\infty} \frac{1}{K^2} (S_T - K)^+ dK && \text{call options} \end{aligned}$$

Therefore, we can make the following remark.

Remark 6.1 (absence of model risk) *The price of a Variance Swap between time u and T is known as soon as we know prices of each vanilla option, e.g. the knowledge of call price for all strike K is enough, because we have a hedging portfolio make up of forward contract and of a static position on call-put options. Thus, prices of Variance Swap have no model risk, i.e. the volatility micro-structure does not change the prices of these securities.*

This remark is crucial, first of all, because this fact ensures a necessary condition to be verified when we search to calibrate all financial models, e.g. perturbative Black Scholes one; secondly, the knowledge of this property has permitted the exchange development over these securities, that has provided a careful pricing, by a real balance between supply and demand, characterized by a tight bid-ask spread.

6.4 Calibration

In this section, we present a procedure for calibrating Perturbative Black Scholes model in accord with Variance Swap securities prices. We start with an evaluation of implicit bias of Variance Swap premium.

6.4.1 Variance Swap constraint

In this subsection, we evaluate the expected value of a Variance Swap security and its bid-ask spread according to Perturbative Black Scholes model. We have the two following theorems.

Theorem 6.1 (Bias of Variance Swap premium)

Under PBS model, the bias of Variance Swap security premium is equal to

$$(6.13) \quad \mathcal{A} \left[\int_0^T \sigma^2(t) dt \right] = 2\bar{\sigma}_0(T) \sqrt{T} \mathcal{A} \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} + \Gamma \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}.$$

Proof: Two strategies are possible, the first is a direct application of bias operator rules, see equation (1.8). The second proof is more financial; we start with identities (6.11) and (6.12), we recall that the bias is a linear operator and is worth zero when it acts on the underlying, since this is unerroneous. Therefore, we find

$$\mathcal{A} \left[\int_0^T \sigma^2(t) dt \right] = \int_0^{S_0} \frac{1}{K^2} \mathcal{A} [(K - S_T)^+] dK + \int_{S_0}^{\infty} \frac{1}{K^2} \mathcal{A} [(S_T - K)^+] dK.$$

Call-put parity shows that the relation for the bias of a call, i.e. the relation (6.7), is true for put too. So, we have to compute

$$\mathcal{A} \left[\int_0^T \sigma^2(t) dt \right] = \int_0^{\infty} \frac{S_0}{K^2} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ \mathcal{A} \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} + \frac{d_1 d_2}{2\bar{\sigma}_0(T) \sqrt{T}} \Gamma \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} \right\} dK.$$

We make the change of variable $y = \ln K$, we integrate and we find, easily, result (6.13). □

Theorem 6.2 (Variance of Variance Swap premium)

Under PBS model, the variance of Variance Swap security premium is equal to

$$(6.14) \quad \Gamma \left[\int_0^T \sigma^2(t) dt \right] = 4\bar{\sigma}_0^2(T) T \Gamma \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}.$$

Proof: The proof follows the same idea of previous theorem, the proof using equation (6.5) requires the employment of sharp operator, see Bouleau [13] or definition 1.2, but the computation follows the same plan of previous theorem. □

Result 6.3 (Variance Swap mid-price)

Relation (6.5) and theorem 6.1 define the price of Variance Swap securities, it is given by

$$(6.15) \quad \mathbb{E} \left[\int_0^T \sigma^2(t) dt \right] = \bar{\sigma}_0^2(T) T + 2\epsilon \bar{\sigma}_0(T) \sqrt{T} \mathcal{A} \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} + \epsilon \Gamma \left[\bar{\varsigma}_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}.$$

Remark 6.2 (Bid-Ask spread of Variance Swap) Relation (6.6) and theorem 6.2 define the bid-ask spread of Variance Swap securities, it is given by

$$(6.16) \quad \text{Bid-Ask spread} = 4\sqrt{\epsilon \bar{\sigma}_0^2(T) T \Gamma \left[\varsigma_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}} \mathcal{N}_\alpha.$$

A fundamental hypothesis of perturbative approach is that the parameter ϵ is small compared with 1, then a starting estimation of cumulated volatility is the square of mid-premium of Variance Swap, i.e. $\mathbb{E} \left[\int_0^T \sigma^2(t) dt \right] \approx \bar{\sigma}_0(T) T$. Relation (6.16) gives us an estimation for $\Gamma \left[\bar{\zeta}_T \sqrt{T} \right]$:

$$\epsilon \Gamma \left[\bar{\zeta}_T \sqrt{T} \right] = \frac{(\text{Bid-Ask spread})^2}{16 \bar{\sigma}_0^2(T) T (\mathcal{N}_\alpha)^2}$$

where the bid-ask spread can be estimated on the market, the cumulated variance is approximated by Variance Swap security premium and the only unknown parameter remains the quantile α . It is well-known that if we fix $\alpha = 16\%$ (respectively $\alpha = 1\%$) we have $\mathcal{N}_\alpha \simeq 1$ (respectively $\mathcal{N}_\alpha \simeq 3$). Therefore, we have an estimation of $\epsilon \Gamma \left[\bar{\zeta}_T \sqrt{T} \right]$ with a precision of one order of magnitude.

6.4.2 Bias estimation using ATM volatility

In this subsection, we estimate the volatility bias anticipated by PBS model. In previous subsection, we have provided a rough estimation for volatility parameter σ_0 and for its variance $\epsilon \Gamma \left[\bar{\zeta}_T \sqrt{T} \right]$. In PBS model, the last parameter is the bias of volatility; we propose to use At-the-money vanilla options to estimate it.

Remark 6.3 *It is a market evidence that the most exchanged vanilla option have a strike around forward money.*

We decide to use this security for the calibration of PBS model. Equations (6.5) and (6.7) give us the expected value of a call in PBS model:

$$(6.17) \quad \text{PBS Price} \left(S_0, K, T, \bar{\sigma}_0(T), \Gamma \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}, \mathcal{A} \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} \right) =$$

$$\text{BS Price} (S_0, K, T, \bar{\sigma}_0(T))$$

$$+ \epsilon S_0 \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \left\{ \mathcal{A} \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} + \frac{d_1 d_2}{2\bar{\sigma}_0 \sqrt{T}} \Gamma \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} \right\}.$$

We choose a traded call with strike K , around the money. We force PBS price to be equal to market one. BS price is given by Black-Scholes formula. Bias $\epsilon \mathcal{A} \left[\bar{\zeta}_T \sqrt{T} \right]$ stays the only unknown parameter and, thanks to the linearity, we have a closed form for it.

Result 6.4 (Variance Swap bias estimation)

Under the (ARB) condition, see section 1.6, we have the following estimation for bias.

$$(6.18) \quad \epsilon \mathcal{A} \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}} = \sqrt{2\pi} e^{\frac{1}{2}d_1^2} \frac{\text{PBS Price} - \text{BS Price}}{S_0} - \frac{\epsilon d_1 d_2}{2\bar{\sigma}_0(T) \sqrt{T}} \Gamma \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}$$

6.4.3 Calibration procedure

In this subsection, we resume the calibration procedure of PBS model using Variance Swap securities and we propose some improvements and remarks.

The basic procedure is the following one:

Algorithm 6.5 (Calibration of Perturbative Black Scholes model)

1. for each maturity T , compute, using vanilla contingent claims prices founded on the market, the price of Variance Swap over the same period, and set the value on cumulated volatility $\bar{\sigma}_0(T) \sqrt{T}$ to be equal to the square root of Variance Swap premium;
2. fix a bearable risk probability $\alpha < 0.5$ (this parameter is the only exogenous one, an economic argument is needed);
3. evaluate the bid-ask spread on the market and, using theorem 6.2, fix the variance of cumulated volatility $\epsilon \in \Gamma \left[\zeta_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}$;
4. choose a really traded vanilla option with a strike around the money and compute the bias of cumulated volatility $\epsilon \in \mathcal{A} \left[\bar{\zeta}_T \sqrt{T} \right] \Big|_{\bar{\sigma}_0(T) \sqrt{T}}$, thanks to equation (6.18).

Remark 6.4 Clearly, this procedure is a rough estimate and it is inconsistent, since PBS price for Variance Swap securities depends on the bias, see equation (6.14). Another drawback concerns the choice of At-The-Money option, calibration result depends deeply on this selection.

In order to avoid the inconsistency of previous calibration procedure, we propose to iterate it until the estimated parameters verify relation (6.5) with a set accuracy. At the end of a loop, we hold the previous value for bias and variance of cumulated volatility and we define a new estimation of this one, thanks to relation (6.5). In order to strengthen the calibration, we propose to fix a basket of at-the-money vanilla options and to fit the bias of volatility over this basket rather than to choose a single option.

6.5 Conclusion

In this chapter we have proposed a calibration procedure for Perturbative Black Scholes model, see chapter 4, based on prices of Variance Swap securities. Variance Swaps are a no-risk-model contingent claims, so their premium represents an implicit constraint to be verified by a calibrated model. In the last section, we have showed an iterative procedure that can be performed quickly, thanks to the closed-forms relations of each step. The only exogenous parameter remains the bearable risk probability accepted by the trader in exchange for the expected return.

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Part III

**Partial Differential Equations and
Physics**

Chapter 7

Linear PDEs and Wavelets

The study of the sensitivity of the solution of a Partial Differential Equation (PDE) with respect to small perturbations of the starting (or terminal) condition is a key problem in applied mathematics. The classical approach, used especially in engineering science, is to define a basis of "perturbation functions" and to evaluate the variation of the PDE solution along each "direction"; this method is known as Gateaux derivative of the solution. But this approach assumes that the perturbation of the starting condition is deterministic, on the other hand the starting condition is often estimated by means of some measurements, i.e., with mathematical language, make a statistic; therefore the result of this estimation is a random variable, with a few known moments, i.e. we know generally the mean, the variance and, maybe, the skewness and the kurtosis.

The probabilistic nature of the uncertainty on each estimation pushes us to give up the Gateaux derivative approach. We apply the methodology suggested by Bouleau, see [7] or the preliminary part of this thesis. This idea yields a representation of small perturbation coherent with the truncated expansion of the perturbed solution by a small random variable. If the estimation of starting (or terminal) condition is good the related uncertainty is very small, we may neglect order higher than the second, i.e. we choose to work with Gaussian distributions.

The analysis of the uncertainty transferred from the terminal function to the PDE's solution requires to specify a representation basis, see section 1.5. We choose to work with a wavelet basis and we will prove some useful properties in this case when the PDE is linear.

We decide to consider a terminal problem with a parabolic linear partial differential equation, i.e. we search a function $Q(t, x)$ depending on two variables, t and x , that verifies a partial differential equation of the first order with respect to the variable t , of the second order with respect to x and linear with respect to the function $Q(t, x)$, moreover the function $Q(t, x)$ verifies a terminal condition $f(x)$ when the time t is equal to a final date T . Parabolic PDEs with terminal condition are commonly used in finance, whereas parabolic PDEs with starting condition describe many physical problems, especially in engineering science. Clearly, the link between the two problems are a reversal of time, i.e. the PDEs in finance are said backward, whereas in physics are said forward. Of course, the results of this chapter are true when forward PDEs are concerned, the choice of backward PDEs is for simplicity's sake and it is justified by an application in finance.

The combination of error theory and wavelets basis justifies some hypotheses, helpful to simplify the computations. However, this approach cannot work when the PDE is not linear, this case is treated in chapters 8 and 9.

Summarizing, we propose a new approach to study the impact of uncertainty on the solution

of a Linear Partial Differential Equation due to an random imprecision on the starting condition; this method permits, first of all, the study of variance of the LPDE solution, i.e. his sensitivity; secondly we can estimate a covariance between the LPDE solution at two different points of time-space domain.

The chapter is organized as follows: Section 1 is a survey of wavelets theory. In section 2, we study the solution of a LPDE using wavelet's decomposition and we present some particular cases where the wavelet's properties play a crucial role. In section 3, we describe the evolution equation for the operators of error theory in the LPDE case. Section 4 is devoted to an introduction to possible applications in finance. The results are summarized in section 5.

7.1 Wavelets theory

In this section we give a short presentation of the theory of wavelets, in order to preserve the self-containedness of this paper, this introduction follows Hardle et al. [18] and Mallat [23]. We fix a function, called father wavelet, $\phi \in L^2(\mathbb{R})$, such that the family $\{\phi_{0,k} = \phi(\cdot - k), k \in \mathbb{Z}\}$ is an orthonormal system¹.

We define the linear space (sub-space of L^2)

$$V_0 = \left\{ f(\cdot) \mid f(\cdot) = \sum_{k=-\infty}^{\infty} c_k \phi_{0,k}(\cdot) \right\}.$$

From this original space we can define a chain of sub-spaces $\{V_i\}_{i \in \mathbb{Z}}$ by the relation

$$f(\cdot) \in V_i \text{ iff } f(2\cdot) \in V_{i-1}$$

these spaces are called "generated" by the function ϕ . Mallat and Meyer introduce the concept of multiresolution analysis in the years 1988-1990.

Definition 7.1 (MRA) *A chain of sub-spaces $\{V_i\}_{i \in \mathbb{Z}}$, "generated" by a function ϕ is called a Multiresolution Analysis [MRA] if the two following conditions are held:*

- $V_i \subset V_{i+1}$ for all i and
- $\bigcup_i V_i$ is dense in L^2

In this case the function ϕ is called the "father Wavelet".

To define an orthogonal basis of the L^2 space we must define a sequence of orthogonal spaces, since the chain $\{V_i\}$ is decreasing sequence. Define W_i the orthogonal complement of V_i into V_{i-1} , for all i ; we find that $V_i = V_0 \oplus \bigoplus_{j=1}^i W_j$ and, thanks to the second property of MRA, we have:

$$V_0 \oplus \bigoplus_{j=1}^{\infty} W_j \text{ is dense in } L^2$$

¹The easy way is to consider a compactly supported ϕ .

We can fix an orthonormal basis $\{\psi_{i,k}\}_{k \in \mathbb{Z}}$ in each space W_i , Mallat and Meyer show that we can fix $\psi_{i,k}(\cdot) = \sqrt{2^{-j}}\psi(2^{-j} \cdot -k)$ where $\psi(\cdot)$ is a function depending of $\phi(\cdot)$, see Mallat [23], page 233 for the explicit relation; the function $\psi(\cdot)$ is called the “mother Wavelet”.

We conclude that the original function $f(x)$ has a unique representation in term of the following series:

$$f(x) = \sum_k \alpha_k \phi_{0,k}(x) + \sum_i \sum_k \beta_{i,k} \psi_{i,k}(x)$$

7.1.1 Daubechies wavelets

The construction of the wavelet basis depends on the choice of the father wavelet; this choice is not constrained, many options are possible; in this section, we present a particular class of wavelets called Daubechies Wavelets, see Daubechies [11].

The first historical wavelet is the Haar’s basis, its father wavelet is $\phi(x) = \mathbf{1}_{[0,1]}$; this basis has some advantages, in particular the boundary support and the quick computation, it is also positivity preserving, see Neveu [25]. Unluckily, the Haar mother wavelet is discontinuous, therefore this basis misapproximates all continuous functions, this roughly approximation induces that the coefficients $\beta_{i,k}$ do not decrease fast with the rescaling index i . Another possibility is the Riesz’s class, but the choice of the Riesz’s bases approach causes uncompactly supported father and mother wavelets, see [18] chapter 6, and it is clear that a function with compact support is easier to treat numerically.

Actually the simplest known class of wavelets with continuous and compactly supported father and mother wavelets is the Daubechies’ wavelets, see [18] chapter 7 or [23] pages 246-251. The definition of Daubechies’ father wavelet starts from its Laplace transform

$$\widehat{\phi}(\omega) = \prod_{j=0}^{\infty} m_0(2^{-j}\omega)$$

where m_0 is a 2π -periodic function, that must verify some constraints, see [18] chapter 5 and 6 or [23] pages 222-231; in particular an usual choice for m_0 is a trigonometric polynomial:

$$m_0(\omega) = \sum_k h_k e^{-ik\omega}$$

The Daubechies’ basis of order p must verify that the function $m_0(\omega)$ has a zero of order p when $\omega = \pi$; we conclude with the equation of the Fourier transform of mother wavelet:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2}} \overline{m_0(2^{-1}\omega + \pi)} e^{-i\frac{\omega}{2}} \widehat{\phi}\left(\frac{\omega}{2}\right)$$

the principal advantages of these bases are:

- *regularity*: the Daubechies’ father and mother wavelets are uniformly γ -Lipschitz where the parameter γ grows with the order p , in particular if p is grater than three the wavelets are differentiable, see Daubechies et al. [12].
- *compact support*: the support of Daubechies’ father and mother wavelets are $2p - 1$ long.

Finally, we have a representation of the function $f(x)$ via a Daubechies' decomposition on wavelets:

$$(7.1) \quad f(x) = \sum_k \alpha_k \phi_{0,k}^D(x) + \sum_i \sum_k \beta_{i,k} \psi_{i,k}^D(x)$$

7.2 Linear partial differential equation

In this section, we want study the interaction between wavelets and linear partial differential equations with a terminal condition $f(x)$. In the previous paragraph, we have decomposed the function $f(x)$ into a Daubechies' wavelets basis; now we study the evolution of this decomposition through a linear partial differential equation and we focus on the properties of this evolution. We consider a Linear Partial Differential Equation

$$(7.2) \quad \begin{cases} \frac{\partial Q}{\partial t}(t, x) + \frac{\sigma^2(x, t)}{2} \frac{\partial^2 Q}{\partial x^2}(t, x) + \mu(x, t) \frac{\partial Q}{\partial x} = 0 \\ Q(T, x) = f(x) \end{cases}$$

The Feynman-Kac formula tells us that the solution of LPDE (7.2) is

$$(7.3) \quad Q(t, x) = \mathbb{E}[f(X_T) | X_t = x]$$

where the Ito process X_t verifies the SDE

$$(7.4) \quad dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t$$

where W_t is a Brownian motion.

Since the linearity of the PDE (7.2), the terminal condition and the solution admits a decomposition on basis, in particular on a wavelets basis. Then, if the terminal condition is written as follows

$$f(x) = \sum_k \alpha_k \phi_{0,k}^D(x) + \sum_i \sum_k \beta_{i,k} \psi_{i,k}^D(x)$$

we can study the evolution of each factor and prove that the solution has a similar decomposition.

Notation 7.1 *In order to simplify the notation, we shift the time coordinate in such a way that we bring the final time T at zero. Clearly each time t smaller than T became negative and the terminal condition became $Q(0, x) = f(x)$.*

7.2.1 Diffusionlets

In this subsection we analyze a particular case in which the wavelets properties play a key role; a seminal study has been done by Shen and Strang, see [28], when the PDE is the heat equation $U_t(t, x) = c\Delta U(t, x)$.

Shen and Strang introduce the notion of mother and father heatlets, given a choice of wavelets basis, these are the heat evolution of the mother and father wavelets.

In a similar way, if we consider an LPDE and a wavelets basis, we can define an object called Diffusionlet:

Definition 7.2 (Diffusionlets) *The solution of an LPDE with a wavelet as terminal condition is called diffusionlet.*

The diffusionlets associated at a LPDE are denoted:

$$(7.5) \quad \begin{aligned} \Phi_{0,k}^D(t, x) &= \mathbb{E} [\phi_{0,k}^D(X_0) | X_t = x] \\ \Psi_{i,k}^D(t, x) &= \mathbb{E} [\psi_{i,k}^D(X_0) | X_t = x] \end{aligned}$$

We can make some remarks:

Remark 7.1 *In Daubechies' case, the father and mother wavelets are compactly supported and bounded, therefore the variance of the solutions Φ^D and Ψ^D are, generally, smaller than the variance of the solution $Q(t, x)$.*

Remark 7.2 *The Daubechies' mother wavelet $\psi^D(x)$ and its rescaling functions $\psi_{i,k}^D(x)$ have p vanishing moments (where p is the order of Daubechies's wavelet), so the associate diffusions depend mainly on the assymetries and we can suppose that the contributions of high order wavelets vanish very quickly as the time evolves.*

Remark 7.3 *The study of the solution of the LPDE (7.5) requires to solve the same LPDE with each wavelet basis as terminal value.*

The least remark underlines the main problem with this approach, we have earned a high precision on the estimation of the solution but the price to pay is that we need to solve many times the same problem in order to have a good estimation. But we introduce a class of LPDE where this difficulty has an easy answer. We consider a special case of the previous LPDE, we fix the function $\mu(t, x) = rx$ and the function $\sigma(t, x)$ to be a power of the variable x .

The LPDE becomes

$$(7.6) \quad \begin{cases} \frac{\partial Q}{\partial t}(t, x) + \frac{\sigma^2 x^{2\lambda}}{2} \frac{\partial^2 Q}{\partial x^2}(t, x) + r x \frac{\partial Q}{\partial x} = 0 \\ Q(0, x) = f(x) \end{cases}$$

and the associate diffusion is

$$(7.7) \quad dX_t = r X_t dt + \sigma X_t^\lambda dW_t.$$

This diffusion is known in finance as the diffusion of a local volatility model, see Dupire [14], and this kind of local volatility is introduced by Hagan and Woodward, see [16], when $\lambda = 1$ we find the classical Black Scholes model, see Black and Scholes [3]. A change of numeraire proves that the good solution of LPDE (7.6) can be written as $Q(t, x) = \tilde{Q}(t, x)e^{-rt}$ where $\tilde{Q}(t, x)$ verifies the LPDE

$$(7.8) \quad \begin{cases} \frac{\partial \tilde{Q}}{\partial t}(t, x) + \frac{\sigma^2 x^{2\lambda}}{2} \frac{\partial^2 \tilde{Q}}{\partial x^2}(t, x) = 0 \\ \tilde{Q}(0, x) = f(x). \end{cases}$$

Remark 7.4 (Self-similarity) *This equation shows a rescaling invariance: if $\tilde{Q}(t, x)$ is a solution of LPDE (7.8) with terminal value $g(x)$, then also $\tilde{Q}(\alpha^{2-2\lambda}t, \alpha x)$ is a solution of LPDE with terminal value $g(\alpha x)$.*

We can prove some properties and theorems of the wavelets diffusion through a LPDE of type (7.6), each results is a generalization of an equivalent theorem on heatlets decomposition, see Shen and Strang [28], the crucial hypothesis used in all theorems is the linearity of the PDE.

Proposition 7.1 (Refinement) *Let $\phi(x)$ and $\psi(x)$ be, respectively, the father and mother wavelets of a wavelets basis. Suppose that the LPDE (7.8) has a unique solution for each terminal function $f(x)$; let $\Phi^D(t, x)$ and $\Psi^D(t, x)$ be the solutions of the LPDE (7.8) with terminal conditions, respectively, the father and the mother wavelets. If*

$$(7.9) \quad \begin{aligned} \phi(x) &= 2 \sum_{n \in \mathbb{Z}} h_n \phi(2x - n) \\ \psi(x) &= 2 \sum_{n \in \mathbb{Z}} g_n \phi(2x - n) \end{aligned}$$

with

$$\sum_{n \in \mathbb{Z}} (h_n^2 + g_n^2) < \infty.$$

Then

$$(7.10) \quad \begin{aligned} \Phi^D(t, x) &= 2 \sum_{n \in \mathbb{Z}} h_n \Phi^D((2^{(2-2\lambda)}t, 2x - n) \\ \Psi^D(t, x) &= 2 \sum_{n \in \mathbb{Z}} g_n \Phi^D(2^{(2-2\lambda)}t, 2x - n). \end{aligned}$$

Proof: It is easy to check that $\Phi^D(2^{(2-2\lambda)}t, 2x - n)$ is the solution of LPDE (7.8) with terminal value $\phi(2x - n)$, see remark 7.4. Then the evolution of the functions $\phi(x)$ and $\psi(x)$ given the left-hand sides of equations (7.10) and the decomposition of the same functions, thanks to relations (7.9) given the right-hand sides; by uniqueness the two solutions must be equal. \square

Theorem 7.2 (Diffusionlets decomposition)

Suppose that $f(x)$ belongs to L^2 and then $f(x)$ admits a decomposition of type (7.1) and suppose that LPDE (7.8) admits a unique solution when the terminal condition is $\tilde{Q}(0, x) = f(x)$ then the solution of LPDE (7.8) with terminal value $f(x)$ is given by

$$(7.11) \quad \tilde{Q}(t, x) = \sum_k \alpha_k \Phi^D(t, x - k) + \sum_i \sum_k \beta_{i,k} \Psi^D(2^{(2-2\lambda)^i} t, 2^i x - k)$$

Proof: Since $f(x) \in L^2(\mathbb{R})$ and $\{\phi_k^D \psi_{i,k}^D(x)\}$ is an orthonormal basis of $L^2(\mathbb{R})$, the wavelets expansion of $f(x)$ converges to $f(x)$ in L^2 -norm. It is easy to check that $\Psi^D(2^{(2-2\lambda)^i} t, 2^i x - k)$ is the evolution of $\psi_{i,k}^D(x)$ and $\Phi^D(t, x - k)$ is the evolution of $\phi_{0,k}^D(x)$, see remark 7.4. By uniqueness of the solution the diffusion of the wavelets expansion of terminal condition $f(x)$ converges to the diffusion of $f(x)$ in L^2 -norm. \square

Remark 7.5 (Diffusionlets Advantages) The key advantage of the diffusionlets is the independence of the initial state. Therefore, we can store the solution of the LPDE (7.8) with the father wavelet as terminal condition, this solution can be estimate with an high degree of precision due to the compactly supported and bounded father wavelet in Daubechies' case. The solution of the LPDE (7.8) with terminal value $f(x)$ can be evaluate using the following strategy:

1. study the decomposition coefficients of the function $f(x)$ into the wavelets basis, using fast wavelet transform tool as an example, see Mallat [24];
2. reconstruct the solution with $f(x)$ as terminal condition using the store solution for the father wavelet, the coefficients estimated and the result of theorem 7.2.

The second, crucial, advantage is the fact that Daubechies' wavelets of order p have p vanishing moments, this fact, combined with the smoothing property of parabolic partial differential equations (property true for LPDE (7.8) far from $x = 0$) forces a fast convergence of the mother wavelet to 0 with time t ; this effect is magnified by the scaling effect when $\lambda < 1$, see equation (7.11). The local volatility model, introduced by Hagan and Woodward [16] assumes λ smaller than 1.

Remark 7.6 (Diffusionlets disadvantage) Due to the linearity of the LPDE (7.8), diffusionlets are not compactly supported when $t \neq 0$. This fact force the necessity to estimate an "essential support" for the diffusion of father and mother wavelets, i.e. the region where the norm of father and mother diffusionlets are bigger than a reference level related to the asked sensitivity of the searched solution.

This "essential support" given a length, depending on time t , useful to determine the number of decomposition elements for each scaling level, i.e. the number of k necessarily considered for a well-estimation of the solution before changing the coefficient i . The smoothing effect, emphasized between the advantages, permits to define an order $I(\epsilon)$ beyond which the remainder is smaller than ϵ . The approximated solution becomes:

$$(7.12) \quad \tilde{Q}(t, x) \simeq \sum_{k=-K_0(\epsilon)}^{K_0(\epsilon)} \alpha_k \Phi^D(t, x - k) + \sum_{i=0}^{I(\epsilon)} \sum_{k=-K_i(\epsilon)}^{K_i(\epsilon)} \beta_{i,k} \Psi^D(2^{(2-2\lambda)^i} t, 2^i x - k)$$

We conclude with a negative remarque on the diffusionlets basis:

Remark 7.7 (non-orthogonality of diffusionlets basis) *The class of functions generated by the diffusion of a wavelets basis is a basis of the L^2 -subspace characterized by the diffusion itself², i.e., for each time s , the smaller subspace of L^2 that contains the functions that can be a solution of the LPDE at time s with a terminal value belongs to L^2 . However, the basis $\{\Phi_k^D(t, x), \Psi_{i,k}^D(t, x)\}_{i,k}$, where $\Phi_k^D(t, x) = \Phi^D(t, x - k)$ and $\Psi_{i,k}^D(t, x) = \Psi^D(2^{(2-2\lambda)^i} t, 2^i x - k)$, is not, generally, orthogonal, especially owing to the scaling factor in time.*

7.3 Sensitivity of LPDE solution

In this section, we suppose that the terminal condition $f(x)$ of a LPDE is erroneous and we study the diffusion of this uncertainty. The starting point is to define an error structure on a functional space, see section 1.5. We use the decomposition of $f(x)$ into a wavelets basis, see equation (7.1); and we set the coefficients α_k and $\beta_{i,k}$ to be random.

We define an error structure on each subspace, generated by each element of the wavelets basis, in accord with hypotheses of independence 1.1 and proportionality 1.3, i.e. we can assume that the error structures on each subspace are independent and the uncertainty is proportional to the estimate parameter α_k or $\beta_{i,k}$ depending on the cases. Now, we can study the variance caused by the uncertainty on the terminal value.

7.3.1 Uncertainty on the solution

In this section, we prove three results.

Proposition 7.3 (Variance of terminal condition)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure that verifies the hypotheses 1.1 and 1.3, we denote $\gamma(k)$ and $\gamma(i, k)$ the eigenvalues of the operator Γ , i.e. $\Gamma[\alpha_k] = \gamma(k) \alpha_k^2$ and $\Gamma[\beta_{i,k}] = \gamma(i, k) \beta_{i,k}^2$. Suppose that this error structure admits a sharp operator, see definition 1.2, and denote $\hat{\alpha}_k$ and $\hat{\beta}_{i,k}$ the copies of α_k and $\beta_{i,k}$. Then the terminal condition $f(x)$ has the following variance:

$$(7.13) \quad \Gamma[f(x)] = \sum_k \gamma(k) \alpha_k^2 [\phi_{0,k}^D(x)]^2 + \sum_i \sum_k \gamma(i, k) \beta_{i,k}^2 [\psi_{i,k}^D(x)]^2$$

²The proof of this fact is very simple, we must check only the independence of each vector of the basis, that follows on the uniqueness of the solution.

Proof: We start with the computation of the sharp of the function $f(x)$, the definition of the sharp operator 1.2 and the hypotheses of the chosen structure give the following relation:

$$\begin{aligned}
 [f(x)]^\# &= \sum_k \alpha_k^\# \phi_{0,k}^D(x) + \sum_i \sum_k \beta_{i,k}^\# \psi_{i,k}^D(x) \\
 (7.14) \quad &= \sum_k \sqrt{\gamma(k)} \alpha_k \hat{\alpha}_k \phi_{0,k}^D(x) + \sum_i \sum_k \sqrt{\gamma(i,k)} \beta_{i,k} \hat{\beta}_{i,k} \psi_{i,k}^D(x)
 \end{aligned}$$

Now the second property of operator sharp gives the result (7.13). □

The previous theorem has an equivalent for the solution of LPDE.

