



Numerical modeling of anisothermal multi-phase flows in petroleum wellbore and reservoir

Loyal LIZAIK

Ph. D. Thesis advisors :

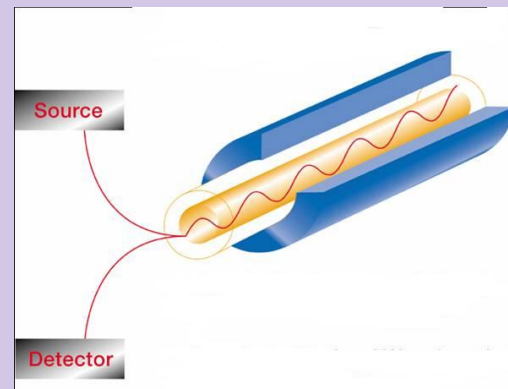
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18th of December 2008

Motivations

Optical fiber sensors

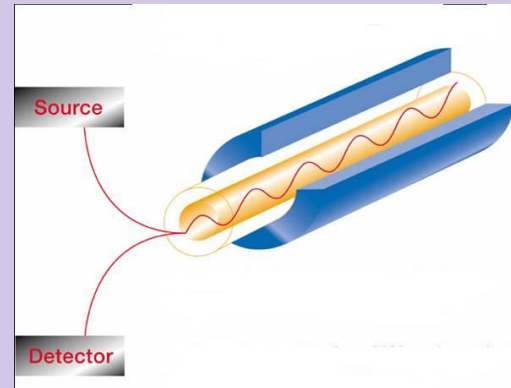
- Increase of subsea wellheads and highly deviated wells
 - ↳ Production log are less easy to be performed
- Emerging of new technologies such as optical fiber sensors
- Temperature measurements continuous in time and all along the well



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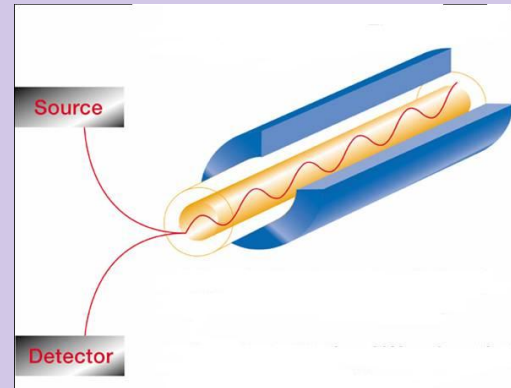
Possible applications

- Estimate virgin reservoir temperature
- Predict flow profiles and the flow rate of each layer

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Interpretation of temperature profiles

↪ Need for an **EXHAUSTIVE ENERGY EQUATION**

Mass conservation law

Single phase flow :

$$\frac{\partial(\phi\rho)}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0$$

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Multi-component multi-phase flow :

$$\sum_{p=1}^{n_p} \left(\frac{\partial}{\partial t} (\phi S_p \rho_p y_{c,p}) + \nabla \cdot (\rho_p y_{c,p} \mathbf{u}_p) \right) = 0, \quad c = 1, \dots, n_c$$

n_c is the number of components

n_p is the number of phases

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Generalized Darcy's law : $\mathbf{u}_p = -k_{rp} \mu_p^{-1} \underline{\mathbf{K}} (\nabla p_p - \rho_p \mathbf{g})$

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Darcy-Forchheimer law : $\mu\mathbf{K}^{-1}\mathbf{u} + F\rho|\mathbf{u}|\mathbf{u} = -(\nabla p - \rho\mathbf{g})$

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Energy conservation law

Single phase flow :

$$\frac{\partial(\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\underline{\tau} \mathbf{u})$$

- Total energy : $E = E_c + E_p + \mathcal{U}$

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- Kinetic energy : $\frac{\partial}{\partial t}(\frac{1}{2}\rho \mathbf{u}^2) + \nabla \cdot (\frac{1}{2}\rho \mathbf{u}^2 \mathbf{u}) = \rho \mathbf{g} \mathbf{u} - \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\nabla : \underline{\tau})$

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- Mass conservation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$
- Enthalpy expression : $\rho \mathcal{U} = \rho \mathcal{H} - p$

$$\Leftrightarrow \rho \frac{\partial \mathcal{H}}{\partial t} - \frac{\partial p}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathcal{H} - \mathbf{u} \cdot \nabla p - \nabla \cdot (\lambda \nabla T) - \underline{\tau} : \nabla \cdot \mathbf{u} = 0$$

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- $\left\{ \begin{array}{l} \frac{\partial \mathcal{H}}{\partial T} = \frac{1-\beta T}{\rho} \quad \beta \text{ coefficient of thermal expansion} \\ \frac{\partial \mathcal{H}}{\partial p} = \frac{(\rho c)_f}{\rho} \quad (\rho c)_f \text{ specific heat capacity of the fluid} \end{array} \right.$

$$\Leftrightarrow (\rho c)_* \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{u} \cdot \nabla T - \nabla \cdot (\lambda \nabla T) - \phi \beta T \frac{\partial p}{\partial t} - \beta T \mathbf{u} \cdot \nabla p - \underline{\tau} : \nabla \cdot \mathbf{u} = 0$$

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Equation considered in Denel's thesis

Energy conservation law

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Multi-component multi-phase flow :

$$\sum_p \left(\frac{\partial}{\partial t} (\phi S_p \rho_p \mathbf{u}_p) + \nabla \cdot (\phi S_p \rho_p \mathbf{u}_p \mathbf{u}_p) + p_p \nabla \cdot \mathbf{u}_p - \nabla \cdot (\lambda_p \nabla T) - \Phi_{\mu,p} \right) = 0$$

- **Equivalent conductivity :** $\lambda = (\lambda_s)^{(1-\phi)} \times (\lambda_w)^{S_w \times \phi} \times (\lambda_o)^{S_o \times \phi} \times (\lambda_g)^{S_g \times \phi}$

$$\sum_p \left(\frac{\partial}{\partial t} (\phi S_p \rho_p \mathbf{u}_p) + \nabla \cdot (\phi S_p \rho_p \mathbf{u}_p \mathbf{u}_p) + p_p \nabla \cdot \mathbf{u}_p \right) - \nabla \cdot (\lambda \nabla T) + \sum_p \mathbf{u}_p \cdot \nabla p_p = 0$$

Outline

* Part I: Coupling of single phase reservoir and wellbore models

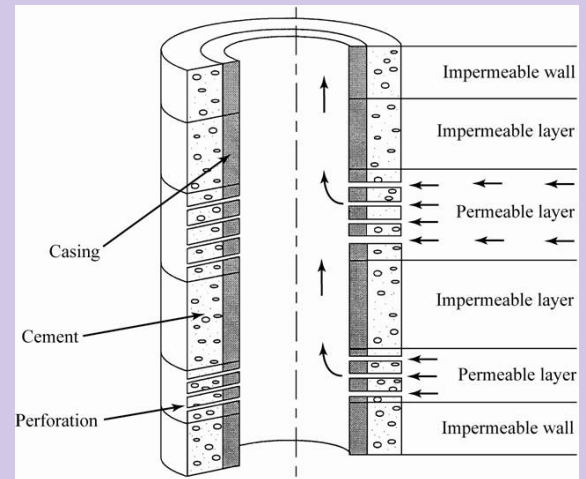
- *Coupling of the two models/Transmission conditions*
- *Analysis of the continuous global problem*
- *Finite element discretization*
- *Numerical results*

* Part II: Multi-component multi-phase model in reservoir

- *Physical modeling*
- *Finite volume discretization*
- *Numerical scheme*
- *Numerical results*

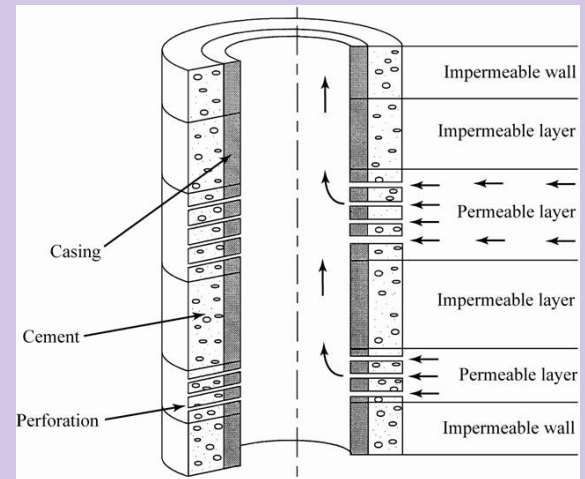
Coupling of single phase reservoir and wellbore models

- Porous media Ω divided into N geological layers Ω_i
- Layers characterized by their own physical and thermodynamic properties
- Layers saturated with a formation water and a monophasic compressible fluid
- Only the monophasic fluid is mobile
- 2D axisymmetric hypothesis



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• Couple Denel's reservoir and wellbore models

B. Denel, Simulation numérique et couplage de modèles thermomécaniques puits-milieux poreux, Thèse de doctorat, Université de Pau, 2004

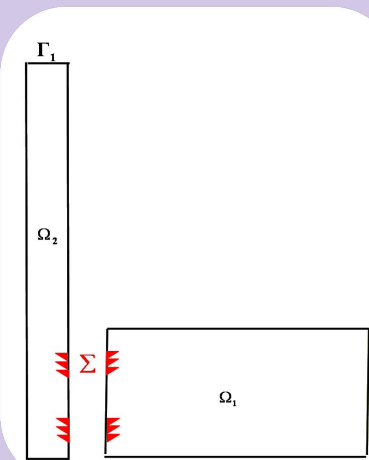
- Semi-discretized reservoir model (*Darcy-Forchheimer*):

$$\left\{ \begin{array}{l} \frac{1}{r} \underline{\mathbf{M}} \mathbf{G}_1 + \nabla p_1 = -\rho_1^{n-1} \mathbf{g} \\ \frac{1}{r\lambda} \mathbf{q}_1 - \nabla T_1 = 0 \\ r \frac{a}{\Delta t} p_1 - r \frac{b}{\Delta t} T_1 + \operatorname{div} \mathbf{G}_1 = r \frac{a}{\Delta t} p_1^{n-1} - r \frac{b}{\Delta t} T_1^{n-1} \\ r \frac{d}{\Delta t} T_1 + \kappa \mathbf{G}_1^{n-1} \cdot \nabla T_1 - r \frac{f}{\Delta t} p_1 + l \mathbf{G}_1^{n-1} \cdot \nabla p_1 - \operatorname{div} \mathbf{q}_1 = r \frac{d}{\Delta t} T_1^{n-1} - r \frac{f}{\Delta t} p_1^{n-1} \end{array} \right.$$

