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THÈSE DE DOCTORAT

présentée par

DANIEL JUTEAU

pour obtenir le grade de

Docteur de Mathématiques de l'Université Paris 7 – Denis Diderot

Correspondance de Springer modulaire et
matrices de décomposition

—
Modular Springer correspondence and
decomposition matrices

Thèse soutenue le 11 décembre 2007, devant le jury composé de :

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Introduction (français)

Contexte et vue d'ensemble

En 1976, Springer a introduit une construction géométrique des représentations irréductibles des groupes de Weyl, qui a eu une profonde influence et de nombreux développements ultérieurs, culminant avec la théorie des faisceaux-caractères de Lusztig, qui permet de calculer les valeurs des caractères de groupes réductifs finis. Dans cette thèse, nous définissons une correspondance de Springer pour les représentations modulaires des groupes de Weyl, et en établissons quelques propriétés, répondant ainsi à une question posée par Springer lui-même.

Représentations modulaires, matrices de décomposition

La théorie des représentations modulaires des groupes finis, initiée et développée par Brauer à partir du début des années 1940, est l'étude des représentations des groupes finis sur un corps de caractéristique $\ell > 0$. Lorsque ℓ divise l'ordre du groupe, la catégorie des représentations n'est plus semi-simple.

Soit W un groupe fini. On fixe une extension finie \mathbb{K} du corps \mathbb{Q}_ℓ des nombres ℓ -adiques. Soit \mathbb{O} son anneau de valuation. On note $\mathfrak{m} = (\varpi)$ l'idéal maximal de \mathbb{O} , et \mathbb{F} le corps résiduel (qui est fini de caractéristique ℓ). Le triplet $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ est ce qu'on appelle un système modulaire, et on suppose qu'il est assez gros pour tous les groupes finis que nous rencontrerons (c'est-à-dire que tous les $\mathbb{K}W$ -modules simples sont absolument simples, et de même pour \mathbb{F}). La lettre \mathbb{E} désignera l'un des anneaux de ce triplet.

Pour une catégorie abélienne \mathcal{A} , on note $K_0(\mathcal{A})$ son groupe de Grothendieck. Lorsque \mathcal{A} est la catégorie des A -modules de type fini, où A est un anneau, on adopte la notation $K_0(A)$. La sous-catégorie pleine formée des objets projectifs de \mathcal{A} sera notée $\text{Proj } \mathcal{A}$.

Le groupe de Grothendieck $K_0(\mathbb{K}W)$ est libre de base $([E])_{E \in \text{Irr } \mathbb{K}W}$, où $\text{Irr } \mathbb{K}W$ désigne l'ensemble des classes d'isomorphisme de \mathbb{K} -modules simples. De même pour $K_0(\mathbb{F}W)$. Pour $F \in \text{Irr } \mathbb{F}W$, on note P_F une enveloppe projective de F . Alors $([P_F])_{F \in \text{Irr } \mathbb{F}W}$ est une base de $K_0(\text{Proj } \mathbb{F}W)$. La réduction modulo \mathfrak{m} définit un isomorphisme de $K_0(\text{Proj } \mathbb{O}W)$ sur $K_0(\text{Proj } \mathbb{F}W)$.

Nous allons définir le triangle cde [Ser67].

$$\begin{array}{ccc} K_0(\text{Proj } \mathbb{F}W) & \xrightarrow{c} & K_0(\mathbb{F}W) \\ & \searrow e & \nearrow d \\ & & K_0(\mathbb{K}W) \end{array}$$

Introduction (français)

Le morphisme c est induit par l'application qui à chaque $\mathbb{F}W$ -module projectif associe sa classe dans $K_0(\mathbb{F}W)$. Le foncteur d'extensions des scalaires à \mathbb{K} induit un morphisme de $K_0(\text{Proj } \mathbb{O}W)$ vers $K_0(\mathbb{K}W)$. Le morphisme e s'obtient en composant avec l'inverse de l'isomorphisme canonique de $K_0(\text{Proj } \mathbb{O}W)$ sur $K_0(\text{Proj } \mathbb{F}W)$. Le morphisme d est un peu plus délicat. Soit E un $\mathbb{K}W$ -module. On peut choisir un réseau $E_{\mathbb{O}}$ dans E stable par W . L'image de $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$ dans $K_0(\mathbb{F}W)$ ne dépend pas du choix du réseau [Ser67]. Le morphisme d est induit par l'application qui à E associe $[\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}]$.

On définit la matrice de décomposition $D^W = (d_{E,F}^W)_{E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W}$ par

$$d([E]) = \sum_{F \in \text{Irr } \mathbb{F}W} d_{E,F}^W [F]$$

Un des plus grands problèmes en théorie des représentations modulaires est de déterminer ces nombres de décomposition $d_{E,F}^W$ explicitement pour des classes intéressantes de groupes finis. Ce problème est ouvert pour le groupe symétrique.

Le triangle cde peut s'interpréter en termes de caractères ordinaires et modulaires (de Brauer). Nous renvoyons à [Ser67]. Lorsque les caractères ordinaires sont connus (c'est le cas pour le groupe symétrique), la connaissance de la matrice de décomposition équivaut à la détermination des caractères de Brauer.

Il y a des variantes de ce problème lorsqu'on sort du cadre des groupes finis. On peut par exemple considérer les représentations modulaires des algèbres de Hecke. Ces algèbres sont des déformations d'algèbres de groupes de réflexions, et jouent un rôle très important dans la théorie des représentations des groupes finis de type de Lie. Elles sont définies de manière générique, et on peut regarder ce qu'il se passe lorsqu'on spécialise un ou plusieurs paramètres.

On peut aussi s'intéresser aux représentations rationnelles d'un groupe réductif en caractéristique positive, ou les représentations d'une algèbre de Lie réductive, ou de groupes quantiques (déformations d'algèbres enveloppantes), etc. Dans ce cas, on considère les multiplicités des simples dans des classes d'objets particuliers dont on connaît les caractères.

Faisceaux pervers et représentations

Depuis trois décennies, l'utilisation de méthodes géométriques a permis des progrès spectaculaires dans de nombreux domaines de la théorie des représentations. Nous nous intéresserons ici tout particulièrement aux représentations des groupes de Weyl et des groupes finis de type de Lie.

Il y a une trentaine d'années, Springer est parvenu à construire géométriquement toutes les représentations irréductibles ordinaires des groupes de Weyl, dans la cohomologie de certaines variétés liées aux éléments nilpotents de l'algèbre de Lie correspondante [Spr76, Spr78]. Cette découverte a eu un retentissement considérable. De nombreuses autres constructions ont été proposées par la suite. Par exemple, Kazhdan et Lusztig ont proposé une approche topologique [KL80b], et Slodowy a construit ces représentations par monodromie [Slo80a]. Au début des années 1980, l'essor de la cohomologie d'inter-

section a permis de réinterpréter la correspondance de Springer en termes de faisceaux pervers [Lus81, BM81].

Lusztig a prolongé ce travail en étudiant une correspondance de Springer généralisée, ainsi que des complexes de cohomologie d'intersection sur un groupe réductif G ou son algèbre de Lie \mathfrak{g} , qu'il appelle complexes admissibles ou faisceaux-caractères [Lus84, Lus85a, Lus85b, Lus85c, Lus86a, Lus86b]. Si G est muni d'une structure \mathbb{F}_q -rationnelle définie par un endomorphisme de Frobenius F , les faisceaux-caractères F -stables donnent lieu à des fonctions centrales sur le groupe fini G^F , qui sont très proches des caractères irréductibles. La matrice de transition entre ces deux bases est décrite par une transformation de Fourier. Ainsi, ces méthodes géométriques ont permis de déterminer les caractères de groupes réductifs finis (au moins lorsque le centre est connexe).

Jusqu'ici, à ma connaissance, on n'a pas utilisé ces méthodes pour étudier les représentations modulaires des groupes de Weyl ou des groupes finis de type de Lie. Pourtant, cela a été le cas dans au moins deux autres situations modulaires. Soergel [Soe00] a converti un problème sur la catégorie \mathcal{O} en un problème sur les faisceaux pervers à coefficients modulo ℓ sur des variétés de Schubert. D'autre part, Mirkovic et Vilonen ont établi une équivalence de catégories entre les représentations rationnelles d'un groupe réductif sur un anneau quelconque Λ et les faisceaux pervers à coefficients Λ sur le dual de Langlands, défini sur \mathbb{C} . La topologie classique permet d'utiliser des coefficients arbitraires. Ce travail permet d'ailleurs de donner une définition intrinsèque du dual de Langlands en construisant sa catégorie de représentations. Nous y reviendrons dans la section « Perspectives ».

Correspondance de Springer modulaire

Il était tentant de chercher un lien entre représentations modulaires des groupes de Weyl et faisceaux pervers modulo ℓ sur les nilpotents. Autrement dit, de chercher à définir une correspondance de Springer modulaire. Il est vrai que la construction de Lusztig-Borho-MacPherson utilise le théorème de décomposition de Gabber [BBD82], qui n'est pas valable en caractéristique ℓ . Mais Hotta et Kashiwara [HK84] ont une approche *via* une transformation de Fourier pour les \mathcal{D} -modules dans le cas où le corps de base est celui des complexes, ce qui évite de recourir au théorème de décomposition. De plus, la transformation de Fourier-Deligne permet de considérer un corps de base de caractéristique p et des coefficients ℓ -adiques [Bry86]. Dans cette thèse, nous définissons une correspondance de Springer en utilisant la transformation de Fourier-Deligne avec des coefficients modulo ℓ . De plus, nous introduisons une matrice de décomposition pour les faisceaux pervers sur les nilpotents, et nous la comparons à la matrice de décomposition du groupe de Weyl. Par ailleurs, nous calculons de façon purement géométrique certains nombres de décomposition. Nous constatons que certaines propriétés des nombres de décomposition des groupes de Weyl peuvent être vues comme le reflet de propriétés géométriques. Par exemple, la règle de suppression de lignes et de colonnes de James peut s'expliquer par une règle similaire de Kraft et Procesi [KP81] sur les singularités nilpotentes, une fois qu'on a déterminé la correspondance de Springer modulaire pour GL_n (ce que nous ferons dans cette thèse).

Contenu détaillé

Préliminaires et exemples

Dans le chapitre 1, nous faisons des rappels sur les faisceaux pervers sur \mathbb{K} , \mathbb{O} et \mathbb{F} et donnons quelques compléments, qui nous seront utiles par la suite. Nous insistons en particulier sur les aspects spécifiques à \mathbb{O} et \mathbb{F} . Par exemple, sur \mathbb{O} nous n'avons pas une perversité autoduale, mais une paire de perversités, p et p_+ , échangées par la dualité. De plus, nous étudions l'interaction entre les paires de torsion et les t -structures (voir à ce sujet [HRS96]), et aussi avec les situations de recollement. Dans ce passage quelque peu technique, on trouvera de nombreux triangles distingués qui seront utilisés par la suite. Le point clé est que la réduction (dérivée) modulo ℓ ne commute pas aux troncations en général. Nous donnons aussi quelques compléments sur les extensions perverses ${}^p j_!$, ${}^p j_{!*}$, ${}^p j_*$ (sur la tête et le socle, et sur le comportement vis-à-vis des multiplicités). Finalement, nous définissons les nombres de décomposition pour les faisceaux pervers. Nous sommes particulièrement intéressés par le cas d'une G -variété ayant un nombre fini d'orbites : c'est le cône nilpotent que nous avons en vue.

Dans le chapitre 2, nous donnons quelques exemples de faisceaux pervers, et en particulier de complexes de cohomologie d'intersection sur \mathbb{E} . Nous rappelons les propriétés des morphismes propres et semi-petits (resp. petits). En particulier, le complexe de cohomologie d'intersection d'une variété admettant une petite résolution est obtenu par image directe.

Ensuite nous introduisons la notion d'équivalence lisse de singularités, et rappelons que la cohomologie d'intersection locale est un invariant pour cette équivalence.

Puis nous étudions les singularités coniques, où la cohomologie d'intersection locale se ramène à un calcul de la cohomologie d'une variété (l'ouvert complémentaire du sommet du cône), et, plus généralement, le cas d'une variété affine munie d'une \mathbb{G}_m -action contractant tout sur l'origine.

Il est naturel de se demander quand le complexe de cohomologie d'intersection se réduit au faisceau constant \mathbb{E} (de telle sorte que la variété vérifie la dualité de Poincaré usuelle). On parle alors de variété \mathbb{E} -lisse. Un exemple typique de variété \mathbb{K} -lisse (resp. \mathbb{F} -lisse) est fourni par le quotient d'une variété lisse par un groupe fini (resp. par un groupe fini d'ordre premier à ℓ).

Dans la dernière section de ce chapitre, nous étudions les singularités simples. Une variété normale X a des singularités rationnelles si on a une résolution $\pi : \tilde{X} \rightarrow X$ avec $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ pour $i > 0$. Sur \mathbb{C} , les surfaces à point double rationnel sont (à équivalence analytique près) les quotients du plan affine par un sous-groupe fini de $SL_2(\mathbb{C})$. Elles sont classifiées par les diagrammes de Dynkin simplement lacés. On peut interpréter les autres types en considérant en plus l'action d'un groupe de symétries. On associe à chaque diagramme de Dynkin Γ un diagramme homogène $\hat{\Gamma}$, et un groupe de symétries $A(\Gamma)$. Dans le cas où Γ est déjà homogène, on a $\hat{\Gamma} = \Gamma$ et $A(\Gamma) = 1$. Soit $\hat{\Phi}$ un système de racines de type $\hat{\Gamma}$. On note $P(\hat{\Phi})$ le réseau des poids, et $Q(\hat{\Phi})$ le réseau radiciel. Soit H le sous-groupe fini de $SL_2(\mathbb{C})$ associé à $\hat{\Gamma}$. On montre que

$$H^2((\mathbb{A}^2 \setminus \{0\})/H, \mathbb{Z}) \simeq P(\hat{\Phi})/Q(\hat{\Phi})$$

avec une action naturelle de $A(\Gamma)$. Grâce aux résultats du chapitre 1, cela nous permet de comparer les faisceaux pervers en caractéristiques 0 et ℓ .

Calculs de nombres de décomposition

Jusqu'au chapitre 4, notre but est de calculer certains nombres de décomposition pour les faisceaux pervers G -équivariants sur la variété nilpotente, par des méthodes géométriques.

Introduisons d'abord quelques notations. Les faisceaux pervers simples sur \mathbb{K} (resp. \mathbb{F}) sont paramétrés par l'ensemble $\mathfrak{N}_{\mathbb{K}}$ (resp. $\mathfrak{N}_{\mathbb{F}}$) des paires (x, ρ) (à conjugaison près) constituées d'un élément nilpotent x et d'un caractère $\rho \in \text{Irr } \mathbb{K}A_G(x)$ (resp. $\rho \in \text{Irr } \mathbb{F}A_G(x)$), où $A_G(x)$ est le groupe fini des composantes du centralisateur de x dans G . On utilisera la notation

$$(d_{(x,\rho),(y,\sigma)})_{(x,\rho) \in \mathfrak{N}_{\mathbb{K}}, (y,\sigma) \in \mathfrak{N}_{\mathbb{F}}}$$

pour la matrice de décomposition de ces faisceaux pervers. Dans le cas de GL_n , tous les $A_G(x)$ sont triviaux, si bien qu'on peut oublier ρ qui est toujours 1, et les orbites nilpotentes sont paramétrées par l'ensemble $\mathfrak{P}_n = \{\lambda \vdash n\}$ des partitions de n . Dans ce cas, la matrice de décomposition sera notée

$$(d_{\lambda,\mu})_{\lambda,\mu \in \mathfrak{P}_n}$$

Quant à la matrice de décomposition pour le groupe de Weyl W , on la notera

$$(d_{E,F}^W)_{E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W}$$

Pour le groupe symétrique \mathfrak{S}_n , les $\mathbb{K}\mathfrak{S}_n$ -modules simples sont les modules de Specht S^λ , pour $\lambda \in \mathfrak{P}_n$. Ils sont définis sur \mathbb{Z} , et munis d'une forme bilinéaire symétrique définie sur \mathbb{Z} . La réduction modulaire du module de Specht, que l'on notera encore S^λ , est donc munie elle aussi d'une forme bilinéaire symétrique. Le quotient de S^λ par le radical de cette forme bilinéaire symétrique est soit nul, soit un $\mathbb{F}\mathfrak{S}_n$ -module simple. L'ensemble des μ tels que ce quotient soit non nul (celui-ci sera alors noté D^μ) est l'ensemble $\mathfrak{P}_n^{\ell\text{-reg}}$ des partitions de n qui sont ℓ -régulières (dont chaque partie est répétée au plus $\ell - 1$ fois). Les D^μ , pour $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, forment un système de représentants des classes d'isomorphisme de $\mathbb{F}\mathfrak{S}_n$ -modules simples. La matrice de décomposition du groupe symétrique \mathfrak{S}_n sera notée plutôt

$$(d_{\lambda,\mu}^{\mathfrak{S}_n})_{\lambda \in \mathfrak{P}_n, \mu \in \mathfrak{P}_n^{\ell\text{-reg}}}$$

Pour l'algèbre de Schur

$$S_{\mathbb{E}}(n) = S_{\mathbb{E}}(n, n) = \text{End}_{\mathbb{E}\mathfrak{S}_n} \left(\bigoplus_{\lambda \vdash n} \text{Ind}_{\mathbb{E}\mathfrak{S}_\lambda}^{\mathbb{E}\mathfrak{S}_n} \mathbb{E} \right)$$

on notera la matrice de décomposition

$$(d_{\lambda,\mu}^{S(n)})_{\lambda,\mu \in \mathfrak{P}_n}$$

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Il est connu que

$$d_{\lambda,\mu}^{S(n)} = d_{\lambda',\mu'}^{\mathfrak{S}_n}$$

pour $\lambda \in \mathfrak{P}_n$, $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, où λ' désigne la partition transposée. Nous allons voir que

$$d_{\lambda,\mu} = d_{\lambda',\mu'}^{\mathfrak{S}_n} = d_{\lambda,\mu}^{S(n)}$$

pour $\lambda \in \mathfrak{P}_n$, $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, et nous conjecturons que

$$d_{\lambda,\mu} = d_{\lambda,\mu}^{S(n)}$$

pour *toutes* les partitions λ, μ de n (voir à ce sujet les remarques dans la dernière section de cette introduction).

Dans le chapitre 3, nous calculons la cohomologie entière de l'orbite nilpotente (non triviale) minimale \mathcal{O}_{\min} dans une algèbre de Lie simple \mathfrak{g} sur le corps des nombres complexes. En réalité, les résultats et méthodes de ce chapitre sont valables pour un corps de base de caractéristique $p > 0$, à condition de travailler avec la cohomologie étale et de prendre pour coefficients les entiers ℓ -adiques.

La cohomologie rationnelle de \mathcal{O}_{\min} est déjà connue. La dimension de \mathcal{O}_{\min} est $d = 2h^\vee - 2$, où h^\vee est le nombre de Coxeter dual. La première moitié de la cohomologie est donnée par

$$\tau_{\leq d-1} \text{RG}(\mathcal{O}_{\min}, \mathbb{Q}) \simeq \bigoplus_{i=1}^k \mathbb{Q}[-2(d_i - 2)]$$

où k est le nombre de racines simples longues, et $d_1 \leq \dots \leq d_k \leq \dots \leq d_n$ sont les degrés de W (n étant le nombre total de racines simples). L'autre moitié s'en déduit par dualité de Poincaré.

C'est donc la torsion qui nous intéresse. Si Φ est le système de racines de \mathfrak{g} , et Φ' le sous-système engendré par les racines simples longues (pour une certaine base), alors la cohomologie moitié de \mathcal{O}_{\min} est

$$H^d(\mathcal{O}_{\min}, \mathbb{Z}) \simeq P^\vee(\Phi')/Q^\vee(\Phi')$$

Nous verrons par la suite que ce résultat est lié à la réduction modulaire de la représentation naturelle du groupe de Weyl W' de Φ' .

En dehors de la cohomologie moitié, nous n'avons pas d'expression uniforme pour la partie de torsion de la cohomologie de \mathcal{O}_{\min} . En revanche, on sait toutefois que c'est le conoyau d'une matrice dont les coefficients sont déterminés explicitement par l'ensemble ordonné des racines longues dans Φ , qui est nivelé par cohauteur (hauteur de la coracine), ce qui nous permet de faire le calcul dans tous les types. En dehors de la cohomologie moitié, nous constatons que les seuls nombres premiers qui interviennent dans la torsion de la cohomologie sont mauvais. Cela revient à dire que les fibres du complexe de cohomologie d'intersection entière sont sans ℓ -torsion pour ℓ bon (uniquement pour la perversité p , précisément pas pour p_+ , où la cohomologie moitié intervient). Nous ignorons si l'on peut trouver une interprétation (peut-être homologique) en termes de théorie des représentations à ces groupes de mauvaise torsion.

Dans le chapitre 4, nous calculons certains nombres de décomposition pour les faisceaux pervers G -équivariants sur la variété nilpotente, en utilisant d'une part les résultats précédents, et d'autre part des résultats géométriques que l'on peut trouver dans la littérature.

Tout d'abord, nous déterminons les nombres de décomposition associés aux classes régulière et sous-régulière (le centralisateur d'un élément sous-régulier peut être non connexe). Comme dans la section sur les singularités simples, on associe au type Γ de G un diagramme homogène $\widehat{\Phi}$ et un groupe de symétries $A := A(\Gamma)$ qui est isomorphe à $A_G(x_{\text{subreg}})$ lorsque G est adjoint. On a

$$d_{(x_{\text{reg}},1),(x_{\text{subreg}},\rho)} = [\mathbb{F} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) : \rho]$$

pour tous les ρ dans $\text{Irr } \mathbb{F}A$. On calcule cette multiplicité dans tous les types, pour chaque nombre premier ℓ et pour chaque $\rho \in \text{Irr } \mathbb{F}A$. En ce qui concerne les classes minimales et triviales, il découle des résultats du chapitre 3 que

$$d_{(x_{\text{min}},1),(0,1)} = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} (P^{\vee}(\Phi')/Q^{\vee}(\Phi'))$$

Nous donnons également cette multiplicité dans tous les types. À titre d'exemple, voyons ce qu'il se passe pour GL_n . On trouve

$$d_{(n),(n-1,1)} = d_{(2^{1^{n-2}}),(1^n)} = \begin{cases} 1 & \text{si } \ell \mid n \\ 0 & \text{sinon} \end{cases}$$

ce qui est compatible avec notre conjecture faisant le lien avec l'algèbre de Schur. Nous avons un autre résultat qui va dans ce sens. Les nombres de décomposition de l'algèbre de Schur vérifient la propriété suivante. Si λ et μ sont deux partitions de n dont les r premières lignes et les s premières colonnes sont identiques, et si λ_1 et μ_1 désignent les partitions (d'un entier n_1 plus petit) obtenues à partir de λ et μ en supprimant ces lignes et ces colonnes, on a

$$d_{\lambda,\mu}^{S(n)} = d_{\lambda_1,\mu_1}^{S(n_1)}$$

Kraft et Procesi ont montré que les singularités des adhérences des orbites nilpotentes dans GL_n vérifient une propriété similaire [KP81]. Avec les mêmes notations, on a

$$\text{codim}_{\overline{\mathcal{O}}_{\lambda_1}} \mathcal{O}_{\mu_1} = \text{codim}_{\overline{\mathcal{O}}_{\lambda}} \mathcal{O}_{\mu} \quad \text{et} \quad \text{Sing}(\overline{\mathcal{O}}_{\lambda_1}, \mathcal{O}_{\mu_1}) = \text{Sing}(\overline{\mathcal{O}}_{\lambda}, \mathcal{O}_{\mu})$$

Nous en déduisons que les nombres de décomposition $d_{\lambda,\mu}$ vérifient la même propriété :

$$d_{\lambda,\mu} = d_{\lambda_1,\mu_1}$$

Si $\lambda > \mu$ sont deux partitions de n adjacentes pour l'ordre de dominance (c'est-à-dire s'il n'existe pas de partition ν telle que $\lambda > \nu > \mu$), Kraft et Procesi utilisent le résultat sur les lignes et les colonnes pour ramener la détermination de la singularité de $\overline{\mathcal{O}}_{\lambda}$ le long de \mathcal{O}_{μ} aux cas extrêmes $(\lambda, \mu) = ((m), (m-1, 1))$ et $(\lambda, \mu) = ((2, 1^{m-2}), (1^m))$, pour

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un m plus petit. Les dégénérescences minimales en type A_n sont donc toutes de types A_m (une singularité simple de type A_m) ou a_m (une singularité minimale de type a_m), pour des m plus petits.

Comme, dans GL_n , tous les $A_G(x)$ sont triviaux, cela suffit pour déterminer le nombre de décomposition $d_{\lambda,\mu}$ lorsque λ et μ sont adjacentes. Dans ce cas, on a bien :

$$d_{\lambda,\mu}^{S(n)} = d_{\lambda,\mu}$$

Kraft et Procesi ont aussi démontré que les singularités des adhérences des orbites nilpotentes dans les types classiques vérifient une règle de suppression des lignes et des colonnes [KP82]. Il faut traiter les cas orthogonaux et symplectiques à la fois. Ils en déduisent le type des singularités des dégénérescences minimales dans ce cas. Ils ne trouvent que des singularités simples et minimales de types classiques, à une exception près. Plus précisément, dans le cas de la codimension deux, on a (à équivalence lisse près) une singularité de type A_k , D_k ou $A_k \cup A_k$, cette dernière étant la réunion non normale de deux singularités simples de type A_k , s'intersectant transversalement en leur point singulier. Lorsque la codimension est strictement supérieure à 2, on a une singularité minimale de type b_k , c_k ou d_k . En supprimant des lignes et des colonnes, on peut toujours se ramener à ces cas irréductibles. Dans cet article, Kraft et Procesi déterminent quelles adhérences d'orbites sont normales dans les types classiques, ce qui était leur but.

Nous pouvons utiliser leurs résultats pour déterminer d'autres nombres de décomposition dans les types classiques, mais pour pouvoir le faire dans tous les cas il faudrait aussi déterminer les systèmes locaux qui interviennent. Quoiqu'il en soit, pour une dégénérescence minimale $\overline{\mathcal{O}} \supset \mathcal{O}'$ en type classique, on peut toujours déterminer la quantité suivante :

$$\sum_{\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})} d_{(x_{\mathcal{O}},1),(x_{\mathcal{O}'},\rho)}$$

(Dans les types classiques, les $A_G(x)$ sont de la forme $(\mathbb{Z}/2)^k$, donc abéliens, et tous les $\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})$ sont de degré 1.) En particulier, on peut dire quand les $d_{(x_{\mathcal{O}},1),(x_{\mathcal{O}'},\rho)}$ sont nuls pour tous les $\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})$. Une étude plus précise devrait pouvoir suffire à déterminer tous les nombres de décomposition de ce type.

Un autre résultat de Kraft et Procesi, concernant la décomposition spéciale de la variété nilpotente [KP89], nous permet de montrer la nullité de certains nombres de décomposition, dans les types classiques, lorsque $\ell \neq 2$. Dans [Lus79], Lusztig a introduit un sous-ensemble de $\text{Irr } \mathbb{K}W$, constitué des représentations dites spéciales. Les classes nilpotentes spéciales sont les classes \mathcal{O} telles que la représentation χ associée à $(\mathcal{O}, \mathbb{K})$ par la correspondance de Springer est spéciale. D'autre part, Spaltenstein a introduit dans [Spa78] une application décroissante de l'ensemble des classes nilpotentes dans lui-même, telle que $d^3 = d$ (c'est une involution sur son image). L'image de d est précisément l'ensemble des classes spéciales. Les variétés localement fermées

$$\widehat{\mathcal{O}} = \overline{\mathcal{O}} \setminus \bigcup_{\substack{\mathcal{O}' \text{ spéciale} \\ \overline{\mathcal{O}'} \subset \overline{\mathcal{O}}}} \overline{\mathcal{O}'}$$

où \mathcal{O} parcourt l'ensemble des classes spéciales, forment une partition de la variété nilpotente \mathcal{N} . Elles sont appelées pièces spéciales. Ainsi, chaque classe nilpotente est incluse dans une unique pièce spéciale. Lusztig a attaché à chaque classe spéciale \mathcal{O} un quotient canonique $\overline{A}_G(x_{\mathcal{O}})$ du groupe fini $A_G(x_{\mathcal{O}})$, et conjecturé que la pièce spéciale $\widehat{\mathcal{O}}$ est le quotient d'une variété lisse par $\overline{A}_G(x_{\mathcal{O}})$. Une conséquence de cette conjecture est que $\widehat{\mathcal{O}}$ est \mathbb{K} -lisse, mais en fait cela donne plus d'information : en particulier, la conjecture implique que $\widehat{\mathcal{O}}$ est \mathbb{F} -lisse dès que ℓ ne divise pas l'ordre de ce groupe $\overline{A}_G(x_{\mathcal{O}})$. Dans [KP89], Kraft et Procesi montrent cette conjecture dans les types classiques. Nous en déduisons que, dans les types classiques, on a

$$d_{(x_{\mathcal{O}},1),(x_{\mathcal{O}'},\rho)} = 0$$

dès que $\ell > 2$, lorsque \mathcal{O} est une classe spéciale, \mathcal{O}' une classe incluse dans la pièce spéciale $\widehat{\mathcal{O}}$, et $\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})$. Une étude plus détaillée permettrait peut-être de déterminer les nombres de décomposition lorsque $\ell = 2$.

Faisons une remarque supplémentaire. Dans un autre article [Kra89], Kraft résout le problème de la normalité des adhérences d'orbites nilpotentes dans G_2 . On y trouve l'information suivante, qui n'est pas couverte par les résultats précédents : $\overline{\mathcal{O}}_{10}$ a une singularité de type A_1 en \mathcal{O}_8 , où \mathcal{O}_i désigne l'unique orbite nilpotente de dimension i dans l'algèbre de Lie \mathfrak{g} d'un groupe simple G de type G_2 . Comme $A_G(x_8) = 1$ (on note x_i est un représentant de \mathcal{O}_i), cela nous permet de déterminer le nombre de décomposition $d_{(x_{10},1),(x_8,1)}$:

$$d_{(x_{10},1),(x_8,1)} = \begin{cases} 1 & \text{si } \ell = 2, \\ 0 & \text{sinon} \end{cases}$$

Une étude plus détaillée de cet article permettrait peut-être de retrouver d'autres nombres de décomposition géométriquement dans G_2 . Quoi qu'il en soit, en utilisant la correspondance de Springer modulaire nous pourrions déterminer toute la matrice de décomposition pour G_2 lorsque $\ell = 3$, et toute la matrice sauf une colonne lorsque $\ell = 2$. Pour $\ell > 3$, ℓ ne divise pas l'ordre du groupe de Weyl, et la matrice de décomposition est l'identité ; je pense que c'est vrai pour dans tous les types, mais il faudra étudier la notion de cuspidalité. Au moins, la partie de la matrice de décomposition correspondant au groupe de Weyl est bien l'identité, comme nous le verrons.

Correspondance de Springer modulaire et matrices de décomposition

Dans la suite de la thèse, nous définissons une correspondance de Springer modulaire et en établissons quelques propriétés, notamment le fait qu'elle préserve les nombres de décomposition. Comme le théorème de décomposition de Gabber [BBD82] n'est pas vrai dans le cadre modulaire, nous nous inspirons de l'approche de Kashiwara et Brylinski [Bry86], utilisant une transformation de Fourier.

Dans le chapitre 5, nous introduisons la transformation de Fourier-Deligne en suivant un article de Laumon [Lau87]. Nous détaillons les preuves, et vérifions que tout se passe bien lorsque les coefficients sont \mathbb{K} , \mathbb{O} ou \mathbb{F} .

Introduction (français)

Le chapitre 6 est le cœur de cette thèse. Nous commençons par rappeler le contexte géométrique de la correspondance de Springer, qui est celui de la résolution simultanée de Grothendieck π des singularités des fibres du quotient adjoint. Prenant la fibre en zéro, on retrouve la résolution de Springer $\pi_{\mathcal{N}}$ du cône nilpotent \mathcal{N} .

Puis nous introduisons les faisceaux pervers $\mathbb{E}\mathcal{K}_{\mathfrak{r}_s}$, $\mathbb{E}\mathcal{K}$ et $\mathbb{E}\mathcal{K}_{\mathcal{N}}$, respectivement sur l'ouvert $\mathfrak{g}_{\mathfrak{r}_s}$ des éléments réguliers semi-simples, sur \mathfrak{g} tout entière, et sur le fermé \mathcal{N} des éléments nilpotents. Nous avons le diagramme commutatif à carrés cartésiens suivant :

$$\begin{array}{ccccc}
 \tilde{\mathfrak{g}}_{\mathfrak{r}_s} & \xrightarrow{\tilde{j}_{\mathfrak{r}_s}} & \tilde{\mathfrak{g}} & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \tilde{\mathcal{N}} \\
 \downarrow \pi_{\mathfrak{r}_s} & & \downarrow \pi & & \downarrow \pi_{\mathcal{N}} \\
 \mathfrak{g}_{\mathfrak{r}_s} & \xrightarrow{j_{\mathfrak{r}_s}} & \mathfrak{g} & \xleftarrow{i_{\mathcal{N}}} & \mathcal{N}
 \end{array}$$

On note r le rang de G , et ν le nombre de racines positives dans Φ . On pose

$$\begin{aligned}
 \mathbb{E}\mathcal{K}_{\mathfrak{r}_s} &= \pi_{\mathfrak{r}_s!} \mathbb{E}_{\tilde{\mathfrak{g}}_{\mathfrak{r}_s}}[2\nu + r] \\
 \mathbb{E}\mathcal{K} &= \pi_! \mathbb{E}_{\tilde{\mathfrak{g}}}[2\nu + r] \\
 \mathbb{E}\mathcal{K}_{\mathcal{N}} &= \pi_{\mathcal{N}!} \mathbb{E}_{\tilde{\mathcal{N}}}[2\nu]
 \end{aligned}$$

On a

$$\begin{aligned}
 \mathbb{E}\mathcal{K} &= {}^p j_{\mathfrak{r}_s! *} \mathbb{E}\mathcal{K}_{\mathfrak{r}_s} \\
 \mathbb{E}\mathcal{K}_{\mathcal{N}} &= i_{\mathcal{N}}^* \mathbb{E}\mathcal{K}[-r]
 \end{aligned}$$

Le morphisme π est propre et petit, génériquement un W -torseur (au-dessus de $\mathfrak{g}_{\mathfrak{r}_s}$), et sa restriction $\pi_{\mathcal{N}}$ aux nilpotents est semi-petite.

Ensuite, nous définissons une correspondance de Springer modulaire utilisant une transformation de Fourier-Deligne. Pour $E \in \text{Irr } \mathbb{K}W$, le faisceau pervers associé par la correspondance de Springer à la Brylinski est $\mathcal{T}(E) = \mathcal{F}({}^p j_{\mathfrak{r}_s! *} (E[2\nu + r]))$. Cela permet de définir une application injective

$$\Psi_{\mathbb{K}} : \text{Irr } \mathbb{K}W \longrightarrow \mathfrak{N}_{\mathbb{K}}$$

On notera $\mathfrak{N}_{\mathbb{K}}^0$ son image. Nous procédons de même pour la correspondance de Springer modulaire. À $F \in \text{Irr } \mathbb{F}W$, on associe

$$\mathcal{T}(F) = \mathcal{F}({}^p j_{\mathfrak{r}_s! *} (F[2\nu + r]))$$

C'est un faisceau pervers simple G -équivariant sur \mathcal{N} . Il est donc de la forme ${}^p \mathcal{J}_{!*}(\mathcal{O}_F, \mathcal{L}_F)$ pour une certaine paire $(\mathcal{O}_F, \mathcal{L}_F)$ appartenant à l'ensemble $\mathfrak{N}_{\mathbb{F}}$ des paires $(\mathcal{O}, \mathcal{L})$ constituées d'une orbite nilpotente \mathcal{O} et d'un \mathbb{F} -système local G -équivariant sur \mathcal{O} . On identifiera $\mathfrak{N}_{\mathbb{F}}$ à l'ensemble des paires (x, ρ) où $x \in \mathcal{N}$ et $\rho \in \text{Irr } \mathbb{F}A_G(x)$, à conjugaison près. On obtient donc une application injective

$$\Psi_{\mathbb{F}} : \text{Irr } \mathbb{F}W \longrightarrow \mathfrak{N}_{\mathbb{F}}$$

On notera $\mathfrak{N}_{\mathbb{F}}^0$ son image.

Ensuite, nous montrons que la matrice de décomposition du groupe de Weyl W peut être extraite de la matrice de décomposition pour les faisceaux pervers G -équivariants sur la variété nilpotente, en ne gardant que les lignes qui sont dans l'image de la correspondance de Springer ordinaire, et les colonnes qui sont dans l'image de la correspondance de Springer modulaire. Plus précisément, nous montrons que, pour tous $E \in \text{Irr } \mathbb{K}W$ et $F \in \text{Irr } \mathbb{F}W$, on a

$$d_{E,F}^W = d_{\Psi_{\mathbb{K}}(E), \Psi_{\mathbb{F}}(F)}$$

Puis nous déterminons la correspondance de Springer modulaire lorsque $G = GL_n$. On a alors :

$$\mathfrak{N}_{\mathbb{F}}^0 = \mathfrak{P}_n^{\ell\text{-res}}$$

où $\mathfrak{P}_n^{\ell\text{-res}}$ est l'ensemble des partitions ℓ -restreintes de n , c'est-à-dire dont la transposée est ℓ -régulière.

$$\forall \lambda \in \mathfrak{P}_n^{\ell\text{-reg}}, \quad \Psi_{\mathbb{F}}(D^\lambda) = \lambda'$$

En particulier, pour $\lambda \in \mathfrak{P}_n$ et $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, on a :

$$d_{\lambda,\mu}^{\mathfrak{S}_n} = d_{\lambda',\mu'}$$

de telle sorte qu'on peut voir la règle de suppression des lignes et des colonnes de James comme une conséquence du résultat géométrique de Kraft et Procesi sur les singularités nilpotentes.

Perspectives

Les thèmes de réflexion pour prolonger ce travail ne manquent pas.

Géométrie des orbites nilpotentes

Cette thèse a révélé de nouveaux liens entre la théorie des représentations des groupes de Weyl et la géométrie des classes nilpotentes. On peut s'attendre à de nouvelles interactions entre les deux domaines.

Par exemple, nous avons remarqué que la règle de suppression de lignes et de colonnes de James peut s'expliquer géométriquement grâce au résultat de Kraft et Procesi sur les singularités nilpotentes.

Du côté des représentations, Donkin a trouvé une généralisation de cette règle [Don85]. Je m'attends à une généralisation similaire du côté de la géométrie des adhérences d'orbites nilpotentes (on devrait trouver une singularité produit).

Détermination de la correspondance de Springer modulaire, ensembles basiques

Une question se pose naturellement à propos de la correspondance de Springer modulaire. Supposons pour d'abord pour simplifier que ℓ ne divise pas les $A_G(x)$. Alors on peut identifier $\mathfrak{N}_{\mathbb{F}}$ et $\mathfrak{N}_{\mathbb{K}}$ à un ensemble de paramètres \mathfrak{P} commun. A-t-on alors $\mathfrak{N}_{\mathbb{F}}^0 \subset \mathfrak{N}_{\mathbb{K}}^0$?

Supposons que ce soit le cas. Dans ce cas, pour chaque $F \in \text{Irr } \mathbb{F}W$, il existe un unique $E \in \text{Irr } \mathbb{K}W$ tel que $\Psi_{\mathbb{K}}(E) = \Psi_{\mathbb{F}}(F)$. Cela permet de construire un ensemble basique pour W et montre de façon géométrique la triangularité de la matrice de décomposition de W .

Même si ℓ divise l'ordre de certains $A_G(x)$, on peut s'en sortir en choisissant un ensemble basique pour chaque $A_G(x)$ (qui est de la forme $(\mathbb{Z}/2)^k$ ou un groupe symétrique \mathfrak{S}_k , $k \leq 5$, pour G adjoint). En fait, pour tous ces groupes, il y a un choix canonique.

Dans l'autre sens, la connaissance d'un ensemble basique pour W et une propriété de triangularité pour un ordre compatible avec les adhérences des orbites associés par la correspondance de Springer permet de déterminer la correspondance de Springer modulaire.

Nous avons pu déterminer la correspondance de Springer modulaire de GL_n , ainsi que pour les groupes de rang inférieur ou égal à trois, pour cette raison (il faut faire attention pour G_2 car il y a une paire cuspidale en caractéristique zéro).

Si on arrivait à montrer que $\mathfrak{N}_{\mathbb{F}}^0 \subset \mathfrak{N}_{\mathbb{K}}^0$, il serait intéressant de déterminer l'ensemble basique qu'on obtient, et de le comparer à l'ensemble basique canonique de [GR01], lorsque celui-ci est défini (c'est-à-dire quand ℓ ne divise pas les $A_G(x)$).

Correspondance de Springer modulaire généralisée, faisceaux-caractères modulaires

Dans la correspondance de Springer originale, $\mathfrak{N}_{\mathbb{K}}^0$ contient toujours les paires de la forme $(\mathcal{O}, \mathbb{K})$, mais en général $\mathfrak{N}_{\mathbb{K}}^0$ est strictement inclus dans $\mathfrak{N}_{\mathbb{K}}$. La principale motivation de Lusztig dans [Lus84] est de comprendre les paires manquantes. Ce travail se poursuit dans la série d'articles sur les faisceaux caractères, qui permet de décrire les caractères des groupes finis de type de Lie.

Il est clair qu'une des premières choses à faire pour continuer le travail de cette thèse est d'étudier les notions d'induction et de restriction, de cuspidalité, de définir une correspondance de Springer généralisée, et de la déterminer dans tous les cas. Peut-être verra-t-on apparaître de nouveaux objets combinatoires pour les types classiques (des ℓ -symboles ?).

J'espère que tout cela débouchera sur une théorie des faisceaux-caractères modulaires permettant d'étudier les représentations modulaires des groupes finis de type de Lie. Dans le dernier chapitre, nous présentons brièvement quelques calculs sur \mathfrak{sl}_2 .

Détermination de fibres de cohomologie d'intersection

La détermination des fibres de cohomologie d'intersection (soit sur \mathbb{O} pour la perversité p_+ , soit sur \mathbb{F}) pour les adhérences d'orbites nilpotentes suffirait pour connaître la matrice de décomposition pour les faisceaux pervers, et donc (si l'on a déterminé la correspondance de Springer modulaire) celle du groupe de Weyl. Dans cette thèse, nous déterminons cette correspondance pour GL_n . On a transformé le problème des matrices de décomposition du groupe symétrique en un problème topologique et géométrique, où il n'est plus fait mention du groupe de Weyl.

Bien sûr, ce problème est sans aucun doute très difficile. En caractéristique zéro, la détermination des fibres de cohomologie d'intersection pour les nilpotents passe par la correspondance de Springer, les formules d'orthogonalité des fonctions de Green (voir l'algorithme que Shoji utilise dans [Sho82] pour le type F_4 , qui est repris dans d'autres travaux comme [BS84] pour les types E_6 , E_7 , E_8 , et généralisé dans [Lus86b, §24]). Il est peu probable qu'il existe un tel algorithme en caractéristique ℓ .

Variétés de Schubert

Les faisceaux pervers à coefficients modulo ℓ n'avaient jamais été utilisés, à ma connaissance, pour étudier directement les représentations modulaires des groupes de Weyl, mais en revanche ils apparaissent dans au moins deux autres contextes en théorie des représentations. Le premier de ces contextes est celui de la théorie de Kazhdan-Lusztig, et donc de variétés de Schubert.

Cette fois, on considère un groupe réductif complexe G sur k de caractéristique ℓ , et on veut en étudier les représentations rationnelles. Pour tout poids λ dans $X(T)$, on a un module induit $\nabla(\lambda)$. S'il est non nul, il a un socle simple $L(\lambda)$, et toutes les représentations simples de G peuvent être construites de cette manière. On veut déterminer les multiplicités $[\nabla(\lambda) : L(\mu)]$ pour des poids λ, μ dans $X(T)$ tels que $\nabla(\lambda)$ et $\nabla(\mu)$ soient non nuls. Lusztig [Lus80] a proposé une conjecture pour ces multiplicités dans le cas $\ell > h$ (l'analogie pour G défini sur \mathbb{C} avait été conjecturée dans [KL79]), faisant un lien avec les faisceaux pervers sur le dual de Langlands \mathcal{G} de G .

À l'époque où Soergel écrivait [Soe00], on savait [AJS94] que cette conjecture était vraie pour ℓ « assez grand », mais, en dehors des types A_1 , A_2 , A_3 , B_2 et G_2 , il n'y avait pas un seul nombre premier ℓ dont on sût s'il était assez grand ! On espère qu'il suffit de prendre ℓ plus grand que le nombre de Coxeter h . Soergel montre que, si $\ell > h$, alors une partie de la conjecture de Lusztig (pour les poids « autour du poids de Steinberg ») est équivalente au fait que $\pi_{s*}^p \mathcal{J}_{!*}(S_w, \mathbb{F})$ est semi-simple pour toute réflexion simple s et tout élément w du groupe de Weyl W , où π_s est le morphisme quotient $\mathcal{G}/\mathcal{B} \rightarrow \mathcal{G}/\mathcal{P}_s$ (on dénote par \mathcal{B} un sous-groupe de Borel de \mathcal{G} , et par \mathcal{P}_s le sous-groupe parabolique minimal contenant \mathcal{B} correspondant à s). Pour \mathbb{K} au lieu de \mathbb{F} , cela résulte du théorème de décomposition. De plus, il définit de manière unique pour chaque x dans W un faisceau pervers indécomposable \mathcal{L}_x , dont les fibres de cohomologie encodent des multiplicités.

À la fin de leur article original [KL80b], Kazhdan et Lusztig mentionnent le cas de Sp_4 . On a deux éléments de longueur trois dans le groupe de Weyl. Parmi les deux variétés de Schubert correspondantes, l'une est lisse, et l'autre a un lieu singulier de codimension deux. Plus précisément, cette dernière est un fibré en droites projectives sur une singularité simple de type A_1 . On sait (et nous le verrons dans cette thèse) que la cohomologie d'intersection se comporte différemment pour $\ell = 2$ dans ce cas. Je remercie Geordie Williamson pour avoir attiré mon attention sur ce point.

Ainsi, on connaissait depuis longtemps des exemples avec de la 2-torsion dans les types non simplement lacés. Ce n'est que récemment que Braden a montré qu'il y avait de la torsion dans les types A_7 et D_4 (il l'a annoncé à la rencontre « Algebraische Gruppen » à Oberwolfach en 2004). Encore plus récemment, Geordie Williamson (un étudiant de

Soergel) est parvenu à obtenir des résultats positifs. Dans [Wil07], ce dernier développe une procédure combinatoire (basée sur le W -graphe), qui montre qu'il n'y a pas de ℓ -torsion, pour ℓ bon et différent de 2, sous certaines conditions qui sont très souvent vérifiées en petit rang. En particulier, il montre que c'est le cas pour tout $\ell \neq 2$ dans les types A_n , $n < 7$. Ainsi, la conjecture de Lusztig (pour les poids autour du poids de Steinberg) est vérifiée pour SL_n , $n \leq 7$, dès que $\ell > n$.

Je pense qu'on pourrait trouver d'autres exemples de torsion dans les variétés de Schubert, en utilisant les résultats de [BP99]. Dans cet article, Brion et Polo décrivent les singularités génériques de certaines variétés de Schubert (paraboliques), et en déduisent en particulier une façon efficace de calculer certaines polynômes de Kazhdan-Lusztig. Même dans le cas où les polynômes étaient déjà connus, Brion et Polo donnent une information géométrique plus précise, qu'on pourrait utiliser pour calculer la torsion dans la cohomologie d'intersection. Dans les cas où leurs résultats s'appliquent, Brion et Polo décrivent la singularité transverse comme l'adhérence de l'orbite d'un vecteur de plus haut poids dans un module de Weyl pour un certain sous-groupe réductif contenant T . On pourrait traiter ces singularités comme on l'a fait pour la classe minimale. Ils décrivent aussi une généralisation avec des multicônes.

Grassmannienne affine

Le deuxième contexte où l'on a déjà utilisé des faisceaux pervers à coefficients quelconques est celui des grassmanniennes affines. Mirković et Vilonen donnent dans [MV] une version géométrique de l'isomorphisme de Satake. Ils construisent une équivalence entre la catégorie des représentations rationnelles d'un groupe réductif G sur un anneau quelconque Λ et une catégorie de faisceaux pervers équivariants sur la grassmannienne affine du dual de Langlands de G , défini sur \mathbb{C} . Étant donné le lien entre la variété nilpotente et la grassmannienne affine pour $G = GL_n$ [Lus81], il me semble maintenant que cela doit impliquer notre conjecture sur la coïncidence de la matrice de décomposition pour les faisceaux pervers sur le cône nilpotent avec celle de l'algèbre de Schur en type A (il y a peut-être quelques compatibilités à vérifier). Je remercie les personnes qui m'ont signalé cet article, à commencer par George Lusztig lui-même. Cependant, je pense qu'il serait aussi intéressant d'explorer l'approche que nous proposons dans le dernier chapitre, qui est un premier pas dans l'étude des faisceaux-caractères modulaires.

L'article de [MV] suggère que les matrices de décomposition de faisceaux pervers équivariants sur la grassmannienne affine a une interprétation en termes de représentations. En ce qui concerne la détermination géométrique concrète de ces nombres de décomposition, dans un type quelconque, cette thèse donne déjà quelques résultats, en utilisant [MOV05]. En effet, la plupart des dégénération minimales sont des singularités Kleiniennes ou minimales, pour lesquelles nos résultats s'appliquent directement. Dans les types non simplement lacés, on trouve aussi des singularités que les auteurs appellent « quasi-minimales », de types qu'ils désignent par ac_2 , ag_2 , et cg_2 . Il serait intéressant de faire les calculs sur les entiers dans ce cas. Par exemple, Malkin, Ostrik et Vybornov conjecturent que les singularités de types a_2 , ac_2 et ag_2 (resp. c_2 et cg_2) sont deux à deux non équivalentes. La cohomologie d'intersection rationnelle ne permet pas de les

séparer. Mais peut-être pourrait-on les séparer en travaillant sur les entiers. De la même manière, on pourrait trouver des preuves plus simples de non-lissité (voir la dernière section de leur article, où il font des calculs de multiplicités équivariantes). Par exemple, les singularités de type c_n et g_2 sont rationnellement lisses mais non \mathbb{F}_2 -lisses. Il faudrait faire les calculs pour les singularités quasi-minimales.

D'une manière générale, je pense que les faisceaux pervers sur les entiers et modulo ℓ sont encore sous-utilisés, et qu'ils seront amenés à jouer un rôle de plus en plus important, notamment en théorie des représentations.

Introduction (français)

Introduction

Context and overview

In 1976, Springer introduced a geometrical construction of irreducible representations of Weyl groups, which had a deep influence and many later developments, which lead to Lusztig's theory of character sheaves, which enables one to calculate character values for finite reductive groups. In this thesis, we define a Springer correspondence for modular representations of Weyl groups, and establish some of its properties, thus answering a question raised by Springer himself.

Modular representations, decomposition matrices

The modular representation theory of finite groups, initiated and developed by Brauer since the early 1940's, is concerned with representations of finite groups over fields of characteristic $\ell > 0$. When ℓ divides the order of the group, the category of representations is no longer semi-simple.

Let W be a finite group. We fix a finite extension \mathbb{K} of the field \mathbb{Q}_ℓ of ℓ -adic numbers. Let \mathbb{O} be its valuation ring. We denote by $\mathfrak{m} = (\varpi)$ the maximal ideal of \mathbb{O} , and by \mathbb{F} the residual field (which is finite of characteristic ℓ). The triplet $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ is called an ℓ -modular system, and we assume that it is large enough for W (that is, we assume that all simple $\mathbb{K}W$ -modules are absolutely simple, and similarly for \mathbb{F}). The letter \mathbb{E} will denote either of the rings of this triplet.

For an abelian category \mathcal{A} , we denote by $K_0(\mathcal{A})$ its Grothendieck group. When \mathcal{A} is the category of finitely generated A -modules, where A is a ring, we adopt the notation $K_0(A)$. The full subcategory consisting of the projective objects of \mathcal{A} will be denoted $\text{Proj } \mathcal{A}$.

The Grothendieck group $K_0(\mathbb{K}W)$ is free with basis $([E])_{E \in \text{Irr } \mathbb{K}W}$, where $\text{Irr } \mathbb{K}W$ denotes the set of isomorphism classes of simple \mathbb{K} -modules; and similarly for $K_0(\mathbb{F}W)$. For $F \in \text{Irr } \mathbb{F}W$, let P_F be a projective cover of F . Then $([P_F])_{F \in \text{Irr } \mathbb{F}W}$ is a basis of $K_0(\text{Proj } \mathbb{F}W)$. Reduction modulo \mathfrak{m} defines an isomorphism from $K_0(\text{Proj } \mathbb{O}W)$ onto $K_0(\text{Proj } \mathbb{F}W)$.

We will define the *cde* triangle [Ser67].

$$\begin{array}{ccc} K_0(\text{Proj } \mathbb{F}W) & \xrightarrow{c} & K_0(\mathbb{F}W) \\ & \searrow e & \nearrow d \\ & & K_0(\mathbb{K}W) \end{array}$$

Introduction

The morphism c is induced by the function which maps every projective $\mathbb{F}W$ -module on its class in $K_0(\mathbb{F}W)$. The functor “extension of scalars to \mathbb{K} ” induces a morphism from $K_0(\text{Proj } \mathbb{O}W)$ into $K_0(\mathbb{K}W)$. The morphism e is obtained by composing with the inverse of the canonical isomorphism from $K_0(\text{Proj } \mathbb{O}W)$ onto $K_0(\text{Proj } \mathbb{F}W)$. The morphism d is a little more difficult to define. Let E be a $\mathbb{K}W$ -module. One can choose a W -stable \mathbb{O} -lattice $E_{\mathbb{O}}$ in E . The image of $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$ in $K_0(\mathbb{F}W)$ does not depend on the choice of the lattice [Ser67]. The morphism d is induced by the function which maps E to $[\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}]$.

We define the decomposition matrix $D^W = (d_{E,F}^W)_{E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W}$ by

$$d([E]) = \sum_{F \in \text{Irr } \mathbb{F}W} d_{E,F}^W [F]$$

One of the main problems in modular representation theory is to determine these decomposition numbers $d_{E,F}^W$ explicitly for interesting classes of finite groups. This problem is open for the symmetric group.

The cde triangle can be interpreted in terms of ordinary and modular (or Brauer) characters. We refer to [Ser67]. When the ordinary characters are known (which is the case for the symmetric group), the knowledge of the decomposition matrix is equivalent to the determination of Brauer characters.

There are variants of this problem when one leaves the framework of finite groups. One can for example consider the modular representations of Hecke algebras. These algebras are deformations of reflection group algebras, and play a key role in the representation theory of finite groups of Lie type. They are defined generically, and one can study what happens for special values of the parameters.

One can also study the rational representations of a reductive group in positive characteristic, or the representations of a reductive Lie algebra, or of quantum groups (deformations of enveloping algebras), etc. In that case, one considers the multiplicities of simple objects in some classes of particular objects, whose characters are known.

Perverse sheaves and representations

In the last three decades, geometric methods have led to dramatic progress in many parts of representation theory. We will be particularly concerned with representations of Weyl groups and finite groups of Lie type.

In 1976, Springer managed to construct geometrically all the irreducible ordinary representations of a Weyl group in the cohomology of certain varieties associated with the nilpotent elements of the corresponding Lie algebra [Spr76, Spr78]. This discovery had a huge impact. Many other constructions were subsequently proposed by other mathematicians. For example, Kazhdan and Lusztig proposed a topological approach [KL80b], and Slodowy constructed these representations by monodromy [Slo80a]. At the beginning of the 1980's, the blossoming of intersection cohomology permitted to reinterpret Springer correspondence in terms of perverse sheaves [Lus81, BM81].

Lusztig extended this work by studying a generalized Springer correspondence, and some intersection cohomology complexes on a reductive group G or its Lie algebra \mathfrak{g} ,

which he calls admissible complexes or character sheaves [Lus84, Lus85a, Lus85b, Lus85c, Lus86a, Lus86b]. If G is endowed with an \mathbb{F}_q -rational structure defined by a Frobenius endomorphism F , then the F -stable character sheaves yield central functions on the finite group G^F , which are very close to the irreducible characters. The transition matrix between these two bases is described by a Fourier transform. Thus, these geometric methods were powerful enough to determine the ordinary characters of finite reductive groups (at least for those with connected centre).

As far as I know, up to now these methods have not been used to study the modular representations of Weyl groups or of finite groups of Lie type. Yet, this was the case in at least two other modular situations. Soergel [Soe00] transformed a problem about the category \mathcal{O} into a problem for perverse sheaves with modular coefficients on Schubert varieties. Secondly, Mirkovic and Vilonen established an equivalence of categories between the rational representations of a reductive group over an arbitrary ring Λ and the perverse sheaves with Λ coefficients on the Langlands dual group, defined over \mathbb{C} . The classical topology allows one to use arbitrary coefficients. By the way, this work gives an intrinsic definition of the Langlands dual group by defining its category of representations. We will come back to this in the section about perspectives.

Modular Springer correspondence

It was tempting to look for a link between modular representations of Weyl groups and perverse sheaves modulo ℓ on the nilpotents. In other words, to try to define a modular Springer correspondence. Of course, the construction of Lusztig-Borho-MacPherson uses Gabber's decomposition theorem [BBD82], which no longer holds in characteristic ℓ . But Hotta and Kashiwara [HK84] have an approach *via* a Fourier transform for \mathcal{D} -modules, in the case of the base field is \mathbb{C} , and this permits to avoid the use of the decomposition theorem. Moreover, the Fourier-Deligne transform allows one to consider a base field of characteristic p and ℓ -adic coefficients [Bry86]. In this thesis, we define a modular Springer correspondence by using a Fourier-Deligne transform with modular coefficients. Besides, we define a decomposition matrix for perverse sheaves on the nilpotents, and we compare it with the decomposition matrix of the Weyl group. Moreover, we calculate certain decomposition numbers in a purely geometric way. We will see that certain properties of decomposition numbers for Weyl groups can be considered as the shadow of some geometric properties. For example, the James's row and column removal rule can be explained by a similar rule about nilpotent singularities obtained by Kraft and Procesi [KP81], once the modular Springer correspondence has been determined (which we will also do in this thesis).