Theorem 7.4 (Variance of LPDE solution)

Under hypotheses of proposition 7.3, the solution of LPDE (7.8) has the following variance:

$$(7.15) \quad \Gamma [\tilde{Q}(t, x)] = \sum_k \gamma(k) \alpha_k^2 [\Phi_{0,k}^D(t, x)]^2 + \sum_i \sum_k \gamma(i, k) \beta_{i,k}^2 [\Psi_{i,k}^D(t, x)]^2$$

Proof: The proof is similar to the previous theorem, we start with the sharp decomposition. Thanks to the diffusionlets decomposition 7.2, we find:

$$\begin{aligned}
 \tilde{Q}^\#(t, x) &= \sum_k \alpha_k^\# \Phi_{0,k}^D(t, x) + \sum_i \sum_k \beta_{i,k}^\# \Psi_{i,k}^D(t, x) \\
 (7.16) \quad &= \sum_k \sqrt{\gamma(k)} \alpha_k \hat{\alpha}_k \Phi_{0,k}^D(t, x) + \sum_i \sum_k \sqrt{\gamma(i, k)} \beta_{i,k} \hat{\beta}_{i,k} \Psi_{i,k}^D(t, x)
 \end{aligned}$$

Now the second property of operator sharp, gives the result (7.15). □

Remark 7.8 *This uncertainty is easy to estimate, since diffusionlets decomposition is independent with respect to the initial state, see remark 7.5.*

Theorem 7.5 (Covariance of LPDE solution)

Under hypotheses of proposition 7.3, the solution of LPDE (7.8) has the following covariance:

$$\begin{aligned}
 \Gamma [\tilde{Q}(t, x), \tilde{Q}(s, y)] &= \sum_k \gamma(k) \alpha_k^2 \Phi_{0,k}^D(t, x) \Phi_{0,k}^D(s, y) \\
 (7.17) \quad &+ \sum_i \sum_k \gamma(i, k) \beta_{i,k}^2 \Psi_{i,k}^D(t, x) \Psi_{i,k}^D(s, y)
 \end{aligned}$$

This theorem is a direct consequence of the previous one.

7.4 Applications to finance

In the domain of mathematical finance, Linear Partial Differential Equations play a key role. Excepted few cases, the equations in a financial model do not have a closed form solution, therefore a numerical approach is mandatory. The Feynman-Kac formula gives us a bridge between the parabolic PDE and SDE, hence it is possible to choose the numerical method, PDE or Monte-Carlo, depending on the time-efficiency; generally PDE approach is used in the case of a low number of variables.

A first application is the Cox-Ingersoll-Ross stochastic differential equation, see [9], in this case the parameter λ is equal to 0.5, the SDE becomes:

$$dX_t = b(a - X_t) dt + \sigma \sqrt{X_t} dW_t$$

when the mean a is zero; the associate PDE is

$$\frac{\partial U}{\partial t}(t, x) - bx \frac{\partial U}{\partial x} + \sigma^2 x \frac{\partial^2 U}{\partial x^2} = 0$$

This case has a relative interest, since an exact characterization of the solution exists, see Lamberton and Lapeyre [20] or Shreve [29]. The principal advantage consists on the possibility to test the wavelets procedure.

A second, more relevant, application is a simplified SABR model; Hagan et al, see [17], emphasize the incoherence of the dynamic behavior of log-normal model, the well-known Black Scholes model, compared to the behavior observed in the marketplace. In order to eliminate this problem, Hagan and Woodward, see [16], propose a local volatility model in which the forward value satisfies

$$dF_t = \sigma_t F_t^\lambda dW_t$$

where λ is a fixed parameter, that takes values between 0 and 1, estimated on the market. This model is the starting point for the SABR model, the SDE is of type (7.4); therefore the procedure described in this article can optimize the procedure of option pricing.

Hagan et al, see [17], emphasize an other relevant aspect, actually market smiles are managed using Dupire's local volatility models, but a local volatility function different to a power introduces an intrinsic "length scale" for the forward price, this inhomogeneous has an hard financial explanation. Therefore the model proposed by Hagan and Woodward has the principal problems of all local volatility model, i.e. a poor equivalent volatility for basket options and an unsmiled forward volatility, a possible solution of this problem is investigate in chapter 4.

7.5 Conclusion

In this chapter, we have investigated the relations between three objects, i.e. the linear partial differential equations, the error theory using Dirichlet forms and wavelets. These three objects have a different origin and application fields separated to date; this study shows how capitalizing the advantages of wavelets bases in order to solve a LPDE and how to exploit the wavelets as a decomposition basis to study the sensitivity of the LPDE.

In a particular case, when the LPDE can be reduced to the form

$$U_t = \sigma^2 x^{2\lambda} U_{xx},$$

we have proved that the properties of wavelets are partly preserved, especially the invariance under a scaling and a translation; these properties permit a fast processing of the general solution of this type of PDE. Considering LPDE is the new key element of the non-lognormal financial models, studied by Hagan et al, see [16] and [17].

The principal new feature is the study of the sensitivity of the PDE solution using error theory by means of Dirichlet forms, introduced by Bouleau, see [7], this methodology has us permitted an evaluation of the sensitivity of the solution with respect to an uncertainty on the terminal value, this inaccuracy on the payoff can model an imprecision on the final spot value, due to the spread bid-ask as an example. This analysis is just a starting point for the study of inaccuracy on terminal value and non-lognormal models.

Chapter 8

Sensitivity of Non-Linear PDEs: Discrete Approach

In this chapter, we analyze the sensitivity of solutions of a partial differential equation when there is an uncertainty on boundary conditions. We propose a new approach, based on the technique developed by Bouleau, to estimate the transmission of this uncertainty through a PDE by means of approximated solution given by a finite element approach.

In order to evaluate the sensitivity of PDE solutions, a classical approach consists to perturb boundary conditions by means of an analytic function added to; then to estimate the solution of PDE with the new boundary conditions; to compute the difference, i.e. the Gateaux derivative of solution, using mathematical terms; and, finally, to repeat this procedure for all functions of a “perturbation” basis. Clearly, this approach has some limits: first of all, it assumes a deterministic nature for uncertainty and this erases all bias effects due to the non-linearity of partial differential equation; the second matter comes from the numerical processing to compute the solution, this is often expensive. How many Gateaux derivatives are necessary in order to give a fine estimation of this sensitivity? That depends, mainly on the number of constraints to be verified, i.e. on the information that we have on boundary conditions; frequently, the numerical computation becomes too expensive.

In this chapter, we propose a new method based on error theory using Dirichlet forms, approach introduced by Bouleau [7]. The principal advantage is to consider a random uncertainty on boundary conditions, this allows us to investigate the variance of solution but also its bias, i.e. the discrepancy between the solution and the mean of all possible solutions depending on different boundary conditions weighed by the probability; this mismatch has a stochastic root, in fact it vanishes in deterministic case. Another interesting feature is the possibility to compute the covariance of the error of the solution at two different points in time-space domain.

In order to simplify the comprehension of our methodology, we use an example, i.e shallow water equations (also known as Saint Venant equations), there are a particular case of Navier-Stokes equations and govern the motion of a single homogeneous incompressible fluid layer in hydrostatic equilibrium, whose depth is small compared to other dimensions. Shallow water equations are commonly used for solving open channel flow problems and have interesting applications in climatology, e.g. these equations describe a simple approximation to the depth averaged dynamics of ocean circulation, see Fraedrich [15]. The choice of these equations depends on the presence

of non-linear terms; as a consequence, the solutions depend sensitively on boundary conditions and grid scale parametrisations which are poorly known from observations, see Fraedrich [15]. The procedure described in this chapter works with any partial differential equation, but, as far as linear partial differential equations are concerned, many other approaches are possible; in particular, we suggest the analysis of errors transmission through the associated stochastic differential equation or the decomposition of uncertainties through a wavelets basis, see chapter 7.

We explain our methodology in the case of finite element approach on space and finite difference on time with explicit scheme, since, from theoretical point of view, the explicit scheme simplifies the mathematical objects and all results are easier to understand than in implicit scheme case; this second case is analyzed in appendix. Nevertheless, the use of finite element approach causes often complicated formulae, hence the reader does not get scared about the complexity and the size of formulae, since, with a good choice of Galerkin's representation functions, many terms are equal to zero and the computation of formulae by a computer is immediate.

The chapter is organized as follows:

Section 1 is dedicated to the analysis of one-dimensional shallow water equations and we find the approximate solution on a mesh by means of finite element approach as well. In section 2, we resume, briefly, two approaches, deterministic and stochastic, to study the sensitivity of solutions, in particular we recall basic ingredients of error theory using Dirichlet form and we prove a key theorem that will be applied afterwards in our analysis. In section 3, we apply error theory to shallow water equations, we show the variance-covariance matrix and the bias of solution at the mesh points. Finally, section 4 resumes and concludes.

8.1 Shallow water equations

In this section, we introduce shallow water equations system. Generally, this system cannot be solved explicitly except under very specific and, for most situations, unrealistic assumptions. Therefore, we apply a numerical technique, the finite element with an explicit scheme on time; we follow the analysis of Hervouet [19]. We consider the following partial differential equation, called one-dimensional shallow water PDE, the generalization in dimension higher than one is evident but formulae become too complicated.

$$(8.1) \quad \left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = q_l \\ \frac{\partial Q}{\partial t} + \frac{\partial Q^2}{\partial x} = -g h \frac{\partial Z_s}{\partial x} + h F + \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial Q}{\partial x} \right) + U_s q_l \\ \text{starting conditions} \quad \forall x \quad h(x, 0), \quad Q(x, 0) \\ \text{boundary conditions} \quad \forall t \quad h(x_0, t), \quad h(x_1, t), \quad Q(x_0, t), \quad Q(x_1, t) \end{array} \right.$$

where

- t is the time variable;

- x is the space variable;
- $h(x, t)$ is the depth of flow in x at time t ;
- $Q(x, t)$ is the discharge in x at time t ;
- q_l is the lateral rate of flow from the layer;
- g is the acceleration due to gravity;
- $Z_s(x, t)$ denotes the height of free-surface side;
- $F(x, t)$ summarizes a local average of all other external forces;
- ν_e is an effective diffusion coefficient that take into account the dispersion and turbulence viscosity;
- U_S is the speed of water coming from the layer;
- x_0 and x_1 denote, respectively, the starting and ending point of channel.

Figure 8.1 describes our problem with the distinction between starting and boundary conditions.

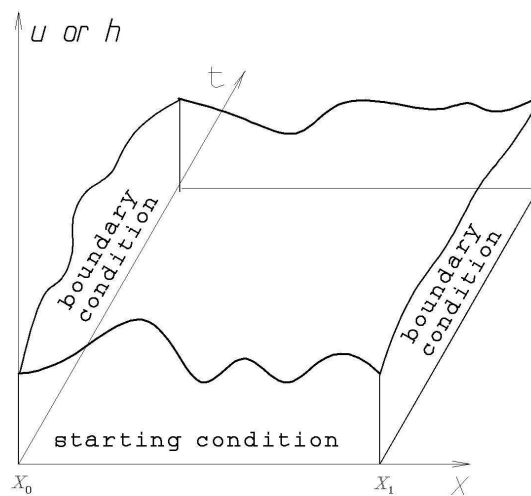


Figure 8.1: Representation of our problem with starting and boundary conditions.

We rewrite previous equations using relation $Q(x, t) = h(x, t) u(x, t)$, where $u(x, t)$ is the velocity of canal water in x at time t .

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{\partial(h u)}{\partial x} = q_l \\ \frac{\partial(h u)}{\partial t} + \frac{\partial(h u^2)}{\partial x} = -g h \frac{\partial Z_s}{\partial x} + h F + \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) + U_s q_l \\ \text{starting conditions} \quad \forall x \quad h(x, 0), \quad u(x, 0) \\ \text{boundary conditions} \quad \forall t \quad h(x_0, t), \quad h(x_1, t), \quad u(x_0, t), \quad u(x_1, t) \end{array} \right.$$

First equation becomes

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = -h \frac{\partial u}{\partial x} + q_l.$$

Second equation can be rewritten as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial Z_s}{\partial x} + F + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) + (U_s - u) \frac{q_l}{h},$$

thanks to first equation. Accordingly, shallow water PDEs become

$$(8.2) \quad \left\{ \begin{array}{l} \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = -h \frac{\partial u}{\partial x} + q_l \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial Z_s}{\partial x} + F + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) + (U_s - u) \frac{q_l}{h}. \end{array} \right.$$

Due to the non-linearity of shallow water PDEs, we use a numerical approximation of these by means of finite elements technique. We start with a discretization on time. We estimate the variables $h(x, t)$ and $u(x, t)$ just at times $t_n = n \Delta T$, where ΔT is a fixed temporal interval; in order to simplify our notations, we denote:

$$\begin{aligned} h_n(x) &= h(x, t_n) \\ u_n(x) &= u(x, t_n) \end{aligned}$$

We approximate all derivative with respect to time t by the classical finite difference approximation:

$$\begin{aligned} \frac{\partial h}{\partial t} \Big|_{t=t_n} &= \frac{h_n - h_{n-1}}{\Delta T} \\ \frac{\partial u}{\partial t} \Big|_{t=t_n} &= \frac{u_n - u_{n-1}}{\Delta T} \end{aligned}$$

We start with the value of couple (u, h) for any x at time 0 and we find the couple (u, h) at time t_n by means of n iterations of the following system.

$$(8.3) \quad \begin{cases} \frac{h_n - h_{n-1}}{\Delta T} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} = q_l \\ \frac{u_n - u_{n-1}}{\Delta T} + u \frac{\partial u}{\partial x} = -g \frac{\partial Z_s}{\partial x} + F + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) + (U_S - u) \frac{q_l}{h} \end{cases}$$

Now, we have to specify the numerical diagram in order to represent variables h and u in the previous equation. A good choice, see Hervouet [19], is to fix two parameters, θ_h and θ_u , better if close to but bigger than one half, and use the following discretization.

$$\begin{aligned} h &\approx \theta_h h_n + (1 - \theta_h) h_{n-1} \\ u &\approx \theta_u u_n + (1 - \theta_u) u_{n-1} \end{aligned}$$

The non specification of values of θ_h and θ_u permits to include two limit cases:

- the explicit scheme, when $\theta_h = \theta_u = 0$;
- the completely implicit scheme, when $\theta_h = \theta_u = 1$.

Explicit scheme has generally a bad rate of convergence, but the study of sensitivity is cheaper in both points of view, numerically and theoretically.

First equation has two non linear terms $h \frac{\partial u}{\partial x}$ and $u \frac{\partial h}{\partial x}$. In order to avoid the presence of a term $h_n u_n$, we approximate the first term by

$$h \frac{\partial u}{\partial x} \approx h_{n-1} \frac{\partial [\theta_u u_n + (1 - \theta_u) u_{n-1}]}{\partial x}$$

and the second by

$$u \frac{\partial h}{\partial x} \approx u_{n-1} \frac{\partial [\theta_h h_n + (1 - \theta_h) h_{n-1}]}{\partial x}.$$

Second equation has a similar problem and we can make the same approximation.

$$u \frac{\partial u}{\partial x} \approx u_{n-1} \frac{\partial [\theta_u u_n + (1 - \theta_u) u_{n-1}]}{\partial x}$$

Now, we study the term due to gravity force $-g \frac{\partial Z_s}{\partial x}$, related to the pressure of water pillar; Z_s denotes the height of free-surface, we have that $Z_s = h + Z_f$, where Z_f is the height of floor. Then, we can write

$$-g \frac{\partial Z_s}{\partial x} = -g \theta_h \frac{\partial (h_n - h_{n-1})}{\partial x} - g \frac{\partial Z_s^{n-1}}{\partial x},$$

where $Z_s^{n-1} = Z_f + h_{n-1}$ is the height of free-surface at time t_{n-1} ; this description is easy to understand when we consider a calm lake with an irregular floor; in this particular case, Z_s^{n-1} does not vary on space, contrary to variables Z_f and h_{n-1} .

We approximate the term $\frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right)$, due to flow viscosity, with

$$\frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) \approx \frac{\partial}{\partial x} \left(\nu_e \frac{\partial u}{\partial x} \right) \approx \frac{\partial}{\partial x} \left[\nu_e \frac{\partial (\theta_u u_n + (1 - \theta_u) u_{n-1})}{\partial x} \right].$$

Finally, we find the following discretized PDE.

$$(8.4) \quad \left\{ \begin{array}{l} \frac{h_n - h_{n-1}}{\Delta T} + h_{n-1} \frac{\partial [\theta_u u_n + (1 - \theta_u) u_{n-1}]}{\partial x} + u_{n-1} \frac{\partial [\theta_h h_n + (1 - \theta_h) h_{n-1}]}{\partial x} = q_l \\ \\ \frac{u_n - u_{n-1}}{\Delta T} + u_{n-1} \frac{\partial [\theta_u u_n + (1 - \theta_u) u_{n-1}]}{\partial x} = -g \theta_h \frac{\partial [h_n - h_{n-1}]}{\partial x} - g \frac{\partial Z_s^{n-1}}{\partial x} + F \\ \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \frac{\partial}{\partial x} \left\{ \nu_e \frac{\partial [\theta_u u_n + (1 - \theta_u) u_{n-1}]}{\partial x} \right\} \\ \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \{ U_S - [\theta_u u_n + (1 - \theta_u) u_{n-1}] \} \frac{q_l}{h_{n-1}} \end{array} \right.$$

Now, we reduce our problem at the study of functions h_n and u_n only at knots of a mesh. In particular, we fix two bases Ψ_i^h and Ψ_i^u in a height-velocity space where the function Ψ_i has been chosen in order to take the value 1 at point i and 0 otherwise; thus, it represents the degree of freedom associated at point i . Therefore, we have the two following decompositions:

$$h_n(x) = \sum_{i=1}^{N_h} h_n^i \Psi_i^h(x)$$

$$u_n(x) = \sum_{i=1}^{N_u} u_n^i \Psi_i^u(x)$$

where h_n^i and u_n^i belong to \mathbb{R} for all i and n ; so, discretized PDE (8.4) becomes:

$$(8.5) \quad \left\{ \begin{array}{l} \left[\sum_{i=1}^{N_h} \frac{\delta h_n^i}{\Delta T} \Psi_i^h \right] + \left(\sum_{i=1}^{N_h} h_{n-1}^i \Psi_i^h \right) \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \\ + \left(\sum_{i=1}^{N_u} u_{n-1}^i \Psi_i^u \right) \left[\sum_{i=1}^{N_h} (\theta_h \delta h_n + h_{n-1}) \frac{\partial \Psi_i^u}{\partial x} \right] = q_l \\ \\ \left[\sum_{i=1}^{N_u} \frac{\delta u_n^i}{\Delta T} \Psi_i^u \right] + \left(\sum_{i=1}^{N_u} u_{n-1}^i \Psi_i^u \right) \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] = \\ -g \left(\sum_{i=1}^{N_h} \theta_h \delta h_n^i \frac{\partial \Psi_i^h}{\partial x} \right) + \frac{\partial}{\partial x} \left\{ \nu_e \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \right\} \\ -g \frac{\partial Z_s^{n-1}}{\partial x} + F + \left\{ U_S - \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \Psi_i^u \right] \right\} \frac{q_l}{\sum_{i=1}^{N_h} h_{n-1}^i \Psi_i^h} \end{array} \right.$$

where we have used notations $\delta u_n^i = u_n^i - u_{n-1}^i$ and $\delta h_n^i = h_n^i - h_{n-1}^i$. Whereas we have used a finite-difference approach in time discretization, in space we prefer a finite element approach as it is usage.

8.1.1 Galerkin-Variational Approach

A variational approach consists in the choice of a finite basis of test functions $\{\phi_j\}_{1 \leq j \leq N}$ and to rewrite an equation of type $E(\cdot) = 0$ in the weak form given by the system of N equations $\int E(\cdot) \phi_j(\cdot) d\Omega$ where Ω is the considered domain; this approach is known as Galerkin's principle. In our particular case, we fix two test functions bases, one $\{\phi_j^h\}_{1 \leq j \leq N_h}$ for height and the other one $\{\phi_j^u\}_{1 \leq j \leq N_u}$ for velocity.

We apply Galerkin's principle at equation (8.5), so we transform the continuity equation in the following system.

$$\begin{aligned} & \int_{\Omega} \left[\sum_{i=1}^{N_h} \frac{\delta h_n^i}{\Delta T} \Psi_i^h \phi_j^h \right] d\Omega + \int_{\Omega} \left(\sum_{i=1}^{N_h} h_{n-1}^i \Psi_i^h \right) \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \phi_j^h d\Omega \\ & + \int_{\Omega} \left(\sum_{i=1}^{N_u} u_{n-1}^i \Psi_i^u \right) \left[\sum_{i=1}^{N_h} (\theta_h \delta h_n + h_{n-1}) \frac{\partial \Psi_i^u}{\partial x} \right] \phi_j^h d\Omega = \sum_{i=1}^{N_h} q_l^i \int_{\Omega} \Psi_i^h \phi_j^h d\Omega \end{aligned}$$

for all j between 1 and N_h . In a similar way we transform the motion equation in the following

system.

$$\begin{aligned} & \int_{\Omega} \left[\sum_{i=1}^{N_u} \frac{\delta u_n^i}{\Delta T} \Psi_i^u \phi_j^u \right] d\Omega + \int_{\Omega} \left(\sum_{i=1}^{N_u} u_{n-1}^i \Psi_i^u \right) \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \phi_j^u d\Omega = \\ & - \int_{\Omega} g \left(\sum_{i=1}^{N_h} \theta_h \delta h_n^i \frac{\partial \Psi_i^h}{\partial x} \right) \phi_j^u d\Omega + \int_{\Omega} \phi_j^u \frac{\partial}{\partial x} \left\{ \nu_e \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \right\} d\Omega \\ & - \int_{\Omega} g \frac{\partial Z_s^{n-1}}{\partial x} \phi_j^u d\Omega + \int_{\Omega} F \phi_j^u d\Omega + \int_{\Omega} \phi_j^u \left\{ U_S - \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \Psi_i^u \right] \right\} \frac{q_l}{\sum_{i=1}^{N_h} h_{n-1}^i \Psi_i^h} d\Omega \end{aligned}$$

for all j between 1 and N_u . In accord with Hervouet [19], we integrate by part a term of continuity equation in order to make explicit boundary conditions called ‘‘weak impermeability’’:

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^{N_h} h_{n-1}^i \Psi_i^h \right) \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \phi_j^h d\Omega = \\ & - \int_{\Omega} \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \Psi_i^u \right] \left[\sum_{i=1}^{N_h} h_{n-1}^i \frac{\partial (\Psi_i^h \phi_j^h)}{\partial x} \right] d\Omega \\ & + \int_{\partial\Omega} \left(\sum_{i=1}^{N_h} h_{n-1}^i \Psi_i^h \right) \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \Psi_i^u \right] \phi_j^h d\partial\Omega \end{aligned}$$

Where $\partial\Omega$ denotes the boundary of domain Ω . Weak impermeability conditions assumes that the boundary integral vanishes. Instead, motion equation exhibit a term with a second derivative, we integrate by part and we find

$$\begin{aligned} & \int_{\Omega} \phi_j^u \frac{\partial}{\partial x} \left\{ \nu_e \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \right\} d\Omega = \\ & - \int_{\Omega} \frac{\partial \phi_j^u}{\partial x} \left\{ \nu_e \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \right\} d\Omega + \int_{\partial\Omega} \phi_j^u \left\{ \nu_e \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \right\} d\partial\Omega. \end{aligned}$$

We approximate the previous boundary integral by the following equation, see Hervouet [19] page 106.

$$\int_{\partial\Omega} \phi_j^u \left\{ \nu_e \left[\sum_{i=1}^{N_u} (\theta_u \delta u_n^i + u_{n-1}^i) \frac{\partial \Psi_i^u}{\partial x} \right] \right\} d\partial\Omega = \int_{\partial\Omega} \phi_j^u \nu_e a \sum_{i=1}^{N_u} u_{n-1}^i \Psi_i^u d\partial\Omega$$

where a is a fixed coefficient. The last term that we may make explicit is the friction F . We rewrite this term using the following approximation.

$$(8.6) \quad \int_{\Omega} F \phi_j^u d\Omega = -\frac{1}{\cos(\alpha)} \frac{C_f}{2h_{n-1}} u_n u_{n-1} \int_{\Omega} \phi_j^u d\Omega$$

where C_f is a friction coefficient, and α the slope angle of floor.

8.1.2 Solution of shallow water PDEs

If we resume, we have that the numerical solution of shallow water equations can be find thanks to the recursive solution of a linear system. Therefore, we state a theorem:

Theorem 8.1 (Numerical solution of shallow water partial differential equations)

Shallow water PDE admits a recursive numerical solution, given by the following system:

$$(8.7) \quad \begin{pmatrix} {}^{n-1}D1 & {}^{n-1}C1 \\ {}^{n-1}C2 & {}^{n-1}D2 \end{pmatrix} \begin{pmatrix} \delta^n H \\ \delta^n U \end{pmatrix} = \begin{pmatrix} {}^{n-1}S1 \\ {}^{n-1}S2 \end{pmatrix}$$

where $\delta^n H$ is the vector of components δh_n^i , $\delta^n U$ is the vector of components δu_n^i . Matrix ${}^{n-1}D1$, ${}^{n-1}D2$, ${}^{n-1}C1$ and ${}^{n-1}C2$ are given by

$$\begin{aligned} {}^{n-1}D1_{j,i} &= \int_{\Omega} \frac{\Psi_i^h \phi_j^h}{\Delta T} + \theta_h \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \\ {}^{n-1}D2_{j,i} &= \int_{\Omega} \frac{\Psi_i^u \phi_j^u}{\Delta T} + \theta_u \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega + \int_{\Omega} \nu_e \frac{\partial \Psi_i^u}{\partial x} \frac{\partial \phi_j^u}{\partial x} d\Omega \\ &\quad + \frac{\delta_{i,j}}{\cos(\alpha)} \frac{C_f}{2h_{n-1}^i} u_{n-1}^i \int_{\Omega} \phi_i^u d\Omega - \int_{\partial\Omega} \nu_e \phi_j^u a \Psi_i^u d\partial\Omega \\ {}^{n-1}C1_{j,i} &= -\theta_u \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} h_{n-1}^k \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \\ {}^{n-1}C2_{j,i} &= q\theta_h \int_{\Omega} \frac{\partial \Psi_i^h}{\partial x} \phi_j^u \end{aligned}$$

and vectors ${}^{n-1}S1$ and ${}^{n-1}S2$ are given by

$$\begin{aligned} {}^{n-1}S1_j &= (\theta_h - 1) \left\{ \sum_{i=1}^{N_h} h_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \right\} \\ &\quad - (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} h_{n-1}^k \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \right\} + \sum_{i=1}^{N_h} q_l^i \int_{\Omega} \Psi_i^h \phi_j^h d\Omega \\ {}^{n-1}S2_j &= \frac{\sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \Psi_i^u \phi_j^u}{\Delta T} + (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \right\} \\ &\quad - g \left\{ \int_{\Omega} \frac{\partial Z_s^n}{\partial x} \phi_j^u d\Omega \right\} + \int_{\Omega} F \phi_j^u d\Omega \end{aligned}$$

and we have the following corollary when we choose an explicit scheme.

Corollary 8.2 (numerical solution of shallow water PDE with an explicit scheme)

Shallow water PDE admits a recursive numerical solution, given by the following system:

$$(8.8) \quad \begin{pmatrix} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{pmatrix} \begin{pmatrix} \delta^n \widetilde{H} \\ \delta^n \widetilde{U} \end{pmatrix} = \begin{pmatrix} \widetilde{S1} \\ \widetilde{S2} \end{pmatrix}$$

where $\delta^n \widetilde{H}$ is the vector of components δh_n^i , ${}^n \widetilde{U}$ is the vector of components δu_n^i . Matrix $\widetilde{D1}$, $\widetilde{C1}$, $\widetilde{C2}$ and $\widetilde{D2}$ are given by

$$\widetilde{D1}_{j,i} = \int_{\Omega} \frac{\Psi_i^h \phi_j^h}{\Delta T} d\Omega$$

$$\widetilde{D2}_{j,i} = \int_{\Omega} \frac{\Psi_i^u \phi_j^u}{\Delta T} d\Omega + \int_{\Omega} \nu_e \frac{\partial \Psi_i^u}{\partial x} \frac{\partial \phi_j^u}{\partial x} d\Omega - \int_{\partial \Omega} \nu_e \phi_j^u a \Psi_i^u d\partial \Omega$$

$${}^{n-1} \widetilde{C1}_{j,i} = 0$$

$${}^{n-1} \widetilde{C2}_{j,i} = 0$$

and vectors ${}^{n-1} \widetilde{S1}$ and ${}^{n-1} \widetilde{S2}$ are given by

$${}^{n-1} \widetilde{S1}_j = - \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} h_{n-1}^i u_{n-1}^k \int_{\Omega} \Psi_k^u \Psi_i^h \frac{\partial \phi_j^h}{\partial x} d\Omega + \sum_{i=1}^{N_h} q_l^i \int_{\Omega} \Psi_i^h \phi_j^h d\Omega$$

$$\begin{aligned} {}^{n-1} \widetilde{S2}_j &= \frac{1}{\Delta T} \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \Psi_i^u \phi_j^u - \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \\ &\quad + \frac{1}{\cos(\alpha)} \frac{C_f}{2h_{n-1}^j} (u_{n-1}^j)^2 \int_{\Omega} \phi_j^u d\Omega - g \int_{\Omega} \frac{\partial Z_s^n}{\partial x} \phi_j^u d\Omega + \int_{\Omega} F \phi_j^u d\Omega \end{aligned}$$

This corollary is a direct consequence of theorem 8.1, where we have take $\Theta_u = \Theta_h = 0$ and the friction term, see equation (8.6) are estimate at the time $n - 1$.

8.2 Sensitivity

In this section, we study the way to evaluate, theoretically, the propagation of an uncertainty. The goal is to compute the impact of an uncertainty on boundary conditions, induced by a physical estimation that are compromised by errors, on PDE solution.

In order to study this sensitivity two approaches are possible:

- the deterministic approach and
- the probabilistic one.

8.2.1 Deterministic approach

The first approach consists to consider a deterministic error on the boundary function $f(x) : \partial\mathcal{D} \rightarrow \mathcal{I}$, where \mathcal{D} is the domain and \mathcal{I} the co-domain. We can model this perturbation by a small function $\epsilon g(x)$, where the parameter ϵ is assumed very small, so we can compute the solution of shallow water PDE with the new boundary condition $f(x) + \epsilon g(x)$. If we denote $(u, h)_f$ the solution of our PDE with boundary function $f(x)$, we can study

$$\lim_{\epsilon \rightarrow 0} \frac{(u, h)_{f+\epsilon g} - (u, h)_f}{\epsilon}$$

This limit is known as Gateaux derivative of function (u, h) at f in the direction g . In order to complete the analytic study of the perturbation, it is enough to define a basis g_n of a ‘‘perturbation-functions’’ space and study the Gateaux derivative of function (u, t) in the direction g_n for each n .

This approach is the classical one, but it has two important limits, the first one is the hypothesis, implicit in an analytic approach, that the correct solution exists and the problem is only a mismatch in the value of boundary conditions. The second limit is the numerical cost of this procedure. What is the number of basis functions g_n required to have a good estimation? Maybe a very high number. What is the numerical cost to estimate the solution with a different boundary condition? Generally the same to estimate the first solution. Now, it is clear that this procedure is very expensive in computation time.

8.2.2 Probabilistic approach

A second approach consists to represent errors as very small random variables; the boundary function becomes $f(x) + \epsilon X_x(\omega)$, where $X_x(\omega) : \partial\mathcal{D} \times \Omega \rightarrow \mathcal{I}$ is a random variable depending on the space-variable x and the realization $\omega \in \Omega$, ϵ is still a small parameter.

First of all, using the language of probability theory, we can study the law of

$$(u, h)_{f+\epsilon X}$$

but this strategy presents three drawbacks:

- *ill-posed problem* : in order to study the law of $(u, h)_{f+\epsilon X}$, we need to know the law of X , unfortunately X is poorly known, statistics can specify some properties of X , like mean, variance and, maybe skewness and kurtosis, but the information is not complete.
- *mathematical framework* : as a matter of fact, X depends on two variables x and ω , so it is a collection of random variables defined on a manifold.
- *nonlinear calculation* : even if the two first problems are solved, our system is non-linear. Therefore, the law of $(u, h)_{f+\epsilon X}$ is hard to evaluate.