- Semi-discretized wellbore model (*Compressible Navier-Stokes*):

$$\left\{ \begin{array}{l} \operatorname{div}(r \mathbf{G}_2) = -r \frac{\rho_2 - \rho_2^n}{\Delta t} \\ \operatorname{div}(r \mathbf{u}_2) = \frac{1}{\rho} (\operatorname{div}(r \mathbf{G}_2) - r \mathbf{G}_2 \cdot \nabla \rho_2) \\ r \rho \frac{\mathbf{u}_2}{\Delta t} + r \mathbf{G}_2 \cdot \nabla \mathbf{u}_2 + r \nabla p_2 - \operatorname{div}(r \underline{\boldsymbol{\tau}}) + \tau_{\theta\theta} \mathbf{e}_r + r F |\mathbf{G}_2| \mathbf{u}_2 = r \rho_2 \mathbf{g} + r \rho_2 \frac{\mathbf{u}_2^n}{\Delta t} \\ \frac{1}{\lambda} \mathbf{q}_2 - \nabla T_2 = 0 \\ r c_v (\rho \frac{T_2}{\Delta t} + \mathbf{G}_2 \cdot \nabla T_2) - \operatorname{div}(r \mathbf{q}_2) = r \rho c_v \frac{T_2^n}{\Delta t} - \frac{1}{2} (\rho \frac{|\mathbf{u}_2|^2 - |\mathbf{u}_2^n|^2}{\Delta t} + \mathbf{G}_2 \cdot \nabla (|\mathbf{u}_2|^2)) - \operatorname{div}(r \rho_2 \mathbf{u}_2) + \operatorname{div}(r \tau_2 \mathbf{u}_2) + r g \mathbf{G}_2 \end{array} \right.$$

$\mathbf{G} = r \rho \mathbf{u}$: mass flux
 $\mathbf{q} = r \lambda \nabla T$: heat flux



Mathematical difficulties related to the coupling

- **A multiscale problem (stiff coupling):**
 - 1 2D axisymmetric reservoir model
 - 2 1.5D wellbore model

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Flow has a **privileged direction**

⇒ Derive a 1.5D model

- Explicit dependency on r :

$$\mathbf{u} = \frac{r}{R} \mathbf{u}_1(z) + \frac{R-r}{R} \mathbf{u}_0(z)$$

$$\mathbf{G} = \begin{pmatrix} \frac{r}{R} \mathbf{G}_1(z) \\ \mathbf{G}_2(z) \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} \frac{r}{R} \mathbf{q}_1(z) \\ \mathbf{q}_2(z) \end{pmatrix}$$

$$\rho = \rho(z) , p = p(z) , T = T(z)$$

- Consider only **one rectangular mesh** in the radial direction

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- **Fixed point method with respect to ρ**

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Evaluation of the specific flux

$$\text{Use } \operatorname{div}(r\mathbf{G}) = -r \frac{r\rho}{\partial t} \text{ and solve } \begin{cases} \text{Find } \mathbf{G} \in \mathbb{W}^* \\ \int_{\Omega_2} \operatorname{div}(r\mathbf{G})\chi dx = - \int_{\Omega_2} r \frac{\rho - \rho^n}{\Delta t} \chi dx, \quad \forall \chi \in M \end{cases}$$

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Evaluation of (\mathbf{u}, p)

$$\text{Use } \rho \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{G} \cdot \nabla \mathbf{u} \text{ and solve } \begin{cases} \text{Find } \mathbf{u} \in \mathbb{V}^*, p \in M \\ m(\mathbf{u}, \mathbf{v}) + n(p, \mathbf{v}) = l_1(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{V}^0 \\ n(q, \mathbf{u}) = l_2(q) \quad \forall q \in M \end{cases}$$

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Evaluation of (\mathbf{q}, T)

$$\text{Use } \rho \mathbf{u} \cdot \nabla T = \mathbf{G} \cdot \nabla T \text{ and solve } \left\{ \begin{array}{l} \text{Find } \mathbf{q} \in \mathbb{H}, T \in M \\ a(\mathbf{q}, \mathbf{w}) + b(T, \mathbf{w}) = f_1(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbb{H} \\ b(S, \mathbf{q}) - c(T, S) = f_2(S) \quad \forall S \in M \end{array} \right.$$

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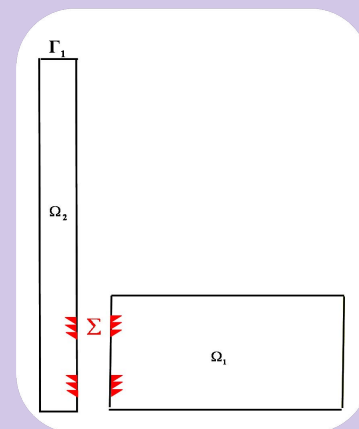
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- Density is not constant in the two domains
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Transmission conditions at the interface

- $[\mathbf{G} \cdot \mathbf{n}] = 0 \Rightarrow \mathbf{G}_1 \cdot \mathbf{n} = R\mathbf{G}_2 \cdot \mathbf{n}$
- $[\sigma \mathbf{n} \cdot \mathbf{n}] = 0 \Rightarrow -p_1 + \tau_{rr} = -p_2$
- $u_2 \cdot t = 0$
 or $u_2 \cdot t = -\frac{\sqrt{k}}{\alpha} \underline{\sigma}_2 \mathbf{n} \cdot t$ **Beavers-Joseph-Saffman**
- $\mathbf{G}_1 \cdot \mathbf{n} = \rho u_2 \cdot \mathbf{n}$
- $[\mathbf{q} \cdot \mathbf{n}] = 0 \Rightarrow \mathbf{q}_1 \cdot \mathbf{n} = R\mathbf{q}_2 \cdot \mathbf{n}$
- $[T] = 0 \Rightarrow T_1 = T_2$



Variational formulation in the reservoir

- We denote : $x_1 = (\mathbf{G}_1, \mathbf{q}_1, p_1, T_1)$
- Functional framework :

$$\begin{aligned} X_1 &= \mathbb{H}(\text{div}, \Omega_1) \times \mathbb{H}(\text{div}, \Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1) \\ X_1^0 &= \{x_1 \in X_1 ; \mathbf{G}_1 \cdot \mathbf{n} = 0 \text{ on } \Upsilon_{\mathbf{G}}, \mathbf{q}_1 \cdot \mathbf{n} = 0 \text{ on } \Upsilon_{\mathbf{q}}\} \\ X_1^* &= \{x_1 \in X_1 ; \mathbf{G}_1 \cdot \mathbf{n} = G^* \text{ on } \Upsilon_{\mathbf{G}}, \mathbf{q}_1 \cdot \mathbf{n} = q^* \text{ on } \Upsilon_{\mathbf{q}}\} \end{aligned}$$

- Variational formulation :

$$\left\{ \begin{array}{l} \text{Find } x_1 \in X_1^* \\ \mathcal{A}_1(x_1, x'_1) = \mathcal{F}_1(x'_1) \quad \forall x'_1 \in X_1^0 \end{array} \right.$$

Weak formulation in the wellbore

- We denote : $x_2 = (\mathbf{G}_2, \mathbf{u}_2, p_2, \mathbf{q}_2, T_2)$
- Functional framework :

$$\begin{aligned} X_2 &= \mathbb{W} \times \mathbb{V} \times M \times \mathbb{H} \times M \\ X_2^* &= \mathbb{W}^* \times \mathbb{V}^* \times M \times \mathbb{H} \times M \\ Y_2 &= M \times \mathbb{V}^0 \times M \times \mathbb{H} \times M \end{aligned}$$

- Weak formulation :

$$\left\{ \begin{array}{l} \text{Find } x_2 \in X_2^* \\ \mathcal{A}_2(x_2, x'_2) = \mathcal{F}_2(x'_2) \quad \forall x'_2 \in Y_2 \end{array} \right.$$

- An integration by part in the reservoir yields the terms :

$$\int_{\Sigma} p_1 \mathbf{G}'_1 \cdot \mathbf{n} \, d\sigma - \int_{\Sigma} T_1 \mathbf{q}'_1 \cdot \mathbf{n} \, d\sigma$$

- An integration by part in the wellbore yields the terms :

$$\int_{\Sigma} R(p_2 - \tau_2 \mathbf{n} \cdot \mathbf{n}) \mathbf{u}'_2 \cdot \mathbf{n} \, d\sigma - \int_{\Sigma} RT_2 \mathbf{q}'_2 \cdot \mathbf{n} \, d\sigma$$

- Dualization by Lagrange multipliers $\Lambda = (\theta, \mu)$:

$$\theta = p_1 = p_2 - \tau_2 \mathbf{n} \cdot \mathbf{n} \quad \mu = T_1 = T_2$$

- Multipliers' spaces:

$$\mathbb{L} = L^2(\Sigma) \times L^2(\Sigma) \quad \mathbb{K} = L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Sigma)$$

- Bilinear forms:

$$\mathcal{I}(\Lambda, x') = \int_{\Sigma} (\mathbf{G}'_1 \cdot \mathbf{n} - R\mathbf{u}'_2 \cdot \mathbf{n}) \theta \, d\sigma - \int_{\Sigma} (\mathbf{q}'_1 \cdot \mathbf{n} - R\mathbf{q}'_2 \cdot \mathbf{n}) \mu \, d\sigma$$

$$\mathcal{J}(\Lambda', x) = \int_{\Sigma} (\mathbf{G}_1 \cdot \mathbf{n} - R\rho_2 \mathbf{u}_2 \cdot \mathbf{n}) \theta' \, d\sigma + \int_{\Sigma} (\mathbf{G}_1 \cdot \mathbf{n} - R\mathbf{G}_2 \cdot \mathbf{n}) \zeta' \, d\sigma - \int_{\Sigma} (\mathbf{q}_1 \cdot \mathbf{n} - R\mathbf{q}_2 \cdot \mathbf{n}) \mu' \, d\sigma$$

- An integration by part in the reservoir yields the terms :

$$\int_{\Sigma} p_1 \mathbf{G}'_1 \cdot \mathbf{n} \, d\sigma - \int_{\Sigma} T_1 \mathbf{q}'_1 \cdot \mathbf{n} \, d\sigma$$

- An integration by part in the wellbore yields the terms :

$$\int_{\Sigma} R(p_2 - \tau_2 \mathbf{n} \cdot \mathbf{n}) \mathbf{u}'_2 \cdot \mathbf{n} \, d\sigma - \int_{\Sigma} RT_2 \mathbf{q}'_2 \cdot \mathbf{n} \, d\sigma$$

- Dualization by Lagrange multipliers $\Lambda = (\theta, \mu)$:

$$\theta = p_1 = p_2 - \tau_2 \mathbf{n} \cdot \mathbf{n} \quad \mu = T_1 = T_2$$

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Weak formulation for the coupled problem