Detailed contents

Preliminaries and examples

In Chapter 1, we review perverse sheaves over \mathbb{K} , \mathbb{O} and \mathbb{F} which we will use subsequently. We particularly insist on the aspects which are specific to \mathbb{O} and \mathbb{F} . For example, over

⊙ there is no self-dual perversity, but a pair of perversities, p and p_+ , exchanged by the duality. Moreover, we study the interaction between torsion pairs and t -structures (about this subject, see [HRS96]), and also with recollement situations. In this part, which is somewhat technical, one can find many distinguished triangles which will be used in the sequel. The key point is that the (derived) reduction modulo ℓ does not commute with truncations in general. We also give some complements about the perverse extensions ${}^p j_!$, ${}^p j_{!*}$, ${}^p j_*$ (on the top and the socle, and on the behavior with respect to multiplicities). Finally, we define decomposition numbers for perverse sheaves. We are particularly interested by the case of a G -variety with a finite number of orbits: we have the nilpotent cone in mind.

In Chapter 2, we give some examples of perverse sheaves, and in particular of intersection cohomology complexes over \mathbb{E} . We recall the properties of proper small (resp. semi-small) morphisms. In particular, the intersection cohomology complex of a variety having a small resolution is obtained by direct image.

Afterwards, we introduce the notion of smooth equivalence of singularities, and recall that the local intersection cohomology is an invariant for that equivalence.

Then we study conical singularities, where the local intersection cohomology is reduced to the calculation of the cohomology of a variety (the open complement to the vertex of the cone), and, more generally, we consider the case of an affine variety endowed with a \mathbb{G}_m -action contracting everything onto the origin.

It is a natural question to ask when the intersection cohomology complex is reduced to the constant sheaf \mathbb{E} (so that the variety satisfies usual Poincaré duality). In that case, we say that the variety is \mathbb{E} -smooth. A typical example of \mathbb{K} -smooth (resp. \mathbb{F} -smooth) variety is given by the quotient of a smooth variety by a finite group (resp. a finite group with order prime to ℓ).

In the last section of this chapter, we study simple singularities. A normal variety X has rational singularities if it has a resolution $\pi : \tilde{X} \rightarrow X$ with $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ for $i > 0$. Over \mathbb{C} , the surfaces with a rational double point are (up to analytic equivalence) the quotients of the affine plane by a finite subgroup of $SL_2(\mathbb{C})$. They are classified by simply-laced Dynkin diagrams. One can interpret the other types by considering the action of a group of symmetries. One associates to each Dynkin diagram Γ a homogeneous diagram $\hat{\Gamma}$, and a group of symmetries $A(\Gamma)$. In the case where Γ is already homogeneous, we have $\hat{\Gamma} = \Gamma$ and $A(\Gamma) = 1$. Let $\hat{\Phi}$ be a root system of type $\hat{\Gamma}$. We denote by $P(\hat{\Phi})$ the weight lattice, and by $Q(\hat{\Phi})$ the root lattice. Let H be the finite group of $SL_2(\mathbb{C})$ associated to $\hat{\Gamma}$. We show that

$$H^2((\mathbb{A}^2 \setminus \{0\})/H, \mathbb{Z}) \simeq P(\hat{\Phi})/Q(\hat{\Phi})$$

with a natural action of $A(\Gamma)$. Thanks to the results of Chapter 1, this allows us to compare perverse sheaves in characteristic 0 and in characteristic ℓ .

Calculation of decomposition numbers

Until Chapter 4, our aim is to calculate certain decomposition numbers for G -equivariant perverse sheaves on the nilpotent variety, by geometrical methods.

Let us first introduce some notation. The simple perverse sheaves over \mathbb{K} (resp. \mathbb{F}) are parametrized by the set $\mathfrak{N}_{\mathbb{K}}$ (resp. $\mathfrak{N}_{\mathbb{F}}$) of pairs (x, ρ) (up to conjugacy) consisting of a nilpotent element x and a character $\rho \in \text{Irr } \mathbb{K}A_G(x)$ (resp. $\rho \in \text{Irr } \mathbb{F}A_G(x)$), where $A_G(x)$ is the finite group of components of the centralizer of x in G . We will denote by

$$(d_{(x,\rho),(y,\sigma)})_{(x,\rho) \in \mathfrak{N}_{\mathbb{K}}, (y,\sigma) \in \mathfrak{N}_{\mathbb{F}}}$$

the decomposition matrix of these perverse sheaves. In the case of GL_n , all the $A_G(x)$ are trivial, so that one can forget about ρ which is always 1, and the nilpotent orbits are parametrized by the set $\mathfrak{P}_n = \{\lambda \vdash n\}$ of all partitions of n . In that case, the decomposition matrix will be denoted by

$$(d_{\lambda,\mu})_{\lambda,\mu \in \mathfrak{P}_n}$$

On the other hand, the decomposition matrix for the Weyl group W will be denoted by

$$(d_{E,F}^W)_{E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W}$$

For the symmetric group \mathfrak{S}_n , the simple $\mathbb{K}\mathfrak{S}_n$ -modules are the Specht modules S^λ , for $\lambda \in \mathfrak{P}_n$. They are defined over \mathbb{Z} , and endowed with a symmetric bilinear form defined over \mathbb{Z} . The modular reduction of the Specht module, which we will still denote by S^λ , is thus also endowed with a symmetric bilinear form. The quotient of S^λ by the radical of that symmetric bilinear form is either zero, or a simple $\mathbb{F}\mathfrak{S}_n$ -module. The set of partitions μ such that this quotient is non-zero (we then denote it by D^μ) is the set $\mathfrak{P}_n^{\ell\text{-reg}}$ of partitions of n which are ℓ -regular (each entry is repeated at most $\ell - 1$ times). The D^μ , for $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, form a complete set of representatives of isomorphism classes of simple $\mathbb{F}\mathfrak{S}_n$ -modules. The decomposition matrix of the symmetric group \mathfrak{S}_n will be rather denoted by

$$(d_{\lambda,\mu}^{\mathfrak{S}_n})_{\lambda \in \mathfrak{P}_n, \mu \in \mathfrak{P}_n^{\ell\text{-reg}}}$$

For the Schur algebra

$$S_{\mathbb{E}}(n) = S_{\mathbb{E}}(n, n) = \text{End}_{\mathbb{E}\mathfrak{S}_n} \left(\bigoplus_{\lambda \vdash n} \text{Ind}_{\mathbb{E}\mathfrak{S}_\lambda}^{\mathbb{E}\mathfrak{S}_n} \mathbb{E} \right)$$

we will denote the decomposition matrix by

$$(d_{\lambda,\mu}^{S(n)})_{\lambda,\mu \in \mathfrak{P}_n}$$

It is known that

$$d_{\lambda,\mu}^{S(n)} = d_{\lambda',\mu'}^{\mathfrak{S}_n}$$

for $\lambda \in \mathfrak{P}_n$, $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, where λ' stands for the conjugate partition. We will see that

$$d_{\lambda,\mu} = d_{\lambda',\mu'}^{\mathfrak{S}_n} = d_{\lambda,\mu}^{S(n)}$$

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for $\lambda \in \mathfrak{P}_n$, $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, and we conjecture that

$$d_{\lambda,\mu} = d_{\lambda,\mu}^{S(n)}$$

for all partitions λ, μ of n (see the remarks in the last section of this introduction).

In Chapter 3, we calculate the integral cohomology of the minimal (non-trivial) nilpotent orbit \mathcal{O}_{\min} in a simple Lie algebra \mathfrak{g} over the fields of complex numbers. Actually, the results and methods of this chapter are still valid for a base field of characteristic $p > 0$, if one works with étale cohomology with coefficients in the ℓ -adic integers.

The rational cohomology of \mathcal{O}_{\min} is already known. The dimension of \mathcal{O}_{\min} is $d = 2h^\vee - 2$, where h^\vee is the dual Coxeter number. The first half of the cohomology is given by

$$\tau_{\leq d-1} \mathrm{R}\Gamma(\mathcal{O}_{\min}, \mathbb{Q}) \simeq \bigoplus_{i=1}^k \mathbb{Q}[-2(d_i - 2)]$$

where k is the number of long simple roots, and $d_1 \leq \dots \leq d_k \leq \dots \leq d_n$ are the degrees of W (and n is the total number of simple roots). The other half can be deduced Poincaré duality.

Therefore, we are mainly interested by the torsion. If Φ is the root system of \mathfrak{g} , and Φ' is the root subsystem generated by the long simple roots (for some choice of basis), then the middle cohomology of \mathcal{O}_{\min} is

$$H^d(\mathcal{O}_{\min}, \mathbb{Z}) \simeq P^\vee(\Phi')/Q^\vee(\Phi')$$

We will see in the sequel that this result is linked to the modular reduction of the natural representation of the Weyl group W' of Φ' .

Apart from the middle cohomology, we have no uniform formula for the torsion part of the cohomology of \mathcal{O}_{\min} . But we know that it is the cokernel of a matrix whose coefficients are determined explicitly by the poset of the long roots in Φ , which is leveled by the coheight (height of the coroot), which allows us to make the calculation in any type. Apart from the middle cohomology, we observe that the prime numbers dividing the torsion are bad. This amounts to say that the intersection cohomology stalks with integer coefficients are without ℓ -torsion when ℓ is good (only for the perversity p , precisely not for p_+ , where the middle cohomology appears. We do not know whether one can find an interpretation (maybe homological) in terms of representation theory to these finite groups of bad torsion.

In Chapter 4, we calculate certain decomposition numbers for G -equivariant perverse sheaves on the nilpotent variety, using on the one hand the preceding results, and on the other hand geometric results that can be found in the literature.

First of all, we determine the decomposition numbers associated to the regular and subregular classes (the centralizer of a subregular nilpotent element is not necessarily connected). As in the section on simple singularities, we associate to the type Γ of G a homogeneous diagram $\widehat{\Phi}$ and a group of symmetries $A := A(\Gamma)$ which is isomorphic to $A_G(x_{\text{subreg}})$ when G is adjoint. We have

$$d_{(x_{\text{reg}}, 1), (x_{\text{subreg}}, \rho)} = [\mathbb{F} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) : \rho]$$

for all ρ in $\text{Irr } \mathbb{F}A$. we calculate this multiplicity in all types, for each prime number ℓ and for each $\rho \in \text{Irr } \mathbb{F}A$. For the minimal and trivial classes, we deduce from the results of Chapter 3 that

$$d_{(x_{\min},1),(0,1)} = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} (P^{\vee}(\Phi')/Q^{\vee}(\Phi'))$$

We also give this multiplicity in all types. As an example, let us see what happens for GL_n . We find

$$d_{(n),(n-1,1)} = d_{(21^{n-2}),(1^n)} = \begin{cases} 1 & \text{if } \ell \mid n \\ 0 & \text{otherwise} \end{cases}$$

which is compatible with our conjecture making a link with the Schur algebra. We have another result in that direction. The decomposition numbers for the Schur algebras satisfy the following property. If λ and μ are two partitions of n whose r first lines and s first columns are identical, and if λ_1 and μ_1 are the partitions (of a smaller integer n_1) obtained from λ and μ by suppressing these lines and columns, we have

$$d_{\lambda,\mu}^{S(n)} = d_{\lambda_1,\mu_1}^{S(n_1)}$$

Kraft and Procesi have shown that the singularities of the closures of nilpotent orbits in GL_n satisfy a similar property [KP81]. With the same notation, we have

$$\text{codim}_{\overline{\mathcal{O}}_{\lambda_1}} \mathcal{O}_{\mu_1} = \text{codim}_{\overline{\mathcal{O}}_{\lambda}} \mathcal{O}_{\mu} \quad \text{and} \quad \text{Sing}(\overline{\mathcal{O}}_{\lambda_1}, \mathcal{O}_{\mu_1}) = \text{Sing}(\overline{\mathcal{O}}_{\lambda}, \mathcal{O}_{\mu})$$

We deduce that the decomposition numbers $d_{\lambda,\mu}$ also satisfy that property:

$$d_{\lambda,\mu} = d_{\lambda_1,\mu_1}$$

If $\lambda > \mu$ are two adjacent partitions of n for the dominance order (that is, if there is no partition ν such that $\lambda > \nu > \mu$), Kraft and Procesi use the result on lines and columns to reduce the determination of the singularity of $\overline{\mathcal{O}}_{\lambda}$ along \mathcal{O}_{μ} to the extreme cases $(\lambda, \mu) = ((m), (m-1, 1))$ and $(\lambda, \mu) = ((2, 1^{m-2}), (1^m))$, for a smaller integer m . The minimal degenerations in type A_n are thus all of type A_m (a simple singularity of type A_m) or a_m (a minimal singularity of type a_m), for smaller integers m .

Since, in GL_n , all the $A_G(x)$ are trivial, this is enough to determine the decomposition number $d_{\lambda,\mu}$ when λ and μ are adjacent. In that case, we have:

$$d_{\lambda,\mu}^{S(n)} = d_{\lambda,\mu}$$

as expected.

Kraft and Procesi have also shown that the singularities of the closures of nilpotent orbits classical types satisfy a row and column removal rule [KP82]. They must deal with orthogonal and symplectic groups simultaneously. They deduce the singularity type of the minimal degenerations in that case. They find only simple and minimal singularities of classical types, with only one exception. More precisely, in the codimension two case, we have (up to smooth equivalence) a singularity of type A_k , D_k or $A_k \cup A_k$, the

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latter being the non-normal union of two simple singularities of type A_k , intersecting transversely at the singular point. When the codimension is greater than 2, we have a minimal singularity of type b_k , c_k or d_k . By suppressing these lines and columns, one can always reduce to these irreducible cases. In this article, Kraft and Procesi determine which orbit closures are normal in classical types, which was their goal.

We can also use their results to determine other decomposition numbers in classical types, but to do so in all cases, one should also determine the local systems which appear. In any case, for a minimal degeneration $\overline{\mathcal{O}} \supset \mathcal{O}'$ classical type, one can always determine the following quantity:

$$\sum_{\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})} d_{(x_{\mathcal{O}},1),(x_{\mathcal{O}'},\rho)}$$

(In classical types, the $A_G(x)$ are of the form $(\mathbb{Z}/2)^k$, and thus abelian, so all the $\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})$ are of degree 1.) In particular, one can tell when the $d_{(x_{\mathcal{O}},1),(x_{\mathcal{O}'},\rho)}$ are zero for all the $\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})$. A more detailed study should be enough to determine all the decomposition numbers of this type.

Another result of Kraft and Procesi, about the special decomposition of the nilpotent variety [KP89], allows us to show that certain decomposition numbers are zero, in the classical types, when $\ell \neq 2$. In [Lus79], Lusztig introduced a subset of $\text{Irr } \mathbb{K}W$, whose elements are called special representations. The special nilpotent classes are the classes \mathcal{O} such that the representation χ associated to $(\mathcal{O}, \mathbb{K})$ by the Springer correspondence is special. On the other hand, Spaltenstein introduced in [Spa78] an order reversing map from the set of nilpotent classes to itself, such that $d^3 = d$ (it is an involution on its image). The image of d is precisely the set of special classes. The locally closed subvarieties

$$\widehat{\mathcal{O}} = \overline{\mathcal{O}} \setminus \bigcup_{\substack{\mathcal{O}' \text{ special} \\ \overline{\mathcal{O}'} \subset \overline{\mathcal{O}}}} \overline{\mathcal{O}'}$$

where \mathcal{O} runs over the set of special classes, form a partition of the nilpotent variety \mathcal{N} . They are called special pieces. Thus each nilpotent class is contained in a unique special piece. Lusztig attached to each special class \mathcal{O} a canonical quotient $\overline{A}_G(x_{\mathcal{O}})$ of the finite group $A_G(x_{\mathcal{O}})$, and conjectured that the special piece $\widehat{\mathcal{O}}$ is the quotient of a smooth variety by $\overline{A}_G(x_{\mathcal{O}})$. A consequence of this conjecture is that $\widehat{\mathcal{O}}$ is \mathbb{K} -smooth, but actually it gives more information: in particular, the conjecture implies that $\widehat{\mathcal{O}}$ is \mathbb{F} -smooth as soon as ℓ does not divide the order of the group $\overline{A}_G(x_{\mathcal{O}})$. In [KP89], Kraft and Procesi show that this conjecture holds for classical types. We deduce that, in classical types, we have

$$d_{(x_{\mathcal{O}},1),(x_{\mathcal{O}'},\rho)} = 0$$

for $\ell > 2$, when \mathcal{O} is a special class, \mathcal{O}' is a class contained in the special piece $\widehat{\mathcal{O}}$, and $\rho \in \text{Irr } \mathbb{F}A_G(x_{\mathcal{O}'})$. A more detailed study could maybe give the decomposition numbers when $\ell = 2$.

Let us make one more remark. In another article [Kra89], Kraft solves the normality problem for closures of nilpotent orbits in G_2 . He gives the following information, which

is not covered by the preceding results: $\overline{\mathcal{O}}_{10}$ has a simple singularity of type A_1 along \mathcal{O}_8 , where \mathcal{O}_i denotes the unique nilpotent class of dimension i in the Lie algebra \mathfrak{g} of a simple group G of type G_2 . Since $A_G(x_8) = 1$ (we denote by x_i a representative of \mathcal{O}_i), this allows us to determine the decomposition number $d_{(x_{10},1),(x_8,1)}$:

$$d_{(x_{10},1),(x_8,1)} = \begin{cases} 1 & \text{if } \ell = 2, \\ 0 & \text{otherwise} \end{cases}$$

A more detailed study of this article would maybe yield more decomposition numbers in a geometrical way. In any case, using the modular Springer correspondence we will be able to determine the whole decomposition matrix when $\ell = 3$, and all the matrix but one column when $\ell = 2$. For $\ell > 3$, ℓ does not divide the order of the Weyl group, and the decomposition matrix is the identity; I think that this holds in any type, but one will need the notion of cuspidality. At least, the part of the decomposition matrix corresponding to the Weyl group is the identity matrix, as we shall see.

Modular Springer correspondence and decomposition matrices

In the sequel of the thesis, we define a modular Springer correspondence and establish some of its properties, notably the fact that it preserves decomposition numbers. Since Gabber's decomposition theorem [BBD82] is no longer true in the modular case, we are inspired by the approach of Kashiwara and Brylinski [Bry86], using a Fourier transform.

In Chapter 5, we introduce the Fourier-Deligne transform, following an article by Laumon [Lau87]. We give detailed proofs, and check that everything is fine when we take \mathbb{K} , \mathbb{O} or \mathbb{F} coefficients.

Chapter 6 is the core of this thesis. First, we recall the geometric context of Springer correspondence, which is Grothendieck's simultaneous resolution π of the singularities of the fibers of the adjoint quotient. Taking the fiber at zero, we recover Springer's resolution $\pi_{\mathcal{N}}$ of the nilpotent cone \mathcal{N} .

Then we introduce the perverse sheaves $\mathbb{E}\mathcal{K}_{\text{rs}}$, $\mathbb{E}\mathcal{K}$ and $\mathbb{E}\mathcal{K}_{\mathcal{N}}$, respectively on the open subvariety \mathfrak{g}_{rs} of the regular semi-simple elements, on \mathfrak{g} itself, and on the closed subvariety \mathcal{N} of nilpotent elements. We have the following diagram with cartesian squares:

$$\begin{array}{ccccc} \tilde{\mathfrak{g}}_{\text{rs}} & \xleftarrow{\tilde{j}_{\text{rs}}} & \tilde{\mathfrak{g}} & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \tilde{\mathcal{N}} \\ \downarrow \pi_{\text{rs}} & \square_{\text{rs}} & \downarrow \pi & \square_{\mathcal{N}} & \downarrow \pi_{\mathcal{N}} \\ \mathfrak{g}_{\text{rs}} & \xleftarrow{j_{\text{rs}}} & \mathfrak{g} & \xleftarrow{i_{\mathcal{N}}} & \mathcal{N} \end{array}$$

Let r be the rank of G , and ν the number of positive roots in Φ . We set

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$$\begin{aligned}\mathbb{E}\mathcal{K}_{\mathfrak{rs}} &= \pi_{\mathfrak{rs}!}\mathbb{E}_{\tilde{\mathfrak{g}}_{\mathfrak{rs}}}[2\nu + r] \\ \mathbb{E}\mathcal{K} &= \pi_!\mathbb{E}_{\tilde{\mathfrak{g}}}[2\nu + r] \\ \mathbb{E}\mathcal{K}_{\mathcal{N}} &= \pi_{\mathcal{N}!}\mathbb{E}_{\mathcal{N}}[2\nu]\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\mathcal{K} &= {}^p j_{\mathfrak{rs}!*}\mathbb{E}\mathcal{K}_{\mathfrak{rs}} \\ \mathbb{E}\mathcal{K}_{\mathcal{N}} &= i_{\mathcal{N}}^*\mathbb{E}\mathcal{K}[-r]\end{aligned}$$

The morphism π is proper and small, generically a W -torsor (above $\mathfrak{g}_{\mathfrak{rs}}$), and its restriction $\pi_{\mathcal{N}}$ to the nilpotents is semi-small.

Afterwards, we define a modular Springer correspondence, using a Fourier-Deligne transform. To $E \in \text{Irr } \mathbb{K}W$, the Springer correspondence à la Brylinski associates the perverse sheaf $\mathcal{T}(E) = \mathcal{F}({}^p j_{\mathfrak{rs}!*}(E[2\nu + r]))$. This defines an injective map

$$\Psi_{\mathbb{K}} : \text{Irr } \mathbb{K}W \longrightarrow \mathfrak{N}_{\mathbb{K}}$$

We will denote by $\mathfrak{N}_{\mathbb{K}}^0$ its image. We proceed similarly for the modular Springer correspondence. To $F \in \text{Irr } \mathbb{F}W$, we associate

$$\mathcal{T}(F) = \mathcal{F}({}^p j_{\mathfrak{rs}!*}(F[2\nu + r]))$$

and this defines an injective map

$$\Psi_{\mathbb{F}} : \text{Irr } \mathbb{F}W \longrightarrow \mathfrak{N}_{\mathbb{F}}$$

We will denote by $\mathfrak{N}_{\mathbb{F}}^0$ its image.

Then, we show that the decomposition matrix of the Weyl group W can be extracted from the decomposition matrix for G -equivariant perverse sheaves on the nilpotent variety, by keeping only the lines which are in the image of the ordinary Springer correspondence, and the columns which are in the image of the modular Springer correspondence. More precisely, we show that, for all $E \in \text{Irr } \mathbb{K}W$ and $F \in \text{Irr } \mathbb{F}W$, we have

$$d_{E,F}^W = d_{\Psi_{\mathbb{K}}(E), \Psi_{\mathbb{F}}(F)}$$

Finally, we determine the modular Springer correspondence when $G = GL_n$. We have:

$$\mathfrak{N}_{\mathbb{F}}^0 = \mathfrak{P}_n^{\ell\text{-res}}$$

where $\mathfrak{P}_n^{\ell\text{-res}}$ is the set of ℓ -restricted partitions of n , that is, whose conjugate is ℓ -regular.

$$\forall \lambda \in \mathfrak{P}_n^{\ell\text{-reg}}, \quad \Psi_{\mathbb{F}}(D^\lambda) = \lambda'$$

In particular, for $\lambda \in \mathfrak{P}_n$ and $\mu \in \mathfrak{P}_n^{\ell\text{-reg}}$, we have:

$$d_{\lambda,\mu}^{\mathfrak{S}_n} = d_{\lambda',\mu'}$$

so that James's row and column removal rule can be seen as a consequence of the geometric result of Kraft and Procesi about nilpotent singularities.

Perspectives

There are many themes I can explore to extend the present work.

Geometry of the nilpotent orbits

This thesis has revealed new links between the representation theory of Weyl group and the geometry of nilpotent classes. One can expect new interactions between these two areas.

For example, we observed that James's row and column removal rule can be explained geometrically by the result of Kraft and Procesi about nilpotent singularities.

On the representation theoretic side, Donkin found a generalization of this rule [Don85]. I expect a similar generalization on the geometrical side (one should find a product singularity).

Determination of the modular Springer correspondence, basic sets

A question arises naturally about the modular Springer correspondence. For simplicity, first suppose that ℓ does not divide the orders of the groups $A_G(x)$. Then one can identify $\mathfrak{N}_{\mathbb{F}}$ and $\mathfrak{N}_{\mathbb{K}}$ to a common set of parameters \mathfrak{P} . Does one have $\mathfrak{N}_{\mathbb{F}}^0 \subset \mathfrak{N}_{\mathbb{K}}^0$ in that case?

Let us assume that it is the case. Then, for each $F \in \text{Irr } \mathbb{F}W$, there is a unique $E \in \text{Irr } \mathbb{K}W$ such that $\Psi_{\mathbb{K}}(E) = \Psi_{\mathbb{F}}(F)$. This defines a basic set for W and shows in a geometrical way the triangularity of the decomposition matrix W .

Even if ℓ divides the order of some $A_G(x)$, this question still makes sense if we choose a basic set for each $A_G(x)$ (which is of the form $(\mathbb{Z}/2)^k$, or a symmetric group \mathfrak{S}_k , $k \leq 5$, for G adjoint). In fact, for all these groups, there is a canonical choice.

In the other direction, the knowledge of a basic set for W and a triangularity property compatible with the order of the orbits through the Springer correspondence allows to determine the modular Springer correspondence.

We could determine the modular Springer correspondence of GL_n , and in rank up to three, for this reason (one has to be careful for G_2 because there is one cuspidal pair in characteristic zero).

If we could show that $\mathfrak{N}_{\mathbb{F}}^0 \subset \mathfrak{N}_{\mathbb{K}}^0$, it would be interesting to determine the basic set that we obtain, and to compare it with the canonical basic set of [GR01], when the latter is well defined (that is, when ℓ does not divide the $A_G(x)$).

Generalized modular Springer correspondence, modular character sheaves

In the original Springer correspondence, $\mathfrak{N}_{\mathbb{K}}^0$ contains all the pairs of the form $(\mathcal{O}, \mathbb{K})$, but in general $\mathfrak{N}_{\mathbb{K}}^0$ is strictly contained in $\mathfrak{N}_{\mathbb{K}}$. The main motivation of Lusztig in [Lus84] is to understand these missing pairs. This work is extended in the series of articles about character sheaves, which allows to compute character values of finite groups of Lie type.

Clearly, one of the first things to do to continue the work of this thesis would be to study the notions of induction and restriction, of cuspidality, and to define a generalized

modular Springer correspondence, and to determine it in all cases. Perhaps some new combinatorial objects could appear for classical types (ℓ -symbols?).

I hope that this will lead to a theory of modular character sheaves, with a link with the modular representation theory of finite groups of Lie type. In the last chapter, we present briefly some calculations for \mathfrak{sl}_2 .

Determination intersection cohomology stalks

The determination of the intersection cohomology stalks (either over \mathbb{O} for the perversity p_+ , or over \mathbb{F}) for the nilpotent orbit closures would be enough to determine the decomposition matrix for perverse sheaves, and thus for the Weyl group (if the modular Springer correspondence has been determined). In this thesis, we determine this correspondence for GL_n . We have translated the problem of the decomposition matrices of the symmetric group into a geometrical and topological problem, where there is no mention of the Weyl group.

Of course, this problem is certainly very difficult. In characteristic zero, the determination of the intersection cohomology stalks for the nilpotents goes through the Springer correspondence and the orthogonality relations for Green functions (see the algorithm that Shoji uses in [Sho82] for the type F_4 , which is used again in other works like [BS84] for the types E_6 , E_7 , E_8 , and generalized in [Lus86b, §24]). It is unlikely that such an algorithm exists in characteristic ℓ .

Schubert varieties

As far as I know, perverse sheaves modulo ℓ had never been used to study directly the modular representations of Weyl groups, but they were used in at least two other contexts in representation theory. The first of these is concerned with Kazhdan-Lusztig theory, and thus Schubert varieties.

This time, we consider a complex reductive group G over k of characteristic ℓ , and we want to study its rational representations. For each weight λ in $X(T)$, we have an induced module $\nabla(\lambda)$. If it is non-zero, then it has a simple socle $L(\lambda)$, and all simple representations of G can be obtained in this way. We want to determine the multiplicities $[\nabla(\lambda) : L(\mu)]$ for weights λ, μ in $X(T)$ such that $\nabla(\lambda)$ and $\nabla(\mu)$ are non-zero. Lusztig [Lus80] proposed a conjecture for these multiplicities in the case $\ell > h$ (the analogue for G defined over \mathbb{C} had been conjectured in [KL79]), making a link with the perverse sheaves with the Langlands dual \mathcal{G} of G .

When Soergel wrote [Soe00], this conjecture was known to be true when ℓ is “large enough” [AJS94]. Nevertheless, apart from the types A_1 , A_2 , A_3 , B_2 and G_2 , there was no single prime number ℓ which was known to be large enough! It is hoped that it is enough to take ℓ greater than the Coxeter number h . Soergel shows that, if $\ell > h$, then part of Lusztig’s conjecture (for the weights “around the Steinberg weight”) is equivalent to the fact that $\pi_{s*} \mathcal{J}_*(S_w, \mathbb{F})$ is semi-simple for each simple reflection s and each element w of the Weyl group W , where π_s is the quotient morphism $\mathcal{G}/\mathcal{B} \rightarrow \mathcal{G}/\mathcal{P}_s$ (we denote by \mathcal{B} a Borel subgroup of \mathcal{G} , and by \mathcal{P}_s the minimal parabolic subgroup containing \mathcal{B}

corresponding to s). For \mathbb{K} instead of \mathbb{F} , this results from the decomposition theorem. Moreover, he defines for each x in W an indecomposable perverse sheaf \mathcal{L}_x , whose cohomology stalks encode multiplicities.

At the end of their original article [KL80b], Kazhdan and Lusztig mention the case of Sp_4 . We have two elements of length three in the Weyl group. Among the two corresponding Schubert varieties, one is smooth, and the other one has a singular locus of codimension two. More precisely, the latter is a \mathbb{P}^1 bundle over a simple singularity of type A_1 . It is known (and we will see this in the thesis) that the intersection cohomology is different for $\ell = 2$ in this case. I thank Geordie Williamson for explaining this to me.

So, examples with 2-torsion have been known for a long time in simply-laced types. Only recently, Braden found examples of 2-torsion in types A_7 and D_4 (he announced this result at the meeting “Algebraische Gruppen” in Oberwolfach in 2004). Even more recently, Geordie Williamson (a student of Soergel) obtained positive results. In [Wil07], he develops a combinatorial procedure (based on the W -graph), which shows that there is no ℓ -torsion, for ℓ good and different from 2, under certain conditions which are very often satisfied in small rank. In particular, he shows that it is the case for all $\ell \neq 2$ in types A_n , $n < 7$. Thus, Lusztig’s conjecture (for the weights around the Steinberg weight) is satisfied for SL_n , $n \leq 7$, as soon as $\ell > n$.

I think that one could find other examples of torsion in Schubert varieties, using the results in [BP99]. In this article, Brion and Polo describe the generic singularities of certain (parabolic) Schubert varieties, and deduce in particular an efficient way to calculate certain Kazhdan-Lusztig polynomials. Even in the cases where these polynomials were already known, Brion and Polo give a more precise geometrical description, that one could use to calculate the torsion in the local intersection cohomology. For the cases where their results apply, Brion and Polo describe the transverse singularity as the closure of the orbit of a highest weight vector in a Weyl module, for a certain reductive subgroup containing T . One could treat these singularities in the same way as the minimal class. They also describe a generalization with multicones.

Affine Grassmannians

The second context where perverse sheaves with arbitrary coefficients were used is about affine Grassmannians. In [MV], Mirković and Vilonen give a geometric version of Satake isomorphism. They construct an equivalence between the category of rational representations of a reductive group G over any ring Λ and a category of equivariant perverse sheaves on the affine Grassmannian of the Langlands dual of G , defined over \mathbb{C} . Given the link between the singularities of the nilpotent variety and the ones of the affine Grassmannian for $G = GL_n$ [Lus81], it seems to me now that this should imply our conjecture about the equality between the decomposition matrix for perverse sheaves on the nilpotent variety and for the Schur algebra (there may be some compatibilities that have to be checked). I thank the mathematicians who told me about this article, and particularly George Lusztig. However, I think it would also be interesting to explore the approach that we propose in the last chapter, which is a first step in the study of modular character sheaves.

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The article of [MV] suggests that decomposition matrices for equivariant perverse sheaves on the affine Grassmannian have a representation theoretic interpretation. This thesis can be used to determine concretely some of these decomposition numbers, using [MOV05]. Indeed, most of the minimal degenerations are either Kleinian or minimal singularities, for which our results apply directly. In non-homogeneous types, one can also find other singularities, that the authors call “quasi-minimal”, of types ac_2 , ag_2 , and cg_2 . It would be interesting to determine the intersection cohomology stalks over the integers in this case. For example, Malkin, Ostrik and Vybornov conjecture that the singularities of types a_2 , ac_2 and ag_2 (resp. c_2 and cg_2) are pairwise non-equivalent. The rational intersection cohomology is not enough to distinguish them. However, they might have a different local intersection cohomology over the integers. One could also obtain simpler proofs for non-smoothness (see the last section of their article, where they calculate equivariant multiplicities). For example, the singularities of type c_n and g_2 are rationally smooth, but not \mathbb{F}_2 -smooth. One should do the calculations for quasi-minimal singularities.

More generally, I think that perverse sheaves over the integers and modulo ℓ are still underused, and that their role will be more and more important in the years to come, notably in representation theory.

Chapter 1

Perverse sheaves over $\mathbb{K}, \mathbb{O}, \mathbb{F}$

1.1 Context

In all this thesis, we fix on the one hand a prime number p and an algebraic closure $\overline{\mathbb{F}}_p$ of the prime field with p elements, and for each power q of p , we denote by \mathbb{F}_q the unique subfield of $\overline{\mathbb{F}}_p$ with q elements. On the other hand, we fix a prime number ℓ distinct from p , and a finite extension \mathbb{K} of the field \mathbb{Q}_ℓ of ℓ -adic numbers, whose valuation ring we denote by \mathbb{O} . Let $\mathfrak{m} = (\varpi)$ be the maximal ideal of \mathbb{O} , and let $\mathbb{F} = \mathbb{O}/\mathfrak{m}$ be its residue field (which is finite of characteristic ℓ). In modular representation theory, a triplet such as $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ is called an ℓ -modular system. The letter \mathbb{E} will often be used to denote either of these three rings.

Let k denote \mathbb{F}_q or $\overline{\mathbb{F}}_p$ (sometimes we will allow k to be the field \mathbb{C} of complex numbers instead). We will consider only separated k -schemes of finite type, and morphisms of k -schemes. Such schemes will be called varieties. If X is a variety, we will say “ \mathbb{E} -sheaves on X ” for “constructible \mathbb{E} -sheaves on X ”. We will denote by $\text{Sh}(X, \mathbb{E})$ the noetherian abelian category of \mathbb{E} -sheaves on X , and by $\text{Loc}(X, \mathbb{E})$ the full subcategory of \mathbb{E} -local systems on X . If X is connected, these correspond to the continuous representations of the étale fundamental group of X at any base point.

Let $D_c^b(X, \mathbb{E})$ be the bounded derived category of \mathbb{E} -sheaves as defined by Deligne. The category $D_c^b(X, \mathbb{E})$ is triangulated, and endowed with a t -structure whose heart is equivalent to the abelian category of \mathbb{E} -sheaves, because the following condition is satisfied [Del80, BBD82].

$$\begin{aligned} &\text{For each finite extension } k' \text{ of } k \text{ contained in } \overline{\mathbb{F}}_p, \\ &\text{the groups } H^i(\text{Gal}(\overline{\mathbb{F}}_p/k'), \mathbb{Z}/\ell), i \in \mathbb{N}, \text{ are finite.} \end{aligned} \tag{1.1}$$

We call this t -structure the *natural* t -structure on $D_c^b(X, \mathbb{E})$. The notion of t -structure will be recalled in the next section. For triangulated categories and derived categories, we refer to [Wei94, KS06].

We have internal operations $\otimes_{\mathbb{E}}^{\mathbb{L}}$ and RHom on $D_c^b(X, \mathbb{E})$, and, if Y is another scheme, for $f : X \rightarrow Y$ a morphism we have triangulated functors

$$\begin{aligned} f_!, f_* &: D_c^b(X, \mathbb{E}) \rightarrow D_c^b(Y, \mathbb{E}) \\ f^*, f^! &: D_c^b(Y, \mathbb{E}) \rightarrow D_c^b(X, \mathbb{E}) \end{aligned}$$

We omit the letter R which is normally used (e.g. Rf_* , $Rf_!$) meaning that we consider derived functors. For the functors between categories of sheaves, we will use a 0 superscript, as in ${}^0f_!$ and 0f_* , following [BBD82].

We will denote by

$$\mathcal{D}_{X,\mathbb{E}} : D_c^b(X, \mathbb{E})^{\text{op}} \rightarrow D_c^b(X, \mathbb{E})$$

the dualizing functor $\mathcal{D}_{X,\mathbb{E}}(-) = \mathbf{R}\underline{\text{Hom}}(-, a^!\mathbb{E})$, where $a : X \rightarrow \text{Spec } k$ is the structural morphism.

We have a modular reduction functor $\mathbb{F} \otimes_{\mathbb{O}}^{\mathbb{L}}(-) : D_c^b(X, \mathbb{O}) \rightarrow D_c^b(X, \mathbb{F})$, which we will simply denote by $\mathbb{F}(-)$. It is triangulated, and it commutes with the functors $f_!$, f_* , f^* , $f^!$ and the duality. Moreover, it maps a torsion-free sheaf to a sheaf, and a torsion sheaf to a complex concentrated in degrees -1 and 0 .

By definition, we have $D_c^b(X, \mathbb{K}) = \mathbb{K} \otimes_{\mathbb{O}} D_c^b(X, \mathbb{O})$, and $\text{Sh}(X, \mathbb{K}) = \mathbb{K} \otimes_{\mathbb{O}} \text{Sh}(X, \mathbb{O})$. The functor $\mathbb{K} \otimes_{\mathbb{O}}(-) : D_c^b(X, \mathbb{O}) \rightarrow D_c^b(X, \mathbb{K})$ is exact.

In this chapter, we are going to recall the construction of the perverse t -structure on $D_c^b(X, \mathbb{E})$ for the middle perversity p (with two versions over \mathbb{O} , where we have two perversities p and p_+ exchanged by the duality). We will recall the main points of the treatment of t -structures and recollement of [BBD82], to which we refer for the details. However, in this work we emphasize the aspects concerning \mathbb{O} -sheaves, and we give some complements.

Before going through all these general constructions, let us already see what these perverse sheaves are. They form an abelian full subcategory ${}^p\mathcal{M}(X, \mathbb{E})$ of $D_c^b(X, \mathbb{E})$. If \mathbb{E} is \mathbb{K} or \mathbb{F} , then this abelian category is artinian and noetherian, and its simple objects are of the form $j_{!*}(\mathcal{L}[\dim V])$, where $j : V \rightarrow X$ is the inclusion of a smooth irreducible subvariety, \mathcal{L} is an irreducible locally constant constructible \mathbb{E} -sheaf on V , and $j_{!*}$ the intermediate extension functor. If $\mathbb{E} = \mathbb{O}$, the abelian category is only noetherian. In any case, ${}^p\mathcal{M}(X, \mathbb{E})$ is the intersection of the full subcategories ${}^pD^{\leq 0}(X, \mathbb{E})$ and ${}^pD^{\geq 0}(X, \mathbb{E})$ of $D_c^b(X, \mathbb{E})$, where, if A is a complex in $\mathcal{D}_c^b(X, \mathbb{E})$, we have

$$A \in {}^pD^{\leq 0}(X, \mathbb{E}) \iff \text{for all points } x \text{ in } X, \mathcal{H}^i i_x^* A = 0 \text{ for all } i > -\dim(x) \quad (1.2)$$

$$A \in {}^pD^{\geq 0}(X, \mathbb{E}) \iff \text{for all points } x \text{ in } X, \mathcal{H}^i i_x^! A = 0 \text{ for all } i < -\dim(x) \quad (1.3)$$

Here the points are not necessarily closed, i_x is the inclusion of x into X , and $\dim(x) = \dim \overline{\{x\}} = \deg \text{tr}(k(x)/k)$.

The pair $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ is a t -structure on $D_c^b(X, \mathbb{E})$, and ${}^p\mathcal{M}(X, \mathbb{E})$ is its *heart*.

When \mathbb{E} is a field (i.e. $\mathbb{E} = \mathbb{K}$ or \mathbb{F}), the duality $\mathcal{D}_{X,\mathbb{E}}$ exchanges ${}^pD^{\leq 0}(X, \mathbb{E})$ and ${}^pD^{\geq 0}(X, \mathbb{E})$, so it induces a self-duality on ${}^p\mathcal{M}(X, \mathbb{E})$.

However, when $\mathbb{E} = \mathbb{O}$, this is no longer true. The perversity p is no longer self-dual. The duality exchanges the t -structure defined by the middle perversity p with the t -structure $({}^{p+}D^{\leq 0}(X, \mathbb{O}), {}^{p+}D^{\geq 0}(X, \mathbb{O}))$ defined by

$$A \in {}^{p+}D^{\leq 0}(X, \mathbb{O}) \iff \text{for all points } x \text{ in } X, \begin{cases} \mathcal{H}^i i_x^* A = 0 \text{ for all } i > -\dim(x) + 1 \\ \mathcal{H}^{-\dim(x)+1} i_x^* A \text{ is torsion} \end{cases} \quad (1.4)$$

$$A \in {}^{p+}D^{\geq 0}(X, \mathbb{O}) \iff \text{for all points } x \text{ in } X, \begin{cases} \mathcal{H}^i i_x^! A = 0 \text{ for all } i < -\dim(x) \\ \mathcal{H}^{-\dim(x)} i_x^! A \text{ is torsion-free} \end{cases} \quad (1.5)$$

The definition of torsion (resp. torsion-free) objects is given in Definition 1.3.2.

We say that this t -structure is defined by the perversity p_+ , and that the duality exchanges p and p_+ . We denote by ${}^{p+}\mathcal{M}(X, \mathbb{O}) = {}^{p+}D^{\leq 0}(X, \mathbb{O}) \cap {}^{p+}D^{\geq 0}(X, \mathbb{O})$ the heart of the t -structure defined by p_+ , and we call its objects p_+ -perverse sheaves, or dual perverse sheaves. This abelian category is only artinian.

The t -structures defined by p and p_+ determine each other (see [BBD82, §3.3]). We have

$$A \in {}^{p+}D^{\leq 0}(X, \mathbb{O}) \iff A \in {}^pD^{\leq 1}(X, \mathbb{O}) \text{ and } {}^pH^1 A \text{ is torsion} \quad (1.6)$$

$$A \in {}^{p+}D^{\geq 0}(X, \mathbb{O}) \iff A \in {}^pD^{\geq 0}(X, \mathbb{O}) \text{ and } {}^pH^0 A \text{ is torsion-free} \quad (1.7)$$

$$A \in {}^pD^{\leq 0}(X, \mathbb{O}) \iff A \in {}^{p+}D^{\leq 0}(X, \mathbb{O}) \text{ and } {}^{p+}H^0 A \text{ is divisible} \quad (1.8)$$

$$A \in {}^pD^{\geq 0}(X, \mathbb{O}) \iff A \in {}^{p+}D^{\geq -1}(X, \mathbb{O}) \text{ and } {}^{p+}H^{-1} A \text{ is torsion} \quad (1.9)$$

If A is p -perverse, then it is also p_+ -perverse if and only if A is torsion-free in ${}^p\mathcal{M}(X, \mathbb{O})$. If A is p_+ -perverse, then A is also p -perverse if and only if A is divisible in ${}^{p+}\mathcal{M}(X, \mathbb{O})$. Thus, if A is both p - and p_+ -perverse, then A is without torsion in ${}^p\mathcal{M}(X, \mathbb{O})$ and divisible in ${}^{p+}\mathcal{M}(X, \mathbb{O})$.

In the next sections, we will recall why $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ (resp. the two versions with p and p_+ if $\mathbb{E} = \mathbb{O}$) is indeed a t -structure on $D_c^b(X, \mathbb{E})$. We refer to [BBD82] for more details, however their treatment of the case $\mathbb{E} = \mathbb{O}$ is quite brief, so we give some complements. The rest of the chapter is organized as follows.

First, we recall the definition of t -categories and their main properties. Then we see how they can be combined with torsion theories. Afterwards, we recall the notion of recollement of t -categories, stressing on some important properties, such as the construction of the perverse extensions ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$ with functors of truncation on the closed part. Then again, we study the connection with torsion theories. Already at this point, we have six possible extensions (the three just mentioned, in the two versions p and p_+). We also study the heads and socles of the extensions ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$, and show that the intermediate extension preserves decomposition numbers.

Then we see how those constructions show that the definitions of the last section give indeed t -structures on the triangulated categories $D_c^b(X, \mathbb{E})$, first fixing a stratification, and then taking the limit. Then, we stick to this case, where we have functors $\mathbb{K} \otimes_{\mathbb{O}}^{\mathbb{L}} (-)$ and $\mathbb{F} \otimes_{\mathbb{O}}^{\mathbb{L}} (-)$ (we did not try to axiomatize this setting), and we study the connection between modular reduction and truncation. If we take a complex A over \mathbb{O} , for each degree we have three places where we can truncate its reduction modulo ϖ , because $\mathcal{H}^i(\mathbb{F}A)$ has pieces coming from $\mathcal{H}_{\text{tors}}^i(A)$, $\mathcal{H}_{\text{free}}^i(A)$ and $\mathcal{H}_{\text{tors}}^{i+1}(A)$. So, in a recollement situation, we have 9 possible truncations.

Finally, we introduce decomposition numbers for perverse sheaves, and particularly in the G -equivariant setting. We have in mind G -equivariant perverse sheaves on the nilpotent variety.

The relation between modular reduction and truncation is really one of the main technical points of this thesis. For example, the fact that the modular reduction does not commute with the intermediate extension means that the reduction of a simple perverse sheaf will not necessarily be simple, that is, that we can have non-trivial decomposition numbers.

1.2 t -categories

Let us begin by recalling the definition of a t -structure on a triangulated category.

Definition 1.2.1 *A t -category is a triangulated category \mathcal{D} , endowed with two strictly full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$, such that, if we let $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[n]$, we have*

(i) *For X in $\mathcal{D}^{\leq 0}$ and Y in $\mathcal{D}^{\geq 1}$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.*

(ii) *$\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.*

(iii) *For each X in \mathcal{D} , there is a distinguished triangle (A, X, B) in \mathcal{D} with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.*

We also say that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t -structure on \mathcal{D} . Its heart is the full subcategory $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Let \mathcal{D} be a t -category.

Proposition 1.2.2 (i) *The inclusion of $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$) in \mathcal{D} has a right adjoint $\tau_{\leq n}$ (resp. a left adjoint $\tau_{\geq n}$).*

(ii) *For all X in \mathcal{D} , there is a unique $d \in \mathrm{Hom}(\tau_{\geq 1}X, \tau_{\leq 0}X[1])$ such that the triangle*

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \xrightarrow{d}$$

is distinguished. Up to unique isomorphism, this is the unique triangle (A, X, B) with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

(iii) *Let $a \leq b$. Then, for any X in \mathcal{D} , there is a unique morphism $\tau_{\geq a}\tau_{\leq b}X \rightarrow \tau_{\leq b}\tau_{\geq a}X$ such that the following diagram is commutative.*

$$\begin{array}{ccccc} \tau_{\leq b}X & \longrightarrow & X & \longrightarrow & \tau_{\geq a}X \\ \downarrow & & & & \uparrow \\ \tau_{\geq a}\tau_{\leq b}X & \xrightarrow{\sim} & & & \tau_{\leq b}\tau_{\geq a}X \end{array}$$

It is an isomorphism.

For example, if \mathcal{A} is an abelian category and \mathcal{D} is its derived category, the natural t -structure on \mathcal{D} is the one for which $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$) is the full subcategory of the complexes K such that $H^i K = 0$ for $i > n$ (resp. $i < n$). For $K = (K^i, d^i : K^i \rightarrow K^{i+1})$ in \mathcal{D} , the truncated complex $\tau_{\leq n} K$ is the subcomplex $\cdots \rightarrow K^{n-1} \rightarrow \text{Ker } d^n \rightarrow 0 \rightarrow \cdots$ of K . The heart is equivalent to the abelian category \mathcal{A} we started with. Note that, in this case, the cone of a morphism $f : A \rightarrow B$ between two objects of \mathcal{A} is a complex concentrated in degrees -1 and 0 . More precisely, we have $H^{-1}(\text{Cone } f) \simeq \text{Ker } f$ and $H^0(\text{Cone } f) \simeq \text{Coker } f$. In particular, we have a triangle $(\text{Ker } f[1], \text{Cone } f, \text{Coker } f)$.

If we abstract the relations between \mathcal{A} and $\mathcal{D}(\mathcal{A})$, we get the notion of admissible abelian subcategory of a triangulated category \mathcal{D} , and a t -structure on \mathcal{D} precisely provides an admissible abelian subcategory by taking the heart.

More precisely, let \mathcal{D} be a triangulated category and \mathcal{C} a full subcategory of \mathcal{D} such that $\text{Hom}^i(A, B) := \text{Hom}(A, B[i])$ is zero for $i < 0$ and A, B in \mathcal{C} . We have the following proposition, which results from the octahedron axiom.

Proposition 1.2.3 *Let $f : X \rightarrow Y$ in \mathcal{C} . We can complete f into a distinguished triangle (X, Y, S) . Suppose S is in a distinguished triangle $(N[1], S, C)$ with N and C in \mathcal{C} . Then the morphisms $N \rightarrow S[-1] \rightarrow X$ and $Y \rightarrow S \rightarrow C$, obtained by composition from the morphisms in the two triangles above, are respectively a kernel and a cokernel for the morphism f in \mathcal{C} .*

Such a morphism will be called \mathcal{C} -admissible. In a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{d}$ on objects in \mathcal{C} , the morphisms f and g are admissible, f is a kernel of g , g is a cokernel of f , and d is uniquely determined by f and g . A short exact sequence in \mathcal{C} will be called *admissible* if it can be obtained from a distinguished triangle in \mathcal{D} by suppressing the degree one morphism.

Proposition 1.2.4 *Suppose \mathcal{C} is stable by finite direct sums. Then the following conditions are equivalent.*

- (i) \mathcal{C} is abelian, and its short exact sequences are admissible.
- (ii) Every morphism of \mathcal{C} is \mathcal{C} -admissible.

A full abelian \mathcal{C} subcategory of \mathcal{D} , such that $\text{Hom}_{\mathcal{D}}^{-1}(\mathcal{C}, \mathcal{C}) = 0$, satisfying the equivalent conditions of the proposition, is called admissible. We will now see that t -structures provide admissible abelian subcategories.

Theorem 1.2.5 *The heart \mathcal{C} of a t -category \mathcal{D} is an admissible abelian subcategory of \mathcal{D} , stable by extensions. The functor $H^0 := \tau_{\geq 0} \tau_{\leq 0} \simeq \tau_{\leq 0} \tau_{\geq 0} : \mathcal{D} \rightarrow \mathcal{C}$ is a cohomological functor.*

Let \mathcal{D}_i ($i = 1, 2$) be two t -categories, and let $\varepsilon_i : \mathcal{C}_i \rightarrow \mathcal{D}_i$ denote the inclusion functors of their hearts. Let $T : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ be a triangulated functor. Then we say that T is right t -exact if $T(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$, left t -exact if $T(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$, and t -exact if it is both left and right exact.

Proposition 1.2.6 1. If T is left (resp. right) t -exact, then the additive functor ${}^pT := H^0 \circ T \circ \varepsilon_1$ is left (resp. right) exact.

2. Let (T^*, T_*) be a pair of adjoint triangulated functors, with $T^* : \mathcal{D}_2 \rightarrow \mathcal{D}_1$ and $T_* : \mathcal{D}_1 \rightarrow \mathcal{D}_2$. Then T^* is right t -exact if and only if T_* is left t -exact, and in that case $({}^pT^*, {}^pT_*)$ is a pair of adjoint functors between \mathcal{C}_1 and \mathcal{C}_2 .

1.3 Torsion theories and t -structures

Definition 1.3.1 Let \mathcal{A} be an abelian category. A torsion theory on \mathcal{A} is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that

(i) for all objects T in \mathcal{T} and F in \mathcal{F} , we have

$$\mathrm{Hom}_{\mathcal{A}}(T, F) = 0 \quad (1.10)$$

(ii) for any object A in \mathcal{A} , there are objects T in \mathcal{T} and F in \mathcal{F} such that there is a short exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0 \quad (1.11)$$

Then the short exact sequence 1.11 is functorial. We obtain functors $(-)_\mathrm{tors} : \mathcal{A} \rightarrow \mathcal{T}$ and $(-)_\mathrm{free} : \mathcal{A} \rightarrow \mathcal{F}$.

Examples of torsion theories arise with \mathbb{O} -linear abelian categories.

Definition 1.3.2 Let \mathcal{A} be an \mathbb{O} -linear abelian category. An object A in \mathcal{A} is torsion if $\varpi^N 1_A$ is zero for some $N \in \mathbb{N}$, and it is torsion-free (resp. divisible) if $\varpi \cdot 1_A$ is a monomorphism (resp. an epimorphism).

Proposition 1.3.3 Let \mathcal{A} be an \mathbb{O} -linear abelian category.

(i) If $T \in \mathcal{A}$ is torsion and $F \in \mathcal{A}$ is torsion-free, then we have

$$\mathrm{Hom}_{\mathcal{A}}(T, F) = 0 \quad (1.12)$$

(ii) If $Q \in \mathcal{A}$ is divisible and $T \in \mathcal{A}$ is torsion, then we have

$$\mathrm{Hom}_{\mathcal{A}}(Q, T) = 0 \quad (1.13)$$

Proof. (i) Let $f \in \mathrm{Hom}_{\mathcal{A}}(T, F)$. Let $N \in \mathbb{N}$ such that $\varpi^N \cdot 1_T = 0$. Then we have $(\varpi^N \cdot 1_F)f = f(\varpi^N \cdot 1_T) = 0$, and consequently $f = 0$, since $\varpi^N \cdot 1_F$ is a monomorphism.

(ii) Let $g \in \mathrm{Hom}_{\mathcal{A}}(Q, T)$. Let $N \in \mathbb{N}$ such that $\varpi^N \cdot 1_T = 0$. Then we have $g(\varpi^N \cdot 1_Q) = (\varpi^N \cdot 1_T)g = 0$, and consequently $g = 0$, since $\varpi^N \cdot 1_Q$ is an epimorphism. \square

Proposition 1.3.4 Let A be an object in \mathcal{A} .

1. If A is noetherian, then A has a greatest torsion subobject A_{tors} , the quotient A/A_{tors} has no torsion and $\mathbb{K}A \simeq \mathbb{K}A/A_{\text{tors}}$.
2. If A is artinian, then A has a greatest divisible subobject A_{div} , the quotient A/A_{div} is a torsion object and we have $\mathbb{K}A \simeq \mathbb{K}A_{\text{div}}$.

Proof. In the first case, the increasing sequence of subobjects $\text{Ker } \varpi^n \cdot 1_A$ must stabilize, so there is an integer N such that $\text{Ker } \varpi^n \cdot 1_A = \text{Ker } \varpi^N \cdot 1_A$ for all $n \geq N$. We set $A_{\text{tors}} := \text{Ker } \varpi^N \cdot 1_A$. This is clearly a torsion object, since it is killed by ϖ^N . Now let T be a torsion subobject of A . It is killed by some ϖ^k , and we can assume $k \geq N$. Thus $T \subset \text{Ker } \varpi^k \cdot 1_A = \text{Ker } \varpi^N \cdot 1_A = A_{\text{tors}}$. This shows that A_{tors} is the greatest torsion subobject of A . We have $\text{Ker } \varpi \cdot 1_{A/A_{\text{tors}}} = \text{Ker } \varpi^{N+1} \cdot 1_A / \text{Ker } \varpi^N \cdot 1_A = 0$ which shows that A/A_{tors} is torsion-free. Applying the exact functor $\mathbb{K} \otimes_{\mathbb{O}} -$ to the short exact sequence $0 \rightarrow A_{\text{tors}} \rightarrow A \rightarrow A/A_{\text{tors}} \rightarrow 0$, we get $\mathbb{K}A \simeq \mathbb{K}A/A_{\text{tors}}$.

In the second case, the decreasing sequence of subobjects $\text{Im } \varpi^n \cdot 1_A$ must stabilize, so there is an integer N such that $\text{Im } \varpi^n \cdot 1_A = \text{Im } \varpi^N \cdot 1_A$ for all $n \geq N$. We set $A_{\text{div}} := \text{Im } \varpi^N \cdot 1_A$. We have $\text{Im } \varpi \cdot 1_{A_{\text{div}}} = \text{Im } \varpi^{N+1} \cdot 1_A = \text{Im } \varpi^N \cdot 1_A = A_{\text{div}}$, thus A_{div} is divisible. We have $\text{Im } \varpi^n \cdot 1_{A/A_{\text{div}}} = \text{Im } \varpi^n \cdot 1_A / \text{Im } \varpi^N \cdot 1_A = 0$ for $n \geq N$. Hence A/A_{div} is a torsion object. Applying the exact functor $\mathbb{K} \otimes_{\mathbb{O}} -$ to the short exact sequence $0 \rightarrow A_{\text{div}} \rightarrow A \rightarrow A/A_{\text{div}} \rightarrow 0$, we get $\mathbb{K}A_{\text{div}} \simeq \mathbb{K}A$. \square

Proposition 1.3.5 *Let \mathcal{A} be an \mathbb{O} -linear abelian category. We denote by \mathcal{T} (resp. \mathcal{F} , \mathcal{Q}) the full subcategory of torsion (resp. torsion-free, divisible) objects in \mathcal{A} . If \mathcal{A} is noetherian (resp. artinian), then $(\mathcal{T}, \mathcal{F})$ (resp. $(\mathcal{Q}, \mathcal{T})$) is a torsion theory on \mathcal{A} .*

Proof. This follows from Propositions 1.3.3 and 1.3.4 \square

We want to discuss the combination of t -structures with torsion theories.