Hence, we turn to a different approach, that blends the advantages of analytic and probabilistic approaches. We consider infinitely small random errors, we expand the perturbation of the solution in a series, like the Taylor formula but, in order to express this rigorously, we use the language of Dirichlet forms as explained in introduction, see part I.

8.2.3 Key theorem

In this section, we show a theorem useful in the rest of our analysis.

Theorem 8.3 (Key theorem)

Let X be a vector in \mathbb{R}^d , the only solution of a linear system $MX = B$, where M is an invertible matrix and B another vector. Suppose that it exists an error structure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma)$ that admits a sharp operator and suppose that the vector B is erroneous, whereas the matrix M is not. Then, the vector X is erroneous and we have the following relations:

$$(8.9) \quad \begin{aligned} MX^\# &= B^\# \\ M \Gamma [X, X^T] M^T &= \hat{\mathbb{E}} [B^\# (B^\#)^T] = \Gamma [B, B^T] \end{aligned}$$

$$M \mathcal{A}[X] = \mathcal{A}[B]$$

where each operator acts at the same time on each component of a vector or a matrix and $()^T$ is the transpose operator.

Proof: Since X is the only solution of the considered system, matrix M admits an inverse M^{-1} ; therefore, we have $X = M^{-1}B$. The inverse verifies the property $M^{-1}M = Id$, then, if the matrix M is non-erroneous, and considering that the matrix Id is also non-erroneous, the matrix M^{-1} must be non-erroneous. At present, the sharp of vector X must verify first relation of (8.9), third relation follows in the same way. Second relation of (8.9) is a consequence of the definition of sharp, see definition 1.2, and the fact that the matrix M^{-1} is non-erroneous.

□

8.3 Application to shallow water equations

In order to get over the difficulty of general study, we restrict our analysis of uncertainty diffusion at a finite dimensional problem applying the same discretization mesh used in the computation of approximated solution of shallow water PDEs, we avoid thus second drawback highlighted in section 8.2.2. Therefore, we have a finite dimensional space, where the dimension is given by the number of knots of the mesh; in each knot we define an error structure, see definition 1.1, and we consider the product of these error structures that is an error structure, see Bouleau [7] pages 56-59 or section 1.3.2.

Why we consider only an error structure associated at knots? The problem of the definition of an error structure on a variety is really intricate and a practicable solution is to consider a map that transforms a classical space, where we are able to define an error structure, into the variety and lastly to take the image of this error structure through the map, see Bouleau [7] chapter 3 or section 1.3.1. But this strategy is generally not practical, since the image of an error structure is often complicated, and, besides, we know the solution of shallow water problem only at mesh knots. Therefore, we do not take great interest in some very local effects, even if we recognize that these can yield some macroscopic impacts.

Hence, we propose to discretize the frontier of the domain using the same mesh introduced to compute the numerical solution of shallow water equations, see section 8.1.2. Now, the number of mesh's knots at the frontier is finite, so we can define an error structure for the frontier in a very easy way, i.e. we define an error structure at each knot of boundary $h(x_0, t)$, $u(x_0, t)$ and $h(x_1, t)$, $u(x_1, t)$, and of starting frontier $h(x, 0)$ and $u(x, 0)$, see figure 8.2; after that, we take the product of all error structure, that it is an error structure, since the product is finite, see Bouleau [7]. In order to define the carré du champ operator of our error structure, we suggest to refer to the work of Bouleau and Chorro [8]. We assume that the considered error structure admits a sharp operator, denoted by $(\cdot)^\#$, and we study uncertainties diffusion from the boundary into the domain, see figure 8.3 for a graphic representation.

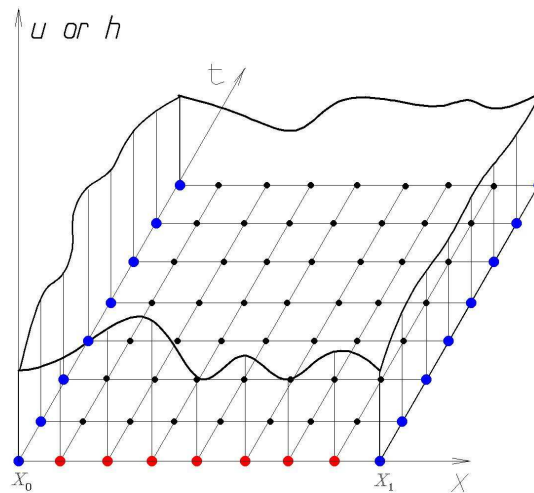


Figure 8.2: Position of mesh's knots of boundary condition (in blue), starting condition (in red) and other mesh's knots (in black).

In the mathematical space given by the knots, we have to solve shallow water system, given by corollary 8.2 and, afterwards, we search to evaluate the variance and the bias of founded solution at a fixed point in space-time (x, t) , belonging to mesh's knots, and the covariance of this solution at two different points (x, t) and (y, s) , thanks to the language of Dirichlet forms. Clearly, no advantage exists to use a refinement or a different mesh, since we will remark that the relations verified by sharp and bias depend on the solution of shallow water system. Using a different mesh, we have to estimate shallow water solution at points different to the knots in first mesh.

In next subsections, we state two theorems, some corollaries and an algorithm to compute, numerically, the variance-covariance matrix and the bias of the numerical solution of shallow water partial differential equations. We recall that the finite element approach generates complicated formulae but very easy to compute by a computer; the complexity raises in the computation of the bias due to the non-linearity of shallow water equations.

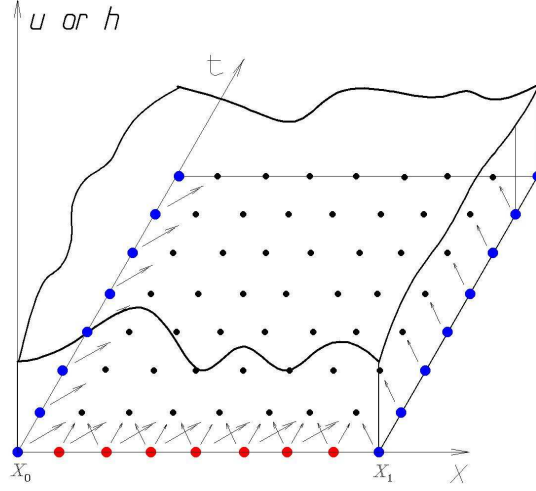


Figure 8.3: First step of uncertainties diffusion.

8.3.1 Variance of shallow water PDEs solution

In this subsection, we apply the technique of error theory using Dirichlet forms at shallow water partial differential equations after discretization. We take advantage of the existence of a sharp operator, see definition 1.2. We study the sharp associated with shallow water PDE solution with explicit scheme in time, see corollary 8.2, we have the following theorem:

Theorem 8.4 (Sharp of shallow water PDE solution with explicit scheme)

In the case of shallow water problem with solution given by corollary 8.2, the sharp of the solution verifies the following system:

$$(8.10) \quad \begin{pmatrix} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{pmatrix} \begin{pmatrix} \delta^n \widetilde{H}^\# \\ \delta^n \widetilde{U}^\# \end{pmatrix} = \begin{pmatrix} {}^{n-1}\widetilde{S1}^\# \\ {}^{n-1}\widetilde{S2}^\# \end{pmatrix}$$

where ${}^{n-1}\widetilde{D1}$, ${}^{n-1}\widetilde{D2}$, ${}^{n-1}\widetilde{C1}$, ${}^{n-1}\widetilde{C2}$, are the same as in corollary 8.2, $\delta^n \widetilde{H}^\#$ is the vector of components $(\delta h_n^i)^\#$, ${}^n \widetilde{U}^\#$ is the vector of components $(u_n^i)^\#$, and vectors ${}^{n-1}\widetilde{S1}^\#$ and ${}^{n-1}\widetilde{S2}^\#$ are given by

$$\begin{aligned}
{}^{n-1}\widetilde{S1}_j^\# &= \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} \left\{ u_{n-1}^k (h_{n-1}^i)^\# + h_{n-1}^i (u_{n-1}^k)^\# \right\} \int_{\Omega} \Psi_k^u \Psi_i^h \frac{\partial \phi_j^h}{\partial x} d\Omega \\
{}^{n-1}\widetilde{S2}_j^\# &= \frac{1}{\Delta T} \sum_{i=1}^{N_u} (u_{n-1}^i)^\# \int_{\Omega} \Psi_i^u \phi_j^u d\Omega \\
&\quad - \sum_{i=1}^{N_u} \sum_{k=1}^{N_u} \left\{ u_{n-1}^k (u_{n-1}^i)^\# + u_{n-1}^i (u_{n-1}^k)^\# \right\} \int_{\Omega} \Psi_k^u \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \\
&\quad - \frac{C_f}{\cos(\alpha)} \frac{u_{n-1}^j}{h_{n-1}^j} \left[2 (u_{n-1}^j)^\# - \frac{(h_{n-1}^j)^\#}{h_{n-1}^j} u_{n-1}^j \right] \int_{\Omega} \phi_j^u d\Omega
\end{aligned}$$

This theorem is a direct consequence of key theorem 8.3. Formally we can inverse the matrix

$$\widetilde{M} = \begin{pmatrix} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{pmatrix}$$

and compute the variance. Thus, we find the following corollary:

Corollary 8.5 (Variance of shallow water PDE solution with explicit scheme)

In the case of shallow water problem with solution given by corollary 8.2, the variance of the solution is given by

$$\begin{aligned}
(8.11) \quad \Gamma [\delta h_n^i] &= \widehat{\mathbb{E}} \left[\left\{ \sum_j \left(\begin{array}{cc} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{array} \right)_{i,j}^{-1} \begin{pmatrix} {}^{n-1}\widetilde{S1}_j^\# \\ {}^{n-1}\widetilde{S2}_j^\# \end{pmatrix} \right\}^2 \right] \\
\Gamma [\delta u_n^i] &= \widehat{\mathbb{E}} \left[\left\{ \sum_j \left(\begin{array}{cc} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{array} \right)_{i+n,j}^{-1} \begin{pmatrix} {}^{n-1}\widetilde{S1}_j^\# \\ {}^{n-1}\widetilde{S2}_j^\# \end{pmatrix} \right\}^2 \right],
\end{aligned}$$

and the matrix of variance-covariance can be write as

$$(8.12) \quad \Gamma \left[\begin{pmatrix} \delta^n \widetilde{H} \\ \delta^n \widetilde{U} \end{pmatrix} \right] = \widetilde{M}^{-1} \widehat{\mathbb{E}} \left[\begin{pmatrix} {}^{n-1}\widetilde{S1}_j^\# \\ {}^{n-1}\widetilde{S2}_j^\# \end{pmatrix} \begin{pmatrix} {}^{n-1}\widetilde{S1}_j^\# \\ {}^{n-1}\widetilde{S2}_j^\# \end{pmatrix}^T \right] (\widetilde{M}^{-1})^T,$$

using the notations of theorem 8.4.

Proof:

This corollary is a direct consequence of theorem 8.4 and properties of sharp operator 1.2, see also the second relation of key theorem 8.3. We need only to compute explicitly the term

$$\widehat{\mathbb{E}} \left[\begin{pmatrix} {}^{n-1}\widetilde{S1}_j^\# \\ {}^{n-1}\widetilde{S2}_j^\# \end{pmatrix} \begin{pmatrix} {}^{n-1}\widetilde{S1}_j^\# \\ {}^{n-1}\widetilde{S2}_j^\# \end{pmatrix}^T \right]$$

we find

$$\begin{aligned}
(8.13) \quad \widehat{\mathbb{E}} \left[{}^{n-1}\widetilde{S1}_l^\# \quad {}^{n-1}\widetilde{S1}_m^\# \right] &= \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} \sum_{j=1}^{N_h} \sum_{q=1}^{N_u} \left\{ u_{n-1}^k u_{n-1}^q \Gamma [h_{n-1}^i, h_{n-1}^j] \right. \\
&\quad + u_{n-1}^k h_{n-1}^j \Gamma [h_{n-1}^i, u_{n-1}^q] + h_{n-1}^i u_{n-1}^q \Gamma [u_{n-1}^k, h_{n-1}^j] \\
&\quad \left. + h_{n-1}^i h_{n-1}^j \Gamma [u_{n-1}^k, u_{n-1}^q] \right\} \\
&\quad \times \int_{\Omega} \Psi_q^u \Psi_j^h \frac{\partial \phi_m^h}{\partial x} d\Omega \int_{\Omega} \Psi_k^u \Psi_i^h \frac{\partial \phi_l^h}{\partial x} d\Omega
\end{aligned}$$

$$\begin{aligned}
(8.14) \quad \widehat{\mathbb{E}} \left[{}^{n-1}\widetilde{S1}_l^\# \quad {}^{n-1}\widetilde{S2}_m^\# \right] &= \frac{1}{\Delta T} \sum_{j=1}^{N_u} \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} \left\{ u_{n-1}^k \Gamma [h_{n-1}^i, u_{n-1}^j] \right. \\
&\quad \left. + h_{n-1}^i \Gamma [u_{n-1}^k, u_{n-1}^j] \right\} \int_{\Omega} \Psi_k^u \Psi_i^h \frac{\partial \phi_l^h}{\partial x} d\Omega \int_{\Omega} \Psi_j^u \phi_m^u d\Omega \\
&\quad - \sum_{i=1}^{N_u} \sum_{k=1}^{N_u} \sum_{j=1}^{N_h} \sum_{q=1}^{N_u} \left\{ u_{n-1}^j u_{n-1}^k \Gamma [u_{n-1}^q, h_{n-1}^i] \right. \\
&\quad \left. + u_{n-1}^j h_{n-1}^i \Gamma [u_{n-1}^q, u_{n-1}^k] \right\} \\
&\quad \times \int_{\Omega} \frac{\partial \Psi_j^u \Psi_q^u}{\partial x} \phi_m^u d\Omega \int_{\Omega} \Psi_k^u \Psi_i^h \frac{\partial \phi_l^h}{\partial x} d\Omega \\
&\quad - \frac{C_f}{\cos(\alpha)} \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} \left(\frac{u_{n-1}^m}{h_{n-1}^m} \right)^2 \left\{ \frac{h_{n-1}^i}{u_{n-1}^m} \Gamma [u_{n-1}^k, u_{n-1}^m] \right. \\
&\quad \left. + \frac{u_{n-1}^k}{u_{n-1}^m} \Gamma [h_{n-1}^i, u_{n-1}^m] - \frac{u_{n-1}^k}{h_{n-1}^m} \Gamma [h_{n-1}^i, h_{n-1}^m] \right. \\
&\quad \left. - \frac{h_{n-1}^i}{h_{n-1}^m} \Gamma [u_{n-1}^k, h_{n-1}^m] \right\} \int_{\Omega} \phi_m^u d\Omega \int_{\Omega} \Psi_k^u \Psi_i^h \frac{\partial \phi_l^h}{\partial x} d\Omega
\end{aligned}$$

$$\begin{aligned}
(8.15) \quad \widehat{\mathbb{E}} \left[{}^{n-1}\widetilde{S}2_l^\# \quad {}^{n-1}\widetilde{S}2_m^\# \right] &= \frac{1}{(\Delta T)^2} \sum_{i,j=1}^{N_u} \Gamma [u_{n-1}^i, u_{n-1}^j] \int_{\Omega} \Psi_i^u \phi_l^u d\Omega \int_{\Omega} \Psi_j^u \phi_m^u d\Omega \\
&- \frac{1}{\Delta T} \sum_{i,j,q=1}^{N_u} u_{n-1}^q \Gamma [u_{n-1}^i, u_{n-1}^j] \int_{\Omega} \Psi_i^u \phi_l^u d\Omega \int_{\Omega} \frac{\partial \Psi_q^u \Psi_j^u}{\partial x} \phi_m^u d\Omega \\
&- \frac{1}{\Delta T} \sum_{i,j,k=1}^{N_u} u_{n-1}^k \Gamma [u_{n-1}^j, u_{n-1}^i] \int_{\Omega} \Psi_j^u \phi_m^u d\Omega \int_{\Omega} \frac{\partial \Psi_k^u \Psi_i^u}{\partial x} \phi_l^u d\Omega \\
&- \frac{C_f}{\Delta T \cos(\alpha)} \left(\frac{u_{n-1}^m}{h_{n-1}^m} \right)^2 \sum_{i=1}^{N_u} \left\{ \frac{2}{u_{n-1}^m} \Gamma [u_{n-1}^i, u_{n-1}^m] - \frac{1}{h_{n-1}^m} \Gamma [u_{n-1}^i, h_{n-1}^m] \right\} \\
&\times \int_{\Omega} \Psi_i^u \phi_l^u d\Omega \int_{\Omega} \phi_m^u d\Omega - \frac{C_f}{\Delta T \cos(\alpha)} \left(\frac{u_{n-1}^l}{h_{n-1}^l} \right)^2 \sum_{j=1}^{N_u} \left\{ \frac{2}{u_{n-1}^l} \Gamma [u_{n-1}^j, u_{n-1}^l] \right. \\
&- \left. \frac{1}{h_{n-1}^l} \Gamma [u_{n-1}^j, h_{n-1}^l] \right\} \int_{\Omega} \Psi_j^u \phi_l^u d\Omega \int_{\Omega} \phi_m^u d\Omega \\
&+ \sum_{i,j,k,q=1}^{N_u} u_{n-1}^j \Gamma [u_{n-1}^i, u_{n-1}^q] \int_{\Omega} \frac{\partial (\Psi_k^u \Psi_i^u)}{\partial x} \phi_l^u d\Omega \int_{\Omega} \frac{\partial (\Psi_q^u \Psi_j^u)}{\partial x} \phi_m^u d\Omega \\
&+ \frac{C_f}{\cos(\alpha)} \left(\frac{u_{n-1}^m}{h_{n-1}^m} \right)^2 \sum_{i,k=1}^{N_u} \left\{ 2 \frac{u_{n-1}^i}{u_{n-1}^m} \Gamma [u_{n-1}^k, u_{n-1}^m] - \frac{u_{n-1}^i}{h_{n-1}^m} \Gamma [u_{n-1}^k, h_{n-1}^m] \right\} \\
&\times \int_{\Omega} \frac{\partial (\Psi_k^u \Psi_i^u)}{\partial x} \phi_l^u d\Omega \int_{\Omega} \phi_m^u d\Omega + \frac{C_f}{\cos(\alpha)} \left(\frac{u_{n-1}^l}{h_{n-1}^l} \right)^2 \sum_{j,q=1}^{N_u} \left\{ 2 \frac{u_{n-1}^j}{u_{n-1}^l} \right. \\
&\times \left. \Gamma [u_{n-1}^q, u_{n-1}^l] - \frac{u_{n-1}^j}{h_{n-1}^l} \Gamma [u_{n-1}^q, h_{n-1}^l] \right\} \int_{\Omega} \frac{\partial (\Psi_q^u \Psi_j^u)}{\partial x} \phi_m^u d\Omega \int_{\Omega} \phi_l^u d\Omega \\
&+ \left(\frac{C_f}{\cos(\alpha)} \right)^2 \left(\frac{u_{n-1}^l}{h_{n-1}^l} \frac{u_{n-1}^m}{h_{n-1}^m} \right)^2 \left\{ 4 \frac{\Gamma [u_{n-1}^l, u_{n-1}^m]}{u_{n-1}^l u_{n-1}^m} - 2 \frac{\Gamma [h_{n-1}^l, u_{n-1}^m]}{h_{n-1}^l u_{n-1}^m} \right. \\
&- \left. 2 \frac{\Gamma [u_{n-1}^l, h_{n-1}^m]}{u_{n-1}^l h_{n-1}^m} + \frac{\Gamma [h_{n-1}^l, h_{n-1}^m]}{h_{n-1}^l h_{n-1}^m} \right\} \int_{\Omega} \phi_l^u d\Omega \int_{\Omega} \phi_m^u d\Omega
\end{aligned}$$

□

8.3.2 Bias of shallow water PDEs solution

In this subsection, we apply the technique of error theory using Dirichlet forms at shallow water partial differential equations after discretization. Therefore, we can compute the bias, see section 1.2, and we have the following theorem.

Theorem 8.6 (Bias of shallow water PDEs with explicit scheme)

Assume the ARB condition hold, see section 1.6. In the case of shallow water problem with solution given by corollary 8.2, the bias of the solution verifies the following system:

$$(8.16) \quad \begin{pmatrix} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{pmatrix} \begin{pmatrix} \mathcal{A} [\delta^n \widetilde{H}] \\ \mathcal{A} [\delta^n \widetilde{U}] \end{pmatrix} = \begin{pmatrix} \mathcal{A} [{}^{n-1} \widetilde{S1}] \\ \mathcal{A} [{}^{n-1} \widetilde{S2}] \end{pmatrix}$$

where

$$\begin{aligned} \mathcal{A} [{}^{n-1} \widetilde{S1}_j] &= \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} \{ \mathcal{A} [h_{n-1}^i] u_{n-1}^k + h_{n-1}^i \mathcal{A} [u_{n-1}^k] + \Gamma [h_{n-1}^i, u_{n-1}^k] \} \\ &\quad \times \int_{\Omega} \Psi_k^u \frac{\partial \phi_j^h}{\partial x} \Psi_i^h d\Omega \\ \mathcal{A} [{}^{n-1} \widetilde{S2}_j] &= \left\{ - \sum_{i=1}^{N_u} \sum_{k=1}^{N_u} \{ \mathcal{A} [u_{n-1}^i] u_{n-1}^k + \mathcal{A} [u_{n-1}^k] u_{n-1}^i + \Gamma [u_{n-1}^i, u_{n-1}^k] \} \right. \\ &\quad \times \int_{\Omega} \Psi_k^u \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \left. \right\} + \frac{1}{\Delta T} \sum_{i=1}^{N_u} \mathcal{A} [u_{n-1}^i] \int_{\Omega} \Psi_i^u \phi_j^u d\Omega \\ &\quad + \frac{1}{\cos(\alpha)} \frac{C_f}{2 h_{n-1}^j} (u_{n-1}^j)^2 \int_{\Omega} \phi_j^u d\Omega \\ &\quad \times \left\{ 2 \frac{\mathcal{A} [u_{n-1}^j]}{u_{n-1}^j} - \frac{\mathcal{A} [h_{n-1}^j]}{h_{n-1}^j} + \frac{\Gamma [u_{n-1}^j]}{(u_{n-1}^j)^2} - \frac{\Gamma [h_{n-1}^j]}{(h_{n-1}^j)^2} - 2 \frac{\Gamma [u_{n-1}^j, h_{n-1}^j]}{h_{n-1}^j u_{n-1}^j} \right\} \end{aligned}$$

Also this theorem comes from key theorem 8.3.

8.3.3 Algorithm

In this subsection, we analyze previous results and we propose an algorithm to evaluate the solution of shallow water partial differential equations, its variance-covariance, and its bias at the same time. We start with some remarks:

Remark 8.1 (Numerical computation) The computation of the solution, the sharp, the variance and finally the bias require the solution of an equations system, see equations 8.8, 8.10, 8.12

and 8.16, but all these systems depend on the same matrix

$$\widetilde{M} = \begin{pmatrix} \widetilde{D1} & \widetilde{C1} \\ \widetilde{C2} & \widetilde{D2} \end{pmatrix},$$

and this matrix is the same at each time step. Therefore, it can be numerically attractive to compute the inverse of matrix \widetilde{M} , this computation is simplified by the fact that the matrix \widetilde{M} is a block diagonal matrix and an accurate choice of test functions permits to restrict the number of diagonals different to zero.

A second remark shows the importance of bias study for a non-linear partial differential equation.

Remark 8.2 (Bias induced by non-linearities) *Theorem 8.6 shows that a non-linear partial differential equations generates compulsorily a bias, even if starting and boundary conditions are unbiased. The presence of a variance and the non-linearity of shallow water partial differential equations cooperate to generate a bias, with a stochastic root, from an unbiased boundary condition.*

We conclude this section with the structure of a proposed algorithm in explicit scheme case:

Algorithm 8.7 (Numerical analysis of a PDE)

1. *Given a non linear PDE, discretize it in time and space, thanks to a Galerkin variational approach and find a system of type 8.8;*
2. *compute, numerically, the matrix \widetilde{M} and its inverse;*
3. *define an error structure on each knot of the mesh, see section 1.2;*
4. *compute the solution of the system using the inverse of matrix \widetilde{M} computed at step 2 and store it;*
5. *compute the variance-covariance matrix of the solution, thanks to relation 8.12 and store it.*
6. *compute the bias of the solution, thanks to relation 8.16 and store it;*
7. *iterate points 4, 5 and 6 for each time of discretization t_n .*

8.4 Conclusion

In this chapter we have studied the sensitivity of the solution of a non-linear partial differential equation with respect to the presence of an uncertainty on starting and boundary conditions. In particular, we have dealt with the problem to define, rigorously, the uncertainty on the frontier and to define a procedure, numerically cheap but coherent and theoretically rich, in order to estimate the transmission of this uncertainty to PDE solution.

We have proposed to use the language of error theory using Dirichlet forms, since it combines the forces of the classical approach, using Gateaux derivatives, and the probabilistic one, with the possibility to evaluate the bias and correlations effects.

We have described our procedure with a classical example, i.e. the shallow water partial differential equations. Clearly, this approach works for all PDE, but shows its power when the PDE is non-linear. In this case, a numerical study is the only practicable way to find the solution. Our approach follows this strategy and we have shown that, if the discretization scheme is explicit on time, it exists an algorithm that permits to evaluate the PDE solution and, at the same time, step by step, the variance-covariance matrix and the bias of its error. The algorithm is relatively cheap, since the solution, the variance-covariance matrix and the bias are solutions of three different linear systems, but characterized by the same matrix.

Appendix 8.A Generalization with implicit scheme

In this appendix, we study how previous results are modified when an implicit scheme in time is used. This case is more complicated in both theoretical and practical point of view. As a matter of fact, when implicit schemes are concerned, the matrix of system (8.7) depends on the solution itself at the previous time step. Therefore, this matrix becomes erroneous. In order to study this case, we prove a more general key theorem. Then, we analyze variance and bias of shallow water partial differential equations.

8.A.1 Key theorem in general case

In this section, we show a generalization of theorem 8.3, when the matrix is erroneous and we highlight a drawback in the study of error diffusion when implicit schemes are concerned.

Theorem 8.8 (Key theorem, a generalisation)

Let X be a vector in \mathbb{R}^d , the only solution of a linear system $MX = B$, where M is an invertible matrix and B another vector. Suppose that it exists an error structure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma)$ that admits a sharp operator and suppose that the vector B and the matrix M are both erroneous. Then, the vector X is erroneous and we have the following relations:

$$\begin{aligned}
 & a) \quad MX^\# + M^\# X = B^\# \\
 (8.17) \quad & b) \quad (M^{-1})^\# = -M^{-1} M^\# M^{-1} \\
 & c) \quad X^\# = M^{-1} B^\# - M^{-1} M^\# M^{-1} B
 \end{aligned}$$

$$\begin{aligned}
\Gamma [X, X^T] &= M^{-1} \Gamma [B, B^T] (M^{-1})^T \\
&\quad - M^{-1} \widehat{\mathbb{E}} \left[B^\# B^T (M^{-1})^T (M^\#)^T \right] (M^{-1})^T \\
(8.18) \quad &\quad + M^{-1} \widehat{\mathbb{E}} \left[M^\# M^{-1} B B^T (M^{-1})^T (M^\#)^T \right] (M^{-1})^T \\
&\quad - M^{-1} \widehat{\mathbb{E}} \left[M^\# M^{-1} B (B^\#)^T \right] (M^{-1})^T
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}[X] &= M^{-1} \mathcal{A}[B] - M^{-1} \mathcal{A}[M] M^{-1} B - M^{-1} \widehat{\mathbb{E}} [M^\# M^{-1} B^\#] \\
(8.19) \quad &\quad + M^{-1} \widehat{\mathbb{E}} [M^\# M^{-1} M^\#] M^{-1} B
\end{aligned}$$

where

$$M^\# = \begin{pmatrix} (M_{1,1})^\# & \cdots & (M_{1,k})^\# \\ \vdots & \ddots & \vdots \\ (M_{k,1})^\# & \cdots & (M_{k,k})^\# \end{pmatrix} \quad \text{and} \quad (M^{-1})^\# = \begin{pmatrix} (M_{1,1}^{-1})^\# & \cdots & (M_{1,k}^{-1})^\# \\ \vdots & \ddots & \vdots \\ (M_{k,1}^{-1})^\# & \cdots & (M_{k,k}^{-1})^\# \end{pmatrix}$$

and similar notation for vectors.

Proof: Since X is the only solution of the considered system, matrix M admits an inverse M^{-1} . Therefore, we have $X = M^{-1}B$. Relation (a) in (8.17) is a direct consequence of sharp definition 1.2 and the linearity of matrix product.

In order to prove identity (b) in (8.17), we recall that the inverse verifies the property $M^{-1} M = Id$ and the fact that this identity can be read as a system of k^2 inner products, where k denotes the row's number of matrix M . As a matter of fact, all these inner products are equal to 0 or 1; as a consequence, $\langle M_i (M^{-1})_j^T \rangle$ is non-erroneous for all i and j , where M_i denotes the i -row of matrix M . Thus, we can apply the sharp operator on $\langle M_i (M^{-1})_j^T \rangle$ and we find $\langle M_i^\# (M^{-1})_j^T \rangle + \langle M_i \left[(M^{-1})_j^T \right]^\# \rangle = 0$. Now, we can reconstruct the relation between matrix $M^\#$ and $(M^{-1})^\#$, i.e. $M^\# M^{-1} + M (M^{-1})^\# = 0$, where 0 is the null matrix. Therefore, we find second identity in (8.17).

Relation (c) in (8.17) follows easily by (b). Equation (8.18) is a consequence on sharp definition and relation (b) in (8.17); while equation (8.19) follows easily by relation (8.18) and the chain rule of bias, see relation (1.8).

□

Remark 8.3 Results of theorem 8.8 are theoretically strong but numerically unserviceable, since matrix $M^\#$ is in fact a functional one, i.e. each element of matrix $M^\#$ is a function of two variables, $\{M_{i,j}\}_{\forall i,j}$ and a second variable, represented by ω , defined on a probability space.

This fact is an important drawback, especially when we search to compute variances and biases using formulae (8.18) and (8.19). This drawback is worsened by the non-commutation between matrix in formulae (8.18) and (8.19).

8.A.2 Variance of shallow water PDEs solution with implicit scheme

In this subsection we study the sharp and the variance of shallow water PDEs solution, when an implicit scheme in time is considered. We propose also a strategy to avoid the drawback explained in the previous subsection.