- Functional framework :

$$\mathbb{X} = \{x = (x_1, x_2) \in X_1 \times X_2; \mathbf{G}_1 \cdot \mathbf{n}, \mathbf{q}_1 \cdot \mathbf{n} \in L^2(\Sigma)\}$$

$$\mathbb{Y} = \{x' = (x'_1, x'_2) \in X_1 \times Y_{2\Sigma}; \mathbf{G}_1 \cdot \mathbf{n}, \mathbf{q}_1 \cdot \mathbf{n} \in L^2(\Sigma)\}$$

$$\mathbb{X}^* = \{(x_1, x_2) \in \mathbb{X}; \mathbf{G}_1 \cdot \mathbf{n} = G^* \text{ on } \Upsilon_G \setminus \Sigma, \mathbf{q}_1 \cdot \mathbf{n} = q^* \text{ on } \Upsilon_q \setminus \Sigma, \mathbf{u}_2 \cdot \mathbf{n} = Q \text{ on } \Gamma_s\}$$

$$\mathbb{Y}^0 = \{x' \in \mathbb{Y}; \mathbf{G}'_1 \cdot \mathbf{n} = 0 \text{ on } \Upsilon_G \setminus \Sigma, \mathbf{q}'_1 \cdot \mathbf{n} = 0 \text{ on } \Upsilon_q \setminus \Sigma, \mathbf{u}'_2 \cdot \mathbf{n} = 0 \text{ on } \Gamma_s\}$$

- Weak formulation :

$$\left\{ \begin{array}{l} \text{Find } x \in \mathbb{X}^*, \Lambda \in \mathbb{L} \\ \mathcal{A}(x, x') + I(\Lambda, x') = \mathcal{F}(x') \quad \forall x' \in \mathbb{Y}^0 \\ \mathcal{J}(\Lambda', x) = 0 \quad \forall \Lambda' \in \mathbb{K} \end{array} \right.$$

Where :

$$\mathcal{A}(x, x') = \mathcal{A}_1(x_1, x'_1) + \mathcal{A}_2(x_2, x'_2) \quad \forall x \in \mathbb{X}, \forall x' \in \mathbb{Y}$$

$$\mathcal{F}(x') = \mathcal{F}_1(x'_1) + \mathcal{F}_2(x'_2) \quad \forall x' \in \mathbb{Y}$$

Analysis of the continuous global problem

Babuška's theorem :

- \mathcal{I} and \mathcal{J} satisfy an inf-sup condition
- The coupled problem reduces to :

$$\begin{cases} \text{Find } x \in \mathbb{J}^* \\ \mathcal{A}(x, x') = \mathcal{F}(x') \end{cases} \quad \forall x' \in \mathbb{I}$$

Where :

$$\mathbb{J}^* = \{x \in \mathbb{X}^*; \mathcal{J}(\Lambda', x) = 0 \quad \forall \Lambda' \in \mathbb{K}\}$$

$$\mathbb{I} = \{x' \in \mathbb{Y}^0; \mathcal{I}(\Lambda, x') = 0 \quad \forall \Lambda \in \mathbb{L}\}$$

Theorem : $\forall x \in \mathbb{J}^0 \setminus \{0\}, \sup_{x' \in \mathbb{I}} \frac{\mathcal{A}(x, x')}{\|x'\|_{\mathbb{Y}}} > 0$

→ UNIQUENESS OF THE SOLUTION

- **Existence : Galerkin's method** (via finite element spaces)

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↔ **UNIQUENESS OF THE SOLUTION**

- **Existence : Galerkin's method** (via finite element spaces)

Proof of the uniqueness

• By putting : $\mathbf{U} = (\mathbf{G}_1, \mathbf{q}_1, \mathbf{G}_2, \mathbf{u}_2, \mathbf{q}_2)$ $\mathbf{U}' = (\mathbf{G}'_1, \mathbf{q}'_1, \chi, \mathbf{u}'_2, \mathbf{q}'_2)$ $s = (p_1, T_1, p_2, T_2)$

• **Non-standard Mixed formulation :**

$$\left\{ \begin{array}{ll} \text{Find } (\mathbf{U}, s) \in \mathbf{U}^* \times \mathcal{S} & \\ \mathbf{A}(\mathbf{U}, \mathbf{U}') + \mathbf{B}(s, \mathbf{U}') = \mathbf{F}_1(\mathbf{U}') & \forall \mathbf{U}' \in \mathbf{T}^0 \\ -\mathbf{B}(s', \mathbf{U}) + \mathbf{C}(s, s') = \mathbf{F}_2(s') & \forall s' \in \mathcal{S} \end{array} \right.$$

Where :

$$\mathbf{A}(\mathbf{U}, \mathbf{U}') = \int_{\Omega_1} \frac{1}{r} M \mathbf{G}_1 \cdot \mathbf{G}'_1 dx + \int_{\Omega_1} \frac{1}{r \lambda_1} \mathbf{q}_1 \cdot \mathbf{q}'_1 dx + \int_{\Omega_2} \chi \operatorname{div}(r \mathbf{G}_2) dx + \int_{\Omega_2} \frac{r}{\lambda_1} \mathbf{q}_2 \cdot \mathbf{q}'_2 dx + a(\mathbf{u}_2, \mathbf{u}'_2) \text{ non symmetric}$$

$$\mathbf{B}(s, \mathbf{U}') = - \int_{\Omega_1} p_1 \operatorname{div} \mathbf{G}'_1 dx + \int_{\Omega_1} T_1 \operatorname{div} \mathbf{q}'_1 dx - \int_{\Omega_2} p_2 \operatorname{div}(r \mathbf{u}'_2) dx + \int_{\Omega_2} T_2 \operatorname{div}(r \mathbf{q}'_2)$$

$$\mathbf{C}(s, s') = \int_{\Omega_1} r \frac{a}{\Delta t} p_1 p'_1 dx - \int_{\Omega_1} r \frac{b}{\Delta t} T_1 p'_1 dx + \int_{\Omega_1} r \frac{d}{\Delta t} T_1 T'_1 dx - \int_{\Omega_1} r \frac{f}{\Delta t} p_1 T'_1 dx + \int_{\Omega_2} r \frac{c_v \rho_2}{\Delta t} T_2 T'_2 dx \text{ non symmetric}$$

The homogeneous problem admits only the trivial solution ??

- Let (\mathbf{U}, s) a solution of :
$$\begin{cases} \mathbf{A}(\mathbf{U}, \mathbf{U}') + \mathbf{B}(s, \mathbf{U}') = 0 & \forall \mathbf{U}' \in \mathbb{T}^0 \\ -\mathbf{B}(s', \mathbf{U}) + \mathbf{C}(s, s') = 0 & \forall s' \in \mathbb{S} \end{cases}$$

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* *There exists $\mathcal{R} : \mathbb{U}^0 \rightarrow \mathbb{T}^0$ linear , continuous and satisfying :*

$$\mathbf{A}(\mathbf{U}, \mathcal{R}\mathbf{U}) > 0, \quad \mathbf{U} - \mathbf{U}' \in \text{Ker}\mathbf{B}$$

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• Let (\mathbf{U}, s) a solution of :

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• $\mathbf{U} = (\mathbf{G}_1, \mathbf{q}_1, \mathbf{G}_2, \mathbf{u}_2, \mathbf{q}_2) \in \mathbb{U}^0$ define $\mathcal{R}\mathbf{U} = \mathbf{U}' = (\mathbf{G}'_1, \mathbf{q}_1, \chi, \mathbf{u}_2, \mathbf{q}_2)$ satisfying :

$$\mathbf{G}'_1 \cdot \mathbf{n} = \frac{1}{\rho_2} \mathbf{G}_1 \cdot \mathbf{n} \text{ on } \Sigma, \quad \text{div} \mathbf{G}'_1 = \text{div} \mathbf{G}_1 \text{ in } \Omega_1, \quad \|\mathbf{G}'_1\|_{0, \Omega_1} + \|\chi\|_{0, \Omega_2} \leq c \|\mathbf{U}\|.$$

$$\hookrightarrow \begin{cases} \mathbf{U}' \in \mathbb{T}^0, \|\mathbf{U}'\| \leq c \|\mathbf{U}\|, \mathbf{B}(s, \mathbf{U}) = \mathbf{B}(s, \mathbf{U}') \\ \mathbf{A}(\mathbf{U}, \mathbf{U}') \geq c \left(\|\mathbf{q}_1\|_{0, \Omega_1}^2 + \|\mathbf{q}_2\|_{0, \Omega_2}^2 \right) + m(\mathbf{u}_2, \mathbf{u}_2) + \int_{\Omega_1} \frac{1}{r} \underline{\mathbf{M}} \mathbf{G}_1 \cdot \mathbf{G}'_1 dx + \int_{\Omega_2} \chi \text{div}(r \mathbf{G}_2) dx \end{cases}$$

• Bound $m(\mathbf{u}_2, \mathbf{u}_2)$ by means of Young's inequality

The homogeneous problem admits only the trivial solution ??

• Let (\mathbf{U}, s) a solution of :

$$\begin{cases} \mathbf{A}(\mathbf{U}, \mathbf{U}') + \mathbf{B}(s, \mathbf{U}') = 0 & \forall \mathbf{U}' \in \mathbb{T}^0 \\ -\mathbf{B}(s', \mathbf{U}) + \mathbf{C}(s, s') = 0 & \forall s' \in \mathbb{S} \end{cases}$$

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- $\mathbf{C}(s, s') = \int_{\Omega_1} r \frac{a}{\Delta t} p_1 p_1' dx - \int_{\Omega_1} r \frac{b}{\Delta t} T_1 p_1' dx + \int_{\Omega_1} r \frac{d}{\Delta t} T_1 T_1' dx - \int_{\Omega_1} r \frac{f}{\Delta t} p_1 T_1' dx + \int_{\Omega_2} r \frac{c_0 \rho_2}{\Delta t} T_2 T_2' dx$

- If $4ad - (b + f)^2 \geq c$ a.e. in Ω_1 , we have:

$$\int_{\Omega_1} r \frac{a}{\Delta t} p_1 p_1 dx - \int_{\Omega_1} r \frac{b}{\Delta t} T_1 p_1 dx + \int_{\Omega_1} r \frac{d}{\Delta t} T_1 T_1 dx - \int_{\Omega_1} r \frac{f}{\Delta t} p_1 T_1 dx \geq \frac{c}{\Delta t} (\|p_1\|_{0,\Omega_1}^2 + \|T_1\|_{0,\Omega_1}^2)$$

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then $\mathbf{U} = \mathbf{0}$ and $(p_1, T_1, T_2) = \mathbf{0}$

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* There exists $\beta > 0$ such that : $\beta_2 \|s\| \leq \sup_{\mathbf{U}' \in \mathbb{T}^0} \frac{\mathbf{B}(s, \mathbf{U}')}{\|\mathbf{U}'\|}$

• **Fortin's trick :**

$$s = (p_1, T_1, p_2, T_2) \in \mathbb{S} \mapsto \mathbf{U}' = (\mathbf{G}'_1, \mathbf{q}_1, \chi, \mathbf{u}_2, \mathbf{q}_2) \in \mathbb{T}^0 \text{ satisfying } \begin{cases} \mathbf{B}(s, \mathbf{U}') \geq c_1 \|s\|^2 \\ \|\mathbf{U}'\| \leq c_2 \|s\| \end{cases}$$

The homogeneous problem admits only the trivial solution ??