Proposition 1.3.6 *Let \mathcal{D} be a triangulated category with a t -structure $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$, with heart \mathcal{C} , truncation functors ${}^p\tau_{\leq i}$ and ${}^p\tau_{\geq i}$, and cohomology functors ${}^pH^i : \mathcal{D} \rightarrow \mathcal{C}$, and suppose that \mathcal{C} is endowed with a torsion theory $(\mathcal{T}, \mathcal{F})$. Then we can define a new t -structure $({}^{p+}\mathcal{D}^{\leq 0}, {}^{p+}\mathcal{D}^{\geq 0})$ on \mathcal{D} by*

$${}^{p+}\mathcal{D}^{\leq 0} = \{A \in {}^p\mathcal{D}^{\leq 1} \mid {}^pH^1(A) \in \mathcal{T}\}$$

$${}^{p+}\mathcal{D}^{\geq 0} = \{A \in {}^p\mathcal{D}^{\geq 0} \mid {}^pH^0(A) \in \mathcal{F}\}$$

Proof. Let us check the three axioms for t -structures given in Definition 1.2.1.

(i) Let $A \in {}^{p+}\mathcal{D}^{\leq 0}$ and $B \in {}^{p+}\mathcal{D}^{\geq 1}$. Then we have

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(A, B) &= \text{Hom}_{\mathcal{D}}({}^p\tau_{\geq 1}A, {}^p\tau_{\leq 1}B) \quad \text{since } A \in {}^p\mathcal{D}^{\leq 1} \text{ and } B \in {}^p\mathcal{D}^{\geq 1} \\ &= \text{Hom}_{\mathcal{C}}({}^pH^1A, {}^pH^1B) = 0 \quad \text{by (1.10), since } {}^pH^1A \in \mathcal{T} \text{ and } {}^pH^1B \in \mathcal{F} \end{aligned}$$

(ii) We have ${}^{p+}\mathcal{D}^{\leq 0} \subset {}^p\mathcal{D}^{\leq 1} \subset {}^{p+}\mathcal{D}^{\leq 1}$ and ${}^{p+}\mathcal{D}^{\geq 0} \supset {}^p\mathcal{D}^{\geq 1} \supset {}^{p+}\mathcal{D}^{\geq 1}$.

(iii) Let $A \in \mathcal{D}$. By (1.11), there are objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that we have a short exact sequence

$$0 \longrightarrow T \longrightarrow {}^p H^1 A \longrightarrow F \longrightarrow 0$$

By [BBD82, Proposition 1.3.15] there is a distinguished triangle

$$A' \xrightarrow{a} A \xrightarrow{b} A'' \xrightarrow{d} A'[1]$$

such that $A' \in {}^p \mathcal{D}^{\leq 1}$ and $A'' \in {}^p \mathcal{D}^{\geq 1}$, ${}^p H^1 A' \simeq T$ and ${}^p H^1 A'' \simeq F$, and thus $A' \in {}^p \mathcal{D}^{\leq 0}$ and $A'' \in {}^p \mathcal{D}^{\geq 1}$. \square

We denote by \mathcal{C}^+ the heart of this new t -structure, by ${}^{p+}H^i : \mathcal{D} \rightarrow \mathcal{C}^+$ the new cohomology functors, and by ${}^{p+}\tau_{\leq i}, {}^{p+}\tau_{\geq i}$ the new truncation functors.

We may also use the following notation. For the notions attached to the initial t -structure, we may drop all the p , and for the new t -structure one may write i_+ instead of i , as follows: $(\mathcal{D}^{\leq i_+}, \mathcal{D}^{\geq i_+})$, H^{i_+} , $\tau_{\leq i_+}$, $\tau_{\geq i_+}$.

Note that \mathcal{C}^+ is endowed with a torsion theory, namely $(\mathcal{F}, \mathcal{T}[-1])$. We can do the same construction, and we find that $\mathcal{C}^{++} = \mathcal{C}[-1]$. We recover the usual shift of t -structures.

By definition, we have functorial distinguished triangles

$$\tau_{\leq i} \longrightarrow \tau_{\leq i_+} \longrightarrow H_{\text{tors}}^{i+1}(-)[-i-1] \quad (1.14)$$

and

$$\tau_{\leq i_+} \longrightarrow \tau_{\leq i+1} \longrightarrow H_{\text{free}}^{i+1}(-)[-i-1] \quad (1.15)$$

Example 1.3.7 *If \mathcal{D} is an \mathbb{O} -linear triangulated category, then its heart \mathcal{C} is also \mathbb{O} -linear. If \mathcal{C} is noetherian (resp. artinian), then it is naturally endowed with a torsion theory by Proposition 1.3.5, and the preceding considerations apply.*

1.4 Recollement

The recollement (gluing) construction consists roughly in a way to construct a t -structure on some derived category of sheaves on a topological space (or a ringed topos) X , given t -structures on derived categories of sheaves on U and on F , where $j : U \rightarrow X$ is an open subset of X , and $i : F \rightarrow X$ its closed complement. This can be done in a very general axiomatic framework [BBD82, §1.4], which can be applied to both the complex topology and the étale topology. The axioms can even be applied to non-topological situations, for example for representations of algebras. Let us recall the definitions and main properties of the recollement procedure.

So let \mathcal{D} , \mathcal{D}_U and \mathcal{D}_F be three triangulated categories, and let $i_* : \mathcal{D}_F \rightarrow \mathcal{D}$ and $j^* : \mathcal{D} \rightarrow \mathcal{D}_U$ be triangulated functors. It is convenient to set $i^! = i_*$ and $j^! = j^*$. We assume that the following conditions are satisfied.

Assumption 1.4.1 (i) i_* has triangulated left and right adjoints, denoted by i^* and $i^!$ respectively.

- (ii) j^* has triangulated left and right adjoints, denoted by $j_!$ and j_* respectively.
- (iii) We have $j^*i_* = 0$. By adjunction, we also have $i^*j_! = 0$ and $i^!j_* = 0$ and, for A in \mathcal{D}_F and B in \mathcal{D}_U , we have

$$\mathrm{Hom}(j_!B, i_*A) = 0 \text{ and } \mathrm{Hom}(i_*A, j_*B) = 0$$

- (iv) For all K in \mathcal{D} , there exists $d : i_*i^*K \rightarrow j_!j^*K[1]$ (resp. $d : j_*j^*K \rightarrow i_*i^!K[1]$), necessarily unique, such that the triangle $j_!j^*K \rightarrow K \rightarrow i_*i^*K \xrightarrow{d}$ (resp. $i_*i^!K \rightarrow K \rightarrow j_*j^*K \xrightarrow{d}$) is distinguished.
- (v) The functors i_* , $j_!$ and j_* are fully faithful: the adjunction morphisms $i^*i_* \rightarrow \mathrm{Id} \rightarrow i^!i_*$ and $j^*j_* \rightarrow \mathrm{Id} \rightarrow j^!j_!$ are isomorphisms.

Whenever we have a diagram

$$\begin{array}{ccc} & \xleftarrow{i^*} & \xleftarrow{j_!} \\ \mathcal{D}_F & \xrightarrow{i_*} \mathcal{D} & \xrightarrow{j^*} \mathcal{D}_U \\ & \xleftarrow{i^!} & \xleftarrow{j_*} \end{array} \quad (1.16)$$

such that the preceding conditions are satisfied, we say that we are in a situation of recollement.

Note that for each recollement situation, there is a dual recollement situation on the opposite derived categories. Recall that the opposite category of a triangulated category \mathcal{T} is also triangulated, with translation functor $[-1]$, and distinguished triangles the triangles (Z, Y, X) , where (X, Y, Z) is a distinguished triangle in \mathcal{T} . One can check that the conditions in 1.4.1 are satisfied for the following diagram, where the roles of i^* and $i^!$ (resp. $j_!$ and j_*) have been exchanged.

$$\begin{array}{ccc} & \xleftarrow{i^!} & \xleftarrow{j_*} \\ \mathcal{D}_F^{\mathrm{op}} & \xrightarrow{i_*} \mathcal{D}^{\mathrm{op}} & \xrightarrow{j^*} \mathcal{D}_U^{\mathrm{op}} \\ & \xleftarrow{i^*} & \xleftarrow{j_!} \end{array} \quad (1.17)$$

We can say that there is a ‘‘formal duality’’ in the axioms of a recollement situation, exchanging the symbols $!$ and $*$. Note that, in the case of $D_c^b(X, \mathbb{E})$, the duality $\mathcal{D}_{X, \mathbb{E}}$ really exchanges these functors.

If $\mathcal{U} \xrightarrow{u} \mathcal{T} \xrightarrow{q} \mathcal{V}$ is a sequence of triangulated functors between triangulated categories such that u identifies \mathcal{U} with a thick subcategory of \mathcal{T} , and q identifies \mathcal{V} with the quotient of \mathcal{T} by the thick subcategory $u(\mathcal{U})$, then we say that the sequence $0 \rightarrow \mathcal{U} \xrightarrow{u} \mathcal{T} \xrightarrow{q} \mathcal{V} \rightarrow 0$ is exact.

Proposition 1.4.2 *The sequences*

$$\begin{aligned} 0 &\longleftarrow \mathcal{D}_{\mathcal{F}} \xleftarrow{i^*} \mathcal{D} \xleftarrow{j_!} \mathcal{D}_U \longleftarrow 0 \\ 0 &\longrightarrow \mathcal{D}_{\mathcal{F}} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}_U \longrightarrow 0 \\ 0 &\longleftarrow \mathcal{D}_{\mathcal{F}} \xleftarrow{i^!} \mathcal{D} \xleftarrow{j_*} \mathcal{D}_U \longleftarrow 0 \end{aligned}$$

are exact.

Suppose we are given a t -structure $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ on \mathcal{D}_U , and a t -structure $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ on \mathcal{D}_F . Let us define

$$\mathcal{D}^{\leq 0} := \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\leq 0} \text{ and } i^*K \in \mathcal{D}_F^{\leq 0}\} \quad (1.18)$$

$$\mathcal{D}^{\geq 0} := \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\geq 0} \text{ and } i^!K \in \mathcal{D}_F^{\geq 0}\} \quad (1.19)$$

Theorem 1.4.3 *With the preceding notations, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t -structure on \mathcal{D} .*

We say that it is obtained from those on \mathcal{D}_U and \mathcal{D}_F by *recollement* (gluing).

Now suppose we are just given a t -structure on \mathcal{D}_F . Then we can apply the recollement procedure to the degenerate t -structure $(\mathcal{D}_U, 0)$ on \mathcal{D}_U and to the given t -structure on \mathcal{D}_F . The functors $\tau_{\leq p}$ ($p \in \mathbb{Z}$) relative to the t -structure obtained on \mathcal{D} will be denoted $\tau_{\leq p}^F$. The functor $\tau_{\leq p}^F$ is right adjoint to the inclusion of the full subcategory of \mathcal{D} whose objects are the X such that i^*X is in $\mathcal{D}_F^{\leq p}$. We have a distinguished triangle $(\tau_{\leq p}^F X, X, i_*\tau_{>p}i^*X)$. The H^p cohomology functors for this t -structure are the $i_*H^pi^*$.

Dually, one can define the functor $\tau_{\geq p}^F$ using the degenerate t -structure $(0, \mathcal{D}_U)$ on \mathcal{D}_U . It is left adjoint to the inclusion of $\{X \in \mathcal{D} \mid i^!X \in \mathcal{D}_F^{\geq p}\}$ in \mathcal{D} , we have distinguished triangles $(i_*\tau_{<p}i^!X, X, \tau_{\geq p}^F X)$, and the H^p are the $i_*H^pi^!$.

Similarly, if we are just given a t -structure on \mathcal{D}_U , and if we endow \mathcal{D}_F with the degenerate t -structure $(\mathcal{D}_F, 0)$ (resp. $(0, \mathcal{D}_F)$), we can define a t -structure on \mathcal{D} for which the functors $\tau_{\leq p}$ (resp. $\tau_{\geq p}$), denoted by $\tau_{\leq p}^U$ (resp. $\tau_{\geq p}^U$), yield distinguished triangles $(\tau_{\leq p}^U X, j_*\tau_{>p}j^*X)$ (resp. $(j_!\tau_{<p}j^*X, X, \tau_{\geq p}^U X)$), and for which the H^p functors are the $j_*H^pj^*$ (resp. $j_!H^pj^*$).

Moreover, we have

$$\tau_{\leq p} = \tau_{\leq p}^F \tau_{\leq p}^U \text{ and } \tau_{\geq p} = \tau_{\geq p}^F \tau_{\geq p}^U \quad (1.20)$$

An *extension* of an object Y of \mathcal{D}_U is an object X of \mathcal{D} endowed with an isomorphism $j^*X \xrightarrow{\sim} Y$. Such an isomorphism induces morphisms $j_!Y \rightarrow X \rightarrow j_*Y$ by adjunction. If an extension X of Y is isomorphic, as an extension, to $\tau_{\geq p}^F j_!Y$ (resp. $\tau_{\leq p}^F j_*Y$), then the isomorphism is unique, and we just write $X = \tau_{\geq p}^F j_!Y$ (resp. $\tau_{\leq p}^F j_*Y$).

Proposition 1.4.4 *Let Y in \mathcal{D}_U and p an integer. There is, up to unique isomorphism, a unique extension X of Y such that i^*X is in $\mathcal{D}_F^{\leq p-1}$ and $i^!X$ is in $\mathcal{D}_F^{\geq p+1}$. It is $\tau_{\leq p-1}^F j_*Y$, and this extension of Y is canonically isomorphic to $\tau_{\geq p+1}^F j_!Y$.*

Let \mathcal{D}_m be the full subcategory of \mathcal{D} consisting in the objects X such that $i^*X \in \mathcal{D}_F^{\leq p-1}$ and $i^!X \in \mathcal{D}_F^{\leq p+1}$. The functor j^* induces an equivalence $\mathcal{D}_m \rightarrow \mathcal{D}_U$, with quasi-inverse $\tau_{\leq p-1}^F j_* = \tau_{\geq p+1}^F j_!$, which will be denoted $j_{!*}$.

Let $\mathcal{C}, \mathcal{C}_U$ and \mathcal{C}_F denote the hearts of the t -categories $\mathcal{D}, \mathcal{D}_U$ and \mathcal{D}_F . We will use the notation pT of Proposition 1.2.6, where T is one of the functors of the recollement diagram 1.16. By definition of the t -structure of \mathcal{D} , j^* is t -exact, i^* is right t -exact, and $i^!$ is left t -exact. Applying Proposition 1.2.6, we get

Proposition 1.4.5 *(i) The functors $j_!$ and i^* are right t -exact, the functors j^* and i_* are t -exact, and the functors j_* and $i^!$ are left t -exact.*

(ii) $({}^p j_!, {}^p j^*, {}^p j_*)$ and $({}^p i^*, {}^p i_*, {}^p i^!)$ form two sequences of adjoint functors.

Proposition 1.4.6 (i) The compositions ${}^p j^* {}^p i_*$, ${}^p i^* {}^p j_!$ and ${}^p i^! {}^p j_*$ are zero. For A in \mathcal{C}_F and B in \mathcal{C}_U , we have

$$\mathrm{Hom}({}^p j_! B, {}^p i_* A) = \mathrm{Hom}({}^p i_* A, {}^p j_* B) = 0$$

(ii) For any object A in \mathcal{C} , we have exact sequences

$${}^p j_! {}^p j^* A \longrightarrow A \longrightarrow {}^p i_* {}^p i^* A \longrightarrow 0 \quad (1.21)$$

$$0 \longrightarrow {}^p i_* {}^p i^! A \longrightarrow A \longrightarrow {}^p j_* {}^p j^* A \quad (1.22)$$

(iii) If we identify \mathcal{C}_F with its essential image by the fully faithful functor ${}^p i_*$, which is a thick subcategory of \mathcal{C} , then for any object A in \mathcal{C} , ${}^p i^* A$ is the largest quotient of A in \mathcal{C}_F , and ${}^p i^! A$ is the largest subobject of A in \mathcal{C}_F .

The functor ${}^p j^*$ identifies \mathcal{C}_U with the quotient of \mathcal{C} by the thick subcategory \mathcal{C}_F .

For any object A in \mathcal{C} , we have exact sequences

$$0 \longrightarrow {}^p i_* H^{-1} i^* A \longrightarrow {}^p j_! {}^p j^* A \longrightarrow A \longrightarrow {}^p i_* {}^p i^* A \longrightarrow 0 \quad (1.23)$$

$$0 \longrightarrow {}^p i_* {}^p i^! A \longrightarrow A \longrightarrow {}^p j_* {}^p j^* A \longrightarrow {}^p i_* H^1 i^! A \longrightarrow 0 \quad (1.24)$$

Since j^* is a quotient functor of triangulated categories, the composition of the adjunction morphisms $j_! j^* \rightarrow \mathrm{Id} \rightarrow j_* j^*$ comes from a unique morphism of functors $j_! \rightarrow j_*$. Applying j^* , we get the identity automorphism of the identity functor.

Similarly, since the functor ${}^p j^*$ is a quotient functor of abelian categories, the composition of the adjunction morphisms ${}^p j_! {}^p j^* \rightarrow \mathrm{Id} \rightarrow {}^p j_* {}^p j^*$ comes from a unique morphism of functors ${}^p j_! \rightarrow {}^p j_*$. Applying ${}^p j^*$, we get the identity automorphism of the identity functor.

Let ${}^p j_{!*}$ be the image of ${}^p j_!$ in ${}^p j_*$. We have a factorization

$$j_! \longrightarrow {}^p j_! \longrightarrow {}^p j_{!*} \longrightarrow {}^p j_* \longrightarrow j_* \quad (1.25)$$

Proposition 1.4.7 We have

$${}^p j_! = \tau_{\geq 0}^F j_! = \tau_{\leq -2}^F j_* \quad (1.26)$$

$${}^p j_{!*} = \tau_{\geq 1}^F j_! = \tau_{\leq -1}^F j_* \quad (1.27)$$

$${}^p j_* = \tau_{\geq 2}^F j_! = \tau_{\leq 0}^F j_* \quad (1.28)$$

For A in \mathcal{C} , the kernel and cokernel of ${}^p j_! A \rightarrow {}^p j_* A$ are in \mathcal{C}_F . More precisely, we have the following Yoneda splice of two short exact sequences.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & i_* H^{-1} i^* j_* A & \longrightarrow & {}^p j_! A & \longrightarrow & {}^p j_* A \longrightarrow i_* H^0 i^* j_* A \longrightarrow 0 \\
 & & & & \searrow & & \nearrow \\
 & & & & & {}^p j_{!*} A & \\
 & & & & \nearrow & & \searrow \\
 0 & & & & & & 0
 \end{array} \quad (1.29)$$

Corollary 1.4.8 For A in \mathcal{C}_U , $j_{!*}A$ is the unique extension X of A in \mathcal{D} such that i^*X is in $\mathcal{D}_F^{\leq -1}$ and $i^!X$ is in $\mathcal{D}_F^{\geq 1}$. Thus it is the unique extension of A in \mathcal{C} with no non-trivial subobject or quotient in \mathcal{C}_F .

Similarly, ${}^p j_! A$ (resp. ${}^p j_* A$) is the unique extension X of A in \mathcal{D} such that i^*X is in $\mathcal{D}_F^{\leq -2}$ (resp. $\mathcal{D}_F^{\leq 0}$) and $i^!X$ is in $\mathcal{D}_F^{\geq 0}$ (resp. $\mathcal{D}_F^{\geq 2}$). In particular, ${}^p j_! A$ (resp. ${}^p j_* A$) has no non-trivial quotient (resp. subobject) in \mathcal{C}_F .

Proposition 1.4.9 The simple objects in \mathcal{C} are the ${}^p i_* S$, with S simple in \mathcal{C}_F , and the $j_{!*} S$, for S simple in \mathcal{C}_U .

1.5 Torsion theories and recollement

Suppose we are in a recollement situation, and that we are given torsion theories $(\mathcal{T}_F, \mathcal{F}_F)$ and $(\mathcal{T}_U, \mathcal{F}_U)$ of \mathcal{C}_F and \mathcal{C}_U . Then we can define a torsion theory $(\mathcal{T}, \mathcal{F})$ on \mathcal{C} by

$$\mathcal{T} = \{K \in \mathcal{C} \mid {}^p i^* K \in \mathcal{T}_F \text{ and } j^* K \in \mathcal{T}_U\} \quad (1.30)$$

$$\mathcal{F} = \{K \in \mathcal{C} \mid {}^p i^! K \in \mathcal{F}_F \text{ and } j^* K \in \mathcal{F}_U\} \quad (1.31)$$

Using these torsion theories on \mathcal{C} , \mathcal{C}_F and \mathcal{C}_U , one can define new t -structures on \mathcal{D} , \mathcal{D}_F and \mathcal{D}_U , with the superscript p_+ . Then the new t -structure on \mathcal{D} is obtained by recollement from the new t -structures on \mathcal{D}_F and \mathcal{D}_U .

Moreover, we have six interesting functors from $\mathcal{C}_U \cap \mathcal{C}_U^+$ to \mathcal{D}

$${}^p j_! = {}^p \tau_{\leq -2}^F j_* = {}^p \tau_{\geq 0}^F j_! \quad (1.32)$$

$${}^{p+} j_! = {}^p \tau_{\leq -2_+}^F j_* = {}^p \tau_{\geq 0_+}^F j_! \quad (1.33)$$

$${}^p j_{!*} = {}^p \tau_{\leq -1}^F j_* = {}^p \tau_{\geq 1}^F j_! \quad (1.34)$$

$${}^{p+} j_{!*} = {}^p \tau_{\leq -1_+}^F j_* = {}^p \tau_{\geq 1_+}^F j_! \quad (1.35)$$

$${}^p j_* = {}^p \tau_{\leq 0}^F j_* = {}^p \tau_{\geq 2}^F j_! \quad (1.36)$$

$${}^{p+} j_* = {}^p \tau_{\leq 0_+}^F j_* = {}^p \tau_{\geq 2_+}^F j_! \quad (1.37)$$

The first of these functors has image in \mathcal{C} , the last one in \mathcal{C}^+ , and the other four in $\mathcal{C} \cap \mathcal{C}^+$.

By 1.14 and 1.15 and the description above, we have functorial triangles

$${}^p j_! \longrightarrow {}^{p+} j_! \longrightarrow {}^p i_* {}^p H_{\text{tors}}^{-1} i^* j_*[1] \rightsquigarrow \quad (1.38)$$

$${}^{p+} j_! \longrightarrow {}^p j_{!*} \longrightarrow {}^p i_* {}^p H_{\text{free}}^{-1} i^* j_*[1] \rightsquigarrow \quad (1.39)$$

$${}^p j_{!*} \longrightarrow {}^{p+} j_{!*} \longrightarrow {}^p i_* {}^p H_{\text{tors}}^0 i^* j_* \rightsquigarrow \quad (1.40)$$

$${}^{p+} j_{!*} \longrightarrow {}^p j_* \longrightarrow {}^p i_* {}^p H_{\text{free}}^0 i^* j_* \rightsquigarrow \quad (1.41)$$

$${}^p j_* \longrightarrow {}^{p+} j_* \longrightarrow {}^p i_* {}^p H_{\text{tors}}^1 i^* j_*[-1] \rightsquigarrow \quad (1.42)$$

1.6 Complements on perverse extensions

Assume we are in a recollement situation with \mathbb{E} -linear categories, where \mathbb{E} is a field, and that \mathcal{C}_F , \mathcal{C}_U and \mathcal{C} are artinian and noetherian. Moreover, we assume that any simple object in these three hearts has a finite-dimensional endomorphism algebra. In general, the functor $j_{!*}$, which we also denote by ${}^p j_{!*}$, is fully faithful and sends a simple on a simple, but is not necessarily exact. However, we will prove two simple but useful results, the first one about heads and socles, and the second one which says that the multiplicities of the simples are preserved by $j_{!*}$.

1.6.1 Top and socle of perverse extensions

Proposition 1.6.1 *Let A be an object of \mathcal{C}_U . Then we have*

$$\text{Soc } {}^p j_* A \simeq \text{Soc } j_{!*} A \simeq j_{!*} \text{Soc } A \quad (1.43)$$

$$\text{Top } {}^p j_! A \simeq \text{Top } j_{!*} A \simeq j_{!*} \text{Top } A \quad (1.44)$$

Proof. Let S be a simple object in \mathcal{C} . Then either $S \simeq j_{!*} j^* S$ or $S \simeq i_* i^* S$.

Suppose we are in the first case. Then we have

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(S, \text{Soc } {}^p j_* A) &\simeq \text{Hom}_{\mathcal{C}}(S, {}^p j_* A) && \text{because the socle is the largest} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(j^* S, A) && \text{semisimple subobject} \\ &&& \text{by adjunction of the pair } (j^*, {}^p j_*) \\ \text{Hom}_{\mathcal{C}}(S, \text{Soc } j_{!*} A) &\simeq \text{Hom}_{\mathcal{C}}(j_{!*} j^* S, j_{!*} A) && \text{by assumption and because the socle} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(j^* S, A) && \text{is the largest semisimple subobject} \\ &&& \text{because } j_{!*} \text{ is fully faithful} \\ \text{Hom}_{\mathcal{C}}(S, j_{!*} \text{Soc } A) &\simeq \text{Hom}_{\mathcal{C}}(j_{!*} j^* S, j_{!*} \text{Soc } A) && \text{by assumption} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(j^* S, \text{Soc } A) && \text{because } j_{!*} \text{ is fully faithful} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(j^* S, A) && \text{because the socle is the largest} \\ &&& \text{semisimple subobject} \\ &&& \text{and } j^* S \text{ is simple} \end{aligned}$$

Taking dimensions and dividing by $\dim \text{End}_{\mathcal{C}(U)}(S)$, we find

$$[\text{Soc } {}^p j_* A : S] = [\text{Soc } j_{!*} A : S] = [j_{!*} \text{Soc } A : S] = [\text{Soc } A : j^* S]$$

Similarly, we have

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(\text{Top } {}^p j_! A, S) &\simeq \text{Hom}_{\mathcal{C}}({}^p j_! A, S) && \text{because the top is the largest} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(A, j^* S) && \text{semisimple quotient} \\ &&& \text{by adjunction of the pair } ({}^p j_!, j^*) \\ \text{Hom}_{\mathcal{C}}(\text{Top } j_{!*} A, S) &\simeq \text{Hom}_{\mathcal{C}}(j_{!*} A, j_{!*} j^* S) && \text{by assumption and because the top} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(A, j^* S) && \text{is the largest semisimple quotient} \\ &&& \text{because } j_{!*} \text{ is fully faithful} \\ \text{Hom}_{\mathcal{C}}(j_{!*} \text{Top } A, S) &\simeq \text{Hom}_{\mathcal{C}}(j_{!*} \text{Top } A, j_{!*} j^* S) && \text{by assumption} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(\text{Top } A, j^* S) && \text{because } j_{!*} \text{ is fully faithful} \\ &\simeq \text{Hom}_{\mathcal{C}_U}(A, j^* S) && \text{because the top is the largest} \\ &&& \text{semisimple quotient} \\ &&& \text{and } j^* S \text{ is simple} \end{aligned}$$

Again, taking dimensions and dividing by $\dim \text{End}_{\mathcal{C}(U)}(S)$, we find

$$[\text{Top } {}^p j_! A : S] = [\text{Top } j_{!*} A : S] = [j_{!*} \text{Top } A : S] = [\text{Top } A : j^* S]$$

Now suppose we are in the second case. The objects ${}^p j_* A$ and $j_{!*} A$ have no non-trivial subobjects in \mathcal{C}_F , hence

$$[\text{Soc } {}^p j_* A : S] = [\text{Soc } j_{!*} A : S] = 0$$

Besides, $\text{Soc } A$ is semisimple, therefore $j_{!*} \text{Soc } A$ is semisimple and has no non-trivial subobject in \mathcal{C}_F , hence

$$[j_{!*} \text{Soc } A : S] = 0$$

Similarly, the objects ${}^p j_* A$ and $j_{!*} A$ have no non-trivial quotients in \mathcal{C}_F , hence

$$[\text{Top } {}^p j_! A : S] = [\text{Top } j_{!*} A : S] = 0$$

Besides, since $\text{Top } A$ is semisimple, and therefore $j_{!*} \text{Top } A$ is semisimple and has no non-trivial quotient in \mathcal{C}_F , hence

$$[j_{!*} \text{Top } A : S] = 0$$

Since $\text{Soc } {}^p j_* A$, $\text{Soc } j_{!*} A$ and $j_{!*} \text{Soc } A$ (respectively $\text{Top } {}^p j_! A$, $\text{Top } j_{!*} A$ and $j_{!*} \text{Top } A$) are semisimple, they are isomorphic if and only if the multiplicity of each simple object is the same in each of them, hence the result. \square

1.6.2 Perverse extensions and multiplicities

Let \mathcal{S} (resp. $\mathcal{S}_U, \mathcal{S}_F$) denote the set of (isomorphism classes of) simple objects in \mathcal{C} (resp. $\mathcal{C}_U, \mathcal{C}_F$). We have $\mathcal{S} = {}^p j_{!*} \mathcal{S}_U \cup {}^p i_* \mathcal{S}_F$. Since these three hearts are assumed to be noetherian and artinian, the multiplicities of the simple objects and the notion of composition length are well-defined. Thus, if B is an object in \mathcal{C} , then we have the following relation in the Grothendieck group $K_0(\mathcal{C})$.

$$[B] = \sum_{T \in \mathcal{S}} [B : T] \cdot [T] \quad (1.45)$$

Proposition 1.6.2 *If B is an object in \mathcal{C} , then we have*

$$[B : {}^p j_{!*} S] = [j^* B : S] \quad (1.46)$$

for all simple objects S in \mathcal{C}_U . In particular, if A is an object in \mathcal{C}_U , then we have

$$[{}^p j_! A : {}^p j_{!*} S] = [{}^p j_! A : {}^p j_{!*} S] = [{}^p j_* A : {}^p j_{!*} S] = [A : S] \quad (1.47)$$

Proof. The functor j^* is exact, and sends a simple object T on a simple a simple object if $T \in {}^p j_{!*} \mathcal{S}_U$, or on zero if $T \in {}^p i_* \mathcal{S}_F$. Moreover, it sends non-isomorphic simple objects in ${}^p j_{!*} \mathcal{S}_U$ on non-isomorphic simple objects in \mathcal{S}_U . Thus, applying j^* to the relation 1.45, we get

$$[j^* B] = \sum_{T \in {}^p j_{!*}(\mathcal{S}_U)} [j^* B : j^* T] \cdot [j^* T] = \sum_{S \in \mathcal{S}_U} [j^* B : S] \cdot [S]$$

hence 1.46, and 1.47 follows. \square

1.7 Perverse t -structures

Let us go back to the setting of Section 1.1. We want to define the t -structure defining the \mathbb{E} -perverse sheaves on X for the middle perversity p (and, in case $\mathbb{E} = \mathbb{O}$, also for the perversity p_+). Let us start with the case $\mathbb{E} = \mathbb{F}$. We will consider pairs $(\mathfrak{X}, \mathfrak{L})$, where

- (i) \mathfrak{X} is a partition of X into finitely many locally closed smooth pieces, called strata, and the closure of a stratum is a union of strata.
- (ii) \mathfrak{L} consists in the following data: for each stratum S in \mathfrak{X} , a finite set $\mathfrak{L}(S)$ of isomorphism classes of irreducible locally constant sheaves of \mathbb{F} -modules over S .
- (iii) For each S in \mathfrak{X} and for each \mathcal{F} in $\mathfrak{L}(S)$, if j denotes the inclusion of S into X , then the $R^n j_* \mathcal{F}$ are $(\mathfrak{X}, \mathfrak{L})$ -constructible, with the definition below.

A sheaf of \mathbb{F} -modules is $(\mathfrak{X}, \mathfrak{L})$ -constructible if and only if its restriction to each stratum S in \mathfrak{X} is locally constant and a finite iterated extension of irreducible locally constant sheaves whose isomorphism class is in $\mathfrak{L}(S)$. We denote by $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$ the full subcategory of $D^b(X, \mathbb{F})$ consisting in the $(\mathfrak{X}, \mathfrak{L})$ -constructible complexes, that is, whose cohomology sheaves are $(\mathfrak{X}, \mathfrak{L})$ -constructible.

We say that $(\mathfrak{X}', \mathfrak{L}')$ refines $(\mathfrak{X}, \mathfrak{L})$ if each stratum S in \mathfrak{X} is a union of strata in \mathfrak{X}' , and all the \mathcal{F} in $\mathfrak{L}(S)$ are $(\mathfrak{X}', \mathfrak{L}')$ -constructible, that is, $(\mathfrak{X}'_{|S}, \mathfrak{L}'_{|\mathfrak{X}'_{|S}})$ -constructible.

The condition (iii) ensures that for $U \xrightarrow{j} V \subset X$ locally closed and unions of strata, the functors j_* , $j_!$ (resp. j^* , $j^!$) send $D_{\mathfrak{X}, \mathfrak{L}}^b(U, \mathbb{F})$ into $D_{\mathfrak{X}, \mathfrak{L}}^b(V, \mathbb{F})$ (resp. $D_{\mathfrak{X}, \mathfrak{L}}^b(V, \mathbb{F})$ into $D_{\mathfrak{X}, \mathfrak{L}}^b(U, \mathbb{F})$). It follows from the constructibility theorem for j_* (SGA 4 $\frac{1}{2}$) that any pair $(\mathfrak{X}', \mathfrak{L}')$ satisfying (i) and (ii) can be refined into a pair $(\mathfrak{X}, \mathfrak{L})$ satisfying (i), (ii) and (iii) (see [BBD82, §2.2.10]).

So let us fix a pair $(\mathfrak{X}, \mathfrak{L})$ as above. Then we define the full subcategories ${}^p D_{\mathfrak{X}, \mathfrak{L}}^{\leq 0}(X, \mathbb{F})$ and ${}^p D_{\mathfrak{X}, \mathfrak{L}}^{\geq 0}(X, \mathbb{F})$ of $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$ by

$$\begin{aligned} A \in {}^p D_{\mathfrak{X}, \mathfrak{L}}^{\leq 0}(X, \mathbb{E}) &\iff \text{for all strata } S \text{ in } \mathfrak{X}, \mathcal{H}^i i_S^* A = 0 \text{ for all } i > -\dim(S) \\ A \in {}^p D_{\mathfrak{X}, \mathfrak{L}}^{\geq 0}(X, \mathbb{E}) &\iff \text{for all strata } S \text{ in } \mathfrak{X}, \mathcal{H}^i i_S^! A = 0 \text{ for all } i < -\dim(S) \end{aligned}$$

for any A in $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$, where i_S is the inclusion of the stratum S .

One can show by induction on the number of strata that this defines a t -structure on $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$, by repeated applications of Theorem 1.4.3. On a stratum, we consider the natural t -structure shifted by $\dim S$, and we glue these t -structures successively.

The t -structure on $D_{\mathfrak{X}', \mathfrak{L}'}^b(X, \mathbb{F})$ for a finer pair $(\mathfrak{X}', \mathfrak{L}')$ induces the same t -structure on $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$, so passing to the limit we obtain a t -structure on $D_c^b(X, \mathbb{F})$.

Over \mathbb{O}/ϖ^n , we proceed similarly. An object K of $D_c^b(X, \mathbb{O}/\varpi^n)$ is $(\mathfrak{X}, \mathfrak{L})$ -constructible if the $\varpi^i \mathcal{H}^j K / \varpi^{i+1} \mathcal{H}^j K$ are $(\mathfrak{X}, \mathfrak{L})$ -constructible as \mathbb{F} -sheaves.

Over \mathbb{O} , since our field k is finite or algebraically closed, we can use Deligne's definition of $D_c^b(X, \mathbb{O})$ as the projective 2-limit of the triangulated categories $D_c^b(X, \mathbb{O}/\varpi^n)$. The assumption insures that it is triangulated. We have triangulated functors $\mathbb{O}/\varpi^n \otimes_{\mathbb{O}}^{\mathbb{L}} (-) : D_c^b(X, \mathbb{O}) \rightarrow D_c^b(X, \mathbb{O}/\varpi^n)$, and in particular $\mathbb{F} \otimes_{\mathbb{O}}^{\mathbb{L}} (-)$. We will often omit from the notation $\otimes_{\mathbb{O}}^{\mathbb{L}}$ and simply write $\mathbb{F}(-)$. The functor $\mathcal{H}^i : D_c^b(X, \mathbb{O}) \rightarrow \text{Sh}(X, \mathbb{O})$ is defined by sending an object K to the projective system of the $\mathcal{H}^i(\mathbb{O}/\varpi^n \otimes_{\mathbb{O}}^{\mathbb{L}} K)$. We have exact sequences

$$0 \longrightarrow \mathbb{O}/\varpi^n \otimes_{\mathbb{O}} \mathcal{H}^i(K) \longrightarrow \mathcal{H}^i(\mathbb{O}/\varpi^n \otimes_{\mathbb{O}}^{\mathbb{L}} K) \longrightarrow \text{Tor}_1^{\mathbb{O}}(\mathbb{O}/\varpi^n, \mathcal{H}^{i+1}(K)) \longrightarrow 0 \quad (1.48)$$

Let $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{O})$ be the full subcategory of $D_c^b(X, \mathbb{F})$ consisting in the objects K such that for some (or any) n , $\mathbb{O}/\varpi^n \otimes_{\mathbb{O}}^{\mathbb{L}} K$ is in $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{O}/\varpi^n)$, or equivalently, such that the $\mathbb{F} \mathcal{H}^i K$ are $(\mathfrak{X}, \mathfrak{L})$ -constructible. We define the t -structure for the perversity p on $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{O})$ as above. Its heart is the abelian category ${}^p \mathcal{M}_{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{O})$. Since it is \mathbb{O} -linear, it is endowed with a natural torsion theory, and we can define another t -structure as in 1.3, and we will say that it is associated to the perversity p_+ . By 1.5, it can also be

obtained by recollement. Passing to the limit, we get two t -structures on $D_c^b(X, \mathbb{O})$, for the perversities p and p_+ , which can be characterized by 1.2, 1.3, 1.4 and 1.5.

An object A of $D_c^b(X, \mathbb{O})$ is in ${}^p D^{\leq 0}(X, \mathbb{O})$ (resp. ${}^{p+} D^{\geq 0}(X, \mathbb{O})$) if and only if $\mathbb{F} A$ is in ${}^p D^{\leq 0}(X, \mathbb{F})$ (resp. ${}^{p+} D^{\geq 0}(X, \mathbb{F})$).

If A is an object in ${}^p \mathcal{M}(X, \mathbb{O})$, then $\mathbb{F} A$ is in ${}^p \mathcal{M}(X, \mathbb{F})$ if and only if A is torsion-free (that is, if and only if A is also p_+ -perverse). Then we have $\mathbb{F} A = \text{Coker } \varpi.1_A$ (the cokernel being taken in ${}^p \mathcal{M}(X, \mathbb{O})$).

Similarly, if A is an object in ${}^{p+} \mathcal{M}(X, \mathbb{O})$, then $\mathbb{F} A$ is in ${}^{p+} \mathcal{M}(X, \mathbb{F})$ if and only if A is divisible (that is, if and only if A is also p -perverse). Then we have $\mathbb{F} A = \text{Ker } \varpi.1_A[1]$ (the kernel being taken in ${}^{p+} \mathcal{M}(X, \mathbb{O})$).

To pass from \mathbb{O} to \mathbb{K} , we simply apply $\mathbb{K} \otimes_{\mathbb{O}} (-)$. Thus $D_c^b(X, \mathbb{K})$ has the same objects as $D_c^b(X, \mathbb{O})$, and $\text{Hom}_{D_c^b(X, \mathbb{K})}(A, B) = \mathbb{K} \otimes_{\mathbb{O}} \text{Hom}_{D_c^b(X, \mathbb{O})}(A, B)$. We write $D_c^b(X, \mathbb{K}) = \mathbb{K} \otimes_{\mathbb{O}} D_c^b(X, \mathbb{O})$. We also have $\text{Sh}(X, \mathbb{K}) = \mathbb{K} \otimes_{\mathbb{O}} \text{Sh}(X, \mathbb{O})$. Then we define the full subcategory $D_{\mathfrak{X}, \mathfrak{Z}}^b(X, \mathbb{K})$ of $D_c^b(X, \mathbb{K})$ as the image of $D_{\mathfrak{X}, \mathfrak{Z}}^b(X, \mathbb{O})$. The t -structures p and p_+ on $D_{\mathfrak{X}, \mathfrak{Z}}^b(X, \mathbb{O})$ give rise to a single t -structure p on $D_{\mathfrak{X}, \mathfrak{Z}}^b(X, \mathbb{K})$, because torsion objects are killed by $\mathbb{K} \otimes_{\mathbb{O}} (-)$. This perverse t -structure can be defined by recollement. Passing to the limit, we get the perverse t -structure on $D_c^b(X, \mathbb{K})$ defined by 1.2 and 1.3. We have ${}^p \mathcal{M}(X, \mathbb{K}) = \mathbb{K} \otimes_{\mathbb{O}} {}^p \mathcal{M}(X, \mathbb{O})$.

1.8 Modular reduction and truncation functors

Modular reduction does not commute with truncation functors. To simplify the notation, we will write $\mathbb{F}(-)$ for $\mathbb{F} \otimes_{\mathbb{O}}^{\mathbb{L}} (-)$.

Proposition 1.8.1 *For $A \in D_c^b(X, \mathbb{O})$ and $i \in \mathbb{Z}$, we have distinguished triangles*

$$\mathbb{F} \tau_{\leq i} A \longrightarrow \tau_{\leq i} \mathbb{F} A \longrightarrow \mathcal{H}^{-1}(\mathbb{F} \mathcal{H}_{\text{tors}}^{i+1} A)[-i] \rightsquigarrow \quad (1.49)$$

$$\tau_{\leq i} \mathbb{F} A \longrightarrow \mathbb{F} \tau_{\leq i_+} A \longrightarrow \mathcal{H}^0(\mathbb{F} \mathcal{H}_{\text{tors}}^{i+1} A)[-i-1] \rightsquigarrow \quad (1.50)$$

$$\mathbb{F} \tau_{\leq i_+} A \longrightarrow \mathbb{F} \tau_{\leq i+1} A \longrightarrow \mathbb{F} \mathcal{H}_{\text{free}}^{i+1} A[-i-1] \rightsquigarrow \quad (1.51)$$

In particular,

$$\mathcal{H}_{\text{tors}}^{i+1} A = 0 \implies \mathbb{F} \tau_{\leq i} A \xrightarrow{\sim} \tau_{\leq i} \mathbb{F} A \xrightarrow{\sim} \mathbb{F} \tau_{\leq i_+} A \quad (1.52)$$

$$\mathcal{H}_{\text{free}}^{i+1} A = 0 \implies \mathbb{F} \tau_{\leq i_+} A \xrightarrow{\sim} \mathbb{F} \tau_{\leq i+1} A \quad (1.53)$$

Proof. We have a distinguished triangle

$$\tau_{\leq i_+} A \rightarrow \tau_{\leq i+1} A \rightarrow \mathcal{H}_{\text{free}}^{i+1} A[-i-1] \rightsquigarrow$$

in $D_c^b(X, \mathbb{O})$. This follows from [BBD82, Prop. 1.3.15], which is proved using the octahedron axiom. Applying $\mathbb{F}(-)$, we get the triangle (1.51), and (1.53) follows.

By definition, we have a distinguished triangle

$$\tau_{\leq i} A \rightarrow \tau_{\leq i_+} A \rightarrow \mathcal{H}_{\text{tors}}^{i+1} A[-i-1] \rightsquigarrow$$

in $D_c^b(X, \mathbb{O})$. Applying $\mathbb{F}(-)$, we get a distinguished triangle in $D_c^b(X, \mathbb{F})$

$$\mathbb{F} \tau_{\leq i} A \rightarrow \mathbb{F} \tau_{\leq i_+} A \rightarrow \mathbb{F} \mathcal{H}_{\text{tors}}^{i+1} A[-i-1] \rightsquigarrow \quad (1.54)$$

On the other hand, we have a distinguished triangle

$$\text{Tor}_1^{\mathbb{O}}(\mathbb{F}, \mathcal{H}_{\text{tors}}^{i+1} A)[-i] \rightarrow \mathbb{F} \mathcal{H}_{\text{tors}}^{i+1} A[-i-1] \rightarrow \mathbb{F} \otimes_{\mathbb{O}} \mathcal{H}_{\text{tors}}^{i+1} A[-i-1] \quad (1.55)$$

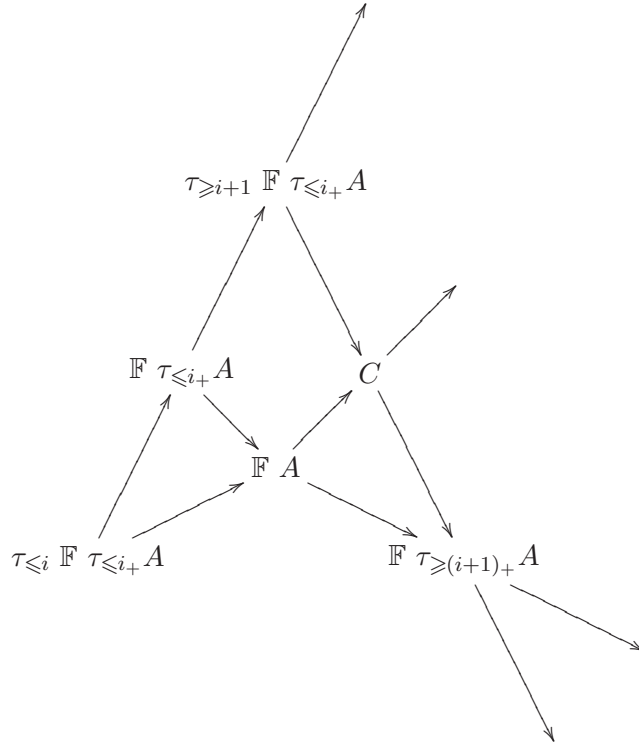
By the TR 4' axiom, we have an octahedron diagram

$$(1.56)$$

for some B in $D_c^b(X, \mathbb{O})$.

The triangle $(\mathbb{F} \tau_{\leq i} A, B, \text{Tor}_1^{\mathbb{O}}(\mathbb{F}, \mathcal{H}_{\text{tors}}^{i+1} A)[-i])$ shows that $B \in D_c^{\leq i}(X, \mathbb{F})$, and then the triangle $(B, \mathbb{F} \tau_{\leq i_+} A, \mathbb{F} \otimes_{\mathbb{O}} \mathcal{H}_{\text{tors}}^{i+1} A[-i-1])$ shows that B is (uniquely) isomorphic to $\tau_{\leq i} \mathbb{F} \tau_{\leq i_+} A$. Let us now show that $\tau_{\leq i} \mathbb{F} \tau_{\leq i_+} A \simeq \tau_{\leq i} \mathbb{F} A$.

By the TR 4 axiom, we have an octahedron diagram



for some C in $D_c^b(X, \mathbb{O})$.

The triangle $(\tau_{\geq i+1} \mathbb{F} \tau_{\leq i_+} A, C, \mathbb{F} \tau_{\geq (i+1)_+} A)$ shows that $C \in D_c^{\geq i+1}(X, \mathbb{F})$, and then the triangle $(\tau_{\leq i} \mathbb{F} \tau_{\leq i_+} A, \mathbb{F} A, C)$ shows that $B \simeq \tau_{\leq i} \mathbb{F} \tau_{\leq i_+} A \simeq \tau_{\leq i} \mathbb{F} A$ and $C \simeq \tau_{\geq i+1} \mathbb{F} A$.

Hence the octahedron diagram (1.56) contains the triangles (1.49) and (1.50). If $\mathcal{H}_{\text{tors}}^{i+1} A = 0$, the diagram reduces to the isomorphisms (1.52). \square

We have the same result if we replace $\tau_{\leq i}$ by ${}^p\tau_{\leq i}$, and \mathcal{H}^i by ${}^p\mathcal{H}^i$. The same remark applies for the functors $\tau_{\leq i}^F$ and ${}^p\tau_{\leq i}^F$.

1.9 Modular reduction and recollement

Let us fix an open subvariety $j : U \rightarrow X$, with closed complement $i : F \rightarrow X$. We want to see how the modular reduction behaves with respect to this recollement situation.

For A in ${}^p\mathcal{M}(U, \mathbb{O}) \cap {}^p\mathcal{M}(U, \mathbb{O})$, we have nine interesting extensions of $\mathbb{F} A$, out of which seven are automatically perverse. These correspond to nine ways to truncate $\mathbb{F} j_* A = j_* \mathbb{F} A$, three for each degree between -2 and 0 . Indeed, each degree is “made of” three parts: the ${}^p\mathcal{H}^0 \mathbb{F}(-)$ of the torsion part of the cohomology of A of the same degree, the reduction of the torsion-free part of the cohomology of A of the same degree, and the ${}^p\mathcal{H}^{-1} \mathbb{F}(-)$ of the torsion part of the next degree (like a Tor_1).

-2		-1			0			1	
$\mathbb{F}^{p j!}$	${}^p j! \mathbb{F}$	$\mathbb{F}^{p+j!}$	$\mathbb{F}^{p j!_*}$	${}^p j!_* \mathbb{F}$	$\mathbb{F}^{p+j!_*}$	$\mathbb{F}^{p j_*}$	${}^p j_* \mathbb{F}$	\mathbb{F}^{p+j_*}	

Using Proposition 1.8.1 with the functors ${}^p \tau_{\leq i}^F$, we obtain the following distinguished triangles.

$$\mathbb{F}^{p j!} \longrightarrow {}^p j! \mathbb{F} \longrightarrow {}^p \mathcal{H}^{-1} \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{tors}}^{-1} i^* j_*[2] \rightsquigarrow \quad (1.57)$$

$${}^p j! \mathbb{F} \longrightarrow \mathbb{F}^{p+j!} \longrightarrow {}^p \mathcal{H}^0 \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{tors}}^{-1} i^* j_*[1] \rightsquigarrow \quad (1.58)$$

$$\mathbb{F}^{p+j!} \longrightarrow \mathbb{F}^{p j!_*} \longrightarrow \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{free}}^{-1} i^* j_*[1] \rightsquigarrow \quad (1.59)$$

$$\mathbb{F}^{p j!_*} \longrightarrow {}^p j!_* \mathbb{F} \longrightarrow {}^p \mathcal{H}^{-1} \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{tors}}^0 i^* j_*[1] \rightsquigarrow \quad (1.60)$$

$${}^p j!_* \mathbb{F} \longrightarrow \mathbb{F}^{p+j!_*} \longrightarrow {}^p \mathcal{H}^0 \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{tors}}^0 i^* j_* \rightsquigarrow \quad (1.61)$$

$$\mathbb{F}^{p+j!_*} \longrightarrow \mathbb{F}^{p j_*} \longrightarrow \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{free}}^0 i^* j_* \rightsquigarrow \quad (1.62)$$

$$\mathbb{F}^{p j_*} \longrightarrow {}^p j_* \mathbb{F} \longrightarrow {}^p \mathcal{H}^{-1} \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{tors}}^1 i^* j_* \rightsquigarrow \quad (1.63)$$

$${}^p j_* \mathbb{F} \longrightarrow \mathbb{F}^{p+j_*} \longrightarrow {}^p \mathcal{H}^0 \mathbb{F}^{p i_*} {}^p \mathcal{H}_{\text{tors}}^1 i^* j_*[-1] \rightsquigarrow \quad (1.64)$$

In particular, for A in ${}^p \mathcal{M}(U, \mathbb{O}) \cap {}^{p+} \mathcal{M}(U, \mathbb{O})$, we have

$${}^p \mathcal{H}_{\text{tors}}^{-1} i^* j_* A = 0 \implies \mathbb{F}^{p j!} A \xrightarrow{\sim} {}^p j! \mathbb{F} A \xrightarrow{\sim} \mathbb{F}^{p+j!} A \quad (1.65)$$

$${}^p \mathcal{H}_{\text{free}}^{-1} i^* j_* A = 0 \implies \mathbb{F}^{p+j!} A \xrightarrow{\sim} \mathbb{F}^{p j!_*} A \quad (1.66)$$

$${}^p \mathcal{H}_{\text{tors}}^0 i^* j_* A = 0 \implies \mathbb{F}^{p j!_*} A \xrightarrow{\sim} {}^p j!_* \mathbb{F} A \xrightarrow{\sim} \mathbb{F}^{p+j!_*} A \quad (1.67)$$

$${}^p \mathcal{H}_{\text{free}}^0 i^* j_* A = 0 \implies \mathbb{F}^{p+j!_*} A \xrightarrow{\sim} \mathbb{F}^{p j_*} A \quad (1.68)$$

$${}^p \mathcal{H}_{\text{tors}}^1 i^* j_* A = 0 \implies \mathbb{F}^{p j_*} A \xrightarrow{\sim} {}^p j_* \mathbb{F} A \xrightarrow{\sim} \mathbb{F}^{p+j_*} A \quad (1.69)$$

1.10 Decomposition numbers

Let X be endowed with a pair $(\mathfrak{X}, \mathfrak{L})$ satisfying the conditions (i), (ii) and (iii) of Section 1.7. Let \mathfrak{P} be the set of pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O} \in \mathfrak{X}$ and $\mathcal{L} \in \mathfrak{L}(\mathcal{O})$. Let $K_0^{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F})$ be the Grothendieck group of the triangulated category $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$.

For $\mathcal{O} \in \mathfrak{X}$, let $j_{\mathcal{O}} : \mathcal{O} \rightarrow X$ denote the inclusion. For $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}$, let us denote by

$${}^0 \mathcal{J}_!(\mathcal{O}, \mathcal{L}) = {}^0 j_{\mathcal{O}!} (\mathcal{L}[\dim \mathcal{O}]) \quad (1.70)$$

the extension by zero of the local system \mathcal{L} , shifted by $\dim \mathcal{O}$. We also introduce the following notation for the three perverse extensions.

$${}^p \mathcal{J}_!(\mathcal{O}, \mathcal{L}) = {}^p j_{\mathcal{O}!} (\mathcal{L}[\dim \mathcal{O}]) \quad (1.71)$$

$${}^p \mathcal{J}_{!_*}(\mathcal{O}, \mathcal{L}) = {}^p j_{\mathcal{O}!_*} (\mathcal{L}[\dim \mathcal{O}]) \quad (1.72)$$

$${}^p \mathcal{J}_*(\mathcal{O}, \mathcal{L}) = {}^p j_{\mathcal{O}*} (\mathcal{L}[\dim \mathcal{O}]) \quad (1.73)$$

We have

$$K_0^{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F}) \simeq K_0(\mathrm{Sh}_{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F})) \simeq K_0({}^p\mathcal{M}_{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F})) \quad (1.74)$$

If $K \in D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{F})$, then we have

$$[K] = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}^i(K)] = \sum_{j \in \mathbb{Z}} (-1)^j [{}^p\mathcal{H}^j(K)]$$

in $K_0^{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F})$.

This Grothendieck group is free over \mathbb{Z} , and admits the following bases

$$\begin{aligned} \mathcal{B}_0 &= ({}^0\mathcal{J}_!(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}} \\ \mathcal{B}_! &= ({}^p\mathcal{J}_!(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}} \\ \mathcal{B}_{!*} &= ({}^p\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}} \\ \mathcal{B}_* &= ({}^p\mathcal{J}_*(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}} \end{aligned}$$

For $C \in K_0^{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F})$, let us define the integers $\chi_{(\mathcal{O}, \mathcal{L})}(C)$, for $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}$, by the relations

$$C = \sum_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}} \chi_{(\mathcal{O}, \mathcal{L})}(C) [{}^0\mathcal{J}_!(\mathcal{O}, \mathcal{L})] \quad (1.75)$$

For $? \in \{!, !*, *\}$, the complex ${}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L})$ extends the shifted local system $\mathcal{L}[\dim \mathcal{O}]$, and is supported on $\overline{\mathcal{O}}$. This implies

$$\chi_{(\mathcal{O}', \mathcal{L}')}({}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L})) = 0 \text{ unless } \overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}} \text{ or } (\mathcal{O}', \mathcal{L}') = (\mathcal{O}, \mathcal{L}) \quad (1.76)$$

and

$$\chi_{(\mathcal{O}, \mathcal{L})}({}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L})) = 1 \quad (1.77)$$

In other words, the three bases $\mathcal{B}_!$, $\mathcal{B}_{!*}$ and \mathcal{B}_* are unitriangular with respect to the basis \mathcal{B}_0 . This implies that they are also unitriangular with respect to each other. In fact, we already knew it by the results of Paragraph 1.6.1, since ${}^p\mathcal{J}_!(\mathcal{O}, \mathcal{L})$ (resp. ${}^p\mathcal{J}_*(\mathcal{O}, \mathcal{L})$) has a top (resp. socle) isomorphic to ${}^p\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L})$, and the radical (resp. the quotient by the socle) is supported on $\overline{\mathcal{O}} \setminus \mathcal{O}$. In particular, for $? \in \{!, *\}$, we have

$$[{}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L}) : {}^p\mathcal{J}_{!*}(\mathcal{O}', \mathcal{L}')] = 0 \text{ unless } \overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}} \text{ or } (\mathcal{O}', \mathcal{L}') = (\mathcal{O}, \mathcal{L}) \quad (1.78)$$

and

$$[{}^p\mathcal{J}_?(\mathcal{O}, \mathcal{L}) : {}^p\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L})] = 1 \quad (1.79)$$

Let $K_0^{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{K})$ be the Grothendieck group of the triangulated category $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{K})$. It can be identified with $K_0(\mathrm{Sh}_{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{K}))$ and $K_0({}^p\mathcal{M}_{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{K}))$ as for the case $\mathbb{E} = \mathbb{F}$.

Now, let K be an object of $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{K})$. If $K_{\mathbb{O}}$ is an object of $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{O})$ such that $\mathbb{K} \otimes_{\mathbb{O}} K_{\mathbb{O}} \simeq K$, we can consider $[\mathbb{F}K_{\mathbb{O}}]$ in $K_0^{\mathfrak{X}, \mathfrak{L}}(X, \mathbb{F})$. This class does not depend on the choice of $K_{\mathbb{O}}$ (note that the modular reduction of a torsion object has a zero class in the Grothendieck group: if we assume, for simplicity, that we have only finite monodromy,

then by dévissage we can reduce to the analogue result for finite groups). In fact, it depends only on the class $[K]$ of K in $K_0^{\mathfrak{X}, \mathcal{L}}(X, \mathbb{K})$. So we have a well-defined morphism

$$d : K_0^{\mathfrak{X}, \mathcal{L}}(X, \mathbb{K}) \longrightarrow K_0^{\mathfrak{X}, \mathcal{L}}(X, \mathbb{F}) \quad (1.80)$$

For $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}$, we can consider the decomposition number $[\mathbb{F}K_{\mathbb{O}} : {}^p\mathcal{J}_!(\mathcal{O}, \mathcal{L})]$, where $K_{\mathbb{O}}$ is any object of $D_{\mathfrak{X}, \mathcal{L}}^b(X, \mathbb{O})$ such that $\mathbb{K}K_{\mathbb{O}} \simeq K$.