Theorem 8.9 (Sharp of shallow water PDE solution with implicit scheme)

In the case of shallow water problem with solution given by theorem 8.1, the sharp of the solution verifies the following system:

$$(8.20) \quad \begin{pmatrix} {}^{n-1}D1 & {}^{n-1}C1 \\ {}^{n-1}C2 & {}^{n-1}D2 \end{pmatrix} \begin{pmatrix} \delta^n H^\# \\ \delta^n U^\# \end{pmatrix} = \begin{pmatrix} {}^{n-1}S1^\# \\ {}^{n-1}S2^\# \end{pmatrix} - \begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix} \begin{pmatrix} \delta^n H \\ {}^n U \end{pmatrix}$$

where $\delta^n H$, ${}^n U$, ${}^{n-1}D1$, ${}^{n-1}D2$, ${}^{n-1}C1$, ${}^{n-1}C2$, ${}^{n-1}S1$ and ${}^{n-1}S2$ are the same as in the theorem 8.1, $\delta^n H^\#$ is the vector of components $(\delta h_n^i)^\#$, ${}^n U^\#$ is the vector of components $(u_n^i)^\#$. Matrix ${}^{n-1}D1^\#$, ${}^{n-1}D2^\#$, ${}^{n-1}C1^\#$, and ${}^{n-1}C2^\#$ are given by

$$\begin{aligned} {}^{n-1}D1_{j,i}^\# &= \theta_h \int_{\Omega} \left(\sum_{k=1}^{N_u} (u_{n-1}^k)^\# \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \\ {}^{n-1}D2_{j,i}^\# &= \theta_u \int_{\Omega} \left(\sum_{k=1}^{N_u} (u_{n-1}^k)^\# \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \\ &\quad + \frac{\delta_{i,j}}{\cos(\alpha)} \frac{C_f}{2h_{n-1}^i} \left[(u_{n-1}^i)^\# - \frac{(h_{n-1}^i)^\#}{h_{n-1}^i} u_{n-1}^i \right] \int_{\Omega} \phi_i^u d\Omega \\ {}^{n-1}C1_{j,i}^\# &= -\theta_u \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} (h_{n-1}^k)^\# \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \\ {}^{n-1}C2_{j,i}^\# &= 0 \end{aligned}$$

and vectors ${}^{n-1}S1^\#$ and ${}^{n-1}S2^\#$ are given by

$$\begin{aligned}
{}^{n-1}S1_j^\# &= (\theta_h - 1) \left\{ \sum_{i=1}^{N_h} (h_{n-1}^i)^\# \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \right\} \\
&+ (\theta_h - 1) \left\{ \sum_{i=1}^{N_h} h_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} (u_{n-1}^k)^\# \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \right\} \\
&- (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} (u_{n-1}^i)^\# \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} h_{n-1}^k \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \right\} \\
&- (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} (h_{n-1}^k)^\# \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \right\} \\
{}^{n-1}S2_j^\# &= \frac{\sum_{i=1}^{N_u} (u_{n-1}^i)^\# \int_{\Omega} \Psi_i^u \phi_j^u}{\Delta T} \\
&+ (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} (u_{n-1}^i)^\# \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \right\} \\
&+ (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} (u_{n-1}^k)^\# \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \right\}
\end{aligned}$$

Proof: This theorem is a direct consequence of key theorem 8.8 and a computation of sharp of each element of matrix $D1$, $D2$, $C1$ and $C2$, and of vectors $S1$ and $S2$.

□

Corollary 8.10 (Variance of shallow water PDEs solution with implicit scheme)

The variance of the error of shallow water PDEs solution is given by iterations through the following system.

$$\begin{aligned}
(8.21) \quad \Gamma \left[\begin{pmatrix} \delta^n H^\# \\ \delta^n U^\# \end{pmatrix}, \begin{pmatrix} \delta^n H^\# \\ \delta^n U^\# \end{pmatrix}^T \right] &= M^{-1} \left\{ \Gamma \left[\begin{pmatrix} {}^{n-1}S1^\# \\ {}^{n-1}S2^\# \end{pmatrix}, \begin{pmatrix} {}^{n-1}S1^\# \\ {}^{n-1}S2^\# \end{pmatrix}^T \right] \right. \\
&\quad - \widehat{\mathbb{E}} \left[\begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix} M^{-1} \right. \\
&\quad \times \begin{pmatrix} {}^{n-1}S1 \\ {}^{n-1}S2 \end{pmatrix} \begin{pmatrix} {}^{n-1}S1^\# \\ {}^{n-1}S2^\# \end{pmatrix}^T \left. \right] \\
&\quad - \widehat{\mathbb{E}} \left[\begin{pmatrix} {}^{n-1}S1^\# \\ {}^{n-1}S2^\# \end{pmatrix} \begin{pmatrix} {}^{n-1}S1 \\ {}^{n-1}S2 \end{pmatrix}^T \right. \\
&\quad \times (M^{-1})^T \begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix}^T \left. \right] \\
&\quad + \widehat{\mathbb{E}} \left[\begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix} M^{-1} \right. \\
&\quad \times \begin{pmatrix} {}^{n-1}S1 \\ {}^{n-1}S2 \end{pmatrix} \begin{pmatrix} {}^{n-1}S1 \\ {}^{n-1}S2 \end{pmatrix}^T (M^{-1})^T \\
&\quad \left. \left. \begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix}^T \right] \right\} (M^{-1})^T
\end{aligned}$$

where

$$M = \begin{pmatrix} {}^{n-1}D1 & {}^{n-1}C1 \\ {}^{n-1}C2 & {}^{n-1}D2 \end{pmatrix}$$

Proof:

This theorem is a direct consequence of key theorem 8.8 and theorem 8.9.

□

The result of corollary 8.10 is theoretically strong but numerically unserviceable, matrix $D1^\#$, $D2^\#$, $C1^\#$ and $C2^\#$ are functional matrix, i.e. each element of matrix $D1^\#$, $D2^\#$, $C1^\#$ and $C2^\#$ is a function of two variables, the vector $\{u_{n-1}^i, h_{n-1}^i\}_{\forall i}$ and a second variable, represented by ω , defined on a probability space. In order to keep the favorable structure of matrix, we make the following hypothesis.

Hypothesis 8.1 (Orthogonality) *Each sharp $\{u_n^i\}_{0 \leq i \leq N_u}$ or $\{h_n^i\}_{0 \leq i \leq N_h}$ admits a decomposition of type:*

$$\{u_n^i\}^\# = f_i^{(u,n)}(\{u_n^j\}_{0 \leq j \leq N_u}) g_i^{(u,n)}(\widehat{\omega})$$

where $f_i^{(u,n)}(\cdot)$ is a function on the original parameter space and $g_i^{(u,n)}(\cdot)$ is a function on the probability space $\{\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}\}$, a similar relation is verified by $\{h_n^i\}^\#$

We study the impact of this hypothesis on the sharp chain rule:

Theorem 8.11 (Preservation of orthogonality)

If we consider the sharp chain rule defined at theorem 8.9 and we assume that hypothesis 8.1 is hold at boundary conditions $\{u_0^i\}_{0 \leq i \leq N_u}$ and $\{h_0^i\}_{0 \leq i \leq N_h}$; then it is verified for any discretized time t_n too.

Proof:

We proof the theorem by induction: we assume the property 8.1 hold at time t_{n-1} , then we remark that each element of the matrix can be separate into a sum of terms:

$$\begin{aligned} {}^{n-1}D1_{j,i}^\# &= \sum_{k=1}^{N_u} g_k^{(u,n-1)}(\widehat{\omega}) \Theta_h \int_{\Omega} f_k^{(u,n)}(\{u_{n-1}^j\}_{0 \leq j \leq N_u}) \Psi_k^u \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \\ &= \sum_{k=1}^{N_u} g_k^{(u,n-1)}(\widehat{\omega}) {}^{n-1}D1_{k,j,i}^{(n-1)} \\ {}^{n-1}D2_{j,i}^\# &= \sum_{k=1}^{N_u} g_k^{(u,n-1)}(\widehat{\omega}) {}^{n-1}D2_{k,j,i}^{(n-1)} \\ {}^{n-1}C1_{j,i}^\# &= \sum_{k=1}^{N_u} g_k^{(u,n-1)}(\widehat{\omega}) {}^{n-1}C1_{k,j,i}^{(n-1)} \\ {}^{n-1}C2_{j,i}^\# &= 0 \end{aligned}$$

We remark that we can defined three-index tensor ${}^{n-1}D1_{k,j,i}$, ${}^{n-1}D2_{k,j,i}$, ${}^{n-1}C1_{k,j,i}$, and ${}^{n-1}C2_{k,j,i}$, such that

$$\begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix} = \sum_{k=1}^{N_u+N_h} g_k^{(x,n-1)} \begin{pmatrix} {}^{n-1}D1_k & {}^{n-1}C1_k \\ {}^{n-1}C2_k & {}^{n-1}D2_k \end{pmatrix}.$$

We make the same work with vectors ${}^{n-1}S1^\#$ and ${}^{n-1}S2^\#$ and we have defined matrix ${}^{n-1}S1_{k,j}$ and ${}^{n-1}S2_{k,j}$, such that

$$\begin{aligned} {}^{n-1}S1_j^\# &= \sum_{k=1}^{N_u+N_h} g_k^{(x,n-1)}(\widehat{\omega}) {}^{n-1}S1_{k,j} \\ {}^{n-1}S2_j^\# &= \sum_{k=1}^{N_u+N_h} g_k^{(x,n-1)}(\widehat{\omega}) {}^{n-1}S2_{k,j} \end{aligned}$$

Now we recall that the sharp at time t_{n-1} is given by the implied relation

$$\begin{pmatrix} {}^{n-1}D1 & {}^{n-1}C1 \\ {}^{n-1}C2 & {}^{n-1}D2 \end{pmatrix} \begin{pmatrix} \delta^n H^\# \\ \delta^n U^\# \end{pmatrix} = \sum_{k=1}^{N_u+N_h} g_k^{(x,n-1)}(\hat{\omega}) \begin{pmatrix} {}^{n-1}S1_k \\ {}^{n-1}S2_k \end{pmatrix} \\ - \sum_{k=1}^{N_u+N_h} g_k^{(x,n-1)}(\hat{\omega}) \begin{pmatrix} {}^{n-1}D1_k & {}^{n-1}C1_k \\ {}^{n-1}C2_k & {}^{n-1}D2_k \end{pmatrix} \begin{pmatrix} \delta^n H \\ \delta^n U \end{pmatrix}.$$

Finally, we can inverse the matrix and switch the order of summation and remark that

$$\begin{pmatrix} \delta^n H^\# \\ \delta^n U^\# \end{pmatrix} = \sum_{k=1}^{N_u+N_h} g_k^{(x,n-1)}(\hat{\omega}) \begin{pmatrix} {}^{n-1}D1 & {}^{n-1}C1 \\ {}^{n-1}C2 & {}^{n-1}D2 \end{pmatrix}^{-1} \left[\begin{pmatrix} {}^{n-1}S1_k \\ {}^{n-1}S2_k \end{pmatrix} \right. \\ \left. - \begin{pmatrix} {}^{n-1}D1_k & {}^{n-1}C1_k \\ {}^{n-1}C2_k & {}^{n-1}D2_k \end{pmatrix} \begin{pmatrix} \delta^n H \\ \delta^n U \end{pmatrix} \right].$$

□

Now, it is clear that property 8.1 is very useful when we search to simplify equation 8.21. As a matter of fact, orthogonality divides the dependence on $\hat{\omega}$ with respect to matrix structure and the expectation $\widehat{\mathbb{E}}$ act only on $g_k^{(x,n-1)}(\hat{\omega})$, each tensor and matrix goes out of expectation. However, we pay dear for this split, each matrix becomes a tensor and vectors become matrix; finally, a last addition is needed.

8.A.3 Bias of shallow water PDEs solution with implicit scheme

Theorem 8.12 (Bias of shallow water PDEs solution with implicit scheme)

Assume the ARB condition hold, see section 1.6. In the case of shallow water problem with solution given by 8.1, the bias of the solution verifies the following system:

$$(8.22) \quad \begin{pmatrix} {}^{n-1}D1 & {}^{n-1}C1 \\ {}^{n-1}C2 & {}^{n-1}D2 \end{pmatrix} \begin{pmatrix} \mathcal{A}[\delta^n H] \\ \mathcal{A}[\delta^n U] \end{pmatrix} = \begin{pmatrix} \mathcal{A}[{}^{n-1}S1] \\ \mathcal{A}[{}^{n-1}S2] \end{pmatrix} - \begin{pmatrix} \mathcal{A}[{}^{n-1}D1] & \mathcal{A}[{}^{n-1}C1] \\ \mathcal{A}[{}^{n-1}C2] & \mathcal{A}[{}^{n-1}D2] \end{pmatrix} \begin{pmatrix} \delta^n H \\ {}^n U \end{pmatrix} \\ - \widehat{\mathbb{E}} \left[\begin{pmatrix} {}^{n-1}D1^\# & {}^{n-1}C1^\# \\ {}^{n-1}C2^\# & {}^{n-1}D2^\# \end{pmatrix} \begin{pmatrix} \delta^n H^\# \\ {}^n U^\# \end{pmatrix} \right]$$

where $\delta^n H$, ${}^n U$, ${}^{n-1}D1$, ${}^{n-1}D2$, ${}^{n-1}C1$, ${}^{n-1}C2$, ${}^{n-1}S1$ and ${}^{n-1}S2$ are the same as in the theorem 8.1; $\delta^n H^\#$, ${}^n U^\#$, ${}^{n-1}D1^\#$, ${}^{n-1}D2^\#$, ${}^{n-1}C1^\#$, ${}^{n-1}C2^\#$ are the same as in the theorem 8.9. $\mathcal{A}[\delta^n H]$ is the vector of components $\mathcal{A}[\delta h_n^i]$ and $\mathcal{A}[{}^n U]$ is the vector of components $\mathcal{A}[u_n^i]$. Finally, matrix $\mathcal{A}[{}^{n-1}D1]$, $\mathcal{A}[{}^{n-1}D2]$, $\mathcal{A}[{}^{n-1}C1]$ and $\mathcal{A}[{}^{n-1}C2]$ are given by

$$\begin{aligned}
\mathcal{A} [{}^{n-1}D1]_{j,i} &= \theta_h \int_{\Omega} \left(\sum_{k=1}^{N_u} \mathcal{A} [u_{n-1}^k] \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \\
\mathcal{A} [{}^{n-1}D2]_{j,i} &= \theta_u \int_{\Omega} \left(\sum_{k=1}^{N_u} \mathcal{A} [u_{n-1}^k] \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \\
&\quad + \frac{\delta_{i,j}}{\cos(\alpha)} \frac{C_f}{2h_{n-1}^i} \left[\mathcal{A} [u_{n-1}^i] - \frac{\mathcal{A} [h_{n-1}^i]}{h_{n-1}^i} u_{n-1}^i \right] \int_{\Omega} \phi_i^u d\Omega \\
&\quad + \frac{\delta_{i,j}}{\cos(\alpha)} \frac{C_f}{2(h_{n-1}^i)^2} \left\{ 2 \frac{\Gamma [h_{n-1}^i]}{h_{n-1}^i} - \Gamma [u_{n-1}^i, h_{n-1}^i] \right\} \int_{\Omega} \phi_i^u d\Omega \\
\mathcal{A} [{}^{n-1}C1]_{j,i} &= -\theta_u \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} \mathcal{A} [h_{n-1}^k] \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \\
\mathcal{A} [{}^{n-1}C2]_{j,i} &= 0
\end{aligned}$$

and vectors $\mathcal{A} [{}^{n-1}S1]$ and $\mathcal{A} [{}^{n-1}S2]$ are given by

$$\begin{aligned}
\mathcal{A} [{}^{n-1}S1]_j &= (\theta_h - 1) \left\{ \sum_{i=1}^{N_h} \mathcal{A} [h_{n-1}^i] \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \right\} \\
&\quad + (\theta_h - 1) \left\{ \sum_{i=1}^{N_h} h_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} \mathcal{A} [u_{n-1}^k] \Psi_k^u \right) \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \right\} \\
&\quad - (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} \mathcal{A} [u_{n-1}^i] \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} h_{n-1}^k \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \right\} \\
&\quad - (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \Psi_i^u \left[\sum_{k=1}^{N_h} \mathcal{A} [h_{n-1}^k] \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} \right] d\Omega \right\} \\
&\quad + (\theta_h - 1) \sum_{i=1}^{N_h} \sum_{k=1}^{N_u} \left\{ \int_{\Omega} \Psi_k^u \frac{\partial \Psi_i^h}{\partial x} \phi_j^h d\Omega \right\} \Gamma [h_{n-1}^i, u_{n-1}^k] \\
&\quad - (\theta_u - 1) \sum_{i=1}^{N_u} \sum_{k=1}^{N_h} \left\{ \int_{\Omega} \Psi_i^u \frac{\partial (\Psi_k^h \phi_j^h)}{\partial x} d\Omega \right\} \Gamma [h_{n-1}^k, u_{n-1}^i]
\end{aligned}$$

$$\begin{aligned}
\mathcal{A} [{}^{n-1}S2]_j &= \frac{\sum_{i=1}^{N_u} \mathcal{A} [u_{n-1}^i] \int_{\Omega} \Psi_i^u \phi_j^u}{\Delta T} \\
&+ (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} \mathcal{A} [u_{n-1}^i] \int_{\Omega} \left(\sum_{k=1}^{N_u} u_{n-1}^k \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \right\} \\
&+ (\theta_u - 1) \left\{ \sum_{i=1}^{N_u} u_{n-1}^i \int_{\Omega} \left(\sum_{k=1}^{N_u} \mathcal{A} [u_{n-1}^k] \Psi_k^u \right) \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \right\} \\
&(\theta_u - 1) \sum_{i=1}^{N_u} \sum_{k=1}^{N_u} \left\{ \int_{\Omega} \Psi_k^u \frac{\partial \Psi_i^u}{\partial x} \phi_j^u d\Omega \right\} \Gamma [u_{n-1}^k, u_{n-1}^i]
\end{aligned}$$

Proof: Theorem 8.12 follows the theorem 8.8, we need just to compute the bias of matrix ${}^{n-1}D1$, ${}^{n-1}D2$, ${}^{n-1}C1$, ${}^{n-1}C2$, ${}^{n-1}S1$, and ${}^{n-1}S2$. A problem exists when we analyze the bias of ${}^{n-1}D2$, due to the presence of a term h_{n-1}^i in the denominator. In order to exceed this problem, we assume that the height of the canal is strictly bigger than a positive value; then all functions in approached solution of Saint Venant PDEs are Lipschitzian with respect to erroneous variables u_{n-1}^i and h_{n-1}^i . Hence the relation of theorem 8.1 is an implicit but Lipschitzian function. The proof ends with the computation of the bias on each element.

□

Chapter 9

Sensitivity of Non-Linear PDEs: Continuum Approach

In this chapter, we analyze the same problem introduced in chapter 8, but using a different approach. We prove that the sharp of the theoretical solution of the shallow water partial differential equations verifies a system of two linear partial differential equations depending on the solution of the former PDE itself. We analyze the behavior of this system and we study some particular cases. Under some hypotheses, we show that the carré du champ verifies a incomplete system of partial differential equations too. We give its numerical solution by means of its Laplace transform.

Finally, we analyze the bias of the theoretical solution of the shallow water PDE; we prove that it verifies a system of two linear partial differential equations depending on the solution of the former PDE itself and its variance. We give its numerical solution by means of its Laplace transform.

Let us mention the analogy of this approach with the celebrated methods of Malliavin and Bismut concerning SDE. The argument of Malliavin is based on the SDE satisfied by the Malliavin derivative (similar to our sharp) and that of Bismut on the SDE satisfied by the derivative with respect to the starting condition. Here we will find PDEs satisfied by the sharp, by the carré du champ and by the bias.

9.1 Introduction

In chapter 8, we have introduced the shallow water problem and showed a numerical method to study the sensitivity of the solution with respect to the starting and boundary conditions. This approach is not unique: various strategies are possible and we can summarized them into two classes, following a classification commonly used in mechanical engineering, see Choi and Kim [10]:

1. discrete approach;
2. continuum approach.

Our analysis in chapter 8 belongs to the first method, while we exploit the second one in this chapter.

In the discrete method, sensitivity is obtained by taking derivatives (i.e. the sharp operator, in our case) of the discrete governing equations. If the derivative is obtained analytically using the discretized matrix, see equation (8.7), the method is named analytical, while if it is obtained using a finite difference method, the approach is named semianalytical.

In the continuum approach, the sharp operator acts on the governing equations before they are discretized. If it is possible to solve the shallow water problem and the sensitivity equations as a continuum problem, then we would have a continuum-continuum method. However, shallow water equations cannot be solved analytically. Thus, we have to discretize our problem; the continuum sharp equation can be solved using the same discretisation mesh used to solve the shallow water equations, see theorem 8.1. This method is called continuum-discrete method.

Since the sharp acts on the shallow water equations before any discretisation takes place, this method provides theoretically more accurate results than the discrete approach, introduced in chapter 8. There are several primary advantages of the continuum approach to sensitivity analysis:

- A rigorous mathematical theory is obtained, without the uncertainty associated with finite-dimensional approximation errors, and
- Explicit partial differential equations for sharp and bias are obtained, and we can find an incomplete system of partial differential equation for variance.

Sharp expression is obtained as a solution of a partial differential equation depending on the solution of shallow water problem and boundary conditions. Therefore, the sharp is obtained in the form of (convolution) integrals and Picard's series, with integrands written in terms of the solution of the former problem and other physical quantities. Since exact solutions to the shallow water problem are generally unknown, an approximation method such as the finite element one is used to evaluate these terms. When finite element analysis is used to compute the solution of a problem, then the same discretisation method and the same mesh have to be used to evaluate the sensitivity in continuum-discrete method, see Choi and Kim [10], pages 21-30.

The key difference between discrete-analytical approach, detailed in chapter 8, and continuum-discrete approach, analyzed later on this chapter, is that the first one provides the exact sensitivity of an approximate model, the second one an approximate sensitivity of the exact model instead. The continuum approach uses the theoretical partial differential equations verified by the sharp, while discrete approach relies on nodal conditions for such information.

A second interesting analysis comes from the bias. In mechanical engineering, there exists a method to increase the accuracy of the approximation of the sensitivity analysis. First-order sensitivity analysis is a linear approximation of the perturbation of a solution in terms of the former problem itself. It is plain that an high-order approximation increases the accuracy of the approximation. In particular, engineers use second-order sensitivity analysis, where the analyzed function ψ depending on structural design \bar{u} is expanded into a Taylor series up to quadratic terms, as

$$\psi(\bar{u} + \delta\bar{u}) \approx \psi(u) + \langle \bar{\nabla}_u \psi, \delta\bar{u} \rangle + \frac{1}{2} (\delta\bar{u})^T \mathbf{H} \delta\bar{u}$$

where $\delta\bar{u}$ is the generic increment of structural design \bar{u} and \mathbf{H} denotes the Hessian matrix of function ψ with respect to vector \bar{u} . In mechanical engineering, second-order sensitivity informa-

tion is very useful for an optimization algorithm, since quadratic forms have good properties for convergence, for instance see Ern and Guermond [13] chapter 3.

It is plain that second-order sensitivity analysis has an equivalent operator in error theory using Dirichlet forms, i.e. the bias. After analyzing the variance of the solution of shallow water equation, we can study how the bias operator acts on the solution and the same two approaches appear, i.e. discrete and continuum methods. However, computing bias of shallow water solution results in quite large computational costs.

This chapter is organized as follows:

In section 2, we resume the shallow water problem introduced in chapter 8. Section 3 is dedicated to the analysis of the variance of the shallow water equations. We show that the sharp of the shallow water theoretical solution verifies a system of two linear PDEs. Under some hypotheses, we prove that the gamma operator of the shallow water solution verifies a incomplete system of PDEs; we show that the PDE verified by the variance of the velocity is autonomous and we write the approximated solution by means of its Laplace transform. Section 4 shows the same study using the characteristic form of shallow water PDE. In section 5, we analyze the bias of the solution; we show that it verifies a system of linear partial differential equations with the same generator of the PDEs verified by the sharp. Using the same hypotheses introduced in section 3, we show that the bias of the velocity can be solved numerically by means of its Laplace transform. Finally, section 6 resumes and concludes.

9.2 Shallow water equations and uncertainties

In order to describe the continuum approach using error theory, we detail it on a particular case, the shallow water problem. The example is the same introduced in chapter 8.

We consider the following partial differential equation, called 1-Dim shallow water PDEs:

$$(9.1) \quad \begin{cases} \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = -h \frac{\partial u}{\partial x} + q_l \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial Z_s}{\partial x} + F + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) + (U_S - u) \frac{q_l}{h} \end{cases}$$

where

- t is the time;
- x is the space variable;
- $h(t, x)$ is the depth of the canal;
- $u(t, x)$ is the velocity of flow;
- q_l is the lateral rate of flow from the layer;
- g is the acceleration of gravity;

- $Z_s(t, x)$ is the height of the free-surface side;
- $F(t, x)$ summarizes a local average of all other external forces excepted the gravity;
- ν_e is an effective diffusion coefficient that takes into account the dispersion and turbulence viscosity;
- U_S is speed of water coming from the layer;
- $u(0, x)$ and $h(0, x)$ denote the starting conditions assumed known but afflicted by uncertainties;
- $u(t, x_0)$, $u(t, x_1)$, $h(t, x_0)$ and $h(t, x_1)$ denote the boundary conditions assumed known but always afflicted by uncertainties.

In our settings, we assume that the starting and boundary conditions are erroneous. That is we assume that it exists an error structure

$$\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \mathbb{D}, \Gamma \right),$$

such that the starting and boundary conditions belong to \mathbb{D} and we assume known:

- the variance-covariance of the starting conditions, i.e. $\Gamma[u(0, x), u(0, y)]$, $\Gamma[h(0, x), h(0, y)]$ and $\Gamma[h(0, x), u(0, y)]$, for all x, y belong to $[x_0, x_1]$;
- the variance-covariance of the two boundary conditions, i.e. $\Gamma[u(t, x_i), u(s, x_j)]$, $\Gamma[h(t, x_i), h(s, x_j)]$, $\Gamma[u(t, x_i), h(s, x_j)]$, for all t, s belong to $[0, T]$, and i and j equal to 0 or 1;
- the covariance between the boundary and starting conditions, e.g. $\Gamma[u(t, x_i), h(0, x)]$.

We suppose also known:

- the biases of the starting conditions, i.e. $\mathcal{A}[u(0, x)]$ and $\mathcal{A}[h(0, x)]$, for all x belongs to $[x_0, x_1]$; and
- the biases of the two boundary conditions, i.e. $\mathcal{A}[u(t, x_i)]$ and $\mathcal{A}[h(t, x_i)]$, for all t belongs to $[0, T]$, and i and j equal to 0 or 1.

Finally, we assume that our error structure admits a sharp operator denoted $(\)^\#$.

9.3 Variance of the shallow water problem

In this section, we consider the PDEs system (9.1) and we prove the existence of an associated system of two partial differential equations for the sharp under some hypothesis. Then we analyze this system of PDEs in order to find a canonical form, we prove that our system cannot admit a decomposition into two autonomous PDEs. For a special choice of our parameters, we prove that the PDE for the sharp of the velocity $u^\#$ is autonomous but the partial equation for the sharp of depth $h^\#$ depends on the sharp of u . We show that the related variance-covariance of h and u solves two partial differential equations, but this system is not complete.

9.3.1 PDE verified by the sharp operator

In this section, we analyze the sharp of the theoretical solution of problem (9.1).

Theorem 9.1 (PDE verified by the sharp)

Let $(u, h)_{(t,x)}$ be the theoretical solution of problem (9.1). Suppose that the starting and boundary conditions are erroneous, and the related error structure admits a sharp operator denoted $(\cdot)^\#$. Suppose that the theoretical solution of problem (9.1) is known and it is differentiable; then the sharp $(u^\#, h^\#)_{(t,x)}$ verifies the following PDEs system, if it exists and is differentiable a.e..

$$(9.2) \quad \left\{ \begin{array}{l} \frac{\partial h^\#}{\partial t} + u \frac{\partial h^\#}{\partial x} = -u^\# \frac{\partial h}{\partial x} - h^\# \frac{\partial u}{\partial x} - h \frac{\partial u^\#}{\partial x} \\ \frac{\partial u^\#}{\partial t} + u \frac{\partial u^\#}{\partial x} = -u^\# \frac{\partial u}{\partial x} + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u^\#}{\partial x} \right) \\ \qquad \qquad \qquad + \frac{1}{h} \frac{\partial}{\partial x} \left(h^\# \nu_e \frac{\partial u}{\partial x} \right) - \frac{h^\#}{h^2} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) \\ \qquad \qquad \qquad - u^\# \frac{q_l}{h} - (U_S - u) \frac{q_l h^\#}{h^2} \end{array} \right.$$

Proof: The seminal argument is to show the following identities

$$\begin{aligned} \left(\frac{\partial h}{\partial x} \right)^\# &= \frac{\partial h^\#}{\partial x} \\ \left(\frac{\partial u}{\partial x} \right)^\# &= \frac{\partial u^\#}{\partial x} \\ \left(\frac{\partial h}{\partial t} \right)^\# &= \frac{\partial h^\#}{\partial t} \\ \left(\frac{\partial u}{\partial t} \right)^\# &= \frac{\partial u^\#}{\partial t}, \end{aligned}$$

i.e., more generally, to prove that the sharp operator commutes with the derivative one. It is clear that left-hand side of previous relations are well-defined, while right-hand side are not, so we assume as an hypothesis of theorem that functions $u^\#$ and $h^\#$ are differentiable. Then we have for instance

$$\left(\frac{\partial h}{\partial x} \right)^\# = \left(\lim_{\epsilon \rightarrow 0} \frac{h(t, x + \epsilon) - h(t, x)}{\epsilon} \right)^\# = \lim_{\epsilon \rightarrow 0} \frac{h^\#(t, x + \epsilon) - h^\#(t, x)}{\epsilon} = \frac{\partial h^\#}{\partial x},$$

thanks to the linearity of the sharp operator. Now, the proof ends with the remark that PDE (9.1) is an implicit relation; thus, we can apply a slight modification of key theorem 8.8 and the linearity of sharp, combined with an easy computation, gives us the PDEs (9.2) verified by the sharp.

□

We have a first interesting remark about an advantage of system (9.2):

Remark 9.1 (Linearity) *System (9.2) is a linear partial differential equations system (LPDES) as function of variables $(u^\#, h^\#)_{(t,x)}$. Besides, PDEs (9.2) are homogeneous, then the solution admits a expansion into a basis.*

Therefore, in order to solve theoretically system (9.2), a practicable strategy is to expand functions $u^\#$ and $h^\#$ into a basis, to find a theoretical solution depending on the solutions u and h of system (9.1) and to discretize them using the approximate solution given by theorem 8.1. However, wavelets basis, introduced in chapter 7, is probably not the best choice given the large class of behavior spanned by solutions of shallow water problem.

A second remark shows the complexity of our problem:

Remark 9.2 (Ill-posed) *System (9.2) depends on the solution $(u, h)_{(t,x)}$ and its derivatives. Therefore, the accuracy of the solution has an impact on the solution of LPDEs (9.2), and the dependence on the derivatives of the solution has an impact on the stability of the convergence. Thus, in order to solve our former problem, see theorem 8.1, it is suitable to use a stabilization technique, this analysis leaves the purposes of our study, interested readers can refer to the book of Ern and Guermond [13] section 5.4.*

System (9.2) can be rewrite in the following way:

$$(9.3) \quad \frac{\partial}{\partial t} \begin{pmatrix} h^\# \\ u^\# \end{pmatrix} = \mathbf{A} \begin{pmatrix} h^\# \\ u^\# \end{pmatrix} + \mathbf{B} \frac{\partial}{\partial x} \begin{pmatrix} h^\# \\ u^\# \end{pmatrix} + \mathbf{C} \frac{\partial^2}{\partial x^2} \begin{pmatrix} h^\# \\ u^\# \end{pmatrix}$$

where the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} are given by:

$$(9.4) \quad \mathbf{A} = \begin{pmatrix} -\frac{\partial u}{\partial x} & -\frac{\partial h}{\partial x} \\ -\frac{\nu_e}{h^2} \frac{\partial u}{\partial x} \frac{\partial h}{\partial x} - (U_S - u) \frac{q_l}{h^2} & -\frac{\partial u}{\partial x} - \frac{q_l}{h} \end{pmatrix}$$

$$(9.5) \quad \mathbf{B} = \begin{pmatrix} -u & -h \\ \frac{\nu_e}{h} \frac{\partial u}{\partial x} & \frac{\nu_e}{h} \frac{\partial h}{\partial x} - u \end{pmatrix}$$

$$(9.6) \quad \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & \nu_e \end{pmatrix}$$

9.3.2 Characterization of the PDE

In this subsection, we analyze system (9.3) in order to characterize the type of PDEs system, a general characterisation of first-order PDEs system can be found for instance in Lopez [22] chapter 2, and more general analysis in Anderson [2] chapter 3 and Renardy and Rogers [26] chapter 2.