• Let (\mathbf{U}, s) a solution of :

$$\begin{cases} \mathbf{A}(\mathbf{U}, \mathbf{U}') + \mathbf{B}(s, \mathbf{U}') = 0 & \forall \mathbf{U}' \in \mathbb{T}^0 \\ -\mathbf{B}(s', \mathbf{U}) + \mathbf{C}(s, s') = 0 & \forall s' \in \mathbb{S} \end{cases}$$

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then $\mathbf{U} = \mathbf{0}$ and $(p_1, T_1, T_2) = \mathbf{0}$

* There exists $\beta > 0$ such that : $\beta_2 \|s\| \leq \sup_{\mathbf{U}' \in \mathbb{T}^0} \frac{\mathbf{B}(s, \mathbf{U}')}{\|\mathbf{U}'\|}$

then $p_2 = 0$

Discrete problem

- Suppose that the two meshes match on the perforations
- Denote by \mathcal{E}_h the set of edges situated on the interface Σ

Finite dimensional spaces

- Conservative variables (specific flux, heat flux) :

$$RT_0 = \left\{ \left(\begin{array}{c} ar + b \\ az + c \end{array} \right) \mid a, b, c \in \mathbb{R} \right\}$$

$$V_h = \{ \mathbf{G} \in H(\text{div}, \Omega); \mathbf{G}_{/K} \in RT_0 \quad \forall K \in \mathcal{T}_h \}$$

- Scalar variables (pressure, temperature, density) :

$$L_h = \{ p \in L^2(\Omega); p_{/K} \in P_0 \quad \forall K \in \mathcal{T}_h \}$$

- Fluid's velocity :

$$X_h = \{ u \in H^1(\Omega); v_{/K} \in Q_1 \quad \forall K \in \mathcal{T}_h \}$$

- Lagrange multipliers on the interface :

$$K_h = \{ \mu \in L^2(\Sigma); \mu \in P_0(e) \quad \forall e \in \mathcal{E}_h \}$$

Discrete formulation

$$\left\{ \begin{array}{l} \text{Find } x_h \in \mathbb{X}_h^*, \Lambda_h \in \mathbb{L}_h \\ \mathcal{A}_h(x_h, x') + \mathcal{I}(\Lambda_h, x') = \mathcal{F}_h(x') \quad \forall x' \in \mathbb{Y}_h \\ \mathcal{J}(\Lambda', x_h) = 0 \quad \forall \Lambda' \in \mathbb{K}_h \end{array} \right.$$

Upwind scheme for convective terms

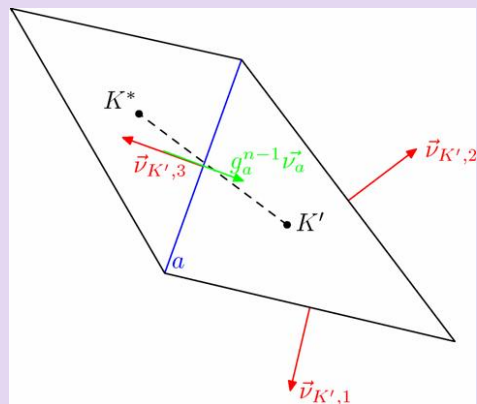
- $\partial K^- = \{e \in \partial K / \mathbf{G}_h^{n-1} \cdot \mathbf{n} < 0\}$

- For any P_0 functions T :

$$\int_K \kappa \mathbf{G}_h^{n-1} \cdot \nabla T \, dx = \sum_{e \in \partial K^-} \kappa (T^* - T_{/K}) \int_e \mathbf{G}_h^{n-1} \cdot \mathbf{n} \, d\sigma \quad \forall T \in L_h$$

- For any Q_1 -continuous functions ϕ and v :

$$\int_K r \mathbf{G}_h^{n-1} \cdot \nabla \phi \, v \, dx = \sum_{e \in \partial K^-} (\phi^* - P_K(\phi)) \int_e r \mathbf{G}_h^{n-1} \cdot \mathbf{n} \, v \, d\sigma \quad \forall K \in \mathcal{T}_h$$



Discrete formulation

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Upwind scheme for convective terms

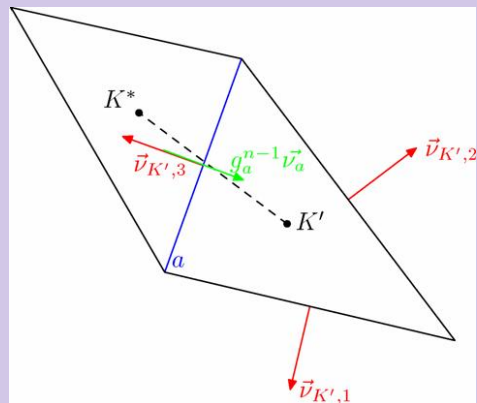
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Theorem : The discrete problem has a unique solution for Δt small enough

Sketch of the proof

- Follow the proof of the continuous case, **with constants independent of h**
- Use Fortin's trick and interpolate continuous functions
- Need an auxiliary result :

For any $(p, \theta) \in M_h \times K_h$, there exists $\mathbf{G} \in V_h$ satisfying :

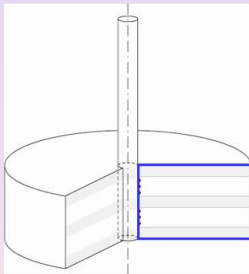
$$\left\{ \begin{array}{l} \mathbf{G} \cdot \mathbf{n} = \theta \text{ on } \Sigma, \quad \mathbf{G} \cdot \mathbf{n} = 0 \text{ on } \Upsilon_{\mathbf{G}} \setminus \Sigma \\ \operatorname{div} \mathbf{G} = p \quad \text{in } \Omega_1 \\ \|\mathbf{G}\|_{H(\operatorname{div}, \Omega_1)} + \|\mathbf{G} \cdot \mathbf{n}\|_{0, \Sigma} \leq c(\|p\|_{0, \Omega_1} + \|\theta\|_{0, \Sigma}) \end{array} \right.$$

Mesh convergence

Test 1: two-layered reservoir

- Two layers with the same properties
- Only the lower one is perforated
- Production of light oil during seven days

$$Q = 500 \text{ m}^3 / j$$

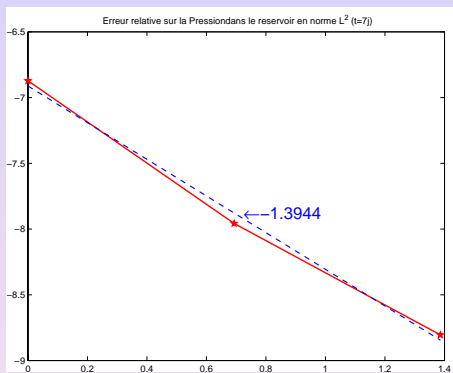


$$k_h = 1000mD \quad k_v = 350mD \quad \phi = 0.28 \quad s_w = 0.15$$

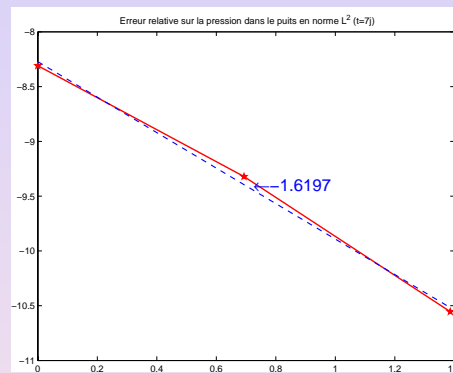
$$p = 400 \text{ bars}$$

$$\vec{q} \cdot \vec{n} = 0$$

Convergence rate for the pressure at t=7 days



(a) order of the error in the reservoir



(b) order of the error in the well

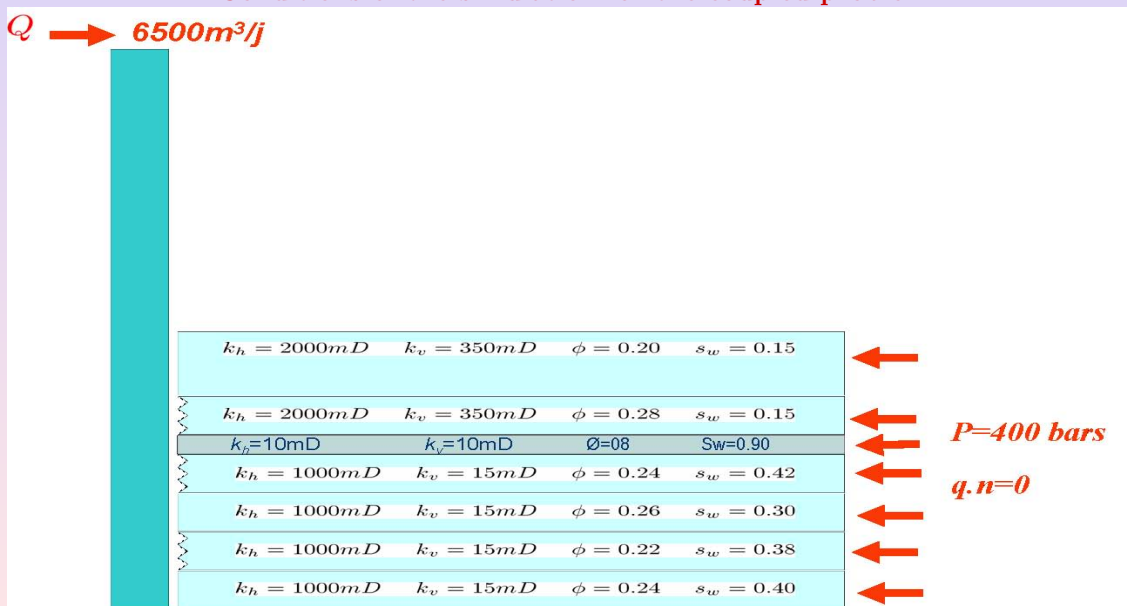
Numerically, $\|p - p_h\|_{0,\Omega} \leq C|h|^\alpha$, with $\alpha \approx 1.39$ in the reservoir and $\alpha \approx 1.61$ in the well