1.11 Equivariance

We now introduce G -equivariant perverse sheaves in the sense of [Lus84, §0], [Let05, §4.2].

Let G be a *connected* algebraic group acting on a variety X . Let $\rho : G \times X \rightarrow X$ be the morphism defining the action, and let $p : G \times X \rightarrow X$ be the second projection. A sheaf F on X is G -equivariant if there is an isomorphism $\alpha : p^*F \xrightarrow{\sim} \rho^*F$. In that case, we can choose α in a unique way such that the induced isomorphism $i^*(\alpha) : F \rightarrow F$ is the identity, where $i : X \rightarrow G \times X$ is defined by $i(x) = (1_G, x)$.

If $f : X \rightarrow Y$ is a G -equivariant morphism, the functors ${}^0f^*$, 0f_* and ${}^0f_!$ take G -equivariant sheaves to G -equivariant sheaves.

Let $\text{Sh}_G(X, \mathbb{E})$ be the category whose objects are the G -equivariant \mathbb{E} -sheaves on X , and such that the morphisms between two objects F_1 and F_2 are the morphisms ϕ in $\text{Sh}(X, \mathbb{E})$ such that the following diagram commutes

$$\begin{array}{ccc} p^*F_1 & \xrightarrow{p^*\phi} & p^*F_2 \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \rho^*F_1 & \xrightarrow{\rho^*\phi} & \rho^*F_2 \end{array}$$

where α_j is the unique isomorphism such that $i^*(\alpha_j)$ is the identity for $j = 1, 2$. Then it turns out that $\text{Sh}_G(X, \mathbb{E})$ is actually a full subcategory of $\text{Sh}(X, \mathbb{E})$.

For a general complex in $D_c^b(X, \mathbb{E})$, the notion of G -equivariance is more delicate. However, for a perverse sheaf we can take the same definition as above, and again the isomorphism α can be normalized with the same condition. If f is a G -equivariant morphism, then the functors ${}^p\mathcal{H}^j f^*$, ${}^p\mathcal{H}^j f^!$, ${}^p\mathcal{H}^j f_*$ and ${}^p\mathcal{H}^j f_!$ take G -equivariant perverse sheaves to G -equivariant perverse sheaves.

We define in the same way the category ${}^p\mathcal{M}_G(X, \mathbb{E})$ of G -equivariant perverse \mathbb{E} -sheaves, and again it is a full subcategory of ${}^p\mathcal{M}(X, \mathbb{E})$. Moreover, it is stable by subquotients. The simple objects in ${}^p\mathcal{M}_G(X, \mathbb{E})$ are the intermediate extensions of irreducible G -equivariant \mathbb{E} -local systems on G -stable locally closed smooth irreducible subvarieties of X .

Suppose \mathbb{E} is a field. If \mathcal{O} is a homogeneous space for G , let x be a point in \mathcal{O} , and let $A_G(x) = C_G(x)/C_G^0(x)$. Then the set of isomorphism classes of irreducible G -equivariant \mathbb{E} -local systems is in bijection with the set $\text{Irr } \mathbb{E}A_G(x)$ of isomorphism classes of irreducible representations of the group algebra $\mathbb{E}A_G(x)$.

Suppose X is a G -variety with finitely many orbits. Then we can take the stratification \mathfrak{X} of X by its G -orbits. The orbits are indeed locally closed by [Spr98, Lemma 2.3.1], and they are smooth. For each G -orbit \mathcal{O} in X , let $x_{\mathcal{O}}$ be a closed point in \mathcal{O} . For $\mathfrak{L}(\mathcal{O})$ we take all the irreducible G -equivariant \mathbb{F} -local systems, so that we can identify $\mathfrak{L}(\mathcal{O})$ with $\text{Irr } \mathbb{F}A_G(x_{\mathcal{O}})$.

Suppose \mathbb{E} is a field. Let $K_0^G(X, \mathbb{E})$ be the Grothendieck group of the triangulated category $D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{E})$. Then we have

$$K_0^G(X, \mathbb{E}) = K_0({}^p\mathcal{M}_G(X, \mathbb{E})) = K_0(\text{Sh}_G(X, \mathbb{E})) \simeq \bigoplus_{\mathcal{O}} K_0(\text{Irr } \mathbb{E}A_G(x_{\mathcal{O}})) \quad (1.81)$$

If $K \in D_{\mathfrak{X}, \mathfrak{L}}^b(X, \mathbb{E})$, then we have

$$[K] = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}^i(K)] = \sum_{j \in \mathbb{Z}} (-1)^j [{}^p\mathcal{H}^j(K)]$$

in $K_0^G(X, \mathbb{E})$.

Let $\mathfrak{P}_{\mathbb{E}}$ be the set of pairs $(\mathcal{O}, \mathcal{L})$ with $\mathcal{O} \in \mathfrak{X}$ and \mathcal{L} an irreducible G -equivariant \mathbb{E} -local system on \mathcal{O} (corresponding to an irreducible representation L of $\mathbb{E}A_G(x_{\mathcal{O}})$). Then we have bases $\mathcal{B}_0^{\mathbb{E}} = ({}^0j_!(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{\mathbb{E}}}$, $\mathcal{B}_!^{\mathbb{E}} = ({}^pj_!(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{\mathbb{E}}}$, $\mathcal{B}_{!*}^{\mathbb{E}} = ({}^pj_{!*}(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{\mathbb{E}}}$, $\mathcal{B}_*^{\mathbb{E}} = ({}^pj_*(\mathcal{O}, \mathcal{L}))_{(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{\mathbb{E}}}$. Note that, if ℓ does not divide the $|A_G(x_{\mathcal{O}})|$, then we can identify $\mathfrak{P}_{\mathbb{K}}$ with $\mathfrak{P}_{\mathbb{F}}$.

The transition matrices from $\mathcal{B}_0^{\mathbb{E}}$ to $\mathcal{B}_?^{\mathbb{E}}$ (for $? \in \{!, !*, *\}$) are unitriangular, and also the transition matrices from $\mathcal{B}_{!*}^{\mathbb{E}}$ to $\mathcal{B}_?^{\mathbb{E}}$ (for $? \in \{!, *\}$).

As in the last section, we have a morphism

$$d : K_0^G(X, \mathbb{K}) \longrightarrow K_0^G(X, \mathbb{F})$$

The matrix of d with respect to the bases $\mathcal{B}_0^{\mathbb{E}}$ is just a product of blocks indexed by the orbits \mathcal{O} , the block corresponding to \mathcal{O} being the decomposition matrix of the finite group $A_G(x_{\mathcal{O}})$. If ℓ does not divide the $|A_G(x_{\mathcal{O}})|$, this is just the identity matrix.

We are interested in the matrix of d in the bases $\mathcal{B}_{!*}^{\mathbb{E}}$. That is, we want to study the decomposition numbers $d_{(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')} = [{}^{\mathbb{F}}\mathcal{J}_{!*}(\mathcal{O}, \mathcal{L}_{\mathbb{O}}) : \mathcal{J}_{!*}(\mathcal{O}', \mathcal{L}')]_{\mathbb{F}}$ for $(\mathcal{O}, \mathcal{L}) \in \mathfrak{P}_{\mathbb{K}}$ and $(\mathcal{O}', \mathcal{L}') \in \mathfrak{P}_{\mathbb{F}}$, where $\mathcal{L}_{\mathbb{O}}$ is an integral form for \mathcal{L} . Recall that, if ℓ does not divide the $|A_G(x)|$, then we can identify $\mathfrak{P}_{\mathbb{K}}$ with $\mathfrak{P}_{\mathbb{F}}$.

We will see in Chapter 6 that, when X is the nilpotent variety \mathcal{N} , part of these numbers can be interpreted as decomposition numbers for the Weyl group, and we expect that the whole decomposition matrix coincides with the decomposition matrix for the Schur algebra. In types other than type A , we would have to make clear which Schur algebra should show up, since several of them appear naturally. Moreover, when ℓ divides some $|A_G(x)|$, we cannot expect to have a correspondence with a quasi-hereditary algebra, so interesting things should happen for these primes.

Chapter 2

Examples

2.1 Semi-small morphisms

Definition 2.1.1 A morphism $\pi : \tilde{X} \rightarrow X$ is semi-small if there is a stratification \mathfrak{X} of X such that for all strata S in \mathfrak{X} , and for all closed points s in S , we have $\dim \pi^{-1}(s) \leq \frac{1}{2} \operatorname{codim}_X(S)$. If moreover these inequalities are strict for all strata of positive codimension, we say that π is small.

Recall that $\operatorname{Loc}(S, \mathbb{E})$ is the full subcategory of $\operatorname{Sh}(X, \mathbb{E})$ consisting in the \mathbb{E} -local systems. It is the heart of the t -category $D_{\operatorname{Loc}}^b(S, \mathbb{E})$ which is the full subcategory of $D_c^b(S, \mathbb{E})$ of objects A such that all the $\mathcal{H}^i A$ are local systems, with the t -structure induced by the natural t -structure on $D_c^b(S, \mathbb{E})$. For $\mathbb{E} = \mathbb{O}$, according to the definition given after Proposition 1.3.6, we have an abelian category $\operatorname{Loc}^+(S, \mathbb{O})$, which is the full subcategory of $D_c^b(S, \mathbb{O})$ consisting in the objects A such that $\mathcal{H}^0 A$ is a torsion-free \mathbb{O} -local system, and $\mathcal{H}^1 A$ is a torsion \mathbb{O} -local system.

Proposition 2.1.2 Let $\pi : \tilde{X} \rightarrow X$ be a surjective, proper and separable morphism, with \tilde{X} smooth of pure dimension d . Let \mathcal{L} be in $\operatorname{Loc}(\tilde{X}, \mathbb{E})$. Let us consider the complex $K = \pi_! \mathcal{L}[d]$.

- (i) If π is semi-small, then $\dim X = d$ and K is p -perverse.
- (ii) If π is small, then $K = {}^p j_{!*} j^* K$ for any inclusion $j : U \rightarrow X$ of a smooth open dense subvariety over which π is étale.

In the case $\mathbb{E} = \mathbb{O}$, we can take \mathcal{L} in $\operatorname{Loc}^+(X, \mathbb{O})$ and replace p by p_+ .

Proof.

- (i) Let us choose stratifications $\mathfrak{X}, \tilde{\mathfrak{X}}$ such that π is stratified relatively to $\mathfrak{X}, \tilde{\mathfrak{X}}$. By refining \mathfrak{X} , we can assume that for any stratum S in \mathfrak{X} , we have $2 \dim(\pi^{-1}(s)) \leq d - \dim S$ for all closed points s in S . Over a stratum of maximal dimension, π is an étale covering, so X is of dimension d .

For the sequel, first assume that \mathbb{E} is \mathbb{K} or \mathbb{F} . For each stratum S in \mathfrak{X} , and for any closed point s in S , the fiber K_s is isomorphic to $\operatorname{R}\Gamma_c(\pi^{-1}(s), \mathcal{L})[d]$, which is concentrated in degrees $[-d, -d + 2 \dim \pi^{-1}(s)] \subset [-d, -\dim S]$. Hence $K \in {}^p D^{\leq 0}(X, \mathbb{E})$.

Now $\mathcal{D}_{X,\mathbb{E}}(K) = \pi_! \mathcal{L}^\vee[d]$, where \mathcal{L}^\vee is the local system dual to \mathcal{L} , so we can apply the same argument to show that $\mathcal{D}_{X,\mathbb{E}}(K) \in {}^pD^{\leq 0}(X, \mathbb{E})$, and thus $K \in {}^pD^{\geq 0}(X, \mathbb{E})$. Consequently, K is perverse.

For $\mathbb{E} = \mathbb{O}$, let us first treat the case of the perversity p . If \mathcal{L} is a torsion-free local system (so that it is p and p_+ -perverse), then the same argument applies, since $\mathcal{D}_{\tilde{X},\mathbb{O}}(\mathcal{L}[d]) = \mathcal{L}^\vee[d]$ is still a local system shifted by d .

If \mathcal{L} is a torsion sheaf, the same argument as above shows that K is in ${}^pD^{\leq 0}(X, \mathbb{O})$. But $\mathcal{D}_{\tilde{X},\mathbb{O}}(\mathcal{L}[d])$ is in degree $-d + 1$, so the same argument shows that, for $s \in S$, K_s is concentrated in degrees $[-d + 1, -\dim S + 1]$, and $\mathcal{H}^{-\dim S + 1}K_s$ is torsion, so $\mathcal{D}_{X,\mathbb{O}}K$ is in ${}^{p+}D^{\leq 0}(X, \mathbb{O})$. This implies that K is in ${}^pD^{\geq 0}(X, \mathbb{O})$. So K is p -perverse in this case.

For a general \mathcal{L} in $\text{Loc}(X, \mathbb{O})$, the result follows from the above, using the distinguished triangle $(\mathcal{L}_{\text{tors}}, \mathcal{L}, \mathcal{L}_{\text{free}})$. For \mathcal{L} in $\text{Loc}^+(X, \mathbb{O})$, the result follows by duality.

- (ii) First assume that \mathbb{E} is \mathbb{K} or \mathbb{F} . If π is small, with the notation above, K_s and $\mathcal{D}_{X,\mathbb{E}}(K)_s$ are concentrated in degrees $[-d, -\dim S - 1]$ on all strata of positive codimension. If S is a stratum of dimension d , then the morphism $\pi_S : \tilde{S} \rightarrow S$ obtained by base change $S \rightarrow X$ is a finite covering, so $K|_S = ((\pi_S)_* \mathcal{L}|_{\tilde{S}}[d])$ is a local system shifted by d . Hence we have $K = j_{!*}j^*K$, where $j : U \rightarrow X$ is the inclusion of the union of all strata of dimension d in \mathfrak{X} .

If $\mathbb{E} = \mathbb{O}$, we can treat the perversities p and p_+ as in the first part of the Proposition.

□

Remark 2.1.3 We will apply this Proposition, in Section 6.2, to the surjective and proper morphisms $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ (which is small) and $\pi_{\mathcal{N}} : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ (which is semi-small), to show that $\mathbb{E}\mathcal{K}$ is an intersection cohomology complex, and that $\mathbb{E}\mathcal{K}_{\mathcal{N}}$ is perverse (see the notation there).

We note that, in the case of a small resolution, the intersection cohomology complex can be obtained by a direct image.

2.2 Equivalent singularities

Definition 2.2.1 Given X and Y two varieties, and two points $x \in X$ and $y \in Y$, we say that the singularity of X at x and the singularity of Y at y are smoothly equivalent, and we write $\text{Sing}(X, x) = \text{Sing}(Y, y)$, if there exist a variety Z , a point $z \in Z$, and two maps $\varphi : Z \rightarrow X$ and $\psi : Z \rightarrow Y$, smooth at z , with $\varphi(z) = x$ and $\psi(z) = y$.

If an algebraic group G acts on X , then $\text{Sing}(X, x)$ depends only on the orbit \mathcal{O} of x . In that case, we write $\text{Sing}(X, \mathcal{O}) := \text{Sing}(X, x)$.

In fact, there is an open subset U of Z containing z where φ and ψ are smooth, so after replacing Z by U , we can assume that φ and ψ are smooth on Z .

We have the following result (it follows from the remarks after Lemma 4.2.6.1. in [BBD82]).

Proposition 2.2.2 *Suppose that $\text{Sing}(X, x) = \text{Sing}(Y, y)$. Then $\mathbf{IC}(X, \mathbb{E})_x \simeq \mathbf{IC}(Y, \mathbb{E})_y$ as \mathbb{E} -modules.*

Remark 2.2.3 Suppose we have a stratification \mathfrak{X} (resp. \mathfrak{Y}) of X (resp. Y) adapted to $\mathbf{IC}(X, \mathbb{E})$ (resp. $\mathbf{IC}(Y, \mathbb{E})$), and let $\mathcal{O}(x)$ (resp. $\mathcal{O}(y)$) be the stratum of x (resp. y). Suppose we know $\mathbf{IC}(X, \mathbb{E})_x$ as a representation of $\mathbb{E}\pi_1(\mathcal{O}(x), x)$. The proposition gives us $\mathbf{IC}(Y, \mathbb{E})_y$ as an \mathbb{E} -module, but not as an $\mathbb{E}\pi_1(\mathcal{O}(y), y)$ -module. To determine the latter structure, one needs more information.

2.3 Cones

Let $X \subset \mathbb{P}^{N-1}$ be a smooth projective variety of dimension $d - 1$. We denote by $\pi : \mathbb{A}^N \setminus \{0\} \rightarrow \mathbb{P}^{N-1}$ the canonical projection. Let $U = \pi^{-1}(X) \subset \mathbb{A}^N \setminus \{0\}$ and $C = \overline{U} = U \cup \{0\} \subset \mathbb{A}^N$. They have dimension d .

We have a smooth open immersion $j : U \hookrightarrow C$ and a closed immersion $i : \{0\} \hookrightarrow C$. If $d > 1$, then j is not affine.

Proposition 2.3.1 *With the preceding notations, we have*

$$i^* j_* \mathbb{E} \simeq \mathbf{R}\Gamma(U, \mathbb{E})$$

Truncating appropriately, one deduces the fiber at 0 of the complexes ${}^p j_* \mathbb{E}[d]$, where $? \in \{!, !*, *\}$, and similarly for p_+ if $\mathbb{E} = \mathbb{O}$.

More generally, we have the following result, which is contained in [KL80a, Lemma 4.5 (a)]. As indicated there, in the complex case, this follows easily from topological considerations.

Proposition 2.3.2 *Let C be an irreducible closed subvariety of \mathbb{A}^N stable under the \mathbb{G}_m -action defined by $\lambda(z_1, \dots, z_N) = (\lambda^{a_1} z_1, \dots, \lambda^{a_N} z_N)$, where $a_1 > 0, \dots, a_N > 0$. Let $j : U = C \setminus \{0\} \rightarrow C$ be the open immersion, and $i : \{0\} \rightarrow C$ the closed immersion. Then we have*

$$i^* j_* \mathbb{E} \simeq \mathbf{R}\Gamma(U, \mathbb{E})$$

So, if U is smooth, the calculation of the intersection cohomology complex stalks for C is reduced to the calculation of the cohomology of U .

2.4 \mathbb{E} -smoothness

2.4.1 Definition and remarks

The following notion was introduced by Deligne for $\mathbb{E} = \overline{\mathbb{Q}}_\ell$ in [Del80].

Definition 2.4.1 *Let X be a k -variety, purely of dimension n , with structural morphism $a : X \rightarrow \mathrm{Spec} k$. Then X is \mathbb{E} -smooth if and only if the adjoint $\mathbb{E}(n)[2n] \rightarrow a^! \mathbb{E}$ of the trace morphism is an isomorphism.*

This condition ensures that X satisfies Poincaré duality with \mathbb{E} coefficients. It is equivalent to the following condition: for all closed points x in X , we have $\mathbb{E}(n)[2n] \xrightarrow{\sim} i_x^! \mathbb{E}$, that is, $H_{\langle x \rangle}^i(X, \mathbb{E})$ is zero if $i \neq n$, and isomorphic to $\mathbb{E}(-n)$ if $i = n$.

Then the shifted constant sheaf $\mathbb{E}_X[n]$ is self-dual (up to twist), and $\mathbb{E}_X[n]$ is an intersection cohomology complex. Indeed, it is a complex extending the shifted constant sheaf on an everywhere dense open subvariety, trivially satisfying the support condition of the intersection cohomology complex. Since it is self-dual, it must be the intersection cohomology complex. If moreover \mathbb{E} is a field, then $\mathbb{E}_X[n]$ is a simple perverse sheaf.

Note that, in general, the fact that $\mathbb{E}_X[n]$ is perverse does not imply that X is \mathbb{E} -smooth. For example, $\mathbb{E}_X[n]$ is perverse if X is a complete intersection [KW01]. But a complete intersection can have several irreducible components intersecting at the same point, or more generally a branched point, and then X cannot be \mathbb{E} -smooth. Moreover, we will shortly see, in Paragraph 2.4.3, that the cone over a smooth projective curve of genus g cannot be \mathbb{E} -smooth if $g > 0$, but on such a surface the shifted constant sheaf is always perverse.

2.4.2 A stability property

Proposition 2.4.2 *Let $\pi : X \rightarrow Y$ be a finite surjective and separable morphism of k -varieties, with X and Y irreducible of dimension n and normal. If X is \mathbb{K} -smooth then Y is also \mathbb{K} -smooth. If X is \mathbb{F} -smooth and ℓ does not divide the cardinality d of the generic fiber of π , then Y is also \mathbb{F} -smooth.*

Proof. Since π is separable, we can choose an open dense subset Y_0 of Y over which π is étale. Let $\pi_0 : X_0 \rightarrow Y_0$ be the morphism deduced by base change. Since π_0 is finite étale, we can find a Galois covering \tilde{X}_0 of X_0 , such that the composite $\tilde{X} \rightarrow Y_0$ is also a Galois covering. Let $H = \mathrm{Gal}(\tilde{X}_0/X_0)$ and $G = \mathrm{Gal}(\tilde{X}_0/Y_0)$. We have $|G : H| = d$. Hence, if $\mathbb{E} = \mathbb{K}$, or $\mathbb{E} = \mathbb{F}$ and $\ell \nmid d$, then the local system $\pi_{0*} \mathbb{E}_{X_0}$, corresponding to the representation $\mathrm{Ind}_{\mathbb{E}H}^{\mathbb{E}G} \mathbb{E}$ of G (which is a finite quotient of the fundamental group of Y_0), has the constant sheaf \mathbb{E} as a direct summand. Hence $\mathcal{J}_{!*}(Y_0, \mathbb{E})$ is a direct summand of $\mathcal{J}_{!*}(Y_0, \pi_{0*} \mathbb{E}_{X_0}) = \pi_* \mathbb{E}_X[n]$ since π is finite (in particular, it is small). But $\pi_* \mathbb{E}_X[n]$ is concentrated in degree $-n$, so $\mathcal{J}_{!*}(Y_0, \mathbb{E})$ is so as well, and it is isomorphic to ${}^0\mathcal{J}_*(Y_0, \mathbb{E})$ which is the shifted constant sheaf $\mathbb{E}_Y[n]$ since Y is normal. \square

In particular, the quotient of a smooth irreducible variety by a finite group H is \mathbb{K} -smooth, and also \mathbb{F} -smooth if ℓ does not divide H . This will be illustrated with the simple singularities in Section 2.5.

2.4.3 Cone over a smooth projective curve

With the notations of Section 2.3, suppose $X \subset \mathbb{P}^N$ is an irreducible smooth projective curve of genus g . Let us denote by H^j , for $j \in \mathbb{Z}$, the cohomology group $H^j(X, \mathbb{Z}_\ell)$ (it is 0 for $j < 0$ or $j > 2$). We have $H^0 \simeq \mathbb{Z}_\ell$, $H^1 \simeq \mathbb{Z}_\ell^{2g}$, and $H^2 \simeq \mathbb{Z}_\ell$ (non-canonically).

We would like to compute the fiber at 0 of the intersection cohomology complex $K = j_{!*}(\mathbb{Z}_\ell[2])$. Since we have a cone singularity, we just have to compute the cohomology of U .

Now, U is the line bundle corresponding to the invertible sheaf $\mathcal{O}(-1)$ on X , minus the null section. We denote by c the first Chern class of $\mathcal{O}(-1)$. We have the Gysin sequence

$$H^{i-2} \xrightarrow{c} H^i \longrightarrow H^i(U, \mathbb{Z}_\ell) \longrightarrow H^{i-1} \xrightarrow{c} H^{i+1}$$

and hence a short exact sequence

$$0 \longrightarrow \text{Coker}(c : H^{i-2} \rightarrow H^i) \longrightarrow H^i(U, \mathbb{Z}_\ell) \longrightarrow \text{Ker}(c : H^{i-1} \rightarrow H^{i+1}) \longrightarrow 0$$

We deduce that the cohomology of U is

$$\text{R}\Gamma(U, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell^{2g}[-1] \oplus (\mathbb{Z}_\ell^{2g} \oplus \mathbb{Z}/c)[-2] \oplus \mathbb{Z}_\ell[-3]$$

(a bounded complex of \mathbb{Z} -modules is quasi-isomorphic to the direct sum of its shifted cohomology sheaves, because \mathbb{Z} is a hereditary ring).

Hence we have

$$\begin{aligned} i^* {}^p j_! \mathbb{Z}_\ell[2] &= i^* {}^{p+} j_! \mathbb{Z}_\ell[2] \simeq \mathbb{Z}_\ell[2] \\ i^* {}^p j_{!*} \mathbb{Z}_\ell[2] &\simeq \mathbb{Z}_\ell[2] \oplus \mathbb{Z}_\ell^{2g}[1] \\ i^* {}^{p+} j_{!*} \mathbb{Z}_\ell[2] &\simeq \mathbb{Z}_\ell[2] \oplus \mathbb{Z}_\ell^{2g}[1] \oplus \mathbb{Z}/c \\ i^* {}^p j_* \mathbb{Z}_\ell[2] &= i^* {}^{p+} j_* \mathbb{Z}_\ell[2] \simeq \mathbb{Z}_\ell[2] \oplus \mathbb{Z}_\ell^{2g}[1] \oplus (\mathbb{Z}/c \oplus \mathbb{Z}_\ell^{2g}) \end{aligned}$$

Thus C cannot rationally smooth (resp. \mathbb{Z}/ℓ -smooth) if $g > 0$. If $g = 0$, then C is rationally smooth, and it is \mathbb{Z}/ℓ smooth if ℓ does not divide c . If one takes $N = 1$ and $X = \mathbb{P}^1$ embedded in \mathbb{P}^1 by the identity, then $C = \mathbb{A}^2$ is actually smooth.

But the constant perverse sheaf $\mathbb{E}_C[2]$ is perverse in any case, since it is equal to ${}^p j_! \mathbb{E}[2]$ (in the case $\mathbb{E} = \mathbb{Z}_\ell$, it is both p and p_+ -perverse).

2.5 Simple singularities

In this section, we will calculate the intersection cohomology complexes over \mathbb{K} , \mathbb{O} and \mathbb{F} for simple singularities, and the corresponding decomposition numbers. We will also consider the case of simple singularities of inhomogeneous type, that is, simple singularities with an associated group of symmetries. For the convenience of the reader, we will recall the main points in the theory of simple singularities, following [Slo80b], to which we refer for more details. We will use the results of this section in Chapter 4, to calculate the decomposition numbers involving the regular class and the subregular class in a

simple Lie algebra. Indeed, by the work of Brieskorn and Slodowy, the singularity of the nilpotent variety along the subregular class is a simple singularity of the corresponding type.

2.5.1 Rational double points

We assume that k is algebraically closed. Let (X, x) be the spectrum of a two-dimensional normal local k -algebra, where x denotes the closed point of X . Then (X, x) is *rational* if there is a resolution $\pi : \tilde{X} \rightarrow X$ of the singularities of X such that the higher direct images of the structural sheaf of \tilde{X} vanish, that is, $R^q\pi_*(\mathcal{O}_{\tilde{X}}) = 0$ for $q > 0$. In fact, this property is independent of the choice of a resolution. The rationality property is stronger than the Cohen-Macaulay property.

If $\pi : \tilde{X} \rightarrow X$ is a resolution, then the reduced exceptional divisor $E = \pi^{-1}(x)_{\text{red}}$ is a finite union of irreducible curves. Since X is a surface, there is a minimal resolution, unique up to isomorphism, through which all other resolutions must factor. For the minimal resolution of a simple singularity, these curves will have a very special configuration.

Let Γ be an irreducible homogeneous Dynkin diagram, with set of vertices Δ . We recall that a Dynkin diagram is *homogeneous*, or *simply-laced*, when the corresponding root system has only roots of the same length. Thus Γ is of type A_n ($n \geq 1$), D_n ($n \geq 4$), E_6 , E_7 or E_8 . The Cartan matrix $C = (n_{\alpha,\beta})_{\alpha,\beta \in \Delta}$ of Γ satisfies $n_{\alpha,\alpha} = 2$ for all α in Δ , and $n_{\alpha,\beta} \in \{0, -1\}$ for all $\alpha \neq \beta$ in Δ .

A resolution $\pi : \tilde{X} \rightarrow X$ of the surface X , as above, has an *exceptional configuration of type* Γ if all the irreducible components of the exceptional divisor E are projective lines, and if there is a bijection $\alpha \mapsto E_\alpha$ from Δ to the set $\text{Irr}(E)$ of these components such that the intersection numbers $E_\alpha \cdot E_\beta$ are given by the opposite of the Cartan matrix C , that is, $E_\alpha \cdot E_\beta = -n_{\alpha,\beta}$ for α and β in Δ . Thus we have a union of projective lines whose normal bundles in \tilde{X} are isomorphic to the cotangent bundle $T^*\mathbb{P}^1$, and two of them intersect transversely in at most one point.

The minimal resolution is characterized by the fact that it has no exceptional curves with self-intersection -1 . Therefore, if the resolution π of the surface X has an exceptional configuration of type Γ , then it is minimal.

Theorem 2.5.1 *The following properties of a normal surface (X, x) are equivalent.*

- (i) (X, x) is rational of embedding dimension 3 at x .
- (ii) (X, x) is rational of multiplicity 2 at x .
- (iii) (X, x) is of multiplicity 2 at x and it can be resolved by successive blowing up of points.
- (iv) The minimal resolution of (X, x) has the exceptional configuration of an irreducible homogeneous Dynkin diagram.

Definition 2.5.2 *If any (hence all) of the properties of the preceding theorem is satisfied, then (X, x) is called a rational double point or a simple singularity.*

Theorem 2.5.3 *Let the characteristic of k be good for the irreducible homogeneous Dynkin diagram Γ . Then there is exactly one rational double point of type Γ up to isomorphism of Henselizations. Representatives of the individual classes are given by the local varieties at $0 \in \mathbb{A}^3$ defined by the equations in the table below.*

In each case, this equation is the unique relation (syzygy) between three suitably chosen generators X, Y, Z of the algebra $k[\mathbb{A}^2]^H$ of the invariant polynomials of \mathbb{A}^2 under the action of a finite subgroup H of SL_2 , given in the same table.

H		$ H $	equation of $\mathbb{A}^2/H \subset \mathbb{A}^3$	Γ
\mathfrak{C}_{n+1}	cyclic	$n + 1$	$X^{n+1} + YZ = 0$	A_n
$\mathfrak{D}_{4(n-2)}$	dihedral	$4(n - 2)$	$X^{n-1} + XY^2 + Z^2 = 0$	D_n
\mathfrak{T}	binary tetrahedral	24	$X^4 + Y^3 + Z^2 = 0$	E_6
\mathfrak{O}	binary octahedral	48	$X^3Y + Y^3 + Z^2 = 0$	E_7
\mathfrak{I}	binary icosahedral	120	$X^5 + Y^3 + Z^2 = 0$	E_8

Moreover, if k is of characteristic 0, these groups are, up to conjugation, the only finite subgroups of SL_2 .

Thus, in good characteristic, every rational double point is, after Henselization at the singular point, isomorphic to the corresponding quotient \mathbb{A}^2/H . When p divides $n + 1$ (resp. $4(n - 2)$), the group \mathfrak{C}_{n+1} (resp. $\mathfrak{D}_{4(n-2)}$) is not reduced. We have the following exact sequences

$$1 \longrightarrow \mathfrak{D}_8 \longrightarrow \mathfrak{T} \longrightarrow \mathfrak{C}_3 \longrightarrow 1 \tag{2.1}$$

$$1 \longrightarrow \mathfrak{T} \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{C}_2 \longrightarrow 1 \tag{2.2}$$

$$1 \longrightarrow \mathfrak{D}_8 \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{S}_3 \longrightarrow 1 \tag{2.3}$$

when the characteristic of k is good for the Dynkin diagram attached to each of the groups involved.

2.5.2 Symmetries on rational double points

To each inhomogeneous irreducible Dynkin diagram Γ we associate a homogeneous diagram $\widehat{\Gamma}$ and a group $A(\Gamma)$ of automorphisms of $\widehat{\Gamma}$, as follows.

Γ	B_n	C_n	F_4	G_2
$\widehat{\Gamma}$	A_{2n-1}	D_{n+1}	E_6	D_4
$A(\Gamma)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	\mathfrak{S}_3

In general, there is a unique (in case $\Gamma = C_3$ or G_2 : up to conjugation by $\text{Aut}(\widehat{\Gamma}) = \mathfrak{S}_3$) faithful action of $A(\Gamma)$ on $\widehat{\Gamma}$. One can see Γ as the quotient of $\widehat{\Gamma}$ by $A(\Gamma)$.

In all cases but $\Gamma = C_3$, the group $A(\Gamma)$ is the full group of automorphisms of $\widehat{\Gamma}$. Note that D_4 is associated to C_3 and G_2 . For a homogeneous diagram, it will be convenient to set $\widehat{\Gamma} = \Gamma$ and $A(\Gamma) = 1$.

A rational double point may be represented as the quotient \mathbb{A}^2/H of \mathbb{A}^2 by a finite subgroup H of SL_2 provided the characteristic of k is good for the corresponding Dynkin diagram. If \widehat{H} is another finite subgroup of SL_2 containing H as a normal subgroup, then the quotient \widehat{H}/H acts naturally on \mathbb{A}^2/H .

Definition 2.5.4 *Let Γ be an inhomogeneous irreducible Dynkin diagram and let the characteristic of k be good for Γ . A couple (X, A) consisting of a normal surface singularity X and a group A of automorphisms of X is called a simple singularity of type Γ if it is isomorphic (after Henselization) to a couple $(\mathbb{A}^2/H, \widehat{H}/H)$ according to the following table.*

Γ	B_n	C_n	F_4	G_2
H	\mathfrak{C}_{2n}	$\mathfrak{D}_{4(n-1)}$	\mathfrak{I}	\mathfrak{D}_8
\widehat{H}	\mathfrak{D}_{4n}	$\mathfrak{D}_{8(n-1)}$	\mathfrak{D}	\mathfrak{D}

Then X is a rational double point of type $\widehat{\Gamma}$ and A is isomorphic to $A(\Gamma)$. The action of A on X lifts in a unique way to an action of A on the resolution \widetilde{X} of X . As A fixes the singular point of X , the exceptional divisor in \widetilde{X} will be stable under A . In this way, we recover the action of A on $\widehat{\Gamma}$. The simple singularities of inhomogeneous type can be characterized in the following way.

Proposition 2.5.5 *Let Γ be a Dynkin diagram of type B_n, C_n, F_4 or G_2 , and let the characteristic of k be good for Γ . Let X be a rational double point of type $\widehat{\Gamma}$ endowed with an action of $A(\Gamma)$, free on the complement of the singular point, and such that the induced action on the dual diagram of the minimal resolution of X coincides with the associated action of $A(\Gamma)$ on $\widehat{\Gamma}$. Then (X, A) is a simple singularity of type Γ .*

2.5.3 Perverse extensions and decomposition numbers

Let Γ be any irreducible Dynkin diagram, and suppose the characteristic of k is good for Γ . Let $\widehat{\Gamma}$ be the associated homogeneous Dynkin diagram, $A(\Gamma)$ the associated symmetry group, and $H \subset \widehat{H}$ the corresponding finite subgroups of SL_2 . We recall that, if Γ is already homogeneous, then we take $\widehat{\Gamma} = \Gamma$, $A(\Gamma) = 1$ and $\widehat{H} = H$. We stratify the simple singularity $X = \mathbb{A}^2/H$ into two strata: the origin $\{0\}$ (the singular point), and its complement U , which is smooth since H acts freely on $\mathbb{A}^2 \setminus \{0\}$. We want to determine the stalks of the three perverse extensions of the (shifted) constant sheaf \mathbb{E} on U , for \mathbb{E} in $(\mathbb{K}, \mathbb{O}, \mathbb{F})$, and for the two perversities p and p_+ in the case $\mathbb{E} = \mathbb{O}$. By the results of Chapter 1, this will allow us to determine a decomposition number.

By the quasi-homogeneous structure of the equation defining X in \mathbb{A}^3 , we have a \mathbb{G}_m -action on X contracting X to the origin. We are in the situation of Proposition 2.3.2, with $C = X$. Thus it is enough to calculate the cohomology of U with \mathbb{O} coefficients. The cases $\mathbb{E} = \mathbb{K}$ or \mathbb{F} will follow.

Let $\widehat{\Phi}$ be the root system corresponding to $\widehat{\Gamma}$, in a real vector space \widehat{V} of dimension equal to the rank n of $\widehat{\Gamma}$. We identify the set $\widehat{\Delta}$ of vertices of $\widehat{\Gamma}$ with a basis of $\widehat{\Phi}$. We denote by $P(\widehat{\Phi})$ and $Q(\widehat{\Phi})$ the weight lattice and the root lattice of \widehat{V} . The finite

abelian group $P(\widehat{\Phi})/Q(\widehat{\Phi})$ is the fundamental group of the corresponding adjoint group, and also the center of the corresponding simply-connected group. Its order is called the connection index of $\widehat{\Phi}$. The coweight lattice $P^\vee(\widehat{\Phi})$ (the weight lattice of the dual root system $\widehat{\Phi}^\vee$ in \widehat{V}^*) is in duality with $Q(\widehat{\Phi})$, and the coroot lattice $Q^\vee(\widehat{\Phi})$ is in duality with $P(\widehat{\Phi})$. Thus the finite abelian group $P^\vee(\widehat{\Phi})/Q^\vee(\widehat{\Phi})$ is dual to $P(\widehat{\Phi})/Q(\widehat{\Phi})$.

Let $\pi : \widetilde{X} \rightarrow X$ be the minimal resolution of X . The exceptional divisor E is the union of projective lines E_α , $\alpha \in \widehat{\Delta}$. Then we have an isomorphism $H^2(\widetilde{X}, \mathbb{O}) \xrightarrow{\sim} \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})$ such that, for each α in $\widehat{\Delta}$, the cohomology class of the subvariety E_α is identified with $1 \otimes \alpha$, and such that the intersection pairing is the opposite of the pullback of the W -invariant pairing on P normalized by the condition $(\alpha, \alpha) = 2$ for α in $\widehat{\Delta}$ [IN99]. Thus the natural map $H_c^2(\widetilde{X}, \mathbb{O}) \rightarrow H_c^2(E, \mathbb{O})$ is identified with the opposite of the map $\mathbb{O} \otimes_{\mathbb{Z}} Q^\vee(\widehat{\Phi}) \rightarrow \mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})$ induced by the inclusion.

By Poincaré duality (U is smooth), it is enough to compute the cohomology with proper support of U , and to do this we will use the long exact sequence in cohomology with proper support for the open subvariety U with closed complement E in \widetilde{X} . The following table gives the $H_c^i(-, \mathbb{O})$ of the three varieties (the first column is deduced from the other two).

i	U	\widetilde{X}	E
0	0	0	\mathbb{O}
1	\mathbb{O}	0	0
2	0	$\mathbb{O} \otimes_{\mathbb{Z}} Q^\vee(\widehat{\Phi})$	$\mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})$
3	$\mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})/Q^\vee(\widehat{\Phi})$	0	0
4	\mathbb{O}	\mathbb{O}	0

By (derived) Poincaré duality, we obtain the cohomology of U .

Proposition 2.5.6 *The cohomology of U is given by*

$$R\Gamma(U, \mathbb{O}) \simeq \mathbb{O} \oplus \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi})[-2] \oplus \mathbb{O}[-3] \quad (2.4)$$

The closed stratum is a point, and for complexes on the point the perverse t -structures for p and p_+ are the usual ones (there is no shift since the point is 0-dimensional). With the notations of Section 2.3, we have

$$H^{-1}i^*j_*(\mathbb{O}[2]) \simeq H^1(U, \mathbb{O}) = 0 \quad (2.5)$$

$$H^0i^*j_*(\mathbb{O}[2]) \simeq H^2(U, \mathbb{O}) \simeq \mathbb{O} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi}) \quad (2.6)$$

$$H^1i^*j_*(\mathbb{O}[2]) \simeq H^3(U, \mathbb{O}) \simeq \mathbb{O} \quad (2.7)$$

By our analysis in the sections 1.5 and 1.8, we obtain the following results.

Proposition 2.5.7 *We keep the preceding notation. In particular, X is a simple singularity of type Γ .*

Over \mathbb{K} , we have canonical isomorphisms

$${}^p j_!(\mathbb{K}[2]) \simeq {}^p j_{!*}(\mathbb{K}[2]) \simeq {}^p j_*(\mathbb{K}[2]) \simeq \mathbb{K}_X[2] \quad (2.8)$$

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In particular, X is \mathbb{K} -smooth.

Over \mathbb{O} , we have canonical isomorphisms

$${}^p j_! (\mathbb{O}[2]) \simeq {}^{p+} j_! (\mathbb{O}[2]) \simeq {}^p j_{!*} (\mathbb{O}[2]) \simeq \mathbb{O}_X[2] \quad (2.9)$$

$${}^{p+} j_{!*} (\mathbb{O}[2]) \simeq {}^p j_* (\mathbb{O}[2]) \simeq {}^{p+} j_* (\mathbb{O}[2]) \quad (2.10)$$

and a short exact sequence in ${}^p \mathcal{M}(X, \mathbb{O})$

$$0 \longrightarrow {}^p j_{!*} (\mathbb{O}[2]) \longrightarrow {}^{p+} j_{!*} (\mathbb{O}[2]) \longrightarrow i_* \mathbb{O} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) \longrightarrow 0 \quad (2.11)$$

Over \mathbb{F} , we have canonical isomorphisms

$$\mathbb{F} {}^p j_! (\mathbb{O}[2]) \xrightarrow{\sim} {}^p j_! (\mathbb{F}[2]) \xrightarrow{\sim} \mathbb{F} {}^{p+} j_! (\mathbb{O}[2]) \xrightarrow{\sim} \mathbb{F} {}^p j_{!*} (\mathbb{O}[2]) \xrightarrow{\sim} \mathbb{F}_X[2] \quad (2.12)$$

$$\mathbb{F} {}^{p+} j_{!*} (\mathbb{O}[2]) \xrightarrow{\sim} \mathbb{F} {}^p j_* (\mathbb{O}[2]) \xrightarrow{\sim} {}^p j_* (\mathbb{F}[2]) \xrightarrow{\sim} \mathbb{F} {}^{p+} j_* (\mathbb{O}[2]) \quad (2.13)$$

and short exact sequences

$$0 \longrightarrow i_* \mathbb{F} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) \longrightarrow \mathbb{F} {}^p j_{!*} (\mathbb{O}[2]) \longrightarrow {}^p j_{!*} (\mathbb{F}[2]) \longrightarrow 0 \quad (2.14)$$

$$0 \longrightarrow {}^p j_{!*} (\mathbb{F}[2]) \longrightarrow \mathbb{F} {}^{p+} j_{!*} (\mathbb{O}[2]) \longrightarrow i_* \mathbb{F} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) \longrightarrow 0 \quad (2.15)$$

We have

$$[\mathbb{F} {}^p j_{!*} (\mathbb{O}[2]) : i_* \mathbb{F}] = [\mathbb{F} {}^{p+} j_{!*} (\mathbb{O}[2]) : i_* \mathbb{F}] = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} (P(\widehat{\Phi})/Q(\widehat{\Phi})) \quad (2.16)$$

In particular, $\mathbb{F} {}^p j_{!*} (\mathbb{O}[2])$ is simple (and equal to $\mathbb{F} {}^{p+} j_{!*} (\mathbb{O}[2])$) if and only if ℓ does not divide the connection index $|P(\widehat{\Phi})/Q(\widehat{\Phi})|$ of $\widehat{\Phi}$. The variety X is \mathbb{F} -smooth under the same condition.

Let us give this decomposition number in each type.

$\widehat{\Gamma}$	$P(\widehat{\Phi})/Q(\widehat{\Phi})$	$[\mathbb{F} {}^p j_{!*} (\mathbb{O}[2]) : i_* \mathbb{F}]$
A_n	$\mathbb{Z}/(n+1)$	1 if $\ell \mid n+1$, 0 otherwise
D_n (n even)	$(\mathbb{Z}/2)^2$	2 if $\ell = 2$, 0 otherwise
D_n (n odd)	$\mathbb{Z}/4$	1 if $\ell = 2$, 0 otherwise
E_6	$\mathbb{Z}/3$	1 if $\ell = 3$, 0 otherwise
E_7	$\mathbb{Z}/2$	1 if $\ell = 2$, 0 otherwise
E_8	0	0

Let us note that for $\Gamma = E_8$, the variety X is \mathbb{F} -smooth for any ℓ . However, it is not smooth, since it has a double point.

What about the action of $A(\Gamma)$? Let us first recall some facts from [Bou68]. Let $\text{Aut}(\widehat{\Phi})$ denote the group of automorphisms of \widehat{V} stabilizing $\widehat{\Phi}$. The subgroup of $\text{Aut}(\widehat{\Phi})$ of the elements stabilizing $\widehat{\Delta}$ is identified with $\text{Aut}(\widehat{\Gamma})$. The Weyl group $W(\widehat{\Phi})$ is a normal subgroup of $\text{Aut}(\widehat{\Phi})$, and $\text{Aut}(\widehat{\Phi})$ is the semi-direct product of $\text{Aut}(\widehat{\Gamma})$ and $W(\widehat{\Phi})$ [Bou68, Chap. VI, §1.5, Prop. 16].

The group $\text{Aut}(\widehat{\Phi})$ stabilizes $P(\widehat{\Phi})$ and $Q(\widehat{\Phi})$, thus it acts on the quotient $P(\widehat{\Phi})/Q(\widehat{\Phi})$. By [Bou68, Chap. VI, §1.10, Prop. 27], the group $W(\widehat{\Phi})$ acts trivially on $P(\widehat{\Phi})/Q(\widehat{\Phi})$. Thus, the quotient group $\text{Aut}(\widehat{\Phi})/W(\widehat{\Phi}) \simeq \text{Aut}(\widehat{\Gamma})$ acts canonically on $P(\widehat{\Phi})/Q(\widehat{\Phi})$.

Now $A(\Gamma)$ acts on X, \widetilde{X}, E and U , and hence on their cohomology (with or without supports). Moreover, the action of $A(\Gamma)$ on $H_c^2(E, \mathbb{O}) \simeq \mathbb{O} \otimes_{\mathbb{Z}} P^\vee(\widehat{\Phi})$ is the one induced by the inclusions $A(\Gamma) \subset \text{Aut}(\widetilde{\Gamma}) \subset \text{Aut}(\widehat{\Phi})$. The inclusions of E and U in \widetilde{X} are $A(\Gamma)$ -equivariant, hence the maps in the long exact sequence in cohomology with compact support that we considered earlier (to calculate $H_c^3(U, \mathbb{O})$) are $A(\Gamma)$ -equivariant. Thus the action of $A(\Gamma)$ on $H_c^3(U, \mathbb{O}) \simeq P^\vee(\widehat{\Phi})/Q^\vee(\widehat{\Phi})$ is induced by the inclusion $A(\Gamma) \subset \text{Aut}(\Gamma) \simeq \text{Aut}(\widehat{\Phi})/W(\widehat{\Phi})$ from the canonical action. It follows that the action of $A(\Gamma)$ on $H^2(U, \Gamma) \simeq P(\widehat{\Phi})/Q(\widehat{\Phi})$ also comes from the canonical action of $\text{Aut}(\widehat{\Phi})/W(\widehat{\Phi})$.

Why do we care about this action, since it does not appear in the proposition? Well, here the closed stratum is a point, so \mathbb{E} -local systems are necessarily trivial. We can regard them as mere \mathbb{E} -modules. However, when we use the results of this section for the subregular class, we will need to determine the local system involved on the regular class. If we just have an equivalence of singularities, we can deal with the \mathbb{E} -module structure, but for the action of the fundamental group we need more information. This is why we also studied the action of $A(\Gamma)$ for inhomogeneous Dynkin diagrams.

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Chapter 3

Cohomology of the minimal nilpotent orbit

This chapter is taken from the preprint [Jut07]. We compute the integral cohomology of the minimal non-trivial nilpotent orbit in a complex simple (or quasi-simple) Lie algebra. We express it in terms of the root system. We find by a uniform approach that the middle cohomology group is isomorphic to the fundamental group of the root subsystem generated by the long simple roots. We compute the rest of the cohomology in a case-by-case analysis, which shows in particular that the primes dividing the torsion of the rest of the cohomology are bad primes.

All the results and proofs of this chapter remain valid for G a quasi-simple reductive group over $\overline{\mathbb{F}}_p$, with p good for G , using the étale cohomology with \mathbb{O} coefficients.

We will use the middle cohomology to determine the decomposition number corresponding to the minimal and trivial orbits in Chapter 4. If Φ is the root system of G , and W its Weyl group, let Φ' be the root subsystem generated by the long simple roots (for some given basis of Φ), and let W' be its Weyl group (a parabolic subgroup, but also a quotient, of W). With the knowledge of the middle cohomology, we will be able to show that this decomposition number $d_{(x_{\min},1),(0,1)}$ is equal to $\dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} P^{\vee}(\Phi')/Q^{\vee}(\Phi')$.

In the Fourier transform approach to the (ordinary) Springer correspondence, the trivial orbit corresponds to the trivial representation of W . When G is of simply-laced type, the minimal orbit corresponds to the natural (reflection) representation of W . In general, the minimal orbit corresponds to the natural representation of W' , lifted to W . In the modular Springer correspondence, which we will define in Chapter 6, the trivial class still corresponds to the trivial representation (in particular, the pair $(0, 1)$ is always in the image of this correspondence). In Section 6.3, we will see that $d_{(x_{\min},1),(0,1)}$ can be interpreted as the corresponding decomposition number for the Weyl group. Note that the trivial representation is involved in the reduction of the natural representation of W' if and only if ℓ divides the determinant of the Cartan matrix of W' , which is precisely the connection index $P^{\vee}(\Phi')/Q^{\vee}(\Phi')$ of Φ' . I wrote this section before proving the equality of decomposition numbers in general, and the wish to find the right decomposition number allowed me to predict that the Cartan matrix of W' (to be precise, without minus signs), should appear in the Gysin sequence for the middle cohomology group ! A GAP session with my supervisor Cédric Bonnafé confirmed this guess. Then it was not difficult to prove it.

Even if we have now a general theorem relating the decomposition numbers for the Weyl group and for the nilpotent variety, I think this chapter is still useful, at least to show that concrete calculations are possible on the geometric side. Besides, we will find

torsion not only in the middle (the part which controls the decomposition number), but also in other places for bad primes. Could this torsion have a representation theoretic interpretation ?

Introduction

Let G be a quasi-simple complex Lie group, with Lie algebra \mathfrak{g} . We denote by \mathcal{N} the nilpotent variety of \mathfrak{g} . The group G acts on \mathcal{N} by the adjoint action, with finitely many orbits. If \mathcal{O} and \mathcal{O}' are two orbits, we write $\mathcal{O} \leq \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$. This defines a partial order on the adjoint orbits. It is well known that there is a unique minimal non-zero orbit \mathcal{O}_{\min} (see for example [CM93], and the introduction of [KP82]). The aim of this article is to compute the integral cohomology of \mathcal{O}_{\min} .

The nilpotent variety \mathcal{N} is a cone in \mathfrak{g} : it is closed under multiplication by a scalar. Let us consider its image $\mathbb{P}(\mathcal{N})$ in $\mathbb{P}(\mathfrak{g})$. It is a closed subvariety of this projective space, so it is a projective variety. Now G acts on $\mathbb{P}(\mathcal{N})$, and the orbits are the $\mathbb{P}(\mathcal{O})$, where \mathcal{O} is a non-trivial adjoint orbit in \mathcal{N} . The orbits of G in $\mathbb{P}(\mathcal{N})$ are ordered in the same way as the non-trivial orbits in \mathcal{N} . Thus $\mathbb{P}(\mathcal{O}_{\min})$ is the minimal orbit in $\mathbb{P}(\mathcal{N})$, and therefore it is closed: we deduce that it is a projective variety. Let $x_{\min} \in \mathcal{O}_{\min}$, and let $P = N_G(\mathbb{C}x_{\min})$ (the letter N stands for normalizer, or setwise stabilizer). Then G/P can be identified to $\mathbb{P}(\mathcal{O}_{\min})$, which is a projective variety. Thus P is a parabolic subgroup of G . Now we have a resolution of singularities (see Section 3.2)

$$G \times_P \mathbb{C}x_{\min} \longrightarrow \overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \{0\}$$

which restricts to an isomorphism

$$G \times_P \mathbb{C}^*x_{\min} \xrightarrow{\sim} \mathcal{O}_{\min}.$$

From this isomorphism, one can already deduce that the dimension of \mathcal{O}_{\min} is equal to one plus the dimension of G/P . If we fix a maximal torus T in G and a Borel subgroup B containing it, we can take for x_{\min} a highest weight vector for the adjoint action on \mathfrak{g} . Then P is the standard parabolic subgroup corresponding to the simple roots orthogonal to the highest root, and the dimension of G/P is the number of positive roots not orthogonal to the highest root, which is $2h - 3$ in the simply-laced types, where h is the Coxeter number (see [Bou68, chap. VI, §1.11, prop. 32]). So the dimension of \mathcal{O}_{\min} is $2h - 2$ in that case. In [Wan99], Wang shows that this formula is still valid if we replace h by the dual Coxeter number h^\vee (which is equal to h only in the simply-laced types).

We found a similar generalization of a result of Carter (see [Car70]), relating the height of a long root to the length of an element of minimal length taking the highest root to that given long root, in the simply-laced case: the result extends to all types, if we take the height of the corresponding coroot instead (see Section 3.1, and Theorem 3.1.14).

To compute the cohomology of \mathcal{O}_{\min} , we will use the Gysin sequence associated to the \mathbb{C}^* -fibration $G \times_P \mathbb{C}^*x_{\min} \longrightarrow G/P$. The Pieri formula of Schubert calculus gives an

3.1 Long roots and distinguished coset representatives

answer in terms of the Bruhat order (see Section 3.2). Thanks to the results of Section 3.1, we translate this in terms of the combinatorics of the root system (see Theorem 3.2.1). As a consequence, we obtain the following results (see Theorem 3.2.2):

Theorem (i) *The middle cohomology of \mathcal{O}_{\min} is given by*

$$H^{2h^\vee-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq P^\vee(\Phi')/Q^\vee(\Phi')$$

where Φ' is the root subsystem of Φ generated by the long simple roots, and $P^\vee(\Phi')$ (resp. $Q^\vee(\Phi')$) is its coweight lattice (resp. its coroot lattice).

(ii) *If ℓ is a good prime for G , then there is no ℓ -torsion in the rest of the cohomology of \mathcal{O}_{\min} .*

Part (i) is obtained by a general argument, while (ii) is obtained by a case-by-case analysis (see Section 3.3, where we give tables for each type).

In Section 3.4, we explain a second method for the type A_{n-1} , based on another resolution of singularities: this time, it is a cotangent bundle on a projective space (which is also a generalized flag variety). This cannot be applied to other types, because the minimal class is a Richardson class only in type A .

Note that we are really interested in the torsion. The rational cohomology must already be known to the experts (see Remark 3.2.4).

3.1 Long roots and distinguished coset representatives

The Weyl group W of an irreducible and reduced root system Φ acts transitively on the set Φ_{lg} of long roots in Φ , hence if α is an element of Φ_{lg} , then the long roots are in bijection with W/W_α , where W_α is the stabilizer of α in W (a parabolic subgroup). Now, if we fix a basis Δ of Φ , and if we choose for α the highest root $\tilde{\alpha}$, we find a relation between the partial orders on W and Φ_{lg} defined by Δ , and between the length of a distinguished coset representative and the (dual) height of the corresponding long root. After this section was written, I realized that the result was already proved by Carter in the simply-laced types in [Car70] (actually, this result is quoted in [Spr76]). We extend it to any type and study more precisely the order relations involved. I also came across [BB05, §4.6], where the depth of a positive root β is defined as the minimal integer k such that there is an element w in W of length k such that $w(\beta) < 0$. By the results of this section, the depth of a positive long root is nothing but the height of the corresponding coroot (and the depth of a positive short root is equal to its height).

For the classical results about root systems that are used throughout this section, the reader may refer to [Bou68, Chapter VI, §1]. It is now available in English [Bou02].

3.1.1 Root systems

Let V be a finite dimensional \mathbb{R} -vector space and Φ a root system in V . We note $V^* = \text{Hom}(V, \mathbb{R})$ and, if $\alpha \in \Phi$, we denote by α^\vee the corresponding coroot and by s_α the reflexion s_{α, α^\vee} of [Bou68, chap. VI, §1.1, déf. 1, (SR_{II})]. Let W be the Weyl group of Φ .

The perfect pairing between V and V^* will be denoted by \langle, \rangle . Let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. In all this section, we will assume that Φ is *irreducible* and *reduced*. Let us fix a scalar product (\mid) on V , invariant under W , such that

$$\min_{\alpha \in \Phi} (\alpha \mid \alpha) = 1.$$

We then define the integer

$$r = \max_{\alpha \in \Phi} (\alpha \mid \alpha).$$

Let us recall that, since Φ is irreducible and reduced, we have $r \in \{1, 2, 3\}$ and $(\alpha \mid \alpha) \in \{1, r\}$ if $\alpha \in \Phi$ (see [Bou68, chap. VI, §1.4, prop. 12]). We define

$$\Phi_{\text{lg}} = \{\alpha \in \Phi \mid (\alpha \mid \alpha) = r\}$$

and

$$\Phi_{\text{sh}} = \{\alpha \in \Phi \mid (\alpha \mid \alpha) < r\} = \Phi \setminus \Phi_{\text{lg}}.$$

If α and β are two roots, then

$$\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha \mid \beta)}{(\beta \mid \beta)}. \quad (3.1)$$

In particular, if α and β belong to Φ , then

$$2(\alpha \mid \beta) \in \mathbb{Z} \quad (3.2)$$

and, if α or β belongs to Φ_{lg} , then

$$2(\alpha \mid \beta) \in r\mathbb{Z} \quad (3.3)$$

The following classical result says that Φ_{lg} is a closed subset of Φ .