In order to analyze our system, we have to prove that its solution exists, the Cauchy-Kowalewsky theorem asserts the local existence of a real analytic solution.

Theorem 9.2 (Cauchy-Kowalewsky)

Let $L_{j,\alpha}(t, x)$ and $f(t, x)$ be real analytic functions on a neighborhood of a point (t_0, x_0) , Then it exists a local solution for any system of the form

$$\frac{\partial^m \phi}{\partial t^m} = \sum_{j=0}^{m-1} \sum_{|\alpha| < m-j} L_{j,\alpha}(t, x) \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^j \phi}{\partial t^j} + f(t, x),$$

and this solution is real analytic.

For a complete analysis see Taylor [30] section 6.4. Clearly, the two partial differential equations in system (9.3) verify this hypothesis if u and h with their derivatives are analytic functions and $h(t, x) \geq K > 0$.

Given the existence of the solution, we have to study the differential operator in (9.3), So we can define

$$(9.7) \quad L = \mathbb{1} \frac{\partial}{\partial t} - \mathbf{B} \frac{\partial}{\partial x} - \mathbf{C} \frac{\partial^2}{\partial x^2}$$

We have to find a transformation with respect to which operator L acts autonomously on the two components. However, it is impossible in our particular case, since there does not exist a basis with respect to which matrices \mathbf{B} and \mathbf{C} become both diagonal. A well-known result in linear algebra says that two matrices admit a common diagonalisation basis if and only if they are diagonalizable and they commute, it is not the case for matrices \mathbf{B} and \mathbf{C} , since the only choice under which $\mathbf{B}\mathbf{C} = \mathbf{C}\mathbf{B}$ is $\nu_e = 0$; but in this case \mathbf{B} is a Jordan canonical form matrix, and it is diagonal if and only if $h \equiv 0$, that is absurd.

The case $\nu_e = 0$ is however interesting; under this hypothesis, our operator L is a first-order partial differential equations system. Furthermore, if $U_S \equiv u$, the equation for u in system (9.2) is autonomous; thus, we can solve the PDE verified by the sharp of u without any analysis of the sharp of h . In this case, second equation in (9.2) is a first-order hyperbolic equation, i.e. $u^\#$ has a wave type transmission with local coefficient $u(t, x)$, similar equations are Klein-Gordon equations. However, the solution of this problem is not explicit.

9.3.3 Formal solution via Laplace transform

In this subsection, we assume $\nu_e = 0$ and $U_S = u$. Under these hypotheses, we have showed that the partial differential equation verified by the sharp of velocity u is the following autonomous PDE.

$$\frac{\partial u^\#}{\partial t} + u \frac{\partial u^\#}{\partial x} = -u^\# \frac{\partial u}{\partial x} - u^\# \frac{q_l}{h}$$

In order to write the solution of this PDE, we can consider its Laplace transform $\mathcal{F}[u^\#](t, y)$ and the related differential equation.

Proposition 9.3 (Laplace transform of $u^\#$)

The Laplace transform $\mathcal{F}[u^\#](t, y)$ of the function $u^\#(t, x)$ verifies the following integro-differential equation:

$$(9.8) \quad \begin{aligned} \frac{\partial \mathcal{F}[u^\#](t, y)}{\partial t} &= -y \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) \mathcal{F}[u^\#](t, z) dz \\ &\quad - q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y-z) \mathcal{F}[u^\#](t, z) dz. \end{aligned}$$

Proof: First of all, we remark that

$$u \frac{\partial u^\#}{\partial x} + u^\# \frac{\partial u}{\partial x} = \frac{\partial (u u^\#)}{\partial x}.$$

We recall the Laplace convolution theorem

$$\mathcal{F}[f g] = \mathcal{F}[f] \star \mathcal{F}[g],$$

and the Laplace transform of a derivative

$$\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = y \mathcal{F}[f].$$

Then, the result of theorem 9.3 comes easily.

□

Now, it is easy to compute numerically

$$(9.9) \quad \begin{aligned} \mathcal{F}[u](t, y) &= \int_{[x_0, x_1]} u(t, x) e^{-xy} dx \\ \mathcal{F}\left[\frac{1}{h}\right](t, y) &= \int_{[x_0, x_1]} \frac{1}{h(t, x)} e^{-xy} dx, \end{aligned}$$

using the approximate solutions given by theorem 8.1. Finally is possible to find an approximate solution for PDE (9.8) and to compute numerically the inverse of Laplace transform. Given the Laplace transform of $u^\#$, we can study the Laplace transform of $h^\#$.

Proposition 9.4 (Laplace transform of $h^\#$)

The Laplace transform $\mathcal{F}[h^\#](t, y)$ of the function $h^\#(t, x)$ verifies the following integro-differential equation:

$$(9.10) \quad \begin{aligned} \frac{\partial \mathcal{F}[h^\#](t, y)}{\partial t} &= -y \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) \mathcal{F}[h^\#](t, z) dz \\ &\quad -y \int_{\mathbb{R}} \mathcal{F}[u^\#](t, y-z) \mathcal{F}[h](t, z) dz \end{aligned}$$

The proof is similar to the previous one. Using this two propositions, we propose an easy algorithm to compute the Laplace transform of the couple $(u^\#, h^\#)_{(t,x)}$. Given the fact that $u(t, x)$ and $h(t, x)$ are known only at the knots of a mesh, see theorem 8.1, we search the Laplace transform of $(u^\#, h^\#)_{(t,x)}$ only at time $t_k = k \Delta T$.

Algorithm 9.5 (Laplace transform of the sharp)

1. Given the shallow water PDE (9.1), with $\nu_e = 0$ and $U_S = u$, discretize it in time and space, thanks to a Galerkin variational approach and find a system of type (8.7);
2. compute numerically the solution of system (8.7) for all $t_k = k \Delta T$;
3. compute numerically $\mathcal{F}[u](t_k, y)$, $\mathcal{F}[h](t_k, y)$ and $\mathcal{F}[h^{-1}](t_k, y)$ for all k ;
4. given the sharp of starting condition $u(0, x)$, compute its Laplace transform $\mathcal{F}[u^\#](0, y)$;
5. compute recursively the Laplace transform $\mathcal{F}[u^\#](t_k, y)$ using the following approximation of integro-differential equation (9.8)

$$(9.11) \quad \begin{aligned} \mathcal{F}[u^\#](t_{k+1}, y) &= \mathcal{F}[u^\#](t_k, y) - \Delta T y \int_{\mathbb{R}} \mathcal{F}[u](t_k, y-z) \mathcal{F}[u^\#](t_k, z) dz \\ &\quad - \Delta T q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t_k, y-z) \mathcal{F}[u^\#](t_k, z) dz \end{aligned}$$

for all k ;

6. given the sharp of starting condition $h(0, x)$, compute its Laplace transform $\mathcal{F}[h^\#](0, y)$;
7. compute recursively the Laplace transform $\mathcal{F}[h^\#](t_k, y)$ using the following approximation of integro-differential equation (9.10)

$$(9.12) \quad \begin{aligned} \mathcal{F}[h^\#](t_{k+1}, y) &= \mathcal{F}[h^\#](t_k, y) - \Delta T y \int_{\mathbb{R}} \mathcal{F}[u](t_k, y-z) \mathcal{F}[h^\#](t_k, z) dz \\ &\quad - \Delta T y \int_{\mathbb{R}} \mathcal{F}[u^\#](t_k, y-z) \mathcal{F}[h](t_k, z) dz \end{aligned}$$

for all k .

9.3.4 PDE verified by the carré du champ operator

In this section, we assume $\nu_e = 0$. We consider the system of partial differential equations (9.2) verified by the sharp of the solution of shallow water equations (9.1), and we search a system of three coupled equations verified by $\Gamma[u]$, $\Gamma[h]$ and $\Gamma[u, h]$. However, only two partial differential equations exist due to the fact that matrix \mathbf{B} is not diagonalizable.

Theorem 9.6 (PDEs verified by the carré du champ operator)

Under the same hypotheses of theorem 9.1, we have the two following partial differential equations verified by the carré du champ of the solutions of shallow water problem.

$$(9.13) \quad \left\{ \begin{array}{l} \frac{1}{2} \frac{\partial \Gamma[u]}{\partial t} + \frac{1}{2} u \frac{\partial \Gamma[u]}{\partial x} = -\Gamma[u] \frac{\partial u}{\partial x} - \Gamma[u] \frac{q_l}{h} - (U_S - u) \frac{q_l \Gamma[h, u]}{h^2} \\ \frac{\partial}{\partial t} \Gamma[h, u] + u \frac{\partial}{\partial x} \Gamma[h, u] = \Gamma[u] \frac{\partial h}{\partial x} - \left(2 \frac{\partial u}{\partial x} - \frac{q_l}{h} \right) \Gamma[u, h] \\ -\frac{1}{2} h \frac{\partial \Gamma[u]}{\partial x} - (U_S - u) \frac{q_l \Gamma[h]}{h^2} \end{array} \right.$$

Proof:

In order to find these partial differential equations, we multiply the two equations in (9.2) by $u^\#$ and $h^\#$; thus, we find four equations. We take the expectation under probability $\widehat{\mathbb{P}}$ and we can easily find the following relations:

$$(9.14) \quad \begin{aligned} \frac{1}{2} \frac{\partial \Gamma[h]}{\partial t} + \frac{1}{2} u \frac{\partial \Gamma[h]}{\partial x} &= -\Gamma[h, u] \frac{\partial h}{\partial x} - \Gamma[h] \frac{\partial u}{\partial x} - h \widehat{\mathbb{E}} \left[h^\# \frac{\partial u^\#}{\partial x} \right] \\ \widehat{\mathbb{E}} \left[u^\# \frac{\partial h^\#}{\partial t} \right] + u \widehat{\mathbb{E}} \left[u^\# \frac{\partial h^\#}{\partial x} \right] &= -\Gamma[u] \frac{\partial h}{\partial x} - \Gamma[u, h] \frac{\partial u}{\partial x} - \frac{1}{2} h \frac{\partial \Gamma[u]}{\partial x} \\ \frac{1}{2} \frac{\partial \Gamma[u]}{\partial t} + \frac{1}{2} u \frac{\partial \Gamma[u]}{\partial x} &= -\Gamma[u] \frac{\partial u}{\partial x} - \Gamma[u] \frac{q_l}{h} - (U_S - u) \frac{q_l \Gamma[h, u]}{h^2} \\ \widehat{\mathbb{E}} \left[h^\# \frac{\partial u^\#}{\partial t} \right] + u \widehat{\mathbb{E}} \left[h^\# \frac{\partial u^\#}{\partial x} \right] &= -\Gamma[h, u] \frac{\partial u}{\partial x} - \Gamma[h, u] \frac{q_l}{h} - (U_S - u) \frac{q_l \Gamma[h]}{h^2}. \end{aligned}$$

Third equation in (9.14) is the partial differential equation verified by $\Gamma[u]$, i.e. first equation in (9.13). In order to find the partial differential equation verified by $\Gamma[u, h]$, we sum second and fourth equations in (9.14) and we find

$$\frac{\partial}{\partial t} \Gamma[h, u] + u \frac{\partial}{\partial x} \Gamma[h, u] = \Gamma[u] \frac{\partial h}{\partial x} - \left(2 \frac{\partial u}{\partial x} - \frac{q_l}{h} \right) \Gamma[u, h] - \frac{1}{2} h \frac{\partial \Gamma[u]}{\partial x} - (U_S - u) \frac{q_l \Gamma[h]}{h^2},$$

i.e. second equation in (9.13).

□

Using the same argument used to write the Laplace transform for $u^\#$, we have the following proposition for the carré du champ of u .

Proposition 9.7 (Laplace transform of the variance of the error on u)

If $U_s = u$, the Laplace transform $\mathcal{F}[\Gamma[u]](t, y)$ of the function $\Gamma[u](t, x)$ verifies the following integro-differential equation

$$(9.15) \quad \begin{aligned} \frac{1}{2} \frac{\partial \mathcal{F}[\Gamma[u]](t, y)}{\partial t} &= - \int_{\mathbb{R}} \mathcal{F}[u](t, y - z) \left(y - \frac{z}{2}\right) \mathcal{F}[\Gamma[u]](t, z) dz \\ &\quad - q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y - z) \mathcal{F}[\Gamma[u]](t, z) dz. \end{aligned}$$

Now, it is possible to solve numerically the differential equation (9.15) using the same approximation proposed in section 9.3.3. Given the Laplace transform of $\Gamma[u]$, we can compute the Laplace transform of $\Gamma[u, h]$.

Proposition 9.8 (Laplace transform of the covariance between errors on u and h)

If $U_s = u$, the Laplace transform $\mathcal{F}[\Gamma[u, h]](t, y)$ of the function $\Gamma[u, h](t, x)$ verifies the following integro-differential equation

$$(9.16) \quad \begin{aligned} \frac{\partial \mathcal{F}[\Gamma[u, h]](t, y)}{\partial t} &= - \int_{\mathbb{R}} \mathcal{F}[u](t, y - z) (2y - z) \mathcal{F}[\Gamma[u, h]](t, z) dz \\ &\quad - \int_{\mathbb{R}} \mathcal{F}[h](t, y - z) \left(y - \frac{z}{2}\right) \mathcal{F}[\Gamma[u]](t, z) dz \\ &\quad + q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y - z) \mathcal{F}[\Gamma[u, h]](t, z) dz. \end{aligned}$$

Analogously to the sharp analysis, we propose an algorithm to evaluate numerically the Laplace transform of the couple $(\Gamma[u], \Gamma[u, h])_{(t, x)}$. Given the fact that $u(t, x)$ and $h(t, x)$ are known only at the knots of a mesh, see theorem 8.1, we search the Laplace transform of $(\Gamma[u], \Gamma[u, h])_{(t, x)}$ only at time $t_k = k \Delta T$.

Algorithm 9.9 (Laplace transform of the variance)

1. Given the shallow water PDE (9.1), with $\nu_e = 0$ and $U_S = u$, discretize it in time and space, thanks to a Galerkin variational approach and find a system of type (8.7);
2. compute numerically the solution of system (8.7) for all $t_k = k \Delta T$;
3. compute numerically $\mathcal{F}[u](t_k, y)$, $\mathcal{F}[h](t_k, y)$ and $\mathcal{F}[h^{-1}](t_k, y)$ for all k ;

4. given the variance of starting condition $u(0, x)$, compute its Laplace transform $\mathcal{F}[\Gamma[u]](0, y)$;
5. compute recursively the Laplace transform $\mathcal{F}[\Gamma[u]](t_k, y)$ using the following approximation of integro-differential equation (9.15)

$$\begin{aligned}
 \mathcal{F}[\Gamma[u]](t_{k+1}, y) &= \mathcal{F}[\Gamma[u]](t_k, y) \\
 &\quad - \Delta T \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) (2y-z) \mathcal{F}[\Gamma[u]](t, z) dz \\
 &\quad - 2 \Delta T q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y-z) \mathcal{F}[\Gamma[u]](t, z) dz
 \end{aligned}
 \tag{9.17}$$

for all k ;

6. given the covariance of starting condition between $u(0, x)$ and $h(0, x)$, compute its Laplace transform $\mathcal{F}[\Gamma[u, h]](0, y)$;
7. compute recursively the Laplace transform $\mathcal{F}[\Gamma[u, h]](t_k, y)$ using the following approximation of integro-differential equation (9.16)

$$\begin{aligned}
 \mathcal{F}[\Gamma[u, h]](t_{k+1}, y) &= \mathcal{F}[\Gamma[u, h]](t, y) \\
 &\quad - \Delta T \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) (2y-z) \mathcal{F}[\Gamma[u, h]](t, z) dz \\
 &\quad - \Delta T \int_{\mathbb{R}} \mathcal{F}[h](t, y-z) \left(y - \frac{z}{2}\right) \mathcal{F}[\Gamma[u]](t, z) dz \\
 &\quad + \Delta T q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y-z) \mathcal{F}[\Gamma[u, h]](t, z) dz
 \end{aligned}
 \tag{9.18}$$

for all k .

However, this algorithm cannot furnish us either the Laplace transform of variance $\Gamma[h]$, or the covariance between our functions at two distinct point on space-time, e.g. $\Gamma[u(t_i, x_m), u(t_j, x_k)]$.

9.3.5 Initialisation using stationary processes

In this subsection, we analyze how to define a continuous starting condition with an intrinsic error structure. We search a model that can be combine a high number of degrees of freedom, in order to represent many situations, with an easy characterisation of the error structure.

Our model is based on the theory of weak-sense stationary processes, for instance see Bouleau [6]. We recall that a process X_τ is said (weak-sense) stationary if its covariance function $\mathbb{E}[X_{\tau+\delta\tau} X_\tau]$ depends only on $\delta\tau$.

A subset of the set of stationary processes can be defined using Wiener integrals. We consider a couple of independent Brownian motions $(B_\alpha^{(1)}, B_\alpha^{(2)})$, indexed by the time α and we consider the two processes

$$(9.19) \quad \begin{aligned} U_x &= \int_0^\infty e^{ix\alpha} f_u(\alpha) dB_\alpha^{(1)} \\ H_x &= \int_0^\infty e^{ix\alpha} f_h(\alpha) dB_\alpha^{(2)}, \end{aligned}$$

defined on $[x_0, x_1]$, where f_u and f_h belong to $L^2(\mathbb{R}^+)$. It is plain that U_x and H_x are independent stochastic processes. It is easy to check, see Bouleau [6] chapter 5, that we have the following autocovariation functions

$$(9.20) \quad \begin{aligned} \mathbb{E} [U_x, \bar{U}_{x+y}] &= \int_0^\infty e^{iy\alpha} f_u^2(\alpha) d\alpha \\ \mathbb{E} [H_x, \bar{H}_{x+y}] &= \int_0^\infty e^{iy\alpha} f_h^2(\alpha) d\alpha \\ \mathbb{E} [U_x, \bar{H}_{x+y}] &= 0. \end{aligned}$$

We can easily define an error structure for the couple $(B_\alpha^{(1)}, B_\alpha^{(2)})$, for instance we fix an Ornstein-Uhlenbeck structure given by the product of two O-U structures on the two Wiener spaces spanned respectively by $B_\alpha^{(1)}$ and $B_\alpha^{(2)}$, for more details see the introductory part of this thesis.

Finally, we can easily define a starting condition for our problem (9.1) using a realization of the couple (U_x, H_x) , and taking the real part.

$$(9.21) \quad \begin{aligned} u(0, x) &= \Re U_x(\bar{\omega}) \\ h(0, x) &= \Re H_x(\bar{\omega}) \end{aligned}$$

This strategy can be easily extended in order to take into account a correlation between the uncertainties on $h(0, x)$ and $u(0, x)$, using the fact that the error on two random variables can be correlated even if the two random variables are independent, see Bouleau [5].

9.4 Analysis via characteristic form

In this section, we analyze how to go beyond the difficulties of our previous analysis, especially the impossibility to find a partial differential equation verified by $\Gamma[h]$. The method that we will discuss in this section is based on the characteristic form of shallow water equations, we follow Fromion and Litrico, see [21] chapter 2.

We simplify our PDE (9.1) assuming $\nu_e = 0$, $q_l = 0$ and $Z_s = h$, i.e. we consider an inviscid fluid in a artificial channel¹. Our PDE becomes

$$(9.22) \quad \begin{cases} \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = -h \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} + F \end{cases}$$

We introduce the celerity $c(t, x) = \sqrt{g h(t, x)}$ and we have the following system for the couple $(u, c)_{(t, x)}$:

$$\begin{cases} 2 \frac{\partial c}{\partial t} + c \frac{\partial u}{\partial x} + 2u \frac{\partial c}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + 2c \frac{\partial c}{\partial x} = F \end{cases}$$

and finally

$$(9.23) \quad \begin{cases} \frac{\partial(u+2c)}{\partial t} + (u+c) \frac{\partial(u+2c)}{\partial x} = F \\ \frac{\partial(u-2c)}{\partial t} + (u-c) \frac{\partial(u-2c)}{\partial x} = -F. \end{cases}$$

We have obtained a system of two PDEs with two new variables

$$\begin{aligned} J_1(t, x) &= u(t, x) + 2c(t, x) \\ J_2(t, x) &= u(t, x) - 2c(t, x). \end{aligned}$$

Two characteristic curves exist for variables J_1 and J_2 and system (9.23) is equivalent to the coupled system

$$(9.24) \quad \begin{cases} \frac{dJ_1}{dt}(t, y_1(t)) = F(t, y_1(t)) \\ \frac{dy_1}{dt} = u(t, y_1(t)) + c(t, y_1(t)) \\ \frac{dJ_2}{dt}(t, y_2(t)) = -F(t, y_2(t)) \\ \frac{dy_2}{dt} = u(t, y_2(t)) + c(t, y_2(t)). \end{cases}$$

¹In this case, the lateral flow from the layer is negligible, the mass is conserved and the bed channel is generally flat.

If we consider that the starting and boundary conditions are erroneous, we have an error structure for the couple $(u, h)_{(t,x)}$, when either x belongs to $\{x_0, x_1\}$ or $t = 0$. This uncertainty is transmitted to the solution $(u, h)_{(t,x)}$. We start our analysis with two interesting remarks.

Remark 9.3 (biased celerity) *Celerity $c(t, x)$ is generally a biased function, since it is a non-linear transformation of water depth $h(t, x)$, that is generally assumed unbiased. Due to the concavity of the square root, celerity is negatively biased.*

$$\mathcal{A}[c] = \frac{g}{2c} \mathcal{A}[h] - \frac{g^2}{8c^3} \Gamma[h]$$

Remark 9.4 (erroneous characteristic curves) *The two characteristic curves defined in system (9.24) are erroneous, since they depend on u and c . This fact is a severe drawback in the use of characteristic method to evaluate sensitivity using the error theory.*

However, characteristic form (9.23) for shallow water PDE is interesting, we have

$$(9.25) \quad \begin{cases} \frac{\partial J_1}{\partial t} + \frac{3J_1 + J_2}{4} \frac{\partial J_1}{\partial x} = F \\ \frac{\partial J_2}{\partial t} + \frac{J_1 + 3J_2}{4} \frac{\partial J_2}{\partial x} = -F. \end{cases}$$

We can easily prove the following theorem.

Theorem 9.10 (PDE verified by the sharp)

Let $(J_1, J_2)_{(t,x)}$ be the theoretical solution of problem (9.25). Suppose that the starting and boundary conditions are erroneous, and the related error structure admits a sharp operator denoted $(\cdot)^\#$. Suppose that the theoretical solution of problem (9.25) is known and it is differentiable; then the sharp $(J_1^\#, J_2^\#)_{(t,x)}$ verifies the following PDEs system, if it exists and is differentiable a.e..

$$(9.26) \quad \begin{cases} \frac{\partial J_1^\#}{\partial t} + \frac{3J_1 + J_2}{4} \frac{\partial J_1^\#}{\partial x} = -\frac{3J_1^\# + J_2^\#}{4} \frac{\partial J_1}{\partial x} \\ \frac{\partial J_2^\#}{\partial t} + \frac{J_1 + 3J_2}{4} \frac{\partial J_2^\#}{\partial x} = -\frac{J_1^\# + 3J_2^\#}{4} \frac{\partial J_2}{\partial x}. \end{cases}$$

However, also in this case, only two partial differential equations exist for the triplet $(\Gamma[J_1], \Gamma[J_2], \Gamma[J_1, J_2])$.

Theorem 9.11 (PDEs verified by the carré du champ operator)

Under the same hypotheses of theorem 9.10, we have the two following partial differential equations verified by the carré du champ of the solutions of shallow water problem.

$$(9.27) \quad \begin{cases} \frac{1}{2} \frac{\partial \Gamma[J_1]}{\partial t} + \frac{3J_1 + J_2}{8} \frac{\partial \Gamma[J_1]}{\partial x} = -\frac{3\Gamma[J_1] + \Gamma[J_1, J_2]}{4} \frac{\partial J_1}{\partial x} \\ \frac{1}{2} \frac{\partial \Gamma[J_2]}{\partial t} + \frac{J_1 + 3J_2}{8} \frac{\partial \Gamma[J_2]}{\partial x} = -\frac{\Gamma[J_1, J_2] + 3\Gamma[J_2]}{4} \frac{\partial J_2}{\partial x}. \end{cases}$$

Besides, system (9.27) is not decoupled.

9.5 Bias of the shallow water problem

In this section, we consider the PDEs system (9.1) and we prove the existence of an associated system of two partial differential equations for the bias under some hypothesis. In accordance with results showed in section 9.3.2, the PDE for the bias of the velocity u is autonomous but the partial equation for the bias of depth h depends on the bias of u .

9.5.1 PDE verified by the bias

In this section, we analyze the bias of the theoretical solution of problem (9.1).

Theorem 9.12 (PDE verified by the bias)

Assume the ARB condition hold, see section 1.6 Let $(u, h)_{(t,x)}$ be the solution of problem (9.1); under the same hypotheses of theorem 9.1, the solution $(u, h)_{(t,x)}$ is biased. If it is differentiable, this bias $(\mathcal{A}[u], \mathcal{A}[h])_{(t,x)}$ verified the following PDEs system:

$$(9.28) \quad \left\{ \begin{array}{l} \frac{\partial \mathcal{A}[h]}{\partial t} + u \frac{\partial \mathcal{A}[h]}{\partial x} = -\mathcal{A}[u] \frac{\partial h}{\partial x} - \mathcal{A}[h] \frac{\partial u}{\partial x} - h \frac{\partial \mathcal{A}[u]}{\partial x} - \frac{\partial \Gamma[h, u]}{\partial x} \\ \frac{\partial \mathcal{A}[u]}{\partial t} + u \frac{\partial \mathcal{A}[u]}{\partial x} = -\mathcal{A}[u] \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial \Gamma[u]}{\partial x} + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial \mathcal{A}[u]}{\partial x} \right) \\ \quad + \frac{1}{h} \frac{\partial}{\partial x} \left(\mathcal{A}[h] \nu_e \frac{\partial u}{\partial x} \right) - \frac{\mathcal{A}[h]}{h^2} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) \\ \quad + \frac{1}{h} \widehat{\mathbb{E}} \left[\frac{\partial}{\partial x} \left(h^\# \nu_e \frac{\partial u^\#}{\partial x} \right) \right] \\ \quad - \frac{1}{h^2} \widehat{\mathbb{E}} \left[h^\# \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u^\#}{\partial x} + h^\# \nu_e \frac{\partial u}{\partial x} \right) \right] \\ \quad + \frac{\Gamma[h]}{h^3} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) - \mathcal{A}[u] \frac{q_l}{h} \\ \quad - (U_S - u) \frac{q_l \mathcal{A}[h]}{h^2} + (U_S - u) \frac{q_l \Gamma[h]}{h^3} \end{array} \right.$$

where $u^\#$ and $h^\#$ are given by theorem 9.1 and $\Gamma[u]$, $\Gamma[u, h]$ and $\Gamma[h]$ can be obtained using the same theorem and the properties of sharp.

Proof: The basic arguments of the proof are the same used in theorem 9.1, i.e. the commutation between bias and derivatives, and the implicit relation of PDE (9.1). We have to make the computation using the chain rule for bias. We have

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{A}[h]}{\partial t} + u \frac{\partial \mathcal{A}[h]}{\partial x} = -\mathcal{A}[u] \frac{\partial h}{\partial x} - \mathcal{A}[h] \frac{\partial u}{\partial x} - h \frac{\partial \mathcal{A}[u]}{\partial x} - \widehat{\mathbb{E}} \left[h^\# \frac{\partial u^\#}{\partial x} \right] - \widehat{\mathbb{E}} \left[\frac{\partial h^\#}{\partial x} u^\# \right] \\ \frac{\partial \mathcal{A}[u]}{\partial t} + u \frac{\partial \mathcal{A}[u]}{\partial x} = -\mathcal{A}[u] \frac{\partial u}{\partial x} - \frac{1}{2} \widehat{\mathbb{E}} \left[\frac{\partial (u^\#)^2}{\partial x} \right] + \frac{1}{h} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial \mathcal{A}[u]}{\partial x} \right) \\ \quad + \frac{1}{h} \frac{\partial}{\partial x} \left(\mathcal{A}[h] \nu_e \frac{\partial u}{\partial x} \right) - \frac{\mathcal{A}[h]}{h^2} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) \\ \quad + \frac{1}{h} \widehat{\mathbb{E}} \left[\frac{\partial}{\partial x} \left(h^\# \nu_e \frac{\partial u^\#}{\partial x} \right) \right] - \frac{1}{h^2} \widehat{\mathbb{E}} \left[h^\# \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u^\#}{\partial x} + h^\# \nu_e \frac{\partial u}{\partial x} \right) \right] \\ \quad + \frac{\Gamma[h]}{h^3} \frac{\partial}{\partial x} \left(h \nu_e \frac{\partial u}{\partial x} \right) - \mathcal{A}[u] \frac{q_l}{h} - (U_S - u) \frac{q_l \mathcal{A}[h]}{h^2} + (U_S - u) \frac{q_l \Gamma[h]}{h^3}. \end{array} \right.$$

Using the fact that the partial derivative can be come out of the expectation and the relations $\Gamma[u, h] = \widehat{\mathbb{E}} [u^\# h^\#]$ and $\Gamma[u] = \widehat{\mathbb{E}} [(u^\#)^2]$, we find the PDE (9.28) verified by the bias.

□

Remark 9.5 (Linearity) *The PDEs system (9.28) is a linear partial differential equations system (LPDES) as function of variables $(\mathcal{A}[u], \mathcal{A}[h])_{(t,x)}$. Besides, PDEs (9.28) are homogeneous, then the solution can be expanded into a basis.*

9.5.2 Analysis of the PDE

We can analyze system (9.28) likewise our work on sharp operator. We remark that operator L , introduced in formula (9.7), is the generator of PDE (9.28) too. Therefore, our analysis about the existence of a solution and the impossibility to transform our problem into two uncoupled PDEs is suited also in the study of bias.

The case $\nu_e = 0$ is still interesting, since system (9.28) becomes

$$(9.29) \quad \left\{ \begin{array}{l} \frac{\partial \mathcal{A}[h]}{\partial t} + u \frac{\partial \mathcal{A}[h]}{\partial x} = -\mathcal{A}[u] \frac{\partial h}{\partial x} - \mathcal{A}[h] \frac{\partial u}{\partial x} - h \frac{\partial \mathcal{A}[u]}{\partial x} - \frac{\partial \Gamma[h, u]}{\partial x} \\ \frac{\partial \mathcal{A}[u]}{\partial t} + u \frac{\partial \mathcal{A}[u]}{\partial x} = -\mathcal{A}[u] \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial \Gamma[u]}{\partial x} - \mathcal{A}[u] \frac{q_l}{h} \\ \quad - (U_S - u) \frac{q_l \mathcal{A}[h]}{h^2} + (U_S - u) \frac{q_l \Gamma[h]}{h^3}. \end{array} \right.$$

Under the hypothesis $U_S = u$, this system becomes

$$(9.30) \quad \begin{cases} \frac{\partial \mathcal{A}[h]}{\partial t} + u \frac{\partial \mathcal{A}[h]}{\partial x} = -\mathcal{A}[u] \frac{\partial h}{\partial x} - \mathcal{A}[h] \frac{\partial u}{\partial x} - h \frac{\partial \mathcal{A}[u]}{\partial x} - \frac{\partial \Gamma[h, u]}{\partial x} \\ \frac{\partial \mathcal{A}[u]}{\partial t} + u \frac{\partial \mathcal{A}[u]}{\partial x} = -\mathcal{A}[u] \frac{\partial u}{\partial x} - \frac{1}{2} \frac{\partial \Gamma[u]}{\partial x} - \mathcal{A}[u] \frac{q_l}{h}. \end{cases}$$

We remark that the second equation is autonomous and depends only on $\Gamma[u]$, as well as u , h and the partial derivative of u with respect to x . In this case we can propose a formal solution using Laplace transform.