Realistic reservoir

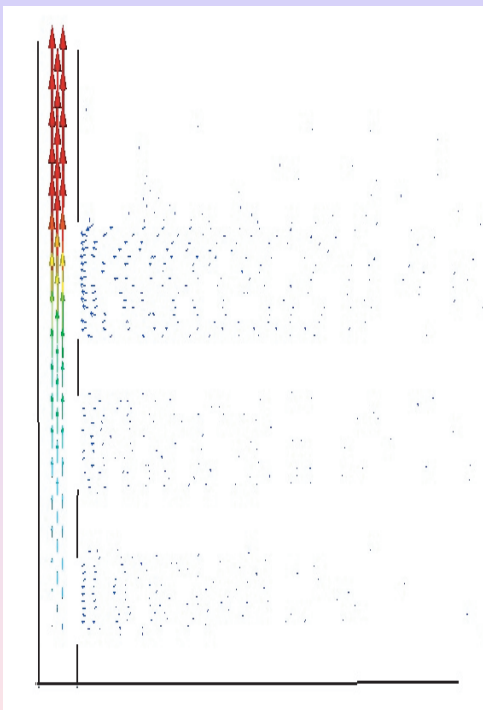
Test 2: realistic reservoir

- Seven layers with different properties
- Production of oil during 28 days

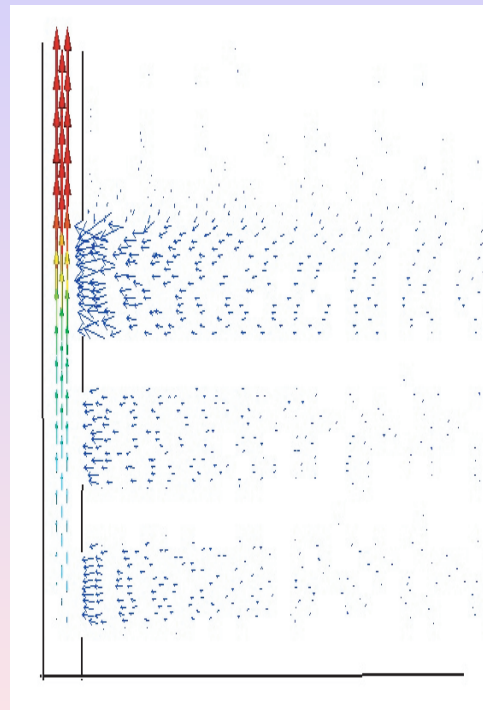
Conditions of the simulation for the coupled problem



Specific flux at the end of the production

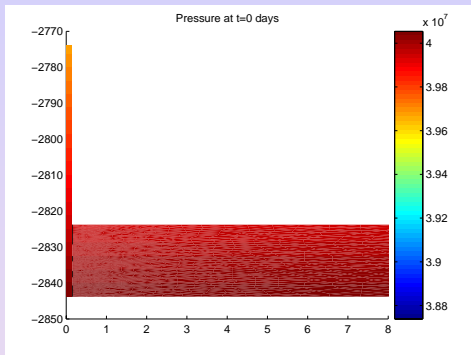


(c) Same scale in the 2 domains

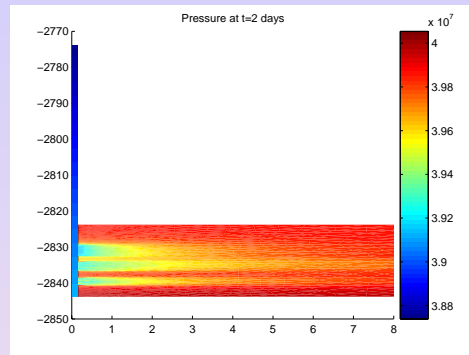


(d) Different scales (ratio equal to 10)

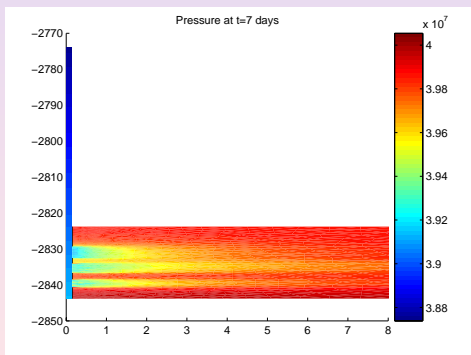
Behaviour of the pressure during one month production



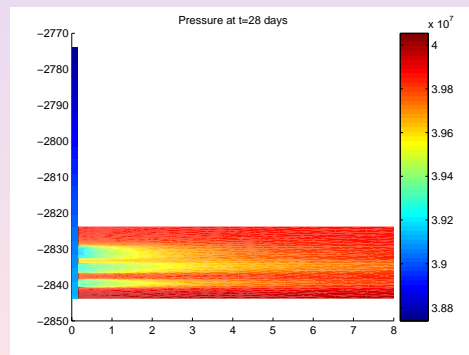
(e) Pressure at t=0 day



(f) Pressure at t=2 days

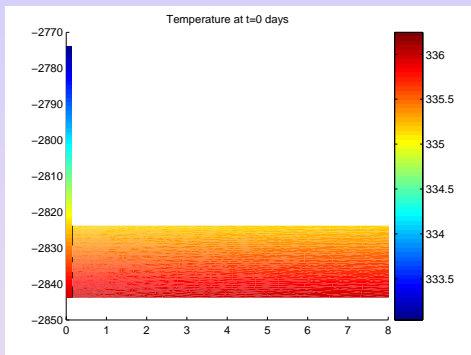


(g) Pressure at t=7 days

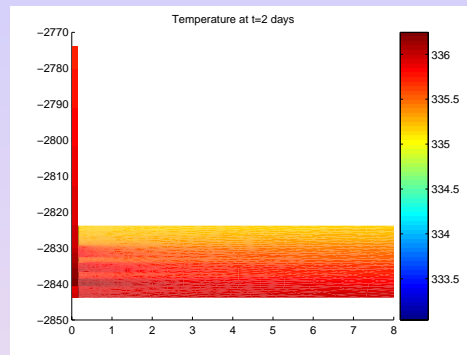


(h) Pressure at t=28 days

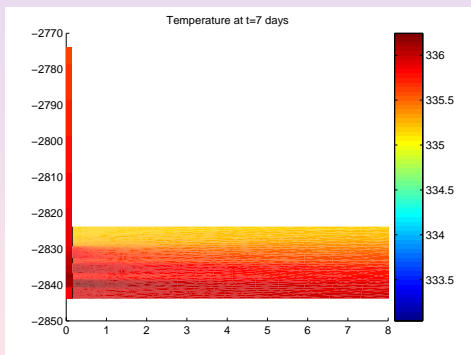
Behaviour of the temperature during one month production



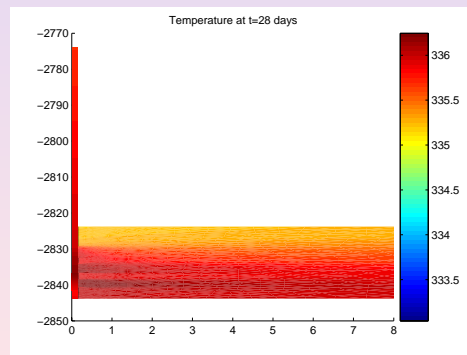
(i) Temperature at t=0 day



(j) Temperature at t=2 days



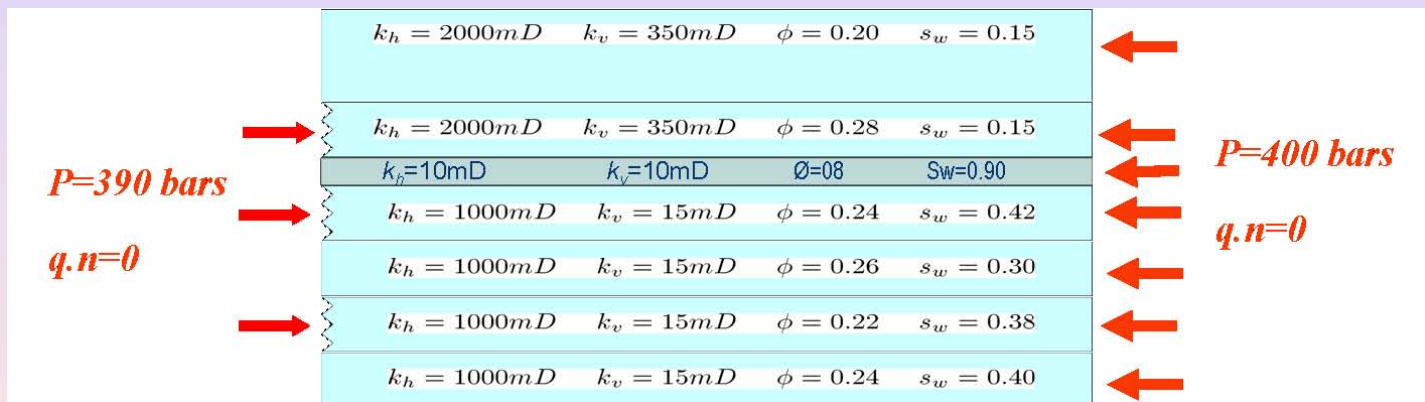
(k) Temperature at t=7 days



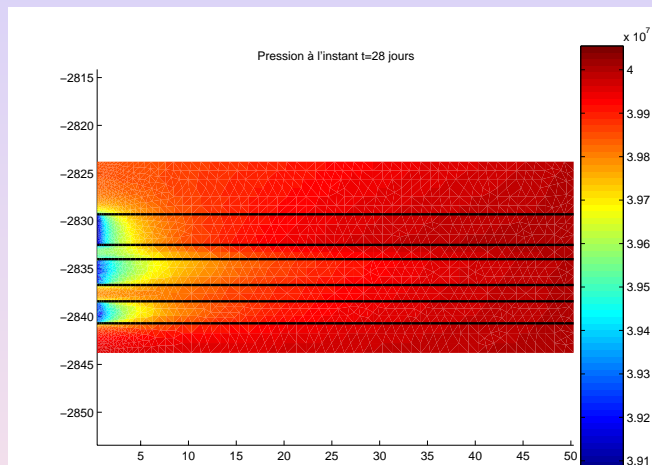
(l) Temperature at t=28 days

Comparison with the separate reservoir and wellbore codes

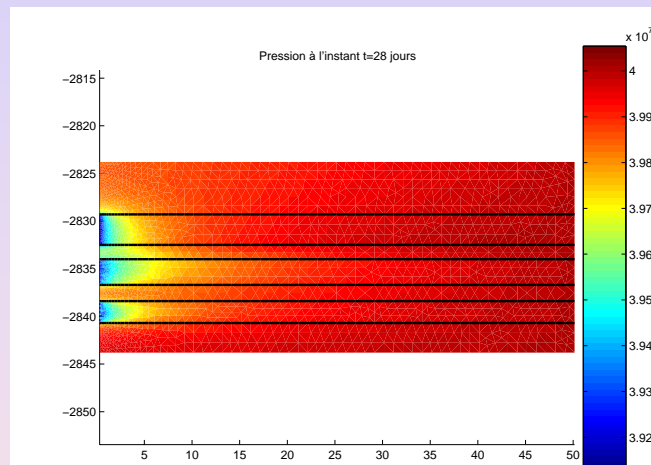
- Previous simulation conditions for the coupled code
- Conditions of the simulation for the sole reservoir model :