Lemma 3.1.1 *If $\alpha, \beta \in \Phi_{\text{lg}}$ are such that $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi_{\text{lg}}$.*

Proof. We have $(\alpha + \beta \mid \alpha + \beta) = (\alpha \mid \alpha) + (\beta \mid \beta) + 2(\alpha \mid \beta)$. Thus, by 3.3, we have $(\alpha + \beta \mid \alpha + \beta) \in r\mathbb{Z}$, which implies the desired result. \square

3.1.2 Basis, positive roots, height

Let us fix a basis Δ of Φ and let Φ^+ be the set of roots $\alpha \in \Phi$ whose coefficients in the basis Δ are non-negative. Let $\Delta_{\text{lg}} = \Phi_{\text{lg}} \cap \Delta$ and $\Delta_{\text{sh}} = \Phi_{\text{sh}} \cap \Delta$. Note that Δ_{lg} need not be a basis of Φ_{lg} . Indeed, Φ_{lg} is a root system of rank equal to the rank of Φ , whereas Δ_{lg} has fewer elements than Δ if Φ is of non-simply-laced type. Let us recall the following well-known result [Bou68, chap. VI, §1, exercice 20 (a)]:

Lemma 3.1.2 *Let $\gamma \in \Phi$ and write $\gamma = \sum_{\alpha \in \Delta} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}$. Then $\gamma \in \Phi_{\text{lg}}$ if and only if r divides all the n_α , $\alpha \in \Delta_{\text{sh}}$.*

3.1 Long roots and distinguished coset representatives

Proof. Let Φ' be the set of roots $\gamma' \in \Phi$ such that, if $\gamma' = \sum_{\alpha \in \Delta} n'_\alpha \alpha$, then r divides n'_α for all $\alpha \in \Delta_{\text{sh}}$. We want to show that $\Phi_{\text{lg}} = \Phi'$.

Suppose that r divides all the n_α , $\alpha \in \Delta_{\text{sh}}$. Then $n_\alpha^2(\alpha|\alpha) \in r\mathbb{Z}$ for all $\alpha \in \Delta$, and by 3.2 and 3.3, we have $2n_\alpha n_\beta(\alpha|\beta) \in r\mathbb{Z}$ for all $(\alpha, \beta) \in \Delta \times \Delta$ such that $\alpha \neq \beta$. Thus $(\gamma|\gamma) \in r\mathbb{Z}$, which implies that $\gamma \in \Phi_{\text{lg}}$. Thus $\Phi' \subset \Phi_{\text{lg}}$.

Since W acts transitively on Φ_{lg} , it suffices to show that W stabilizes Φ' . In other words, it is enough to show that, if $\alpha \in \Delta$ and $\gamma \in \Phi'$, then $s_\alpha(\gamma) \in \Phi'$. But $s_\alpha(\gamma) = \gamma - \langle \gamma, \alpha^\vee \rangle \alpha$. If $\alpha \in \Delta_{\text{lg}}$, then $s_\alpha(\gamma) \in \Phi'$ because $\gamma \in \Phi'$. If $\alpha \in \Delta_{\text{sh}}$, then $\langle \gamma, \alpha^\vee \rangle = 2(\gamma|\alpha) \in r\mathbb{Z}$ because $\gamma \in \Phi' \subset \Phi_{\text{lg}}$ (see 3.1 and 3.3). Thus $s_\alpha(\gamma) \in \Phi'$. \square

If $\gamma = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \Phi$, the *height* of γ (denoted by $\text{ht}(\gamma)$) is defined by $\text{ht}(\gamma) = \sum_{\alpha \in \Delta} n_\alpha$. One defines the height of a coroot similarly.

If γ is long, we have

$$\gamma^\vee = \sum_{\alpha \in \Delta_{\text{lg}}} n_\alpha \alpha^\vee + \frac{1}{r} \sum_{\alpha \in \Delta_{\text{sh}}} n_\alpha \alpha^\vee.$$

Let

$$\text{ht}^\vee(\gamma) := \text{ht}(\gamma^\vee) = \sum_{\alpha \in \Delta_{\text{lg}}} n_\alpha + \frac{1}{r} \sum_{\alpha \in \Delta_{\text{sh}}} n_\alpha.$$

In particular, the right-hand side of the last equation is an integer, which is also a consequence of Lemma 3.1.2.

If α and β are long roots such that $\alpha + \beta$ is a (long) root, then $(\alpha + \beta)^\vee = \alpha^\vee + \beta^\vee$, so ht^\vee is additive on long roots.

3.1.3 Length

Let $l : W \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$ be the *length* function associated to Δ : if we let

$$N(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in -\Phi^+\},$$

then we have

$$l(w) = |N(w)|. \quad (3.4)$$

If $\alpha \in \Phi^+$ and if $w \in W$, then we have

$$l(ws_\alpha) > l(w) \text{ if and only if } w(\alpha) \in \Phi^+. \quad (3.5)$$

Replacing w by w^{-1} , and using the fact that an element of W has the same length as its inverse, we get

$$l(s_\alpha w) > l(w) \text{ if and only if } w^{-1}(\alpha) \in \Phi^+. \quad (3.6)$$

More generally, it is easy to show that, if x and y belong to W , then

$$N(xy) = N(y) \dot{+} y^{-1}N(x) \quad (3.7)$$

where $\dot{+}$ denotes the symmetric difference (there are four cases to consider), and therefore

$$l(xy) = l(x) + l(y) \quad \text{if and only if } N(y) \subset N(xy). \quad (3.8)$$

Let w_0 be the longest element of W . Recall that

$$l(w_0w) = l(ww_0) = l(w_0) - l(w) \quad (3.9)$$

for all $w \in W$. If I is a subset of Δ , we denote by Φ_I the set of the roots α which belong to the sub-vector space of V generated by I and we let

$$\Phi_I^+ = \Phi_I \cap \Phi^+ \quad \text{and} \quad W_I = \langle s_\alpha \mid \alpha \in I \rangle.$$

We also define

$$X_I = \{w \in W \mid w(\Phi_I^+) \subset \Phi^+\}.$$

Let us recall that X_I is a set of coset representatives of W/W_I and that $w \in X_I$ if and only if w is of minimal length in wW_I . Moreover, we have

$$l(xw) = l(x) + l(w) \quad (3.10)$$

if $x \in X_I$ and $w \in W_I$. We denote by w_I the longest element of W_I . Then w_0w_I is the longest element of X_I (this can be easily deduced from 3.9 and 3.10). Finally, if i is an integer, we denote by W^i the set of elements of W of length i , and similarly X_I^i is the set of elements of X_I of length i . To conclude this section, we shall prove the following result, which should be well known:

Lemma 3.1.3 *If $\beta \in \Phi_{\text{lg}}^+$, then $l(s_\beta) = 2 \text{ht}^\vee(\beta) - 1$.*

Proof. We shall prove the result by induction on $\text{ht}^\vee(\beta)$. The case where $\text{ht}^\vee(\beta) = 1$ is clear. Suppose $\text{ht}^\vee(\beta) > 1$ and suppose the result holds for all positive long roots whose dual height is strictly smaller.

First, there exists a $\gamma \in \Delta$ such that $\beta - \gamma \in \Phi^+$ (see [Bou68, chap. VI, §1.6, prop. 19]). Let $\alpha = \beta - \gamma$. There are two possibilities:

- If $\gamma \in \Delta_{\text{lg}}$, then $\alpha = \beta - \gamma \in \Phi_{\text{lg}}$ by Lemma 3.1.1. Moreover, $\text{ht}^\vee(\alpha) = \text{ht}^\vee(\beta) - 1$. Thus $l(s_\alpha) = 2 \text{ht}^\vee(\alpha) - 1$. We have $(\alpha|\gamma) \neq 0$ (otherwise $\beta = \alpha + \gamma$ would be of squared length $2r$, which is impossible). By [Bou68, chap. VI, §1.3], we have $(\alpha|\gamma) = -r/2$. Thus $\beta = s_\gamma(\alpha) = s_\alpha(\gamma)$, and $s_\beta = s_\gamma s_\alpha s_\gamma$. Since $s_\alpha(\gamma) > 0$, we have $l(s_\beta s_\alpha) = l(s_\beta) + 1$ (see 3.5). Since $s_\gamma s_\alpha(\gamma) = s_\gamma(\beta) = \alpha > 0$, we have $l(s_\gamma s_\alpha s_\gamma) = l(s_\alpha s_\gamma) + 1 = l(s_\alpha) + 2$ (see 3.5), as expected.

- If $\gamma \in \Delta_{\text{sh}}$, then, by [Bou68, chap. VI, §1.3], we have $\alpha = \beta - r\gamma \in \Phi_{\text{lg}}^+$, $(\alpha|\gamma) = -r/2$, and $\text{ht}^\vee(\alpha) = \text{ht}^\vee(\beta) - 1$. As in the first case, we have $\beta = s_\gamma(\alpha)$. Thus $s_\beta = s_\gamma s_\alpha s_\gamma$ and the same argument applies. \square

Remark 3.1.4 By duality, if $\beta \in \Phi_{\text{sh}}^+$, we have

$$l(s_\beta) = 2 \text{ht}(\beta) - 1.$$

3.1.4 Highest root

Let $\tilde{\alpha}$ be the highest root of Φ relatively to Δ (see [Bou68, chap. VI, §1.8, prop. 25]). It is of height $h - 1$, where h is the Coxeter number of Φ . The dual Coxeter number h^\vee can be defined as $1 + \text{ht}^\vee(\tilde{\alpha})$. Let us recall the following facts:

$$\tilde{\alpha} \in \Phi_{\text{lg}} \quad (3.11)$$

and

$$\text{If } \alpha \in \Phi^+ \setminus \{\tilde{\alpha}\}, \text{ then } \langle \alpha, \tilde{\alpha}^\vee \rangle \in \{0, 1\}. \quad (3.12)$$

In particular,

$$\text{If } \alpha \in \Phi^+, \text{ then } \langle \tilde{\alpha}, \alpha^\vee \rangle \geq 0 \quad (3.13)$$

and

$$\tilde{\alpha} \in \bar{C}, \quad (3.14)$$

where C is the chamber associated to Δ .

From now on, \tilde{I} will denote the subset of Δ defined by

$$\tilde{I} = \{\alpha \in \Delta \mid (\tilde{\alpha}|\alpha) = 0\}. \quad (3.15)$$

By construction, \tilde{I} is stable under any automorphism of V stabilizing Δ . In particular, it is stable under $-w_0$. By 3.13, we have

$$\Phi_{\tilde{I}} = \{\alpha \in \Phi \mid (\tilde{\alpha}|\alpha) = 0\}. \quad (3.16)$$

From 3.16 and [Bou68, chap. V, §3.3, prop. 2], we deduce that

$$W_{\tilde{I}} = \{w \in W \mid w(\tilde{\alpha}) = \tilde{\alpha}\}. \quad (3.17)$$

Note that w_0 and $w_{\tilde{I}}$ commute (because $-w_0(\tilde{I}) = \tilde{I}$). We have

$$N(w_0w_{\tilde{I}}) = \Phi^+ \setminus \Phi_{\tilde{I}}^+. \quad (3.18)$$

Let us now consider the map $W \rightarrow \Phi_{\text{lg}}$, $w \mapsto w(\tilde{\alpha})$. It is surjective [Bou68, chap. VI, §1.3, prop. 11] and thus induces a bijection $W/W_{\tilde{I}} \rightarrow \Phi_{\text{lg}}$ by 3.17. It follows that the map

$$\begin{array}{ccc} X_{\tilde{I}} & \longrightarrow & \Phi_{\text{lg}} \\ x & \longmapsto & x(\tilde{\alpha}) \end{array} \quad (3.19)$$

is a bijection. If $\alpha \in \Phi_{\text{lg}}$, we will denote by x_α the unique element of $X_{\tilde{I}}$ such that $x_\alpha(\tilde{\alpha}) = \alpha$. We have

$$x_\alpha s_{\tilde{\alpha}} = s_\alpha x_\alpha. \quad (3.20)$$

Lemma 3.1.5 *We have $w_0w_{\tilde{I}} = w_{\tilde{I}}w_0 = s_{\tilde{\alpha}}$.*

Proof. We have already noticed that w_0 and $w_{\bar{I}}$ commute.

In view of [Bou68, chap. VI, §1, exercice 16], it suffices to show that $N(w_0w_{\bar{I}}) = N(s_{\tilde{\alpha}})$, that is, $N(s_{\tilde{\alpha}}) = \Phi^+ \setminus \Phi_{\bar{I}}^+$ (see 3.18). First, if $\alpha \in \Phi_{\bar{I}}^+$, then $s_{\tilde{\alpha}}(\alpha) = \alpha$, so that $\alpha \notin N(s_{\tilde{\alpha}})$. This shows that $N(s_{\tilde{\alpha}}) \subset \Phi^+ \setminus \Phi_{\bar{I}}^+$.

Let us show the other inclusion. If $\alpha \in \Phi^+ \setminus \Phi_{\bar{I}}^+$, then $\langle \tilde{\alpha}, \alpha^\vee \rangle > 0$ by 3.13 and 3.16. In particular, $s_{\tilde{\alpha}}(\alpha) = \alpha - \langle \alpha, \tilde{\alpha}^\vee \rangle \tilde{\alpha}$ cannot belong to Φ^+ since $\tilde{\alpha}$ is the highest root. \square

Proposition 3.1.6 *Let $\alpha \in \Phi_{\text{lg}}^+$. Then we have*

$$l(x_\alpha s_{\tilde{\alpha}}) = l(s_{\tilde{\alpha}}) - l(x_\alpha)$$

Proof. We have

$$\begin{aligned} l(x_\alpha s_{\tilde{\alpha}}) &= l(x_\alpha w_{\bar{I}} w_0) && \text{by Lemma 3.1.5} \\ &= l(w_0) - l(x_\alpha w_{\bar{I}}) && \text{by 3.9} \\ &= l(w_0) - l(w_{\bar{I}}) - l(x_\alpha) && \text{by 3.10} \\ &= l(w_0 w_{\bar{I}}) - l(x_\alpha) && \text{by 3.9} \\ &= l(s_{\tilde{\alpha}}) - l(x_\alpha) && \text{by Lemma 3.1.5.} \end{aligned}$$

\square

Proposition 3.1.7 *If $\alpha \in \Phi_{\text{lg}}^+$, then $x_{-\alpha} = s_\alpha x_\alpha$ and $l(x_{-\alpha}) = l(s_\alpha x_\alpha) = l(s_\alpha) + l(x_\alpha)$.*

Proof. We have $s_\alpha x_\alpha(\tilde{\alpha}) = s_\alpha(\alpha) = -\alpha$, so to show that $x_{-\alpha} = s_\alpha x_\alpha$, it is enough to show that $s_\alpha x_\alpha \in X_{\bar{I}}$. But, if $\beta \in \Phi_{\bar{I}}^+$, we have (see 3.20) $s_\alpha x_\alpha(\beta) = x_\alpha s_{\tilde{\alpha}}(\beta) = x_\alpha(\beta) \in \Phi^+$. Hence the first result.

Let us now show that the lengths add up. By 3.8, it is enough to show that $N(x_\alpha) \subset N(s_\alpha x_\alpha)$. Let then $\beta \in N(x_\alpha)$. Since $x_\alpha \in X_{\bar{I}}$, β cannot be in $\Phi_{\bar{I}}^+$. Thus $\langle \beta, \tilde{\alpha}^\vee \rangle > 0$. Therefore, $\langle x_\alpha(\beta), \alpha^\vee \rangle > 0$. Now, we have $s_\alpha x_\alpha(\beta) = x_\alpha(\beta) - \langle x_\alpha(\beta), \alpha^\vee \rangle \alpha < 0$ (remember that $x_\alpha(\beta) < 0$ since $\beta \in N(x_\alpha)$). \square

Proposition 3.1.8 *For $\alpha \in \Phi_{\text{lg}}^+$, we have*

$$\begin{aligned} l(x_\alpha) &= \frac{l(s_{\tilde{\alpha}}) - l(s_\alpha)}{2} = \text{ht}^\vee(\tilde{\alpha}) - \text{ht}^\vee(\alpha) \\ l(x_{-\alpha}) &= \frac{l(s_{\tilde{\alpha}}) + l(s_\alpha)}{2} = \text{ht}^\vee(\tilde{\alpha}) + \text{ht}^\vee(\alpha) - 1 \end{aligned}$$

Proof. This follows from Propositions 3.1.6 and 3.1.7, and 3.20. \square

3.1.5 Orders

The choice of Δ determines an order relation on V . For $x, y \in V$, we have $y \leq x$ if and only if $y - x$ is a linear combination of the simple roots with non-negative coefficients.

For $\alpha \in \Phi_{\text{lg}}$, it will be convenient to define the *level* $L(\alpha)$ of α as follows:

$$L(\alpha) = \begin{cases} \text{ht}^\vee(\tilde{\alpha}) - \text{ht}^\vee(\alpha) & \text{if } \alpha > 0 \\ \text{ht}^\vee(\tilde{\alpha}) - \text{ht}^\vee(\alpha) - 1 & \text{if } \alpha < 0 \end{cases} \quad (3.21)$$

If i is an integer, let Φ_{lg}^i be the set of long roots of level i . Then Proposition 3.1.8 says that the bijection 3.19 maps $X_{\bar{I}}^i$ onto Φ_{lg}^i .

For $\gamma \in \Phi^+$, we write

$$\beta \xrightarrow{\gamma} \alpha \text{ if and only if } \alpha = s_\gamma(\beta) \text{ and } L(\alpha) = L(\beta) + 1. \quad (3.22)$$

In that case, we have $\beta - \alpha = \langle \beta, \gamma^\vee \rangle \gamma > 0$, so $\beta > \alpha$.

If α and β are two long roots, we say that there is a *path* from β to α , and we write $\alpha \preceq \beta$, if and only if there exists a sequence $(\beta_0, \beta_1, \dots, \beta_k)$ of long roots, and a sequence $(\gamma_1, \dots, \gamma_k)$ of positive roots, such that

$$\beta = \beta_0 \xrightarrow{\gamma_1} \beta_1 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_k} \beta_k = \alpha. \quad (3.23)$$

In that case, we have $L(\beta_i) = L(\beta) + i$ for $i \in \{0, \dots, k\}$. If moreover all the roots γ_i are simple, we say that there is a *simple path* from β to α .

On the other hand, we have the Bruhat order on W defined by the set of simple reflections $S = \{s_\alpha \mid \alpha \in \Delta\}$. If w and w' belong to W , we write $w \rightarrow w'$ if $w' = s_\gamma w$ and $l(w') = l(w) + 1$, for some positive root γ . In that case, we write $w \xrightarrow{\gamma} w'$ (the positive root γ is uniquely determined). The Bruhat order \leq is the reflexive and transitive closure of the relation \rightarrow . On $X_{\bar{I}}$, we will consider the restriction of the Bruhat order on W .

Let us now consider the action of a simple reflection on a long root.

Lemma 3.1.9 *Let $\beta \in \Phi_{\text{lg}}$ and $\gamma \in \Delta$. Let $\alpha = s_\gamma(\beta)$.*

- (i) *If $\beta \in \Delta_{\text{lg}}$ and $(\beta|\gamma) > 0$, then $\gamma = \beta$, $\alpha = -\beta$ and $\langle \beta, \gamma^\vee \rangle = 2$.*
- (ii) *If $\beta \in -\Delta_{\text{lg}}$ and $(\beta|\gamma) < 0$, then $\gamma = -\beta$, $\alpha = -\beta$ and $\langle \beta, \gamma^\vee \rangle = -2$.*
- (iii) *Otherwise, α and $\langle \beta, \gamma^\vee \rangle$ are given by the following table:*

	$\gamma \in \Delta_{\text{lg}}$		$\gamma \in \Delta_{\text{sh}}$	
$(\beta \gamma) > 0$	$\alpha = \beta - \gamma$	$\langle \beta, \gamma^\vee \rangle = 1$	$\alpha = \beta - r\gamma$	$\langle \beta, \gamma^\vee \rangle = r$
$(\beta \gamma) = 0$	$\alpha = \beta$	$\langle \beta, \gamma^\vee \rangle = 0$	$\alpha = \beta$	$\langle \beta, \gamma^\vee \rangle = 0$
$(\beta \gamma) < 0$	$\alpha = \beta + \gamma$	$\langle \beta, \gamma^\vee \rangle = -1$	$\alpha = \beta + r\gamma$	$\langle \beta, \gamma^\vee \rangle = -r$

- (i) *If $(\beta|\gamma) > 0$ then $L(\alpha) = L(\beta) + 1$, so that $\beta \xrightarrow{\gamma} \alpha$.*
- (ii) *If $(\beta|\gamma) = 0$ then $L(\alpha) = L(\beta)$, and in fact $\alpha = \beta$.*

(iii) If $(\beta|\gamma) < 0$ then $L(\alpha) = L(\beta) - 1$, so that $\alpha \xrightarrow{\gamma} \beta$.

Proof. Part 1 follows from inspection of the possible cases in [Bou68, Chapitre VI, §1.3].

Part 2 is a consequence of part 1. Note that there is a special case when we go from positive roots to negative roots, and *vice versa*. This is the reason why there are two cases in the definition of the level. \square

To go from a long simple root to the opposite of a long simple root, one sometimes needs a non-simple reflection.

Lemma 3.1.10 *Let $\beta \in \Delta_{\text{lg}}$, $\alpha \in -\Delta_{\text{lg}}$ and $\gamma \in \Phi^+$. Then $\beta \xrightarrow{\gamma} \alpha$ if and only if we are in one of the following cases:*

(i) $\alpha = -\beta$ and $\gamma = \beta$. In this case, $\langle \beta, \gamma^\vee \rangle = 2$.

(ii) $\beta + (-\alpha)$ is a root and $\gamma = \beta + (-\alpha)$. In this case, $\langle \beta, \gamma^\vee \rangle = 1$.

Proof. This is straightforward. \square

But otherwise, one can use simple roots at each step.

Proposition 3.1.11 *Let α and β be two long roots such that $\alpha \leq \beta$. Write $\beta = \sum_{\sigma \in J} n_\sigma \sigma$, and $\alpha = \sum_{\tau \in K} m_\tau \tau$, where J (resp. K) is a non-empty subset of Δ , and the n_σ (resp. the m_τ) are non-zero integers, all of the same sign.*

(i) If $0 < \alpha \leq \beta$, then there is a simple path from β to α .

(ii) If $\alpha < 0 < \beta$, then there is a simple path from β to α if and only if there is a long root which belongs to both J and K . Moreover, there is a path from β to α if and only if there is a long root σ in J , and a long root τ in K , such that $(\sigma|\tau) \neq 0$.

(iii) If $\alpha \leq \beta < 0$, then there is a simple path from β to α .

Proof. We will prove (i) by induction on $m = \text{ht}^\vee(\beta) - \text{ht}^\vee(\alpha)$.

If $m = 0$, then $\beta = \alpha$, and there is nothing to prove.

So we may assume that $m > 0$ and that the results holds for $m - 1$. Thus $\alpha < \beta$ and we have

$$\beta - \alpha = \sum_{\gamma \in J} n_\gamma \gamma$$

where J is a non-empty subset of Δ , and the n_γ , $\gamma \in J$, are positive integers. We have

$$(\beta - \alpha | \beta - \alpha) = \sum_{\gamma \in J} n_\gamma (\beta|\gamma) - \sum_{\gamma \in J} n_\gamma (\alpha|\gamma) > 0.$$

So there is a γ in J such that $(\beta|\gamma) > 0$ or $(\alpha|\gamma) < 0$. In the first case, let $\beta' = s_\gamma(\beta)$. It is a long root. If γ is long (resp. short), then $\beta' = \beta - \gamma$ (resp. $\beta' = \beta - r\gamma$), so that

3.1 Long roots and distinguished coset representatives

$\alpha \leq \beta' < \beta$ (see Lemma 3.1.2). We have $\beta \xrightarrow{\gamma} \beta'$ and $\text{ht}^\vee(\beta') = \text{ht}^\vee(\beta) - 1$, so we can conclude by the induction hypothesis. The second case is similar: if $\alpha' = s_\gamma(\alpha) \in \Phi_{\text{lg}}$, then $\alpha < \alpha' \leq \beta$, $\alpha' \xrightarrow{\gamma} \alpha$, $\text{ht}^\vee(\alpha') = \text{ht}^\vee(\alpha) + 1$, and we can conclude by the induction hypothesis. This proves (i).

Now (iii) follows, applying (i) to $-\alpha$ and $-\beta$ and using the symmetry -1 .

Let us prove (ii). If there is a long simple root σ which belongs to J and K , we have $\alpha \leq -\sigma < \sigma \leq \beta$. Using (i), we find a simple path from β to σ , then we have $\sigma \xrightarrow{\sigma} -\sigma$, and using (iii) we find a simple path from $-\sigma$ to α . So there is a simple path from β to α .

Suppose there is a long root γ in J , and a long root γ' in K , such that $(\sigma|\tau) \neq 0$. Then either we are in the preceding case, or there are long simple roots $\sigma \in J$ and $\tau \in K$, such that $\alpha \leq -\tau < \sigma \leq \beta$ and $\gamma = \sigma + \tau$ is a root. By Lemma 3.1.10, we have $\sigma \xrightarrow{\gamma} -\tau$. Using (i) and (iii), we can find simple paths from β to σ and from $-\tau$ to α . So there is a path from β to α .

Now suppose there is a path from β to α . In this path, we must have a unique step of the form $\sigma \xrightarrow{\gamma} -\tau$, with σ and τ in Δ_{lg} . We have $\sigma \in J$, $\tau \in K$, and $(\sigma|\tau) \neq 0$. If moreover it is a simple path from β to α , then we must have $\tau = -\sigma$. This completes the proof. \square

The preceding analysis can be used to study the length and the reduced expressions of some elements of W .

Proposition 3.1.12 *Let α and β be two long roots. If x is an element of W such that $x(\beta) = \alpha$, then we have $l(x) \geq |L(\alpha) - L(\beta)|$.*

Moreover, there is an $x \in W$ such that $x(\beta) = \alpha$ and $l(x) = |L(\alpha) - L(\beta)|$ if and only if α and β are linked by a simple path, either from β to α , or from α to β . In this case, there is only one such x , and we denote it by $x_{\alpha\beta}$. The reduced expressions of $x_{\alpha\beta}$ correspond bijectively to the simple paths from β to α .

If $\alpha \leq \beta \leq \gamma$ are such that $x_{\alpha\beta}$ and $x_{\beta\gamma}$ are defined, then $x_{\alpha\gamma}$ is defined, and we have $x_{\alpha\gamma} = x_{\alpha\beta}x_{\beta\gamma}$ with $l(x_{\alpha\gamma}) = l(x_{\alpha\beta}) + l(x_{\beta\gamma})$.

The element $x_{-\alpha,\alpha}$ is defined for all $\alpha \in \Phi_{\text{lg}}^+$, and is equal to s_α .

The element $x_{\alpha,\tilde{\alpha}}$ is defined for all $\alpha \in \Phi_{\text{lg}}$, and is equal to x_α .

Proof. Let $(s_{\gamma_k}, \dots, s_{\gamma_1})$ be a reduced expression of x , where $k = l(x)$. For $i \in \{0, \dots, k\}$, let $\beta_i = s_{\gamma_i} \dots s_{\gamma_1}(\beta)$. For each $i \in \{0, \dots, k-1\}$, we have $|L(\beta_{i+1}) - L(\beta_i)| \leq 1$ by Lemma 3.1.9. Then we have

$$|L(\alpha) - L(\beta)| \leq \sum_{i=0}^{k-1} |L(\beta_{i+1}) - L(\beta_i)| \leq k = l(x)$$

If we have an equality, then all the $L(\beta_{i+1}) - L(\beta_i)$ must be of absolute value one and of the same sign, so either they are all equal to 1, or they are all equal to -1 . Thus, either we have a simple path from β to α , or we have a simple path from β to α .

Suppose there is a simple path from β to α . Let $(\beta_0, \dots, \beta_k)$ be a sequence of long roots, and $(\gamma_1, \dots, \gamma_k)$ a sequence of simple roots, such that

$$\beta = \beta_0 \xrightarrow{\gamma_1} \beta_1 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_k} \beta_k = \alpha.$$

Let $x = s_{\gamma_k} \dots s_{\gamma_1}$. Then we have $x(\beta) = \alpha$, and $l(x) \leq k$. But we have seen that $l(x) \geq L(\alpha) - L(\beta) = k$. So we have equality. The case where there is a simple path from α to β is similar.

Let $\alpha \in \Phi_{\text{lg}}$. If $\alpha > 0$, we have $0 < \alpha \leq \tilde{\alpha}$, so by Proposition 3.1.11 (i), there is a simple path from $\tilde{\alpha}$ to α . If $\alpha < 0$, we have $\alpha < -\alpha \leq \tilde{\alpha}$, so by Proposition 3.1.11 (i) and (ii), there is also a simple path from $\tilde{\alpha}$ to α in this case. Let x be the product of the simple reflections it involves. Then $l(x) = L(\alpha)$, so x is of minimal length in $xW_{\tilde{I}}$, and $x \in X_{\tilde{I}}$. Thus $x = x_\alpha$ is uniquely determined, and $x_{\alpha, \tilde{\alpha}}$ is defined. It is equal to x_α and is of length $L(\alpha)$.

Let α and β be two long roots such there is a simple path from β to α , and let x be the product of the simple reflections it involves. We have $xx_\beta(\tilde{\alpha}) = x(\beta) = \alpha$, and it is of length $L(\alpha)$, so it is of minimal length in its coset modulo $W_{\tilde{I}}$. Thus $xx_\beta = x_\alpha$, and $x = x_\alpha x_\beta^{-1}$ is uniquely determined. Therefore, $x_{\alpha\beta}$ is defined and equal to $x_\alpha x_\beta^{-1}$. Any simple path from β to α gives rise to a reduced expression of $x_{\alpha\beta}$, and every reduced expression of $x_{\alpha\beta}$ gives rise to a simple path from β to α . These are inverse bijections.

If $\alpha \leq \beta \leq \gamma$ are such that $x_{\alpha\beta}$ and $x_{\beta\gamma}$ are defined, one can show that $x_{\alpha\gamma}$ is defined, and that we have $x_{\alpha\gamma} = x_{\alpha\beta}x_{\beta\gamma}$ with $l(x_{\alpha\gamma}) = l(x_{\alpha\beta}) + l(x_{\beta\gamma})$, by concatenating simple paths from γ to β and from β to α .

If α is a positive long root, then there is a simple path from α to $-\alpha$. We can choose a symmetric path (so that the simple reflections form a palindrome). So $x_{-\alpha, \alpha}$ is defined, and is a reflection: it must be s_α . It is of length $L(-\alpha) - L(\alpha) = 2\text{ht}^\vee(\alpha) - 1$. \square

Remark 3.1.13 We have seen in the proof that, if $\alpha \in \Phi_{\text{lg}}^+$, then

$$l(s_\alpha) = l(x_{-\alpha, \alpha}) = L(-\alpha) - L(\alpha) = 2\text{ht}^\vee(\alpha) - 1$$

and, if $\alpha \in \Phi_{\text{lg}}$, then

$$l(x_\alpha) = l(x_{\alpha, \tilde{\alpha}}) = L(\alpha)$$

thus we have a second proof of Lemma 3.1.3 and Proposition 3.1.8. Similarly, the formulas $x_{\alpha\gamma} = x_{\alpha\beta}x_{\beta\gamma}$ and $l(x_{\alpha\gamma}) = l(x_{\alpha\beta}) + l(x_{\beta\gamma})$, applied to the triple $(-\alpha, \alpha, \tilde{\alpha})$, give another proof of Proposition 3.1.7.

To conclude this section, let us summarize the results which we will use in the sequel.

Theorem 3.1.14 *The bijection 3.19 is an anti-isomorphism between the posets $(\Phi_{\text{lg}}, \preceq)$ and $(X_{\tilde{I}}, \leq)$ (these orders were defined at the beginning of 3.1.5), and a root of level i corresponds to an element of length i in $X_{\tilde{I}}$.*

3.2 Resolution of singularities, Gysin sequence

If β and α are long roots, and γ is a positive root, then we have

$$\beta \xrightarrow{\gamma} \alpha \quad \text{if and only if} \quad x_\beta \xrightarrow{\gamma} x_\alpha$$

(these relations have been defined at the beginning of 3.1.5).

Moreover, in the above situation, the integer $\partial_{\alpha\beta} = \langle \beta, \gamma^\vee \rangle$ is determined as follows:

- (i) if $\beta \in \Delta_{\text{lg}}$ and $\alpha \in -\Delta_{\text{lg}}$, then $\partial_{\alpha\beta}$ is equal to 2 if $\alpha = -\beta$, and to -1 if $\beta + (-\alpha)$ is a root;
- (ii) otherwise, $\partial_{\alpha\beta}$ is equal to 1 if γ is long, and to r if γ is short (where $r = \max_{\alpha \in \Phi} (\alpha | \alpha)$).

If β and α are two long roots such that $L(\alpha) = L(\beta) + 1$, then we set $\partial_{\alpha\beta} = 0$ if there is no simple root γ such that $\beta \xrightarrow{\gamma} \alpha$.

The numbers $\partial_{\alpha\beta}$ will appear in Theorem 3.2.1 as the coefficients of the matrices of some maps appearing in the Gysin sequence associated to the \mathbb{C}^* -fibration $\mathcal{O}_{\min} \simeq G \times_P \mathbb{C}^* x_{\min}$ over G/P , giving the cohomology of \mathcal{O}_{\min} . By Theorem 3.1.14, these coefficients are explicitly determined in terms of the combinatorics of the root system.

3.2 Resolution of singularities, Gysin sequence

Let us choose a maximal torus T of G , with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. We then denote by $X(T)$ its group of characters, and $X^\vee(T)$ its group of cocharacters. For each $\alpha \in \Phi$, there is a closed subgroup U_α of G , and an isomorphism $u_\alpha : \mathbb{G}_a \rightarrow U_\alpha$ such that, for all $t \in T$ and for all $\lambda \in \mathbb{C}$, we have $tu_\alpha(\lambda)t^{-1} = u_\alpha(\alpha(t)\lambda)$. We are in the set-up of 3.1.1, with Φ equal to the root system of (G, T) in $V = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We denote $X(T) \times_{\mathbb{Z}} \mathbb{Q}$ by $V_{\mathbb{Q}}$, and the symmetric algebra $S(V_{\mathbb{Q}})$ by S .

There is a root subspace decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$$

where \mathfrak{g}_α is the (one-dimensional) weight subspace $\{x \in \mathfrak{g} \mid \forall t \in T, \text{Ad}(t).x = \alpha(t)x\}$. We denote by e_α a non-zero vector in \mathfrak{g}_α . Thus we have $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$.

Let $W = N_G(T)/T$ be the Weyl group. It acts on $X(T)$, and hence on $V_{\mathbb{Q}}$ and S .

Let us now fix a Borel subgroup B of G containing T , with Lie algebra \mathfrak{b} . This choice determines a basis Δ , the subset of positive roots Φ^+ , and the height (and dual height) function, as in 3.1.2, the length function l as in 3.1.3, the highest root $\tilde{\alpha}$ and the subset \tilde{I} of Δ as in 3.1.4, and the orders on Φ_{lg} and $X_{\tilde{I}}$ as in 3.1.5. So we can apply all the notations and results of Section 3.1.

Let H be a closed subgroup of G , and X a variety with a left H -action. Then H acts on $G \times X$ on the right by $(g, x).h = (gh, h^{-1}x)$. If the canonical morphism $G \rightarrow G/H$ has local sections, then the quotient variety $(G \times X)/H$ exists (see [Spr98, §5.5]). The quotient is denoted by $G \times_H X$. One has a morphism $G \times_H X \rightarrow G/H$ with local

sections, whose fibers are isomorphic to X . The quotient is the *fiber bundle over G/H associated to X* . We denote the image of (g, x) in this quotient by $g *_H x$, or simply $g * x$ if the context is clear. Note that G acts on the left on $G \times_H X$, by $g' . g *_H x = g' g *_H x$.

In 3.2.1, we describe the cohomology of G/B , both in terms Chern classes of line bundles and in terms of fundamental classes of Schubert varieties, and we state the Pieri formula (see [BGG73, Dem73, Hil82] for a description of Schubert calculus). In 3.2.2, we explain how this generalizes to the parabolic case. In 3.2.3, we give an algorithm to compute the cohomology of any line bundle minus the zero section, on any generalized flag variety. To do this, we need the Gysin sequence (see for example [BT82, Hus94], or [Mil80] in the étale case). In 3.2.4, we will see that the computation of the cohomology of \mathcal{O}_{\min} is a particular case. Using the results of Section 3.1, we give a description in terms of the combinatorics of the root system.

3.2.1 Line bundles on G/B , cohomology of G/B

Let $\mathcal{B} = G/B$ be the flag variety. It is a smooth projective variety of dimension $|\Phi^+|$. The map $G \rightarrow G/B$ has local sections (see [Spr98, §8.5]). If α is a character of T , one can lift it to B : let \mathbb{C}_α be the corresponding one-dimensional representation of B . We can then form the G -equivariant line bundle

$$\mathcal{L}(\alpha) = G \times_B \mathbb{C}_\alpha \rightarrow G/B. \quad (3.24)$$

Let $c(\alpha) \in H^2(G/B, \mathbb{Z})$ denote the first Chern class of $\mathcal{L}(\alpha)$. Then $c : X(T) \rightarrow H^2(G/B, \mathbb{Z})$ is a morphism of \mathbb{Z} -modules. It extends to a morphism of \mathbb{Q} -algebras, which we still denote by $c : S \rightarrow H^*(G/B, \mathbb{Q})$. The latter is surjective and has kernel \mathcal{I} , where \mathcal{I} is the ideal of S generated by the W -invariant homogeneous elements in S of positive degree. So it induces an isomorphism of \mathbb{Q} -algebras

$$\bar{c} : S/\mathcal{I} \simeq H^*(G/B, \mathbb{Q}) \quad (3.25)$$

which doubles degrees.

The algebra S/\mathcal{I} is called the coinvariant algebra. As a representation of W , it is isomorphic to the regular representation. We also have an action of W on $H^*(G/B, \mathbb{Q})$, because G/B is homotopic to G/T , and W acts on the right on G/T by the formula $gT.w = gnT$, where $n \in N_G(T)$ is a representative of $w \in W$, and $g \in G$. One can show that \bar{c} commutes with the actions of W .

On the other hand, we have the Bruhat decomposition [Spr98, §8.5]

$$G/B = \bigsqcup_{w \in W} C(w) \quad (3.26)$$

where the $C(w) = BwB/B \simeq \mathbb{C}^{l(w)}$ are the Schubert cells. Their closures are the Schubert varieties $S(w) = \overline{C(w)}$. Thus the cohomology of G/B is concentrated in even degrees, and $H^{2i}(G/B, \mathbb{Z})$ is free with basis $(Y_w)_{w \in W^i}$, where Y_w is the cohomology class of the Schubert variety $S(w_0w)$ (which is of codimension $l(w) = i$). The object of Schubert calculus is to describe the multiplicative structure of $H^*(G/B, \mathbb{Z})$ in these

terms (see [BGG73, Dem73, Hil82]). We will only need the following result (known as the Pieri formula, or Chevalley formula): if $w \in W$ and $\alpha \in X(T)$, then

$$c(\alpha) \cdot Y_w = \sum_{w \xrightarrow{\gamma} w'} \langle w(\alpha), \gamma^\vee \rangle Y_{w'} \quad (3.27)$$

3.2.2 Parabolic invariants

Let I be a subset of Δ . Let P_I be the parabolic subgroup of G containing B corresponding to I . It is generated by B and the subgroups $U_{-\alpha}$, for $\alpha \in I$. Its unipotent radical U_{P_I} is generated by the U_α , $\alpha \in \Phi^+ \setminus \Phi_I^+$. And it has a Levi complement L_I , which is generated by T and the U_α , $\alpha \in \Phi_I$. One can generalize the preceding constructions to the parabolic case.

If $\alpha \in X(T)^{W_I}$ (that is, if α is a character orthogonal to I), then we can form the G -equivariant line bundle

$$\mathcal{L}_I(\alpha) = G \times_{P_I} \mathbb{C}_\alpha \longrightarrow G/P_I \quad (3.28)$$

because the character α of T , invariant by W_I , can be extended to L_I and lifted to P_I .

We have a surjective morphism $q_I : G/B \longrightarrow G/P_I$, which induces an injection

$$q_I^* : H^*(G/P_I, \mathbb{Z}) \hookrightarrow H^*(G/B, \mathbb{Z})$$

in cohomology, which identifies $H^*(G/P_I, \mathbb{Z})$ with $H^*(G/B, \mathbb{Z})^{W_I}$.

The isomorphism \bar{c} restricts to

$$(S/\mathcal{I})^{W_I} \simeq H^*(G/P_I, \mathbb{Q}) \quad (3.29)$$

We have cartesian square

$$\begin{array}{ccc} \mathcal{L}(\alpha) & \longrightarrow & \mathcal{L}_I(\alpha) \\ \downarrow & & \downarrow \\ G/B & \xrightarrow{q_I} & G/P_I \end{array}$$

That is, the pullback by q_I of $\mathcal{L}_I(\alpha)$ is $\mathcal{L}(\alpha)$. By functoriality of Chern classes, we have $q_I^*(c_I(\alpha)) = c(\alpha)$.

We still have a Bruhat decomposition

$$G/P_I = \bigsqcup_{w \in X_I} C_I(w) \quad (3.30)$$

where $C_I(w) = BwP_I/P_I \simeq \mathbb{C}^{l(w)}$ for w in X_I . We note

$$S_I(w) = \overline{C_I(w)} \text{ and } Y_{I,w} = \overline{Bw_0wP_I/P_I} = \overline{Bw_0ww_I P_I/P_I} = [S_I(w_0ww_I)] \text{ for } w \text{ in } X_I$$

Note that

$$\text{if } w \text{ is in } X_I, \text{ then } w_0ww_I \text{ is also in } X_I \quad (3.31)$$

since for any root β in Φ_I^+ , we have $w_I(\beta) \in \Phi_I^-$, hence $ww_I(\beta)$ is also negative, and thus $w_0ww_I(\beta)$ is positive. Moreover, we have

$$\text{if } w \in X_I, \text{ then } l(w_0ww_I) = l(w_0) - l(ww_I) = l(w_0) - l(w_I) - l(w) = \dim G/P_I - l(w) \quad (3.32)$$

so that $Y_{I,w} \in H^{2l(w)}(G/P_I, \mathbb{Z})$. We have $q_I^*(Y_{I,w}) = Y_w$.

The cohomology of G/P_I is concentrated in even degrees, and $H^{2i}(G/P_I, \mathbb{Z})$ is free with basis $(Y_{I,w})_{w \in X_I^i}$. The cohomology ring $H^*(G/P_I, \mathbb{Z})$ is identified *via* q_I^* to a subring of $H^*(G/B, \mathbb{Z})$, so the Pieri formula can now be written as follows. If $w \in X_I$ and $\alpha \in X(T)^{W_I}$, then we have

$$c_I(\alpha) \cdot Y_{I,w} = \sum_{w \xrightarrow{\gamma} w' \in X_I} \langle w(\alpha), \gamma^\vee \rangle Y_{I,w'} \quad (3.33)$$

3.2.3 Cohomology of a \mathbb{C}^* -fiber bundle on G/P_I

Let I be a subset of Δ , and α be a W_I -invariant character of T . Let us consider

$$\mathcal{L}_I^*(\alpha) = G \times_{P_I} \mathbb{C}_\alpha^* \longrightarrow G/P_I, \quad (3.34)$$

that is, the line bundle $\mathcal{L}_I(\alpha)$ minus the zero section. In the sequel, we will have to calculate the cohomology of $\mathcal{L}_I^*(\tilde{\alpha})$, but we can explain how to calculate the cohomology of $\mathcal{L}_I^*(\alpha)$ for any given I and α (the point is that the answer for the middle cohomology will turn out to be nicer in our particular case, thanks to the results of Section 3.1).

We have the Gysin exact sequence

$$H^{n-2}(G/P_I, \mathbb{Z}) \xrightarrow{c_I(\alpha)} H^n(G/P_I, \mathbb{Z}) \longrightarrow H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \longrightarrow H^{n-1}(G/P_I, \mathbb{Z}) \xrightarrow{c_I(\alpha)} H^{n+1}(G/P_I, \mathbb{Z})$$

where $c_I(\alpha)$ means the multiplication by $c_I(\alpha)$, so we have a short exact sequence

$$0 \longrightarrow \text{Coker}(c_I(\alpha) : H^{n-2} \rightarrow H^n) \longrightarrow H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \longrightarrow \text{Ker}(c_I(\alpha) : H^{n-1} \rightarrow H^{n+1}) \longrightarrow 0$$

where H^j stands for $H^j(G/P_I, \mathbb{Z})$. Thus all the cohomology of $\mathcal{L}_I^*(\alpha)$ can be explicitly computed, thanks to the results of 3.2.2.

Let us now assume that α is dominant and regular for P_I , so that $\mathcal{L}_I(\alpha)$ is ample. Then, by the hard Lefschetz theorem, $c_I(\alpha) : \mathbb{Q} \otimes_{\mathbb{Z}} H^{n-2} \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} H^n$ is injective for $n \leq d_I = \dim \mathcal{L}_I^*(\alpha) = \dim G/P_I + 1$, and surjective for $n \geq d_I$. By the way, we see that we could immediately determine the rational cohomology of \mathcal{O}_{\min} , using only the results in this paragraph and the cohomology of G/P_I .

But we can say more. The cohomology of G/P_I is free and concentrated in even degrees. In fact, $c_I(\alpha) : H^{n-2} \rightarrow H^n$ is injective for $n \leq d_I$, and has free kernel and finite cokernel for $n \geq d_I$.

We have

$$\text{if } n \text{ is even, then } H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \simeq \text{Coker}(c_I(\alpha) : H^{n-2} \rightarrow H^n) \quad (3.35)$$

which is finite for $n \geq d_I$, and

$$\text{if } n \text{ is odd, then } H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \simeq \text{Ker}(c_I(\alpha) : H^{n-1} \rightarrow H^{n+1}) \quad (3.36)$$

which is free (it is zero if $n \leq d_I - 1$).

3.2.4 Resolution of singularities

Let \tilde{I} be the subset of Δ defined in 3.15. There is a resolution of singularities (see for example the introduction of [KP82])

$$\begin{aligned} \mathcal{L}_{\tilde{I}}(\tilde{\alpha}) = G \times_{P_{\tilde{I}}} \mathbb{C}_{\tilde{\alpha}} &\longrightarrow \overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \{0\} \\ g * \lambda &\longmapsto \text{Ad } g.(\lambda.e_{\tilde{\alpha}}) \end{aligned} \quad (3.37)$$

It is the one mentioned in the introduction, with $P = P_{\tilde{I}}$ and $x_{\min} = e_{\tilde{\alpha}}$. It induces an isomorphism

$$\mathcal{L}_{\tilde{I}}^*(\tilde{\alpha}) = G \times_{P_{\tilde{I}}} \mathbb{C}_{\tilde{\alpha}}^* \simeq \mathcal{O}_{\min} \quad (3.38)$$

Set $d = d_{\tilde{I}} = 2h^\vee - 2$. For all integers j , let H^j denote $H^j(G/P_{\tilde{I}}, \mathbb{Z})$. For $\alpha \in \Phi_{\text{lg}}$, we have $x_\alpha \in X_{\tilde{I}}$, so $Z_\alpha := Y_{\tilde{I}, x_\alpha}$ is an element of $H^{2i}(G/P_{\tilde{I}}, \mathbb{Z})$, where $i = l(x_\alpha) = L(\alpha)$. Then $H^*(G/P_{\tilde{I}}, \mathbb{Z})$ is concentrated in even degrees, and $H^{2i}(G/P_{\tilde{I}}, \mathbb{Z})$ is free with basis $(Z_\alpha)_{\alpha \in \Phi_{\text{lg}}^i}$. Combining Theorem 3.1.14 and the analysis of 3.2.3, we get the following description of the cohomology of \mathcal{O}_{\min} (the highest root $\tilde{\alpha}$ is dominant and regular with respect to $P_{\tilde{I}}$).

Theorem 3.2.1 *We have*

$$\text{if } n \text{ is even, then } H^n(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Coker}(c_{\tilde{I}}(\tilde{\alpha}) : H^{n-2} \rightarrow H^n) \quad (3.39)$$

which is finite for $n \geq d$, and

$$\text{if } n \text{ is odd, then } H^n(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Ker}(c_{\tilde{I}}(\tilde{\alpha}) : H^{n-1} \rightarrow H^{n+1}) \quad (3.40)$$

which is free (it is zero if $n \leq d - 1$).

Moreover, if $\beta \in \Phi_{\text{lg}}^i$, then we have

$$c_{\tilde{I}}(\tilde{\alpha}).Z_\beta = \sum_{\beta \xrightarrow{\gamma} \alpha} \langle \beta, \gamma^\vee \rangle Z_\alpha = \sum_{\alpha \in \Phi_{\text{lg}}^{i+1}} \partial_{\alpha\beta} Z_\beta \quad (3.41)$$

where the $\partial_{\alpha\beta}$ are the integers defined in Theorem 3.1.14.

As a consequence, we obtain the following results.

Theorem 3.2.2 (i) *The middle cohomology of \mathcal{O}_{\min} is given by*

$$H^{2h^\vee-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq P^\vee(\Phi')/Q^\vee(\Phi')$$

where Φ' is the root subsystem of Φ generated by Δ_{lg} , and $P^\vee(\Phi')$ (resp. $Q^\vee(\Phi')$) is its coweight lattice (resp. its coroot lattice).

(ii) *The rest of the cohomology of \mathcal{O}_{\min} is as described in Section 3.3. In particular, if ℓ is a good prime for G , then there is no ℓ -torsion in the rest of the cohomology of \mathcal{O}_{\min} .*

Proof. The map $c_{\tilde{\alpha}} : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ is described as follows. By Theorem 3.2.1, the cohomology group $H^{2h^\vee-4}$ is free with basis $(Z_\beta)_{\beta \in \Delta_{\text{lg}}}$ (the long roots of level $h^\vee - 2$ are of the long roots dual height 1, so they are the long simple roots). Similarly, $H^{2h^\vee-4}$ is free with basis $(Z_{-\alpha})_{\alpha \in \Delta_{\text{lg}}}$. Besides, the matrix of $c_{\tilde{\alpha}} : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ in these bases is $(\partial_{-\alpha,\beta})_{\alpha,\beta \in \Delta_{\text{lg}}}$. We have $\partial_{-\alpha,\alpha} = 2$ for $\alpha \in \Delta_{\text{lg}}$, and for distinct α and β we have $\partial_{-\alpha,\beta} = 1$ if $\alpha + \beta$ is a (long) root, 0 otherwise.

Thus the matrix of $c_{\tilde{\alpha}} : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ is the Cartan matrix of Φ' without minus signs. This matrix is equivalent to the Cartan matrix of Φ' : since the Dynkin diagram of Φ' is a tree, it is bipartite. We can write $\Delta_{\text{lg}} = J \cup K$, so that all the edges in the Dynkin diagram of Φ' link an element of J to an element of K . If we replace the $Z_{\pm\alpha}$, $\alpha \in J$, by their opposites, then the matrix of $c_{\tilde{\alpha}} : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ becomes the Cartan matrix of Φ' .

Now, the Cartan matrix of Φ' is transposed to the matrix of the inclusion of $Q(\Phi')$ in $P(\Phi')$ in the bases Δ_{lg} and $(\varpi_\alpha)_{\alpha \in \Delta_{\text{lg}}}$ (see [Bou68, Chap. VI, §1.10]), so it is in fact the matrix of the inclusion of $Q^\vee(\Phi')$ in $P^\vee(\Phi')$ in the bases $(\beta^\vee)_{\beta \in \Delta_{\text{lg}}}$ and $(\varpi_{\alpha^\vee})_{\alpha \in \Delta_{\text{lg}}}$.

The middle cohomology group $H^{2h^\vee-2}(\mathcal{O}_{\text{min}}, \mathbb{Z})$ is isomorphic to the cokernel of the map $c_{\tilde{\alpha}} : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$. This proves (i).

Part (ii) follows from a case-by-case analysis which will be done in Section 3.3. \square

Remark 3.2.3 Besides, we have $\partial_{\alpha\beta} = \partial_{-\beta,-\alpha}$, so the maps “multiplication by $c_{\tilde{\alpha}}$ ” in complementary degrees are transposed to each other. This accounts for the fact that \mathcal{O}_{min} satisfies Poincaré duality, since \mathcal{O}_{min} is homeomorphic to \mathbb{R}_*^+ times a smooth compact manifold of (real) dimension $2h^\vee - 5$ (since we deal with integral coefficients, one should take the derived dual for the Poincaré duality).

Remark 3.2.4 For the first half of the rational cohomology of \mathcal{O}_{min} , we find

$$\bigoplus_{i=1}^k \mathbb{Q}[-2(d_i - 2)]$$

where k is the number of long simple roots, and $d_1 \leq \dots \leq d_k \leq \dots \leq d_n$ are the degrees of W (n being the total number of simple roots). This can be observed case by case, or related to the corresponding Springer representation. The other half is determined by Poincaré duality.

3.3 Case-by-case analysis

In the preceding section, we have explained how to compute the cohomology of the minimal class in any given type in terms of root systems, and we found a description of the middle cohomology with a general proof. However, for the rest of the cohomology, we need a case-by-case analysis. It will appear that the primes dividing the torsion of the rest of the cohomology are bad. We have no *a priori* explanation for this fact. Note

that, for the type A , we have an alternative method, which will be explained in the next section.

For all types, first we give the Dynkin diagram, to fix the numbering $(\alpha_i)_{1 \leq i \leq r}$ of the vertices, where r denotes the semisimple rank of \mathfrak{g} , and to show the part \bar{I} of Δ (see 3.15). The corresponding vertices are represented in black. They are exactly those that are not linked to the additional vertex in the extended Dynkin diagram.

Then we give a diagram whose vertices are the positive long roots; whenever $\beta \xrightarrow{\gamma} \alpha$, we put an edge between β (above) and α (below), and the multiplicity of the edge is equal to $\partial_{\alpha\beta} = \langle \beta, \gamma^\vee \rangle$. In this diagram, the long root $\sum_{i=1}^r n_i \alpha_i$ (where the n_i are non-negative integers) is denoted by $n_1 \dots n_r$. The roots in a given line appear in lexicographic order.

For $1 \leq i \leq d-1$, let \mathcal{D}_i be the matrix of the map $c_{\bar{I}}(\tilde{\alpha}) : H^{2i-2} \rightarrow H^{2i}$ in the bases Φ_{lg}^{i-1} and Φ_{lg}^i (the roots being ordered in lexicographic order, as in the diagram). We give the matrices \mathcal{D}_i for $i = 1 \dots h^\vee - 2$. The matrix $\mathcal{D}_{h^\vee-1}$ is equal to the Cartan matrix without minus signs of the root system Φ' (corresponding to Δ_{lg}). The last matrices can be deduced from the first ones by symmetry, since (by Remark 3.2.3) we have $\mathcal{D}_{d-i} = {}^t \mathcal{D}_i$.

Then we give the cohomology of the minimal class with \mathbb{Z} coefficients (one just has to compute the elementary divisors of the matrices \mathcal{D}_i).

It will be useful to introduce some notation for the matrices in classical types. Let k be an integer. We set

$$M(k) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix} \quad N(k) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

where $M(k)$ is a square matrix of size k , and $N(k)$ is of size $(k+1) \times k$.

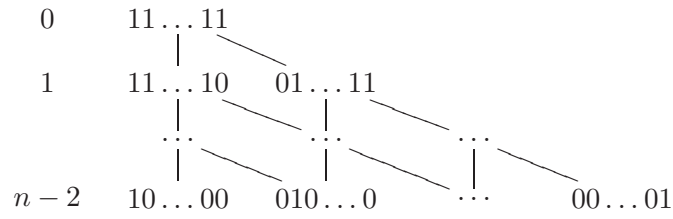
Now let k and l be non-negative integers. For i and j any integers, we define a $k \times l$ matrix $E_{i,j}(k,l)$ as follows. If (i,j) is not in the range $[1,k] \times [1,l]$, then we set $E_{i,j}(k,l) = 0$, otherwise it will denote the $k \times l$ matrix whose only non-zero entry is a 1 in the intersection of line i and column j . If the size of the matrix is clear from the context, we will simply write $E_{i,j}$.

First, the calculations of the elementary divisors of the matrices \mathcal{D}_i were done with GAP3 (see [S⁺97]). We used the data on roots systems of the package CHEVIE. But actually, all the calculations can be done by hand.

3.3.1 Type A_{n-1}



We have $h = h^\vee = n$ and $d = 2n - 2$.



The odd cohomology of $G/P_{\bar{1}}$ is zero, and we have

$$H^{2i}(G/P_{\bar{1}}) = \begin{cases} \mathbb{Z}^{i+1} & \text{if } 0 \leq i \leq n-2 \\ \mathbb{Z}^{2n-2-i} & \text{if } n-1 \leq i \leq 2n-3 \\ 0 & \text{otherwise} \end{cases}$$

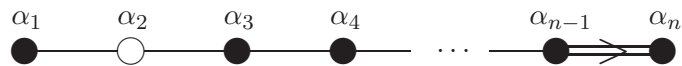
For $1 \leq i \leq n-2$, we have $\mathcal{D}_i = N(i)$; the cokernel is isomorphic to \mathbb{Z} . We have

$$\mathcal{D}_{n-1} = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix}$$

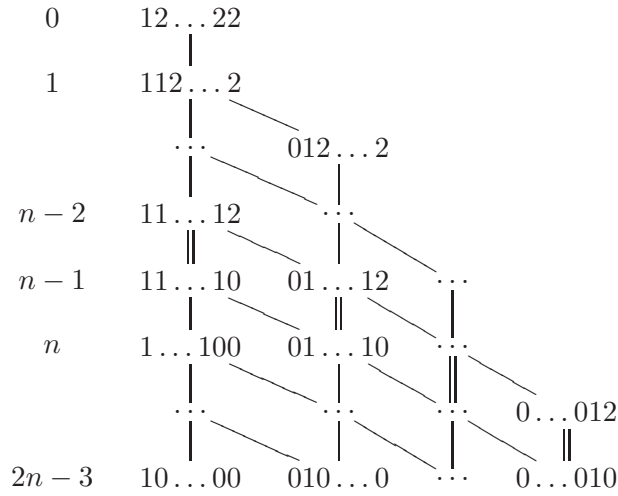
Its cokernel is isomorphic to \mathbb{Z}/n . The last matrices are transposed to the first ones, so the corresponding maps are surjective. From this, we deduce the cohomology of \mathcal{O}_{\min} in type A_{n-1} . We will see another method in Section 3.4.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq 2n-4 \text{ and } i \text{ is even,} \\ & \text{or } 2n-1 \leq i \leq 4n-5 \text{ and } i \text{ is odd} \\ \mathbb{Z}/n & \text{if } i = 2n-2 \\ 0 & \text{otherwise} \end{cases}$$

3.3.2 Type B_n



We have $h = 2n$, $h^\vee = 2n - 1$, and $d = 4n - 4$.