Proposition 9.13 (Laplace transform of the bias of u)

The Laplace transform $\mathcal{F}[\mathcal{A}[u]](t, y)$ of the function $\mathcal{A}[u](t, x)$ verifies the following integro-differential equation depending on the Laplace transform of $\Gamma[u](t, x)$

$$(9.31) \quad \begin{aligned} \frac{\partial \mathcal{F}[\mathcal{A}[u]]}{\partial t}(t, y) &= -y \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) \mathcal{F}[\mathcal{A}[u]](t, z) dz \\ &\quad - \frac{1}{2} y \mathcal{F}[\Gamma[u]](t, y) \\ &\quad - q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y-z) \mathcal{F}[\mathcal{A}[u]](t, z) dz. \end{aligned}$$

The proof follows the same argument proposed in proposition 9.3. Now, it is possible to solve numerically the integro-differential equation (9.31).

In spite of the difficulty to estimate $\Gamma[h]$, we can compute the Laplace transform of $\mathcal{A}[h]$.

Proposition 9.14 (Laplace transform of the bias of h)

The Laplace transform $\mathcal{F}[\mathcal{A}[h]](t, y)$ of the function $\mathcal{A}[h](t, x)$ verifies the following integro-differential equation depending on the Laplace transform of $\Gamma[u, h](t, x)$ and $\mathcal{A}[u](t, x)$

$$(9.32) \quad \begin{aligned} \frac{\partial \mathcal{F}[\mathcal{A}[h]]}{\partial t}(t, y) &= -y \int_{\mathbb{R}} \mathcal{F}[h](t, y-z) \mathcal{F}[\mathcal{A}[u]](t, z) dz \\ &\quad - y \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) \mathcal{F}[\mathcal{A}[h]](t, z) dz \\ &\quad - y \mathcal{F}[\Gamma[u, h]](t, y). \end{aligned}$$

We conclude our analysis with the algorithm to evaluate the bias.

Algorithm 9.15 (Laplace transform of variance)

1. Perform algorithm 9.9;

2. given the bias of starting condition $u(0, x)$, compute its Laplace transform $\mathcal{F}[\mathcal{A}[u]](0, y)$;
3. compute recursively the Laplace transform $\mathcal{F}[\mathcal{A}[u]](t_k, y)$ using the following approximation of integro-differential equation (9.31)

$$\begin{aligned}
 \mathcal{F}[\mathcal{A}[u]](t_{k+1}, y) &= \mathcal{F}[\mathcal{A}[u]](t_k, y) \\
 &\quad - \Delta T y \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) \mathcal{F}[\mathcal{A}[u]](t, z) dz \\
 (9.33) \quad &\quad - \frac{1}{2} \Delta T y \mathcal{F}[\Gamma[u]](t, y) \\
 &\quad - \Delta T q_l \int_{\mathbb{R}} \mathcal{F}\left[\frac{1}{h}\right](t, y-z) \mathcal{F}[\mathcal{A}[u]](t, z) dz
 \end{aligned}$$

for all k ;

4. given the bias of starting condition $h(0, x)$, compute its Laplace transform $\mathcal{F}[\mathcal{A}[h]](0, y)$;
5. compute recursively the Laplace transform $\mathcal{F}[\mathcal{A}[u]](t_k, y)$ using the following approximation of integro-differential equation (9.32)

$$\begin{aligned}
 \mathcal{F}[\mathcal{A}[h]](t_{k+1}, y) &= \mathcal{F}[\mathcal{A}[h]](t_k, y) \\
 &\quad - \Delta T y \int_{\mathbb{R}} \mathcal{F}[h](t, y-z) \mathcal{F}[\mathcal{A}[u]](t, z) dz \\
 (9.34) \quad &\quad - \Delta T y \int_{\mathbb{R}} \mathcal{F}[u](t, y-z) \mathcal{F}[\mathcal{A}[h]](t, z) dz \\
 &\quad - \Delta T y \mathcal{F}[\Gamma[u, h]](t, y)
 \end{aligned}$$

for all k .

9.6 Conclusion

In this chapter, we have analyzed the impact of an uncertainty on the starting and boundary conditions of a nonlinear partial differential equation. We have considered a particular case, the shallow water equations used to model the dynamics of open channel flow. This chapter and the previous one are twin, in the sense that they study the same problem using two different strategies. While we have studied a discrete approach in chapter 8, we have showed, in this chapter, that a continuum approach can be applied.

We have consider the shallow water PDEs system and we have analyzed the variance and the bias of the theoretical solution. We have showed that the sharp of the theoretical solution solves a

system of two linear partial differential equations depending on the solution of the shallow water problem itself. Unfortunately, this system cannot be decoupled into two different autonomous partial differential equations. However, we have proved, under some hypotheses, that the Laplace transform of the sharp of the solution can be computed numerically using an integro-differential equation. We have also analyzed the variance of the solution, we have proved that it verifies a incomplete system of partial differential equations, and the variance of the velocity can be estimated numerically using another integro-differential equation.

Finally, we have analyzed the bias, we have showed that it verifies a system of linear partial differential equations characterized by the same generator of the sharp PDEs. We have proved also that the bias of the velocity can be evaluated numerically using an integro-differential equation verified by its Laplace transform.

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Part IV

Stochastic Partial Differential Equations and Climatology

Chapter 10

Uncertainty on Starting Condition

In this chapter, we analyze how an uncertainty on the starting condition of a stochastic partial differential equation is transferred to its solution. We model the noise on the starting condition following the technique developed by Bouleau. We prove that the variance and the bias of the solution can be easily estimated using two linear stochastic partial differential equations depending on the solution of SPDE itself.

10.1 Introduction

Stochastic partial differential equations appeared in the mid-1960s to model random phenomena analyzed in biology, e.g. the evolution of populations, and in physics, e.g. the waves propagation in random media. Nowadays, SPDEs have taken a crucial role in climatology, after the seminal paper of Hasselmann [23]: this methodology is used to create Stochastic Climate Models, a new and prolific branch in climatology. In all previous studies, the stochastic nature is imposed only at evolution equation, whereas the starting condition is assumed to be deterministic. However, the starting condition of a problem is generally not perfectly known, see Fraedrich [20]. As an example, when climate models are concerned, it is not so easy to define the situation of earth today, even if we limit our study to some macro parameters.

The aim of our analysis is not to introduce a new particular stochastic model in physics, biology or climatology, but to propose a method to take into account the presence of a perturbation in starting condition.

Using the recent technique of error theory using Dirichlet forms, see Bouleau [5] or the opening part of this thesis, we suppose that all perturbations are very small: this fact allows us to expand the perturbation in a series and to stop it at the two first corrections, i.e. bias and variance. Error theory using the language of Dirichlet forms defines a correct mathematical framework to analyze how an uncertainty passes through a stochastic partial differential equation.

The structure of this chapter is the following. Section 2 aims at describing a survey of stochastic partial differential equations theory. Section 3 shows how the two mathematical tools, i.e. SPDE and error theory using Dirichlet forms, can interact in order to describe the diffusion of uncertainty through an SPDE. Finally, section 4 resumes and concludes.

10.2 Stochastic partial differential equations

We consider a general parabolic stochastic partial differential equation for all (t, x) belonging to $\mathbb{R}^+ \times \mathbb{R}^d$,

$$(10.1) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = \sigma(t, x, u(t, x)) \dot{C}(t, x) + \beta(t, x, u(t, x)) \\ u(0, x) = f(x) \end{cases}$$

where, Δ denotes the Laplacian operator in \mathbb{R}^d , $\dot{C}(t, x)$ is a Gaussian noise white-in-time, see Dalang et al. [13], with spatial correlation function $k(\cdot, \cdot)$, i.e.

$$\mathbb{E}[\dot{C}(t, x) \dot{C}(s, y)] = \delta_0(t - s) k(x, y)$$

and $\sigma, \beta : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are functions that satisfy the classical properties, see Da Prato and Zabczyk [14], as well Lipschitz in the last variable. This problem admits a unique mild solution, see Da Prato [15] and Zabczyk [33], which is the following.

$$(10.2) \quad \begin{aligned} u(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \left[\sigma(s, y, u(s, y)) \dot{C}(s, y) + \beta(s, y, u(s, y)) \right] ds dy \\ &+ \int_{\mathbb{R}^d} G(t, x - y) f(y) dy \end{aligned}$$

where $G(t, x)$ is the Green function of the associated PDE, in our case

$$(10.3) \quad G(t, x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}.$$

It is clear that a notion of stochastic integral is needed for the term involving the noise and we decide to follow the approach of Da Prato and Zabczyk, see [14].

10.2.1 Gaussian noise

We fix a measurable space (E, \mathcal{E}, μ) where μ is a σ -finite measure.

Definition 10.1 (White noise)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and \mathbb{A} be the set of the subsets A of E such that $\mu(A)$ is finite. Then the white noise associated with μ is a function W defined on \mathbb{A} with values in Ω such that the following properties are verified.

- $\forall A \in \mathbb{A}$, $W(A)$ has a gaussian law $\mathcal{N}(0, \mu(A))$;
- $\forall A, B$ such that $A \cap B = \emptyset$, the two random variable $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

Consistency theorem of Kolmogorov and Bochner's theorem assure the existence of this process on \mathbb{A} , see Khoshnevisan in [13] and Schwartz [28] theorem XIX chapter VII.

In our particular case, we can fix $(E, \mathcal{E}, \mu) = (\mathbb{R}^+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d), dt dx)$ and we can define the stochastic integral.

Definition 10.2 (Stochastic integral)

We fix a basis A_i of disjoint subsets of E such that $\mu(A_i) < \infty$ and let $h(t, x)$ be a simple function in $\mathbb{R}^+ \times \mathbb{R}^d$, i.e.

$$h(t, x) = \sum \eta_i \mathbb{I}_{A_i}(t, x).$$

where η_i are constants. Then we define:

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} h(t, x) W(dt, dx) \equiv \sum_i \eta_i W(A_i).$$

Thanks to the fact that this integral is an isometry between the simple functions space into the $L^2(\Omega)$ space, we can extend the integral at the closure of simple functions space, i.e. the $L^2(\mathbb{R}^+ \times \mathbb{R}^d)$ -space.

Definition of white noise can be generalized to Gaussian noises which are white-in-time and colored in space, i.e. there is a correlation function between the noise at two different points in space; however, we keep the white noise hypothesis in time, thus the time correlation is a delta function, see Dalang et al. [13] or Sanz-Sole [27].

Generally, the spatial Gaussian noise is denoted by the covariance function $g(x, y) \delta(t - s)$, a Gaussian noise is said homogenous if $g(x, y) = \bar{g}(x - y)$, see Sanz-Sole [27] chapter 6, and the Fourier transform of \bar{g} , existing thanks to Bochner's theorem, is called the spectral measure of the noise.

10.2.2 Particular cases

Before the analysis of the general case, given by SPDE (10.1), we have to analyze some interesting particular cases. The first one is the homogenous diffusion where $\sigma \equiv 1$ and $\beta \equiv 0$. In this case, the mild solution (10.2) is the strong solution

$$(10.4) \quad \widehat{u}(t, x) = \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) C(ds, dy) + \int_{\mathbb{R}^d} G(t, x - y) f(y) dy,$$

if and only if the stochastic integral is well-defined, i.e. the Fourier transform of Green function $G(t, x)$ belongs to $L^2(\nu)$ where ν is the spectral measure of the noise, see Dalang in [13] page 47 or Sanz-Sole [27] lemma 6.1 page 80.

A second interesting case is additive noise, when σ is a constant but we release the constraint on β , thus the mild solution (10.2) becomes

$$(10.5) \quad \begin{aligned} \widetilde{u}(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \beta(s, y, \widetilde{u}(s, y)) ds dy + \int_{\mathbb{R}^d} G(t, x - y) f(y) dy. \end{aligned}$$

Also in this case, the stochastic integral is well-defined under the same hypothesis of the homogeneous diffusion, whereas the solution can be estimated thanks to a Picard iteration scheme:

$$\tilde{u}_0(t, x) = \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) C(ds, dy) + \int_{\mathbb{R}^d} G(t, x-y) f(y) dy$$

and

$$(10.6) \quad \begin{aligned} \tilde{u}_n(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta(s, y, \tilde{u}_{n-1}(s, y)) ds dy \\ &+ \int_{\mathbb{R}^d} G(t, x-y) f(y) dy. \end{aligned}$$

It holds that $\tilde{u}_n(t, x)$ converges in L^p -norm to $\tilde{u}(t, x)$ for any $p \geq 2$, see Nualart in [13].

10.2.3 General case

In the general case, the solution of SPDE (10.1) is given by the limit of the following Picard iteration

$$u_0(t, x) = \int_{\mathbb{R}^d} G(t, x-y) f(y) dy$$

and

$$(10.7) \quad \begin{aligned} u_n(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \sigma(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta(s, y, u_{n-1}(s, y)) ds dy \\ &+ \int_{\mathbb{R}^d} G(t, x-y) f(y) dy, \end{aligned}$$

and the solution converges to $u(t, x)$ in L^p -norm for any $p \geq 2$, see Nualart in [13].

We have a last interesting case: the linear multiplicative noise. We assume $\sigma(t, x, u) = \bar{\sigma}(t, x) u$ and $\beta(t, x, u) = \bar{\beta}(t, x) u$, thus we have a better approximation for $u(t, x)$.

Theorem 10.1 (Semi-linear case)

Let $\bar{u}(t, x)$ be the solution of the SPDE

$$(10.8) \quad \begin{cases} \frac{\partial \bar{u}}{\partial t} - \Delta \bar{u} = \bar{\sigma}(t, x) \bar{u}(t, x) \dot{C}(t, x) + \bar{\beta}(t, x) \bar{u}(t, x) \\ \bar{u}(0, x) = f(x) \end{cases}$$

where $\dot{C}(t, x)$ is a Gaussian noise with spectral measure ν . Suppose that, for all t , the Fourier transform of Green function $G(t, x)$ belongs to $L^2(\nu \times ds)$, where ds denotes the Lebesgue measure over the interval $[0, t]$ and suppose that the product of the Fourier transform of $G(t, x)$ and the Fourier transform of $\beta(t, x)$ belongs to $L^1(dx \times ds)$. Then the solution $\bar{u}(t, x)$ is given by

$$(10.9) \quad \bar{u}(t, x) = \sum_{m=0}^{\infty} I_m(t, x)$$

where

$$(10.10) \quad \begin{aligned} I_0(t, x) &= \int_{\mathbb{R}^d} G(t, x - y) f(y) dy \\ I_{m+1}(t, x) &= \int_{[0, t]} \int_{\mathbb{R}^d} G(t - s, x - y) I_m(s, y) [C(ds, dy) + \beta(s, y) ds dy] \end{aligned}$$

and the convergence is uniform in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

Proof: We start with the remark¹ that the Picard series associated with SPDE (10.8) contains only linear operator with respect to u . Therefore, if we denote w the starting condition, we have that the solution can be written as

$$u = w + L(w) + L^2(w) + \dots$$

where

$$(10.11) \quad L(\cdot) = \int_{[0, t]} \int_{\mathbb{R}^d} G(t - s, x - y) \cdot [C(ds, dy) + \beta(s, y) ds dy]$$

It is now clear that series (10.10) verifies the integral form of SPDE (10.8). We need only to proof that

$$|L^n(w)|_{L^2} \rightarrow 0$$

That is an easy consequence of a straight modification of Gronwall lemma see Walsh [32] lemma 3.3 pages 316-318.

□

¹We have found this proof on the handwritten note of a course given by Robert Dalang at EPFL. We thank Daniel Conus.

10.3 Uncertainties in SPDEs due to the starting condition

In this section, we analyze how an uncertainty on the starting condition is transmitted to the solution of a stochastic partial differential equation. We have shown, in section 2, four cases of SPDEs, i.e. homogeneous, additive noise, multiplicative noise and general case; now we analyze the impact of uncertainty in each previous case.

We consider that the starting condition function $f(x)$ is characterized by an uncertainty on its values, we define an error structure for the space of function L^p from \mathbb{R}^d into \mathbb{R} , we follow the approach of Bouleau, see [5] pages 83-85 or section 1.5 in opening part. We consider a basis $\phi_n(x)$ of the function space and we represent the function $f(x)$ using the vector of coefficients a_n ,

$$f(x) = \sum_n a_n \phi_n(x),$$

with setting the coefficients a_n to be random with an error structure on each sub-space, then we have an error structure on the function space thanks to an infinite product of structures, see Bouleau [5] pages 59-65 or section 1.3.2.

Remark 10.1 (Correlation) *We do not assume any independence between two error structures related with two different sub-spaces, since we can perform the computation without this hypothesis. However, under the hypothesis of \mathbb{D} -independence, see Bouleau [8], the numerical evaluations of variance-covariance and bias are more simple.*

We assume that this error structure admits a sharp operator, so we have the representation

$$(10.12) \quad f^\#(x) = \sum_n a_n^\# \phi_n(x),$$

the related variance-covariance and the bias

$$(10.13) \quad \begin{aligned} \Gamma[f(x), f(y)] &= \sum_{n,m} \phi_n(x) \phi_m(y) \Gamma[a_n, a_m] \\ A[f(x)] &= \sum_n \phi_n(x) A[a_n]. \end{aligned}$$

For sake of simplicity, we assume that only a finite number of variables a_n are erroneous, this hypothesis permits to make the proof easier. On the other hand, this assumption is very restrictive and a large part of our results remains true without. A possible way to force a finite number of erroneous variables, without external hypothesis, is presented in Bouleau [5] page 84, see also result 1.11. In the next two subsections, we study the variance-covariance and the bias of the solution.

Before the analysis of the variance and bias of SPDE (10.1), we introduce two useful lemmas.

Lemma 10.2 (Sharp of a stochastic integral)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a error structure equipped with the sharp operator $()^\#$.

Let $Y(t, x) \in L^2(dt \times \nu(dx))$, where $dt \times \nu(dx)$ is the measure associated at the noise $\dot{C}(t, x)$ using Bochner's theorem, be an erroneous function belonging to \mathbb{D} . We assume that the noise $\dot{C}(t, x)$ is not erroneous.

Then

$$\int_{[0, t] \times \mathbb{R}^d} Y(s, y) C(ds, dy)$$

belongs to \mathbb{D} and we have

$$\left(\int_{[0, t] \times \mathbb{R}^d} Y(s, y) C(ds, dy) \right)^\# = \int_{[0, t] \times \mathbb{R}^d} Y^\#(s, y) C(ds, dy)$$

Proof: We recall that stochastic integrals are defined as the limit of an L^2 -series. Therefore, the proof of this lemma is a straight modification of lemmas 1.2.1.1 and 1.2.2.1 in Bouleau and Hirsh [4] chapter IV, see also Bouleau [5] pages 171-173.

□

We have a similar lemma in the case of the bias.

Lemma 10.3 (Bias of a stochastic integral)

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}, \Gamma)$ be a error structure and let \mathcal{A} be its generator with domain \mathcal{DA} .

Let $Y(t, x) \in L^2(dt \times \nu(dx))$, where $dt \times \nu(dx)$ is the measure associated at the noise $\dot{C}(t, x)$ using Bochner's theorem, be an erroneous function belonging to \mathcal{DA} . We assume that the noise $\dot{C}(t, x)$ is not erroneous.

Then

$$\int_{[0, t] \times \mathbb{R}^d} Y(s, y) C(ds, dy)$$

belongs to \mathcal{DA} and we have

$$\mathcal{A} \left[\int_{[0, t] \times \mathbb{R}^d} Y(s, y) C(ds, dy) \right] = \int_{[0, t] \times \mathbb{R}^d} \mathcal{A}[Y](s, y) C(ds, dy)$$

Proof: We recall that stochastic integrals are defined as the limit of an L^2 -series. Therefore, the proof of this lemma is a straight modification of the theorem 3.16, see appendix 3.A.

□

10.3.1 Variance and covariance of solution

In this subsection, we apply operator Γ on the solution $u(t, x)$ of SPDE (10.1), we use the sharp operator and we start with the specification of the SPDE verified by sharp operator.

Theorem 10.4 (SPDE verified by sharp operator)

Under hypothesis that functions $\sigma(t, x, u)$ and $\beta(t, x, u)$ belong to C^1 w.r.t. the variable u and the functions with their derivatives are bounded and Lipschitzian. The sharp of the solution of SPDE (10.1) verifies the following SPDE.

$$(10.14) \quad \begin{cases} \frac{\partial u^\#}{\partial t} - \Delta u^\# = \frac{\partial \sigma}{\partial u}(t, x, u(t, x)) u^\#(t, x) \dot{C}(t, x) + \frac{\partial \beta}{\partial u}(t, x, u(t, x)) u^\# \\ u^\#(0, x) = f^\#(x) \end{cases}$$

and its solution exists and is unique.

Proof: The proof is analogue, following a different approach, as the proof of theorem 6.2 in Sanz-Sole [27] and theorem 13 in Dalang [12].

We give the proof following the sequence of particular cases of section 10.2, that is we compute the sharp of mild solutions $\hat{u}(t, x)$, $\tilde{u}(t, x)$ and $u(t, x)$. The seminal idea of this proof is to apply sharp operator on the solution or on its Picard iteration and to recognize the SPDE verified by the limit.

In homogeneous diffusion case, we can apply sharp operator directly on solution (10.4), so we find

$$\hat{u}^\#(t, x) = \int_{\mathbb{R}^d} G(t, x - y) f^\#(y) dy,$$

thanks to the properties of sharp operator, see definition 1.2. In this case, the stochastic integral is not perturbed. Thus, the sharp operator has no impact on the stochastic integral and we can use error theory using Dirichlet form without any further proof. Clearly, the previous result is the strong solution of SPDE (10.14) when $\sigma = 1$ and $\beta = 0$.

When a non-homogeneous drift exists, i.e. when we study the case of an additive noise, see SPDE (10.5), we can apply the sharp operator on Picard series (10.6), so we find

$$\tilde{u}_0^\#(t, x) = \int_{\mathbb{R}^d} G(t, x - y) f^\#(y) dy$$

and

$$(10.15) \quad \begin{aligned} \tilde{u}_n^\#(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial \beta}{\partial u}(s, y, \tilde{u}_{n-1}(s, y)) \tilde{u}_{n-1}^\#(s, y) ds dy \\ &+ \int_{\mathbb{R}^d} G(t, x - y) f^\#(y) dy. \end{aligned}$$

We have to prove that this series converges to a fix point. We remark that $\tilde{u}_n(s, y)$ converges to $\tilde{u}(s, y)$ in L^p -norm, $p \geq 2$, see Nualart in [13]. So, we can change \tilde{u}_n with \tilde{u} thanks to Lipschitzian coefficients and the difference remains controlled. Now, we use the contraction property, thanks to bounded parameters hypothesis. Clearly, the limit of this series is the sharp of the solution $\tilde{u}^\#(t, x)$ and verifies the equation

$$\tilde{u}^\#(t, x) = \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, \tilde{u}(s, u)) \tilde{u}^\#(s, y) ds dy + \int_{\mathbb{R}^d} G(t, x-y) f(y)^\# dy,$$

that is the integral form of SPDE (10.14), when $\sigma \equiv 1$ and $\beta \equiv 0$. As well in this case, the stochastic integral does not produce any effect, the error theory using Dirichlet forms assures the well-posedness of SPDE (10.14) and its solution.

Finally, we analyze the general case given by SPDE (10.1). We use the same strategy seen in additive noise case. We apply the sharp operator to Picard scheme (10.7) and we find

$$u_0^\#(t, x) = \int_{\mathbb{R}^d} G(t, x-y) f^\#(y) dy$$

and

$$(10.16) \quad \begin{aligned} u_n^\#(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) ds dy \\ &+ \int_{\mathbb{R}^d} G(t, x-y) f^\#(y) dy \end{aligned}$$

and we use a fix point argument to assure the existence of solution and to verify SPDE (10.14). The difference with respect to the previous cases is the presence of a stochastic integral depending on the sharp of the SPDE solution. We verify that for any n the integral

$$\int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) C(ds, dy)$$

is well-posed, using the fact that $\sigma_u(t, x, u)$ is bounded and $G(t, x)$ belongs to test-functions space. Therefore, the Gronwall lemma assures the convergence of Picard series, see Da Prato [15] chapter 3.

□

Theorem 10.4 presents an interesting similarity with two results; one of Da Prato, see [15] page 64, about the derivative of the solution of a SPDE with respect to the initial datum; the other one of Sanz-Sole, see [27] proposition 7.1 pages 95-120, about Malliavin derivative of a SPDE solution. Theorem 10.4 has a direct consequence, we have an easy representation of the sharp, we state this property in the following corollary.

Corollary 10.5 (Series for solution sharp)

Under hypotheses of theorems 10.1 and 10.4, we have the following series for the sharp.

$$(10.17) \quad u^\#(t, x) = \sum_{m=0}^{\infty} I_m^{(u^\#)}(t, x)$$

where

$$(10.18) \quad \begin{aligned} I_0^{(u^\#)}(t, x) &= \int_{\mathbb{R}^d} G(t, x - y) f^\#(y) dy \\ I_{m+1}^{(u^\#)}(t, x) &= \int_{[0, t]} \int_{\mathbb{R}^d} G(t - s, x - y) I_m^{(u^\#)}(s, y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) C(ds, dy) \\ &\quad + \int_{[0, t]} \int_{\mathbb{R}^d} G(t - s, x - y) I_m^{(u^\#)}(s, y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) ds dy \end{aligned}$$

Proof: We remark that the SPDE (10.14) verified by the sharp of the solution is linear, the statement of the lemma is now a direct consequence of the theorem 10.1. □

We can rewrite the sharp using decomposition (10.12).

Result 10.6 (Sharp of SPDE solution)

The sharp of the solution of SPDE (10.1) admits the following decomposition.

$$u^\#(t, x) = \sum_{n, m} a_n^\# J_m^{(n)}(t, x)$$

where

$$(10.19) \quad \begin{aligned} J_0^{(n)}(t, x) &= \int_{\mathbb{R}^d} G(t, x - y) \phi_n(y) dy \\ J_{m+1}^{(n)}(t, x) &= \int_{[0, t]} \int_{\mathbb{R}^d} G(t - s, x - y) J_m^{(n)}(s, y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) C(ds, dy) \\ &\quad + \int_{[0, t]} \int_{\mathbb{R}^d} G(t - s, x - y) J_m^{(n)}(s, y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) ds dy \end{aligned}$$

Proof:

This result is a direct consequence of the corollary 10.5, the linearity of the sharp operator and the linearity of SPDE (10.14) verified by the sharp, an easy proof by induction is needed. □

Now we can make a remark.

Remark 10.2 (Choice of the basis) *Relation (10.19) shows the importance of the choice of the basis $\phi_n(x)$. As a matter of fact, it stands to reason that the computation of $J_m^{(n)}(t, x)$ for many n becomes numerically expensive. Therefore, the essential thing is that the expansion over the basis $\phi_n(x)$ could be cut off. In this sense, a good choice can be a wavelets basis, see Scotti [30] or chapter 7 for a complete analysis.*

Thanks to result 10.6, we have the following characterization for the variance of the error of the solution.

Result 10.7 (Gamma of the solution)

The variance-covariance of the error of the solution is given by

$$(10.20) \quad \Gamma[u(t, x), u(s, y)] = \sum_{j, k, n, m} \Gamma[a_n, a_k] J_m^{(n)}(t, x) J_j^{(k)}(s, y)$$

Proof: This result is a direct consequence of the result 10.6 and the properties of sharp operator, see definition 1.2. □

We conclude our analysis with two remarks.

Remark 10.3 (Linearity) *Equation (10.20) shows that variance-covariance operator admits an easy decomposition into two terms. The first one $\Gamma[a_n, a_k]$ is the covariance between the two erroneous coefficients of the decomposition of starting function $f(x)$ into the basis $\phi_n(x)$. The second term $J_m^{(n)}(t, x) J_j^{(k)}(s, y)$ catches the evolution of the solution through the SPDE but it is unrelated with the error on the function $f(x)$.*

This decomposition depends crucially on the linearity of SPDE (10.14) verified by the sharp, besides this characteristic of the sharp is intrinsic, in PDE analysis the sharp is a generalization of the tangent linear problem, see Choi [10] or Talagrand et al. [31] for a description of tangent linear problem and Scotti [29] or part III in this thesis for the study of uncertainty diffusion through a PDE.

Remark 10.4 (Independence) *If we assume that error structures on each sub-space, on which the functions space has been split, are independent, then the relation for variance-covariance (10.20) becomes more simple.*

$$(10.21) \quad \Gamma[u(t, x), u(s, y)] = \sum_{j, n, m} \Gamma[a_n] J_m^{(n)}(t, x) J_j^{(n)}(s, y)$$

Therefore, a good choice for the basis ϕ_n would be a basis that exploits the information about the uncertainty on starting condition.

10.3.2 Bias of solution

Given the function of variance-covariance we can study the bias.

Theorem 10.8 (SPDE verified by bias operator)

Under the hypothesis that functions $\sigma(t, x, u)$ and $\beta(t, x, u)$ belong to C^2 w.r.t. the variable u and these functions with their derivatives are bounded and Lipschitz. We assume the ARB condition hold, see section 1.6. The bias of the solution of SPDE (10.1) verifies the following SPDE.

$$(10.22) \quad \left\{ \begin{array}{l} \frac{\partial A[u]}{\partial t} - \Delta A[u] = \frac{\partial \sigma}{\partial u}(t, x, u(t, x)) A[u](t, x) \dot{C}(t, x) \\ \quad \quad \quad \quad \quad \quad + \frac{\partial \beta}{\partial u}(t, x, u(t, x)) A[u](t, x) \\ \quad \quad \quad \quad \quad \quad + \frac{1}{2} \frac{\partial^2 \sigma}{\partial u^2}(t, x, u(t, x)) \Gamma[u](t, x) \dot{C}(t, x) \\ \quad \quad \quad \quad \quad \quad + \frac{1}{2} \frac{\partial^2 \beta}{\partial u^2}(t, x, u(t, x)) \Gamma[u](t, x) \\ A[u](0, x) = A[f](x) \end{array} \right.$$

where $\Gamma[u](t, x)$ is given by equation (10.20). Moreover, the solution of this SPDE exists and it is unique.

Proof: This theorem is a slight variant of theorem 10.4. We apply the bias operator A on the Picard iteration (10.7). Using bias chain rule (1.8), we find

$$A[u_0](t, x) = \int_{\mathbb{R}^d} G(t, x - y) A[f](y) dy$$

and

$$\begin{aligned} A[u_n](t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y)) A[u_{n-1}] C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial \beta}{\partial u}(s, y, u_{n-1}(s, y)) A[u_{n-1}](s, y) ds dy \\ &+ \frac{1}{2} \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial^2 \sigma}{\partial u^2}(s, y, u_{n-1}(s, y)) \Gamma[u_{n-1}](s, y) C(ds, dy) \\ &+ \frac{1}{2} \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial^2 \beta}{\partial u^2}(s, y, u_{n-1}(s, y)) \Gamma[u_{n-1}](s, y) ds dy \\ &+ \int_{\mathbb{R}^d} G(t, x - y) A[f](y) dy \end{aligned}$$

where $\Gamma[u_{n-1}](s, y)$ is given by Picard iteration (10.16) and the properties of the sharp operator, see definition 1.2. But $u_n(t, x)$ converges in L^p -norm to $u(t, x)$, therefore, we can control the error given by the exchange between $u_n(t, x)$ and $u(t, x)$ in Picard iteration (10.23), thanks to the Lipschitzian coefficients. Following the same idea we can exchange $\Gamma[u_{n-1}](s, y)$ with $\Gamma[u](s, y)$, given by result 10.7.