Pressure maps in the reservoir at $t = 28$ days

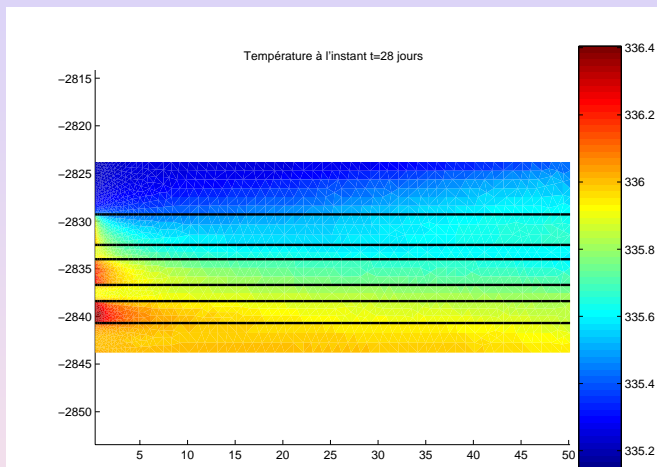


(m) Pressure given by reservoir code

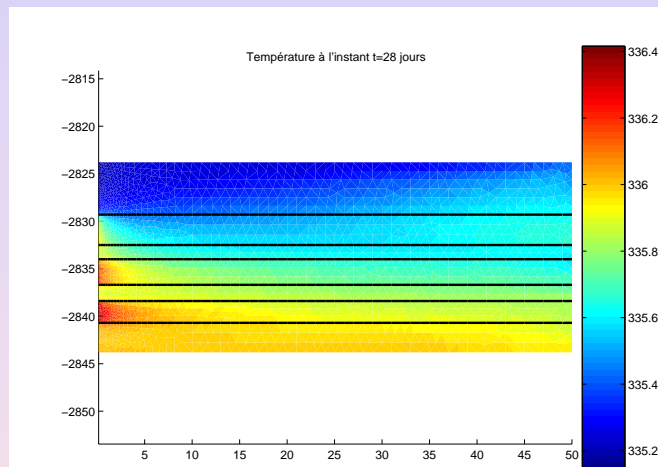


(n) Pressure given by coupled code

Temperature maps in the reservoir at $t = 28$ days

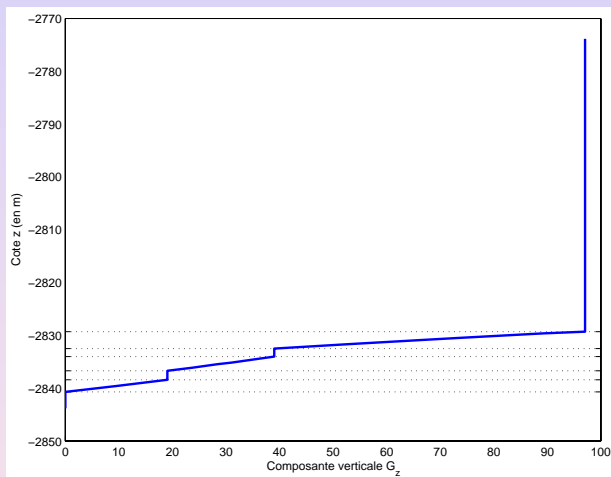


(o) Temperature given by reservoir code

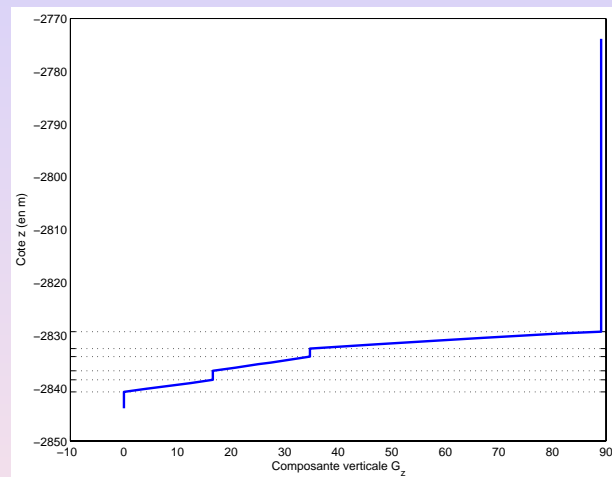


(p) Temperature given by coupled code

Vertical mass fluxes in the wellbore at $t = 28$ days



(q) G_z given by wellbore code



(r) G_z given by coupled code

Multi-component multi-phase model in reservoir

- Three phases (p) : water(w), oil(o) and gas (g)
- n_c components: water, heavy hydrocarbons, light hydrocarbons, methan....
- n_h hydrocarbon components ($n_h = n_c - 1$)

	\bar{w}	n_1	n_2	n_h
w	×					
o		×	×	×	×	×
g		×	×	×	×	×

- 3D / Porous medium Ω with n_W wells

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o		×	×	×	×	×
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Governing equations

- Mass conservation equation for each component c :

$$\mathcal{F}_c = \sum_{p=o,g,w} \left(\frac{\partial}{\partial t} (\phi S_p \rho_p y_{c,p}) + \nabla \cdot (\rho_p \mathbf{u}_p y_{c,p}) \right) = 0$$

\mathbf{u}_p is given by the generalized Darcy law : $\mathbf{u}_p = -k_{rp} \mu_p^{-1} \underline{\mathbf{K}} (\nabla p_p - \rho_p \mathbf{g})$

- Energy equation :

$$\mathcal{F}_T = \frac{\partial}{\partial t} \left[\sum_{p=o,g,w} (\phi S_p \rho_p \mathcal{H}_p - p_p) + (1 - \phi) \rho_s \mathcal{H}_s \right] + \sum_{p=o,w,g} \nabla \cdot (\phi S_p \rho_p \mathcal{H}_p \mathbf{u}_p) - \nabla \cdot (\lambda \nabla T) + \sum_{p=o,g,w} \mathbf{u}_p \cdot \nabla p_p = 0$$

- Capillary pressure constraints :

$$p_{c,ow} = p_o - p_w \quad (\text{oil-water capillary pressure})$$

$$p_{c,go} = p_g - p_o \quad (\text{gas-oil capillary pressure})$$

Capillary pressures are measured in laboratories

- Saturation constraint : $\sum_{p=1}^{n_p} S_p = 1$
- Component mole fraction constraints : $\sum_{c=1}^{n_c} y_{c,p} = 1 \quad \forall p = w, o, g$
- Phase equilibrium relation for each hydrocarbon component c in oil and gas phases:

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* **Number of equations :**

<u>Type</u>	<u>Number</u>
Mass conservation	$n_h + 1$
Energy equation	1
Capillary pressure constraints	2
Saturation constraint	1
Component mole fraction constraints	2
Equilibrium relation equations	n_h
Total	$2n_h + 7$

Primary and secondary variables

- According to Gibb's phase rule, the number of **primary variables** is equal to :

$$(n_c + 2 - n_{phase}) + (n_{phase} - 1) = n_c + 1$$

- Use linear constraint equations to remove two pressures, one saturation and two component mole fractions

$$\rightarrow 2n_h + 2 \text{ non-linear equations and variables left}$$

- Multiple choices for the selection of primary variables and equations leading to different models

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Coats Model

- Primary equations are the $n_c + 1$ mass and energy balance equations ($\mathbf{F}_p = \{\mathcal{F}_c, \mathcal{F}_T\}$)
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- Primary variables X_p are :

$\rho_o, \rho_g, \rho_w, \rho_{cs}, \rho_{gs}, \rho_{ws}$ when both oil and gas phases are present

$\rho_o, \rho_w, \rho_{cs}, \rho_{ws}$ when only oil phase is present

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Flash calculations are used to check the state of hydrocarbon phases in gridblocks

- *Phase disappearance for a gridblock with two hydrocarbon phases*
If either S_o or S_g is negative, the corresponding hydrocarbon phase has disappeared
→ set the negative saturation to zero and reassign saturations and mole fractions
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 &= \sum_{\sigma \in \varepsilon_K} k_{K,\sigma} \frac{v_{\sigma} - v_K}{d_{K,\sigma}} \operatorname{mes}(\sigma) \\
 &= - \sum_{\sigma \in \varepsilon_K} F_{K,\sigma}^v
 \end{aligned}$$

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 \int_K \operatorname{div}(\underline{\mathbf{K}}\nabla v) dx &= \sum_{\sigma \in \varepsilon_K} \int_{\sigma} \underline{\mathbf{K}}\nabla v \cdot \mathbf{n}_{K,\sigma} d\sigma \\
 &= \sum_{\sigma \in \varepsilon_K} k_{K,\sigma} \frac{v_{\sigma} - v_K}{d_{K,\sigma}} \operatorname{mes}(\sigma) \\
 &= - \sum_{\sigma \in \varepsilon_K} F_{K,\sigma}^v = \sum_{\sigma \in \varepsilon_K} \tau_{\sigma} (v_L - v_K)
 \end{aligned}$$

- Conservation of fluxes over the face σ : $F_{K,\sigma}^v = -F_{L,\sigma}^v$
- τ_{σ} is the transmissibility : $\tau_{\sigma} = \operatorname{mes}(\sigma) \frac{k_{K,\sigma} k_{L,\sigma}}{k_{K,\sigma} d_{L,\sigma} + k_{L,\sigma} d_{K,\sigma}}$.