There is a gap at each even line (the length of the line increases by one). The diagram can be a little bit misleading if n is even: in that case, there is a gap at the line $n - 2$. Let us now describe the matrices \mathcal{D}_i .

First suppose $1 \leq i \leq n - 2$. If i is odd, then we have $\mathcal{D}_i = M\left(\frac{i+1}{2}\right)$ (an isomorphism); if i is even, then we have $\mathcal{D}_i = N\left(\frac{i}{2}\right)$ and the cokernel is isomorphic to \mathbb{Z} .

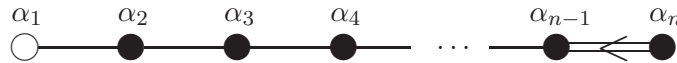
Now suppose $n - 1 \leq i \leq 2n - 3$. If i is odd, then we have $\mathcal{D}_i = M\left(\frac{i+1}{2}\right) + E_{i+2-n, i+2-n}$ and the cokernel is isomorphic to $\mathbb{Z}/2$. If i is even, then we have $\mathcal{D}_i = N\left(\frac{i}{2}\right) + E_{i+2-n, i+2-n}$ and the cokernel is isomorphic to \mathbb{Z} .

The long simple roots generate a root system of type A_{n-1} . Thus the matrix \mathcal{D}_{2n-2} is the Cartan matrix without minus signs of type A_{n-1} , which has cokernel \mathbb{Z}/n .

So the cohomology of \mathcal{O}_{\min} is described as follows.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq 4n - 8 \text{ and } i \equiv 0 \pmod{4}, \\ & \text{or } 4n - 1 \leq i \leq 8n - 9 \text{ and } i \equiv -1 \pmod{4} \\ \mathbb{Z}/2 & 2n - 2 \leq i \leq 6n - 6 \text{ and } i \equiv 2 \pmod{4} \\ \mathbb{Z}/n & \text{if } i = 4n - 4 \\ 0 & \text{otherwise} \end{cases}$$

3.3.3 Type C_n



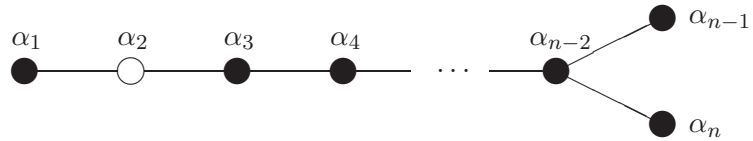
We have $h = 2n$, $h^\vee = n + 1$, and $d = 2n$. The root system Φ' is of type A_1 . Its Cartan matrix is (2).

$$\begin{array}{ccc}
 0 & 22 \dots 21 & \\
 & \parallel & \\
 1 & 02 \dots 21 & \\
 & \parallel & \\
 & \vdots & \\
 & \parallel & \\
 n-2 & 0 \dots 021 & \\
 & \parallel & \\
 n-1 & 00 \dots 01 &
 \end{array}$$

The matrices \mathcal{D}_i are all equal to (2).

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } 4n - 1 \\ \mathbb{Z}/2 & \text{if } 2 \leq i \leq 4n - 2 \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

3.3.4 Type D_n

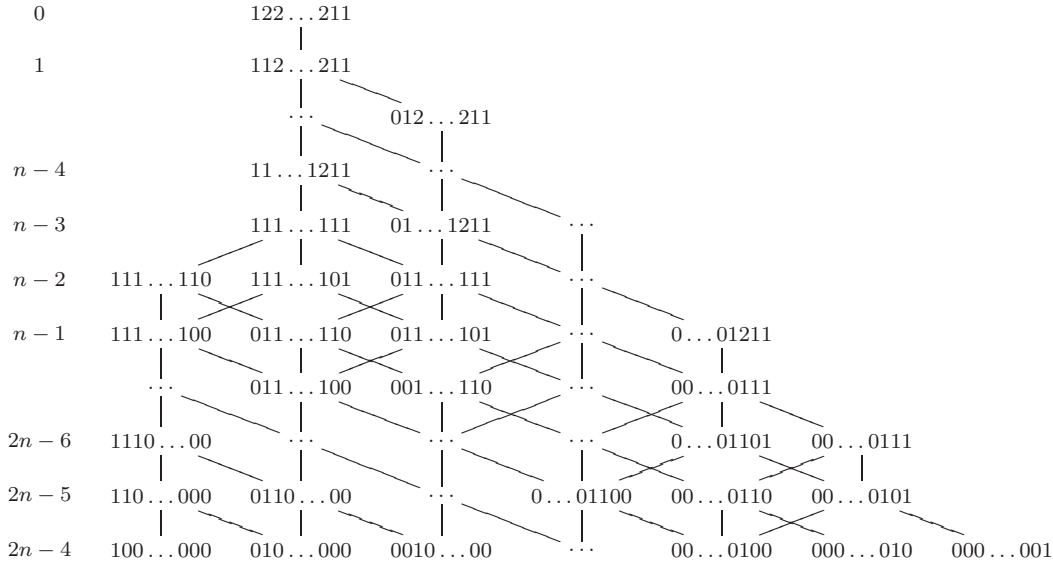


We have $h = h^\vee = 2n - 2$, and $d = 4n - 6$. We have

$$\mathcal{D}_{2n-3} = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 & 0 \\ \vdots & \ddots & 1 & 2 & 1 & 1 \\ 0 & \dots & 0 & 1 & 2 & 0 \\ 0 & \dots & 0 & 1 & 0 & 2 \end{pmatrix}$$

Its cokernel is $(\mathbb{Z}/2)^2$ when n is even, $\mathbb{Z}/4$ when n is odd.

As in the B_n case, the reader should be warned that there is a gap at line $n - 4$ if n is even. Besides, not all dots are meaningful. The entries $0 \dots 01211$ and $00 \dots 0111$ are on the right diagonal, but usually they are not on the lines $n - 1$ and n .



First suppose $i \leq i \leq n - 3$. We have

$$\mathcal{D}_i = \begin{cases} M\left(\frac{i+1}{2}\right) & \text{if } i \text{ is odd} \\ N\left(\frac{i}{2}\right) & \text{if } i \text{ is even} \end{cases}$$

Then the cokernel is zero if i is odd, \mathbb{Z} if i is even.

Let V be the $1 \times \frac{n-1}{2}$ matrix $(1, 0, \dots, 0)$. We have

$$\mathcal{D}_{n-2} = \begin{cases} \begin{pmatrix} V \\ N\left(\frac{n-2}{2}\right) \end{pmatrix} & \text{if } n \text{ is even} \\ \begin{pmatrix} V \\ M\left(\frac{n-1}{2}\right) \end{pmatrix} & \text{if } n \text{ is odd} \end{cases}$$

The cokernel is \mathbb{Z}^2 if n is even, \mathbb{Z} if n is odd.

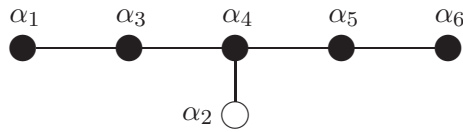
Now suppose $n - 1 \leq i \leq 2n - 4$. We have

$$\mathcal{D}_i = \begin{cases} M\left(\frac{i+3}{2}\right) + E_{i+2-n, i+3-n} - E_{i+3-n, i+3-n} + E_{i+3-n, i+4-n} & \text{if } i \text{ is odd} \\ N\left(\frac{i+2}{2}\right) + E_{i+2-n, i+3-n} - E_{i+3-n, i+3-n} + E_{i+3-n, i+4-n} & \text{if } i \text{ is even} \end{cases}$$

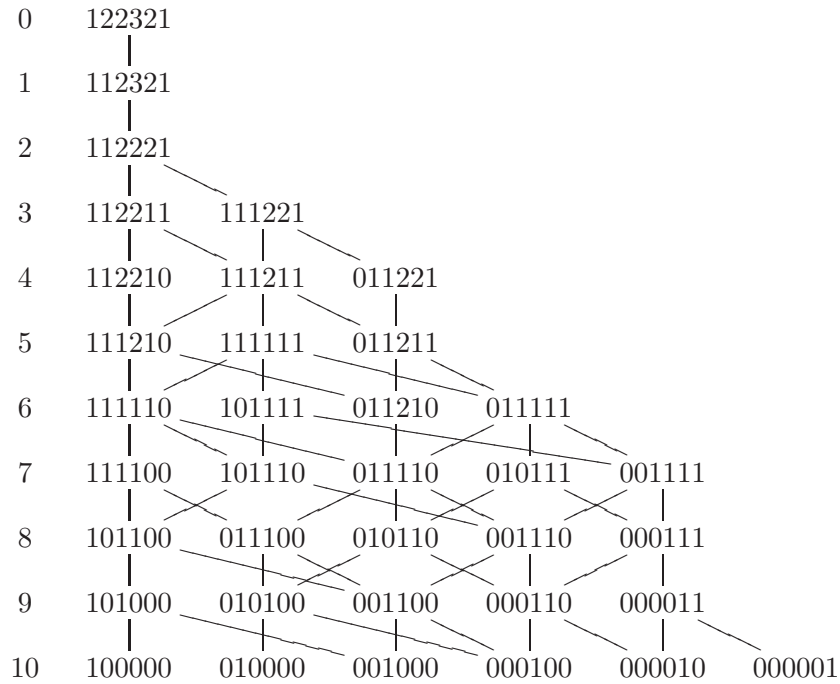
Then the cokernel is $\mathbb{Z}/2$ if i is odd, \mathbb{Z} if i is even.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq 4n - 8 \text{ and } i \equiv 0 \pmod{4}, \\ & \text{or } 4n - 5 \leq i \leq 8n - 13 \text{ and } i \equiv -1 \pmod{4}; \\ \mathbb{Z}/2 & \text{if } 2n - 4 < i < 4n - 6 \text{ and } i \equiv 2 \pmod{4}; \\ & \text{or } 4n - 6 < i < 6n - 8 \text{ and } i \equiv 2 \pmod{4}; \\ (\mathbb{Z}/2)^2 & \text{if } i = 4n - 6 \text{ and } n \text{ is even;} \\ \mathbb{Z}/4 & \text{if } i = 4n - 6 \text{ and } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases} \oplus \mathbb{Z} \begin{cases} \text{if } i = 2n - 4 \\ \text{or } i = 6n - 7. \end{cases}$$

3.3.5 Type E_6



We have $h = h^\vee = 12$, and $d = 22$. The Cartan matrix has cokernel isomorphic to $\mathbb{Z}/3$.



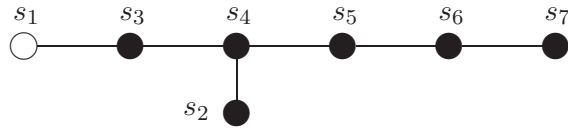
$$\mathcal{D}_1 = \mathcal{D}_2 = (1) \quad \mathcal{D}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathcal{D}_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathcal{D}_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{D}_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_8 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

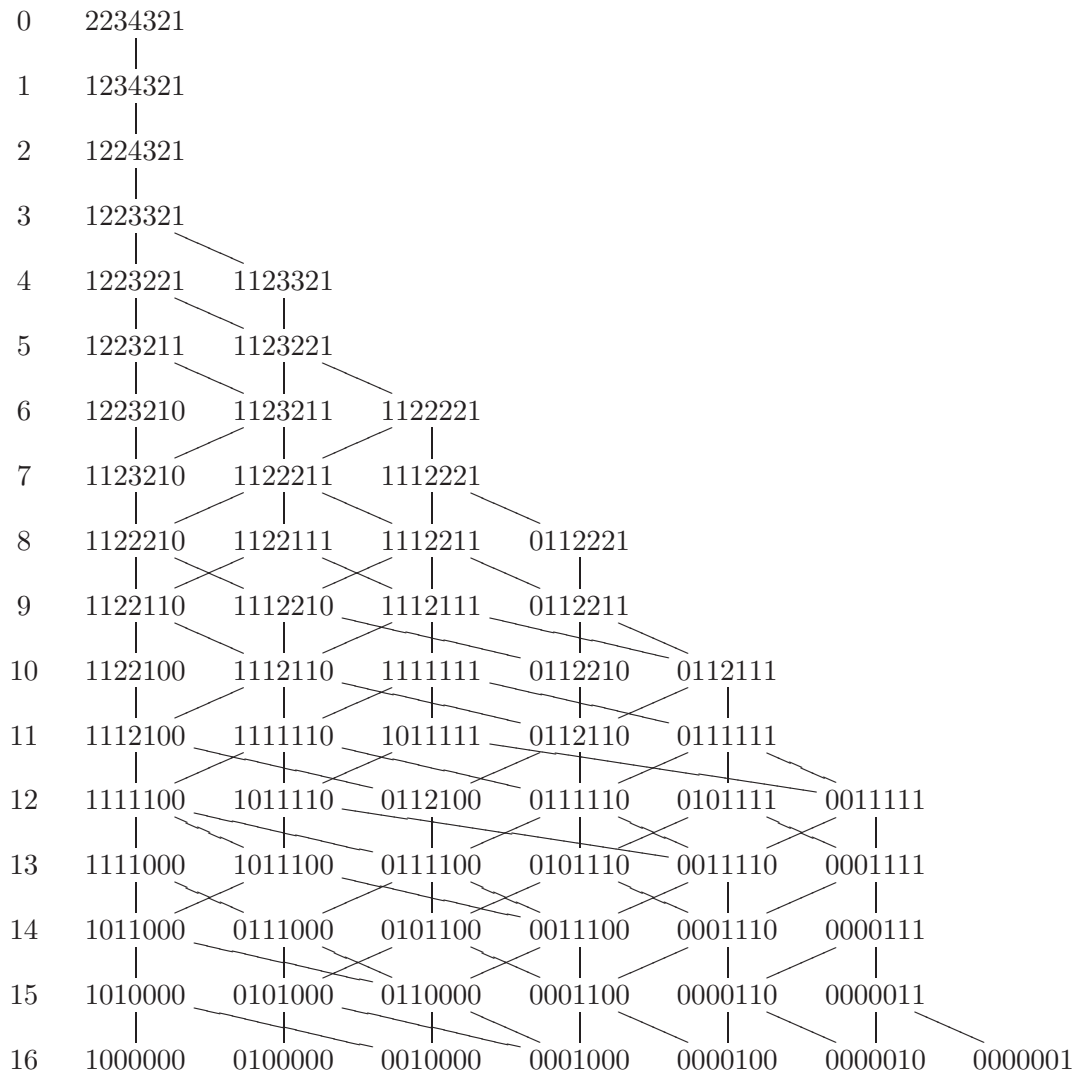
$$\mathcal{D}_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0, 6, 8, 12, 14, 20, 23, 29, 31, 35, 37, 43 \\ \mathbb{Z}/3 & \text{for } i = 16, 22, 28 \\ \mathbb{Z}/2 & \text{for } i = 18, 26 \\ 0 & \text{otherwise} \end{cases}$$

3.3.6 Type E_7



We have $h = h^\vee = 18$, and $d = 34$. The Cartan matrix has cokernel isomorphic to $\mathbb{Z}/2$.



$$\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = (1) \quad \mathcal{D}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathcal{D}_5 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathcal{D}_6 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_7 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_8 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_9 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

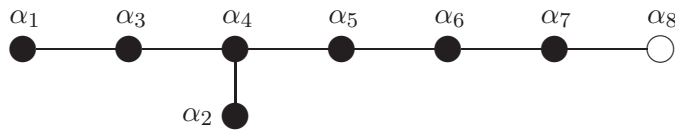
$$\mathcal{D}_{11} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{12} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{14} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

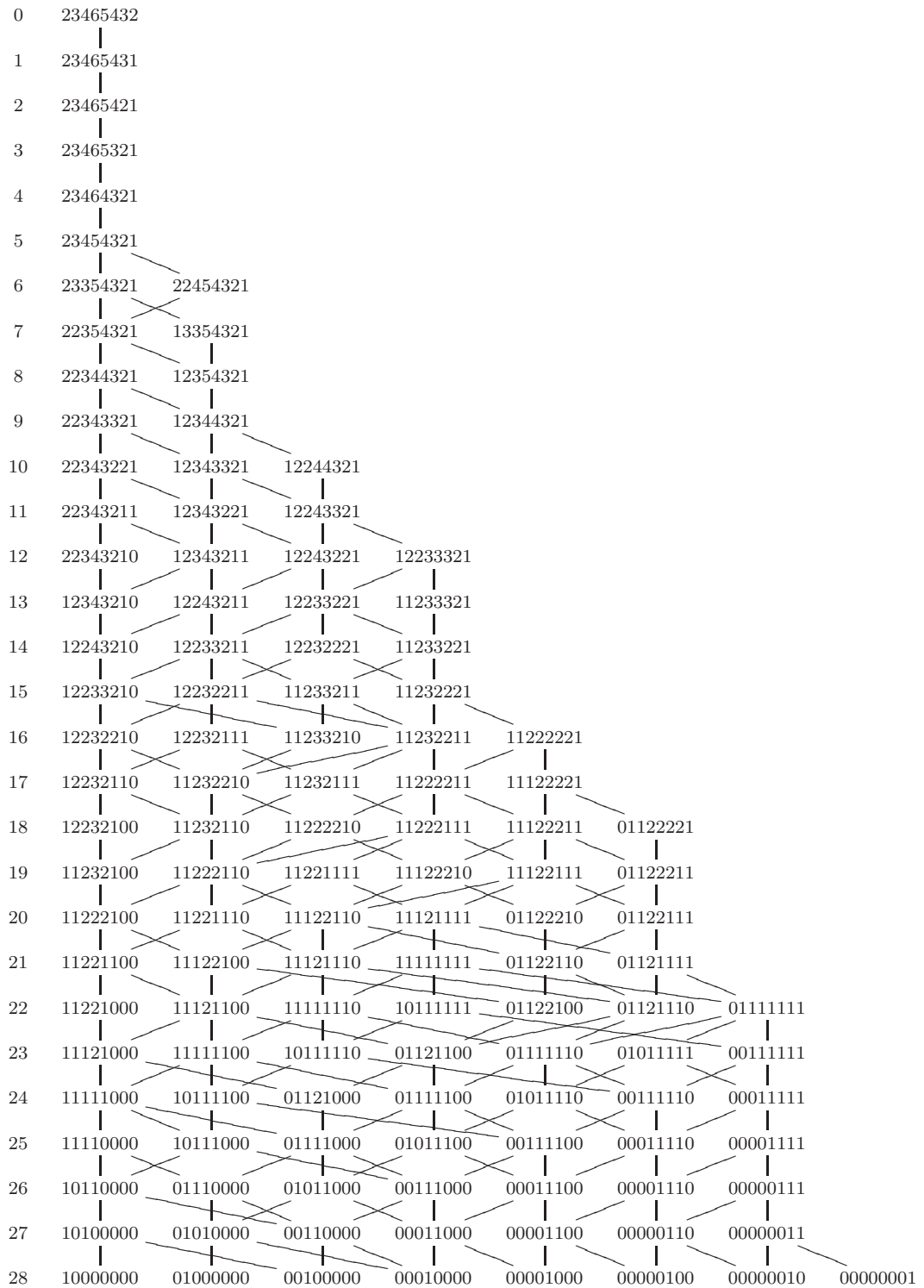
$$\mathcal{D}_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{16} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0, 8, 12, 16, 20, 24, 32, \\ & 35, 43, 47, 51, 55, 59, 67 \\ \mathbb{Z}/2 & \text{for } i = 18, 26, 30, 34, 38, 42, 50 \\ \mathbb{Z}/3 & \text{for } i = 28, 40 \\ 0 & \text{otherwise} \end{cases}$$

3.3.7 Type E_8

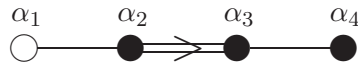


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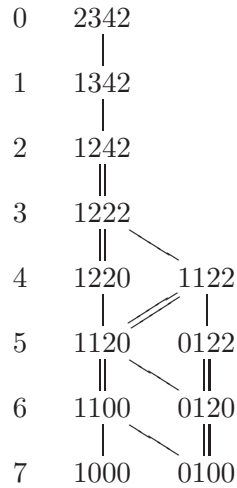


$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0, 12, 20, 24, 32, 36, 44, 56, \\ & \quad 59, 71, 79, 83, 91, 95, 103, 115 \\ \mathbb{Z}/2 & \text{for } i = 30, 42, 50, 54, 62, 66, 74, 86 \\ \mathbb{Z}/3 & \text{for } i = 40, 52, 64, 76 \\ \mathbb{Z}/5 & \text{for } i = 48, 68 \\ 0 & \text{otherwise} \end{cases}$$

3.3.8 Type F_4



We have $h = 12$, $h^\vee = 9$ and $d = 16$.



We have

$$\mathcal{D}_1 = \mathcal{D}_2 = (1) \quad \mathcal{D}_3 = (2) \quad \mathcal{D}_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathcal{D}_5 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \mathcal{D}_6 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \quad \mathcal{D}_7 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

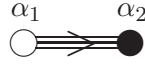
The type of Φ' is A_2 , so we have

$$\mathcal{D}_8 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The matrices of the last differentials are transposed to the first ones.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0, 8, 23, 31 \\ \mathbb{Z}/2 & \text{for } i = 6, 14, 18, 26 \\ \mathbb{Z}/4 & \text{for } i = 12, 20 \\ \mathbb{Z}/3 & \text{for } i = 16 \\ 0 & \text{otherwise} \end{cases}$$

3.3.9 Type G_2



We have $h = 6$, $h^\vee = 4$, and $d = 6$. The root system Φ' is of type A_1 . Its Cartan matrix has cokernel $\mathbb{Z}/2$.

$$\begin{array}{cc} 0 & 23 \\ & | \\ 1 & 13 \\ & || \\ 2 & 10 \end{array}$$

We have

$$\mathcal{D}_1 = (1) \quad \mathcal{D}_2 = (3) \quad \mathcal{D}_3 = (2) \quad \mathcal{D}_4 = (3) \quad \mathcal{D}_5 = (1)$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0, 11 \\ \mathbb{Z}/3 & \text{for } i = 4, 8 \\ \mathbb{Z}/2 & \text{for } i = 6 \\ 0 & \text{otherwise} \end{cases}$$

3.4 Another method for type A

Here we will explain a method which applies only in type A. This is because the minimal class is a Richardson class only in type A.

So suppose we are in type A_{n-1} . We can assume $G = GL_n$. The minimal class corresponds to the partition $(2, 1^{n-2})$. It consists of the nilpotent matrices of rank 1 in \mathfrak{gl}_n , or, in other words, the matrices of rank 1 and trace 0.

Let us consider the set E of pairs $([v], x) \in \mathbb{P}^{n-1} \times \mathfrak{gl}_n$ such that $\text{Im}(x) \subset \mathbb{C}v$ (so x is either zero or of rank 1). Together with the natural projection, this is a vector bundle on \mathbb{P}^{n-1} , corresponding to the locally free sheaf $\mathcal{E} = \mathcal{O}(-1)^n$ (we have one copy of the tautological bundle for each column).

There is a trace morphism $\text{Tr} : \mathcal{E} \rightarrow \mathcal{O}$. Let \mathcal{F} be its kernel, and let F be the corresponding sub-vector bundle of E . Then F consists of the pairs $([v], x)$ such that x is either zero or a nilpotent matrix of rank 1 with image $\mathbb{C}v$. The second projection gives a morphism $\pi : E \rightarrow \overline{\mathcal{O}_{\min}}$, which is a resolution of singularities, with exceptional fiber the null section. So we have an isomorphism from F minus the null section onto \mathcal{O}_{\min} .

As before, we have a Gysin exact sequence

$$H^{i-2n+2} \xrightarrow{c} H^i \longrightarrow H^i(\mathcal{O}_{\min}, \mathbb{Z}) \longrightarrow H^{i-2n+3} \xrightarrow{c} H^{i+1}$$

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where H^j stands for $H^j(\mathbb{P}^{n-1}, \mathbb{Z})$ and c is the multiplication by the last Chern class c of F . Thus $H^i(\mathcal{O}_{\min}, \mathbb{Z})$ fits in a short exact sequence

$$0 \longrightarrow \text{Coker}(c : H^{i-2n+2} \rightarrow H^i) \longrightarrow H^i(\mathcal{O}_{\min}, \mathbb{Z}) \longrightarrow \text{Ker}(c : H^{i-2n+3} \rightarrow H^{i+1}) \longrightarrow 0$$

We denote by $y \in H^2(\mathbb{P}^{n-1}, \mathbb{Z})$ the first Chern class of $\mathcal{O}(-1)$. We have $H^*(\mathbb{P}^{n-1}, \mathbb{Z}) \simeq \mathbb{Z}[y]/(y^n)$ as a ring. In particular, the cohomology of \mathbb{P}^{n-1} is free and concentrated in even degrees.

For $0 \leq i \leq 2n-4$, we have $H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq H^i$ which is isomorphic to \mathbb{Z} if i is even, and to 0 if i is odd. We have $H^{2n-3}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Ker}(c : H^0 \rightarrow H^{2n-2})$ and $H^{2n-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Coker}(c : H^0 \rightarrow H^{2n-2})$. For $2n-1 \leq i \leq 4n-5$, we have $H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq H^{i-2n+3}$ which is isomorphic to \mathbb{Z} if i is odd, and to 0 if i is even.

We have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} = \mathcal{O}(-1)^n \longrightarrow \mathcal{O} \longrightarrow 0$$

The total Chern class of F is thus

$$(1 + y)^n = \sum_{i=0}^{n-1} \binom{n}{i} y^i$$

by multiplicativity (remember that $y^n = 0$). So its last Chern class c is ny^{n-1} .

In fact, F can be identified with the cotangent bundle $T^*(G/Q)$, where Q is the parabolic subgroup which stabilizes a line in \mathbb{C}^n , and $G/Q \simeq \mathbb{P}^{n-1}$; then we can use the fact that the Euler characteristic of \mathbb{P}^{n-1} is n .

We can now determine the two remaining cohomology groups.

$$H^{2n-3}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Ker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = 0$$

and

$$H^{2n-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Coker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = \mathbb{Z}/n$$

Thus we find the same result as in Section 3.3 for the cohomology of \mathcal{O}_{\min} in type A_{n-1} .

Chapter 4

Some decomposition numbers

In this chapter, we compute in a geometrical way certain decomposition numbers for the G -equivariant perverse sheaves on the nilpotent variety, using the results of the preceding chapters, and geometrical results about nilpotent classes that can be found in the literature.

In Chapter 6, we will prove that part of this decomposition matrix is the decomposition matrix for the Weyl group (see Theorem 6.3.2), and we expect that the whole matrix is the one of the Schur algebra, at least in type A (in the other types one would have to give a precise meaning to the Schur algebra to be used). The decomposition numbers we find are in accordance with this expectation.

We made these calculations to have some examples and to get used to perverse sheaves over \mathbb{K} , \mathbb{O} and \mathbb{F} , and to find evidence for Theorem 6.3.2 before it was proved. Now that this theorem is proved, this chapter also shows that some calculations are feasible on the geometrical side. It should be clear, however, that further calculations would be increasingly difficult, so it is not clear whether Theorem 6.3.2 will help to compute the decomposition matrices of the symmetric group.

In any case, we can see *a posteriori* that the calculations in this chapter reveal new connections between the geometry of nilpotent orbits and the representations of Weyl groups. Next developments in this directions could include, in type A , the treatment of two-column (resp. two-row) partitions, and the transfer of a generalization of the row and column removal rule from the group side to the geometrical side.

4.1 Subregular class

We assume that G is simple and adjoint of type Γ , and that the characteristic of k is 0 or greater than $4h - 2$ (where h is the Coxeter number). This is a serious restriction on p , but it does not matter so much for our purposes. Note that, on the other hand, we make no concessions on ℓ (the only restriction is $\ell \neq p$).

Let \mathcal{O}_{reg} (resp. $\mathcal{O}_{\text{subreg}}$) be the regular (resp. subregular) orbit in \mathcal{N} . The orbit $\mathcal{O}_{\text{subreg}}$ is the unique open dense orbit in $\mathcal{N} \setminus \mathcal{O}_{\text{reg}}$ (we assume that \mathfrak{g} is simple). It is of codimension 2 in \mathcal{N} . Let $x_{\text{reg}} \in \mathcal{O}_{\text{reg}}$ and $x_{\text{subreg}} \in \mathcal{O}_{\text{subreg}}$.

The centralizer of x_{reg} in G is a connected unipotent subgroup, hence $A_G(x_{\text{reg}}) = 1$. The unipotent radical of the centralizer in G of x_{subreg} has a reductive complement C given by the following table.

Γ	$A_n (n > 1)$	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$C(x)$	\mathbb{G}_m	$\mathbb{G}_m \rtimes \mathbb{Z}/2$	$\mathbb{Z}/2$	1	1	1	1	$\mathbb{Z}/2$	\mathfrak{S}_3

In type A_1 , the subregular class is just the trivial class, so in this case the centralizer is $G = PSL_2$ itself, which is reductive.

We have $A_G(x_{\text{subreg}}) \simeq C/C^0$. This group is isomorphic to the associated symmetry group $A(\Gamma)$ introduced in section 2.5.

Let X be the intersection $X = S \cap \mathcal{N}$ of a transverse slice S to the orbit $\mathcal{O}_{\text{subreg}}$ of x_{subreg} with the nilpotent variety \mathcal{N} . The group C acts on X . We can find a section A of $C/C^0 \simeq A_G \simeq A(\Gamma)$ in C . In homogeneous types, A is trivial. If $\Gamma = C_n, F_4$ or G_2 , then $A = C$. If $\Gamma = B_n$, take $\{1, s\}$ where s is a nontrivial involution (in this case, A is well-defined up to conjugation by $C^0 = \mathbb{G}_m$).

Theorem 4.1.1 [Bri71, Slo80a] *We keep the preceding notation. The surface X has a rational double point of type $\widehat{\Gamma}$ at x_{subreg} . Thus $\text{Sing}(\overline{\mathcal{O}}_{\text{reg}}, \mathcal{O}_{\text{subreg}}) = \widehat{\Gamma}$.*

Moreover the couple (X, A) is a simple singularity of type Γ .

In fact, the first part of the theorem is already true when the characteristic of k is very good for G . This part is enough to calculate the decomposition numbers $d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)}$ for homogeneous types (then $A = 1$), and even some more decomposition numbers $d_{(x_{\text{reg}},1),(x_{\text{subreg}},\rho)}$ for the other types. Actually, what can be deduced in all types is the following relation:

$$\sum_{\rho \in \text{Irr } \mathbb{F}A} \rho(1) \cdot d_{(x_{\text{reg}},1),(x_{\text{subreg}},\rho)} = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi})$$

This is enough, for example, to determine for which ℓ we have $d_{(x_{\text{reg}},1),(x_{\text{subreg}},\rho)} = 0$ for all ρ (those which do not divide the connection index of $\widehat{\Phi}$).

Anyway, the second part of the theorem will allow us to deal with the local systems involved on $\mathcal{O}_{\text{subreg}}$.

Let $j_{\text{reg}} : \mathcal{O}_{\text{reg}} \hookrightarrow \mathcal{O}_{\text{reg}} \cup \mathcal{O}_{\text{subreg}}$ be the open immersion, and $i_{\text{subreg}} : \mathcal{O}_{\text{subreg}} \hookrightarrow \mathcal{O}_{\text{reg}} \cup \mathcal{O}_{\text{subreg}}$ the closed complement. Finally, let j be the open inclusion of $\mathcal{O}_{\text{subreg}} \cup \mathcal{O}_{\text{reg}}$ into \mathcal{N} . Applying the functor j^* , we see that

$$\begin{aligned} d_{(x_{\text{reg}},1),(x_{\text{subreg}},\rho)} &= [\mathbb{F}^p \mathcal{J}_{!*}(\mathcal{O}_{\text{reg}}, \mathbb{O}) : {}^p \mathcal{J}_{!*}(\mathcal{O}_{\text{subreg}}, \rho)] \\ &= [\mathbb{F} j_{\text{reg}!*}(\mathbb{O}[2\nu]) : i_{\text{subreg}*}(\rho[2\nu + 2])] \end{aligned}$$

By Slodowy's theorem and the analysis of Section 2.5, we obtain the following result.

Theorem 4.1.2 *We have*

$$d_{(x_{\text{reg}},1),(x_{\text{subreg}},\rho)} = [\mathbb{F} \otimes_{\mathbb{Z}} P(\widehat{\Phi})/Q(\widehat{\Phi}) : \rho] \tag{4.1}$$

for all ρ in $\text{Irr } \mathbb{F}A$.

For homogeneous types, we recover the decomposition numbers described in section 2.5. Let us describe in detail all the other possibilities. The action of $\text{Aut}(\widehat{\Phi})/W(\widehat{\Phi})$ on $P(\widehat{\Phi})/Q(\widehat{\Phi})$ is described in all types in [Bou68, Chap. VI, §4].

In the types B_n , C_n and F_4 , we have $A \simeq \mathbb{Z}/2$. When $\ell = 2$, we have $\text{Irr } \mathbb{F}A = \{1\}$. In this case, we would not even need to know the actual action, since for our purposes we only need the class in the Grothendieck group $K_0(\mathbb{F}A) \simeq \mathbb{Z}$, that is, the dimension. When ℓ is not 2, we have $\text{Irr } \mathbb{F}A = \{1, \varepsilon\}$, where ε is the unique non-trivial character of $\mathbb{Z}/2$.

4.1.1 Case $\Gamma = B_n$

We have $\widehat{\Gamma} = A_{2n-1}$ and $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq \mathbb{Z}/2n$. The non-trivial element of $A \simeq \mathbb{Z}/2$ acts by -1 . Thus we have

$$\begin{aligned} \text{If } \ell = 2, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 1 \\ \text{If } 2 \neq \ell \mid n, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 0 \quad \text{and} \quad d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = 1 \\ \text{If } 2 \neq \ell \nmid n, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 0 \quad \text{and} \quad d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = 0 \end{aligned}$$

4.1.2 Case $\Gamma = C_n$

We have $\widehat{\Gamma} = D_{n+1}$.

If n is even, then we have $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq \mathbb{Z}/4$, and the nontrivial element of $A \simeq \mathbb{Z}/2$ acts by -1 .

If n is odd, then we have $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq (\mathbb{Z}/2)^2$, and the nontrivial element of $A \simeq \mathbb{Z}/2$ acts by exchanging two nonzero elements.

Thus we have

$$\begin{aligned} \text{If } \ell = 2 \text{ and } n \text{ is even,} & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 1 \\ \text{If } \ell = 2 \text{ and } n \text{ is odd,} & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 2 \\ \text{If } \ell \neq 2, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = 0 \end{aligned}$$

4.1.3 Case $\Gamma = F_4$

We have $\widehat{\Gamma} = E_6$ and $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq \mathbb{Z}/3$. The nontrivial element of $A \simeq \mathbb{Z}/2$ acts by -1 . Thus we have

$$\begin{aligned} \text{If } \ell = 2, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 0 \\ \text{If } \ell = 3, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 0 \quad \text{and} \quad d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = 1 \\ \text{If } \ell > 3, & \quad \text{then } d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 0 \quad \text{and} \quad d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = 0 \end{aligned}$$

4.1.4 Case $\Gamma = G_2$

We have $\widehat{\Gamma} = D_4$ and $P(\widehat{\Phi})/Q(\widehat{\Phi}) \simeq (\mathbb{Z}/2)^2$. The group $A \simeq \mathfrak{S}_3$ acts by permuting the three non-zero elements. Let us denote the sign character by ε (it is nontrivial when

$\ell \neq 2$), and the degree two character by ψ (it remains irreducible for $\ell = 2$, but for $\ell = 3$ it decomposes as $1 + \varepsilon$). We have

$$\begin{aligned} \text{If } \ell = 2, \text{ then } & d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = 0 \quad \text{and} \quad d_{(x_{\text{reg}},1),(x_{\text{subreg}},\psi)} = 1 \\ \text{If } \ell = 3, \text{ then } & d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = 0 \\ \text{If } \ell > 3, \text{ then } & d_{(x_{\text{reg}},1),(x_{\text{subreg}},1)} = d_{(x_{\text{reg}},1),(x_{\text{subreg}},\varepsilon)} = d_{(x_{\text{reg}},1),(x_{\text{subreg}},\psi)} = 0 \end{aligned}$$

4.2 Minimal class

We assume that G is simple. Thus there is a unique minimal non-zero nilpotent orbit \mathcal{O}_{\min} in \mathfrak{g} (corresponding to the highest weight of the adjoint representation, the highest root $\tilde{\alpha}$). It is of dimension $d = 2h^\vee - 2$, where h^\vee is the dual Coxeter number (see Chapter 3).

Its closure $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$ is a cone with origin 0. Let $j_{\min} : \mathcal{O}_{\min} \rightarrow \overline{\mathcal{O}}_{\min}$ be the open immersion, and $i_0 : \{0\} \rightarrow \overline{\mathcal{O}}_{\min}$ the closed complement. By Proposition 2.3.1, we have

$$i_0^* j_{\min*}(\mathbb{O}[d]) \simeq \bigoplus_i H^{i+d}(\mathcal{O}_{\min}, \mathbb{O})[-i]$$

In Chapter 3, we have calculated all the cohomology of \mathcal{O}_{\min} when the base field is \mathbb{C} , and by the comparison theorems we can deduce the cohomology in the étale case. However we will only need the following results. Remember that Φ' is the root subsystem of Φ generated by the long simple roots.

$$H^{-1} i_0^* j_{\min*}(\mathbb{O}[d]) = H^{2h^\vee-3}(\mathcal{O}_{\min}, \mathbb{O}) = 0 \quad (4.2)$$

$$H^0 i_0^* j_{\min*}(\mathbb{O}[d]) = H^{2h^\vee-2}(\mathcal{O}_{\min}, \mathbb{O}) = \mathbb{O} \otimes_{\mathbb{Z}} (P^\vee(\Phi')/Q^\vee(\Phi')) \quad (4.3)$$

$$H^1 i_0^* j_{\min*}(\mathbb{O}[d]) = H^{2h^\vee-1}(\mathcal{O}_{\min}, \mathbb{O}) \text{ is torsion-free} \quad (4.4)$$

By the distinguished triangles in Sections 1.5 and 1.8, we obtain the following results.

Theorem 4.2.1 *Over \mathbb{O} , we have canonical isomorphisms*

$${}^p j_{\min!}(\mathbb{O}[d]) \simeq {}^{p+} j_{\min!}(\mathbb{O}[d]) \simeq {}^p j_{\min!*}(\mathbb{O}[d]) \quad (4.5)$$

$${}^{p+} j_{\min!*}(\mathbb{O}[d]) \simeq {}^p j_{\min*}(\mathbb{O}[d]) \simeq {}^{p+} j_{\min*}(\mathbb{O}[d]) \quad (4.6)$$

and a short exact sequence

$$0 \longrightarrow {}^p j_{\min!*}(\mathbb{O}[d]) \longrightarrow {}^{p+} j_{\min!*}(\mathbb{O}[d]) \longrightarrow i_{0*} \mathbb{O} \otimes_{\mathbb{Z}} (P^\vee(\Phi')/Q^\vee(\Phi')) \longrightarrow 0 \quad (4.7)$$

Over \mathbb{F} , we have canonical isomorphisms

$$\mathbb{F} {}^p j_{\min!}(\mathbb{O}[d]) \xrightarrow{\sim} {}^p j_{\min!}(\mathbb{F}[d]) \xrightarrow{\sim} \mathbb{F} {}^{p+} j_{\min!}(\mathbb{O}[d]) \xrightarrow{\sim} \mathbb{F} {}^p j_{\min!*}(\mathbb{O}[d]) \quad (4.8)$$

$$\mathbb{F} {}^{p+} j_{\min!*}(\mathbb{O}[d]) \xrightarrow{\sim} \mathbb{F} {}^p j_{\min*}(\mathbb{O}[d]) \xrightarrow{\sim} {}^p j_{\min*}(\mathbb{F}[d]) \xrightarrow{\sim} \mathbb{F} {}^{p+} j_{\min*}(\mathbb{O}[d]) \quad (4.9)$$

and short exact sequences

$$0 \longrightarrow i_{0*} \mathbb{F} \otimes_{\mathbb{Z}} (P^\vee(\Phi')/Q^\vee(\Phi')) \longrightarrow \mathbb{F} {}^p j_{\min!} (\mathbb{O}[d]) \longrightarrow {}^p j_{\min!} (\mathbb{F}[d]) \longrightarrow 0 \quad (4.10)$$

$$0 \longrightarrow {}^p j_{\min!} (\mathbb{F}[d]) \longrightarrow \mathbb{F} {}^{p+} j_{\min!} (\mathbb{O}[d]) \longrightarrow i_{0*} \mathbb{F} \otimes_{\mathbb{Z}} (P^\vee(\Phi')/Q^\vee(\Phi')) \longrightarrow 0 \quad (4.11)$$

We have

$$[\mathbb{F} {}^p j_{\min!} (\mathbb{O}[d]) : i_{0*} \mathbb{F}] = [\mathbb{F} {}^{p+} j_{\min!} (\mathbb{O}[d]) : i_{0*} \mathbb{F}] = \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} (P^\vee(\Phi')/Q^\vee(\Phi')) \quad (4.12)$$

In particular, $\mathbb{F} {}^p j_{\min!} (\mathbb{O}[d])$ is simple (and equal to $\mathbb{F} {}^{p+} j_{\min!} (\mathbb{O}[d])$) if and only if ℓ does not divide the connection index of Φ' .

Let us give this decomposition number in each type. We denote the singularity of $\overline{\mathcal{O}}_{\min}$ at the origin by the lower case letter γ corresponding to the type Γ of \mathfrak{g} .

γ	Γ'	$P^\vee(\Phi')/Q^\vee(\Phi')$	$d_{(x_{\min},1),(0,1)}$
a_n	A_n	$\mathbb{Z}/(n+1)$	1 if $\ell \mid n+1$, 0 otherwise
b_n	A_{n-1}	\mathbb{Z}/n	1 if $\ell \mid n$, 0 otherwise
c_n	A_1	$\mathbb{Z}/2$	1 if $\ell = 2$, 0 otherwise
d_n (n even)	D_n	$(\mathbb{Z}/2)^2$	2 if $\ell = 2$, 0 otherwise
d_n (n odd)	D_n	$\mathbb{Z}/4$	1 if $\ell = 2$, 0 otherwise
e_6	E_6	$\mathbb{Z}/3$	1 if $\ell = 3$, 0 otherwise
e_7	E_7	$\mathbb{Z}/2$	1 if $\ell = 2$, 0 otherwise
e_8	E_8	0	0
f_4	A_2	$\mathbb{Z}/3$	1 if $\ell = 3$, 0 otherwise
g_2	A_1	$\mathbb{Z}/2$	1 if $\ell = 2$, 0 otherwise

Note that the singularities c_n (for $n \geq 1$, including $c_1 = a_1 = A_1$ and $c_2 = b_2$) and g_2 are \mathbb{K} -smooth but not \mathbb{F}_2 -smooth.

4.3 Rows and columns

In this section, $G = GL_n$. The nilpotent orbits are parametrized by the partitions of n , and the order defined by the inclusions of closures of orbits corresponds to the usual dominance order on partitions. Let us introduce some notation. The decomposition number $[\mathbb{F} \mathcal{J}_*(\mathcal{O}_\lambda, \mathbb{O}) : \mathcal{J}_*(\mathcal{O}_\mu, \mathbb{F})]$ will be denoted by $d_{\lambda,\mu}$, and we introduce the ‘‘characteristic functions’’

$$\chi_{\lambda,\mu} = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{K}} \mathcal{H}_{x_\mu}^i \mathcal{J}_*(\mathcal{O}_\lambda, \mathbb{K}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{F}} \mathcal{H}_{x_\mu}^i \mathbb{F} \mathcal{J}_*(\mathcal{O}_\lambda, \mathbb{O})$$

and

$$\phi_{\nu,\mu} = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{F}} \mathcal{H}_{x_\mu}^i \mathcal{J}_*(\mathcal{O}_\nu, \mathbb{F})$$

These form triangular systems, in the sense that $\chi_{\lambda,\mu}$ can be non-zero only if $\mu \leq \lambda$, and $\phi_{\nu,\mu}$ can be non-zero only if $\mu \leq \nu$. We have

$$\chi_{\lambda,\mu} = \sum_{\nu} d_{\lambda,\nu} \phi_{\nu,\mu}$$

and $d_{\lambda,\nu}$ can be non-zero only if $\nu \leq \lambda$. Moreover, we have $\chi_{\lambda,\lambda} = \phi_{\lambda,\lambda} = 1$, and $d_{\lambda,\lambda} = 1$.

Kraft and Procesi found a row and column removal rule for the singularities of the closures of the nilpotent orbits in type A_{n-1} [KP81]. Actually, they state the result when the base field is \mathbb{C} , but it is certainly true when p is very good.

Proposition 4.3.1 *Let $\mathcal{O}_{\mu} \subset \overline{\mathcal{O}}_{\lambda}$ be a degeneration of nilpotent orbits in \mathfrak{gl}_n and assume that the first r rows and the first s columns of λ and μ coincide. Denote by λ_1 and μ_1 the Young diagrams obtained from λ and μ by erasing these rows and columns. Then $\mathcal{O}_{\mu_1} \subset \overline{\mathcal{O}}_{\lambda_1}$, and we have*

$$\text{codim}_{\overline{\mathcal{O}}_{\lambda_1}} \mathcal{O}_{\mu_1} = \text{codim}_{\overline{\mathcal{O}}_{\lambda}} \mathcal{O}_{\mu} \quad \text{and} \quad \text{Sing}(\overline{\mathcal{O}}_{\lambda_1}, \mathcal{O}_{\mu_1}) = \text{Sing}(\overline{\mathcal{O}}_{\lambda}, \mathcal{O}_{\mu}) \quad (4.13)$$

All the partitions ν in the interval $[\mu, \lambda] = \{\nu \mid \mu \leq \nu \leq \lambda\}$ have the same first r rows and first s columns. For ν in $[\mu, \lambda]$, let us denote by ν_1 the partition obtained from ν by erasing them.

The proposition implies that, for all $\eta \leq \zeta$ in $[\mu, \lambda]$, the local intersection cohomology of $\overline{\mathcal{O}}_{\eta_1}$ at $\overline{\mathcal{O}}_{\zeta_1}$ is the same as the local intersection cohomology of $\overline{\mathcal{O}}_{\eta}$ at $\overline{\mathcal{O}}_{\zeta}$, both over \mathbb{K} and over \mathbb{F} , and thus $\chi_{\eta,\zeta} = \chi_{\eta_1,\zeta_1}$ and $\phi_{\eta,\zeta} = \phi_{\eta_1,\zeta_1}$.

Since the decomposition numbers can be deduced from this information (for GL_n , we only have trivial $A_G(x)$), we find a row and column removal rule for the decomposition numbers for GL_n -equivariant perverse sheaves on the nilpotent variety of GL_n .

Proposition 4.3.2 *With the notations above, we have*

$$[\mathbb{F} \mathcal{J}_{!*}(\mathcal{O}_{\lambda}, \mathbb{O}) : \mathcal{J}_{!*}(\mathcal{O}_{\mu}, \mathbb{F})] = [\mathbb{F} \mathcal{J}_{!*}(\mathcal{O}_{\lambda_1}, \mathbb{O}) : \mathcal{J}_{!*}(\mathcal{O}_{\mu_1}, \mathbb{F})] \quad (4.14)$$

Proof. The decomposition numbers $(d_{\eta,\zeta})_{\mu \leq \zeta \leq \eta \leq \lambda}$ are the unique solution of the triangular linear system

$$\chi_{\eta,\zeta} = \sum_{\mu \leq \nu \leq \lambda} d_{\eta,\nu} \phi_{\nu,\zeta}$$

whereas the decomposition numbers $(d_{\eta_1,\zeta_1})_{\mu_1 \leq \zeta_1 \leq \eta_1 \leq \lambda_1}$ are the unique solution of the triangular linear system

$$\chi_{\eta_1,\zeta_1} = \sum_{\mu \leq \nu \leq \lambda} d_{\eta_1,\nu_1} \phi_{\nu_1,\zeta_1}$$

Since we have $\chi_{\eta,\zeta} = \chi_{\eta_1,\zeta_1}$ and $\phi_{\eta,\zeta} = \phi_{\eta_1,\zeta_1}$, the two systems are identical, so that $d_{\eta,\zeta} = d_{\eta_1,\zeta_1}$ for all $\zeta \leq \eta$ in $[\mu, \lambda]$. In particular, we have $d_{\lambda,\mu} = d_{\lambda_1,\mu_1}$. \square

So we have a rule similar to the one for decomposition matrices for the symmetric groups, and even for the Schur algebra. Now, if we have two adjacent partitions, using this rule we can reduce to the extreme cases of the minimal or subregular class for a smaller general linear group. So we get the following result.

Corollary 4.3.3 *The decomposition numbers for the GL_n -equivariant perverse sheaves on the nilpotent variety and for the Schur algebra coincide for adjacent partitions.*

In 6.3 we will see that, at least for the part of the decomposition matrix corresponding to the symmetric group, this is not a coincidence. We hope to find an explanation for the whole decomposition matrix of the Schur algebra as well. We will come back to this later.

In [KP82] Kraft and Procesi obtain a similar result for all classical types, so the same rule should apply for perverse sheaves in the classical types. The only problem that is left is to deal with the local systems involved, when the $A_G(x)$ are non-trivial. In many cases, we have enough information (see the introduction, and the tables in Chapter 7).

4.4 Special classes

We will use another geometrical result by Kraft and Procesi, contained in [KP89].

In [Lus79], Lusztig introduced the special representations of a finite Weyl group. The *special unipotent classes* of a simple group are the unipotent classes C such that the representation of W corresponding to the pair $(C, 1)$ is *special*.

On the other hand, Spaltenstein introduced an order-reversing map d from the set of unipotent classes to itself, such that $d^3 = d$ (it is an involution on its image) in [Spa78]. The image of d consists exactly in the special unipotent classes, and the locally closed subvarieties

$$\widehat{C} = \overline{C} \setminus \bigcup_{\substack{C' \text{ special} \\ \overline{C'} \subset \overline{C}}} \overline{C'}$$

where C runs through the special classes, form a partition of the unipotent variety (any unipotent class is contained in a \widehat{C} for a unique special class C).

If p is good for G (or if the base field is \mathbb{C}), then we can use a G -equivariant homeomorphism from the unipotent variety to the nilpotent variety to transport these notions to nilpotent orbits.

In type A , all the unipotent classes (resp. nilpotent orbits) are special, so this section gives information only for the other classical types.

Theorem 4.4.1 *Let C be a special unipotent conjugacy class of a classical group. Define \widehat{C} as above. Then \widehat{C} consists of 2^d conjugacy classes, where d is the number of irreducible components of $\widehat{C} \setminus C$. There is a smooth variety Y with an action of the group $(\mathbb{Z}/2)^d$, and an isomorphism*

$$Y/(\mathbb{Z}/2)^d \xrightarrow{\sim} \widehat{C} \tag{4.15}$$

which identifies the stratification of \widehat{C} with the stratification of the quotient by isotropy groups. (These are the 2^d subproducts of $(\mathbb{Z}/2)^d$). In particular \widehat{C} is a rational homology manifold.

But actually their result gives more information. By Proposition 2.4.2, the variety \widehat{C} is not only \mathbb{K} -smooth, but also \mathbb{F} -smooth for $\ell \neq 2$.

If p is good, the result can be transferred to the nilpotent variety, to fit with our context. We obtain the following result.

Proposition 4.4.2 *Assume p is good. Let \mathcal{O} be a special nilpotent orbit in a classical group. Let*

$$\widehat{\mathcal{O}} = \overline{\mathcal{O}} \setminus \bigcup_{\substack{\mathcal{O}' \text{ special} \\ \overline{\mathcal{O}'} \subset \overline{\mathcal{O}}}} \overline{\mathcal{O}'}$$

Assume $\ell \neq 2$. Then $\widehat{\mathcal{O}}$ is \mathbb{F} -smooth and, for all orbits $\mathcal{O}' \subset \widehat{\mathcal{O}} \setminus \mathcal{O}$, we have

$$[\mathbb{F}\mathcal{J}_!^*(\mathcal{O}, \mathbb{O}) : \mathcal{J}_!^*(\mathcal{O}', \mathbb{F})] = 0$$

Chapter 5

Fourier-Deligne transform

Our aim is to define a Springer correspondence modulo ℓ . The fact that the Weyl group W acts on the cohomology spaces $H^i(\mathcal{B}_x, \mathbb{E})$ is true for $\mathbb{E} = \mathbb{K}, \mathbb{O}$ or \mathbb{F} , with the same proof. So we have “Springer representations”. But the main point, in characteristic zero, is that all $E \in \text{Irr } \mathbb{K}W$ appear in such representations, and that we can associate to each E a pair $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is a nilpotent orbit and \mathcal{L} is a G -equivariant simple perverse sheaf on \mathcal{N} .

With \mathbb{K} coefficients, there are many versions of the Springer correspondence. We had to choose a version which can be adapted to \mathbb{F} coefficients. The main obstacle is that Gabber’s decomposition theorem [BBD82] is no longer true, already for a finite étale covering. In this case taking the direct image is like inducing from $\mathbb{F}H$ to $\mathbb{F}K$ where K is a finite group and H is a subgroup of K . If we start with the constant sheaf \mathbb{F} , we get the local system corresponding to the representation $\mathbb{F}[K/H]$ of K , which is not semi-simple if ℓ divides $|G : H|$.

However, the approach of Hotta-Kashiwara [HK84] uses a Fourier transform for \mathcal{D} -modules in the complex case, thus avoiding the decomposition theorem. Brylinski [Bry86] adapted this method to the case of a base field of characteristic p , with ℓ -adic coefficients, using the Fourier-Deligne transform. This transform actually makes sense over \mathbb{K}, \mathbb{O} and \mathbb{F} . We are going to use it in chapter 6 to define a modular Springer correspondence. It would be interesting to understand better the modular version of the Lusztig-Bohro-MacPherson approach [BM81, Lus81, Lus84], where a restriction to the nilpotent variety is used. We hope that, in type A , it will be possible to involve the Schur algebra.

In this chapter, we define Fourier-Deligne transforms and study their properties, following Laumon [Lau87], but with coefficients \mathbb{E} instead of $\overline{\mathbb{Q}}_\ell$. No particular difficulty arises. We shall give more details than in Laumon’s article. In particular, when we have a chain of functorial isomorphisms, we will justify each step. However, we do not give a proof for the most difficult result, Theorem 5.3.1 (SUPP), since it appears in [KL85] with torsion coefficients.

The use of the Fourier-Deligne transform is the reason why we had to consider a base field of characteristic p . Otherwise, we could have used complex varieties instead. From now on, k is the finite field \mathbb{F}_q . Remember that, in section 1.1, we assumed that all schemes we consider are varieties, that is, separated schemes of finite type over k .

5.1 Preliminaries

Since we will justify each step in the proofs, we are going to use abbreviations for the main theorems or properties that we use.

If we have a commutative square of varieties

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \pi' \downarrow & \square & \downarrow \pi \\ S' & \xrightarrow{f} & S \end{array}$$

then we have canonical isomorphisms

$$\begin{array}{ll} f'^* \pi^* \simeq \pi'^* f^* & \text{COM}^*(\square) \\ \pi_* f'_* \simeq f_* \pi'_* & \text{COM}_*(\square) \\ \pi_! f'_! \simeq f_! \pi'_! & \text{COM}_!(\square) \\ f'^! \pi^! \simeq \pi'^! f^! & \text{COM}^!(\square) \end{array}$$

For example, we have $f'^* \pi^* \simeq (\pi \circ f')^* \simeq (f \circ \pi')^* \simeq \pi'^* f^*$, and similarly for the other relations. We will use similar notations for other kinds of diagrams, like triangles.

If moreover the square \square is cartesian, then we can apply the Proper Base Change Theorem (because of the assumptions on the schemes we made in 1.1). Thus we have a canonical isomorphism

$$f^* \pi_! \simeq \pi'_! f'^* \quad \text{PBCT}(\square)$$

In other words, the functor $\pi_!$ commutes with any base change. This theorem is stated and proved in [GV72, XVII, Th. 5.2.6] for \mathbb{E} -sheaves when \mathbb{E} is a finite ring of order prime to p . We will freely use it for the derived categories $D_c^b(X, \mathbb{E})$, where $\mathbb{E} = \mathbb{K}, \mathbb{O}$ or \mathbb{F} . Recall that we omit the R for the derived functors associated to a morphism.

The same remarks apply for the Projection Formula [GV72, XVII, Prop. 5.2.9]. Let $\pi : X \rightarrow S$ be a morphism. Then we have a functorial isomorphism

$$L \otimes_{\mathbb{E}}^{\mathbb{L}} \pi_!(K) \simeq \pi_!(\pi^* L \otimes_{\mathbb{E}}^{\mathbb{L}} K) \quad \text{PROJ}(\pi)$$

for (K, L) in $D_c^b(X, \mathbb{E}) \times D_c^b(S, \mathbb{E})$.

We also have a “distributivity” functorial isomorphism

$$\pi^*(K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} K_2) \simeq \pi^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \pi^* K_2 \quad \text{DISTR}(\pi)$$

for (K_1, K_2) in $D_c^b(S, \mathbb{E}) \times D_c^b(S, \mathbb{E})$.

More generally, suppose we have a commutative diagram (with cartesian square)

$$\begin{array}{ccc}
 & Z & \\
 \pi_1 \swarrow & \downarrow \pi & \searrow \pi_2 \\
 & X_1 \times_S X_2 & \\
 p_1 \swarrow & & \searrow p_2 \\
 X_1 & & X_2 \\
 q_1 \swarrow & & \searrow q_2 \\
 & S &
 \end{array}$$

For (K_1, K_2) in $D_c^b(X_1, \mathbb{E}) \times D_c^b(X_2, \mathbb{E})$, we set

$$K_1 \boxtimes_S^{\mathbb{L}} K_2 := p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} p_2^* K_2$$

Then we have a functorial isomorphism

$$\pi^*(K_1 \boxtimes_S^{\mathbb{L}} K_2) \simeq \pi_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \pi_2^* K_2$$

for (K_1, K_2) in $D_c^b(X_1, \mathbb{E}) \times D_c^b(X_2, \mathbb{E})$. Indeed, we have

$$\pi^*(K_1 \boxtimes_S^{\mathbb{L}} K_2) = \pi^*(p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} p_2^* K_2) = \pi^* p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \pi^* p_2^* K_2 = \pi_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \pi_2^* K_2$$

(using the definition, $\text{DISTR}(\pi)$, and COM^* of the two triangles).