Now we have a Picard iteration depending only on the term $A[u_{n-1}](s, y)$. The iteration is linear in $A[u_{n-1}](s, y)$ and all coefficients are bounded.

$$A[u_0](t, x) = \int_{\mathbb{R}^d} G(t, x - y) A[f](y) dy$$

and

$$\begin{aligned}
(10.23) \quad A[u_n](t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) A[u_{n-1}] C(ds, dy) \\
&+ \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) A[u_{n-1}](s, y) ds dy \\
&+ \frac{1}{2} \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial^2 \sigma}{\partial u^2}(s, y, u(s, y)) \Gamma[u](s, y) C(ds, dy) \\
&+ \frac{1}{2} \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \frac{\partial^2 \beta}{\partial u^2}(s, y, u(s, y)) \Gamma[u](s, y) ds dy \\
&+ \int_{\mathbb{R}^d} G(t, x - y) A[f](y) dy
\end{aligned}$$

The Gronwall lemma assures the convergence of the Picard iteration to a unique solution $A[u](t, x)$ and this solution verifies the stochastic partial differential equation (10.22).

□

We make an interesting remark.

Remark 10.5 (Biased solution) Equation (10.22) shows that the solution of a stochastic partial differential equation can be biased even if the starting condition is unbiased. It is sufficient to suppose that the SPDE is non-linear, i.e. $\sigma(t, x, u)$ or $\beta(t, x, u)$ have a non-zero second derivative with respect to u . In this case, SPDE (10.22), verified by the bias, shows an exogenous term proportional to variance of the solution. Therefore, the study of the bias is very important and can modify the behavior of the model.

We can remark that the first term of Picard iteration (10.23) has an interesting structure, it admits in particular the following decomposition.

$$(10.24) \quad A[u_0](t, x) = \sum_n A[a_n] J_0^{(n)}(t, x)$$

where $J_0^{(n)}(t, x)$ is given by equation (10.19). This property can be generalized at each term of Picard iteration (10.23). We introduce the following notation.

Definition 10.3

We define a series of functions $\Psi_m^{(n)}(t, x)$ using the following relations.
If $A[a_n] \neq 0$ then

$$\begin{aligned}
\Psi_m^{(n)}(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) J_{m-1}^{(n)}(s, y) C(ds, dy) \\
&+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) J_{m-1}^{(n)}(s, y) ds dy \\
&+ \sum_k \Theta_k^{(n)} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \sigma}{\partial u^2}(s, y, u(s, y)) \\
(10.25) \quad &* J_{m-1}^{(n)}(s, y) J_{m-1}^{(k)}(s, y) C(ds, dy) + \sum_k \Theta_k^{(n)} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \\
&* \frac{\partial^2 \beta}{\partial u^2}(s, y, u(s, y)) J_{m-1}^{(n)}(s, y) J_{m-1}^{(k)}(s, y) ds dy \\
&+ \int_{\mathbb{R}^d} G(t, x-y) \phi_n(y) dy
\end{aligned}$$

where

$$(10.26) \quad \Theta_k^{(n)} = \frac{\Gamma[a_n, a_k]}{2 A[a_n]}$$

otherwise, if $A[a_n] = 0$, then

$$\begin{aligned}
\Psi_m^{(n)}(t, x) &= \sum_k \theta_k^{(n)} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \sigma}{\partial u^2}(s, y, u(s, y)) \\
(10.27) \quad &* J_{m-1}^{(n)}(s, y) J_{m-1}^{(k)}(s, y) C(ds, dy) + \sum_k \theta_k^{(n)} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \\
&* \frac{\partial^2 \beta}{\partial u^2}(s, y, u(s, y)) J_{m-1}^{(n)}(s, y) J_{m-1}^{(k)}(s, y) ds dy
\end{aligned}$$

where

$$(10.28) \quad \theta_k^{(n)} = \frac{\Gamma[a_n, a_k]}{2}$$

Thanks to these definitions, we have a characterization for the solution of SPDE (10.22), i.e. for the bias of the solution of SPDE (10.1).

Theorem 10.9 (Series for bias)*The series of functions*

$$(10.29) \quad \Xi_m(t, x) = \sum_n \{A[a_n] \Psi_m^{(n)}(t, x) \mathbb{I}_{A[a_n] \neq 0} + \Psi_m^{(n)}(t, x) \mathbb{I}_{A[a_n] = 0}\}$$

converges to $A[u]$ when m goes to infinity in L^2 -norm.

Proof: We remark that the sum over n involves only a finite number of elements, thanks to the hypothesis of a finite number of erroneous coefficients a_n .

The crucial argument of this proof is to remark that $\Xi_m(t, x)$ is a rewriting of Picard iteration of SPDE (10.22). In particular, we recall that this SPDE is linear with respect to the bias $A[u_n]$. The starting condition admits a decomposition, see equation (10.13), also the operator Γ admits the same decomposition, see equation (10.20). Thus, each term of the SPDE (10.22) can be separated into a sum. Clearly, Picard iteration preserves this property, and the decomposition gives birth to the functions $\Psi_m^{(n)}(t, x)$.

The proof ending using the fact that Picard iteration converges to the solution of SPDE (10.22) in L^2 -norm.

□

We conclude our analysis with a particular case:

Remark 10.6 (Independence) *If we assume that error structures on each sub-space, on which the functions space has been split, are independent, then the decomposition, given by functions $\Psi_m^{(n)}(t, x)$, splits the bias of the solution into independent sub-spaces; in the sense that the two coefficients $\Theta_k^{(n)}$ and $\theta_k^{(n)}$, defined in equations (10.26) and (10.28) respectively, become proportional to a Dirac delta $\delta_{n,k}$. In this case, the computation becomes more easy.*

10.4 Conclusion

In this chapter, we have studied how an uncertainty on the starting condition passes on the solution of a stochastic partial differential equation. We have considered a stochastic partial differential equation of heat diffusion type, with a colored noise, in order to simplify the proof of the well-posedness of solutions of SPDEs.

In order to describe the uncertainty on starting condition and to compute the uncertainty on the solution, we turn to error theory using Dirichlet forms, technique introduced by Bouleau. We have assumed that the uncertainty is very small with respect to the values taken by the function, thus we have used a hieratic strategy, we have computed the SPDE solution without uncertainty, then the variance-covariance of this solution and finally the bias induced by non-linearities.

We have find that variance-covariance can be easily estimated thanks to the sharp, a linear version of the standard deviation of the uncertainty, that verifies a linear parabolic stochastic partial differential equation. We have proved that the variance admits a decomposition in a series.

The study of the bias has allowed to show the existence of a stochastic partial differential equation verified by the bias of the solution. Furthermore, we have proved that the bias of the solution can be decomposed into a sum of terms that lives into the subspaces used to break up the starting condition.

We have also showed that the bias exists even if the starting condition are unbiased, it is enough that the stochastic partial equation would be non-linear. This result is very interesting, as a matter of fact, the bias induced only by the variance has a purely probabilistic origin and this fact modifies the behavior of the model below the stochastic partial differential equation.

In physics and climatology, noises play a central role. However, the starting condition has to be estimated too and this estimation is afflicted by an uncertainty, it is possible to evaluate its variance thanks to the Fisher information matrix, see Bouleau and Chorro [7] for an analysis of the relation between information matrix and error theory. The variance and the bias induced by this uncertainty have an important role in the forecast of a model.

Our analysis is a first study in the combination of the error theory using Dirichlet forms and stochastic partial differential equations, many others ways has to be examined, in particular, we work on the uncertainty in the functional structure of the SPDE, i.e. it is possible to consider an uncertainty on the functions σ and β , see chapter 11. Another interesting analysis will be the study of the transmission of an uncertainty on the diffusion coefficient, in this case the error will be propagated through the Green function. Finally, a more theoretical study will be the definition of an error structure on the colored noise itself.

Chapter 11

Uncertainty on Diffusion Parameters

In this chapter, we analyze the impact of an uncertainty on the functional coefficients of a stochastic partial differential equation using the technique of the error theory developed by Bouleau. There are two functional coefficients in a SPDE, the drift and the noise coefficient. A second interesting case is the analysis of a perturbation on the rate of diffusion (propagation) related to the Green's function of the parabolic (hyperbolic) stochastic partial differential equation. This second case is, in our opinion, the most interesting, since it enables us to evaluate the impact of a "noised Green's function". Moreover, all parameters in a physical or financial model must be estimated and these estimations are afflicted by an uncertainty. In particular, the uncertainty on the diffusion rate must play, always in our opinion, a crucial role in forecast model for climate. In order to propose our methodology, we choose a particular case, i.e. parabolic stochastic partial differential equations, since this case is more easy to treat thanks to the diffusive behavior of the related Green's function.

This chapter is organized as follows: Section 1 introduces parabolic stochastic partial differential equations and their mild solutions, when we consider a constant diffusion rate different to 1. Section 2 analyzes the diffusion of the uncertainties on drift and noise coefficient through an SPDE to its solution. In particular, we show that the sharp and bias of solution verify two SPDEs depending on the solution itself. Section 3 shows how error theory using Dirichlet forms may modelize a perturbation on the rate of diffusion of a stochastic partial differential equation, we show that the sharp verifies, formally, a new type of stochastic partial differential equation. Finally, section 4 resumes and concludes.

11.1 Parabolic stochastic partial differential equations

We consider a general parabolic stochastic partial differential equation for all (t, x) belonging to $\mathbb{R}^+ \times \mathbb{R}^d$,

$$(11.1) \quad \begin{cases} \frac{\partial u}{\partial t} - c\Delta u = \sigma(t, x, u(t, x)) \dot{C}(t, x) + \beta(t, x, u(t, x)) \\ u(0, x) = f(x) \end{cases}$$

where, Δ denotes the Laplacian operator in \mathbb{R}^d , c is a parameter used to describe the diffusion rate of the related Green's function, $\dot{C}(t, x)$ is a Gaussian noise white-in-time, see Dalang et al. [13], with spatial correlation function $k(\cdot, \cdot)$, i.e.

$$\mathbb{E}[\dot{C}(t, x) \dot{C}(s, y)] = \delta_0(t - s) k(x, y)$$

and $\sigma, \beta : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are functions that satisfy the classical properties, see Da Prato and Zabczyk [14], as well Lipschitz in the last variable. This problem admits a unique mild solution, see Da Prato [15] and Zabczyk [33], which is the following,

$$(11.2) \quad \begin{aligned} u(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \sigma(s, y, u(s, y)) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \beta(s, y, u(s, y)) ds dy \\ &+ \int_{\mathbb{R}^d} G_c(t, x - y) f(y) dy, \end{aligned}$$

where $G_c(t, x)$ is the Green function of the associated PDE, in our case

$$(11.3) \quad G_c(t, x) = (2\pi ct)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2ct}}.$$

The stochastic integral in relation (11.2) is defined in accord with classical theory, see Da Prato and Zabczyk [14] or section 10.2.1 in this thesis. The solution of SPDE (11.1) is given by the limit of the following Picard iteration,

$$(11.4) \quad \begin{aligned} u_0(t, x) &= \int_{\mathbb{R}^d} G_c(t, x - y) f(y) dy \\ &\text{and} \\ u_n(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \sigma(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \beta(s, y, u_{n-1}(s, y)) ds dy \\ &+ \int_{\mathbb{R}^d} G_c(t, x - y) f(y) dy. \end{aligned}$$

The solution converges to $u(t, x)$ in L^p -norm for any $p \geq 2$, see Nualart in [13], this result is an easy generalization of results of chapter 10 after the easy re-normalization $t \rightarrow ct$.

In the analysis of the uncertainties on drift and noise coefficients we fix $c = 1$. However, when we study the impact of a perturbation on rate diffusion, we have clearly to preserve this parameter.

11.2 Uncertainties in SPDEs due to functional coefficients

In this section, we analyze how an uncertainty on the functional coefficients is transmitted to the solution of a stochastic partial differential equation. We have showed, in previous chapter, the solution of a parabolic stochastic partial differential equation, using a Picard's series, and the impact of an error on the starting condition. Now, we analyze the impact of uncertainty on functional coefficients, driving the SPDE.

We consider that drift function $\beta(t, x, u)$ and volatility $\sigma(t, x, u)$ are characterized by an uncertainty on their value, we define an error structure for the space of function L^p from $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} , we follow the approach of Bouleau, see [5] pages 83-85 or section 1.5 in introductory part of this thesis.

We consider two bases $\beta_n(t, x, u)$ and $\sigma_n(t, x, u)$ of the considered function space; clearly, the first one will be used to develop the drift, the second one the volatility; we keep two distinguished bases in order to preserve the generality, it stands to reason that we can take the same basis for drift and volatility.

We represent drift function $\beta(t, x, u)$ and volatility one $\sigma(t, x, u)$ using the coefficients b_k and s_k respectively.

$$(11.5) \quad \begin{aligned} \beta(t, x, u) &= \sum_k b_k \beta_k(t, x, u) \\ \sigma(t, x, u) &= \sum_n s_n \sigma_n(t, x, u) \end{aligned}$$

with setting coefficients $\{b_k\}_{k \in \mathbb{N}}$ and $\{s_k\}_{k \in \mathbb{N}}$ to be random with an error structure on each sub-space spanned by functions $\beta_k(t, x, u)$ and $\sigma_k(t, x, u)$. Then, we have an error structure on the function space using an infinite product of structures, see Bouleau [5] pages 59-65 or section 1.3.2 in introductory part of this thesis.

Remark 11.1 (Correlation) *We do not assume any independence between two error structures related with two different sub-spaces, since we can perform the computation without this hypothesis. In a similar way, we can suppose that coefficients $\{b_k\}_{k \in \mathbb{N}}$ and $\{s_k\}_{k \in \mathbb{N}}$ are correlated among themselves. However, under the hypothesis of \mathbb{D} -independence, see Bouleau [8], the numerical evaluations of variance-covariance and bias are more simple and the study of bias is helped as well.*

We assume that this error structure admits a sharp operator, so we have the following representations.

$$(11.6) \quad \begin{aligned} \beta^\#(t, x, u) &= \sum_k b_k^\# \beta_k(t, x, u) \\ \sigma^\#(t, x, u) &= \sum_k s_k^\# \sigma_k(t, x, u) \end{aligned}$$

For sake of simplicity, we assume that only a finite number of coefficients $\{b_k\}_{k \in \mathbb{N}}$ and $\{s_k\}_{k \in \mathbb{N}}$ are erroneous, this hypothesis enables to make the proof easier. On the other hand, this assumption is very restrictive and many of our results remain true without. A possible way to force a finite number of erroneous variables, without external hypothesis, is presented in Bouleau [5] page 84, see also result 1.11 in introductory part.

We can also represent the bias on coefficients estimations:

$$(11.7) \quad \begin{aligned} \mathcal{A}[\beta](t, x, u) &= \sum_k \mathcal{A}[b_k] \beta_k(t, x, u) \\ \mathcal{A}[\sigma](t, x, u) &= \sum_k \mathcal{A}[s_k] \sigma_k(t, x, u) \end{aligned}$$

In the next two subsections, we study the variance-covariance and bias of the solution.

11.2.1 Variance and covariance of solution

In this subsection, we apply the operator Γ on solution $u(t, x)$ of SPDE (10.1), we use the sharp operator and we start with the specification of the SPDE verified by the sharp operator.

Theorem 11.1 (SPDE verified by sharp operator)

Under hypothesis that functions $\sigma_k(t, x, u)$ and $\beta_k(t, x, u)$ belong to C^1 w.r.t. the variable u and the functions with their derivatives are bounded and Lipschitzian, for all $k \in \mathbb{N}$. The sharp of the solution of SPDE (11.1) solves the following SPDE.

$$(11.8) \quad \left\{ \begin{aligned} \frac{\partial u^\#}{\partial t} - \Delta u^\# &= \left[\frac{\partial \sigma}{\partial u}(t, x, u(t, x)) u^\#(t, x) + \sum_k s_k^\# \sigma_k(t, x, u(t, x)) \right] \dot{C}(t, x) \\ &+ \frac{\partial \beta}{\partial u}(t, x, u(t, x)) u^\# + \sum_k b_k^\# \beta_k(t, x, u(t, x)) \\ u^\#(0, x) &= 0 \end{aligned} \right.$$

and its solution exists and is unique.

Proof: Our proof follows the ideas used in theorem 10.4, we simplify our proof admitting all results of convergence, since the basic arguments are the same. We concentrate on the computation of the Picard's iteration scheme verified by the sharp.

First of all, we remark that the starting condition is unerroneous in our setting. Therefore, the starting condition for SPDE (11.8) verified by the sharp must be equal to zero.

A second point is the following trivial relations for first derivative of functions $\sigma(t, x, u)$ and $\beta(t, x, u)$.

$$(11.9) \quad \begin{aligned} \frac{\partial \beta}{\partial u}(t, x, u) &= \sum_k b_k \frac{\partial \beta_k}{\partial u}(t, x, u) \\ \frac{\partial \sigma}{\partial u}(t, x, u) &= \sum_k s_k \frac{\partial \sigma_k}{\partial u}(t, x, u) \end{aligned}$$

Now, our proof follows the same arguments used in previous chapter, i.e. we apply the sharp operator on Picard's iteration (11.4) and we search to recognize the SPDE verified by the limit. We recall that Picard's iteration (11.4) associated with SPDE (11.1) is given by

$$\begin{aligned} u_0(t, x) &= \int_{\mathbb{R}^d} G(t, x - y) f(y) dy \\ \text{and} \\ u_n(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \sigma(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &\quad + \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \beta(s, y, u_{n-1}(s, y)) ds dy \\ &\quad + \int_{\mathbb{R}^d} G(t, x - y) f(y) dy, \end{aligned}$$

that can be rewritten using relations (11.5)

$$(11.10) \quad \begin{aligned} u_0(t, x) &= \int_{\mathbb{R}^d} G(t, x - y) f(y) dy \\ \text{and} \\ u_n(t, x) &= \sum_n s_n \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \sigma_n(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &\quad + \sum_n b_n \int_{[0, t] \times \mathbb{R}^d} G(t - s, x - y) \beta_n(s, y, u_{n-1}(s, y)) ds dy \\ &\quad + \int_{\mathbb{R}^d} G(t, x - y) f(y) dy. \end{aligned}$$

When we apply sharp operator, we find the following Picard's iteration.

$$u_0^\#(t, x) = 0$$

and

$$\begin{aligned} u_n^\#(t, x) &= \sum_k s_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \sigma_k(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &+ \sum_k s_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma_k}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\ &+ \sum_k b_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta_k(s, y, u_{n-1}(s, y)) ds dy \\ &+ \sum_k b_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta_k}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) ds dy \end{aligned}$$

Second and fourth integrals are well-defined since first derivative of $\sigma_k(tx, u)$ and $\beta_k(tx, u)$ with respect to u satisfy Lipschitz condition and $u_{n-1}^\#(s, y)$ is controlled by iteration. We can easily rewrite second and fourth integrals using relations (11.9).

$$u_0^\#(t, x) = 0$$

and

$$\begin{aligned} u_n^\#(t, x) &= \sum_k s_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \sigma_k(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\ &+ \sum_k b_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta_k(s, y, u_{n-1}(s, y)) ds dy \\ &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) ds dy \end{aligned}$$

Using the hypothesis of a finite number of erroneous coefficients, we have that sums on first and third integrals are finite. Thanks to the convergence of $u_{n-1}(s, y)$ to $u(s, y)$ in L^p -norm, we can replace it.

$$u_0^\#(t, x) = 0$$

and

$$\begin{aligned}
 (11.11) \quad u_n^\#(t, x) &= \sum_k s_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \sigma_k(s, y, u(s, y)) C(ds, dy) \\
 &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\
 &+ \sum_k b_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta_k(s, y, u(s, y)) ds dy \\
 &+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) u_{n-1}^\#(s, y) ds dy
 \end{aligned}$$

Now, we can recognize the Picard iteration of stochastic partial differential equation (11.8), the solution exists and it is unique since (11.8) is a parabolic linear SPDE with bounded and Lipschitz coefficients.

□

Theorem 11.1 as an interesting consequence:

Remark 11.2 (Affine SPDE) *Stochastic partial differential equation (11.8) is of type affine since, given the knowledge of $u(t, x)$, it is possible to rewrite it in the following way:*

$$\left\{ \begin{array}{l} \frac{\partial u^\#}{\partial t} - \Delta u^\# = [\bar{\sigma}_1(t, x) u^\#(t, x) + \bar{\sigma}_2(t, x)] \dot{C}(t, x) \\ \qquad \qquad \qquad + \bar{\beta}_1(t, x) u^\#(t, x) + \bar{\beta}_2(t, x) \\ u^\#(0, x) = 0 \end{array} \right.$$

Theorem 10.1 can be slightly modified in order to give us a series for the sharp of the solution.

Corollary 11.2 (Series for solution sharp)

Under hypotheses of theorems 10.1 and 11.1, we have the following series for the sharp of solution (11.2).

$$(11.12) \quad u^\#(t, x) = \sum_{m=0}^{\infty} I_m^{(u^\#)}(t, x)$$

where

$$\begin{aligned}
I_0^{(u^\#)}(t, x) &= \sum_k s_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \sigma_k(s, y, u(s, y)) C(ds, dy) \\
&\quad + \sum_k b_k^\# \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta_k(s, y, u(s, y)) ds dy \\
(11.13) \quad I_{m+1}^{(u^\#)}(t, x) &= \int_{[0, t]} \int_{\mathbb{R}^d} G(t-s, x-y) I_m^{(u^\#)}(s, y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) C(ds, dy) \\
&\quad + \int_{[0, t]} \int_{\mathbb{R}^d} G(t-s, x-y) I_m^{(u^\#)}(s, y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) ds dy.
\end{aligned}$$

In a similar way of the analysis in chapter 10, we can split terms $I_m^{(u^\#)}(s, y)$ into a sum.

Result 11.3 (Sharp of SPDE solution)

The sharp of the solution of SPDE (11.1) can be decomposed as

$$u^\# = \sum_{k, n} s_k^\# J_{n, k}^{(\sigma)}(t, x) + \sum_{k, n} b_k^\# J_{n, k}^{(\beta)}(t, x),$$

where

$$\begin{aligned}
J_{0, k}^{(\sigma)}(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \sigma_k(s, y, u(s, y)) C(ds, dy) \\
J_{0, k}^{(\beta)}(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \beta_k(s, y, u(s, y)) ds dy \\
(11.14) \quad J_{n+1, k}^{(\sigma)}(t, x) &= \int_{[0, t]} \int_{\mathbb{R}^d} G(t-s, x-y) J_{n, k}^{(\sigma)}(t, x) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) C(ds, dy) \\
&\quad + \int_{[0, t]} \int_{\mathbb{R}^d} G(t-s, x-y) J_{n, k}^{(\sigma)}(t, x) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) ds dy \\
J_{n+1, k}^{(\beta)}(t, x) &= \int_{[0, t]} \int_{\mathbb{R}^d} G(t-s, x-y) J_{n, k}^{(\beta)}(t, x) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) C(ds, dy) \\
&\quad + \int_{[0, t]} \int_{\mathbb{R}^d} G(t-s, x-y) J_{n, k}^{(\beta)}(t, x) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) ds dy.
\end{aligned}$$

Proof: This result can be easily proved by induction.

□

Thanks to decomposition (11.14), we can write explicitly the variance of the solution.

Result 11.4 (Gamma of the solution)

The variance-covariance operator acts on the solution of SPDE (11.1) in the following way.

$$\begin{aligned}
 \Gamma[u(t, x), u(s, y)] &= \sum_{j, k, n m} \Gamma[s_k, s_j] J_{n, k}^{(\sigma)}(t, x) J_{m, j}^{(\sigma)}(s, y) \\
 &+ \sum_{j, k, n m} \Gamma[b_k, s_j] J_{n, k}^{(\beta)}(t, x) J_{m, j}^{(\sigma)}(s, y) \\
 &+ \sum_{j, k, n m} \Gamma[s_k, b_j] J_{n, k}^{(\sigma)}(t, x) J_{m, j}^{(\beta)}(s, y) \\
 &+ \sum_{j, k, n m} \Gamma[b_k, b_j] J_{n, k}^{(\beta)}(t, x) J_{m, j}^{(\beta)}(s, y)
 \end{aligned}
 \tag{11.15}$$

We conclude our analysis with the same two remarks underlined in our analysis in chapter 10.

Remark 11.3 (Linearity) Equation (11.15) shows that variance-covariance operator admits an easy decomposition a sum of four products of two terms. The first one is the covariance between the two erroneous coefficients of the decomposition of functional coefficients $\beta(t, x, u)$ and $\sigma(t, x, u)$. The second term catches the evolution of the solution through the SPDE but it is unrelated with the uncertainties on functional coefficients $\beta(t, x, u)$ and $\sigma(t, x, u)$. This decomposition depends crucially on the linearity of SPDE (11.8) verified by the sharp.

Remark 11.4 (Independence) We can analyze two types of independence:

- independence among the uncertainties on functions $\beta(t, x, u)$ and $\sigma(t, x, u)$, and
- independence among the uncertainties on functions decompositions $\beta_k(t, x, u)$ and $\sigma_k(t, x, u)$.

Under first hypothesis, we have

$$\Gamma[s_k, b_j] = \Gamma[b_k, s_j] = 0$$

for all k and j . Under the second one we have clearly

$$\begin{aligned}
 \Gamma[s_k, s_j] &\propto \delta_{k, j} \\
 \Gamma[b_k, b_j] &\propto \delta_{k, j}
 \end{aligned}$$

but also

$$\Gamma[b_k, s_j] \propto \delta_{k, j},$$

since it is possible that we have chosen a unique basis in order to expand function $\beta(t, x, u)$ and $\sigma(t, x, u)$, thus a correlation can exist between uncertainties on functions $\beta_k(t, x, u)$ and $\sigma_k(t, x, u)$ with the same index.

Under the first type of independence, variance-covariance (11.15) can be simplified into

$$\begin{aligned} \Gamma[u(t, x), u(s, y)] &= \sum_{j, k, n m} \Gamma[s_k, s_j] J_{n, k}^{(\sigma)}(t, x) J_{m, j}^{(\sigma)}(s, y) \\ &+ \sum_{j, k, n m} \Gamma[b_k, b_j] J_{n, k}^{(\beta)}(t, x) J_{m, j}^{(\beta)}(s, y). \end{aligned}$$

Under the second hypothesis we find

$$\begin{aligned} \Gamma[u(t, x), u(s, y)] &= \sum_{k, n m} \Gamma[s_k] J_{n, k}^{(\sigma)}(t, x) J_{m, k}^{(\sigma)}(s, y) \\ &+ \sum_{k, n m} \Gamma[b_k, s_k] \left\{ J_{n, k}^{(\beta)}(t, x) J_{m, k}^{(\sigma)}(s, y) + J_{n, k}^{(\sigma)}(t, x) J_{m, k}^{(\beta)}(s, y) \right\} \\ &+ \sum_{k, n m} \Gamma[b_k] J_{n, k}^{(\beta)}(t, x) J_{m, k}^{(\beta)}(s, y). \end{aligned}$$

Under both of them, we have

$$\begin{aligned} \Gamma[u(t, x), u(s, y)] &= \sum_{k, n m} \Gamma[s_k] J_{n, k}^{(\sigma)}(t, x) J_{m, j}^{(\sigma)}(s, y) \\ &+ \sum_{k, n m} \Gamma[b_k] J_{n, k}^{(\beta)}(t, x) J_{m, k}^{(\beta)}(s, y). \end{aligned}$$

Therefore, a good choice of the representation basis enables us an optimization in variance computation.

11.2.2 Bias of solution

Given the function of variance-covariance we can study the bias.

Theorem 11.5 (SPDE verified by bias operator)

Under the hypothesis that functions $\sigma_k(t, x, u)$ and $\beta_k(t, x, u)$ belong to C^2 w.r.t. the variable u , for all k , and these functions with their derivatives are bounded and Lipschitz. We assume the ARB condition hold, see section 1.6. The bias of the solution of SPDE (11.1) verifies the following SPDE.

$$(11.16) \quad \left\{ \begin{aligned} \frac{\partial \mathcal{A}[u]}{\partial t} - \Delta A[u] &= \frac{\partial \sigma}{\partial u}(t, x, u(t, x)) \mathcal{A}[u](t, x) \dot{C}(t, x) \\ &+ \frac{\partial \beta}{\partial u}(t, x, u(t, x)) \mathcal{A}[u](t, x) \\ &+ \frac{1}{2} \frac{\partial^2 \sigma}{\partial u^2}(t, x, u(t, x)) \Gamma[u](t, x) \dot{C}(t, x) \\ &+ \frac{1}{2} \frac{\partial^2 \beta}{\partial u^2}(t, x, u(t, x)) \Gamma[u](t, x) \\ &+ \mathcal{A}[\sigma](t, x, u(t, x)) \dot{C}(t, x) + \mathcal{A}[\beta](t, x, u(t, x)) \\ &+ \sum_{k,j,n} \frac{\partial \sigma_k}{\partial u}(s, y, u(s, y)) \left\{ \Gamma[s_k, s_j] J_{n,j}^{(\sigma)}(t, x) \right. \\ &+ \left. \Gamma[s_k, b_j] J_{n,j}^{(\beta)}(t, x) \right\} \dot{C}(t, x) + \sum_{k,j,n} \frac{\partial \beta_k}{\partial u}(s, y, u(s, y)) \\ &\star \left\{ \Gamma[b_k, s_j] J_{n,j}^{(\sigma)}(t, x) + \Gamma[b_k, b_j] J_{n,j}^{(\beta)}(t, x) \right\} \\ &\mathcal{A}[u](0, x) = 0 \end{aligned} \right.$$

where $\Gamma[u](t, x)$ is given by equation (11.15). $A[\sigma](t, x, u(t, x))$ and $A[\beta](t, x, u(t, x))$ denotes, respectively, $\sum_k A[s_k] \sigma_k(t, x, u(t, x))$ and $\sum_k A[b_k] \beta_k(t, x, u(t, x))$, in accord with relations (11.7). $J_{n,j}^{(\sigma)}(t, x)$ and $J_{n,j}^{(\beta)}(t, x)$ are given by relations (11.14).

Moreover, the solution of this SPDE exists and it is unique.