Finite volume discretization

- Equations are integrated over each gridblock V
- For diffusive terms :

$$\int_K \operatorname{div}(\underline{\mathbf{K}}\nabla v) dx = - \sum_{\sigma \in \varepsilon_K} F_{K,\sigma}^v = \sum_{\sigma \in \varepsilon_K} \tau_\sigma (v_L - v_K)$$

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$$\begin{aligned} \int_K \mathbf{u} \cdot \nabla v \, dx &= \int_{\partial K} v \mathbf{u} \cdot \mathbf{n} \, d\sigma - \int_K v_K \operatorname{div}(\mathbf{u}) \, dx \\ &\approx \int_{\partial K^+} v_K \mathbf{u} \cdot \mathbf{n} \, d\sigma + \int_{\partial K^-} v_{ext} \mathbf{u} \cdot \mathbf{n} \, d\sigma - \int_{\partial K} v_K \mathbf{u} \cdot \mathbf{n} \, d\sigma \\ &= \sum_{\sigma \in \partial K^-} (v_{ext} - v_K) \int_{\sigma} \mathbf{u} \cdot \mathbf{n}_{K,\sigma} \, d\sigma \end{aligned}$$

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- For diffusive terms :

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- In our equation, \mathbf{u} is equal to $\underline{\mathbf{K}}\nabla p$:

$$\int_K \underline{\mathbf{K}}\nabla p \cdot \nabla v \, dx = - \sum_{\sigma \in \partial K^-} (v_{ext} - v_K) F_{K,\sigma}^p$$

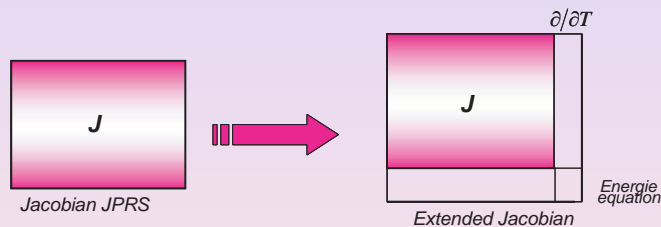
Discretized system

$$\left. \begin{aligned}
 & \forall K \in \mathcal{T}_h, \\
 E_K^c &= V_K \frac{(\phi \sum_p S_p \rho_p y_{c,p})_K - (\phi \sum_p S_p \rho_p y_{c,p})_K^{n-1}}{\Delta t} + Q_{lim,K}^c \\
 & - \sum_p \left(\sum_{\{\sigma\}=\bar{K}\bar{L}} \tau_\sigma (\Lambda_p y_{c,p})_{K/L} (p_{p,L} - p_{p,K} - \gamma(Z_L - Z_K)) \right) = 0, \quad c = 1, \dots, n_c \\
 E_K^T &= V_K \frac{(\sum_p (\phi S_p \rho_p \mathcal{H}_p - p_p) + (1-\phi) \rho_s \mathcal{H}_s)_K - (\sum_p (\phi S_p \rho_p \mathcal{H}_p - p_p) + (1-\phi) \rho_s \mathcal{H}_s)_K^{n-1}}{\Delta t} \\
 & - \sum_p \left(\sum_{\{\sigma\}=\bar{K}\bar{L}} \tau_\sigma (\Lambda_p H_p)_{K/L} (p_{p,L} - p_{p,K} - \gamma(Z_L - Z_K)) \right) + Q_{lim,K}^T \\
 & - \sum_p \sum_{\substack{\{\sigma\}=\bar{K}\bar{L} \\ \sigma \in \partial K^-}} \tau_\sigma (\phi^{-1} S_p^{-1} \mu_p^{-1} k_{rp})_{K/L} (p_{p,L} - p_{p,K} - \gamma(Z_L - Z_K)) (p_{p,L} - p_{p,K}) \\
 & - \sum_{\{\sigma\}=\bar{K}\bar{L}} \tau'_\sigma \lambda_{K/L} (T_L - T_K) + Q_{lim,K}^\lambda = 0
 \end{aligned} \right\}$$

Numerical scheme

- Extend an existing isothermal simulator GPRS (General Purpose Reservoir Simulator), developed at the University of Stanford
- Iterative Newton Raphson method :

$$\mathbf{J}\Delta\mathbf{X} = -\mathbf{F}(\mathbf{X}^n) \text{ with } \mathbf{J} = \frac{\partial\mathbf{F}}{\partial\mathbf{X}}(\mathbf{X}^n)$$



Numerical scheme

- Extend an existing isothermal simulator GPRS (General Purpose Reservoir Simulator), developed at the University of Stanford
- The jacobian matrix of the full system can be written as :

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{F}_p}{\partial \mathbf{X}_p} & \frac{\partial \mathbf{F}_p}{\partial \mathbf{X}_s} \\ \frac{\partial \mathbf{F}_s}{\partial \mathbf{X}_p} & \frac{\partial \mathbf{F}_s}{\partial \mathbf{X}_s} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad -\mathbf{F} = \begin{bmatrix} -\mathbf{F}_p \\ -\mathbf{F}_s \end{bmatrix}$$

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- First, extract primary variables :

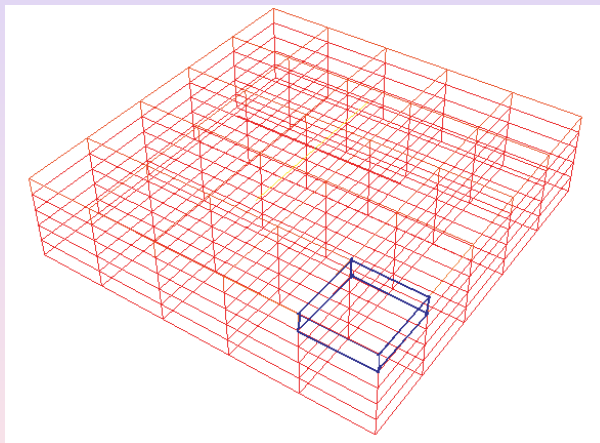
$$(A - B D^{-1} C) \Delta \mathbf{X}_p = (-\mathbf{F}_p + B D^{-1} \mathbf{F}_s)$$

- Next, update secondary ones :

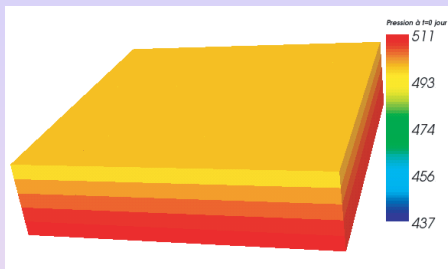
$$\Delta \mathbf{X}_s = -D^{-1} \mathbf{F}_s - (D^{-1} C) \Delta \mathbf{X}_p$$

Comparison with isothermal GPRS

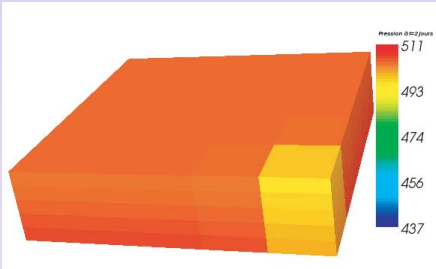
- Reservoir of dimensions $5000\text{ft} \times 5000\text{ft} \times 50\text{ft}$
- Three components: methane CH_4 , butane C_4H_{10} and heptane C_7H_{16}
- Production during 50 days by imposing a bottom hole pressure of 300 psi



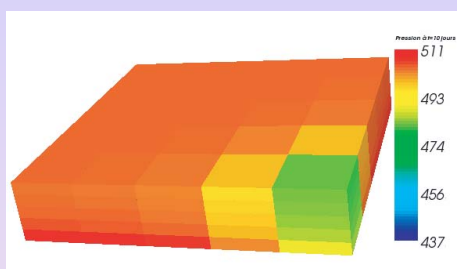
Behaviour of the pressure during 50 days production



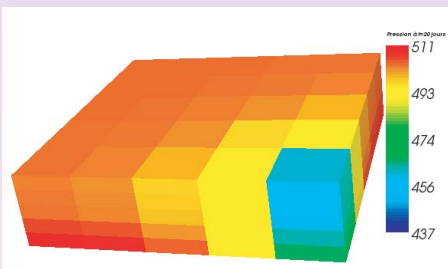
(a) Pressure at t=0 day



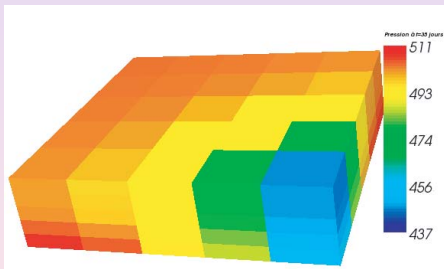
(b) Pressure at t=2 days



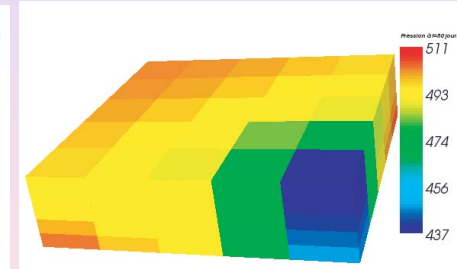
(c) Pressure at t=10 days



(d) Pressure at t=20 days

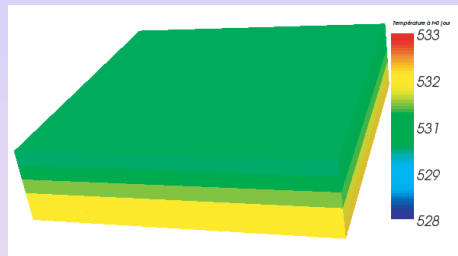


(e) Pressure at t=35 days

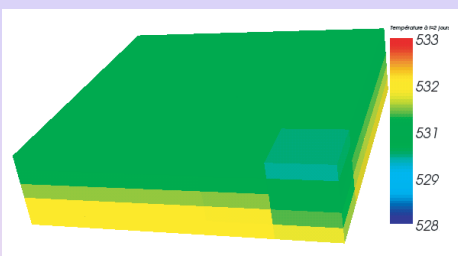


(f) Pressure at t=50 days

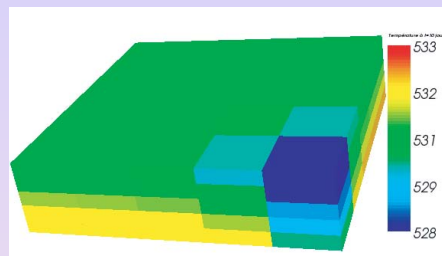
Behaviour of the temperature during 50 days production



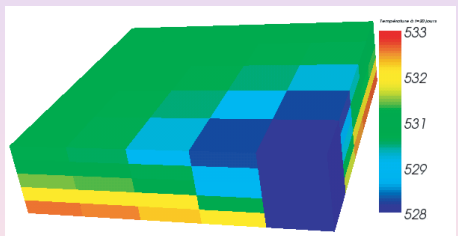
(g) Temperature at t=0 day



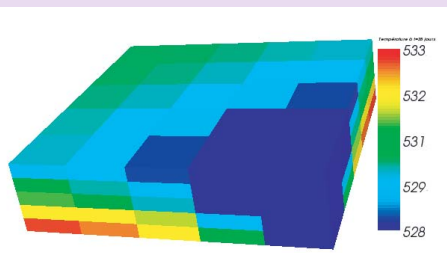
(h) Temperature at t=2 days



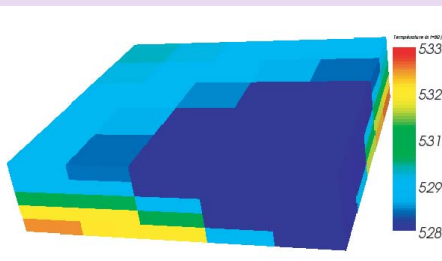
(i) Temperature at t=10 days



(j) Temperature at t=20 days

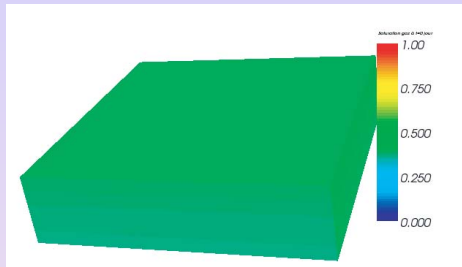


(k) Temperature at t=35 days

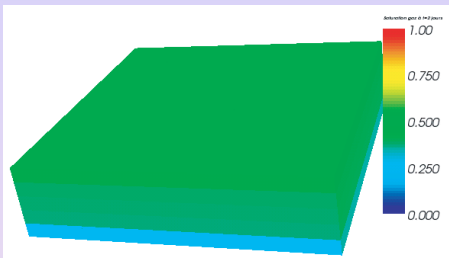


(l) Temperature at t=50 days

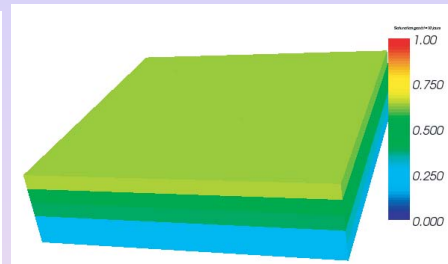
Behaviour of the gas saturation during 50 days production