If we use the notation $\pi = (\pi_1, \pi_2)$, this can be rewritten as follows:

$$(\pi_1, \pi_2)^*(K_1 \boxtimes_S^{\mathbb{L}} K_2) \simeq \pi_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \pi_2^* K_2 \quad \text{DISTR}(\pi_1, \pi_2)$$

We will also need the Künneth formula [GV72, XVII, Th. 5.4.3]. Suppose we have a commutative diagram

$$\begin{array}{ccccc}
 & Y_1 \times_S Y_2 & & & \\
 \pi_1 \swarrow & \downarrow f & \searrow \pi_2 & & \\
 Y_1 & X_1 \times_S X_2 & Y_2 & & \\
 f_1 \downarrow & p_1 \swarrow & \searrow p_2 & \downarrow f_2 & \\
 X_1 & & X_2 & & \\
 q_1 \swarrow & & \searrow q_2 & & \\
 & S & & &
 \end{array}$$

Then we have a functorial isomorphism

$$f_{1!} K_1 \boxtimes_S^{\mathbb{L}} f_{2!} K_2 \simeq f_!(K_1 \boxtimes_S^{\mathbb{L}} K_2) \quad \text{KUNNETH}$$

for (K_1, K_2) in $D_c^b(Y_1, \mathbb{E}) \times D_c^b(Y_2, \mathbb{E})$.

Finally, we will need the duality theorems [KW01, II.7] Let $f : Y \rightarrow X$ be a morphism. Then we have a functorial isomorphism

$$\underline{\mathrm{RHom}}(f_!L, K) \simeq f_* \underline{\mathrm{RHom}}(L, f^!K) \quad \text{DUAL}(f)$$

for (K, L) in $D_c^b(X, \mathbb{E}) \times D_c^b(Y, \mathbb{E})$. As a consequence, we have the second duality theorem: we have a functorial isomorphism

$$f^! \underline{\mathrm{RHom}}(K_1, K_2) \simeq \underline{\mathrm{RHom}}(f^*K_1, f^!K_2) \quad \text{DUAL}_2(f)$$

5.2 Definition, first properties and examples

5.2.1 Definition

Let us assume that \mathbb{E}^\times contains a primitive root of unity of order p . We fix a non-trivial character $\psi : \mathbb{F}_p \rightarrow \mathbb{E}^\times$, that is, a primitive root $\psi(1)$ of order p in \mathbb{E}^\times . Composing with $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, we get a character of \mathbb{F}_q . Let \mathcal{L}_ψ be the locally constant \mathbb{E} -sheaf of rank 1 on \mathbb{G}_a associated to ψ (the corresponding Artin-Schreier local system). It is endowed with a “rigidification” at the origin:

$$(\mathcal{L}_\psi)|_0 \simeq \mathbb{E} \quad \text{RIG}$$

Moreover, we have an isomorphism

$$\mathcal{L}_\psi \boxtimes^{\mathbb{L}} \mathcal{L}_\psi \simeq m^* \mathcal{L}_\psi \quad \text{ADD}$$

where $m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the addition morphism. If $f, g : X \rightarrow \mathbb{A}^1$ are two morphisms, we set $f + g = m \circ (f, g)$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow f & \downarrow (f,g) & \searrow g & \\
 & & \mathbb{A}^1 \times \mathbb{A}^1 & & \\
 & \swarrow \mathrm{pr}_1 & \downarrow m & \searrow \mathrm{pr}_2 & \\
 \mathbb{A}^1 & & \mathbb{A}^1 & & \mathbb{A}^1
 \end{array}$$

Then the following chain of isomorphisms:

$$\begin{aligned}
 (f + g)^* \mathcal{L}_\psi &\simeq (f, g)^* m^* \mathcal{L}_\psi \\
 &\simeq (f, g)^* (\mathcal{L}_\psi \boxtimes^{\mathbb{L}} \mathcal{L}_\psi) \quad \text{by ADD} \\
 &\simeq f^* \mathcal{L}_\psi \otimes_{\mathbb{E}}^{\mathbb{L}} g^* \mathcal{L}_\psi \quad \text{by DISTR}(f, g)
 \end{aligned}$$

When we use this result, we will simply refer to it as ADD.

Now let S be a variety, and $E \xrightarrow{\pi} S$ a vector bundle of constant rank $r \geq 1$. We denote by $E' \xrightarrow{\pi'} S$ its dual vector bundle, by $\mu : E \times_S E' \rightarrow \mathbb{A}^1$ the canonical pairing,

and by $\text{pr} : E \times_S E' \rightarrow E$ and $\text{pr}' : E \times_S E' \rightarrow E'$ the canonical projections. So we have the following diagram.

$$\begin{array}{ccccc}
 & & E \times_S E' & \xrightarrow{\mu} & \mathbb{A}^1 \\
 & \swarrow \text{pr} & & \searrow \text{pr}' & \\
 E & & \square & & E' \\
 & \searrow \pi & & \swarrow \pi' & \\
 & & S & &
 \end{array}$$

Definition 5.2.1 *The Fourier-Deligne transform for $E \xrightarrow{\pi} S$, associated to the character ψ , is the triangulated functor*

$$\mathcal{F}_\psi : D_c^b(E, \mathbb{E}) \longrightarrow D_c^b(E', \mathbb{E})$$

defined by

$$\mathcal{F}_\psi(K) = \text{pr}'_!(\text{pr}^* K \otimes_{\mathbb{E}} \mu^* \mathcal{L}_\psi)[r]$$

In the sequel, we will drop the indices ψ from the notations \mathcal{F}_ψ and \mathcal{L}_ψ when no confusion may arise.

5.2.2 First properties

Let $E'' \xrightarrow{\pi''} S$ be the bidual vector bundle of $E \xrightarrow{\pi} S$ and $a : E \xrightarrow{\sim} E''$ the S -isomorphism defined by $a(e) = -\mu(e, -)$ (that is, the opposite of the canonical S -isomorphism). We will denote by $\sigma : S \rightarrow E$, $\sigma' : S \rightarrow E'$ and $\sigma'' : S \rightarrow E''$ the respective null sections of π , π' and π'' . Finally, we will denote by $s : E \times_S E \rightarrow E$ the addition of the vector bundle $E \xrightarrow{\pi} S$ and by $-1_E : E \rightarrow E$ the opposite for this addition.

The following Proposition is the analogue of the fact that the Fourier transform of the constant function is a Dirac distribution supported at the origin in classical Fourier analysis. By the function/sheaf dictionary, this becomes a functorial isomorphism, to which we will refer as DIRAC.

Proposition 5.2.2 *We have a functorial isomorphism*

$$\mathcal{F}(\pi^* L[r]) \simeq \sigma'_* L(-r) \quad (\text{DIRAC})$$

for all objects L in $D_c^b(S, \mathbb{E})$.

Proof. We have the following diagram

$$\begin{array}{ccccc}
 & & \mathbb{A}^1 & & \\
 & i_0 \nearrow & \uparrow \mu & & \\
 0 & \square_0 & E \times_S E' & & \\
 \uparrow p_0 & \text{pr} \searrow & \downarrow \text{pr}' & & \\
 E & \xrightarrow{(1_E, \sigma' \pi)} & E' & & \\
 \downarrow \pi & \square_{\sigma'} & \downarrow \pi' & & \\
 & & S & &
 \end{array}$$

where \square denotes the cartesian square containing pr and π' , and $\square_{\sigma'}$ the cartesian square containing $(1_E, \sigma' \pi)$ and σ' . The square \square_0 containing i_0 and $(1_E, \sigma' \pi)$ is commutative. Moreover, σ' is a section of π' , and $(1_E, \sigma' \pi)$ is a section of pr .

We begin by showing that $\text{pr}'_! \mu^* \mathcal{L} = \sigma'_* \mathbb{E}_S[-2r](-r)$. Let us first prove that $\text{pr}'_! \mu^* \mathcal{L}$ is concentrated on the closed subset $\sigma'(S)$ of E' . Let $e' \in E' - \sigma'(S)$. Set $s = \pi'(e')$. We have a commutative diagram with cartesian square

$$\begin{array}{ccc}
 & \mu(-, e') \nearrow & \mathbb{A}^1 \\
 E_s \times e' & \xrightarrow{i_{E_s \times e'}} & E \times_S E' \\
 \text{pr}'_{e'} \downarrow & \square_{e'} & \downarrow \text{pr}' \\
 e' & \xrightarrow{i_{e'}} & E'
 \end{array}$$

Fix a basis (e'_1, \dots, e'_r) of E'_s such that $e'_1 = e'$, and let (e_1, \dots, e_r) be the dual basis of E_s . We have

$$\begin{aligned}
 (\text{pr}'_! \mu^* \mathcal{L})_{e'} &= \text{R}\Gamma_c(E_s \times e', i_{E_s \times e'}^* \mu^* \mathcal{L}) && \text{by PBCT}(\square_{e'}) \\
 &= \text{R}\Gamma_c(E_s, \mu(-, e')^* \mathcal{L}) && \text{by COM}^*(\Delta_{e'}) \\
 &= \text{R}\Gamma_c((\mathbb{A}^1)^r, \mathcal{L} \boxtimes^{\mathbb{L}} \mathbb{E} \boxtimes^{\mathbb{L}} \dots \boxtimes^{\mathbb{L}} \mathbb{E}) && \text{by the choice of the basis} \\
 &= \text{R}\Gamma_c(\mathbb{A}^1, \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} \text{R}\Gamma_c(\mathbb{A}^1, \mathbb{E}) \otimes_{\mathbb{E}}^{\mathbb{L}} \dots \otimes_{\mathbb{E}}^{\mathbb{L}} \text{R}\Gamma_c(\mathbb{A}^1, \mathbb{E}) && \text{by KUNNETH} \\
 &= 0
 \end{aligned}$$

since $\text{R}\Gamma_c(\mathbb{A}^1, \mathcal{L}) = 0$. Thus $\text{pr}'_! \mu^* \mathcal{L}$ is concentrated in the closed subset $\sigma'(S)$ of E' , and we have

$$\begin{aligned}
 \text{pr}'_! \mu^* \mathcal{L} &= \sigma'_* \sigma'^* \text{pr}'_! \mu^* \mathcal{L} && \text{by the above} \\
 &= \sigma'_* \pi_! (1_E, \sigma' \pi)^* \mu^* \mathcal{L} && \text{by PBCT}(\square_{\sigma'}) \\
 &= \sigma'_* \pi_! p_0^* i_0^* \mathcal{L} && \text{by COM}^*(\square_0) \\
 &= \sigma'_* \pi_! p_0^* \mathbb{E} && \text{by RIG} \\
 &= \sigma'_* \pi_! \mathbb{E}_E \\
 &= \sigma'_* \mathbb{E}_S[-2r](-r)
 \end{aligned} \tag{5.1}$$

5.2 Definition, first properties and examples

We are now ready to estimate the Fourier-Deligne transform of $\pi^*L[r]$.

$$\begin{aligned}
 \mathcal{F}(\pi^*L[r]) &= \mathrm{pr}'_1(\mathrm{pr}^* \pi^*L[r] \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})[r] && \text{by definition} \\
 &= \mathrm{pr}'_1(\mathrm{pr}'^* \pi'^*L \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})[2r] && \text{by COM}^*(\square) \\
 &= \pi'^*L \otimes_{\mathbb{E}}^{\mathbb{L}} \mathrm{pr}'_1 \mu^* \mathcal{L}[2r] && \text{by PROJ}(\mathrm{pr}') \\
 &= \pi'^*L \otimes_{\mathbb{E}}^{\mathbb{L}} \sigma'_* \mathbb{E}_S(-r) && \text{by the above} \\
 &= \sigma'_*(\sigma'^* \pi'^*L \otimes_{\mathbb{E}}^{\mathbb{L}} \mathbb{E}_S)(-r) && \text{by PROJ}(\sigma') \\
 &= \sigma'_*L(-r)
 \end{aligned}$$

□

Theorem 5.2.3 *Let \mathcal{F}' be the Fourier-Deligne transform, associated to the character ψ , of the vector bundle $E' \xrightarrow{\pi'} S$. Then we have a functorial isomorphism*

$$\mathcal{F}' \circ \mathcal{F}(K) \simeq a_*K(-r) \quad (\mathrm{INV})$$

for all objects K in $D_c^b(E, \mathbb{E})$.

Proof. We have a commutative diagram (with plain arrows) with cartesian squares $\square_{1,2,3}$

$$\begin{array}{ccccc}
 & & E \times_S E'' & & \\
 & \xleftarrow{\mathrm{pr}_1^{13}} & & \xrightarrow{\mathrm{pr}_3^{13}} & \\
 & & E & & E'' \\
 & & \square_1 & & \square_{3,3'} \\
 & & \uparrow \mathrm{pr}_{13} & & \\
 & & E \times_S E' \times_S E'' & & \\
 & \xleftarrow{\mathrm{pr}_{12}} & & \xrightarrow{\mathrm{pr}_{23}} & \\
 & & E \times_S E' & & E' \times_S E'' \\
 & & \square_2 & & \\
 & & \downarrow \mathrm{pr}_2^{12} & & \downarrow \mathrm{pr}_2^{23} \\
 & & E' & &
 \end{array}$$

and the square \square'_3 containing the dashed arrows α and β is also cartesian, with $\alpha(e, e', e'') = (e', e'' - a(e))$ and $\beta(e, e'') = e'' - a(e)$. We have $\mu \mathrm{pr}_{12} + \mu' \mathrm{pr}_{23} = \mu' \alpha$, where $\mu' : E' \times_S E'' \rightarrow \mathbb{A}^1$ is the canonical pairing. Indeed, we have

$$\mu' \circ \alpha(e, e', e'') = \mu'(e', e'' - a(e)) = \mu'(e', e'') - \mu'(e', a(e)) = \mu'(e', e'') + \mu(e, e')$$

Therefore we have

$$\begin{aligned}
 \mathrm{pr}_{12}^* \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} \mathrm{pr}_{23}^* \mu'^* \mathcal{L} &= (\mu \mathrm{pr}_{12})^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} (\mu' \mathrm{pr}_{23})^* \mathcal{L} \\
 &= (\mu \mathrm{pr}_{12} + \mu' \mathrm{pr}_{23})^* \mathcal{L} && \text{by ADD} \\
 &= (\mu' \alpha)^* \mathcal{L} \\
 &= \alpha^* \mu'^* \mathcal{L}
 \end{aligned}$$

Let us now start to calculate $\mathcal{F}' \circ \mathcal{F}(K)$.

$$\begin{aligned}
 \mathcal{F}' \circ \mathcal{F}(K) &= \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_2^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{12} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L})[2r] && \text{by definition} \\
 &= \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_{23} \! \! \! \dashv \! \! \! \mathrm{pr}_{12} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{12} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L})[2r] && \text{by PBCT}(\square_2) \\
 &= \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_{12} \! \! \! \dashv \! \! \! \mathrm{pr}_1^{12} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L})[2r] && \text{by DISTR} \\
 &= \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_{13} \! \! \! \dashv \! \! \! \mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L})[2r] && \text{by COM}^*(\square_1) \\
 &= \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! \mathrm{pr}_{23} \! \! \! \dashv \! \! \! ((\mathrm{pr}_{13} \! \! \! \dashv \! \! \! \mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} \mathrm{pr}_{23} \! \! \! \dashv \! \! \! \mu'^* \mathcal{L})[2r] && \text{by PROJ}(\mathrm{pr}_{23}) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! \mathrm{pr}_{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_{13} \! \! \! \dashv \! \! \! \mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} \mathrm{pr}_{23} \! \! \! \dashv \! \! \! \mu'^* \mathcal{L})[2r] && \text{by COM}_1(\square_3) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \mathrm{pr}_{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_{12} \! \! \! \dashv \! \! \! \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} \mathrm{pr}_{23} \! \! \! \dashv \! \! \! \mu'^* \mathcal{L})) [2r] && \text{by PROJ}(\mathrm{pr}_{13}) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \mathrm{pr}_{13} \! \! \! \dashv \! \! \! \alpha^* \mu'^* \mathcal{L}) [2r] && \text{by the above} \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! \mu'^* \mathcal{L}) [2r] && \text{by PBCT}(\square'_3) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* (\mathbb{E}_{E''}[r] \otimes_{\mathbb{E}}^{\mathbb{L}} \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! \mu'^* \mathcal{L}) [r]) && \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! \mathbb{E}_{E''}[r] \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L}) [r]) && \text{by PROJ}(\mathrm{pr}_3^{23}) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathbb{E}_{E' \times_S E''}[r] \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L}) [r]) && \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* \mathrm{pr}_3^{23} \! \! \! \dashv \! \! \! (\mathrm{pr}_2^{23} \! \! \! \dashv \! \! \! \mathbb{E}_{E'}[r] \otimes_{\mathbb{E}}^{\mathbb{L}} \mu'^* \mathcal{L}) [r]) && \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* \mathcal{F}'(\mathbb{E}_{E'}[r])) && \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! \beta^* \sigma'' \mathbb{E}_S(-r)) && \text{by DIRAC}
 \end{aligned}$$

Now, we have a cartesian square (which we will denote by \square)

$$\begin{array}{ccc}
 E & \xrightarrow{\pi} & S \\
 \delta \downarrow & & \downarrow \sigma'' \\
 E \times_S E'' & \xrightarrow{\beta} & E''
 \end{array}$$

where $\delta(e) = (e, a(e))$, and σ'' (hence δ) is finite and proper since it is a closed immersion. Thus we have

$$\begin{aligned}
 \mathcal{F}' \circ \mathcal{F}(K) &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \delta_* \pi^* \mathbb{E}_S(-r)) && \text{by PBCT}(\square) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! (\mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \delta_* \mathbb{E}_E(-r)) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! \delta_* (\delta^* \mathrm{pr}_1^{13} \! \! \! \dashv \! \! \! K \otimes_{\mathbb{E}}^{\mathbb{L}} \mathbb{E}_E(-r)) && \text{by PROJ}(\delta) \\
 &= \mathrm{pr}_3^{13} \! \! \! \dashv \! \! \! \delta_* (K(-r)) && \text{since } \mathrm{pr}_1^{13} \circ \delta = 1_E \\
 &= a_*(K(-r)) && \text{since } \mathrm{pr}_3^{13} \circ \delta = a
 \end{aligned}$$

The proof is complete. \square

Corollary 5.2.4 *The triangulated functor \mathcal{F} is an equivalence of triangulated categories from $D_c^b(E, \mathbb{E})$ to $D_c^b(E', \mathbb{E})$, with quasi-inverse $a^* \mathcal{F}'(-)(r)$.*

Theorem 5.2.5 *Let $f : E_1 \rightarrow E_2$ be a morphism of vector bundles over S , with constant ranks r_1 and r_2 respectively, and let $f' : E'_2 \rightarrow E'_1$ denote the transposed morphism. Then we have a functorial isomorphism*

$$\mathcal{F}_2(f! K_1) \simeq f'^* \mathcal{F}_1(K_1)[r_2 - r_1] \quad (\text{MOR})$$

for K_1 in $D_c^b(E_1, \mathbb{E})$.

Proof. First, by adjunction of f and f' , we have

$$\mu_2 \circ (f \times_S 1_{E'_2}) = \mu_1 \circ (1_{E_1} \times_S f')$$

where μ_i , $i = 1, 2$, is the pairing $E_i \times_S E'_i \rightarrow \mathbb{A}^1$. That is, the following diagram, which we will denote by \square , is commutative.

$$\begin{array}{ccc} E_1 \times_S E'_2 & \xrightarrow{1_{E_1} \times_S f'} & E_1 \times_S E'_1 \\ f \times_S 1_{E'_2} \downarrow & & \downarrow \mu_1 \\ E_2 \times_S E'_2 & \xrightarrow{\mu_2} & \mathbb{A}^1 \end{array}$$

We also have a commutative diagram with cartesian squares and commutative triangles

$$\begin{array}{ccccc} & & E_1 \times_S E'_1 & & \\ & \text{pr}_1 \nearrow \Delta & & \searrow \Delta' & \text{pr}'_1 \\ & E_1 \times_S E'_2 & \square_1 & E_2 \times_S E'_1 & \\ & p_1 \nearrow & f \times_S 1_{E'_2} \searrow & 1_{E_2} \times_S f' \nearrow & p'_1 \searrow \\ E_1 & & E_2 \times_S E'_2 & & E'_1 \\ & f \searrow & & \nearrow f' & \\ & E_2 & & E'_2 & \end{array}$$

The result is proved by successive applications of the proper base change theorem and of the projection formula, following the diagram:

$$\begin{aligned} & \mathcal{F}_2(f! K_1) \\ &= \text{pr}'_{2!} (\text{pr}_2^* f! K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_2^* \mathcal{L})[r_2] && \text{by definition} \\ &= \text{pr}'_{2!} ((f \times_S 1_{E'_2})! p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_2^* \mathcal{L})[r_2] && \text{by PBCT}(\square_2) \\ &= \text{pr}'_{2!} (f \times_S 1_{E'_2})! (p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} (f \times_S 1_{E'_2})^* \mu_2^* \mathcal{L})[r_2] && \text{by PROJ}(f \times_S 1_{E'_2}) \\ &= \text{pr}'_{2!} (f \times_S 1_{E'_2})! ((1_{E_1} \times_S f')^* \text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} (1_{E_1} \times_S f')^* \mu_1^* \mathcal{L})[r_2] && \text{by COM}^*(\square, \Delta) \\ &= \text{pr}'_{2!} (f \times_S 1_{E'_2})! (1_{E_1} \times_S f')^* (\text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_1^* \mathcal{L})[r_2] && \text{by DISTR}(1_{E_1} \times_S f') \\ &= f'^* p'_1! (f \times_S 1_{E'_1})! (\text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_1^* \mathcal{L})[r_2] && \text{by PBCT}(\square_1, \square'_2) \\ &= f'^* \text{pr}'_{1!} (\text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_1^* \mathcal{L})[r_2] && \text{by COM}_!(\Delta') \\ &= f'^* \mathcal{F}_1(K_1)[r_2 - r_1] \end{aligned}$$

The proof is complete. □

Corollary 5.2.6 *We have a functorial isomorphism*

$$\pi'_! \mathcal{F}(K) \simeq \sigma^* K(-r)[-r] \quad (\text{EV})$$

for K in $D_c^b(E, \mathbb{E})$.

Proof. Consider the morphism $\pi' : E' \rightarrow S$ of vector bundles over S . The transposed morphism is $\sigma'' : S \rightarrow E''$. We can apply Theorem 5.2.5 (MOR) to $E_1 = E'$, $E_2 = S$, $f = \pi'$ and $K_1 = \mathcal{F}(K)$. Then we have $\mathcal{F}_1 = \mathcal{F}'$ and $\mathcal{F}_2 = \text{Id}_{D_c^b(S, \mathbb{E})}$. We get

$$\pi'_! \mathcal{F}(K) = \mathcal{F}_2(\pi'_! \mathcal{F}(K)) = \sigma''^* \mathcal{F}' \circ \mathcal{F}(K)[-r] = \sigma^* a^* a_* K(-r)[-r] = \sigma^* K(-r)[-r]$$

□

Definition 5.2.7 *The convolution product for $E \xrightarrow{\pi} S$ is the internal operation*

$$* : D_c^b(E, \mathbb{E}) \times D_c^b(E, \mathbb{E}) \longrightarrow D_c^b(E, \mathbb{E})$$

defined by

$$K_1 * K_2 = s_!(K_1 \boxtimes_S^{\mathbb{L}} K_2)$$

Proposition 5.2.8 *We have a functorial isomorphism*

$$\mathcal{F}(K_1 * K_2) \simeq \mathcal{F}(K_1) \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K_2)[-r] \quad (\text{CONV})$$

for (K_1, K_2) in $D_c^b(E, \mathbb{E}) \times D_c^b(E, \mathbb{E})$.

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
 & & E \times_S E & & \\
 & & \uparrow P & & \\
 & p_1 & & p_2 & \\
 E & \xleftarrow{\square_1} & E \times_S E' \times_S E \times_S E' & \xrightarrow{\square_2} & E \\
 \uparrow \text{pr} & & \downarrow P' & & \uparrow \text{pr} \\
 E \times_S E' & \xleftarrow{P_1} & E' \times_S E' & \xrightarrow{P_2} & E \times_S E' \\
 \downarrow \text{pr}' & & \downarrow p'_1 & & \downarrow \text{pr}' \\
 E' & \xleftarrow{\square'_1} & E' \times_S E' & \xrightarrow{\square'_2} & E' \\
 & & \downarrow p'_2 & & \\
 & & E' & & \\
 & \xleftarrow{\pi'} & S & \xrightarrow{\pi'} &
 \end{array}$$

5.2 Definition, first properties and examples

Let $M : E \times_S E' \times_S E \times_S E' \rightarrow \mathbb{A}^1$ be the pairing defined by $M(e_1, e'_1, e_2, e'_2) = \mu(e_1, e'_1) + \mu(e_2, e'_2)$. Thus we have $M = \mu P_1 + \mu P_2$. By ADD, we deduce that

$$M^* \mathcal{L} = P_1^* \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} P_2^* \mu^* \mathcal{L}$$

Let us still denote by \mathcal{F} the Fourier-Deligne transform for $E \times_S E \xrightarrow{\pi \times_S \pi} S$. Then we have

$$\mathcal{F}(\tilde{K}) = P'_1 (P^* \tilde{K} \otimes_{\mathbb{E}}^{\mathbb{L}} M^* \mathcal{L}) = P'_1 (P^* \tilde{K} \otimes_{\mathbb{E}}^{\mathbb{L}} P_1^* \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} P_2^* \mu^* \mathcal{L})$$

for \tilde{K} in $D_c^b(E \times_S E)$

We are now ready to prove the proposition. The crucial point relies on the Künneth formula.

$$\begin{aligned} & \mathcal{F}(K_1) \boxtimes_S^{\mathbb{L}} \mathcal{F}(K_2) \\ &= \text{pr}'_1 (\text{pr}^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \boxtimes_S^{\mathbb{L}} \text{pr}'_1 (\text{pr}^* K_2 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) [2r] && \text{by definition} \\ &= P'_1 ((\text{pr}^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \boxtimes_S^{\mathbb{L}} (\text{pr}^* K_2 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})) [2r] && \text{by KUNNETH} \\ &= P'_1 (P_1^* (\text{pr}^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \otimes_{\mathbb{E}}^{\mathbb{L}} P_2^* (\text{pr}^* K_2 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})) [2r] && \text{by definition} \\ &= P'_1 (P_1^* \text{pr}^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} P_1^* \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} P_2^* \text{pr}^* K_2 \otimes_{\mathbb{E}}^{\mathbb{L}} P_2^* \mu^* \mathcal{L}) [2r] && \text{by DISTR}(P_{1,2}) \\ &= P'_1 (P^* p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} P^* p_2^* K_2 \otimes_{\mathbb{E}}^{\mathbb{L}} P_1^* \mu^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} P_2^* \mu^* \mathcal{L}) [2r] && \text{by COM}^*(\square_{1,2}) \\ &= P'_1 (P^* (p_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} p_2^* K_2) \otimes M^* \mathcal{L}) [2r] && \text{by DISTR}(P) \text{ and the above} \\ &= \mathcal{F}(K_1 \boxtimes_S^{\mathbb{L}} K_2) \end{aligned}$$

Now, we just need to apply Theorem 5.2.5 (MOR) to the morphism $s : E \times_S E \rightarrow E$ and the complex $K_1 \boxtimes_S^{\mathbb{L}} K_2$. So we take $E_1 = E \times_S E$, $E_2 = E$, $f = s$ and $K_1 := K_1 \boxtimes_S^{\mathbb{L}} K_2$. Let us remark that $s' : E' \rightarrow E' \times_S E'$ is the diagonal embedding $s'(e') = (e', e')$, so that $p'_i \circ s' = 1_{E'}$ for $i = 1, 2$. We get

$$\begin{aligned} \mathcal{F}(K_1 * K_2) &= \mathcal{F}(s_! (K_1 \boxtimes_S^{\mathbb{L}} K_2)) && \text{by definition} \\ &= s'^* \mathcal{F}(K_1 \boxtimes_S^{\mathbb{L}} K_2)[-r] && \text{by MOR} \\ &= s'^* (\mathcal{F}(K_1) \boxtimes_S^{\mathbb{L}} \mathcal{F}(K_2))[-r] && \text{by the above} \\ &= s'^* (p_1'^* \mathcal{F}(K_1) \otimes_{\mathbb{E}}^{\mathbb{L}} p_2'^* \mathcal{F}(K_2))[-r] && \text{by definition} \\ &= s'^* p_1'^* \mathcal{F}(K_1) \otimes_{\mathbb{E}}^{\mathbb{L}} s'^* p_2'^* \mathcal{F}(K_2)[-r] && \text{by DISTR}(s') \\ &= \mathcal{F}(K_1) \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K_2)[-r] && \text{since } p'_i \circ s' = 1_{E'}, i = 1, 2 \end{aligned}$$

The proof is complete. □

Proposition 5.2.9 *We have a “Plancherel” functorial isomorphism*

$$\pi'_1(\mathcal{F}(K_1) \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K_2)) \simeq \pi_1(K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} (-1_E)^* K_2)(-r) \quad (\text{PL})$$

for (K_1, K_2) in $D_c^b(E, \mathbb{E}) \times D_c^b(E, \mathbb{E})$.

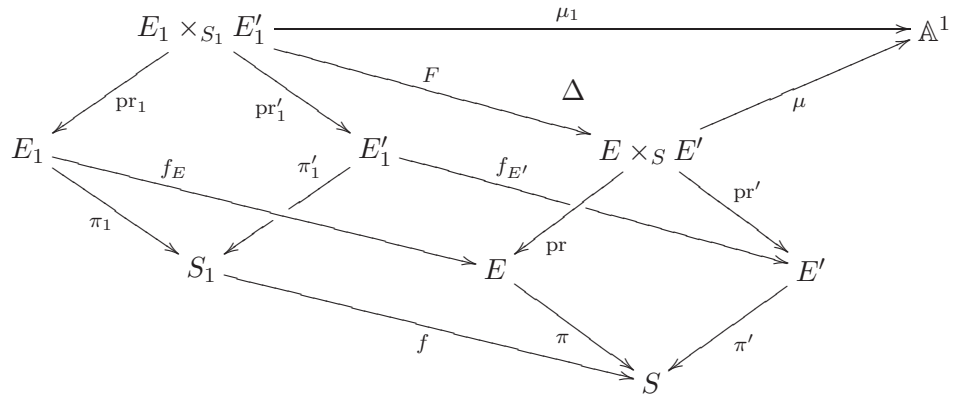
Proof. We have a cartesian square (which we will denote by \square)

$$\begin{array}{ccc} E & \xrightarrow{(1_E, -1_E)} & E \times_S E \\ \pi \downarrow & & \downarrow s \\ S & \xrightarrow{\sigma} & E \end{array}$$

$$\begin{aligned}
 \pi'_1 (\mathcal{F}(K_1) \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K_2)) &= \pi'_1 \mathcal{F}(K_1 * K_2)[r] && \text{by CONV} \\
 &= \sigma^*(K_1 * K_2)(-r) && \text{by EV} \\
 &= \sigma^*_{S_1} (K_1 \boxtimes_S^{\mathbb{L}} K_2)(-r) && \text{by definition} \\
 &= \pi_! (1_E, -1_E)^*(K_1 \boxtimes_S^{\mathbb{L}} K_2)(-r) && \text{by PBCT}(\square) \\
 &= \pi_! (K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} (-1_E)^* K_2)(-r) && \text{by DISTR}'
 \end{aligned}$$

□

Proposition 5.2.10 *The formation of $\mathcal{F}(K)$, for an object K in $D_c^b(E, \mathbb{E})$, commutes with any base change $S_1 \rightarrow S$. That is, if we fix the notations by the diagram*



then we have a functorial isomorphism

$$\mathcal{F}_1(f_E^* K) \simeq f_{E'}^* \mathcal{F}(K) \quad (\text{BC}^*)$$

for K in $D_c^b(E, \mathbb{E})$, where \mathcal{F}_1 denotes the Fourier-Deligne transform associated to $E_1 \xrightarrow{\pi_1} S$.

Proof. Let us denote by \square_{pr} (respectively $\square_{\text{pr}'}$) the cartesian square containing F and pr (respectively pr'). Then we have

$$\begin{aligned}
 \mathcal{F}_1(f_E^* K) &= \text{pr}'_{1!} (\text{pr}_1^* f_E^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_1^* \mathcal{L})[r] && \text{by definition} \\
 &= \text{pr}'_{1!} (F^* \text{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} F^* \mu^* \mathcal{L})[r] && \text{by COM}^*(\square_{\text{pr}}, \Delta) \\
 &= \text{pr}'_{1!} F^* (\text{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})[r] && \text{by DISTR}(F) \\
 &= f_{E'}^* \text{pr}'_{1!} (\text{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})[r] && \text{by PBCT}(\square_{\text{pr}'}) \\
 &= f_{E'}^* \mathcal{F}(K) && \text{by definition}
 \end{aligned}$$

□

5.2.3 Examples

Proposition 5.2.11 *Let $i : F \hookrightarrow E$ be a sub-vector bundle over S , with constant rank r_F . We denote by $i^\perp : F^\perp \hookrightarrow E'$ the orthogonal of F in E' . Then we have a canonical isomorphism*

$$\mathcal{F}(i_* \mathbb{E}_F[r_F]) \simeq i_*^\perp \mathbb{E}_{F^\perp}(-r_F)[r - r_F] \quad (\text{SUB})$$

Proof. First remark that we have a cartesian square (which we will denote by \square)

$$\begin{array}{ccc} F^\perp & \xrightarrow{\pi_{F^\perp}} & S \\ i^\perp \downarrow & & \downarrow \sigma'_F \\ E' & \xrightarrow{i'} & F' = E'/F^\perp \end{array}$$

Let us denote by \mathcal{F}_F the Fourier-Deligne transform associated to $F \xrightarrow{\pi_F} S$. Then we have

$$\begin{aligned} \mathcal{F}(i_* \mathbb{E}_F[r_F]) &= i'^* \mathcal{F}_F(\pi_F^* \mathbb{E}_S[r_F])[r - r_F] && \text{by MOR} \\ &= i'^* \sigma'_{F*} \mathbb{E}_S(-r_F)[r - r_F] && \text{by DIRAC} \\ &= i_*^\perp \pi_{F^\perp}^* \mathbb{E}_S(-r_F)[r - r_F] && \text{by PBCT}(\square) \\ &= i_*^\perp \mathbb{E}_{F^\perp}(-r_F)[r - r_F] \end{aligned}$$

□

Proposition 5.2.12 *Let $e \in E(S)$ be a section of $E \xrightarrow{\pi} S$. Let $\tau_e : E \xrightarrow{\sim} E$ denote the translation by e . Finally, let $\mu_e = \mu(e, -) : E' \rightarrow \mathbb{A}^1$. Then we have a functorial isomorphism*

$$\mathcal{F}(\tau_{e*} K) \simeq \mu_e^* \mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K) \quad (\text{TRANS})$$

for K in $D_c^b(E, \mathbb{E})$.

Proof. Let us first show that $\tau_{e*} K = (e_* \mathbb{E}_S) * K$. We have a commutative diagram

$$\begin{array}{ccccc} E \times_S E & \xrightarrow{s} & E & & \\ \swarrow p_1 & \searrow p_2 & \downarrow (e\pi, 1_E) & \nearrow \tau_e & \\ E & & E & & \\ \swarrow e & \searrow \pi & \downarrow \pi & & \\ & & S & & \end{array}$$

We denote by \square_e the cartesian square containing π and e , and by Δ_e the commutative

triangle corresponding to the relation $\tau_e = s \circ (e\pi, 1_E)$. Now we have

$$\begin{aligned}
 & (e_*\mathbb{E}_S) * K \\
 &= s_! (e_*\mathbb{E}_S \boxtimes_S^{\mathbb{L}} K) && \text{by definition} \\
 &= s_! (p_1^*e_*\mathbb{E}_S \otimes_{\mathbb{E}}^{\mathbb{L}} p_2^*K) && \text{by definition} \\
 &= s_! ((e\pi, 1_E)_*\pi^*\mathbb{E}_S \otimes_{\mathbb{E}}^{\mathbb{L}} p_2^*K) && \text{by PBCT}(\square_e) \\
 &= s_! (e\pi, 1_E)_*(\pi^*\mathbb{E}_S \otimes_{\mathbb{E}}^{\mathbb{L}} (e\pi, 1_E)^*p_2^*K) && \text{by PROJ}((e\pi, 1_E)) \\
 &= \tau_{e_*}K && \text{by COM}_1(\Delta_e) \text{ and } p_2(e\pi, 1_E) = 1_E
 \end{aligned}$$

Secondly, let us show that $\mathcal{F}(e_*\mathbb{E}_S) = \mu_e^*\mathcal{L}[r]$. We have a commutative diagram

$$\begin{array}{ccccc}
 & E \times_S E' & \xrightarrow{\mu} & & \mathbb{A}^1 \\
 & \swarrow \text{pr} & \searrow (e\pi', 1_{E'}) & & \nearrow \mu_e \\
 E & & & E' & \\
 & \searrow e & & \swarrow \pi' & \\
 & & S & &
 \end{array}$$

We denote by \square'_e the cartesian square containing π' and e , and by Δ'_e the commutative triangle corresponding to the relation $\mu_e = \mu \circ (e\pi', 1_{E'})$. Now we have

$$\begin{aligned}
 & \mathcal{F}(e_*\mathbb{E}_S) \\
 &= \text{pr}'_! (\text{pr}^*e_*\mathbb{E}_S \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^*\mathcal{L})[r] && \text{by definition} \\
 &= \text{pr}'_! ((e\pi', 1_{E'})_*\pi'^*\mathbb{E}_S \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^*\mathcal{L})[r] && \text{by PBCT}(\square'_e) \\
 &= \text{pr}'_! (e\pi', 1_{E'})_*(\mathbb{E}_{E'} \otimes_{\mathbb{E}}^{\mathbb{L}} (e\pi', 1_{E'})^*\mu^*\mathcal{L})[r] && \text{by PROJ}((e\pi', 1_{E'})) \\
 &= \mu_e^*\mathcal{L}[r] && \text{by COM}_1(\Delta'_e) \text{ and } \mu(e\pi', 1_{E'}) = \mu_e
 \end{aligned}$$

Finally, the result follows by Proposition 5.2.8 (CONV).

$$\mathcal{F}(\tau_{e_*}K) = \mathcal{F}((e_*\mathbb{E}_S) * K) = \mathcal{F}(e_*\mathbb{E}_S) \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K)[-r] = \mu_e^*\mathcal{L} \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{F}(K)$$

□

Proposition 5.2.13 *Let G be a smooth affine group scheme over S , acting linearly on the vector bundle $E \xrightarrow{\pi} S$, let K and L be two objects in $D_c^b(E, \mathbb{E})$, and let M be an object in $D_c^b(G, \mathbb{E})$. We denote by $m : G \times_S E \rightarrow E$ the action of G on E , and by $m' : G \times_S E' \rightarrow E'$ the contragredient action, defined by $m'(g, e') = {}^t g^{-1}.e'$. Then each isomorphism*

$$m^*K \simeq M \boxtimes_S^{\mathbb{L}} L$$

in $D_c^b(G \times_S E, \mathbb{E})$ induces canonically an isomorphism

$$m'^*\mathcal{F}(K) \simeq M \boxtimes_S^{\mathbb{L}} \mathcal{F}(L) \quad (G\text{-EQ})$$

in $D_c^b(G \times_S E', \mathbb{E})$.

Proof.

We have a base change diagram

$$\begin{array}{ccccc}
 & G \times_S E \times_S E' & \xrightarrow{\mu_G} & & \mathbb{A}^1 \\
 & \swarrow (1_G, \text{pr}) & \searrow P & \Delta & \swarrow \mu \\
 G \times_S E & & G \times_S E' & \xrightarrow{p_{E'}} & E \times_S E' \\
 \swarrow p_E & \searrow p'_G & \swarrow p_E & \searrow \text{pr}' & \swarrow \text{pr}' \\
 & G & \xrightarrow{p_G} & E & \xrightarrow{\pi} & S \\
 & \swarrow p_G & \searrow \pi_G & \swarrow \pi & \searrow \pi' & \\
 & & & & & E' \\
 & & & & & \swarrow \pi' \\
 & & & & & S
 \end{array}$$

and commutative triangles

$$\begin{array}{ccc}
 G \times_S E & \xrightarrow{(p_G, m)} & G \times_S E \\
 \searrow m & \Delta_G & \swarrow p_E \\
 & E &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G \times_S E' & \xrightarrow{(p'_G, m')} & G \times_S E' \\
 \searrow m' & \Delta'_G & \swarrow p_{E'} \\
 & E' &
 \end{array}$$

We will use Theorem 5.2.5 (MOR) for the morphism of G -vector bundles

$$(p_G, m)^{-1} : G \times_S E' \longrightarrow G \times_S E$$

whose transposed morphism is

$$(p'_G, m') : G \times_S E \longrightarrow G \times_S E'$$

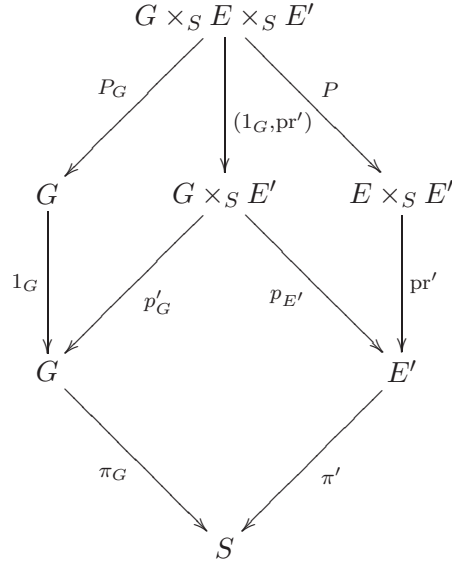
Both are isomorphisms. We will use the fact that the functor $(p_G, m)_!^{-1}$ is an equivalence, isomorphic to $(p_G, m)_*^{-1}$ and to $(p_G, m)^*$.

$$\begin{aligned}
 m^* \mathcal{F}(K) &= (p'_G, m')^* p_{E'}^* \mathcal{F}(K) && \text{by COM}^*(\Delta'_G) \\
 &= (p'_G, m')^* \mathcal{F}_{G \times_S E'}(p_{E'}^* K) && \text{by BC}^* \\
 &= \mathcal{F}_{G \times_S E}((p_G, m)_!^{-1} p_{E'}^* K) && \text{by MOR} \\
 &= \mathcal{F}_{G \times_S E}((p_G, m)^* p_{E'}^* K) && \text{by the above} \\
 &= \mathcal{F}_{G \times_S E}(m^* K) && \text{by COM}^*(\Delta_G)
 \end{aligned}$$

Now assume we are given an isomorphism $\phi : m^* K \xrightarrow{\sim} M \boxtimes_S^{\mathbb{L}} L$. Then ϕ induces an isomorphism

$$\begin{aligned}
 \mathcal{F}_{G \times_S E}(m^* K) &\simeq \mathcal{F}_{G \times_S E}(M \boxtimes_S^{\mathbb{L}} L) && \text{induced by } \phi \\
 &= (1_G, \text{pr}')_! ((1_G, \text{pr})^*(M \boxtimes_S^{\mathbb{L}} L) \otimes_{\mathbb{E}} P^* \mu^* \mathcal{L})[r] && \text{by COM}^*(\Delta) \\
 &= (1_G, \text{pr}')_! ((M \boxtimes_S^{\mathbb{L}} \text{pr}^* L) \otimes_{\mathbb{E}} (\mathbb{E}_G \boxtimes_S^{\mathbb{L}} \mu^* \mathcal{L})) [r] && \text{by DISTR}(1_G, \text{pr}) \\
 &= (1_G, \text{pr}')_! (M \boxtimes_S^{\mathbb{L}} (\text{pr}^* L \otimes_{\mathbb{E}} \mu^* \mathcal{L})) [r] \\
 &= M \boxtimes_S^{\mathbb{L}} \text{pr}'_! (\text{pr}^* L \otimes_{\mathbb{E}} \mu^* \mathcal{L}) [r] && \text{by KUNNETH} \\
 &= M \boxtimes_S^{\mathbb{L}} \mathcal{F}(L)
 \end{aligned}$$

We have applied the Künneth formula to the following diagram.



□

Proposition 5.2.14 *Let $f : S_1 \rightarrow S$ be an \mathbb{F}_q -morphism of finite type. With the notations of Proposition 5.2.10 (BC*), we have a functorial isomorphism*

$$\mathcal{F}(f_{E!} K_1) \simeq f_{E'!} \mathcal{F}_1(K_1) \quad (\text{BC}_!)$$

for K_1 in $D_c^b(E_1, \mathbb{E})$.

Proof. We have

$$\begin{aligned}
 \mathcal{F}(f_{E!} K_1) &= \text{pr}'_! (\text{pr}^* f_{E!} K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})[r] && \text{by definition} \\
 &= \text{pr}'_! (F_! \text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L})[r] && \text{by PBCT}(\square_{\text{pr}}) \\
 &= \text{pr}'_! F_! (\text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} F^* \mu^* \mathcal{L})[r] && \text{by PROJ}(F) \\
 &= f_{E'!} \text{pr}'_{1!} (\text{pr}_1^* K_1 \otimes_{\mathbb{E}}^{\mathbb{L}} \mu_1^* \mathcal{L})[r] && \text{by COM}_1(\square_{\text{pr}'}, \Delta) \\
 &= f_{E'!} \mathcal{F}_1(K_1) && \text{by definition}
 \end{aligned}$$

□

5.3 Fourier-Deligne transform and duality

We keep the preceding notations. For the proof of the following fundamental theorem, we refer to [KL85]. Katz and Laumon state the result for $\overline{\mathbb{Q}}_\ell$, but for the proof they make a reduction to torsion coefficients, and prove it in that context. The crucial point is the one-dimensional case.

Theorem 5.3.1 For any object K in $D_c^b(E, \mathbb{E})$, the support forgetting morphism

$$\mathrm{pr}'_! (\mathrm{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \longrightarrow \mathrm{pr}'_* (\mathrm{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}) \quad (\mathrm{SUPP})$$

is an isomorphism.

This theorem has the following corollaries.

Theorem 5.3.2 We have a functorial isomorphism

$$\mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}_\psi(K), \pi^! L) \simeq \mathcal{F}_{\psi^{-1}}(\mathrm{R}\underline{\mathrm{Hom}}(K, \pi^! L))(r)$$

for (K, L) in $D_c^b(E, \mathbb{E})^{op} \times D_c^b(S, \mathbb{E})$.

Proof. First recall that \mathcal{L}_ψ is a rank one local system on \mathbb{A}^1 . Applying μ^* to the relation $\mathcal{L}_\psi \otimes_{\mathbb{E}}^{\mathbb{L}} \mathcal{L}_{\psi^{-1}} \simeq \mathbb{E}_{\mathbb{A}^1}$ and using DISTR, we find that $\mu^* \mathcal{L}_\psi$ is a rank one local system with inverse $\mu^* \mathcal{L}_{\psi^{-1}}$. So $(-)\otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_\psi$ is an automorphism of $D_c^b(E \times_S E')$, with quasi-inverse $(-)\otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_{\psi^{-1}}$.

We have

$$\begin{aligned} & \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}_\psi(K), \pi^! L) \\ &= \mathrm{R}\underline{\mathrm{Hom}}(\mathrm{pr}'_! (\mathrm{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_\psi), \pi^! L)[-r] && \text{by definition} \\ &= \mathrm{pr}'_* \mathrm{R}\underline{\mathrm{Hom}}(\mathrm{pr}^* K \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_\psi, \mathrm{pr}'_! \pi^! L)[-r] && \text{by DUAL}(\mathrm{pr}') \\ &= \mathrm{pr}'_* (\mathrm{R}\underline{\mathrm{Hom}}(\mathrm{pr}^* K, \mathrm{pr}'_! \pi^! L) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_{\psi^{-1}})[-r] && \text{by the above and COM}^!(\square) \\ &= \mathrm{pr}'_* (\mathrm{pr}^! \mathrm{R}\underline{\mathrm{Hom}}(K, \pi^! L) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_{\psi^{-1}})[-r] && \text{by DUAL}_2(\mathrm{pr}) \\ &= \mathrm{pr}'_* (\mathrm{pr}^* \mathrm{R}\underline{\mathrm{Hom}}(K, \pi^! L) \otimes_{\mathbb{E}}^{\mathbb{L}} \mu^* \mathcal{L}_{\psi^{-1}})r && \text{since pr is smooth} \\ &= \mathcal{F}_{\psi^{-1}}(\mathrm{R}\underline{\mathrm{Hom}}(K, \pi^! L))(r) && \text{by SUPP} \end{aligned}$$

□

Remember that, if X is a variety, we denote by $\mathcal{D}_{X, \mathbb{E}}$ the duality functor of $D_c^b(X, \mathbb{E})$ (see section 1.1). If $a : X \rightarrow \mathrm{Spec} k$ is the structural morphism, we denote by $D_{X, \mathbb{E}}$ the dualizing complex $a^! \mathbb{E}$.

Corollary 5.3.3 We have a functorial isomorphism

$$\mathcal{D}_{E', \mathbb{E}}(\mathcal{F}_\psi(K)) \simeq \mathcal{F}_{\psi^{-1}}(\mathcal{D}_{E, \mathbb{E}}(K))(r)$$

for K in $D_c^b(E, \mathbb{E})^{op}$.

Proof. We have

$$\begin{aligned} \mathcal{D}_{E', \mathbb{E}}(\mathcal{F}_\psi(K)) &= \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}_\psi(K), \pi^! D_{S, \mathbb{E}}) \\ &= \mathcal{F}_{\psi^{-1}} \mathrm{R}\underline{\mathrm{Hom}}(K, \pi^! D_{S, \mathbb{E}})(r) \\ &= \mathcal{F}_{\psi^{-1}}(\mathcal{D}_{E, \mathbb{E}}(K))(r) \end{aligned}$$

□

Theorem 5.3.4 \mathcal{F} maps ${}^p\mathcal{M}(E, \mathbb{E})$ onto ${}^p\mathcal{M}(E', \mathbb{E})$. The functor

$$\mathcal{F} : {}^p\mathcal{M}(E, \mathbb{E}) \longrightarrow {}^p\mathcal{M}(E', \mathbb{E}) \quad (\text{EQUIV})$$

is an equivalence of abelian categories, with quasi-inverse $a^*\mathcal{F}'(-)(r)$.

Proof. Since pr is smooth, purely of relative dimension r , the functor $\text{pr}^*(-)[r]$ is t -exact [BBD82, 4.2.5] and, since pr' is affine, the functor $\text{pr}'_!$ is left t -exact, whereas the functor pr'_* is right t -exact. By Theorem 5.3.1, we deduce that \mathcal{F} is t -exact.

The second assertion follows from the first and Theorem 5.2.3 (INV). □

Corollary 5.3.5 Suppose $\mathbb{E} = \mathbb{K}$ or \mathbb{F} . Then \mathcal{F} transforms simple \mathbb{E} -perverse sheaves on E into simple \mathbb{E} -perverse sheaves on E' .

Chapter 6

Springer correspondence and decomposition matrices

6.1 The geometric context

6.1.1 Notation

Let G be a connected semisimple linear algebraic group of rank r over k , and let \mathfrak{g} be its Lie algebra. Let us fix a Borel subgroup B of G , with unipotent radical U , and a maximal torus T contained in B . We denote by \mathfrak{b} , \mathfrak{u} and \mathfrak{t} the corresponding Lie algebras. The characters of T form a free abelian group $X(T)$ of rank r . The Weyl group $W = N_G(T)/T$ acts as a reflection group on $V = \mathbb{Q} \otimes_{\mathbb{Z}} X(T)$.

Let $\Phi \subset X(T)$ be the root system of (G, T) , Φ^+ the set of positive roots defined by B , and Δ the corresponding basis. We denote by ν_G (or just ν) the cardinality of Φ^+ . Then $\dim G = 2\nu + r$, $\dim B = \nu + r$, $\dim T = r$ and $\dim U = \nu$.

6.1.2 The finite quotient map

Let $\phi : \mathfrak{t} \rightarrow \mathfrak{t}/W$ be the quotient map, corresponding to the inclusion $k[\mathfrak{t}]^W \hookrightarrow k[\mathfrak{t}]$. It is finite and surjective. For $t \in \mathfrak{t}$, we will also denote $\phi(t)$ by \bar{t} .

Let us assume that p is not a torsion prime for \mathfrak{g} . Then $k[\mathfrak{t}]^W = k[\phi_1, \dots, \phi_r]$ for some algebraically independent homogeneous polynomials ϕ_1, \dots, ϕ_r of degrees $d_1 \leq \dots \leq d_r$, and we have $d_i = m_i + 1$, where the m_i are the exponents of W (see [Dem73]). Then \mathfrak{t}/W can be identified with \mathbb{A}^r and ϕ with (ϕ_1, \dots, ϕ_r) .

For example, if $G = SL_n$, we can identify \mathfrak{t} with the hyperplane $\{(x_1, \dots, x_n) \mid x_1 + \dots + x_n = 0\}$ of k^n , and we can take $\phi_i = \sigma_{i+1}$, for $i = 2, \dots, n-1$ (here $r = n-1$), that is, the $i+1$ st elementary symmetric function of k^n , restricted to this hyperplane (σ_1 does not appear, since its restriction vanishes).

6.1.3 The adjoint quotient

The Chevalley restriction theorem says that the restriction map $k[\mathfrak{g}]^G \rightarrow k[\mathfrak{t}]^W$ is an isomorphism, so $k[\mathfrak{g}]^G$ is also generated by r homogeneous algebraically independent polynomials χ_1, \dots, χ_r . With a suitable ordering, they have the same degrees $d_1 \leq \dots \leq d_r$ as the ϕ_i 's.

Hence we have a morphism $\chi = (\chi_1, \dots, \chi_r) : \mathfrak{g} \rightarrow \mathfrak{g}/G \simeq \mathfrak{t}/W \simeq \mathbb{A}^r$. It is called the Steinberg map, or the adjoint quotient. In the last section, we could have taken $\phi_i = \chi_i|_{\mathfrak{t}}$.

The morphism χ has been extensively studied (see [Slo80b] and the references therein). First, it is flat, and its schematic fibers are irreducible, reduced and normal complete intersections, of codimension r in \mathfrak{g} . If $t \in \mathfrak{t}$, let $\mathfrak{g}_{\bar{t}}$ be the fiber $\chi^{-1}(\bar{t})$. It is the union of finitely many classes. It contains exactly one class of regular elements, which is open and dense in $\mathfrak{g}_{\bar{t}}$, and whose complement has codimension ≥ 2 in $\mathfrak{g}_{\bar{t}}$. This regular class consists exactly in the nonsingular points of $\mathfrak{g}_{\bar{t}}$. So \mathfrak{t}/W parametrizes the classes of regular elements. The fiber $\mathfrak{g}_{\bar{t}}$ also contains exactly one class of semisimple elements, the orbit of t , which is the only closed class in $\mathfrak{g}_{\bar{t}}$, and which lies in the closure of every other class in $\mathfrak{g}_{\bar{t}}$.

In fact, χ can be interpreted as the map which sends x to the intersection of the class of x_s with \mathfrak{t} , which is a W -orbit.

For example, for $G = SL_n$, we can define the $\chi_i : \mathfrak{sl}_n \rightarrow k$ by the formula

$$\det(\xi - x) = \xi^n + \sum_{i=0}^{n-1} (-1)^{i+1} \chi_i(x) \xi^{n-i-1} \in k[\xi]$$

for $x \in \mathfrak{sl}_n$. So $\chi(x)$ can be interpreted as the characteristic polynomial of x . Restricting χ_i to \mathfrak{t} , we recover the previous ϕ_i .

6.1.4 Springer's resolution of the nilpotent variety

Let \mathcal{N} be the closed subvariety of \mathfrak{g} consisting in its nilpotent elements. It is the fiber $\mathfrak{g}_0 = \chi^{-1}(0)$. In particular, it is a complete intersection in \mathfrak{g} , given by the equations $\chi_1(x) = \dots = \chi_r(x) = 0$. It is singular. We are going to describe Springer's resolution of the nilpotent variety.

The set \mathcal{B} of Borel subalgebras of \mathfrak{g} is a homogeneous space under G , in bijection with G/B , since the normalizer of \mathfrak{b} in G is B . Hence \mathcal{B} is endowed with a structure of smooth projective variety, of dimension ν .

Let $\tilde{\mathcal{N}} = G \times^B \mathfrak{u} \simeq \{(x, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}'\}$. It is a smooth variety: the second projection makes it a vector bundle over \mathcal{B} . Actually it can be identified to the cotangent bundle $T^*\mathcal{B}$, since $T\mathcal{B} = T(G/B) = G \times^B \mathfrak{g}/\mathfrak{b}$ and $\mathfrak{u} = \mathfrak{b}^\perp$. Now let $\pi_{\tilde{\mathcal{N}}}$ be the first projection. Since $\tilde{\mathcal{N}}$ is closed in $\mathcal{N} \times \mathcal{B}$ and \mathcal{B} is projective, the morphism $\pi_{\tilde{\mathcal{N}}}$ is projective. Moreover, it is an isomorphism over the open dense subvariety of \mathcal{N} consisting in the regular nilpotent elements. Hence $\pi_{\tilde{\mathcal{N}}}$ is indeed a resolution of \mathcal{N} .

6.1.5 Grothendieck's simultaneous resolution of the adjoint quotient

In the last paragraph, we have seen the resolution of the fiber $\chi^{-1}(0)$. We are now going to explain Grothendieck's simultaneous resolution, which gives resolutions for all the fibers of χ simultaneously.

So let $\tilde{\mathfrak{g}} = G \times^B \mathfrak{b} \simeq \{(x, \mathfrak{b}') \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}'\}$. We define $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ by $\pi(g * x) = \text{Ad}(g).x$ (in the identification with pairs (x, \mathfrak{b}') , this is just the first projection). Then

the commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g} \\ \theta \downarrow & & \downarrow \chi \\ \mathfrak{t} & \xrightarrow{\phi} & \mathfrak{t}/W \end{array}$$

where θ is the composition $G \times^B \mathfrak{b} \rightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \xrightarrow{\sim} \mathfrak{t}$, is a simultaneous resolution of the singularities of the flat morphism χ . That is, θ is smooth, ϕ is finite surjective, π is proper, and π induces a resolution of singularities $\theta^{-1}(t) \rightarrow \chi^{-1}(\phi(t))$ for all $t \in \mathfrak{t}$.