Proof:

The arguments of our proof are similar to the one of theorem 10.8, we give only the computations without any further argument about convergence. We start our proof applying bias operator \mathcal{A} to Picard iteration (11.4), we have fixed $c = 1$ for sake of simplicity. First term $u_0(t, x)$ is unbiased since Green's function $G(t, x)$ and starting condition $f(x)$ are unerroneous. So we find

$$\mathcal{A}[u_0(t, x)] = 0$$

and

$$\begin{aligned} \mathcal{A}[u_n(t, x)] &= \mathcal{A} \left[\int_{[0,t] \times \mathbb{R}^d} G(t-s, x-y) \sigma(s, y, u_{n-1}(s, y)) C(ds, dy) \right] \\ &+ \mathcal{A} \left[\int_{[0,t] \times \mathbb{R}^d} G_c(t-s, x-y) \beta(s, y, u_{n-1}(s, y)) ds dy \right]. \end{aligned}$$

we can easily exchange bias operator with integrals and, using relations (11.5), we find

$$\begin{aligned} \mathcal{A}[u_n(t, x)] &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A} \left[\sum_k s_k \sigma_k(s, y, u_{n-1}(s, y)) \right] C(ds, dy) \\ &\quad + \int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \mathcal{A} \left[\sum_k b_k \beta_k(s, y, u_{n-1}(s, y)) \right] ds dy, \end{aligned}$$

since Green's function $G(t, x)$ and colored noise $\dot{C}(s, y)$ are not erroneous. Using bias operator properties, see section 1.2, we can rewrite the previous equation.

$$\begin{aligned} \mathcal{A}[u_n(t, x)] &= \sum_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A}[s_k] \sigma_k(s, y, u_{n-1}(s, y)) C(ds, dy) \\ &\quad + \sum_k s_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma_k}{\partial u}(s, y, u_{n-1}(s, y)) \mathcal{A}[u_{n-1}(s, y)] C(ds, dy) \\ &\quad + \frac{1}{2} \sum_k s_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \sigma_k}{\partial u^2}(s, y, u_{n-1}(s, y)) \Gamma[u_{n-1}(s, y)] C(ds, dy) \\ &\quad + \sum_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma_k}{\partial u}(s, y, u_{n-1}(s, y)) \widehat{\mathbb{E}} \left[s_k^\# u_{n-1}^\#(s, y) \right] C(ds, dy) \\ &\quad + \sum_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A}[b_k] \beta_k(s, y, u_{n-1}(s, y)) ds dy \\ &\quad + \sum_k b_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta_k}{\partial u}(s, y, u_{n-1}(s, y)) \mathcal{A}[u_{n-1}(s, y)] ds dy \\ &\quad + \frac{1}{2} \sum_k b_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \beta_k}{\partial u^2}(s, y, u_{n-1}(s, y)) \Gamma[u_{n-1}(s, y)] ds dy \\ &\quad + \sum_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta_k}{\partial u}(s, y, u_{n-1}(s, y)) \widehat{\mathbb{E}} \left[b_k^\# u_{n-1}^\#(s, y) \right] ds dy \end{aligned}$$

Using the convergence of $\Gamma[u_{n-1}(s, y)]$ to $\Gamma[u(s, y)]$ in L^2 -norm, see theorem 11.1 and result 11.4, we can exchange them, and the difference is controlled. We have to evaluate two terms, i.e.

$$\begin{aligned} &\widehat{\mathbb{E}} \left[b_k^\# u_{n-1}^\#(s, y) \right] \\ &\widehat{\mathbb{E}} \left[s_k^\# u_{n-1}^\#(s, y) \right], \end{aligned}$$

using result 11.3, we have

$$\begin{aligned}\widehat{\mathbb{E}} \left[b_k^\# u^\#(s, y) \right] &= \sum_{j, n} \Gamma[b_k, s_j] J_{n, j}^{(\sigma)}(s, y) + \sum_{j, n} \Gamma[b_k, b_j] J_{n, j}^{(\beta)}(s, y) \\ \widehat{\mathbb{E}} \left[s_k^\# u^\#(s, y) \right] &= \sum_{j, n} \Gamma[s_k, s_j] J_{n, j}^{(\sigma)}(s, y) + \sum_{j, n} \Gamma[s_k, b_j] J_{n, j}^{(\beta)}(s, y),\end{aligned}$$

where $J_{n, j}^{(\beta)}(t, x)$ and $J_{n, j}^{(\sigma)}(t, x)$ are given by relation (11.14). The convergence of $u_{n-1}^\#(s, y)$ to $u^\#(s, y)$, guaranteed by theorem 11.1, enables us to exchange them. In a similar way we can exchange $u_{n-1}(s, y)$ with $u(s, y)$, using the convergence of Picard iteration to the SPDE solution. Therefore, we can write the following Picard iteration for the bias of the SPDE solution.

$$\begin{aligned}\mathcal{A}[u_n(t, x)] &= \sum_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A}[s_k] \sigma_k(s, y, u(s, y)) C(ds, dy) \\ &+ \sum_k s_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma_k}{\partial u}(s, y, u(s, y)) \mathcal{A}[u_{n-1}(s, y)] C(ds, dy) \\ &+ \frac{1}{2} \sum_k s_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \sigma_k}{\partial u^2}(s, y, u(s, y)) \Gamma[u(s, y)] C(ds, dy) \\ &+ \sum_{k, j, n} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma_k}{\partial u}(s, y, u(s, y)) \left\{ \Gamma[s_k, s_j] J_{n, j}^{(\sigma)}(s, y) \right. \\ &+ \left. \Gamma[s_k, b_j] J_{n, j}^{(\beta)}(s, y) \right\} C(ds, dy) \\ &+ \sum_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A}[b_k] \beta_k(s, y, u(s, y)) ds dy \\ &+ \sum_k b_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta_k}{\partial u}(s, y, u(s, y)) \mathcal{A}[u_{n-1}(s, y)] ds dy \\ &+ \frac{1}{2} \sum_k b_k \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \beta_k}{\partial u^2}(s, y, u(s, y)) \Gamma[u(s, y)] ds dy \\ &+ \sum_{k, j, n} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta_k}{\partial u}(s, y, u(s, y)) \left\{ \Gamma[b_k, s_j] J_{n, j}^{(\sigma)}(s, y) \right. \\ &+ \left. \Gamma[b_k, b_j] J_{n, j}^{(\beta)}(s, y) \right\} ds dy\end{aligned}$$

We can simplify our Picard iteration using relations (11.5), (11.7), (11.9) and its generalisation to second derivative. Thus, we find

(11.17)

$$\begin{aligned}
\mathcal{A}[u_n(t, x)] &= \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A}[\sigma](s, y, u(s, y)) C(ds, dy) \\
&+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) \mathcal{A}[u_{n-1}(s, y)] C(ds, dy) \\
&+ \frac{1}{2} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \sigma}{\partial u^2}(s, y, u(s, y)) \Gamma[u(s, y)] C(ds, dy) \\
&+ \sum_{k, j, n} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \sigma_k}{\partial u}(s, y, u(s, y)) \left\{ \Gamma[s_k, s_j] J_{n, j}^{(\sigma)}(s, y) \right. \\
&\left. + \Gamma[s_k, b_j] J_{n, j}^{(\beta)}(s, y) \right\} C(ds, dy) \\
&+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \mathcal{A}[\beta](s, y, u(s, y)) ds dy \\
&+ \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) \mathcal{A}[u_{n-1}(s, y)] ds dy \\
&+ \frac{1}{2} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial^2 \beta}{\partial u^2}(s, y, u(s, y)) \Gamma[u(s, y)] ds dy \\
&+ \sum_{k, j, n} \int_{[0, t] \times \mathbb{R}^d} G(t-s, x-y) \frac{\partial \beta_k}{\partial u}(s, y, u(s, y)) \left\{ \Gamma[b_k, s_j] J_{n, j}^{(\sigma)}(s, y) \right. \\
&\left. + \Gamma[b_k, b_j] J_{n, j}^{(\beta)}(s, y) \right\} ds dy
\end{aligned}$$

We recognize stochastic partial differential equation (11.16) that is an affine SPDE with boundary and Lipschitz coefficients. Therefore, its solution exists and it is unique, e.g. see Da Prato [15] or Zabczyk [33].

□

11.3 Uncertainties in SPDEs Green function

In this section, we analyze how an uncertainty in the diffusion coefficient is transmitted to the solution of the stochastic partial differential equation. We suppose that the diffusion coefficient c is erroneous and we define an error structure for this parameter. Since the fact that c is a one-dimensional parameter, the Dirichlet form is defined in \mathbb{R} and Hamza, [22], has defined the class of coherent Dirichlet forms in \mathbb{R} , we reduce our analysis at the following type of error structure

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, \mathbb{D}, u(x) \rightarrow \eta(x) [u'(x)]^2)$$

where μ is a probability measure absolutely continuous with respect to the Lebesgue one and we assume that this error structure admits a sharp operator denoted $(\cdot)^\#$.

We start our analysis with the study of the erroneous Green function:

Proposition 11.6 (Erroneous Green function)

The solution of the problem

$$(11.18) \quad \begin{cases} \frac{\partial G_c}{\partial t}(t, x) - c \Delta G_c(t, x) = 0 \\ G_c(0, x) = \delta_{x=0} \end{cases}$$

when the parameter c is erroneous, is characterized by the following sharp, variance and bias:

$$(11.19) \quad \begin{aligned} G_c^\#(t, x) &= G_c(t, x) \left[\frac{|x|^2}{2tc} - \frac{d}{2} \right] \frac{c^\#}{c} \\ \Gamma[G_c(t, x), G_c(s, y)] &= G_c(t, x) G_c(s, y) \left[\frac{|x|^2}{2tc} - \frac{d}{2} \right] \left[\frac{|y|^2}{2sc} - \frac{d}{2} \right] \frac{\Gamma[c]}{c^2} \\ \mathcal{A}[G_c(t, x)] &= G_c(t, x) \left[\frac{|x|^2}{2tc} - \frac{d}{2} \right] \frac{\mathcal{A}[c]}{c} \\ &\quad + \frac{1}{2} \left\{ \frac{|x|^4}{4c^2 t^2} - \frac{|x|^2}{ct} \left(1 + \frac{d}{2} \right) + \frac{d^2}{4} + \frac{d}{2} \right\} \frac{\Gamma[c]}{c^2} \end{aligned}$$

Proof: Clearly function (11.3) is the solution of PDE (11.18), accordingly with the properties of the sharp, see definition 1.2, an easy computation gives us the sharp of the function $G_c(t, x)$, the carré du champ and the bias.

□

We have a second interesting characterization of the sharp of Green function $G_c(t, x)$.

Proposition 11.7 (PDE verified by sharp)

The sharp of Green function, see relation (11.19), verifies the following partial differential equation.

$$(11.20) \quad \begin{cases} \frac{\partial G_c^\#}{\partial t}(t, x) - c^\# \Delta G_c(t, x) - c \Delta G_c^\#(t, x) = 0 \\ G_c^\#(0, x) = 0 \end{cases}$$

Proof: An easy computation gives us:

$$(11.21) \quad \bar{\nabla} G_c = \frac{G_c}{2tc} \bar{x}$$

$$\Delta G_c = \frac{G_c}{ct} \left[\frac{|x|^2}{2ct} - \frac{d}{2} \right]$$

$$(11.22) \quad \frac{\partial G_c^\#}{\partial t} = \frac{\partial G_c}{\partial t} \left[\frac{|x|^2}{2tc} - \frac{d}{2} \right] \frac{c^\#}{c} - G_c \frac{|x|^2 c^\#}{2c^2 t^2}$$

$$\bar{\nabla} G_c^\# = \left[\frac{|x|^2}{2ct} - \frac{d}{2} \right] \frac{c^\#}{c} \bar{\nabla} G_c + G_c \frac{c^\#}{2c^2 t} \bar{x}$$

$$\Delta G_c^\# = \left[\frac{|x|^2}{2tc} - \frac{d}{2} \right] \frac{c^\#}{c} \Delta G_c - G_c \frac{c^\#}{c^2 t} \left[\frac{|x|^2}{tc} - \frac{d}{2} \right]$$

Using the first identity in (11.18), we find that $G_c^\#$ verifies partial differential equation (11.20). □

We can make some remarks about the uncertainty on Green function $G_c(t, x)$:

Remark 11.5 (Isotropy) *The isotropy of function $G_c(t, x)$ is transferred to its sharp and its bias; as a matter of fact, only the distance with respect to the origin is relevant*

Remark 11.6 (Typical length) *First relation in (11.19) exhibits an interesting typical length $\sqrt{d}tc$; if we rescale our problem using*

$$(11.23) \quad \lambda^2 = \frac{|x|^2}{d} \frac{t}{c},$$

we find that the ratio between $G_c^\#(t, \lambda)$ and $c^\#$ is negative when $\lambda < 1$ and positive otherwise. Therefore, at each time t , our space is divided into two areas, a circle and its complement. The boundary circumference has no variance (but it will be biased). Two points of the same area are positively correlated, whereas two points of different areas are negatively correlated.

Remark 11.7 (Unbiased curve) *It is clear, seeing formulae (11.19), that it is impossible to find a locus of unerroneous points, i.e. points where function $G_c(t, x)$ is unbiased and with zero variance. However, there is a locus of points where function $G_c(t, x)$ is unbiased*

$$(11.24) \quad \frac{2c \mathcal{A}[c]}{\Gamma[c]} = \frac{d^2}{4} (\lambda^2 - 1) - d - \frac{d}{2(\lambda^2 - 1)},$$

where we have used notation (11.23). It is plain that the equation is almost linear around this point.

11.3.1 Variance and covariance of solution

Using the results on Green function, we can study the sharp operator applied to the solution $u(t, x)$ of stochastic partial differential equation (11.1). We show that the sharp verifies a new type of SPDE.

We start with an hypothesis.

Hypothesis 11.1 (Convergence)

We assume that

$$(11.25) \quad \sup_{t \in [0, t]} G_c(t, x) \left[\frac{|x|^2}{2tc} - \frac{d}{2} \right] < \infty$$

for all $x \in \mathbb{R}^d$, and

$$(11.26) \quad \int_{[0, t] \times \mathbb{R}^d} \nu(dx) \left| \mathcal{F} \left[G_c(t, x) \left(\frac{|x|^2}{2tc} - \frac{d}{2} \right) \right] \right|^2 dt < \infty,$$

where \mathcal{F} denotes the Fourier transform and ν is the non-negative tempered measure associated with the noise $C(x, t)$ via Bochner's theorem, see Schwartz [28] theorem XIX chapter VII.

Then we have the following theorem.

Theorem 11.8 (SPDE verified by sharp operator)

Under the hypothesis that functions $\sigma(t, x, u)$ and $\beta(t, x, u)$ belong to C^2 w.r.t. the variable u , and these functions with their derivatives are bounded and Lipschitz, and, finally, under hypothesis 11.1; the sharp of the solution of SPDE (11.1) verifies the following stochastic partial differential equation.

$$(11.27) \quad \begin{cases} \frac{\partial u^\#}{\partial t} - c \Delta u^\# - c^\# \Delta u = \sigma(t, x, u(t, x)) \dot{C}(t, x) + \beta(t, x, u(t, x)) \\ u(0, x) = 0 \end{cases}$$

In the sense that its solution is the mild one

$$\begin{aligned}
(11.28) \quad u^\#(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) u^\#(s, y) C(ds, dy) \\
&+ \int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) u^\#(s, y) ds dy \\
&+ \frac{c^\#}{c} \left\{ \int_{\mathbb{R}^d} G_c(t, x-y) \left[\frac{|x-y|^2}{2tc} - \frac{d}{2} \right] f(y) dy \right. \\
&+ \int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \beta(s, y, u(s, y)) ds dy \\
&\left. + \int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \sigma(s, y, u(s, y)) C(ds, dy) \right\}.
\end{aligned}$$

In order to prove this theorem we need the following easy lemma.

Lemma 11.9

Under hypothesis 11.1 and if functions $\sigma(t, x, u)$ and $\beta(t, x, u)$ are bounded and Lipschitz, we have that the convolution integrals

$$\begin{aligned}
(11.29) \quad &\int_{\mathbb{R}^d} G_c(t, x-y) \left[\frac{|x-y|^2}{2tc} - \frac{d}{2} \right] f(y) dy, \\
&\int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \beta(s, y, u(s, y)) ds dy \text{ and} \\
&\int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \sigma(s, y, u(s, y)) C(ds, dy)
\end{aligned}$$

are well-defined, for all $n \in \mathbb{N}$, and we have two following convergences in L^2 -norm.

$$\begin{aligned}
&\int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \sigma(s, y, u_n(s, y)) C(ds, dy) \rightarrow \\
&\int_{[0, t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \sigma(s, y, u(s, y)) C(ds, dy)
\end{aligned}$$

and

$$\begin{aligned} & \int_{[0,t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \beta(s, y, u_n(s, u)) \, ds \, dy \rightarrow \\ & \int_{[0,t] \times \mathbb{R}^d} G_c(t-s, x-y) \left[\frac{|x-y|^2}{2(t-s)c} - \frac{d}{2} \right] \beta(s, y, u(s, u)) \, ds \, dy \end{aligned}$$

Proof: The proof of this lemma is straightforward. Hypothesis 11.1 is hypothesis D in Sanz-Sole, see [27] page 80. Thus the third convolution integral in (11.29) is well-defined according with theorem 6.2 in [27] page 85. The two others are well-defined using the fact that function $G_c(t, x)$ belongs to the space of test functions, and all other functions are bounded.

The two convergences are guaranteed by the hypothesis of Lipschitz function $\sigma(t, x, u)$ and $\beta(t, x, u)$ with respect to u , combined with the fact that $u_n(t, x)$ converges to $u(t, x)$ in L^p -norm, $p \geq 2$ and, finally, using the same proof of theorem 6.2 in [27] page 85. □

Now, we can prove theorem 11.8.

Proof: Our proof of theorem 11.8 follows the ideas used for theorem 10.4, we simplify our proof admitting all results of convergence showed in this theorem, since the basic arguments are the same. We concentrate on the computation of Picard iteration scheme.

We consider Picard iteration scheme (11.4) related to SPDE (11.1). We apply sharp operator and we use the linearity of sharp, see definition 1.2. Therefore, we find the following Picard iteration scheme.

$$\begin{aligned} u_0^\#(t, x) &= \int_{\mathbb{R}^d} G_c^\#(t, x-y) f(y) \, dy \\ &\text{and} \\ u_n^\#(t, x) &= \int_{[0,t] \times \mathbb{R}^d} G_c^\#(t-s, x-y) \sigma(s, y, u_{n-1}(s, y)) C(ds, dy) \\ (11.30) \quad &+ \int_{[0,t] \times \mathbb{R}^d} G_c(t-s, x-y) \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\ &+ \int_{[0,t] \times \mathbb{R}^d} G_c^\#(t-s, x-y) \beta(s, y, u_{n-1}(s, y)) \, ds \, dy \\ &+ \int_{[0,t] \times \mathbb{R}^d} G_c(t-s, x-y) \frac{\partial \beta}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) \, ds \, dy \\ &+ \int_{\mathbb{R}^d} G_c^\#(t, x-y) f(y) \, dy, \end{aligned}$$

where we have used the fact that starting condition $f(x)$ and noise $C(t, x)$ are not erroneous. We use the result of theorem 11.6 and in particular the formula for $G_c^\#(t, x)$, see equation 11.19. So, we find

(11.31)

$$u_0^\#(t, x) = c^\# \int_{\mathbb{R}^d} G_c(t, x - y) \left[\frac{|x - y|^2}{2t c^2} - \frac{d}{2} \right] f(y) dy$$

and

$$\begin{aligned} u_n^\#(t, x) &= \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \frac{\partial \sigma}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\ &+ \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \frac{\partial \beta}{\partial u}(s, y, u_{n-1}(s, y)) u_{n-1}^\#(s, y) ds dy \\ &+ \frac{c^\#}{c} \left\{ \int_{\mathbb{R}^d} G_c(t, x - y) \left[\frac{|x - y|^2}{2t c} - \frac{d}{2} \right] f(y) dy \right. \\ &+ \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \left[\frac{|x - y|^2}{2(t - s)c} - \frac{d}{2} \right] \beta(s, y, u_{n-1}(s, y)) ds dy \\ &\left. + \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \left[\frac{|x - y|^2}{2(t - s)c} - \frac{d}{2} \right] \sigma(s, y, u_{n-1}(s, y)) C(ds, dy) \right\} \end{aligned}$$

Now, we remark that the n-th term of Picard series (11.31) has two convolution integrals depending on classical Green function $G_c(t, x)$ and three convolution integrals, only one for the first term, depending on a new exogenous Green function

$$(11.32) \quad \widehat{G}_c(t, x) = G_c(t, x) \left[\frac{|x|^2}{2t c} - \frac{d}{2} \right]$$

and we can remark that these integrals are well-defined using lemma 11.9.

Now, we know that $u_n(t, x)$ converges to $u(t, x)$ in L^p -norm, $p \geq 2$; using the fact that first derivative of functions $\sigma(s, y, u)$ and $\beta(s, y, u)$ are bounded and Lipschitz, we can exchange the two terms in the two first convolution integrals of series (11.31).

Therefore, we have

$$u_0^\#(t, x) = c^\# \int_{\mathbb{R}^d} G_c(t, x - y) \left[\frac{|x - y|^2}{2t c^2} - \frac{d}{2} \right] f(y) dy$$

and

$$\begin{aligned} u_n^\#(t, x) = & \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \frac{\partial \sigma}{\partial u}(s, y, u(s, y)) u_{n-1}^\#(s, y) C(ds, dy) \\ & + \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \frac{\partial \beta}{\partial u}(s, y, u(s, y)) u_{n-1}^\#(s, y) ds dy \\ & + \frac{c^\#}{c} \left\{ \int_{\mathbb{R}^d} G_c(t, x - y) \left[\frac{|x - y|^2}{2t c} - \frac{d}{2} \right] f(y) dy \right. \\ & + \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \left[\frac{|x - y|^2}{2(t - s)c} - \frac{d}{2} \right] \beta(s, y, u(s, y)) ds dy \\ & \left. + \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \left[\frac{|x - y|^2}{2(t - s)c} - \frac{d}{2} \right] \sigma(s, y, u(s, y)) C(ds, dy) \right\}, \end{aligned}$$

that is the Picard serie of the mild solution (11.28).

□

We highlight the connection between SPDE (11.27) and its mild solution (11.28) in the following remark.

Remark 11.8 (Link between SPDE (11.27) and its mild solution) *First of all, we recall that the sharp of Green function $G_c(t, x)$ verifies PDE (11.20), thus we have a good first reason to use SPDE (11.27) in order to represent mild solution (11.28).*

Moreover, we analyze the last term in mild solution (11.28), it is composed of three convolution integrals depending on the solution $u(t, x)$ of original SPDE (11.1). Using the integral form of mild solution (11.2) and the relation (11.22) for the Laplacian of the Green function, we can define, formally, how the Laplacian operator acts on the solution $u(t, x)$:

$$\begin{aligned} (11.33) \quad c \Delta u(t, x) \equiv & \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \left[\frac{|x - y|^2}{2(t - s)c} - \frac{d}{2} \right] \sigma(s, y, u(s, y)) C(ds, dy) \\ & + \int_{[0, t] \times \mathbb{R}^d} G_c(t - s, x - y) \left[\frac{|x - y|^2}{2(t - s)c} - \frac{d}{2} \right] \beta(s, y, u(s, y)) ds dy \\ & + \int_{\mathbb{R}^d} G_c(t, x - y) \left[\frac{|x - y|^2}{2t c} - \frac{d}{2} \right] f(y) dy. \end{aligned}$$

11.4 Applications of stochastic partial differential equations

In this section, we present a survey of some different fields of application of stochastic partial differential equations. We have divided our analysis in four subsections depending on the domains of application.

11.4.1 Climatology: evolution models

Many applications of stochastic partial differential equations exist in climatology. For instance, we introduce the quasi-geostrophic model, see Pedlosky [25], [26] and Duan et al. [17].

The quasi-geostrophic model is a simplified geophysical model devided as an approximation of the shallow water equations at asymptotically high rotation rate.

The flow stream function $\phi(t, x, y)$ verifies the following equation

$$(11.34) \quad \begin{aligned} \frac{\partial \Delta \phi}{\partial t} + J(\phi, \Delta \phi) + \beta \frac{\partial \phi}{\partial x} &= \nu \Delta^2 \phi - r \Delta \phi + \dot{W}_2 \text{ on } \mathcal{D} \\ \phi(t, x, y) &= 0 \text{ on } \partial \mathcal{D} \\ \frac{\partial}{\partial n} \Delta \phi &= \dot{W}_1 \text{ on } \partial \mathcal{D} \end{aligned}$$

where

- D is the space domain,
- β is the gradient of Coriolis parameter,
- ν denotes the viscosity constant,
- r is the Ekman dissipation constant,
- $J(f, g)$ is the Jacobian operator, which is defined by $J(f, g) = f_x g_y - g_x f_y$,
- n is outward normal vector on the boundary $\partial \mathcal{D}$,
- W_1 is a white noise, used to model the starting condition, while
- W_2 is a white noise, used to describe the wind forcing.

It is possible to transform problem (11.34) into a classical parabolic stochastic partial differential equation.

$$(11.35) \quad \begin{aligned} \frac{\partial u}{\partial t}(t, x, y) + J(u, G(u)) + \beta \frac{\partial G(u)}{\partial x} &= \nu \Delta u(t, x, y) - r u(t, x, y) + \dot{W}_2 \text{ on } \mathcal{D} \\ \frac{\partial u}{\partial n} &= \dot{W}_1 \text{ on } \partial \mathcal{D} \end{aligned}$$

where $G(\cdot)$ is the inverse operator of the value problem $\Delta\phi = \cdot$ with Dirichlet conditions $\phi|_{\partial\mathcal{D}} = 0$.

11.4.2 Population genetics

Fleming has proposed, in his papers [18] and [19], a stochastic model to describe the proportion of a gene in a population living in a bounded habitat.

Fleming’s problem is to estimate the evolution of the frequency of gene A, given the hypothesis that only two type of genes exist, i.e. A or B. This frequency depends on the time and, clearly, on the position. For instance, we can assume that gene A is a mutation of gene B, occurred at time $t = 0$ in a specific region, e.g. in Australia. What is the frequency of gene A in a different region, e.g. Europe after one-hundred years?

Fleming describe frequency $u(t, x)$ of gene A as a stochastic process verifying the following parabolic stochastic partial differential equation

$$\frac{\partial u}{\partial t}(t, x) = \nu \Delta u(t, x) + \alpha \sqrt{u(t, x) [1 - u(t, x)]} \dot{W}(t, x),$$

where ν and α are constant and $W(t, x)$ is a white noise.

The choice of a parabolic SPDE is justified by the diffusion behavior of a gene in a population, while noise coefficient $\sqrt{u [1 - u]}$ is chosen in order to verify two evident constraints, that is absence of noise when only one gene is survived.

In this model, ν depends on the emigration rate in the population, while α^2 on the cross-over effects.

11.4.3 Finance: interest rate theory

Many works exist on the application of stochastic partial differential equation in term structure models, e.g. Bjork [9], Cont [11] and Goldys and Musiela [21].

The model introduced by Cont [11] is the following, he models the curve of forward rates $f(t, x)$, i.e. the interest rate for a loan signed at time t with duration $x = T - t$, T denotes the maturity. Cont assumes that the forward rates curve $f(t, x)$ can be written as

$$f(t, x) = r(t) + s(t) [Y(x) + X(t, x)],$$

where

- $r(t)$ is the short interest rate, i.e. generally the overnight interest rate;
- $s(t)$ denotes the spread between short and long interest rates, the latter is often the interest rate with maturity thirty years;
- $Y(x)$ is a deterministic function describing the shape of equilibrium profile of term structure; and finally,
- $X(t, x)$ is a stochastic process that verifies the following parabolic stochastic partial differential equation:

$$\frac{\partial X}{\partial t}(t, x) = \frac{k}{2} \frac{\partial^2 X}{\partial x^2}(t, x) + \frac{\partial X}{\partial x}(t, x) + \sigma_0 \dot{W}(t, x),$$

where k and σ_0 are constant and $W(t, x)$ is a white noise.

Results are very interesting, even if the high number of degrees of freedom causes some problems in calibration.

11.4.4 Insurance: mortality risk

The idea of evolution equations with an infinite number of degrees of freedom begins being applied in insurance modeling, especially for mortality risk. We follow interesting article [3], where a bidimensional approach to mortality risk is presented. The evolution of mortality is an interesting financial problem for life insurance companies and pension schemes. As a matter of fact, pension schemes are one of the most important problem in developed countries, since life expectancy becomes longer. It is clear that the residual lifetime of a person is a random variable depending on the age of insured and many other endogenous variables, e.g. life manner, diet and job; and exogenous, e.g. wars, diseases, pollution, etc. Disaggregate impact of each variable is hard to evaluate, whether no statistical estimations exists (pollution) or data are shielded by privacy laws (life manner).

Biffis and Millosovich focus on the death's stochastic intensity $\mu_x(t)$ of a representative insured aged x at time 0. Their representation permits to evaluate the probability of a residual lifetime τ^x bigger that a level T at time t :

$$\mathbb{P}[\tau^x > T | \mathcal{F}_t] = \mathbb{I}_{\tau^x > t} \mathbb{E} \left[e^{-\int_t^T \mu_x(s) ds} \mid \mathcal{F}_t \right].$$

In this framework, death's intensity $\mu_x(t)$ plays the role of the forward rate in a term structure model, see subsection 11.4.3.

Now we have to define how evolves our intensity $\mu_x(t)$. It is clear that our intensity depends crucially on the age x . As a matter of fact, the death's risk for an insured between time t and $t + \delta t$ is sensibly higher for an elder man than for a young. Therefore, intensity $\mu_x(t)$ is a random field, defined on $\mathbb{R}^+ \times \mathbb{R}^+$. Biffis and Millosovich propose three examples of stochastic intensity fields:

Gaussian intensity: Intensity of mortality can be defined as

$$\mu_x(t) = \delta(t, x) + C(t, x)$$

where $\delta(t, x)$ is a deterministic function and $C(t, x)$ a Gaussian colored noise with covariance function $c(t, x)$. It stands to reason that Gaussian framework has at least a drawback, i.e. mortality intensity can be negative with strictly positive probability.

χ^2 intensity: In order to get out of this difficulty, Biffis and Millosovich propose to consider a non-negative field:

$$\mu_x(t) = \delta(t, x) + \| C(t, x) \|^2$$

where $C(t, x) = [C_1(t, x), C_2(t, x), \dots, C_n(t, x)]$ is a vector of independent gaussian colored noises. The square permits to define a positive random field called, for similitude, a χ^2 -random field, with n degrees of freedom.

Evolving intensity: A more general framework uses evolution equations. Mortality intensity can be defined as

$$\mu_x(t) = \eta_0(t, x) + \eta_1(t, x) Y_{(t,x)}$$

where $Y_{(t,x)}$ follows

$$(11.36) \quad \frac{\partial Y_{(t,x)}}{\partial t} = \delta(t, x, Y_{(t,x)})dt + \sigma(t, x, Y_{(t,x)})dW(t, x)$$

where $W(t, x)$ is a Brownian sheet, see Khoshnevisan in [13].

It is plain that another example can be added. As a matter of fact, equation (11.36) does not enable to mix mortality intensity of people with different age x . A parabolic stochastic partial differential equation enables diffusion effects among generations, helpful to represent relationship impacts. An hyperbolic stochastic partial differential equation, see Dalang in [13], permits wave propagations among people with different age, this effect is very useful to describe war's impacts or degenerative diseases.

In my opinion, stochastic partial differential equation will play a crucial role in life insurance.

11.5 Conclusion

In this chapter, we have analyzed two types of uncertainties on a stochastic partial differential equation and the impacts on its solution.

The first case is when the two functional coefficients of a SPDE, i.e. the drift and the noise coefficient, are erroneous. We have developed the two coefficients on a functional basis, we have assumed that the errors on functions are cached on the expansion coefficients. We have proved that the sharp of the solution verifies a linear stochastic partial differential equation depending on the solution itself. We have showed that the sharp can be decomposed into a sum of terms depending on the decomposition of functional coefficients; also the variance has been computed and we have showed a quasi-closed form for it. Finally, we have studied the bias of the solution, we have proved that the bias verifies a linear stochastic partial differential equation too. We have rewrite the bias in a quasi-closed formula.

The second case is when the coefficient of the diffusion in the Green's function is erroneous. We have analyzed the impact of this uncertainty on the Green's function of the SPDE, we have proved that the sharp of the Green's function verifies a partial differential equation. We have showed that the sharp of the SPDE solution verifies a new type of linear stochastic partial differential equation. We have underlined some, probably stronger, conditions to force the existence of the sharp.

Finally, we have analyzed some applications of stochastic partial differential equations in climatology, genetics, finance and insurance.

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