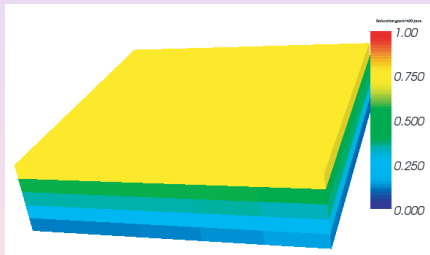
(m) Saturation of gas phase at $t=0$ day



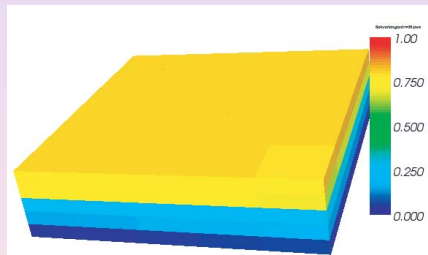
(n) Saturation of gas phase at $t=2$ days



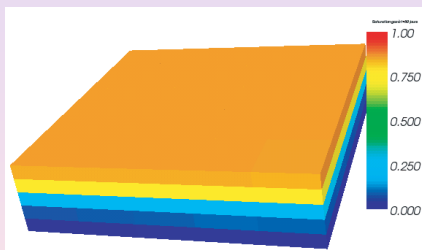
(o) Saturation of gas phase at $t=10$ days



(p) Saturation of gas phase at $t=20$ days

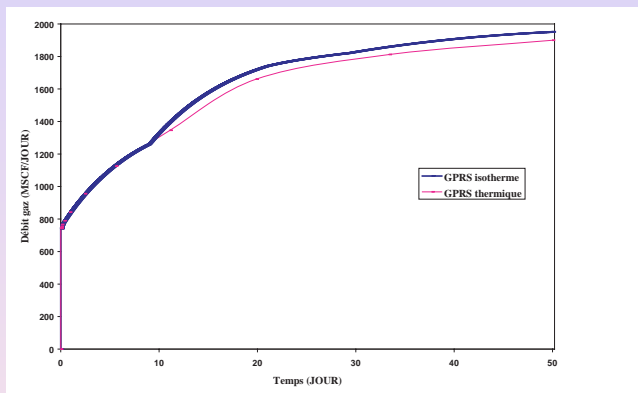


(q) Saturation of gas phase at $t=35$ days

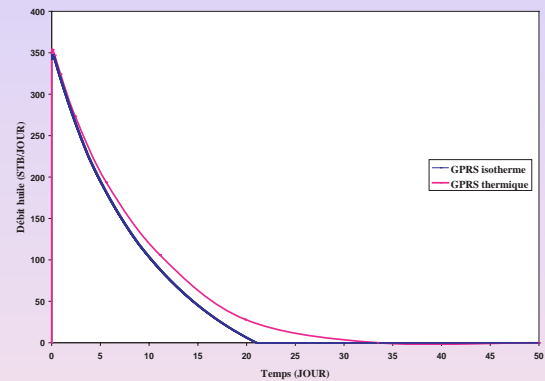


(r) Saturation of gas phase at $t=50$ days

Comparison of production rates

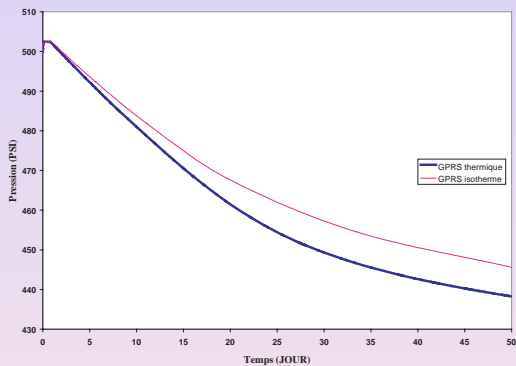


(s) Gas production rate (MSCF/DAY)

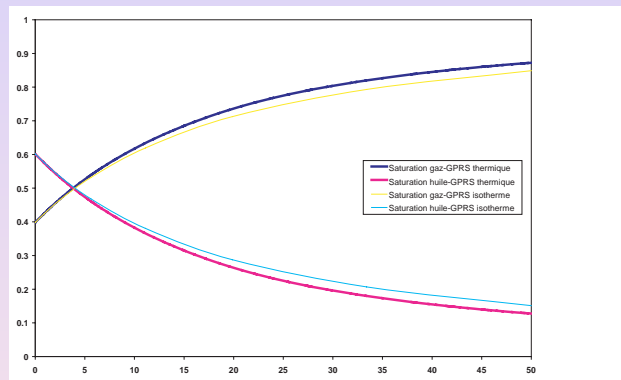


(t) Oil production rate (STB/DAY)

Comparison of pressure and saturations in the well block



(u) Pressure in *psia*

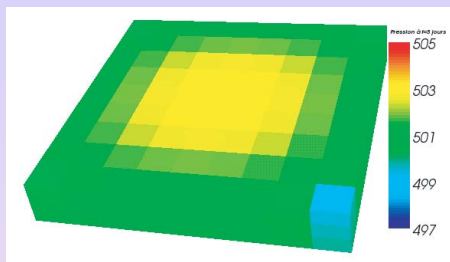


(v) Saturations of oil and gas phases

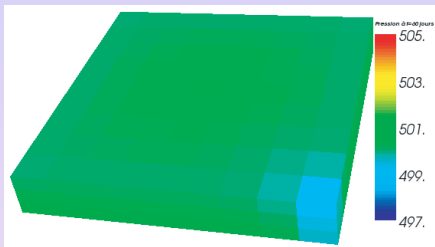
Sensibility vs. boundary conditions

- Reservoir of dimensions $9000\text{ft} \times 9000\text{ft} \times 30\text{ft}$
- Two components: methan CH_4 and butan C_4H_{10}
- Production of gas during 90 days by imposing a constant flowrate

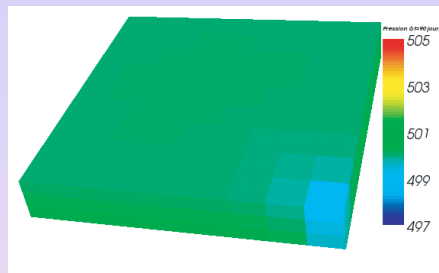
Behaviour of the pressure : constant pressure on the exterior boundary



(a) Pressure at t=5 days

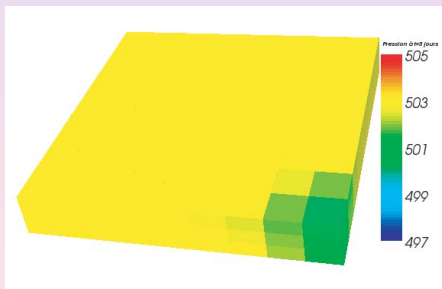


(b) Pressure at t=60 days

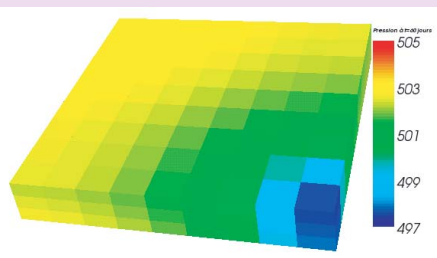


(c) Pressure at t=90 days

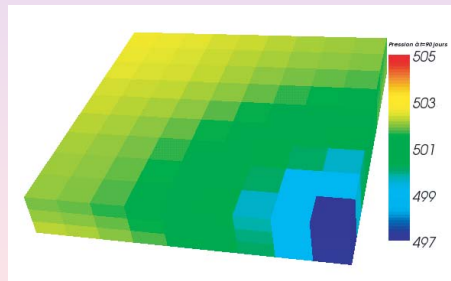
Behaviour of the pressure : no flow on the exterior boundary



(d) Pressure at t=5 days

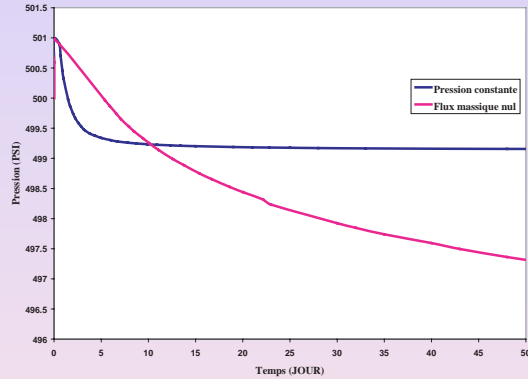


(e) Pressure at t=60 days

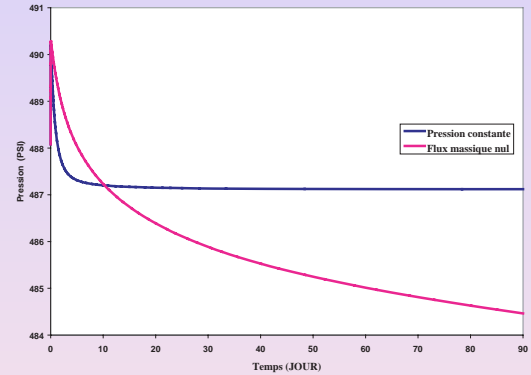


(f) Pressure at t=90 days

Comparison of pressures in the well block and in the well



(g) Pressures in the well block



(h) Pressures in the well

Perspectives

- ① Extend multi-phase simulator in order to treat steam injection
- ② Develop a multi-phase anisothermal wellbore model and couple it with the reservoir
- ③ Solve inverse problems
 - * Determine initial temperature
 - * Determine flow profiles
- ④ Consider deviated wells

Thank you for your attention