6.2 Springer correspondence for $\mathbb{E}W$

6.2.1 The perverse sheaves \mathcal{K}_{rs} , \mathcal{K} and $\mathcal{K}_{\mathcal{N}}$

Let us consider the following commutative diagram with cartesian squares.

$$\begin{array}{ccccc} \tilde{\mathfrak{g}}_{rs} & \xleftarrow{\tilde{j}_{rs}} & \tilde{\mathfrak{g}} & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \tilde{\mathcal{N}} \\ \pi_{rs} \downarrow & \square_{rs} & \downarrow \pi & \square_{\mathcal{N}} & \downarrow \pi_{\mathcal{N}} \\ \mathfrak{g}_{rs} & \xleftarrow{j_{rs}} & \mathfrak{g} & \xleftarrow{i_{\mathcal{N}}} & \mathcal{N} \end{array}$$

Let us define the complex

$$\mathcal{K} = \pi_! \mathbb{O}_{\tilde{\mathfrak{g}}} [2\nu + r]$$

Let \mathbb{E} be \mathbb{K} , \mathbb{O} or \mathbb{F} . Since modular reduction commutes with direct images with proper support, we have $\mathbb{E}\mathcal{K} = \pi_! \mathbb{E}_{\tilde{\mathfrak{g}}} [2\nu + r]$. By the proper base change theorem, the fiber at a point x in \mathfrak{g} of $\mathbb{E}\mathcal{K}$ is given by $(\mathbb{E}\mathcal{K})_x = \mathrm{R}\Gamma_c(\mathcal{B}_x, \mathbb{O})$.

Let $\mathcal{K}_{rs} = j_{rs}^* \mathcal{K}$ and $\mathcal{K}_{\mathcal{N}} = i_{\mathcal{N}}^* \mathcal{K}[-r]$. By the proper base change theorem and the commutation between modular reduction and inverse images, we have

$$\begin{aligned} \mathbb{E}\mathcal{K}_{rs} &= j_{rs}^* \mathbb{E}\mathcal{K} = \pi_{rs*} \mathbb{E}_{\tilde{\mathfrak{g}}_{rs}} [2\nu + r] \\ \mathbb{E}\mathcal{K}_{\mathcal{N}} &= i_{\mathcal{N}}^* \mathbb{E}\mathcal{K}[-r] = \pi_{\mathcal{N}!} \mathbb{E}_{\tilde{\mathcal{N}}} [2\nu] \end{aligned}$$

The morphism π is proper and small, hence $\mathbb{E}\mathcal{K}$ is an intersection cohomology complex by Proposition 2.1.2. Actually π is étale over the open subvariety \mathfrak{g}_{rs} . More precisely, the morphism π_{rs} obtained after the base change j_{rs} is a Galois finite étale covering, with Galois group W , so we have $\mathbb{E}\mathcal{K} = j_{rs!} \mathbb{E}\mathcal{K}_{rs} = j_{rs!} (\mathbb{E}W [2\nu + r])$. Note that, if $\mathbb{E} = \mathbb{O}$, we have ${}^{p+}j_{rs!} \mathcal{K}_{rs} = \mathcal{D}_{\mathfrak{g}, \mathbb{O}} ({}^p j_{rs!} \mathcal{K}_{rs}) = \mathcal{D}_{\mathfrak{g}, \mathbb{O}} (\mathcal{K}) = \mathcal{K}$ so it does not matter whether we use p or p_+ (we have used the fact that the regular representation is self-dual, and that \mathcal{K} is self-dual because π is proper).

Thus the endomorphism algebra of $\mathbb{E}\mathcal{K}_{rs}$ is the group algebra $\mathbb{E}W$. Since the functor $j_{rs!}$ is fully faithful, it induces an isomorphism $\mathrm{End}(\mathcal{K}_{rs}) = \mathbb{E}W \xrightarrow{\sim} \mathrm{End}(\mathcal{K})$. In particular, we have an action of $\mathbb{E}W$ on the stalks $\mathcal{H}_x^i(\mathbb{E}\mathcal{K}) = H^{i+2\nu+r}(\mathcal{B}_x, \mathbb{E})$.

When $\mathbb{E} = \mathbb{K}$, the group algebra $\mathbb{K}W$ is semisimple. The perverse sheaves

$$\mathbb{K}\mathcal{K}_{\text{rs}} = \bigoplus_{E \in \text{Irr } \mathbb{K}W} (E [2\nu + r])^{\dim E}$$

and

$$\mathbb{K}\mathcal{K} = \bigoplus_{E \in \text{Irr } \mathbb{K}W} j_{\text{rs}!_*}(E [2\nu + r])^{\dim E}$$

are semisimple.

If ℓ does not divide the order of the Weyl group W , we have a similar decomposition for $\mathbb{E} = \mathbb{O}$ or \mathbb{F} . However, we are mostly interested in the case where ℓ divides $|W|$. Then $\mathbb{F}\mathcal{K}_{\text{rs}}$ and $\mathbb{F}\mathcal{K}$ are not semisimple. More precisely, we have decompositions

$$\begin{aligned} \mathbb{O}W &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (P_F)^{\dim F} \\ \mathbb{F}W &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (\mathbb{F}P_F)^{\dim F} \end{aligned}$$

where P_F is a projective indecomposable $\mathbb{O}W$ -module such that $\mathbb{F}P_F$ is a projective cover of F . Besides, $\mathbb{F}P_F$ has head and socle isomorphic to F .

Hence we have a similar decomposition for \mathcal{K}_{rs} , and its modular reduction.

$$\begin{aligned} \mathcal{K}_{\text{rs}} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (P_F [2\nu + r])^{\dim F} \\ \mathbb{F}\mathcal{K}_{\text{rs}} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} (\mathbb{F}P_F [2\nu + r])^{\dim F} \end{aligned}$$

These are decompositions into indecomposable summands, and the indecomposable summand $\mathbb{F}P_F [2\nu + r]$ has head and socle isomorphic to $F [2\nu + r]$. By Proposition 1.6.1, applying $j_{\text{rs}!_*}$ we get decompositions into indecomposable summands, and the indecomposable summand $j_{\text{rs}!_*}(\mathbb{F}P_F [2\nu + r])$ has head and socle isomorphic to $j_{\text{rs}!_*}(F [2\nu + r])$.

$$\begin{aligned} \mathcal{K} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} j_{\text{rs}!_*}(P_F [2\nu + r])^{\dim F} \\ \mathbb{F}\mathcal{K} &= \bigoplus_{F \in \text{Irr } \mathbb{F}W} j_{\text{rs}!_*}(\mathbb{F}P_F [2\nu + r])^{\dim F} \end{aligned}$$

The morphism $\pi_{\mathcal{N}}$ is proper and semi-small, hence $\mathbb{E}\mathcal{K}_{\mathcal{N}}$ is perverse. The functor $i_{\mathcal{N}}^*(-)[-r]$ induces a morphism

$$\text{res} : \text{End}(\mathbb{E}\mathcal{K}) \longrightarrow \text{End}(\mathbb{E}\mathcal{K}_{\mathcal{N}}) \tag{6.1}$$

6.2.2 Springer correspondence for $\mathbb{K}W$ by restriction

Theorem 6.2.1 (Lusztig) *If $\mathbb{E} = \mathbb{K}$, then the morphism res in (6.1) is an isomorphism.*

The Weyl group W acts on G/T by $gT.w = gn_wT$, where n_w is any representative of w in $N_G(T)$. So W acts naturally on the cohomology complex $R\Gamma(G/T, \mathbb{E})$. Since the projection $G/T \rightarrow G/B$ is a locally trivial U -fibration, it induces an isomorphism $R\Gamma(G/B, \mathbb{E}) \simeq R\Gamma(G/T, \mathbb{E})$, and thus there is a natural action of W on $R\Gamma(G/B, \mathbb{E})$. When $\mathbb{E} = \mathbb{K}$, one can show that the action on the cohomology is the regular representation. On the other hand, the stalk at 0 of \mathcal{K} is isomorphic to $R\Gamma(G/B, \mathbb{E})$, and $\mathbb{E}W$ acts on it through $\mathbb{E}W \simeq \text{End}(\mathcal{K}) \xrightarrow{\text{res}} \text{End}(\mathcal{K}_{\mathcal{N}})$. In fact, when $\mathbb{E} = \mathbb{K}$, the two actions on the cohomology coincide. Since the regular representation is faithful, this implies that the morphism res is injective.

Then one can show that the two algebras have the same dimension, to prove that res is an isomorphism.

We have

$$\mathbb{K}\mathcal{K} = \bigoplus_{E \in \text{Irr } \mathbb{K}W} j_{\text{rs}!_*} \underline{E}^{\dim E}$$

and $i_{\mathcal{N}}^*(\mathbb{K}\mathcal{K})[-r] = \mathbb{K}\mathcal{K}_{\mathcal{N}}$. In fact, the restriction functor $i_{\mathcal{N}}^*[-r]$ sends each simple constituent $j_{\text{rs}!_*} \underline{E}$ on a simple object. The assignment

$$\mathcal{R} : E \mapsto i_{\mathcal{N}}^* j_{\text{rs}!_*} \underline{E}[-r]$$

is an injective map from $\text{Irr } \mathbb{K}W$ to the simple G -equivariant perverse sheaves on \mathcal{N} , which are parametrized by the pairs $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is a nilpotent orbit, and \mathcal{L} is an irreducible G -equivariant \mathbb{K} -local system on \mathcal{O} . This is the Springer correspondence (by restriction).

We said G was semisimple, but everything can be done for a reductive group instead, as GL_n . For $G = GL_n$, the Specht module S^λ is sent to ${}^p\mathcal{J}_{!_*}(\mathcal{O}_\lambda, \mathbb{K})$, where \mathcal{O}_λ is the nilpotent orbit corresponding to the partition λ by the Jordan normal form.

6.2.3 The Fourier-Deligne transform of $\mathbb{E}\mathcal{K}$

We assume that there exists a non-degenerate G -invariant symmetric bilinear form μ on \mathfrak{g} , so that we can identify \mathfrak{g} with its dual. This is the case, for example, if p is very good for G (take the Killing form), or if $G = GL_n$ (take $\mu(X, Y) = \text{tr}(XY)$). For a more detailed discussion, see [Let05].

Lemma 6.2.2 *The root subspace \mathfrak{g}_α is orthogonal to \mathfrak{t} and to all the root subspaces \mathfrak{g}_β with $\beta \neq -\alpha$.*

Proof. Let $x \in \mathfrak{t}$. For $t \in T$. We have $\mu(x, e_\alpha) = \mu(\text{Ad}(t).x, \text{Ad}(t).e_\alpha) = \alpha(t)\mu(x, e_\alpha)$. Since $\alpha \neq 0$, we can choose t so that $\alpha(t) \neq 1$, and thus $\mu(x, e_\alpha) = 0$.

Now let β be a root different from $-\alpha$. We have $\mu(e_\beta, e_\alpha) = \mu(\text{Ad}(t).e_\beta, \text{Ad}(t).e_\alpha) = \alpha(t)\beta(t)\mu(e_\beta, e_\alpha)$. Since $\beta \neq -\alpha$, we may choose t so that $\alpha(t)\beta(t) \neq 1$, and thus $\mu(e_\beta, e_\alpha) = 0$. \square

Corollary 6.2.3 *The orthogonal of \mathfrak{b} is \mathfrak{u} .*

Proof. By the preceding lemma, \mathfrak{b} is orthogonal to \mathfrak{u} , and we have $\dim \mathfrak{b} + \dim \mathfrak{u} = 2\nu + r = \dim \mathfrak{g}$, hence the result, since μ is non-degenerate. \square

Let \mathcal{F} be the Fourier-Deligne transform associated to $p : \mathfrak{g} \rightarrow \text{Spec } k$ (any vector space can be considered as a vector bundle over a point). Since we identify \mathfrak{g} with \mathfrak{g}' , the functor \mathcal{F} is an auto-equivalence of the triangulated category $D_c^b(\mathfrak{g}, \mathbb{E})$. The application a of INV, which was defined as the opposite of the canonical isomorphism from a vector bundle to its bidual, is now multiplication by -1 .

We will need to consider the base change $f : \mathcal{B} \rightarrow \text{Spec } k$. We will denote by $\mathcal{F}_{\mathcal{B}}$ the Fourier-Deligne transform associated to $p_{\mathcal{B}} : \mathcal{B} \times \mathfrak{g} \rightarrow \mathcal{B}$. We have

$$\begin{array}{ccccc}
 G \times_B \mathfrak{g} & \longleftarrow & G \times_B \mathfrak{b} & \longleftarrow & G \times_B \mathfrak{n} \\
 \parallel & & \parallel & & \parallel \\
 \mathcal{B} \times \mathfrak{g} & \xleftarrow{i} & \tilde{\mathfrak{g}} & \xleftarrow{i_{\tilde{\mathcal{N}}}} & \tilde{\mathcal{N}} \\
 \downarrow p_{\mathcal{B}} & \searrow F & \downarrow \pi & \square_{\mathcal{N}} & \downarrow \pi_{\mathcal{N}} \\
 \mathcal{B} & & \mathfrak{g} & \xleftarrow{i_{\mathcal{N}}} & \mathcal{N} \\
 & \searrow f & \downarrow p & & \\
 & & \text{Spec } k & &
 \end{array}$$

We have

$$\begin{aligned}
 \mathcal{F}(\mathbb{E}\mathcal{K}) &= \mathcal{F}(\pi_! \mathbb{E}_{\tilde{\mathfrak{g}}}[2\nu + r]) \\
 &= \mathcal{F}(F_! i_* \mathbb{E}_{\tilde{\mathfrak{g}}}[\nu + r])[\nu] && \text{by COM}_1(\Delta) \\
 &= F_! \mathcal{F}_{\mathcal{B}}(i_* \mathbb{E}_{\tilde{\mathfrak{g}}}[\nu + r])[\nu] && \text{by BC}_1(f) \\
 &= F_! i_* i_{\tilde{\mathcal{N}}}^* \mathbb{E}_{\tilde{\mathcal{N}}}(-\nu - r)[\nu][\nu] && \text{by SUB} \\
 &= i_{\mathcal{N}*} \pi_{\mathcal{N}!} \mathbb{E}_{\tilde{\mathcal{N}}}(-\nu - r)[2\nu] && \text{by COM}_1(\Delta, \square_{\mathcal{N}}) \\
 &= i_{\mathcal{N}*} \mathbb{E}_{\mathcal{N}}(-\nu - r)
 \end{aligned}$$

Applying \mathcal{F} and using Theorem 5.2.3 (INV), we get

$$a_* \mathbb{E}\mathcal{K}(-2\nu - r) = \mathcal{F}(i_{\mathcal{N}*} \mathbb{E}_{\mathcal{N}}(-\nu - r))$$

But $a_* \mathbb{E}\mathcal{K} \simeq \mathbb{E}\mathcal{K}$ since $\mathbb{E}\mathcal{K}$ is monodromic, so we have

$$\mathcal{F}(i_{\mathcal{N}*} \mathbb{E}_{\mathcal{N}}) \simeq \mathbb{E}\mathcal{K}(-\nu)$$

Theorem 6.2.4 *We have*

$$\begin{aligned}
 \mathcal{F}(\mathbb{E}\mathcal{K}) &\simeq i_{\mathcal{N}*} \mathbb{E}_{\mathcal{N}}(-\nu - r) \\
 \mathcal{F}(i_{\mathcal{N}*} \mathbb{E}_{\mathcal{N}}) &\simeq \mathbb{E}\mathcal{K}(-\nu)
 \end{aligned}$$

Note that this proves a second time that $\mathcal{K}_{\mathcal{N}}$ is perverse.

Corollary 6.2.5 *The functors $j_{rs!*$, $\mathcal{F}(-)(\nu + r)$ and $i_{\mathcal{N}*}$ induce isomorphisms*

$$\mathbb{E}W = \text{End}(\mathbb{E}\mathcal{K}_{rs}) \xrightarrow{\sim} \text{End}(\mathbb{E}\mathcal{K}) \xrightarrow{\sim} \text{End}(i_{\mathcal{N}*} \mathbb{E}\mathcal{K}_{\mathcal{N}}) \xleftarrow{\sim} \text{End}(\mathbb{E}\mathcal{K}_{\mathcal{N}})$$

For $E \in \mathbb{E}W\text{-mod}$, let $\mathcal{T}(E) = \mathcal{F}j_{rs!*}(E[2\nu + r])(\nu + r)$. By the theorem above, we have $\mathcal{T}(\mathbb{E}W) = \mathcal{K}_{\mathcal{N}}$. More correctly, we should say that $\mathcal{T}(\mathbb{E}W)$ is supported on \mathcal{N} , and write $\mathcal{T}(\mathbb{E}W) = i_{\mathcal{N}*}\mathcal{K}_{\mathcal{N}}$, but we identify the perverse sheaves on \mathcal{N} with their extension by zero on \mathfrak{g} .

Corollary 6.2.6 *The perverse sheaf $\mathbb{K}\mathcal{K}_{\mathcal{N}}$ is semisimple, and we have the decomposition*

$$\mathbb{K}\mathcal{K}_{\mathcal{N}} = \bigoplus_{E \in \text{Irr } \mathbb{K}W} \mathcal{T}(E)^{\dim E}$$

Similarly, we have decompositions into indecomposable summands

$$\mathcal{K}_{\mathcal{N}} = \bigoplus_{F \in \text{Irr } \mathbb{F}W} \mathcal{T}(P_F)^{\dim F}$$

and

$$\mathbb{F}\mathcal{K}_{\mathcal{N}} = \bigoplus_{F \in \text{Irr } \mathbb{F}W} \mathcal{T}(\mathbb{F}P_F)^{\dim F}$$

The indecomposable summand $\mathcal{T}(\mathbb{F}P_F)$ has head and socle isomorphic to $\mathcal{T}(F)$.

6.2.4 Springer correspondence by Fourier-Deligne transform

Springer correspondence for $\mathbb{K}W$

Let E be a simple $\mathbb{K}W$ -module. Then $\mathcal{T}(E)$ is a direct summand of $\mathcal{K}_{\mathcal{N}}$. Hence it is a G -equivariant simple perverse sheaf supported on \mathcal{N} , so it is of the form $\mathcal{J}_{!*}(\mathcal{O}_E, \mathcal{L}_E)$ for some adjoint orbit \mathcal{O}_E in \mathcal{N} , and some irreducible G -equivariant local system on \mathcal{O}_E . We may thus associate to the simple $\mathbb{K}W$ -module E the pair $(\mathcal{O}_E, \mathcal{L}_E)$ or equivalently the pair (x_E, ρ_E) (up to G -conjugacy), where x_E is a representative of the orbit \mathcal{O}_E , and ρ_E is the irreducible character of $A_G(x_E)$ corresponding to \mathcal{L}_E .

Let $\mathfrak{N}_{\mathbb{K}}(G)$ (or simply $\mathfrak{N}_{\mathbb{K}}$) be the set of all pairs $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is an adjoint orbit in \mathcal{N} , and \mathcal{L} is an irreducible G -equivariant local system on \mathcal{O} (over \mathbb{K}). This finite set parametrizes the simple G -equivariant simple perverse sheaves on \mathcal{N} .

Let us denote by $\Psi_{\mathbb{K}} : \text{Irr } \mathbb{K}W \rightarrow \mathfrak{N}_{\mathbb{K}}$ be the map defined above, and let $\mathfrak{N}_{\mathbb{K}}^0$ be its image. Then $\Psi_{\mathbb{K}}$ induces a bijection from $\text{Irr } \mathbb{K}W$ to $\mathfrak{N}_{\mathbb{K}}^0$ (that is, $\Psi_{\mathbb{K}}$ is injective). Indeed, if we know $\Psi(E)$ (or equivalently, $\mathcal{T}(E)$) one can recover $j_{rs!*}(\mathcal{L}(E)[2\nu + r])$ by applying Theorem 5.2.3 (INV), and then restricting to \mathfrak{g}_{rs} we get the local system we started with, and hence the representation E .

Theorem 6.2.7 *The map $\Psi_{\mathbb{K}}$ defined above induces a bijection $\text{Irr } \mathbb{K}W \xrightarrow{\sim} \mathfrak{N}_{\mathbb{K}}^0$.*

Springer correspondence for $\mathbb{F}W$

Let F be a simple $\mathbb{F}W$ -module. Then, by Proposition 1.6.1, $\mathcal{T}(F)$ is the head (and also the socle) of $\mathcal{T}(\mathbb{F}P_F)$, which is a direct summand of $\mathcal{K}_{\mathcal{N}}$. Hence $\mathcal{T}(F)$ is supported on \mathcal{N} , and $\mathcal{T}(F) = \mathcal{J}_{!*}(\mathcal{O}_F, \mathcal{L}_F)$ for some pair $\Psi_{\mathbb{F}}(F) = (\mathcal{O}_F, \mathcal{L}_F)$ in the set $\mathfrak{N}_{\mathbb{F}}(G)$ (or simply $\mathfrak{N}_{\mathbb{F}}$) of all pairs $(\mathcal{O}, \mathcal{L})$, where \mathcal{O} is an adjoint orbit in \mathcal{N} , and \mathcal{L} is an irreducible G -equivariant local system on \mathcal{O} (over \mathbb{F}). We will denote by $\mathfrak{N}_{\mathbb{F}}^0$ the image of $\Psi_{\mathbb{F}} : \text{Irr } \mathbb{F}W \rightarrow \mathfrak{N}_{\mathbb{F}}$. Again, $\Psi_{\mathbb{F}}$ is injective.

Theorem 6.2.8 *The map $\Psi_{\mathbb{F}}$ defined above induces a bijection $\text{Irr } \mathbb{F}W \xrightarrow{\sim} \mathfrak{N}_{\mathbb{F}}^0$.*

Let us remark that if ℓ does not divide the order of any of the finite groups $A_G(x)$, $x \in \mathcal{N}$, then all the group algebras $\mathbb{F}A_G(x)$ are semisimple, so for each x there is a natural bijection $\text{Irr } \mathbb{K}A_G(x) \xrightarrow{\sim} \text{Irr } \mathbb{F}A_G(x)$, and thus there is a natural bijection $\mathfrak{N}_{\mathbb{K}} \xrightarrow{\sim} \mathfrak{N}_{\mathbb{F}}$.

6.3 Decomposition matrices

6.3.1 Comparison of e maps

Theorem 6.3.1 *Let $F \in \text{Irr } \mathbb{F}W$. Then $\mathcal{T}(\mathbb{K}P_F)$ is supported on \mathcal{N} , and for each $E \in \text{Irr } \mathbb{K}W$ we have*

$$[\mathbb{K}P_F : E] = [\mathcal{T}(\mathbb{K}P_F) : \mathcal{T}(E)]$$

Proof. We have

$$\bigoplus_{F \in \mathbb{F}W} \mathcal{T}(\mathbb{K}P_F)^{\dim F} = \mathcal{T}(\mathbb{K}W) = \mathbb{K}\mathcal{K}_{\mathcal{N}}$$

hence $\mathcal{T}(\mathbb{K}P_F)$ is supported on \mathcal{N} .

$$\begin{aligned} [\mathbb{K}P_F : E] &= [j_{rs!*} \mathbb{K}P_F : j_{rs!*} E] && \text{by Prop. 1.6.2} \\ &= [\mathcal{F}j_{rs!*} \mathbb{K}P_F(\nu + r) : \mathcal{F}j_{rs!*} E(\nu + r)] \\ &= [\mathbb{K}\mathcal{T}(P_F) : \mathcal{T}(E)] \end{aligned}$$

□

6.3.2 Comparison of d maps

If $E \in \text{Irr } \mathbb{K}W$ and $F \in \text{Irr } \mathbb{F}W$, let $d_{E,F}^W$ be the corresponding decomposition number.

Theorem 6.3.2 *Let $E \in \text{Irr } \mathbb{K}W$, and let $E_{\mathbb{O}}$ be an integral form for E . Then $\mathcal{T}(E_{\mathbb{O}})$ is supported on \mathcal{N} , and for each $F \in \text{Irr } \mathbb{F}W$ we have*

$$[\mathbb{F}E_{\mathbb{O}} : F] = [\mathbb{F}\mathcal{T}(E_{\mathbb{O}}) : \mathcal{T}(F)]$$

Thus

$$d_{\Psi_{\mathbb{K}}(E), \Psi_{\mathbb{F}}(F)} = d_{E,F}^W$$

Proof.

We have a short exact sequence

$$0 \longrightarrow T \longrightarrow \mathbb{F}j_{rs!_*}\underline{E}_\mathbb{O} \longrightarrow j_{rs!_*}\mathbb{F}\underline{E}_\mathbb{O} \longrightarrow 0 \quad (6.2)$$

with T supported on $\mathfrak{g} - \mathfrak{g}_{rs}$.

We have $\mathbb{K}\mathcal{T}(E_\mathbb{O}) = \mathcal{T}(E)$, so it is supported by \mathcal{N} . Let $j : \mathfrak{g} \setminus \mathcal{N} \rightarrow \mathfrak{g}$ be the open immersion. By what we have just said, $j^*\mathcal{T}(E_\mathbb{O})$ is a torsion perverse sheaf. Hence ${}^p j_! j^*\mathcal{T}(E_\mathbb{O})$ is torsion, and thus the adjunction morphism ${}^p j_! j^*\mathcal{T}(E_\mathbb{O}) \rightarrow \mathcal{T}(E_\mathbb{O})$ is zero, because $\mathcal{T}(E_\mathbb{O})$ is torsion-free. But the identity of $j^*\mathcal{T}(E_\mathbb{O})$ factors through $j^*\mathcal{T}(E_\mathbb{O}) \rightarrow j^* {}^p j_! j^*\mathcal{T}(E_\mathbb{O}) \rightarrow j^*\mathcal{T}(E_\mathbb{O})$, so $j^*\mathcal{T}(E_\mathbb{O})$ is zero, that is, $\mathcal{T}(E_\mathbb{O})$ is supported on \mathcal{N} .

We have

$$\begin{aligned} & [\mathbb{F}E_\mathbb{O} : F] \\ &= [j_{rs!_*}\mathbb{F}\underline{E}_\mathbb{O} : j_{rs!_*}\underline{E}] && \text{by Prop. 1.6.2} \\ &= [\mathbb{F}j_{rs!_*}\underline{E}_\mathbb{O} : j_{rs!_*}\underline{E}] && \text{by (6.2) and } [T : j_{rs!_*}\underline{E}] = 0 \\ &= [\mathbb{F}\mathcal{F}(j_{rs!_*}\underline{E}_\mathbb{O})(\nu + r) : \mathcal{F}(j_{rs!_*}\underline{E})(\nu + r)] && \text{by EQUIV and } \mathcal{F}\mathbb{F} = \mathbb{F}\mathcal{F} \\ &= [\mathbb{F}\mathcal{T}(E_\mathbb{O}) : \mathcal{T}(F)] \end{aligned}$$

□

This theorem means that we can obtain the decomposition matrix of the Weyl group W by extracting certain rows (the image $\mathfrak{N}_\mathbb{K}^0$ of the ordinary Springer correspondence) and certain columns (the image $\mathfrak{N}_\mathbb{F}^0$ of the modular Springer correspondence) of the decomposition matrix for G -equivariant perverse sheaves on the nilpotent variety \mathcal{N} .

6.4 Modular Springer correspondence for GL_n

For the symmetric group \mathfrak{S}_n , we have a Specht module theory compatible with the order on the nilpotent orbits through the Springer correspondence. This is enough to determine the modular Springer correspondence for $G = GL_n$.

Since all the groups $A_G(x)$ are trivial, we only need to parametrize nilpotent orbits, which is done using the Jordan normal form. So $\mathfrak{N}_\mathbb{K} = \mathfrak{N}_\mathbb{F}$ is the set \mathfrak{P}_n of partitions of n . If λ is a partition of n , we denote by \mathcal{O}_λ the corresponding orbit, and by x_λ an element of this orbit. Moreover, to simplify the notation, we set $\mathbf{IC}(\lambda, \mathbb{E}) = {}^p \mathcal{J}_{!*}(\mathcal{O}_\lambda, \mathbb{E})$. In characteristic zero, it is known that $\mathfrak{N}_\mathbb{K}^0 = \mathfrak{P}_n$, and $\mathcal{T}(S^\lambda) = \mathbf{IC}(\lambda', \mathbb{K})$ for λ in \mathfrak{P}_n .

Theorem 6.4.1 *Suppose $G = GL_n$. If μ is an ℓ -regular partition, then we have*

$$\mathcal{T}(D^\mu) = \mathbf{IC}(\mu', \mathbb{F})$$

where μ' is the partition dual to μ . Thus we have

$$\mathfrak{N}_\mathbb{F}^0 = \{\lambda \vdash n \mid \lambda \text{ is } \ell\text{-restricted}\}$$

and, for two partitions λ and μ , with μ ℓ -regular, we have $d_{\lambda, \mu}^W = d_{\lambda', \mu'}$.

Proof. We prove by induction on $\lambda \in \mathfrak{P}_n$ that λ is in $\mathfrak{N}_{\mathbb{F}}^0$ if and only if λ is ℓ -restricted, and that, in this case, $\mathcal{T}(D^{\lambda'}) = \mathbf{IC}(\lambda, \mathbb{F})$.

First, note that the partition $\lambda = (1^n)$ is always ℓ -restricted, and that $\mathcal{T}(D^{(n)}) = \mathcal{F}(\mathbb{F}_{\mathfrak{g}}[2\nu + r]) = \mathbf{IC}((1^n), \mathbb{F})$ by DIRAC.

Now assume the claim has been proved for all $\nu < \lambda$. If λ is in $\mathfrak{N}_{\mathbb{F}}^0$, let μ be the ℓ -regular partition such that $\Psi_{\mathbb{F}}(D^{\mu}) = \lambda$. We have

$$d_{\lambda', \mu}^{\mathfrak{S}_n} = d_{\Psi_{\mathbb{K}}(S^{\lambda'}), \Psi_{\mathbb{F}}(D^{\mu})} = d_{\lambda, \lambda} = 1 \neq 0$$

hence $\lambda' \leq \mu$, and thus $\mu' \leq \lambda$. If equality holds, we are done, since μ' is ℓ -restricted and $\Psi_{\mathbb{F}}(D^{\lambda'}) = \Psi_{\mathbb{F}}(D^{\mu}) = \lambda$. But a strict inequality $\mu' < \lambda$ would lead to a contradiction. Indeed, by the induction hypothesis, we would have $\Psi_{\mathbb{F}}(D^{\mu}) = \mu'$, and this would contradict the choice of μ ($\Psi_{\mathbb{F}}(D^{\mu}) = \lambda$).

In the other direction, let us assume that λ is ℓ -restricted. Let $\mu = \Psi_{\mathbb{F}}(D^{\lambda'})$. Then we have

$$d_{\lambda, \mu} = d_{\Psi_{\mathbb{K}}(S^{\lambda'}), \Psi_{\mathbb{F}}(D^{\lambda'})} = d_{\lambda', \lambda'}^{\mathfrak{S}_n} = 1 \neq 0$$

hence

$$\mu \leq \lambda$$

We cannot have $\mu < \lambda$, because this would imply $\Psi_{\mathbb{F}}(\lambda') = \mu = \Psi_{\mathbb{F}}(D^{\mu'})$ by induction, hence $\lambda' = \mu'$ since $\Psi_{\mathbb{F}}$ is injective, and $\lambda = \mu$. Thus we must have $\lambda = \mu = \Psi_{\mathbb{F}}(D^{\lambda'})$. \square

By Proposition 4.3.2, we can now say that the result of Kraft and Procesi implies James's row and column removal rule.

Chapter 7

Tables

In this chapter, we give tables for types of rank at most 3. In each case, we give the character table of W (including ordinary and modular characters) and the (ordinary and modular) Springer correspondence. Be aware that, to get the correspondence obtained by the Lusztig-Borho-MacPherson approach, one should tensor all the representations of W by the sign representation. The ordinary characters and the conjugacy classes of W (resp. the nilpotent orbits) are labeled in the usual way, as for example in the book [Car85]. I used the tables given there for the ordinary Springer correspondence (but I had to tensor by the sign character).

Afterwards, we give the decomposition matrix for G -equivariant perverse sheaves on the nilpotent variety (we assume G is simple of adjoint type, and that p is very good for G). In favorable cases, it is complete. There are more indeterminacies for $\ell = 2$.

The rows correspond to pairs in $\mathfrak{N}_{\mathbb{K}}$, while the columns correspond to pairs in $\mathfrak{N}_{\mathbb{F}}$. When the label of a row or a column is just a nilpotent orbit, then the default local system is the constant one. For the others, we use ε for the sign character, and ψ for the irreducible character of degree 2 of \mathfrak{S}_3 .

The entries that we were able to determine geometrically in chapter 4 are in italics (see [KP82] for the description of the minimal degenerations in classical types). The rows and columns corresponding to the Weyl group (the image of the ordinary resp. modular Springer correspondence) have bold labels, and the corresponding entries are underlined and (if you can see this document in colors) red. The zeros above the diagonal are represented by dots. The unknown entries are left blank.

7.1 Type A_1

		cardinal order class	1 1 1^2	1 2 2
orbit	char	a -function		
1^2	χ_2	0	1	1
2	χ_{1^2}	1	1	-1
1^2	ϕ_2	$\bar{\chi}_2$	1	1

$$\ell = 2$$

	1^2	2
1^2	<u>1</u>	.
2	<u>1</u>	1

7.2 Type A_2

		cardinal order class	1 1 1^3	1 2 21	2 3 3
orbit	char	a -function			
1^3	χ_3	0	1	1	1
21	χ_{21}	1	1	-1	1
3	χ_{1^3}	3	2	0	-1
1^3	ϕ_3	$\bar{\chi}_3$	1		1
21	ϕ_{21}	$\bar{\chi}_{21}$	2		-1
1^3	ϕ_3	$\bar{\chi}_3$	1	1	
21	ϕ_{21}	$\bar{\chi}_{1^3}$	1	-1	

 $\ell = 2$

	1^3	21	3
1^3	<u>1</u>	.	.
21	<u>0</u>	<u>1</u>	.
3	<u>1</u>	<u>0</u>	1

 $\ell = 3$

	1^3	21	3
1^3	<u>1</u>	.	.
21	<u>1</u>	<u>1</u>	.
3	<u>0</u>	<u>1</u>	1

7.3 Type A_3

		cardinal order class	1 1 1^4	6 2 21^2	3 2 2^2	3 3 31	6 4 4
orbit	char	a -function					
1^4	χ_4	0	1	1	1	1	1
21^2	χ_{31}	1	3	1	-1	0	-1
2^2	χ_{2^2}	2	2	0	2	-1	0
31	χ_{21^2}	3	3	-1	-1	0	1
4	χ_{1^4}	6	1	-1	1	1	-1
1^4	ϕ_3	$\bar{\chi}_3$	1		1		
21^2	ϕ_{21}	$\bar{\chi}_{21}$	2		-1		
1^4	ϕ_4	$\bar{\chi}_4$	1	1	1		1
21^2	ϕ_{31}	$\bar{\chi}_{31}$	3	1	-1		-1
2^2	ϕ_{2^2}	$\bar{\chi}_{1^4}$	1	-1	1		-1
31	ϕ_{21^2}	$\bar{\chi}_{21^2}$	3	-1	-1		1

$\ell = 2$

	1^4	21^2	2^2	31	4
1^4	<u>1</u>
21^2	<u>1</u>	<u>1</u>	.	.	.
2^2	<u>0</u>	<u>1</u>	1	.	.
31	<u>1</u>	<u>1</u>	1	1	.
4	<u>1</u>	<u>0</u>	(1)	1	1

The (1) is the decomposition number of the Schur algebra, so it is expected to be the right decomposition number for perverse sheaves.

$\ell = 3$

	1^4	21^2	2^2	31	4
1^4	<u>1</u>
21^2	<u>0</u>	<u>1</u>	.	.	.
2^2	<u>1</u>	<u>0</u>	<u>1</u>	.	.
31	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	.
4	<u>0</u>	<u>0</u>	<u>1</u>	<u>0</u>	1

7.4 Type B_2

		cardinal order class	1 1 $1^2, -$	2 2 1, 1	1 2 $-, 1^2$	2 2 2, -	2 4 $-, 2$
orbit	char	a -function					
1^5	$\chi_{2,-}$	0	1	1	1	1	1
$2^2 1$	$\chi_{1^2,-}$	1	1	1	1	-1	-1
$3 1^2$	$\chi_{1,1}$	1	2	.	-2	.	.
$ \varepsilon$	$\chi_{-,2}$	1	1	-1	1	1	-1
$\bar{5}$	$\chi_{-,1^2}$	4	1	-1	1	-1	1
1^5	$\phi_{2,-}$	$\bar{\chi}_{2,-}$	1				

 $\ell = 2$

	1^5	$2^2 1$	$3 1^2$	5
1^5	<u>1</u>	.	.	.
$2^2 1$	<u>1</u>	1	.	.
$3 1^2$	<u>2</u>	1	1	.
$ \varepsilon$	<u>1</u>		1	.
$\bar{5}$	<u>1</u>		1	1

For C_2 , we have the same geometry and the same Weyl group. One only needs to replace the labels of the nilpotent orbits by 1^4 , $2 1^2$, 2^2 , 4 .

7.5 Type B_3

		cardinal order class	1 1 $1^3, -$	3 2 $1^2, 1$	3 2 $1, 1^2$	1 2 $-, 1^3$	6 2 $21, -$	6 4 $1, 2$	6 2 $2, 1$	6 4 $-, 21$	8 3 $3, -$	8 6 $-, 3$
orbit	char	a -function										
1^7	$\chi_{3,-}$	0	1	1	1	1	1	1	1	1	1	1
$2^2 1^3$	$\chi_{21,-}$	1	2	2	2	2	-1	-1
31^4	$\chi_{2,1}$	1	3	1	-1	-3	1	1	-1	-1	.	.
$ \varepsilon$	$\chi_{-,3}$	1	1	-1	1	-1	1	-1	-1	1	1	-1
32^2	$\chi_{1,2}$	2	3	-1	-1	3	1	-1	1	-1	.	.
$3^2 1$	$\chi_{1^2,1}$	3	3	1	-1	-3	-1	-1	1	1	.	.
$ \varepsilon$	$\chi_{1^3,-}$	4	1	1	1	1	-1	-1	-1	-1	1	1
51^2	$\chi_{1,1^2}$	4	3	-1	-1	3	-1	1	-1	1	.	.
$ \varepsilon$	$\chi_{-,21}$	4	2	-2	2	-2	-1	1
7	$\chi_{-,1^3}$	9	1	-1	1	-1	-1	1	1	-1	1	-1
1^6	$\phi_{3,-}$	$\bar{\chi}_{3,-}$	1								1	
$2^2 1^3$	$\phi_{21,-}$	$\bar{\chi}_{21,-}$	2								-1	
1^7	$\phi_{3,-}$	$\bar{\chi}_{3,-}$	1	1	1	1	1	1	1	1		
$2^2 1^3$	$\phi_{21,-}$	$\bar{\chi}_{1^3,-}$	1	1	1	1	-1	-1	-1	-1		
31^4	$\phi_{2,1}$	$\bar{\chi}_{2,1}$	3	1	-1	-3	1	1	-1	-1		
$ \varepsilon$	$\phi_{-,3}$	$\bar{\chi}_{-,3}$	1	-1	1	-1	1	-1	-1	1		
32^2	$\phi_{1,2}$	$\bar{\chi}_{1,2}$	3	-1	-1	3	1	-1	1	-1		
$3^2 1$	$\phi_{1^2,1}$	$\bar{\chi}_{1^2,1}$	3	1	-1	-3	-1	-1	1	1		
51^2	$\phi_{1,1^2}$	$\bar{\chi}_{1,1^2}$	3	-1	-1	3	-1	1	-1	1		
$ \varepsilon$	$\phi_{-,21}$	$\bar{\chi}_{-,1^3}$	1	-1	1	-1	-1	1	1	-1		

$\ell = 2$

	1^7	$2^2 1^3$	31^4	32^2	$3^2 1$	51^2	7
1^7	<u>1</u>
$2^2 1^3$	<u>0</u>	<u>1</u>
31^4	<u>1</u>	<u>1</u>	1
$ \varepsilon$	<u>1</u>	<u>0</u>	1
32^2	<u>1</u>	<u>1</u>	2	1	.	.	.
$3^2 1$	<u>1</u>	<u>1</u>		1	1	.	.
$ \varepsilon$	<u>1</u>	<u>0</u>			1	.	.
51^2	<u>1</u>	<u>1</u>			1	1	.
$ \varepsilon$	<u>0</u>	<u>1</u>				1	.
7	<u>1</u>	<u>0</u>				1	1

 $\ell = 3$

	1^7	$2^2 1^3$	31^4	ε	32^2	$3^2 1$	ε	51^2	ε	7
1^7	<u>1</u>
$2^2 1^3$	<u>1</u>	<u>1</u>
31^4	<u>0</u>	<u>0</u>	<u>1</u>
$ \varepsilon$	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>
32^2	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>
$3^2 1$	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>
$ \varepsilon$	<u>0</u>	<u>1</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	1	.	.	.
51^2	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	0	<u>1</u>	.	.
$ \varepsilon$	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	<u>0</u>	<u>0</u>		<u>0</u>	<u>1</u>	.
7	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>		<u>0</u>	<u>1</u>	1

7.6 Type C_3

		cardinal order class	1 1 $1^3, -$	3 2 $1^2, 1$	3 2 $1, 1^2$	1 2 $-, 1^3$	6 2 $21, -$	6 4 $1, 2$	6 2 $2, 1$	6 4 $-, 21$	8 3 $3, -$	8 6 $-, 3$
orbit	char	a -function										
1^6	$\chi_{3,-}$	0	1	1	1	1	1	1	1	1	1	1
21^4	$\chi_{-,3}$	1	1	-1	1	-1	1	-1	-1	1	1	-1
$2^2 1^2$	$\chi_{2,1}$	1	3	1	-1	-3	1	1	-1	-1	.	.
$ \varepsilon$	$\chi_{21,-}$	1	2	2	2	2	-1	-1
2^3	$\chi_{1,2}$	2	3	-1	-1	3	1	-1	1	-1	.	.
3^2	$\chi_{1^2,1}$	3	3	1	-1	-3	-1	-1	1	1	.	.
41^2	$\chi_{-,21}$	4	2	-2	2	-2	-1	1
42	$\chi_{1,1^2}$	4	3	-1	-1	3	-1	1	-1	1	.	.
$ \varepsilon$	$\chi_{1^3,-}$	4	1	1	1	1	-1	-1	-1	-1	1	1
6	$\chi_{-,1^3}$	9	1	-1	1	-1	-1	1	1	-1	1	-1
1^6	$\phi_{3,-}$	$\bar{\chi}_{3,-}$	1								1	
$2^2 1^2$	$\phi_{2,1}$	$\bar{\chi}_{21,-}$	2								-1	
1^6	$\phi_{3,-}$	$\bar{\chi}_{3,-}$	1	1	1	1	1	1	1	1		
21^4	$\phi_{-,3}$	$\bar{\chi}_{-,3}$	1	-1	1	-1	1	-1	-1	1		
$2^2 1^2$	$\phi_{2,1}$	$\bar{\chi}_{2,1}$	3	1	-1	-3	1	1	-1	-1		
$ \varepsilon$	$\phi_{21,-}$	$\bar{\chi}_{1^3,-}$	1	1	1	1	-1	-1	-1	-1		
2^3	$\phi_{1,2}$	$\bar{\chi}_{1,2}$	3	-1	-1	3	1	-1	1	-1		
3^2	$\phi_{1^2,1}$	$\bar{\chi}_{1^2,1}$	3	1	-1	-3	-1	-1	1	1		
41^2	$\phi_{-,21}$	$\bar{\chi}_{-,1^3}$	1	-1	1	-1	-1	1	1	-1		
$42, 1$	$\phi_{1,1^2}$	$\bar{\chi}_{1,1^2}$	3	-1	-1	3	-1	1	-1	1		

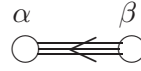
$\ell = 2$

	1^6	21^4	2^21^2	2^3	3^2	41^2	42	6
1^6	<u>1</u>
21^4	<u>1</u>	1
2^21^2	<u>1</u>	1	<u>1</u>
$ \varepsilon$	<u>0</u>		<u>1</u>
2^3	<u>1</u>		<u>1</u>	1
3^2	<u>1</u>		<u>1</u>	1	1	.	.	.
41^2	<u>0</u>		<u>1</u>	1	0	1	.	.
42	<u>1</u>		<u>1</u>		1	1	1	.
$ \varepsilon$	<u>1</u>		<u>0</u>				1	.
6	<u>1</u>		<u>0</u>				2	1

 $\ell = 3$

	1^6	21^4	2^21^2	ε	2^3	3^2	41^2	42	ε	6
1^6	<u>1</u>
21^4	<u>0</u>	<u>1</u>
2^21^2	<u>0</u>	<u>0</u>	<u>1</u>
$ \varepsilon$	<u>1</u>	<u>0</u>	<u>0</u>	<u>1</u>
2^3	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>
3^2	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>
41^2	<u>0</u>	<u>1</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	.	.	.
42	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	.	.
$ \varepsilon$	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	1	.
6	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	<u>0</u>	0	1

7.7 Type G_2



		cardinal order class	1	3	3	2	2	1
			1	2	2	6	3	2
			1	s_α	s_β	$s_\alpha s_\beta$	$(s_\alpha s_\beta)^2$	-1
orbit	char	a -function						
1	$\chi_{1,0}$	0	1	1	1	1	1	1
A_1	$\chi'_{1,3}$	1	1	1	-1	-1	1	-1
\tilde{A}_1	$\chi_{2,2}$	1	2	0	0	-1	-1	2
$G_2(a_1)$	$\chi_{2,1}$	1	2	0	0	1	-1	-2
$ \psi$	$\chi''_{1,3}$	1	1	-1	1	-1	1	-1
G_2	$\chi_{1,6}$	6	1	-1	-1	1	1	1
1	$\phi_{1,0}$	$\bar{\chi}_{1,0}$	1				1	
\tilde{A}_1	$\phi_{2,2}$	$\bar{\chi}_{2,2}$	2				-1	
1	$\phi_{1,0}$	$\bar{\chi}_{1,0}$	1	1	1			1
A_1	$\phi'_{1,3}$	$\bar{\chi}'_{1,3}$	1	1	-1			-1
\tilde{A}_1	$\phi_{2,2}$	$\bar{\chi}_{1,6}$	1	-1	-1			1
$G_2(a_1)$	$\phi_{2,1}$	$\bar{\chi}''_{1,3}$	1	-1	1			-1

$\ell = 2$

	1	A_1	\tilde{A}_1	$G_2(a_1)$	ψ	G_2
1	<u>1</u>
A_1	<u>1</u>	1
\tilde{A}_1	<u>0</u>		<u>1</u>	.	.	.
$G_2(a_1)$	<u>0</u>		<u>1</u>	1	.	.
$ \psi$	<u>1</u>		<u>0</u>	0	1	.
$ \varepsilon$	0	0	0	1	0	.
G_2	<u>1</u>		<u>0</u>	0	1	1

$\ell = 3$

	1	A_1	\tilde{A}_1	$G_2(a_1)$	ε	G_2
1	<u>1</u>
A_1	<u>0</u>	<u>1</u>
\tilde{A}_1	<u>1</u>	<u>0</u>	<u>1</u>	.	.	.
$G_2(a_1)$	<u>0</u>	<u>1</u>	<u>0</u>	<u>1</u>	.	.
$ \psi$	<u>0</u>	<u>0</u>	<u>0</u>	<u>1</u>	1	.
$ \varepsilon$	0	0	0	0	1	.
G_2	<u>0</u>	<u>0</u>	<u>1</u>	<u>0</u>	0	1

Chapter 8

Character sheaves on \mathfrak{sl}_2

Up to now, we have been paying a lot of attention to $\mathcal{K}_{\mathcal{N}}$, but not so much to \mathcal{K} itself. We are going to give a full description in the case of $G = SL_2$, and we will give a hint of what we might do in type A_n to construct a Springer correspondence involving the Schur algebra.

The general idea is to replace the regular representation of W , which is $\text{Ind}_{\mathbb{E}1}^{\mathbb{E}W} \mathbb{E}$, by a direct sum of all permutation modules over standard parabolic subgroups:

$$\bigoplus_{\lambda \in \mathfrak{P}} \text{Ind}_{\mathbb{E}\mathfrak{S}_{\lambda}}^{\mathbb{E}\mathfrak{S}_n} \mathbb{E}$$

whose indecomposable summands are the Young modules Y^{λ} , with some multiplicities, and whose endomorphism algebra is the Schur algebra $S_{\mathbb{E}}(n) = S_{\mathbb{E}}(n, n)$. This corresponds to a (shifted) local system $\mathbb{E}\mathcal{K}'_{\text{rs}}$ on \mathfrak{g}_{rs} , and we call $\mathbb{E}\mathcal{K}'$ its intermediate extension on \mathfrak{g} . We still have $\text{End}(\mathbb{E}\mathcal{K}') \simeq S_{\mathbb{E}}(n)$. I conjecture that the restriction functor $i_{\mathcal{N}}^*[-1]$ sends $\mathbb{E}\mathcal{K}'$ to a perverse sheaf on \mathcal{N} with the same endomorphism algebra, which would enable us to make a direct link between the Schur algebra and the G -equivariant perverse sheaves on \mathcal{N} , and between both decomposition matrices. Let us see what happens for $\mathfrak{g} = \mathfrak{sl}_2$.

Apart from the open stratum \mathfrak{g}_{rs} , we just have the two nilpotent orbits $\mathcal{O}_{\text{reg}} = \mathcal{O}_{\text{min}} = \mathcal{O}_{(2)}$ and $\mathcal{O}_{\text{triv}} = \mathcal{O}_{\text{subreg}} = \mathcal{O}_{(1^2)} = \{0\}$. On \mathfrak{g}_{rs} , we will only consider local systems which become trivial after a pullback by π_{rs} . We have $W = \mathfrak{S}_2$. The local system $\pi_{\text{rs}*}\mathbb{E}$ corresponds to the regular representation $\mathbb{E}\mathfrak{S}_2$.

In characteristic 0, the group algebra is semi-simple, and the perverse sheaf $\mathbb{K}\mathcal{K}_{\text{rs}}$ splits as the sum of the constant perverse sheaf C^{rs} and the shifted local system $C_{\varepsilon}^{\text{rs}}$ corresponding to the sign representation of \mathfrak{S}_2 . These two simple components are sent by $j_{\text{rs}!}$ on two simple perverse sheaves on \mathfrak{g} , the constant perverse sheaf C (since \mathfrak{g} is smooth), and the other one, C_{ε} . Let us denote by A the simple perverse sheaf supported on $\{0\}$, and by B the simple perverse sheaf ${}^p\mathcal{J}_{!}(\mathcal{O}_{\text{reg}}, \mathbb{K})$. Since $\mathcal{F}(C) = A$, we must have $\mathcal{F}(C_{\varepsilon}) = B$. This gives the Springer correspondence for \mathfrak{sl}_2 by Fourier transform.

Let us make tables for the stalks of the perverse sheaves involved. We have a line for each stratum, and one column for each cohomology degree. If x is a point of a given stratum \mathcal{O} and i is an integer, the corresponding entry in the table of a perverse sheaf \mathcal{A} will be the class of $\mathcal{H}_x^i \mathcal{A}$, seen as a representation of a suitable group $A(\mathcal{O})$, in the Grothendieck group of $\mathbb{E}A(\mathcal{O})$. There is a column χ describing the alternating sum of the stalks of each stratum.

Let us first describe $\mathbb{K}\mathcal{K}$. So, over \mathfrak{g}_{rs} , we have the regular representation of \mathfrak{S}_2 . Over \mathcal{O}_{reg} , the fibers are single points, so the cohomology of $\mathcal{B}_{x_{\text{reg}}}$ is just \mathbb{K} . But we have $\mathcal{B}_0 = \mathcal{B} = G/B = \mathbb{P}^1$. We get the following table for $\mathbb{K}\mathcal{K}$.

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	$-1 - \varepsilon$	$\mathbb{K} \oplus \mathbb{K}_\varepsilon$.	.	.
\mathcal{O}_{reg}	2	-1	\mathbb{K}	.	.	.
$\mathcal{O}_{\text{triv}}$	0	-2	\mathbb{K}	.	\mathbb{K}	.

It is the direct sum of the two simple perverse sheaves C

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	-1	\mathbb{K}	.	.	.
\mathcal{O}_{reg}	2	-1	\mathbb{K}	.	.	.
$\mathcal{O}_{\text{triv}}$	0	-1	\mathbb{K}	.	.	.

and C_ε , which we deduce by subtraction (we have a direct sum !)

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	$-\varepsilon$	\mathbb{K}_ε	.	.	.
\mathcal{O}_{reg}	2	0
$\mathcal{O}_{\text{triv}}$	0	-1	.	.	\mathbb{K}	.

The simple G -equivariant perverse sheaves on \mathcal{N} are $B = {}^p\mathcal{J}_{!*}(\mathcal{O}_{\text{reg}}, \mathbb{K})$,

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}_{reg}	2	1	.	\mathbb{K}	.	.
$\mathcal{O}_{\text{triv}}$	0	1	.	\mathbb{K}	.	.

$A = {}^p\mathcal{J}_{!*}(\mathcal{O}_{\text{triv}}, \mathbb{K})$

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}_{reg}	2	0
$\mathcal{O}_{\text{triv}}$	0	1	.	.	.	\mathbb{K}

and the cuspidal $B_\varepsilon = {}^p\mathcal{J}_{!*}(\mathcal{O}_{\text{reg}}, \mathbb{K}_\varepsilon)$, which is clean (its intermediate extension is just the extension by zero), and stable by the Fourier-Deligne transform, by the general theory

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}_{reg}	2	1	.	\mathbb{K}_ε	.	.
$\mathcal{O}_{\text{triv}}$	0	1

We can check that, applying $i_{\mathcal{N}}^*[-1]$ to \mathcal{K} , we recover $\mathcal{K}_{\mathcal{N}}$. This functor sends C to B and C_ε to A . There is a twist by the sign character between the two versions of the Springer representations (by Fourier-Deligne transform, and by restriction).

So, to summarize the situation over \mathbb{K} , we have

\mathfrak{g}_{rs}	\mathfrak{g}	\mathcal{N}
$C^{\text{rs}} \oplus C_{\varepsilon}^{\text{rs}}$	$C \oplus C_{\varepsilon}$	$B \oplus A$

If $\ell \neq 2$, the situation over \mathbb{F} is similar. Now let us assume that $\ell = 2$. Then the sign representation becomes trivial. The regular representation is an extension of the trivial by the trivial, so $\mathbb{F}\mathcal{K}_{\text{rs}}$ is an extension of the constant c_{rs} (reduction of C_{rs}) by itself.

Now $\mathbb{F}\mathcal{K}$ is as follows

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	-2	$\mathbb{F}\mathfrak{S}_2$.	.	.
\mathcal{O}_{reg}	2	-1	\mathbb{F}	.	.	.
$\mathcal{O}_{\text{triv}}$	0	-2	\mathbb{F}	.	\mathbb{F}	.

It must be made of the simple perverse sheaves c , b and c , where c is the constant on \mathfrak{g} (it is the reduction of C , and has the same table with \mathbb{F} instead of \mathbb{K}), a is the constant on the origin (the reduction of A), and $b = {}^p\mathcal{J}_{!*}(\mathcal{O}_{\text{reg}}, \mathbb{F})$ has the following table

Stratum	Dimension	χ	-3	-2	-1	0
\mathfrak{g}_{rs}	3	0
\mathcal{O}^{reg}	2	1	.	\mathbb{F}	.	.
$\mathcal{O}^{\text{triv}}$	0	0	.	\mathbb{F}	\mathbb{F}	.

Looking at the χ functions, we see that $[\mathbb{F}\mathcal{K}] = 2[c] + [b]$ in the Grothendieck group of ${}^p\mathcal{M}_G(\mathcal{N}, \mathbb{F})$. We know that the top and the socle of $\mathbb{F}\mathcal{K}$ must be c , the intermediate extension of c , and that b cannot appear either in the top nor in the socle. Thus there is only one possible Loewy structure:

$$\mathbb{F}\mathcal{K} = \begin{array}{c} c \\ b \\ c \end{array}$$

Similarly, we find

$$\mathbb{F}\mathcal{K}_{\mathcal{N}} = \begin{array}{c} a \\ b \\ a \end{array}$$

Thus, as we already know, $\mathcal{F}(c) = a$, but we also deduce that $\mathcal{F}(b) = b$.

The restriction functor $i_{\mathcal{N}}^*$ sends c onto the reduction of B , which has the following Loewy structure (by Section 2.5):

$$\begin{array}{c} b \\ a \end{array}$$

The reduction of C_{ε} has structure

$$\begin{array}{c} c \\ b \end{array}$$

and it restricts to a (the reduction of A , which is the restriction of C_ε).

So we have the following situation.

\mathfrak{g}_{rs}	\mathfrak{g}	\mathcal{N}
c_{rs}	c	a
c_{rs}	b	b
	c	a

And we check that we can get $\mathcal{K}_{\mathcal{N}}$ either by Fourier-Deligne transform, or by restriction. For the Springer correspondence, b is missing. It appears neither in the top nor in the socle.

Now, let us see what happens with the sum of induced modules. What we lack here is the induction from \mathfrak{S}_2 to \mathfrak{S}_2 , which gives the trivial module, and hence the constant perverse sheaf c_{rs} . So, by restriction, we would get

\mathfrak{g}_{rs}	\mathfrak{g}	\mathcal{N}
$c_{\text{rs}} \oplus c_{\text{rs}}$	c	a
	$c \oplus b$	$b \oplus b$
	c	a

We can hope that, in general, each intermediate extension of a Young module will restrict to an indecomposable on \mathcal{N} with simple top, and that all the simple GL_n -equivariant perverse sheaves appear in this way. We would thus obtain a correspondence involving all the partitions of n , and we would certainly explain why the decomposition matrix for GL_n -equivariant perverse sheaves on \mathcal{N} must be the decomposition matrix for the Schur algebra. Of course, if we looked at SL_n , there would be supercuspidality phenomena, on top of that.

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